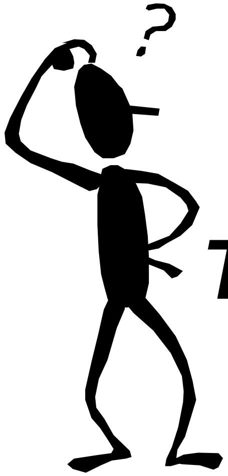


Chapter 3
Transfer Functions and Stability

3.1 1st-order system

Consider the first order difference equation

$$x(k) = \alpha x(k-1) + \beta u(k-1)$$

We have

$$X(z) = \alpha z^{-1} X(z) + \beta z^{-1} U(z)$$

$$X(z) = \frac{\beta z^{-1}}{1-\alpha z^{-1}} U(z) = \frac{\beta}{z-\alpha} U(z)$$

$$= H(z)U(z)$$

where

$$H(z) = \frac{\beta}{z-\alpha}$$

$$x_{k+1} = \alpha x_k + \beta u_k$$

$$x_1 = \alpha x_0 + \beta u_0$$

$$x_2 = \alpha x_1 + \beta u_1 = \alpha^2 x_0 + \alpha\beta u_0 + \beta u_1$$

$$x_3 = \alpha x_2 + \beta u_2 = \alpha^3 x_0 + \alpha^2 \beta u_0 + \alpha\beta u_1 + \beta u_2$$

$$\vdots$$

$$\begin{aligned} x_k &= \alpha x_{k-1} + \beta u_{k-1} \\ &= \alpha^k x_0 + \alpha^{k-1}\beta u_0 + \alpha^{k-2}\beta u_1 + \cdots + \beta u_{k-1} \\ &= \alpha^k x_0 + \sum_{j=0}^{k-1} \alpha^j \beta u_{k-1-j} \end{aligned}$$

Suspose :

(i) an input sequence where $u(k)=0$ for all k < 0.

(ii) zero initial conditions: $x(0)=0$.

$$x(k) = \alpha^{k-1}\beta u_0 + \alpha^{k-2}\beta u_1 + \cdots + \beta u_{k-1}$$

$$= \sum_{j=0}^{k-1} \alpha^{k-1-j}\beta u_j$$

$$= \sum_{j=0}^{k-1} h(k-j)u(j), \qquad h(i) = \alpha^{i-1}\beta, \qquad i \geq 1$$

$$= \sum_{j=0}^{k} h(k-j)u(j), \qquad \text{adding } \ h(0) = 0 \ !$$

Convolution sum in time domain!

$$x(k) = \alpha^{k-1}\beta u_0 + \alpha^{k-2}\beta u_1 + \cdots + \beta u_{k-1}$$

$$= \sum_{j=0}^{k} h(k-j)u(j)$$

$$h(0)=0,\ h(k)=\alpha^{k-1}\beta,\ k\geq 1,\ h(k)=0,\ k<0.$$

$$H(z) = Z\{h(k)\} = \sum_{k=0}^{\infty} h(k)z^{-k}$$

$$= 0 + \sum_{k=1}^{\infty} \alpha^{k-1}\beta z^{-k} = \mathrm{z}^{-1}\beta \sum_{k=1}^{\infty} \alpha^{k-1} z^{-(k-1)}$$

$$= \mathrm{z}^{-1}\beta \sum_{l=0}^{\infty} \alpha^{l} z^{-l} = \mathrm{z}^{-1}\beta \frac{z}{z-\alpha} = \frac{\beta}{z-\alpha}$$

3.2 The Discrete Transfer Functions

For a general linear discrete system:

$$u_k + a_1 u_{k-1} + a_2 u_{k-2} + \cdots + a_n u_{k-n} = b_0 e_k + b_1 e_{k-1} + \cdots + b_m e_{k-m}$$

it follows from time shift property of the z-transform that

$$U(z) + a_1 z^{-1} U(z) + \cdots + a_n z^{-n} U(z)$$
$$= b_0 E(z) + b_1 z^{-1} E(z) + \cdots + b_m z^{-m} E(z)$$

The discrete transfer function is defined as

$$H(z) := \frac{U(z)}{E(z)} := \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

If $n>=m$, write this as

$$H(z) = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + \cdots + a_n} := \frac{b(z)}{a(z)}$$

The input-output relation is expressed as

$$U(z) = H(z)E(z)$$

Because $H(z)$ is a rational function of a complex variable, we say that the places in z where $b(z) = 0$ are zeros of transfer function, and the places in z where $a(z) = 0$ are poles of $H(z)$.

