Q3

(a) Consider two multirate systems shown in Fig. Q3a-1.

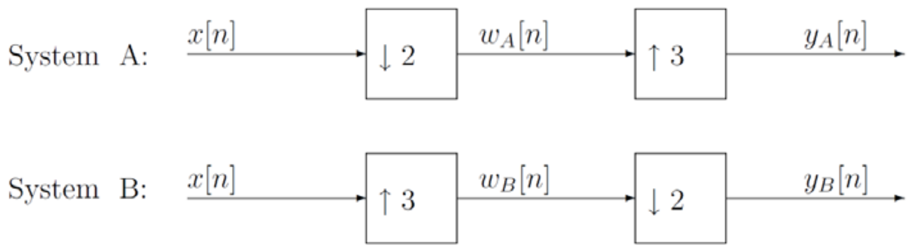


Fig. Q3a-1

i) For an input signal $x[n]$ whose Fourier transform $X(e^{j\omega})$ is shown in Fig. Q3a-2, sketch $W_A(e^{j\omega})$ and $Y_A(e^{j\omega})$, i.e., the Fourier transforms of $w_A[n]$ and $y_A[n]$.

(6 marks)

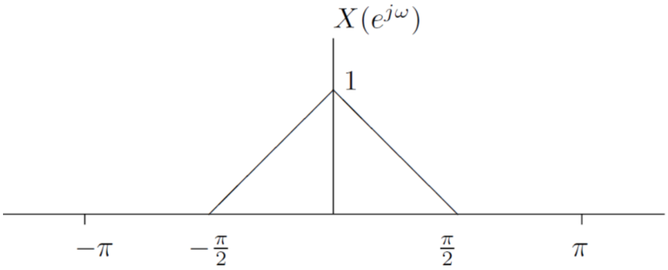
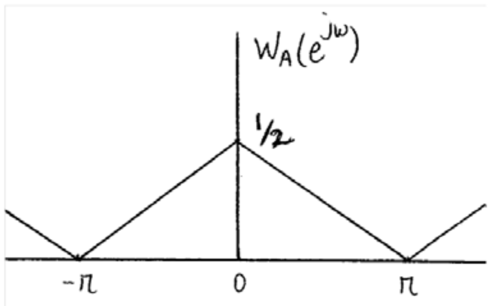
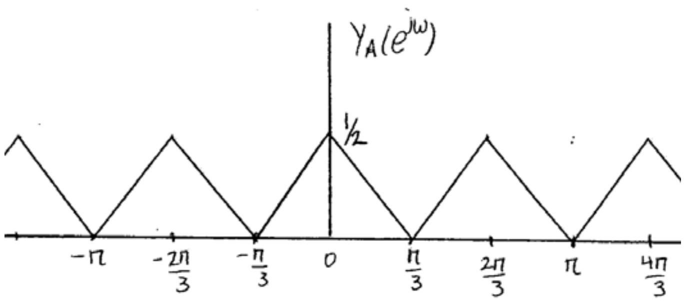


Fig. Q3a-2

Answer:





ii) Let $X(e^{j\omega})$ denote the Fourier transform of an arbitrary input $x[n]$. Express $Y_B(e^{j\omega})$ in terms of $X(e^{j\omega})$. Your answer should be in the form of an equation, not a sketch.

(6 marks)

Answer:

$$W_B(e^{j\omega}) = X(e^{j\omega L}) \xrightarrow{L=3} W_B(e^{j\omega}) = X(e^{j3\omega})$$
$$Y_B(e^{j\omega}) = \frac{1}{2}\left[W_B(e^{j\omega/2}) + W_B(e^{j(\omega-2\pi)/2})\right] = \frac{1}{2}\left[X(e^{j3\omega/2}) + X(-e^{j3\omega/2})\right]$$

iii) For the same input $x[n]$, what is the relationship between the two outputs $y_A[n]$ and $y_B[n]$? Use one sentence to justify your answer.

(3 marks)

Answer:

$y_A[n] = y_B[n]$.

This is because the two systems are equivalent, since L=3 and M=2 are coprime.

(b) Now consider the multirate system shown in Fig. Q3b. Find an expression for $y[n]$ in terms of $x[n]$ by simplifying the system.

