EE3731C: Signal Processing Methods

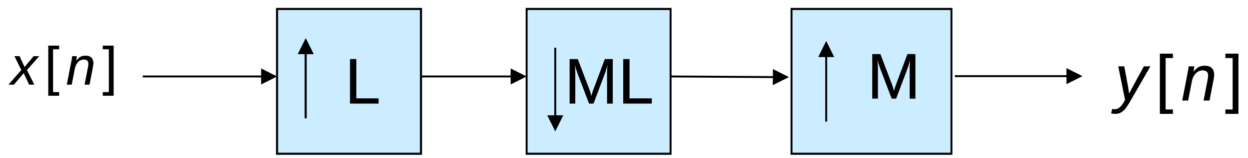
Tutorial II-5

National University of Singapore

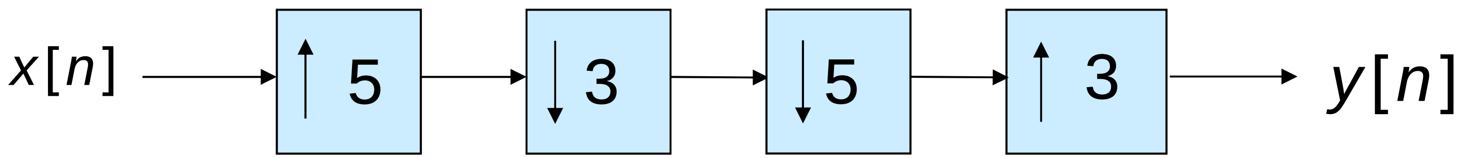
Question #1

Simplify the following multirate systems as much as you can.

i)

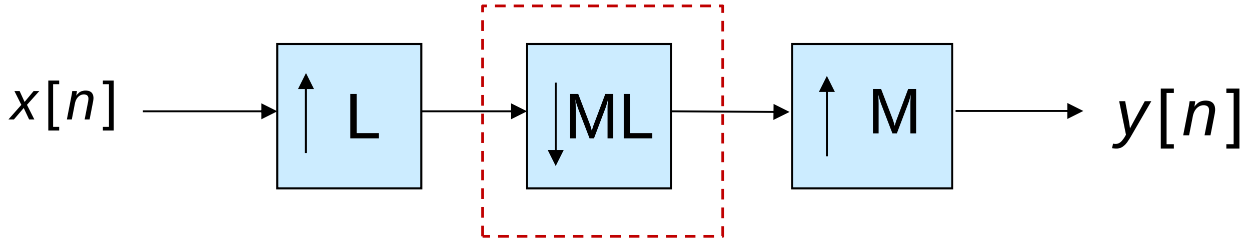


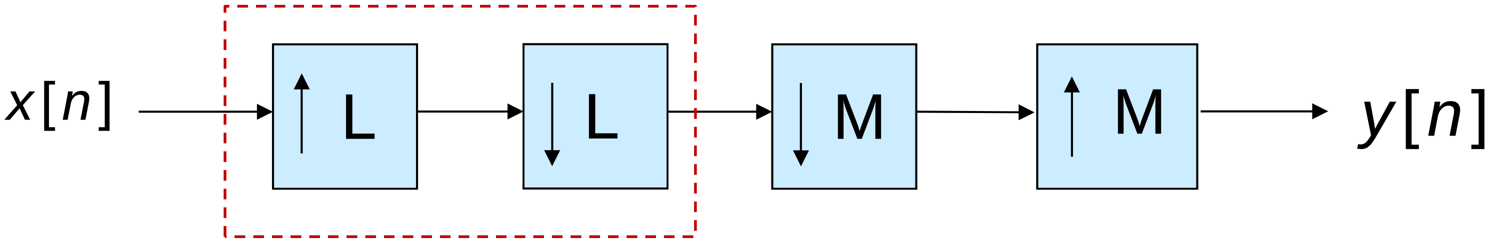
ii)



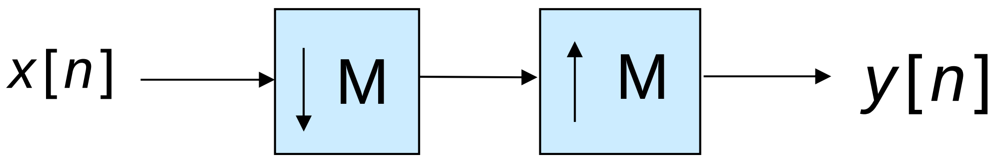
Question #1: Solution

i)





