EE3731C:
Signal Processing Methods

Lecture II-3:
Random Signals I

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Outline

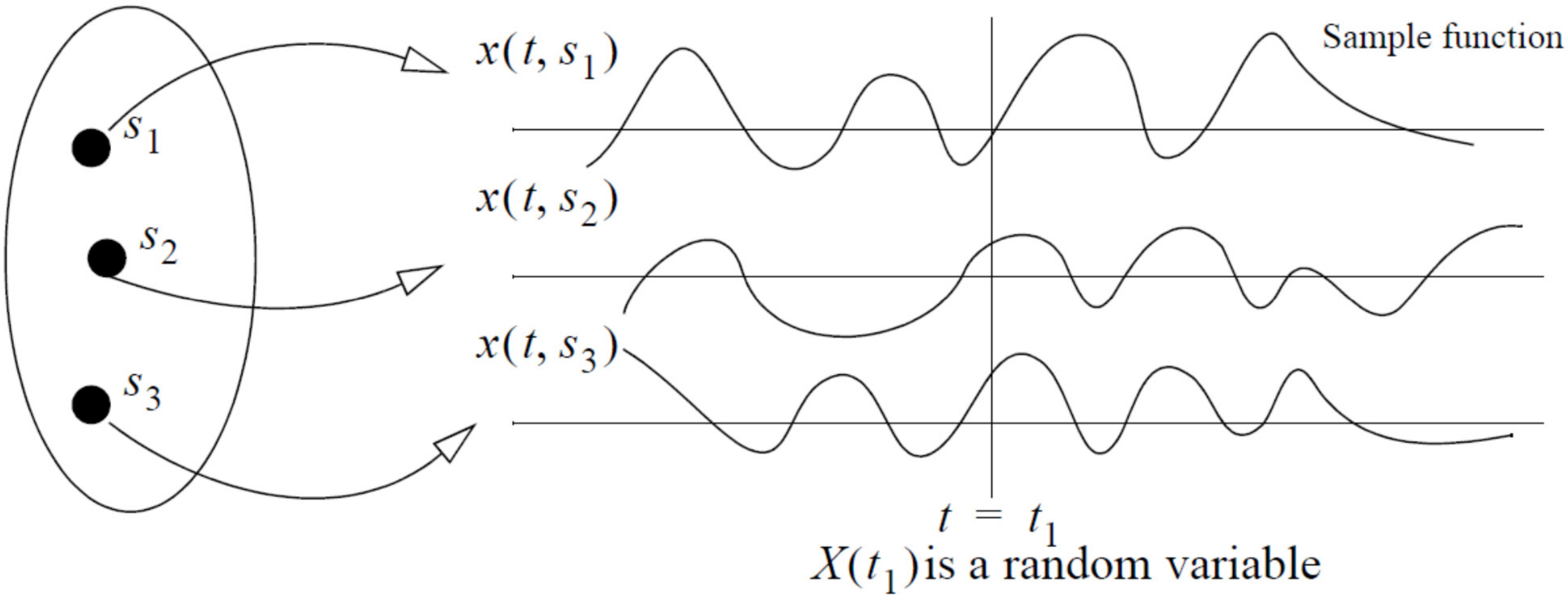
- Introduction to Random Signals
- Time Averages
- Ensemble Averages
- Stationary Processes
- Ergodic Processes
- Auto-correlation Function
- Cross-correlation Function

Deterministic & Nondeterministic (Random) Signal

- A deterministic signal: its value at any given instant of time can be
  - computed using a closed form mathematical function of time, OR
  - predicted from the knowledge of a few past values of the signal.
- A signal that does not satisfy the above condition is nondeterministic (random).

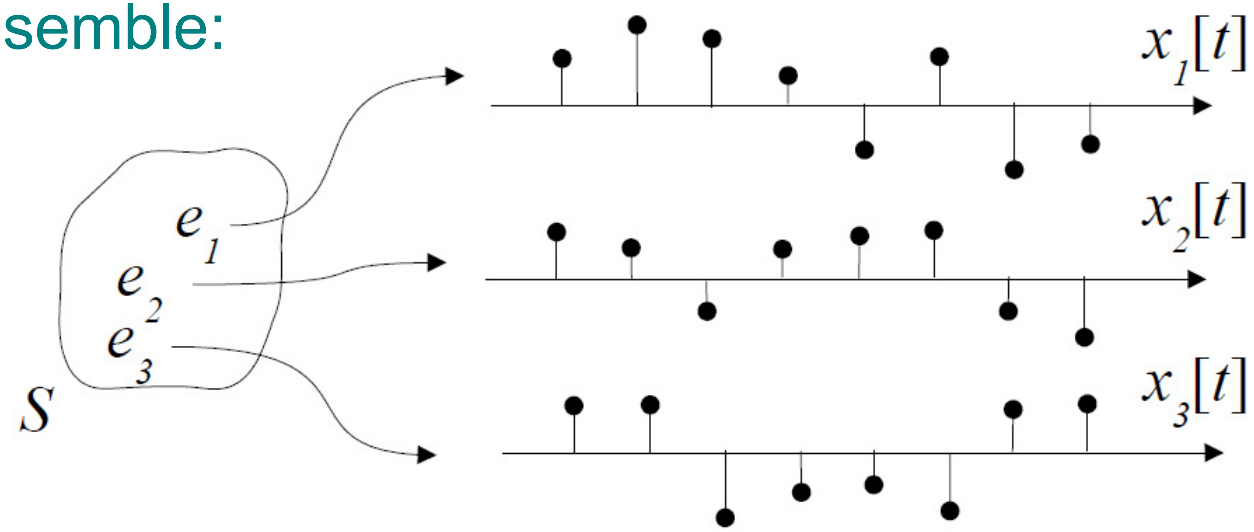
Stochastic (Random) Processes

- A stochastic (random) process is a random quantity that changes over time.
- A typical random process is specified by $X(t, s)$ where s is an outcome and t is time.



Stochastic (Random) Processes

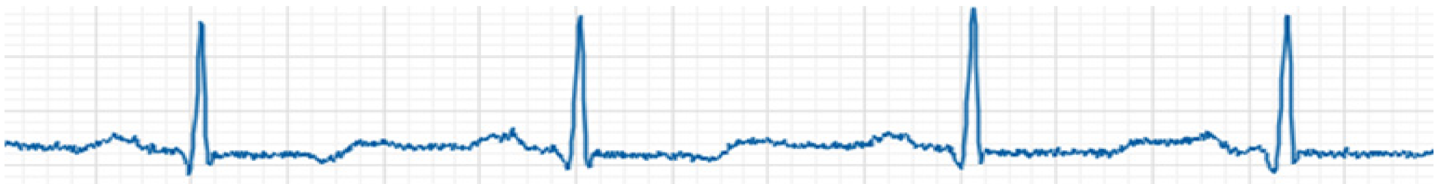
- A sample function is the time function associated with outcome of an experiment.
- The ensemble of a stochastic process is the set of all possible time functions that can result from an experiment.
  - Sampling a random process repeatedly gives an ensemble:



Random Processes vs. Random Variables

- For a random variable, the outcome of a random experiment is mapped onto variable, e.g., a number.
  - –E.g., outcome of tossing a die

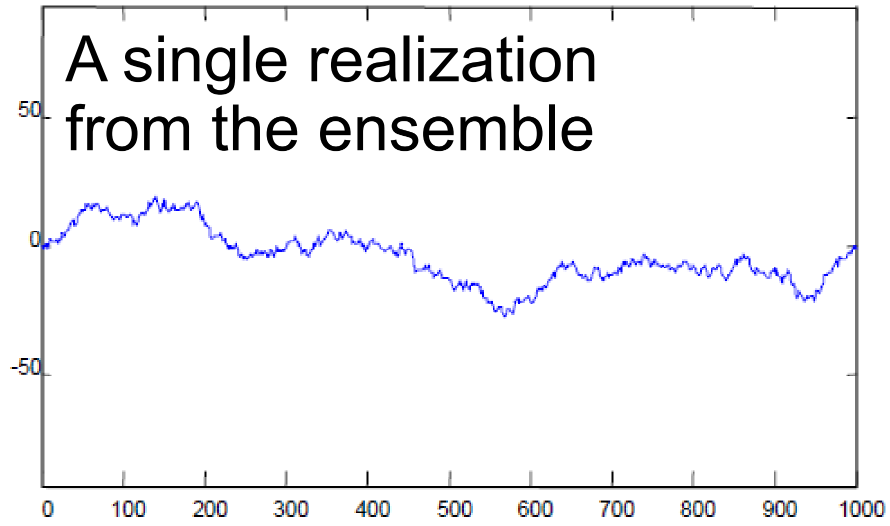
- For a random process, the outcome of a random experiment is mapped onto a waveform that is a function of time.
  - –E.g., recording of an electrocardiogram (ECG) in an environment with random noise.

