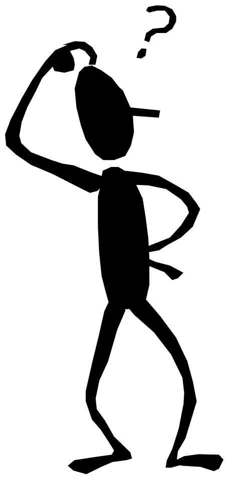


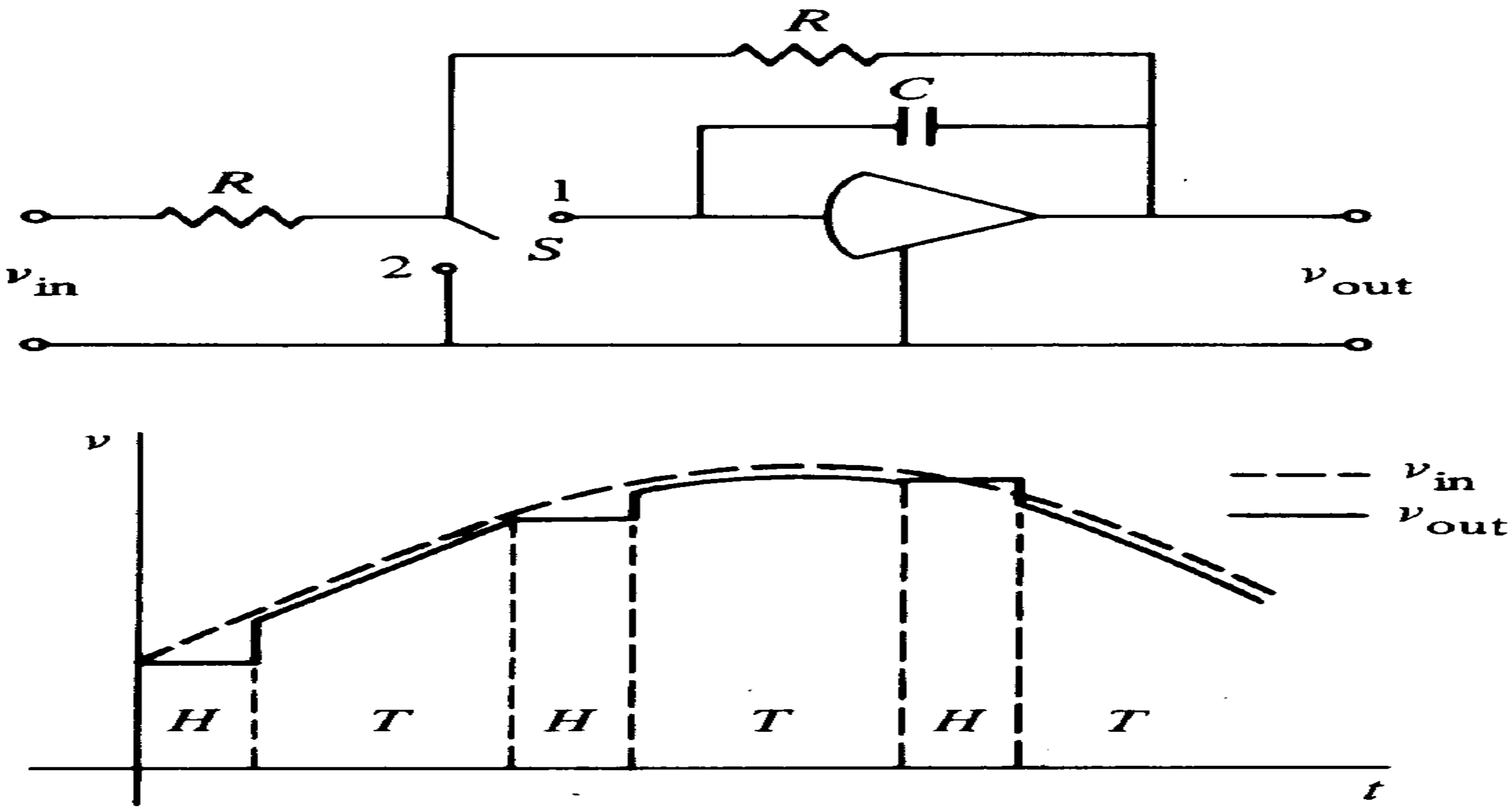
Chapter 4
Sampling and Filtering

So far studied

- Difference equations
- z-transforms
- Inverse z-transforms
- Properties of z-transforms
- Discrete transfer functions
- Stability
- Requires sampling and holding

What are the roles of sampling and holding in digital control?

Analog-to-digital converter

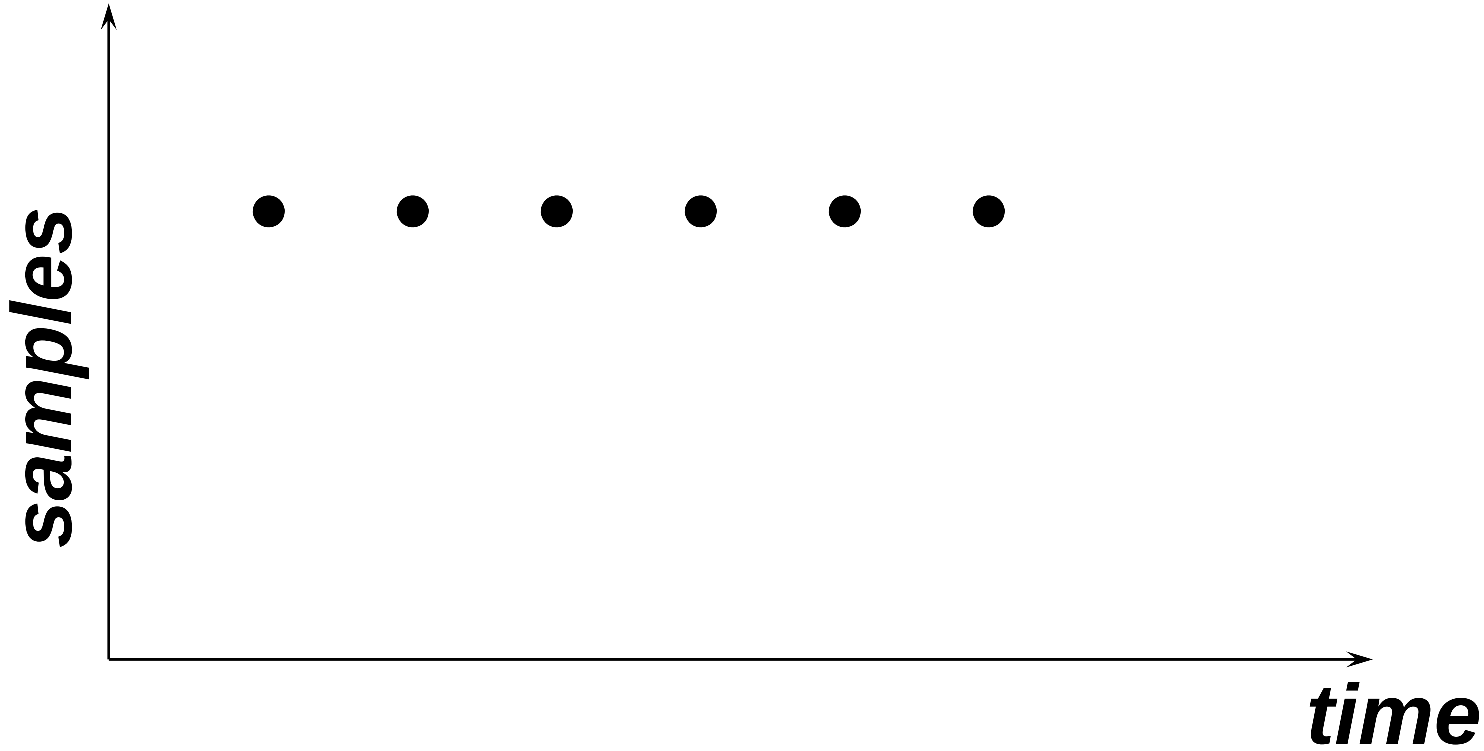


H = Hold; S in position 2.
T = Track; S in position 1.

