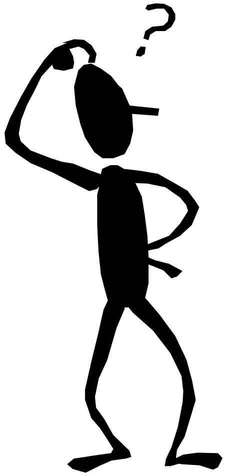
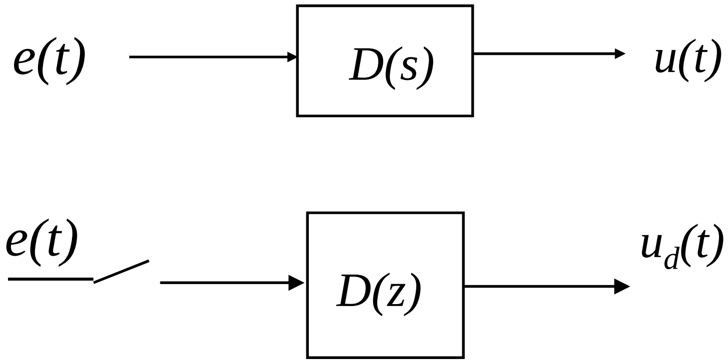


Chapter 6
Discrete Equivalents to Continuous Transfer Functions

What is equivalence?



Design D(s) in s-domain

Discretize to get $D(z)$

For the same input $e(t)$, does $u_d(t)$ approximate $u(t)$?
What kind of approximation should we seek?

Can the spectrum of $u_d(t)$ be made to approximate that of $u(t)$? If so, how do we go about constructing $D(z)$ from $D(s)$ such that the two spectra approximate one another?

6.1 Rational Approximation

The relationship between s and z domains is

$$z = e^{sT}$$

which may be approximated by rational functions.

Recall that $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Then possible approximations are:

3 possible approximations

$$z = e^{sT} \approx 1 + sT$$

$$z = e^{sT} = \frac{1}{e^{-sT}} \approx \frac{1}{1 - sT}$$

$$z = e^{sT} = \frac{e^{sT/2}}{e^{-sT/2}} \approx \frac{1 + sT/2}{1 - sT/2}$$

Apply one of them to make:

$$D(s) \Leftrightarrow D(z)$$

6.2 Numerical Integration

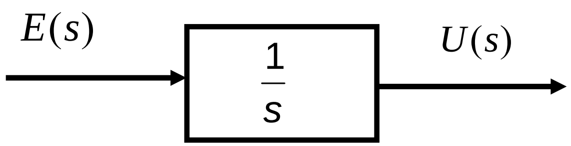
Consider an integrator:

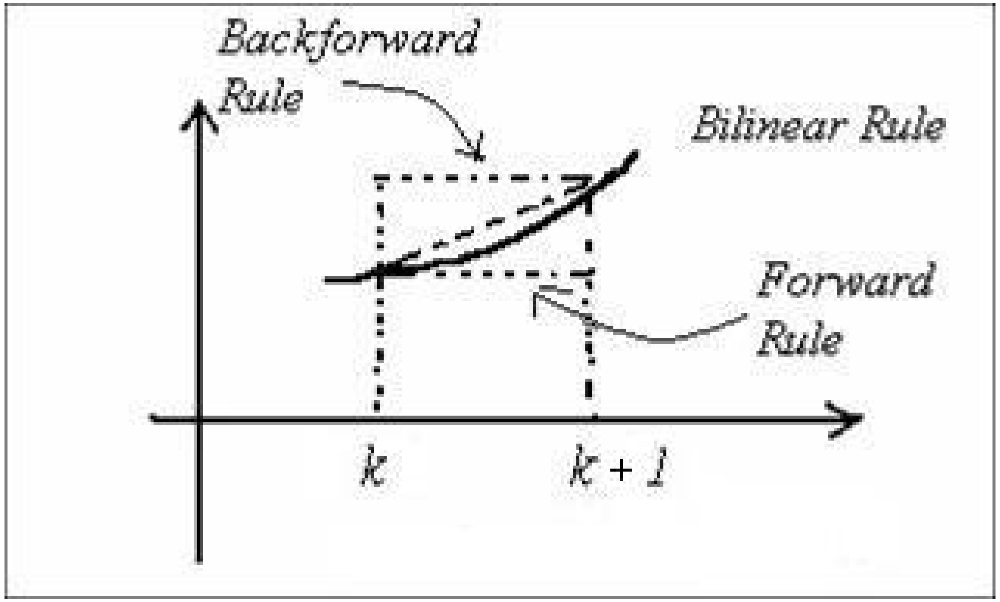
$$u(t) = \int_0^t e(\tau)d\tau \quad \xleftrightarrow{\text{LT}} \quad U(s) = \frac{1}{s}E(s)$$

Then

$$u(t+T) = \int_0^{t+T} e(\tau)d\tau = u(t) + \int_t^{t+T} e(\tau)d\tau$$

3 ways to approximate this area





$$\int_{kT}^{(k+1)T} e(t)dt = \begin{cases} Te\left(kT\right) \\ Te\left(\left(k+1\right)T\right) \\ \dfrac{T}{2}\left[e\left(kT\right)+e\left(\left(k+1\right)T\right)\right] \end{cases}$$

1st method

$$u(t+T)=u(t)+Te(t)$$

$$zU(z)=U(z)+TE(z)$$

$$\frac{U(z)}{E(z)}=\frac{T}{z-1} \qquad \rightarrow \frac{U(s)}{E(s)}=\frac{1}{s}$$

$$\boxed{\frac{1}{s}\approx\frac{T}{z-1} \qquad \text{or} \qquad z\approx 1+sT}$$

Looks familiar??

