EE3304 (Part II)
Digital Control Systems
Chapter Four
Frequency Domain Design

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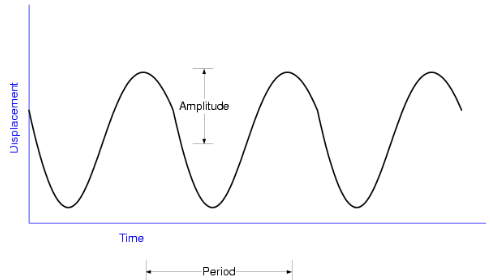
4.1. What is frequency response?

What is the output when the input is sinusoid?



$$\longrightarrow \boxed{H(s)} \longrightarrow \quad ???$$

- The sinusoid can be represented as $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

$$e^{st} \longrightarrow \boxed{H(s)} \longrightarrow H(s)e^{st}$$

$$s = j\omega \quad \Longrightarrow \quad e^{j\omega t} \longrightarrow \boxed{H(s)} \longrightarrow H(j\omega)e^{j\omega t}$$

$$H(j\omega)e^{j\omega t} = |H(j\omega)|\, e^{j\angle H(j\omega)} e^{j\omega t} = |H(j\omega)|\, e^{j(\omega t + \angle H(j\omega))}$$

$H(j\omega)$ completely determines the frequency response.

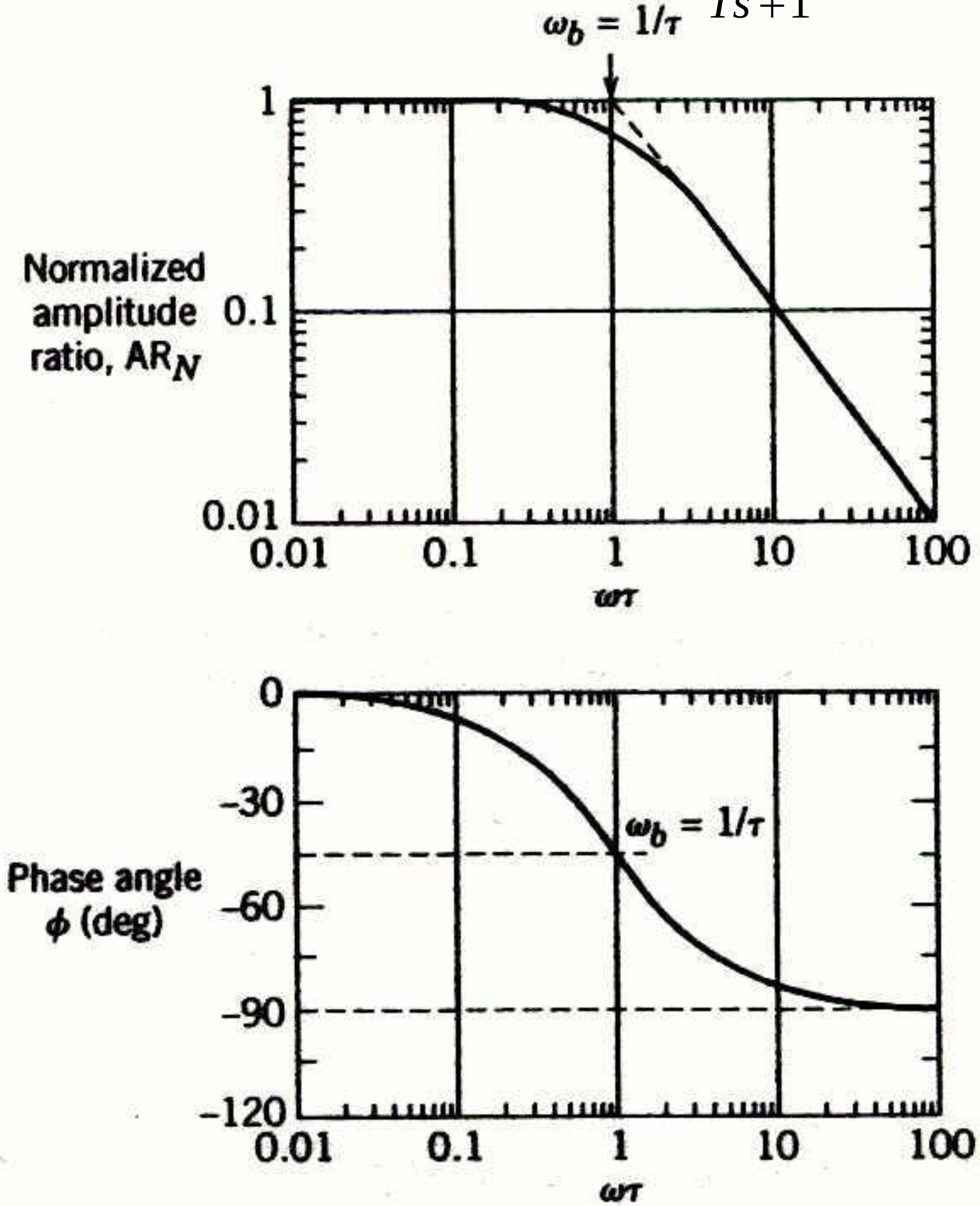
Is it meaningful to talk about frequency response for unstable system? No.

How to plot out the frequency response $H(j\omega)$?

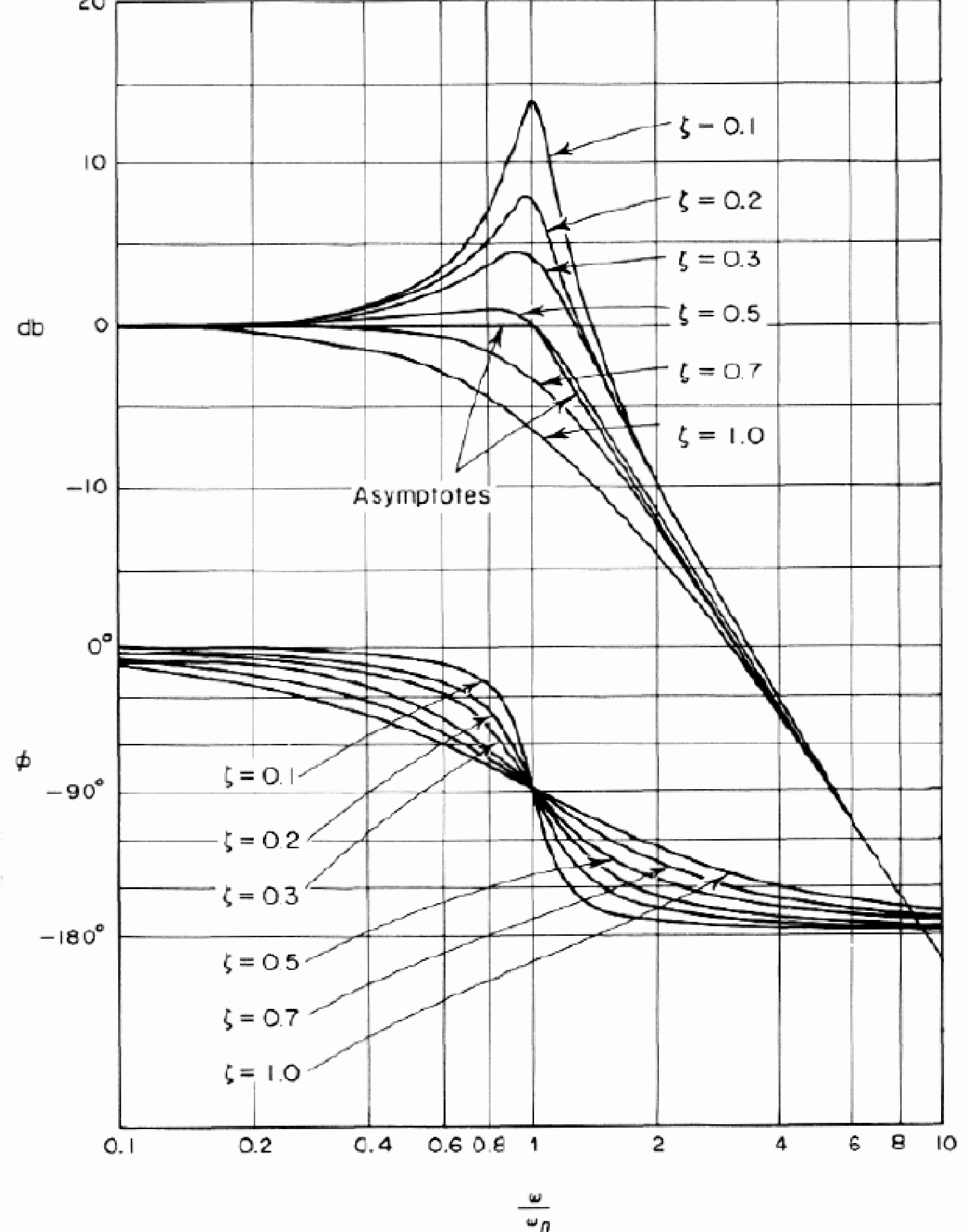
There are two ways.

4.2. *Bode diagrams for continuous-time systems* ___ *a revisit*

First order system $\dfrac{1}{Ts+1}$



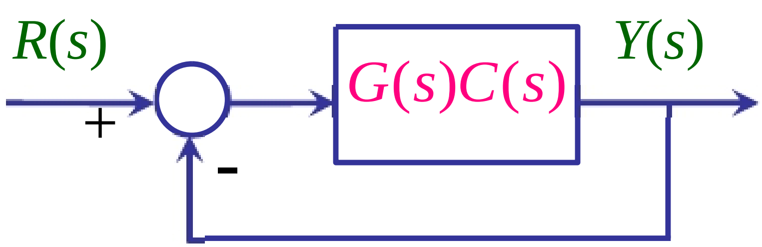
[Second order system](#) $\dfrac{\omega_n^2}{s^2+2\varsigma\omega_n s+\omega_n^2}$

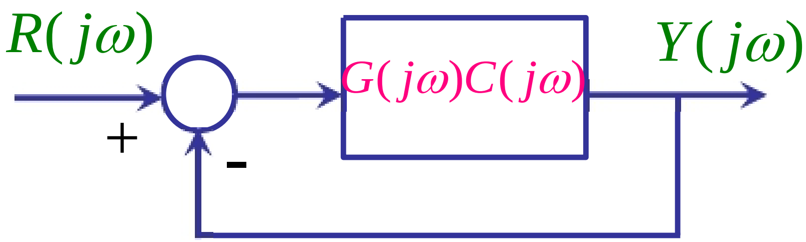


4.2. *Bode diagrams for continuous-time systems*

1. Bode diagram is the frequency responses (both magnitude response and phase response with respect to frequency).

2. Consider a standard unity feedback system, which has a closed-loop transfer function $H(s)$. Bode diagram is obtained with s replaced by $j\omega$.





3. If the open-loop transfer function $G(s)C(s)$ is stable, then its frequency responses (or Bode diagram), $G(j\omega)C(j\omega)$, can be easily used to determine the stability of the closed-loop system $H(s)$.

$$\boxed{H(s) = \frac{Y(s)}{R(s)} = \frac{G(s)C(s)}{1+G(s)C(s)}}$$

4. As such, it is often to plot the Bode diagram for the open-loop system $G(s)C(s)$. Magnitude & phase responses are defined as:

$$\left.\begin{array}{l}\text{Magnitude response} = |G(j\omega)C(j\omega)| \\ \text{Phase response} = \angle G(j\omega)C(j\omega)\end{array}\right| \quad -\infty < \omega < \infty$$

4.2. *Bode diagrams for continuous-time systems*

$$\left.\begin{array}{l}\text{Magnitude response} = \left|G(j\omega)C(j\omega)\right| \\ \text{Phase response} = \angle \mathrm{G}(\mathrm{j}\omega)\mathrm{C}(\mathrm{j}\omega)\end{array}\right| \quad -\infty < \omega < \infty$$

<u>Bode' plots are really log-log plots, why?</u>

1. They collapse a wide range of frequencies (on the horizontal axis) and a wide range of gains (on the vertical axis) into a viewable whole.

2. Since it is log plot, complicated plot can be broken down into summation of simple shapes!

$$\log\left|G_1(j\omega)G_2(j\omega)\right| = \log\left|G_1(j\omega)\right| + \log\left|G_2(j\omega)\right|$$

This enables us to use asymptotes to approximate Bode plot easily.

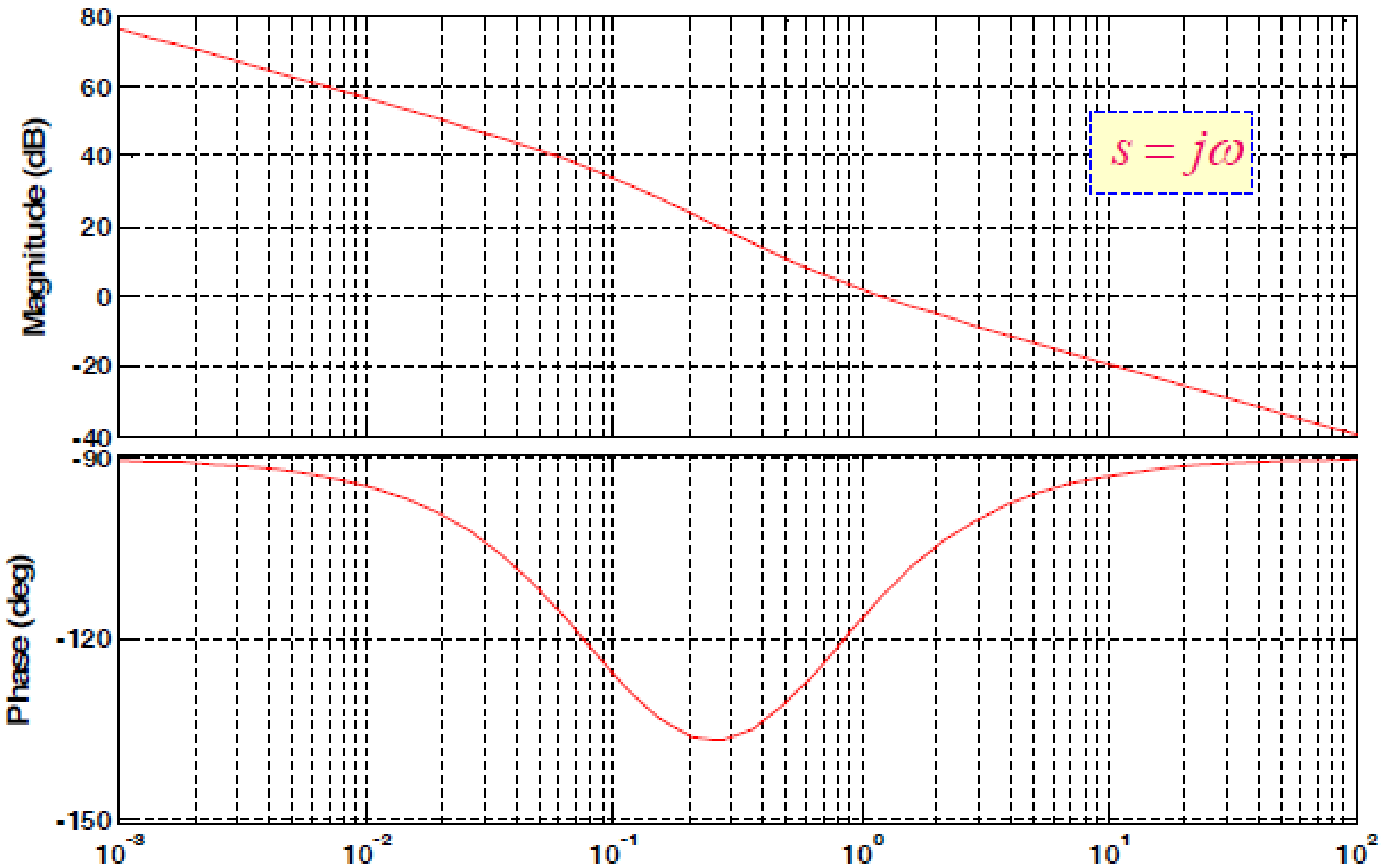
3. The unit is decibel (dB) $\quad 20\log_{10}\left|G(j\omega)\right|$

4. You may have used Bode' plots without knowing it. Stereo equipment - amplifiers, speakers, microphones, headsets, etc. - often have frequency response specifications, and when you buy that kind of equipment, you may have seen a Bode' plot used to communicate frequency response specifications.