4.1 Sampling

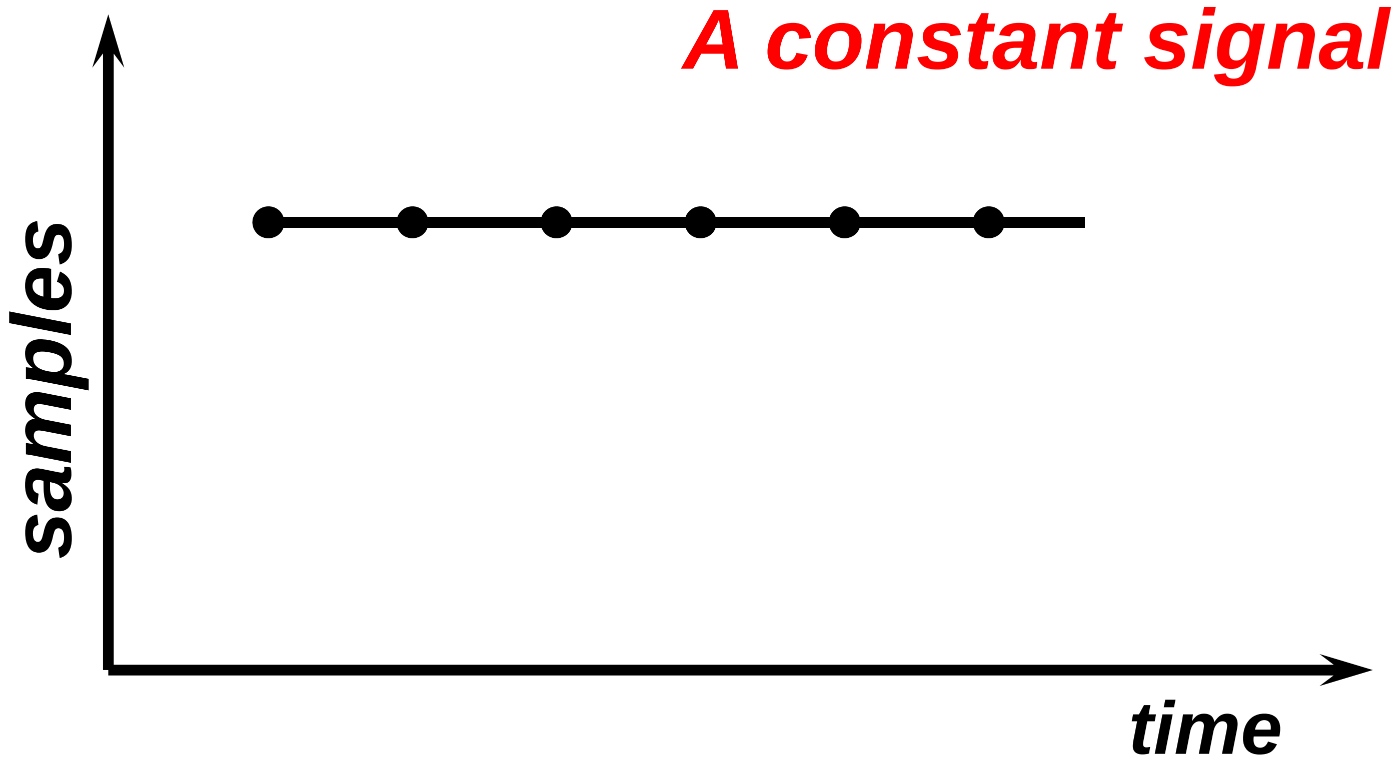
- How fast should we sample?
- What information does sampling process keep/lose?
- The information is frequency response.
- Compare the frequency response of the continuous signal with that of the sampled signal!
- By-product 1: Problems of aliasing
- By-product 2: relationship between the Laplace and z-transforms

How fast should we sample?

