EE3304 DIGITAL CONTROL SYSTEMS PART II
TUTORIAL ONE

Q1. Consider the analog control system shown in Fig. 1 , where the first-order plant a/(s+a) is controlled by the analog integral controller K/s. Obtain the velocity error constant for the system and the steady-state error in the response to a unit ramp input.

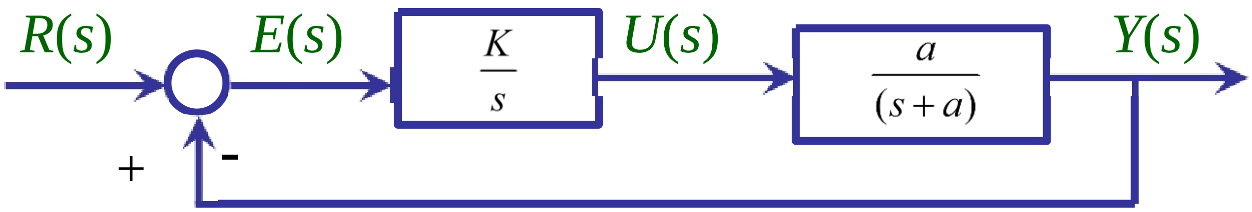


Figure 1 Analog Control System

Next, consider equivalent digital control system. Using equivalent digital controllers obtained by (1) the backward rule and (2) Tustin's rule (i.e., the bilinear rule), show that the velocity error constants for the equivalent digital control systems are the same as that for the analog control system.

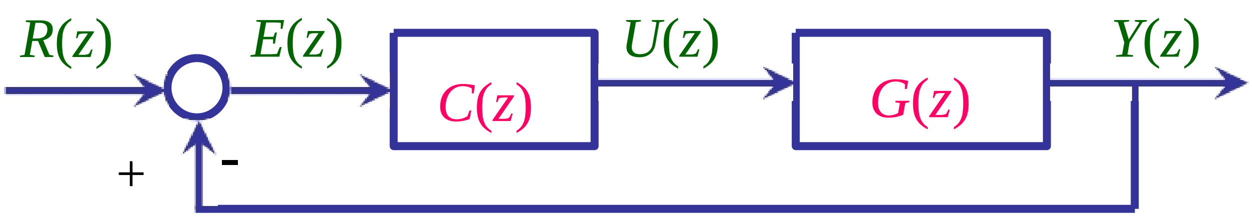


Figure 2 Equivalent Digital Control System

Solution. For the system shown in Fig. 1 the velocity error constant is

$$\beta_v = \lim_{s\to 0}(sC(s)G(s)) = \lim_{s\to 0}(s\frac{K}{s}\frac{a}{s+a}) = K$$

and the steady state error in the unit ramp response is

$$e_{ss} = \frac{1}{\beta_v} = \frac{1}{K}$$

Next, consider the digital control of the given plant. First, we need to get the discrete transfer function of the plant with ZOH:

$$G(z) = (1 - z^{-1})Z\{\frac{G(s)}{s}\} = (1 - z^{-1})Z\{\frac{a}{s(s+a)}\}$$

$$= (1 - z^{-1})Z\{\frac{1}{s} - \frac{1}{s+a}\}$$

$$= (1 - z^{-1})(\frac{z}{z-1} - \frac{z}{z - e^{-aT}})$$

$$= \frac{1 - e^{-aT}}{z - e^{-aT}}$$

You can also obtain the above answer using Table 2.1.
The equivalent digital controller based on the backward rule is

$$C_1(z) = \frac{K}{s}\Big|_{s=\frac{1-z^{-1}}{T}} = \frac{KTz}{z-1}$$

The velocity error constant for this closed loop control system becomes

$$\beta_v = \lim_{z \to 1} \frac{(z-1)}{T} G(z) C_1(z)$$

$$= \lim_{z \to 1} \frac{(z-1)}{T} \frac{1 - e^{-aT}}{z - e^{-aT}} \frac{KTz}{z-1} = K$$

The equivalent digital controller based on the bilinear rule is

$$C_2(z) = \frac{K}{s}\Big|_{s=\frac{2}{T}\frac{z-1}{z+1}} = \frac{KT(z+1)}{2(z-1)}$$

The velocity error constant for this closed loop control system becomes

$$\beta_v = \lim_{z \to 1} \frac{(z-1)}{T} G(z) C_2(z)$$

$$= \lim_{z \to 1} \frac{(z-1)}{T} \frac{1 - e^{-aT}}{z - e^{-aT}} \frac{KT(z+1)}{2(z-1)} = K$$

Thus, the velocity error constants are the same for the analog and digital control systems. The steady state error in the unit ramp response for the digital control system is the same as that for the analog controller,

$$e_{ss} = \frac{1}{\beta_v} = \frac{1}{K}$$

Q2. Consider a unity feedback system with the closed-loop transfer function

$$H(z) = \frac{K(z+c)}{z^2 + az + b}$$

Assume that the open-loop transfer function G(z) involves an integrator, meaning that it has a simple pole at z=1. (The system is of Type 1). Show that the steady-state error in the unit ramp response is given by

$$e_{ss} = \frac{1}{\beta_v} = T\left(\frac{2+a}{1+a+b} - \frac{1}{1+c}\right)$$

where T is the sampling period.