This method is also known as the forward rectangular rule or Euler's rule.

2nd method

$$u(t+T) = u(t) + Te(t+T)$$

$$zU(z) = U(z) + TzE(z)$$

$$\frac{U(z)}{E(z)} = \frac{Tz}{z-1}$$

$$\boxed{\frac{1}{s} \approx \frac{zT}{z-1} \quad \text{or} \quad z \approx \frac{1}{1-sT}}$$

This method is also known as the backward rectangular rule.

3rd method

$$u(t+T)=u(t)+\frac{T}{2}\{e(t)+e(t+T)\}$$

$$zU(z)=U(z)+\frac{T}{2}\{E(z)+zE(z)\}$$

$$\frac{U(z)}{E(z)}=\frac{T}{2}\frac{z+1}{z-1}$$

$$\boxed{\frac{1}{s}\approx\frac{T}{2}\frac{z+1}{z-1} \quad \text{or} \quad z\approx\frac{1+sT/2}{1-sT/2}}$$

This method is also known as the trapezoidal or Tustin's rule or the bilinear rule.

In Summary

Rule	D → C	C → D
Forward	$z = 1 + sT$	$s = \frac{z-1}{T}$
Backward	$z = \frac{1}{1 - sT}$	$s = \frac{z-1}{zT}$
Trapezoidal	$z = \frac{1 + sT/2}{1 - sT/2}$	$s = \frac{2}{T}\frac{z-1}{z+1}$

How do we make use of this approximations to find approximate discrete transfer functions from continuous time models?

Example : $$G(s) = \frac{K}{s^2 + s + 1}$$

Forward rule : $$G_d(z) = \frac{K}{\frac{(z-1)^2}{T^2} + \frac{z-1}{T} + 1}$$

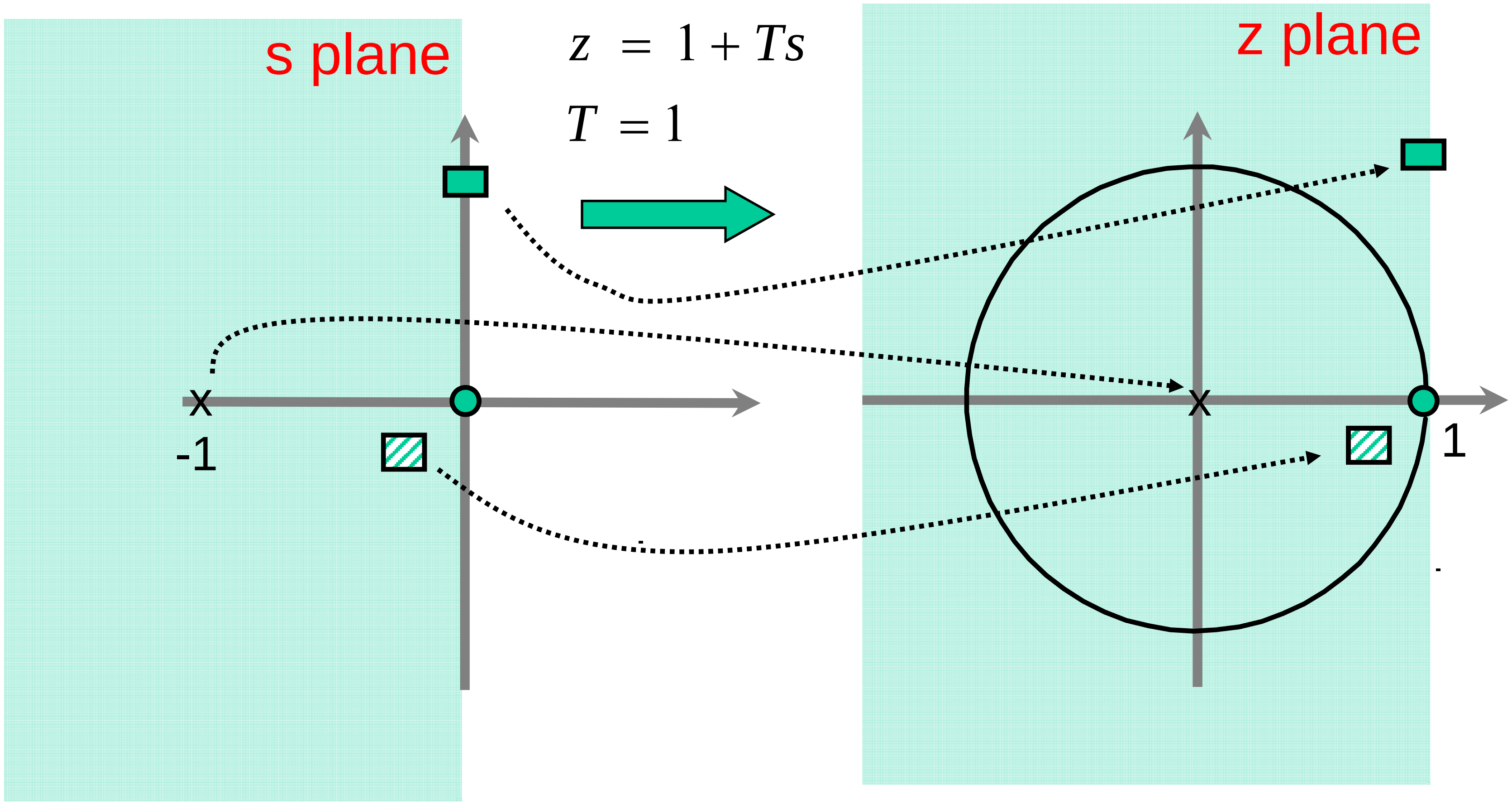
Backward rule : $$G_d(z) = \frac{K}{\frac{(z-1)^2}{z^2T^2} + \frac{z-1}{zT} + 1}$$

Trapezoidal's rule : $$G_d(z) = \frac{K}{\left(\frac{2}{T}\right)^2 \frac{(z-1)^2}{(z+1)^2} + \left(\frac{2}{T}\right) \frac{z-1}{z+1} + 1}$$

Which method is better? What are the differences in three z-models? What are their implications?