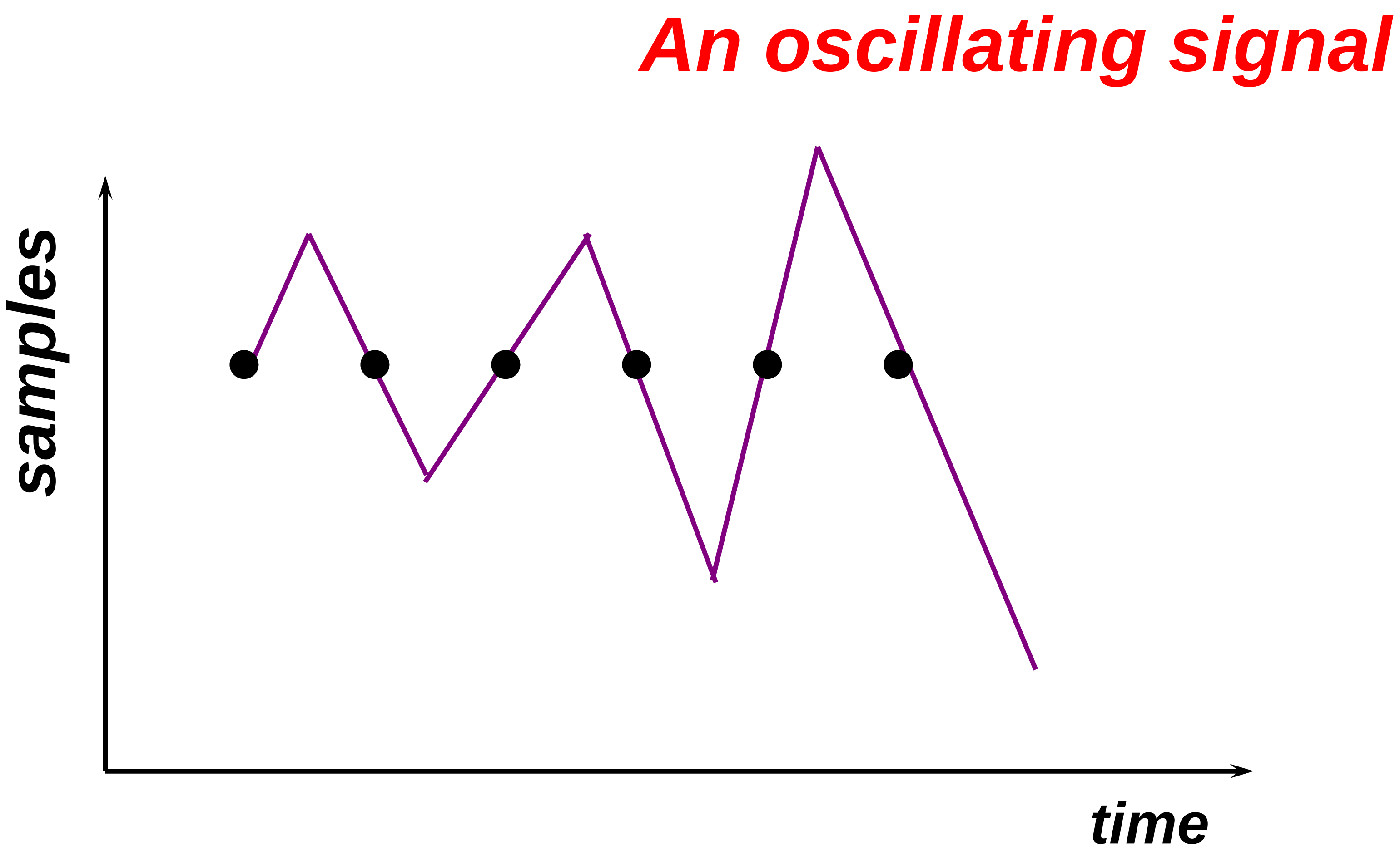


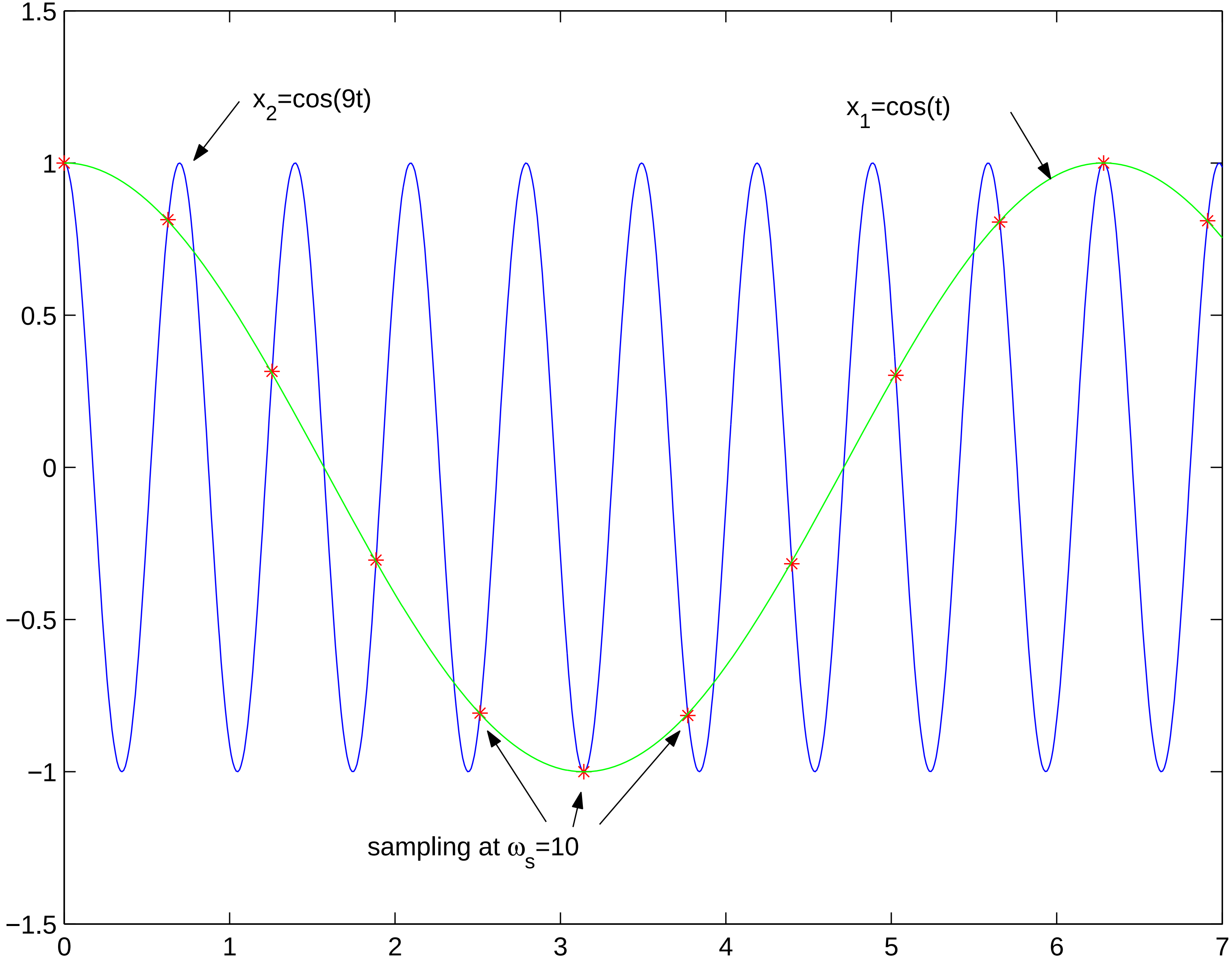
Can you tell what the original waveform is? WHY?

One possibility



Another possibility





x2=cos(9t)
x1=cos(t)
sampling at ωs=10
1.5
1
0.5
0
−0.5
−1
−1.5
0
1
2
3
4
5
6
7

Observations:

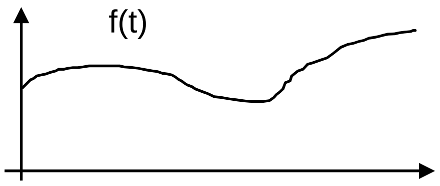
- How fast to sample depends on how fast the analog signal varies with time.
- there could be undersampling or oversampling!

Then, what are the effects of sampling?

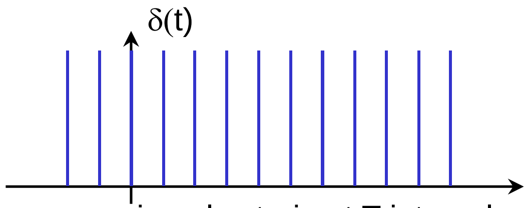
Key is to represent sampling mathematically
to help us quantify the effects of sampling

What can you think of to do so?

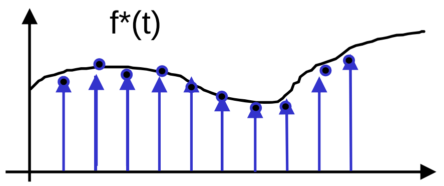
Representing sampled signals



signal to be sampled



impulse train at T intervals



sampled measurements

general $f(t)$

$$\sum_{k=-\infty}^{k=\infty} \delta(t-kT)$$

$$f^*(t)$$

The sampled signal is a string of impulses with amplitude *f(t)*

$$f^*(t) = \sum_{k=-\infty}^{k=\infty} f(t)\delta(t-kT)$$

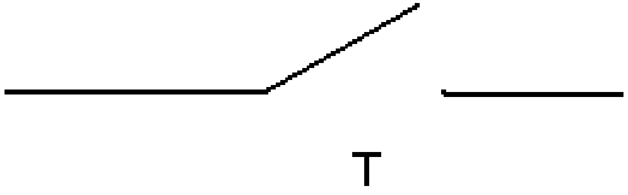
Property of the impulse function, δ(t)

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt = f(\tau)$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Sampler Model:

$f(t)$



$$f^{*}(t) = \sum_{k=-\infty}^{\infty} f(t)\delta(t-kT)$$

$$f^{*}(t) = \sum_{k=-\infty}^{\infty} f(t)\delta(t-kT)$$

Idea: Compare frequency response of *f(t)* with that of $f^{*}(t)$ which depends on *T*, and find the largest *T* such that they have the same frequency response in the interested frequency range

Laplace Transform of a sampled signal

Laplace transform of *f*(t)* where *f(t)* is a general signal

$$F^*(s) = \int_{-\infty}^{\infty} f^*(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{k=\infty} f(t)\delta(t-kT) e^{-st} dt$$

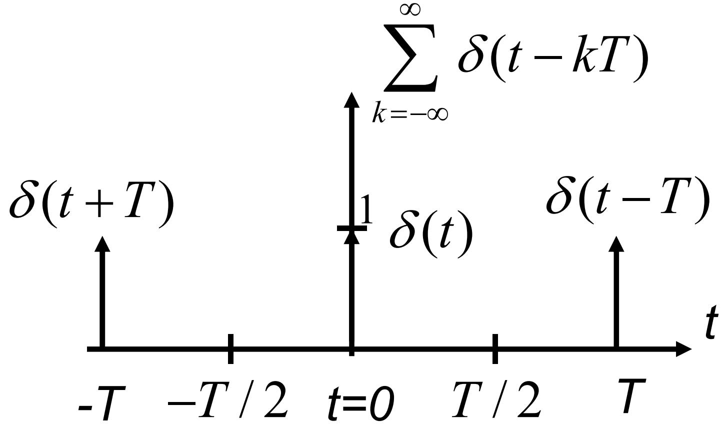
What is its spectrum?

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$$

Using Fourier analysis

$$\text{Periodic } g(t)=\sum_{n=-\infty}^{\infty} c_n e^{j2\pi nt/T}, \quad c_n=\frac{1}{T}\int_{-T/2}^{T/2} g(t)e^{-j2\pi nt/T}\,dt$$

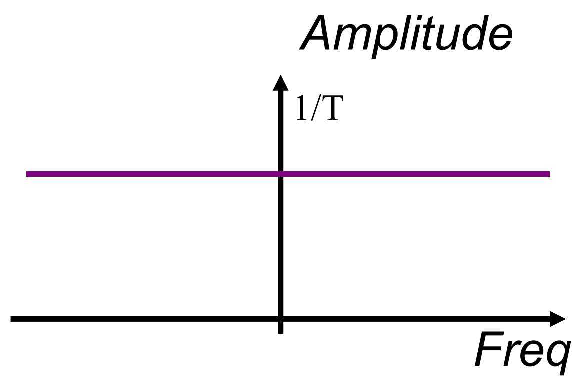
$$\Delta(\mathrm{t})=\sum_{\mathrm{k}=-\infty}^{\infty}\delta(\mathrm{t}-\mathrm{kT}) \quad \text{is periodic.}$$



$$c_n=\frac{1}{T}\int_{-T/2}^{T/2}\sum_{k=-\infty}^{\infty}\delta(t-kT)e^{-j2\pi nt/T}\,dt$$

$$=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-j2\pi nt/T}\,dt=\frac{1}{T}$$

$$\text{Therefore} \quad \sum_{k=-\infty}^{\infty}\delta(t-kT)=\frac{1}{T}\sum_{n=-\infty}^{\infty}e^{j2\pi nt/T}$$



Spectrum of *Δ(t)*

Spectrum of Sampled Signal

- The impulse train is periodic. A fourier series exists.

