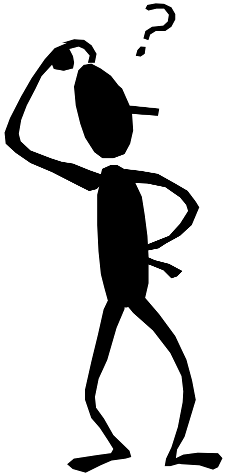
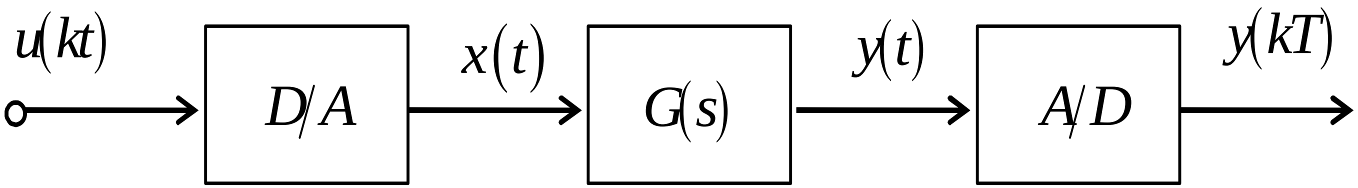


Chapter 5

Sampled Data Control System

5.1 Discrete Models of Sampled Data Systems



The sampled-data system

Task: We here want to compute the discrete transfer function between the samples from digital computer to the D/A converter and samples picked up by the A/D converter. To this end, assume that the D/A is a zero-order holder (ZOH).

Method: Apply the unit pulse input,

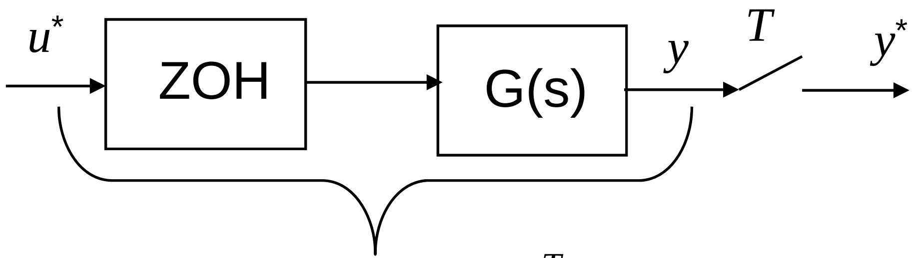
$$u(k) = \begin{cases} 1, & \text{for } k = 0, \\ 0, & \text{for } k \neq 0. \end{cases}$$

Then the corresponding ZOH output is

$$1(t) - 1(t - T).$$

The Laplace transform of the plant output $Y(s)$ is given by

$$Y(s) = G(s) L\left\{1(t) - 1(t - T)\right\} = G(s)\frac{1 - e^{-Ts}}{s}.$$



$$G_{ZOH}(s)G(s) = \frac{1 - e^{-Ts}}{s} G(s)$$

The required discrete transfer function is

$$G(z) = Z\{y(kT)\} = Z\left\{\left\{L^{-1}\{Y(s)\}\right\}\Big|_{t=Tk}\right\} \triangleq Z\big(Y(s)\big).$$

$$G(z) = Z\left\{\frac{1-e^{-Ts}}{s}G(s)\right\}$$

$$= Z\left\{\frac{G(s)}{s}\right\} - Z\left\{\frac{e^{-Ts}G(s)}{s}\right\}$$

$$= Z\left\{\frac{G(s)}{s}\right\} - z^{-1}Z\left\{\frac{G(s)}{s}\right\}$$

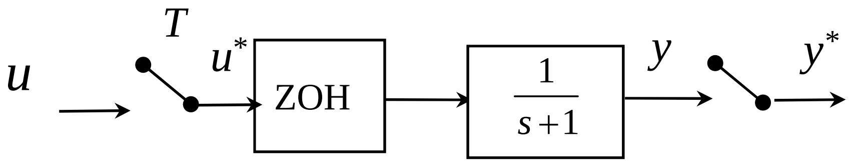
because e^{-Ts} is exactly a delay of one period.

$$G(z) = (1-z^{-1})Z\left\{\frac{G(s)}{s}\right\}.$$

.

