Tutorial 4

EE3731C – Signal Processing Methods

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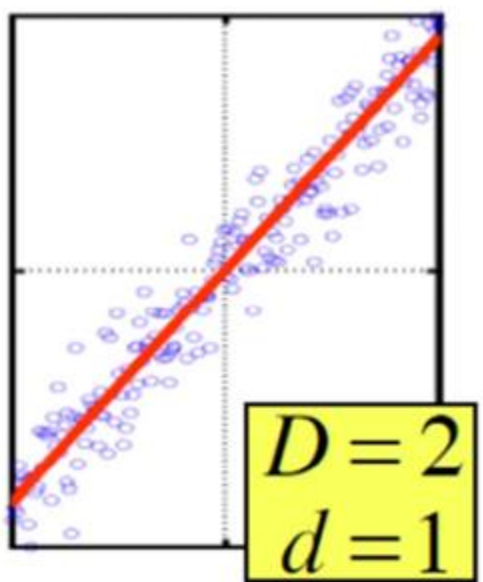
- Expectation
- Variance
- Covariance (Matrix)

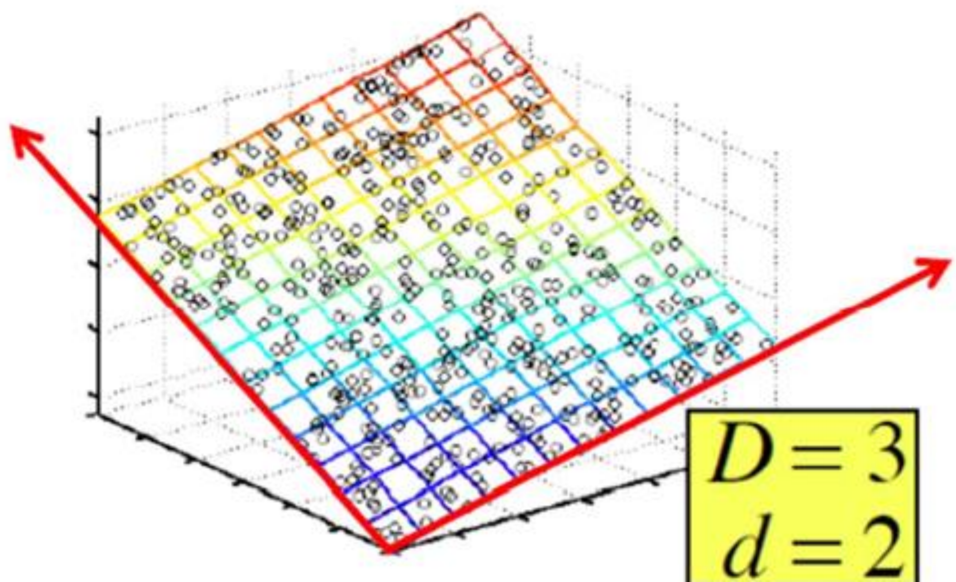
- Orthogonality
- Correlation
- Independence

- Key Properties of expectation, variance and covariance (matrix)
- Not every matrix can be a covariance matrix
- Differentiate orthogonality, correlation and independence

Dimensionality Reduction

- Project the data into lower-dimensional space
  - Assumption: data approximately lie in a lower-dimensional space => *Preserve structure.*
  - Less computation, easier interpretation.





Principal Component Analysis

- Assume data are centered
- For a projection direction $\boldsymbol{v}$, variance of projected data $Var(\boldsymbol{v}^T\boldsymbol{x})$:

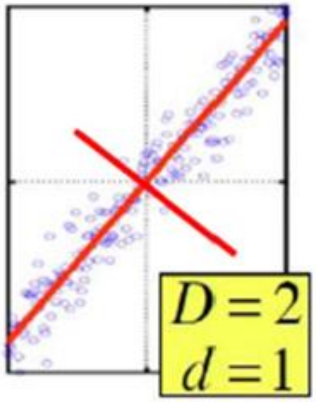
$$\frac{1}{n-1}\sum_{i=1}^{n}(\boldsymbol{v}^T\boldsymbol{x}_i)^2 = \frac{1}{n-1}\boldsymbol{v}^T\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{v}$$

  – $\boldsymbol{X}\boldsymbol{X}^T$: sample covariance

- Maximize the variance of projected data

$$\max_{\boldsymbol{v}} \boldsymbol{v}^T\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{v} \quad \text{s.t. } \boldsymbol{v}^T\boldsymbol{v} = 1$$

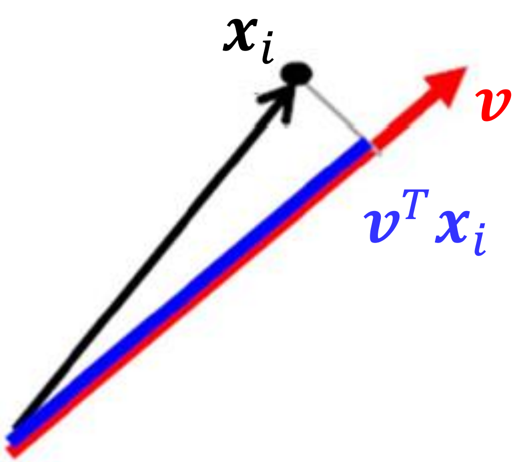
How to solve this?