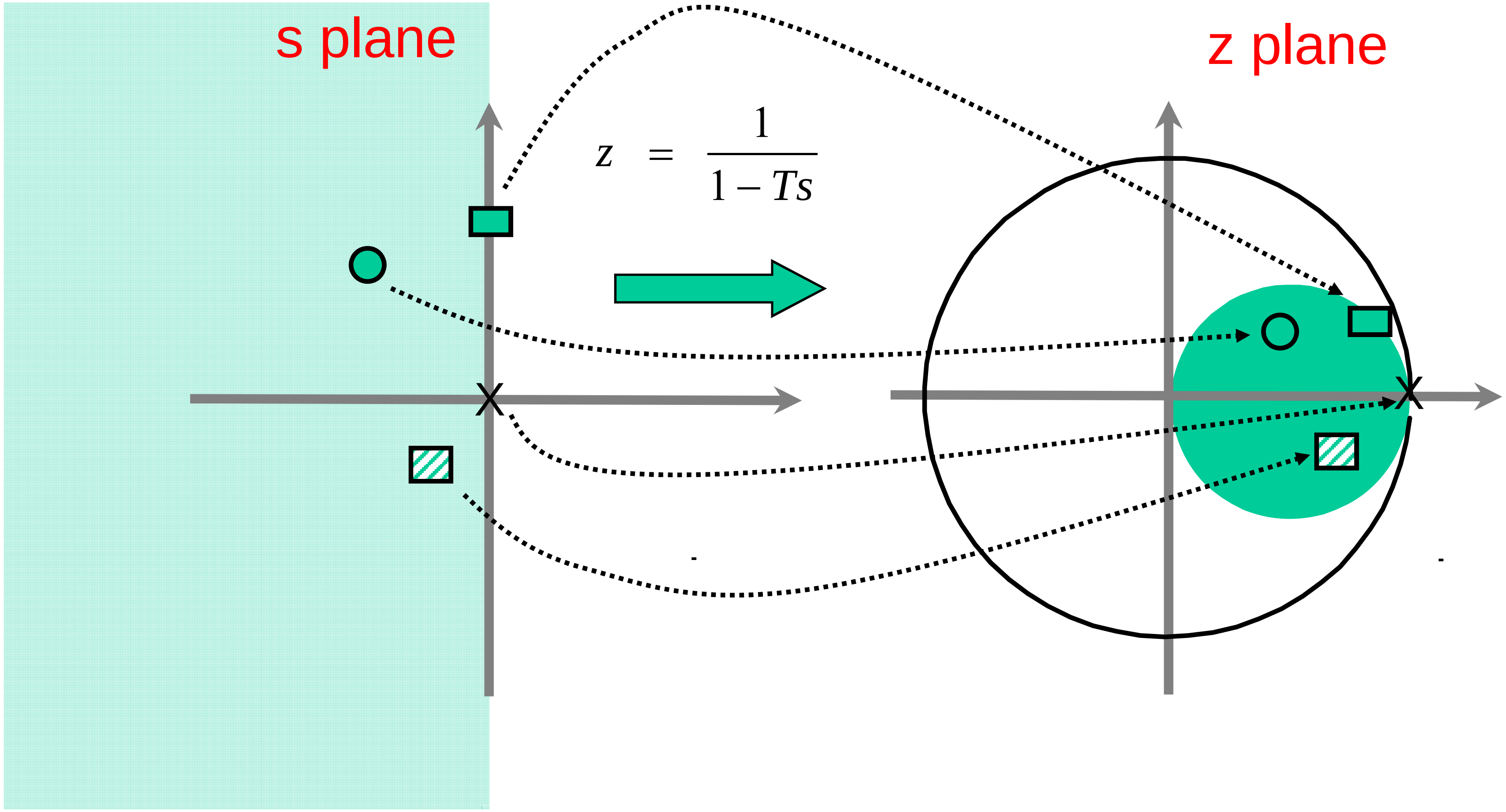
Stability Analysis

Forward Rectangular rule



Stable s-models map into stable z-models? No. s=-3 to z=-2

Backward Rectangular rule



Stable s-models map into stable z-models? Yes.

$$z = \frac{1}{1 - sT}$$

$$z - \frac{1}{2} = \frac{1}{1 - sT} - \frac{1}{2}$$

$$= \frac{1}{2}\frac{1 + sT}{1 - sT}$$

At $s=j\omega$

$$z - \frac{1}{2} = \frac{1}{2}\frac{1 + j\omega T}{1 - j\omega T}$$

$$\left| z - \frac{1}{2} \right| = \frac{1}{2}$$

Therefore, $s=j\omega$ maps into a circle centered at 0.5 with radius of 0.5.