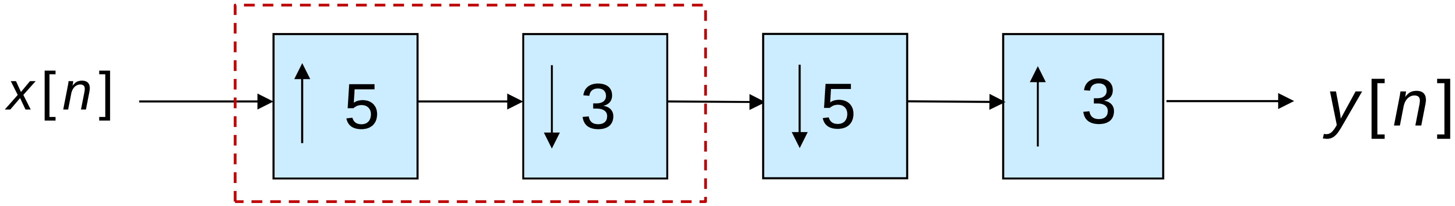
Doing nothing



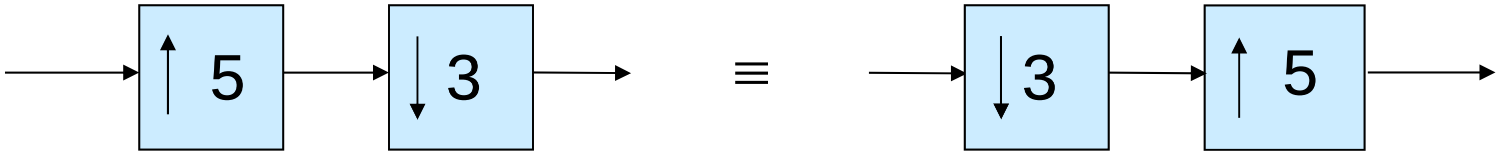
Cannot simplify further

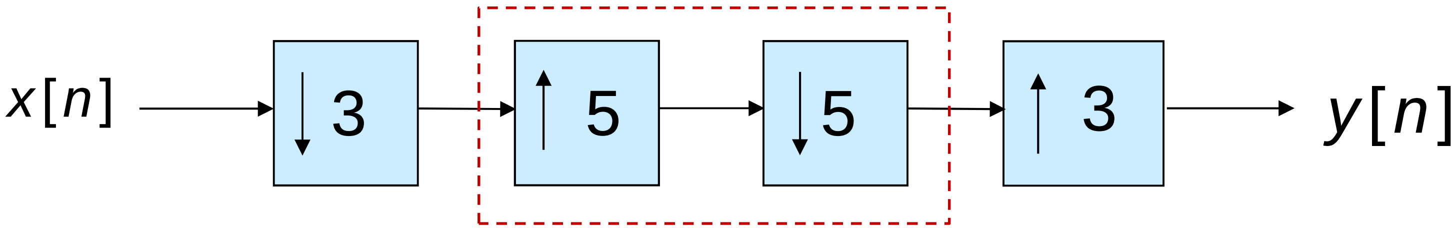
Question #1: Solution

ii)