Plant discrete
Transfer function
with a ZOH

An Example



$$\frac{Y(z)}{U(z)} = \left(1 - z^{-1}\right) Z\left\{\frac{G(s)}{s}\right\}$$

$$= \left(\frac{z-1}{z}\right) Z\left\{\frac{1}{s(s+1)}\right\}$$

$$= \frac{1-e^{-T}}{z-e^{-T}}$$

$$\frac{z(1-e^{-T})}{(z-1)(z-e^{-T})}$$

5.2 Block-Diagram Analysis

If a continuous signal is $f(t)$, the sampled signal is given by

$$f^*(t) = \sum_{-\infty}^{+\infty} f(t)\delta(t - kT)$$

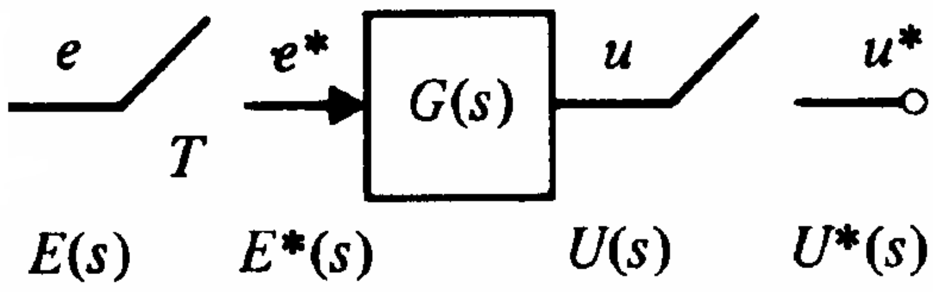
The Laplace transform of $f^*(t)$ is

$$F^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(s - jn\omega_s)$$

where $F(s)$ is the transform of $f(t)$. Note that $F^*(s)$ is periodic but $F(s)$ is usually not.

We assume from now on that the sampling rate is chosen and fixed.

Now consider the system.



$$U(s) = G(s)E^*(s)$$

If the transform of the signal to be sampled is a product of a transform that is already periodic (such as $E^*(s)$) and one that is not, then we have the most important relation:

$$U^*(s) = \left(G(s)E^*(s)\right)^* = G^*(s)E^*(s)$$

- To show

$$\left(G(s)E^*(s)\right)^* = G^*(s)E^*(s)$$

- we see

$$E^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} E(s - jk\omega_s)$$

$$\therefore E^*(s - jn\omega_s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} E(s - jn\omega_s - jk\omega_s)$$

$$\overset{n+k=l}{=\!=\!=\!=} \frac{1}{T}\sum_{l=-\infty}^{\infty} E(s - jl\omega_s)$$

$$= E^*(s)$$

So,

$$\left(G(s)E^*(s)\right)^* = \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s-jk\omega_s)E^*(s-jk\omega_s)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s-jk\omega_s)E^*(s)$$

$$= \frac{1}{T}E^*(s)\sum_{k=-\infty}^{\infty} G(s-jk\omega_s)$$

$$= E^*(s)G^*(s)$$

The Laplace transform of a sampled signal $Y^*(s)$ is related to the corresponding z-transform via

$$Y(z) = Y^*(s)\Big|_{z=e^{sT}}$$

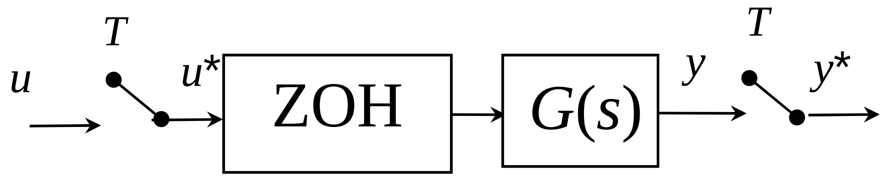
Apply this to

$$U^*(s) = G^*(s)E^*(s)$$

We get

$$U(z) = G(z)E(z)$$

Example 1



$$Y(s) = \frac{1-e^{-Ts}}{s} G(s) U^*(s)$$

$$Y^*(s) = \left(\frac{1-e^{-Ts}}{s} G(s) U^*(s) \right)^* = \left(\frac{1-e^{-Ts}}{s} G(s) \right)^* U^*(s)$$

$$= \left(1-e^{-Ts}\right) \left(\frac{G(s)}{s} \right)^* U^*(s)$$

$$Y(z) = (1-z^{-1}) Z\left(\frac{G(s)}{s} \right) U(z) = G(z) U(z), G(z) = (1-z^{-1}) Z\left(\frac{G(s)}{s} \right)$$

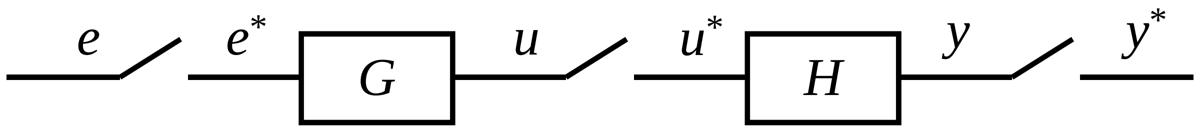
Where

$$\left(1-e^{-Ts}\right)\Big|_{s\to s+j\omega_s} = 1-e^{-T(s+j\omega_s)}$$

$$= 1-e^{-Ts-jT\omega_s} = 1-e^{-Ts}e^{-jT\omega_s}$$

$$= 1-e^{-Ts}e^{-j2\pi} = 1-e^{-Ts},$$

is periodic.