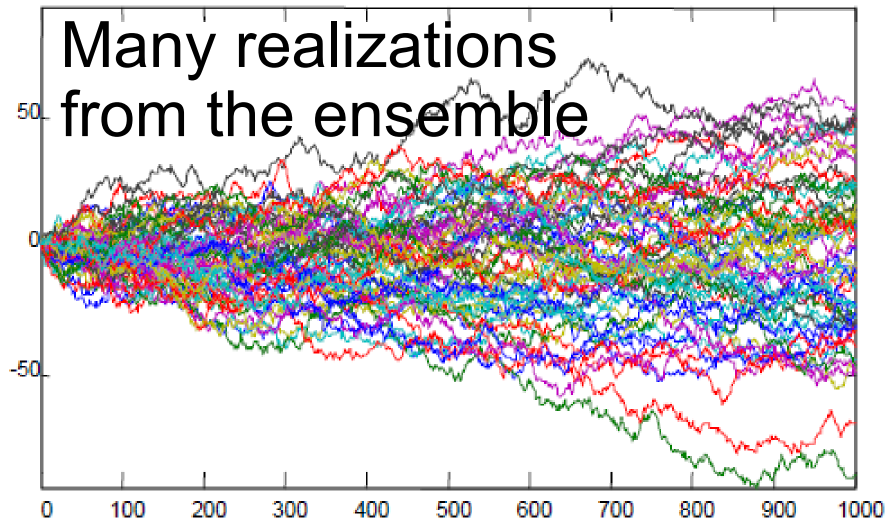


Realization vs. Ensemble

- Ensemble: the set of all realizations

When can one infer the statistical properties of the ensemble from the statistical properties of a single realization?





Example: Ensemble of Coin Tossers

Consider *N* people, each independently having written down a long random binary string of ones and zeros, with each entry chosen independently of any other entry in their string (similar to a sequence of independent coin tosses).

- The random process comprises this ensemble of strings.
- A realization of the process is obtained by randomly selecting a person (and therefore one of the *N* strings of ones and zeros).

Time Average vs. Ensemble Average

- Time average of a random process

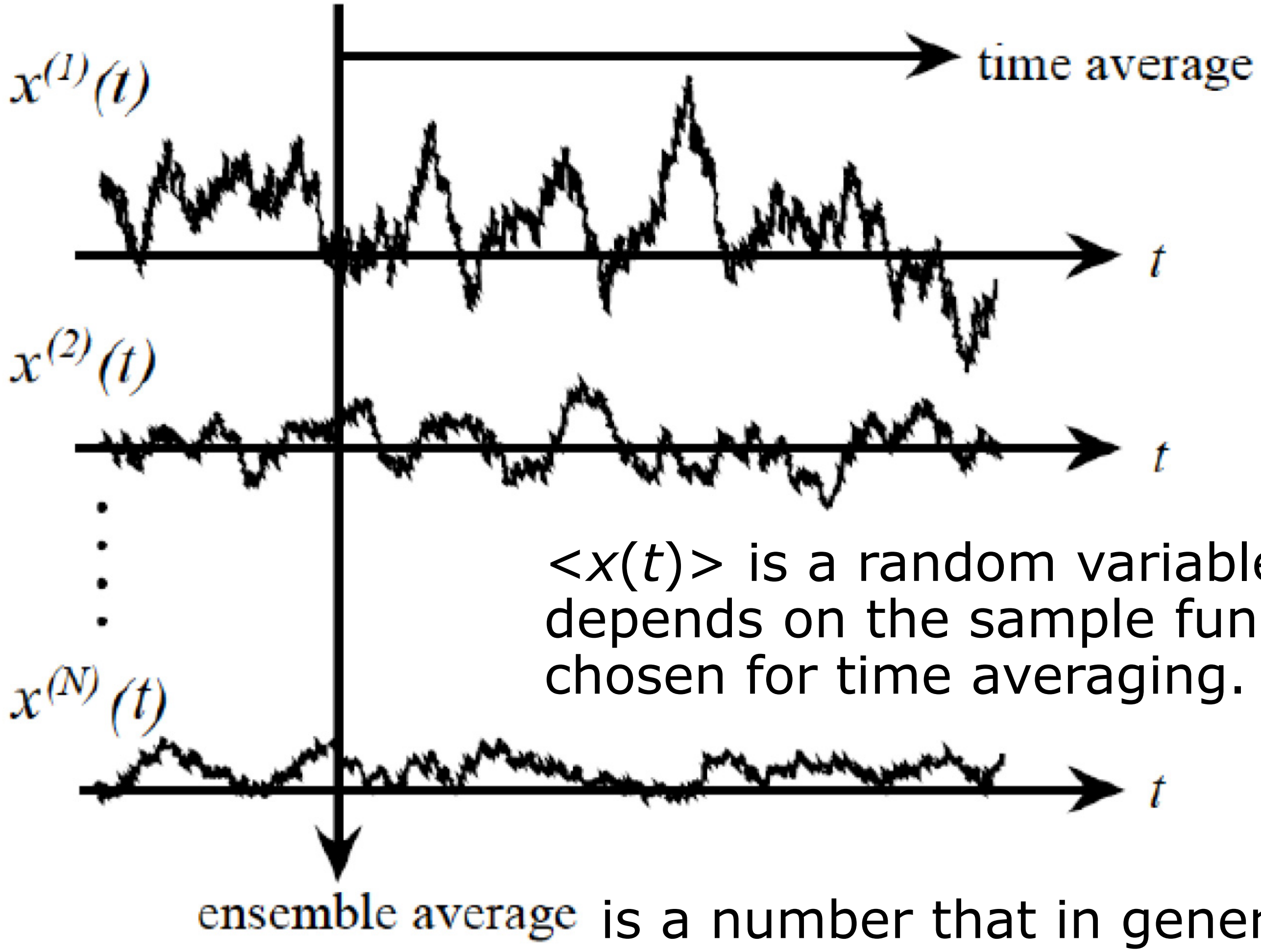
$$< x(t) > \triangleq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$$

