EE3731C – Signal Processing Methods

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Differential Equations

- A differential equation (DE) is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders.
  - Ordinary vs. partial
  - Linear vs. nonlinear

Differential Equations

- Almost all the elementary and numerous advanced parts of theoretical physics are formulated in terms of differential equations.
  - Newton's Laws
  - Maxwell equations
- Since the dynamics of many physical systems involve just two derivatives, DE of second order occur most frequently in physics.
  - e.g. acceleration in classical mechanics $F = ma = m\frac{d^2x}{dt^2}$

Differential and Difference Equations

- Differential equations are great for modeling situations where there is a continually changing population or value.
- If the change happens incrementally rather than continuously then differential equations have their shortcomings.
- Difference equations relate to differential equations as discrete mathematics relates to continuous mathematics.

Difference Equations in Computer Science

- Arise when determining the cost of an algorithm in big-O notation.
- Readily handled by program
  - A standard approach to solving a nasty differential equation is to convert it to an approximately equivalent difference equation.

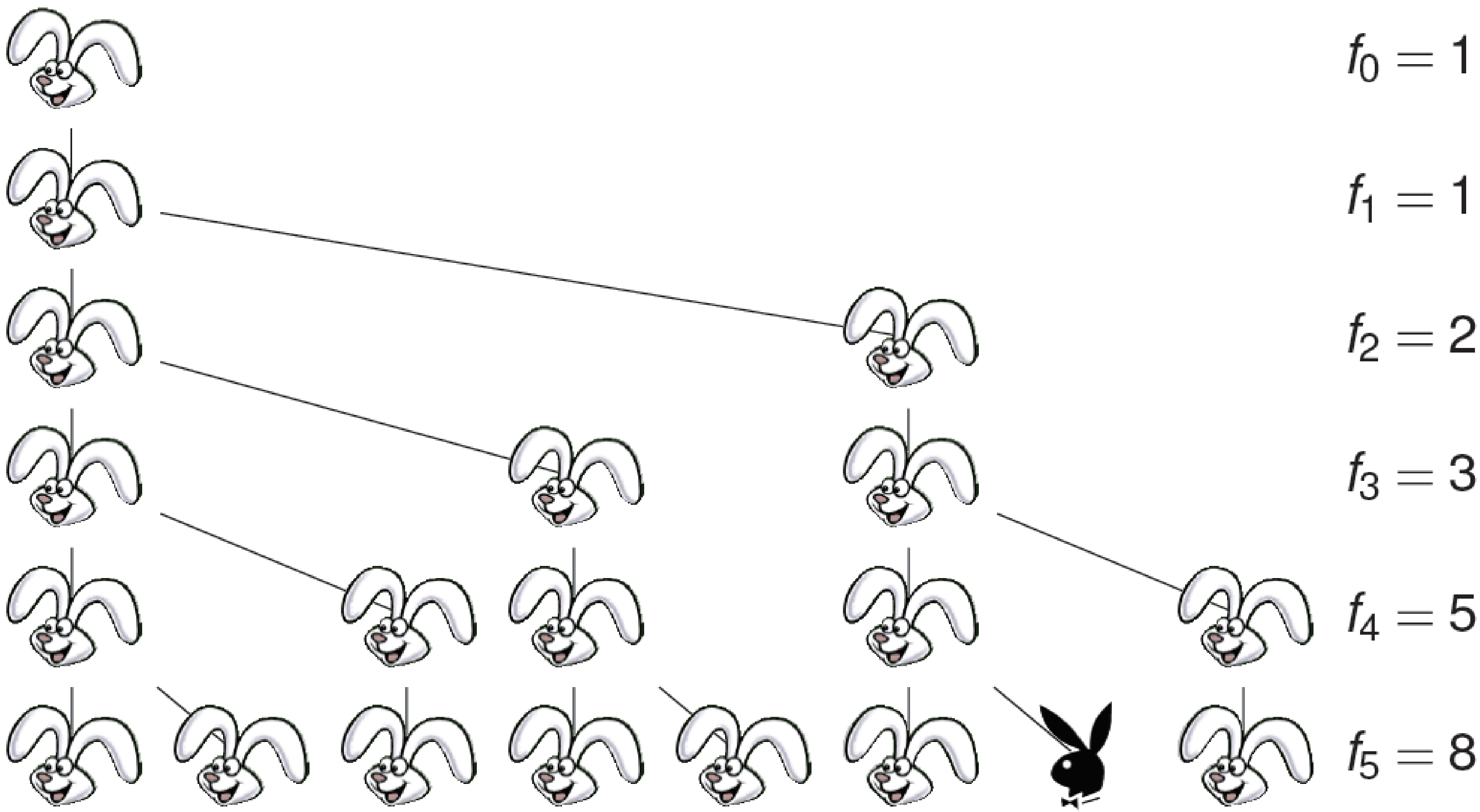
Recurrence Relation

- Initial terms are given.
- Each further term of the sequence is defined as a function of the preceding terms.

