EE3731C – Signal Processing Methods

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Refreshing Memory about Linear Algebra

Vector

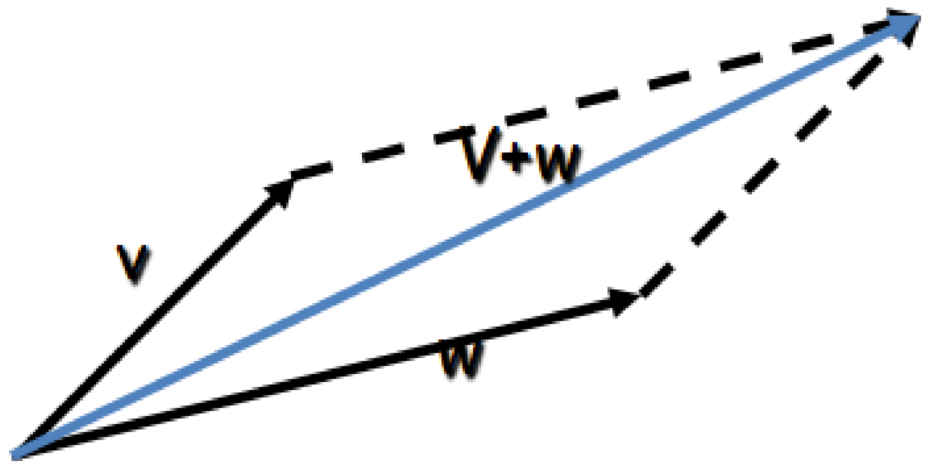
$$\mathbf{v} = (x_1, x_2, \ldots, x_n)$$

$$\|\mathbf{v}\| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

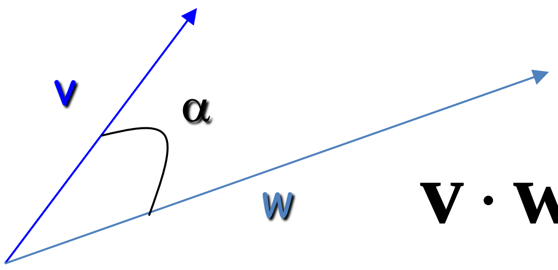
If $\|\mathbf{v}\| = 1$, $\mathbf{v}$ is a unit vector

Vector Addition

$$\mathbf{v} + \mathbf{w} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$



Inner (Dot, Scalar) Product



The inner product is a SCALAR!

$$\mathbf{v} \cdot \mathbf{w} = (x_1, x_2) \cdot (y_1, y_2) = x_1 \cdot y_1 + x_2 \cdot y_2$$

Algebraically, it is the sum of the products of the corresponding entries of the two sequences of numbers.

$$\mathbf{v} \cdot \mathbf{w} = (x_1, x_2) \cdot (y_1, y_2) = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cos\alpha$$

$$\mathbf{v} \cdot \mathbf{w} = 0 \Leftrightarrow \mathbf{v} \perp \mathbf{w}$$

Geometrically, it is the product of the magnitudes of the two vectors and the cosine of the angle between them.

Points

- Representing points:

$$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

  - $x = (x,y,z)\cdot(1,0,0)$
  - $y = (x,y,z)\cdot(0,1,0)$
  - $z = (x,y,z)\cdot(0,0,1)$

Lines

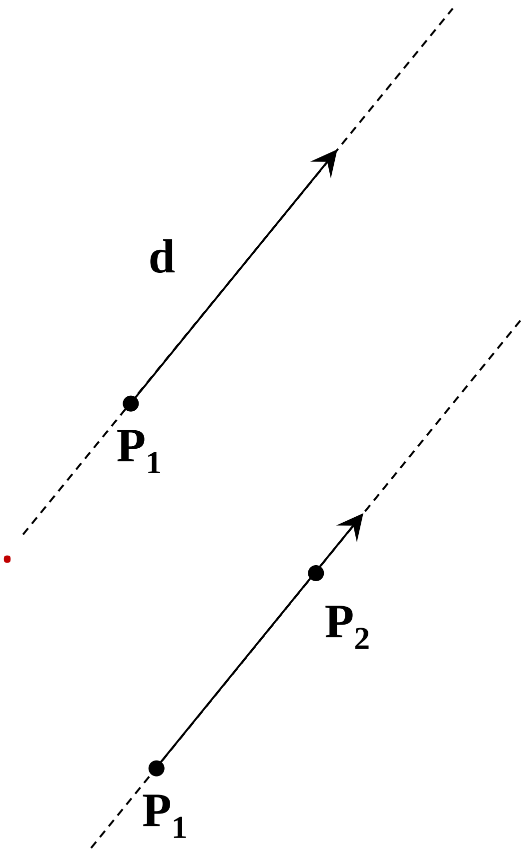
- Line: y = mx + a

- Line: sum of a point and a vector

  $\mathbf{P} = \mathbf{P_1} + \alpha\mathbf{d}$ (where **d** is a column vector)

  $(\mathbf{P} - \mathbf{P_1}) = \alpha\mathbf{d}$

  Example with Matlab: $\mathbf{P_1} = (1,1)$, $\mathbf{d} = (2,3)$.