Computing PCA

- Subtract off the mean (do not forget to add back)
- Form the covariance matrix (be aware of the properties of a covariance matrix)
- Calculate the eigenvectors and eigenvalues of the covariance matrix (remember the orthonormal constraint of eigenvectors)
- Rearrange the eigenvectors and eigenvalues
- Select a subset of eigenvectors as basis vectors

Projecting onto the PCs



- $\boldsymbol{x}_i$: a D-dimensional data point
- $\boldsymbol{v}$: 1st PC
- $\boldsymbol{v}^T\boldsymbol{x}_i$: projection of $\boldsymbol{x}_i$ onto the 1st PC

Reconstruction error: $e_i = ||\boldsymbol{x_i} - (\boldsymbol{v}^T\boldsymbol{x}_i)\boldsymbol{v}||^2$

Computing Eigenvalues and Eigenvectors

- $\boldsymbol{A}\boldsymbol{v}_i = \lambda\boldsymbol{v}_i$

$(\boldsymbol{A} - \lambda_i \mathbf{I})\boldsymbol{v}_i = 0$

$\det(\boldsymbol{A} - \lambda_i \mathbf{I}) = 0$

- For example, $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$

=> $(a-\lambda)(d-\lambda) - bc = 0$

=> $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$

The first PC is the corresponding eigenvector of the largest eigenvalue.

PCA Analysis on Images

- Representation of raw image data

- PCA for dimension reduction

Recurrence Relation

- Initial terms are given.
- Each further term of the sequence is defined as a function of the preceding terms.

- A difference equation is an equation for a sequence written in terms of that sequence and shiftings of it.

Difference Equations

- Plot the equation over time
- Get an analytical solution for the difference equation if it is available
- Determine the equilibrium values
- Analyze stability of equilibria

- First order difference equation: $x_{k+1} = ax_k + b$
- Analytical solution:

If $a \neq 1$, $x_k = a^k x_0 + \frac{\mathrm{b}(1-a^k)}{1-\mathrm{a}}$, or $x_k = \frac{\mathrm{b}}{1-\mathrm{a}} + \left[x_0 - \frac{\mathrm{b}}{1-\mathrm{a}}\right] a^k$,

Otherwise if $a = 1$, $x_k = x_0 + kb$.

- Equilibrium value: $\frac{\mathrm{b}}{1-\mathrm{a}}$, if $a \neq 1$
- Stability criteria - the equilibrium is
  - stable if $|a| < 1$
  - unstable if $|a| > 1$
  - x_k will oscillate if $\mathrm{a} < 0$
  - x_k will change monotonically if $\mathrm{a} > 0$
  - ...

Please try not to memorize it. For any a and b, can also follow:

- Start with a^k times an undetermined constant c
- Find x^*
- Add the two and pick c to match x_0 when k = 0.

Induction

- The simplest and most common form of mathematical induction infers that a statement involving a natural number k holds for all values of k. The proof consists of two steps:
  - The **base case**: prove that the statement holds for the first natural number k. Usually, k = 0 or k = 1.
  - The **inductive step**: prove that, if the statement holds for some natural number k, then the statement holds for $k + 1$.

Testing Example

- A: show that the sequence defined by $y_k = 2^{k+1} - 1$, satisfies the difference equation y_{k+1}=$2y_k + 1$, $y_0 = 1$.

- B: for difference equation y_{k+1}=$2y_k + 1$, $y_0 = 1$, find its analytic solutions.

- Make a conjecture about the stability of a fixed point to a nonlinear system $x_{k+1} = g(x_k)$.

Solution: an equilibrium solution x^*

is stable if $|g'(x^*)| < 1$, and

unstable if $|g'(x^*)| > 1$.

The application (and intuition) of it is important, and proof (shown in the next slide) is not required.

Proof: The goal is to prove that the errors between x_k and x^* $\mathrm{e}_k = x_k - x^*$ tend to 0 as $k \to \infty$.

Toward the goal, we estimate e_{k+1} in terms of e_k:

$$\mathrm{e}_{k+1} = x_{k+1} - x^* = \mathrm{g}(x_k) - \mathrm{g}(x^*) = \mathrm{g}'(\mathrm{y})\left(x_k - x^*\right) = \mathrm{g}'(\mathrm{y})\,\mathrm{e}_k$$

for some y lying between x_k and x^*.

By continuity, if $|\mathrm{g}'(\mathrm{x}^*)| < 1$, we can choose $\delta > 0$ and $|\mathrm{g}'(\mathrm{x}^*)| < \sigma < 1$ such that $|\mathrm{g}'(\mathrm{y})| \le \sigma < 1$ whenever $|y - x^*| < \delta$.

Suppose $|e_k| = |x_k - x^*| < \delta$, then $|x_{k+1} - x^*| \le \sigma |x_k - x^*|$, so $|e_{k+1}| \le \sigma |e_k|$.

Provided the initial iterate satisfies $|e_0| = |x_0 - x^*| < \delta$, the subsequent errors are bounded by $e_k \le \sigma^k e_0$.

Hence, $e_k = |x_k - x^*| \to 0$ as $k \to \infty$. This completes the proof.