Tutorial 1: Questions

1.

(a)) Let

$$x(k) = r^k \sin(k\theta)\mathbf{1}(k)$$

Show

$$Z\left[x(k)\right] = \frac{zr\sin\theta}{z^2 - 2r(\cos\theta)z + r^2}, \qquad |z| > r$$

(b) Find the z-transform of the sampled signal from $y(t) = \alpha t 1(t)$.

2. Consider the difference equation

$$u(k+2) = 0.25u(k)$$

(a) Assume that a solution $u(k) = A_i z^k$ and find the characteristic equation in *z*.

(b) Find the characteristic equation's roots z_1 and z_2 and decide if the solutions are stable.

(c) Assume a general solution of the form

$$u(k) = A_1 z_1^k + A_2 z_2^k$$

and find A_1 and A_2 to match the initial conditions $u(0) = 0,\ \mathrm{u}(1) = 1$.

3. Find the z-transform of

$$r(k) = \begin{cases} 0 & k\text{:even} \\ 2 & k\text{:odd} \end{cases}$$

4. Prove the following properties of the z-transform:

(a) Addition

$$Z\left\{ f_1(t) + f_2(t) \right\} = F_1(z) + F_2(z)$$

(b) Multiplication by a constant

$$Z\left\{ \alpha f(t) \right\} = \alpha F(z)$$

(c) Real translation

$$Z\left\{ f\left(t+nT\right)\right\} = z^{n}F\left(z\right)$$

(d) Complex translation

$$Z\left\{e^{-\alpha t} f\left(t\right)\right\} = F\left(ze^{\alpha T}\right)$$

5. Given the z-transform of $g(t)$ as

$$G\left(z\right)=\frac{1-3z^{-1}+3z^{-2}}{\left(1-0.5z^{-1}\right)\left(1-0.8z^{-1}\right)}.$$

Find the value of $g(k)$ for the first 3 sampling periods.

6. (a) Let $x(k) = k1(k)$ where $1(k)$ is the unit step. Show

$$Z\left\{x(k)\right\} = \frac{z}{\left(z-1\right)^{2}}.$$

(b) For the same $x(k)$ as in (a), look at

$$x(k) - x(k-1) = \{k \ \ -(k-1)\} = 1$$

$$Z\{\ x(k) - x(k-1)\} = (1 - z^{-1})X(z) = \frac{z}{z-1}$$

$$X(z) = \frac{z^2}{(z-1)^2}$$

which is different from the result in (a). Find where are wrong with the above derivation.

7. Find the inverse z-transform of

$$F(z) = \frac{z(z+1)}{(z-1)(z^2-z+1)}$$

by the following methods:

(a) Power series expansion;

(b) Partial fraction expansion.

8. Suppose that a sequence $f(k)$ has the z-transform

$$F(z) = \frac{1-0.2z^{-1}}{(1+0.6z^{-1})(1-0.3z^{-1})(1-z^{-1})}$$

(a) What is *f*(*k*)?

(b) What is the steady state value of the sequence?

9. The z-transform of a signal is given by

$$F(z) = \frac{z}{z^2 + 2z + 1}.$$

Obtain the first 4 samples of the signal, *f*(*k*), *k* = 0, 1, 2, 3.

10. Consider a signal with the transform

$$U(z) = \frac{z}{(z-1)(z-2)}$$

(a) What value is given by the final-value theorem applied to $u(z)$?

(b) Find the final value of $u(k)$ by computing *u*(*k*) actually.

(c) Explain why the above two differ.

11. Solve the difference equation using the z-transform

$$y(k) - 3y(k-1) + 2y(k-2) = r(k)$$

$$r(k) = \begin{cases} 1 & k = 0,1 \\ 0 & k \geq 2, k < 0 \end{cases}$$

Tutorial 1: Solutions

1. Find the z-transform of the following signals:

(a) Let

$$x(k) = r^k \sin(k\theta)\mathbf{1}(k)$$

Show

$$Z[x(k)] = \frac{zr\sin\theta}{z^2 - 2r(\cos\theta)z + r^2}, \qquad |z| > r$$

(b) Find the z-transform of the sampled signal from $y(t) = \alpha t 1(t)$.

