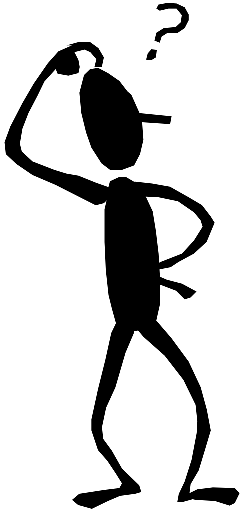


Lecture 2
Discrete Systems

2.1 Difference Equations

In a digital control system, the control computer receives the output measurement from the plant, $y(k)$, at discrete time, k; and sends to the plant its control action, $u(k)$, at discrete time, k.

The computer is controlling an object with input sequence $\{u(k)\}$ and output sequence $\{y(k)\}$. **How to represent their relationship?**

In a continuous system, dynamics or behaviour of its input-output relationship,

$$\{u(t) \rightarrow y(t): t \in [0, \infty)\}$$

is described by differential equation.

In a discrete system, dynamics or behaviour of its input-output relationship,

$$\{u(k) \rightarrow y(k): k = 0, 1, 2, \ldots\}$$

is described by difference equation.

A discrete system's current output *y(k)* is a function of input signal up to the *k*-th time instant, *u(0), u(1), u(2), …, u(k)* and the output signal prior to that time *y(0), y(1), …, y(k-1)*:

$$y(k) = f\left(u(0), \cdots, u(k), y(0), \cdots, y(k-1)\right). \qquad (1)$$

If f is linear:

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) - \cdots - a_n y(k-n) + b_0 u(k) + b_1 u(k-1) + \cdots + b_m u(k-m). \qquad (2)$$

(2) is called a linear difference equation (LDE) of order *n*.

To solve a LDE, we need a starting k and some initial conditions. For example,

$$y(k) = y(k-1) + y(k-2). \tag{3}$$

Let $k = 2$, we must know $y(0)$ and $y(1)$ to compute $y(2)$, $y(3)$, …. If $y(0) = y(1) = 1$, then $y(2) = 2$, $y(3) = 3$, $y(4) = 5$ and $y(5) = 8$, ... This is the numerical solution.

For analytical solution, recall that solutions of the form Ae^{st} are used for continuous time case. For discrete case,

$$t = Tk, \quad e^{st} = e^{sTk} = (e^{sT})^k := z^k$$

Assume $y(k) = Az^k$ in (3), we get

$$Az^k = Az^{k-1} + Az^{k-2}$$

or

$$z^2 = z + 1$$

This polynomial has two solutions

$$z_1 = 1/2 + \sqrt{5}/2$$

$$z_2 = 1/2 - \sqrt{5}/2$$

Thus a general solution to (3) has the form:

$$y(k) = A_1 z_1^k + A_2 z_2^k$$

The unknowns A_1 and A_2 can be solved from the given initial conditions. If $y(0) = 1$ and $y(1) = 1$, we get