We know

$$U(s)=E^*(s)G(s)\rightarrow U^*(s)=E^*(s)G^*(s)$$

Similarly,

$$Y(s)=H(s)U^*(s)\rightarrow Y^*\left(s\right)=H^*\left(s\right)U^*\left(s\right)$$

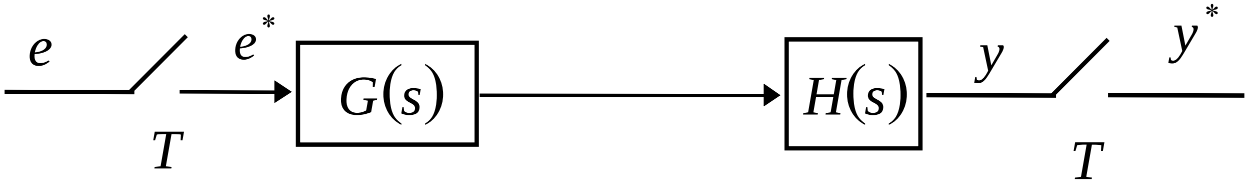
Thus,

$$Y^*\left(s\right)=H^*\left(s\right)G^*\left(s\right)E^*\left(s\right)$$

or

$$Y\left(z\right)=H\left(z\right)G\left(z\right)E\left(z\right)$$

For the system



$$Y(s) = H(s)G(s)E^*(s)$$

$$Y^*(s) = \left(H(s)G(s)\right)^* E^*(s)$$

Usually

$$\left(H(s)G(s)\right)^* \neq H^*(s)G^*(s)$$

Example 2

$$G(s) = \frac{1}{s}, H(s) = \frac{1}{s}$$

$$G(z) = \frac{z}{z-1}, H(z) = \frac{z}{z-1}$$

$$G(z)H(z) = \frac{z^2}{(z-1)^2}$$

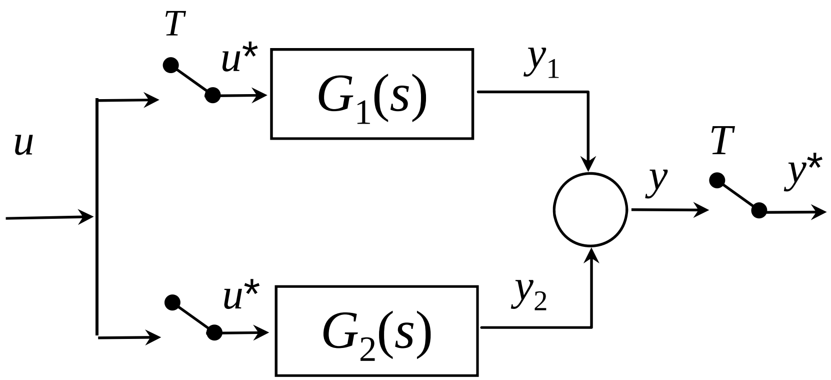
However,

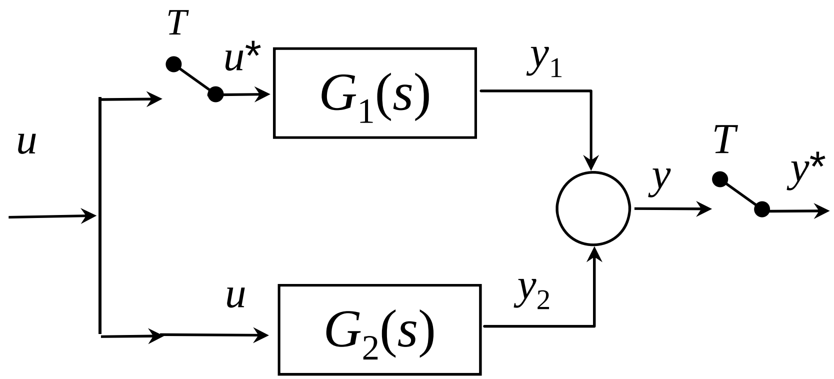
$$G(s)H(s) = \frac{1}{s^2}$$

$$Z\{GH\} = \frac{Tz}{(z-1)^2}$$

Conclusion: Overall transfer function of a system is dependent on position of samplers or ZOH.