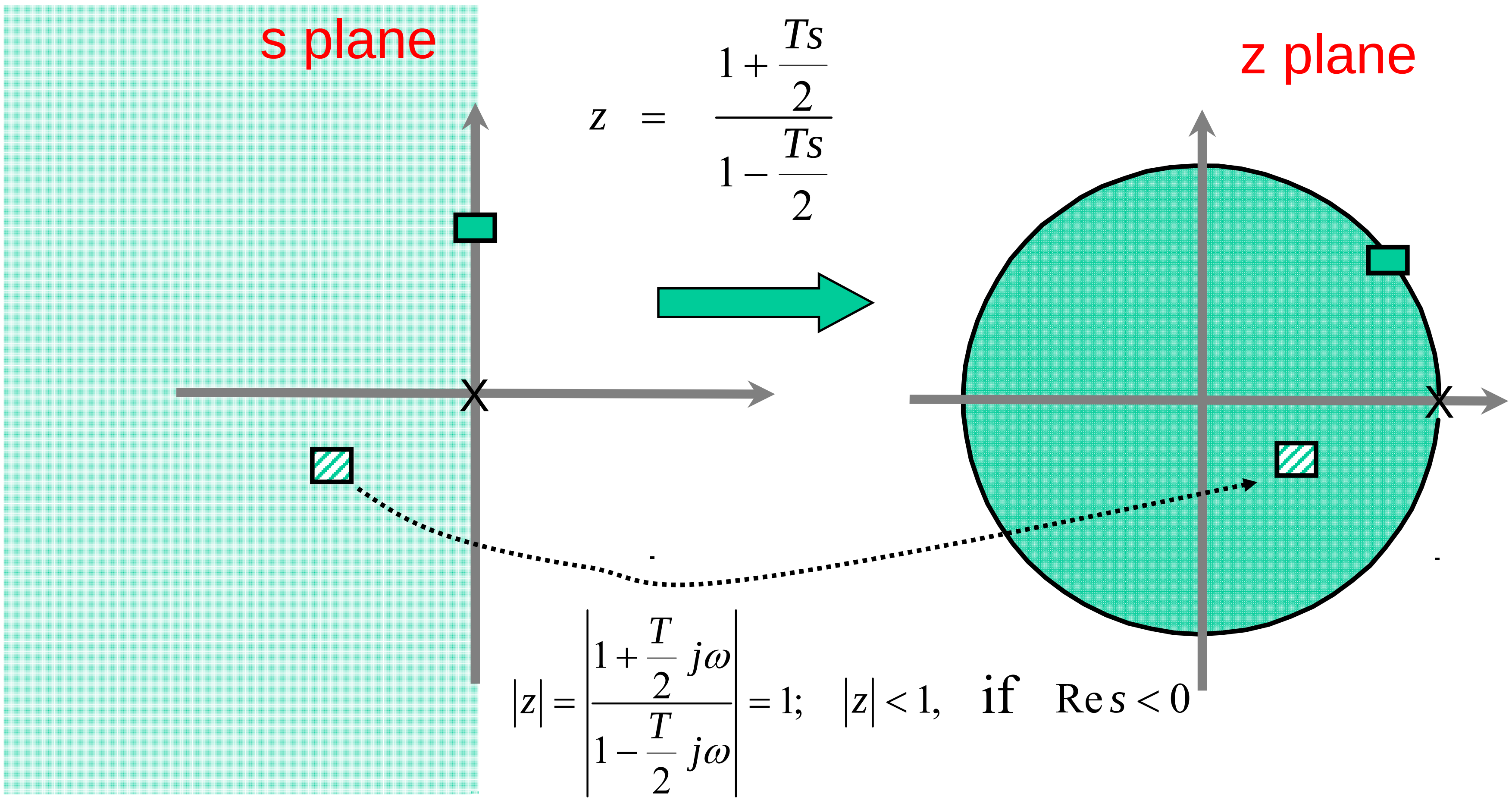
For $s=-\sigma+j\omega$

$$z - \frac{1}{2} = \frac{1}{2}\frac{1 - \sigma T + j\omega T}{1 + \sigma T - j\omega T}$$

$$\left| z - \frac{1}{2} \right| < \frac{1}{2}$$

Therefore, the left half of s-plane maps into the interior of the circle of z-plane centered at 0.5 with radius 0.5

