Tutorial 2---Questions

1. Consider a discrete system

$$y(k) = 1.3y(k-1) - 0.4y(k-2) + 2u(k).$$

(a) Find its transfer function.

(b) Check its stability.

(c) What is the steady state of the output unit step response?

2. The discrete transfer function of a process is given by

$$\frac{Y(z)}{U(z)} = \frac{5(z+0.6)}{z^2 - z + 0.41}$$

(a) Calculate the response $y(k)$ to a unit step change in $u(k)$.

(b) What is the steady state value of $y(k)$?

3. Calculate and sketch the unit step responses of the discrete transfer functions shown below for the first 6 sampling instants. What conclusions can you make about the effect of the pole-zero locations?

(a) $\dfrac{1}{1+0.7z^{-1}}$

(b) $\dfrac{1}{1-0.7z^{-1}}$

(c) $\dfrac{1-0.5z^{-1}}{\left(1+0.7z^{-1}\right)\left(1-0.3z^{-1}\right)}$

4. The open loop transfer function of a discrete time system is given by

$$G(z)=\frac{1}{z^{2}-\frac{3}{4}z+\frac{1}{8}}$$

Find the output time response when the input is the unit pulse.

5. Consider the difference equation given by

$$y(k+1)+0.5y(k)=x(k)$$

Obtain the response $y(k)$ when the input $x(k)$ is a unit step sequence.

6. Check the following for stability

   (a) $u(k)=0.5u(k-1)-0.3u(k-2)$

   (b) $u(k)=1.6u(k-1)-u(k-2)$

   (c) $u(k)=0.8u(k-1)+0.4u(k-2)$

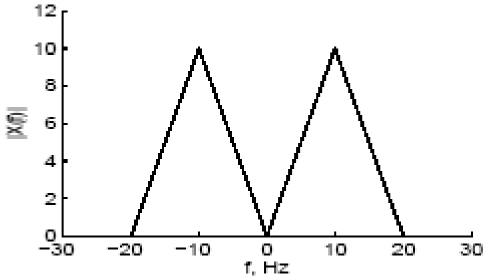
7. If the pole of a digital system is at z=-0.8, what is its equivalent pole in the s-domain, assuming a sampling frequency of 1 Hz?

8. An amplitude modulated signal

$$f(t)=\sin(4\omega_0 t)\cos(2\omega_0 t)$$

is sampled with $T = \pi / 3\omega_0$ seconds. Sketch its two-sided spectrum of f(t). List all frequency components in the sampled signal.

9. Consider the spectra of a continuous signal below. Plot the spectrum of the sampled signal for the three sampling frequencies $f_s = 30, 40$ and 60 Hz. Label all amplitudes and frequencies of interest. Which of the three sampling frequencies are acceptable?



10. A signal given by

$$f(t) = 2\sin 2\pi t \cos 4\pi t - \sin 6\pi t$$

is sampled at 2 Hz. What are the possible frequency components in the sampled signal?

11. A continuous time signal has a period of 2.5 sec. It is sampled at a period of 2 sec. What is the period of the sampled signal?

12. (a) A continuous signal given by

$$f(t) = 2\sin 2t - 3\sin 6t$$

is sampled at $\omega_s = 10.$ Give 5 frequency components in the sampled signal.

(b) If the poles of a continuous system are at $s = -2 \pm j1$, what are their equivalent poles in the z-domain, assuming a sampling frequency of 1 Hz?

13. (a) What is the zero-order hold and why is it the zero-order holder preferred in practice?

(b) Find the zero-order hold discrete equivalent to

$$G(s) = \frac{s}{s^2 + a^2}.$$

Tutorial 2: Question & Solution

1. Consider a discrete system,

$$y(k) = 1.3y(k-1) - 0.4y(k-2) + 2u(k).$$

(a) Find its transfer function.

(b) Check its stability.

(c) What is the steady state of the output unit step response?

Solution:

(a) Z-transform

$$Y(z) = 1.3z^{-1}Y(z) - 0.4z^{-2}Y(z) + 2U(z)$$

$$\frac{Y(z)}{U(z)} = \frac{2}{1-1.3z^{-1}+0.4z^{-2}} = \frac{2z^2}{z^2-1.3z+0.4}$$

(b) The poles are at

$$z_{1,2} = \frac{1}{2}\left[1.3 \pm \sqrt{1.3^2 - 4\times 0.4}\right] = 0.5, 0.8$$

$\left|z_{1,2}\right| < 1$, the system is stable.

