EE3304 DIGITAL CONTROL SYSTEMS PART II
TUTORIAL TWO

Q1. The transfer function of a pure derivative control is $C(z) = k_d \frac{z-1}{zT}$. It is obvious that the pole at z = 0 adds some destabilizing time lag. Can we remove this time lag by using derivative control of the form $C(z) = k_d \frac{z-1}{T}$? Support your answer with the necessary analysis based on the difference equation.

Solution:

We cannot use derivative control of the form $C(z) = k_d \frac{z-1}{T}$. Let $u(k)$ and $e(k)$ be output and input of such controller at time k. We have:

$$U(z) = k_d \frac{z-1}{T} E(z)$$

which implies:

$$Tu(k) = k_d [e(k+1) - e(k)] .$$

It shows that the controller is not causal i.e., it needs to know future outputs.

Q2. A revised PID is

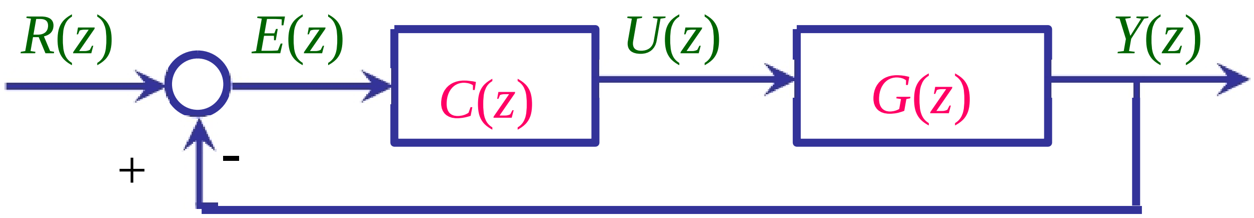
$$\frac{U(s)}{E(s)} = k_p + k_i \frac{1}{s} + k_d \frac{s}{1 + \frac{s}{N}}$$

where N is a positive integer number.

1) Find its discrete equivalent using backward difference.

2) For a process , $G(z) = \frac{z+b}{(z-a)^2}, |a| < 1$, calculate the position and velocity error constants, hence the steady state errors.

Solution:



1)

$$\frac{U(s)}{E(s)} = k_p + k_i \frac{1}{s} + k_d \frac{s}{1+\frac{s}{N}}$$

$$s \rightarrow \frac{1-z^{-1}}{T}$$

$$\frac{U(z)}{E(z)} = k_p + k_i \frac{1}{\frac{1-z^{-1}}{T}} + k_d \frac{\frac{1-z^{-1}}{T}}{1+\frac{\frac{1-z^{-1}}{T}}{N}}$$

$$= k_p + k_i \frac{T\,z}{z-1} + k_d \frac{N(z-1)}{NT\,z+z-1}$$

$$\text{Let } C(z) = \frac{U(z)}{E(z)}$$

2) Steady state error: for unit step input $R(z) = \frac{z}{z-1}$

$$e(\infty) = \lim_{z\to 1}(z-1)\frac{1}{1+G(z)C(z)}\frac{z}{z-1} = \lim_{z\to 1}\frac{1}{1+G(z)C(z)} = \frac{1}{1+\beta_p}$$

$$\beta_p = \lim_{z\to 1} G(z)C(z) = \lim_{z\to 1}\frac{z+b}{(z-a)^2}\left(k_p + k_i\frac{T\,z}{z-1} + k_d\frac{N(z-1)}{NT\,z+z-1}\right) = \infty$$

It is a type-1 system, the position error constant is infinity, and the steady state error for step is 0.

For ramp input, $R(z) = \frac{Tz}{(z-1)^2}$

$$e(\infty) = \lim_{z\to 1}(z-1)\frac{1}{1+G(z)C(z)}\frac{Tz}{(z-1)^2} = \frac{1}{\beta_v}$$

$$\beta_v = \lim_{z\to 1}\frac{z-1}{T}G(z)C(z) = \lim_{z\to 1}\frac{z-1}{T}\frac{z+b}{(z-a)^2}\left(k_p + k_i\frac{T\,z}{z-1} + k_d\frac{N(z-1)}{NT\,z+z-1}\right)$$

$$= k_i\frac{1+b}{(1-a)^2}$$

$$e(\infty) = \frac{(1-a)^2}{k_i(1+b)}$$

It is a type-1 system, the velocity error constant is finite, and the steady state error for ramp reference is not zero.

