Tutorial 3---Questions

1. Compute ZOH discrete equivalent G(z) to G(s):

(a) $\dfrac{k}{s}$,

(b) $\dfrac{3}{s(s+3)}$,

(c) $\dfrac{3}{(s+1)(s+3)}$,

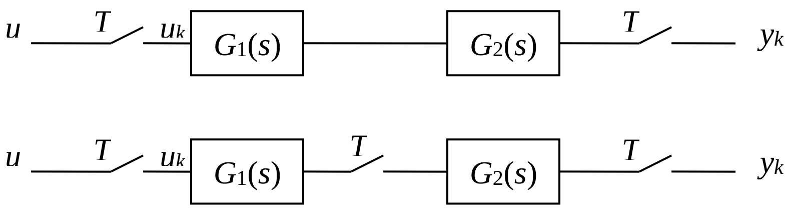
(d) $\dfrac{1-s}{s^2}$

2. Consider the following block diagrams.

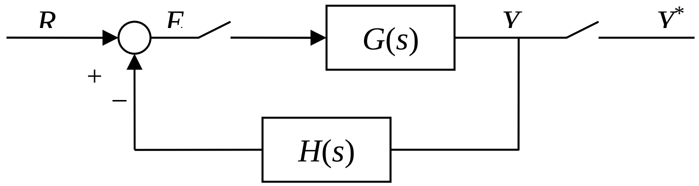
$$G_1(s) = \frac{1}{s(s+2)}, \quad G_2(s) = \frac{1}{(s+0.5)(s+1.5)}$$

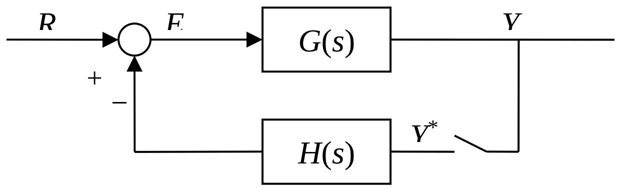
Find the discrete transfer function from u_k to y_k in both cases below.

EE3304 Digital Control System Tutorial 3
Question

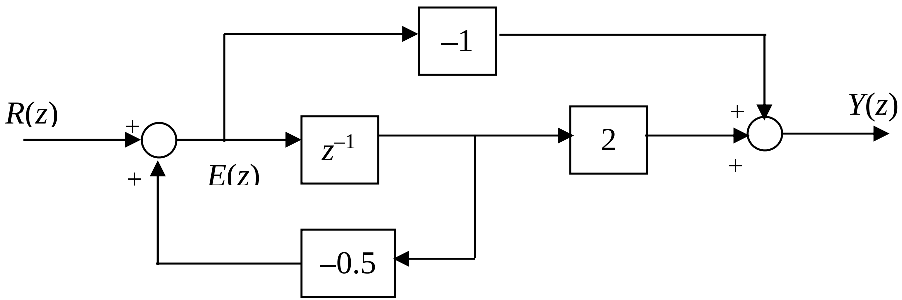


3. For each of the following configurations, obtain the discrete transfer function from the reference input to the output.





4. Consider the discrete time system shown below



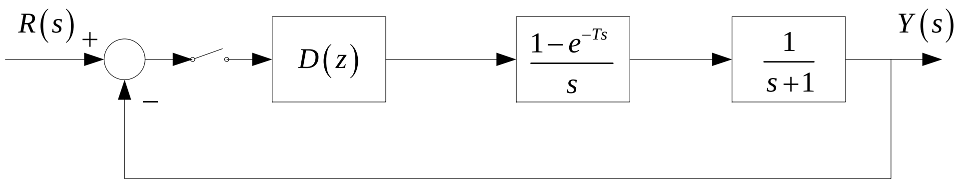
Find the closed loop transfer function from $R(z)$ to $Y(z)$.

5. The difference equation for a process has been identified as

$$y(k)-0.9y(k-1)=0.2u(k-3)$$

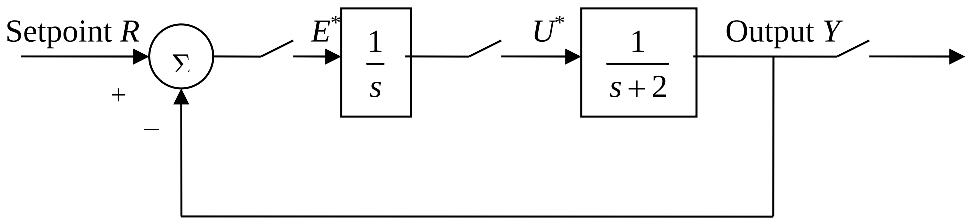
where k is the sampling index, y is the controlled variable and u is the manipulated variable. If a proportional controller of gain K is used, write down the difference equation describing the closed loop system. Find the discrete transfer function of the closed loop system.

6. Consider the system below with $D(z)=1$.



(a) With the sampler and zero-order hold removed, write down the closed loop system differential equation.
(b) Using the backward rectangular rule for numerical integration with $T=0.25s$, obtain the discrete transfer function of the closed loop system and evaluate the unit step response.
(c) Obtain the discrete transfer function of the closed loop system and evaluate the unit step response for the digital system above with the sampler and hold.

7. Consider the following digital system:

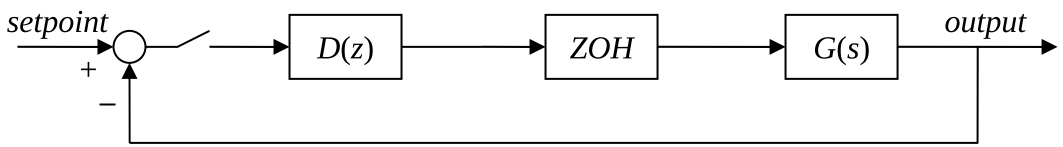


(a) Find the closed-loop discrete transfer function, $\dfrac{Y(z)}{R(z)}$.

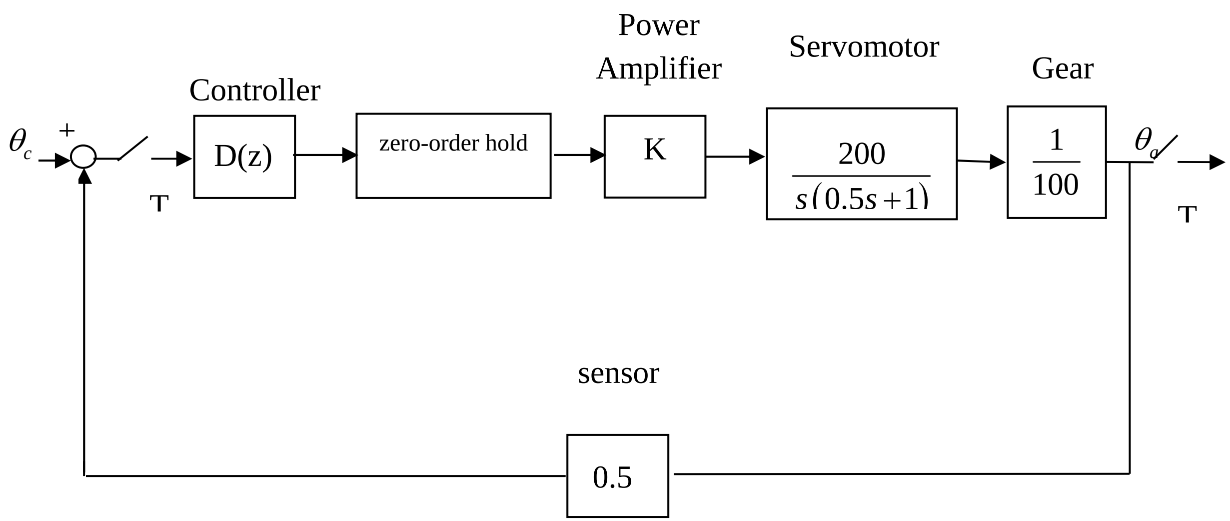
(b) Let $T = 0.1$, obtain the output unit pulse response.