Trapezoidal rule



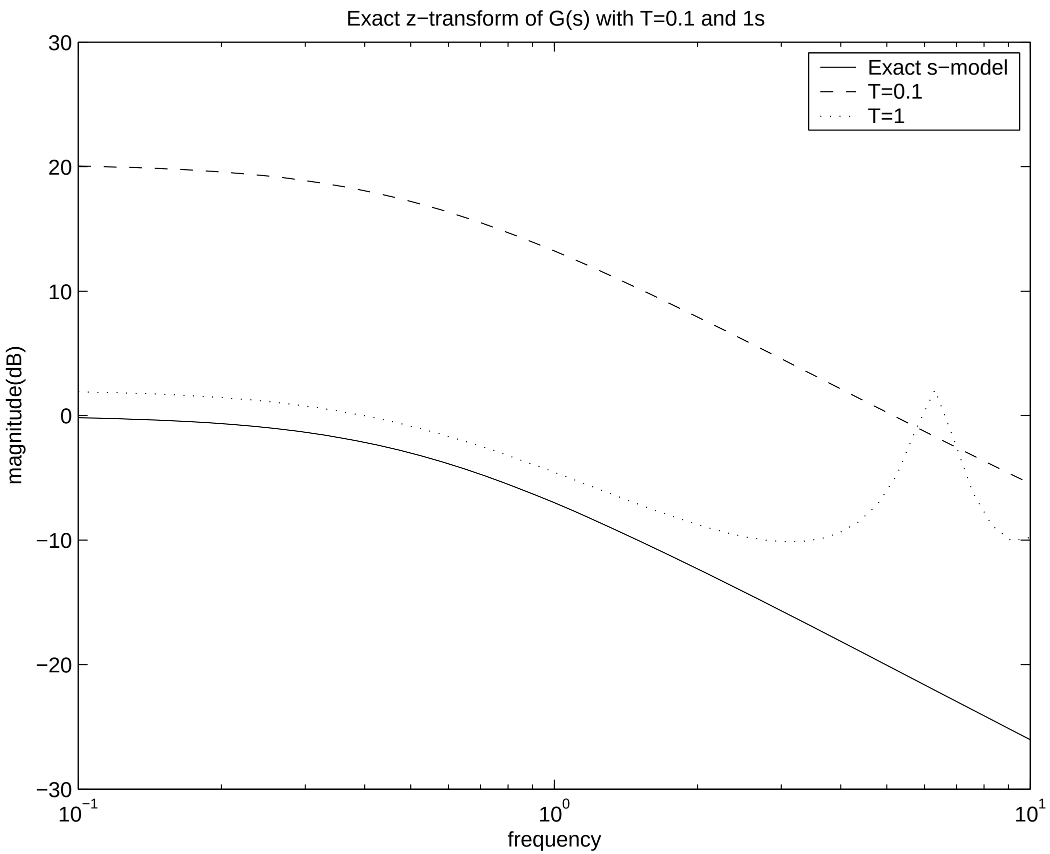
$$z = \frac{1+\frac{Ts}{2}}{1-\frac{Ts}{2}}$$

$$|z| = \left|\frac{1+\frac{T}{2}j\omega}{1-\frac{T}{2}j\omega}\right| = 1; \quad |z| < 1, \quad \text{if} \quad \operatorname{Re} s < 0$$

Stable s-models map into stable z-models? Yes! Exactly.

