Tutorial 2

EE3731C – Signal Processing Methods

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Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$$

- To find the coefficients, multiply both sides by *cosmx*, or *sinmx*, and integrating it over [-π, π]

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx$$

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

Hint: Orthogonality relations of sine and cosine functions

Fourier Transform

- For a continuous function $f(x)$ of a single variable x representing time (or distance), its Fourier Transform is
  - $F(u)=\int_{-\infty}^{\infty} f(x)e^{-i2\pi ux}dx$
    - u: temporal (spatial) frequency

- In general, $F(u)$ is a complex number even if the original data is real.

- What is the relationship between the Fourier series and the Fourier transform?

Solution: both the Fourier series and the Fourier transform show the frequency spectra of signal. The Fourier series is useful for periodic signals. The Fourier transform can provide the frequency spectrum of either periodic or non-periodic signals.

Properties of Fourier Transform

Assume $FT\big(f(x)\big) = F(u), FT\big(g(x)\big) = G(u)$

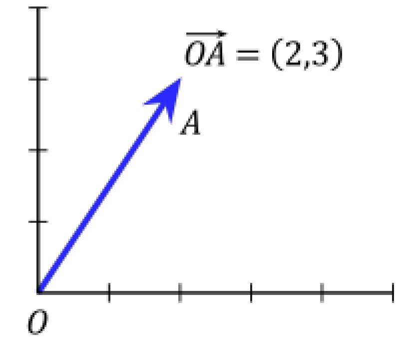
- Linearity property
  - $FT(af(x) + bg(x)) = aF(u) + bG(u)$
- Shift property
  - $FT\big(f(x-a)\big) = e^{-i2\pi ua}F(u)$
- Scaling property
  - $FT\big(f(cx)\big) = \frac{F(\frac{u}{c})}{|c|}$
- Convolution property
  - $FT(f(x) * g(x)) = F(u)G(u)$

- What does the scaling property tell you about the relationship between the time rate-of-change in a signal and its frequency spectrum? How does this relate to system requirements?

Solution: the time rate of change in a signal and its frequency bandwidth are directly proportional. Systems that must operate on rapidly changing signals generally require wide bandwidth capability.

It is often desirable to design signals that are both narrow in time and narrow in frequency, so we need to make trade-offs between the time domain and frequency domain.

Vectors vs. Points



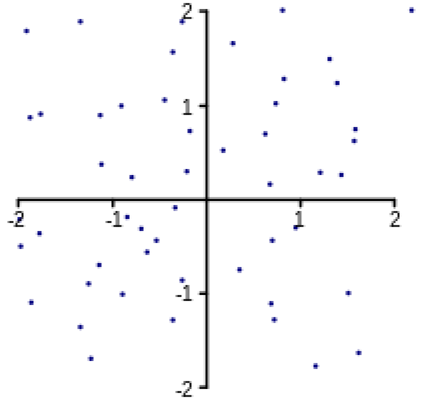
- Vector
  - Has a magnitude and direction.
  - Fundamental in physical science: an example is velocity, the magnitude of which is speed.
  - The coordinate representation of vectors allows the algebraic features of vectors to be expressed in a convenient numerical fashion.

$$\mathbf{v} + \mathbf{w} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

$$\mathbf{v} \cdot \mathbf{w} = (x_1, x_2) \cdot (y_1, y_2) = x_1 \cdot y_1 + x_2 \cdot y_2$$

Vectors vs. Points

- Point
  - A primitive notion upon which other concepts may be defined.
  - In geometry, points are zero-dimensional; i.e., they do not have volume, area, length, or any other higher-dimensional analogue.