8. In the figure below, let $G(s) = \dfrac{3}{(1+2s)(1+5s)}$ and $D(z) = 4$. The sampling period is 1 time unit. Is the closed loop system stable?

Question



9. The block diagram of a robot arm joint control system is given below. Let $T = 0.1s$, $K = 2$ and $D(z) = 1$.

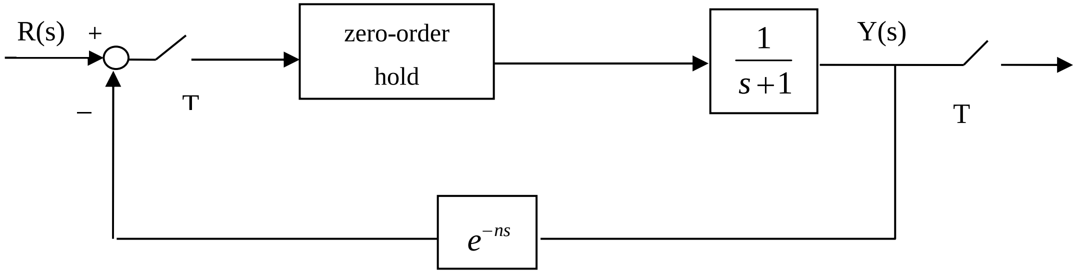


(a) Determine the discrete-time transfer function of the closed-loop system, $\dfrac{\theta_a(z)}{\theta_c(z)}$.

(b) Check the closed-loop stability.

(c) What is the final value of the output when the input is a unit step?

10. Consider the discrete-time control system shown below where n is a natural number. Let the sampling period $T = 1$ s.

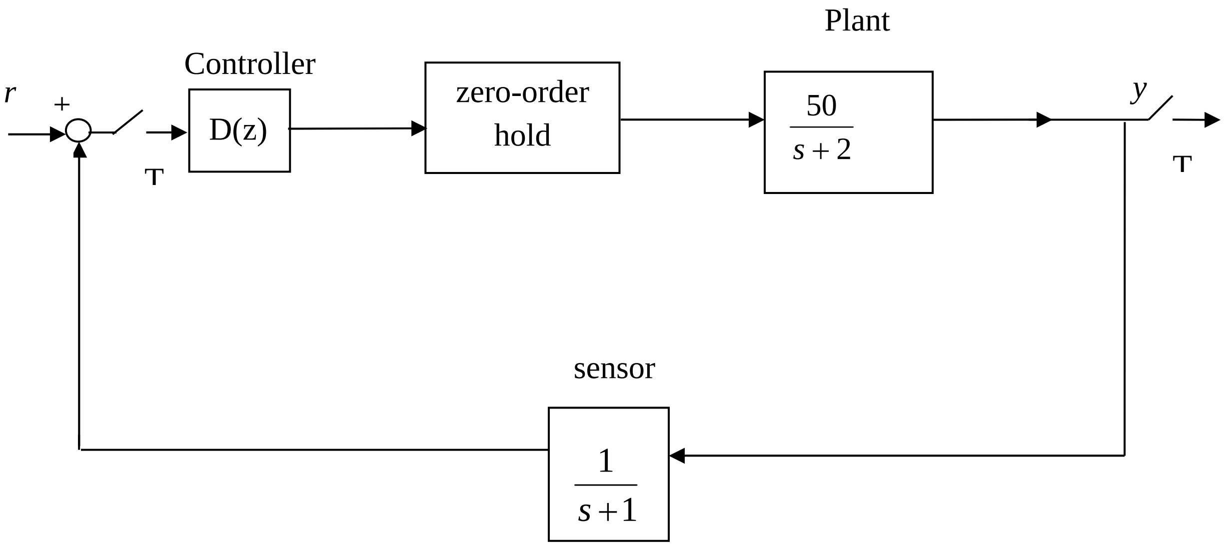


(a) Find the discrete-time closed-loop transfer function of the system, $\dfrac{Y(z)}{R(z)}$.

(b) Comment on the stability of the discrete-time system when n = 1.

11. The block diagram of a digital control system is given below. Let $T = 0.1s$ and $D(z) = z - 2$. Hint:

$$Z\left[\left(\frac{1-e^{-Ts}}{s}\right)\left(\frac{1}{s+1}\right)\left(\frac{50}{s+2}\right)\right]=\frac{0.228z+0.205}{(z-0.819)(z-0.905)}.$$



(a) Determine the discrete-time transfer function of the closed-loop system, $\frac{Y(z)}{R(z)}$.

(b) Find the output time response when the input is the unit pulse.

(c) Is the closed-loop stable and why?

12. Consider a plant

$$\frac{X(s)}{U(s)} = \frac{a}{s-a}$$

controlled discretely with a ZOH, this yields

$$x_{k+1} = e^{aT} x_k + (e^{aT} - 1) u_k \ .$$

Assume control is

$$u_k = -k x_k$$

$a = 1, b = 2$, compute the gain k that yields a z-plane root at $z = e^{-bT}$ for T = 0.1, 1, 2, and 5. Compute percent error in k that will result in an unstable system for T = 1, 2, 5.

13. The following transfer function is a lead network designed to add about 60 degree phase lead at $\omega_1 = 3$ radian/second:

$$H(s) = \frac{s+1}{0.1s+1}$$

For each of the following design methods, compute the z-plane pole and zero locations and the amount of phase lead given by the equivalent network at $z_1 = e^{j\omega_1 t}$ for T=0.25 sec.

(a) Forward rule
(b) Backward rule

(c) Bilinear rule
(d) Zero-pole mapping
(e) Zero-order-hold equivalent

14. Consider the second-order bandpass filter:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{2s}{s^2 + 2s + 100}$$

Determine the discrete transfer function and its difference equation using:

(a) Forward rule
(b) Backward rule
(c) Trapezoidal rule
(d) Zero-pole mapping
(e) Zero order Hold equivalent

Tutorial 3: Questions & Solutions

1. Compute ZOH discrete equivalent G(z) to G(s):

(a) $\frac{k}{s}$,

(b) $\frac{3}{s(s+3)}$,

(c) $\frac{3}{(s+1)(s+3)}$,

(d) $\frac{1-s}{s^2}$

Solution:
The formula is

$$G(z) = \left(1 - z^{-1}\right) \; Z\left\{\frac{G(s)}{s}\right\}$$

(a) $Z\left\{\frac{G(s)}{s}\right\} = Z\left\{\frac{k}{s^2}\right\}$, By Z-transform table,

$$Z\left\{\frac{1}{s^2}\right\} = \frac{Tz}{(z-1)^2}$$

$$\therefore \quad \mathrm{G}(\mathrm{z}) = \left(1-z^{-1}\right) Z\left\{\frac{G(s)}{s}\right\} = \left(1-z^{-1}\right)\frac{kTz}{(z-1)^2} = \frac{kT}{z-1}$$

