EE3731C:
Signal Processing Methods

Review of Part II

National University of Singapore

Summary of Topics

- Multirate Digital Signal Processing
- Introduction to Wavelet Transform
- Random Signals I
  - Time averages, ensemble averages, autocorrelation functions, cross-correlation functions, etc.
- Random Signals II
  - Random signals and linear systems, power spectra, cross spectra, Wiener filters, etc.
- Linear Stochastic Models
- Example Applications

Time and Frequency Domain View

- Up-sampling $\quad x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_u[n]$

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \cdots \\ 0, & \text{otherwise} \end{cases}$$

$$X_u(z) = X\left(z^L\right) \xrightarrow{z=e^{j\omega}} X_u\left(e^{j\omega}\right) = X\left(e^{j\omega L}\right)$$

- Down-sampling $\quad x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n]$

$$x_d[n] = x[nM]$$

$$X_d(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(z^{1/M} W_M^k\right) \xrightarrow{z=e^{j\omega}} X_d\left(e^{j\omega}\right) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j(\omega - 2\pi k)/M}\right)$$

Multirate Identities

- Interchange of Filtering and Up-sampling

$$x[n] \rightarrow \boxed{\uparrow L} \rightarrow \boxed{H(z^L)} \rightarrow y_1[n] \equiv x[n] \rightarrow \boxed{H(z)} \rightarrow \boxed{\uparrow L} \rightarrow y_2[n]$$

- Interchange of Filtering and Down-sampling

$$x[n] \rightarrow \boxed{\downarrow M} \rightarrow \boxed{H(z)} \rightarrow y_1[n] \equiv x[n] \rightarrow \boxed{H(z^M)} \rightarrow \boxed{\downarrow M} \rightarrow y_2[n]$$

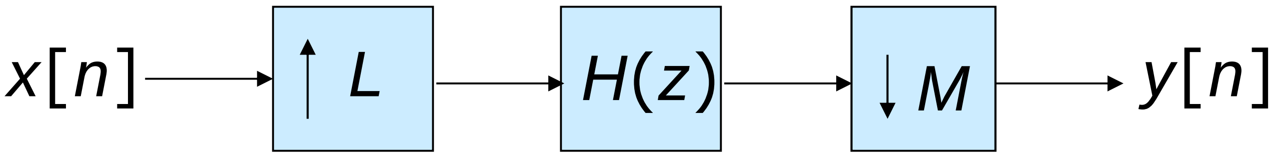
- Interchange of Up-sampling and Down-sampling

$$x[n] \rightarrow \boxed{\downarrow M} \rightarrow \boxed{\uparrow L} \rightarrow y_1[n] \equiv x[n] \rightarrow \boxed{\uparrow L} \rightarrow \boxed{\downarrow M} \rightarrow y_2[n]$$

if and only if *M* and *L* are relatively prime, i.e., *M* and *L* share no common positive factors except 1

Sampling Rate Conversion

- How to change a sampling rate by a non-integer factor of L/M?



– The signal is first interpolated by a factor of L
– Then decimated by a factor of M
– Before decimation the signal must be filtered by a low-pass filter which acts as not only an interpolator but also an anti-aliasing filter.
– When either M or $L = 1$, it reduces to interpolation or decimation filtering.

Filters in Multirate Systems

- Interpolation filter

f_s: the input sampling rate

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{H(z)} \longrightarrow y[n] \qquad \omega_c = \frac{\pi}{L} \qquad f_c = \frac{f_s}{2L}$$