Example: Bode diagrams of the open-loop system in the speed control system example

$$G(s)C(s) = \frac{1}{1000s+100} \cdot \frac{1050s+670}{s} = \frac{670(\frac{1050}{670}s+1)}{100s(10s+1)}$$

$$G(j\omega)C(j\omega) = 6.7\frac{(\frac{j\omega}{670/1050}+1)}{j\omega(\frac{j\omega}{0.1}+1)}$$



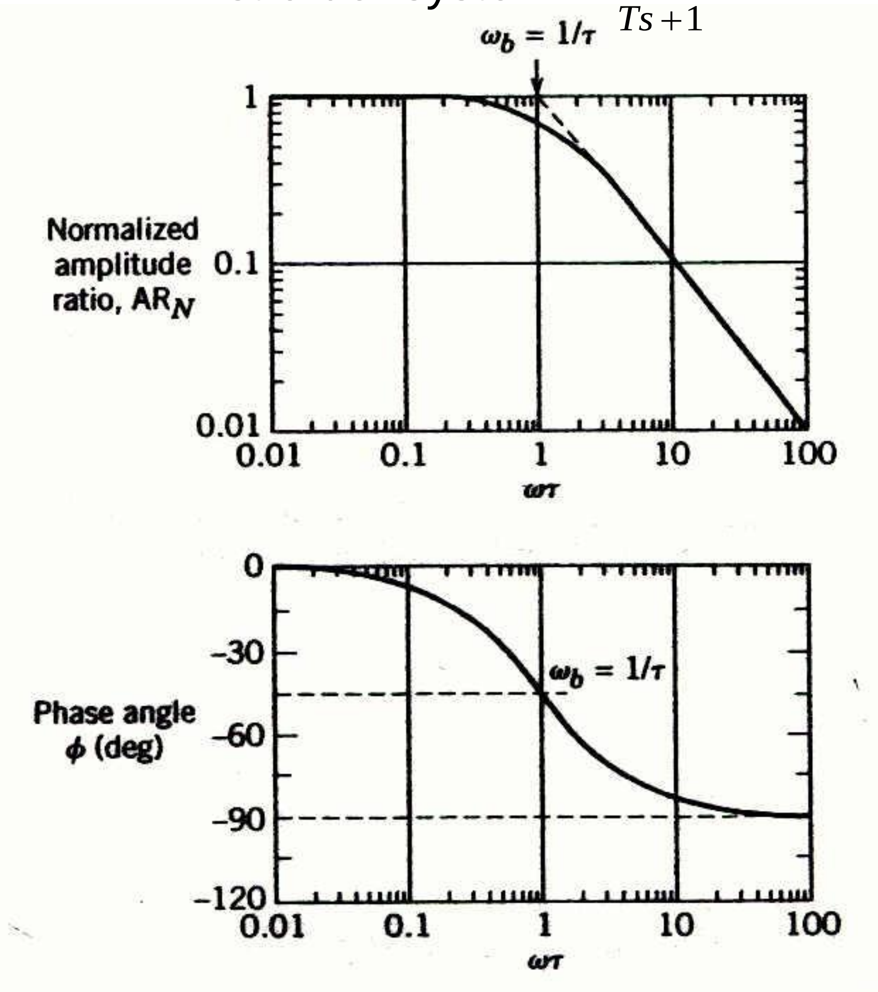
In MATLAB, the command “bode” will plot out the Bode diagram.

What is the second way to plot out the frequency response?

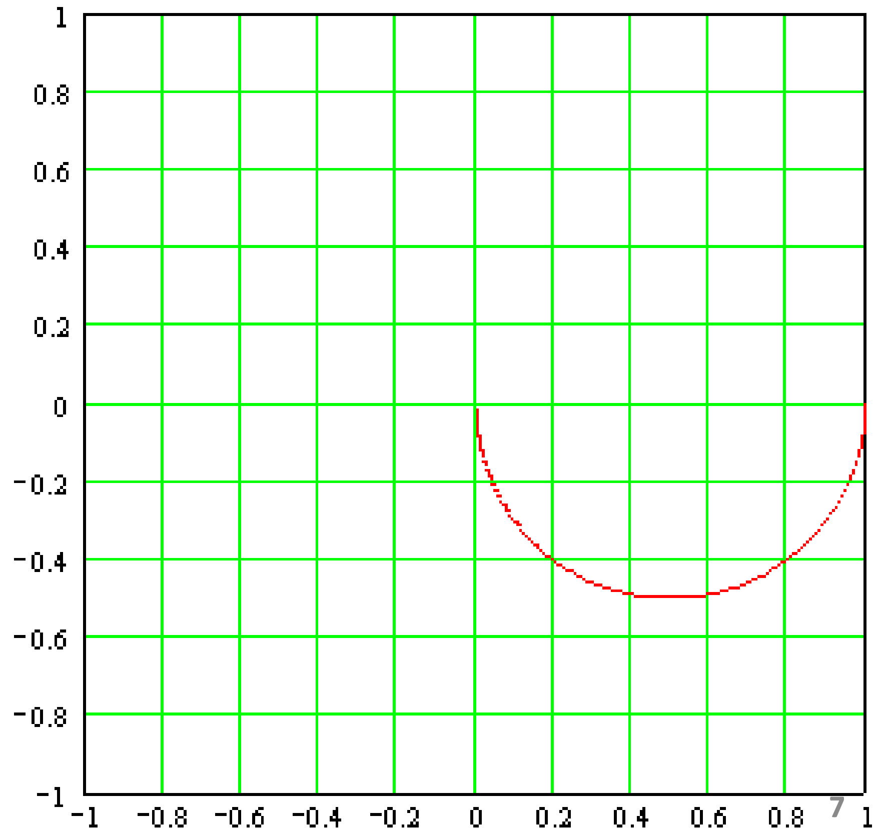
4-3. *Nyquist plot in continuous systems* —— *a revisit*

Nyquist plot is to draw the frequency response of the open-loop system in a single complex plane instead of separating the magnitude and phase responses into two individual diagrams as in the Bode plot.

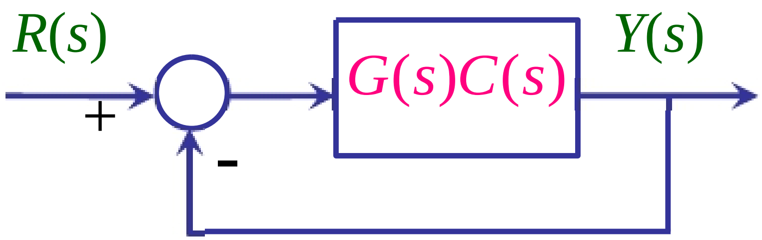
First order system $\frac{1}{Ts+1}$



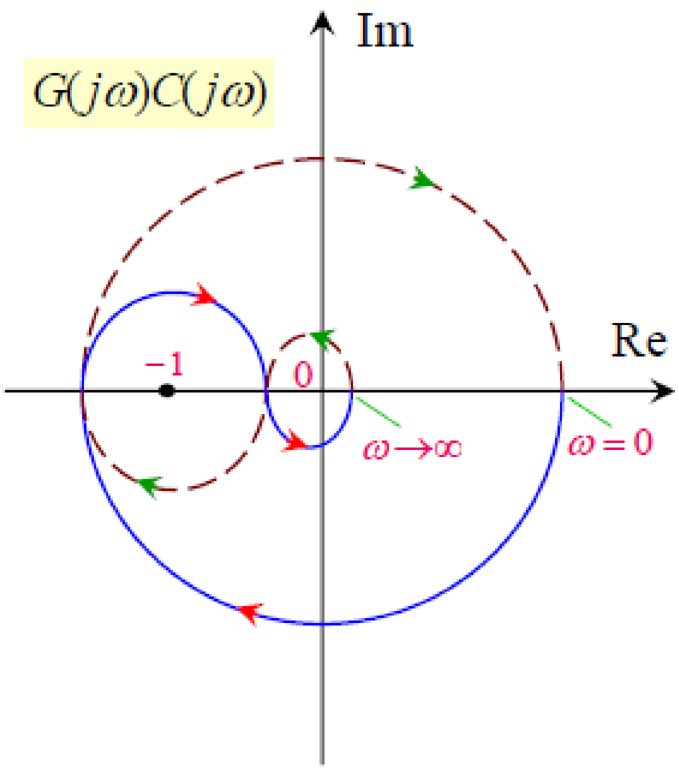
Nyquist Plot



The Nyquist plot can be used to determine the stability of the closed-loop system!



$$H(s) = \frac{G(s)C(s)}{1+G(s)C(s)}$$



<u>Nyquist Stability Criterion</u>

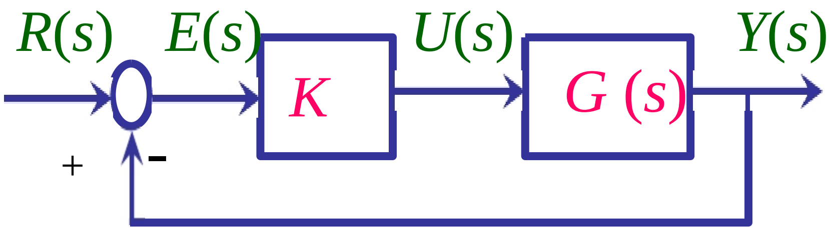
Let N be the number of clockwise encirclements of the point $(-1,0)$ in the Nyquist plot, P be the number of unstable poles of the open-Loop system $G(s)C(s)$, Z be the number of unstable poles of The closed-loop system $H(s)$, then

$$N = Z - P$$

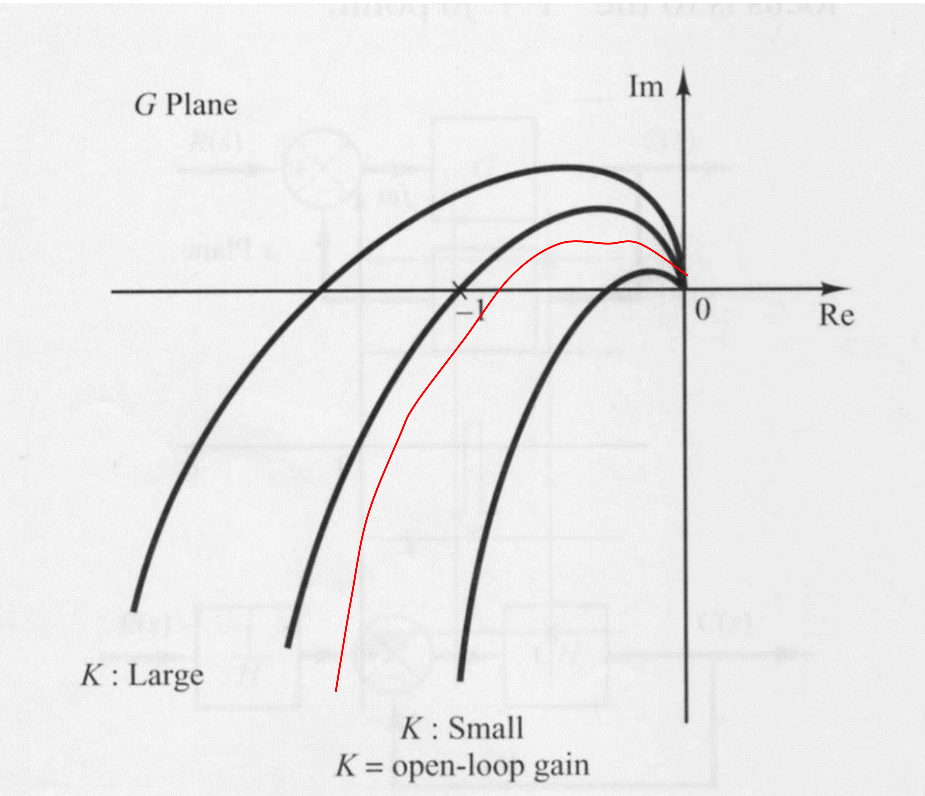
If $G(s)C(s)$ is stable, $P = 0$, then N has to be zero in order to guarantee the stability of the closed-loop system $H(s)$.

If the open-loop is stable, then the Nyquist plot should not encircle the point (-1,0)!

4-4. *Relative Stability*



In designing a control system, we require that the system be stable. Furthermore, it is necessary that the system has adequate relative stability



The left figure shows the Nyquist plot for three different values of the open loop gain K. Assume the open loop is stable.

For a large value of the gain K, is the closed-loop system stable?

No

As the gain is decreased to a certain value, the plot passes through the critical point (-1,0). This means that with this gain value the system is on the verge of instability, and the system will exhibit sustained oscillations.

For a small value of the gain K, the system is stable.

In general, the closer the Nyquist plot to the point (-1,0), the more oscillatory is the system response.

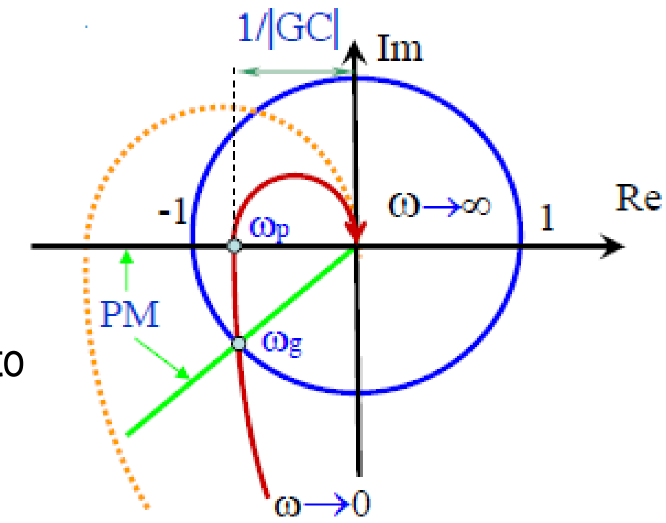
Therefore, the closeness of the plot to the point (-1,0) can be used as the measure of the margin of stability.

How to measure the closeness of the Nyquist plot to the critical point (-1,0)?

How close is the magnitude to the critical point (-1,0)?

Gain Margin

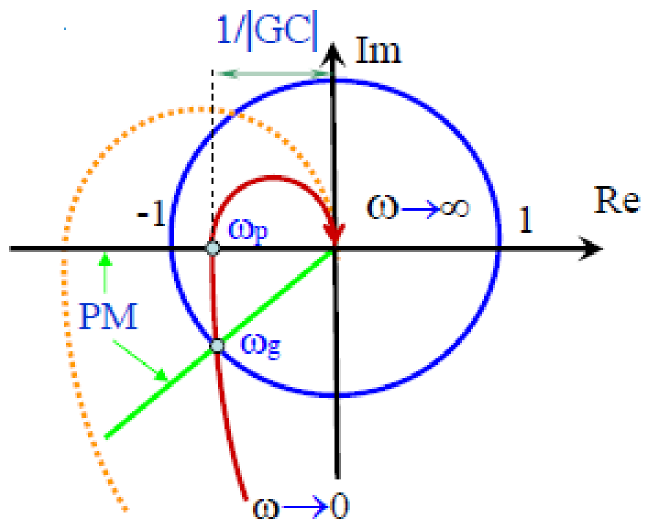
Gain margin is the maximum additional gain that can be applied to the closed-loop system such that it remains stable.



How close is the phase (angle) to the critical point (-1,0)?

Phase Margin

Phase margin is the maximum phase lag that the closed-loop system can tolerate without losing the stability.



Remark: Gain margin is the maximum additional gain that can be applied to the closed-loop system such that it remains stable. Similarly, phase margin is the maximum phase lag that the closed-loop system can tolerate without losing the stability.