Q3. A simple proportional controller, K is used to control a first order plant, $\frac{1}{s+1}$, as shown in Fig. 1. It can be easily verified that the closed loop system is stable for any positive gain. Replace the controller with the digital proportional controller. Consider two cases. For the first case, assuming the sampling period T is fixed, is it possible to make the system unstable by changing the gain K? For the second case, assuming the gain K is fixed, is it possible to make the system unstable by changing the sampling period T?

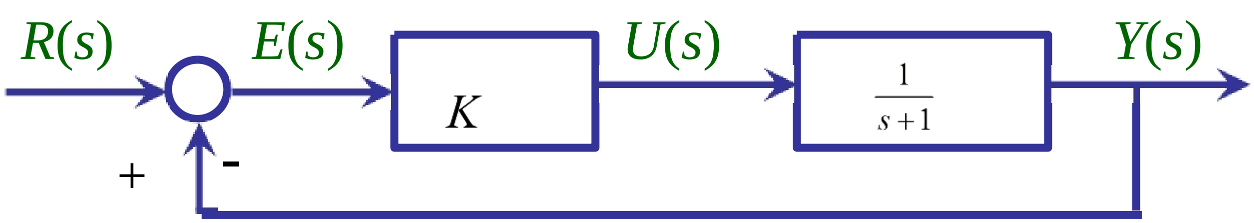


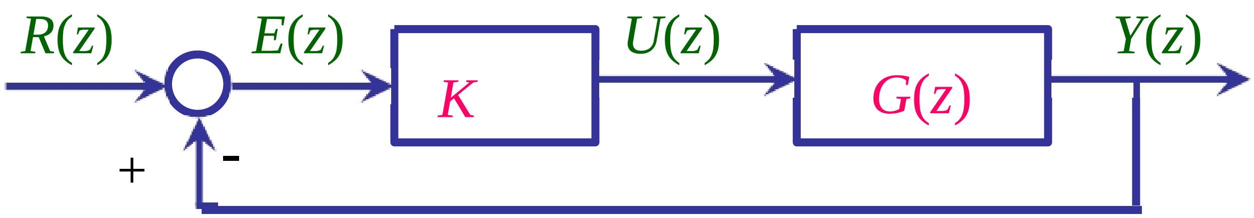
Figure 1 Digital control system for Q3

Solution. For the analog control system, the closed loop transfer function is

$$H(s) = \frac{\dfrac{K}{s+1}}{1+\dfrac{K}{s+1}} = \frac{K}{s+1+K}$$

Obviously, for any positive gain, K>0, the closed loop is stable.

For the digital control system, the whole system becomes



Let's first find out the discrete-time transfer function for the first order system. Using Table 2.1, we have

$$G(z) = \frac{1 - e^{-T}}{z - e^{-T}}$$

The closed loop transfer function is then

$$H(z) = \frac{KG(z)}{1 + KG(z)} = \frac{K\dfrac{1 - e^{-T}}{z - e^{-T}}}{1 + K\dfrac{1 - e^{-T}}{z - e^{-T}}}$$

$$= \frac{K(1 - e^{-T})}{z - e^{-T} + K(1 - e^{-T})}$$

The pole is $z = e^{-T} - K(1 - e^{-T})$

The system will be unstable if |z|>1, which implies that

$$e^{-T} - K(1 - e^{-T}) > 1$$

$$or \quad e^{-T} - K(1 - e^{-T}) < -1$$

It follows that

$$K < -1$$

$$or \qquad K > \frac{1 + e^{-T}}{1 - e^{-T}} > 1$$

From this example, we can see that the closed loop can become unstable after sampling even if the analog system is stable. In particular, it is dangerous to use large gain in the sampled system. For the second case, assume the gain K is positive and fixed, the closed loop is unstable if

$$z = e^{-T} - K(1 - e^{-T}) < -1$$

We get

$$T > \ln\frac{(K+1)}{(K-1)}$$

Therefore, if the sampling period is too big, the closed loop system may become unstable for K>1. If the sampling period is very small, the response of the system is similar to that of the analog system, and the system is always stable.