Solution. Since it is unity feedback system, then the closed loop transfer function, H(z), is related to open loop transfer function, G(z), by

$$H(z) = \frac{G(z)}{1+G(z)}$$

from which we get

$$G(z) = \frac{H(z)}{1-H(z)} = \frac{\dfrac{K(z+c)}{z^2+az+b}}{1-\dfrac{K(z+c)}{z^2+az+b}}$$

$$= \frac{K(z+c)}{z^2+az+b-K(z+c)} = \frac{K(z+c)}{z^2+(a-K)z+b-Kc}$$

Since the system is of type 1(there is one integrator), G(z) can also be expressed as

$$G(z) = \frac{K(z+c)}{(z-1)(z+d)} = \frac{K(z+c)}{z^2+(d-1)z-d}$$

Compare this with the previous one, we have

$$d-1=a-K$$
$$-d=b-Kc$$

which results in

$$K=\frac{1+a+b}{1+c}$$
$$d=\frac{(1+a)c-b}{1+c}$$

From the definition of the velocity error constant, we have

$$\beta_v=\lim_{z\to 1}\frac{(z-1)}{T}G(z)$$
$$=\lim_{z\to 1}\frac{(z-1)}{T}\frac{K(z+c)}{(z-1)(z+d)}=\frac{K(1+c)}{T(1+d)}$$
$$=\frac{(1+a+b)(1+c)}{T(1+2c+ac-b)}$$

Hence the steady-state error in the unit ramp response is

$$e_{ss}=\frac{1}{\beta_v}=\frac{T(1+2c+ac-b)}{(1+a+b)(1+c)}$$
$$=T\frac{(1+2c+ac-b)}{(1+a+b)(1+c)}=T\frac{(1+c)(2+a)-(1+a+b)}{(1+a+b)(1+c)}=T(\frac{2+a}{1+a+b}-\frac{1}{1+c})$$

Q3. A process is modeled as $G(z) = \dfrac{1}{z+1}$. Design an appropriate digital controller such that:

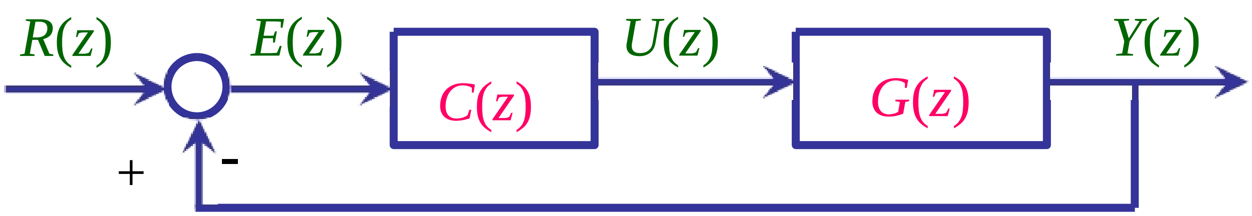
1) place the closed-loop poles within a circle z $\le$ 0.8;

2) achieve a finite steady state error for ramp input;

3) minimize the steady state error for ramp input.

Show that the steady state error is proportional to the sampling period. What kind of closed-loop response will we have?

Solution:



We have the transfer function of the closed loop system:

$$H(z) = \frac{G(z)C(z)}{1+G(z)C(z)}$$

$$E(z) = \frac{R(z)}{1+G(z)C(z)}$$

$$e(\infty) = \lim_{z\to 1}(z-1)E(z) = \lim_{z\to 1}(z-1)\frac{R(z)}{1+G(z)C(z)}$$

From the definition of velocity error constant,

$$\beta_v = \lim_{z\to 1}\frac{(z-1)}{T}G(z)C(z) = \lim_{z\to 1}\frac{(z-1)}{T}\frac{1}{(z+1)}C(z)$$

A finite velocity error implies $C(z)$ has only one integrator. Let the controller be a general first order controller,

$$C(z) = \frac{az+b}{z-1}$$

We have the velocity error constant,

$$\beta_v = \lim_{z\to 1}\frac{(z-1)}{T}\frac{(az+b)}{(z-1)(z+1)} = \frac{a+b}{2T}$$

which results in the steady state error

$$e(\infty) = \frac{1}{\beta_v} = \frac{2T}{a+b}$$

It is obvious that the steady state error is proportional to the sampling period T. And the error is minimal when the |a+b| is maximal.