(b) $G(s) = \frac{3}{s(s+3)}$

$$\frac{G(s)}{s} = \frac{3}{s^2(s+3)}$$

$$Z\left\{\frac{G(s)}{s}\right\} = \frac{z\left[\left(3T-1+e^{-3T}\right)z + \left(1-e^{-3T}-3Te^{-3T}\right)\right]}{3(z-1)^2\left(z-e^{-3T}\right)}$$

$$\therefore \quad \mathrm{G}(\mathrm{z}) = \frac{\left(3T-1+e^{-3T}\right)z + \left(1-e^{-3T}-3Te^{-3T}\right)}{3(z-1)\left(z-e^{-3T}\right)}$$

(c) $G(s)=\dfrac{3}{(s+1)(s+3)}$. Partial fraction expansion gives

$$\frac{G(s)}{s}=\frac{3}{s(s+1)(s+3)}=\frac{1}{s}+\frac{-1.5}{s+1}+\frac{0.5}{s+3}$$

$$Z\left\{\frac{G(s)}{s}\right\}=\frac{z}{z-1}-\frac{1.5z}{z-e^{-T}}+\frac{0.5z}{z-e^{-3T}}$$

$$\therefore \quad \mathrm{G}(\mathrm{z})=\frac{z-1}{z}Z\left\{\frac{G(s)}{s}\right\}$$

$$=\frac{\left(z-e^{-T}\right)\left(z-e^{-3T}\right)-1.5(z-1)\left(z-e^{-3T}\right)+0.5(z-1)\left(z-e^{-T}\right)}{\left(z-e^{-T}\right)\left(z-e^{-3T}\right)}.$$

(d) $G(s)=\dfrac{1-s}{s^2}$

$$\frac{G(s)}{s}=\frac{1-s}{s^3}=\frac{1}{s^3}-\frac{1}{s^2}$$

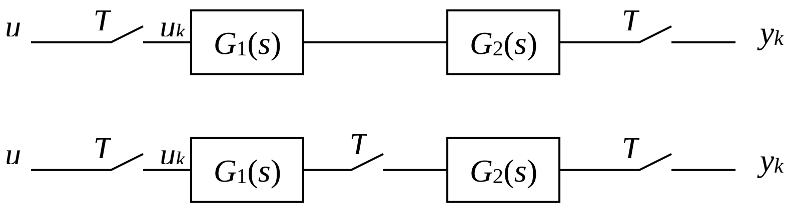
$$Z\left\{\frac{G(s)}{s}\right\}=\frac{T^2}{2}\frac{z(z+1)}{(z-1)^3}-\frac{Tz}{(z-1)^2}$$

$$G(z)=\frac{z-1}{z}Z\left\{\frac{G(s)}{s}\right\}=\frac{T^2(z+1)-2T(z-1)}{2(z-1)^2}$$

2. Let

$$G_1(s)=\frac{1}{s(s+2)},\quad G_2(s)=\frac{1}{(s+0.5)(s+1.5)}$$

Find the discrete transfer function from u_k to y_k in both cases below.



Solution:

For the first case, assuming T = 1,

$$\frac{Y(z)}{U(z)} = Z\left\{G_1(s)G_2(s)\right\} \neq G_1(z)G_2(z)$$

$$G_1(s)G_2(s) = \frac{1}{s(s+2)(s+0.5)(s+1.5)}$$

$$= \frac{2}{3s} - \frac{2}{3(s+2)} - \frac{4}{3(s+0.5)} + \frac{4}{3(s+1.5)}$$

$$Z\left\{G_1(s)G_2(s)\right\}$$

$$= \frac{2}{3}\frac{z}{z-1} - \frac{2}{3}\frac{z}{z-e^{-2T}} - \frac{4}{3}\frac{z}{z-e^{-0.5T}} + \frac{4}{3}\frac{z}{z-e^{-1.5T}}$$

$$= \frac{2z}{3}\left(\frac{1}{z-1} - \frac{1}{z-0.135} - \frac{2}{z-0.606} + \frac{2}{z-0.223}\right)$$

$$= \frac{0.198z^3 + 0.305z^2 + 0.027z}{3z^4 - 5.892z^3 + 3.633z^2 - 0.796z + 0.055}$$

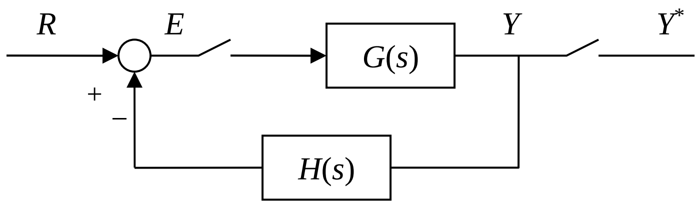
For the second case,

$$G_1(s)=\frac{1}{s(s+2)},\quad G_1(z)=\frac{0.432z}{(z-1)(z-0.135)}$$

$$G_2(s)=\frac{1}{(s+0.5)(s+1.5)},\quad G_2(z)=\frac{0.383z}{(z-0.606)(z-0.223)}$$

$$G_1(z)G_2(z)=\frac{0.165z^2}{(z-1)(z-0.135)(z-0.606)(z-0.223)}$$

3. For each of the following configurations, obtain the discrete transfer function from the reference input to the output.



Solution:

Question

$$
\begin{aligned}
E(s) &= R(s) - H(s)Y(s) \\
&= R(s) - H(s)G(s)E^*(s)
\end{aligned}
$$

$$E^*(s) = R^*(s) - \left[H(s)G(s)\right]^* E^*(s)$$

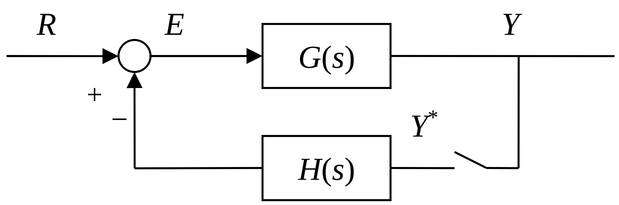
$$E^*(s) = \frac{R^*(s)}{1+\left[H(s)G(s)\right]^*}$$

$$Y(s) = G(s)E^*(s)$$

$$
\begin{aligned}
Y^*(s) &= G^*(s)E^*(s) \\
&= \frac{G^*(s)R^*(s)}{1+\left[H(s)G(s)\right]^*}
\end{aligned}
$$

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1+Z\left\{H(s)G(s)\right\}}$$

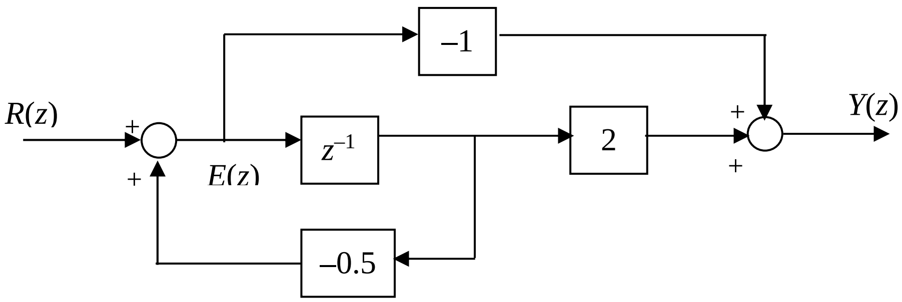
Question



$$Y(s) = G(s)E(s)$$
$$= G(s)R(s) - G(s)H(s)Y^*(s)$$
$$Y^*(s) = \left[G(s)R(s)\right]^* - \left[G(s)H(s)\right]^* Y^*(s)$$
$$Y^*(s) = \frac{\left[G(s)R(s)\right]^*}{1+\left[G(s)H(s)\right]^*}$$
$$Y(z) = \frac{Z\left\{G(s)R(s)\right\}}{1+Z\left\{G(s)H(s)\right\}}$$

Hence there does not exist a transfer function between $R(z)$ and $Y(z)$.

