EE3731C:
Signal Processing Methods

Lecture II-2:
Introduction to Wavelet Transform

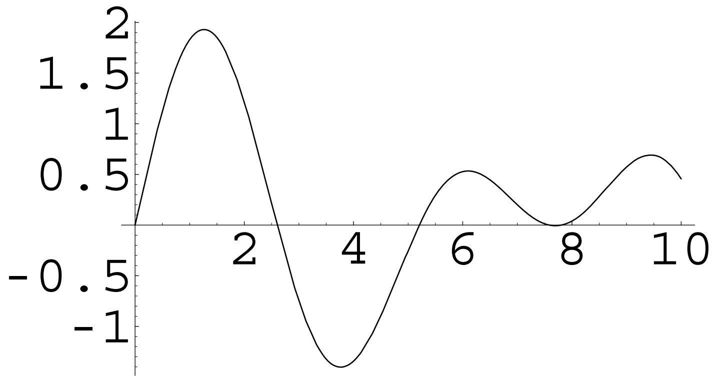
National University of Singapore

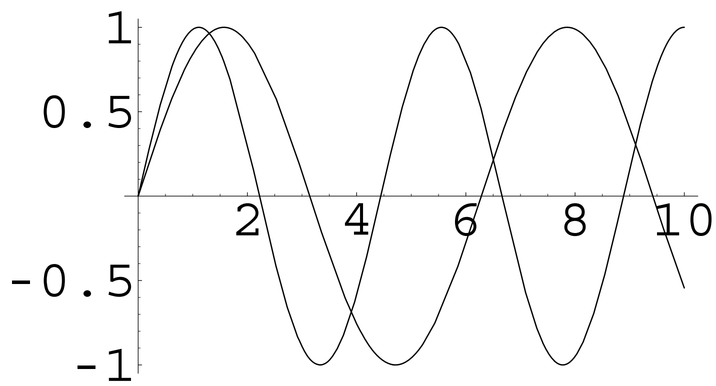
Outline

- Motivation
- What are Wavelets?
- Continuous Wavelet Transform (CWT)
- Discrete Wavelet Transform (DWT)
- Example Applications of Wavelet Transform

Drawbacks of Fourier Transform

- A signal can be viewed from two different standpoints: time domain and frequency domain.
- The Fourier Transform method is used to decompose a signal into its global frequency components - loss of time information.

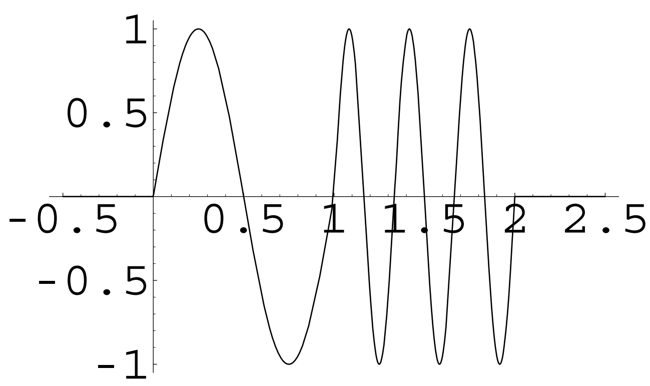


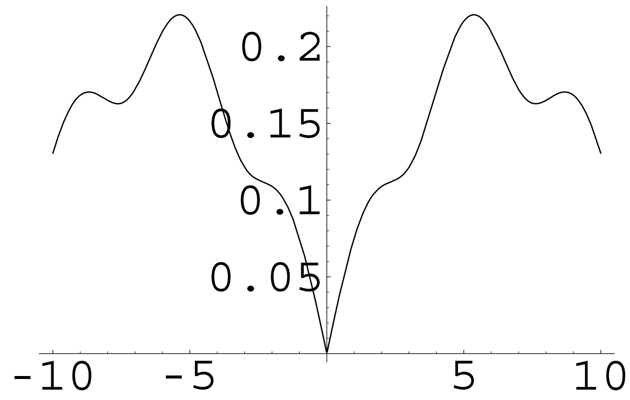


FT is very helpful for the analysis of this signal.

Drawbacks of Fourier Transform

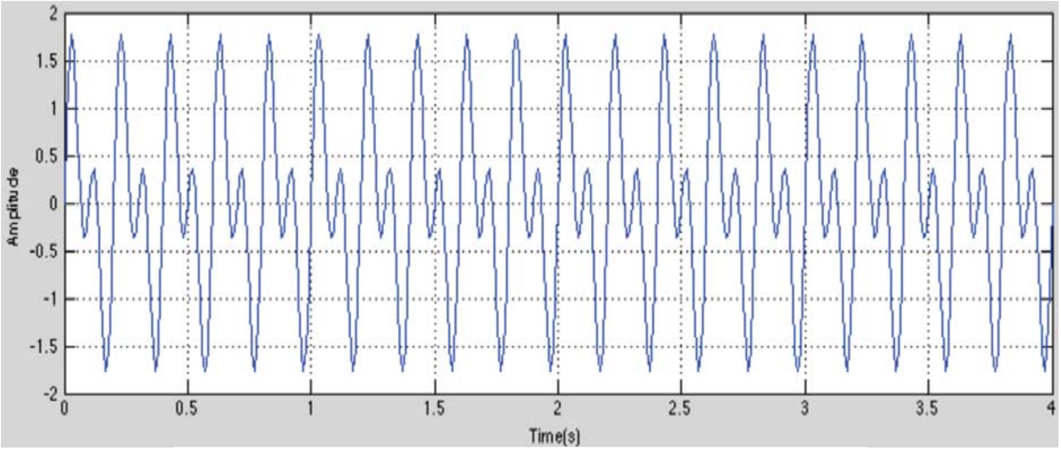
- The Fourier Transform method is used to decompose a signal into its global frequency components - loss of time information.
- FT is not suitable for non-stationary signals whose spectral characteristics vary with time.

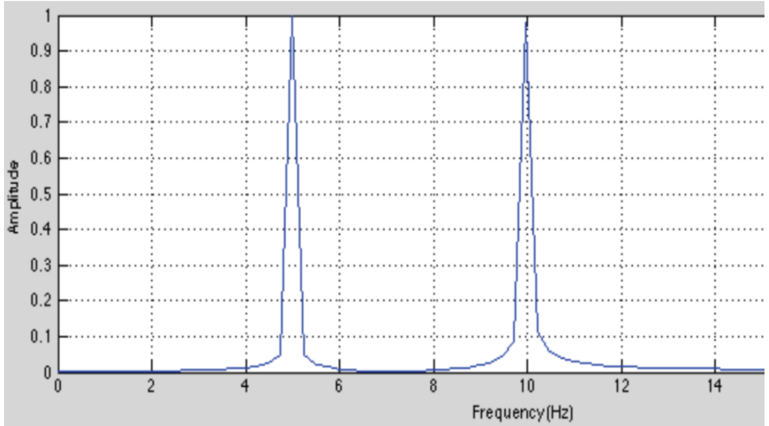




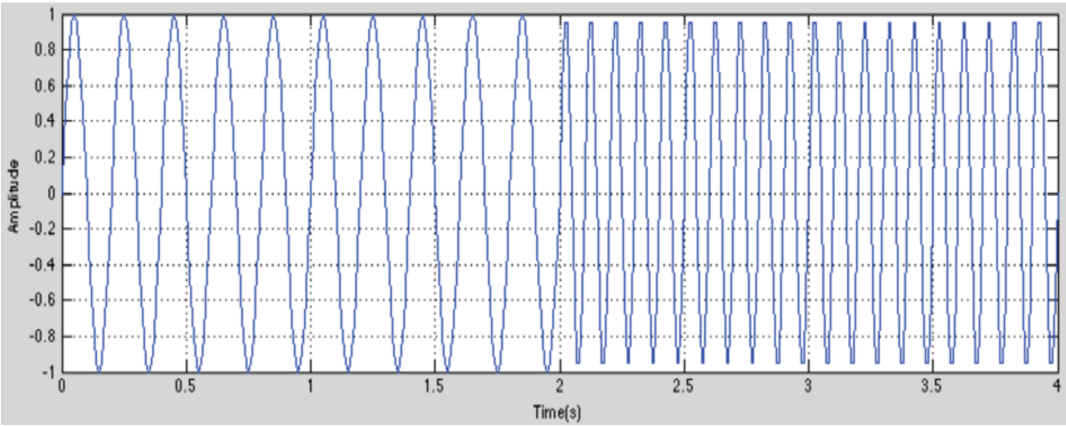
FT is unable to pick out local frequency content. Time varying frequencies are very common in music, speech, etc.

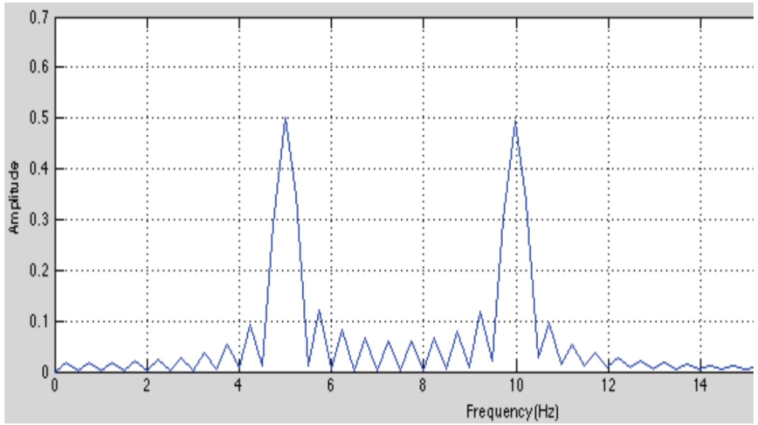
Fourier Transform: Example





$$f(t) = \sin(2\pi 5t) + \sin(2\pi 10t)$$





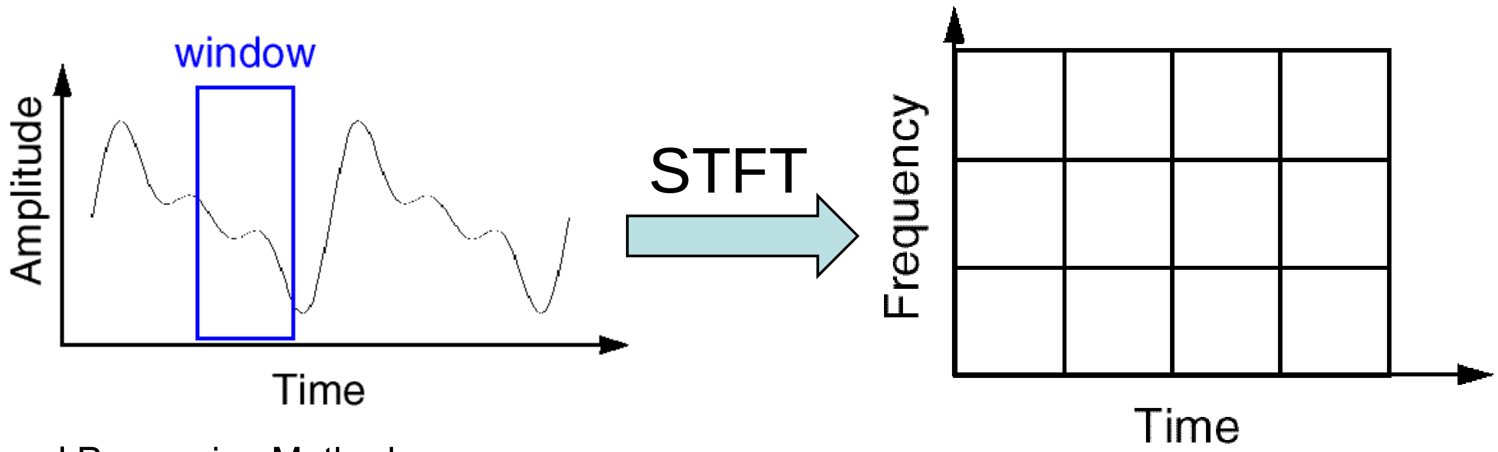
$$f(t) = \sin(2\pi 5t) \quad 0 \le t \le 2$$

$$f(t) = \sin(2\pi 10t) \quad 2 \le t \le 4$$

Two very distinct signals with spectra of very similar shape

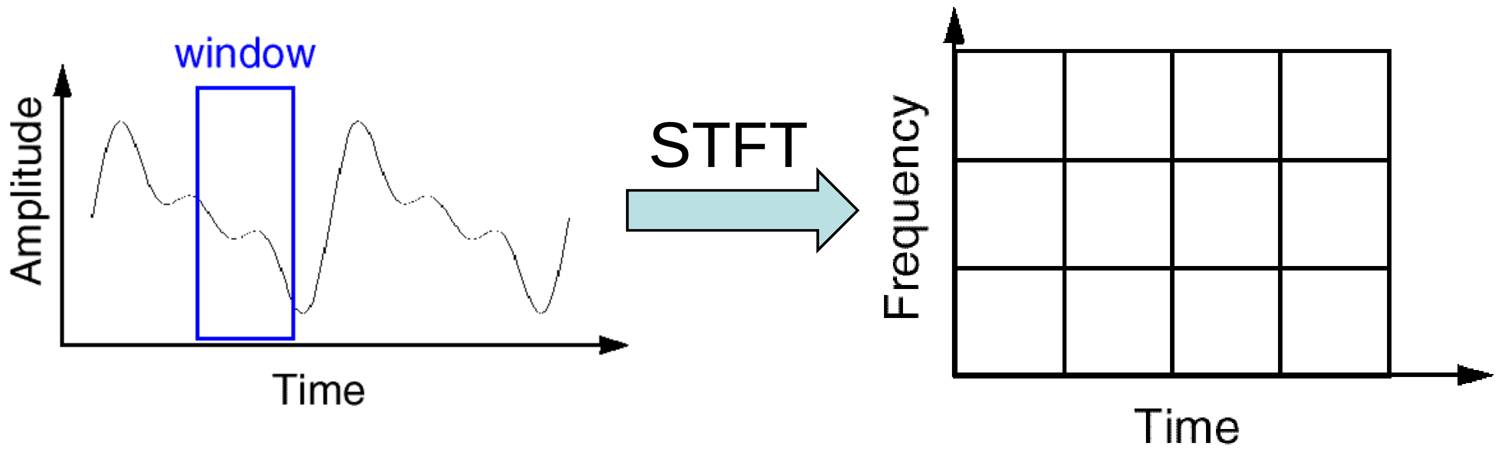
Short-Time Fourier Transform (STFT)

- How can we capture local effects (transients)? Use a window!
- Dennis Gabor (1946) adapted the FT to analyze only a small section of the signal at a time (*windowing* the signal)
- STFT is a function of two variables: time and frequency.



