EE3304 DIGITAL CONTROL SYSTEMS PART II
TUTORIAL THREE

Q1. Compare the phase lead and lag compensators, discuss their differences in terms of transfer functions (pole-zero relation), and Bode plots (phase, gain crossover frequency). For a process given by

$$G(s) = k\frac{(a-s)^2}{(1+s)^2}$$

with $a = 1$ and k is a positive constant, which phase compensator is appropriate to apply if the desired gain crossover frequency is 1 *rad/sec*?

Solution:
Phase lead compensation is commonly used for improving stability margins. The phase lead compensation increases both the magnitude and phase in Bode diagram, and hence increases the system bandwidth. Thus the system has a faster speed to respond. However, such as system using phase lead compensation may be subjected to high-frequency noise problems due to its large bandwidth.

Phase lag compensator reduces the system gain at higher frequencies without reducing the system gain at lower frequencies. The gain crossover frequency is reduced which results in smaller system bandwidth and thus slower response speed. The total DC gain (at low frequency) can be increased, and thereby steady-state accuracy can be improved .Because of the reduced high frequency gain, any high-frequency noises can be attenuated.

For the process given by

$$G(s) = k\frac{(a-s)^2}{(1+s)^2}$$

Let's compute the phase of C(w) at the gain crossover frequency, $\omega_g = 1$

$$G(j\omega_g) = k\frac{(1-j\omega_g)^2}{(1+j\omega_g)^2} = k\frac{(1-j)^2}{(1+j)^2}$$

$$\measuredangle(1-j) = -45^\circ$$

$$\measuredangle(1+j) = +45^\circ$$

$$\measuredangle G(j\omega_g) = 2\measuredangle(1-j) - 2\measuredangle(1+j) = -180^\circ$$

Therefore the phase margin for the uncompensated system is zero. To make the closed loop stable, we need to increase the phase margin at the gain crossover frequency. Since the lead compensator increases the phase for all the frequencies while the lag compensator decreases the phase, we should use lead compensator for this case.

Q2. A phase lead compensator is to be designed.

(i) The desired phase margin is 50°, and the settling time is 3 seconds. Find the minimum damping ratio and the approximate bandwidth ($\omega_b \approx \omega_n$). If the damping ratio is chosen to be 0.77, is a sampling interval of 0.1 seconds adequate for the emulation based design?

(ii) Derive the desired characteristic polynomials and desired closed-loop transfer functions (both *s* domain and *z* domain).

Solution:

(i) From the approximate relation between damping ratio and phase margin

$$\zeta \cong \frac{PM}{100}$$

We get the minimal damping ratio is 0.5.

Let $\zeta = 0.77$. Since the settling time is 3 seconds, we have

$$t_s \cong \frac{4.6}{\zeta\omega_n} = 3$$

$$\omega_n = \frac{4.6}{3\zeta} = 2$$

The bandwidth is also around 2 rad/s=0.32Hz. The sampling frequency 1/0.1=10 Hz is around 30 times the bandwidth. Hence it is adequate.

(ii) From the desired damping ratio and natural frequency, we get the desired closed loop transfer function for the analog system is

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{4}{s^2 + 3.08s + 4}$$

Let the sampling period T=0.1, the desired closed loop transfer function for the sampled system can be obtained from Table 2.1 as

$$H(z) = \frac{0.018z + 0.016}{z^2 - 1.7z + 0.734}$$

Q3. A process has the Bode plots shown in Fig.1.

(i) Find the necessary gain *k* of the lead compensator such that the static error constant is 3.2.

(ii) Find the phase lead needed when the gain crossover frequency is chosen to be 2 rad/sec, where the desired phase margin is 50°.