Q4. An integral controller, $\frac{K}{s}$ is used to control a first order plant, $\frac{1}{s+1}$, as shown in Fig. 2. Find out the range of K for the analog controller such that the closed loop is stable. Assume the sampling period is T=1, replace the analog controller with the digital integral controller which is obtained by backward rule. Find out the range of K such that the closed loop is stable.

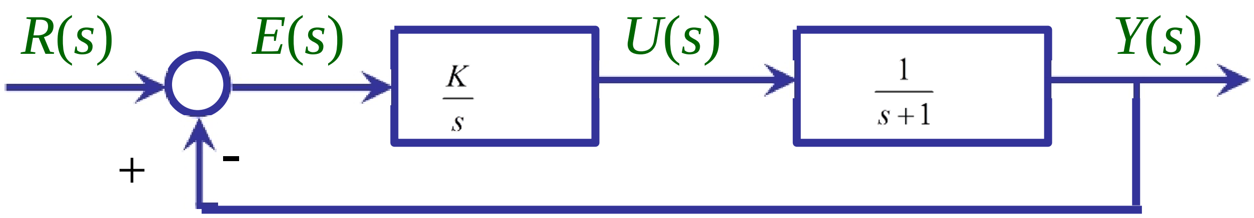
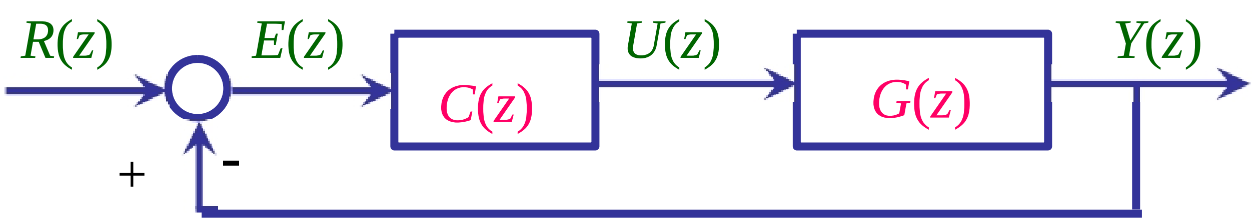


Figure 2 Digital control system for Q4

Solution. For the analog control system, the closed loop transfer function is

$$H(s) = \frac{\frac{K}{s}\frac{1}{s+1}}{1+\frac{K}{s}\frac{1}{s+1}} = \frac{K}{s^2+s+K}$$

Obviously, for any positive gain, K>0, the closed loop is stable.
For the digital control system, the whole system becomes



Let's first find out the discrete-time transfer function for the first order system. Using Table 2.1, we have

$$G(z) = \frac{1-e^{-T}}{z-e^{-T}} = \frac{1-e^{-1}}{z-e^{-1}}$$

The equivalent digital controller based on the backward rule is

$$C_1(z) = \frac{K}{s}\Big|_{s=\frac{1-z^{-1}}{T}} = \frac{Kz}{z-1}$$

The closed loop transfer function is then

$$H(z) = \frac{C_1(z)G(z)}{1+C_1(z)G(z)} = \frac{\frac{Kz}{z-1}\frac{1-e^{-1}}{z-e^{-1}}}{1+\frac{Kz}{z-1}\frac{1-e^{-1}}{z-e^{-1}}}$$

$$= \frac{Kz(1-e^{-1})}{(z-1)(z-e^{-1})+Kz(1-e^{-1})}$$

$$= \frac{Kz(1-e^{-1})}{z^2+[K(1-e^{-1})-(1+e^{-1})]z+e^{-1}}$$

Assume the two poles are z_1 and z_2, then we have

$$z_1+z_2 = 1+e^{-1}-K(1-e^{-1})$$
$$z_1 z_2 = e^{-1}$$

Since $e^{-1}<1$, the system is stable if the two poles are complex conjugates. Also the two poles have the same algebraic sign. If the gain K is large, it is possible to have two negative poles, one with small magnitude and one with big magnitude. Let’s find out the critical gain when the system is marginally stable. At that point, one of the poles is -1, which implies that -1 is the root of $z^2+[K(1-e^{-1})-(1+e^{-1})]z+e^{-1}=0$. We have

$$1-[K(1-e^{-1})-(1+e^{-1})]+e^{-1}=0 \rightarrow \quad K=\frac{2(1+e^{-1})}{1-e^{-1}}$$

Therefore the stability range of the controller gain is $0<K<\dfrac{2(1+e^{-1})}{1-e^{-1}}$

Once again, we can see that large gain may result in unstable system even if the analog system is stable.