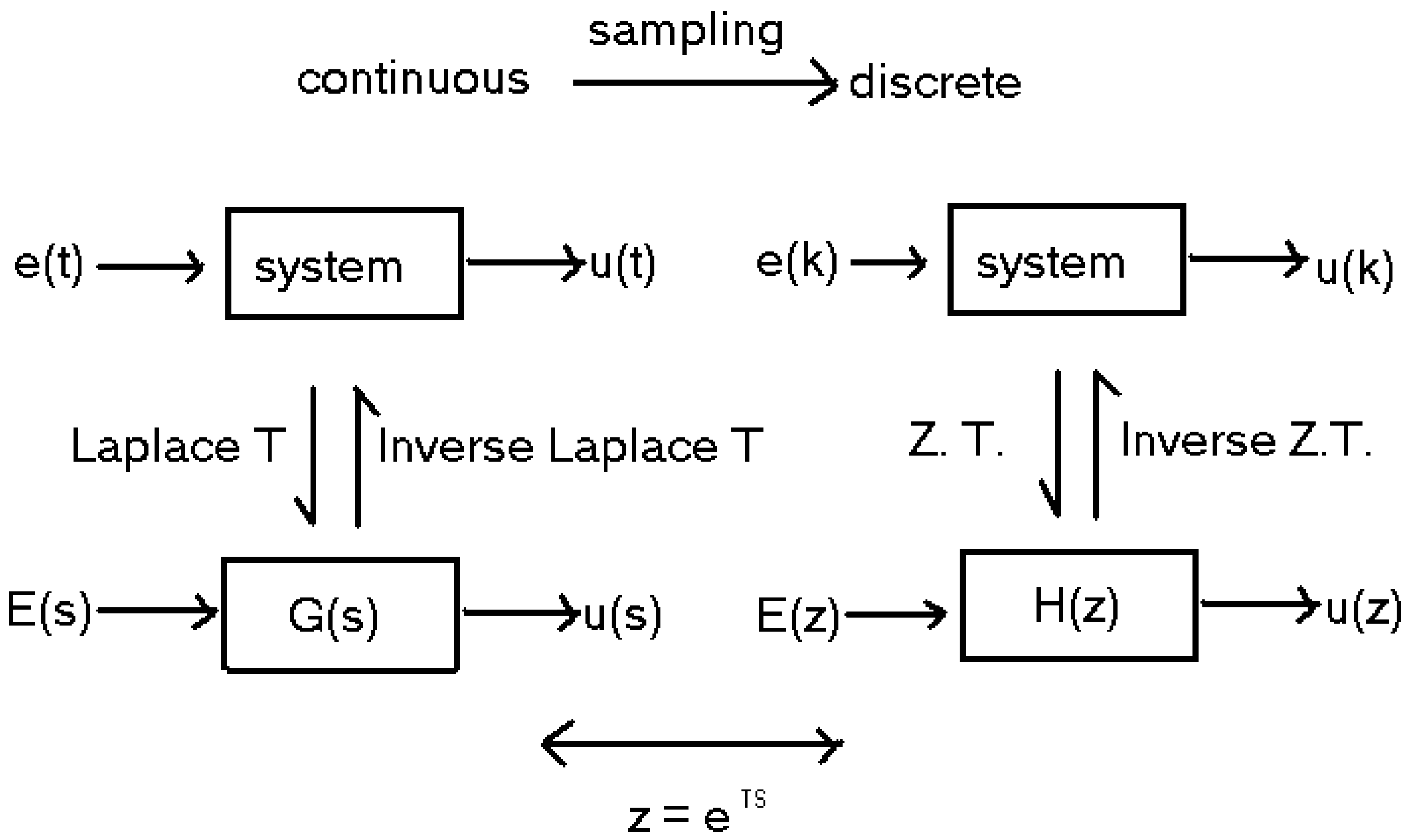
$$A_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}, \qquad A_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}.$$

We can analyze and design continuous systems without solving differential equations with help of

Laplace Transform

Can we do the same for discrete systems?

Yes, with help of z-Transform


sampling
continuous
discrete
e(t)
system
u(t)
e(k)
system
u(k)
Laplace T
Inverse Laplace T
Z. T.
Inverse Z.T.
E(s)
G(s)
u(s)
E(z)
H(z)
u(z)
z = e TS

2.2 The z-Transform

To obtain corresponding transforms for discrete time sequences rather than continuous time signals, we use the z-transform.

Given a sequence, {… x(-2), x(-1), x(0), x(1), x(2), …}, we define z-transform as

$$X(z) = Z\{x(k)\} = \sum_{k=-\infty}^{k=\infty} x(k) z^{-k}$$

Due to the infinite sum, convergence is an important issue. Ideally, the region of convergence (ROC) should be stated. ROC refers to the region on the complex plane on which the transform exists