$$\sum_{k=-\infty}^{k=\infty} \delta(t-kT) = \frac{1}{T} \sum_{n=-\infty}^{n=\infty} e^{j(2\pi n/T)t}$$

- Laplace transform of the sampled signal.

$$F^*(s) = \int_{-\infty}^{\infty} f^*(t) e^{-st} dt = \int_{-\infty}^{\infty} f(t) \left\{ \frac{1}{T} \sum_{n=-\infty}^{n=\infty} e^{j(2\pi n/T)t} \right\} e^{-st} dt$$

$$= \frac{1}{T} \sum_{n=-\infty}^{n=\infty} \int_{-\infty}^{\infty} f(t) e^{j(2\pi n/T)t} e^{-st} dt$$

$$= \frac{1}{T} \sum_{n=-\infty}^{n=\infty} \int_{-\infty}^{\infty} f(t) e^{-(s-j2\pi n/T)t} dt = \frac{1}{T} \sum_{n=-\infty}^{n=\infty} F(s - j2\pi n/T)$$

$$= \frac{1}{T} \sum_{n=-\infty}^{n=\infty} F(s - jn\omega_s), \text{ with } \omega_s = \frac{2\pi}{T}, \text{ radian sampling frequency,}$$

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$$

$$F^*(s) = \frac{1}{T}\sum_{n=-\infty}^{n=\infty} F(s - jn\omega_s)$$

$F(j\omega)$ has frequency of ω.

ω could be a set of discrete values or a range of continuous values

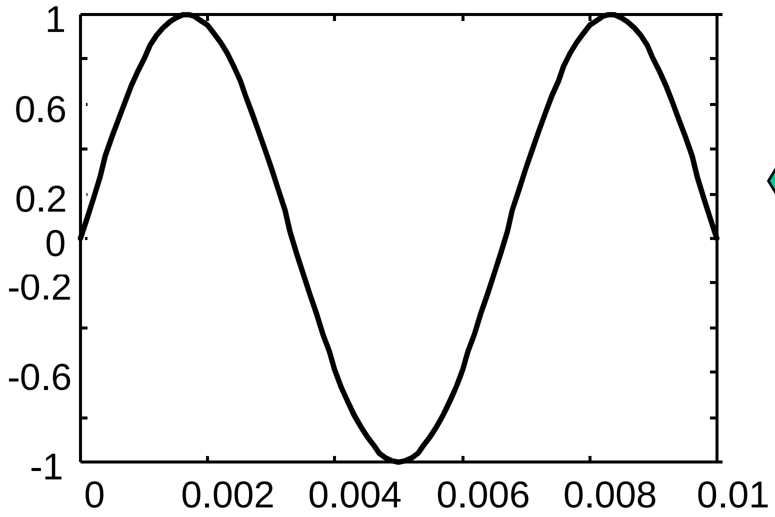
$F^*(j\omega)$ has a duplication of $F(j\omega)$ for each interval of ω_s.

$F^*(j\omega)$ has frequencies of $(\omega - n\omega_s)$, $n = 0, \pm 1, \pm 2, \cdots, \infty$.

$$\omega_d = \omega_c - n\omega_s,\ n = 0, \pm 1, \pm 2, \cdots$$

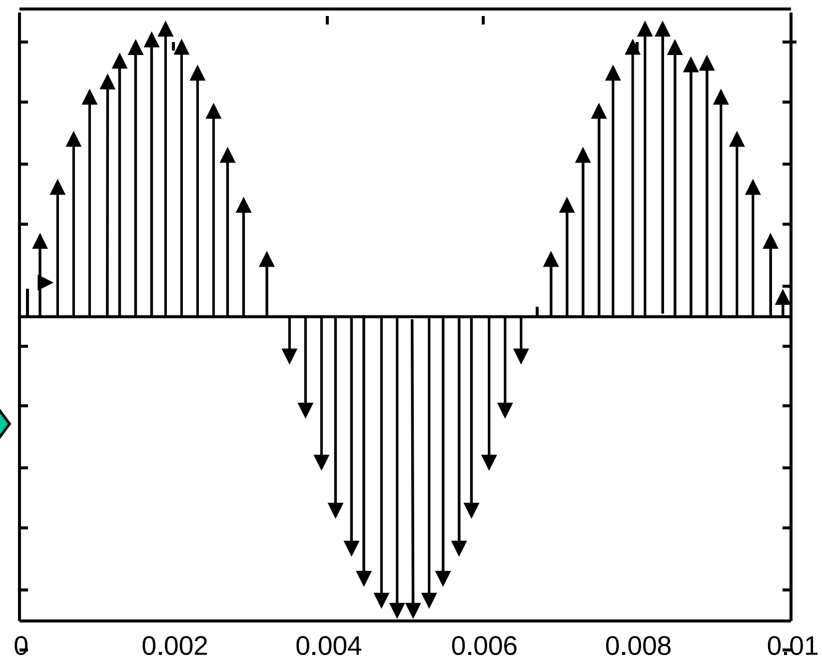
frequency in $f^*(t)$ frequency in $f(t)$

4.2 Aliasing and Sampling Rate Selection



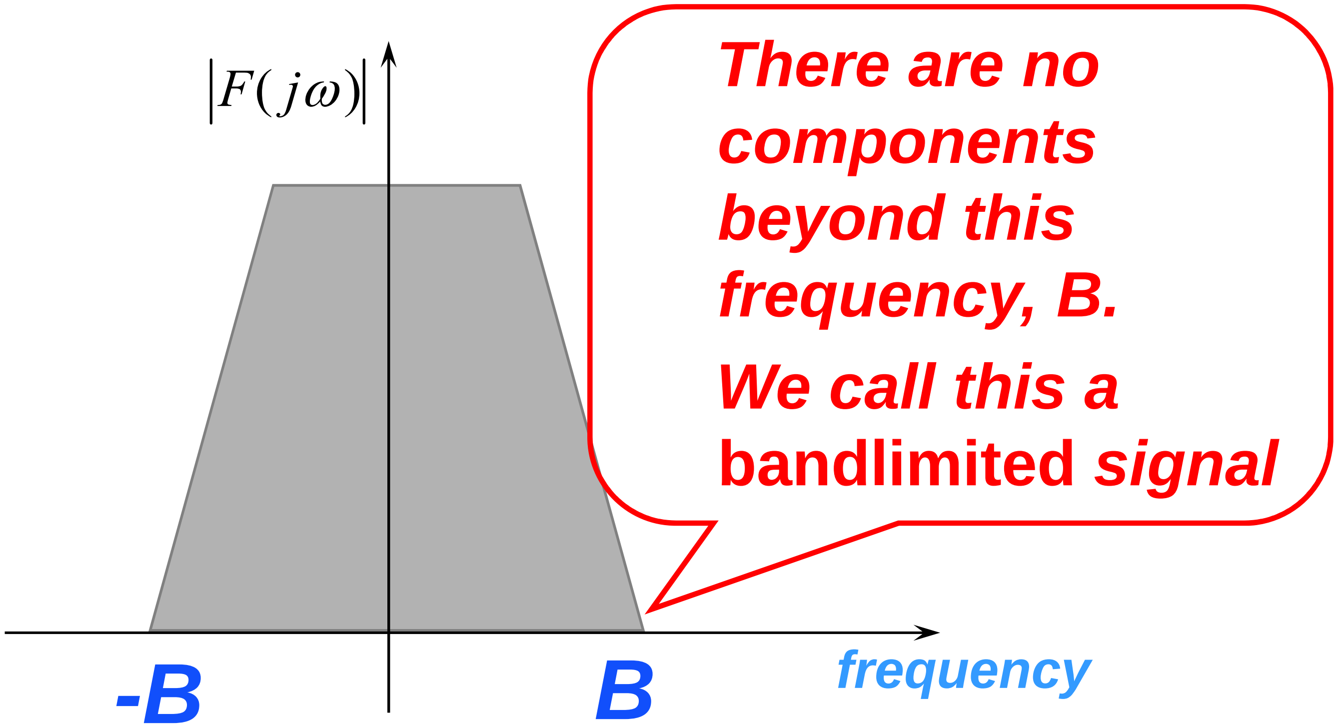
$f(t)$ is a single frequency signal, $\omega=300\pi$.