Mathematically, at the phase cross over frequency, where $\angle G(j\omega_p)C(j\omega_p) = -180^\circ$

$$\left|GM \times G(j\omega_p)C(j\omega_p)\right| = 1$$

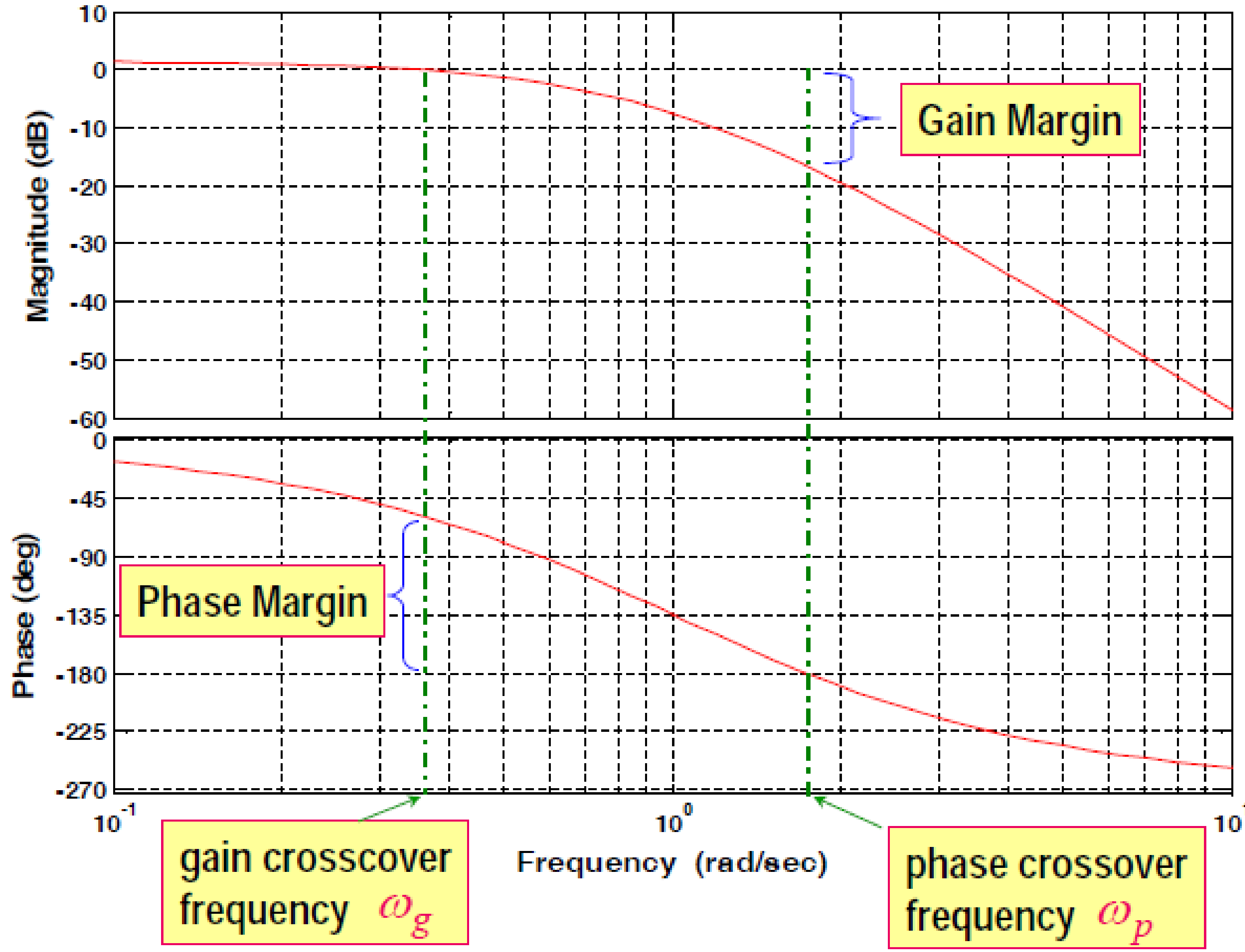
$$GM = \frac{1}{\left|G(j\omega_p)C(j\omega_p)\right|}$$

At gain cross over frequency, where $\left|G(j\omega_g)C(j\omega_g)\right| = 1$

$$\angle G(j\omega_g)C(j\omega_g) - PM = -180^\circ$$

$$PM = \angle G(j\omega_g)C(j\omega_g) + 180^\circ$$

Gain and phase margins in the Bode diagram



Proper phase and gain margins ensure us against variations in the system components.

For satisfactory performance, the phase margin should be between 30º and 60º
And the gain margin should be greater than 6dB.

Notes: Major roles of Bode Diagram for control system design.

1. From open-loop response, obtain closed-loop stability condition from PM and GM.

2. From PM and GM, tell how stable the closed-loop system is.

3. The low-frequency region (the region far below the gain crossover frequency) indicates the steady-state behavior of the closed-loop system. The higher DC gain, the better the steady state performance.

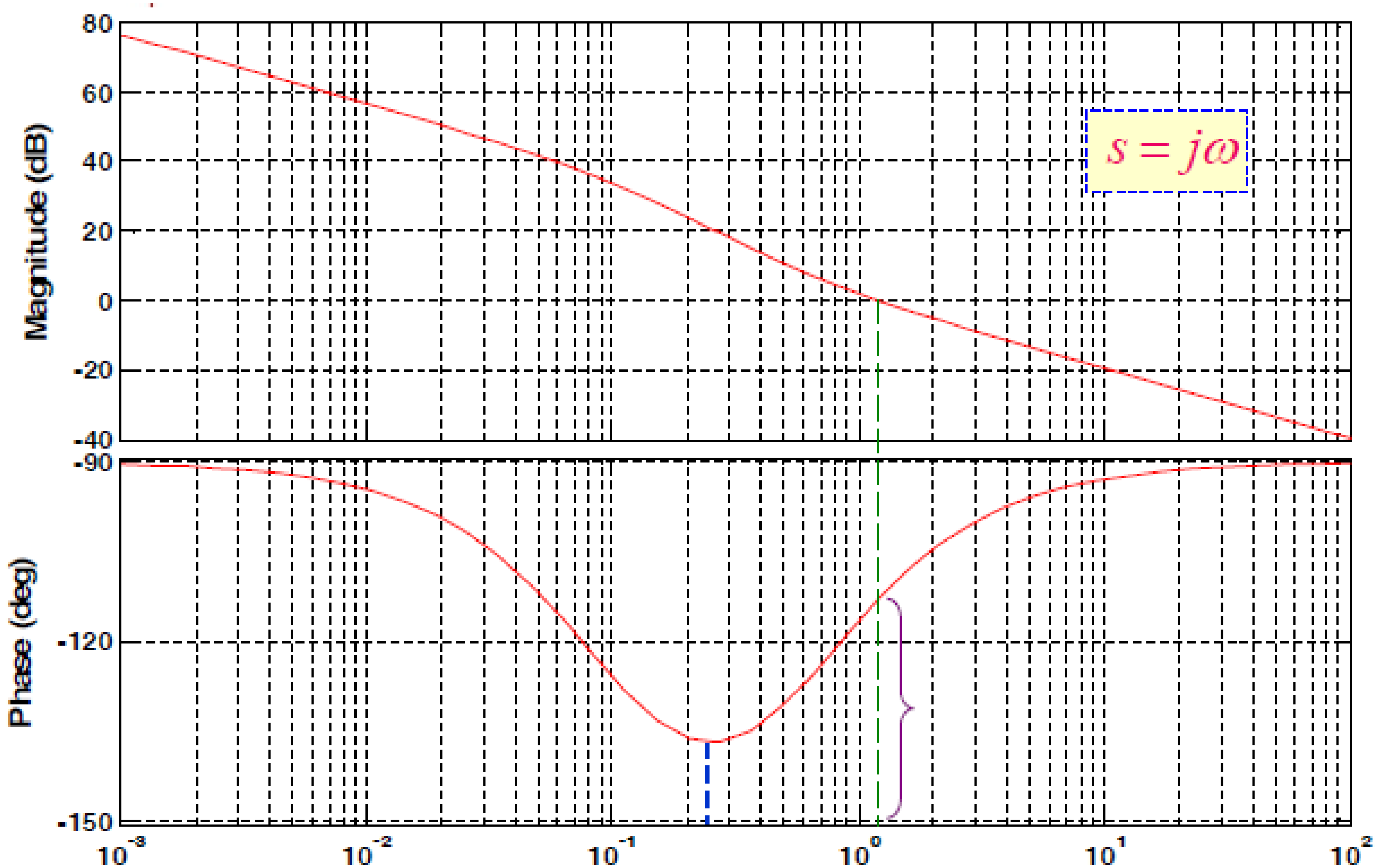
$$e(\infty)=\frac{1}{1+G(0)C(0)}$$

where DC gain of the open loop is $G(0)C(0)$.

4. The medium-frequency region (the region near the (-1,0) point) indicates the relative stability.

5. The high-frequency region indicates the complexity of the system.

$$G(s)C(s) = \frac{1}{1000s+100} \cdot \frac{1050s+670}{s} = \frac{1050s+670}{1000s^2+100s}$$



<u>What is the phase cross-over frequency? What is the gain margin?</u>

There is no phase cross-over frequency. The GM is infinity!

<u>What is the gain cross-over frequency? What is the phase margin?</u>

The PM is above 60^{o}

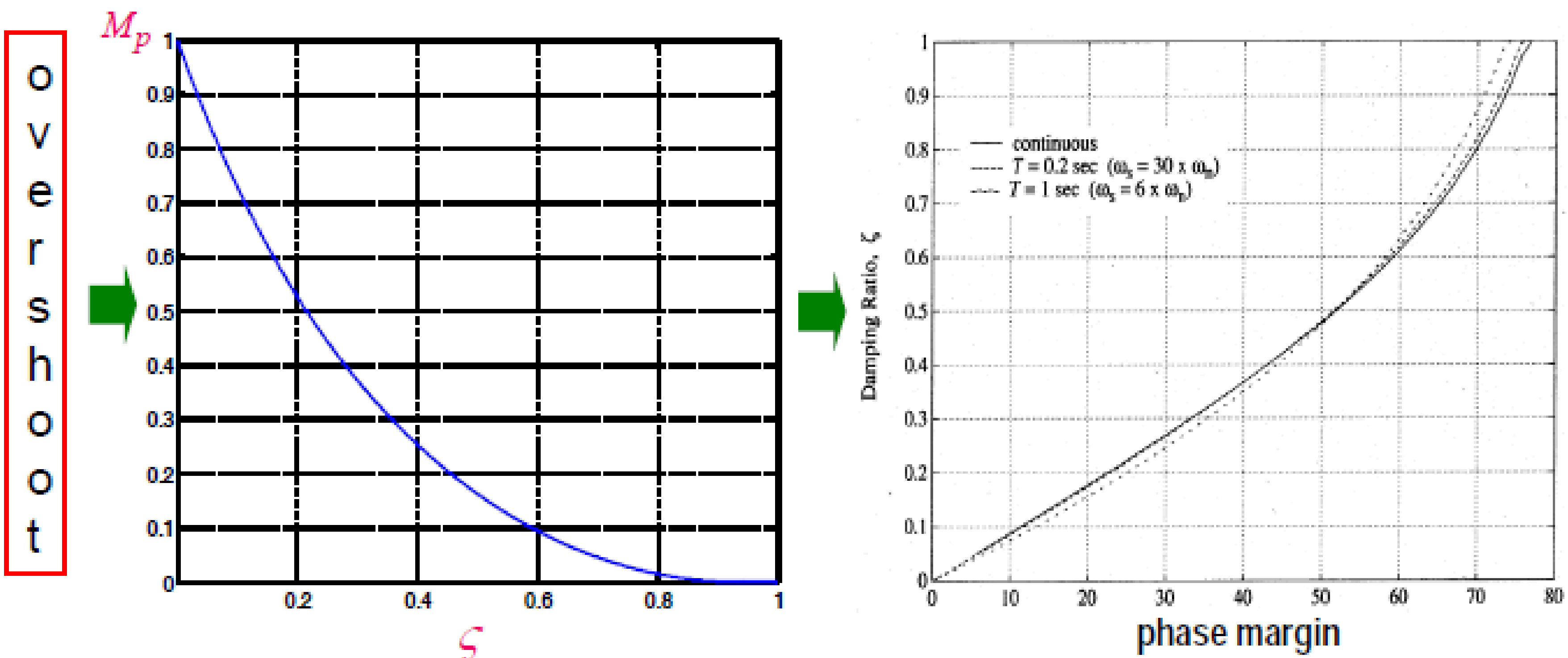
<u>Is the system stable?</u>

It is very stable or robust as PM is above 60 degrees and GM is infinity.

4.6. *Frequency domain design specifications*

It is important to note that transient-response performance (overshoot, rise time and settling time etc) is usually most important for control system design.

The time domain design specifications can be translated into the requirement on phase margin and some other frequency domain properties. In particular, we have the following relationships for typical second order systems:



What is the approximate equation relating damping ratio and PM?

$$\zeta \cong \frac{PM}{100}$$

Why do we want to design the control system in frequency domain?

Design in the frequency domain is simple and straightforward. The Bode diagram indicates clearly the manner in which the system should be modified, although the exact quantitative prediction of the transient-response characteristics cannot be made.

The frequency-response approach can be applied to systems whose dynamic characteristics are given in the form of experimental frequency-response data. Therefore it can be model-free in some sense.

Although the correlation between the transient response and frequency response is indirect, the frequency domain specifications can be conveniently met in the Bode diagram approach.

After the open-loop has been designed by the frequency-response method, the transient-response must be checked to see whether the designed system satisfies the requirements in the time domain. If it does not, then the compensator must be modified.

Requirements on the Open-loop Frequency Response

In many applications, compensation is essentially a compromise between steady-state accuracy and relative stability.

To have a high value of the position or velocity error constants and yet satisfactory relative stability, it is necessary to reshape the open-loop frequency-response curve.

For satisfactory performance, the phase margin should be between 30^o and 60^o and the gain margin should be greater than 6dB.

Normally, do we want a high gain in the low frequency region (G(0)C(0)) or not?

DC gain (G(0)C(0)) corresponds to the position error constant. The gain in the low frequency region should be large enough to assure good steady state accuracy, since the steady state error is given as 1/(1+K).

Do we want a high gain in the high frequency region or not?

For the high-frequency region, the gain should be attenuated as rapidly as possible to minimize the effects of noise.

4.7. *Frequency domain design methods*

A common approach to the frequency domain design is that we first adjust the open-loop gain so that the requirement on the steady-state accuracy is met.

Then the magnitude and phase curves of the uncompensated open loop (with the open-loop gain just adjusted) is plotted. If the specifications on the phase margin and gain margin are not satisfied, then a suitable compensator that will re-shape the open-loop transfer function is determined.

There are three commonly used frequency domain design methods, namely

1. Lead Compensator – which can be viewed as filtered PD control,

2. Lag Compensator – which can be viewed as modified PI control,

3. Lag-Lead Compensator.

Break

State-of-the-art control systems

RHEX

Big Dog

Little Dog

Another Big Dog

Petman

4.8. *Lead Compensator*

A PD controller is $C(s) = K_d s + K_p$. Adding a first order low pass filter

$$C(s) = \frac{K_d s + K_p}{cs+1} = k\frac{\tau s+1}{\alpha\tau s+1}$$

$$\boxed{\begin{aligned} k &= K_p \\ \tau &= K_d / K_p \\ \alpha &= K_p c / K_d \end{aligned}}$$

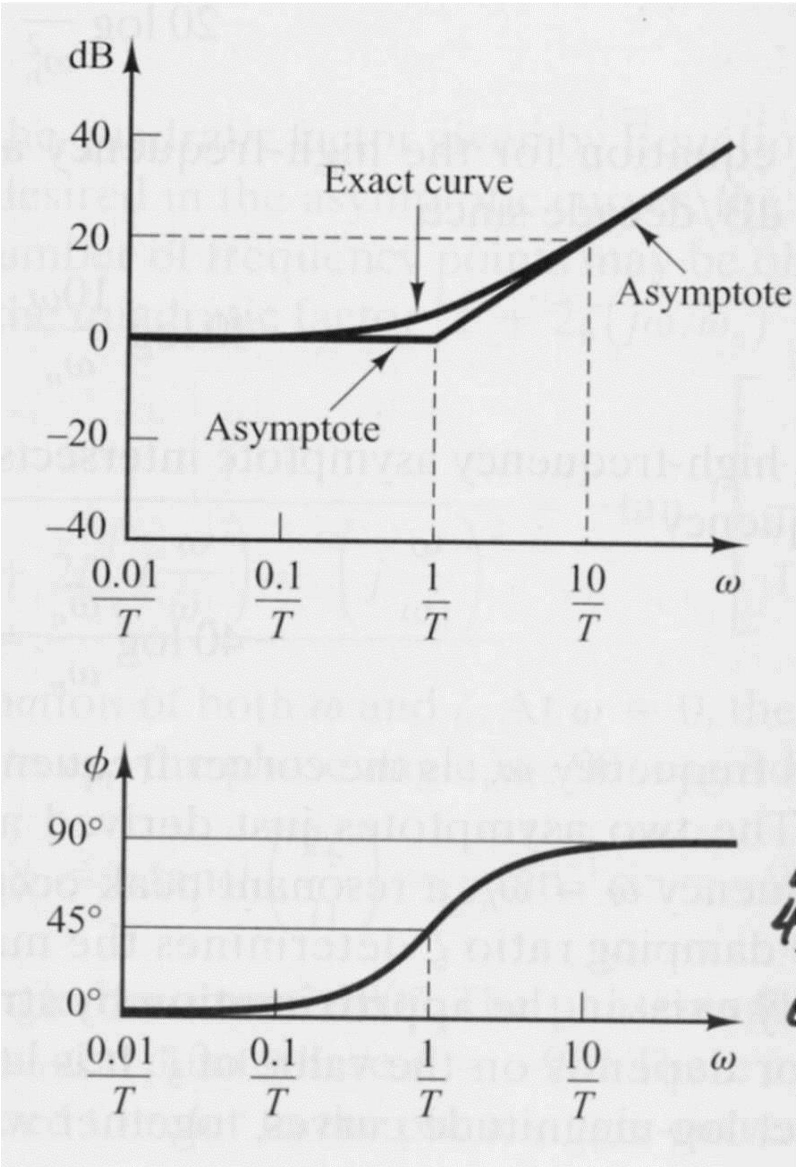
will lead to a lead compensator, if $\alpha < 1$.

Note that PD controller stabilizes the closed-loop. Therefore we can expect a similar role from a lead compensator.

The frequency response of PD controller

$$C(j\omega) = K_p + K_d j\omega = K_p(Tj\omega + 1) = K_p(\frac{j\omega}{1/T} + 1)$$

The frequency response of a typical PD controller (Ts+1)



What would happen if the signal is noisy?