Solution:

(a) Let

$$x(k) = r^k \sin(k\theta)\mathbf{1}(k)$$

x(k) can also be written as

$$x(k) = r^k \frac{e^{jk\theta} - e^{-jk\theta}}{2j} \mathbf{1}(k)$$

$$= \frac{1}{2j}\left(re^{j\theta}\right)^k \mathbf{1}(k) - \frac{1}{2j}\left(re^{-j\theta}\right)^k \mathbf{1}(k)$$

therefore,

$$Z\left[x(k)\right] = \frac{1}{2j}\left(\frac{z}{z - re^{j\theta}} - \frac{z}{z - re^{-j\theta}}\right)$$

$$= \frac{1}{2j} \cdot \frac{z\left(z - re^{-j\theta}\right) - z\left(z - re^{j\theta}\right)}{\left(z - re^{-j\theta}\right)\left(z - re^{j\theta}\right)}$$

$$\therefore Z\left[x(k)\right] = \frac{zr\sin\theta}{z^2 - 2r\left(\cos\theta\right)z + r^2}$$

(b) $y(kT) = \alpha kT1(k)$.

$$
\begin{aligned}
Y(z) &= \sum_{k=0}^{\infty} \alpha k T z^{-k} \\
&= \alpha T\left(z^{-1} + 2z^{-2} + 3z^{-3} + \cdots\right). \\
zY(z) &= \alpha T\left(1 + 2z^{-1} + 3z^{-2} + \cdots\right). \\
zY(z) - Y(z) &= \alpha T\left(1 + z^{-1} + z^{-2} + \cdots\right) \\
&= \frac{\alpha T}{1 - z^{-1}} = \frac{\alpha T z}{z-1}. \\
Y(z) &= \frac{\alpha T z}{(z-1)^2}.
\end{aligned}
$$

2. Consider the difference equation

$$u(k+2) = 0.25u(k)$$

(a) Assume that a solution $u(k) = Az^k$ and find the characteristic equation in *z*.

(b) Find the characteristic equation's roots z_1 and z_2 and decide if the solutions are stable.

(c) Assume a general solution of the form

$$u(k) = A_1 z_1^k + A_2 z_2^k$$

and find A_1 and A_2 to match the initial conditions $u(0) = 0,\ \mathrm{u}(1) = 1$.

Solution:

(a)

$$u(k) = Az^k \quad \text{yields}$$

$$Az^{k+2} = 0.25Az^k$$

$$\Rightarrow$$

$$z^2 - 0.25 = 0$$

(b) The roots are

$$z_1 = 0.5,\quad \mathrm{z}_2 = -0.5.$$

both are inside the unit circle. The system is stable.

(c) if $u(k) = A_1 z_1^k + A_2 z_2^k$ matches the conditions:

$$u(0) = 0 = A_1 + A_2$$
$$u(1) = 1 = A_1 z_1 + A_2 z_2$$

W

e obtain

$$A_1 = 1$$
$$A_2 = -1$$

3. Find the z-transform of ***r(k)=0* for *k*<0** and

$$r(k) = \begin{cases} 0 & k\text{:even} \\ 2 & k\text{:odd} \end{cases}$$

Solution:

Z-transform of $r(k)$ is given by

$$
\begin{aligned}
R(z) &= \sum_{k=0}^{\infty} r(k)\, z^{-k} \\
&= 2z^{-1} + 2z^{-3} + 2z^{-5} + 2z^{-7} + \ldots \\
&= 2z^{-1}(1 + z^{-2} + z^{-4} + z^{-6} + \ldots) \\
&= 2z^{-1}[1 + (z^{-2}) + (z^{-2})^2 + (z^{-2})^3 + \ldots] \\
&= \frac{2z^{-1}}{1 - z^{-2}} \\
&= \frac{2z}{z^2 - 1}
\end{aligned}
$$