(c) Use the final value theorem

$$y(\infty) = \lim_{z\to 1}(z-1)Y(z) = \lim_{z\to 1}\frac{2}{z^2 - 1.3z + 0.4} = 20$$

2. The discrete transfer function of a process is given by

$$\frac{Y(z)}{U(z)} = \frac{5(z+0.6)}{z^2 - z + 0.41}$$

(a) Calculate the response $y(k)$ to a unit step change in $u(k)$.

(b) What is the steady state value of $y(k)$?

Solution:

(a) Unit step: $U(z)=\dfrac{z}{z-1}$. Therefore

$$Y(z)=\frac{5(z+0.6)}{z^2-z+0.41}\frac{z}{z-1}$$

Partial factorizing and taking inverse Z-transform,

$$\begin{aligned}
Y(z) &= \frac{5z-19.5z^2}{z^2-z+0.41}+\frac{19.5z}{z-1} \\
&= \frac{11.426e^{j2.594}z}{z\text{ - }0.64e^{j0.675}}+\frac{11.426e^{-j2.594}z}{z\text{ - }0.64e^{-j0.675}}+\frac{19.5z}{z-1}
\end{aligned}$$

$$\begin{aligned}
y(k) &= 11.426e^{j2.594}\left(0.64e^{j0.675}\right)^k\cdot 1(\mathrm{k}) \\
&\quad + 11.426e^{-j2.594}\left(0.64e^{-j0.675}\right)^k\cdot 1(\mathrm{k})+\ 19.5\cdot 1(k) \\
&= 11.426(0.64)^k\left(e^{j2.594+j0.675k}+e^{-j2.594-j0.675k}\right)\cdot 1(\mathrm{k})+\ 19.5\cdot 1(k) \\
&= 22.85(0.64)^k\cos(0.675k+2.594)\cdot 1(k)\ +\ 19.5\cdot 1(k)
\end{aligned}$$

(b) Steady state value of $y(k)$ is therefore 19.5. One may apply final value theorem to verify this result.

3. Calculate and sketch the unit step responses of the discrete transfer functions shown below for the first 6 sampling instants. What conclusions can you make about the effect of the pole-zero locations?

(a) $\dfrac{1}{1+0.7z^{-1}}$

(b) $\dfrac{1}{1-0.7z^{-1}}$

(c) $\dfrac{1-0.5z^{-1}}{\left(1+0.7z^{-1}\right)\left(1-0.3z^{-1}\right)}$

Solution:

(a)

$$G(z)=\frac{1}{1+0.7z^{-1}}=\frac{z}{z+0.7}$$

The system is stable with its pole at $z=-0.7$. The output is given by

Question

$$\begin{aligned} Y(z) &= \frac{z}{z+0.7}\frac{z}{z-1} \\ &= \frac{10}{17}\frac{z}{z-1} + \frac{7}{17}\frac{z}{z+0.7} \end{aligned}$$

$$y(k) = \frac{10}{17}\cdot 1(k) + \frac{7}{17}(-0.7)^k \cdot 1(k)$$

Calculating the first 6 samples:

$$\{y(k)\} = \{1, 0.3, 0.79, 0.45, 0.69, 0.52, \cdots, 10/17\}$$

Figure 1 shows the response is oscillatory. This is the characteristic of a system with some stable pole with a negative real part in the z-domain. The steady state value is 10/17.

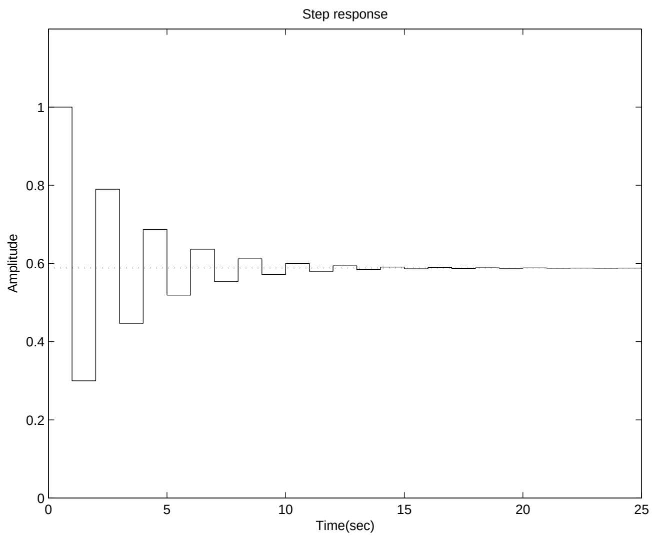


Figure 1: Step response for pole at z=-0.7

(b)