We can now give a physical meaning to the variable z. Suppose all other coefficients to be zero except $b_1 = 1$. Then the LDE reduces to

$$u_k = e_{k-1}$$

The present output u_k equals the input delayed by one period. In this case

$$H(z) = z^{-1}, \; U(z) = z^{-1} E(z).$$

we see that a transfer function of z^{-1} is a delay of one time unit.

Time-domain meaning to an arbitrary transfer function

Recall that $U(z) = H(z)\,E(z)$. Let $e(k)$ be the discrete unit pulse

$$e_k = \delta_k = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

The z-transform of unit discrete pulse is

$$Z(e_k) = \sum_{k=0}^{\infty} e_k z^{-k} = e_k z^{-k} \Big|_{k=0} = 1$$

Thus, $U(z) = H(z)$. The transfer function $H(z)$ is seen to be the z-transform of the output response to a unit-pulse input, $\{h_k\}$, which is called the unit-pulse response.

For input $E(z)$ other than $E(z) = 1$, we can get the response u_k by multiplying the infinite polynomials of $H(z)$ $E(z)$ as $U(z) = H(z)E(z)$. Usually, we assume $h_k = 0$ for $k<0$ and $k=0$ being the starting time for e_k:

$$H(z) = \sum_{k=0}^{\infty} h_k z^{-k}$$

$$E(z) = \sum_{k=0}^{\infty} e_k z^{-k}$$

$$U(z) = \sum_{k=0}^{\infty} u_k z^{-k} = H(z)E(z) = (\sum_{k=0}^{\infty} h_k z^{-k})(\sum_{k=0}^{\infty} e_k z^{-k})$$

Equalizing the coefficients of the same power on both sides gives

$$\sum_{k=0}^{\infty} u_k z^{-k} = \left(\sum_{k=0}^{\infty} h_k z^{-k}\right)\left(\sum_{k=0}^{\infty} e_k z^{-k}\right)$$

$$= \left(h_0 + h_1 z^{-1} + h_2 z^{-2} + \cdots\right)\left(e_0 + e_1 z^{-1} + e_2 z^{-2} + \cdots\right)$$

Coefficient of

$$\begin{aligned}
&z^0 : && u_0 = h_0 e_0 \\
&z^{-1} : && u_1 = h_0 e_1 + h_1 e_0 \\
&z^{-2} : && u_2 = h_0 e_2 + h_1 e_1 + h_2 e_0 \\
&z^{-3} : && \ldots
\end{aligned}$$

In general, we have

$$u_k = \sum_{j=0}^{k} h_{k-j} e_j, \text{ if } h_k = e_k = 0, \quad k < 0.$$

Otherwise

$$u_k = \sum_{j=-\infty}^{\infty} h_{k-j} e_j$$

This is the discrete convolution sum and is the analog of convolution integral in continuous system.

Conversely, let $x(k) = \sum_{j=0}^{k} h(k-j)u(j)$.

If $h(k) = 0,\ k < 0$, then $x(k) = \sum_{j=0}^{k} h(k-j)u(j) = \sum_{j=0}^{\infty} h(k-j)u(j)$.

$$X(z) = \sum_{k=0}^{\infty} x(k)z^{-k} = \sum_{k=0}^{\infty} z^{-k}\left[\sum_{j=0}^{\infty} h(k-j)u(j)\right]$$

$$= \sum_{j=0}^{\infty} u(j)\left[\sum_{k=0}^{\infty} h(k-j)z^{-k}\right]$$

$$= \sum_{j=0}^{\infty} u(j)z^{-j}H(z) = H(z)U(z)$$

H(z) is the z-transform of the sequence, *h(k)*.

$\boxed{\textbf{\textit{h(k) is one to one correspondence with H(z)}}}$

In continuous time

In s-domain:

$$Y(s) = G(s)U(s)$$

In time domain:

$$y(t) = \int_0^t g(\tau)u(t-\tau)\,d\tau$$

Continuous time convolution integral

In continuous time,

- $g(t)$ is the output response to the unit impulse input
- $G(s)$ is the transfer function, or Laplace transform of $g(t)$
- Output, $y(t)$, is given by (**continuous)** convolution integral
- Output $Y(s) = G(s)U(s)$

In discrete time,

- $h(k)$ is the output response to the unit pulse input
- $H(z)$ is the discrete transfer function, or z-transform of $h(k)$
- Output, $u(k)$, is given by **(discrete)** convolution sum
- Output $U(z) = H(z)E(z)$

In Summary, ...