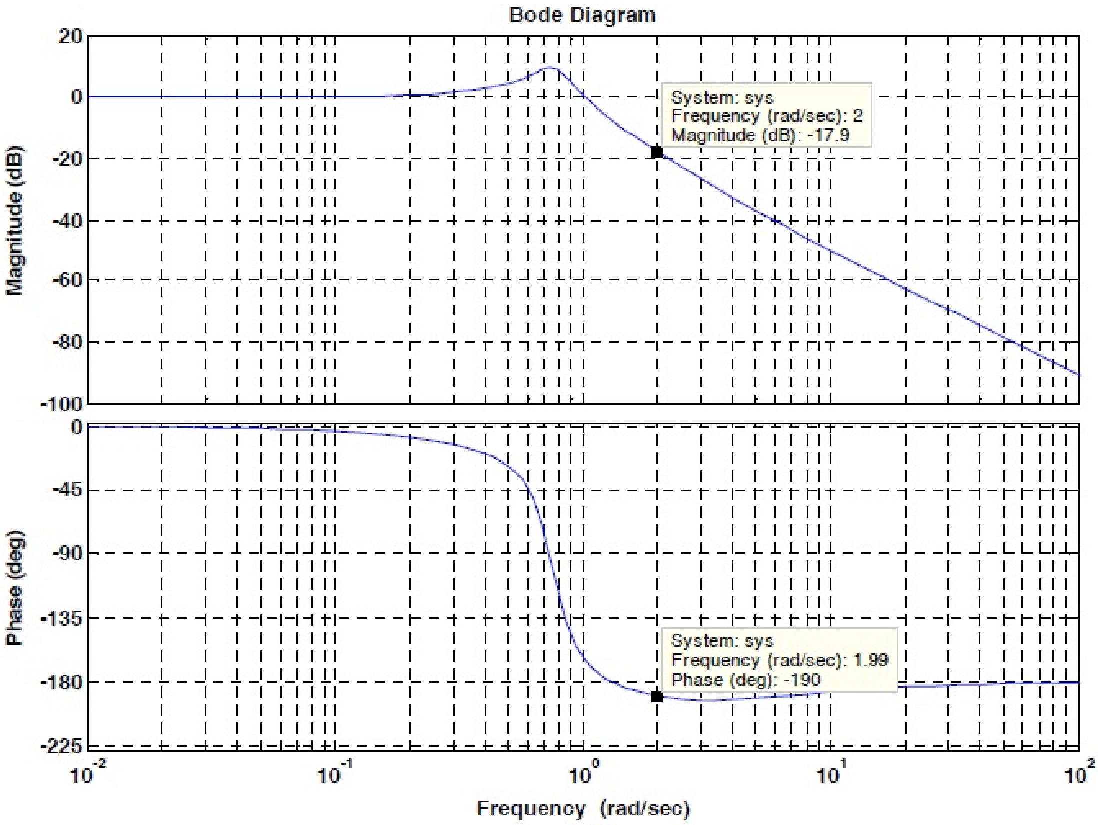


Figure 1 Bode plot for Q3

Solution:

(i) Since the static gain for the uncompensated system G(s) is G(0)=1 (0 dB) as shown in the Bode diagram, and the static error constant is kG(0)=3.2, we get k = 3.2 ;

(ii) The phase margin of the uncompensated system G(s) at the chosen gain crossover frequency is 180° -190°=-10°. To achieve the desired phase margin of 50°, the phase lead needed is

$$\phi_m = 50° - (-10°) = 60°$$

Note that in above calculation, we assume that the gain crossover frequency for the compensated system is known. In practice, this gain crossover frequency is not given before the design of the lead compensator in most of the applications.

Q4. A digital controller is given as

$$C(z) = \frac{az - b}{z}, \qquad a > 0$$

where b is a constant. Find the range of b such that $C(z)$ becomes (i) a proportional controller, (ii) a derivative controller, and (iii) a lead compensator. (You may use the w-transform where necessary, and assume that the sampling period is 1).

Solution:

(i) For proportional controller, $b = 0$;

(ii) For derivative controller, $b = a$;

(iii) For lead compensator, we need to get the transfer function is s-domain. Use w-transform, we have

$$z = \frac{\frac{2}{T} + w}{\frac{2}{T} - w} = \frac{2 + w}{2 - w}$$

Then we have

$$C(w) = C(z)\,|_{z = \frac{2+w}{2-w}}$$

$$= \frac{a\dfrac{2+w}{2-w} - b}{\dfrac{2+w}{2-w}}$$

$$= \frac{(a+b)w + 2a - 2b}{w+2}$$

$$= (a-b)\frac{\frac{1}{2}\frac{a+b}{a-b}w + 1}{(\frac{1}{2}w + 1)}$$

Compare it with the standard form of the lead compensator:

$$C(s) = k\frac{\tau s + 1}{\alpha\tau s + 1}, \alpha < 1$$

We require that

$$\frac{a+b}{a-b} > 1 and \quad (a-b) > 0$$

$$\rightarrow \quad a > b > 0$$

Q5. A process has the Bode plots shown in Fig.1. Design a phase-lead compensator such that

(i) the steady state error for unit step input is less than 10%..