A Famous Math Problem

"*A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also. Because the abovewritten pair in the first month bore, you will double it; there will be two pairs in one month.*"

Leonardo of Pisa
(1170s or 1180s–1250)
a/k/a Fibonacci


$f_0 = 1$
$f_1 = 1$
$f_2 = 2$
$f_3 = 3$
$f_4 = 5$
$f_5 = 8$

An Equation

- Let f_n be the number of pairs of rabbits in month n, each new month we have
  - The same rabbits as last month
  - Every pair of rabbits at least one month old producing a new pair of rabbits

So $f_n = f_{n-1} + f_{n-2}, f_0 = 1, f_1 = 1$

Difference Equation

- A difference equation is an equation for a sequence written in terms of that sequence and shiftings of it.

- Example:

The fibonacci sequence satisfies the difference equation $f_n = f_{n-1} + f_{n-2}, f_0 = 1, f_1 = 1.$

Objectives

- Plot the equation over time
- Get an analytical solution for the difference equation if it is available
- Determine the equilibrium values
- Analyze stability of equilibria

Typical Cases

- $x_{k+1} = ax_k$ exponential growth
- $x_{k+1} =$ a a horizontal line
- $x_{k+1} = x_k +$ a a straight sloping line

- First order difference equation:

$$x_{k+1} = ax_k + b$$

Equilibrium

- At equilibrium, $x_{k+1} = x_k = x^*$

- Example: for first order difference equation $x_{k+1} = ax_k + b$
  - At equilibrium, $x^* = ax^* + b$

  => $x^* = b/(1-a)$ (if $a \neq 1$)

Analytical Solution

- $x_1 = ax_0 + b$
- $x_2 = ax_1 + b = a(ax_0 + b) + b = a^2x_0 + (\mathrm{a} + 1)\mathrm{b}$
- $x_3 = ax_2 + b = a(a^2x_0 + (\mathrm{a} + 1)\mathrm{b}) + b$
  $= a^3x_0 + (a^2 + \mathrm{a} + 1)\mathrm{b}$
- …
- $x_k = a^kx_0 + \left(a^{k-1} + \cdots + \mathrm{a} + 1\right)\mathrm{b}$

Analytical Solution

- $x_k = a^k x_0 + \left(a^{k-1} + \cdots + \mathrm{a} + 1\right)\mathrm{b}$

Since $a^{n-1} + \cdots + \mathrm{a} + 1 = \frac{1-a^k}{1-\mathrm{a}}$,

we have $x_k = a^k x_0 + \frac{\mathrm{b}\left(1-a^k\right)}{1-\mathrm{a}}$ (if $a \neq 1$)

or $x_k = \frac{\mathrm{b}}{1-\mathrm{a}} + [x_0 - \frac{\mathrm{b}}{1-\mathrm{a}}]\, a^k$,

($\frac{\mathrm{b}}{1-\mathrm{a}}$ is the equilibrium value).

Otherwise if $a = 1, x_k = x_0 + kb$.

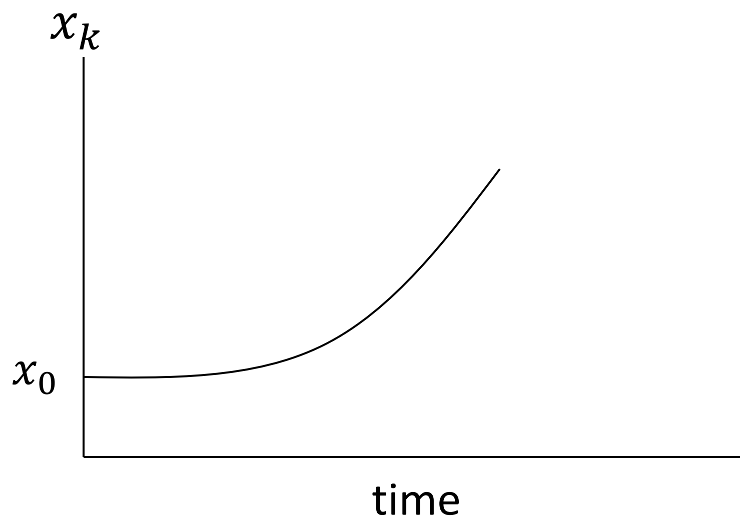
Equilibrium and Stability Criteria

$$x_{k+1} = ax_k + b, x^* = b/(1 - a)$$

- The difference equation has a unique equilibrium when $a \neq 1$.
- The equilibrium is
  - stable if $|a| < 1$
  - unstable if $|a| > 1$
  - x_k will oscillate if $\mathrm{a} < 0$
  - x_k will change monotonically if $\mathrm{a} > 0$
  - …

