EE3731C:
Signal Processing Methods

Tutorial II-3

National University of Singapore

Question #1

A one dimensional random walk starts at $S_0 = 0$ and at each step moves by ±1 with equal probability. To define this walk formally, take independent random variables $Z_1, Z_2, \ldots,$ each of which is 1 with probability 1/2 and −1 with probability 1/2, and set

$$S_n = Z_1 + Z_2 + \ldots + Z_n$$

This sequence $\{S_n\}$ is called the simple random walk on the integers.

(i) Find $E[S_n]$ and $E[S_n^2]$.

(ii) What is the root mean square displacement of the walk after n steps?

Question #1: Solution

The PDF of Z_i is $P(Z_i) = \begin{cases} 1/2 & \text{if } Z_i = 1 \\ 1/2 & \text{if } Z_i = -1 \end{cases}$

So $$E[Z_i] = \frac{1}{2} \times 1 + \frac{1}{2} \times (-1) = 0$$

$$\boxed{S_n = Z_1 + Z_2 + \cdots + Z_n}$$

$$E[Z_i^2] = \frac{1}{2} \times (1)^2 + \frac{1}{2} \times (-1)^2 = 1$$

$$E[Z_i Z_j] = 0, \ \forall\, i \neq j$$

$$E[S_n] = E[Z_1 + Z_2 + \cdots + Z_n] = \sum_{i=1}^{n} E[Z_i] = 0$$

$$E[S_n^2] = E\left[(Z_1 + Z_2 + \cdots + Z_n)^2\right] = \sum_{i=1}^{n} E[Z_i^2] = n$$

The root mean square displacement of the walk after *n* steps is $\sqrt{n}$

Question #2

Let $w[n]$ denote a Gaussian white noise sequence with mean zero and variance 1. Determine the mean and autocorrelation functions of $x[n]$ in the following cases.

(i) $x[n] = 0.8\, x[n-1] + 0.5\, w[n]$

(ii) $x[n] = 0.4w[n] - 0.1w[n-1]$

Hints: Since $x[n]$ is generated by the same rule at all times, $x[n]$ will be stationary. When $x[n\text{-}1]$ is a function past inputs and past noise values, it is independent of $w[n]$ because the noise is independent.

Question #2: Solution

(i) $x[n] = 0.8\, x[n-1] + 0.5\, w[n]$

$$\begin{aligned} m_x &= E\{0.8x[n-1]+0.5w[n]\} \\ &= 0.8E\{x[n-1]\}+0.5E\{w[n]\} \\ &= 0.8m_x \end{aligned}$$

$$\begin{cases} E\{w[n]\}=0 \\ E\{w[n]w[n+m]\}=\delta[m] \end{cases}$$

$$m_x = 0.8m_x \Rightarrow m_x = 0$$

Since $x[n-1]$ is a function past inputs and past noise values, and is therefore independent of $w[n]$.

$$E\{x[n-1]w[n]\} = E\{x[n-1]\}E\{w[n]\} = 0$$

Question #2: Solution

(i) $x[n] = 0.8\, x[n-1] + 0.5\, w[n]$

$$R_x[m] = E\{(0.8x[n-1] + 0.5w[n])(0.8x[n+m-1] + 0.5w[n+m])\}$$
$$= 0.64R_x[m] + 0.25R_w[m] + 0.4R_{wx}[m-1] + 0.4R_{xw}[m+1]$$

$$0.36R_x[m] = 0.25R_w[m] + 0.4R_{wx}[m-1] + 0.4R_{xw}[m+1]$$

$$R_x[m] = \frac{25}{36}\delta[m] + \frac{10}{9}R_{wx}[m-1] + \frac{10}{9}R_{xw}[m+1]$$

$$E\{x[n-1]w[n]\} = E\{x[n-1]\}E\{w[n]\} = 0$$

$$R_{xw}[m] = 0 \quad \forall m > 0$$
$$R_{wx}[m] = 0 \quad \forall m < 0$$

$$R_{xw}[m] = R_{wx}[-m]$$

Question #2: Solution

(i) $x[n] = 0.8\, x[n-1] + 0.5\, w[n]$

$$m=0:\quad R_{wx}[0]=E\{w[n]x[n]\}=E\{w[n](0.8x[n-1]+0.5w[n])\}=0.5$$

$$m>0:\quad R_{wx}[m]=E\{w[n]x[n+m]\}$$
$$=E\{[w[n](0.8x[n+m-1]+0.5w[n+m])]\}=0.8R_{wx}[m-1]$$

$$R_{wx}[m]=\begin{cases}0.5\times(0.8)^m, & m\geq 0\\ 0, & \text{otherwise}\end{cases}\Rightarrow R_{wx}[m]=0.5\times(0.8)^m u[m]$$

$$R_x[m]=\frac{25}{36}\delta[m]+\frac{10}{9}R_{wx}[m-1]+\frac{10}{9}R_{xw}[m+1]\qquad R_{xw}[m]=R_{wx}[-m]$$

$$R_x[m]=\frac{25}{36}\delta[m]+\frac{5}{9}\times\left[(0.8)^{(m-1)}u[m-1]+(0.8)^{-(m+1)}u[-(m+1)]\right]=\frac{25}{36}(0.8)^{|m|}$$

Question #2: Solution

(ii) $x[n] = 0.4w[n] - 0.1w[n-1]$

$$m_x = E\{0.4w[n] - 0.1w[n-1]\} = 0.4E\{w[n]\} - 0.1E\{w[n-1]\} = 0$$

$$\begin{aligned}R_x[m] &= E\{(0.4w[n] - 0.1w[n-1])(0.4w[n+m] - 0.1w[n+m-1])\}\\ &= 0.16R_w[m] - 0.04R_w[m-1] - 0.04R_w[m+1] + 0.01R_w[m]\\ &= 0.17R_w[m] - 0.04R_w[m-1] - 0.04R_w[m+1]\\ &= 0.17\delta[m] - 0.04\delta[m-1] - 0.04\delta[m+1]\end{aligned}$$

Question #3

Let $w[n]$ denote a Gaussian white noise sequence with mean zero and variance 1. Find the power spectrum of the random process $x[n]$ in the following cases.