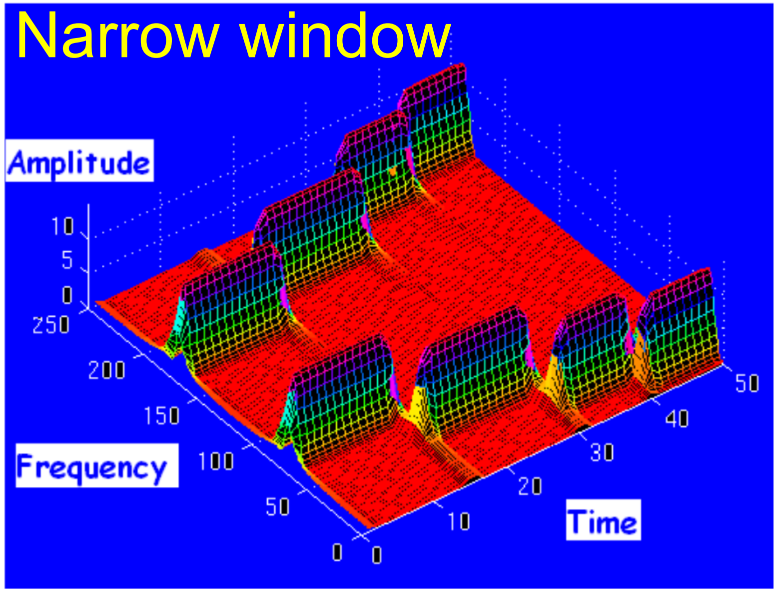
STFT

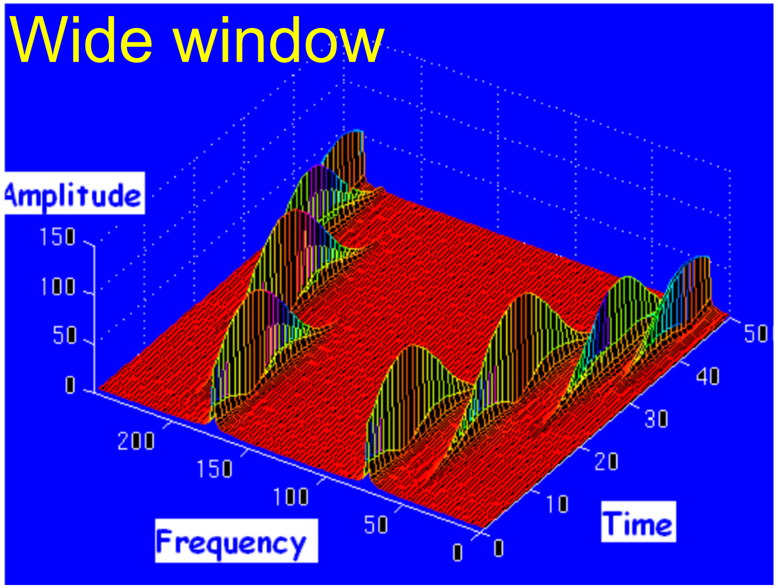
- A compromise between time-based and frequency-based views of a signal
- Provides information about time and frequency (i.e., what frequencies occur when), but with limited precision
- Time/Frequency localization depends on the size of the window.



Drawbacks of STFT

- Once you choose a particular window size, it will be the same for all frequencies.
- Dilemma of Resolution:
  - Narrow window → poor frequency resolution
  - Wide window → poor time resolution



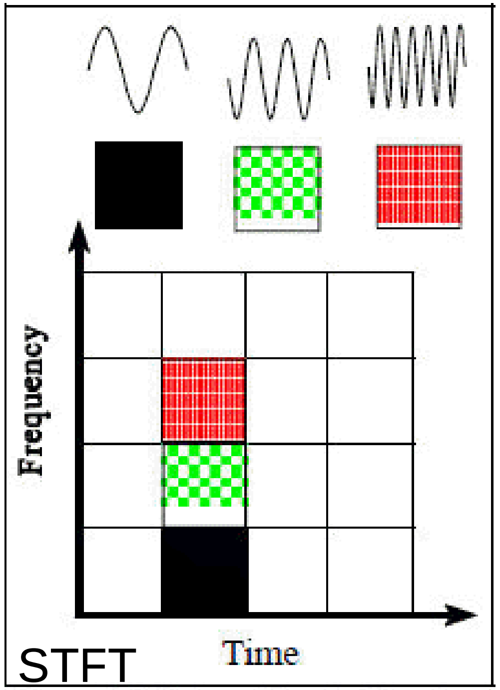


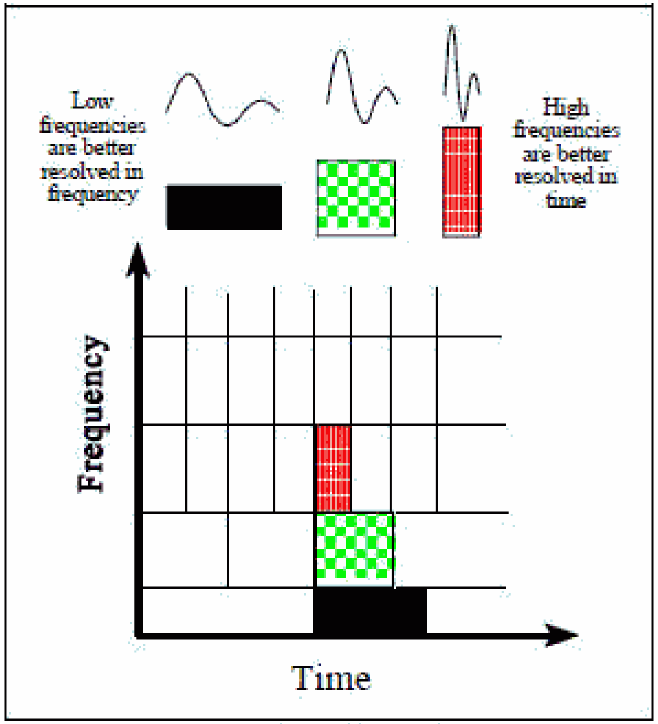
Drawbacks of STFT

- Many signals require a more flexible approach - vary the window size to determine more accurately either time or frequency.
- Solution: Wavelet Transform
  - An alternative approach to STFT to overcome the resolution problem
  - Analyze the signal at different frequencies with different resolutions
  - Good time (frequency) resolution and poor frequency (time) resolution at high (low) frequencies
  - More suitable for shorter duration of higher frequency and longer duration of lower frequency components

Wavelet Transform

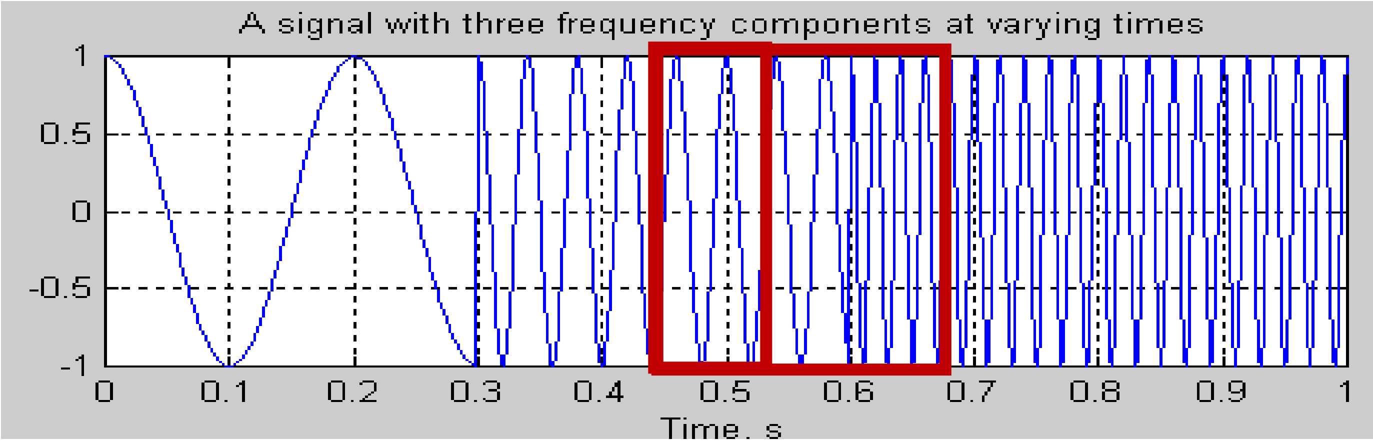
- Overcomes the preset resolution problem:
  - –Use narrower windows at high frequencies for better time resolution.
  - –Use wider windows at low frequencies for better frequency resolution.



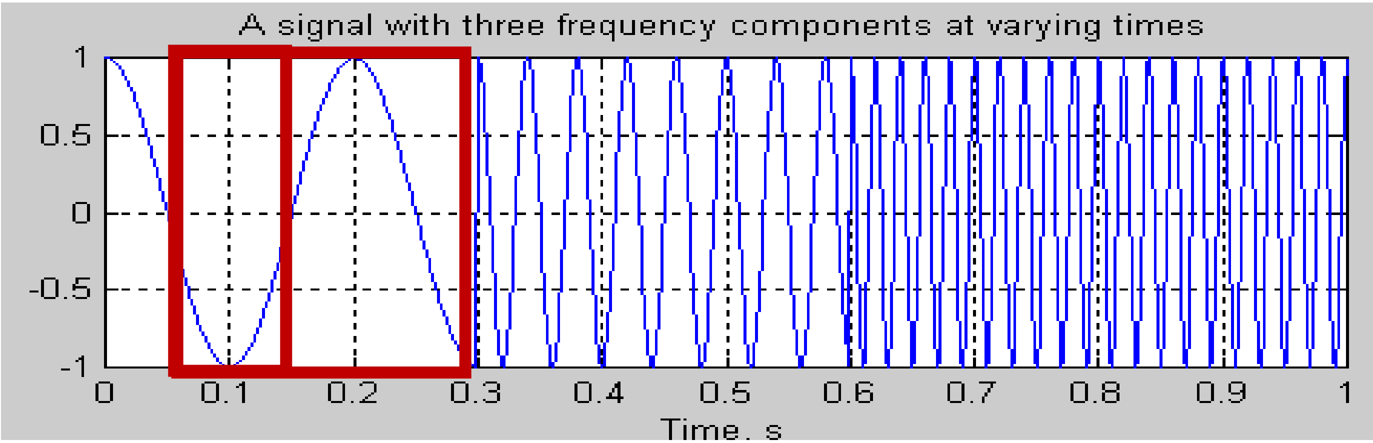


Wavelet Transform

Wavelet Transform

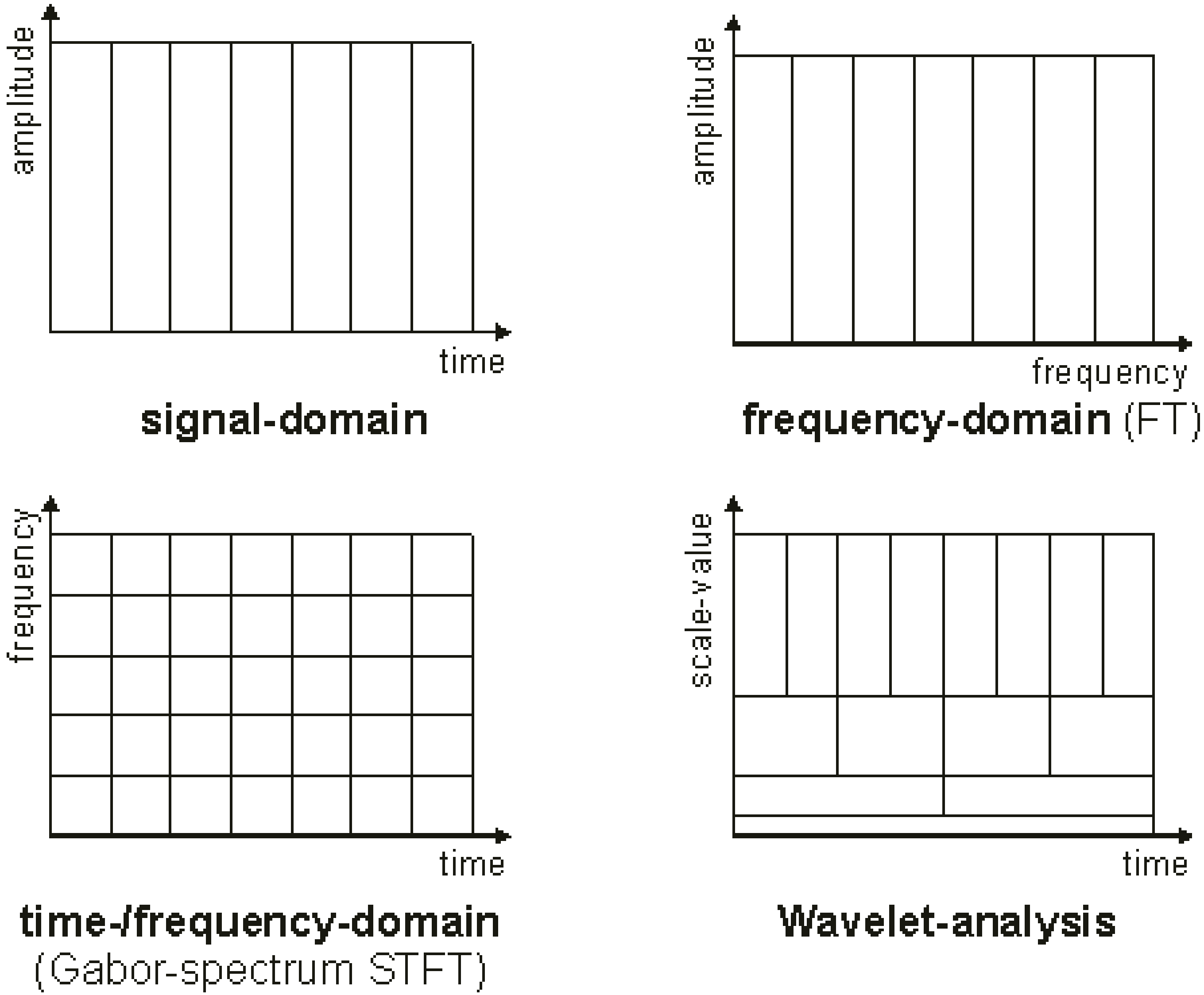


Wide windows do not provide good localization at high frequencies.



Narrow windows do not provide good localization at low frequencies.

Wavelet Transform



Wavelet History

- First wavelet (Haar wavelet) by Alfred Haar in 1909
- Great development in the early 1980's for analysis of seismic signals in geophysics
- Formalized in early 1980's by Morlet, Grossmann and Goupillaud
- Further significant work by Meyer, Mallat, Daubechies and Chui
- Applications: mathematics, physics, digital signal processing, vision, numerical analysis, geophysics, astronomy FBI fingerprint compression, JPEG2000 standard, trend detection of financial data

Wavelet Transforms: Overview

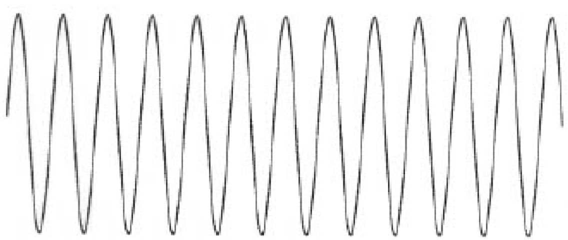
- Convert a signal into a series of wavelets (small waves)
- Provide a way to analyze waveforms in both frequency and time
- Allow signals to be stored more efficiently than by Fourier transform
- Well-suited for approximating data with sharp discontinuities
- "The Forest & the Trees"
  - Notice gross features with a large window
  - Notice small features with a small window

What are Wavelets?