$\omega_s=10000\pi$

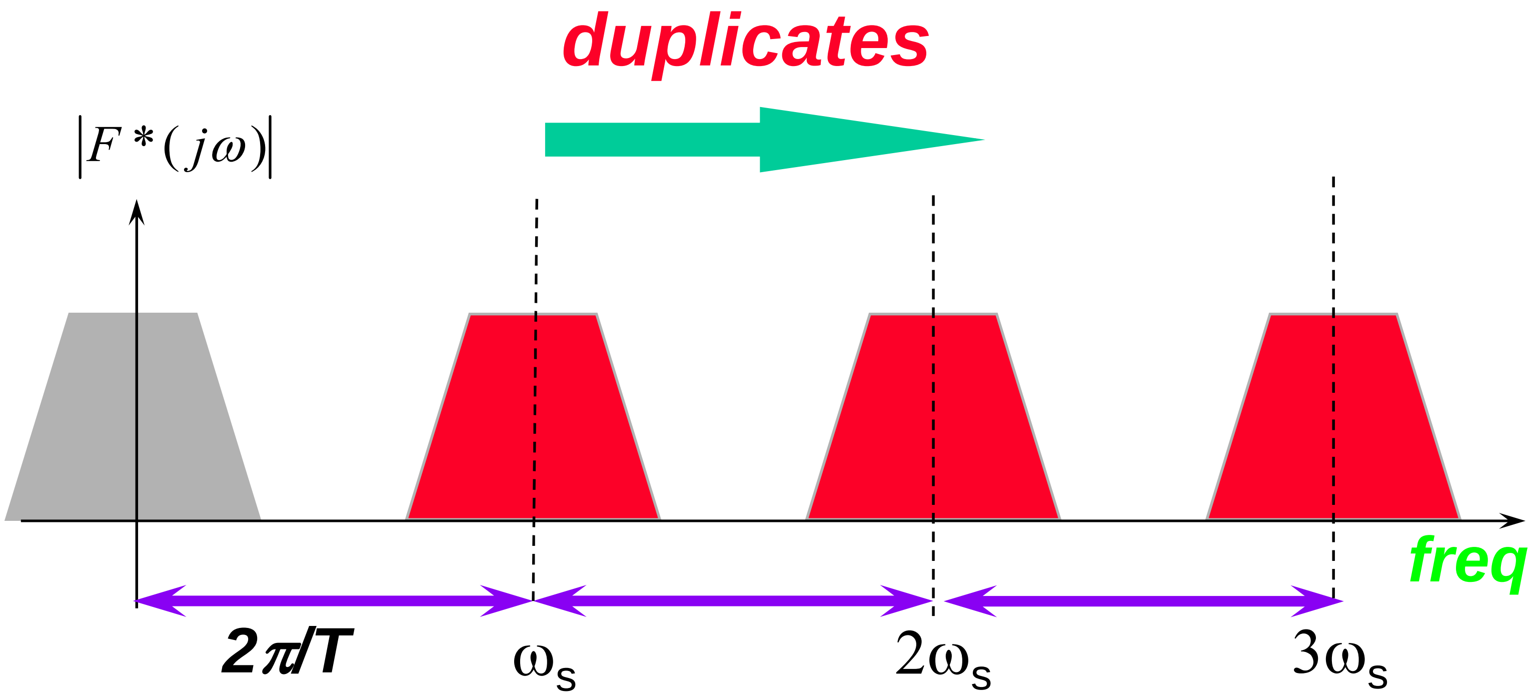


$f^*(t)$ has many frequencies, $\omega=300\pi\pm n10000\pi,\ n=0,1,\ldots$

For more general bandlimited signals, whose frequency components are distributed as shown :



Spectrum of the sampled signal



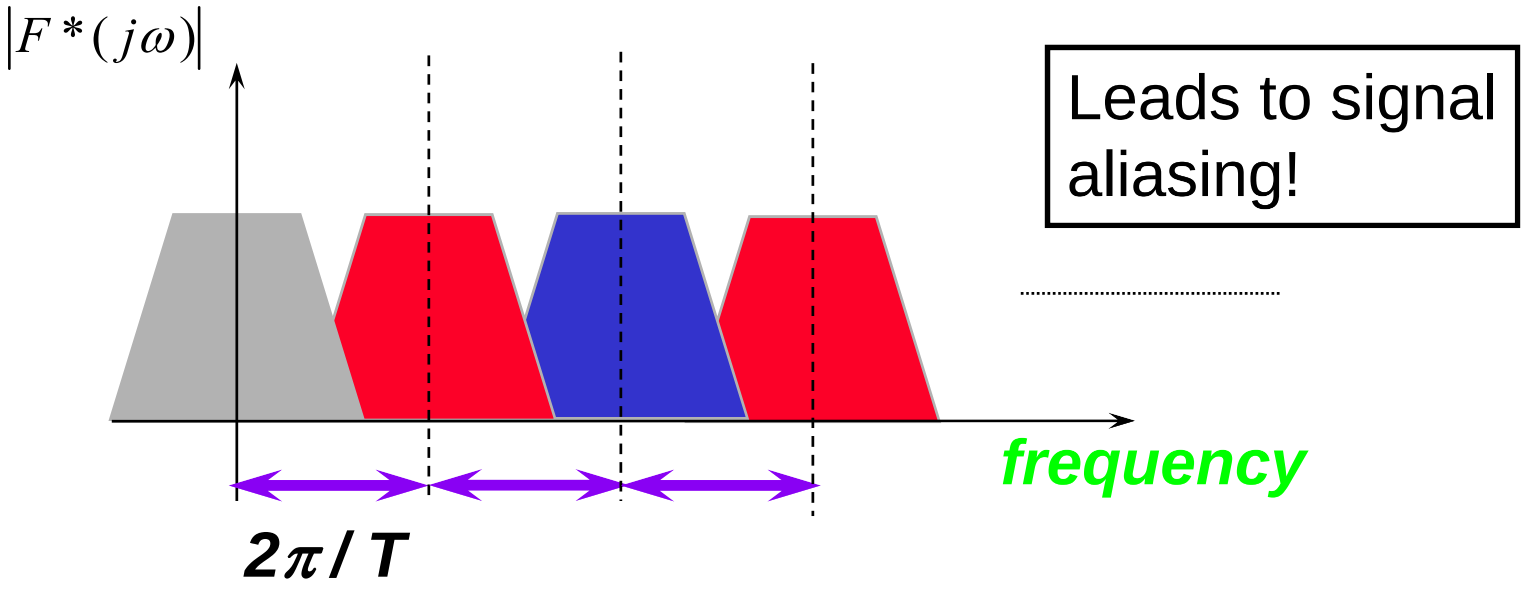
The baseband has been duplicated and shifted.

Each duplicated band is now centered around $n\omega_s$

If the bands come too close .....

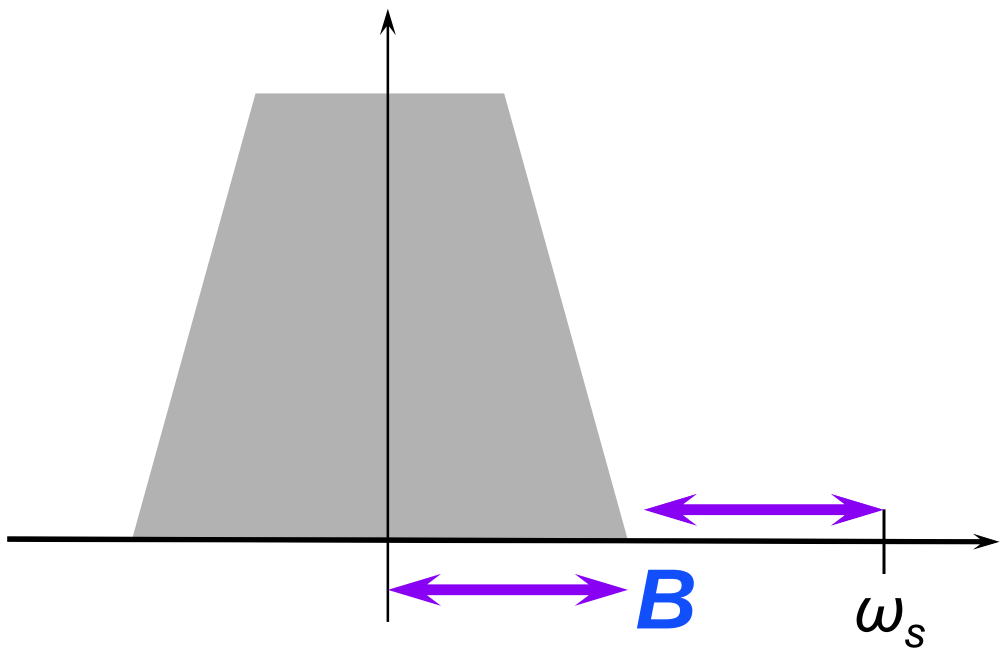
They overlap when the sampling frequency is less than twice the maximum frequency, 2B.