As an example, let $x(t) = e^{-at}\boldsymbol{1(t)}$, then $x(k) = e^{-akT}\boldsymbol{1(k)}$, and its z-transform is

$$\sum_{-\infty}^{+\infty} x(k) z^{-k} = \sum_{0}^{\infty} e^{-akT} z^{-k} = \sum_{0}^{\infty} \left(e^{-aT} z^{-1}\right)^{k}$$

$$= \frac{1}{1 - e^{-aT} z^{-1}} = \frac{z}{z - e^{-aT}}, \qquad |z| > e^{-aT},$$

due to

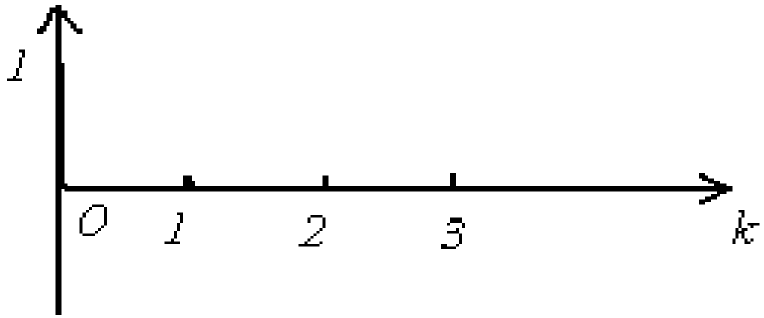
$$1 + \alpha + \alpha^2 + \cdots = \frac{1}{1-\alpha}, |\alpha| < 1.$$

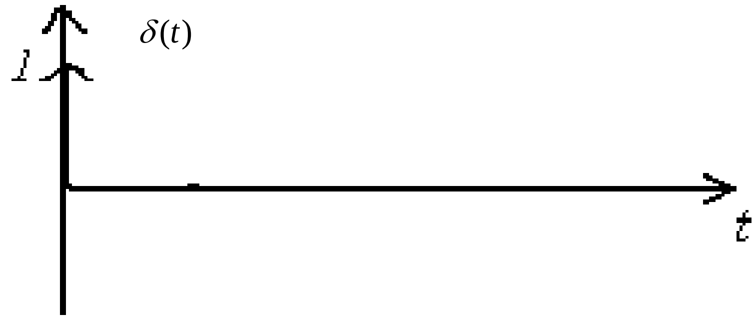
Typical Signals

(1) The unit pulse : The unit pulse is defined by

$$\delta_k := \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

$x(t) = \delta(t) \qquad X(s) = 1$



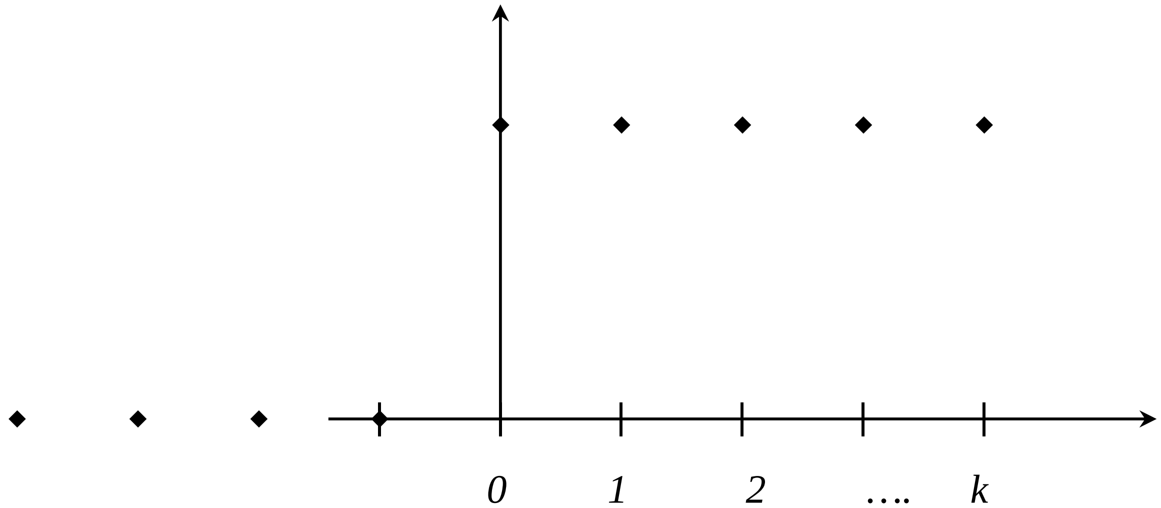


$$Z(\delta_k) = \sum_{-\infty}^{\infty} \delta_k z^{-k} = 1$$

(2) The unit step: It is given by

$$\mathbf{1}(k) := \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$x(t) = \mathbf{1}(t) \qquad X(s) = \frac{1}{s}$$