- Wavelets are functions that “wave” above and below the x-axis.
- Wavelets have:
  - varying frequency
  - limited duration
  - an average value of zero

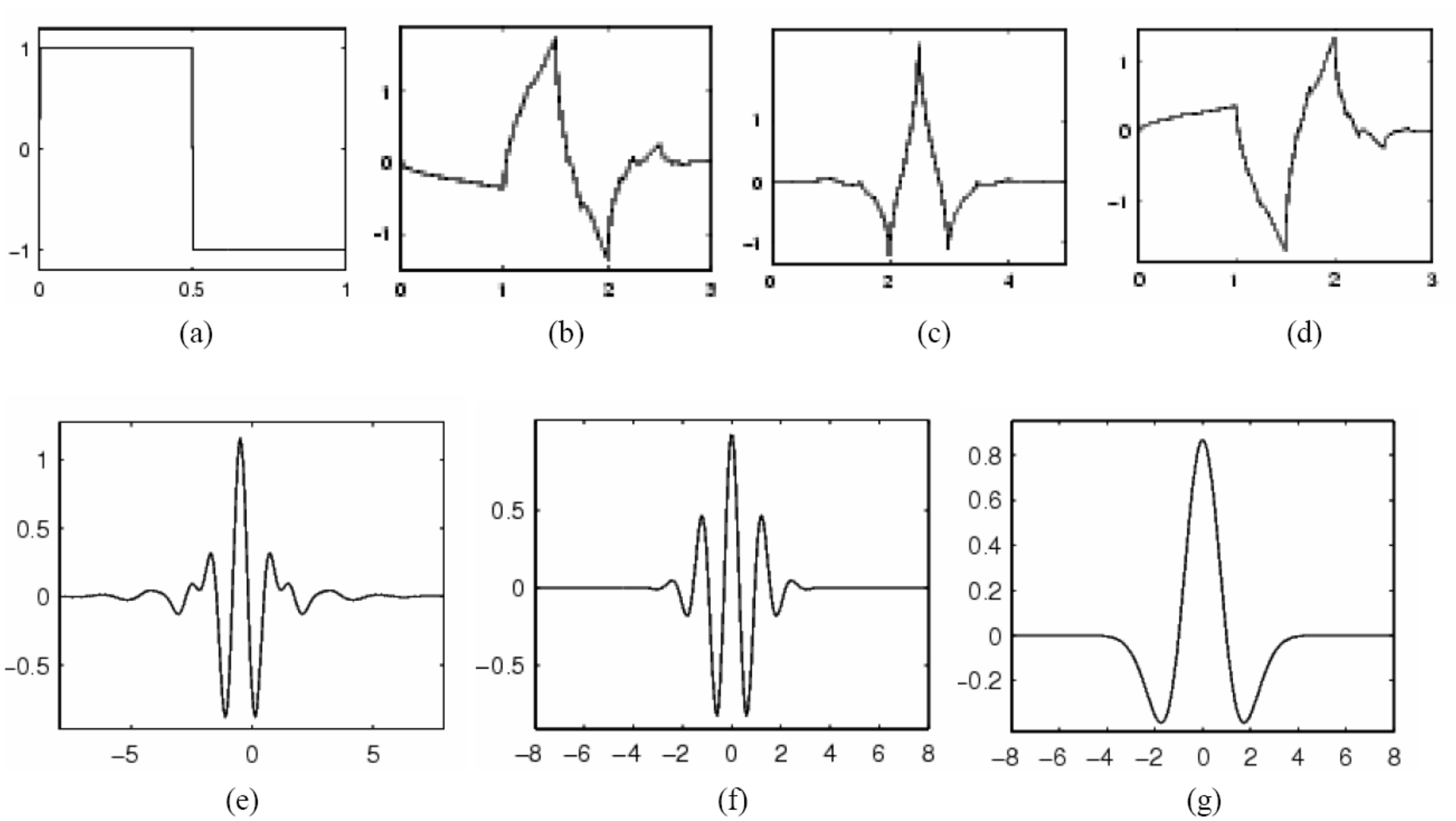
$$\int_{-\infty}^{\infty} \psi(t)\, dt = \Psi(0) = 0$$

Sinusoid wave



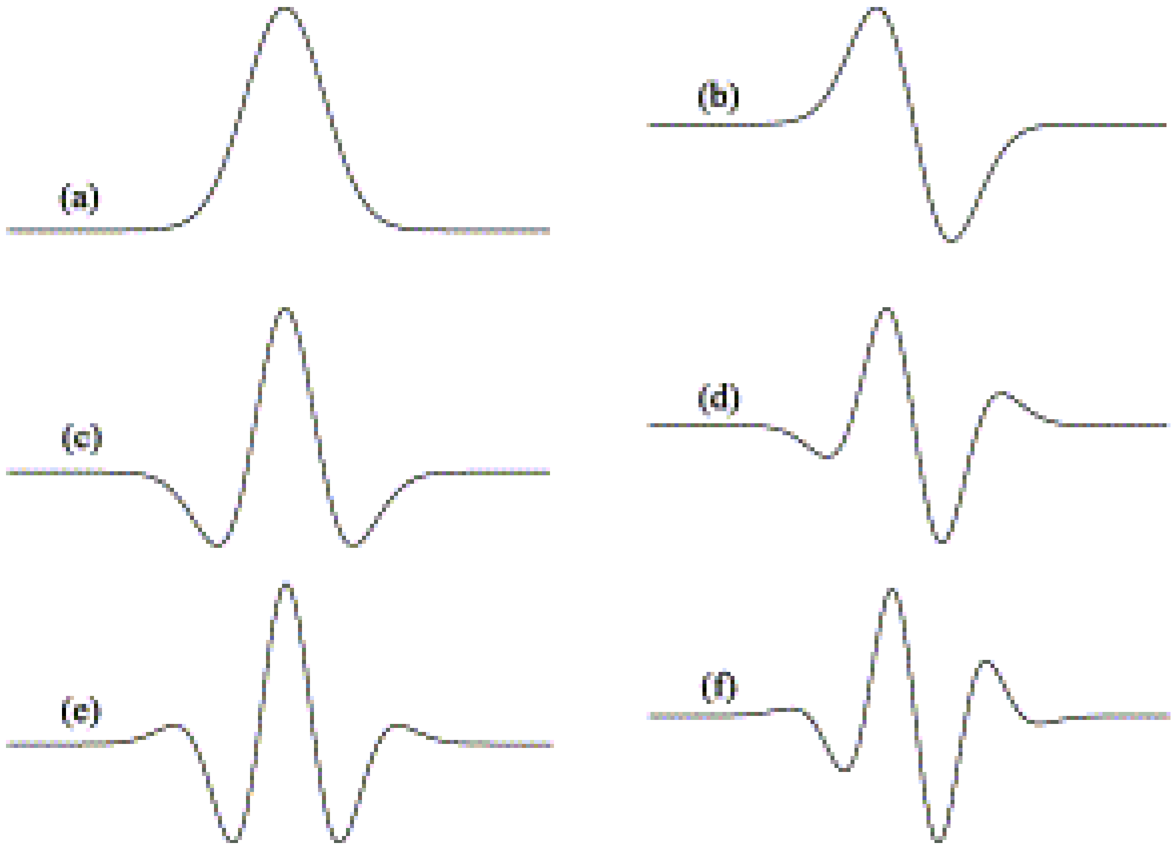
Wavelet

Examples of Wavelets



(a) Haar (b) Daubechies4 (c) Coiflet1 (d) Symlet2 (e) Meyer (f) Morlet (g) Mexican Hat

Gaussian Wavelets

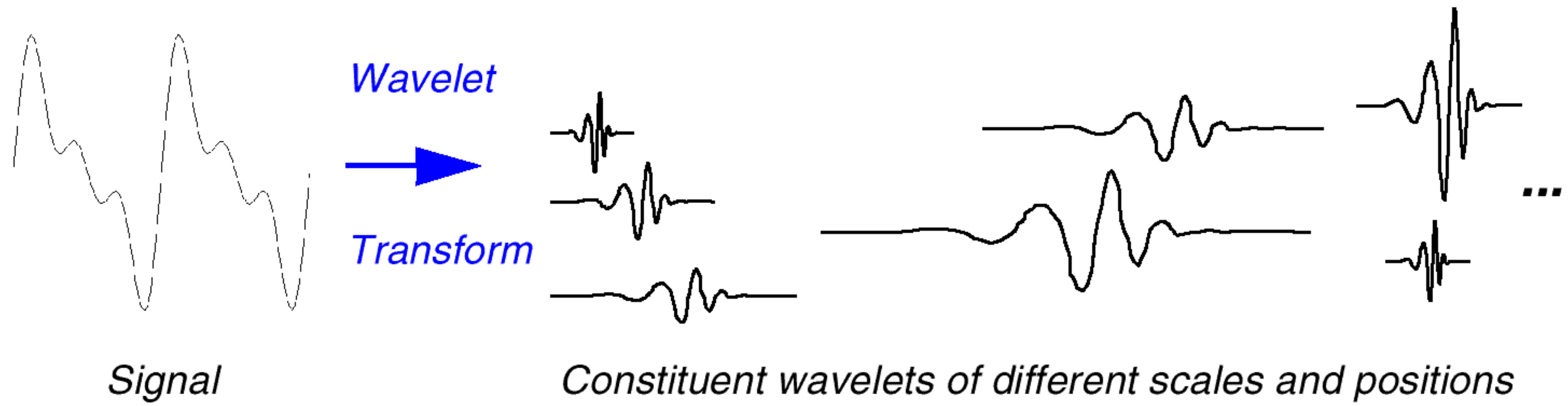


(a) Gaussian function. (b)–(f) are Gaussian wavelet functions which correspond to the first- to fifth-order derivatives of the Gaussian function, respectively.

[PSI,X] = gauswavf(LB,UB,N,P) returns values of the *P*-th derivative of the Gaussian function on an *N* point regular grid for the interval [LB,UB].

Continuous Wavelet Transform

- Wavelets are used as basis functions in representing the original signal.
- CWT results in Wavelet coefficients
  - Multiplying each coefficient by the scaled and shifted mother wavelet yields the constituent wavelets of the original signal.

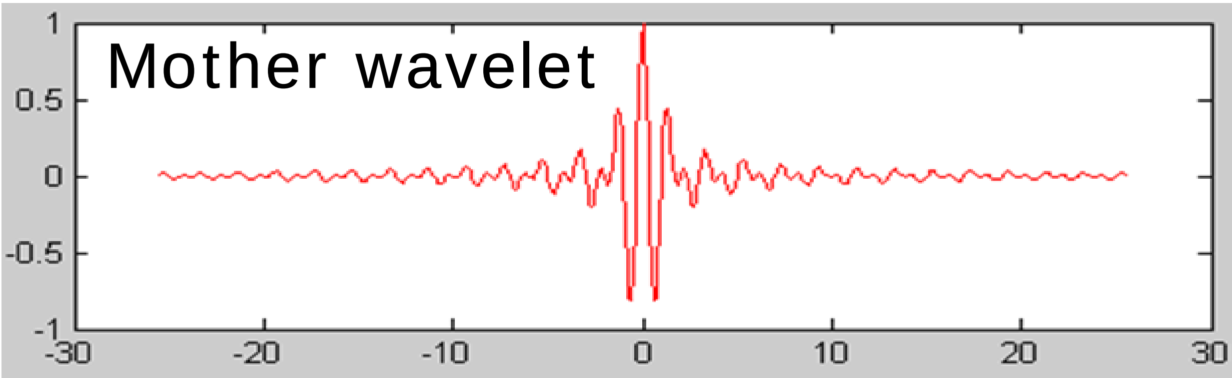


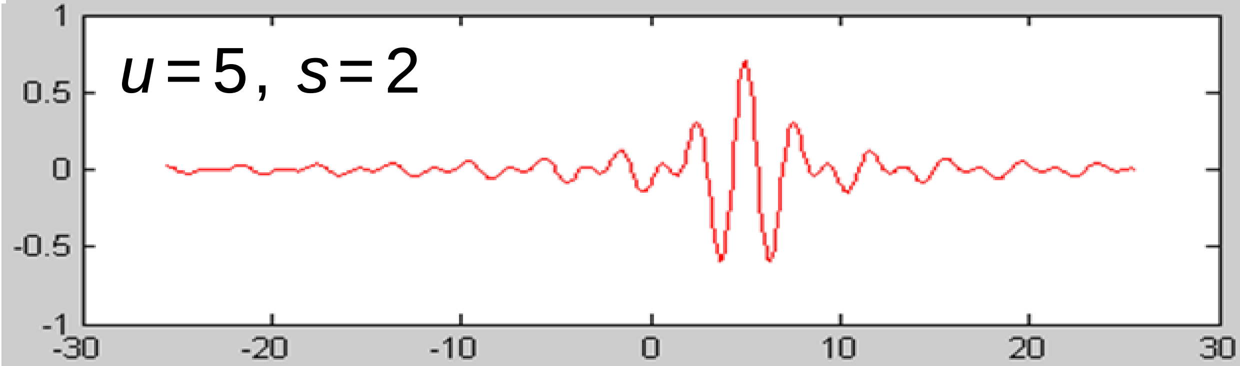
Wavelet Family

- Wavelets derived from *mother wavelet* $\psi(t)$

$$\psi_{u,s} = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)$$

- u – shift
- s – scale

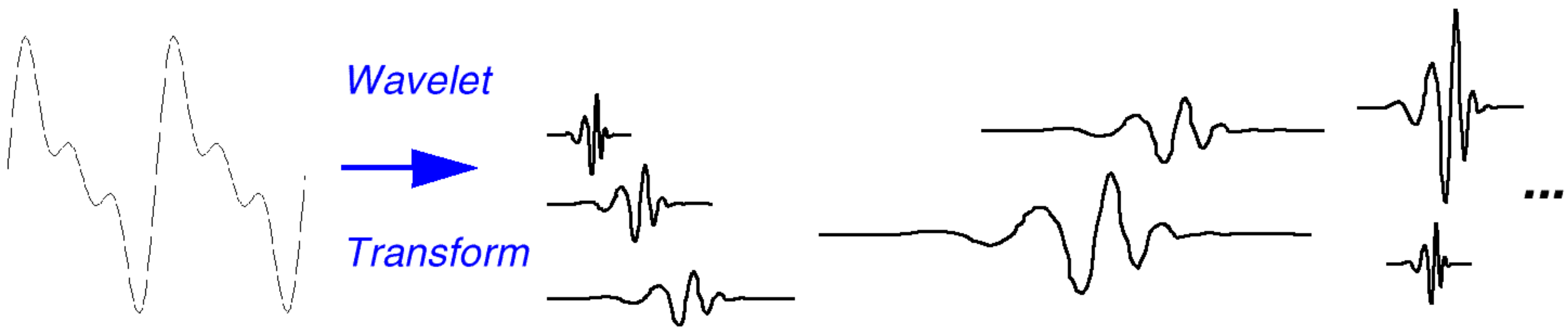




Continuous Wavelet Transform

CWT: the sum over all time of the signal, multiplied by scaled and shifted versions of the wavelet function

$$Wf(u,s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \psi^{*}(\frac{t-u}{s}) dt$$

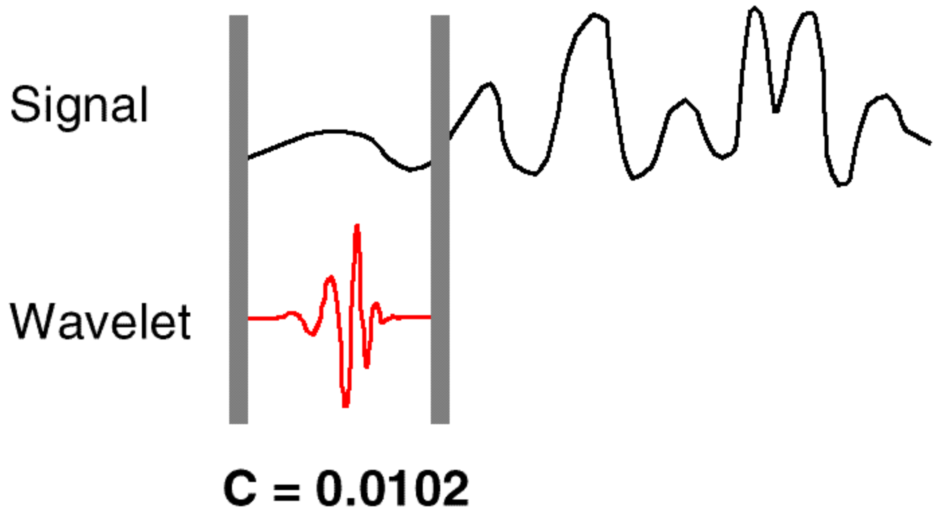


Signal *Constituent wavelets of different scales and positions*

Can ignore the complex conjugate if real wavelets are used.

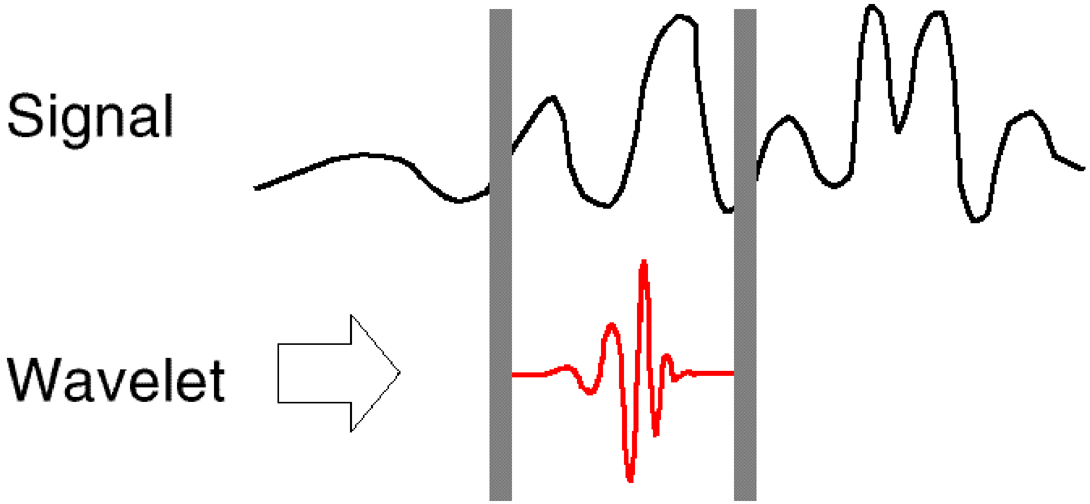
CWT: Main Steps

1. Take a wavelet and compare it to a section at the start of the original signal
2. Calculate a number, C, that represents how closely correlated the wavelet is with this section of the signal. The higher C is, the more the similarity.