Example 3





$$\begin{aligned} Y(s) &= Y_2(s) + Y_1(s) \\ &= G_2(s)U^*(s) + G_1(s)U^*(s) \end{aligned}$$

$$Y^*(s) = G_2{}^*(s)U^*(s) + G_1(s)^*U^*(s)$$

$$Y(z) = G_2(z)U(z) + G_1(z)U(z)$$

$$\frac{Y(z)}{U(z)} = G_2(z) + G_1(z)$$

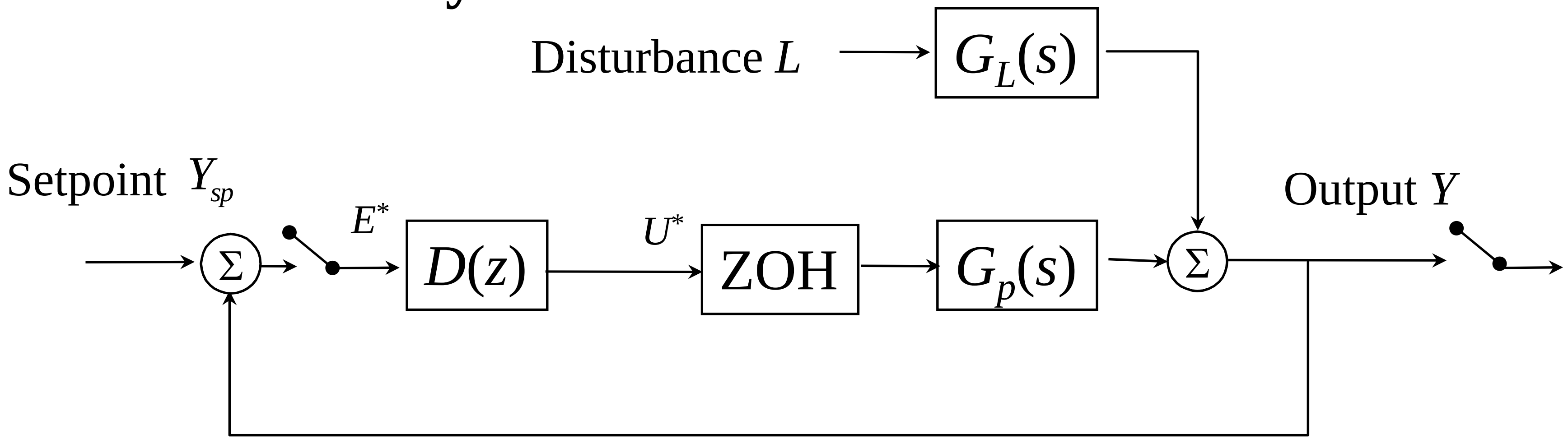
$$\begin{aligned} Y(s) &= Y_2(s) + Y_1(s) \\ &= G_2(s)U(s) + G_1(s)U^*(s) \end{aligned}$$

$$Y^*(s) = \left(G_2(s)U(s)\right)^* + G_1(s)^*U^*(s)$$

$$Y(z) = Z\left(G_2(s)U(s)\right) + G_1(z)U(z)$$

$\boxed{\text{transfer function not separatable!}}$

5.3 Control Systems



$$Y(s) = \frac{1-e^{-Ts}}{s} G_p(s) U^*(s) + G_L(s) L(s)$$

$$U^*(s) = D^*(s) E^*(s)$$

$$E^*(s) = Y_{sp}{}^*(s) - Y^*(s)$$

The Servo Problem

Consider set-point with no load disturbance ($L(s) = 0$)

$$Y(s) = \frac{1-e^{-Ts}}{s} G_p(s) U^*(s), \quad U^*(s) = D^*(s) E^*(s), \quad E^*(s) = Y_{sp}{}^*(s) - Y^*(s)$$