4. Consider the discrete time system shown below



Find the closed loop transfer function from *R*(*z*) to *Y*(*z*).

Solution:
The closed loop transfer function is given by:

$$
\begin{aligned}
E(z) &= R(z) - 0.5z^{-1}E(z) \\
(1+0.5z^{-1})E(z) &= R(z) \\
E(z) &= \frac{1}{1+0.5z^{-1}}R(z) \\
Y(z) &= -E(z) + 2z^{-1}E(z) \\
&= \frac{2z^{-1}-1}{1+0.5z^{-1}}R(z) \\
\frac{Y(z)}{R(z)} &= \frac{2z^{-1}-1}{1+0.5z^{-1}}
\end{aligned}
$$

5.The difference equation for a process has been identified as

$$y(k) - 0.9y(k-1) = 0.2u(k-3)$$

where k is the sampling index, y is the controlled variable and u is the manipulated variable. If a proportional controller of gain K is used, write down the difference equation describing the closed loop system. Find the discrete transfer function of the closed loop system.

Solution:

Controller:

$$u(k) = K\left[y_{sp}(k) - y(k)\right]$$

Therefore

$$y(k) = 0.9y(k-1) + 0.2K\left[y_{sp}(k-3) - y(k-3)\right]$$

Hence closed loop difference equation is given by

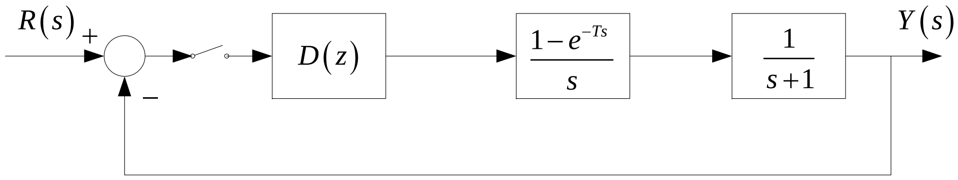
$$y(k) = 0.9y(k-1) - 0.2Ky(k-3) + 0.2Ky_{sp}(k-3)$$

Taking z-transform throughout on both sides

$$Y(z) = z^{-1}0.9Y(z) - 0.2Kz^{-3}Y(z) + 0.2Kz^{-3}Y_{sp}(z)$$

$$\frac{Y(z)}{Y_{sp}(z)} = \frac{z^{-3}0.2K}{1 - 0.9z^{-1} + 0.2Kz^{-3}}$$

6. Consider the system below with $D(z) = 1$.



(a) With the sampler and zero-order hold removed, write down the closed loop system differential equation and evaluate the unit step response.
(b) Using the backward rectangular rule for numerical integration with $T = 0.25s$, obtain the discrete transfer function of the closed loop system and evaluate the unit step response.
(c) Obtain the discrete transfer function of the closed loop system and evaluate the unit step response for the digital system above with the sampler and hold.

Solution:

(a) In the Laplace domain, the closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{G(s)D(s)}{1+G(s)D(s)} = \frac{\dfrac{1}{s+1}}{1+\dfrac{1}{s+1}} = \frac{1}{s+2}$$

In continuous time domain:

$$\dot{y}(t) + 2y(t) = r(t)$$

Its response is shown by the solid line plot in Figure 1.

(b) Discretizing with the backwards rectangular rule:

$$s = \frac{1 - z^{-1}}{T}$$

under $T = 0.25s$ yields

$$\frac{Y(z)}{R(z)} = \frac{1}{6 - 4z^{-1}}.$$

To obtain the unit step response,

$$Y(z) = \frac{1}{6 - 4z^{-1}} \frac{z}{z - 1}$$

we take the inverse Z-transform:

$$Y(z) = \frac{1}{2}\frac{z}{z-1} - \frac{1}{3}\frac{z}{z - 2/3}$$

$$y(k) = \frac{1}{2} \cdot 1(k) - \frac{1}{3}\left(\frac{2}{3}\right)^k \cdot 1(k)$$

The response is shown by the solid line plot in Figure 1.

(c) Discretizing with the ZOH,

$$G(z) = \left(1 - z^{-1}\right) Z\left\{\frac{G(s)}{s}\right\}$$
$$= \left(1 - z^{-1}\right) Z\left\{\frac{1}{s(s+1)}\right\}$$
$$= \frac{1 - e^{-T}}{z - e^{-T}}$$

The closed loop transfer function is

$$G_{CL}(z) = \frac{D(z)G(z)}{1 + D(z)G(z)}$$
$$= \frac{1 - e^{-T}}{z - \left(2e^{-T} - 1\right)}$$

$$Y(z) = \frac{1-e^{-T}}{z-\left(2e^{-T}-1\right)}\frac{z}{z-1}$$

$$= \frac{1}{2}\frac{z}{z-1} - \frac{1}{2}\frac{z}{z-\left(2e^{-T}-1\right)}$$

$$y(k) = \frac{1}{2} - \frac{1}{2}\left(2e^{-T}-1\right)^{k}$$

For $T = 0.25$, the response is shown in the dash line plot in Figure 1.

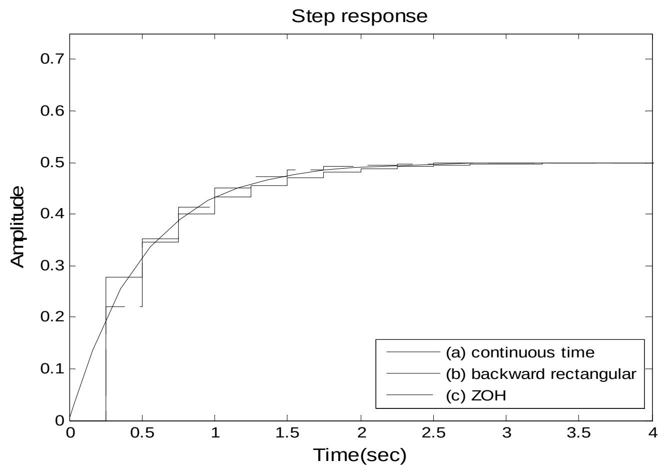
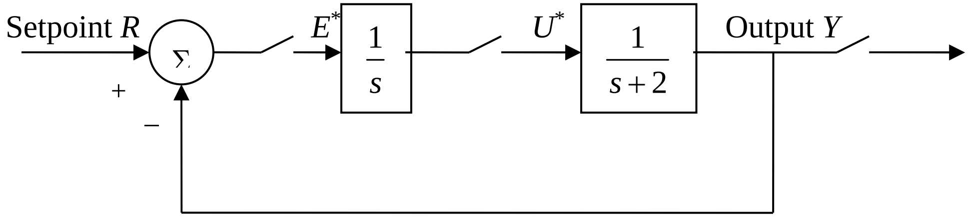


Figure 1: Step response

7. Consider the following digital system:



(a) Find the closed-loop discrete transfer function, $\dfrac{Y(z)}{R(z)}$.

