Tutorial 1 - EE3731C Signal Processing Methods

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Solutions

1. a. Sutstitute:

$$y = cx, dy = cdx$$

If $c > 0$,

$$\begin{aligned} FT(f(cx)) &= \int_{-\infty}^{+\infty} f(cx)e^{-i2\pi ux}dx \\ &= \frac{1}{c}\int_{-\infty}^{+\infty} f(y)e^{-i2\pi \frac{u}{c} y}dy \\ &= \frac{1}{c}F(\frac{u}{c}) \end{aligned}$$

If $c < 0$,

$$\begin{aligned} FT(f(cx)) &= \int_{-\infty}^{+\infty} f(cx)e^{-i2\pi ux}dx \\ &= -\frac{1}{c}\int_{-\infty}^{+\infty} f(y)e^{-i2\pi \frac{u}{c} y}dy \\ &= -\frac{1}{c}F(\frac{u}{c}) \end{aligned}$$

Hence,

$$FT(f(cx)) = \frac{F(\frac{u}{c})}{|c|}$$

This is a general feature of Fourier transform. Compressing one of the $f(x)$ and $F(u)$ will stretch the other and vice versa.

b.

$$\begin{aligned}
f(x) * g(x) &= \int_{-\infty}^{+\infty} f(t)g(x-t)dt \\
FT(f(x) * g(x)) &= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(t)g(x-t)dt \right) e^{-i2\pi ux}dx \\
&= \int_{-\infty}^{+\infty} f(t) \left(\int_{-\infty}^{+\infty} g(x-t)e^{-i2\pi ux}dx \right) dt \\
&= \int_{-\infty}^{+\infty} f(t)e^{-i2\pi ut}G(u)dt \\
&= F(u) \cdot G(u)
\end{aligned}$$

2. a.

$$\begin{aligned}
\cos\phi &= \frac{e^{i\phi} + e^{-i\phi}}{2} \\
\cos(2\pi Ax) &= \frac{e^{i2\pi Ax} + e^{-i2\pi Ax}}{2} \\
FT(\cos(2\pi Ax)) &= \int_{-\infty}^{+\infty} \frac{e^{i2\pi Ax} + e^{-i2\pi Ax}}{2} e^{-i2\pi ux}dx \\
&= \frac{1}{2} \left(\int_{-\infty}^{+\infty} e^{-i2\pi(u-A)x}dx + \int_{-\infty}^{+\infty} e^{-i2\pi(u+A)x}dx \right) \\
&= \frac{1}{2}[\delta(u-A) + \delta(u+A)]
\end{aligned}$$

b.

$$\begin{aligned}
\sin\phi &= \frac{e^{i\phi} - e^{-i\phi}}{2i} \\
\sin(2\pi Ax) &= \frac{e^{i2\pi Ax} - e^{-i2\pi Ax}}{2i} \\
FT(\sin(2\pi Ax)) &= \int_{-\infty}^{+\infty} \frac{e^{i2\pi Ax} - e^{-i2\pi Ax}}{2i} e^{-i2\pi ux}dx \\
&= \frac{1}{2i} \left(\int_{-\infty}^{+\infty} e^{-i2\pi(u-A)x}dx - \int_{-\infty}^{+\infty} e^{-i2\pi(u+A)x}dx \right) \\
&= \frac{1}{2i}[\delta(u-A) - \delta(u+A)]
\end{aligned}$$

3.

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} 1dx + \frac{1}{\pi}\int_{-\pi}^{0}(-1)dx \\
&= 0 \\
a_m &= \frac{1}{\pi}\int_{0}^{\pi} \cos mxdx + \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cos mxdx \\
&= \left.\frac{1}{m\pi}\sin mx\right|_{0}^{\pi} + \left.\frac{1}{m\pi}\sin mx\right|_{-\pi}^{0} \\
&= \frac{1}{m\pi}(\sin m\pi - \sin 0 + \sin 0 + \sin m\pi) \\
&= 0 \\
b_m &= \frac{1}{\pi}\int_{0}^{\pi} \sin mxdx + \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin mxdx \\
&= \left.\frac{1}{m\pi}\cos mx\right|_{0}^{\pi} + \left.\frac{1}{m\pi}\cos mx\right|_{-\pi}^{0} \\
&= \frac{1}{m\pi}(-\cos m\pi + \cos 0 + \cos 0 - \cos m\pi) \\
&= \frac{2}{m\pi}(1-\cos m\pi) \\
&= \begin{cases} \frac{4}{m\pi}, & m = 1,3,5\ldots \\ 0, & m = 2,4,6\ldots \end{cases} \\
f(x) &= \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right)
\end{aligned}
$$

Proof of 2D Rotation Matrix (Lecture 2)

$$
\begin{aligned}
x &= r\cos\alpha \\
y &= r\sin\alpha \\
x' &= r\cos(\theta+\alpha) \\
&= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\
y' &= r\sin(\theta+\alpha) \\
&= r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \\
x' &= x\cos\theta - y\sin\theta \\
y' &= y\cos\theta + x\sin\theta
\end{aligned}
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