$\mathrm{a} > 1,\ x_0 > b/(1-a)$

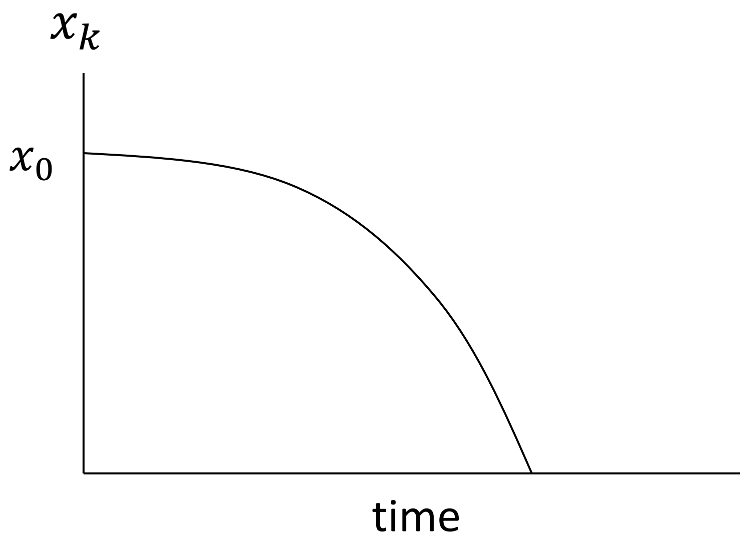
Increasing, monotonic, unbounded



$$x_k = \frac{\mathrm{b}}{1-\mathrm{a}} + \left[x_0 - \frac{\mathrm{b}}{1-\mathrm{a}}\right] a^k$$

$a > 1, x_0 < b/(1-a)$

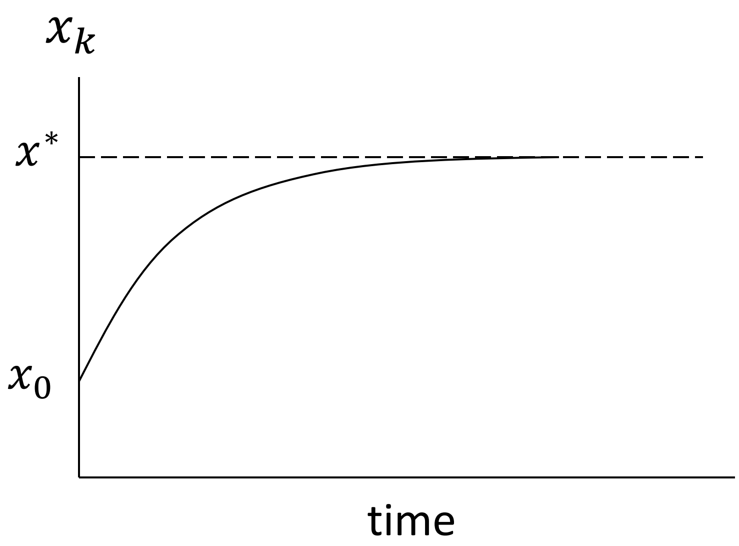
Decreasing, monotonic, unbounded



$$x_k = \frac{b}{1-a} + \left[x_0 - \frac{b}{1-a}\right] a^k$$

$0 < \mathrm{a} < 1, x_0 < b/(1-a)$

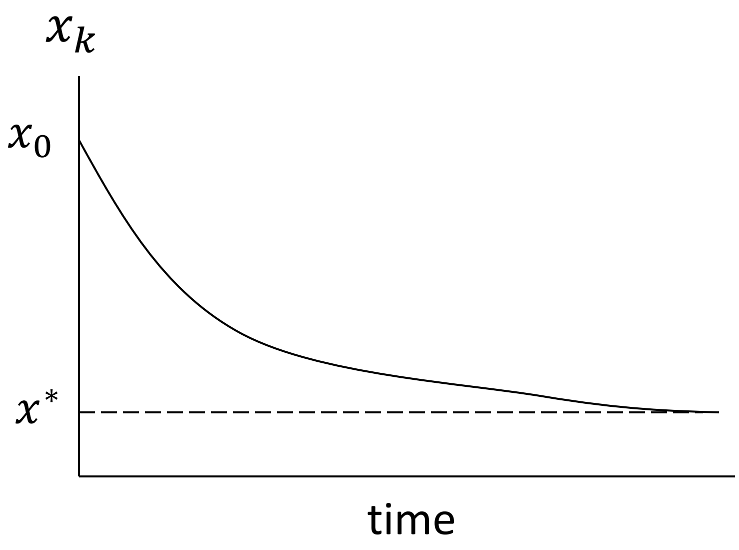
Increasing, monotonic, bounded, convergent



$$x_k = \frac{\mathrm{b}}{1-\mathrm{a}} + \left[x_0 - \frac{\mathrm{b}}{1-\mathrm{a}}\right] a^k$$