(ii) the desired phase margin is 40°.

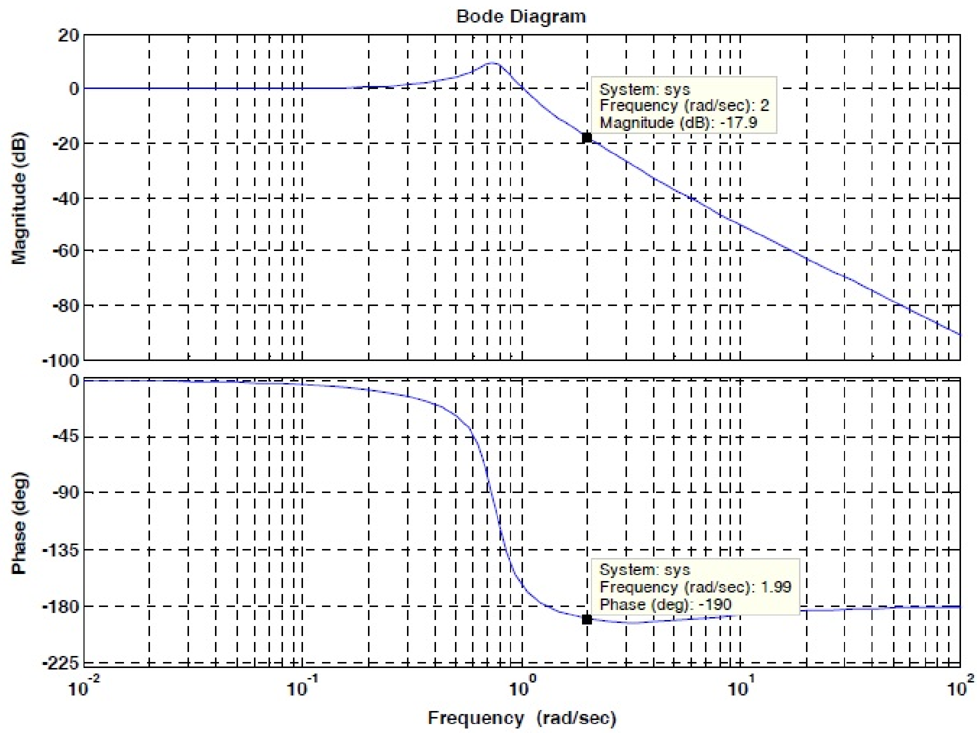


Figure 1 Bode plot for Q5

Solution:

Lead Compensator:

$$C(s) = k\frac{\tau s+1}{\alpha\tau s+1}, \alpha < 1$$

Step 1. Determine open-loop gain k to satisfy requirements on steady state error.

The DC gain of the original plant is 1. The steady state error for unit step input is 1/(1+K). In order to have a steady state error less than 10%, K is at least 9. Choose k=10 for the lead compensator.

Step 2. Find new open-loop crossover frequency ω_g from kG(jw).

From the Bode plot, the magnitude curve is shifted up by 20 dB, then the new crossover frequency (where |G(jw)| is -20 dB) is around 2 rad/sec.

Step 3. Evaluate the PM of kG(jw) at the new crossover frequency $\omega_g = 2$, i.e.

$$\phi = 180^o + \angle kG(j\omega_g) = 180^o - 190^o = -10^o$$

Therefore, the system is unstable!

Step 4. Determine the necessary phase-lead angle to be added to the system. Allow for some extra angle (5°to 12°), because the addition of the lead compensation shifts the gain crossover frequency to the right and decrease the phase margin. For instance we can choose 10°

$$\phi_m = \phi_{desired} - \phi + 10^\circ = 40^\circ - (-10^\circ) + 10^\circ = 60^\circ$$

Step 5. Compute the attenuation factor,

$$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m} = \frac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}} = 7 - 4\sqrt{3} = 0.0718$$

Step 6. Compute $\frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{0.0718}} = 3.732 \Rightarrow 11.44dB$

Find out the frequency where $|kG(j\omega)|$ is equal to $-20\log(1/\sqrt{\alpha})$=-11.44 dB, ω_m. That is where $|G(j\omega)|$ is -11.44-20=-31.44dB, which is around 4.

$$\omega_m = 4$$

and then compute

$$\tau = \frac{1}{\sqrt{\alpha}\,\omega_m} = \frac{3.732}{4} = 0.933$$

Overall, we have the lead compensator,

$$C(s) = 10\frac{0.933s+1}{0.067s+1}$$

Q6. The Bode plots of a plant are shown in Fig.2. Find the phase margin from the Bode plots. In order to achieve a phase margin of 50 and meanwhile keep the bandwidth above 2 rad/sec, should we use a lead compensator or a lag compensator? Briefly explain the reason for your selection.