$$< x[n] > \triangleq \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n]$$

- Ensemble average of a random process

$$E\left(x(t)\right) \triangleq \int_{-\infty}^{\infty} X\ f_x(X;\ t)\, dX$$

Time Average vs. Ensemble Average

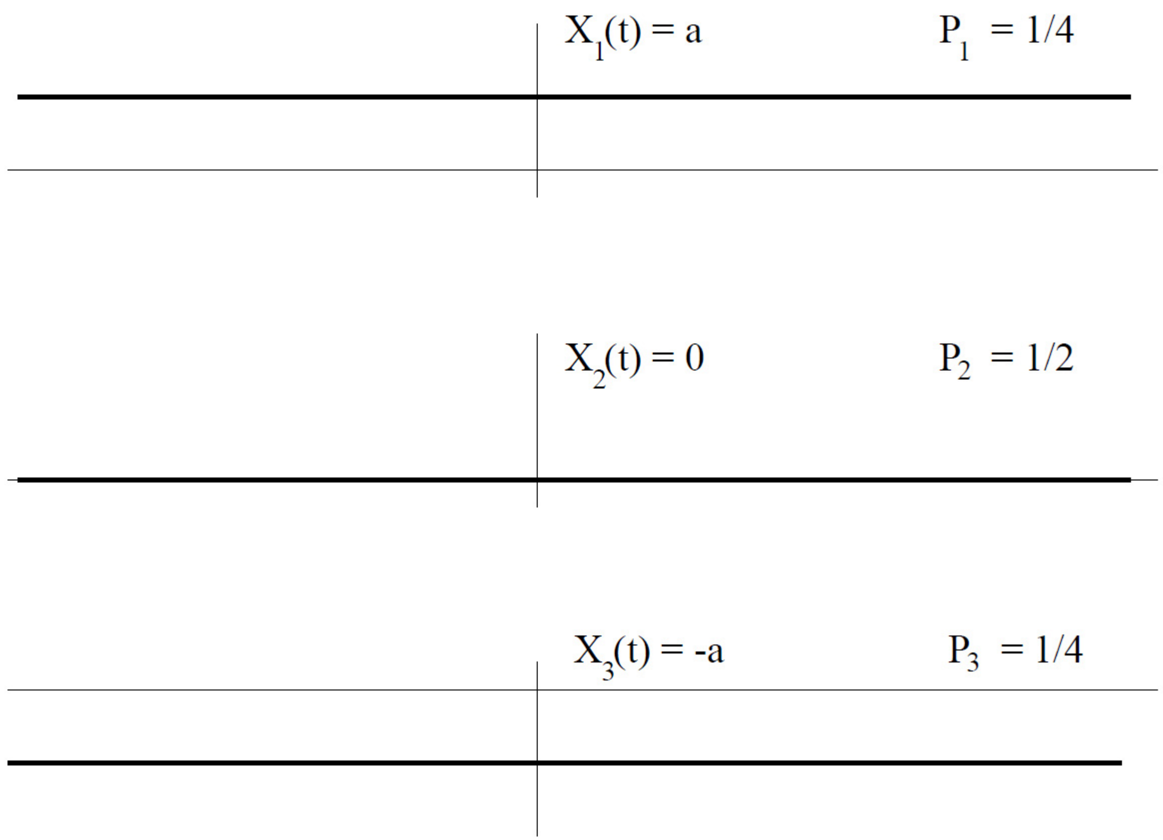


$\langle x(t) \rangle$ is a random variable, since it depends on the sample function chosen for time averaging.

is a number that in general may be a function of the time variable t.

Example

Compute the time average and ensemble average of the random process shown below



Example

Time average

$$\langle X(t) \rangle = \begin{cases} a & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -a & \text{with probability } 1/4 \end{cases}$$

Ensemble average

$$\begin{aligned} \mathrm{E}[X(t)] &= X_1(t)P_1 + X_2(t)P_2 + X_3(t)P_3 \\ &= a \cdot 1/4 + 0 \cdot 1/2 + (-a) \cdot 1/4 = 0 \end{aligned}$$

Important Concepts

- Stationarity:
  - Statistical properties of the random process vary or remain constant over time?
- Ergodicity:
  - Time-averages over a single realization approach statistical expectations over the ensemble?
- Autocorrelation Function:
  - the 2^{nd} order statistical properties of a random process
- Cross-correlation Function:
  - the 2^{nd} order statistical relationships between two random processes

Stationary Processes

- Stationarity is an important statistical property of a random process.
- Strict Sense Stationary (SSS)
  – A very restrictive class of random processes
- Wide Sense Stationary (WSS)
  – A much border class of random processes
- SSS implies WSS, but the converse is not true.

Strict Sense Stationary (SSS)

A random process, *x*[*n*], is SSS if the following property holds:

$$p\left(x[i] = x_0, \ldots, x[i+N-1] = x_{N-1}\right) = p\left(x[j] = x_0, \ldots, x[j+N-1] = x_{N-1}\right)$$

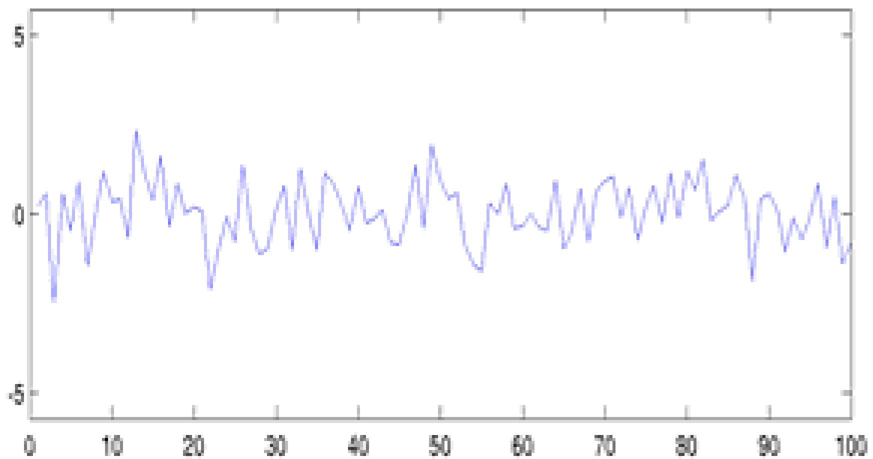
for all i, j, N and $\{x_0, x_1, \cdots, x_{N-1}\}$

- The signal statistics do not inform us about index or time.
- Example: Gaussian *independent* and *identically* distributed (i.i.d.) process
- All i.i.d. processes are SSS.

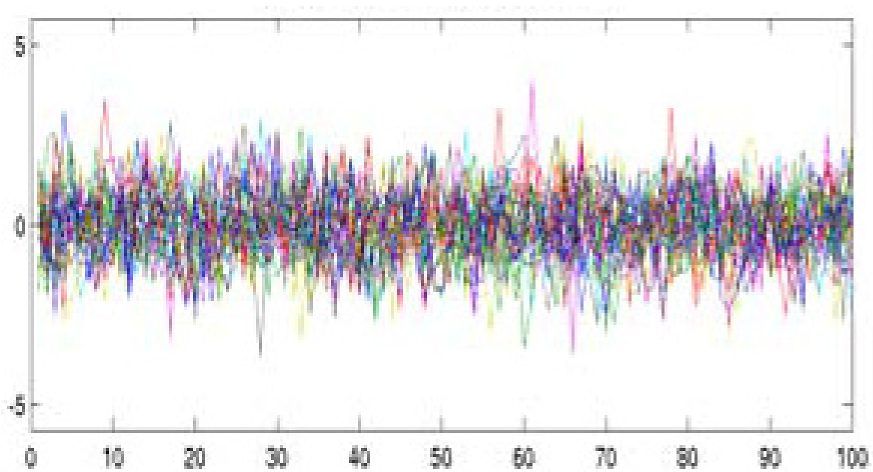
Gaussian I.I.D. Process

- $x[n]$ are sampled independently and identically from the same marginal density

$$x\left[n\right] \sim N\left(x;0,\sigma^2\right)$$



Single realization



Multiple realizations

Wide Sense Stationary (WSS)

- WSS only considers up to the 2nd order statistics of a random process.
- A random process $x[n]$ is WSS if the following properties hold:

$$\begin{aligned} E\{x[n]\} &= \mu_x \quad \forall i \\ E\{x[n]x[n+m]\} &= R_x[m] \quad \text{depends only on } m \end{aligned}$$

- $R_x[m]$: auto-correlation function
- SSS implies WSS.
- WSS implies SSS only when all of the joint densities of the random process are Gaussian

Gaussian Random Walk

- The evolution equation for (Gaussian) random walk is specified in terms of a linear difference equation