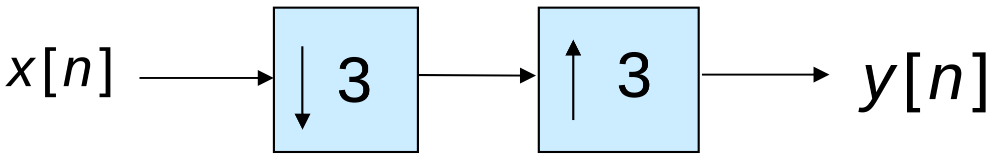


3 and 5 are relatively prime





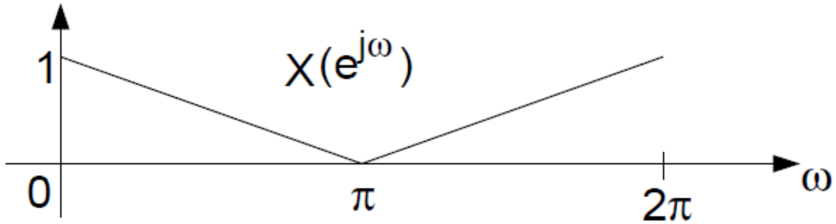
Doing nothing



Cannot simplify further

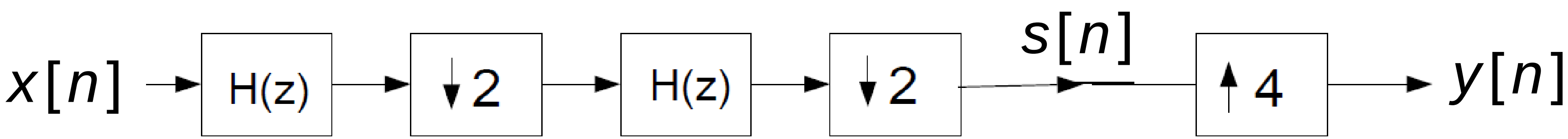
Question #2

Consider a sequence $x[n]$ whose Fourier transform is shown below.



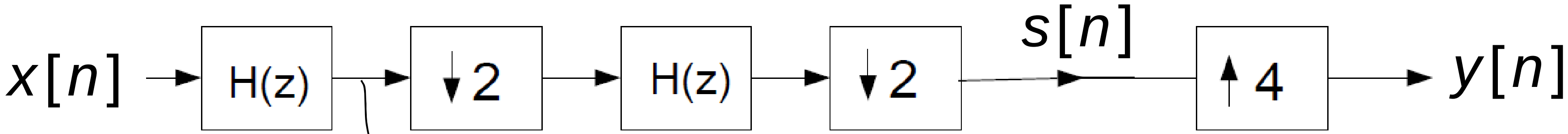
The frequency response of the filter is

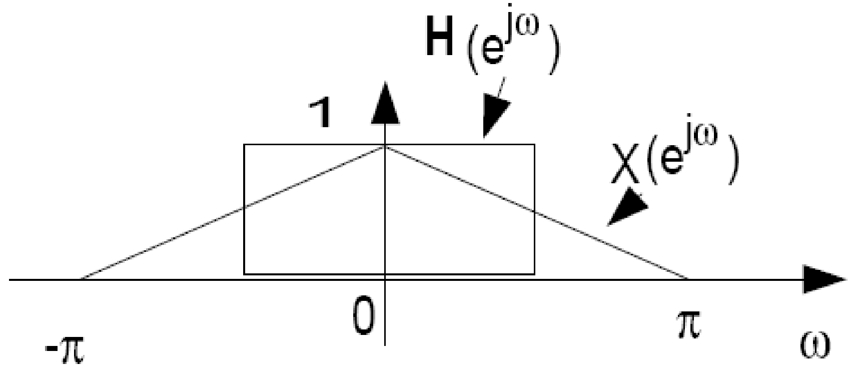
$$H\left(e^{j\omega}\right) = \begin{cases} 1 & |\omega| < \pi/2 \\ 0 & \pi/2 \le |\omega| \le \pi \end{cases}$$



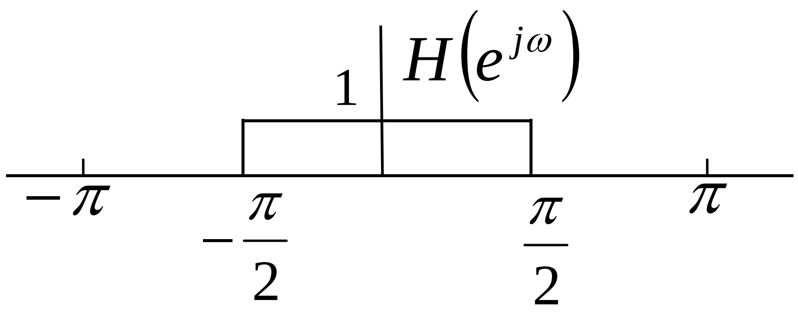
Plot $Y(e^{j\omega})$ and $S(e^{j\omega})$