(5 marks)

$$x[n] \longrightarrow \boxed{\uparrow 2} \longrightarrow \boxed{\uparrow 3} \longrightarrow \boxed{\downarrow 12} \longrightarrow \boxed{\uparrow 2} \longrightarrow y[n]$$

Fig. Q3b

Answer:

The system can be simplified to

$$\longrightarrow \boxed{\downarrow 2} \longrightarrow \boxed{\uparrow 2} \longrightarrow$$

Hence, $y[n] = x[n]$, if n is even; $y[n] = 0$, if n is odd.

(c) You are asked to convert the sampling rate of an audio signal from 44.1 kHz to 48 kHz using the system shown in Fig.Q3c. Specify your choices of *M* and *L*, as well as, the gain and cutoff frequency of the lowpass filter *H*(*z*).

(5 marks)

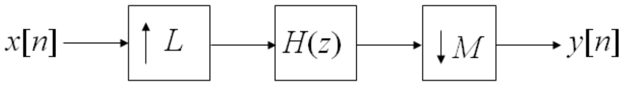


Fig. Q3c

Answer:

L=160, *M*=147, Gain=*L*=160, cutoff =π /160.

Q4

(a) A zero mean noise process *x*[*n*] has the following autocorrelation function

$$R_x(m) = \left(\frac{1}{2}\right)^{|m|}$$

We process $x[n]$ with a linear time-invariant system that satisfies the following difference equation:

$$y[n] = x[n+1] + x[n-1].$$

i) What is the autocorrelation function $R_y[m]$ of the process $y[n]$?

(6 marks)

Answer:

$$R_y[m] = E(y[n]y[n+m]) = E\{(x[n+1]+x[n-1])(x[n+1+m]+x[n+m-1])\}$$

$$R_y[m] = 2R_x[m] + R_x[m-2] + R_x[m+2]$$

ii) What is the power spectral density $S_y\left(e^{j\omega}\right)$ of the process $y[n]$?

(Hint: $a^n u[n] \leftrightarrow \dfrac{1}{1-az^{-1}}$, $|a|<1; -a^n u[-n-1] \leftrightarrow \dfrac{1}{1-az^{-1}}$, $|a|>1$)

(6 marks)

Answer:

Take the z-transform of $R_x(m) = \left(\frac{1}{2}\right)^{|m|}$

$$R_x(m) = \left(\frac{1}{2}\right)^{|m|} = \left(\frac{1}{2}\right)^{m} u(m) + \left(\frac{1}{2}\right)^{-m} u(-m-1) = \left(\frac{1}{2}\right)^{m} u(m) + 2^m u(-m-1)$$

$$S_x(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}} = \frac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)} = \frac{\frac{3}{4}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{z}{2}\right)}$$

$$S_x\left(e^{j\omega}\right) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}} = \frac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)} = \frac{\frac{3}{4}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{e^{j\omega}}{2}\right)}$$

Take the z-transform of the auto-correlation function
$R_y[m] = 2R_x[m] + R_x[m-2] + R_x[m+2]$:

$$S_y\left(e^{j\omega}\right) = \frac{\frac{3}{4}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{e^{j\omega}}{2}\right)}\left(2+e^{-j2\omega}+e^{j2\omega}\right) = \frac{\frac{3}{2}(1+\cos 2\omega)}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{e^{j\omega}}{2}\right)}$$

(b) Consider the power spectral density function given by:

$$S_x\left(e^{j\omega}\right) = 5 + 4\cos(\omega)$$

i) Find the corresponding autocorrelation function $R_x[m]$.

(4 marks)

Answer:

$$S_x\left(e^{j\omega}\right) = 5 + 4\cos(\omega) = 5 + 2e^{j\omega} + 2e^{-j\omega}$$

$$R_x[m] = 5\,\delta[\mathrm{m}] + 2\,\delta[\mathrm{m}-1] + 2\,\delta[\mathrm{m}+1]$$

ii) Find a linear filter whose output process has the same autocorrelation function $R_x[m]$, when excited by white noise of zero mean and unit variance.

(6 marks)

Answer:

Spectral factorization:

$$S_x(z) = 5 + 2z + 2z^{-1} = \left(1+2z^{-1}\right)(1+2z) = H(z)H\left(z^{-1}\right)$$

Causal filter:

$$H(z) = 1 + 2z^{-1}$$

iii) What is the variance of the output process?

(3 marks)

Answer:

$\sigma_x^2 = r_x(0) = 5.$