Solutions (Quiz 1 & 2) - EE3731C Signal Processing Methods

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Quiz 1

1-1 (2 pt) For data (i) lying in a a-dimensional space, and (ii) with each dimension having b levels, what is the number of states?

(d) b^a

1-2 (2 pt) For a 600×800 grayscale image where each pixel is represented as an integer $\in [0, 255]$, what is the number of different images that can be generated?

(a) $256^{600\times 800}$

1-3 (2 pt) Which of the following is commonly used to reduce computation for high-dimensional data?

(b) Principal Component Analysis

For matrix $A = \begin{bmatrix} a & 100 \\ 1 & a \end{bmatrix}$,

2-1 (4 pt) calculate the eigenvalues of A;

$$\begin{aligned} &A - \lambda I = \begin{bmatrix} a-\lambda & 100 \\ 1 & a-\lambda \end{bmatrix} \\ &\det(A - \lambda I) = 0 \\ &\to (a-\lambda)^2 - 100 = 0 \\ &\to a = \pm 10 \end{aligned}$$

2-2 (3 pt) discuss whether A can be a covariance matrix when (i) $a = 1$; (ii) $a = 100$.

A cannot be a covariance matrix in either case, since A is not symmetric.

3-1 (4 pt) For difference equation $y_{k+1} = y_k - 1, y_0 = 0$, find its analytical solution.

$$\begin{aligned} x_k &= a^k x_0 + (a^{k-1} + \cdots + a + 1)b \qquad (a = 1, b = -1) \\ &= x_0 - k \\ &= -k \end{aligned}$$

3-2 (3 pt) For difference equation $y_{k+1} = y_k + (-1)^k, y_0 = 0$, the maximum value of the output y_k is

(a) 1

$$y_k = \begin{cases} 1 & k = 1, 3, 5, \ldots \\ 0 & k = 0, 2, 4, \ldots \end{cases}$$

Quiz 2

1-1 (2 pt) Which of the following can be applied to transform non-periodic signals to the frequency domain?

(b) Fourier transform

1-2 (2 pt) f and g are two periodic functions with period 2π, which of the following describes $h = f+g$?

(c) h is a periodic function with period 2π

1-3 (2 pt) Assume the Fourier transform of a function f is $FT(f) = F$, what does $F(0)$ (known as the DC component, referring to the constant, zero-frequency part of the signal) mean?

(c) mean(f)

2 (6 pt)

$$x_{k+1} = \frac{3}{2}x_k(1 - x_k)$$

2-1 (3 pt) Find its equilibrium $x^*(x^* \neq 0)$;

At equilibrium, $x^* = \frac{3}{2}x^*(1 - x^*)$, since $x^* \neq 0$,

$$\begin{aligned} 1 &= \frac{3}{2}(1 - x^*) \\ \rightarrow x^* &= 1 - \frac{2}{3} = \frac{1}{3} \end{aligned}$$

2-2 (3 pt) discuss the stability of x^*.

$$\begin{aligned} g(x) &= \frac{3}{2}x(1 - x) \\ g'(x) &= \frac{3}{2}(1 - 2x) \\ |g'(x^*)| &= \frac{3}{2}(1 - 2 \times \frac{1}{3}) = \frac{1}{2} < 1 \end{aligned}$$

The equilibrium x is stable.

3 (8 pt) Given a set of data points $(0, 0), (1, 5), (5, 1)$,

3-1 (4 pt) What are the mean and the covariance matrix of the data set? Note: when calculating the covariance matrix, use $1/n$ (n the number of data points) to normalize the data.

The sample mean is

$$\mu = \frac{1}{3}(0 + 1 + 5, 0 + 5 + 1) = (2, 2)$$

After translation the data matrix is

$$X = \begin{pmatrix} -2 & -1 & 3 \\ -2 & 3 & -1 \end{pmatrix}$$

The covariance matrix is

$$\frac{1}{3}XX^T = \frac{1}{3}\begin{pmatrix} -2 & -1 & 3 \\ -2 & 3 & -1 \end{pmatrix}\begin{pmatrix} -2 & -2 \\ -1 & 3 \\ 3 & -1 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 7 & -1 \\ -1 & 7 \end{pmatrix}$$

3-2 (4 pt) Compute the first PC of the given data set.

$$\begin{aligned} &\det\left[\begin{pmatrix} 7-\lambda & -1 \\ -1 & 7-\lambda \end{pmatrix}\right] = 0 \\ &\rightarrow (7 - \lambda)^2 = 1 \\ &\rightarrow \lambda = 6 \quad \text{or} \quad 8 \end{aligned}$$

When $\lambda = 8$,

$$\begin{aligned}
&\begin{pmatrix} 7-\lambda & -1 \\ -1 & 7-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 && (v_1^2 + v_2^2 = 1) \\
\rightarrow & v_1 + v_2 = 0 && (v_1^2 + v_2^2 = 1) \\
\rightarrow & v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}
\end{aligned}$$

Negation is also correct.