4. Prove the following properties of the z-transform:

(a) Addition

$$Z\left\{ f_1(t) + f_2(t) \right\} = F_1(z) + F_2(z)$$

(b) Multiplication by a constant

$$Z\left\{ \alpha f(t) \right\} = \alpha F(z)$$

(c) Time- shift

$$Z\left\{ f\left(t+nT\right)\right\} = z^{n}F\left(z\right)$$

(d) Multiplication by a exponential function

$$Z\left\{e^{-\alpha t} f\left(t\right)\right\} = F\left(ze^{\alpha T}\right)$$

Solution:

(a)

$$\begin{aligned} Z\left\{ f_1\left(k\right) \pm f_2\left(k\right)\right\} &= \sum_{k=-\infty}^{\infty} \left[f_1\left(k\right) \pm f_2\left(k\right)\right] z^{-k} \\ &= \sum_{k=-\infty}^{\infty} f_1\left(k\right) z^{-k} \pm \sum_{k=-\infty}^{\infty} f_2\left(k\right) z^{-k} \\ &= F_1\left(z\right) \pm F_2\left(z\right) \end{aligned}$$

(b)

$$\begin{aligned} Z\left\{\alpha f\left(k\right)\right\} &= \sum_{k=-\infty}^{\infty} \alpha f\left(k\right) z^{-k} \\ &= \alpha \sum_{k=-\infty}^{\infty} f\left(k\right) z^{-k} \\ &= \alpha F\left(z\right) \end{aligned}$$

(c)

$$\begin{aligned} Z\left\{ f\left(k+n\right)\right\} &= \sum_{k=-\infty}^{\infty} f\left(k+n\right) z^{-k} \\ &\underline{\underline{k+n=m}} \sum_{m=-\infty}^{\infty} f\left(m\right) z^{n} z^{-m} \\ &= z^{n} F\left(z\right) \end{aligned} \qquad (3)$$

Equation (3) is the result when the two-sided z-transform (k: –∞ ~ +∞) is used. When the unilateral definition (k: 0 ~ +∞) is used, we get the following result.

$$\begin{aligned} Z\left\{ f\left(k+n\right)\right\} &= \sum_{k=0}^{\infty} f\left(k+n\right) z^{-k} \\ &= f\left(n\right) + f\left(n+1\right) z^{-1} + f\left(n+2\right) z^{-2} + \cdots \\ &= z^{n}\left\{ f\left(n\right) z^{-n} + f\left(n+1\right) z^{-(n+1)} + \cdots \right\} \\ &= z^{n}\left\{ -\sum_{k=0}^{n-1} f\left(k\right) z^{-k} + \sum_{k=0}^{n-1} f\left(k\right) z^{-k} + f\left(n\right) z^{-n} + f\left(n+1\right) z^{-(n+1)} + \cdots \right\} \\ &= z^{n}\left\{ F\left(z\right) - \sum_{k=0}^{n-1} f\left(kT\right) z^{-k} \right\} \end{aligned}$$

(d)

$$
\begin{aligned}
Z\left\{e^{-\alpha t} f(k)\right\} &= \sum_{k=-\infty}^{\infty} e^{-\alpha kT} f(k) z^{-k} \\
&= \sum_{k=-\infty}^{\infty} f(k)\left(e^{\alpha T} z\right)^{-k} \\
&= F\left(e^{\alpha T} z\right)
\end{aligned}
$$

5. Given the z-transform of $g(k)$ as

$$G(z) = \frac{1-3z^{-1}+3z^{-2}}{\left(1-0.5z^{-1}\right)\left(1-0.8z^{-1}\right)}.$$

Find the value of $g(k)$ for the first 3 sampling periods.