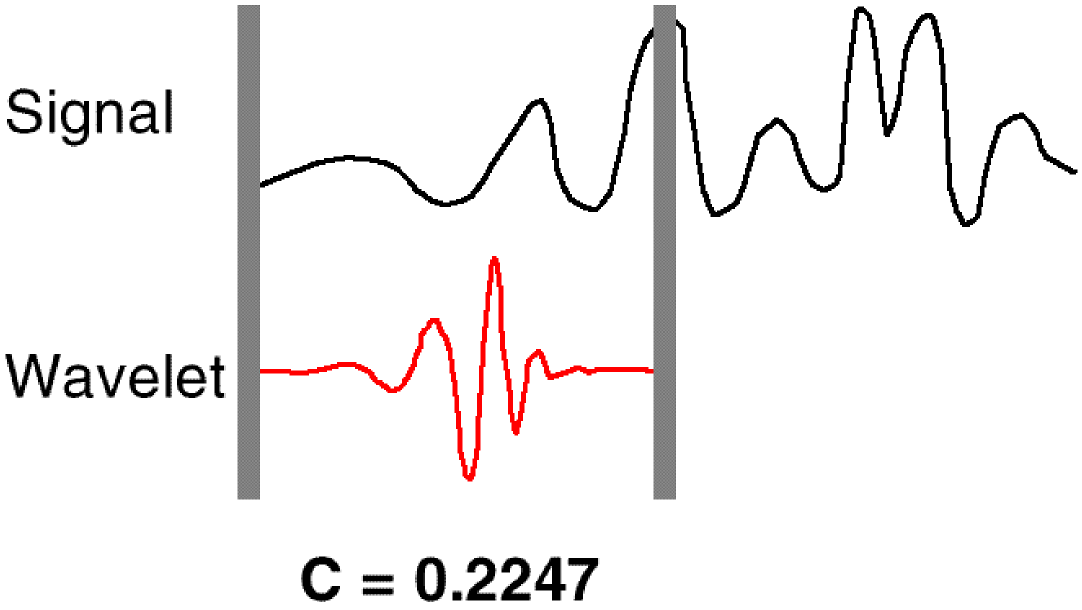
CWT: Main Steps (cont'd)

3. Shift the wavelet to the right and repeat steps 1 and 2 until the whole signal has been covered.



CWT: Main Steps (cont'd)

4. Scale the wavelet and repeat steps 1-3



5. Repeat steps 1-4 for all scales

Inverse CWT

CWT: decomposition of the signal into scaled and shifted versions of the mother wavelet

$$Wf(u,s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \psi^*\left(\frac{t-u}{s}\right) dt$$

Inverse CWT: reconstruction of the original signal as the sum of wavelets of different scales, *s*, and positions, *u*

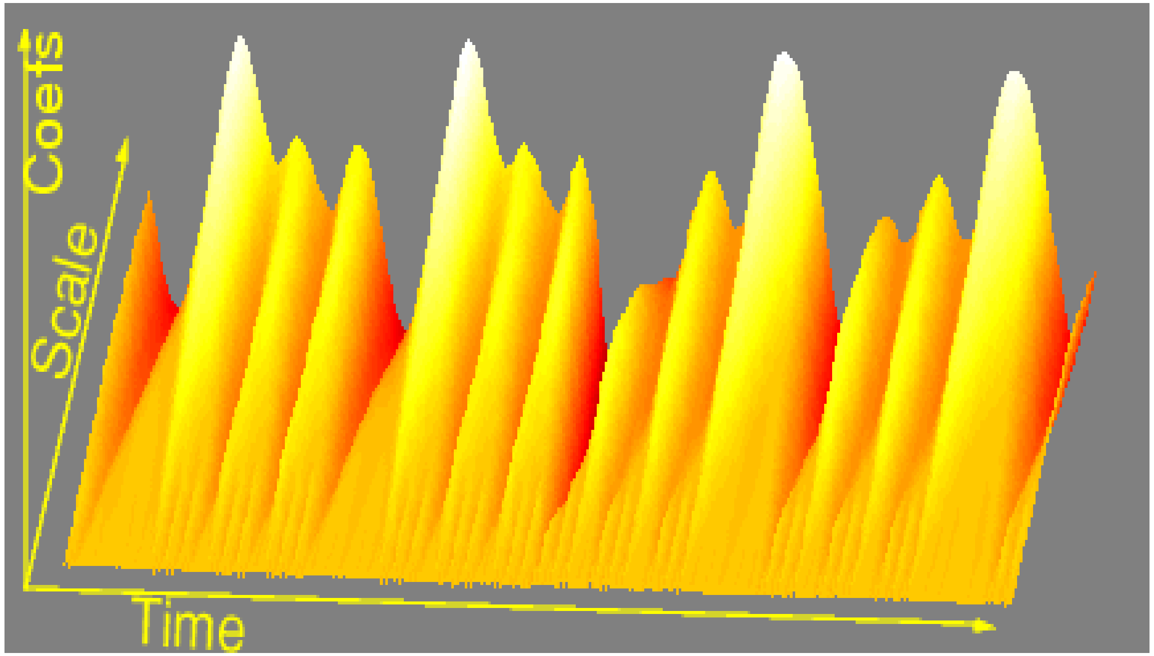
$$f(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Wf(u,s) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) du ds$$

double integral!

Coefficients of CWT

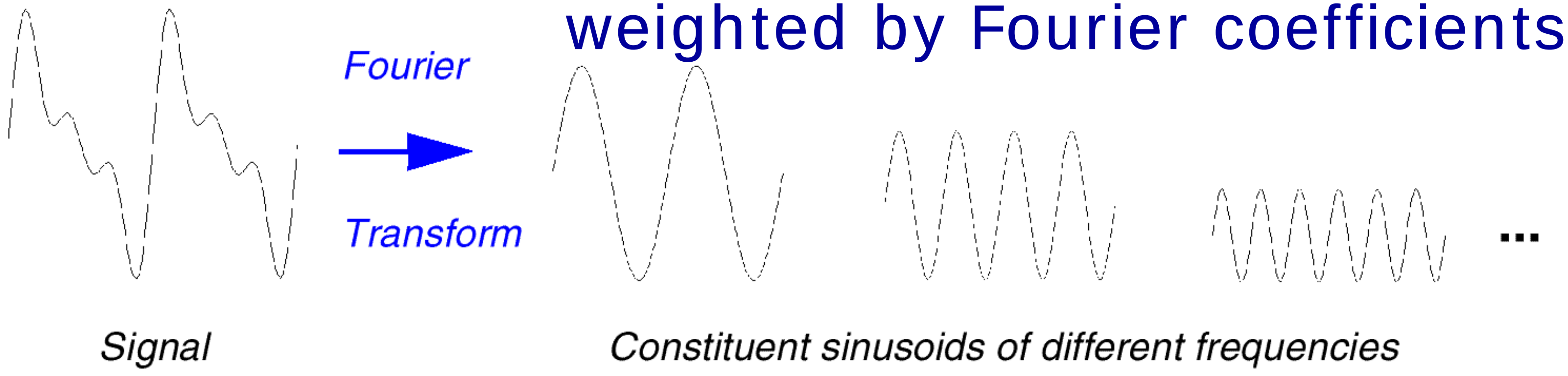
Wavelet analysis produces a time-scale view of the input signal or image.

$$Wf(u,s) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}}\psi^*\left(\frac{t-u}{s}\right)dt$$

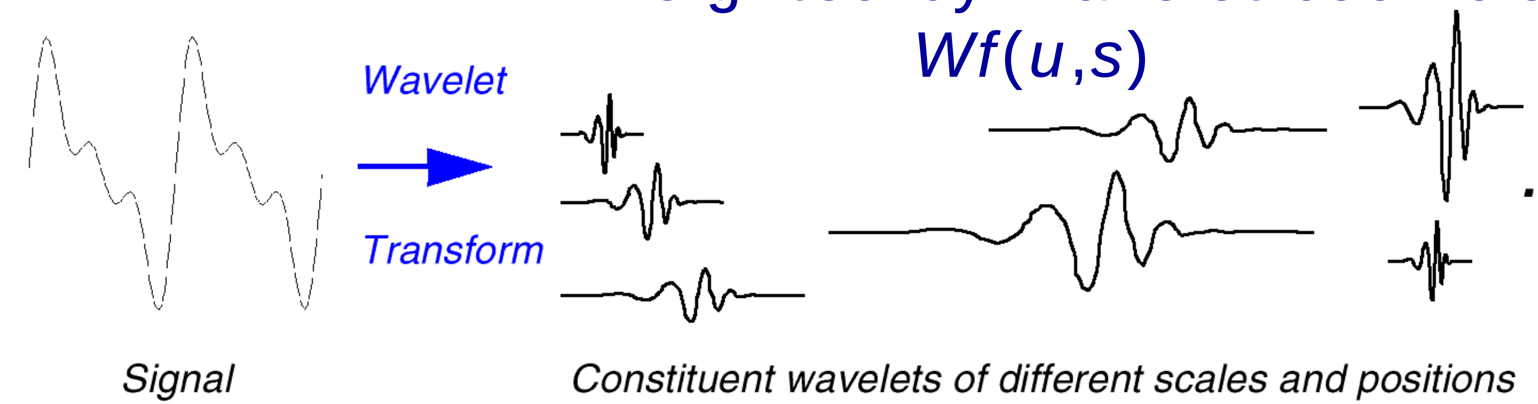


FT Vs. CWT

weighted by Fourier coefficients



weighted by Wavelet coefficients $Wf(u,s)$



Individual wavelet functions are *localized in space/time.*
Fourier sine and cosine functions are not.

Discrete Wavelet Transform (DWT)

- Problems of CWT:
  - Redundancy of wavelet coefficients (scaled versions are not an orthonormal basis)
  - Infinite number of wavelet components
- DWT: Sample at discrete times and scales to reduce redundancy, but still be able to reconstruct the signal

$$a_{jk} = \sum_t f(t)\psi^*_{jk}(t) \qquad \text{(forward DWT)}$$

$$f(t) = \sum_k \sum_j a_{jk}\psi_{jk}(t) \qquad \text{(inverse DWT)}$$

where $\psi_{jk}(t) = 2^{j/2}\psi\left(2^j t - k\right)$

DFT Vs. DWT

- FT expansion:

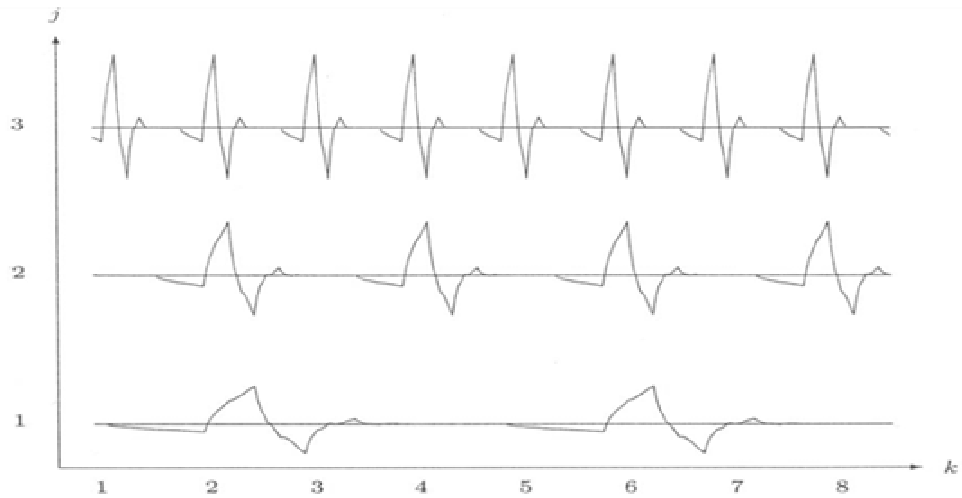
one parameter basis

$$f(x)=\sum_{u=0}^{N-1}F(u)e^{\frac{j2\pi ux}{N}},\quad \text{or}\quad f(t)=\sum_{l}a_l\psi_l(t)$$

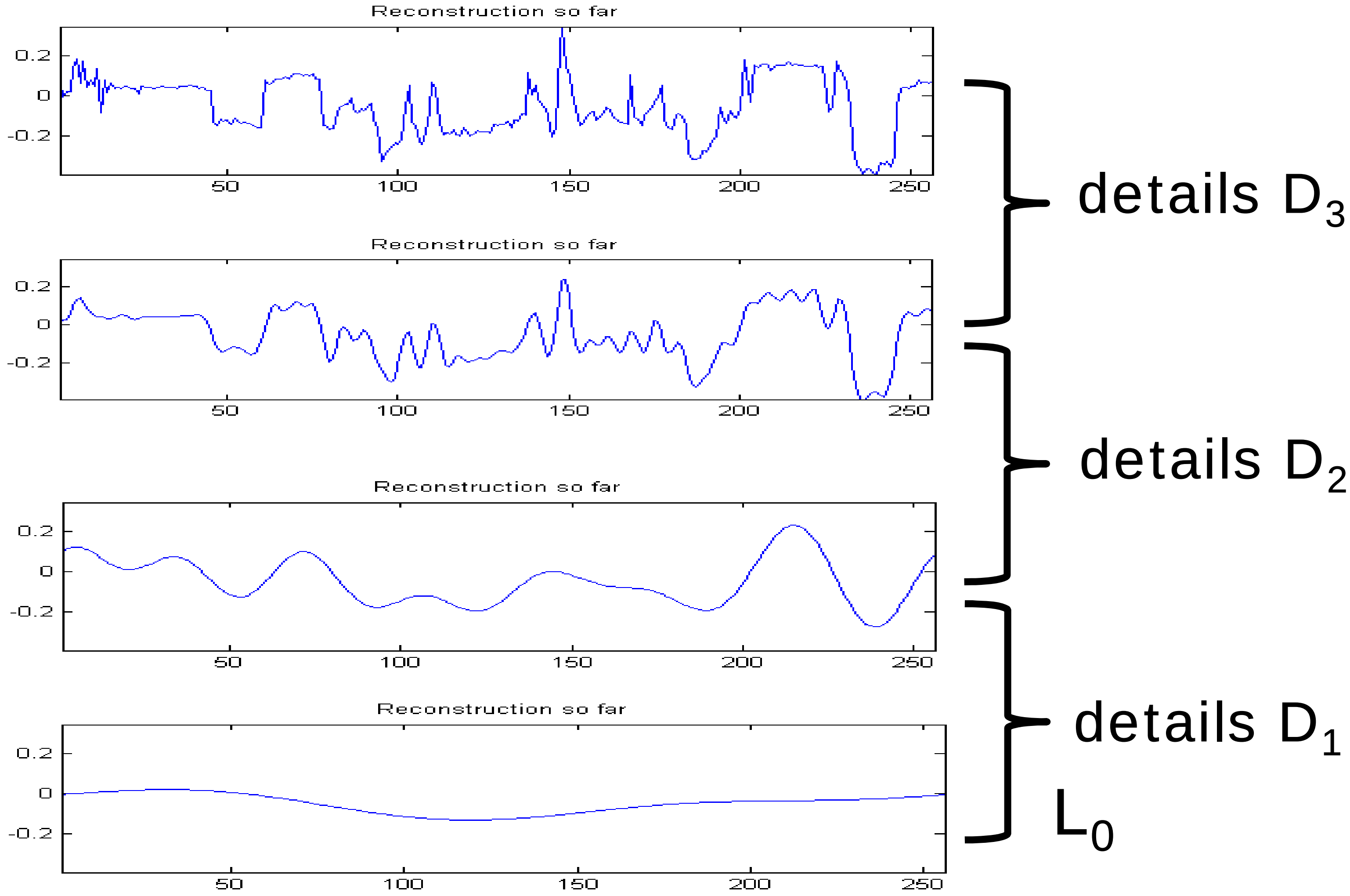
- WT expansion:

$$f(t)=\sum_{k}\sum_{j}a_{jk}\psi_{jk}(t)$$

two parameter basis

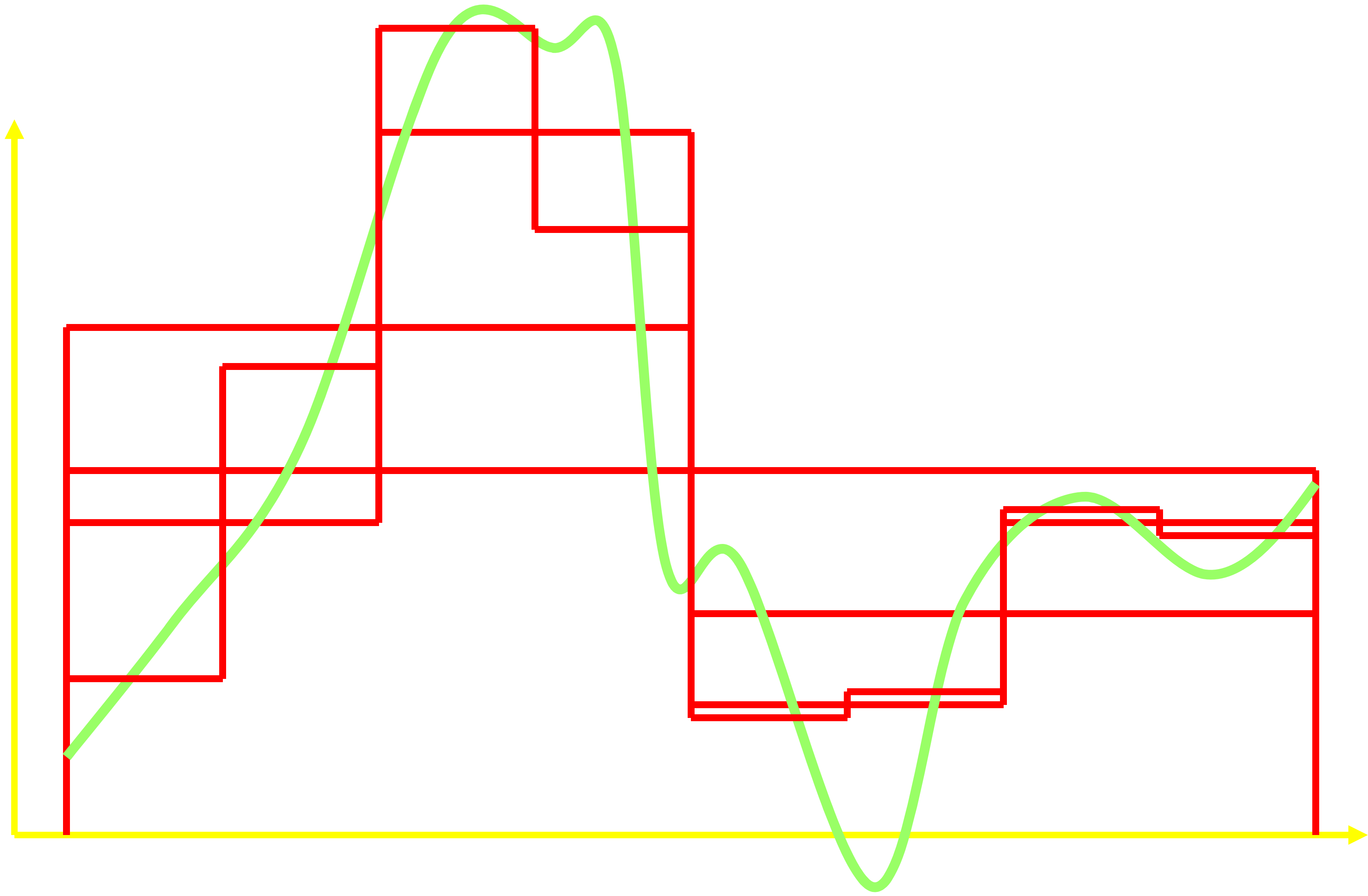


Multiresolution Representation



Efficient representation: L_0 D_1 D_2 D_3

Haar Transform



Haar Wavelets

- Mother scaling function:

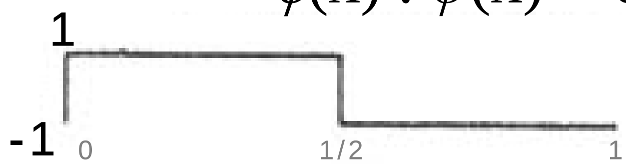
$$\phi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & otherwise \end{cases}$$

compute average

- Mother wavelet function:

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2 \\ -1 & \text{if } 1/2 \le x < 1 \\ 0 & otherwise \end{cases}$$

$$\phi(\text{x}) \,.\, \psi(\text{x}) = 0$$

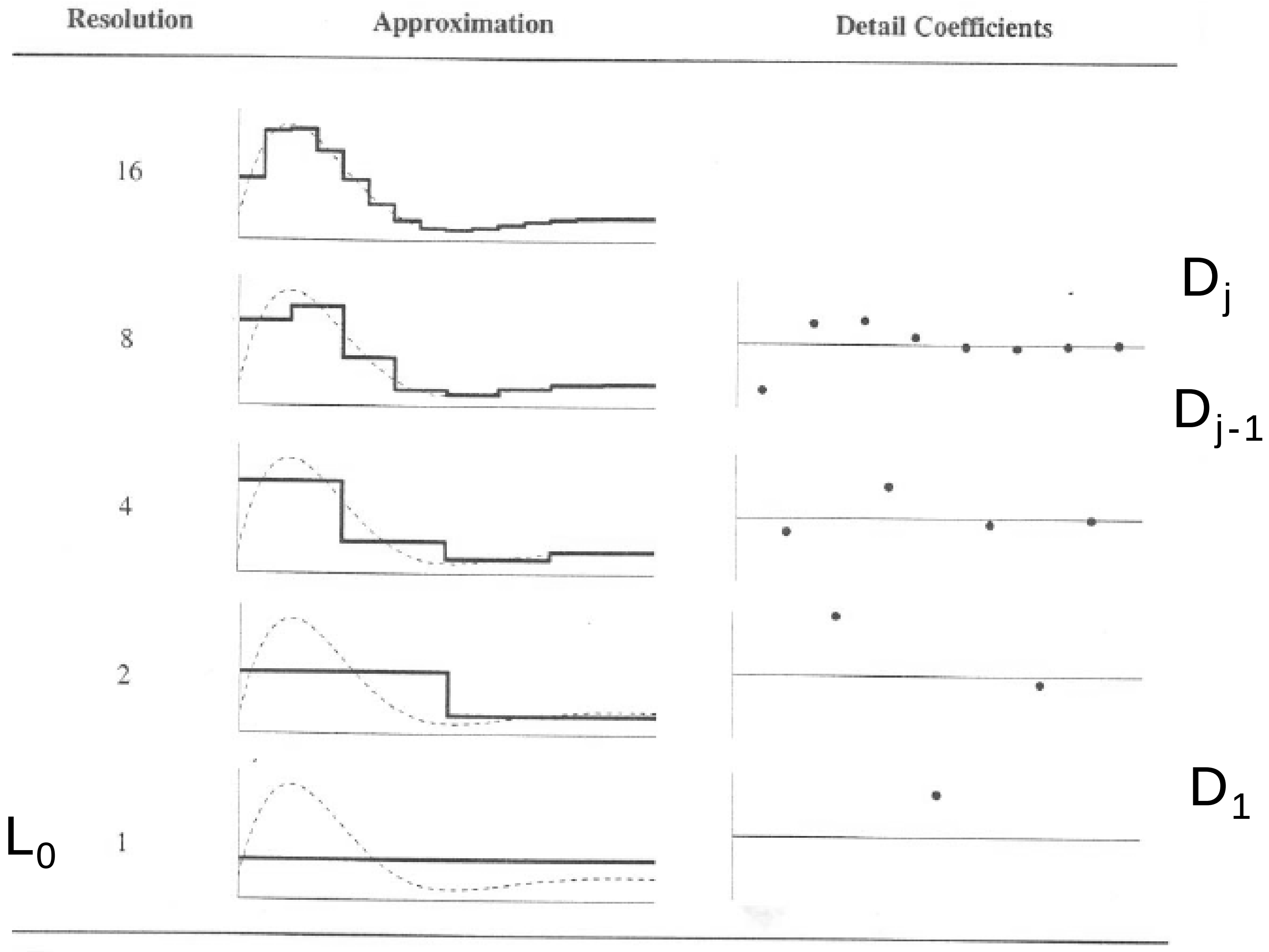


compute details

- Wavelet expansion:

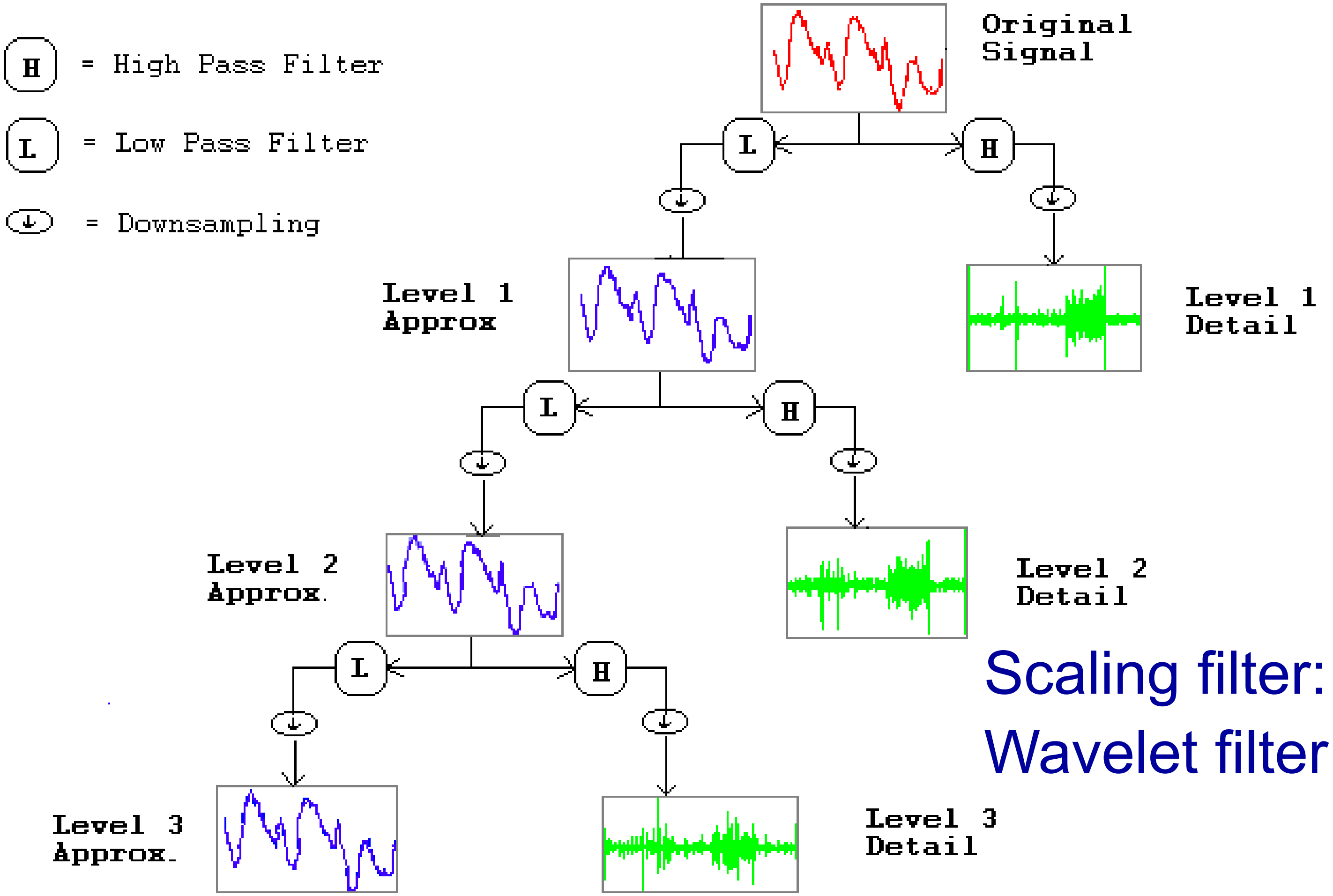
$$f(t) = \sum_k c_k \varphi(t-k) + \sum_k \sum_j d_{jk} \psi(2^j t - k)$$

Haar Wavelets



Multiresolution Representation

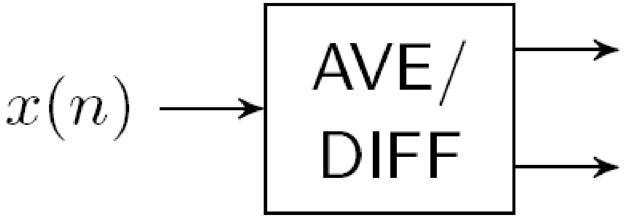
DWT breaks down the signal into detail signals and an approximation.



Scaling filter: low pass

Wavelet filter: high pass

Example: Haar Wavelets



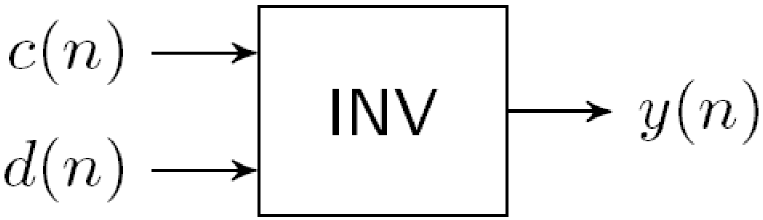
$$c(n) = 0.5\,x(2n) + 0.5\,x(2n+1)$$

$$d(n) = 0.5\,x(2n) - 0.5\,x(2n+1)$$

Basis vectors: [1/2, 1/2] and [1/2, -1/2]

Note that the normalized basis vectors are:

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$



$$y(2n) = c(n) + d(n)$$

$$y(2n+1) = c(n) - d(n)$$

Example: Haar Wavelets

- Given a 1D "image" with 4 pixels:

[9 7 3 5]

low-pass, down-sampling

$[8 \quad 4]$

high-pass, down-sampling

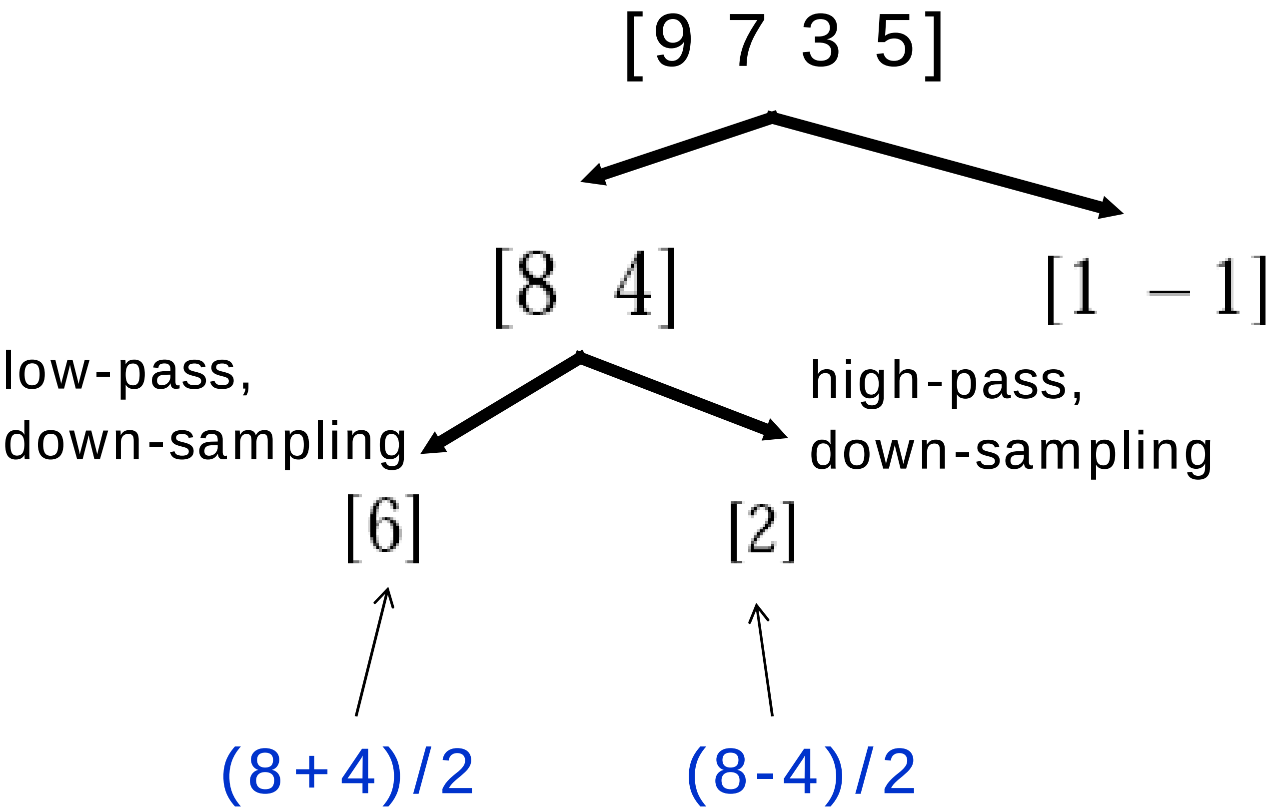
$[1 \quad -1]$

(9+7)/2 (3+5)/2 (9-7)/2 (3-5)/2

Average the four pixels to get a two-pixel lower resolution image [8 4]

To recover the original four pixels from the two averaged pixels, store detail coefficients [1 -1]

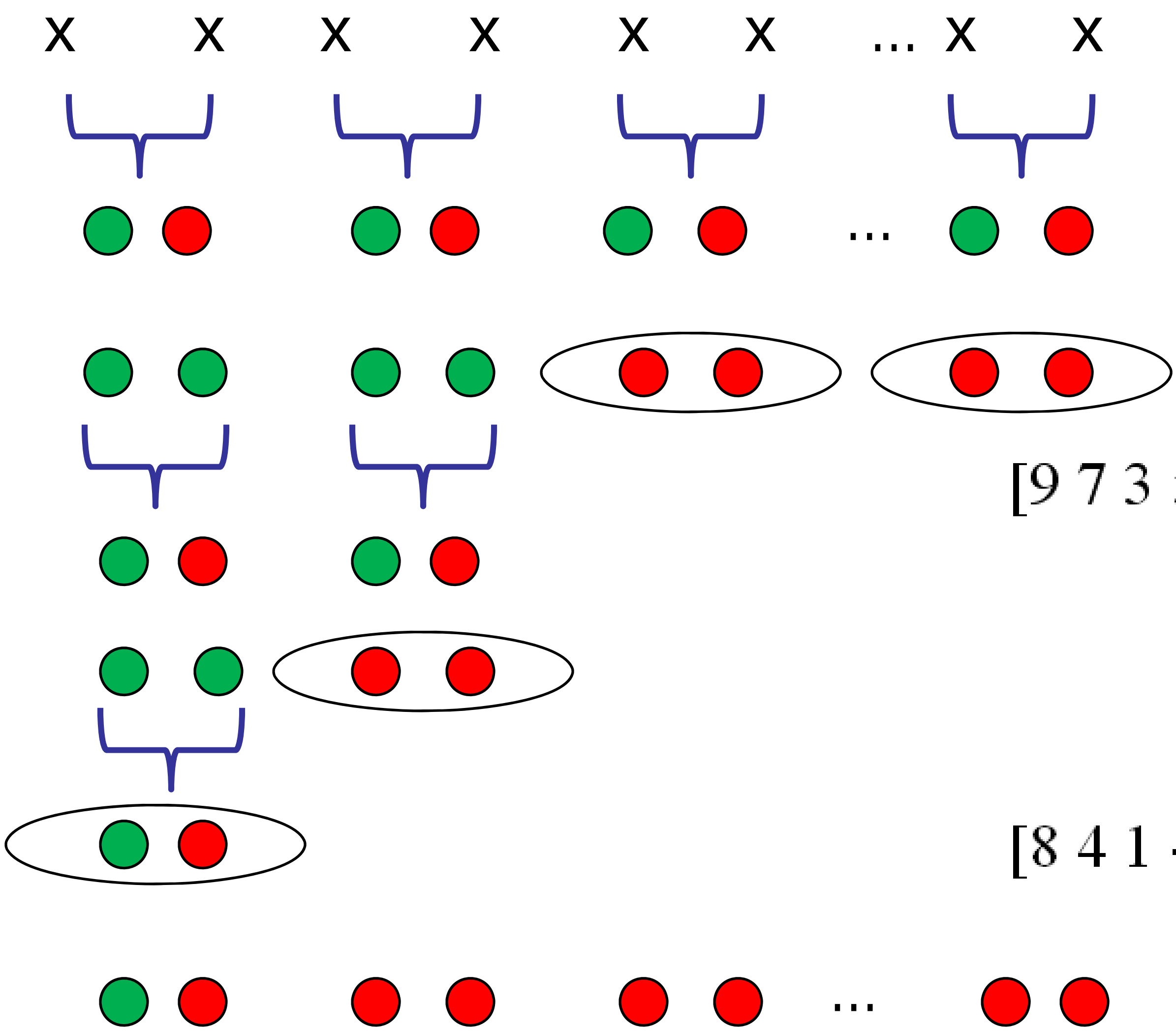
Example: Haar Wavelets



The Haar wavelet transform of [9 7 3 5] is:

[6 2 1 -1]

Example: Haar Wavelets



average

detail

[9 7 3 5]

$(9+7)/2 = 8$

$(9-7)/2 = 1$

$(3+5)/2 = 4$

$(3-5)/2 = -1$

[8 4 1 -1]

$(8+4)/2 = 6$

$(8-4)/2 = 2$

[6 2 1 -1]

Example: Haar Wavelets

- Recall that the Haar decomposition of the original four-pixel image [9 7 3 5] is:

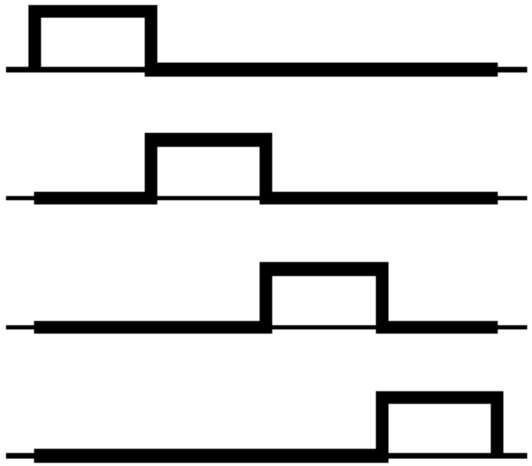
$$[6 \;\; 2 \;\; 1 \;\; -1]$$

- We can reconstruct the original image to a certain resolution by adding or subtracting the detail coefficients from the lower-resolution versions.

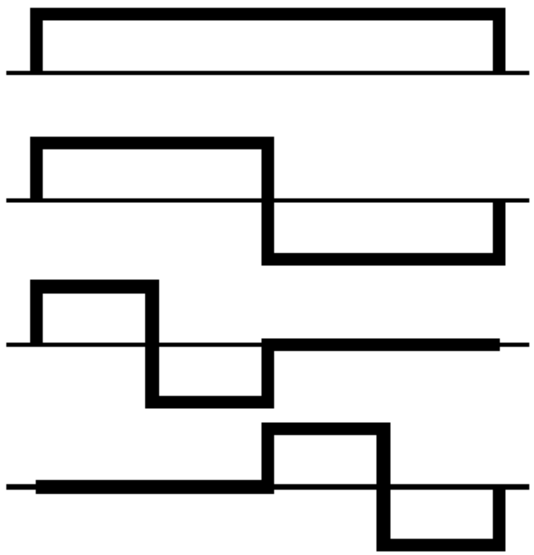
$$[6 \;\; 2 \;\; 1 \;\; -1] \xrightarrow{2} [8 \;\; 4] \xrightarrow{1 \;\; -1} [9 \;\; 7 \;\; 3 \;\; 5]$$

Interpretation

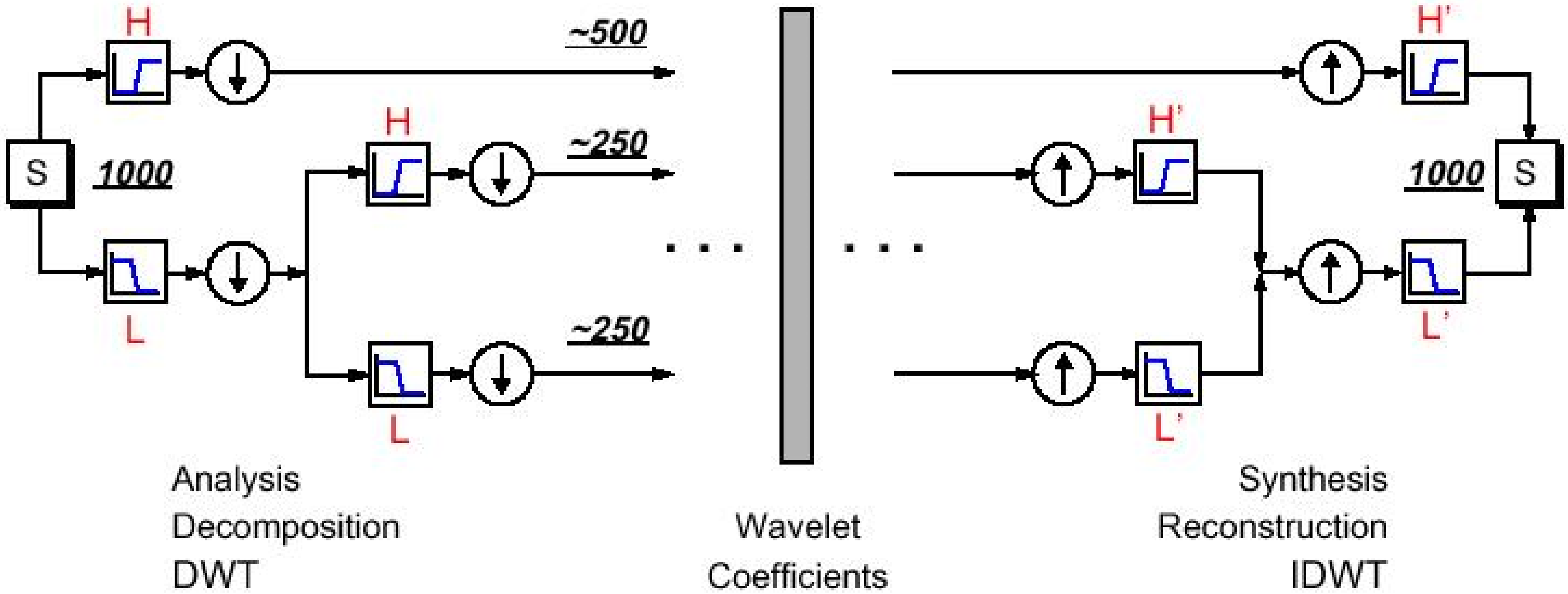
$$
\begin{aligned}
[9\ 7\ 3\ 5] &= 9 \times [1, 0, 0, 0] \\
&+ 7 \times [0, 1, 0, 0] \\
&+ 3 \times [0, 0, 1, 0] \\
&+ 5 \times [0, 0, 0, 1]
\end{aligned}
$$



$$
\begin{aligned}
[9\ 7\ 3\ 5] &= 6 \times [1, 1, 1, 1] \\
&+ 2 \times [1, 1, -1, -1] \\
&+ 1 \times [1, -1, 0, 0] \\
&+ -1 \times [0, 0, 1, -1]
\end{aligned}
$$



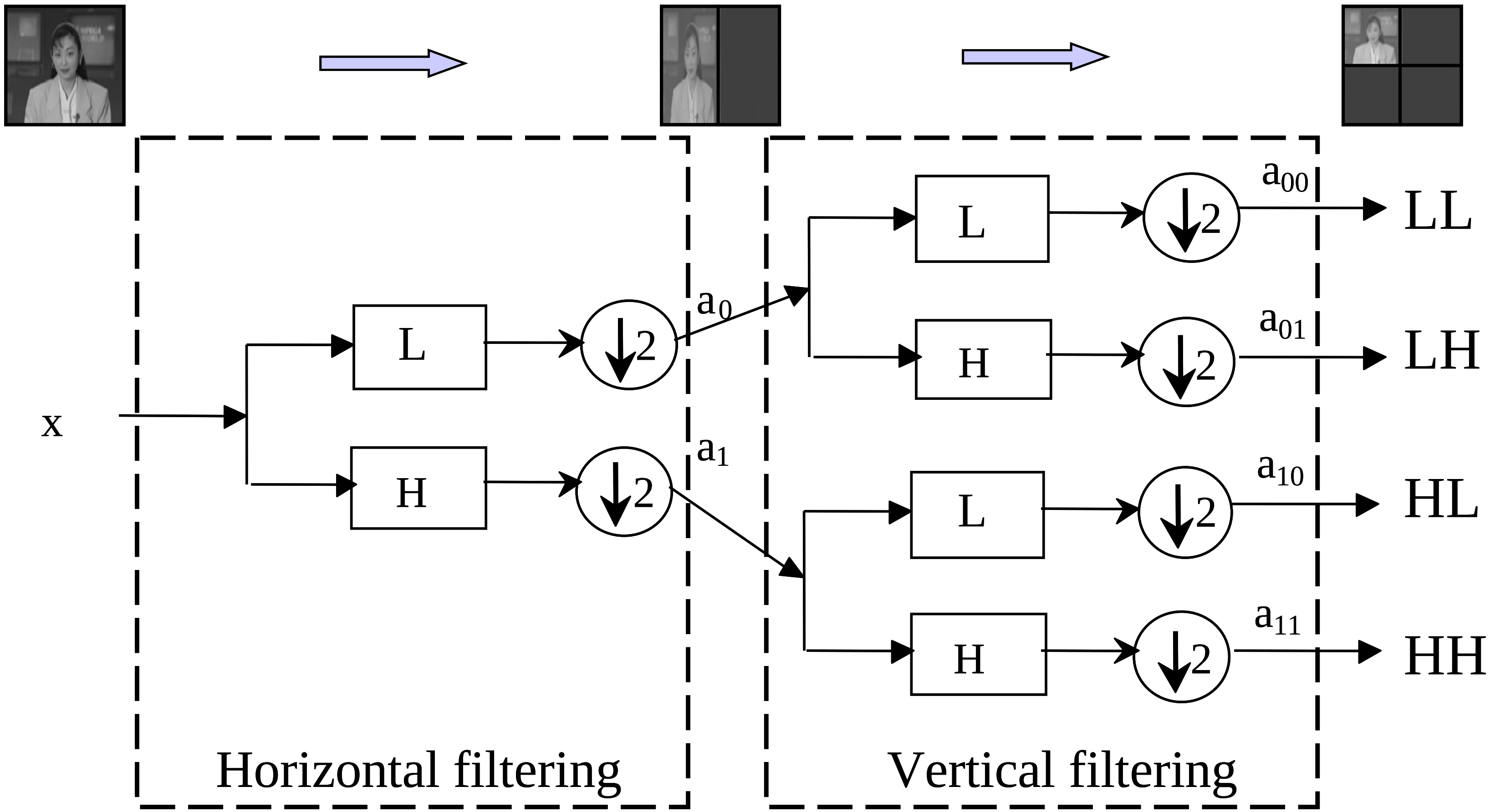
Successive Wavelet/Subband Decomposition