- Decimation filter

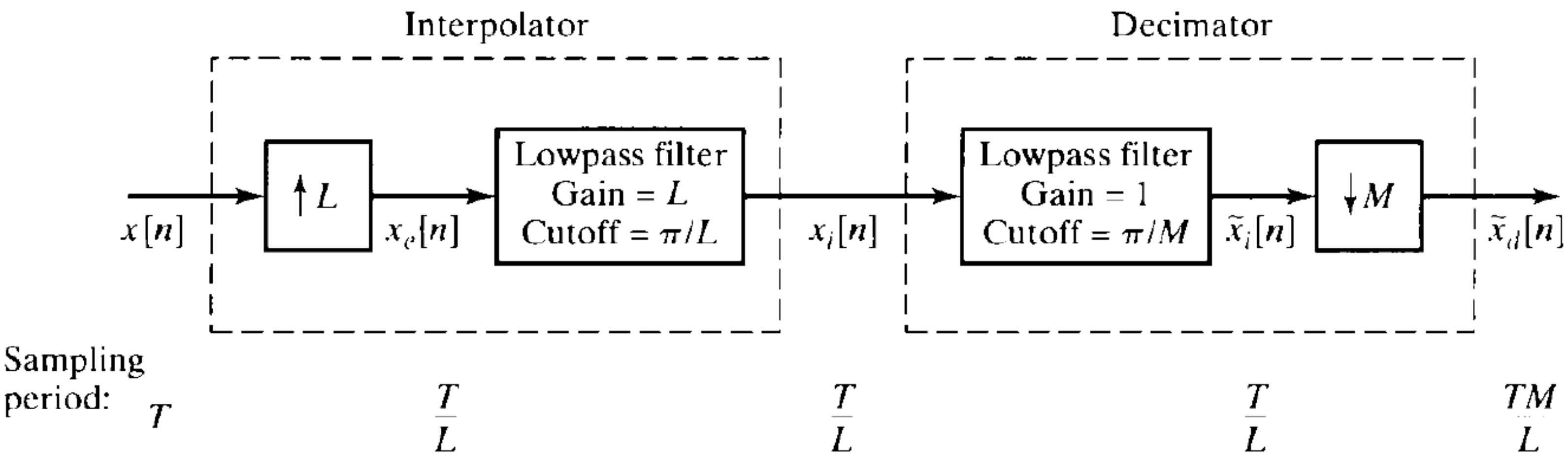
$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow M} \longrightarrow y[n] \qquad \omega_c = \frac{\pi}{M} \qquad f_c = \frac{f_s}{2M}$$

- Decimation and interpolation filter

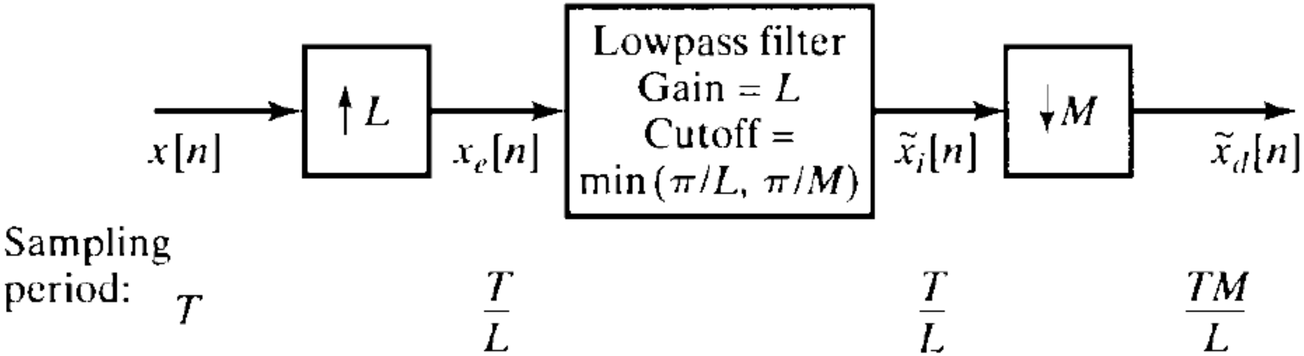
$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow M} \longrightarrow y[n]$$

$$\omega_c = \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right) \qquad f_c = \min\left(\frac{f_s}{2L}, \frac{f_s}{2M}\right)$$

Sampling Rate Conversion



System for changing the sampling rate by a noninteger factor



Simplified system: the decimation and interpolation filters are combined.