Solution:

First method: By long division

$$
\begin{array}{r}
1-1.7z^{-1}+0.39z^{-2}+\cdots \\
1-1.3z^{-1}+0.4z^{-2}\,\overline{\big)\,1-3.0z^{-1}+3.00z^{-2}} \\
1-1.3z^{-1}+0.40z^{-2} \\
\hline
-1.7z^{-1}+2.60z^{-2} \\
\vdots
\end{array}
$$

Coefficients of the quotient gives the sampled values of $g(k)$. Why?

$$G(z)=1-1.7z^{-1}+0.39z^{-2}+\cdots=\sum_{k=0}^{\infty} g(k)z^{-k}.$$

$$g(0)=1,\;\; g(1)=-1.7,\;\; g(2)=0.39,\;\; \mathrm{g}(3)=\cdots$$

Second method: Partial fraction. Multiply numerator and denominator by z^2.

$$\begin{aligned} G(z) &= \frac{z^2 - 3z + 3}{z^2 - 1.3z + 0.4} \\ &= 1 - \frac{1.7z - 2.6}{z^2 - 1.3z + 0.4} \\ &= 1 - \frac{1.7z - 2.6}{(z - 0.5)(z - 0.8)} \\ &= 1 - \left[\frac{5.833}{z - 0.5} - \frac{4.133}{z - 0.8} \right] \\ &= 1 - z^{-1} \left[\frac{5.833z}{z - 0.5} - \frac{4.133z}{z - 0.8} \right]. \end{aligned}$$

Taking inverse z-transform,

$$g(k) = \delta_k - 5.833(0.5)^{k-1} 1(k-1) + 4.133(0.8)^{k-1} 1(k-1),$$

where $1(k-1)$ is the delayed unit step function, noting

$$Z^{-1}\left(\frac{z}{z - 0.5}\right) = (0.5)^k 1(k)$$

$$Z^{-1}\left(\frac{1}{z - 0.5}\right) = Z^{-1}\left(z^{-1} \frac{z}{z - 0.5}\right) = (0.5)^k 1(k) \big|_{k \leftarrow k-1} = (0.5)^{k-1} 1(k-1)$$

Therefore, interpreting δ_k = 1 for k = 0 and zero for all other k, the sample value of g(k) are:

$$g(0)=1,\ \ g(1)=-1.7,\ \ g(2)=0.39,\ \ g(3)=\cdots$$

3rd method:

$$G(z)=\frac{z^2-3z+3}{z^2-1.3z+0.4}$$
$$\frac{G(z)}{z}=\frac{z^2-3z+3}{z(z-0.5)(z-0.8)}$$
$$=\frac{7.5}{z}-\frac{11.66}{z-0.5}+\frac{5.166}{z-0.8}$$
$$G(z)=7.5-\frac{11.66z}{z-0.5}+\frac{5.166z}{z-0.8}.$$

Taking inverse z-transform,

$$g(k)=7.5\delta_k-11.66(0.5)^k 1(k)+5.16(0.8)^k 1(k)$$
$$g(0)=1,\ \ g(1)=-1.7,\ \ g(2)=0.39,\ \ g(3)=\cdots$$

Both $g(k)$ are equivalent to each other because

$$
\begin{aligned}
g(k) &= 7.5\delta_k - 11.66(0.5)^k 1(k) + 5.16(0.8)^k 1(k) \\
&= 7.5\delta_k - 11.66\delta_k - 11.66 \times 0.5 \times (0.5)^{k-1} 1(k-1) + 5.16\delta_k \\
&\quad + 5.16 \times 0.8 \times (0.8)^{k-1} 1(k-1) \\
&= \delta_k - 5.833(0.5)^{k-1} 1(k-1) + 4.133(0.8)^{k-1} 1(k-1)
\end{aligned}
$$

which has made use of

$$
\begin{aligned}
&(0.5)^k 1(k) = (0.5)^k [1(k) - 1(k-1) + 1(k-1)] \\
&= (0.5)^k [\delta(k) + 1(k-1)] = (0.5)^k \delta(k) + (0.5)^k 1(k-1) \\
&= \delta(k) + 0.5(0.5)^{k-1} 1(k-1)]
\end{aligned}
$$

6. (a) Let $x(k) = k1(k)$ where $1(k)$ is the unit step. Show

$$
Z\{x(k)\} = \frac{z}{(z-1)^2}.
$$

(b) For the same $x(k)$ as in (a), look at

$$x(k) - x(k-1) = \{k \ - (k-1)\} = 1$$

$$Z\{\ x(k) - x(k-1)\} = (1 - z^{-1})X(z) = \frac{z}{z-1}$$

$$X(z) = \frac{z^2}{(z-1)^2}$$

which is different from the result in (a). Find where are wrong with the above derivation.