Q5. Design a digital PD controller C(z) for the plant whose transfer function is $\frac{1}{s^2}$ as shown in Fig. 3. It is desired that the damping ratio ζ of the closed-loop system be 0.5 and the natural frequency, ω_n, be 4. Choose appropriate sampling period for the digital PD controller, and design the controller.

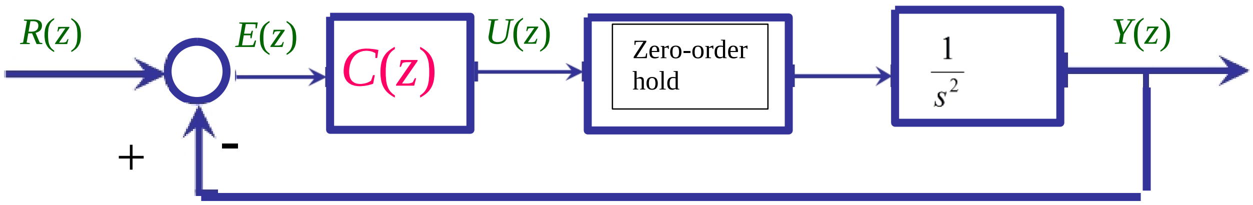


Figure 3 Digital control system for Q5

Solution.

Let's use emulation design for this problem. First, let's try to get the desired poles of the reference model.

The characteristic polynomial in continuous time domain has the form of:

$$s^2 + 2\varsigma\omega_n s + \omega_n^2$$

From the performance specification, we know $\zeta = 0.5$ and $\omega_n = 4$. Therefore, the desired characteristic polynomial is

$$s^2 + 4s + 16$$

Let the analog PD controller be

$$C(s) = K_p + K_d s$$

Then the closed-loop transfer function is

$$H(s) = \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{\frac{(K_p + K_d s)}{s^2}}{1 + \frac{(K_p + K_d s)}{s^2}}$$

$$= \frac{K_p + K_d s}{s^2 + K_d s + K_p}$$

Compared it with the desired polynomial, we get

$$K_p = 16$$
$$K_d = 4$$

Use the backward rule to discretize the analog controller we have

$$C(z) = K_p + K_d \frac{z-1}{Tz} = \frac{(K_d + K_p T)z - K_d}{Tz}$$

We need to choose the appropriate sampling frequency for the digital controller. For this purpose, let's find out the bandwidth of the closed loop analog system.

The closed loop is

$$H(s) = \frac{K_p + K_d s}{s^2 + K_d s + K_p} = \frac{4s+16}{s^2+4s+16}$$

For second order system, the bandwidth can be estimated by the natural frequency

$$\omega_b \approx \omega_n = 4rad/\sec = 0.64Hz$$

Let's try two different sampling frequencies and then compare.

(1) choose a sampling rate at least **30** times the bandwidth

$$T = 0.05 < \frac{1}{30\omega_b} = 0.052$$

(2) choose a sampling rate at least **6** times the bandwidth

$$T = 0.2 < \frac{1}{6\omega_b} = 0.26$$

The poles of the continuous system are:

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -2 \pm j2\sqrt{3}$$

which can be translated into the poles in the z-domain:

$$z = e^{Ts} \Rightarrow z = e^{-2T \pm j2\sqrt{3}T}$$

For T=0.05, the desired poles are: $0.8913 \pm 0.1559j$

For T=0.2, the desired poles are: $0.5158 \pm 0.4281j$

Next, let's try to get the discrete transfer function of G(s). From Table 2.1, we have

$$G(s) = \frac{1}{s^2} \quad \rightarrow G(z) = \frac{T^2(z+1)}{2(z-1)^2}$$

The digital PD controller can be obtained by backward rule as

$$C(z) = K_p + K_d \frac{z-1}{Tz} = \frac{(K_d + K_p T)z - K_d}{Tz}$$

Then the closed-loop transfer function for the sampled system is

$$H(z) = \frac{C(z)G(z)}{1 + C(z)G(z)} = \frac{\frac{(K_d + K_p T)z - K_d}{Tz} \frac{T^2(z+1)}{2(z-1)^2}}{1 + \frac{(K_d + K_p T)z - K_d}{Tz} \frac{T^2(z+1)}{2(z-1)^2}}$$