Example Questions

- Given a multirate system, simplify the system as much as you can using multirate identities
- Given a multirate system and the Fourier transform of the input signal, plot the Fourier transform of the output signal
- Given a multirate system, derive the relationship between the input and output
- How to set the cut-off frequency to avoid aliasing?

Random Signals/Processes

- Stationarity:
  - Statistical properties of the random process vary or remain constant over time?
- Ergodicity:
  - Time-averages over a single realization approach statistical expectations over the ensemble?
- Autocorrelation Function:
  - the 2nd order statistical properties of a random process
- Cross-correlation Function:
  - the 2nd order statistical relationships between two random processes

Autocorrelation Function

For a WSS random process, the autocorrelation function does not depend on the time index *n*, but only on the time separation *m*, hence:

$$R_x[m] = E\{x[n]x[n+m]\}$$

Properties:

- Even function: $R_x[-m] = R_x[m]$
- Mean power: $R_x[0] = E\{x^2[n]\}$
- Always peaks at the origin $R_x[0] \geq |R_x[m]|$

Cross-correlation Function

If two processes are jointly WSS, the expected value of the product of a random variable from one random process with a time-shifted, random variable from a different random process

$$R_{xy}[m] = E\{x[n]y[n+m]\}$$

Properties:

$$R_{xy}[-m] = R_{yx}[m]$$

$$R_x[0]R_y[0] \geq \left|R_{xy}[m]\right|^2$$

Example Questions

- Determine whether a random process is wide sense stationary
- Determine whether a random process is ergodic in mean and covariance
- Given a random process, find its mean and autocorrelation function
- Given two random processes (e.g., input and output of an LTI system), find the cross-correlation function

Response of an LTI System

Given a WSS ergodic random process, $x[n]$, and LTI system with impulse response $h[n]$ we can compute the time-average of the output $y[n]$.

$$x[n] \longrightarrow \boxed{h[n] \quad \leftrightarrow \quad H[k]} \longrightarrow y[n] \qquad m_y = m_x H[0]$$

LTI

$$R_y[m] = R_x[m] * h[m] * h[-m]$$

$$\boxed{S_x\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} R_x[k] e^{-j\omega k}}$$

$$S_y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) H\left(e^{-j\omega}\right) S_x\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right|^2 S_x\left(e^{j\omega}\right)$$

$$R_{xy}[m] = h[m] * R_x[m] \qquad S_{xy}\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) S_x\left(e^{j\omega}\right)$$

$$R_{yx}[m] = h[-m] * R_x[m] \qquad S_{yx}\left(e^{j\omega}\right) = H\left(e^{-j\omega}\right) S_x\left(e^{j\omega}\right)$$

Power Spectrum: Summary

$$S_x(f) = \sum_{k=-\infty}^{\infty} R_x[k] e^{-j2\pi fk}$$

$$R_{xy}[n] = \sum_{k=-\infty}^{\infty} h[k] R_x[n-k]$$
$$= h[n] * R_x[n]$$

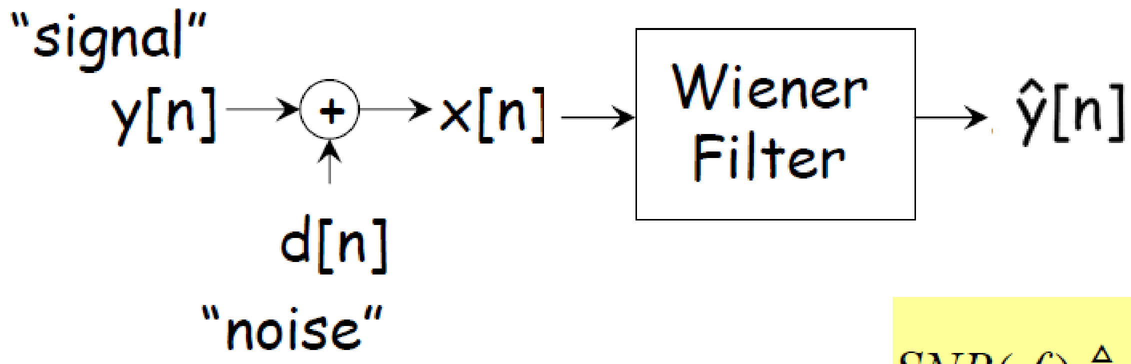
$$R_{xy}[n] \leftrightarrow S_{xy}(f)$$
$$h[n] \leftrightarrow H(f)$$
$$R_x[n] \leftrightarrow S_x(f)$$