$0 < \mathrm{a} < 1, x_0 > b/(1-a)$

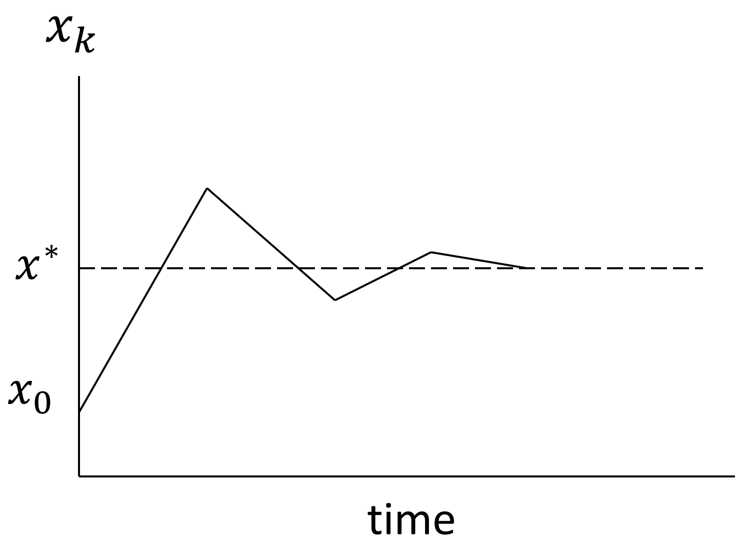
Decreasing, monotonic, bounded, convergent



$$x_k = \frac{\mathrm{b}}{1-\mathrm{a}} + \left[x_0 - \frac{\mathrm{b}}{1-\mathrm{a}}\right] a^k$$

$-1 < a < 0$

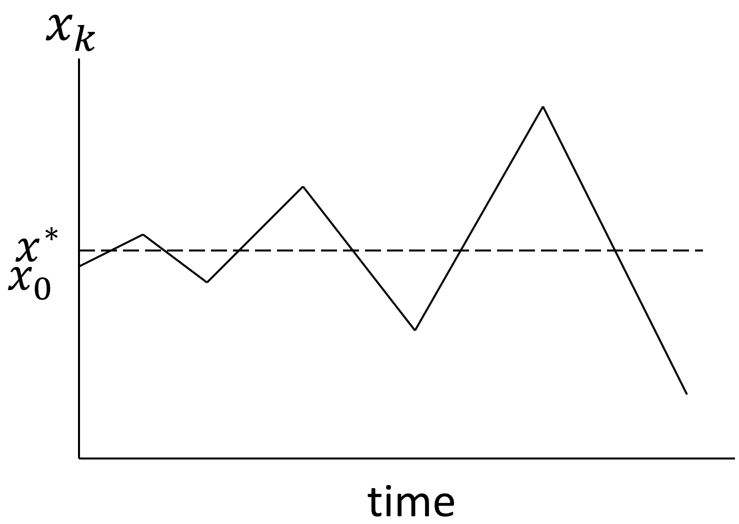
Bounded, oscillatory, convergent



$$x_k = \frac{\mathrm{b}}{1-\mathrm{a}} + \left[x_0 - \frac{\mathrm{b}}{1-\mathrm{a}}\right] a^k$$