The PD controller will amplify the high frequency noise!

This is the main reason that PD controller alone is rarely used in practice.

We need to add a low pass filter to filer out the noise! This is the motivation for the lead compensator.

The frequency response of lead compensator

$$C(j\omega) = k\frac{\tau j\omega + 1}{\alpha\tau j\omega + 1}, \alpha < 1$$

Let's try to use the asymptotes to plot out the frequency response.

The first step is to figure out the corner frequencies:

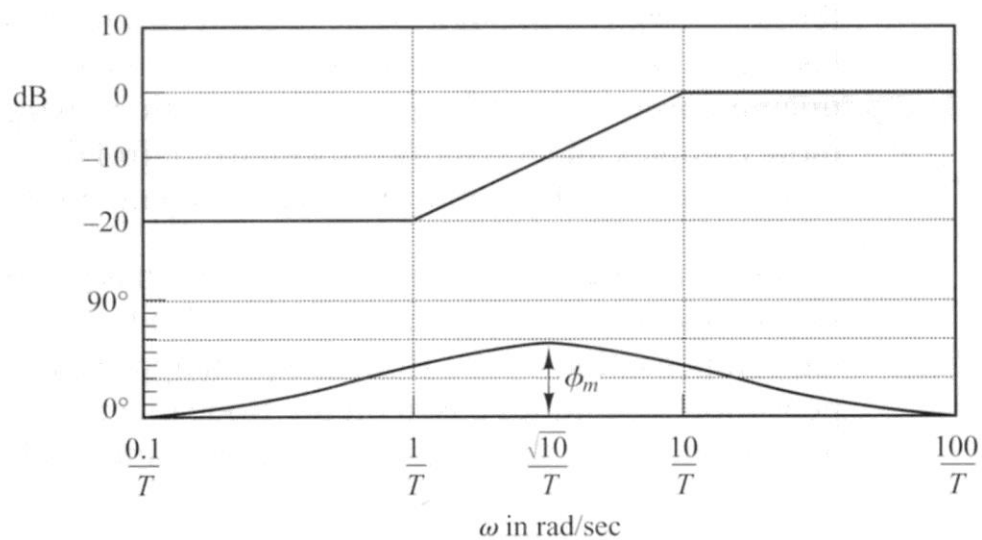
$$C(j\omega) = k\frac{\tau j\omega + 1}{\alpha\tau j\omega + 1} = k\frac{\frac{j\omega}{1/\tau}+1}{\frac{j\omega}{1/\alpha\tau}+1} = k\frac{\frac{j\omega}{\omega_1}+1}{\frac{j\omega}{\omega_2}+1}, \alpha < 1$$

$$\omega_1 = \frac{1}{\tau}, \omega_2 = \frac{1}{\alpha\tau}, \alpha < 1$$

$$\omega_1(zero) < \omega_2(pole)$$

If it is zero, the asymptote has positive slope.

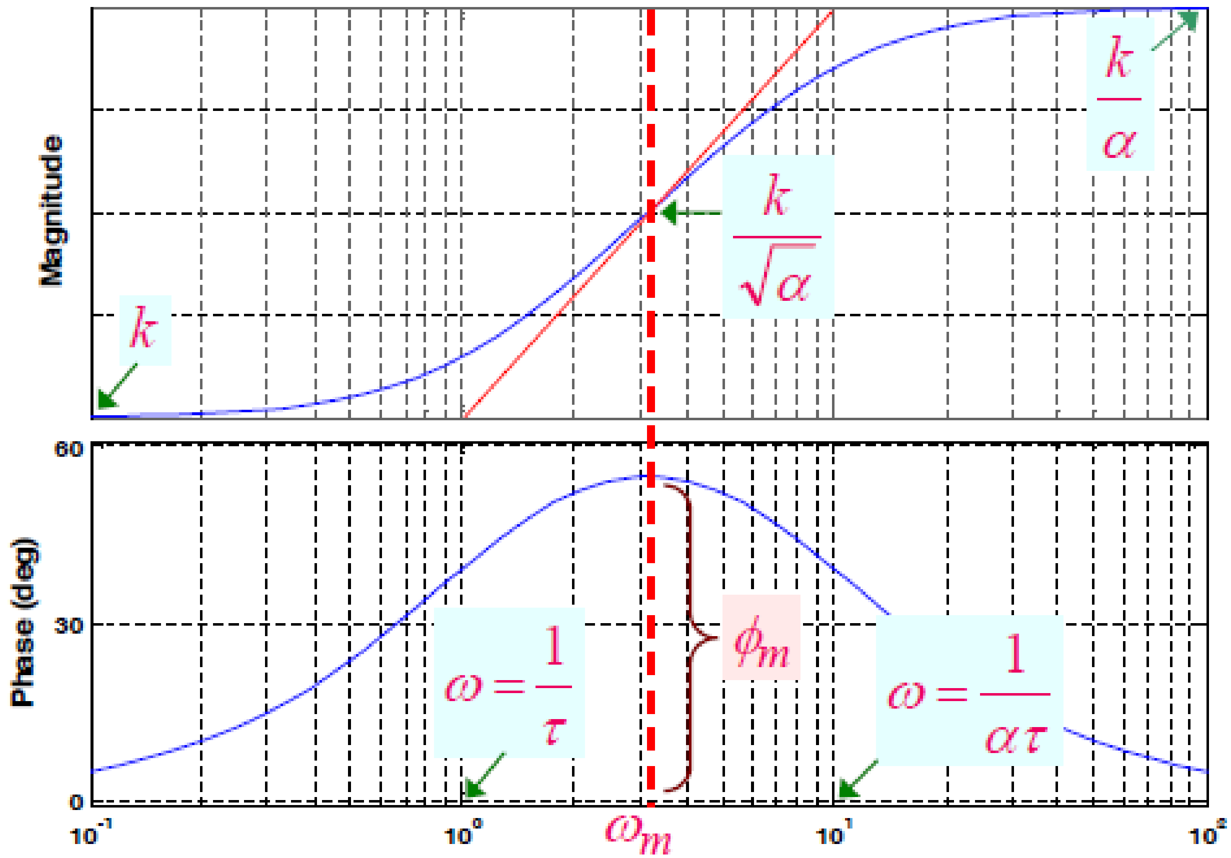
If it is pole, the asymptote has negative slope.



The frequency response of lead compensator

$$C(j\omega) = k\frac{\tau j\omega + 1}{\alpha\tau j\omega + 1}, \alpha < 1$$

If α is very small, then it is an approximation of PD controller.



$$\sin\phi_m = \frac{1-\alpha}{1+\alpha}$$

$$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}$$

$$\tau = \frac{1}{\omega_m\sqrt{\alpha}}, \omega_m = \frac{1}{\tau\sqrt{\alpha}}$$

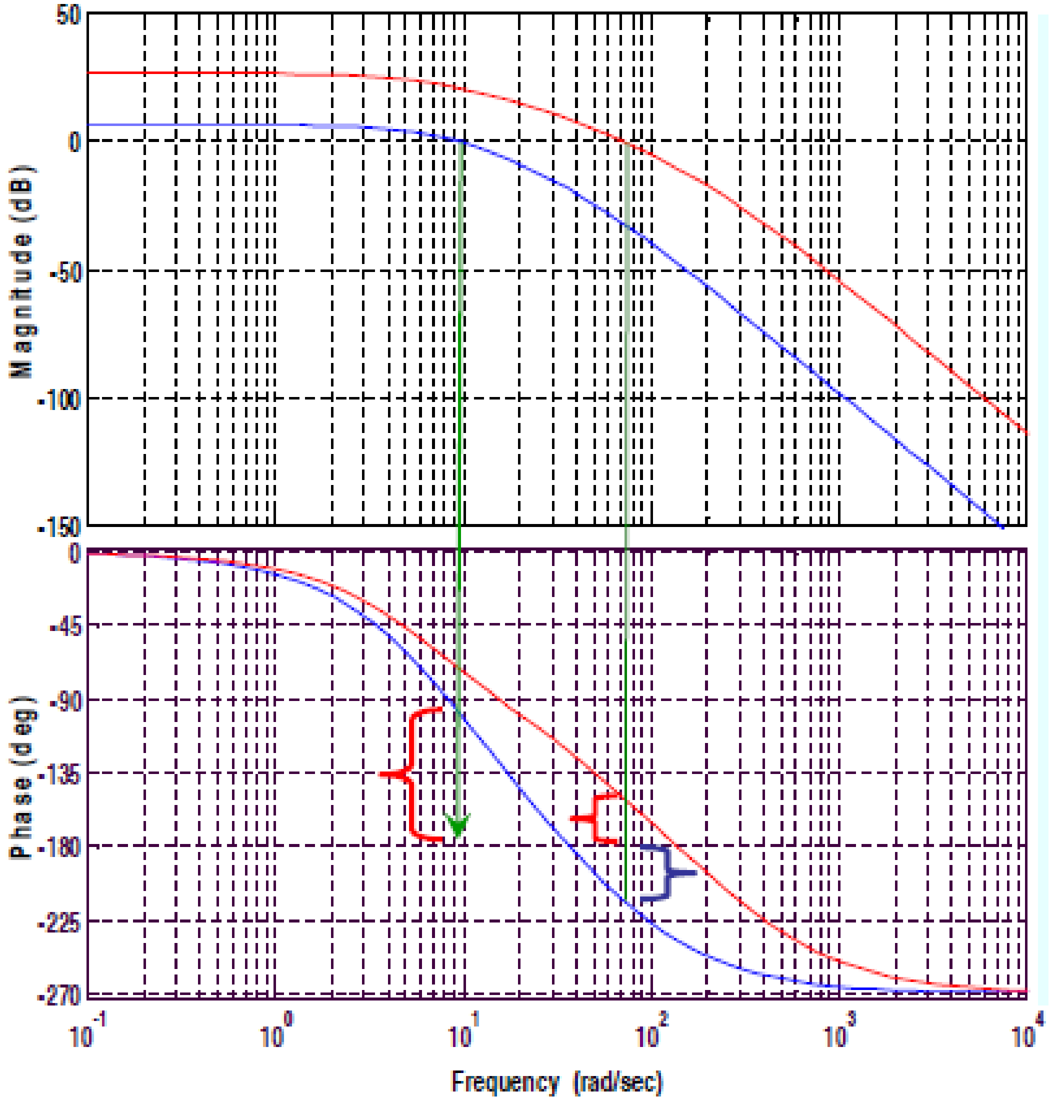
Here k and ϕ_m are determined from design specifications; ω_m is selected to be the gain cross-over frequency

What is the effect of the lead compensator C(s) on the original plant G(s)?

Will it lift up the Bode plot, or lower down?

It would lift up both the magnitude and the phase!

Uncompensated and lead-compensated systems



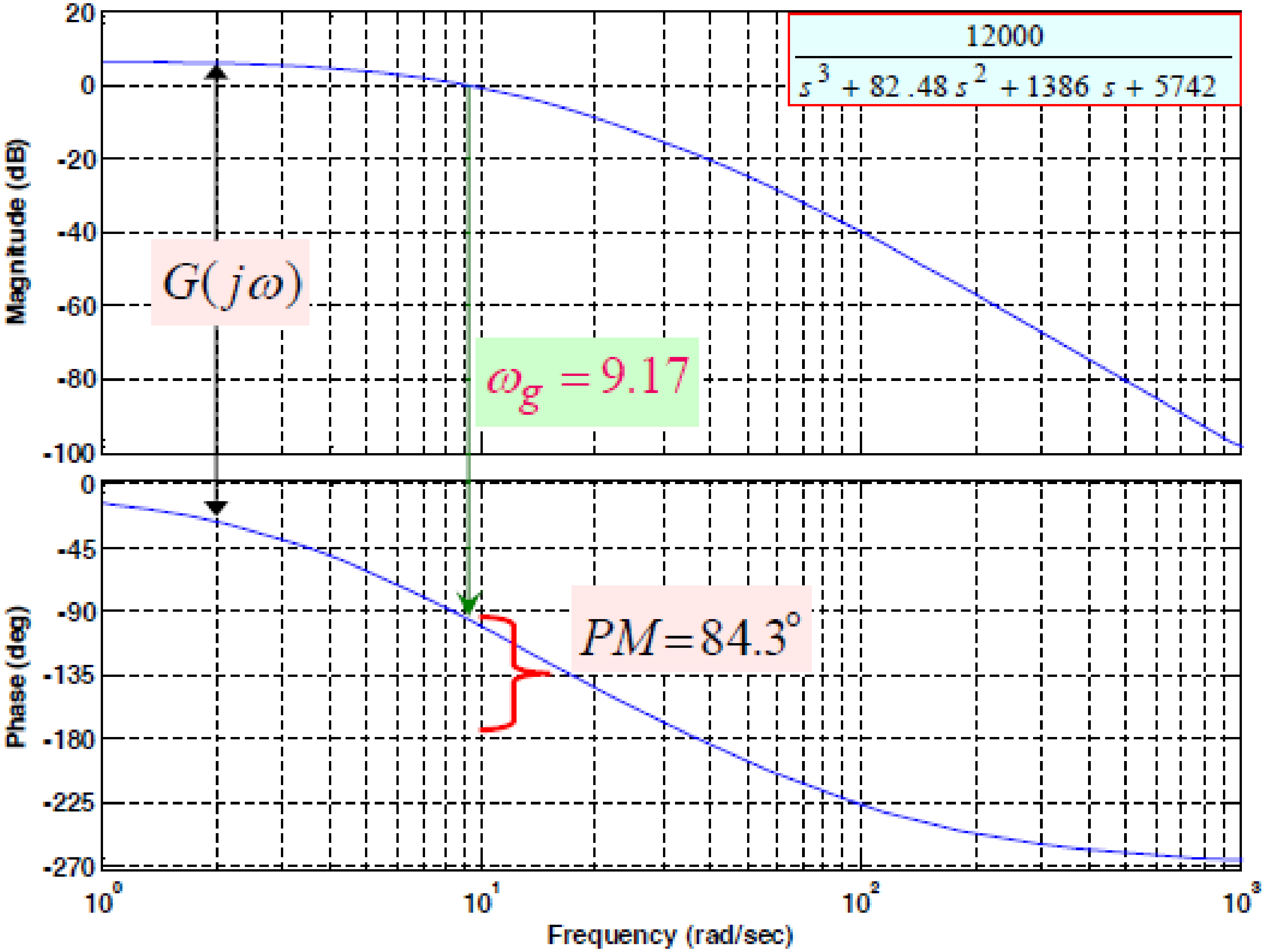
Key Idea:

1) enlarge the gain crossover frequency ω_g or increasing the gain k,
2) add additional phase ϕ_m to improve phase margin.

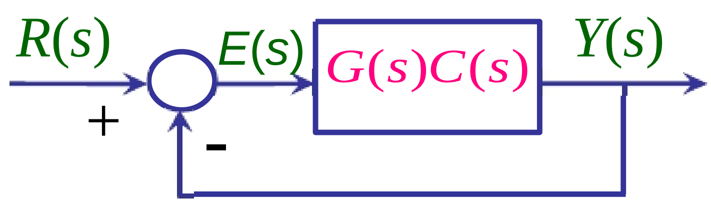
Performance:

1) k ↑ reduces steady error
2) ω_g ↑ leads to faster response
3) PM ↑ leads to more stable CL

Example: original plant : adequate phase margin but inadequate DC gain



For a unity feedback system:



What is the steady state error given the DC gain K (K=G(0)C(0))?