Its z-transform is

$$Z[\mathbf{1}(k)] = \sum_{-\infty}^{\infty} \mathbf{1}(k) z^{-k} = \sum_{0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}, \quad |z| > 1.$$

It has the pole at $z = 1$. Recall that the Laplace transform of the unit step function is $1/s$. And thus a pole at $s = 0$ for a continuous signal corresponds in some way to a pole at $z = 1$ for discrete signals.

$$s = 0 \quad \rightarrow \quad z = 1$$

which is also verified using

$$e^{Ts} = z$$

(3) The unit ramp:

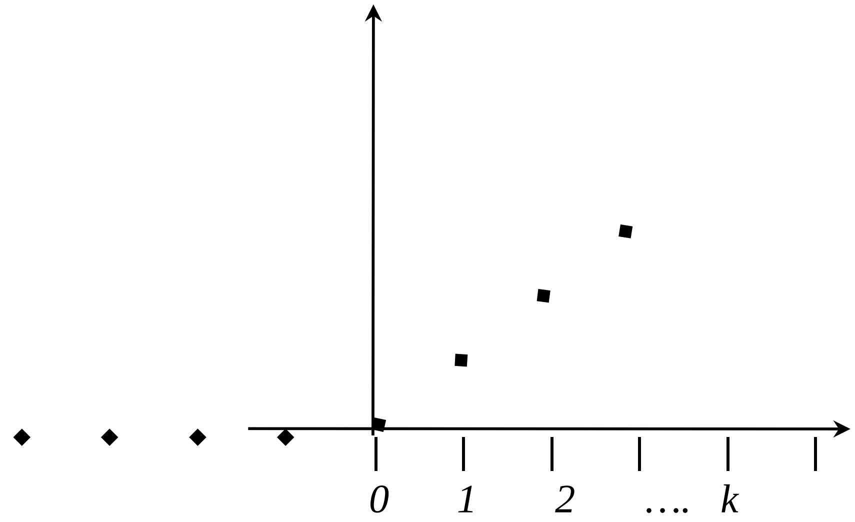
$$x(k) := \begin{cases} k, & k \ge 0, \\ 0, & k < 0. \end{cases} \qquad x(t) = t, \quad X(s) = \frac{1}{s^2}$$