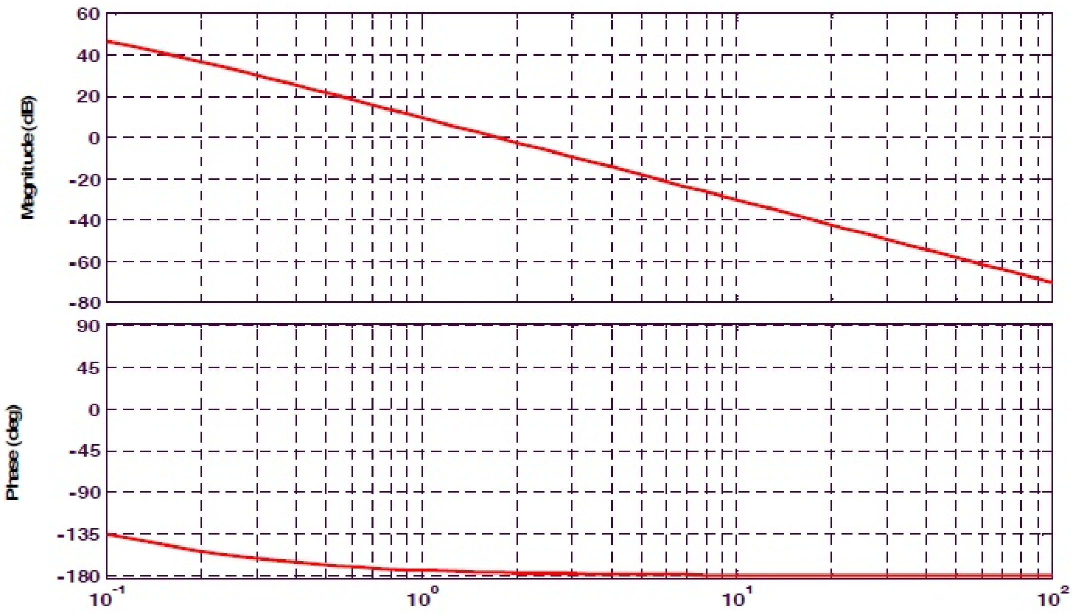


Figure 2 Bode Plot for Q6

Solution:

The Lead Compensator should be used. The phase margin of the system is already very low, and even if we can decrease the gain crossover frequency to 0.1 using lag compensator, the phase margin is only 45. Since the phase-lag compensator would decrease the phase further, it is impossible to get a phase margin of 50 with a gain cross-over frequency higher than 0.1.

In order to have a higher bandwidth (which is close to the gain cross over frequency), we should use lead compensator to increase the phase margin.

Q7. Design a digital lead compensator for the system shown in Fig. 4.

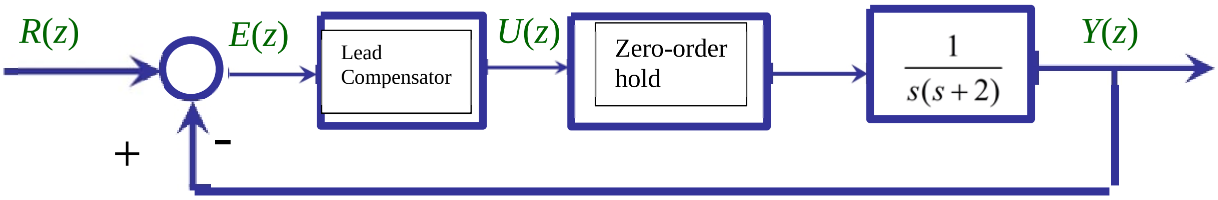


Figure 3 Digital control system for Q7

Use the frequency domain design method in the w-domain. The design specifications are that the phase margin be 50^o, the gain margin be at least 10dB, and the velocity error constant be 5. The sampling period is T=0.1.