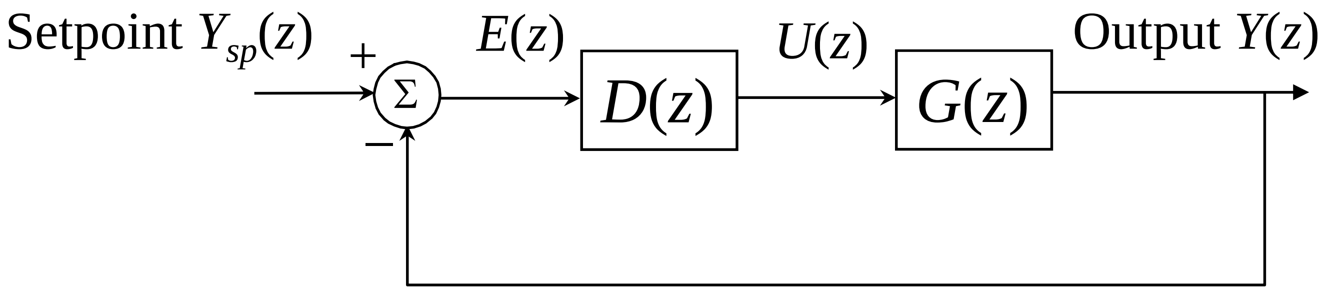
$$Y^*(s) = \left[\frac{1-e^{-Ts}}{s} G_p(s)\right]^* D^*(s) E^*(s) = \left[\frac{1-e^{-Ts}}{s} G_p(s)\right]^* D^*(s) \left[Y_{sp}{}^*(s) - Y^*(s)\right]$$

$$\frac{Y^*(s)}{Y_{sp}{}^*(s)} = \frac{\left[\frac{1-e^{-Ts}}{s} G_p(s)\right]^* D^*(s)}{1 + \left[\frac{1-e^{-Ts}}{s} G_p(s)\right]^* D^*(s)}$$

$$\frac{Y(z)}{Y_{sp}(z)} = \frac{Z\left[\frac{1-e^{-Ts}}{s} G_p(s)\right] D(z)}{1 + Z\left[\frac{1-e^{-Ts}}{s} G_p(s)\right] D(z)}$$

$$H(z) = \frac{G(z) D(z)}{1 + G(z) D(z)}$$

Block diagram for servo problem



block diagram manipulation exactly
as for continuous time systems

For example, let the plant

$$G_p(s) = \frac{a}{s+a}.$$

The plant plus ZOH has the transfer function

$$G(z) = \left(1 - z^{-1}\right) Z\left\{\frac{G(s)}{s}\right\}$$

$$= \left(1 - z^{-1}\right) Z\left\{\frac{1}{s} - \frac{1}{s+a}\right\}$$

$$= \left(1 - z^{-1}\right)\left[\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}\right]$$

If $e^{-aT} = \frac{1}{2}$, this reduces to

$$G(z) = \frac{0.5}{z-0.5}$$

Let the controller be a discrete integrator:

$$u(kT) = u(kT - T) + K_0 e(kT)$$

Then z-transfer function of controller is

$$D(z) = \frac{U(z)}{E(z)} = K_0 \big/ \left(1 - z^{-1}\right) = \frac{K_0 z}{z - 1}$$

Thus, we have

$$H(z) = \frac{G(z)D(z)}{1 + G(z)D(z)} = \frac{\dfrac{0.5}{z-0.5}\dfrac{K_0 z}{z-1}}{1 + \dfrac{0.5}{z-0.5}\dfrac{K_0 z}{z-1}} = \frac{0.5 K_0 z}{z^2 + (0.5K_0 - 1.5)z + 0.5}.$$

The Regulator Problem

Consider load disturbance with no set-point (Y_{sp}= 0)

$$Y(s)=\frac{1-e^{-Ts}}{s}G_p(s)U^*(s)+G_L(s)L(s)$$

$$U^*(s)=D^*(s)E^*(s)=-D^*(s)Y^*(s)$$

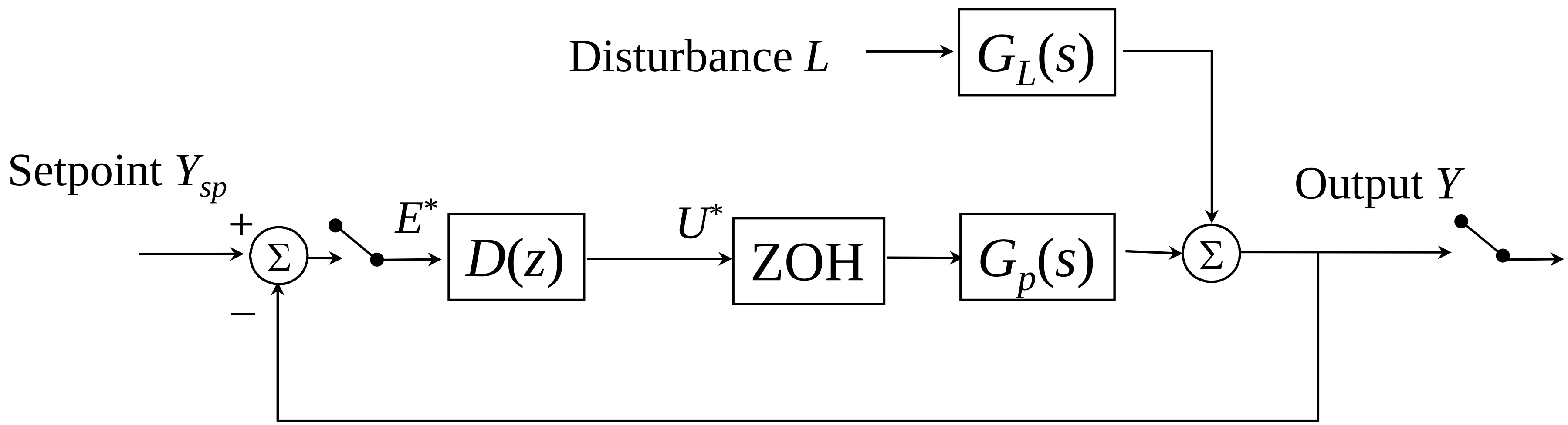
$$Y^*(s)=-\left[\frac{1-e^{-Ts}}{s}G_p(s)\right]^* D^*(s)Y^*(s)+\left[G_L(s)L(s)\right]^*$$