Spectrum Analysis

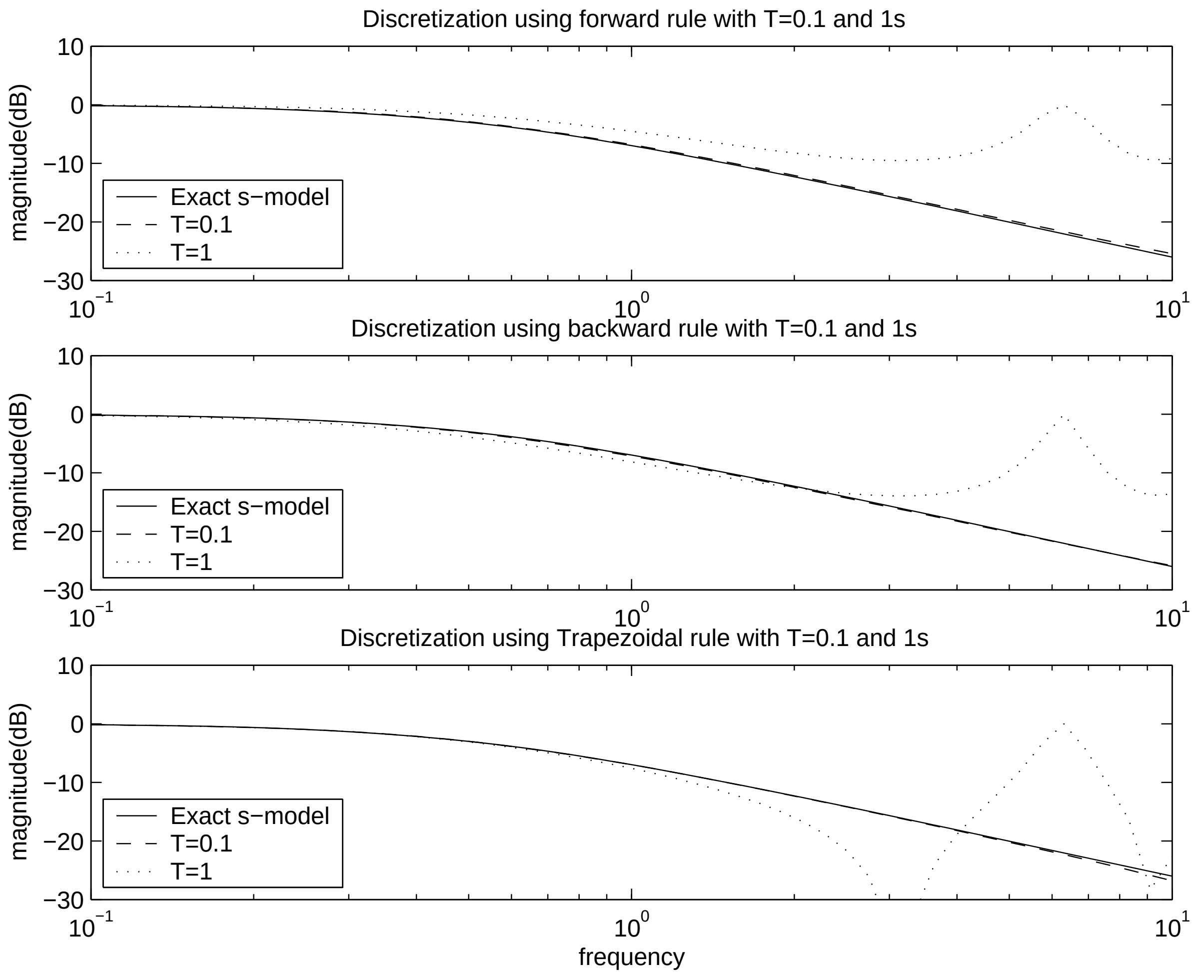
z-transform model



$$G(s) = \frac{1}{2s+1}$$

$$G(z) = \frac{0.5z}{z - e^{-0.5T}}$$

Three approximation rules



$$G(s) = \frac{1}{2s+1}$$

$$G_{fwd}(z) = \frac{T}{2(z-1)+T}$$

$$G_{bwd}(z) = \frac{zT}{z(T+2)-2}$$

$$G_{tpz}(z) = \frac{zT+T}{z(T+4)+T-4}$$

Analysis of Trapezoidal Rule

Continuous System

$$\frac{U(s)}{E(s)} = K\frac{T_1 s+1}{T_2 s+1}$$

$$\frac{U(j\omega)}{E(j\omega)} = K\frac{T_1 j\omega+1}{T_2 j\omega+1}$$

$$\left.\frac{U(j\omega)}{E(j\omega)}\right|_{\omega=0} = K$$

Discretized System

$$\frac{U(z)}{E(z)} = K\frac{T_1\frac{2\left(1-z^{-1}\right)}{T\left(1+z^{-1}\right)}+1}{T_2\frac{2\left(1-z^{-1}\right)}{T\left(1+z^{-1}\right)}+1}$$

$$= K\frac{T+2T_1+\left(T-2T_1\right)z^{-1}}{T+2T_2+\left(T-2T_2\right)z^{-1}}$$

$$\frac{U^*(j\omega)}{E^*(j\omega)} = K\frac{T+2T_1+\left(T-2T_1\right)e^{-j\omega T}}{T+2T_2+\left(T-2T_2\right)e^{-j\omega T}}$$

(i) At DC,

$$\left.\frac{U^*(j\omega)}{E^*(j\omega)}\right|_{\omega=0} = K, \text{ no distortion.}$$

(ii) Assuming fast sampling, then at low frequencies,

$$\left.\frac{U^*(j\omega)}{E^*(j\omega)}\right|_{T,\omega T\, small} \approx K\frac{T+2T_1+\left(T-2T_1\right)\left(1-j\omega T\right)}{T+2T_2+\left(T-2T_2\right)\left(1-j\omega T\right)}$$

$$= K\frac{2TT_1 j\omega + 2T - j\omega T^2}{2TT_2 j\omega + 2T - j\omega T^2}$$

$$\approx K\frac{T_1 j\omega + 1}{T_2 j\omega + 1}, \qquad \text{minimal distortion.}$$

(iii) At high frequencies, the distortion becomes significant

6.3 Zero-Pole mapping

$G(s)$ is converted to $G(z)$ using $z=e^{st}$ with the following rules.

i. If $G(s)$ has a pole at $s=-a$, then $G(z)$ has a pole at $z=e^{-aT}$;
   If $G(s)$ has a pole at $s=-a+jb$ then $G(z)$ has a pole at
   $z=e^{-aT}e^{jbT}$
ii. All finite zeros are mapped in the same way;
iii. Static gain matching is imposed:

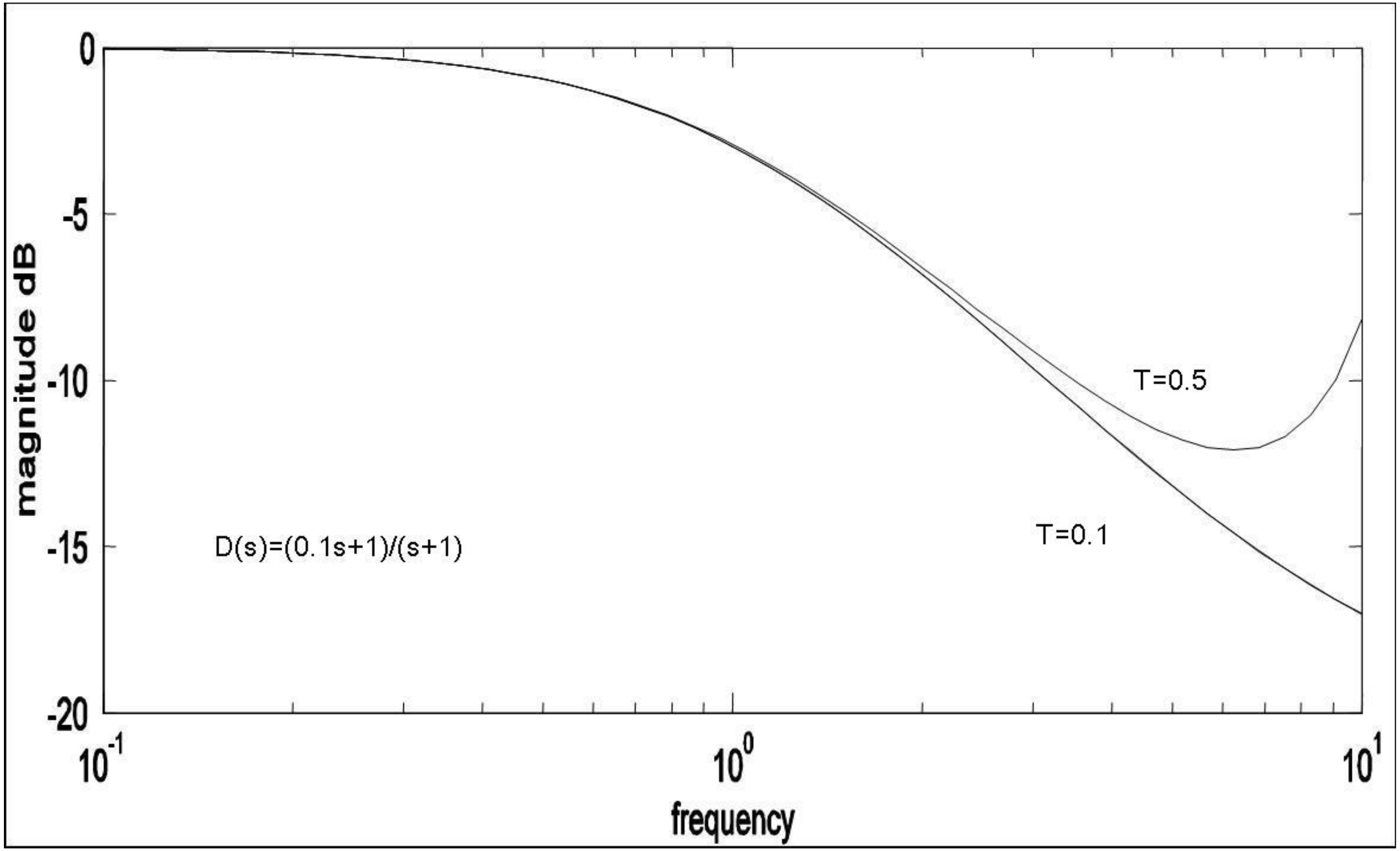
$$G(s)\big|_{s=0} = G(z)\big|_{z=1}$$

Example 1

$$\frac{U(s)}{E(s)} = K\frac{T_1 s+1}{T_2 s+1}$$

- One finite zero $\quad s=-\frac{1}{T_1} \quad \Rightarrow \quad z=e^{sT}=e^{-\frac{T}{T_1}}$
- One finite pole $\quad s=-\frac{1}{T_2} \quad \Rightarrow \quad z=e^{sT}=e^{-\frac{T}{T_2}}$
- Select K' so that dc gains match at z=1 $\quad \frac{U(z)}{E(z)} = K'\frac{z-e^{-\frac{T}{T_1}}}{z-e^{-\frac{T}{T_2}}}$

$$K'\frac{1-e^{-\frac{T}{T_1}}}{1-e^{-\frac{T}{T_2}}} = K, K' = \frac{1-e^{-\frac{T}{T_2}}}{1-e^{-\frac{T}{T_1}}}K.$$



The original frequency response is almost same as for *T*=0.1. Some inaccuracies when sampling frequency is low. Overall, not a bad approximation.

Example 2.

$$G(s) = \frac{a}{s+a}$$

Application of rules (i-ii) gives

$$G(z) = K\frac{1}{z - e^{-aT}}$$

Rule (iii) requires

$$1 = K\frac{1}{1 - e^{-aT}}, \qquad K = 1 - e^{-aT}$$

so that

$$G(z) = \frac{1 - e^{-aT}}{z - e^{-aT}}$$

6.4 Zero-Order Hold Equivalent

The controller is a dynamic system like the plant. So the discretization method for the plant with zero-order hold may also work for the controller.

$$\frac{U(s)}{E(s)} = D(s)$$

$$\frac{U(z)}{E(z)} = \left(1 - z^{-1}\right) Z\left\{\frac{D(s)}{s}\right\}$$

Example 1. $G(s)=\dfrac{1}{s^2}$

Then $Z\left\{\dfrac{G(s)}{s}\right\}=Z\left\{\dfrac{1}{s^3}\right\}=\dfrac{T^2 z(z+1)}{2(z-1)^3}$

and $G(z)=\left(1-z^{-1}\right)Z\left\{\dfrac{G(s)}{s}\right\}=\dfrac{T^2(z+1)}{2(z-1)^2}$