(b) Let T = 0.1, obtain the output unit pulse response.

Solution:

(a)

$$Z\left\{\frac{1}{s}\right\} = \frac{z}{z-1} \triangleq G_1, \quad Z\left\{\frac{1}{s+2}\right\} = \frac{z}{z-e^{-2T}} \triangleq G_2$$

$$\frac{Y(z)}{R(z)} = \frac{G_1G_2}{1+G_1G_2} = \frac{z^2}{(z-1)\left(z-e^{-2T}\right)+z^2}$$

$$= \frac{z^2}{2z^2-\left(1+e^{-2T}\right)z+e^{-2T}}$$

(b) $R(z) = 1$, $T = 0.1$,

$$Y(z) = \frac{z^2}{2z^2-1.8187z+0.8187}$$

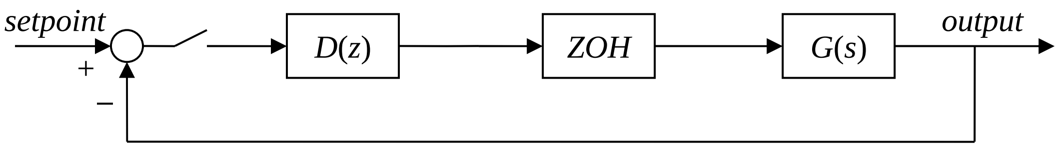
$$= \frac{0.5z^2}{z^2-0.9093z+0.4093} = \frac{0.5\{z[(z-0.9093/2)+0.9093/2]\}}{z^2-0.9093z+0.4093}$$

$$= \frac{0.5z(z-0.4547)}{z^2-0.9093z+0.4093} + \frac{0.2274z}{z^2-0.9093z+0.4093}$$

$$\therefore y(k) = Z^{-1}\{Y(z)\}$$

$$= 0.5(0.6401)^k\cos(0.7809k) + 0.5047(0.6401)^k\sin(0.7809k)$$

8. In the figure below, let $G(s) = \dfrac{3}{(1+2s)(1+5s)}$ and $D(z)$ = 4. The sampling period is 1 time unit. Is the closed loop system stable?



Solution:

$$G(z) = \left(1 - z^{-1}\right) Z\left\{\frac{G(s)}{s}\right\}$$

Question

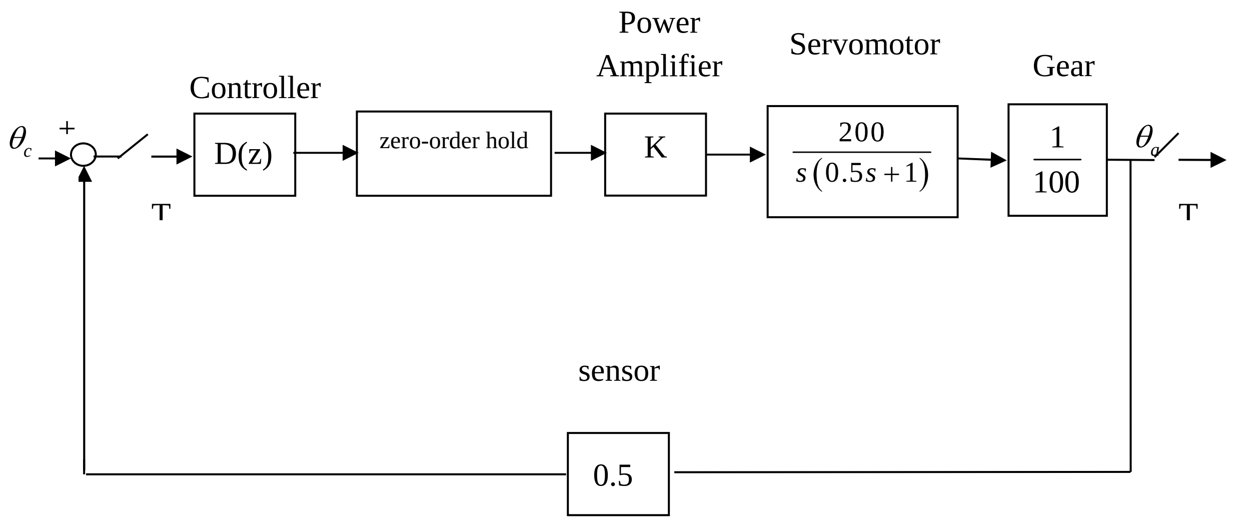
$$\frac{G(s)}{s}=\frac{3}{s(1+2s)(1+5s)}$$
$$=\frac{3}{s}+\frac{4}{1+2s}-\frac{25}{1+5s}$$
$$\longleftrightarrow\frac{3z}{z-1}+\frac{2z}{z-e^{-0.5}}-\frac{5z}{z-0.818}$$
$$G(z)=\frac{0.11z+0.1084}{(z-0.6)(z-0.818)}$$

The closed loop characteristic equation is

$$1+D(z)G(z)=0$$
$$(z-0.6)(z-0.818)+4(0.11z+0.1084)=0$$
$$z^2-0.978z+0.9244=0$$
$$z=0.489\pm j0.8278$$
$$|z|=\sqrt{0.489^2+0.8278^2}=\sqrt{0.9244}=0.9614<1$$

So, the closed loop is stable.

9. The block diagram of a robot arm joint control system is given below. Let $T = 0.1s$, $K = 2$ and $D(z) = 1$.



(a) Determine the discrete-time transfer function of the closed-loop system, $\dfrac{\theta_a(z)}{\theta_c(z)}$.

(b) Check the closed-loop stability.

(c) What is the final value of the output when the input is a unit step?

Solution:

(a) We have,

$$\theta_a = \left(1 - z^{-1}\right) Z\left\{K \frac{200}{s^2\left(0.5s+1\right)} \times \frac{1}{100}\right\} \cdot D(z) \cdot \left(\theta_c - 0.5\theta_a\right)$$

With $T = 0.1s$, $K = 2$ and $D(z) = 1$,

$$\theta_a = \left(\theta_c - 0.5\theta_a\right)\left(1 - z^{-1}\right) \frac{4z\left[\left(0.2 - 1 + e^{-0.2}\right)z + 1 - e^{-0.2} - 0.2e^{-0.2}\right]}{2\left(z-1\right)^2\left(z - e^{-0.2}\right)}$$

$$\theta_a = \left(\theta_c - 0.5\theta_a\right)\left(1 - z^{-1}\right) \frac{0.0374z\left(z + 0.9358\right)}{\left(z-1\right)^2\left(z - 0.8187\right)}$$

$$\theta_a = \left(\theta_c - 0.5\theta_a\right) \frac{0.0374\left(z + 0.9358\right)}{\left(z-1\right)\left(z - 0.8187\right)}$$

$$\left(z^2 - 1.8z + 0.8362\right)\theta_a = 0.0374\left(z + 0.9358\right)\theta_c$$

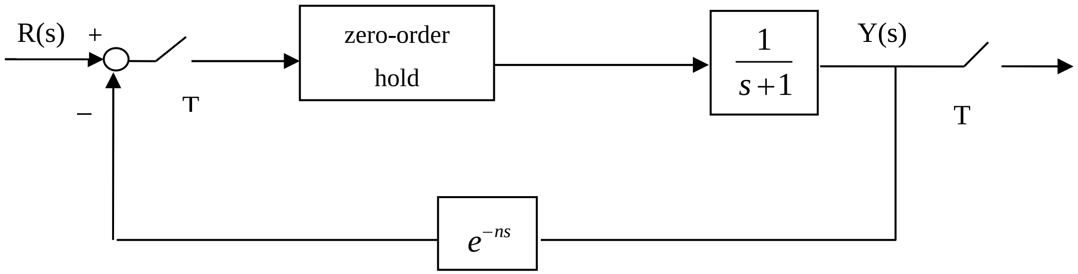
$$\therefore \frac{\theta_a}{\theta_c} = \frac{0.0374\left(z + 0.9358\right)}{z^2 - 1.8z + 0.8362}$$

(b) The closed-loop system has a pair of poles at $z = 0.9 \pm 0.1619j$, thus it is stable.