Solution. Let's first get the discrete transfer function of the plant,

$$G(z) = (1-z^{-1})Z\{\frac{G(s)}{s}\} = (1-z^{-1})Z\{\frac{1}{s^2(s+2)}\}$$

$$= (1-z^{-1})Z\{\frac{1}{2}\frac{1}{s^2} + \frac{1}{4}\frac{1}{s+2} - \frac{1}{4}\frac{1}{s}\}$$

$$= (1-z^{-1})(\frac{1}{2}\frac{Tz}{(z-1)^2} + \frac{1}{4}\frac{z}{z-e^{-2T}} - \frac{1}{4}\frac{z}{z-1})$$

$$= \frac{1}{2}\frac{T}{z-1} + \frac{1}{4}\frac{z-1}{z-e^{-2T}} - \frac{1}{4}$$

$$= \frac{1}{4}\frac{(2T+e^{-2T}-1)z+1-(1+2T)e^{-2T}}{(z-1)(z-e^{-2T})}$$

$$= \frac{0.004683(z+0.9355)}{(z-1)(z-0.8187)}$$

You can also use Table 2.1 to get the above answer.

Let's transform G(z) into G(w) by using the bilinear transformation:

$$z = \frac{\frac{2}{T}+w}{\frac{2}{T}-w} = \frac{20+w}{20-w}$$

We have

$$G(w) = \frac{0.004683(z+0.9355)}{(z-1)(z-0.8187)}\bigg|_{z=\frac{20+w}{20-w}} = \frac{0.004683(\frac{20+w}{20-w}+0.9355)}{(\frac{20+w}{20-w}-1)(\frac{20+w}{20-w}-0.8187)}$$

$$= \frac{0.5(1+0.001666w)(1-0.05w)}{w(1+0.5016w)}$$

$$= \frac{0.5(1-0.048w-0.0000833w^2)}{w(1+0.5016w)}$$

Compared to the original G(s), they are quite close if we ignore those small coefficients in the numerator.

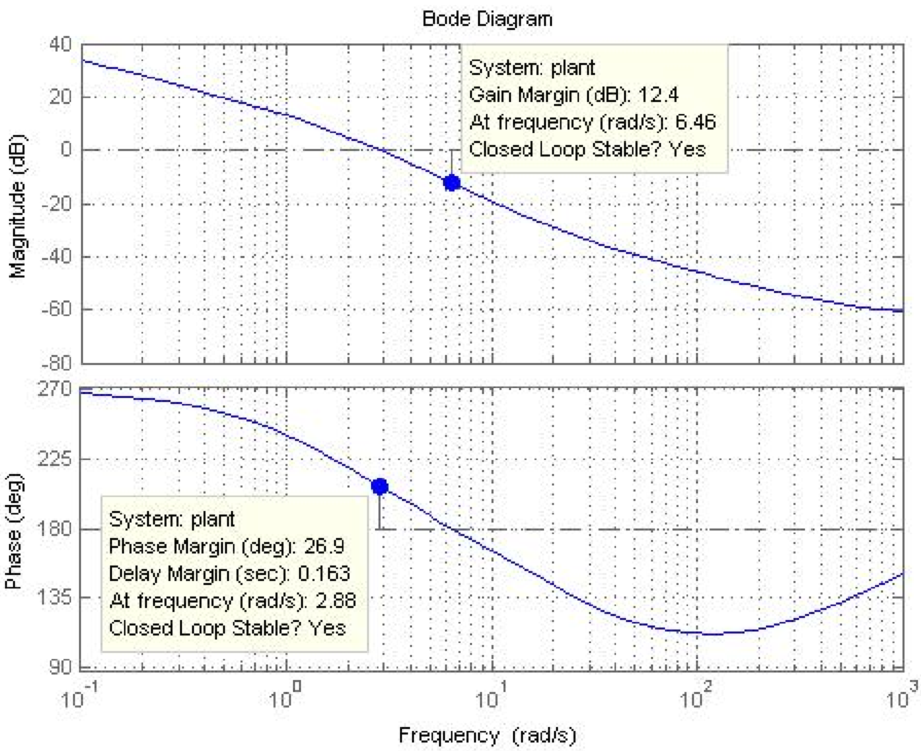
Let the lead compensator be

$$C(w) = k\frac{\tau w+1}{\alpha\tau w+1}, \alpha < 1$$

Let's first determine the necessary gain k. The required velocity error constant is 5, hence

$$\lim_{w\to 0} wC(w)G(w) = 0.5k = 5$$

From which we determine that k=10. The Bode diagram of 10G(w) is shown below.