- Line: <u>Affine sum*</u> of two points

  $\mathbf{P} = \alpha_1\mathbf{P_1} + \alpha_2\mathbf{P_2}$, where $\alpha_1 + \alpha_2 = 1$

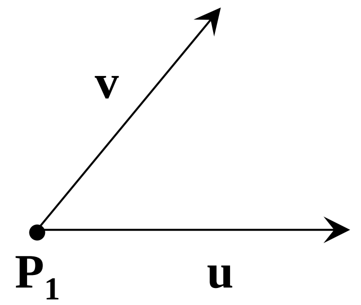
*An affine sum is a linear combination in which the sum of the coefficients is 1.

Line Segment: For $0 < \alpha_1, \alpha_2 < 1$, **P** lies between $\mathbf{P_1}$ and $\mathbf{P_2}$

Plane and Triangle

- Plane: sum of a point and two vectors

  $\mathbf{P} = \mathbf{P_1} + \alpha\mathbf{u} + \beta\mathbf{v}$



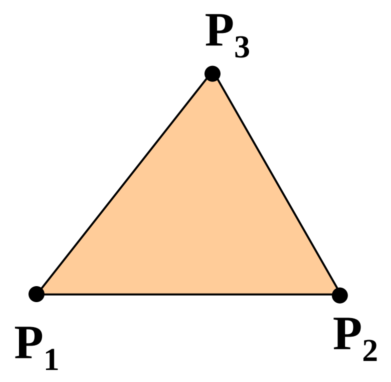
- Triangle: Affine sum of three points

  with $\alpha_i \geq 0$

  $\mathbf{P} = \alpha_1\mathbf{P_1} + \alpha_2\mathbf{P_2} + \alpha_3\mathbf{P_3}$,

  where $\alpha_1 + \alpha_2 + \alpha_3 = 1$

  $\mathbf{P}$ lies between $\mathbf{P_1}$, $\mathbf{P_2}$, $\mathbf{P_3}$

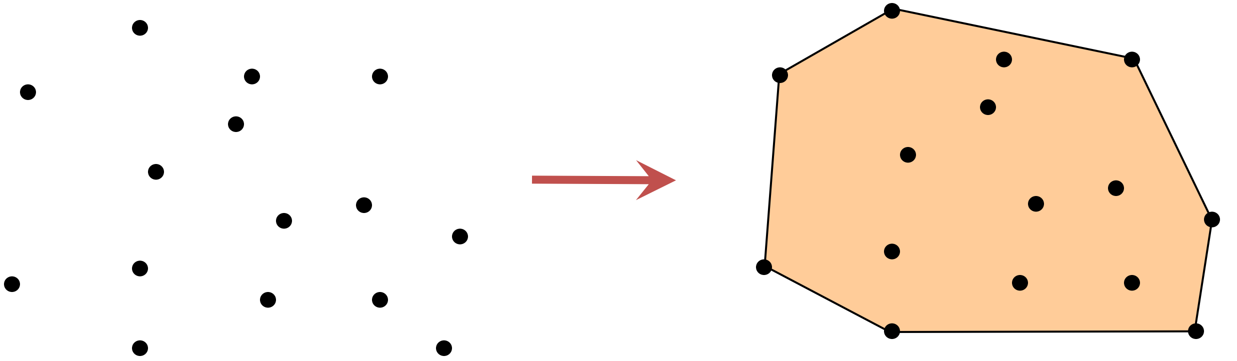


Matlab Illustrations

Generalizing ...

Affine Sum of arbitrary number of points: Convex Hull

$\mathbf{P} = \alpha_1\mathbf{P_1} + \alpha_2\mathbf{P_2} + \ldots + \alpha_n\mathbf{P_n}$, where $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$ and $\alpha_i > 0$



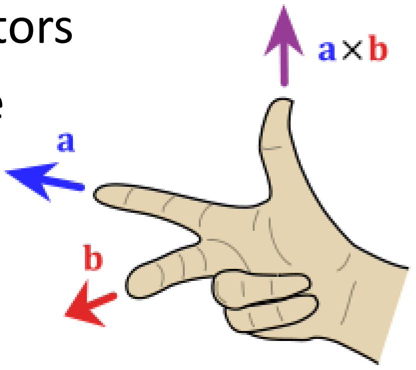
Matlab Illustrations

Cross (Vector) Product

- Results in a VECTOR which is perpendicular to both and therefore normal to the plane containing them.

$$\mathbf{v} \times \mathbf{w} = (x_1, x_2) \times (y_1, y_2) = \| \mathbf{v} \| \times \| \mathbf{w} \| \sin \alpha \mathbf{n}$$

