Tutorial 3 - EE3731C Signal Processing Methods

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Solutions

1.

$$
\begin{aligned}
A - \lambda I = & \begin{bmatrix} 1-\lambda & 1 \\ -4 & 6-\lambda \end{bmatrix} \\
det(A - \lambda I) = & 0 \\
& \rightarrow (1-\lambda)(6-\lambda) + 4 = \lambda^2 - 7\lambda + 10 = 0 \\
& \rightarrow \lambda = 2, 5
\end{aligned}
$$

$$\lambda = 2 \Rightarrow A - \lambda I = \begin{bmatrix} -1 & 1 \\ -4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore, $v_1 = v_2, and \overrightarrow{v}$ is any multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\lambda = 5 \Rightarrow A - \lambda I = \begin{bmatrix} -4 & 1 \\ -4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore, $4v_1 = v_2, and \overrightarrow{v}$ is any multiple of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Overall, the eigenvalues are $2, 5$ and the corresponding eigenvectors are $a \begin{bmatrix} 1 \\ 1 \end{bmatrix}, a \begin{bmatrix} 1 \\ 4 \end{bmatrix} (a \neq 0)$

2.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & a \\ a & 1-\lambda \end{bmatrix}$$

(a) $det(A - \lambda I) = 0 \rightarrow (1-\lambda)^2 - a^2 = 0$, so the eigenvalues are $\lambda = 1 \pm a$.

(b) The symmetric matrix A is a covariance matrix iff all eigenvalues are non-negative. Therefore, $-1 \leq a \leq 1$

3. For data set (i),

(a) The mean and covariance matrix are

$$\mu = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \Sigma = \frac{14}{3}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

note: here when calculating the covariance matrix Σ, $\frac{1}{n}$ is used to normalized the data, and one can also use $\frac{1}{n-1}$. In fact if to perform PCA, a "c" to denote the constant is sufficient as it does not affect the solutions.

The first principle component $\overrightarrow{v}$ is the eigenvector of Σ with the largest eigenvalue, so $\overrightarrow{v} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$, (note that $||\overrightarrow{v}|| = 1$, so the negation is also correct)

(b) When mapped to the 1D space, the three data points are

$$\begin{aligned} [1-3, 1-3]\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} &= -2\sqrt{2}, \\ [2-3, 2-3]\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} &= -\sqrt{2}, \\ [6-3, 6-3]\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} &= 3\sqrt{2} \end{aligned}$$

(c) When mapped back to the 2D space, the three data points are

$$\begin{aligned} -2\sqrt{2}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ -\sqrt{2}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} &= \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \\ 3\sqrt{2}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} &= \begin{bmatrix} 6 \\ 6 \end{bmatrix} \end{aligned}$$

Hence the reconstruction error (i.e. the squared distance between the original and the reconstructed data points) is 0.

For data set (ii),

(a) The mean and covariance matrix are

$$\mu = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \frac{5}{4} & -1 \\ -1 & \frac{5}{4} \end{bmatrix}$$

The first principle component v_1 is the eigenvector of Σ with the largest eigenvalue, so $\overrightarrow{v} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$, (note that $||\overrightarrow{v}|| = 1$, so the negation is also correct)

(b) When mapped to the 1D space, the four data points are

$$
\begin{aligned}
[-1-\frac{1}{2}, 2-\frac{1}{2}]\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} &= \frac{3\sqrt{2}}{2}, \\
[0-\frac{1}{2}, 0-\frac{1}{2}]\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} &= 0, \\
[1-\frac{1}{2}, 1-\frac{1}{2}]\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} &= 0, \\
[2-\frac{1}{2}, -1-\frac{1}{2}]\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} &= -\frac{3\sqrt{2}}{2}
\end{aligned}
$$

(c) When mapped back to the 2D space, the four data points are

$$
\begin{aligned}
\frac{3\sqrt{2}}{2}\begin{bmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} &= \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \\
0\begin{bmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} &= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \\
0\begin{bmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} &= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \\
\frac{-3\sqrt{2}}{2}\begin{bmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} &= \begin{bmatrix} 2 \\ -1 \end{bmatrix}
\end{aligned}
$$

Hence the reconstruction error (i.e. the squared distance between the original and the reconstructed data points) is

$$
\left\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\|^2 = 1.
$$