The resulting waveform is distorted.



The Sampling Theorem

- If the sampling frequency $\left(\omega_s = \frac{2\pi}{T}\right)$ is strictly greater than twice the highest frequency component, ie $\omega_s > 2B$, it is possible to reconstruct the original signal from the samples



Clarifying

- The sampling frequency can be represented by either $\omega_s = 2\pi/T$ *radian/second,* or $f_s = 1/T$ *Hz.*
- $\omega_s = 2\pi f_s$
- Use either

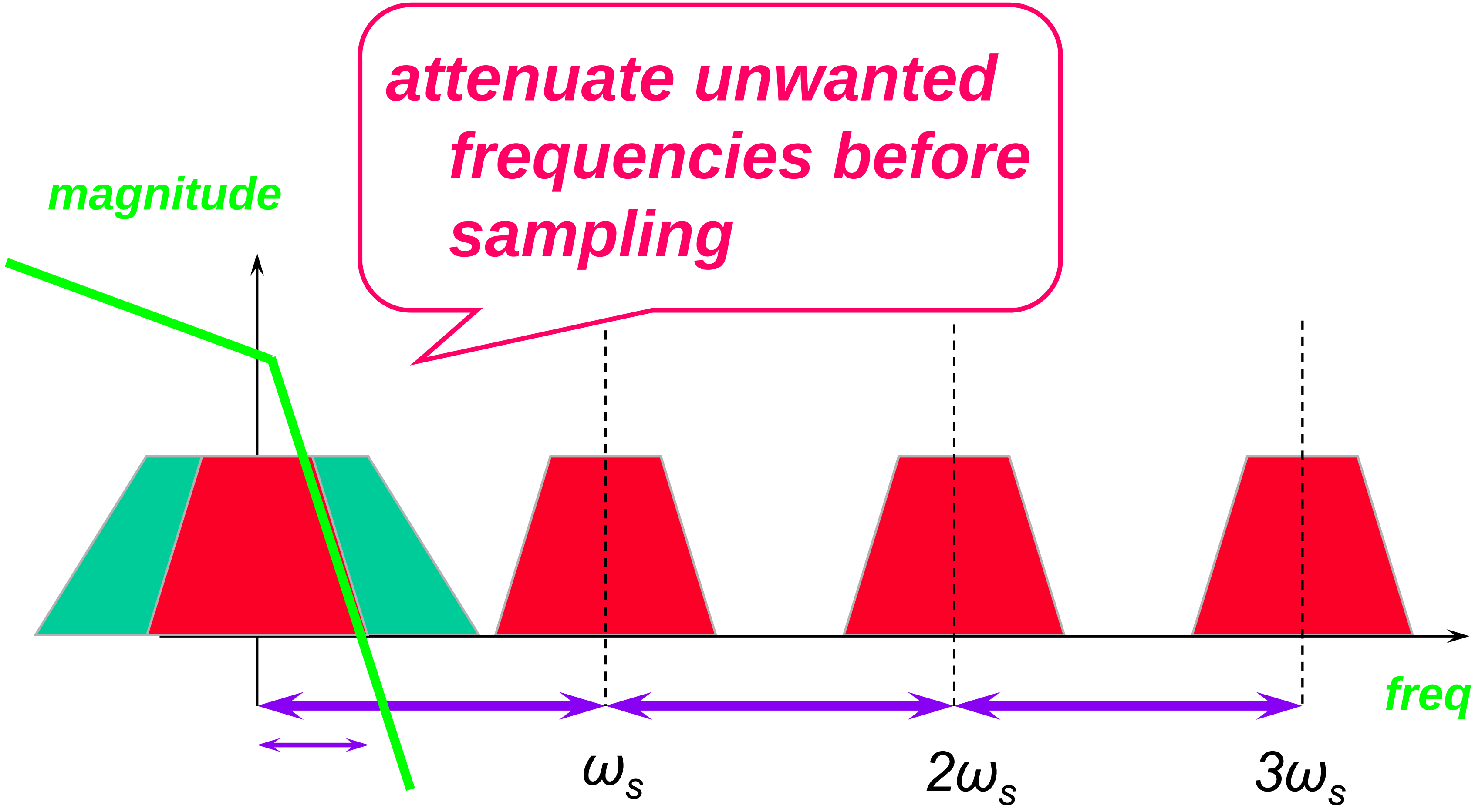
  $f_s > 2B$ (if B is measured in *Hz*) or

  $\omega_s > 2B$ (if B is measured in *radian/second*)

as long as (remember to do so) the same unit is used in respective cases.

Anti Aliasing Filter

- To get rid of the high frequencies - we use a filter



Sample Rate Selection

Based on the sampling theorem, the sampling rate ω_S must be at least twice the signal bandwidth ω_b

$$\frac{\omega_S}{\omega_b} > 2 .$$

In practice,

$$6 \leq \frac{\omega_S}{\omega_b} \leq 40$$

To attenuate measurement noise, a filter is typically placed between the sensor and the sampler and chosen as

$$G_p(s) = \frac{\omega_p}{s + \omega_p}$$

ω_p may be selected as

$$\frac{\omega_p}{\omega_b} \approx 2$$

4.3 Relationship between s and z

- Laplace transform of $f^*(t)$ where $f(t)$ is a general signal

$$F^*(s) = \int_{-\infty}^{\infty} f^*(\tau) e^{-s\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{k=\infty} f(\tau)\delta(\tau - kT) e^{-s\tau} d\tau$$

$$= \sum_{k=-\infty}^{k=\infty} \left[\int_{-\infty}^{\infty} (f(\tau)e^{-s\tau})\delta(\tau - kT) d\tau\right]$$

$$= \sum_{k=-\infty}^{k=\infty} f(kT)\ e^{-skT}$$

- Relationship to z transform

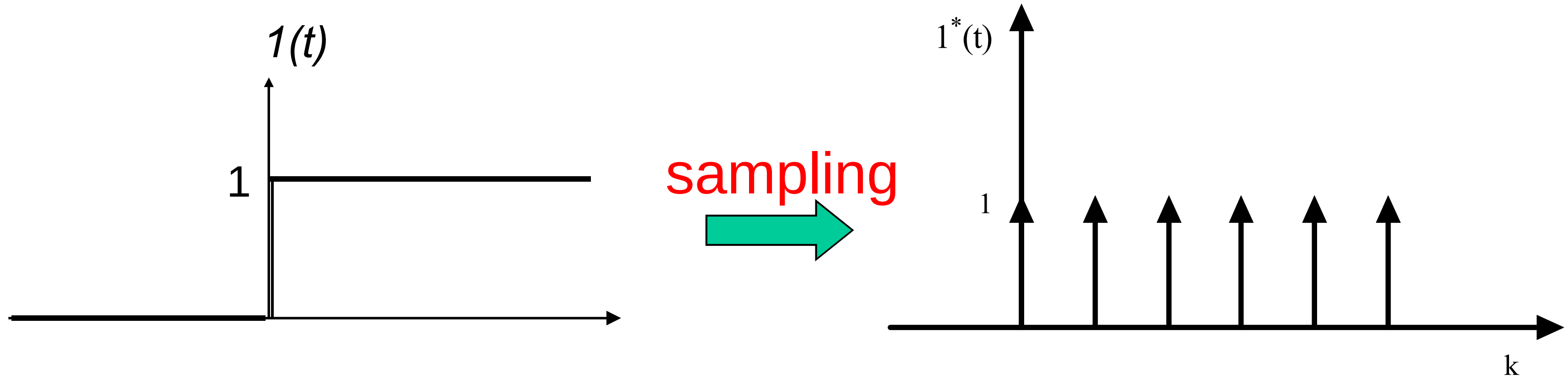
$$= \sum_{k=-\infty}^{k=\infty} f(kT)(e^{sT})^{-k} = F(z) \ \text{ when } z = e^{sT}$$

The Laplace transform of *f*(t)* is also z-transform of *f(kT)* if $z=e^{sT}$.

$$F^*(s) = \sum_{k=-\infty}^{k=\infty} f(kT)(e^{sT})^{-k} = F(z) \; when \;\; z = e^{sT}$$

For example, consider sampling a unit step signal, *1(t)*

$$1^*(t) = \sum_{k=0}^{\infty} \delta(t - kT)$$



Taking Laplace Transform,

$$L\{u^*(t)\} = U^*(s) = \sum_{k=0}^{\infty} 1e^{-kTs} = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}} = U(z)$$

Conclude that the Laplace Transform of *f*(t)* is z-transform of *f(kT)* if

$$z = e^{sT}$$

Important relation to remember since this allows us to interprete the z-variable in the frequency domain. How?