From Matlab Wavelet Toolbox Documentation

Successive lowpass/highpass filtering and down-sampling

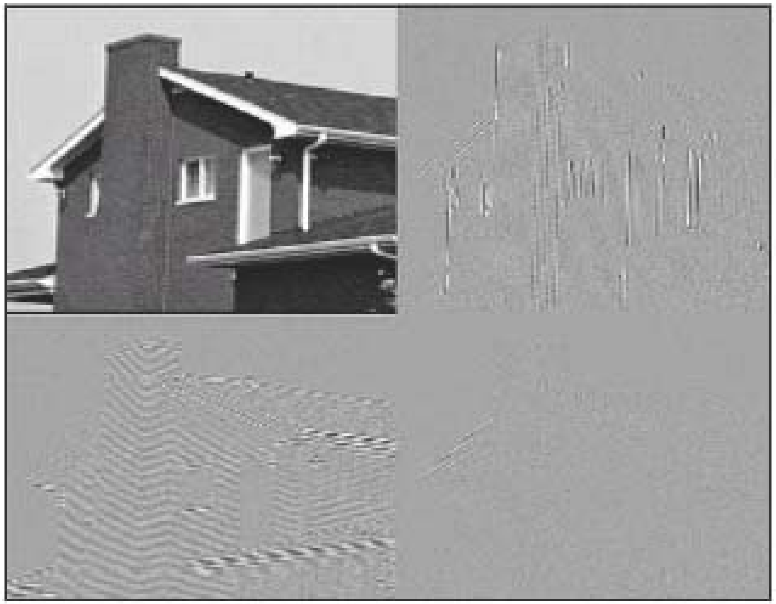
2D Wavelet Transform



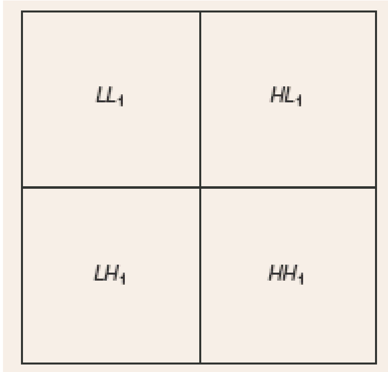
From: http://research.microsoft.com/en-us/um/people/jinl/paper_2002/msri_jpeg.htm

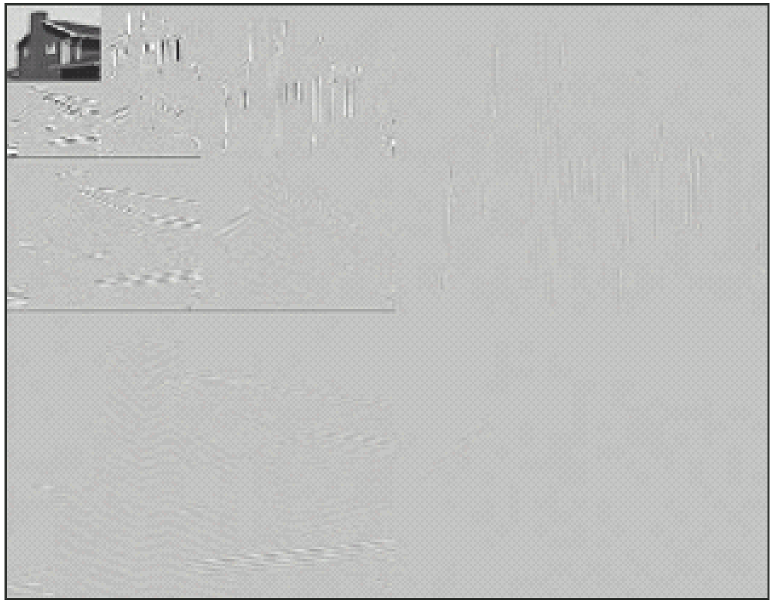
2D Example

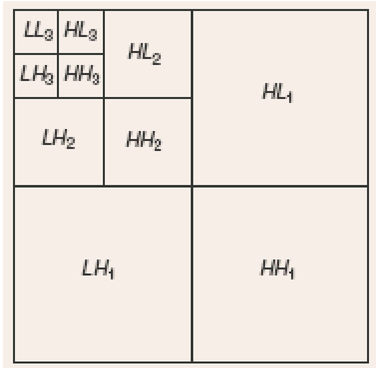
▲ 2. Original image used for demonstrating the 2-D wavelet transform.



▲ 3. A one-level ($K = 1$), 2-D wavelet transform using the symmetric wavelet transform with the 9/7 Daubechies coefficients (the high-frequency bands have been enhanced to show detail).



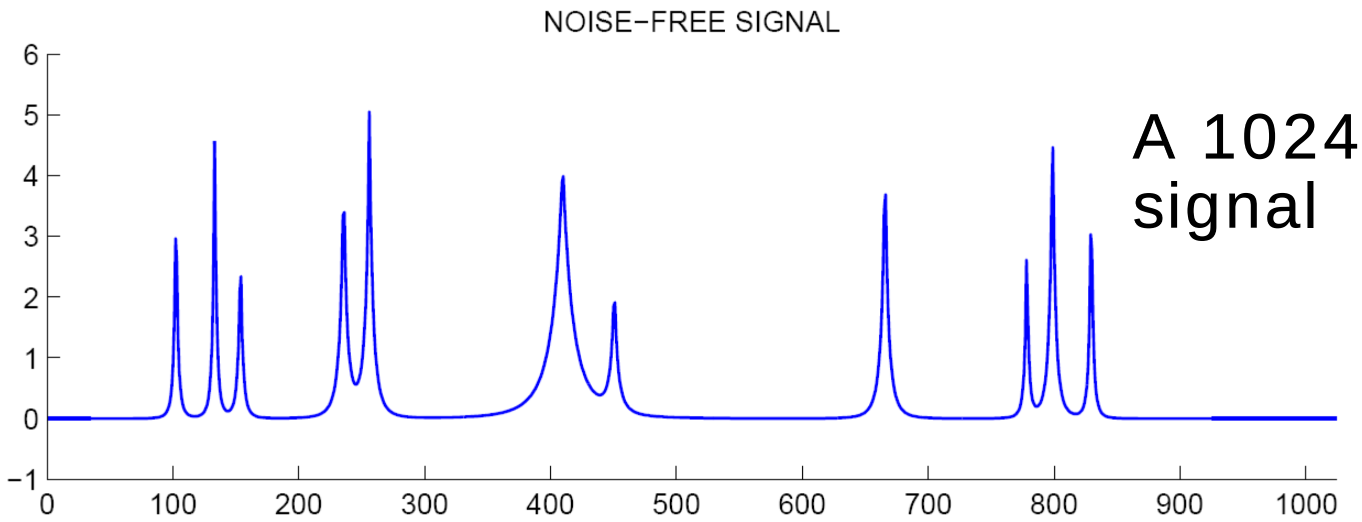




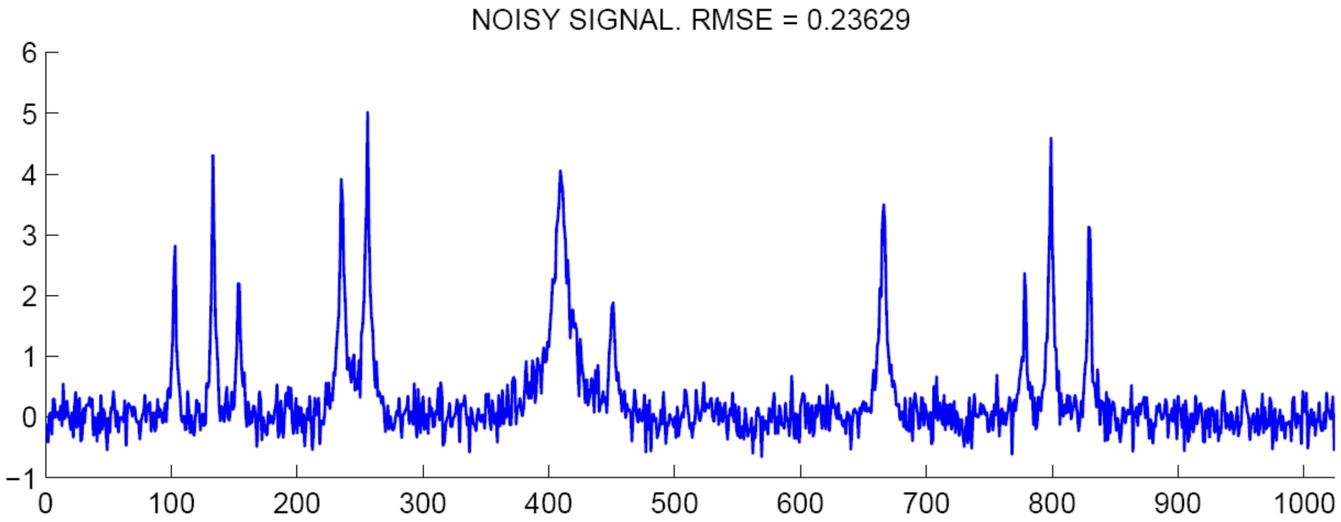
Example Applications

- Noise/trend reduction
- Data and image compression
- Transient detection
- Pattern recognition
- Texture analysis
- Image retrieval
- Many others ...

Noise Reduction



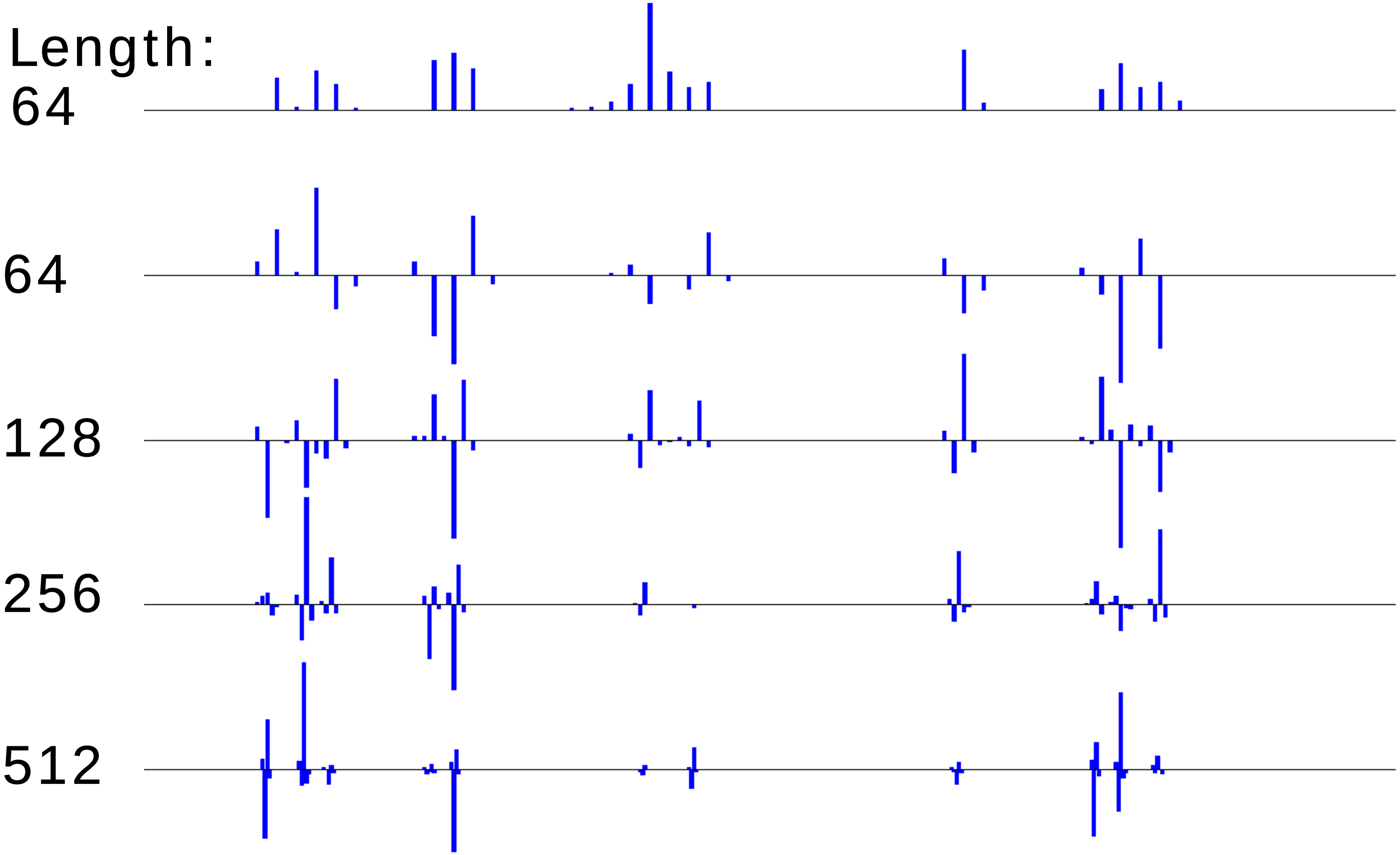
A 1024-point signal



Wavelets for Noise Reduction

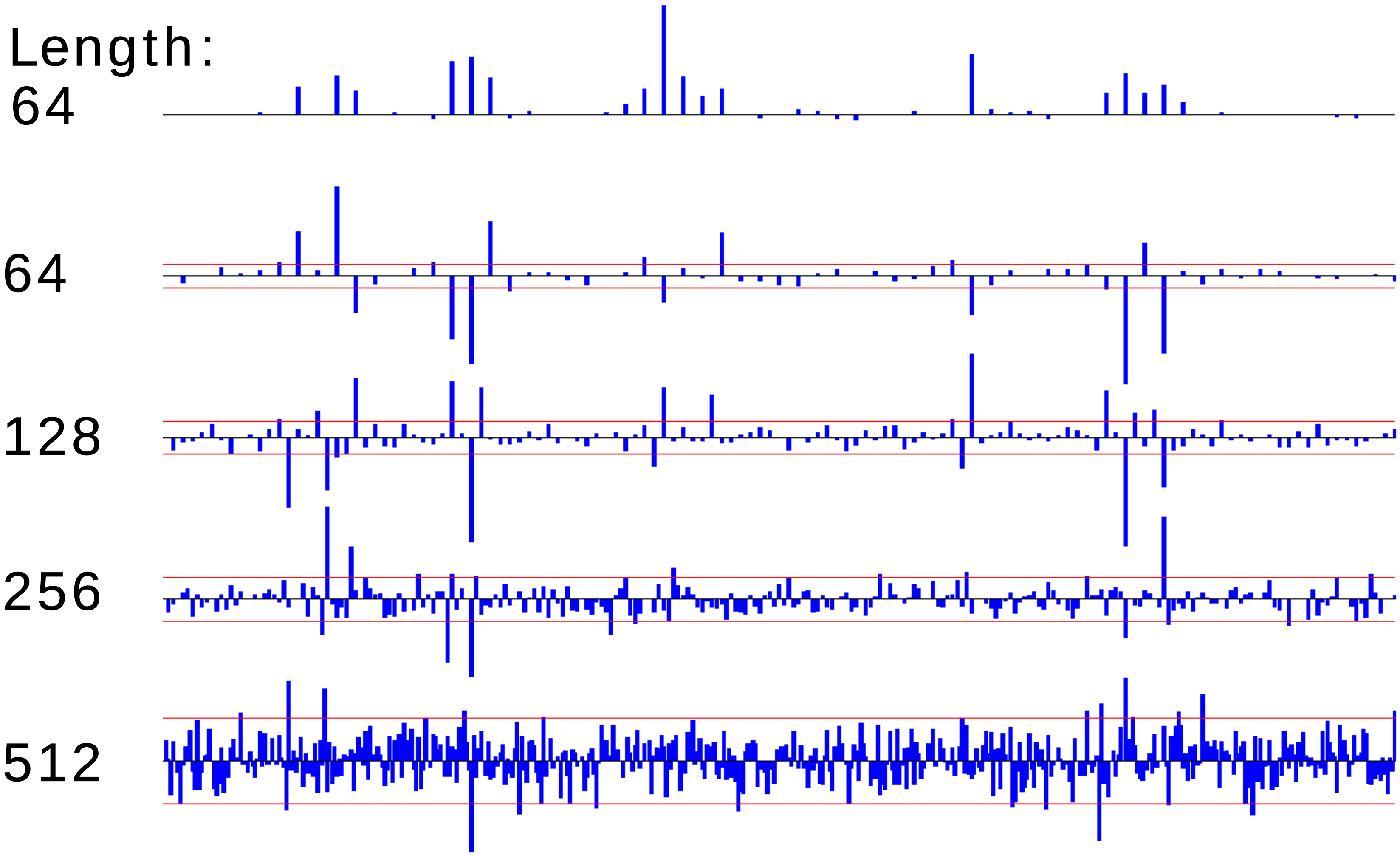
- White noise is independent random fluctuations at each sample of a function
- White noise distributes itself uniformly across all coefficients of a wavelet transform
- Wavelet transforms of the noise-free signal tend to concentrate most of the “energy” in a small number of coefficients
- Throw out the small coefficients and you have removed (most of) the noise
- Little knowledge about noise characteristics required!