$a < -1$

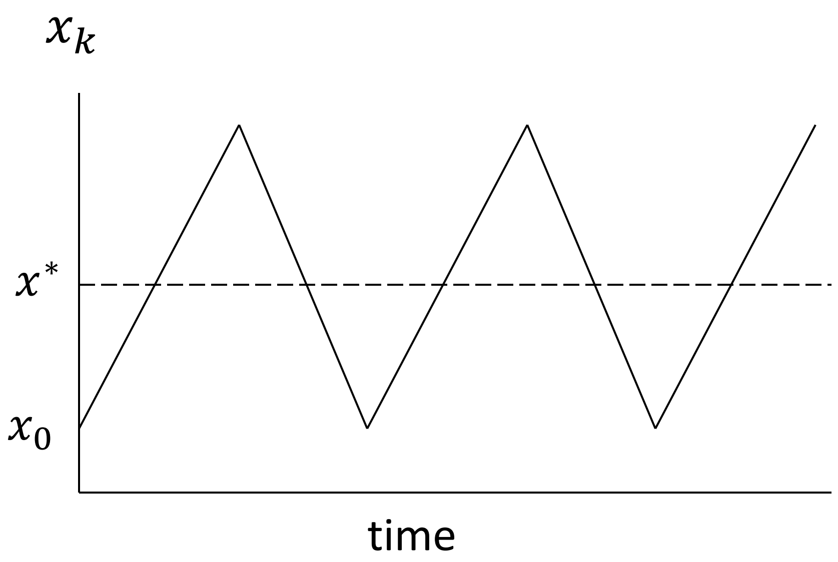
Unbounded, oscillatory, divergent



$$x_k = \frac{b}{1-a} + \left[x_0 - \frac{b}{1-a}\right] a^k$$

$a = -1$

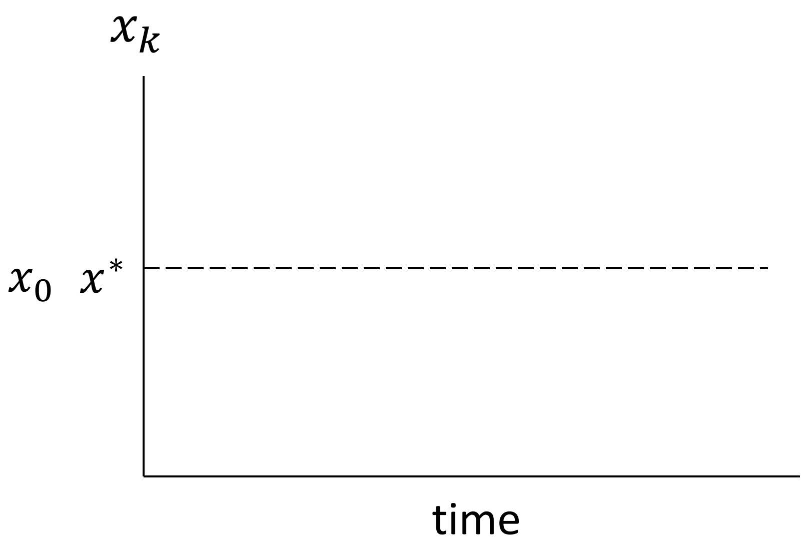
Finite, bounded oscillatory



$$x_k = \frac{b}{1-a} + \left[x_0 - \frac{b}{1-a}\right] a^k$$

$a = 1, b = 0$

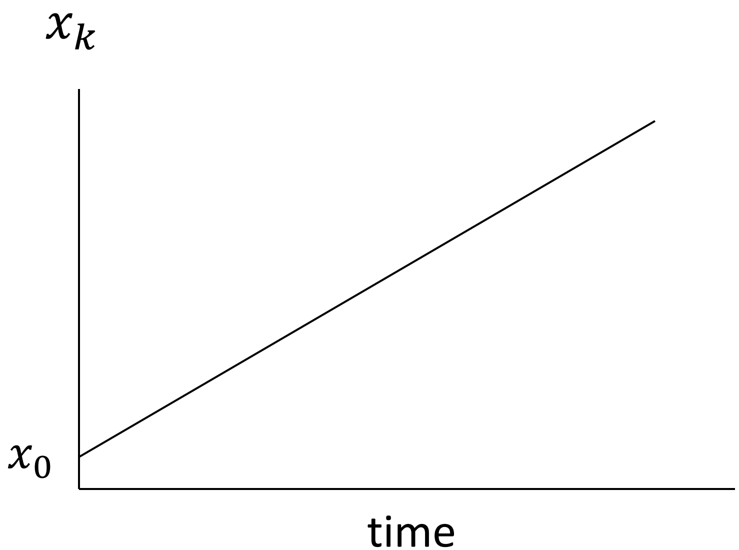
Constant



$$x_k = x_0 + kb$$

$$\mathrm{a} = 1, b > 0$$

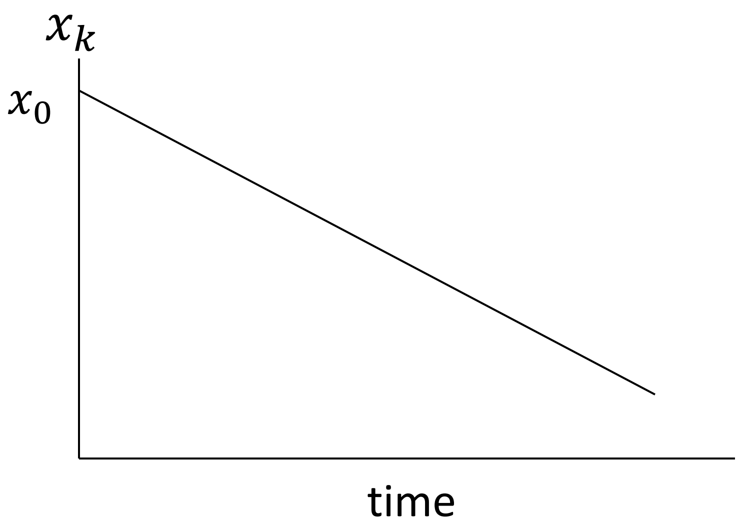
Constant increasing



$$x_k = x_0 + kb$$

$\mathrm{a} = 1, b < 0$

Constant decreasing



$$x_k = x_0 + kb$$

Rules of Interpretation

- $|a| > 1$ unbounded

  [repelled from line $b/(1-a)$]
- $|a| < 1$ bounded

  [attracted or convergent to $b/(1-a)$]
- $a < 0$ oscillatory
- $a > 0$ monotonic
- $a = -1$ bounded oscillatory

All of this can be deduced from the solution

$$x_n = \frac{\mathrm{b}}{1-\mathrm{a}} + \left[x_0 - \frac{\mathrm{b}}{1-\mathrm{a}}\right] a^n \text{, for } a \neq 1$$

Special Cases

- $\mathrm{a} = 1, b = 0$ constant
- $\mathrm{a} = 1, b > 0$ constant increasing
- $\mathrm{a} = 1, b < 0$ constant decreasing

Testing Example I

- A: show that the sequence defined by $y_k = 2^{k+1} - 1$, satisfies the difference equation y_{k+1}=$2y_k + 1$, $y_0 = 1$.