$$X(z) = Z\{x(k)\} = \sum_{k=-\infty}^{k=\infty} x(k) z^{-k}$$

$$= \sum_{k=0}^{k=\infty} kz^{-k}$$

$$= 0 + z^{-1} + 2z^{-2} + \cdots + kz^{-k} + \cdots$$

$$= \frac{z}{(z-1)^2}.$$



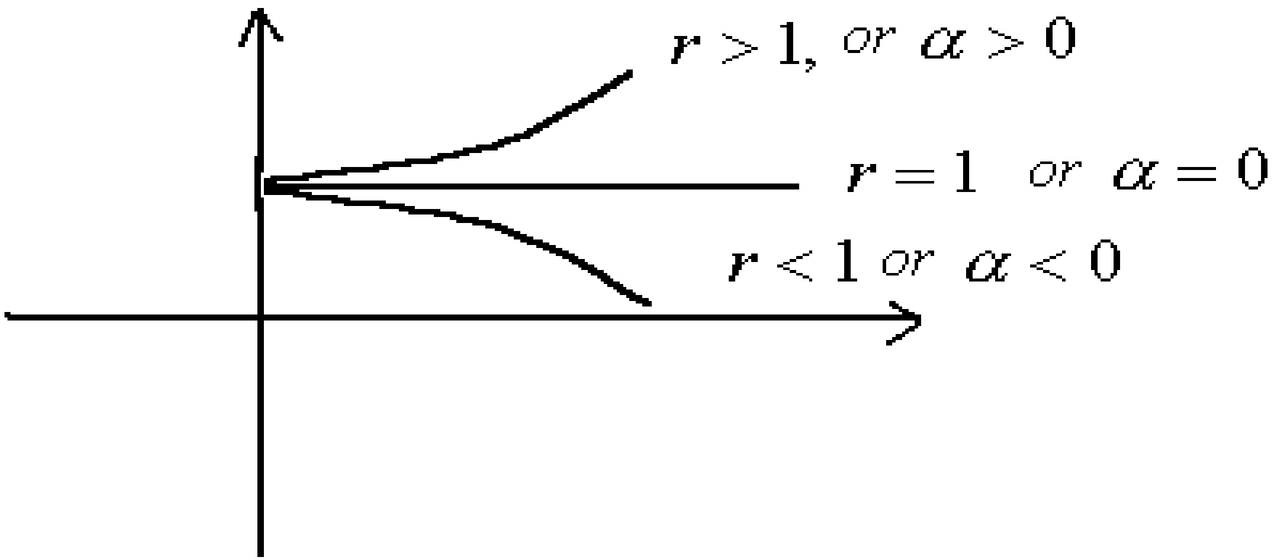
Prove it. Hint: Look at *x(k+1)-x(k)*

(4) Exponential: Let the signal be

$$x(k) := \begin{cases} r^k, & k \geq 0, \\ 0, & k < 0. \end{cases}$$

or $x(k) = r^k \mathbf{1}(k)$

$$x(t) = e^{\alpha t} \qquad X(s) = \frac{1}{s - \alpha} \qquad x(k) = e^{\alpha T k} = r^k, r = e^{\alpha T}$$



Now compute

$$Z[x(k)] = \sum_{k=0}^{\infty} r^k z^{-k} = \frac{z}{z-r}, \qquad |z| > |r|$$

with its pole at $z = r$. One sees

• this signal grows without bound if $|r| > 1$. This indicates that a z-transform with a pole outside $|z| = 1$ corresponds to a growing signal, and it is not bounded.

• this signal decays to zero if $|r|<1$. This indicates that a z-transform with a pole inside $|z| = 1$ corresponds to a decaying signal, and it is bounded.

(5) General Sinusoid: Let

$$x(k) = r^k \cos k\theta \ \mathbf{1}(k) = r^k \left(\frac{e^{jk\theta} + e^{-jk\theta}}{2} \right) \mathbf{1}(k)$$

It follows that

$$Z[x(k)] = \frac{z(z - r\cos\theta)}{z^2 - 2r(\cos\theta)z + r^2}, \qquad |z| > r$$

One may note

$$x(k) = \begin{cases} r^k, & \text{if } \theta = 0 \\ \mathbf{1}(\mathrm{k}), & \text{if } \theta = 0 \text{ and } r = 1 \end{cases}$$

2.3 Properties of the z-transform

1) Linearity

$$Z\{ax_1(k)+bx_2(k)\} = aX_1(z)+bX_2(z)$$

2) Time shifting

$$\begin{aligned} Z\{x(k-1)\} &= \cdots + x(-2)z^1 + x(-1)z^0 + x(0)z^{-1} + x(1)z^{-2} + \cdots \\ &= z^{-1}\left[\cdots + x(-2)z^2 + x(-1)z^1 + x(0)z^0 + x(1)z^{-1} + \cdots\right] \\ &= z^{-1}Z\{x(k)\} \end{aligned}$$

In general,

$$Z\{x(k-k_0)\} = z^{-k_0}Z\{x(k)\} = z^{-k_0}X(z)$$

We can think of z as a kind of shift operator

3) The initial value theorem

- Given the Laplace transform of a signal, $X(s)$, the initial value is given by

$$x(0) = \lim_{s\to\infty} sX(s)$$

- Given the z transform of a sequence, $X(z)$, if $x(k) = 0$, $k < 0$, and $x(k)$ is bounded otherwise, then,

$$X(z) = \sum_{0}^{\infty} x(k) z^{-k}$$

$$= x(0) + \sum_{1}^{\infty} x(k) z^{-k}$$

$$\boxed{x(0) = \lim_{z\to\infty} X(z)}$$

4) The Final Value Theorem

- Given the Laplace Transform of a signal, $X(s)$, the final value is given by

$$x(\infty) = \lim_{s \to 0} sX(s)$$

In both cases, we assume that the signal has the final value.