Solution:

(a)

$$\begin{aligned} X(z) = Z\{x(k)\} &= \sum_{k=-\infty}^{k=\infty} x(k)z^{-k} \\ &= \sum_{k=0}^{k=\infty} kz^{-k} \\ &= 0 + z^{-1} + 2z^{-2} + \cdots + kz^{-k} + \cdots \\ &= \frac{z}{(z-1)^2}. \end{aligned}$$

(b) One sees

$$x(k)-x(k-1)=\{k1(k)-(k-1)1(k-1)\}=0,\ 1,\ 1,\ \ldots$$

$$Z\{\ x(k)-x(k-1)\}=(1-z^{-1})X(z)$$

$$Z\{x(k)-x(k-1)\}=\sum_{k=0}^{\infty}[x(k)-x(k-1)]z^{-k}$$

$$=0+z^{-1}+z^{-2}+\cdots+z^{-k}+\cdots$$

$$=z^{-1}(1+z^{-1}+\cdots+z^{-(k-1)}+\cdots$$

$$=z^{-1}\frac{z}{z-1}=\frac{1}{z-1}$$

$$X(z)=\frac{z}{(z-1)^2}$$

which is the same as in (a).

7. Find the inverse z-transform of

$$F(z)=\frac{z(z+1)}{(z-1)(z^2-z+1)}$$

by the following methods:

(a) Power series expansion;

(b) Partial fraction expansion.

Solution:

By power series expansion:

$$\begin{array}{r@{}l} & z^{-1}+3z^{-2}+4z^{-3}+3z^{-4}+\cdots \\ \cline{2-2} z^3-2z^2+2z-1 \big) & z^2 + z \end{array}$$

By partial fraction expansion:

$$\begin{aligned} \frac{F(z)}{z} &= \frac{z+1}{(z-1)\left(z^2-z+1\right)} \\ &= \frac{2}{z-1}-\frac{2z-1}{z^2-z+1}. \\ F(z) &= \frac{2z}{z-1}-\frac{2z(z-0.5)}{z^2-z+1}. \end{aligned} \qquad (1)$$

Consider the z-transform pair:

$$e^{-akT}\cos bkT \cdot 1(k) \longleftrightarrow \frac{z\left(z-e^{-aT}\cos bT\right)}{z^2-(2e^{-aT}\cos bT)z+e^{-2aT}}. \qquad (2)$$

Comparing the second term in (1) and (2), we have

$$e^{-2aT} = 1 \Rightarrow a = 0$$
$$\cos bT = 0.5$$
$$bT = \frac{\pi}{3}$$

Therefore,

$$f(k) = 2 \times 1(k) - 2\cos(\frac{\pi}{3}k)1(k)$$

1(k) is the unit step function.

8. Suppose that a sequence $f(k)$ has the z-transform

$$F(z) = \frac{1 - 0.2z^{-1}}{\left(1 + 0.6z^{-1}\right)\left(1 - 0.3z^{-1}\right)\left(1 - z^{-1}\right)}$$

(a) What is $f(k)$?

(b) What is the steady state value of the sequence?