(c) Apply the final value theorem,

$$\theta_a(\infty) = \lim_{z\to 1}(z-1)\frac{z}{z-1}\frac{0.0374(z+0.9358)}{z^2-1.8z+0.8362} = 2.0$$

10. Consider the discrete-time control system shown below where n is a natural number. Let the sampling period $T = 1$ s.



(a) Find the discrete-time closed-loop transfer function of the system, $\dfrac{Y(z)}{R(z)}$.

(b) Comment on the stability of the discrete-time system when n = 1.

Solution:

(a) with $T = 1$ s,

$$Y = \left(R - z^{-n}Y\right)\cdot\left(1 - z^{-1}\right)Z\left\{\frac{1}{s\left(s+1\right)}\right\}$$

$$Y = \left(R - z^{-n}Y\right)\frac{1-e^{-1}}{z-e^{-1}}$$

$$\frac{Y}{R} = \frac{1-e^{-1}}{z - e^{-1} + z^{-n}\left(1-e^{-1}\right)}$$

$$\therefore \frac{Y}{R} = \frac{0.6321z^{n}}{z^{n+1} - 0.3679z^{n} + 0.6321}$$

(b) when $n = 1$, the transfer function becomes,

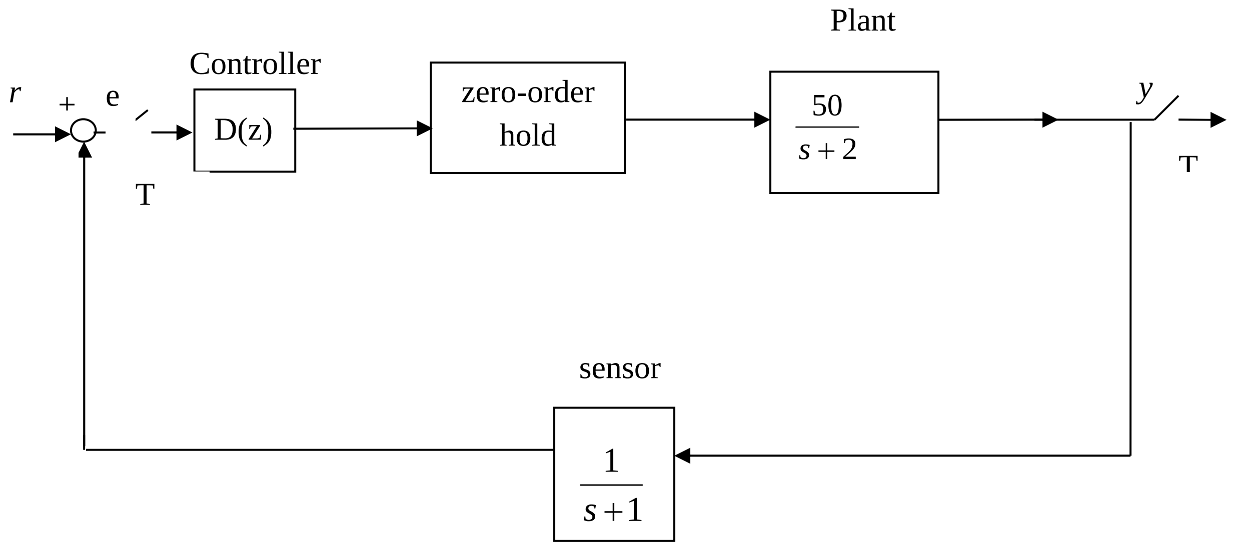
$$\frac{Y}{R} = \frac{0.6321z}{z^{2} - 0.3679z + 0.6321}$$

It has a pair of poles at $z = 0.1840 \pm 0.7735j$. Magnitude of the poles is smaller than 1. So the system is stable.

11. *The block diagram of a digital control system is given below. Let $T = 0.1s$ and $D(z) = z - 2$. Hint:

Question

$$Z\left[\left(\frac{1-e^{-Ts}}{s}\right)\left(\frac{1}{s+1}\right)\left(\frac{50}{s+2}\right)\right]=\frac{0.228z+0.205}{(z-0.819)(z-0.905)}.$$



(a) Determine the discrete-time transfer function of the closed-loop system, $\frac{Y(z)}{R(z)}$.

(b) Find the output time response when the input is the unit pulse.

(c) Is the closed-loop stable and why?

Solution:

(a) We have

$$Y(s) = \left(\frac{1-e^{-TS}}{s}\right)\left(\frac{50}{s+2}\right) D^*(s)E^*(s)$$

$$Y^*(s) = \left[\frac{1-e^{-TS}}{s}\left(\frac{50}{s+2}\right)\right]^* D^*(s)E^*(s)$$

$$E(s) \ = R(s) - \left(\frac{1}{s+1}\right)\left(\frac{1-e^{-Ts}}{s}\right)\left(\frac{50}{s+2}\right) D^*(s)E^*(s)$$

$$E^*(s) \ = R^*(s) - \left[\left(\frac{1}{s+1}\right)\left(\frac{1-e^{-Ts}}{s}\right)\left(\frac{50}{s+2}\right)\right]^* D^*(s)E^*(s)$$

$$E^*(s)\left\{1 + \left[\left(\frac{1-e^{-Ts}}{s}\right)\left(\frac{1}{s+1}\right)\left(\frac{50}{s+2}\right)\right]^* D^*(s)\right\} = R^*(s)$$

Question

$$Y^*(s) = \frac{\left[\frac{1-e^{-TS}}{s}\left(\frac{50}{s+10}\right)\right]^* D^*(s)}{1+\left[\left(\frac{1-e^{-Ts}}{s}\right)\left(\frac{1}{s+1}\right)\left(\frac{50}{s+2}\right)\right]^* D^*(s)} R^*(s)$$

$$\frac{Y(z)}{R(z)} = \frac{Z\left[\frac{1-e^{-TS}}{s}\left(\frac{50}{s+10}\right)\right] D(z)}{1+Z\left[\left(\frac{1-e^{-Ts}}{s}\right)\left(\frac{1}{s+1}\right)\left(\frac{50}{s+2}\right)\right] D(z)}$$