$$G(z)=\frac{1}{1-0.7z^{-1}}=\frac{z}{z-0.7}$$

$$Y(z)=\frac{z}{z-0.7}\frac{z}{z-1}$$

$$y(k)=\frac{10}{3}\cdot 1(k)-\frac{7}{3}(0.7)^{k}\cdot 1(k)$$

Calculating the first few samples:

$$\{y(k)\} = \{1, 1.7, 2.19, 2.53, 2.77, 2.94 \cdots, 10/3\}$$

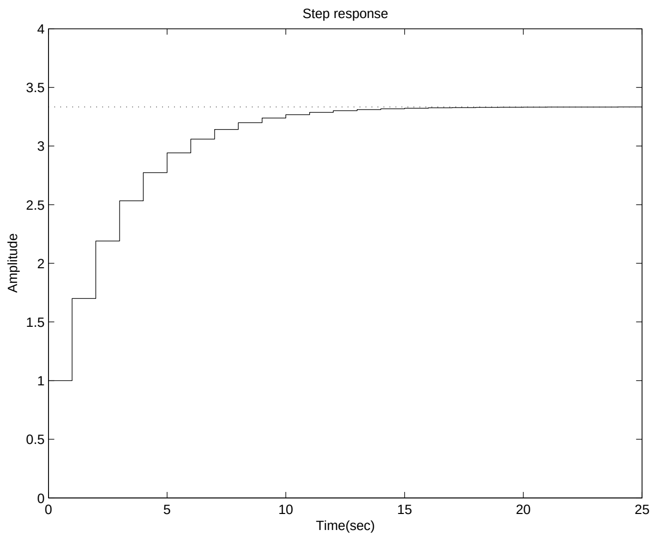


Figure 2: Step response for pole at z=0.7

Figure 2 shows the response is exponentially growing but bounded. This is the characteristic of a system with its all poles with a positive real part in the z-domain. The steady state value is 10/3.

(c)

Question

$$G(z)=\frac{1-0.5z^{-1}}{\left(1+0.7z^{-1}\right)\left(1-0.3z^{-1}\right)}$$

$$\begin{aligned}Y(z)\quad &=\frac{z(z-0.5)}{(z+0.7)(z-0.3)}\frac{z}{z-1}\\ &=0.42\frac{z}{z-1}+0.09\frac{z}{z-0.3}+0.49\frac{z}{z+0.7}\end{aligned}$$

$$\{y(k)\}=\{1,0.104,0.6682,0.254,0.538,0.338\cdots,0.42\}$$

Figure shows that the response is oscillatory but bounded since the poles are at $z=-0.7$ and $z=0.3$. The steady state value is 0.42.

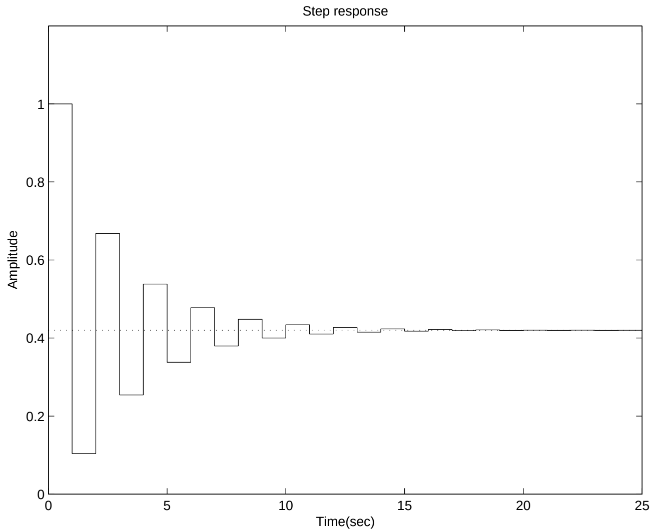


Figure 3: Step response for pole at z=-0.7,0.3

4. The open loop transfer function of a discrete time system is given by

$$G(z) = \frac{1}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

Find the output time response when the input is the unit pulse.