Four-level Wavelet Representation



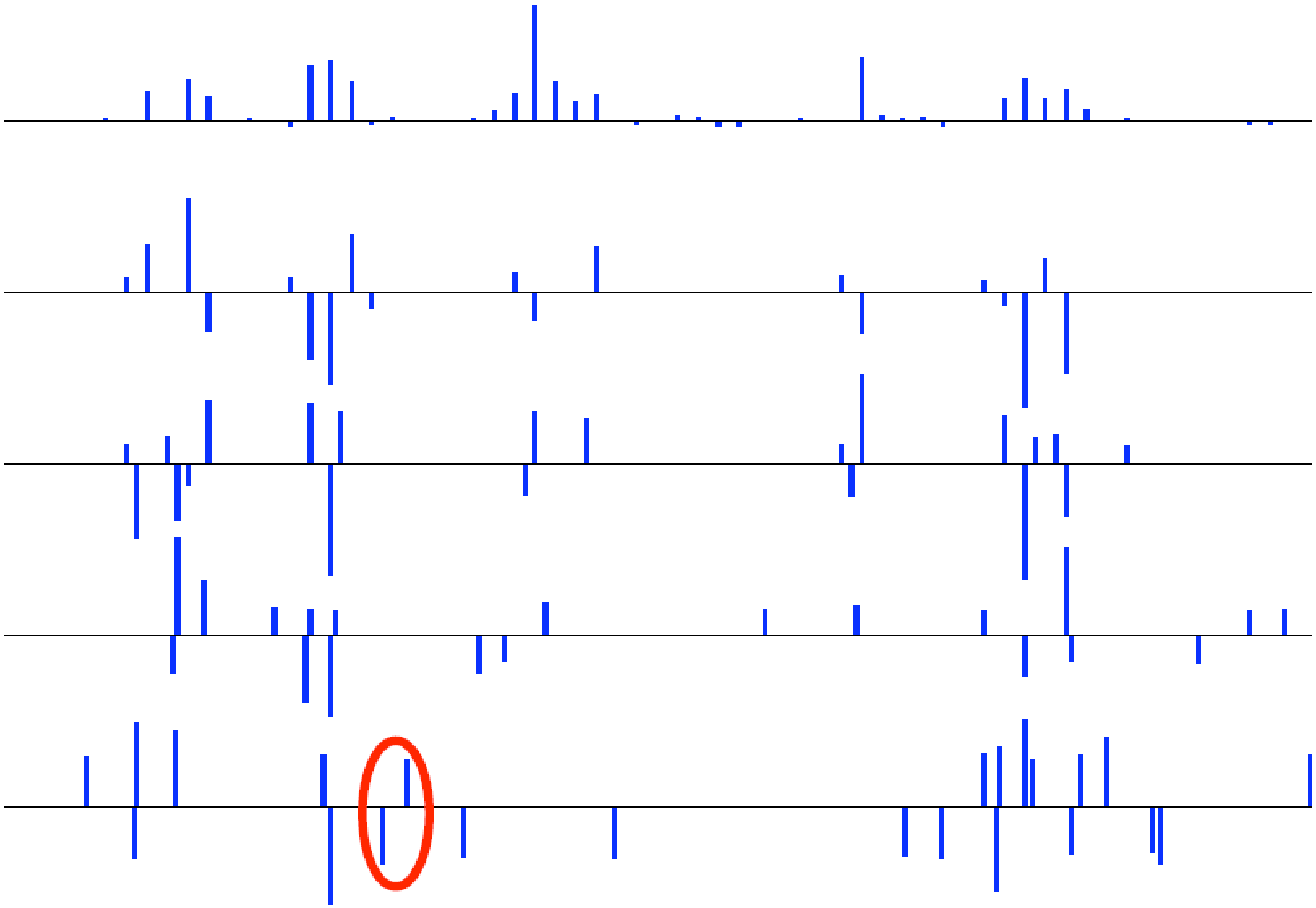
It consists of five vectors computed with a Daubechies-like wavelet transform; many values are small in absolute value; the wavelet representation of the noise-free signal is sparse.

Four-level Wavelet Representation



It consists of five vectors of lengths 64, 64, 128, 256, and 512;
The wavelet representation of the noisy signal is noisy.

After Thresholding



The most basic wavelet-based method for noise reduction is to simply set the small values in the wavelet representation to zero. For example, set those values within the red lines to zero.

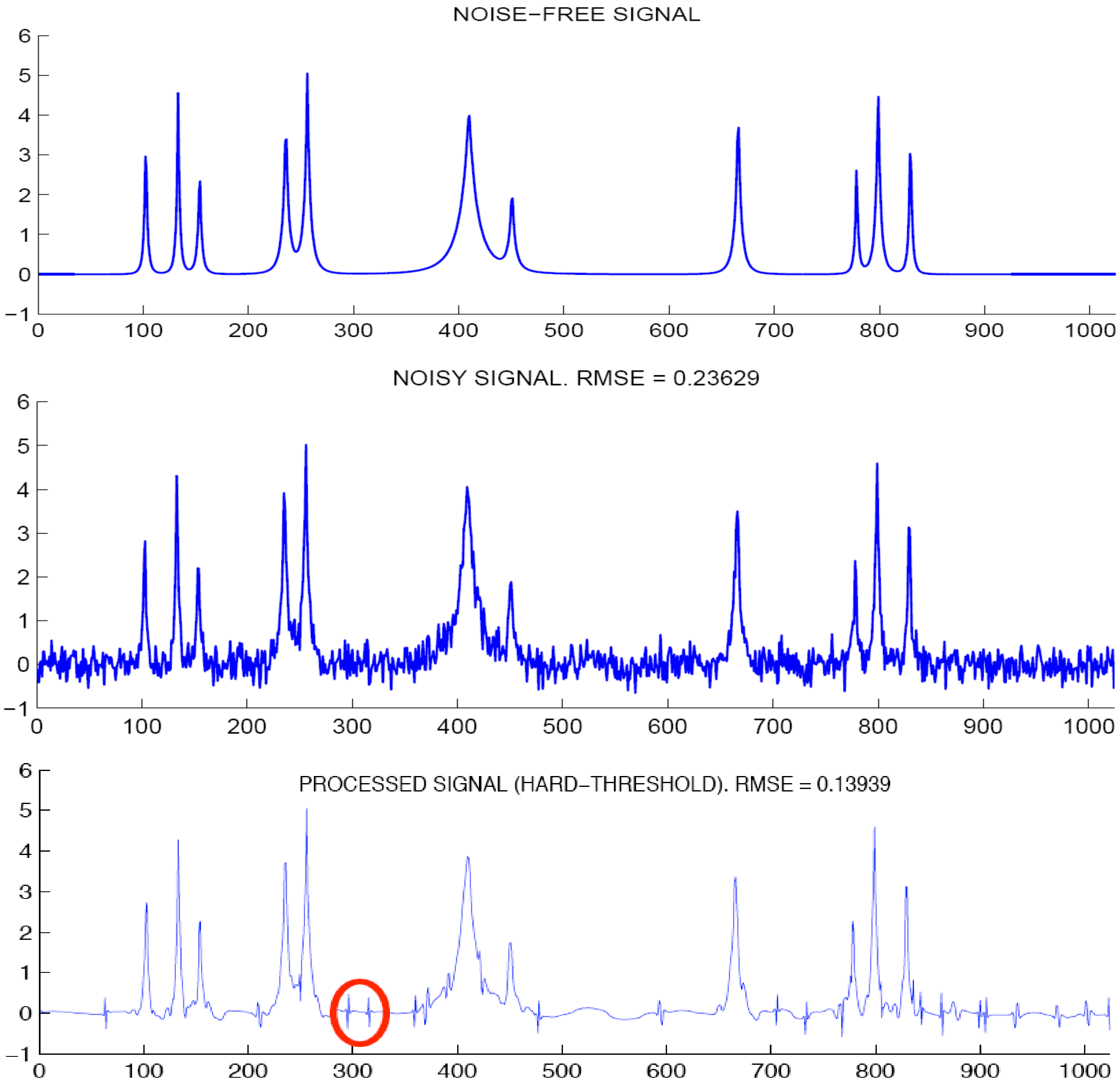
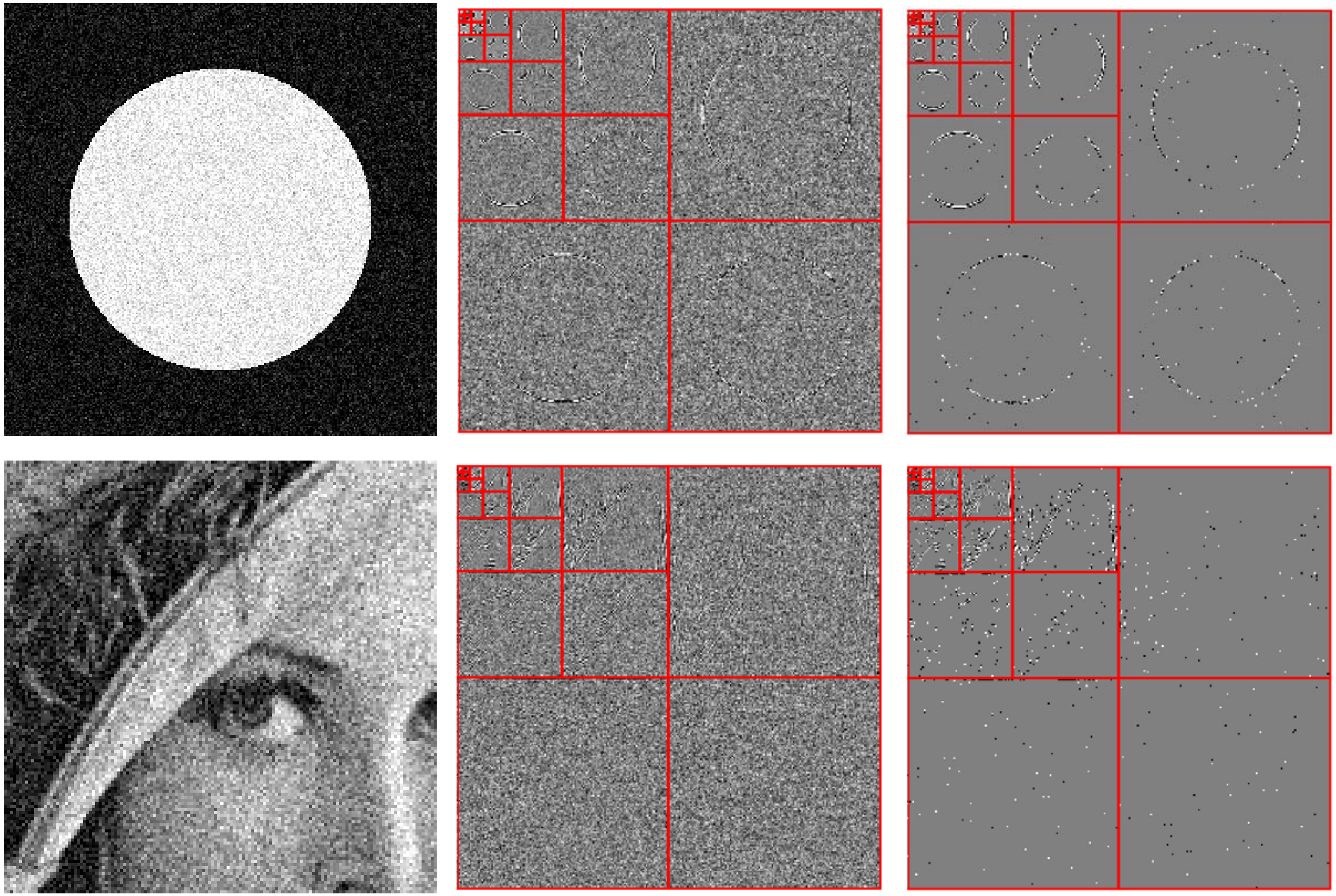

NOISE−FREE SIGNAL
NOISY SIGNAL. RMSE = 0.23629
PROCESSED SIGNAL (HARD−THRESHOLD). RMSE = 0.13939
6
5
4
3
2
1
0
−1
0
100
200
300
400
500
600
700
800
900
1000

Image Denoising

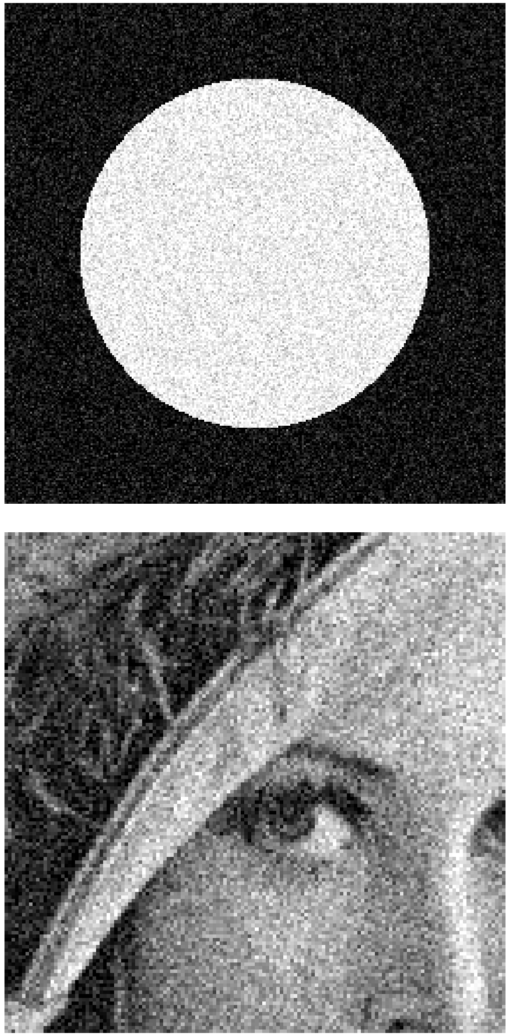


Original image

Noisy wavelet coefficients

Remaining wavelet coefficients

Image Denoising



Original image

Using orthogonal wavelets

Using translation invariant wavelets

Image Compression

JPEG:
DCT-based

(a) (b)

Fig. 20. Reconstructed images compressed at 0.125 bpp by means of (a) JPEG and (b) JPEG2000

JPEG 2000:
Wavelet-
Based

(a) (b)

Fig. 21. Reconstructed images compressed at 0.25 bpp by means of (a) JPEG and (b) JPEG2000

Applications

"Most of the basic wavelet theory has been done... The future of wavelets lies in the as-yet uncharted territory of *applications*." – Amara Graps, 1995

WAVELAB

Available at:

http://www-stat.stanford.edu/~wavelab/

Resources

- Book: Wavelets and Subband Coding
  http://www.waveletsandsubbandcoding.org/
- Tutorials
  - An Introduction to Wavelets (http://www.amara.com/IEEEwave/IEEEwavelet.html)
  - The Wavelet tutorial by Robi Polikar (http://users.rowan.edu/~polikar/WAVELETS/WTtutorial.html)
- Software
  - Matlab Wavelet Toolbox
  - Rice Wavelet Toolbox for Matlab
  - Wavelab at Stanford