$$H(f) = \frac{S_{xy}(f)}{S_x(f)}$$

$$S_y(f) \;=\; |H(f)|^2 S_x(f)$$

Each frequency is independent of all other frequencies.

Wiener Filter for Noise Removal



Uncorrelated noise:

$R_{yd}[n] = 0 \longleftrightarrow S_{yd}(f) = 0$

$$H(f) = \frac{S_{xy}(f)}{S_x(f)} = \frac{S_y(f)}{S_y(f) + S_d(f)}$$

$$SNR(f) \triangleq \frac{S_y(f)}{S_d(f)}$$

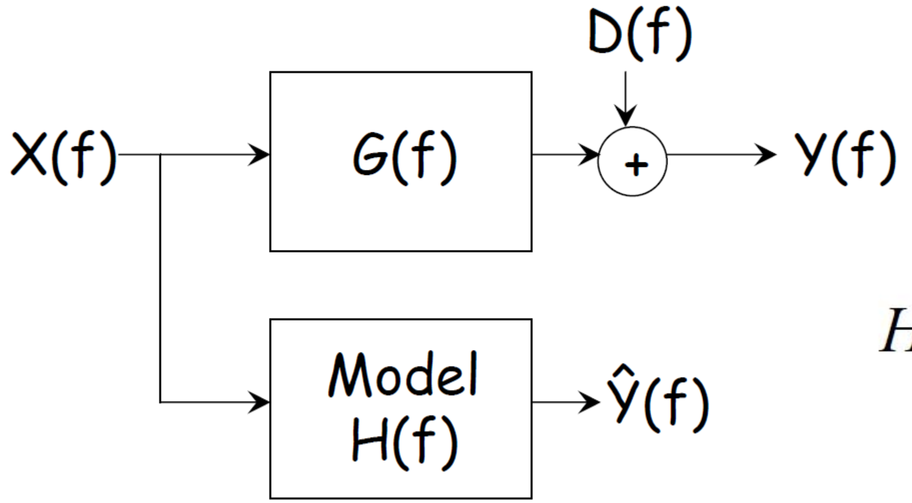
$$H(f) = \frac{S_y(f)}{S_y(f) + S_d(f)} = \frac{SNR(f)}{SNR(f) + 1}$$

$SNR(f) \gg 1$: $H(f) \approx 1$

$SNR(f) \ll 1$: $H(f) \approx 0$

Wiener Filter for System Identification

We can only observe y[n] which is the output of the LTI system + noise



d[*n*] is uncorrelated with *x*[*n*].

$$H(f) = \frac{S_{xy}(f)}{S_x(f)}$$

$$= \frac{S_x(f)G(f) + S_{xd}(f)}{S_x(f)}$$

$$= \frac{S_x(f)G(f)}{S_x(f)} = G(f)$$

We can still recover the LTI system exactly!

Example: Causal Wiener Filter

$$\begin{bmatrix} R_x[0] & R_x[1] & \cdots & R_x[N-1] \\ R_x[1] & R_x[0] & & R_x[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ R_x[N-1] & R_x[N-2] & \cdots & R_x[0] \end{bmatrix} \begin{bmatrix} h[0] \\ \vdots \\ \vdots \\ h[N-1] \end{bmatrix} = \begin{bmatrix} R_{xy}[0] \\ \vdots \\ \vdots \\ R_{xy}[N-1] \end{bmatrix}$$