Solution:

When the input is the unit pulse, the output of the system is simply the inverse z-transform of $G(z)$. Thus,

$$
\begin{aligned}
g(k) &= Z^{-1}\left\{\frac{1}{z^2-\frac{3}{4}z+\frac{1}{8}}\right\} \\
&= Z^{-1}\left\{\frac{1}{(z-\frac{1}{2})(z-\frac{1}{4})}\right\} \\
&= Z^{-1}\left\{\frac{4}{z-\frac{1}{2}}-\frac{4}{z-\frac{1}{4}}\right\} \\
&= 4\left(\frac{1}{2}\right)^{k-1} 1(k-1) - 4\left(\frac{1}{4}\right)^{k-1} 1(k-1)
\end{aligned}
$$

5. Consider the difference equation given by

Question

$$y(k+1)+0.5y(k)=x(k)$$

Obtain the response $y(k)$ when the input $x(k)$ is a unit step sequence.

Solution:

Taking z-transform on both sides,

$$zY(z)+0.5Y(z)=U(z)$$

When the input $u(k)$ is a unit step sequence,

$$U(z)=\frac{z}{z-1}$$

Thus

$$
\begin{aligned}
Y(z) &= \frac{z}{(z-1)(z+0.5)} \\
&= z\left\{\frac{1}{(z-1)(z+0.5)}\right\} \\
&= z\left[\frac{2}{3}\frac{1}{z-1} - \frac{2}{3}\frac{1}{z+0.5}\right]
\end{aligned}
$$

This leads to

$$y(k) = \frac{2}{3}1(k) - \frac{2}{3}(-0.5)^k 1(k).$$

6. Check the following for stability

(a) $u(k) = 0.5u(k-1) - 0.3u(k-2)$

(b) $u(k) = 1.6u(k-1) - u(k-2)$

(c) $u(k) = 0.8u(k-1) + 0.4u(k-2)$

Solution:

(a) The characteristic equation (CE) is

$$z^2 - 0.5z + 0.3 = 0$$

The roots of it are

$$z_1 = 0.25 + j0.4873$$
$$z_2 = 0.25 - j0.4873$$

They are inside the unit circle. The system is stable.

(b) The CE is

$$z^2 - 1.6z + 1 = 0$$

The roots are

$$z_1 = 0.8 + j0.6$$
$$z_2 = 0.8 - j0.6$$

one sees $|z_1| = 1$ is on the unit circle. The system is unstable.

(c) The CE is

$$z^2 - 0.8z - 0.4 = 0$$

The roots are:

$$z_1 = 1.1483$$
$$z_2 = -0.3483$$

$|z_1| > 1$ is outside the unit circle. The system is unstable.

7. If the pole of a digital system is at z=-0.8, what is its equivalent pole in the s-domain, assuming a sampling frequency of 1 Hz?

Solution:

The z-domain poles map through the transformation $z = e^{sT}$ or $s = \frac{1}{T}\ln z$. Hence in this case, the equivalent s-poles are obtained from

$$\begin{aligned} s &= \frac{1}{T}\ln z = \frac{1}{T}\ln\left(|z| e^{j\angle z}\right) \\ &= \ln|z| + j\angle z \quad \text{since } T = 1 \\ &= \ln 0.8 \pm j\pi \quad \text{since } \angle z = \pm\pi \\ &= -0.223 \pm \mathrm{j}\pi \end{aligned}$$

8. An amplitude modulated signal

$$f(t) = \sin(4\omega_0 t)\cos(2\omega_0 t)$$

is sampled with $T = \pi / 3\omega_0$ seconds. Sketch its two-sided spectrum of f(t). List all frequency components in the sampled signal.