$$Y^*(s)=\frac{\left[G_L(s)L(s)\right]^*}{1+\left[\frac{1-e^{-Ts}}{s}G_p(s)\right]^* D^*(s)}$$

$$Y(z)=\frac{Z\left\{G_L(s)L(s)\right\}}{1+G(z)D(z)}$$

We cannot separate a transfer function from $L(z)$ to $Y(z)$!

Stability of the Closed Loop



Closed-loop is said to be stable if the output sequence, $y(kT)$, is bounded for any bounded input sequences, $y_{sp}(kT)$ and $l(kT)$.

Conditions for Stability

- We have shown that

$$
\begin{aligned}
Y(z) &= \frac{G(z)D(z)}{1+G(z)D(z)} Y_{sp}(z) + \frac{Z\{G_L(s)L(s)\}}{1+G(z)D(z)} \\
&= Y_1(z) + Y_2(z)
\end{aligned}
$$

$$1+G(z)D(z)=0$$

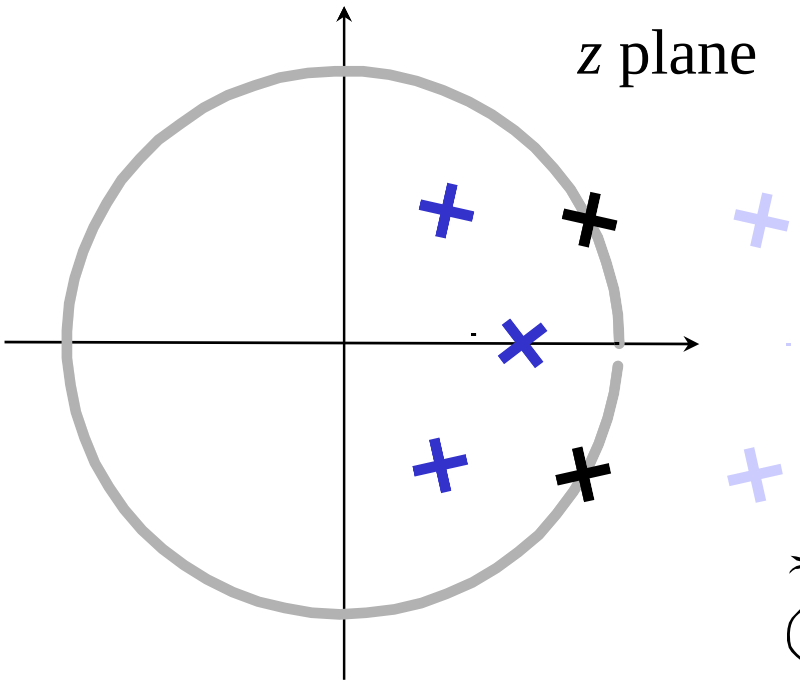
CL characteristic equation should have no roots on or outside the unit circle in the *z* plane for stability

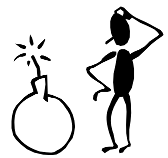
Test for Closed-loop Stability

- Find all the roots of the CL characteristic equation

$$1 + G(z)D(z) = 0$$

- Check if all the roots lie within the unit circle, i.e., have magnitudes less than one
- Yes: stable; otherwise, unstable