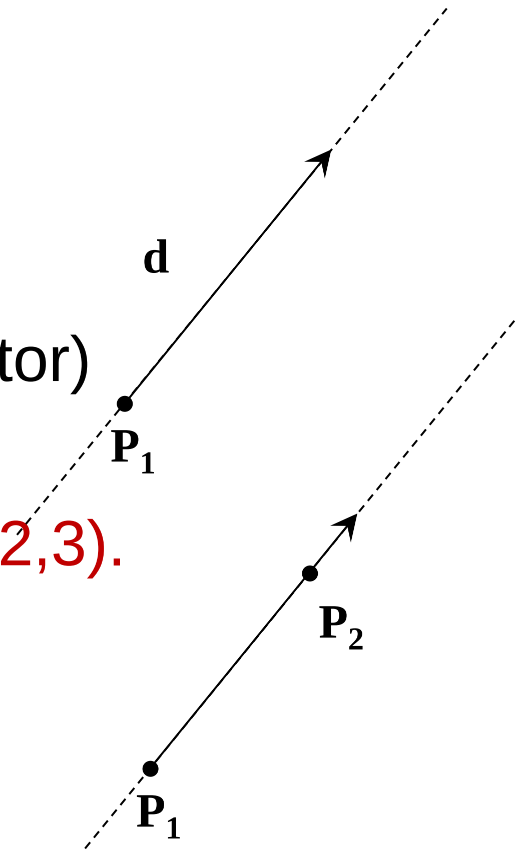
Lines

- Line: y = mx + a
- Line: sum of a point and a vector

  $\mathbf{P} = \mathbf{P_1} + \alpha\mathbf{d}$ (where **d** is a column vector)

  $(\mathbf{P} - \mathbf{P_1}) = \alpha\mathbf{d}$

  Example with Matlab: $\mathbf{P_1} = (1,1)$, $\mathbf{d} = (2,3)$.



- Line: <u>Affine sum*</u> of two points

  $\mathbf{P} = \alpha_1\mathbf{P_1} + \alpha_2\mathbf{P_2}$, where $\alpha_1 + \alpha_2 = 1$

*An affine sum is a linear combination in which the sum of the coefficients is 1.

Line Segment: For $0 < \alpha_1, \alpha_2 < 1$, **P** lies between $\mathbf{P_1}$ and $\mathbf{P_2}$

Geometric Objects as Points and Vectors

For any P on the line: **P** = **P**$_1$ + α**d** or **P** = α_1**P**$_1$ + α_2**P**$_2$

For the latter, Since $\alpha_1 + \alpha_2 = 1$, **P** = α_1**P**$_1$ + α_2**P**$_2$ = α_1**P**$_1$ + $(1-\alpha_1)$**P**$_2$ =α_1 (**P**$_1$–**P**$_2$)+**P**$_2$

Example with Matlab: **P**$_1$ = (1,1), **d** = (2,3); **P**$_1$ = (1,1), **P**$_2$ = (3,4)

Geometric objects	Points	Vectors
A point	One point	NA
A line	Two points	A point and a vector (or two mutually opposite vectors)
A plane	Three points not lying on one line	A point and two linearly independent vectors

Draw a line (segment) that passes through two points

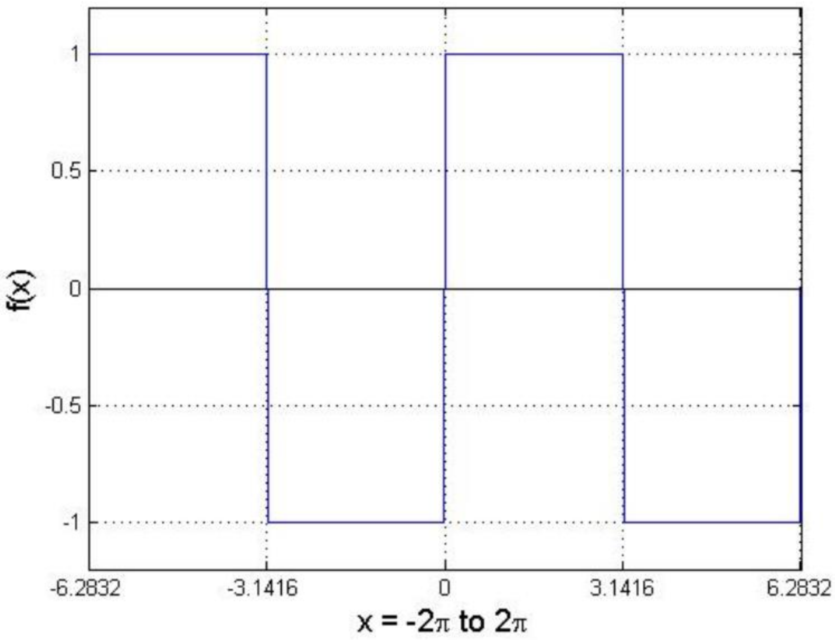
```
p1=[1;1];p2=[3;4]; % two points that determine a line
figure;
plot(p1(1), p1(2),'r*');
hold on;
plot(p2(1), p2(2),'r*');
t=linspace(0,1); % when the coefficient is between 0 and 1
px=1+2*t; py=1+3*t;
plot(px,py,'r-','LineWidth',2);

t=linspace(-1,0); % when the coefficient is between -1 and 0
px=1+2*t; py=1+3*t;
plot(px,py,'b-','LineWidth',2);

t=linspace(1,2); % when the coefficient is between 1 and 2
px=1+2*t; py=1+3*t;
plot(px,py,'g-','LineWidth',2);
```