- Given the z transform of a sequence, $X(z)$, the final value is given by

$$x(\infty) = \lim_{z \to 1} (z-1)X(z)$$

if $x(k) = 0$, $k < 0$.

It is applicable if (z-1)X(z) is has all poles inside the unit circle

Proof:

$$X(z) = \sum_{-\infty}^{\infty} x(k) z^{-k}$$

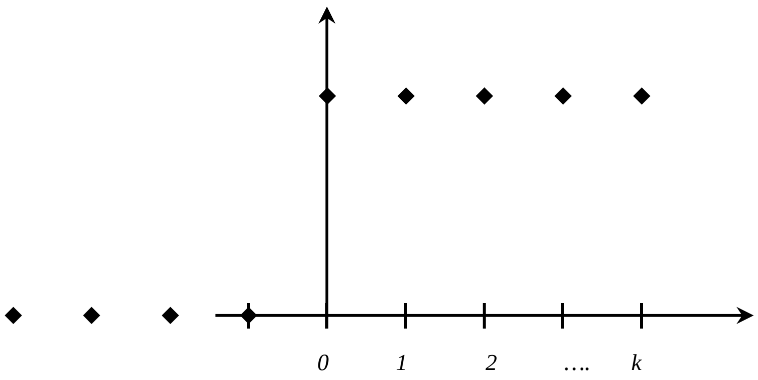
$$zX(z) = \sum_{-\infty}^{\infty} x(k+1) z^{-k}$$

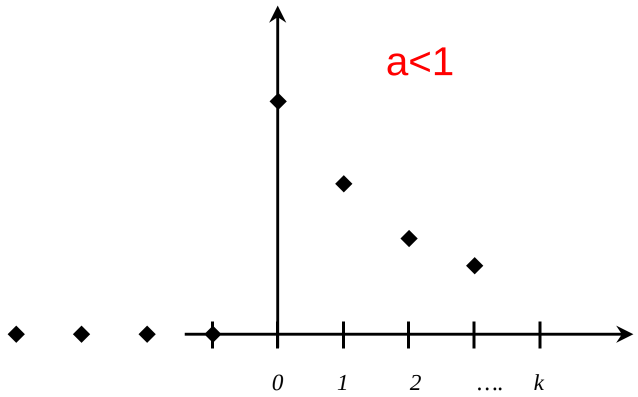
$$zX(z) - X(z) = \sum_{-\infty}^{\infty} \{x(k+1) - x(k)\} z^{-k}$$

$$\begin{aligned} \lim_{z \to 1} (z-1) X(z) &= \sum_{-\infty}^{\infty} \{x(k+1) - x(k)\} \\ &= x(0) + x(1) + x(2) + \cdots + x(\infty) + x(\infty + 1) \\ &\quad - x(0) - x(1) - x(2) - \cdots - x(\infty) \\ &= x(\infty + 1) = x(\infty) \end{aligned}$$

Hence FVT : $\lim_{z \to 1} (z-1) X(z) = x(\infty)$

Examples





$$X(z) = \frac{z}{z-1}$$

$$x(0) = \lim_{z\to\infty} \frac{z}{z-1} = \lim_{z\to\infty} \frac{1}{1-\frac{1}{z}} = 1$$

$$x(\infty) = \lim_{z\to 1}(z-1)\frac{z}{z-1} = 1$$

$$X(z) = \frac{z}{z-a}$$

$$x(0) = \lim_{z\to\infty} \frac{z}{z-a} = \lim_{z\to\infty} \frac{1}{1-\frac{a}{z}} = 1$$

$$x(\infty) = \lim_{z\to 1}(z-1)\frac{z}{z-a} = 0$$

Thus far, only looked at z-transforms of signals.