- We have a tool which works just like the Laplace transform and transfer functions in the continuous time.
- The discrete transfer function is the ratio:

$$\frac{\text{z transform of output sequence}}{\text{z transform of input sequence}}$$

Or it is also the z-transform of the unit pulse response, *h(k).*

Examples of Discrete transfer functions

1) Differentiator

- In our room temperature example, we approximated

$$y(t) = \frac{d\theta(t)}{dt} \approx \frac{\theta(t+T) - \theta(t)}{T}$$

$$y(k) = \frac{\theta(k+1) - \theta(k)}{T}$$

- Taking z transform

$$Y(z) = \frac{z\Theta(z) - \Theta(z)}{T}$$

$$\frac{Y(z)}{\Theta(z)} = \frac{z-1}{T}$$

2) Integrator

$$y(t)=\int_{t=0}^{t}\theta(t)dt,\quad and,\quad y(kT+T)=\int_{t=0}^{t=kT}\theta(t)dt+\int_{t=kT}^{t=kT+T}\theta(t)dt$$

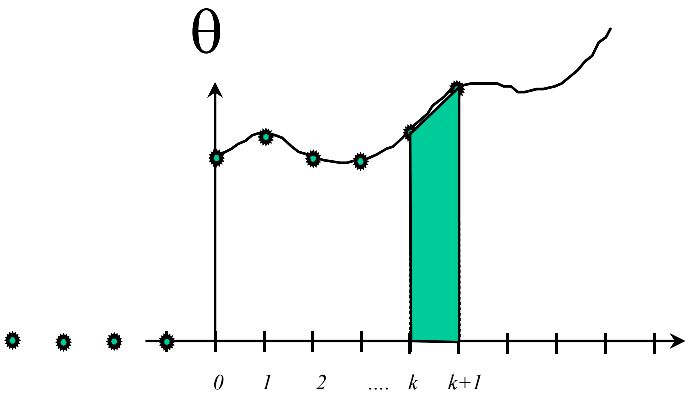
$$\therefore y(k+1)=y(k)+\int_{t=kT}^{t=kT+T}\theta(t)dt\approx y(k)+T\theta(k)\ ,\quad or,\quad y(k)+T\theta(k+1)$$

- Taking the z-transforms, we have

$$zY(z)=Y(z)+T\Theta(z)\qquad or\qquad zY(z)=Y(z)+Tz\Theta(z)$$

$$\frac{Y(z)}{\Theta(z)}=\frac{T}{z-1}\qquad or\qquad \frac{Y(z)}{\Theta(z)}=\frac{Tz}{z-1}$$

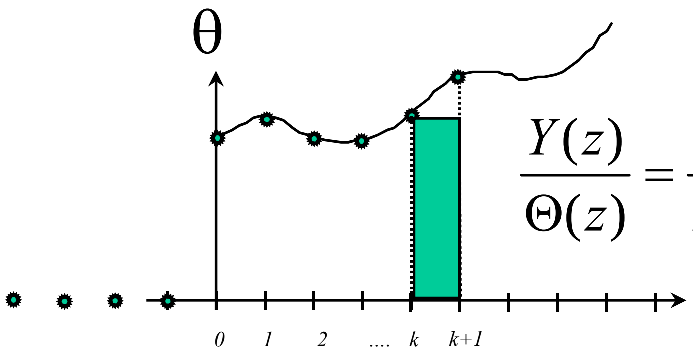
A pictorial view



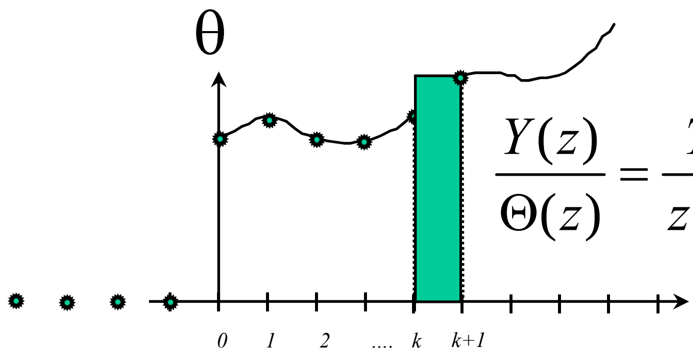
The key is how we represent the shaded area.

The two possibilities we considered were

θ(k)T or *θ(k+1)T*



$$\frac{Y(z)}{\Theta(z)} = \frac{T}{z-1}$$



$$\frac{Y(z)}{\Theta(z)} = \frac{Tz}{z-1}$$

Yet another method

$$y(k+1) = y(k) + \frac{1}{2}T[\theta(k+1) + \theta(k)]$$

- Takes the area of a trapezium
- More accurate than the other two methods to approximate the area under the curve
- Will learn more about this later

3.3 Stability

Definition: A system is said to be BIBO stable if for every bounded input, its output is also bounded.