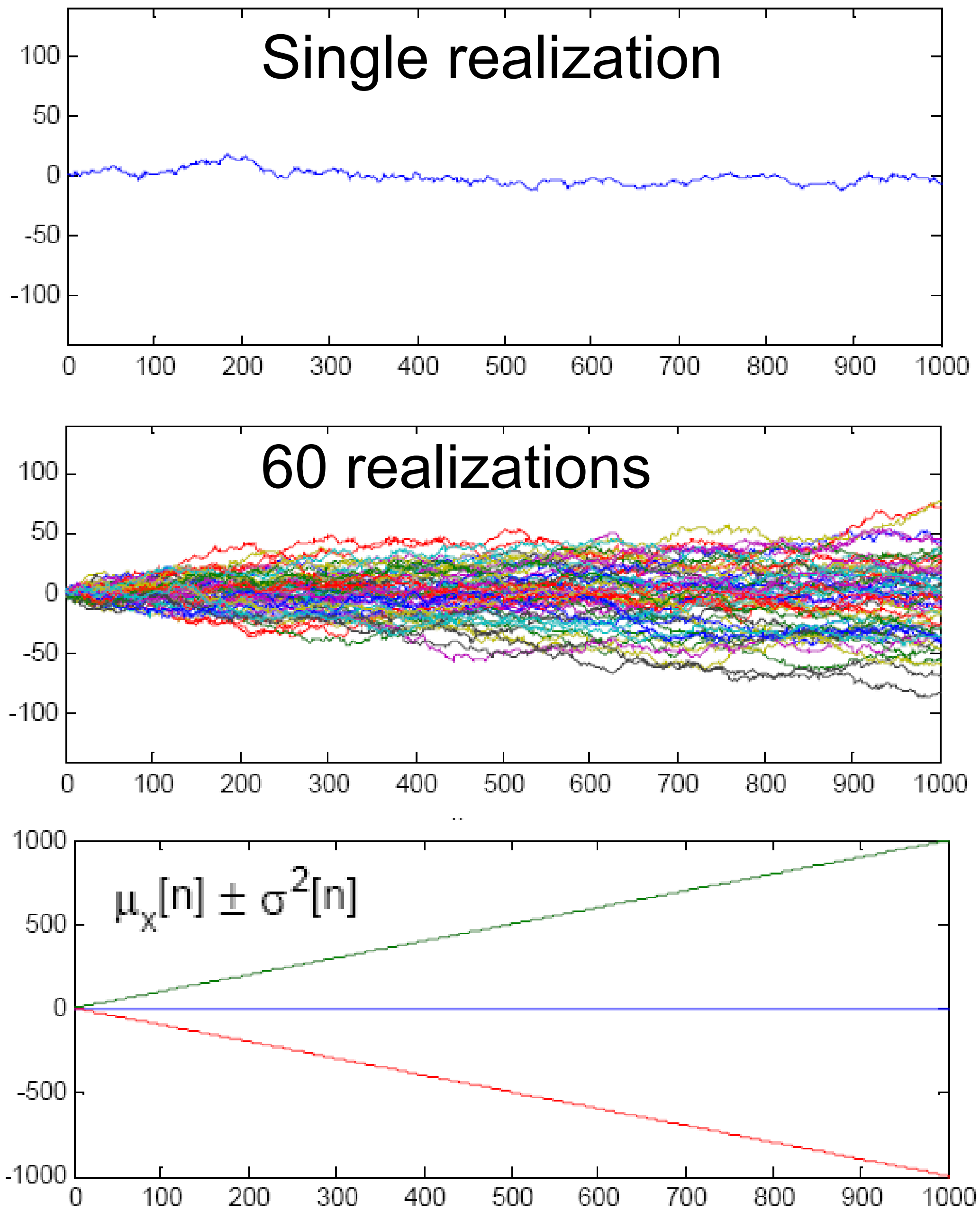
$$
\begin{aligned}
y[0] &= 0 \\
y[n] &= y[n-1] + x[n] \\
x[n] &\sim N\left(x; 0, \sigma^2\right)
\end{aligned}
$$

- Is this random process stationary?

Gaussian Random Walk

$$y[0] = 0$$
$$y[n] = y[n-1] + x[n]$$
$$x[n] \sim N\left(x; 0, \sigma^2\right)$$

The mean is constant over all time, but the variance grows linearly with time!



Signal Statistics using Time-Averages

- The mean (average) of a signal

$$\mu_x = \langle x[n] \rangle$$

- The mean power of a signal:

$$P_x = \left\langle x[n]^2 \right\rangle$$

- The AC power:

$$\sigma_x^2 = \left\langle (x[n] - \mu_x)^2 \right\rangle$$

- Relationship:

$$P_x = \sigma_x^2 + \mu_x^2$$

Ergodicity

- Time averages of an ergodic random process approximate their statistical averages.

$$\langle x[n]^p \rangle = \mathrm{E}\{x[n]^p\} \qquad \text{for all } p$$

- The left hand side is an average over all indices (or time) for a specific realization.
- The right hand side is an expectation over all realizations at a specific index (or time).
- Generally, SSS implies Ergodicity.
  - However, not all strictly stationary random processes are ergodic. Can you give one example?

Mean and Covariance Ergodicity

- Mean ergodicity - the statistical average equals the time average:

$$\begin{aligned} \langle x[n] \rangle &= \mathsf{E}\{x[n]\} \\ &= \mu_x \end{aligned}$$

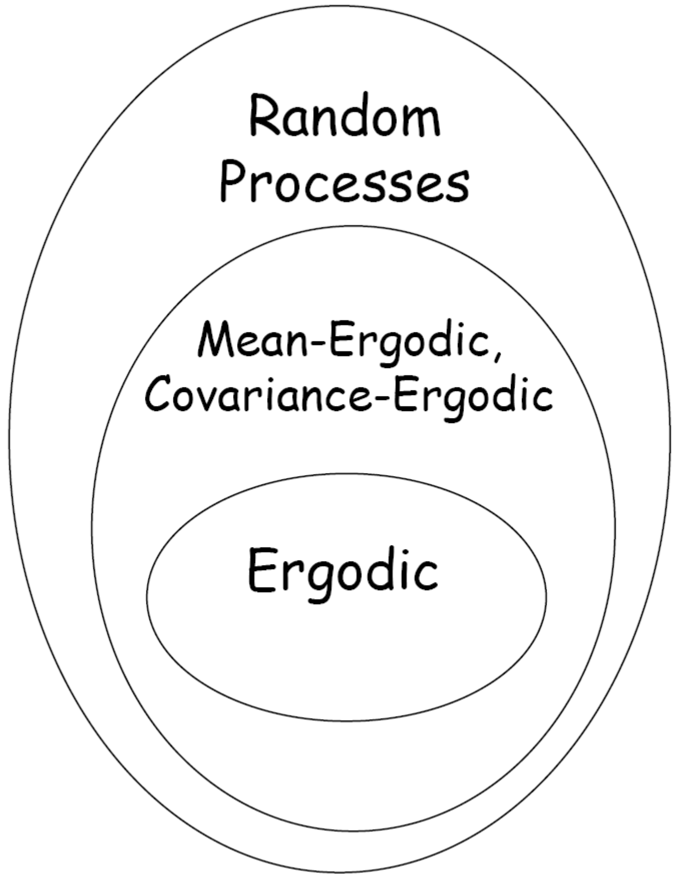
- Covariance ergodicity - the variance over time is equal to the variance over realizations

$$\left\langle (x[n] - \mu_x)^2 \right\rangle = \mathsf{E}\left\{(x[n] - \mu_x)^2\right\}$$