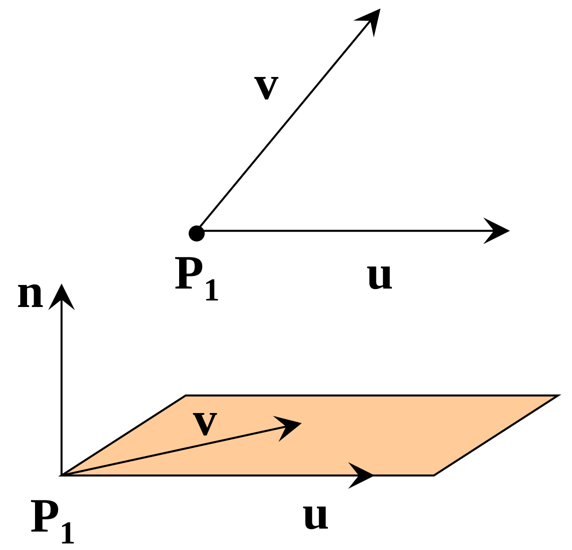
  - α is the smaller angle between the two vectors
  - **n** is a unit vector perpendicular to the plane



Finding the direction of the cross product by the right-hand rule

Normal

- In geometry, an object such as a line or vector is called a **normal** to another object if they are perpendicular to each other.
- A surface normal can be calculated as the vector cross product of two (non-parallel) edges of the polygon.



- If **n** is orthogonal to **u** and **v** , then **n** = **u** x **v**

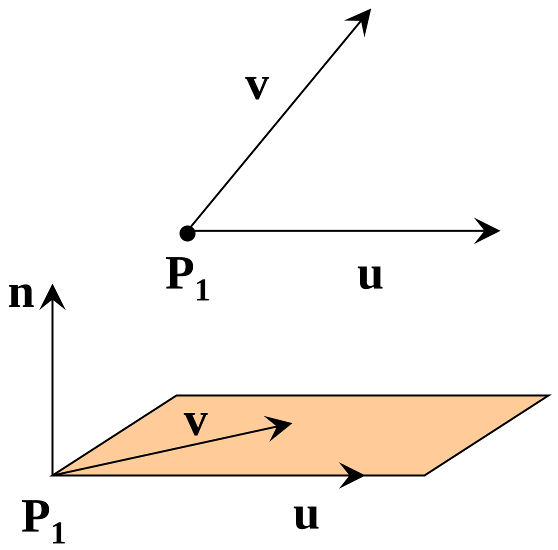
  Plane: sum of a point and two vectors
  - $\mathbf{P} = \mathbf{P_1} + \alpha\mathbf{u} + \beta\mathbf{v}$
  - $\mathbf{P} - \mathbf{P_1} = \alpha\mathbf{u} + \beta\mathbf{v}$

- So we have $\mathbf{n}^T \cdot (\mathbf{P} - \mathbf{P_1}) = \alpha\mathbf{n}^T \cdot\mathbf{u} + \beta\mathbf{n}^T \cdot\mathbf{v} = 0$
  - e.g. **u** =[1, 0], **v** =[0, 1], the only solution is **n** = [0, 0]
  - But if **u** =[1, 0, 0], **v** =[0, 1, 0], the solution is **n** = [0, 0, 1]

Implicit Equation of a Plane

$\mathbf{n}^{\mathbf{T}} \cdot (\mathbf{P} - \mathbf{P_1}) = 0$

Let $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\mathbf{P} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\mathbf{P_1} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$



Then, the equation of a plane becomes:

$a\,(x - x_1) + b\,(y - y_1) + c\,(z - z_1) = 0$

$a\,x + b\,y + c\,z + d = 0$

Thus, the coefficients of x, y, z in a plane equation define the normal.

Matrix

- Matrix: a rectangular array of numbers, symbols, or expressions, arranged in rows and columns
- Element (entry): an individual item in a matrix

$$
\mathbf{A}_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{31} & a_{32} & \cdots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}
$$

n by m matrix

n rows, m columns

Matrix Operations

$$\mathbf{A}_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{31} & a_{32} & \cdots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

- Addition

$$\mathbf{C}_{n\times m} = \mathbf{A}_{n\times m} + \mathbf{B}_{n\times m}$$

$$c_{ij} = a_{ij} + b_{ij}$$

A and **B** must have the same dimensions

- Matrix Multiplication

$$\mathbf{C}_{n\times p} = \mathbf{A}_{n\times m}\mathbf{B}_{m\times p}$$

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

A and **B** must have compatible dimensions

$$\mathbf{A}_{n\times n}\mathbf{B}_{n\times n} \neq \mathbf{B}_{n\times n}\mathbf{A}_{n\times n}$$

- Identity Matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \ddots & 0 \\ 0 & 1 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & 1 \end{pmatrix} \quad \mathbf{IA} = \mathbf{AI} = \mathbf{A}$$

Matrix Operations

$$\mathbf{A}_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{31} & a_{32} & \cdots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