How to get stability condition,…

View from s-z mapping:

$$z = e^{Ts}$$

Let the poles of continuous system *G(s)* be

$$s_{pi} = \alpha_i + j\beta_i$$

G(s) is stable if for all *i*, there hold

$$\alpha_i < 0$$

Then

$$z_{p_i} = e^{Ts_{p_i}} = e^{\alpha_i T} e^{jT\beta_i}$$

$$\left|z_{p_i}\right| = \left|e^{\alpha_i T}\right| < 1$$

View from *h(k):* given rational *H(z),* one expands

$$\frac{H(z)}{z} = \sum_i \frac{\lambda_i}{z - z_{p_i}}$$

Then

$$H(z) = \sum_i \frac{\lambda_i z}{z - z_{p_i}}$$

$$h(k) = \sum_i \lambda_i \left(z_{p_i} \right)^k$$

$$\text{if} \quad \left| z_{p_i} \right| < 1, \quad h(k) \xrightarrow{k \to \infty} 0 .$$

Let

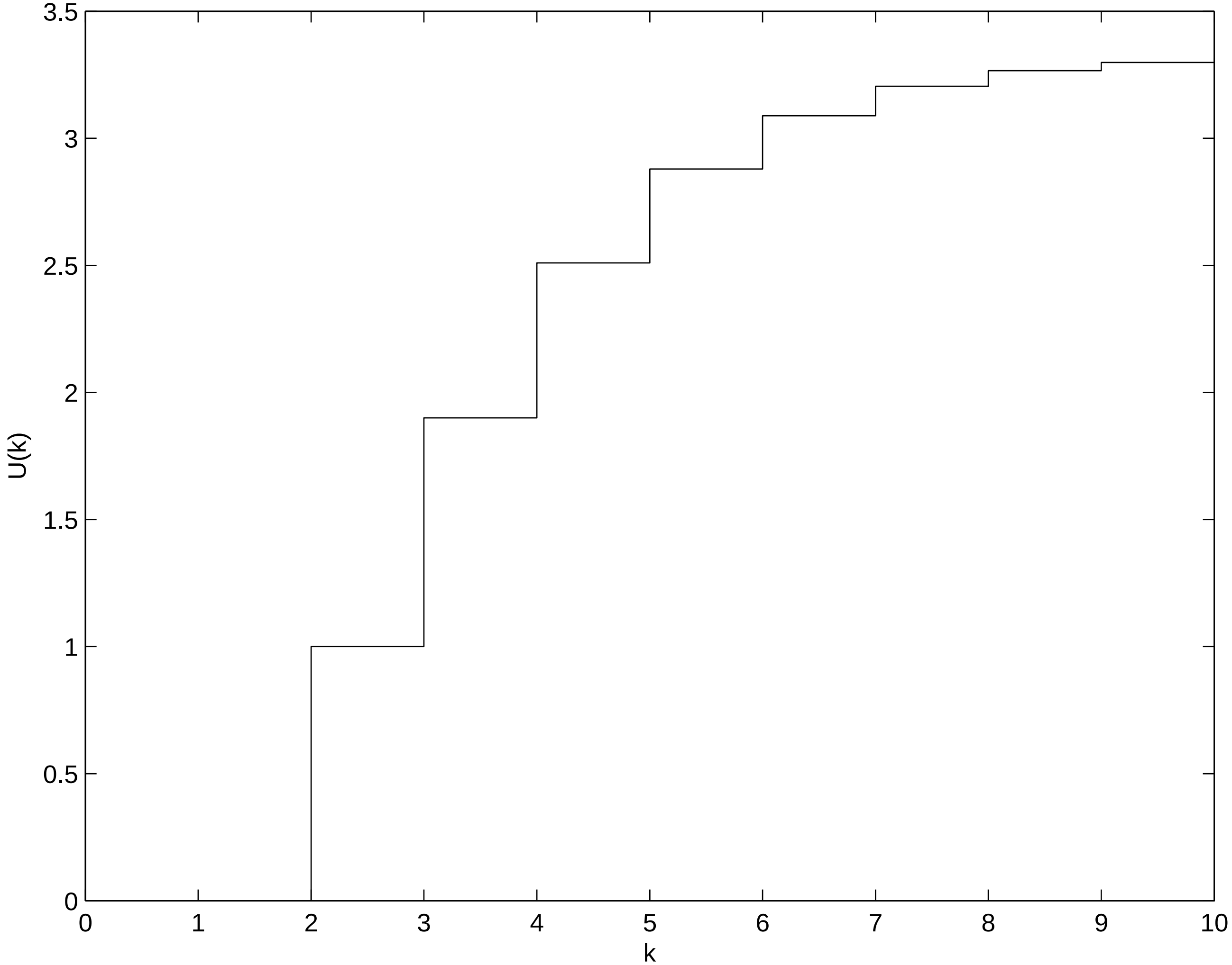
$$H(z) = \frac{1}{z^2 - 0.9z + 0.2}$$

H(z) has two poles at

$$z_1 = 0.5 \quad \text{and} \quad z_2 = 0.4$$

both are inside the unit circle (i.e. magnitude less than 1),
Step response of *H(z)* is shown below.