2.4 Inverse Z-Transform

- Given $X(z)$, what is the sequence $x(k)$?

- Similar to inverse Laplace Transforms, we have

$$x(k) = \frac{1}{2\pi j} \oint_C X(z) z^{k-1} dz$$

We will not usually use this.
A Partial fraction expansion
and a look up table is simpler

Reference: Any maths book on complex variables and Cauchy integral theorems

Easier ways for inversion

- Power series expansion
  - long division to obtain sequence

$$F(z) = \frac{1 - 2z^{-1}}{1 - z^{-1} + z^{-2}}, \qquad (1 - z^{-1} + z^{-2})\overline{\big)1 - 2z^{-1}}^{\;1 - z^{-1} \ldots}$$

- Partial fraction expansion
  - find analytical solution with help of known Z-transform pairs such as

$$Z\{1(k)\} = \frac{z}{z-1}, \quad Z\{r^k\} = \frac{z}{z-r}$$

Example

Given the *z*-transform

$$X(z) = \frac{z}{z^2 - 3z + 2}$$

We get

$$X(z) = \frac{z}{(z-1)(z-2)} = \frac{-z}{z-1} + \frac{z}{z-2}$$

Then

$$x(k) = Z^{-1}[X(z)] = Z^{-1}\left[\frac{-z}{z-1}\right] + Z^{-1}\left[\frac{z}{z-2}\right]$$

$$= (-1 + 2^k)1(k).$$

We also can use long division to obtain sequence. We have

$$\require{enclose}(z^2 - 3z + 2)\enclose{longdiv}{z}\quad z^{-1} + 3z^{-2} + 7z^{-3}..$$

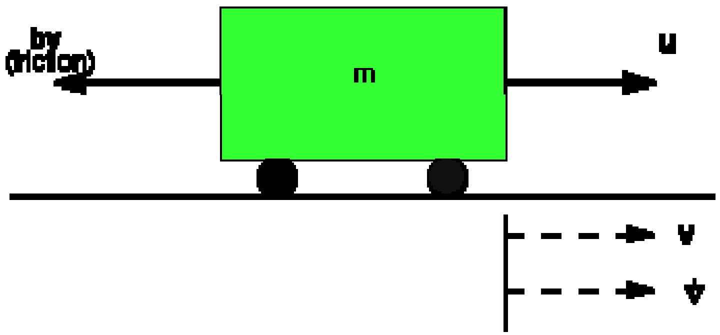
and then get

$$X(z) = \frac{1}{z} + \frac{3}{z^2} + \frac{7}{z^3} + \cdots = x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \frac{x(3)}{z^3} + \cdots$$

Then

$$\begin{aligned} x(0) &= 0, \\ x(1) &= 1, \\ x(2) &= 3, \\ x(3) &= 7, \\ &\vdots \end{aligned}$$