Using T=0.1, we find

$$Z\left[\frac{1-e^{-TS}}{s}\left(\frac{50}{s+2}\right)\right] = \frac{z-1}{z} Z\left(\frac{50}{s(s+2)}\right)$$

$$= \frac{z-1}{z} Z\left(\frac{25}{s} + \frac{-25}{s+2}\right)$$

$$= \frac{z-1}{z}\left(\frac{25z}{z-1} + \frac{-25z}{z-0.819}\right)$$

$$= \frac{4.533}{z-0.819}$$

$$Z\left[\left(\frac{1-e^{-Ts}}{s}\right)\left(\frac{1}{s+1}\right)\left(\frac{50}{s+2}\right)\right]=\frac{z-1}{z}Z\left(\frac{25}{s}+\frac{25}{s+2}-\frac{50}{s+1}\right)$$
$$=\frac{z-1}{z}\left(\frac{25z}{z-1}+\frac{25z}{z-0.819}+\frac{-50z}{z-0.905}\right)$$
$$=\frac{0.228z+0.205}{(z-0.819)(z-0.905)}$$

Substituting back, we get

$$\frac{Y(z)}{R(z)}=\frac{Z\left[\frac{1-e^{-TS}}{s}\left(\frac{50}{s+10}\right)\right]D(z)}{1+Z\left[\left(\frac{1-e^{-Ts}}{s}\right)\left(\frac{1}{s+1}\right)\left(\frac{50}{s+2}\right)\right]\ D(z)}$$
$$=\frac{\frac{4.533}{z-0.819}(z-2)}{1+\frac{0.228z+0.205}{(z-0.819)(z-0.905)}(z-2)}$$

Finally, we obtain

$$\frac{Y(z)}{R(z)}=\frac{4.533(z-0.905)(z-2)}{1.228z^2-1.974z+0.331}.$$

(b) The z-transform of the unit pulse is $R(z)=1$. Therefore,

$$
\begin{aligned}
Y(z) &= \frac{4.533(z-0.905)(z-2)}{1.228z^2 - 1.974z + 0.331} \\
&= \frac{\frac{4.533}{1.228}(1.228z^2 - 1.974z + 0.331)}{1.228z^2 - 1.974z + 0.331} + \frac{-5.882z + 6.983}{1.228z^2 - 1.974z + 0.331} \\
&= 3.691 + \frac{-0.9}{z-1.417} + \frac{-3.895}{z-0.190} \\
&= 3.691 + z^{-1}\left(\frac{-0.9z}{z-1.417} + \frac{-3.895z}{z-0.190}\right)
\end{aligned}
$$

So, the time response of the output is

$$
y(k) = 3.691\delta(k) + \left[-0.9(1.417)^{k-1} - 3.895(0.190)^{k-1}\right]1(k-1)
$$

(c) The closed-loop system has two poles at z= 1.417 and z=0.190. Because one of the poles has its magnitude greater than 1, thus the closed-loop system is unstable.

12. Consider a plant

Question

$$\frac{X(s)}{U(s)} = \frac{a}{s-a}$$

controlled discretely with a ZOH, this yields

$$x_{k+1} = e^{aT} x_k + (e^{aT} - 1) u_k \ .$$

Assume control is

$$u_k = -k x_k$$

$a = 1, b = 2$, compute the gain k that yields a z-plane root at $z = e^{-bT}$ for T = 0.1, 1, 2, and 5. Compute percent error in k that will result in an unstable system for T = 1, 2, 5.

Solution: The closed-loop difference equation is

$$x_{k+1} = e^{aT} x_k + (e^{aT} - 1)(-k x_k).$$

The CE is

$$z - e^{aT} + k\left(e^{aT} - 1\right) = 0 \, .$$

As required, its root, $z = e^{aT} - k\left(e^{aT} - 1\right)$ should be at

$$z = e^{-bT}$$

which gives rise to the gain k as

$$k_0 = \frac{e^{aT} - e^{-bT}}{e^{aT} - 1}.$$

We thus have

T	e^{aT}	e^{-bT}	k_0
0.1	1.105	0.82	2.72
1.0	2.72	0.135	1.50
2.0	7.389	0.0183	1.154
5.0	148.4	0.000454	1.0068

To study sensitivity, let $k = k_0(1+\Delta k)$. For each set of T and k_0 given in the table, we compute Δk from the CE such that the resulting root is at z = 1, that is, solve CE:

$$[z - e^{aT} + k_0(1+\Delta k)\left(e^{aT} - 1\right)]_{z=1} = 0$$

for Δk . The result is

T	Δk
1.0	33%
2.0	13.3%
5.0	0.675%

One sees that the system is very sensitive to error in k if T is large, say T = 5.

13. The following transfer function is a lead network designed to add about 60 degree phase lead at $\omega_1 = 3$ radian/second:

$$H(s) = \frac{s+1}{0.1s+1}$$

For each of the following design methods, compute the z-plane pole and zero locations and the amount of phase lead given by the equivalent network at $z_1 = e^{j\omega_1 t}$ for T=0.25 sec.

(a) Forward rule
(b) Backward rule
(c) Bilinear rule
(d) Zero-pole mapping

(e) Zero-order-hold equivalent

Solution:
(a) Forward rule:

$$H_f(z) = H(s)\big|_{s \to \frac{z-1}{T}}$$

$$= \frac{(z-1)+0.25}{0.1(z-1)+0.25} = \frac{z-0.75}{0.1(z+1.5)} = \frac{10(z-0.75)}{(z+1.5)}$$

Hence it has a pole at $z = -1.5$ and a zero at $z = 0.75$. Since the magnitude of the pole is >1 the discrete system is unstable although the s-plane pole is stable.

At $\omega_1 = 3$ rad/sec; $z_1 = e^{j\omega_1 T} = e^{j3x0.25} = 0.7317 + j0.6818$

$$H_f(z_1) = \frac{10(0.7317 + j0.6816 - 0.75)}{(0.7317 + j0.6816 + 1.5)} = 0.7782 + j2.8165$$

giving a phase lead of 74.6 degree.

(b) Backward rule:

$$H_b(z) = H(s)\big|_{s \to \frac{z-1}{Tz}}$$

$$= \frac{(z-1)+Tz}{0.1(z-1)+TZ} = \frac{1.25z-1}{0.35z-0.1} = \frac{3.571(z-0.8)}{(z-0.286)}$$

Hence it has a pole at $z = 0.286$ and a zero at z=0.8. Since the magnitude of the pole is <1 the discrete system is stable. Note also the s-plane pole is stable.

At $\omega_1 = 3$ rad/sec; $z_1 = e^{j\omega_1 T} = e^{j3x0.25} = 0.7317 + j0.6818$

$$H_b(z_1) = \frac{3.571(0.7317 + j0.6816 - 0.8)}{(0.7317 + j0.6816 - 0.286)} = 2.338 + j1.886$$

giving a phase lead of 38.9 degree.

(c) Bilinear rule:

$$H_t(z) = H(s)\Big|_{s\to\frac{2(Z-1)}{T(z+1)}}$$

$$= \frac{2(z-1)+T(z+1)}{0.1(z-1)+T(z+1)} = \frac{2.25z-1.75}{0.45z+0.05} = \frac{5(z-0.7778)}{(z+0.111)}$$

Hence it has a pole at $z = -0.111$ and a zero at z=0.778. Since the magnitude of the pole is <1 the discrete system is stable.

$$H_t(z_1) = \frac{5(0.7317 + j0.6816 - 0.779)}{(0.7317 + j0.6816 + 0.111)} = 1.811 + j2.579$$

giving a phase lead of 54.9 degree.

(d) Zero-pole mapping

$H(s) = \dfrac{s+1}{0.1s+1}$ gives a s-plane pole at $s = -10$ and a zero at $s = -1$.