- B: for difference equation y_{k+1}=$2y_k + 1$, $y_0 = 1$, find its analytic solutions.

Testing Example II

- Fill out the first few terms of the sequence that satisfies $y_{k+1} = \frac{y_k}{1+y_k}, y_1 = 1$, find the solutions.

Induction

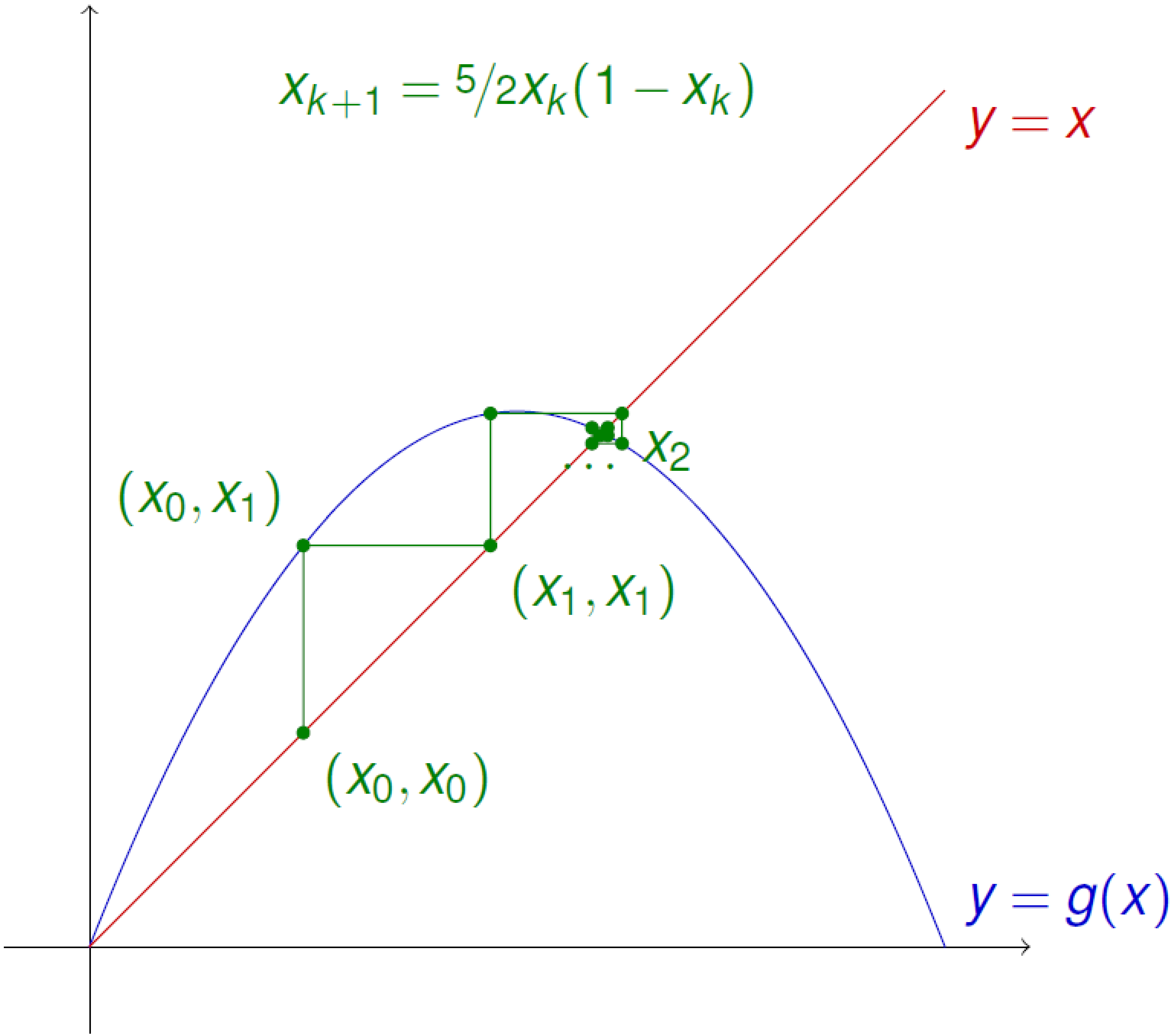
- The simplest and most common form of mathematical induction infers that a statement involving a natural number k holds for all values of k. The proof consists of two steps:
  - The **base case**: prove that the statement holds for the first natural number k. Usually, k = 0 or k = 1.
  - The **inductive step**: prove that, if the statement holds for some natural number k, then the statement holds for $k + 1$.