If $R_x[m] = \delta[m]$, then the system is simplified to

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} h[0] \\ \vdots \\ \vdots \\ h[N-1] \end{bmatrix} = \begin{bmatrix} R_{xy}[0] \\ \vdots \\ \vdots \\ R_{xy}[N-1] \end{bmatrix}$$

Causal Wiener filter: $\boxed{h[n] = \begin{cases} R_{xy}[n] & n \geq 0 \\ 0 & n < 0 \end{cases}}$

Linear Stochastic Processes

- Random processes generated by filtering white noise with a linear shift invariant filter that has a rational transfer function
  - –Autoregressive moving average (ARMA) process
  - –Autoregressive (AR) process (all pole)
  - –Moving average (MA) process (all zero)

$$w[n] \longrightarrow \boxed{H(z)} \longrightarrow x[n]$$

$$R_w[m] = \sigma_w^2 \delta[m]$$

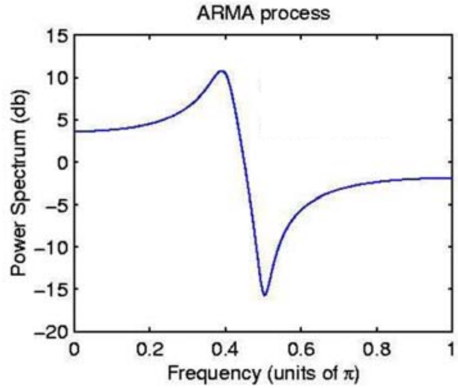
$$P_w\left(e^{j\omega}\right) = \sigma_w^2$$

$$H(z) = \frac{B_q(z)}{A_p(z)} = \frac{\sum_{k=0}^{q} b_q[k] z^{-k}}{1 + \sum_{k=1}^{p} a_p[k] z^{-k}}$$

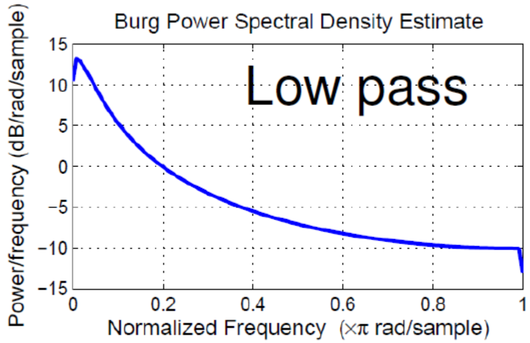
Consider real case

Difference Equations

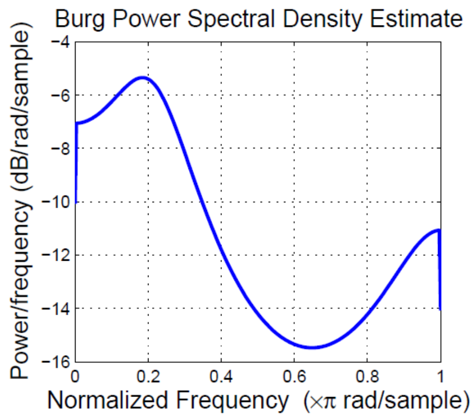
- ARMA: $x[n] + \sum_{k=1}^{p} a_p[k] x[n-k] = \sum_{k=0}^{q} b_q[k] w[n-k]$
  - Can model PSD with both peaks and valleys



- AR: $x[n] + \sum_{k=1}^{p} a_p[k] x[n-k] = w[n]$
  - models peak PSD better



- MA: $x[n] = \sum_{k=0}^{q} b_q[k] w[n-k]$
  - models valley PSD better



AR Modelling

$$\begin{bmatrix} R_x[0] & R_x[1] & \cdots & R_x[p] \\ R_x[1] & R_x[0] & & R_x[p-1] \\ \vdots & \vdots & \ddots & \vdots \\ R_x[p] & R_x[p-1] & \cdots & R_x[0] \end{bmatrix} \begin{bmatrix} 1 \\ a_p[1] \\ \vdots \\ a_p[p] \end{bmatrix} = \begin{bmatrix} \sigma_w^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