Solution:

(a) Find the inverse z-transform of $F(z)$ first.

$$\begin{aligned} F(z) &= \frac{z^3 - 0.2z^2}{(z+0.6)(z-0.3)(z-1)} \\ &= z\left[\frac{A}{z+0.6} + \frac{B}{z-0.3} + \frac{C}{z-1}\right] \\ &= z\left[\frac{0.33}{z+0.6} - \frac{0.0476}{z-0.3} + \frac{0.71}{z-1}\right] \\ &= \frac{0.33z}{z+0.6} - \frac{0.0476z}{z-0.3} + \frac{0.71z}{z-1} \end{aligned}$$

Inverse z-transform gives

$$f(k) = 0.33(-0.6)^k 1(k) - 0.0476(0.3)^k 1(k) + 0.71 \times 1(k)$$

(b) Before applying final value theorem to obtain the steady state value, need to first check that $(z - 1)F(z)$ is stable. This can be determined by examining the poles of $(z - 1)F(z)$ which are at $z = -0.6, 0.3$. $(z - 1)F(z)$ is stable. Applying final value theorem,

$$
\begin{aligned}
f(\infty) &= \lim_{z\to 1}(z-1)F(z) \\
&= \lim_{z\to 1}(z-1)\left[\frac{z^3-0.2z^2}{(z+0.6)(z-0.3)(z-1)}\right] \\
&= 0.71
\end{aligned}
$$

Can you verify that this is correct?

9. The z-transform of a signal is given by

$$F(z) = \frac{z}{z^2+2z+1}.$$

Obtain the first 4 samples of the signal, $f(k)$, k = 0, 1, 2, 3.

Solution:

Since only samples are required, you may use the long division method to obtain the samples directly.

$$
\begin{array}{r}
z^{-1}-2z^{-2}+3z^{-3} \\
z^2+2z+1\,\overline{\big)\,z\qquad\qquad\qquad} \\
\underline{z+2+z^{-1}} \\
-2-z^{-1} \\
\underline{-2-4z^{-1}-2z^{-2}} \\
3z^{-1}+2z^{-2} \\
\underline{3z^{-1}+6z^{-2}+3z^{-3}} \\
\vdots
\end{array}
$$

Thus comparing the quotient to the coefficients of the z-transform, we get sample sequence as $\{0, 1, -2, 3\}$. The first term is zero because the quotient starts with z^{-1} term.

10. Consider a signal with the transform

$$U(z)=\frac{z}{(z-1)(z-2)}$$

(a) What value is given by the final-value theorem applied to $U(z)$?

(b) What is $u(k)$ when k goes to infinity by computing $u(k)$ actually.

(c) Explain why the above two differ.

Solution:

(a) $\lim_{z\to 1}(z-1)U(z)=\lim_{z\to 1}\frac{z}{z-2}=-1$

(b) $U(z)=\frac{z}{(z-1)(z-2)}=\frac{-z}{z-1}+\frac{z}{z-2}$

$$u(k)=-1+2^k, \qquad u(\infty)=\infty$$

(c) The final value Theorem does not apply if $(z-1)U(z)$ has poles on or outside the unit circle. In this case, $(z-1)U(z)=\frac{z}{z-2}$ is unstable, indeed.

11. Solve the difference equation using the z-transform

$$y(k)-3y(k-1)+2y(k-2)=r(k)$$

$$r(k)=\begin{cases}1 & k=0,1\\ 0 & k\geq 2, k<0\end{cases}$$

Solution:

Take z-transform on both sides,

$$Y(z) - 3z^{-1}Y(z) + 2z^{-2}Y(z) = R(z)$$

$$Y(z) = \frac{1}{1 - 3z^{-1} + 2z^{-2}} R(z)$$

In this case, $R(z)$ is not a unit step input. $R(z)$ has to be calculated from the $r(k)$ that is given.

Taking z-transform of $r(k)$,

$$\begin{aligned} R(z) &= \sum_{k=0}^{\infty} r(k) z^{-k} \\ &= 1 + 1z^{-1} + 0z^{-2} + 0z^{-3} + \cdots \\ &= 1 + z^{-1} \end{aligned}$$

Hence $Y(z)$ is given by

$$Y(z) = \frac{1 + z^{-1}}{(1 - z^{-1})(1 - 2z^{-1})} = \frac{-2z}{z - 1} + \frac{3z}{z - 2} \ .$$

Take the inverse:

$$y(k) = -2 \times 1(k) + 3 \times 2^{k} \times 1(k)$$