Relationships between Laplace, Fourier and Z-Transforms

Laplace transform $s = \sigma + j\omega$

Fourier transform $s = j\omega$

z-transform $z = e^{sT}$ or $s = \frac{1}{T}\ln z$

Given G(s) or $G_D(z)$, how do we obtain its frequency response?

G(s) ⇒ Replace *s* by $s = j\omega$

$G_D(z)$ ⇒ Replace *z* by $z = e^{j\omega T}$

4.4 Filtering

- Convert discrete signal to continuous signal. Commonly called holder

  C>>>D: sampling; D>>>C: Filtering

- Ideal holder: perfect but not realizable
- Zero-order holder: hold the signal after each sampling instant till the next. Reasonable and realizable.

- What are they and how do they behavior?

Ideal holder

Recall

$$F^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(s - jn\omega_s), \ \omega_\mathrm{s} = \frac{2\pi}{T}$$

Suppose $\omega_s > 2B$. Then there is no overlapping of frequency bands, enabling reconstruction of the signal from its samples. The ideal holder to recover $f(t)$ from $f^*(t)$ precisely is given by

$$L(j\omega) = \begin{cases} T, & |\omega| < \dfrac{\pi}{T} \\ 0, & |\omega| \geq \dfrac{\pi}{T} \end{cases}$$

because the filter output Y would then recover $F(j\omega)$:

$$Y(j\omega) = L(j\omega)F^*(j\omega) = F(j\omega)$$

By the inverse transform, the filter impulse response is

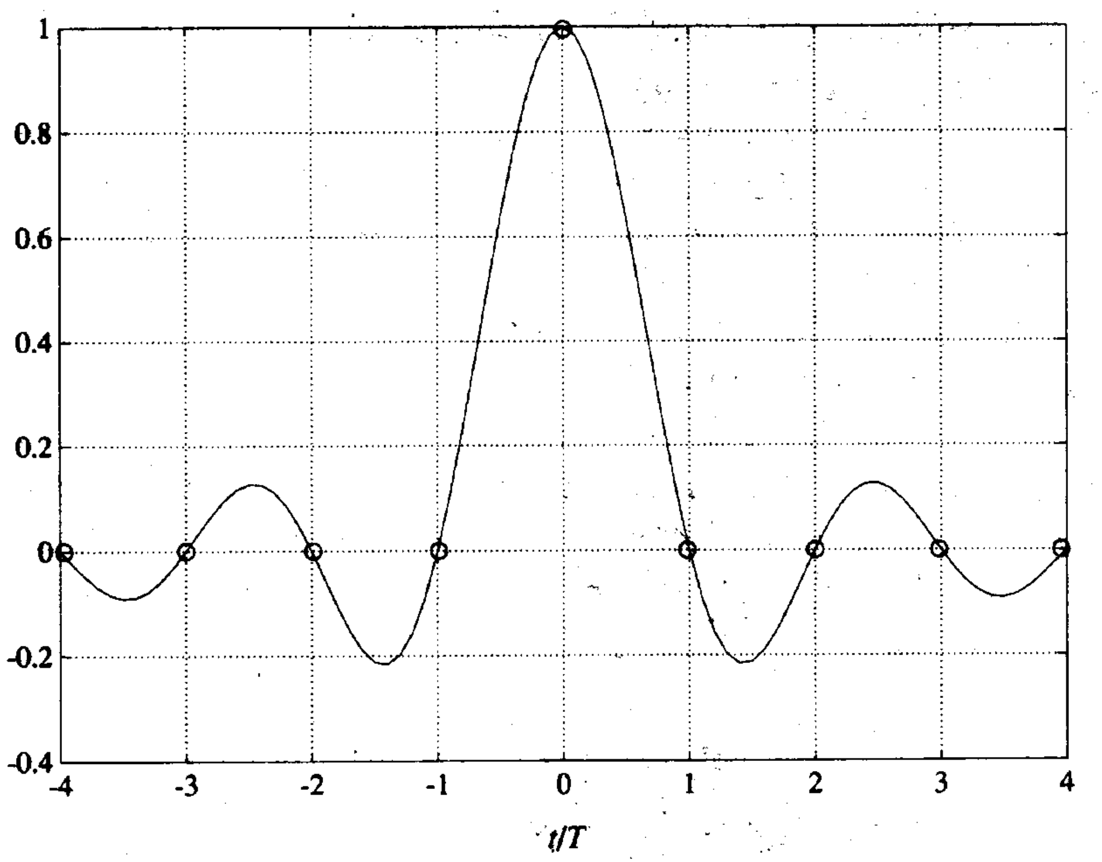
$$l(t) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} T e^{j\omega t} d\omega = \frac{T}{2\pi}\frac{e^{j\omega t}}{jt}\Bigg|_{-\pi/T}^{\pi/T}$$

$$= \frac{T}{2\pi jt}\left(e^{j(\pi t/T)} - e^{-j(\pi t/T)}\right) = \frac{\sin(\pi t/T)}{\pi t/T}$$

The ideal low-pass filter $L(j\omega)$ is noncausal because $l(t)$ is nonzero for $t<0$. $l(t)$ starts at $t=-\infty$ when the impulse that triggers it does not occur until $t=0$!

Not physically realizable!

Plot of the impulse response of the ideal low-pass filter



From:

$$Y(j\omega) = L(j\omega)F^*(j\omega)$$

The continuous signal
is recovered
via their convolution:

$$y(t) = \int_{-\infty}^{\infty} f^*(\tau) l(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} [f(\tau)\sum_{i=-\infty}^{\infty}\delta(\tau - iT)]l(t-\tau)d\tau$$

$$= \sum_{i=-\infty}^{\infty} f(iT) l(t-iT)$$

Let the present time be *t=kT*.
Then, *iT* is future time for i>*k*
iT is past time for *i*<*k*.

$$y(kT) = \sum_{i=-\infty}^{k} f(iT) l(kT - iT)$$

$$+ \sum_{i=k+1}^{\infty} f(iT) l(kT - iT)$$

$$\sum_{i=-\infty}^{k} f\left(iT\right) l(kT - iT)$$

In the first term above:

f(iT) is the past to present signal due to $i<=k$, $iT<=kT$; available!

l(kT-iT) is *l(t)* for $t=kT-iT>0$ due to $k>=i$,

So this term is ok for implementation

$$\sum_{i=k+1}^{\infty} f(iT)\, l(kT - iT)$$

In the 2nd term above:

- $f(iT)$ is the future signal due to $i>k$, $iT>kT$; NOT available at now!
- $l(kT\text{-}iT)$ is $l(t)$ for $t= kT\text{-}iT<0$ due to $k<i$.

This term (so for the whole $f(kT)$) cannot be evaluated with info up to the present since the ideal holder $l(t)$ is non-zero for t<0, or non-causal. The exception is for a causal holder which meets $l(t)=0$ for t<0, and then, the 2nd term is zero.

Note that

- *a* system is causal if $l(t)=0$ for $t<0$; non-causal, otherwise.
- A non-causal system is thus not realizable.

Zero-order holder

The samples taken from continuous signal *f(t)* are represented by

$$f^*(t) = \sum_{k=-\infty}^{\infty} f(t)\delta(t-kT)$$

The zero-order holder is defined as the means to extrapolate impulses to piecewise constants:

$$f_h(t) = f(kT)\ , \quad kT \le t < kT + T$$

Zero-order hold (ZOH):

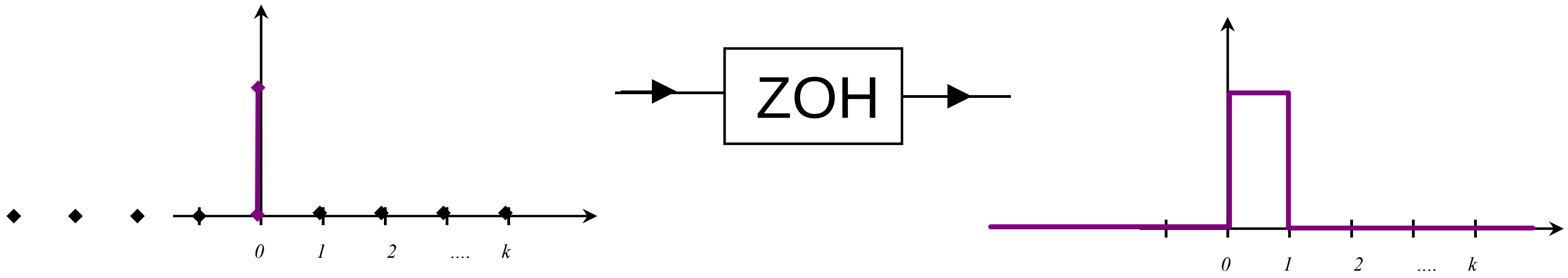
$$f_h(t) = f(kT)\ , \quad kT \le t < kT + T$$

is realizable as it uses past information only.