The system is stable.



3.5
3
2.5
2
1.5
1
0.5
0
U(k)
0
1
2
3
4
5
6
7
8
9
10
k

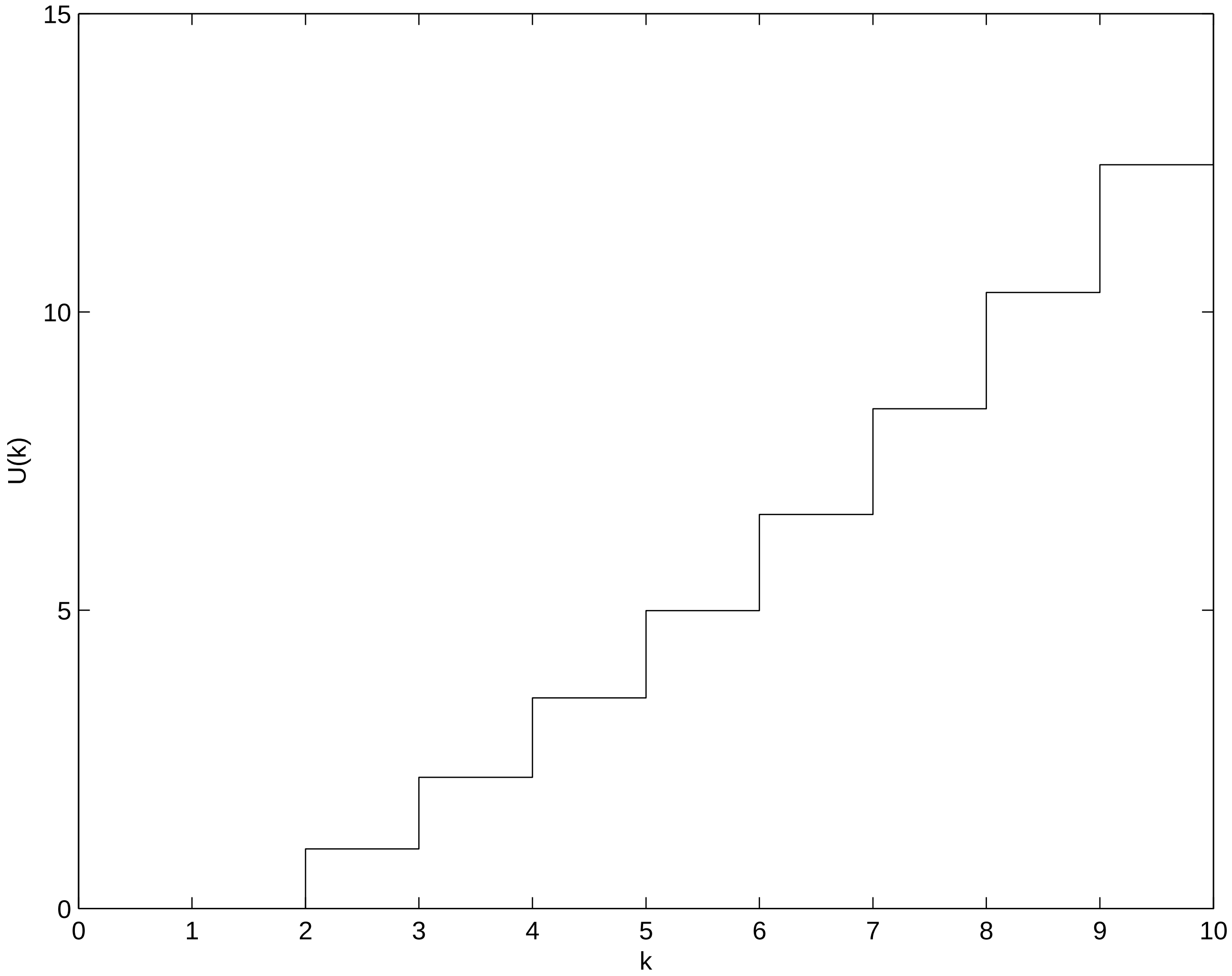
Let

$$H(z)=\frac{1}{z^2-1.2z+0.11}$$

$H(z)$ has two poles at $z_1 = 1.1$ and $z_2 = 0.1$.
z_1 is outside the unit circle.

See the figure below for its step response.