Ergodicity

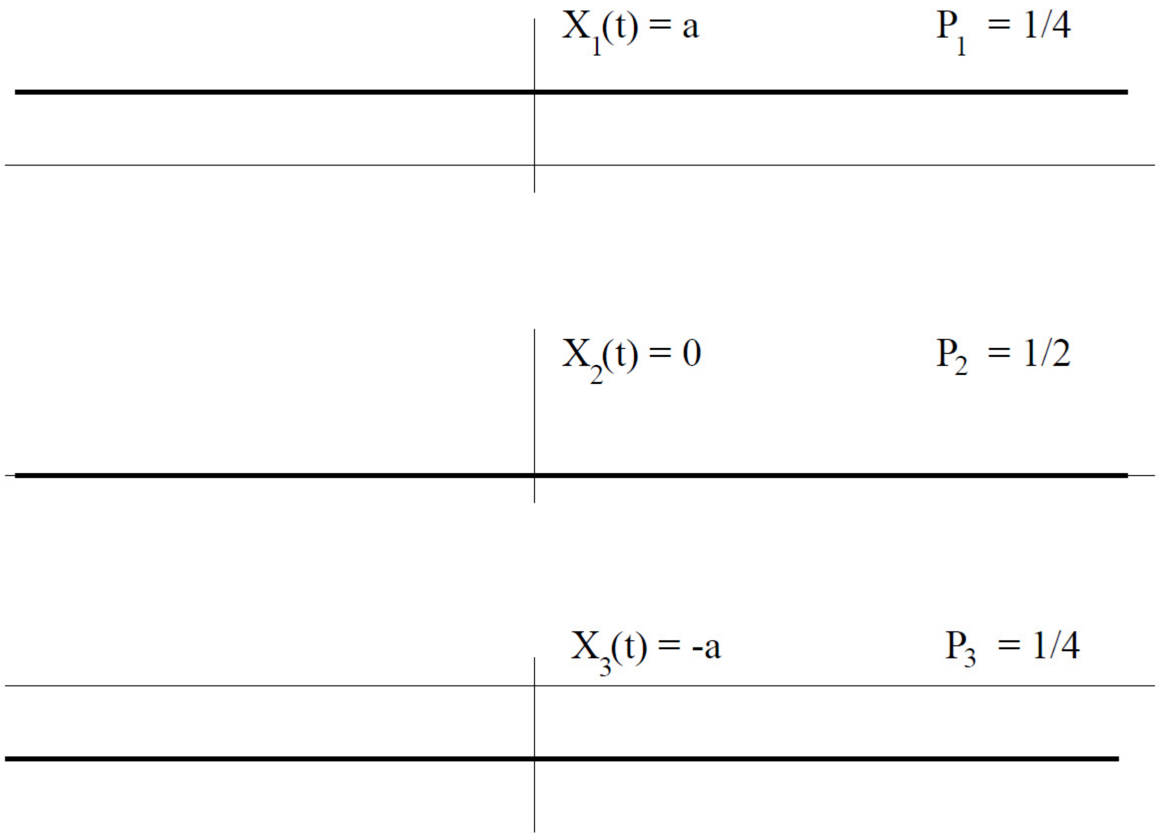
- Time-averages converge to statistical averages.

- Consequently, for stationary ergodic processes we can estimate statistical properties from the time-average of a single realization.



Example

For the random process shown below, determine whether it is ergodic.



Autocorrelation Function

The expected value of the product of a random variable or signal realization with a time-shifted version of itself

- Can be defined as: $R_x[n_1, n_2] = E\{x[n_1]x[n_2]\}$

- Or as a function of index *n* and shift *m*

Two choices of definition:

$$R_x[n, m] = E\{x[n]x[n+m]\}$$

$$R_x[\underbrace{n}_{\text{time index}}, \underbrace{m}_{\text{time shift}}] = E\{x[n]x[n-m]\}$$

Gaussian Random Walk

- Alternatively

$$
\begin{aligned}
y[0] &= 0 \\
y[n] &= y[n-1] + x[n] \\
x[n] &\sim N\left(x; 0, \sigma^2\right)
\end{aligned}
$$

$$
y[n] \triangleq \begin{cases} 0 & \text{if } n \leq 0 \\ \sum_{i=1}^{n} x[i] & \text{if } n > 0 \end{cases}
$$

$$
f_x(X;\ n) = \frac{1}{\sqrt{2\pi}\ \sigma} e^{-\frac{X^2}{2\ \sigma^2}}
$$

- Autocorrelation Function

$$
R_y[n,m] = E\{y[n]y[n+m]\}
$$

$$
= E\left\{\sum_{i=1}^{n} x[i] \sum_{j=1}^{n+m} x[j]\right\} = \sum_{i=1}^{n} \sum_{j=1}^{n+m} E\{x[i]x[j]\}
$$

$$
E\{x[i]x[j]\} = \begin{cases} 0, & \text{if } i \neq j \\ \sigma^2 & \text{if } i = j \end{cases} \implies R_y[n,m] = \min(n, n+m)\sigma^2
$$

Autocorrelation Function

For a WSS random process, the autocorrelation function does not depend on the time index *n*, but only on the time separation *m*, hence:

$$R_x[m] = E\{x[n]x[n+m]\}$$

Properties:

- Even function: $R_x[-m] = R_x[m]$
- Mean power: $R_x[0] = E\{x^2[n]\}$
- Always peaks at the origin $R_x[0] \geq |R_x[m]|$

Autocorrelation Function

Proof of $R_x[0] \geq |R_x[m]|$

$$E\left\{\left(x[n] \pm x[n+m]\right)^2\right\} \geq 0$$

$$E\left\{\left(x[n]\right)^2\right\} + E\left\{\left(x[n+m]\right)^2\right\} \pm 2E\left\{x[n]x[n+m]\right\} \geq 0$$

$$2\left(R_x[0] \pm R_x[m]\right) \geq 0$$

$$R_x[0] \geq \pm R_x[m]$$

$$R_x[0] \geq |R_x[m]|$$

Sample Autocorrelation

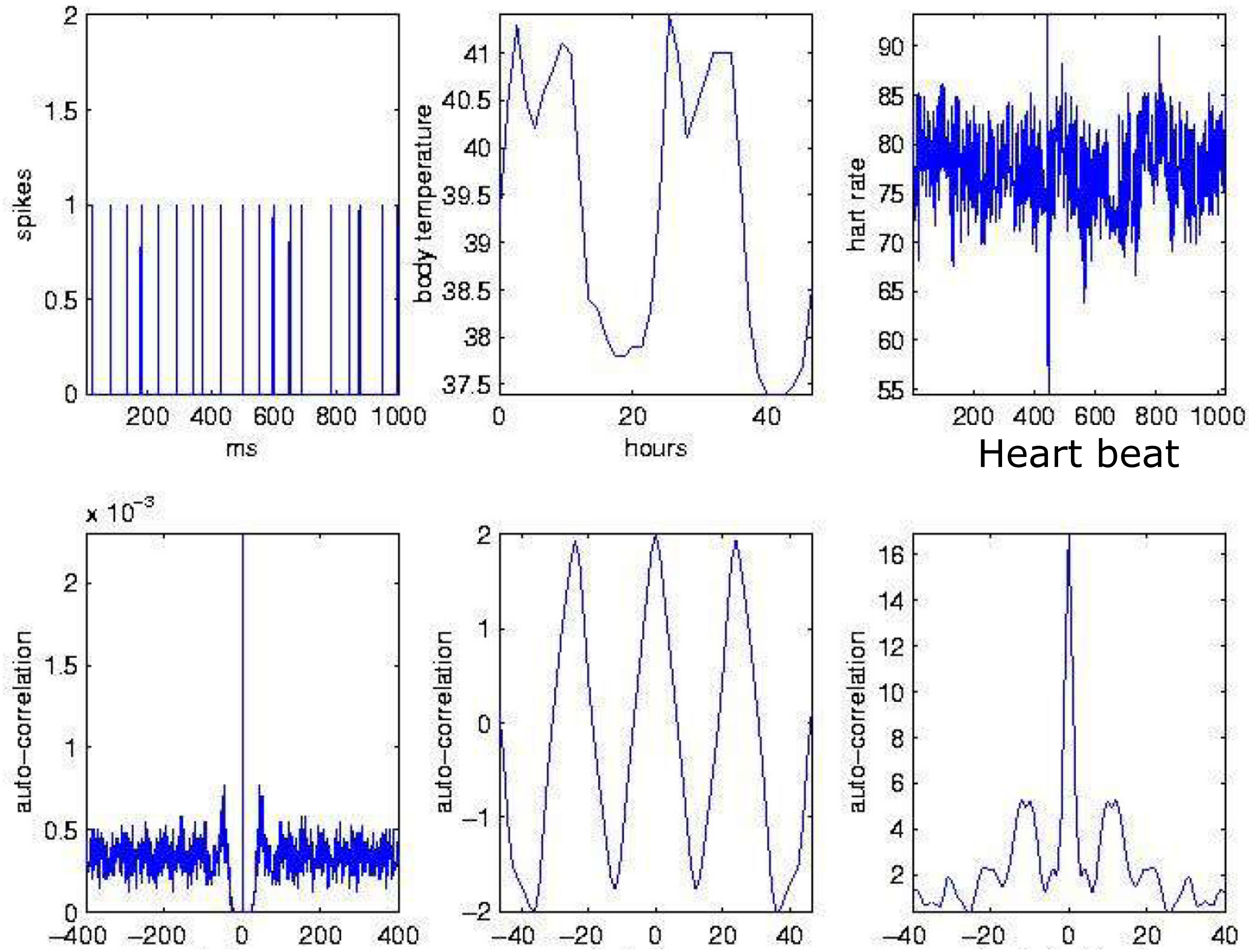
Assuming ergodic WSS, a direct estimate of the autocorrelation is the sample auto-correlation