- Transpose

$$\mathbf{C}_{m\times n} = \mathbf{A}^T{}_{n\times m}$$

$$c_{ij} = a_{ji}$$

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$$

If $\mathbf{A}^T = \mathbf{A}$, **A** is symmetric

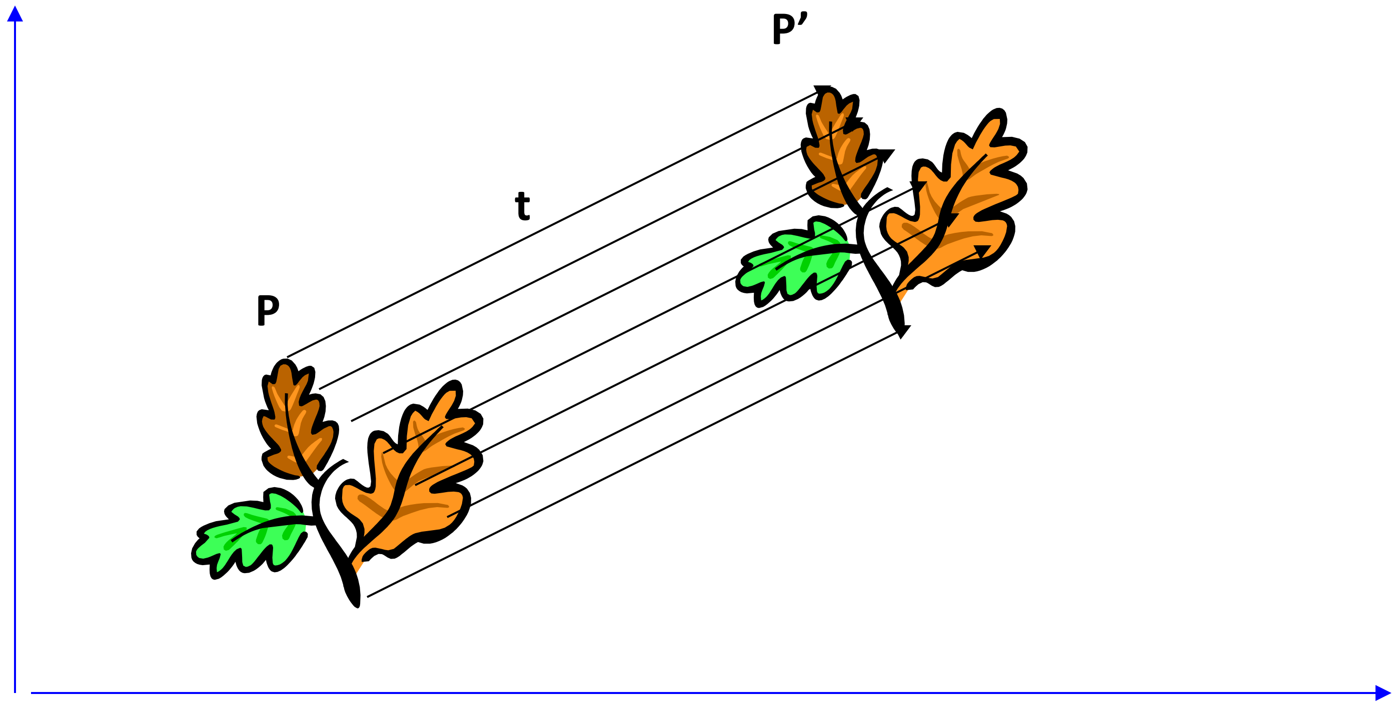
Determinant

A must be square

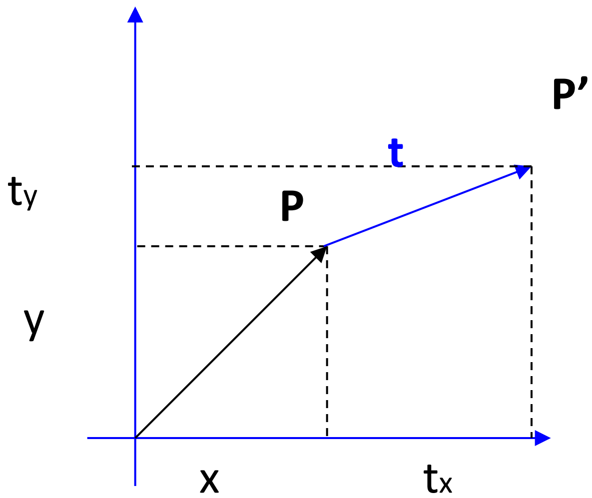
$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