The system is unstable.


15
10
5
0
U(k)
0
1
2
3
4
5
6
7
8
9
10
k

Satellite system

The attitude of a satellite can be described by the equation

$$J\frac{d\theta^2}{dt^2}=\tau$$

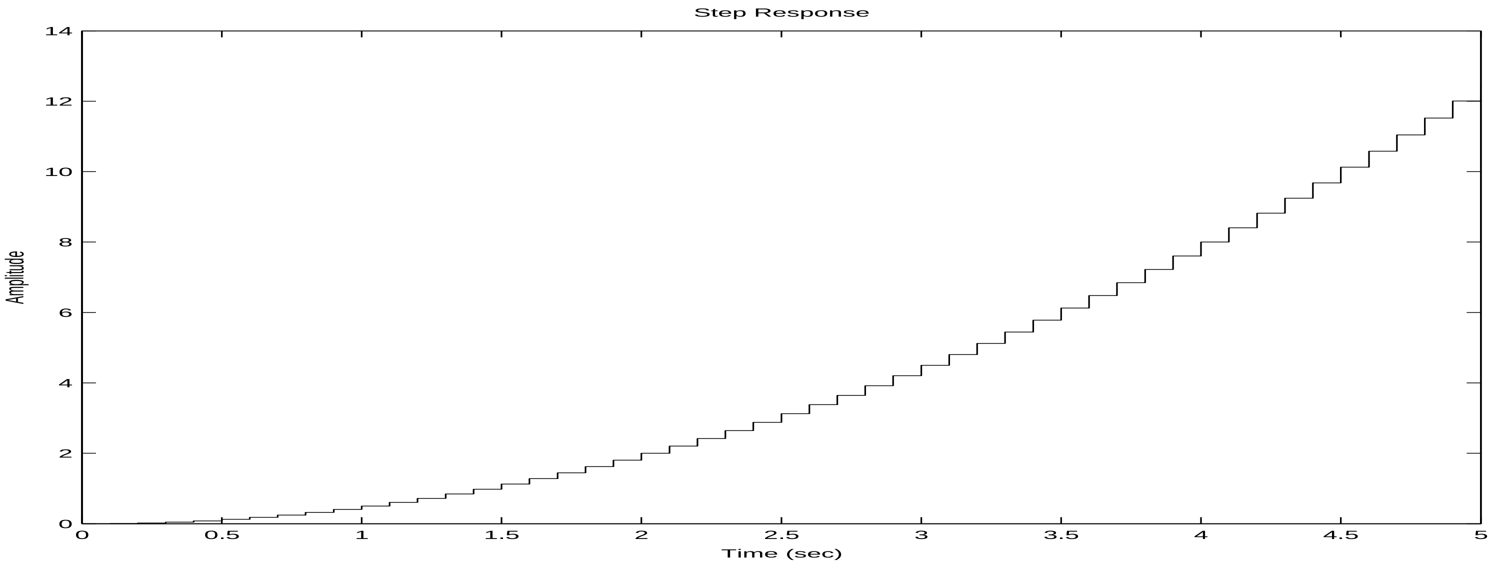
where θ is the attitude angle, τ is the control torque, and J is the moment of inertia. Suppose that $J=1$ and the transfer function is

$$\frac{\theta(s)}{\tau(s)}=\frac{1}{s^2}$$

With *T=0.1*, the zero-hold equivalent method (Chapter 7) leads to the discrete transfer function as

$$\frac{\Theta(z)}{\tau(z)} = \frac{0.005z + 0.005}{z^2 - 2z + 1}.$$

The discrete transfer function has two poles at z=1 so that it is unstable. The step response is given in the following figure.



Stability Theorem: If a discrete system with the transfer function $H(z)$ *has* all its poles, z_k, strictly inside the unit circle, i.e., $| z_k | < 1$ for all k, then the system is stable; If at least one pole is on or outside the unit circle, the corresponding system is not stable.

Going from s to z …

Example: $G(s) = \frac{1}{s^2 + s + 1}$ Poles at $s = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

s-poles are mapped to z-poles via $z=e^{sT}$. Suppose sampling frequency is $\omega_s = 10\pi\ rad/s$, or $T = 2\pi/\omega_s = 0.2s$.