$$\text{Given } x[n], \text{ for } n = 0,1,\cdots,N-1$$

$$\hat{R}_x[m] = \frac{1}{N-m-1}\sum_{n=0}^{N-m-1} x[n]x[n+m]$$

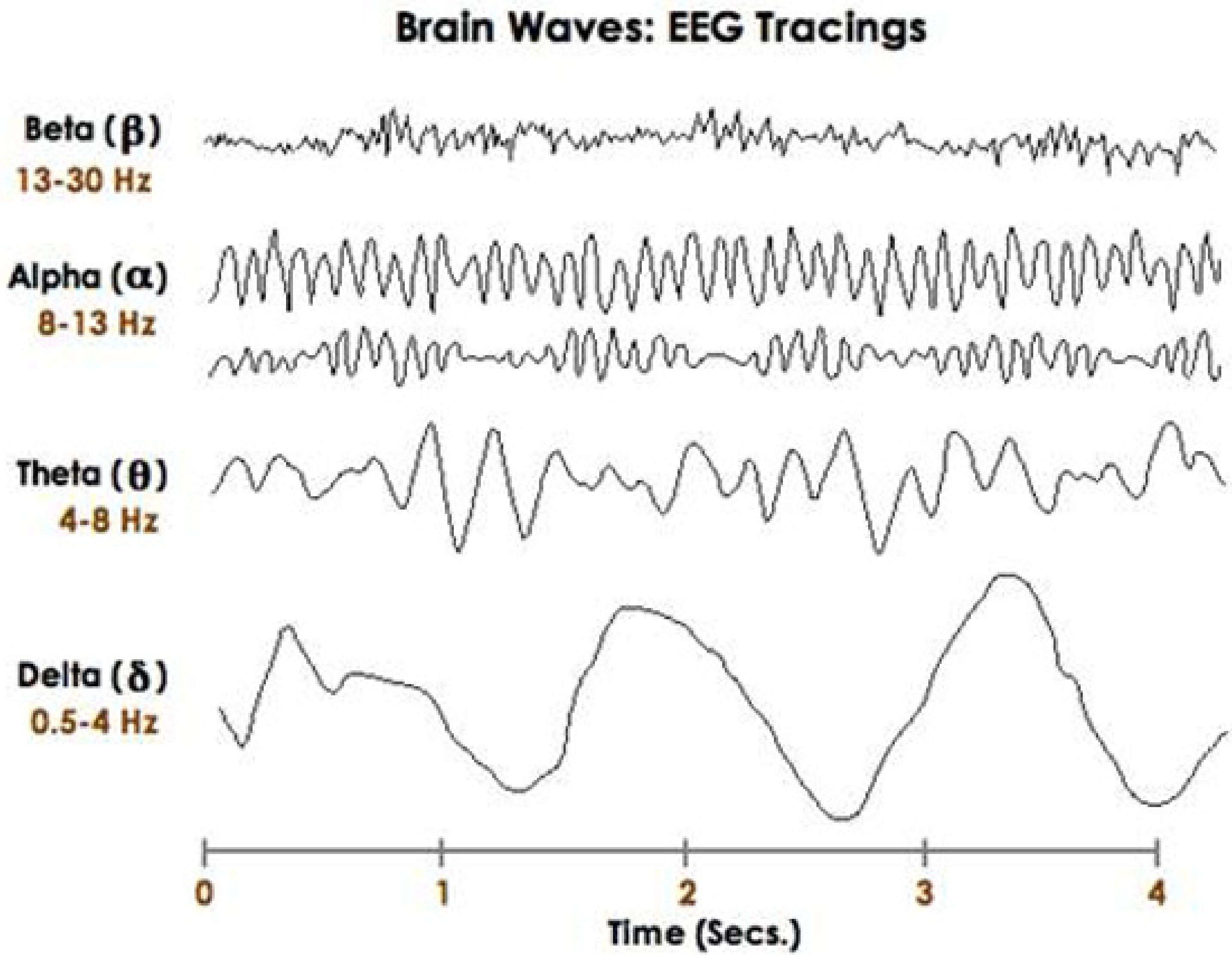
Note that correlation is convolution with opposite sign!

Examples

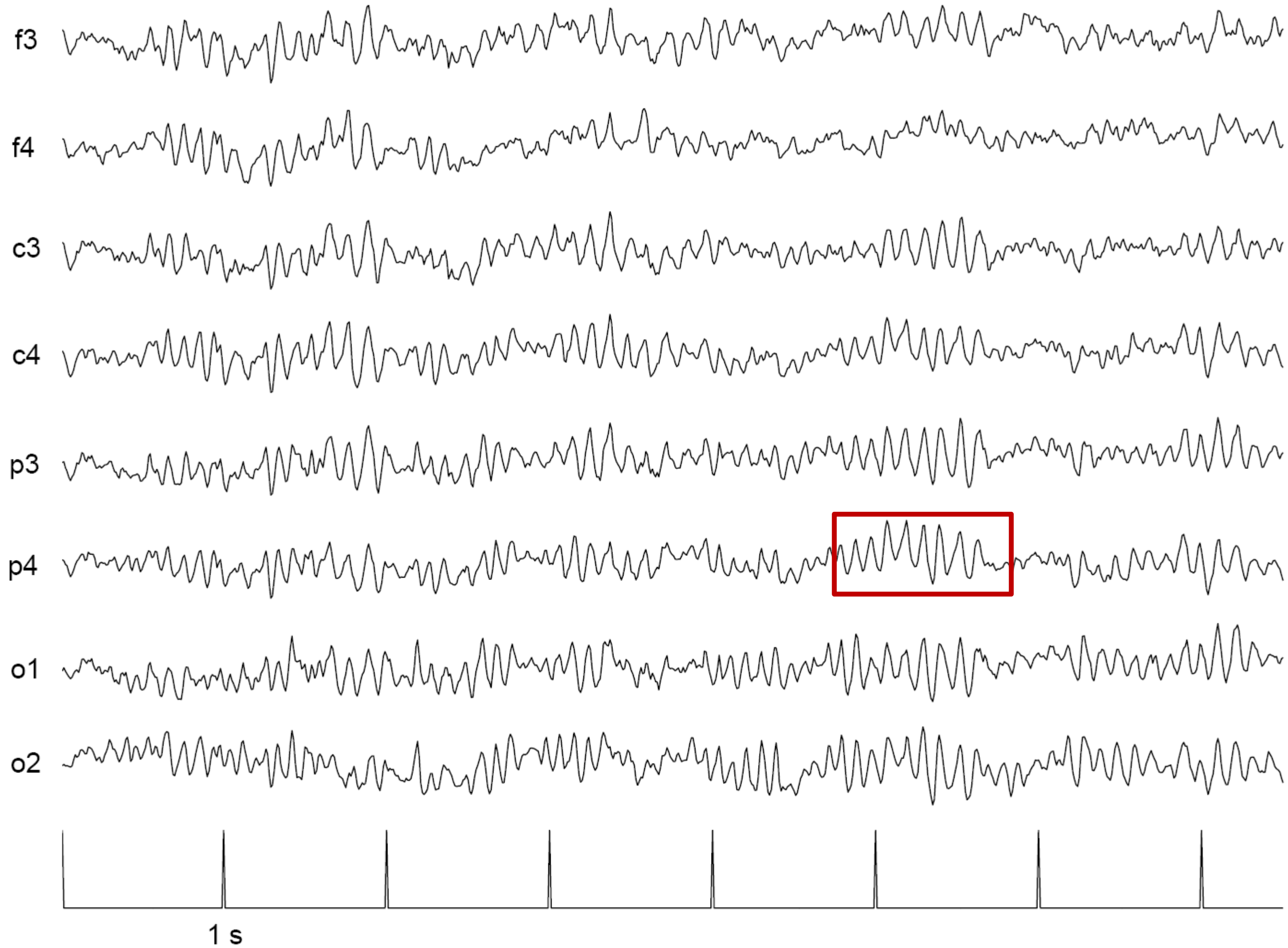


spikes
ms
body temperature
hours
hart rate
Heart beat
x 10^-3
auto-correlation
auto-correlation
auto-correlation