2D Translation



2D Translation using Matrices



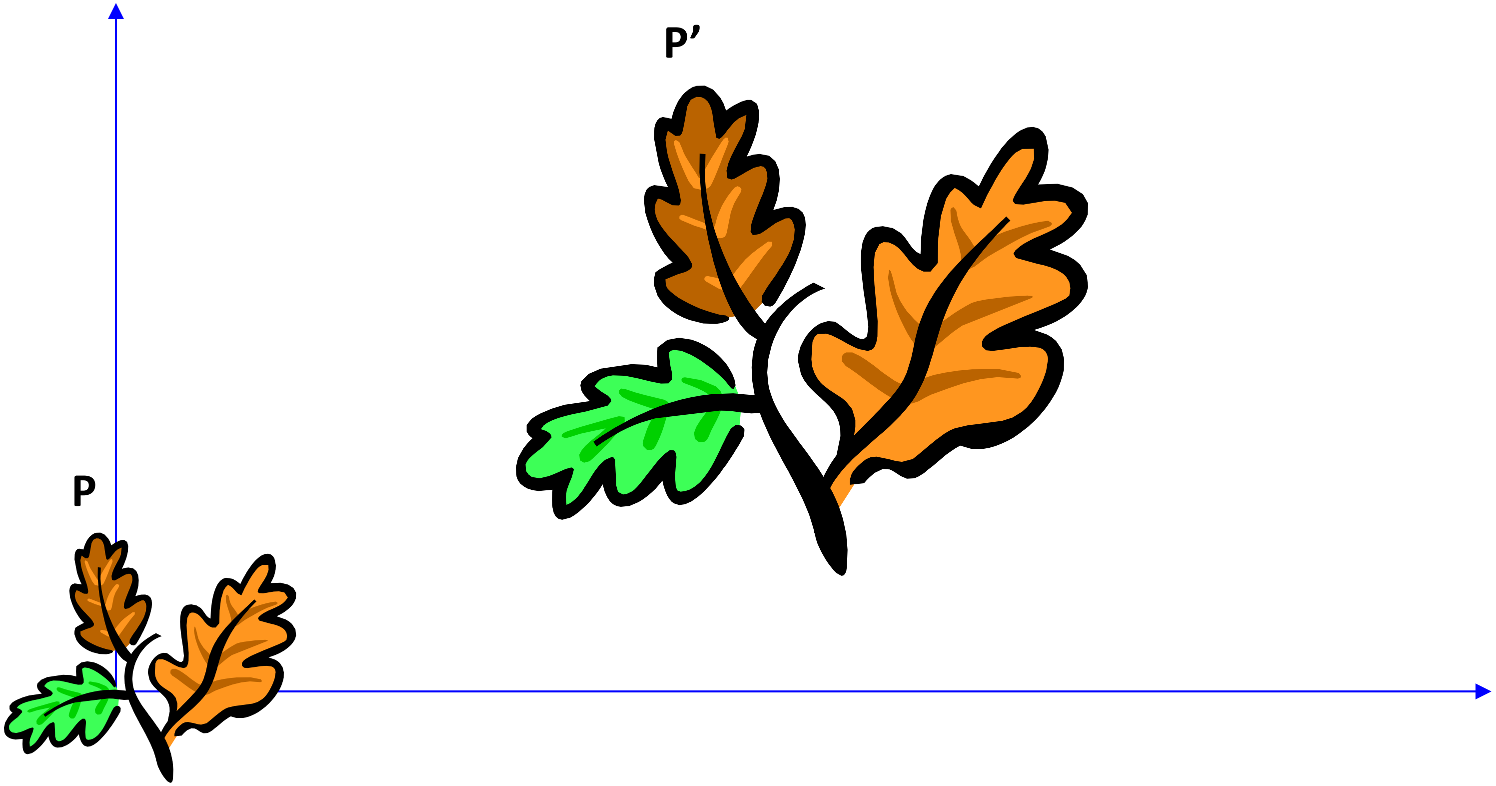
$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

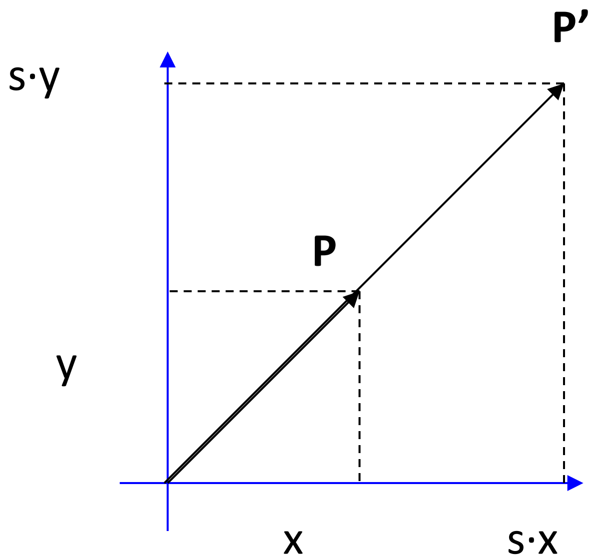
t P

$$\mathbf{P}' \rightarrow \begin{bmatrix} x+t_x \\ y+t_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & \boxed{t_x} \\ 0 & 1 & t_y \end{bmatrix} \cdot \begin{bmatrix} \boxed{x} \\ y \\ 1 \end{bmatrix}$$