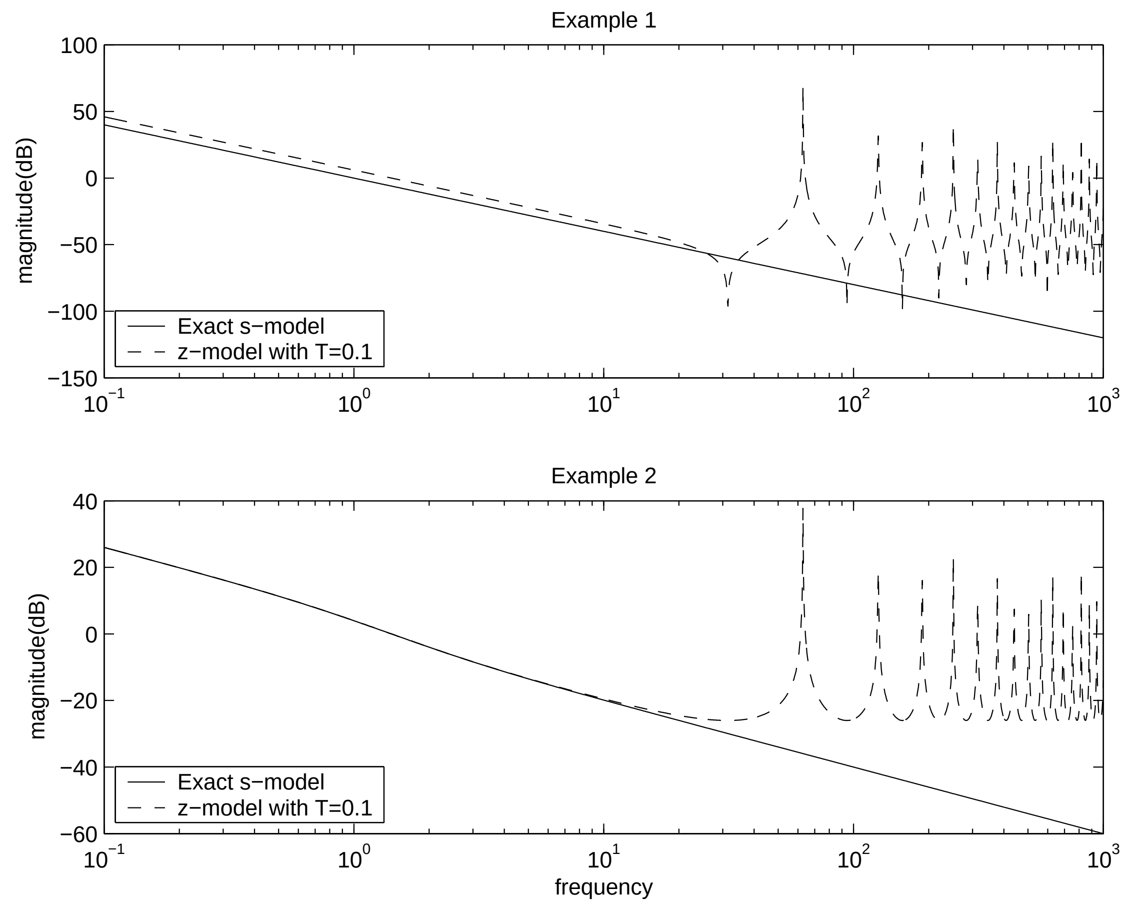
Example 2. $G(s)=\dfrac{s+2}{s(s+1)}$

Then $Z\left\{\dfrac{G(s)}{s}\right\}=Z\left\{\dfrac{2}{s^2}-\dfrac{1}{s}+\dfrac{1}{s+1}\right\}=\dfrac{2Tz}{(z-1)^2}-\dfrac{z}{z-1}+\dfrac{z}{z-e^{-T}}$

$$=\frac{\left(2T+e^{-T}-1\right)z+1-e^{-T}-2Te^{-T}}{(z-1)^2\left(z-e^{-T}\right)}z$$

and $G(z)=\left(1-z^{-1}\right)Z\left\{\dfrac{G(s)}{s}\right\}=\dfrac{\left(2T+e^{-T}-1\right)z+1-e^{-T}-2Te^{-T}}{(z-1)\left(z-e^{-T}\right)}$

The Bode plots of Examples 1 and 2



Summary of Discrete equivalents

- Numerical Integration

Rational Approximation

Forward rectangular

$$z = e^{sT} \approx 1 + sT$$

Backward rectangular

$$z = e^{sT} = \frac{1}{e^{-sT}} \approx \frac{1}{1 - sT}$$

Trapezoidal

$$z = e^{sT} = \frac{e^{sT/2}}{e^{-sT/2}} \approx \frac{1 + sT/2}{1 - sT/2}$$

- Pole-Zero Mapping ⟸ Conversion from s to z

- Hold Equivalence ⟸ ZOH discretization

Antenna system

When disturbances such as wind are neglected, the equation of antenna motion is

$$J\ddot{\theta} + B\dot{\theta} = u,$$

Where *u* is the net torque from the drive motor. Let us define *B/J=a* and we have the transfer function as follows

$$\frac{\theta(s)}{u(s)} = \frac{1/B}{s(s/a+1)}.$$

Assume the sampling time is *T* . If the forward rule is used, we have the discrete transfer function:

$$\frac{\Theta(z)}{U(z)} = \frac{1/B}{\left(\frac{z-1}{T}\right)\left(\frac{z-1}{aT}+1\right)} = \frac{aT^2/B}{z^2+(aT-2)z+1-aT}.$$

If the backward rule is used, we have

$$\frac{\Theta(z)}{U(z)}=\frac{1/B}{\left(\dfrac{z-1}{zT}\right)\left(\dfrac{z-1}{zTa}+1\right)}=\frac{aT^2z^2/B}{(1+Ta)z^2-(2+Ta)z+1}.$$

If the trapezoidal rule is used, we have

$$\frac{\Theta(z)}{U(z)}=\frac{1/B}{\left(\dfrac{2}{T}\dfrac{z-1}{z+1}\right)\left(\dfrac{2}{aT}\dfrac{z-1}{z+1}+1\right)}=\frac{aT^2(z+1)^2/B}{(4+2aT)z^2-8z+4-2aT}.$$

If the zero-pole matching method is used, we have

$$\frac{\Theta(z)}{U(z)} = \frac{K}{(z-1)\left(z-e^{-aT}\right)},$$

Note that since the system has a pole at the origin, the static gain is infinity for both continuous and discrete cases, and thus automatically matched. We then choose to match the velocity constant:

$$sG(s)\Big|_{s=0} = (z-1)G(z)\Big|_{z=1} \rightarrow K = \left.\frac{\dfrac{1/B}{(s/a+1)}}{\dfrac{1}{\left(z-e^{-aT}\right)}}\right|_{z=1,s=0} .$$

If the zero-order hold equivalent method is used, we have

$$\frac{\Theta(z)}{U(z)} = K\frac{(z+b)}{(z-1)(z-e^{-aT})},$$

where

$$K = \frac{aT-1+e^{-aT}}{aB}, \quad b = \frac{1-e^{-at}-aTe^{-aT}}{aT-1+e^{-aT}}.$$