EE3731C - Note 1.0

Problem: $X_{k+1} = f(X_k)$, $k = 0, 1, 2...$ where $f(x)$ is a continuously differentiale function at $x = x_{eq}$, (x_{eq}: equilibrium point). Given $f(x_{eq}) = x_{eq}$, and $|f'(x_{eq})| < 1$ .Prove that X_k is stable around x_{eq}

Solution:
Let $E_k = X_k - x_{eq}$, we will show that for values of E_k near 0, E_k is bounded as $k \to \infty$

Since $f'(x_{eq}) < 1$, we can always find $\varepsilon > 0$, small enough, such that $f'(x_{eq}) + \varepsilon < 1$

Moreover, by definition, $\lim_{x \to x_{eq}} \frac{f(x)-f(x_{eq})}{x-x_{eq}} = f'(x_{eq})$. $\forall \varepsilon$, we can $\delta > 0$, small enough, such that $\forall x$ near x_{eq}, $|x - x_{eq}| < \delta$:

$$
\begin{aligned}
\left| \frac{f(x) - f(x_{eq})}{x - x_{eq}} - f'(x_{eq}) \right| &< \varepsilon \\
\left| \frac{f(x) - x_{eq}}{x - x_{eq}} \right| &< f'(x_{eq}) + \varepsilon \\
\left| \frac{f(x) - x_{eq}}{x - x_{eq}} \right| &< 1
\end{aligned}
$$

For values of X_k that near to x_{eq}, $|X_k - x_{eq}| < \delta$, we have

$$
\begin{aligned}
\left| \frac{f(X_k) - x_{eq}}{X_k - x_{eq}} \right| &< 1 \\
\frac{|X_{k+1} - x_{eq}|}{|X_k - x_{eq}|} &< 1 \\
\frac{|E_{k+1}|}{|E_k|} &< 1
\end{aligned}
$$

Therefore, when $k \to \infty$

$$|E_k| > |E_{k+1}| > |E_{k+2}| > ...$$

E_k is bounded around 0.