(i) $x[n] = 0.8\,x[n-1] + 0.5\,w[n]$

(ii) $x[n] = 0.4w[n] - 0.1w[n-1]$

Two alternative solutions:

- Find the autocorrelation function and take its Fourier transform.
- Find the transfer function H(z) and use the following relationship to obtain the power spectrum.

$$\boxed{S_y\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right|^2 S_x\left(e^{j\omega}\right)}$$

Question #3: Solution

(i) $x[n] = 0.8\, x[n-1] + 0.5\, w[n]$ $\quad S_w(e^{j\omega}) = 1$

Take Z transform of both sides, we get:

$$X(z) = 0.8X(z)z^{-1} + 0.5W(z)$$

$$X(z) - 0.8X(z)z^{-1} = 0.5W(z)$$

$$H(z) = \frac{X(z)}{W(z)} = \frac{0.5}{1-0.8z^{-1}} \qquad \left|H(e^{j\omega})\right|^2 = H(e^{j\omega})H(e^{-j\omega})$$

$$S_x(e^{j\omega}) = \left|H(e^{j\omega})\right|^2 S_w(e^{j\omega}) = \left|H(e^{j\omega})\right|^2 = \frac{0.5}{(1-0.8e^{-j\omega})}\frac{0.5}{(1-0.8e^{j\omega})}$$

$$S_x(e^{j\omega}) = \frac{0.25}{(1-0.8e^{-j\omega})(1-0.8e^{j\omega})} = \frac{0.25}{1.64-1.6\cos(\omega)}$$

Question #3: Solution

(ii) $x[n] = 0.4w[n] - 0.1w[n-1]$

Take Z transform of both sides, we get:

$$X(z) = 0.4W(z) - 0.1W(z)z^{-1}$$

$$X(z) = \left(0.4 - 0.1z^{-1}\right)W(z)$$

$$H(z) = \frac{X(z)}{W(z)} = 0.4 - 0.1z^{-1} \qquad S_w\left(e^{j\omega}\right) = 1$$

$$S_x\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right|^2 S_w\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right|^2 = \left(0.4 - 0.1e^{-j\omega}\right)\left(0.4 - 0.1e^{+j\omega}\right)$$

$$S_x\left(e^{j\omega}\right) = 0.17 - 0.04e^{-j\omega} - 0.04e^{+j\omega} = 0.17 - 0.08\cos(\omega)$$

Question #4

Let a WSS random process $x[n]$ be the input to an LTI system with known impulse response $h[n]$. Suppose that the input $x[n]$ is not observable, but we can measure the output $y[n]$.

i. Propose an estimate of the power spectrum of $y[n]$ (i.e., describe the processing you would apply to the observed output y[n] to estimate the power spectrum $S_y(e^{j\omega})$). Hint: Consider using the DFT to estimate the "power" of the output at each frequency.

ii. How would you use the estimate of $S_y(e^{j\omega})$ to estimate the power spectrum of the unobservable input $x[n]$?

Question #4: Solution

i. Propose an estimate of the power spectrum of $y[n]$.

Suppose $y[n]$ is of length N

$$S_y\left(e^{j\omega}\right)=\frac{1}{N}\left|\sum_{n=0}^{N-1} y[n]e^{-j\omega n}\right|^2$$

Averaged Periodogram:

1) Divide $y[n]$ into K non-overlapping segments of length M.
2) For each segment, compute its M-point DFT to obtain its periodogram.

$$S_y\left(e^{j\omega_k}\right)=\frac{1}{M}\left|\sum_{n=0}^{M-1} y[n]e^{-j\omega_k n}\right|^2, \ \omega_k=\frac{2\pi k}{M}$$

3) Average the periodograms over all K segments to obtain the averaged periodogram.

Question #4: Solution

ii. How would you use the estimate of $S_y(e^{j\omega})$ to estimate the power spectrum of the unobservable input $x[n]$?

$$S_y(f) = |H(f)|^2 S_x(f)$$

$$S_y\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right|^2 S_x\left(e^{j\omega}\right)$$

Since the impulse response $h[n]$ is known, we can compute

$$\left|H\left(e^{j\omega}\right)\right|^2 = \left|\sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}\right|^2$$

Question #5

A zero mean noise process $x[n]$ has the following autocorrelation function

$$R_x(m) = \left(\frac{1}{2}\right)^{|m|}$$

We process $x[n]$ with a linear time-invariant system that satisfies the following difference equation:

$$y[n] = x[n+1] + x[n-1]$$

What is the autocorrelation function of y[n]?

Question #5: Solution

$$y[n] = x[n+1] + x[n-1] \qquad\qquad R_x(m) = \left(\frac{1}{2}\right)^{|m|}$$

Autocorrelation function of y[*n*]:

$$\begin{aligned} R_y[m] &= E\{y[n]y[n+m]\} \\ &= E\{(x[n+1] + x[n-1])(x[n+m+1] + x[n+m-1])\} \\ &= 2R_x[m] + R_x[m-2] + R_x[m+2] \end{aligned}$$

Alternatively, $R_y[m] = R_x[m] * h[m] * h[-m]$

$$h[m] = \delta[m+1] + \delta[m-1]$$