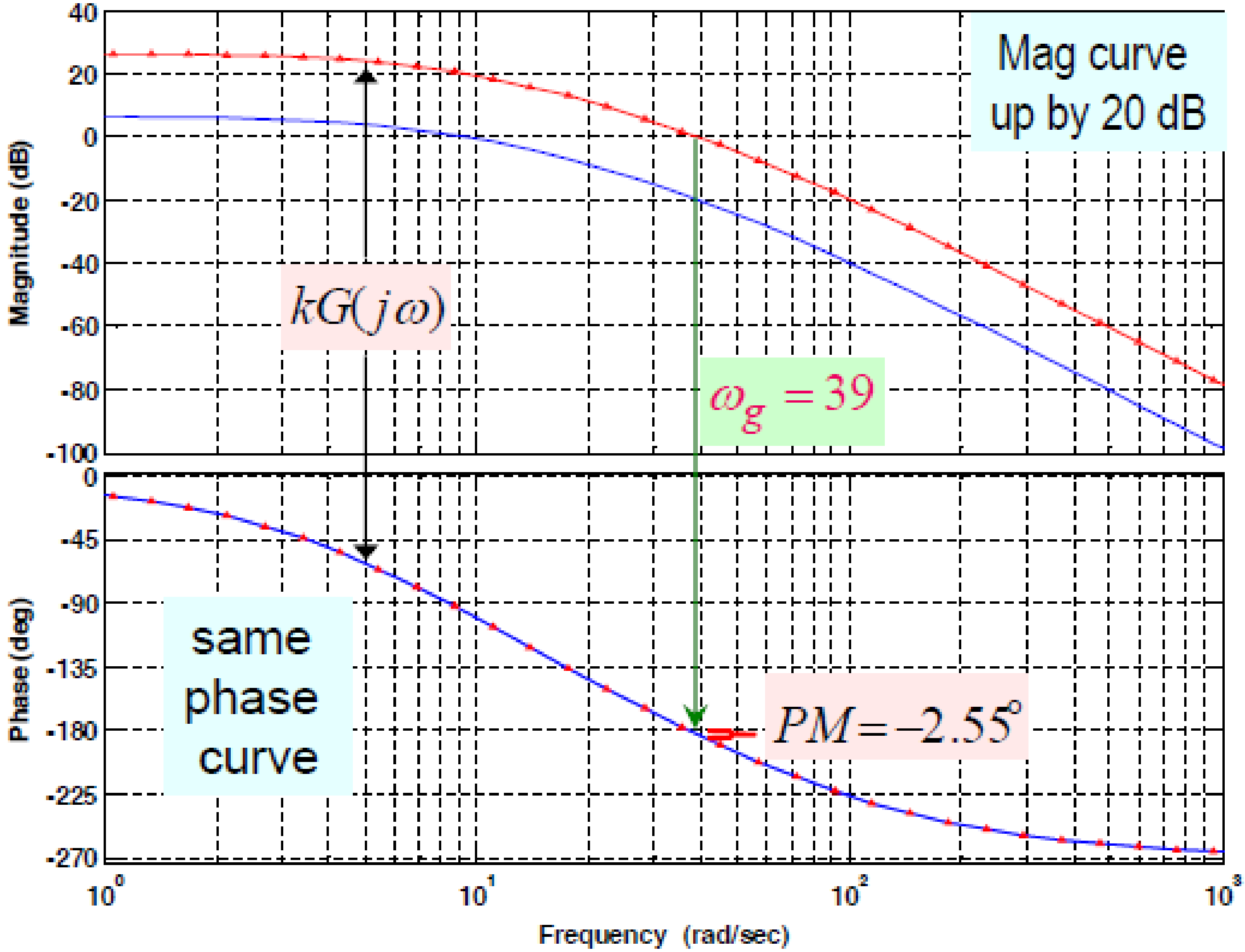
The steady state error of the closed-loop is 1/(1+K).

We need to increase the DC gain to improve the steady-state performance.

What would happen if we want to improve the DC gain by adding a gain directly , KG(s)?

How would the magnitude plot change? It will lift up the magnitude plot.

How would the phase plot change? The phase plot will not be affected at all.



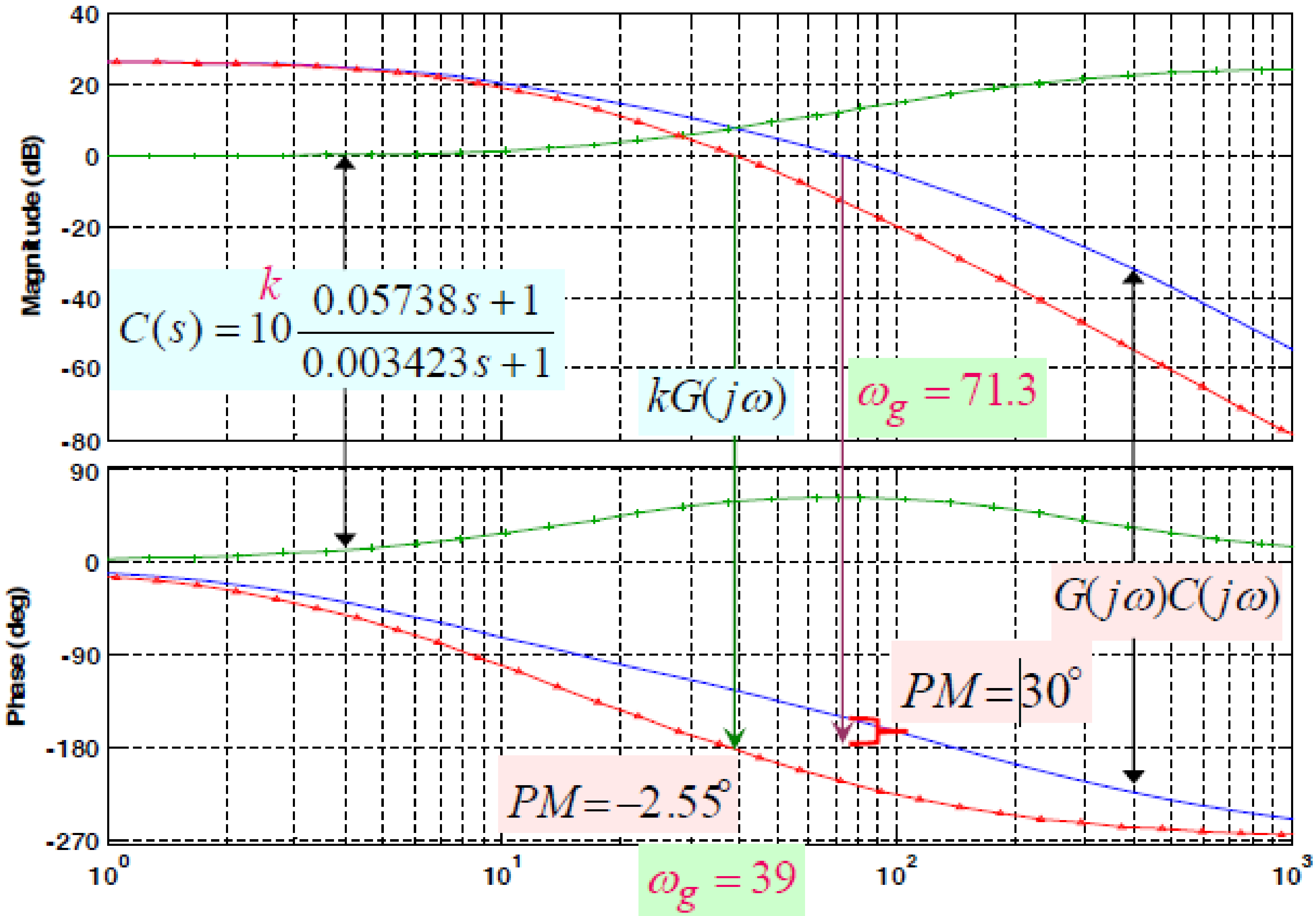
Only adding a DC gain (k=10) : negative phase margin

Is the closed loop system stable?

No. It is dangerous if we simply use a big gain!

What we need is a controller which lifts up both the magnitude and the phase!

Lead compensation : adding leading phase around ω_g



4.9. *Lead compensation design procedure – a trial and error approach*

Step 1. Determine open-loop gain k to satisfy requirements on steady state error.

Step 2. Find new open-loop crossover frequency ω_g from $kG(j\omega)$

Step 3. Evaluate the PM of $kG(j\omega)$ at the new crossover frequency, i.e.

$$\phi = 180^o + \angle kG(j\omega_g)$$

Compare the PM with the desired one $\phi_{desired}$, and estimate what is the phase to be added.

From the Bode plot of the lead compensator, the maximum phase of the lead compensator ϕ_m, should be added at the gain cross-over frequency of the compensated system, ω_m.

But do we know the gain cross-over frequency of the compensated system, ω_m, at this step?

No. We only know ω_g, which is the gain crossover frequency of $kG(j\omega)$, the uncompensated system. Therefore, we need to estimate ω_m.

The simplest way is to use the gain crossover frequency of the uncompensated system ω_g, to approximate that of the compensated system ω_m, and guess

$$\phi_m \approx \phi_{desired} - \phi$$

4.9. *Lead compensation design procedure – a trial and error approach*

Step 4. Determine the necessary phase-lead angle to be added to the system. Allow for some extra angle (5° to 12°), because the addition of the lead compensation shifts the gain crossover frequency to the right and further decrease the phase margin.

Since we do not know what the new crossover frequency would be at this step, we need to take a guess on the extra phase to add.

$$\phi_m = \phi_{desired} - \phi + (5^\circ \rightarrow 12^\circ)$$

Step 5. Compute

$$\boxed{\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}}$$

Step 6. Now we are ready to find out the new gain crossover frequency ω_m, where

$$|C(j\omega_m)kG(j\omega_m)|=1.$$

$$20\log_{10}|C(j\omega_m)|+20\log_{10}|kG(j\omega_m)|=0$$

$$20\log_{10}|kG(j\omega_m)|=-20\log_{10}|C(j\omega_m)|$$

From the Bode plot of the lead compensator, we know that at ω_m

$$|C(j\omega_m)|=\frac{1}{\sqrt{\alpha}} \quad \Rightarrow \quad 20\log_{10}(\frac{1}{\sqrt{\alpha}})\,dB$$

We need to find out the frequency where the magnitude plot of $kG(j\omega)$ is

$$-20\log_{10}(\frac{1}{\sqrt{\alpha}})\,dB$$

Once ω_m is determined, we then compute

$$\boxed{\tau=\frac{1}{\sqrt{\alpha}\omega_m}}$$

Step 4 Extra PM (5° to 12°) is chosen trial and error. Thus this is a trial and error design.

Thus the actual design will be done iteratively. That is the reason why another step is needed.

Step 7. Verify the design using MATLAB. Redo if necessary.

Design example of *lead compensator*

The plant: $G(s)=\dfrac{4}{s(s+2)}$

The lead compensator: $G_c(s)=k\dfrac{\tau s+1}{\alpha\tau s+1}$

Design specifications:

1. The velocity error constant is 20
2. The phase margin is at least 50^o
3. The gain margin is at least 10 dB.

Step 1. Determine open-loop gain *k* to satisfy requirements on steady state error.

The velocity error constant is given

$$\beta_v=\lim_{s\to 0} sG_c(s)G(s)=\lim_{s\to 0} k\frac{\tau s+1}{\alpha\tau s+1}\frac{4}{s+2}=2k=20$$

$\Rightarrow$ k=10

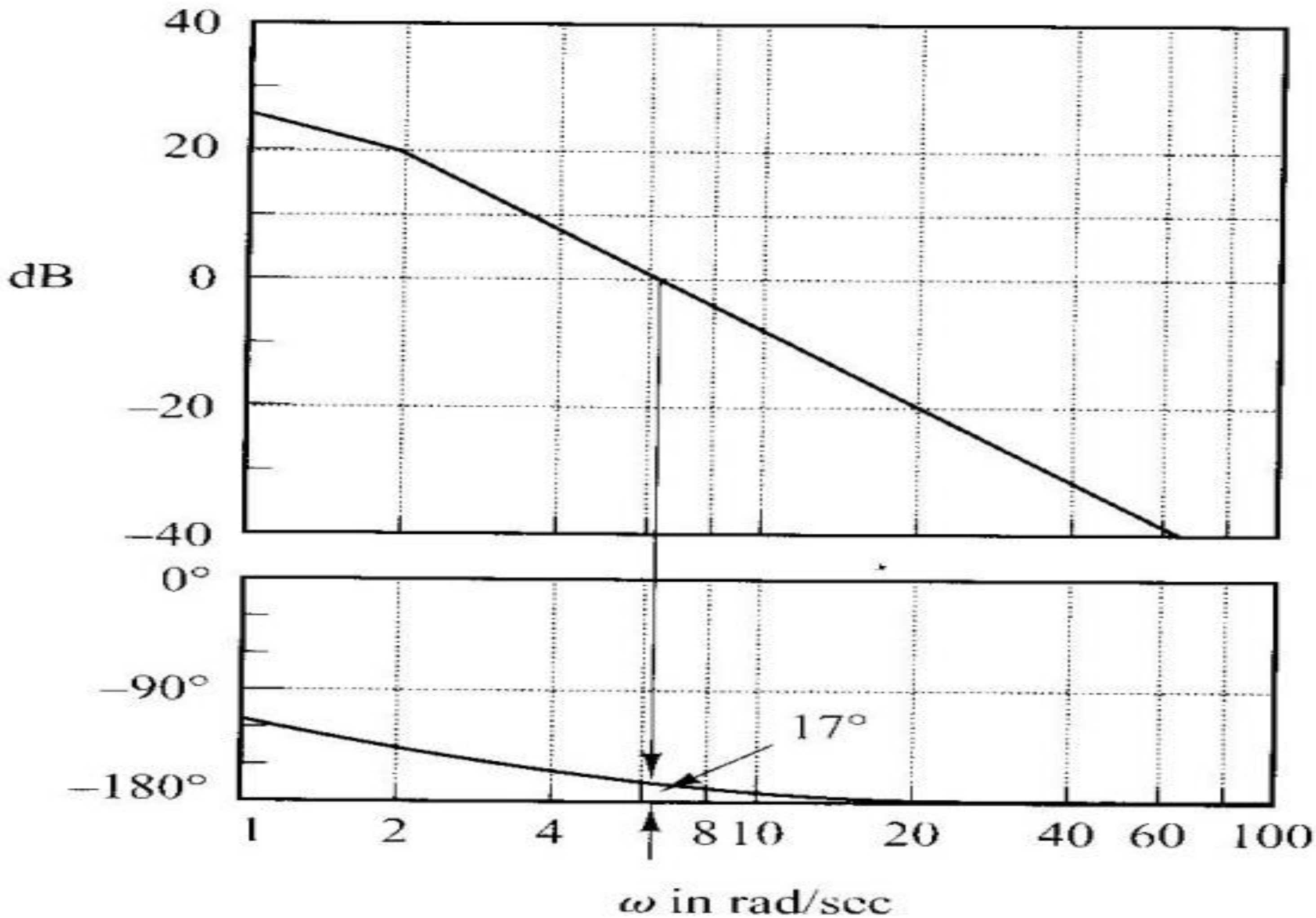
Design example of *lead compensator*

Step 2. Find new gain open-loop crossover frequency from $kG(j\omega)$

$$\omega_g = 6.3$$

Step 3. Evaluate the PM of $kG(j\omega)$ at the new crossover frequency, i.e.

$$\phi = 180^o + \angle kG(j\omega_g) = 17^o$$



Design example of *lead compensator*

Step 4. Determine the necessary phase-lead angle to be added to the system. Allow for some extra angle (5° to 12°), because the addition of the lead compensation shifts the gain crossover frequency to the right and further decrease the phase margin.

$$\phi_m = \phi_{desired} - \phi + 5^\circ = 50^o - 17^o + 5^\circ = 38^o$$

Step 5. Compute

$$\boxed{\alpha = \frac{1 - \sin\phi_m}{1 + \sin\phi_m} = 0.24}$$

The magnitude of the lead compensator at the frequency, ω_m, is

$$\frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{0.24}} = \frac{1}{0.49} \Rightarrow \text{6.2 dB}$$

Step 6. Find out the frequency where $|kG(j\omega)|$ is equal to -6.2 dB, such that

$$|G_c(j\omega_m)G(j\omega_m)| = 1 \rightarrow 0dB.$$

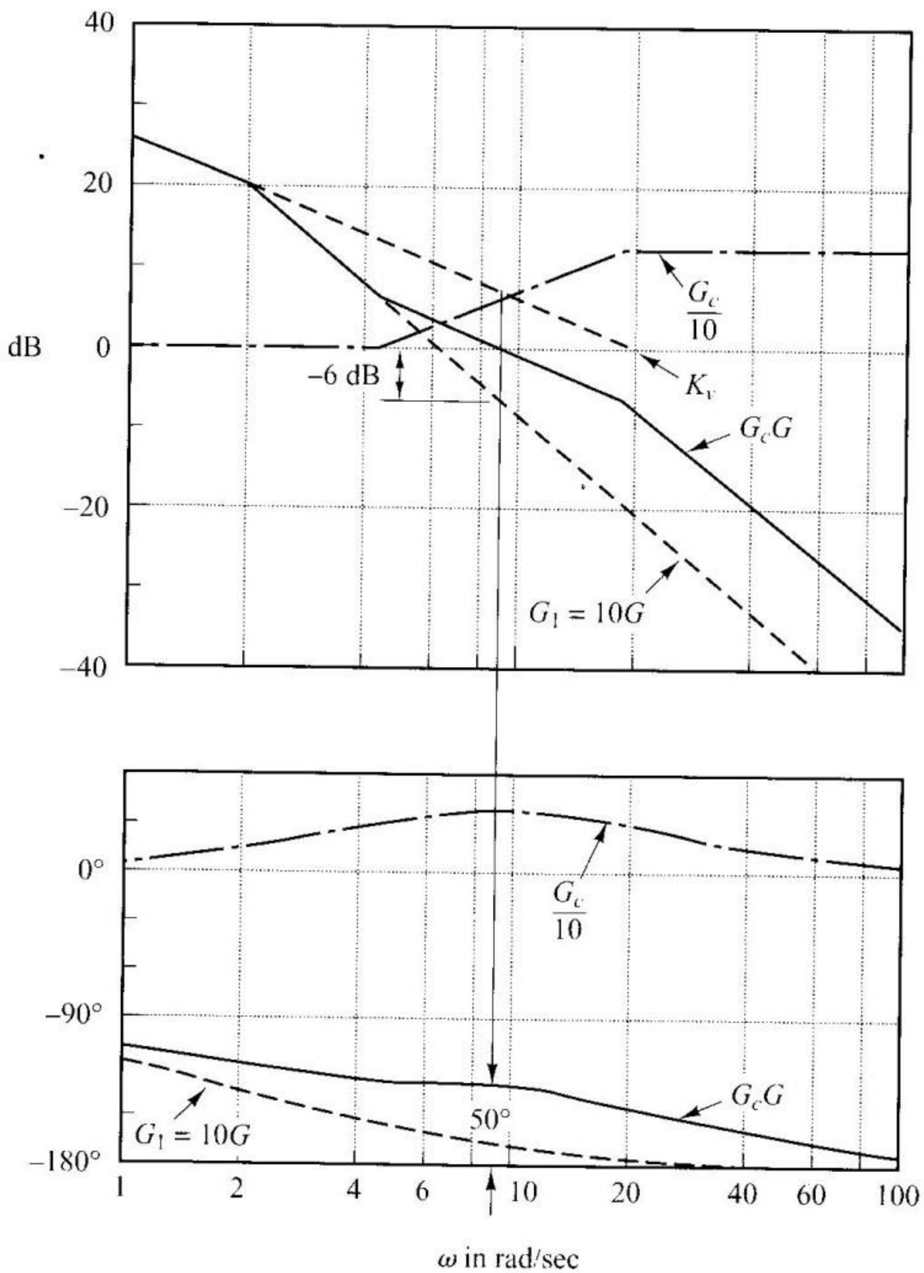
The gain crossover frequency $\omega_m = 9$

and then compute

$$\boxed{\tau = \frac{1}{\sqrt{\alpha}\omega_m} = \frac{1}{4.41} = 0.227}$$

Design example of *lead compensator*

The overall system: $G_c(s)G(s) = k\dfrac{\tau s+1}{\alpha\tau s+1}\dfrac{4}{s(s+2)} = 41.7\dfrac{s+4.41}{s+18.4}\dfrac{4}{s(s+2)}$



The phase margin: 50^o

What is the gain margin?