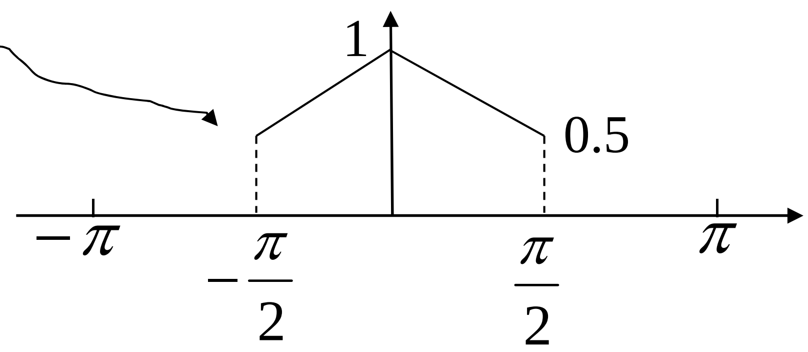
Question #2: Solution



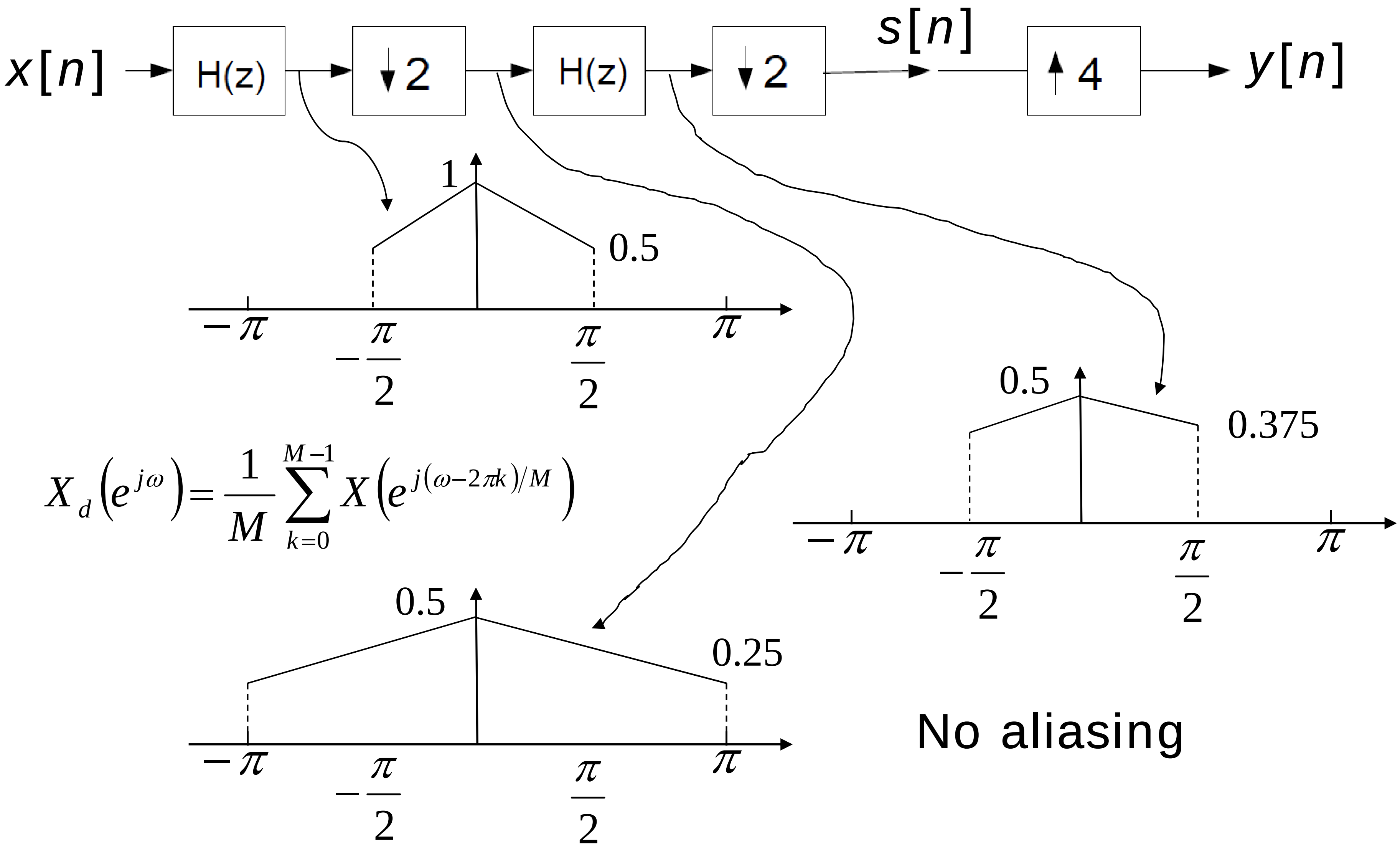


$$H\left(e^{j\omega}\right)=\begin{cases}1 & |\omega|<\pi/2\\ 0 & \pi/2\le|\omega|\le\pi\end{cases}$$