Mapping the s-plane pole to the z-plane using : $z = e^{sT}$.
The corresponding z-plane pole and zero are at $z = e^{-10x0.25}$ and $z = e^{-0.25}$ respectively.

$$H_{pz}(z) = \frac{K(z - e^{-0.25})}{z - e^{-10x0.25}} = \frac{K(z - 0.779)}{z - 0.0821}$$

Mapping the DC gain: $H(s)\,\big|_{s\to 0} = H(z)\,\big|_{z\to 1}$

$$1 = \frac{0.221K}{0.918} \Rightarrow K = 4.15$$

Hence: $H_{pz}(z) = \dfrac{4.15(z-0.779)}{z-0.0821}$

$$H_{pz}(z_1) = \frac{4.15(0.7317 + j0.6816 - 0.779)}{(0.7317 + j0.6816 - 0.0821)} = 2.202 + j2.411$$

giving a phase lead of 47.6 degree.

(e) Zero-order-hold equivalent

$$H_{zoh}(z) = (1 - z^{-1}) Z\{\frac{H(s)}{s}\}$$

$$= (1 - z^{-1}) Z\{\frac{s+1}{s(0.1s+1)}\} = (1 - z^{-1}) Z\{\frac{1}{s} + \frac{9}{s+10}\}$$

$$= (1 - z^{-1})\{\frac{z}{z-1} + \frac{9z}{z-0.0821}\} = \frac{10(z-0.90821)}{z-0.0821}$$

Hence it has a pole at $z = -0.0821$ and a zero at z=0.90821. Since the magnitude of the pole is <1 the discrete system is stable.

$$H_{ho}(z_1) = \frac{10(0.7317 + j0.6816 - 0.9082)}{(0.7317 + j0.6816 - 0.0821)} = 3.984 + j6.349$$

giving a phase lead of 58.1 degree.

14. Consider the second-order bandpass filter:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{2s}{s^2 + 2s + 100}$$

Determine the discrete transfer function and its difference equation using:

(a) Forward rule
(b) Backward rule
(c) Trapezoidal rule
(d) Zero-pole mapping

(e) Zero order Hold equivalent

Solution :
Take a reasonable sampling frequency of 10 Hz, or T=0.1 s.
(a) Forward rule
Substitute $s = (z-1)/T$

We have $$H(z) = \frac{Y(z)}{U(z)} = \frac{2[\frac{z-1}{T}]}{[\frac{z-1}{T}]^2 + 2[\frac{z-1}{T}] + 100}$$

$$= \frac{Y(z)}{U(z)} = \frac{0.2(z-1)}{z^2 - 1.8z + 1.8}$$

Hence the filter has a zero at z = 1 and complex poles at z= $0.45 \pm j0.497$.
And the difference equation of transfer function is:

$$y(k+2) = 1.8y(k+1) - 1.8y(k) + 0.2u(k+1) - 0.2u(k).$$

(b) Backward rule
Substitute $s = (z-1)/Tz$

We have $$H(z) = \frac{Y(z)}{U(z)} = \frac{2[\frac{z-1}{Tz}]}{[\frac{z-1}{Tz}]^2 + 2[\frac{z-1}{Tz}] + 100}$$

$$= \frac{Y(z)}{U(z)} = \frac{0.0909z(z-1)}{z^2-z+0.4545}$$

Hence the filter has a zero at z = 1 and z=0 and complex poles at z= 0.5 ± j0.4522.
And the difference equation of transfer function is:

$$y(k+2) = y(k+1) - 0.4545y(k) + 0.0909u(k+2) - 0.0909u(k+1)$$

(c) Trapezoidal rule
Substitute $s = 2(z-1)/T(z+1)$

We have $H(z) = \frac{Y(z)}{U(z)} = \frac{2[\frac{2}{T}(\frac{z-1}{z+1})]}{[\frac{2}{T}(\frac{z-1}{z+1})]^2+2[\frac{2}{T}(\frac{z-1}{z+1})]+100}$

$$= \frac{Y(z)}{U(z)} = \frac{0.074(z-1)(z+1)}{z^2-1.111z+0.8519}$$

Hence the filter has zero at z=1 and $z = -1$ and complex poles at z=0.5556 ± j0.737.
And the difference equation of transfer function is:

$$y(k+2) = 1.111y(k+1) - 0.8519y(k) + 0.074u(k+2) - 0.074u(k).$$

(d) Zero-Pole mapping

Poles of $H(s)$ are at $s_{1,2} = -1 \pm j9.95$ and zero of $H(s)$ is at $s = 0$,

Mapping the above poles and zeros to the z plane using $z = e^{sT}$ would yield:

Discrete poles: $z_{1,2} = 0.4927 \pm j0.7558$

Discrete zero: $z = 1$

Hence

$$H(z) = \frac{K(z-1)}{z^2 - 0.9853z + 0.8186}$$

Remark on gain matching. In general, there are several rules on gain matching, which varies with situations:

1) match static gain if the system has no pole or zero at the origin. This is the case put in my notes as the formal procedure on Slice 20 of Chapter 7. The static gain is the limit of G(s) when s goes to zero, and is also called as the position constant.

2) match the velocity constant, which is defined as the limit of sG(s) when s goes to zero, when the system has a pole at the origin. This is the case shown on slice 31 of Chapter 7.

3) match the gain at a chosen frequency if the system has a zero at the origin, which is the present case.

These rules were in my first version of lecture notes. Since they are not neat, students felt rather troublesome to learn at the first place. Thus, I now present the simplest case to avoid confusion and complexity at the first place and then give other cases in subsequent examples.

Let turn back to our question. This belongs to case 3) above since $H(s) = 0$ for $s = 0$. We may choose to match the gain at $\omega = 10\ rad/s$. $\left| H(s) \right|_{s=j10} = \left| H(z) \right|_{z=e^{j10\times 0.1}}$ gives

$$1 = K(6.2807) \quad \text{hence} \quad K = 0.1592.$$

The discrete transfer function $H(z)=\frac{Y(z)}{U(z)}=\frac{0.1592(z-1)}{z^2-0.9853z+0.8186}$

And the difference equation of transfer function is:

$$y(k+2)=0.9853y(k+1)-0.8186y(k)+0.1592u(k+1)-0.1592u\left(k\right)$$

(e) Zero order Hold Equivalent is

$$H(z)=\frac{Y(z)}{U(z)}=(1-z^{-1})Z\{\frac{H(s)}{s}\}$$

$$\frac{H(s)}{s}=\frac{2}{s^2+2s+100}=\frac{2}{(s+1)^2+9.95^2}$$

From $\dfrac{b}{(s+a)^2+b^2}\longleftrightarrow\dfrac{ze^{-aT}\sin bT}{z^2-\left(2\cos bT\right)e^{-aT}z+e^{-2aT}}$

One sees that a=1 and b=9.95. With T= 0.1s,

$$e^{-aT}=0.9048$$

$$\sin bT=0.8388$$

$$\cos bT=0.5445$$

$$Z\{\frac{H(s)}{s}\} = \frac{2(0.9048)(0.8338)z}{9.95(z^2 - 0.9853z + 0.8186)}$$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{0.1526(z-1)}{z^2 - 0.9853z + 0.8186}$$

The discrete poles are at $z = 0.4927 \pm j0.7588$ and zero at z=1.

And the difference equation of transfer function is:

$$y(k+2) = 0.9853y(k+1) - 0.8186y(k) + 0.1526u(k+1) - 0.1526u(k)$$