Infinity!

All the design specifications are met!

So far, everything we talked about the frequency domain design is for continuous time system, What about the digital control system?

4.11. *Discrete-time Lead compensation*

The frequency domain design methods for continuous-time systems and sampled discrete-time systems are basically the same. The real plant is continuous in nature.

If the continuous-time plant model G(s) is known, then the controller C(s) can be designed first, and then use bilinear transformation to convert it into digital controller.

But what if the continuous-time model G(s) is not known, and only the discrete-time model G(z) is given? Can we design a digital lead compensator directly in z-domain?

No. Bode diagram is not easy to plot in discrete-time. Lead compensator is normally designed in continuous-time.

We need to estimate the original TF of the continuous-time system. Given G(z), how to get back G(s)? Have you learned that in Part I?

For discrete-time systems G(z), the idea is to use a so-called w-transformation (actually it is a bilinear transformation) to transform the discrete-time system into a w-domain system, which is pretty much the same as a continuous-time system.

Everything we have learnt from the compensation design for continuous-time systems can be directly applied to yield a necessary compensator in w-domain, which can be transformed back to a discrete-time version through an inverse w-transformation (an inverse bilinear transformation).

given $G(z)$

$$G(w) = G(z)\Big|_{z=\frac{\frac{2}{T}+w}{\frac{2}{T}-w}}$$

design compensator $C(w)$ to meet all specifications (treat w as s)

$$C(z) = C(w)\Big|_{w=\frac{2}{T}\cdot\frac{z-1}{z+1}}$$

4.12. Frequency Domain *Design example of digital lead compensator*

An electric motor can be modeled with a sampling period $T = 0.1\text{s}$ as follows:

$$G(z) = \frac{0.0001z + 0.0001}{(z-0.99)(z-0.995)}$$

Design a digital control system with a lead compensator such that the resulting system output tracks a step reference, with an overshoot less than 20% and steady state error less than 3% .

Converting this to w-plane, we have

$$\boxed{G(w) = G(z)\Big|_{z=\frac{2/T+w}{2/T-w}} = \frac{0.0001\dfrac{20+w}{20-w} + 0.0001}{\left(\dfrac{20+w}{20-w} - 0.99\right)\left(\dfrac{20+w}{20-w} - 0.995\right)} = \frac{-0.004w + 0.08}{3.97w^2 + 0.598w + 0.02}}$$

Step 0. Select a lead compensator

$$C(w) = k\frac{\tau w + 1}{\alpha\tau w + 1}, \alpha < 1$$

In frequency domain, by replacing w by $j\omega$, the compensator is

$$C(j\omega) = k\frac{\tau j\omega + 1}{\alpha\tau j\omega + 1}$$

Step 1. Determination of static gain k

First check the adequacy of the plant DC gain

$$G(w)\Big|_{w=0} = \frac{-0.004\times 0 + 0.08}{3.97\times 0^2 + 0.598\times 0 + 0.02} = 4$$

Steady state error is $e(\infty) = \frac{1}{1+4} = 20\%$

Additional gain is needed.

From the requirement on steady state error,

$$e(\infty)=\frac{1}{1+\beta_p}=0.03 \quad \Rightarrow \quad \beta_p=\frac{1}{0.03}-1\approx 32$$

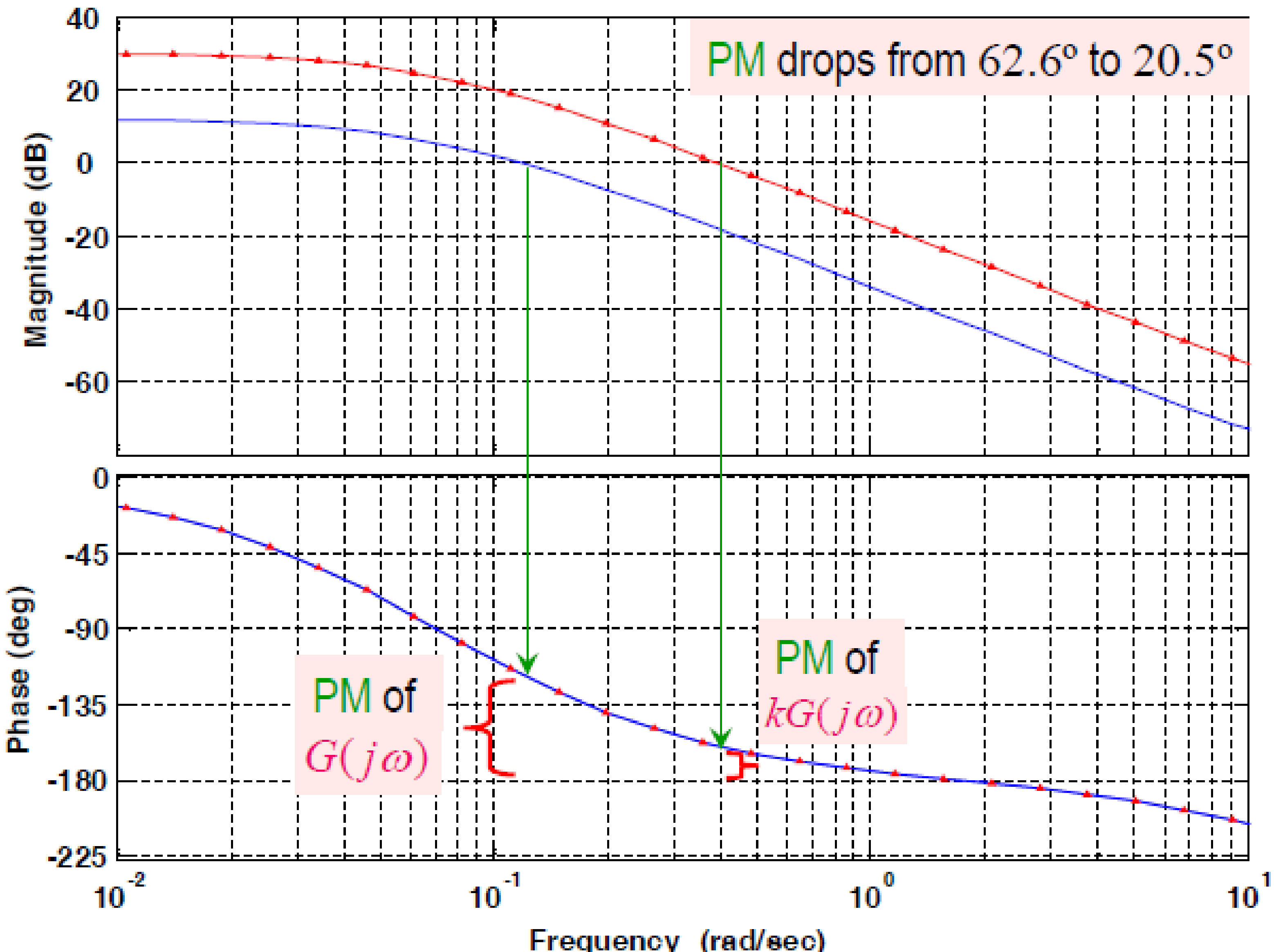
$$\Rightarrow \quad \beta_p=G(w)C(w)\Big|_{w=0}=4\cdot k=32$$

$$\Downarrow$$

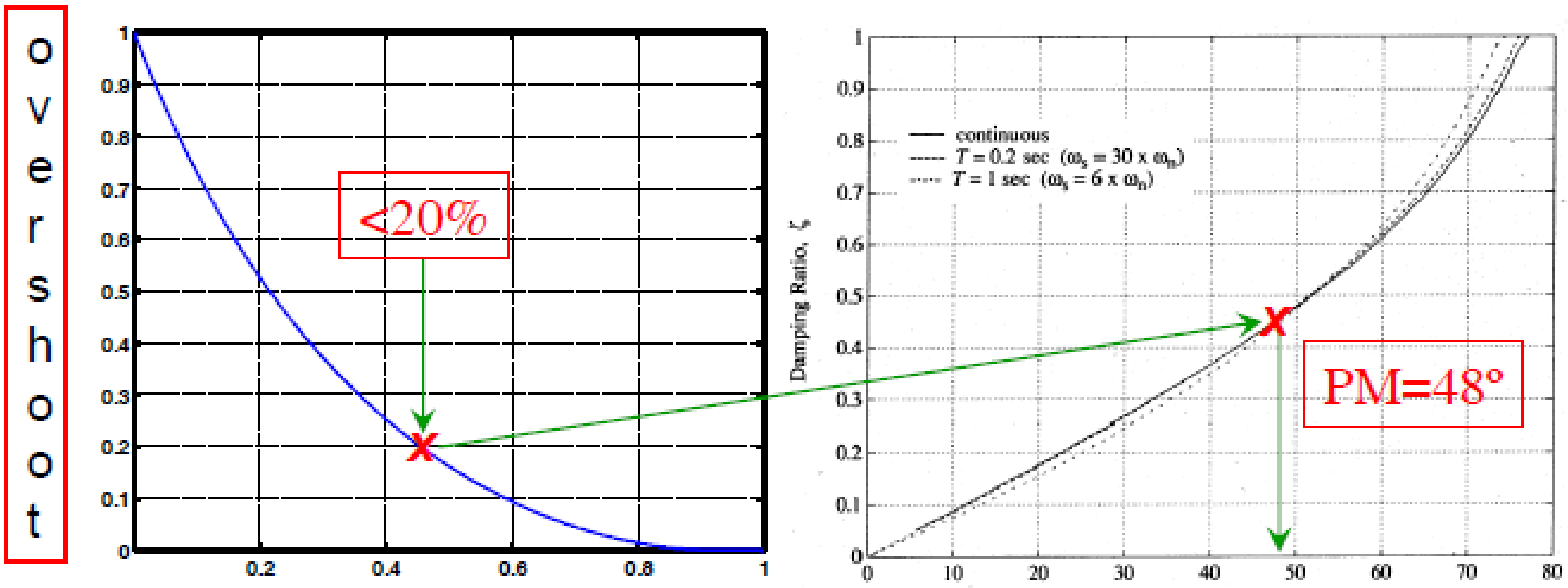
$$k=8 \quad \Rightarrow \quad 20\log_{10}8=18dB$$

Step 2. Determine the needed phase lead

– Evaluate PM of $kG(j\omega)$



The needed phase lead can be calculated below:



$\varsigma = 0.46$

Choose $\phi_{desired} = 50^\circ$

About 30° phase lead is needed.

$$\phi_m = \phi_{desired} - \phi + 7^\circ = 50^o - 20.5^o + 7^\circ = 36.5^o$$

Compute $$\boxed{\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m} = 0.254}$$

The magnitude of the lead compensator at the frequency, ω_m, is

$$\frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{0.254}} = \frac{1}{0.504} \Rightarrow$$ 6.0 dB

Find out the frequency where $|kG(j\omega)|$ is equal to -6.0 dB, i.e., where the magnitude of the compensated system is 1 (0dB),

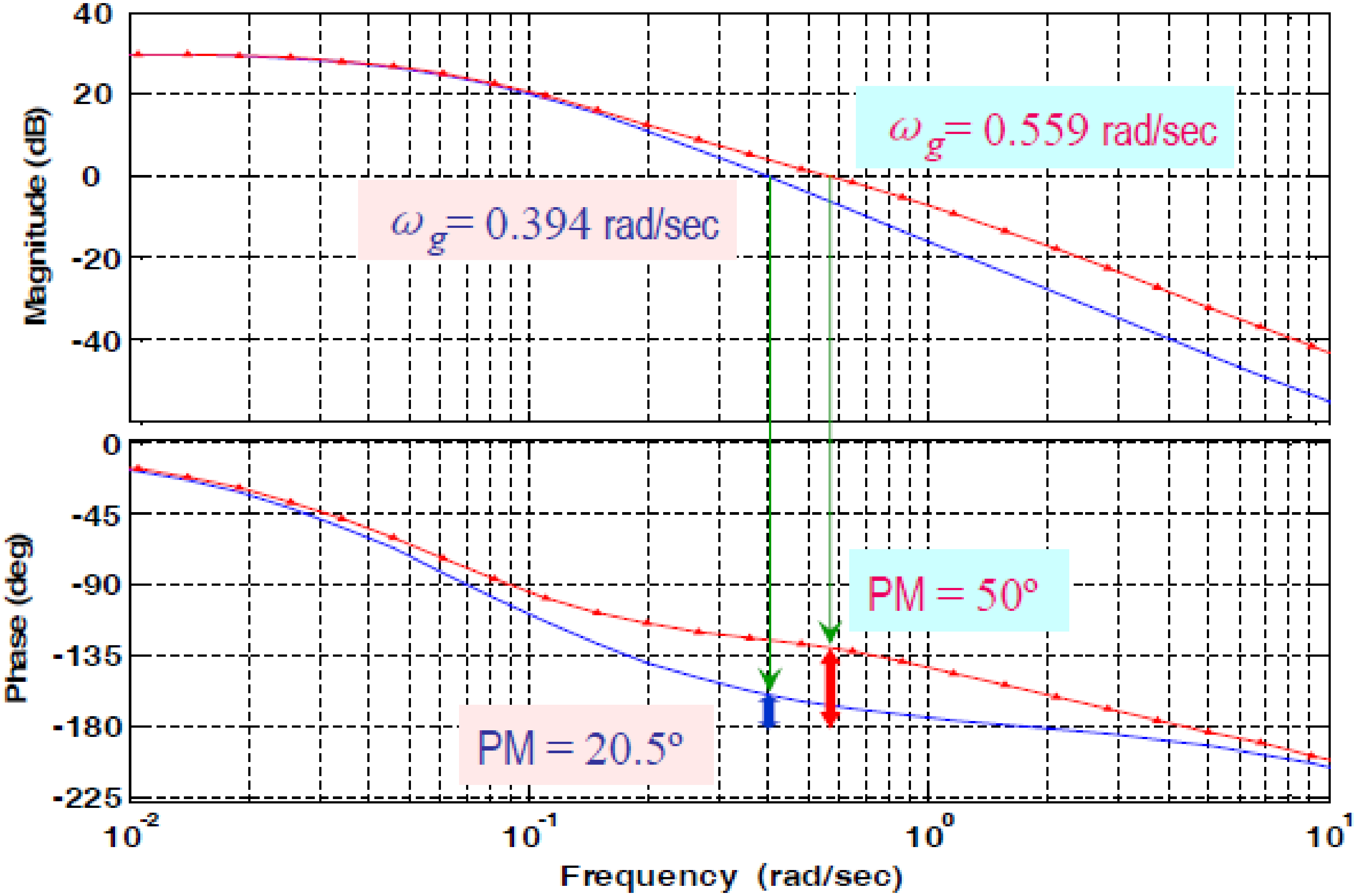
<u>The gain crossover frequency</u> $$\omega_m = 0.56$$

Then compute $$\boxed{\tau = \frac{1}{\sqrt{\alpha}\,\omega_m} = 3.54}$$

Finally we obtain the lead compensator in w-domain

$$C(w) = k\frac{\tau w+1}{\alpha\tau w+1} = 8\frac{3.54w+1}{0.9w+1}$$

After compensation, PM is improved. Bandwidth is also improved.