EEG: α, β, δ, and θ Waves

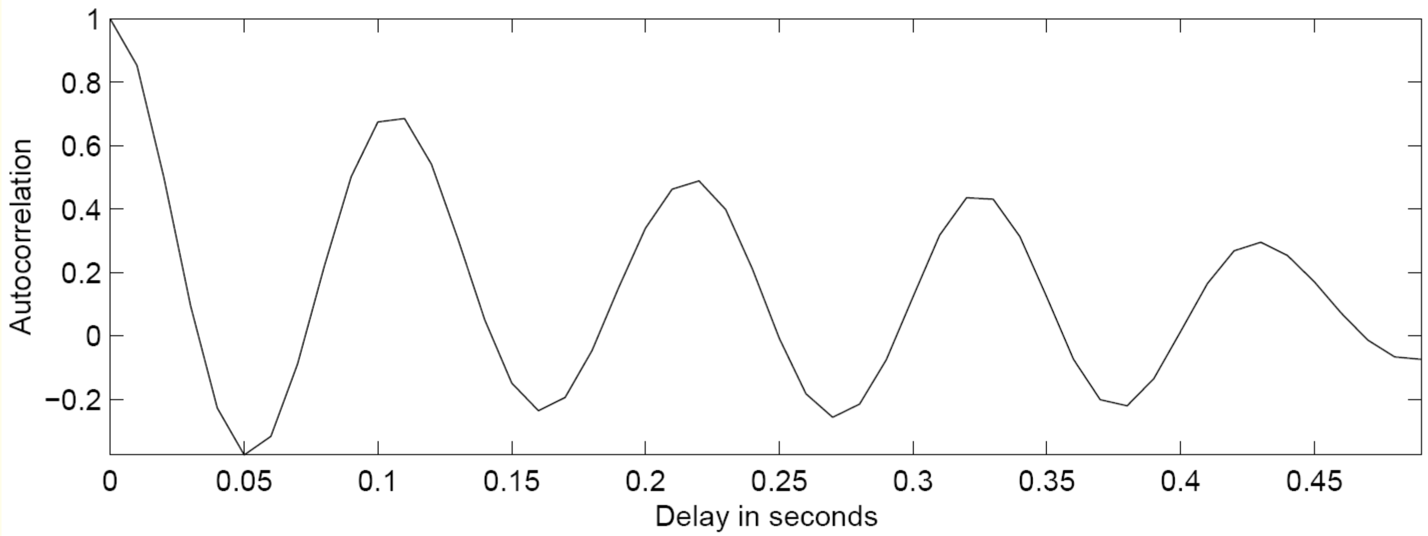


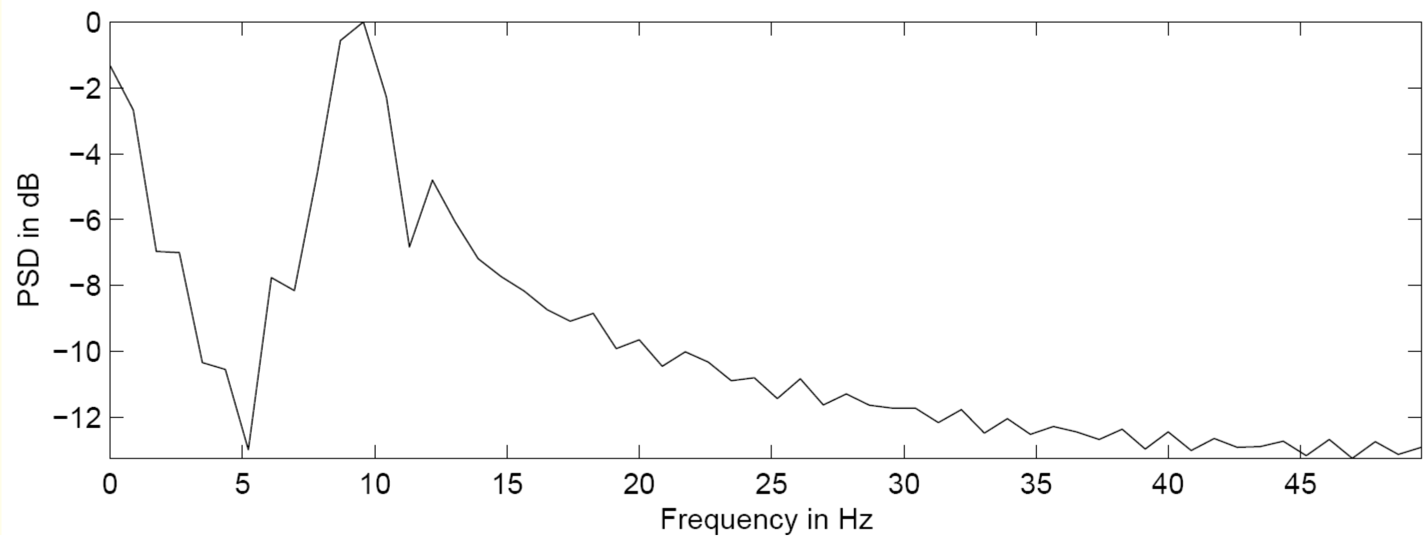
Eight Channels of EEG



The subject displays alpha rhythm.

Auto-Correlation (p4)





Cross-correlation Function

If two processes are jointly WSS, the expected value of the product of a random variable from one random process with a time-shifted, random variable from a different random process