If we omit the first equation, we get

$$\begin{bmatrix} R_x[0] & R_x[1] & \cdots & R_x[p-1] \\ R_x[1] & R_x[0] & & R_x[p-2] \\ \vdots & \vdots & \ddots & \vdots \\ R_x[p-1] & R_x[p-2] & \cdots & R_x[0] \end{bmatrix} \begin{bmatrix} a_p[1] \\ a_p[2] \\ \vdots \\ a_p[p] \end{bmatrix} = -\begin{bmatrix} R_x[1] \\ R_x[2] \\ \vdots \\ R_x[p] \end{bmatrix}$$

Yule-Walker equations in matrix-vector notation: $\mathbf{R}_x\mathbf{a} = -\mathbf{r}_x$ $\qquad$ $\mathbf{a} = -\mathbf{R}_x^{-1}\mathbf{r}_x$

Linear Prediction with an AR Model

- Assume that a time series can be modelled by a *p*-order model.
  - x[n] is a function of the *p* previous values plus an error term e[n]

$$x[n] = -\sum_{k=1}^{p} a[k]x[n-k] + e[n]$$

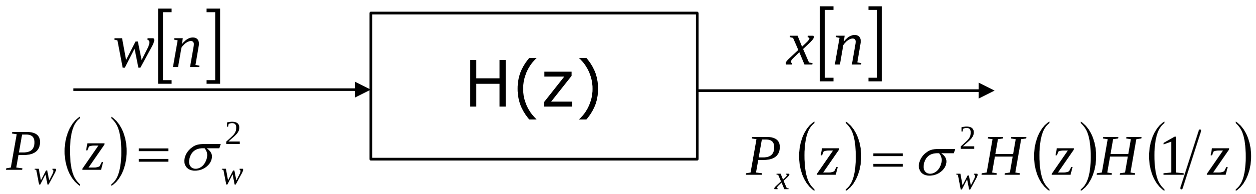
  - Prediction: $\hat{x}[n] = -\sum_{k=1}^{p} a[k]x[n-k]$

- The prediction error is: $e[n] = x[n] - \hat{x}[n]$

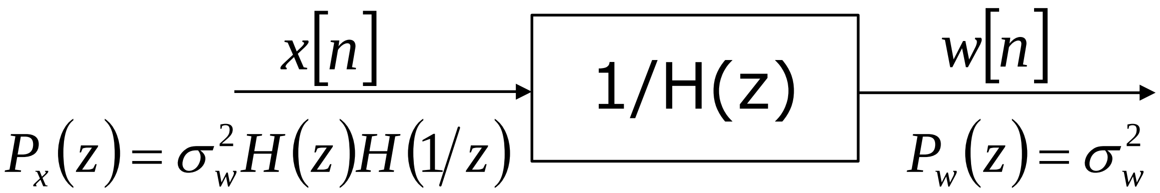
  Minimize the Mean Square Error (MSE), $E\left\{(e[n])^2\right\}$ we obtain the same linear equations as the AR model parameters - Yule-Walker equations

Whitening Filter

- A random process $x[n]$ is represented as the output of a filter H(z) with an input of white noise $w[n]$.



- A whitening filter for the random process $x[n]$ is the inverse of H(z), i.e., 1/H(z).



Different random processes have different whitening filters.

Example Questions

- Given an LTI system and the power spectrum of the input random signal, find the power spectrum of the output
- Given the power spectrum of a random process, design a linear filter such that when the input is a white noise the output has the same autocorrelation function as the given random process
- Given the power spectrum of an input random process, design a whitening filter such that the output is a white noise

Exam Matters

- Final Exam (60%)
  - –Tuesday, 03-Dec-2013 (Evening)
  - –Closed book
  - –One A4-size double-sided hand-written formula sheet is allowed
- IVLE Forum
  - –Some questions and answers will be posted
- Consultation Session (TBD)
  - –During the week of 25-29 Nov

Thank you & Good Luck!