$$= \frac{T((K_d + K_p T)z - K_d)(z+1)}{2z(z-1)^2 + T((K_d + K_p T)z - K_d)(z+1)}$$

$$= \frac{T((K_d + K_p T)z - K_d)(z+1)}{2z^3 + (T(K_d + K_p T) - 4)z^2 + (K_p T^2 + 2)z - K_d T}$$

$$= \frac{T((4+16T)z - 4)(z+1)}{2z^3 + (T(4+16T) - 4)z^2 + (16T^2 + 2)z - 4T}$$

When the sampling T=0.05, the poles are: $0.8776 \pm 0.1753j$, 0.1249

Compared to the desired poles: $0.8913 \pm 0.1559j$, the dominant poles are very close to the desired ones, and the fast mode will converge to zero very fast. Therefore, we can expect that the performance of the digital controller can match the performance of the analog system as we use sufficiently high sampling frequency.

When the sampling T=0.2, the poles are: $0.4317 \pm 0.8797j$, 0.4165

Compared to the desired poles: $0.5158 \pm 0.4281j$, the dominant poles are still close, but not as close as the previous one. Therefore, we expect the performance is still acceptable, while might not be as good as the one with higher sampling frequency. However, since the sampling frequency is 4 times lower, the cost of the controller would be lower.

Q6. Consider the digital control system shown in Fig. 4. Design a digital PI controller C(z) such that the system output will exhibit a deadbeat response to a unit step input. The sampling period is T=1.

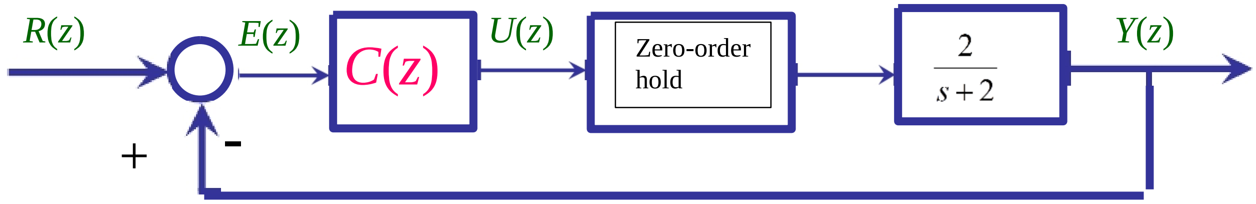


Figure 4 Digital control system for Q6

Solution. First**,** since it is deadbeat control, the desired closed-loop characteristic polynomial is:

$$z^2$$

Next, let's try to get the discrete transfer function of G(s). From Table 2.1, we have

$$G(z) = \frac{1-e^{-2T}}{z-e^{-2T}} = \frac{1-e^{-2}}{z-e^{-2}}$$

Let the digital PI controller be

$$C(z) = K_p + K_i \frac{Tz}{z-1} = \frac{(K_p + K_i T)z - K_p}{z-1}$$

Then the closed-loop transfer function is

$$H(z) = \frac{C(z)G(z)}{1+C(z)G(z)} = \frac{\frac{(K_p+K_i)z-K_p}{z-1}\frac{1-e^{-2}}{z-e^{-2}}}{1+\frac{(K_p+K_i)z-K_p}{z-1}\frac{1-e^{-2}}{z-e^{-2}}}$$

$$= \frac{(1-e^{-2})[(K_p+K_i)z-K_p]}{(z-e^{-2})(z-1)+(1-e^{-2})[(K_p+K_i)z-K_p]}$$

$$= \frac{(1-e^{-2})[(K_p+K_i)z-K_p]}{z^2+((1-e^{-2})(K_p+K_i)-(1+e^{-2}))z+e^{-2}-(1-e^{-2})K_p}$$

Compared with the desired closed-loop characteristic polynomial: z^2
We have

$$(1-e^{-2})(K_p+K_i)-(1+e^{-2})=0$$
$$e^{-2}-(1-e^{-2})K_p = 0$$

which results in

$$K_p = \frac{1}{e^2 - 1} = 0.1565$$

$$K_i = \frac{e^2}{e^2 - 1} = 1.1565$$

The simulated output is shown in figure below. It is obvious that the output reaches 1 in just two steps (at t=2) and keeps there.