Scaling



Scaling Equation



$$\mathbf{P} = (x, y)$$

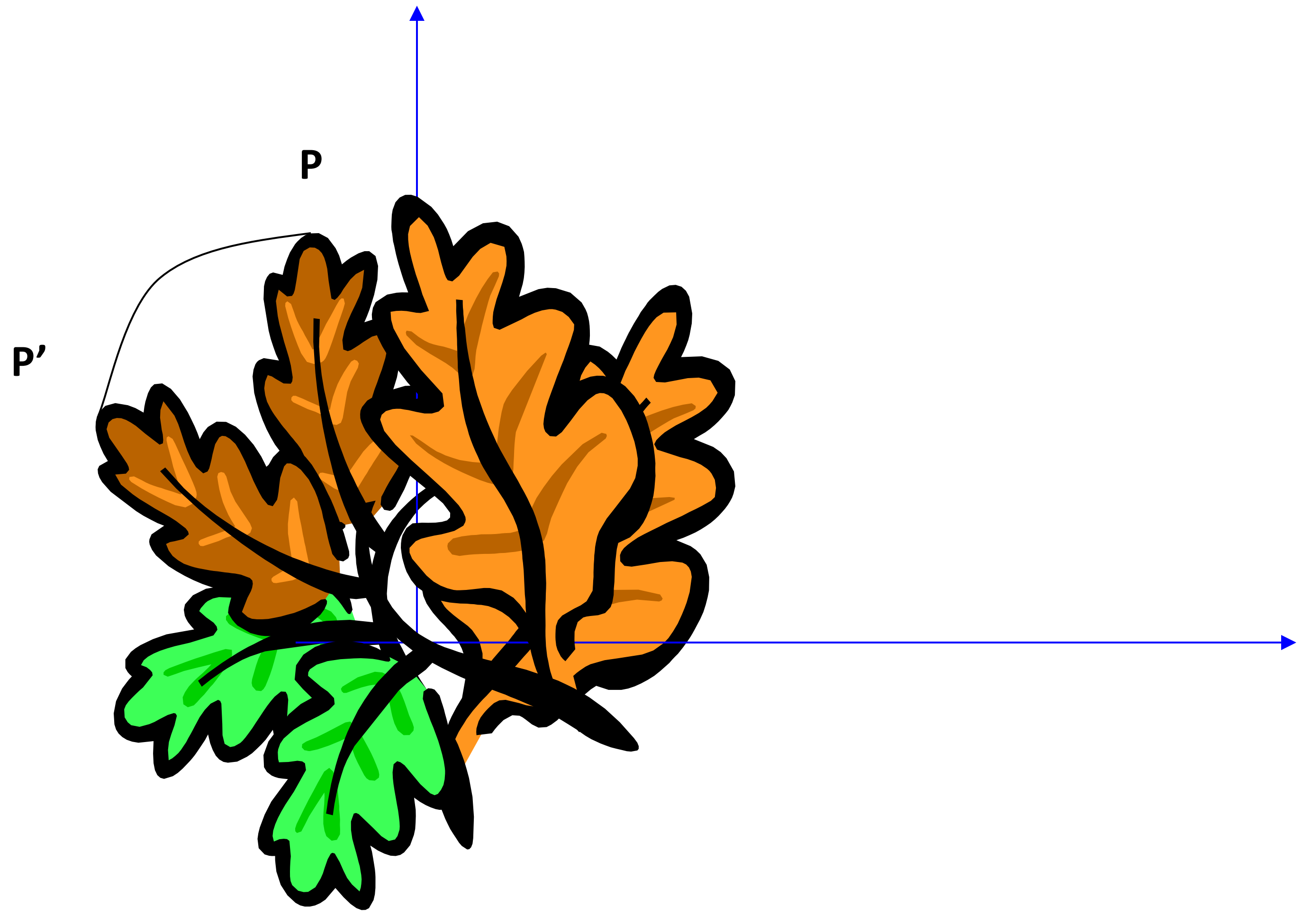
$$\mathbf{P}' = (sx, sy)$$

$$\mathbf{P}' = s \cdot \mathbf{P}$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} sx \\ sy \end{bmatrix} = \underbrace{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

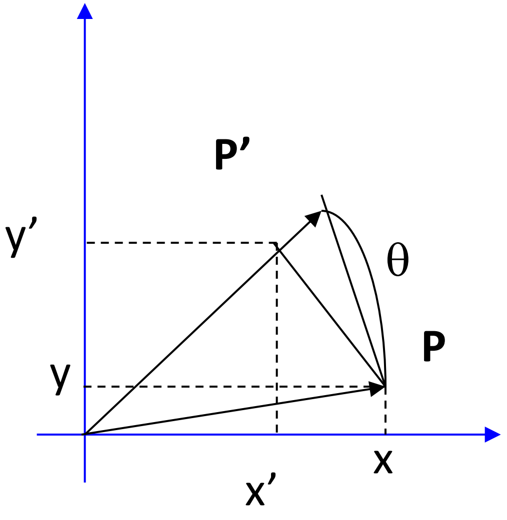
$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

Rotation



Rotation Equations

Counter-clockwise rotation by an angle θ



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{R} \cdot \mathbf{P}$$

Simple 3D Rotation

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & . & . & . & x_n \\ y_1 & y_2 & & & & y_n \\ z_1 & z_2 & & & & z_n \end{pmatrix}$$

Rotation about the z axis.