An early example has CL transfer function

$$H(z) = \frac{0.5K_0 z}{z^2 + (0.5K_0 - 1.5)z + 0.5}.$$

Its CL characteristic polynomial is

$$z^2 + (0.5K_0 - 1.5)z + 0.5$$

Let K_0 = 1. There are two roots at

$$0.5 \pm 0.5i \quad \text{stable}$$

Let K_0 = –1. Then, two roots are at

$$1 \pm \frac{\sqrt{2}}{2} i \quad \text{unstable}$$

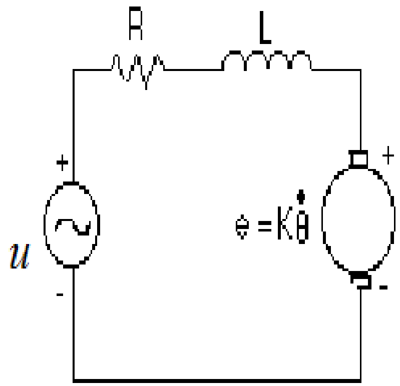
Design Approaches to Digital Systems

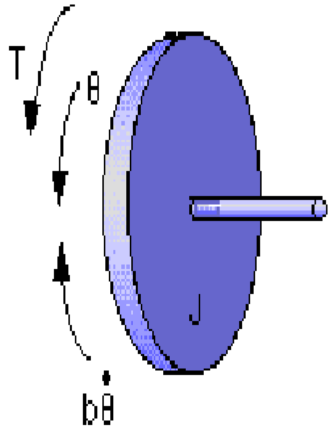
- 1^{st} approach — Based on continuous time plant model, $G(s)$, obtain the controller $D(s)$. Then discretize $D(s)$ to $D(z)$ for implementation;

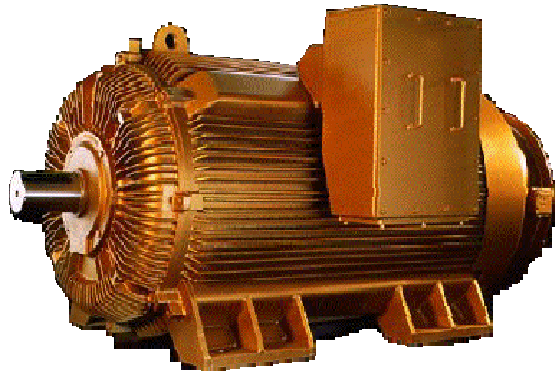
- 2^{nd} approach — Discretize plant, $G(s)$ to $G(z)$. Then design the controller $D(z)$ in discrete time. Proceed to implementation.

Whichever approach is used, discretization is involved. Are there other ways of discretization that are easier or less tedious mathematically? Is exact z-transform necessary all the time?

DC Motor Speed Control







1. The plant model

The model of the DC motor is

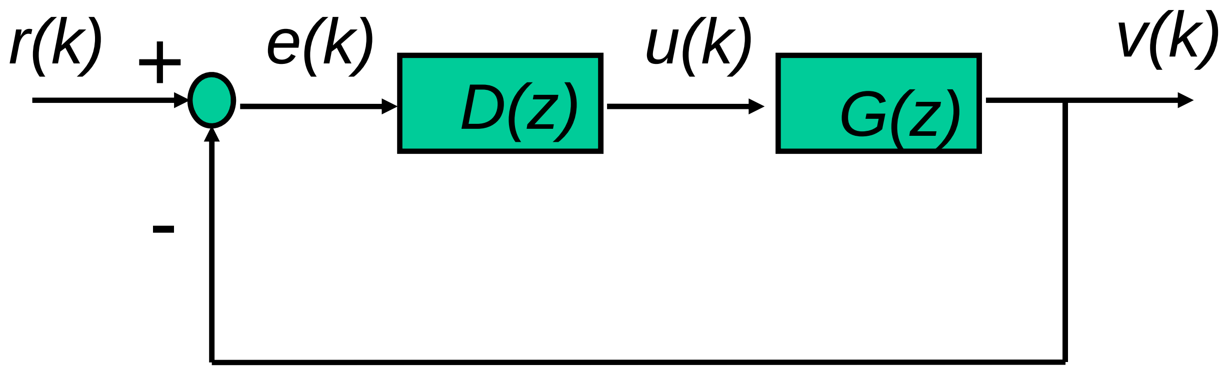
$$\frac{v(s)}{u(s)} = \frac{K}{(Js+b)(Ls+R)+K^2},$$

*electrical resistance *(R) = 1 ohm,*
*electrical inductance *(L) = 0.5 H,*
*electromotive force constant *(K_e=K_t) = 0.01 Nm/Amp,*
*moment of inertia of the rotor *(J) = 0.01 kg*m^2/s^2,*
*damping ratio of the mechanical system *(b) = 0.1 Nms,*
*input (*u*): Source Voltage,
*output (v=$\dot{\theta}$): Rotating speed,
*The rotor and shaft are assumed to be rigid.