Solution:

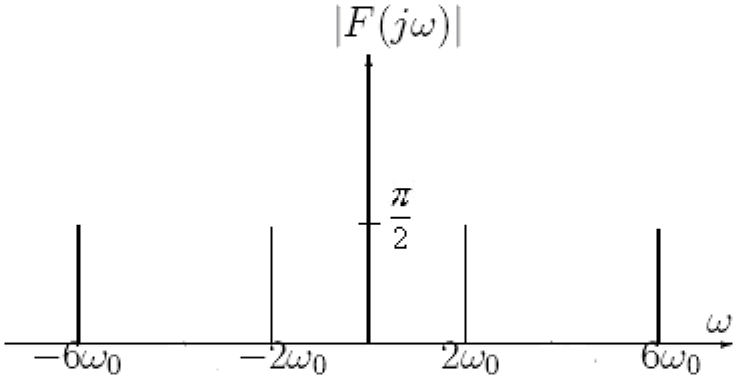
The sampling frequency is

$$\omega_s = \frac{2\pi}{T} = 6\omega_0 \ rad/s\,.$$

The original signal can be converted into the following form

$$f(t) = \sin(4\omega_0 t)\cos(2\omega_0 t)$$
$$= \frac{1}{2}\left[\sin(6\omega_0 t) + \sin(2\omega_0 t)\right]$$

There are 2 frequency components in $f(t)$: $2\omega_0$ and $6\omega_0$. (Hence in order to avoid aliasing, sampling frequency must be at least $12\omega_0$. Since in this case, we are sampling at only $6\omega_0$, we expect to see some aliasing.) The two-sided spectrum of f(t) is as follows:



$$F^*(s) = \frac{1}{T}\sum_{n=-\infty}^{n=\infty} F(s - jn\omega_s)$$

Let ω_c be the frequency components in $f(t)$ and ω_d the frequency components in $f^*(t)$. Then,

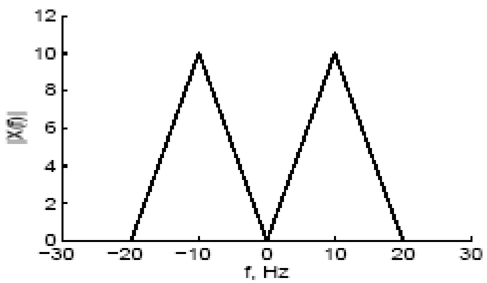
$$\omega_d = \omega_c \pm n\omega_s, \omega_c = \pm 2\omega_0, \pm 6\omega_0, \omega_s = 6\omega_0,$$

$$n = 0, \quad \omega_d = \omega_c = -6\omega_0, -2\omega_0, 2\omega_0, 6\omega_0$$

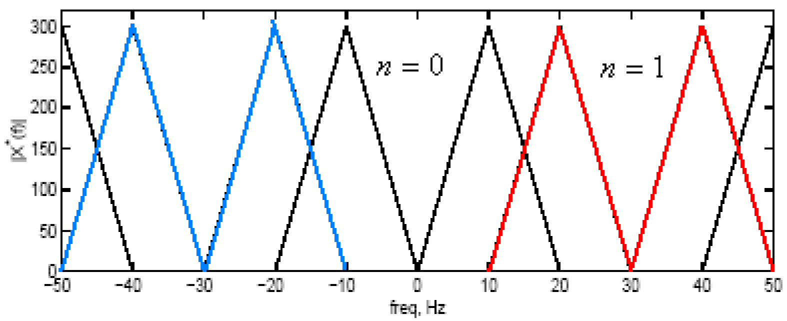
$$n = 1, \quad \omega_d = \omega_c + \omega_s = 0, 4\omega_0, 8\omega_0, 12\omega_0 \qquad \ldots$$

$$n = 2, \quad \omega_d = \omega_c + 2\omega_s = 6\omega_0, 10\omega_0, 14\omega_0, 18\omega_0$$

9. Consider the spectra of a continuous signal below. Plot the spectrum of the sampled signal for the three sampling frequencies $f_s = 30, 40$ and 60 Hz. Label all amplitudes and frequencies of interest. Which of the three sampling frequencies are acceptable?