- Write the Fourier series for the following periodic function:

$$f(x)=\begin{cases} 1, 0 < x < \pi \\ -1, -\pi < x < 0 \end{cases}$$



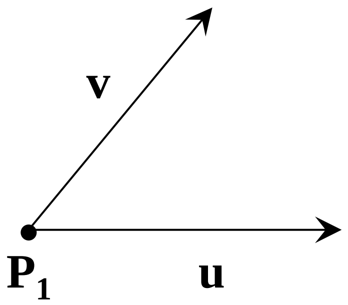
Illustrate the Fourier series

```
x=linspace(-2*pi, 2*pi);
sum=zeros(1,100);
for i=1:2:11 % set a large number to approach the step function
      y=4/pi*(sin(i*x)/i);
      sum=sum+y;
      figure;
      plot(x,y,'b','LineWidth',2); % plot each sin wave
      hold on;
      plot(x,sum,'r','LineWidth',2); % plot the summed sin wave
      xlabel('t = -2\pi to 2\pi','FontSize',12);
      ylabel('sum(sin(mx))','FontSize',12);
end
```

Planes and Triangles

- Plane: sum of a point and two vectors

  $\mathbf{P} = \mathbf{P_1} + \alpha\mathbf{u} + \beta\mathbf{v}$



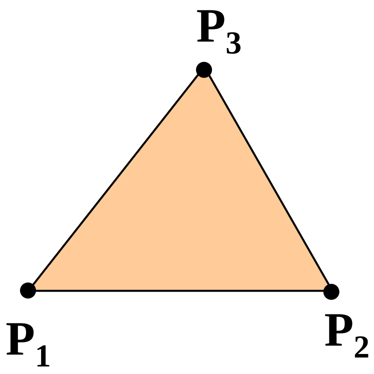
- Triangle: Affine sum of three points

  with $\alpha_i \geq 0$

  $\mathbf{P} = \alpha_1\mathbf{P_1} + \alpha_2\mathbf{P_2} + \alpha_3\mathbf{P_3}$,

  where $\alpha_1 + \alpha_2 + \alpha_3 = 1$

  P lies between $\mathbf{P_1}$, $\mathbf{P_2}$, $\mathbf{P_3}$



Matlab Illustrations

Draw a triangle with three fixed points

```
figure;
hold on;
for i=1:1:10000  % get one point with each iteration
    p1=[0;0];p2=[2;0];p3=[1;1]; % the three fixed points
    a1=rand;a2=rand;a3=rand; % get three random numbers between 0
and 1
    s=a1+a2+a3; % calculate the normalization factor
    a1=a1/s; a2=a2/s; a3=a3/s; % normalize the three random
numbers so they sum up to 1
    px=a1*p1(1)+a2*p2(1)+a3*p3(1);
    py=a1*p1(2)+a2*p2(2)+a3*p3(2);
    plot(px,py,'r+');
end
```

Example 2.1: Eigenvectors of $\boldsymbol{A} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ and transformation of a circle by $\boldsymbol{A}$.

Transform a circle with a transformation matrix A and find eigenvectors

```
x=0;y=0;r=1; % parameters for the input signal (circle)
angle=0:0.01:2*pi;
px=r*cos(angle);
py=r*sin(angle);
figure;
plot(px,py,'b','LineWidth',2); % plot the input signal
hold on;
X=[px; py];
A=[1 0.5;0.5 1]; % transformation matrix
Z=A*X;  % transform the input signal to get the output
signal
plot(Z(1,:),Z(2,:),'r','LineWidth',2); % plot the
transformed signal

[V,D]=eig(A); % calculate the eigenvectors and eigenvalues
of the transformation matrix
```