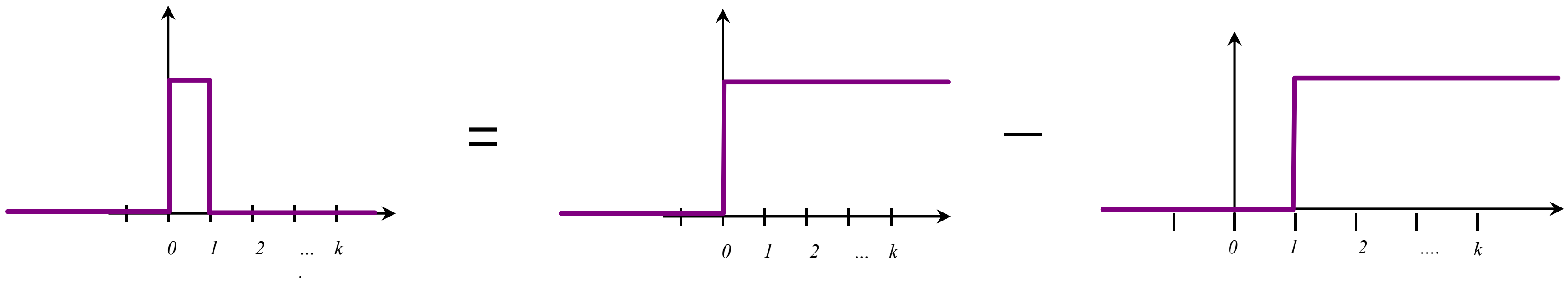
Its transfer function is computed by determining its impulse response. If the input is $f^*(t) = \delta(t)$, then $f_h(t)$ is a pulse of height 1 and duration T, i.e.

$$f_h(t) = 1(t) - 1(t - T)$$

- Consider the impulse response of the ZOH



- The unit pulse is the difference between the step function and the delayed step function



$$f_h(t) = 1(t) - 1(t - T)$$

The transfer function of ZOH is thus given by

$$G_{ZOH}(s) = \frac{1}{s} - \frac{e^{-Ts}}{s} = \frac{1 - e^{-Ts}}{s}$$

Analysis of the ZOH

- The ZOH has a transfer function

$$G_{ZOH} = \frac{1 - e^{-Ts}}{s}$$

- The corresponding frequency response is

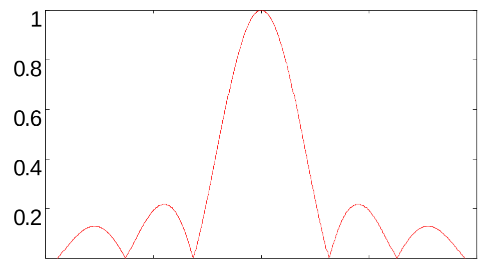
$$G_{ZOH} = \frac{1 - e^{-Tj\omega}}{j\omega}$$

$$= e^{-j\omega T/2} \left\{ \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j} \right\} \frac{2j}{j\omega}$$

$$= T e^{-j\omega T/2} \frac{\sin(\omega T/2)}{(\omega T/2)}$$

- G_{ZOH} has magnitude (close to what we want)

$$\left|G_{ZOH}\right| = \left|T\frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}}\right|$$



- a phase lag (which we do not want!)

$$\angle G_{ZOH}(j\omega) = -\frac{\omega T}{2}$$

a time delay of *T/2*

Time delay/phase lag tends to destabilize closed-loop.

Alternative views

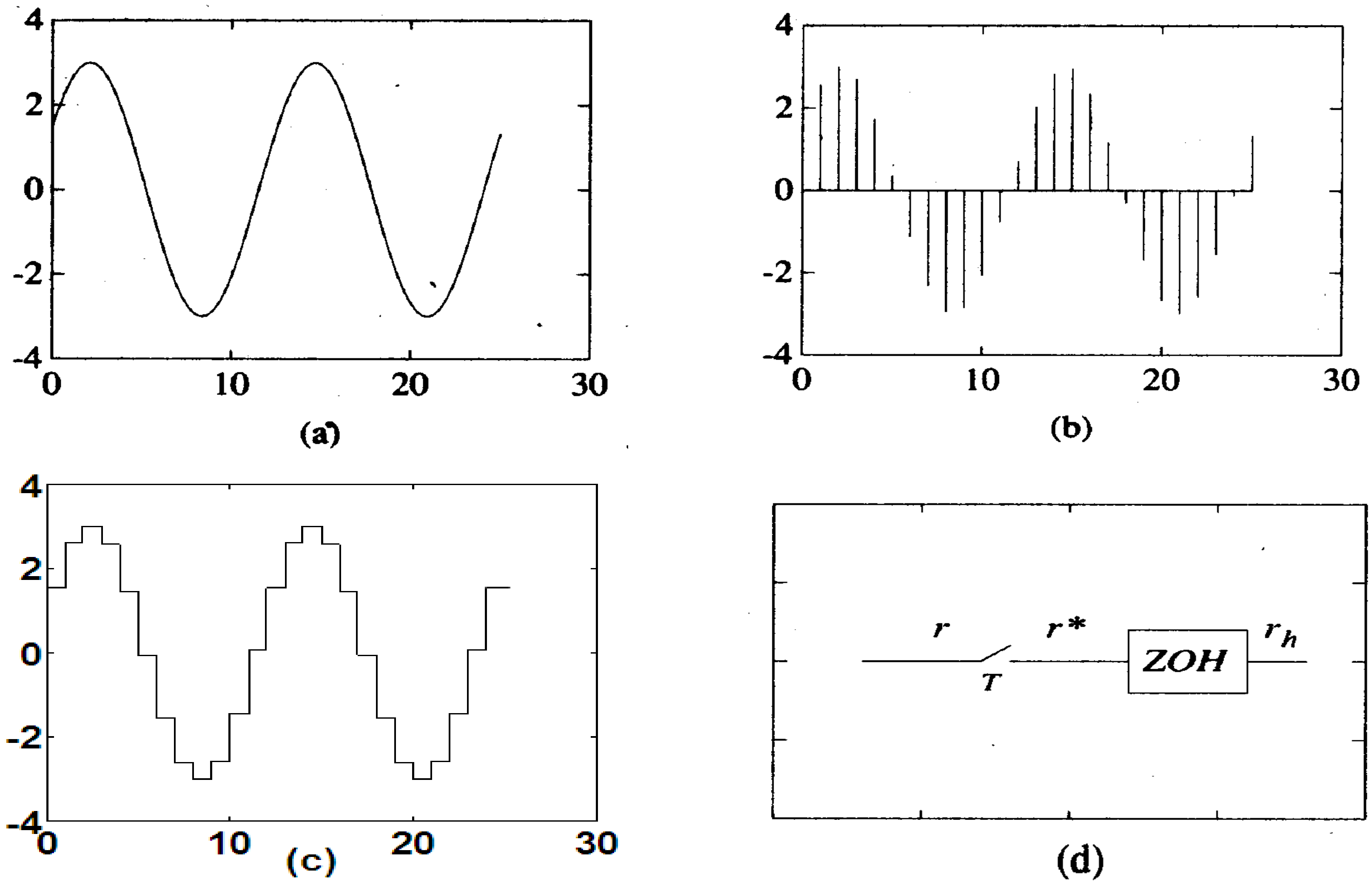
- Mathematical interpretation

$$1 - e^{-Ts} \approx 1 - \left[1 - Ts + \frac{T^2 s^2}{2!} - \cdots \right]$$

$$\frac{1 - e^{-Ts}}{s} \approx T - \frac{T^2 s}{2!} +$$

$$\approx T\left[1 - \frac{sT}{2} \right] \approx Te^{\frac{-sT}{2}}$$

- Physical interpretation of delay



The sample and hold, showing typical signals. (a) Input signal f; (b) sampled signal f^*; (c) output signal f_h; (d) sampler and holder.