Full 3D Rotation

$$\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

Any rotation can be expressed as combination of three rotations about three axes.

$$\mathbf{R}\mathbf{R}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Rows (and columns) of **R** are orthonormal vectors.
- **R** has determinant 1 (not -1).

- Orthogonal: $\boldsymbol{v}_i \boldsymbol{v}_j = 0$, if $\mathrm{i} \neq j$
  - Geometrically perpendicular
  - Statistically uncorrelated in terms of the second-order statistics

- Orthonormal: $\boldsymbol{v}_i \boldsymbol{v}_j = 0$, if $\mathrm{i} \neq j$; $|\boldsymbol{v}_i| = 1$

Eigenanalysis

Linear Systems

- All linear filtering and signal transformation methods can be casted in the form of

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$$

  - $\boldsymbol{x}$: input signal vector
  - $\boldsymbol{y}$: output signal vector
  - $\boldsymbol{A}$: the system that transforms the input signal vector to the output one

Analysis of $\boldsymbol{A}$ is important!

Eigenvalue and Eigenvectors

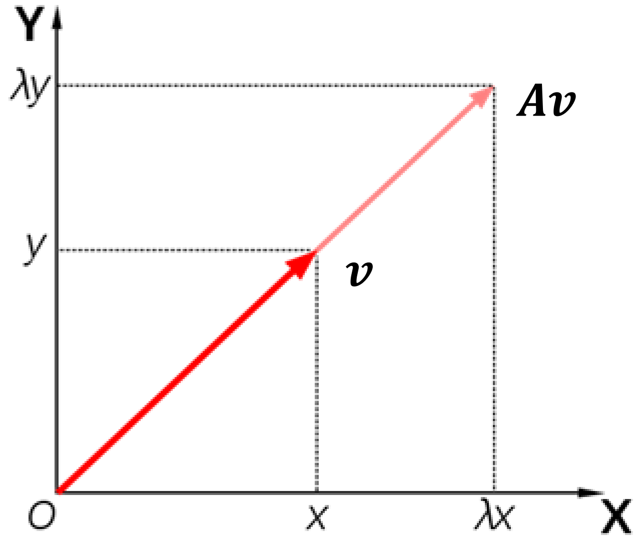
- An **eigenvector** of a square matrix $\boldsymbol{A}$ is a non-zero vector $\boldsymbol{v}$ that, when the matrix is multiplied by $\boldsymbol{v}$, yields a constant multiple of $\boldsymbol{v}$:

$$\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$$

  – λ is the eigenvalue of $\boldsymbol{A}$ corresponding to $\boldsymbol{v}$.

Eigen

- Defining characteristic of an eigenvector: when an eigenvector of a matrix is transformed through the matrix,
  - the direction of the eigenvector is unchanged
  - only the magnitude of the eigenvector is changed