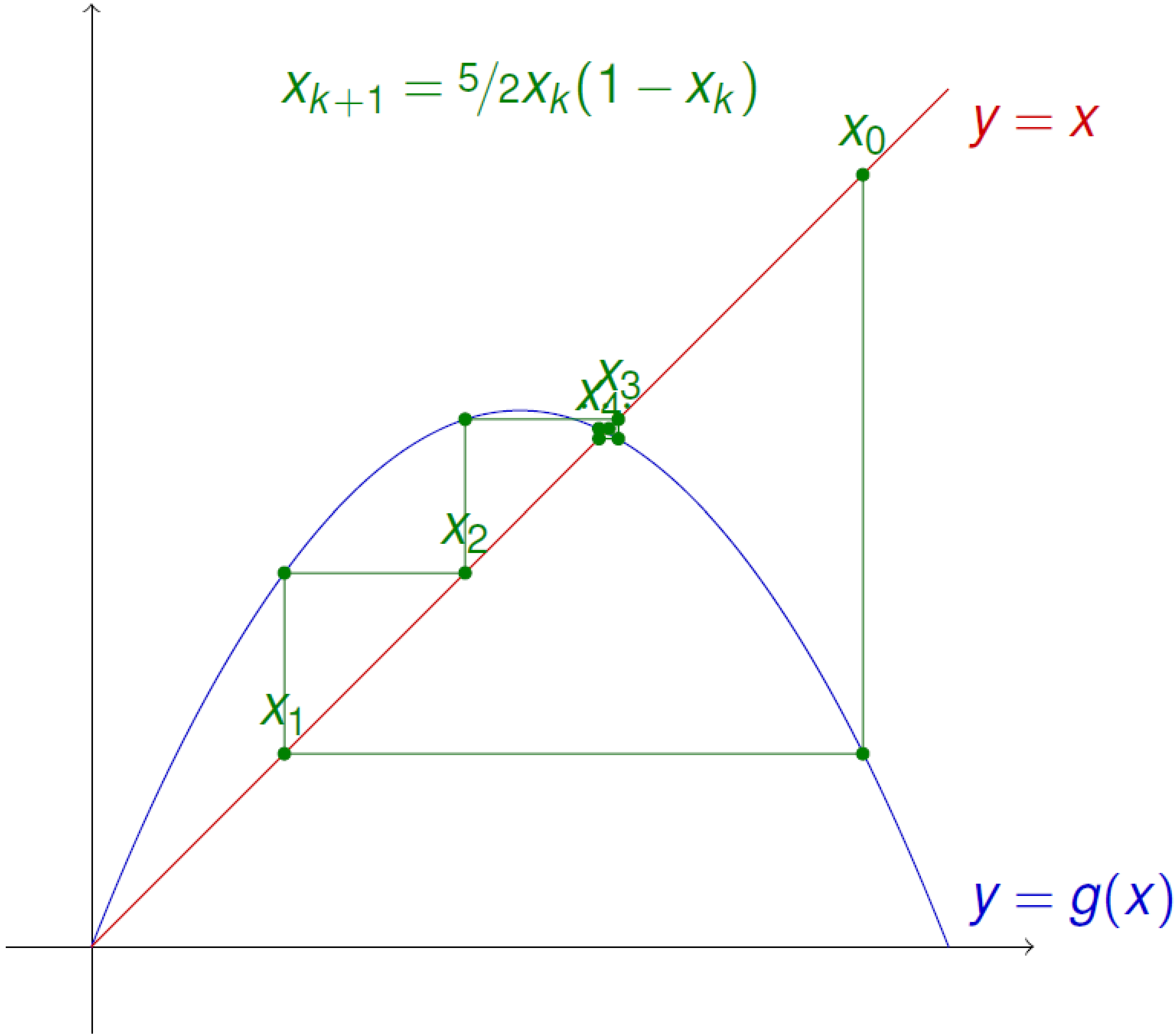
Analyze the Equilibrium

- Example: Cobweb diagrams
  - A visual tool used in the dynamical systems field of mathematics to investigate the qualitative behaviour of iterated functions, such as the logistic map.
  - Logistic map: a recurrence relation of degree 2, often cited as an archetypal example of how complex, chaotic behaviour can arise from very simple non-linear dynamical equations.