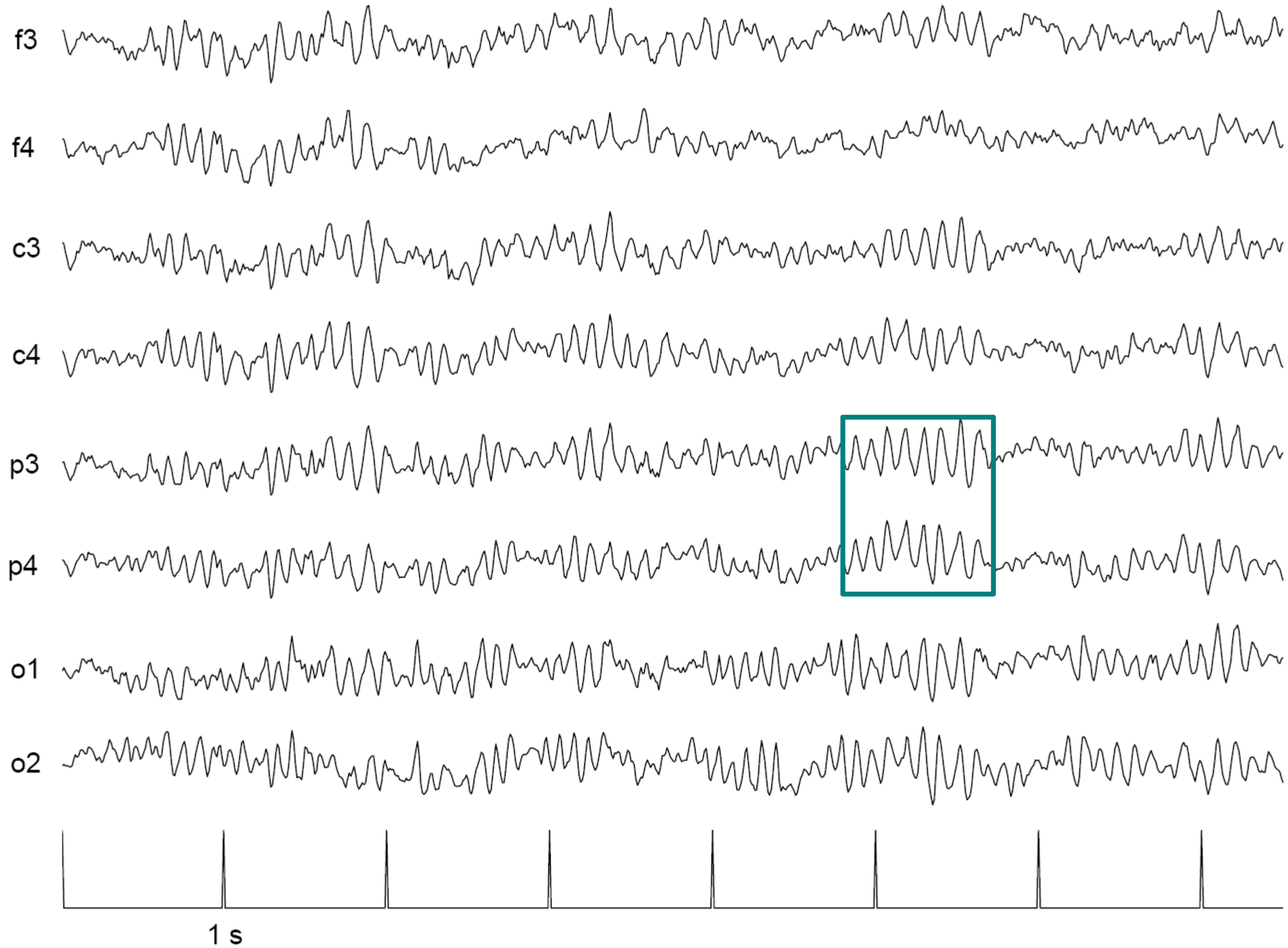
$$R_{xy}[m] = E\{x[n]y[n+m]\}$$

Properties:

$$R_{xy}[-m] = R_{yx}[m]$$

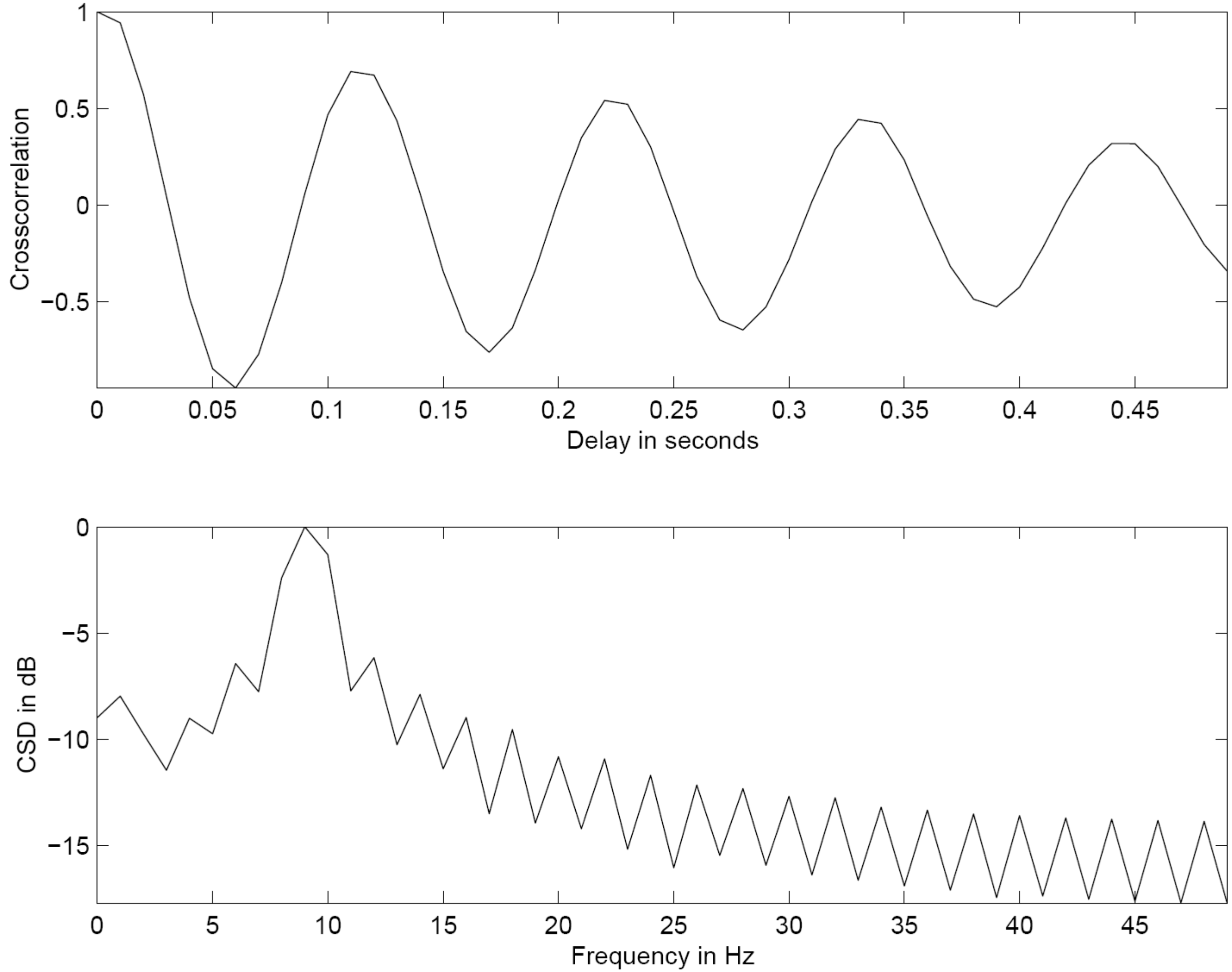
$$R_x[0]R_y[0] \geq \left|R_{xy}[m]\right|^2$$

Eight Channels of EEG



The subject displays alpha rhythm.

Cross-Correlation (p3 and p4)



Random Processes & LTI Systems

$$x[n] \longrightarrow \boxed{H(f)} \longrightarrow y[n]$$

- $H(f)$ is LTI
- $s[n]$ is a random process

$$s[n] \longrightarrow \Sigma \longrightarrow x[n] \longrightarrow \boxed{H(f)} \longrightarrow y[n]$$

$$w[n] \uparrow \Sigma$$

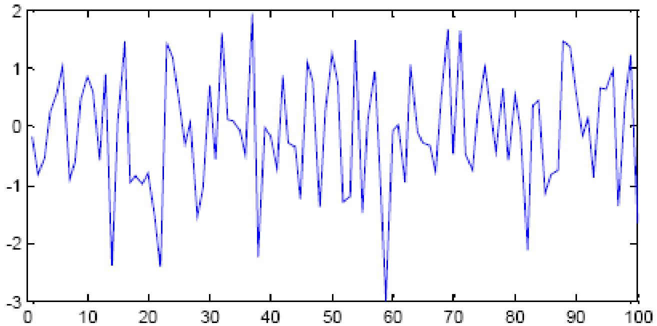
- $H(f)$ is LTI
- $w[n]$ is a random process, i.e., noise
- $s[n]$ is either a deterministic signal or a random process

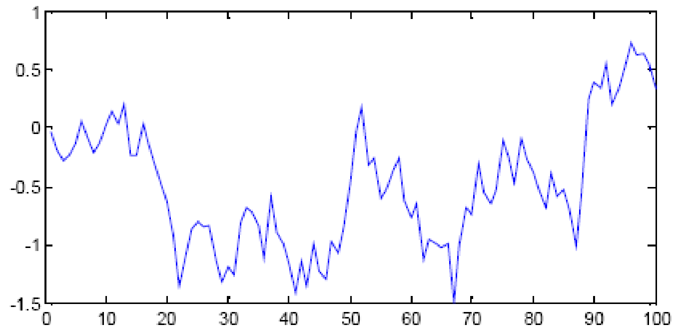
What can we say about y[n]?

Random Sequences & LTI Systems

$$x[n] \longrightarrow \boxed{h[n] \;\leftrightarrow\; H[k]} \longrightarrow y[n]$$

LTI





2nd order statistical properties

$$x[n] \;\Rightarrow\; R_x[m]$$
$$R_x[m], h[n] \;\Rightarrow\; R_y[m]$$

Acknowledgements

- Chapter 11 - RANDOM SIGNALS: BASIC PROPERTIES. Bertrand Delgutte. Course materials for HST.582J / 6.555J / 16.456J, Biomedical Signal and Image Processing, Spring2007. MIT OpenCourseWare (http://ocw.mit.edu), Massachusetts Institute of Technology.
- John Fisher. Course materials for HST.582J / 6.555J / 16.456J, Biomedical Signal and Image Processing, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu), Massachusetts Institute of Technology.