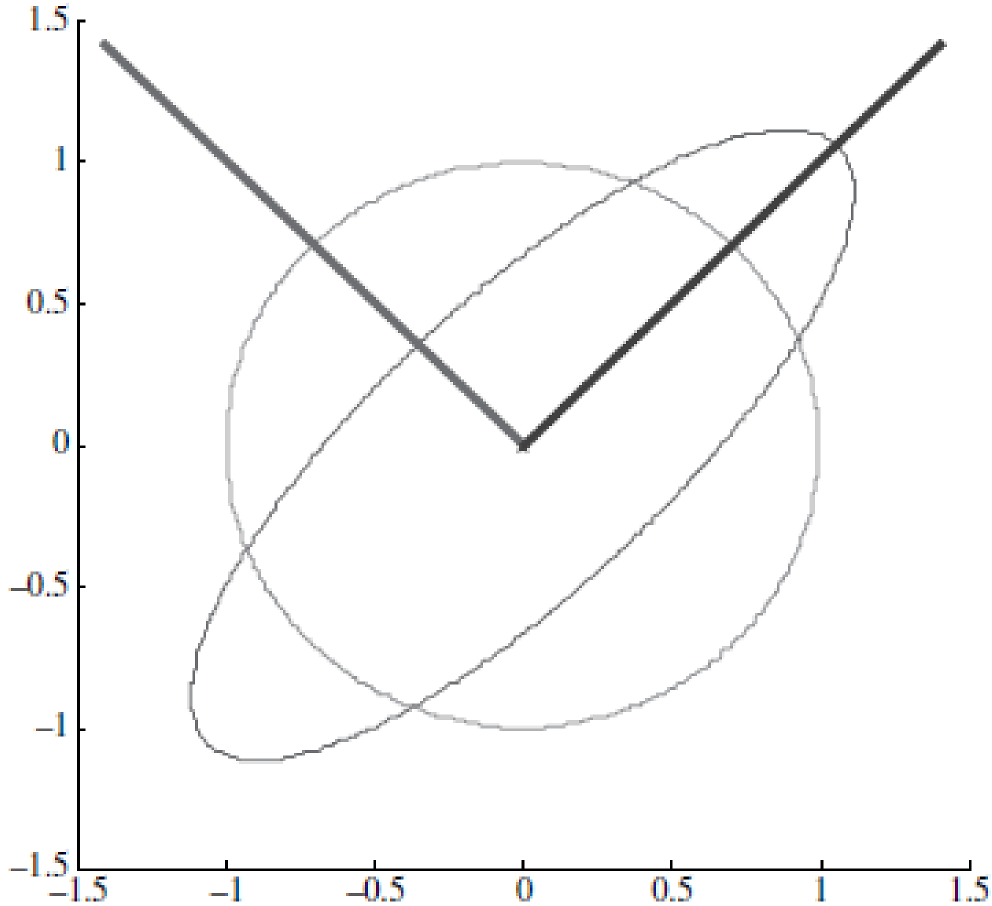


- Eigen: “peculiar to”, or “characteristic” (of a matrix)

Eigenvector Analysis

- The eigenvector analysis of a matrix finds a set of characteristic orthonormal **eigenvectors** and their magnitudes called **eigenvalues**.
- A matrix can be decomposed and expressed in terms of its eigenvectors and eigenvalues.
  - Eigenvectors: lend themselves to independent processing and easier manipulation and interpretation.
  - Eigenvalues: represent the variance or power of the process along the corresponding eigenvectors. The bigger an eigenvalue the more significant the corresponding eigenvector.

Example 2.1: Eigenvectors of $\boldsymbol{A} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ and transformation of a circle by $\boldsymbol{A}$.



The eigenvectors are $[0.7071, 07071]^T$ and $[-0.7071, 0.7071]^T$

How to get eigenvalues and eignenvectors?

In old days (10 years ago), we do hand calculations…, whoever do mental calculation fast is the smartest…

Now we use Matlab...

If you choose not to use matlab…

Computing Eigenvalues and Eigenvectors

- $\boldsymbol{A}\boldsymbol{v}_i = \lambda \boldsymbol{v}_i$

$(\boldsymbol{A} - \lambda_i \mathbf{I})\boldsymbol{v}_i = 0$

$\det(\boldsymbol{A} - \lambda_i \mathbf{I}) = 0$

- For example, $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$

=> $(a - \lambda)(d - \lambda) - bc = 0$

=> $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$

In compact matrix notation:

- $\boldsymbol{A}\boldsymbol{v}_1 + \cdots + \boldsymbol{A}\boldsymbol{v}_N = \lambda\boldsymbol{v}_1$+…+ $\lambda\boldsymbol{v}_N$
- $\boldsymbol{A}[\boldsymbol{v}_1, \ldots \boldsymbol{v}_N] = [\boldsymbol{v}_1, \ldots \boldsymbol{v}_N]\begin{bmatrix} \lambda_1 & 0 & \ldots 0 \\ 0 & \lambda_2 & \ldots 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots \lambda_N \end{bmatrix}$
- $\boldsymbol{A}\boldsymbol{V}_R = \boldsymbol{V}_R\Lambda$
  - $\boldsymbol{V}_R$: right eigenvectors
  - Λ: diagonal eigenvalue matrix
- $\boldsymbol{A} = \boldsymbol{V}_R\Lambda\boldsymbol{V}_R{}^{-1}$

In compact matrix notation:

- $\boldsymbol{V}_L\boldsymbol{A} = \Lambda\boldsymbol{V}_L$ => $\boldsymbol{V}_L\boldsymbol{A}\boldsymbol{V}_R = \Lambda\boldsymbol{V}_L\boldsymbol{V}_R$
- $\boldsymbol{A}\boldsymbol{V}_R = \boldsymbol{V}_R\Lambda$ =>$\boldsymbol{V}_L\boldsymbol{A}\boldsymbol{V}_R = \boldsymbol{V}_L\boldsymbol{V}_R\Lambda$

=> $\Lambda\boldsymbol{V}_L\boldsymbol{V}_R$ = $\boldsymbol{V}_L\boldsymbol{V}_R\Lambda$

- Λ is diagonal => $\boldsymbol{V}_L\boldsymbol{V}_R = \mathbf{I}$
- $\boldsymbol{A} = \boldsymbol{V}_R\Lambda\boldsymbol{V}_R{}^{-1}$ => $\boldsymbol{A} = \boldsymbol{V}_R\Lambda\boldsymbol{V}_L$
- For a symmetric matrix:

$$\boldsymbol{A} = \boldsymbol{V}\,\Lambda\boldsymbol{V}^T = \boldsymbol{V}^T\Lambda\,\boldsymbol{V}$$

Messages

- The determination of the eigenvectors and eigenvalues of a matrix is important in signal processing for matrix diagonalisation.
- Each eigenvector is paired with a corresponding eigenvalue.
- Mathematically, two different kinds of eigenvectors need to be distinguished: left eigenvectors and right ones.

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Some of the materials are from the Internet and the textbooks