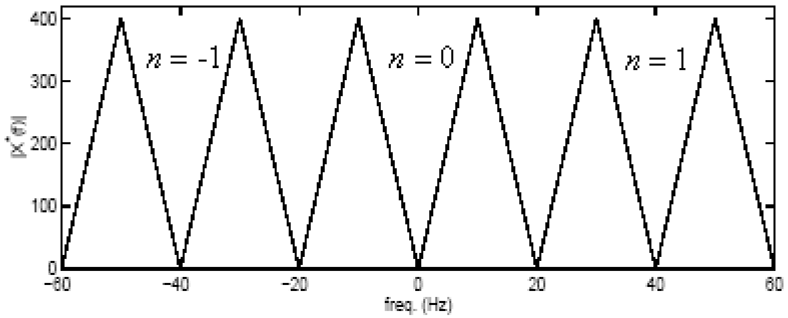


Solution:

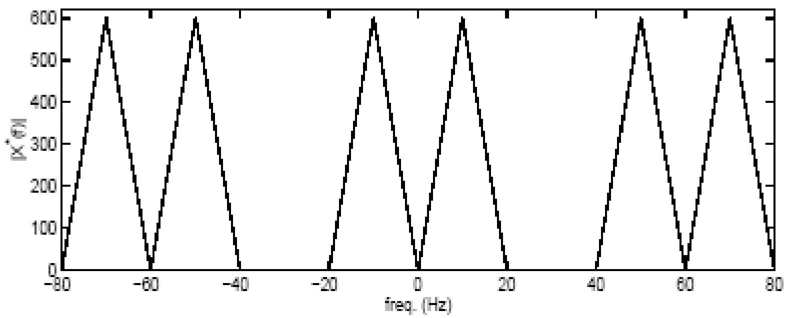


EE3304 Digital Control System Tutorial 2
Question

Sampling frequency of 30 Hz



Sampling frequency of 40 Hz



Sampling frequency of 60 Hz

The plots have peak amplitudes of $10 \times {}^{1}\!/_{T_s}$ where $T_s = {}^{1}\!/_{f_s}$ since the original spectrum has a peak amplitude of 10. The acceptable frequencies are $40Hz$ and $60Hz$ because aliasing do not occur with these sampling frequencies.

10. A signal given by

$$f(t) = 2\sin 2\pi t \cos 4\pi t - \sin 6\pi t$$

is sampled at 2 Hz. What are the possible frequency components in the sampled signal?

Solution:

Rewriting $f(t)$ as

$$\begin{aligned} f(t) &= \sin(6\pi t) - \sin(2\pi t) - \sin(6\pi t) \\ &= -\sin(2\pi t) \end{aligned}$$

Hence $f(t)$ only contains a sinusoidal signal of 1 Hz. After sampling at 2 Hz, the frequency components should be 1 Hz, 3 Hz, 5 Hz. et al.

11. A continuous time signal has a period of 2.5 sec. It is sampled at a period of 2 sec. What is the period of the sampled signal?

Solution:

The continuous time signal has a frequency of $f_c = 1/T_c = 0.4$ Hz. The sampling frequency is $f_s = 1/T_s = 0.5$ Hz. Hence the sampled signal has aliasing frequencies:

$$f_d = f_c \pm n f_s = \pm 0.4 \pm n 0.5 = 0.4, 0.9, \ldots; (-0.4 + 0.5) = 0.1, 0.6, \ldots$$

The smallest frequency is 0.1 Hz which has a period of 10 sec. 10 sec is the period of the sampled signal . Why?

Consider $\sin(2\pi t)$ and $\sin(\pi t) = \sin(\frac{2\pi t}{2})$, what is their common period?