2. Control specifications

The design requirements for 1 rad/sec step input are:

- Settling time: Less than 5 seconds ,
- Overshoot: Less than 5% ,
- Steady-state error: Less than 1%.



3. ZOH discrete model

Substituting the given parameters, we have

$$G(s) = \frac{v(s)}{u(s)} = \frac{0.01}{0.005s^2 + 0.06s + 0.1001}.$$

Choose the sampling time as 0.12s. The plant plus zero-order holder has the discrete transfer function:

$$G(z) = Z\left\{\frac{1-e^{-Ts}}{s}G(s)\right\}\bigg|_{T=0.12} = \frac{0.0092z + 0.0057}{z^2 - 1.0877z + 0.2369}.$$

4. Controller design

Recall that continuous system open loop needs an integrator (pole at origin) for zero steady state error for step response. The discrete case needs a pole at *z=1*. Note that the open loop *G(z)* has poles at *0.7865* and *0.3012* (stable poles). Using the stable pole-zero cancellation, we choose the controller as

$$D(z) = \frac{k\left(z^2 - 1.0877z + 0.2369\right)}{z-1}.$$

Then the characteristic equation for the closed loop is

$$1 + \frac{k\left(0.0092z + 0.0057\right)}{z-1} = 0.$$

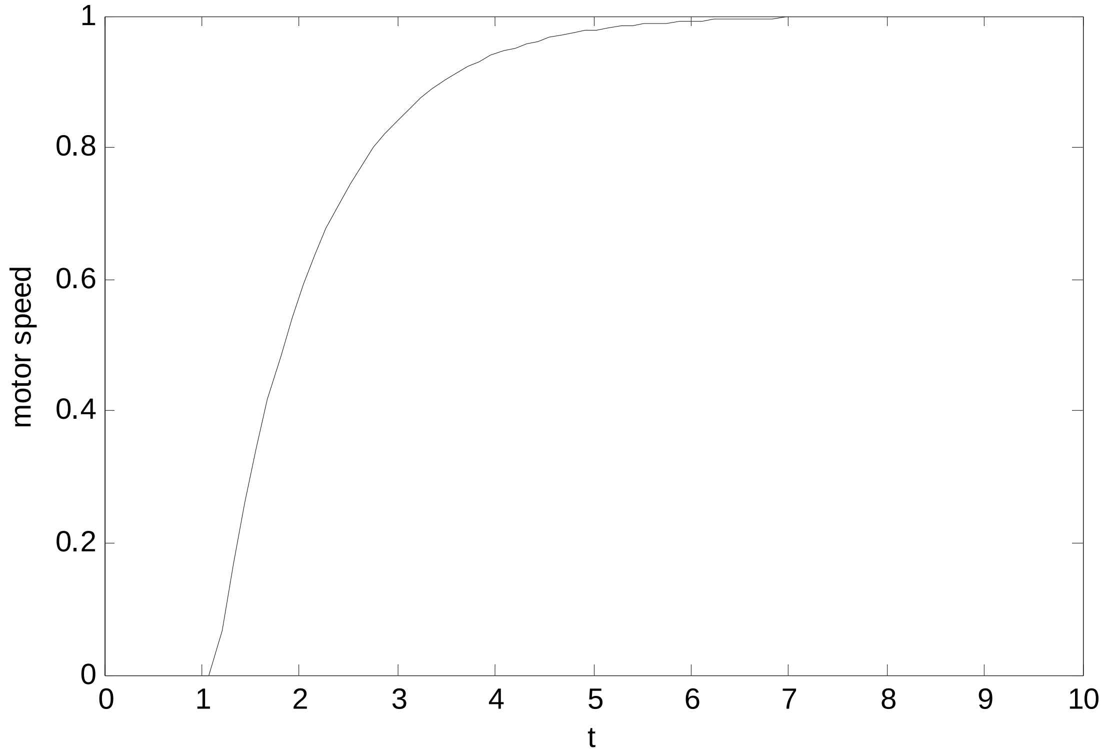
For a continuous first order system with pole at a, settling time is given by

$$t_{settling} = -\frac{4}{a}.$$

According to the specification, settling time should be less than 5 s.

This requires the pole to be located $s < -0.8$. Via $z = e^{sT}, = e^{(-0.8 \text{ x } 0.12)}$, the discrete pole should be located at *z<0.908*. Set the pole at *z=0.9*. Then from the characteristic equation, we have *k=7.153*.

5. Closed-loop performance



The step response of the control system is given in the above figure. The plot shows that the settling time is less than 5 seconds and the percent overshoot is 0. In addition, the steady state error is zero. Therefore this response satisfies all of the design requirements.