EE3731C: Signal Processing Methods

Tutorial II-4

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Question #1

Consider a random process $x[n]$ generated by filtering a white noise that has has a zero mean and a unit variance with the system below:

$$H(z) = \frac{1}{z - 0.5}$$

Find the power spectrum of x[n].

Question #1: Solution

The power spectrum of the white noise with a zero mean and a unit variance is:

$$S_w\left(e^{j\omega}\right)=1$$

The power spectrum of x[n] is given by:

$$S_x\left(e^{j\omega}\right)=H\left(e^{j\omega}\right)H\left(e^{-j\omega}\right)S_w\left(e^{j\omega}\right)=\left|H\left(e^{j\omega}\right)\right|^2 S_w\left(e^{j\omega}\right)$$

$$H(z)=\frac{1}{z-0.5} \qquad H\left(e^{j\omega}\right)=\frac{1}{e^{j\omega}-0.5}$$

$$S_x\left(e^{j\omega}\right)=\frac{1}{\left(e^{j\omega}-0.5\right)}\frac{1}{\left(e^{-j\omega}-0.5\right)}=\frac{1}{1.25-\cos\omega}$$

Question #2

Consider the random process $x[n]$ which has an autocorrelation function given by

$$R_x[k] = 17\delta[k] + 4(\delta[k-1] + \delta[k+1])$$

i. Find a linear filter such that the output has the same autocorrelation function when the input is white noise of zero mean and unit variance.

ii. Find a whitening filter that converts $x[n]$ into white noise.

Question #2: Solution

i. Find a linear filter such that the output has the same autocorrelation function when the input is white noise of zero mean and unit variance.

$$w[n] \longrightarrow \boxed{\mathrm{H(z)}} \longrightarrow x[n]$$

$$P_w(z) = 1 \qquad\qquad P_x(z) = H(z)H(1/z)$$

$$R_x[k] = 17\delta[k] + 4(\delta[k-1] + \delta[k+1])$$

$$P_x(z) = 17 + 4z^{-1} + 4z$$

factorization

$$P_x(z) = \left(4 + z^{-1}\right)(4 + z)$$

$$P_x(z) = H(z)H\left(z^{-1}\right)$$

$$H(z) = 4 + z^{-1}$$

Question #2: Solution

$$R_x[k] = 17\delta[k] + 4\left(\delta[k-1] + \delta[k+1]\right)$$

ii. Find a whitening filter that converts $x[n]$ into white noise.

$$\xrightarrow[P_x(z) = H(z)H(1/z)]{x[n]} \boxed{1/H(z)} \xrightarrow[P_w(z) = 1]{w[n]}$$

$$P_x(z) = \left(4 + z^{-1}\right)\left(4 + z\right) \qquad H(z) = 4 + z^{-1}$$

$$P_x(z) = H(z)H\left(z^{-1}\right) \qquad \frac{1}{H(z)} = \frac{1}{4 + z^{-1}} = \frac{1/4}{1 + \frac{1}{4}z^{-1}}$$

Question #3

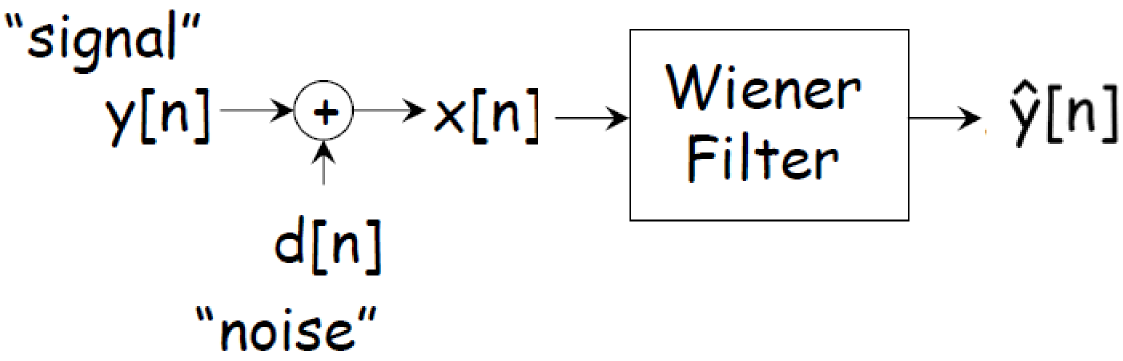
Let $S_s(\omega)$ denote the power spectral density of $s[n]$. Consider the situation in which we observe a realization of signal $s[n]$ in additive Gaussian noise $w[n]$, i.e.,

$$x[n] = s[n] + w[n]$$

Assume that the variance of $w[n]$ is 0.5 and $w[n]$ is independent of $s[n]$. Design an optimal Wiener filter for estimating $s[n]$ from the measured noisy signal $x[n]$. Give an expression of the Wiener filter in the frequency domain.

Question #3: Solution

$$x[n] = s[n] + w[n]$$



$$H(f) = \frac{S_{xy}(f)}{S_x(f)} = \frac{S_y(f)}{S_y(f) + S_d(f)}$$

$$H(\omega) = \frac{S_s(\omega)}{S_s(\omega) + S_w(\omega)} = \frac{S_s(\omega)}{S_s(\omega) + 0.5}$$

Question #4

Consider the power spectral density function given by:

$$S_x\left(e^{j\omega}\right) = 5 + 4\cos(\omega)$$

i. Find the corresponding autocorrelation function of $x[n]$
ii. Find a linear filter whose output process has the same autocorrelation function when excited by white noise of zero mean and unit variance.
iii. What is the variance of the output process?

Question #4: Solution

Consider the power spectral density function given by:

$$S_x\left(e^{j\omega}\right) = 5 + 4\cos(\omega)$$

i. Find the corresponding autocorrelation function of $x[n]$

$$S_x\left(e^{j\omega}\right) = 5 + 4\cos(\omega) = 5 + 2e^{j\omega} + 2e^{-j\omega}$$

Taking the inverse transform, we have

$$R_x[m] = 5\,\delta[\mathrm{m}] + 2\,\delta[\mathrm{m}-1] + 2\,\delta[\mathrm{m}+1]$$

Question #4: Solution

Consider the power spectral density function given by:

$$S_x\left(e^{j\omega}\right) = 5 + 4\cos(\omega)$$

ii. Find a linear filter whose output process has the same autocorrelation function when excited by white noise of zero mean and unit variance.

Spectral factorization:

$$S_x(z) = 5 + 2z + 2z^{-1} = \left(1 + 2z^{-1}\right)\left(1 + 2z\right) = H(z)H\left(z^{-1}\right)$$

Causal filter: $H(z) = 1 + 2z^{-1}$

Question #4: Solution

Consider the power spectral density function given by:

$$S_x\left(e^{j\omega}\right) = 5 + 4\cos(\omega)$$

iii. What is the variance of the output process?

The autocorrelation function is:

$$R_x[m] = 5\,\delta[\mathrm{m}] + 2\,\delta[\mathrm{m}-1] + 2\,\delta[\mathrm{m}+1]$$

The variance is:

$$\sigma_x^2 = R_x[0] = 5$$