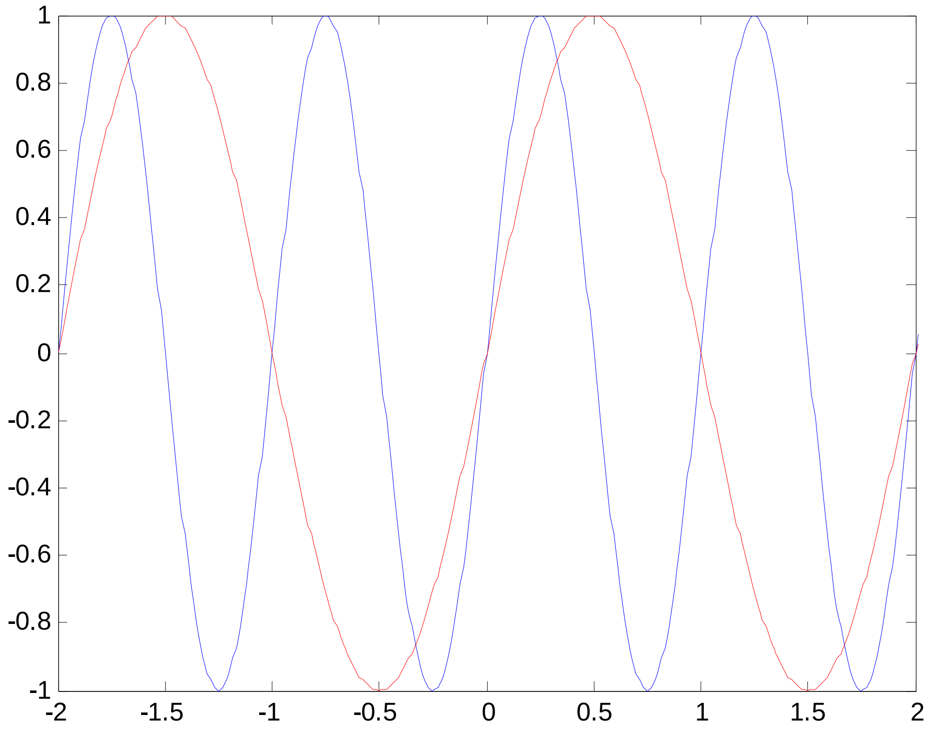
- $\sin(2\pi t)$ has its period: T=1 and frequency: f=1 Hz, see the blue curve in Figure below
- $\sin(\pi t) = \sin(\frac{2\pi t}{2})$ has T=2 and f=0.5, see the red curve in Figure below

One notes that

- $\sin(2\pi t)$ is of a higher frequency or faster-changing than $\sin(\pi t)$
- since $\sin(2\pi(t+2)) = \sin(2\pi t + 4\pi) = \sin(2\pi t)$, T=2 is also a period of $\sin(2\pi t)$. So T=2 is the common period for both $\sin(2\pi t)$ and $\sin(\pi t)$, and the corresponding frequency is 0.5 Hz. So, $\sin(\pi t)$, the slower function, matters.

One concludes that

- if the period of one sinusoidal function is a multiple of another sinusoidal function, then the common period of two is the larger of two
- Equivalently, if the frequency of one sinusoidal function is a multiple of another sinusoidal function, then the common frequency of two is the smaller of two.
- This can be extended to three or more sinusoidal functions.



12. (a) A continuous signal given by

$$f(t) = 2\sin 2t - 3\sin 6t$$

is sampled at $\omega_s = 10$. Give 5 frequency components in the sampled signal.

(b) If the poles of a continuous system are at $s = -2 \pm j1$, what are their equivalent poles in the z-domain, assuming a sampling frequency of 1 Hz?

Solution:

(a) $f(t)$ contains frequency components $\omega_c = 2, 6 rad/s$. After sampling at $\omega_s = 10$, the frequency components should be

$$\omega_d = \omega_c + k\omega_s = 2, 6, 12, 16, 22, \ldots$$

(b) The z-domain poles map through the transformation $z = e^{sT}$. Hence in this case, T=1 and the equivalent z-poles are obtained from

$$z = e^{sT} = e^{-2\pm j1} = e^{-2}e^{\pm j1} = 0.0731 \pm j0.1139$$

13. (a) What is the zero-order hold and why is it the zero-order holder preferred in practice?

(b) Find the zero-order hold discrete equivalent to

$$G(s) = \frac{s}{s^2 + a^2}.$$

Solution:

(a) The samples taken from the continuous signal $f(t)$ are represented by

$$f^*(t) = \sum_{k=-\infty}^{\infty} f(t)\delta(t - kT)$$

The zero-order holder is defined as the means to extrapolate impulses to piecewise constants:

$$f_h(t) = f(kT), \quad kT \le t \le kT + T$$

It is preferred as it is simple, realizable, has reasonable performance.

(b)

$$G(z) = \left(1 - z^{-1}\right) Z\left\{\frac{G(s)}{s}\right\}$$

$$\frac{G(s)}{s} = \frac{1}{s^2 + a^2} = \frac{1}{a}\frac{a}{s^2 + a^2}$$

$$\frac{a}{s^2 + a^2} \longleftrightarrow \frac{z \sin aT}{z^2 - (2\cos aT)z + 1}$$

$$G(z) = \frac{1}{a}\frac{(\sin aT)(z-1)}{z^2 - (2\cos aT)z + 1}$$