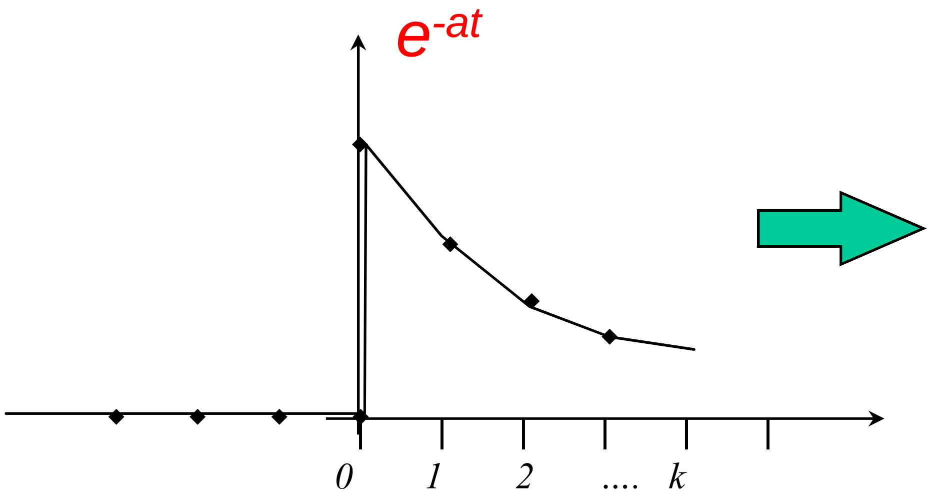
z-domain poles at $z = e^{sT}$

$$= e^{(-0.5 \pm j0.5\sqrt{3})0.2}$$

$$= 0.8913 \pm j0.1559$$

Its magnitude is 0.9048<1 ⇨ stable z-poles

Signal Analysis 1: real poles



$e^{-akT}=r^k$

sampled sequence

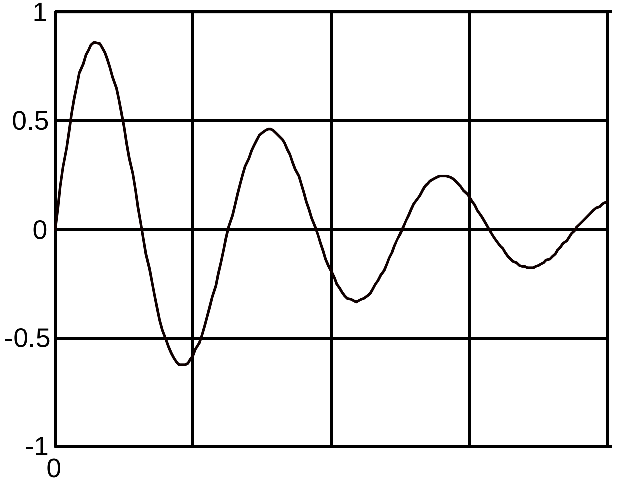
$$L\left\{e^{-at}\right\}=\frac{1}{s+a}$$

$$Z\left\{r^k\right\}=\frac{z}{z-r}, \quad r=e^{-aT}$$

- pole at $-a$
- $a<0$ *for* unbounded signal
- $a=0$ for step function
- $a>0$ for bounded signal
- large *positive a for* fast decay

- pole at $z=r=e^{-aT}$
- $a<0$, *then*, $r>1$ for unbounded signal
- $a=0$, *then*, $r=1$ for step function
- $a>0$, *then* $r<1$ for bounded signal
- large *positive a, then* small r for fast decay

Signal Analysis 2: complex poles



$$e^{-at}\sin(bt)\ \ t \geq 0$$

when sampled gives

$$e^{-aTk}\sin(bTk),\ \ t = kT$$

$$r^{k}\sin(k\theta),\ \ r = e^{-aT},\ \ \theta = bT$$

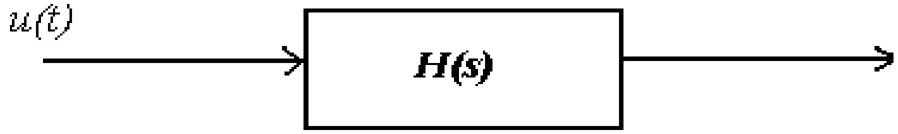
$$\frac{b}{(s+a)^2+b^2} \Longleftrightarrow \frac{r\sin(\theta)z}{z^2-2r\cos(\theta)z+r^2}$$

- poles at $-a+jb$, $-a-jb$
- $a<0$ for unbounded signal
- $a>0$ for bounded signal
- large a for fast decay
- large b for higher frequency

- poles at $r\cos(\theta) +/- \ j\, r\sin(\theta)$
- r>1 for unbounded signal
- r<1 for bounded signal
- Small r for fast decay
- large θ for higher frequency