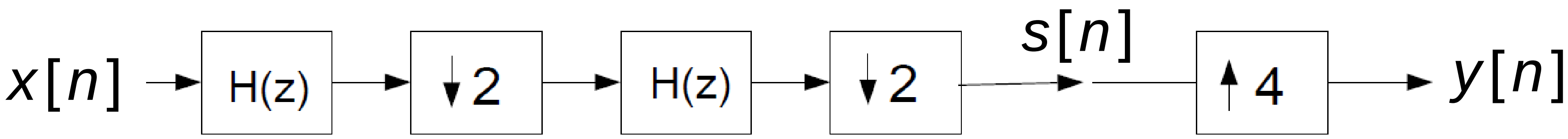
Question #2: Solution

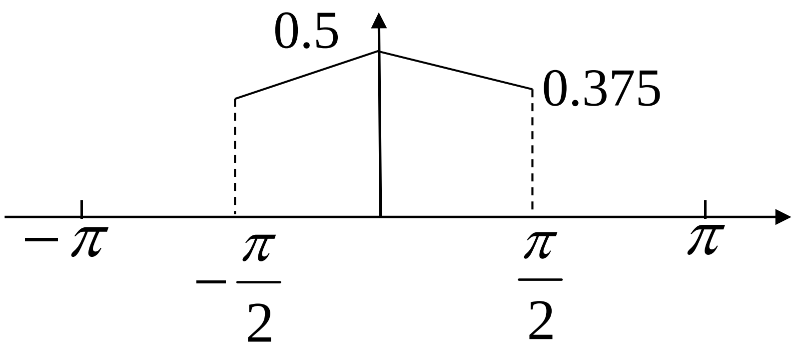


$$X_d\left(e^{j\omega}\right) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j(\omega-2\pi k)/M}\right)$$

No aliasing

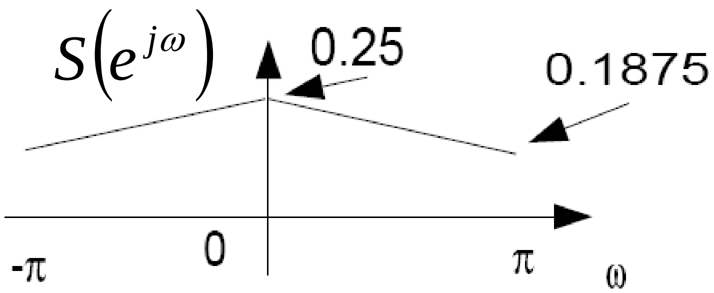
Question #2: Solution

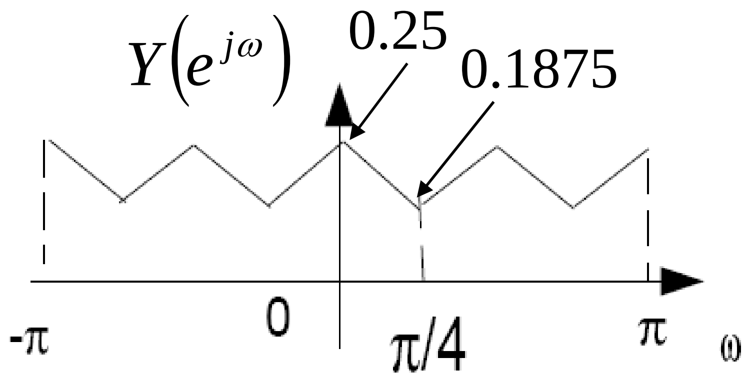




$$X_d\left(e^{j\omega}\right)=\frac{1}{M}\sum_{k=0}^{M-1}X\left(e^{j(\omega-2\pi k)/M}\right)$$

$$X_u\left(e^{j\omega}\right)=X\left(e^{j\omega L}\right)$$





Question #3

Let θ denote a random variable that is uniformly distributed on the interval from 0 to 2π, and let $e[n]$ be a sequence of zero-mean random variables that are uncorrelated with each other and also uncorrelated with θ (i.e., $e[n]$ represents white noise). The signal is given by:

$$x[n] = A\cos(\omega_0 n + \theta) + e[n],$$

where A and ω_0 are known constant values.