In the next, let's use the stability condition that the poles are inside the circle with the radius of 0.8. The closed-loop TF is

$$H(z) = \frac{G(z)C(z)}{1+G(z)C(z)} = \frac{\frac{1}{z+1}\frac{az+b}{z-1}}{1+\frac{1}{z+1}\frac{az+b}{z-1}} = \frac{az+b}{z^2+az+b-1}$$

Let the two roots of the denominator (the poles) be z_1 and z_2. We have:

$$z_1 + z_2 = -a$$
$$z_1 z_2 = b-1$$

From the condition that the poles: $|z| \le 0.8$, we have

$$|z_1 + z_2| = |a| \le |z_1| + |z_2| \le 1.6$$
$$|z_1 z_2| = |b-1| \le |z_1||z_2| \le 0.8^2 = 0.64$$

Then we have

$$-1.6 \le a \le 1.6$$
$$-0.64 \le b-1 \le 0.64 \rightarrow \quad 0.36 \le b \le 1.64$$

It is evident that the maximum of |a+b| is obtained with a=1.6 and b=1.64.

The controller is:

$$C(z) = \frac{1.6z+1.64}{z-1}.$$

The closed loop is a type-1 system. The steady state error for unit step input is zero, the steady state error for ramp input is finite, $\frac{T}{1.62}$.

Q.4 Derive a desired closed-loop characteristic polynomial for discrete systems such that the resulting closed-loop system response has an overshoot of less than 10% and a settling time of less than 10 seconds for a step reference input, where the sampling period is T=2 seconds.

Solution:

The characteristic polynomial in continuous time domain has the form of:

$$s^2 + 2\varsigma\omega_n s + \omega_n^2$$

Overshoot less than 10%: $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} < 10\% \Rightarrow \zeta > 0.5912$. Let $\zeta = 0.6$

Settling time less than 10 seconds: $t_s = \dfrac{4.6}{\zeta\omega_n} < 10 \Rightarrow \omega_n = \dfrac{4.6}{10\zeta} = 0.77$.

The poles of the continuous system are:

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -0.462 \pm j0.616$$

which can be translated into the poles in the z-domain:

$$z = e^{Ts} \Rightarrow z = 0.1319\ \pm\ \text{j}0.3744;$$

Finally, the desired closed-loop characteristic polynomial is:

$$(z - z_p)(z - \overline{z}_p) = z^2 - 0.2638z + 0.1576\cdot$$

Q.5 A digital controller $C(z) = \dfrac{-z+1.5}{z-1}$ is designed for a plant $G(z) = \dfrac{1}{z+1}$ in a unity feedback system. Select an appropriate sampling period T such that the closed-loop response satisfies $t_s \le \dfrac{0.1\times 9.2}{\ln 2} = 1.3273$.

Solution:

We have the transfer function of the closed loop system:

$$H(z) = \frac{G(z)C(z)}{1+G(z)C(z)} = \frac{\dfrac{1}{z+1}\dfrac{-z+1.5}{z-1}}{1+\dfrac{1}{z+1}\dfrac{-z+1.5}{z-1}} = \frac{-z+1.5}{z^2 - z + 0.5}$$

Get the roots of $z^2 - z + 0.5 = 0$

The poles of the closed loop is $z = 0.5 \pm 0.5j$. We can get

$$|z| = \frac{1}{\sqrt{2}}$$

Assume the corresponding continuous-time CP is

$$s^2 + 2\varsigma\omega_n s + \omega_n^2$$

And we know that the poles of the discrete-time system can be expressed as

$$z = e^{Ts} = e^{-T\zeta\omega_n \pm jT\omega_n\sqrt{1-\zeta^2}} = e^{-T\zeta\omega_n} e^{\pm jT\omega_n\sqrt{1-\zeta^2}}$$

Therefore

$$|z| = e^{-T\zeta\omega_n} = \frac{1}{\sqrt{2}}$$

We get

$$T = \frac{\ln(\sqrt{2})}{\zeta\omega_n} = \frac{\ln 2}{2\zeta\omega_n}$$

From the condition on settling time,

$$t_s = \frac{4.6}{\zeta\omega_n} \le \frac{0.1\times 9.2}{\ln 2} \Rightarrow \zeta\omega_n \ge \frac{\ln 2}{0.2}$$

Overall, we have

$$T \le 0.1$$