Car control system

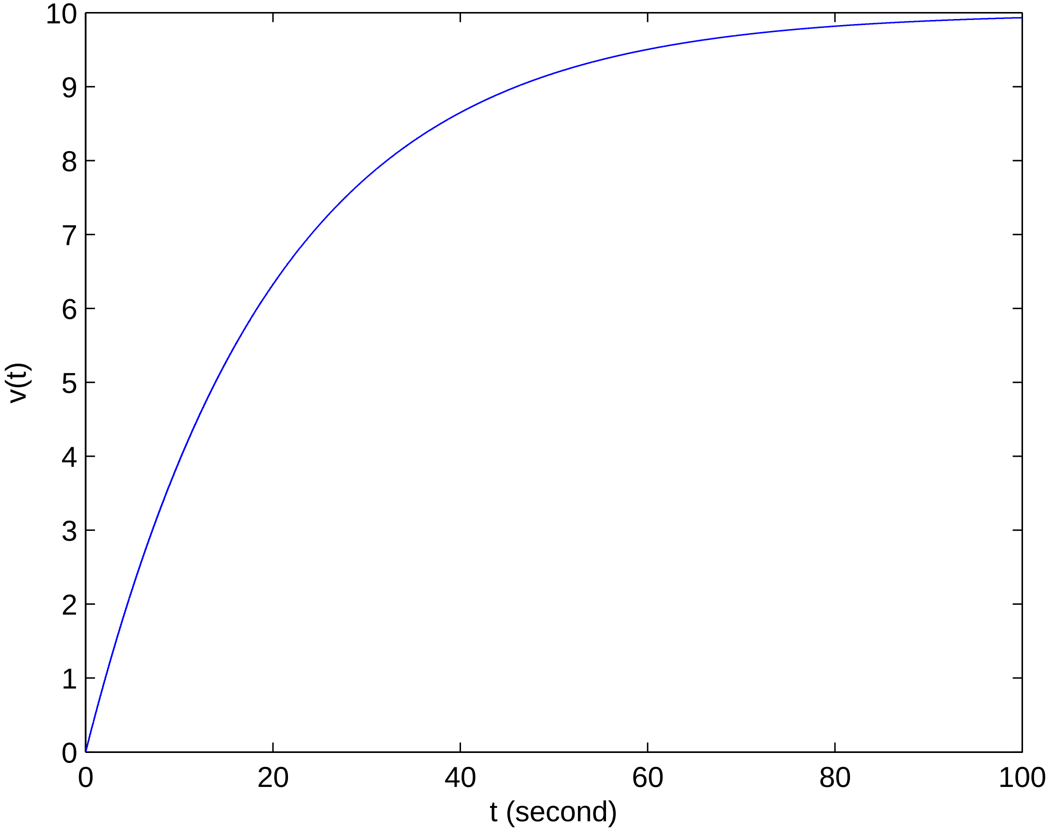


If the inertia of the wheels is neglected and the friction is proportional to the car's speed but opposes the motion of the car, then the system is reduced to the simple mass and damper one, for which Newton's law gives motion equation:

$$m\dot{v} + bv = \tau,$$

where τ is the force from the engine and *m = 1000kg, b = 50Nsec/m*.

For $\tau = 5001(t)$ *and* $v(0)=0$, its output response $v(t)$ is shown in the following figure.



Let us discretize the system with the sampling time of 0.01 sec.

By the zero-order hold equivalent method (to be covered in future topics), we can replace the differential equation by the following difference equation,

$$v(k+1) = 0.9995v(k) + 0.000001 \times 9.998\tau(k)$$

For τ (k) =*500*, *k=0,1,2,…, v(0)=0, v(k)* is calculated and listed in Table 1.

Z-transform of *v(k)* is

$$V(z) = \frac{0.0050z}{(z-0.9995)(z-1)}.$$

$$V(z)=\frac{0.0050z}{(z-0.9995)(z-1)}.$$

We get

$$V(z)=\frac{10z}{z-1}+\frac{-10z}{z-0.9995}.$$

Then

$$v(k)=Z^{-1}\left[V(z)\right]=Z^{-1}\left[\frac{10z}{z-1}\right]+Z^{-1}\left[\frac{-10z}{z-0.9995}\right]$$

$$=\left(10-10\times 0.9995^{k}\right)1(k).$$

We have

$$\begin{array}{r}0.005z^{-1}+0.01z^{-2}+0.015z^{-3}..\\ (z^2-1.9995z+0.9995)\overline{\big)0.005z\qquad\qquad\qquad\qquad}\end{array}$$

and then get

$$V(z)=\frac{0.005}{z}+\frac{0.01}{z^2}+\frac{0.015}{z^3}+\cdots$$

Then

$$\begin{aligned}&v(0)=0,\\&v(1)=0.005,\\&v(2)=0.01,\\&v(3)=0.015,\\&\vdots\end{aligned}$$

Table 1

t	k	v(t)	v(k)
0.01	1	0.0050	0.0050
0.02	2	0.0100	0.0100
0.03	3	0.0150	0.1500
0.04	4	0.0200	0.0200
0.05	5	0.0250	0.0249
0.06	6	0.0300	0.0299
0.07	7	0.0349	0.0349
0.08	8	0.0399	0.0399
0.09	9	0.0449	0.0448
0.1	10	0.0499	0.0498
…	…	…	…