Discretization

The compensator can be converted back in z-domain using the inverse bilinear transform

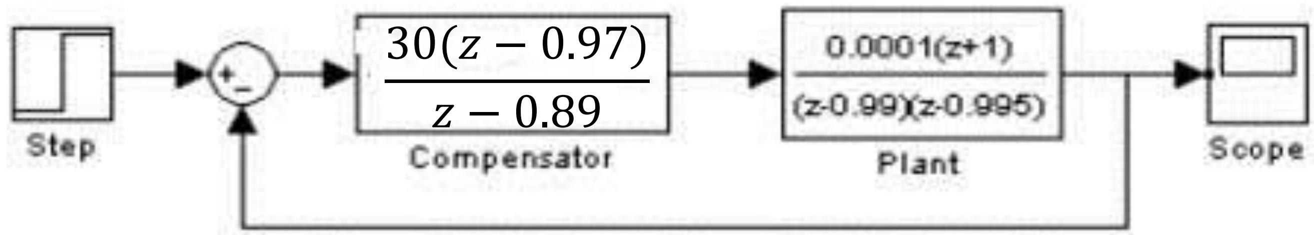
$$\boxed{\begin{aligned} C(z) &= C(w)\Big|_{w=\frac{2}{T}\cdot\frac{z-1}{z+1}} \\ &= 8\,\frac{3.54w+1}{0.9w+1}\Bigg|_{w=\frac{2}{0.1}\cdot\frac{z-1}{z+1}} \\ &= 30\,\frac{z-0.97}{z-0.89} \end{aligned}}$$

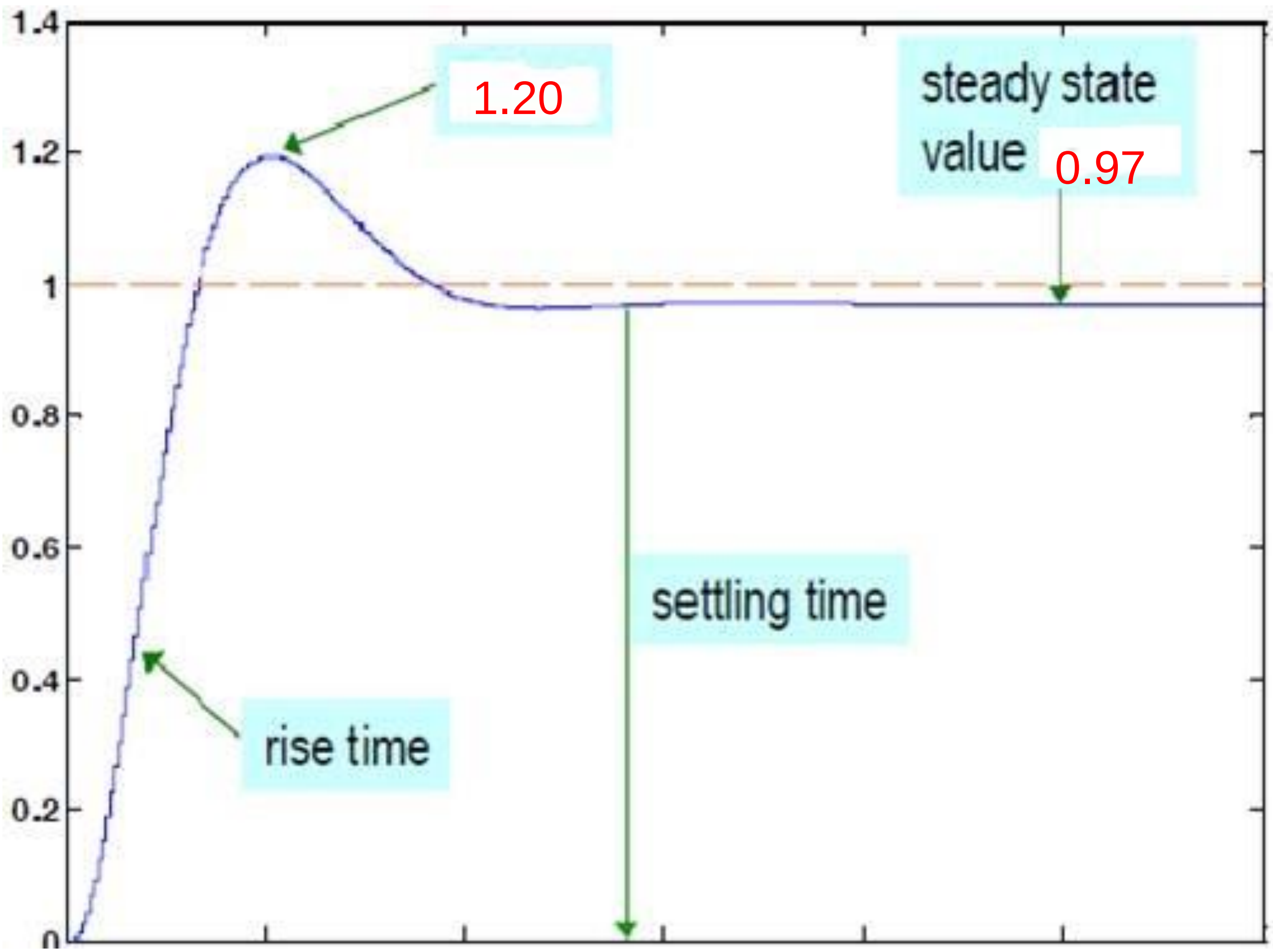
Verification

The above is the digital lead compensator. However, nothing is certain without verification. We need to first verify our design in frequency domain. More importantly, it should also meet the design specifications in time domain.

- Verification in *w*-domain (done)
- Verification in *time*-domain

Verification in *time*-domain





Note:

Steady state error is 3.0%,
settling time is 14 s,
and overshoot is 20%.

All the objectives are met!

Break

State-of-the-art control systems

RHEX

Big Dog

Little Dog

Another Big Dog

Petman

4. 13. *Lag compensation*

Lag compensation has the same type of transfer function as Lead compensator

$$C(j\omega)=k\frac{\tau j\omega+1}{\alpha\tau j\omega+1},\alpha>1$$

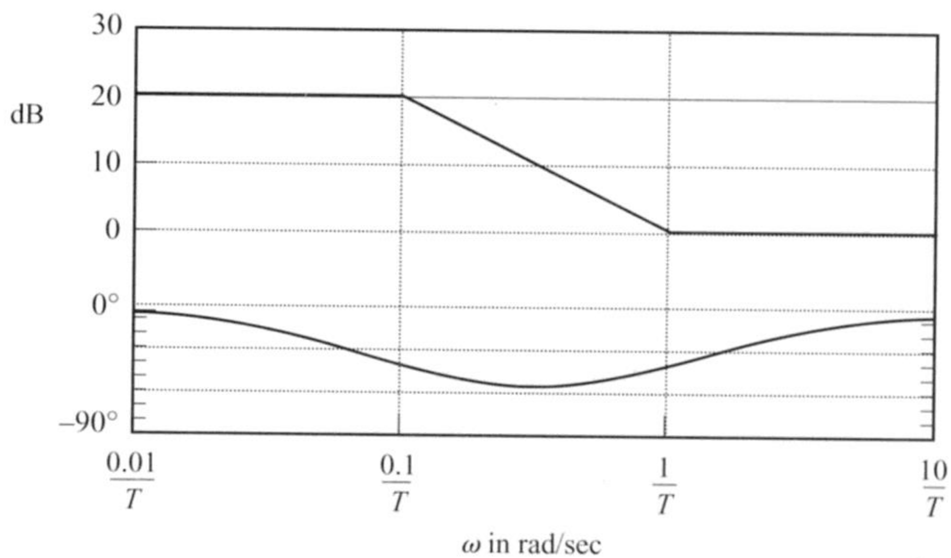
The only difference is that $\alpha>1$

Let's try to use the asymptotes to draw the frequency response.

The first step is to figure out the corner frequencies:

$$C(j\omega)=k\frac{\tau j\omega+1}{\alpha\tau j\omega+1}=k\frac{\frac{j\omega}{1/\tau}+1}{\frac{j\omega}{1/\alpha\tau}+1},\alpha>1 \quad \Rightarrow \quad \omega_1=\frac{1}{\tau},\omega_2=\frac{1}{\alpha\tau},\alpha>1$$

$$\omega_1(zero)>\omega_2(pole)$$



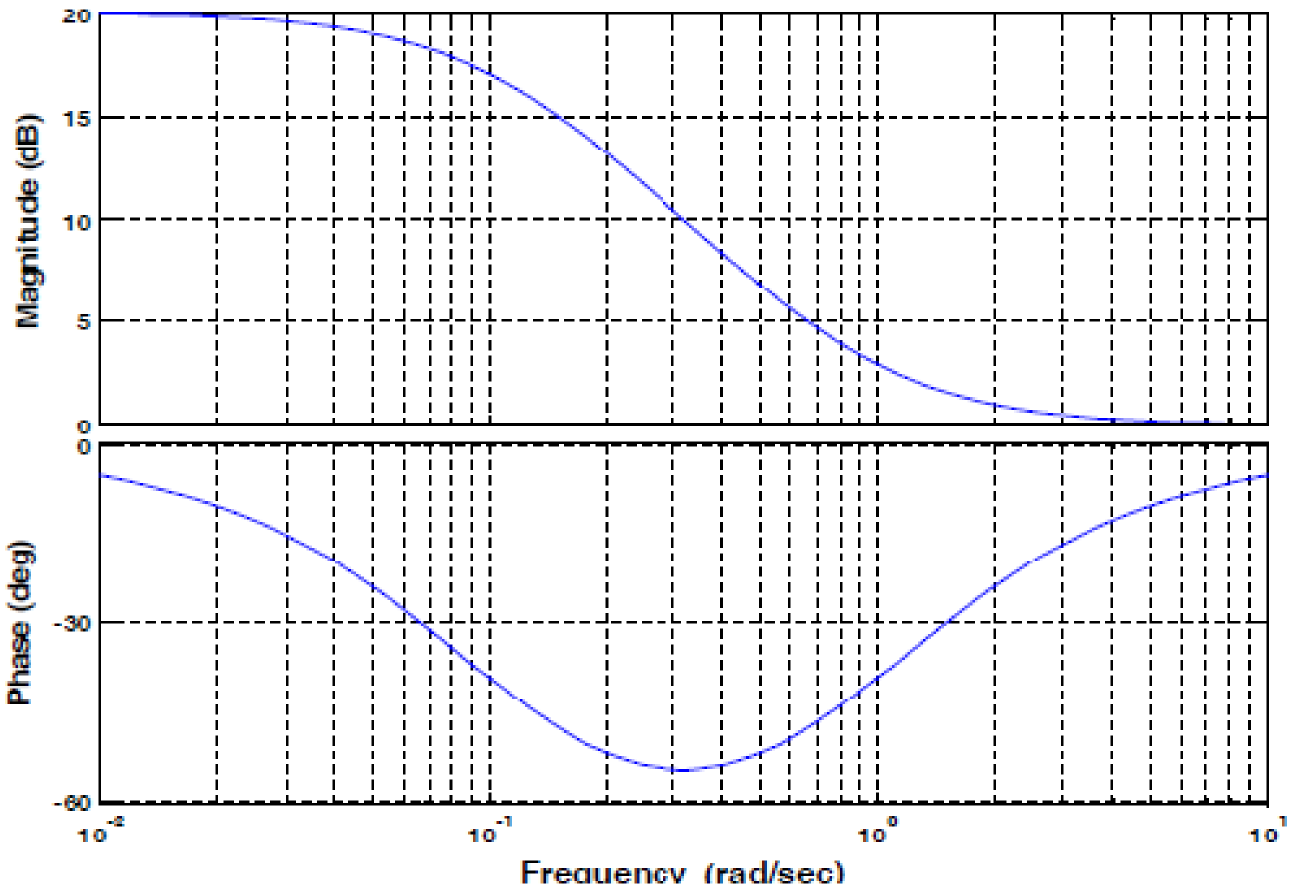
4. 13. *Lag compensation*

Lag compensation has the same type of transfer function as Lead compensator

$$C(j\omega) = k\frac{\tau j\omega + 1}{\alpha\tau j\omega + 1}, \alpha > 1$$

The only difference is that $\alpha > 1$

For a large α , it is an approximation of PI control.

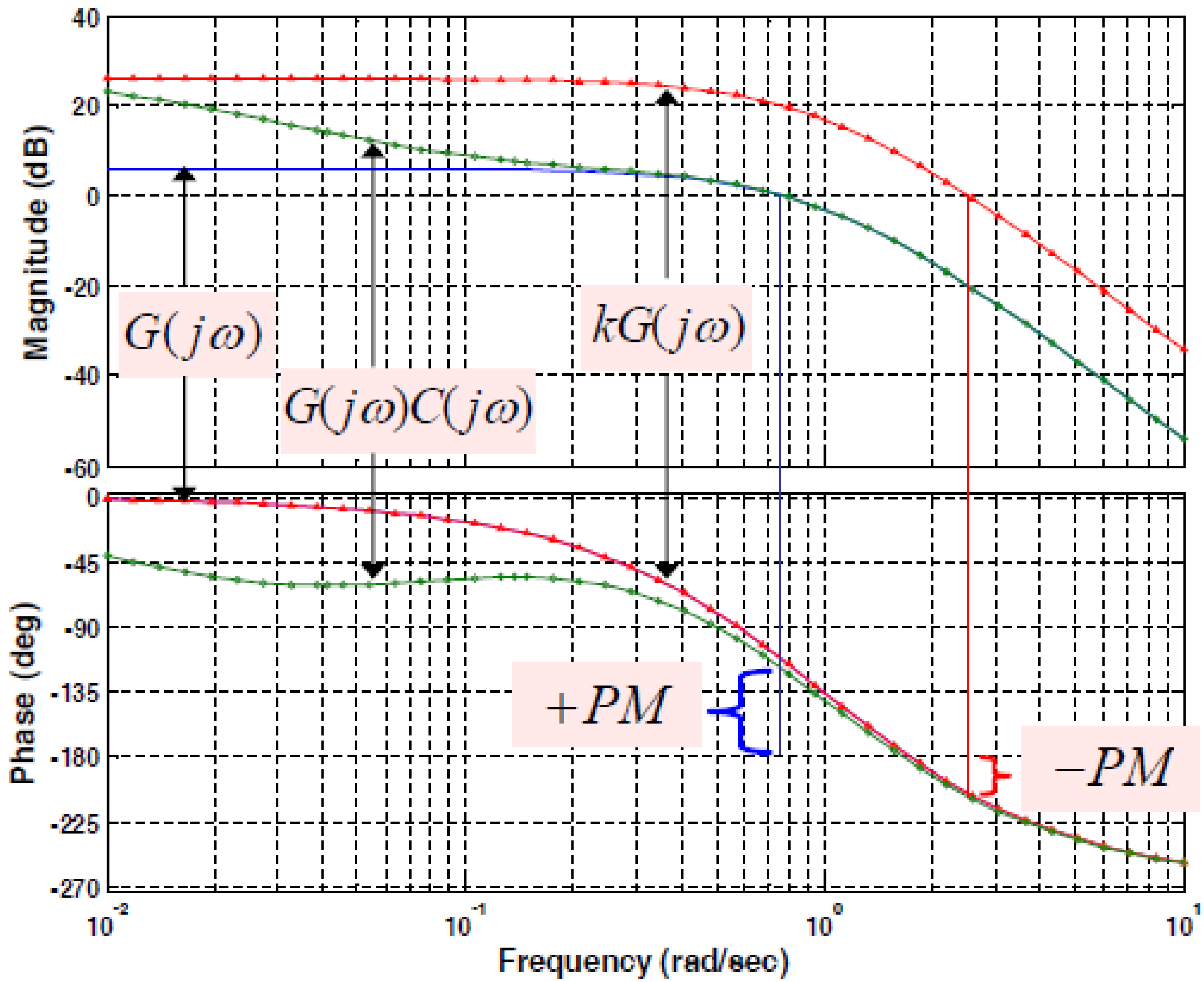


Key Feature:

Lag compensator is essentially a low pass filter. Therefore it can permits high gain at low frequencies and reduces gain in higher critical range of frequencies so as to improve the phase margin

When do we need this?

Look at the following example.



Key Idea:

Lag compensator reduces the gain crossover frequency to where the phase margin is required.