The gain margin is 12.4 dB, which satisfies the requirement. However, the phase margin is about 27^o . The desired phase margin is 50^o. Choose the extra phase to be 12^o, we have

$$\phi_m = \phi_{desired} - \phi + 12^\circ = 50^o - 27^o + 12^\circ = 35^o$$

Then we can compute

$$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m} = 0.271$$

Then we have

$$\frac{1}{\sqrt{\alpha}} = 1.92 \Rightarrow 5.67dB$$

Find out the frequency where $|10G(j\omega)|$ is equal to $-20\log(1/\sqrt{\alpha})$ =-5.67 dB, ω_m, which is around 4.2.

$$\omega_m = 4.2$$

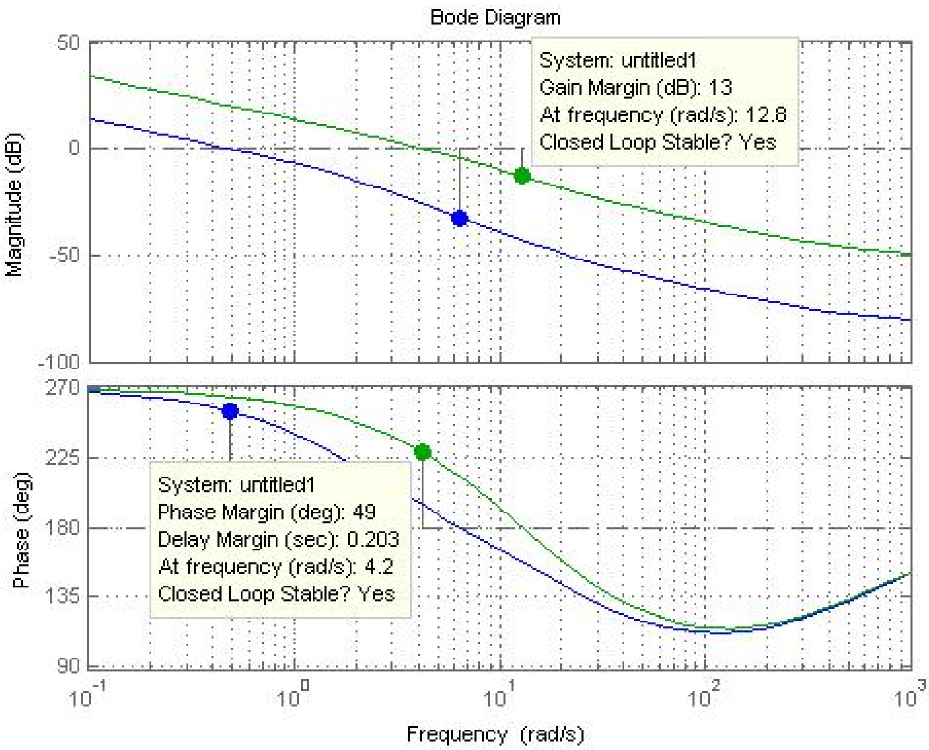
and then compute

$$\tau = \frac{1}{\sqrt{\alpha}\omega_m} = \frac{1.92}{4.2} = 0.457$$

Overall, we have the lead compensator,

$$C(w) = 10\frac{0.457w+1}{0.124w+1}$$

The bode diagram of the uncompensated system G(w), and the compensated one, C(w)G(w) are shown in figure below.



The gain margin is 13 dB, and the phase margin is 49^o, which is close to the desired one (50^o).

Next, we need to transform the lead compensator C(w) into digital lead compensator C(z). Then

$$C(z) = 10\frac{0.457w+1}{0.124w+1}\bigg|_{w=\frac{2}{T}\frac{z-1}{z+1}=20\frac{z-1}{z+1}}$$

$$= 10\frac{0.457\times 20\dfrac{z-1}{z+1}+1}{0.124\times 20\dfrac{z-1}{z+1}+1}$$

$$= \frac{29.14z-23.4}{z-0.425}$$