3.4 Frequency Response

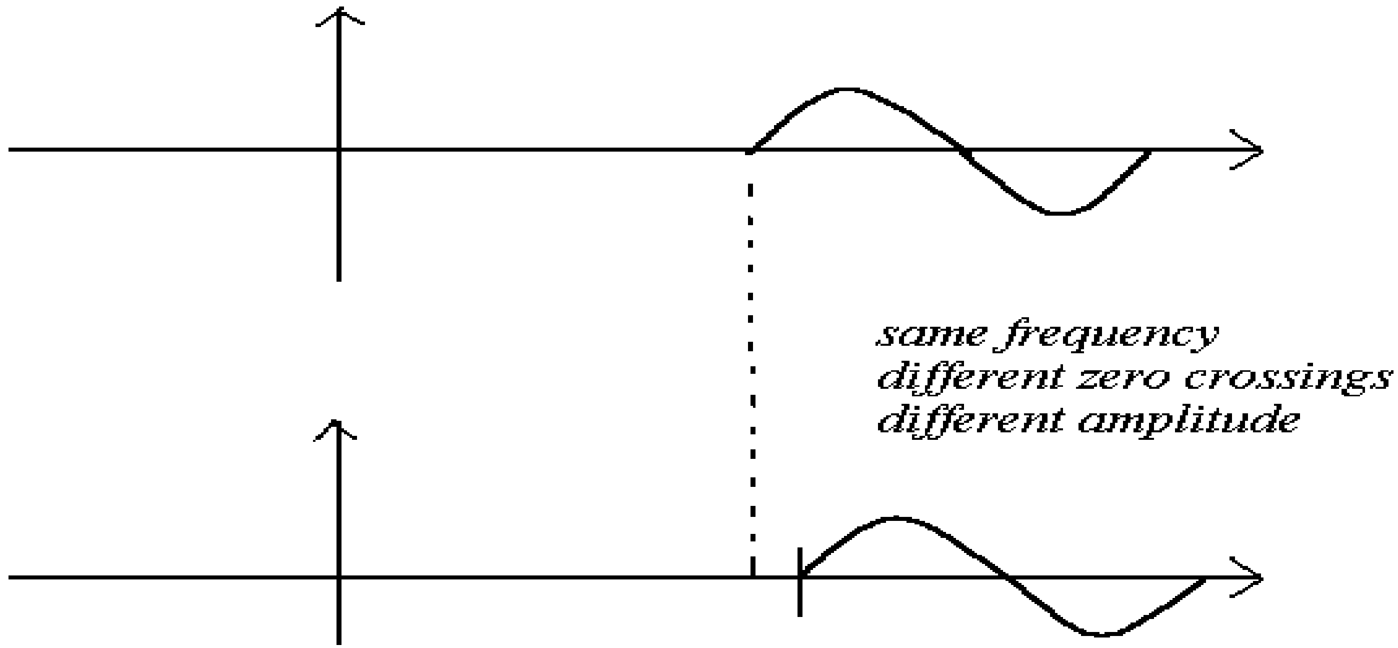
Continuous system: transfer function *H(s)* >>> frequency response $H(j\omega) = A(\omega)\, e^{j\varphi(\omega)}$; Physically, if a sinusoid at frequency ω_0 is applied to a stable, linear, time-invariant system, the response is a transient plus sinusoidal steady state at the same frequency, ω_0 , as the input. The steady-state response to a unit-amplitude sinusoidal signal has amplitude $A(\omega_0)$ and phase $\varphi(\omega_0)$, related to the input signal.



$$y(t) = y_t(t) + y_s(t)$$

$$u(t) = \bar{u} \cos \omega_0 t$$

$$y_s(t) = \bar{y} \cos(\omega_0 t + \varphi)$$



$$\frac{\bar{y}}{\bar{u}} = \left| H(jw_0) \right| \qquad \varphi = \underline{\angle H(jw_0)}$$

Discrete system is almost same: from transfer function $H(z)$ to frequency response $H(e^{j\omega T}) = A(\omega T)e^{j\varphi(\omega T)}$

Let the system be

$$U(z) = H(z)E(z)$$

If $e(k) = \cos(\omega_0 Tk)1(k)$, $u(k)$ has the steady state response:

$$u_{ss}(kT) = A\cos(\omega_0 Tk + \varphi)$$

which, of course, are samples at kT instants on a sinusoid of amplitude A, phase φ, and frequency ω_0.

Caution: it should be noticed here that although a sinusoid of frequency ω_0 could be passed through the samples of $u_{ss}(kT)$, there are other continuous sinusoids of frequency $\omega_0 + l2\pi / T$ for integer l which also pass through these points.

Convention: define the discrete frequency response of a transfer function $H(z)$ to sinusoids of frequency ω_0 as $H\left(e^{j\omega_0 T}\right)$

$$H(s) \to H(j\omega)$$

$$H(z) \xrightarrow{z=e^{Ts},\, s=j\omega} H(e^{j\omega T})$$