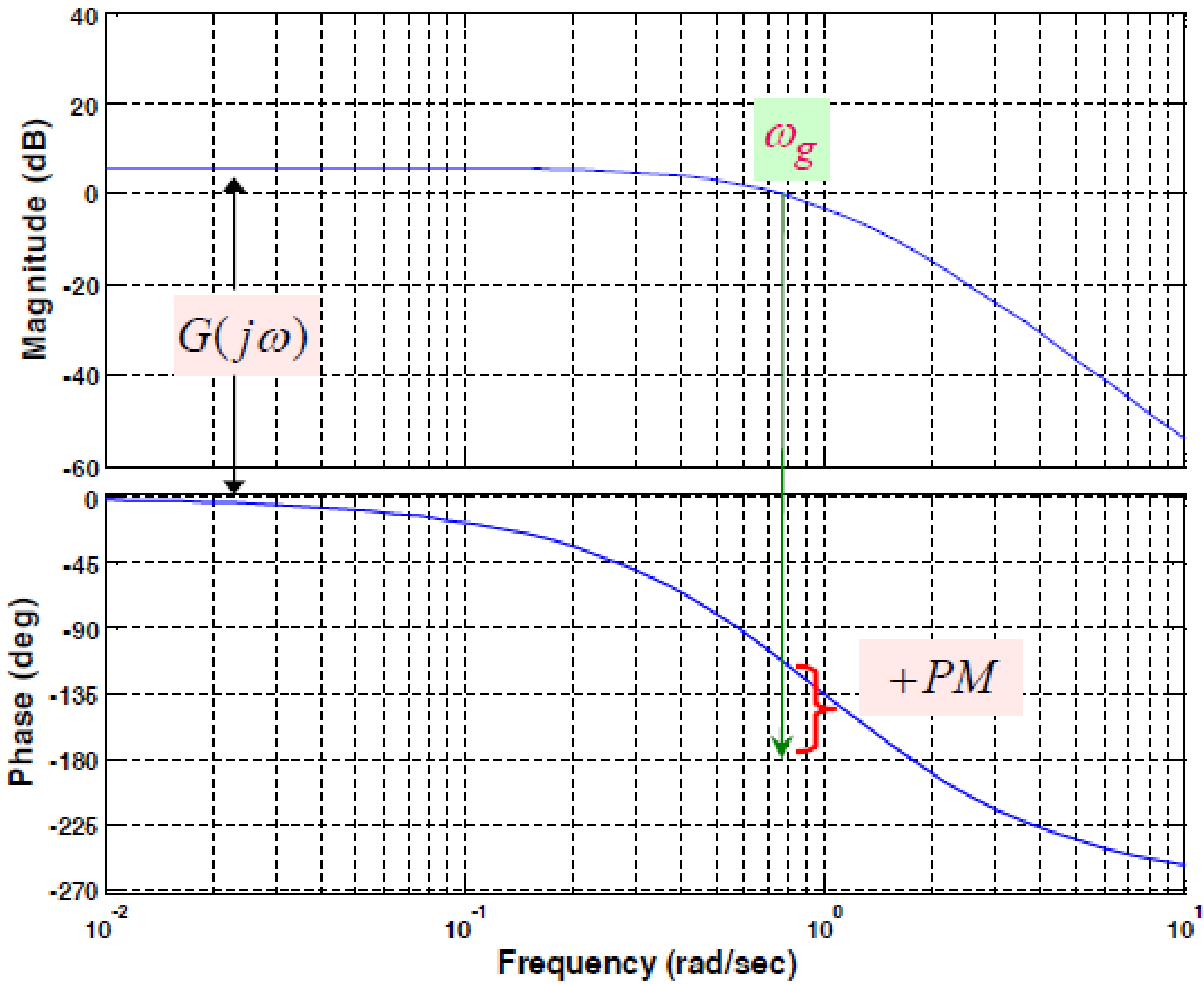
It can retain the high gain at low frequencies.

The phase-lag is of no use for compensation.

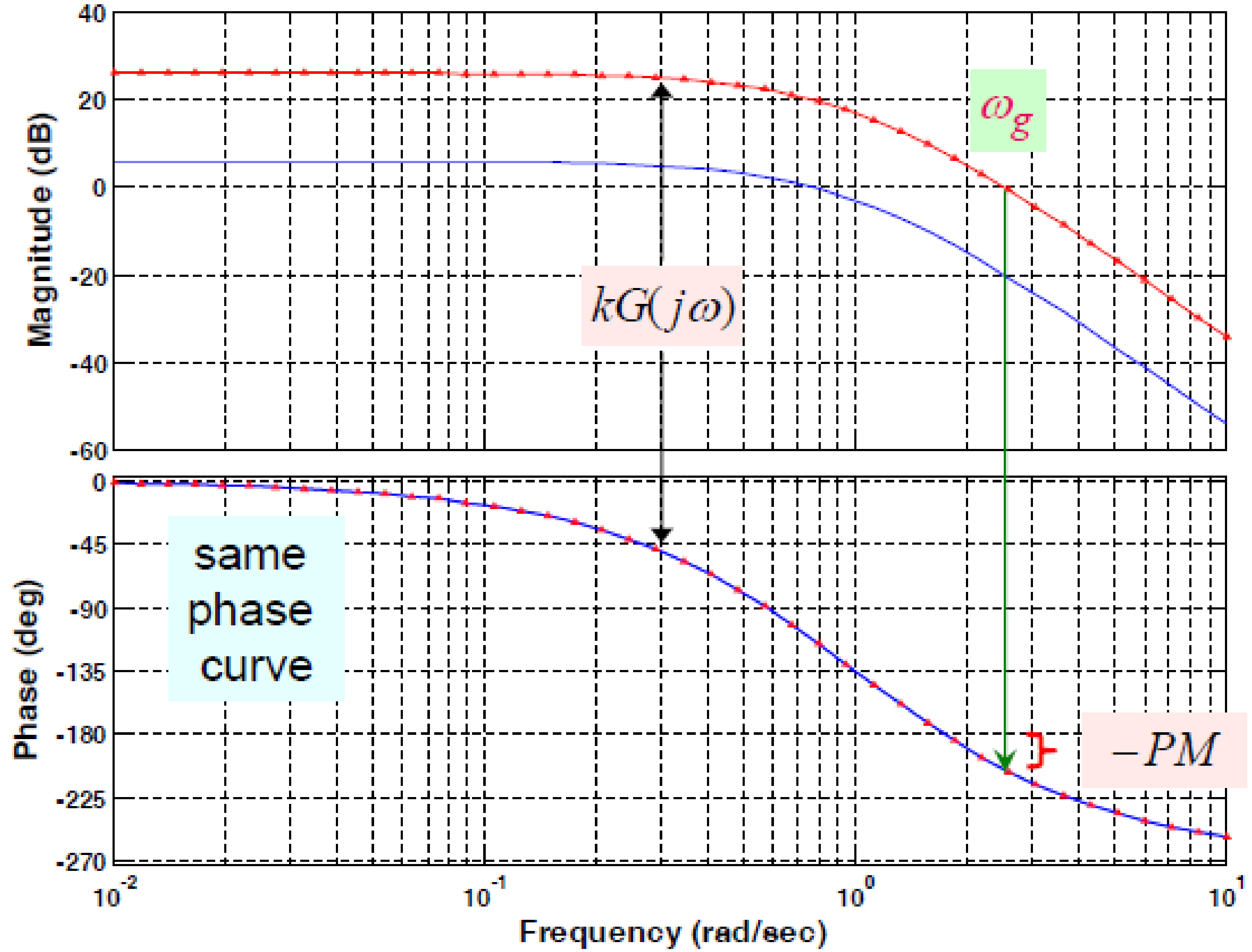
<u>Can we apply lead compensator to increase the PM for this example?</u>

Yes. The Lead compensation relies on increasing the phase at the gain crossover frequency.

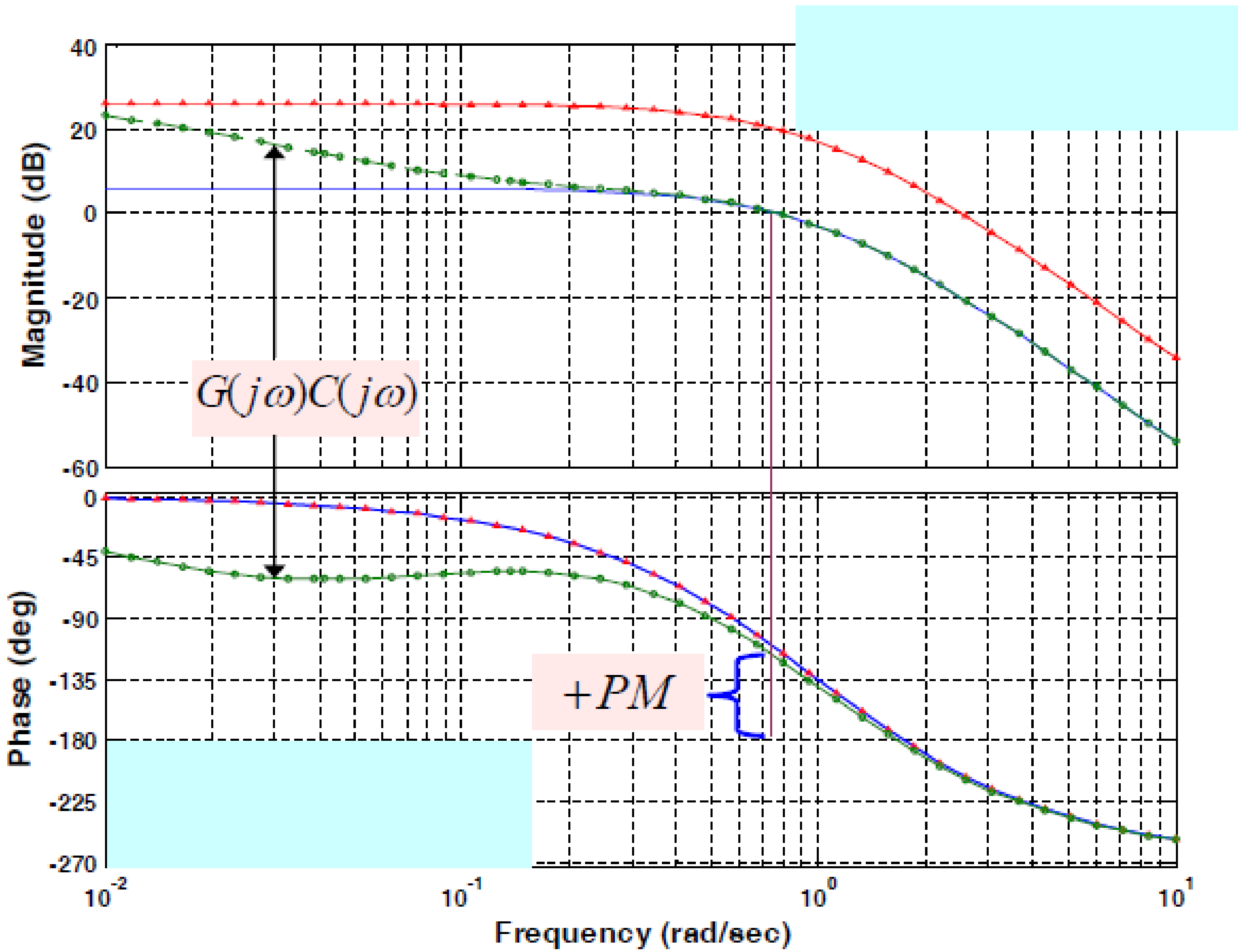
Original plant : adequate phase margin but inadequate DC gain



Only adding gain ($k>1$) : negative phase margin



Lag compensation : increasing low frequency gain only



Comparison of Lead and Lag Compensation

1. Lead compensation achieves the desired result through the merits of its phase-lead contribution, where as lag compensation accomplishes the result through the merits of its attenuation property at high frequencies. In some design problems both lag and lead compensation may satisfy the specifications.

2. Lead compensation is commonly used for improving the stability margins. Lead compensation yields a higher gain crossover frequency, which results in larger bandwidth, and hence faster response. If fast response is desired, lead compensation should be employed. If noise signals are present, then a large bandwidth may not be suitable.

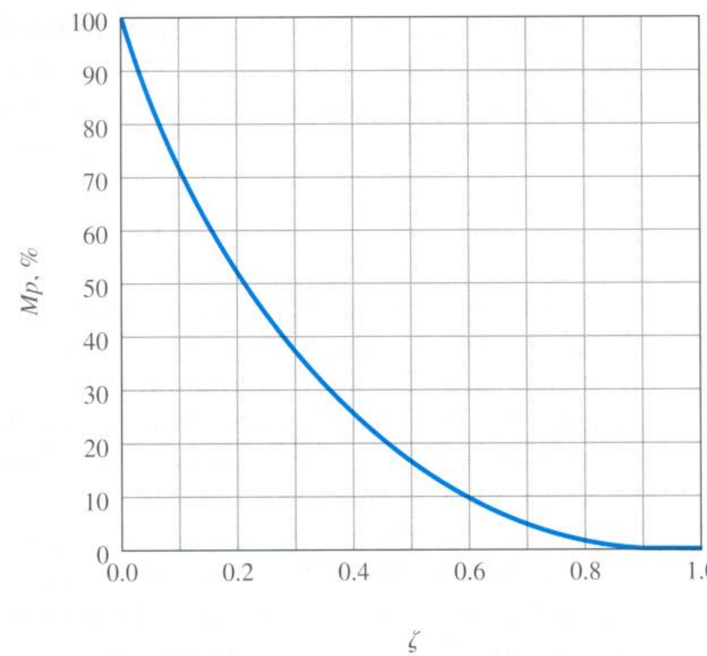
3. Lag compensation reduces the system gain at higher frequencies without affecting the gain at lower frequencies. Since the bandwidth is reduced, the system has a slower speed to respond. The total DC gain can be increased, and thereby steady-state accuracy can be improved. Also any high-frequency noises can be attenuated.

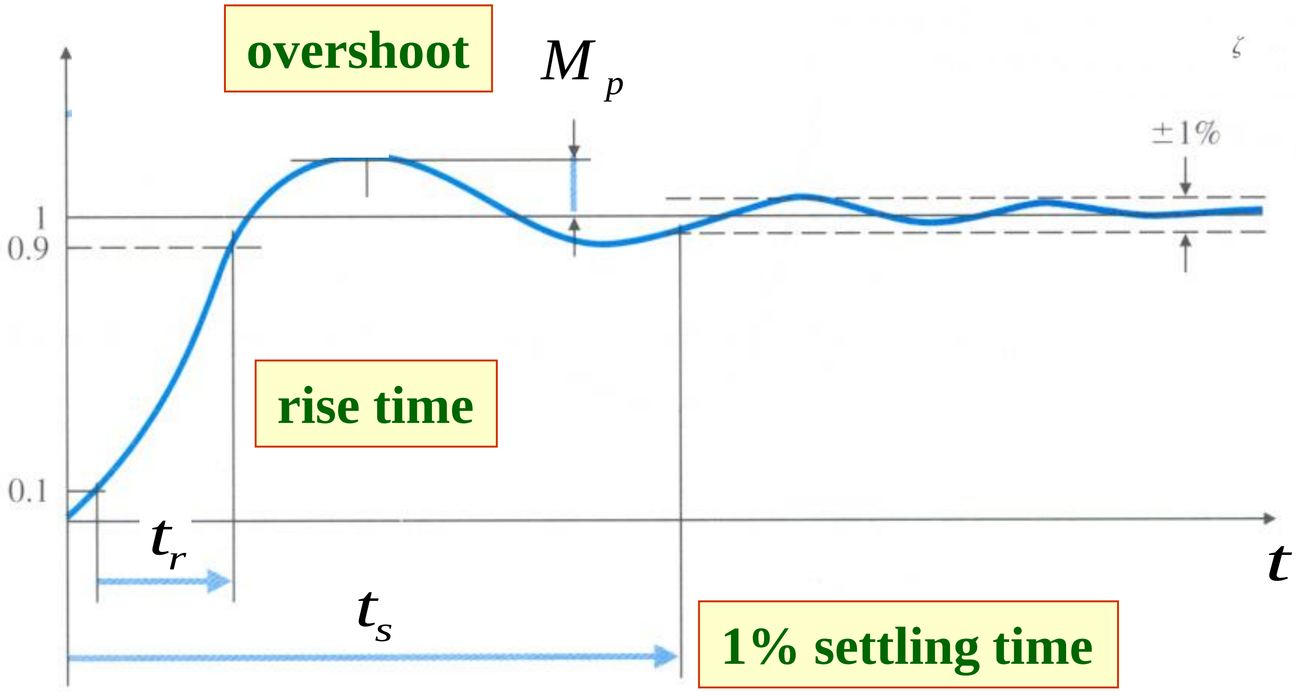
4. Although a large number of practical compensation tasks can be achieved by the simple compensators, more advanced controllers are needed for complicated systems.

Q & A…

THANK YOU !

$R(s) = 1/s$, $r = 1$ → $$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$ → $Y(s)$





$$t_r \cong \frac{1.8}{\omega_n}$$

$$t_s \cong \frac{4.6}{\zeta\omega_n}$$

return