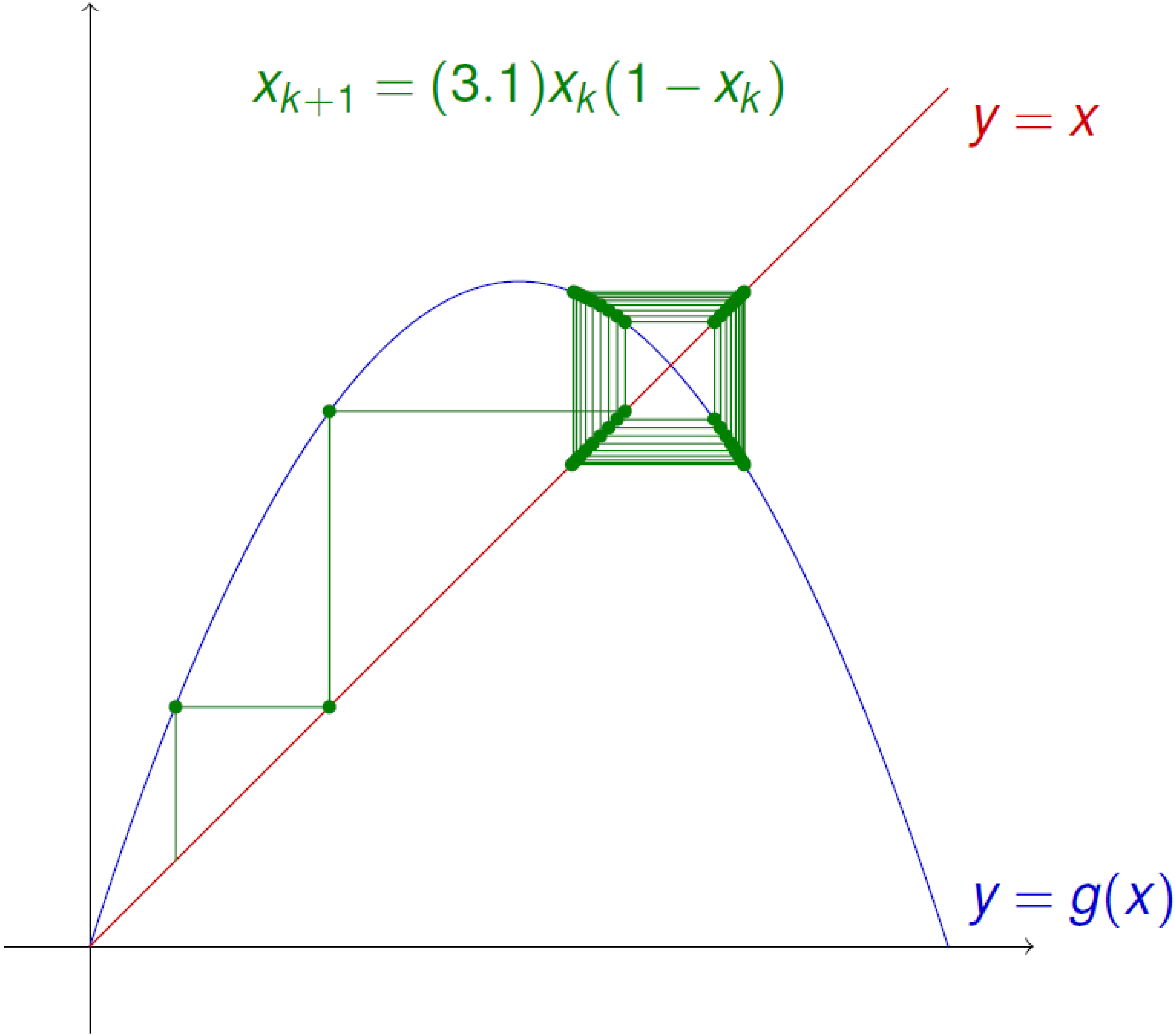
Analyze the Equilibrium

- Example: Cobweb diagrams
  - Idea: use graphics to identify and classify equilibria of the difference equation $x_{k+1} = g(x_k)$.
  - Method:
    - Draw the graphs $\mathrm{y} = g(\mathrm{x})$ and $\mathrm{y} = \mathrm{x}$
    - Pick a point (x_0, x_0) on the line
    - Move vertically to $(x_0, f(x_0))$
    - Move horizontally to (x_1, x_1)
    - Repeat .


$x_{k+1} = {}^5\!/_2 x_k(1 - x_k)$
$y = x$
x_2
$\dots$
(x_0, x_1)
(x_1, x_1)
(x_0, x_0)
$y = g(x)$



$x_{k+1} = {}^5/_2 x_k(1 - x_k)$
$y = x$
x_0
x_1
x_2
x_3
x_4
$y = g(x)$



$x_{k+1} = (3.1)x_k(1 - x_k)$
$y = x$
$y = g(x)$

- Equilibria of the difference equation $x_{k+1} = g(x_k)$ are solutions to the equation $x = g(x)$.
  - If an equilibrium is stable, nearby points will spiral towards it.
  - If an equilibrium is unstable, nearby points will spiral away from it.

- Given the equation $x_{k+1} = rx_k(1 - x_k)$, find the equilibria in terms of r.

Solution: since $x_{k+1} = x_k = x^*$,
solving $x^* = rx^*(1 - x^*)$ gives $x^* = 0$ and $1 - 1/r$.

- Make a conjecture about the stability of a fixed point to a nonlinear system $x_{k+1} = g(x_k)$.

Solution: an equilibrium solution x^*

is stable if $|g'(x^*)| < 1$, and

unstable if $|g'(x^*)| > 1$.

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Some of the materials are from the Internet and the textbooks