Determine the mean and autocorrelation functions of $x[n]$.

Question #3: Solution

Determine the mean and autocorrelation functions of $x[n] = A\cos(\omega_0 n + \theta) + e[n]$

Mean:

$$E\{x[n]\} = E\{A\cos(\omega_0 n + \theta) + e[n]\} = E\{A\cos(\omega_0 n + \theta)\} + E\{e[n]\} = 0$$

Autocorrelation:

The cosine signal and white noise are uncorrelated.

$$\begin{aligned} R_x[m] &= E\{x[n]x[n+m]\} \\ &= E\{(A\cos(\omega_0 n + \theta) + e[n])(A\cos(\omega_0 (n+m) + \theta) + e[n+m])\} \\ &= E\{A\cos(\omega_0 n + \theta)A\cos(\omega_0 (n+m) + \theta)\} + E\{e[n]e[n+m]\} \\ &= \frac{A^2}{2}\cos(\omega_0 m) + \sigma_e^2 \delta[m] \end{aligned}$$

$$\sigma_e^2 = E\{(e[n])^2\}$$

Question #4

A zero mean noise process $x[n]$ has the following autocorrelation function

$$R_x(m) = \left(\frac{1}{2}\right)^{|m|}$$

We process $x[n]$ with a linear time-invariant system that satisfies the following difference equation:

$$y[n] = x[n+1] + x[n-1]$$

What is the power spectral density of $y[n]$?

Question #4: Solution

Find the power spectral density of $x[n]$

$$R_x(m)=\left(\frac{1}{2}\right)^{|m|}=\left(\frac{1}{2}\right)^{m}u(m)+\left(\frac{1}{2}\right)^{-m}u(-m-1)=\left(\frac{1}{2}\right)^{m}u(m)+2^{m}u(-m-1)$$

$$S_x(z)=\frac{1}{1-\frac{1}{2}z^{-1}}-\frac{1}{1-2z^{-1}}=\frac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}=\frac{\frac{3}{4}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{z}{2}\right)}$$

$$S_x\left(e^{j\omega}\right)=\frac{1}{1-\frac{1}{2}z^{-1}}-\frac{1}{1-2z^{-1}}=\frac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}=\frac{\frac{3}{4}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{e^{j\omega}}{2}\right)}$$

Question #4: Solution

Find the transfer function:

$$y[n] = x[n+1] + x[n-1]$$

$$Y(z) = X(z)z + X(z)z^{-1}$$

$$H(z) = \frac{Y(z)}{X(z)} = z + z^{-1}$$

Frequency response: $H(e^{j\omega}) = e^{j\omega} + e^{-j\omega} = 2\cos\omega$

The spectrum of y[*n*]

$$S_y(e^{j\omega}) = \frac{\frac{3}{4}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{e^{j\omega}}{2}\right)}\left(e^{-j\omega}+e^{j\omega}\right)^2 = \frac{3(\cos\omega)^2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{e^{j\omega}}{2}\right)}$$

Question #4: Solution

Alternatively, first find the autocorrelation function of y[n] as:

$$R_y[m] = E(y[n]y[n+m]) = E\{(x[n+1]+x[n-1])(x[n+1+m]+x[n+m-1])\}$$

$$R_y[m] = 2R_x[m] + R_x[m-2] + R_x[m+2]$$

Then take the Fourier transform

$$S_y\left(e^{j\omega}\right) = \frac{\frac{3}{4}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{e^{j\omega}}{2}\right)}\left(2+e^{-j2\omega}+e^{j2\omega}\right) = \frac{\frac{3}{2}(1+\cos 2\omega)}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{e^{j\omega}}{2}\right)}$$

Note that $1+\cos 2\omega = 2(\cos\omega)^2$