EE3331C/EE3331E Feedback Control Systems
L3: Laplace Transform & Transfer Functions

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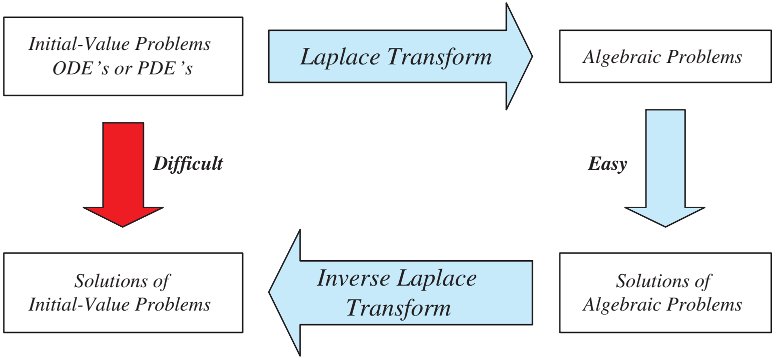
Outline I

Outline II

Motivations

- Laplace Transform converts integral and differential equations into algebraic equations, allows us to analyze
  - linear constant coefficient ordinary differential equations
  - complicated circuits with sources, Ls, Rs, and Cs
  - complicated systems with integrators, differentiators, gains.
  - system behaviour without having to solve differential equations



Definition

- The Laplace Transform of a signal (function) f is the function $F = \mathcal{L}(f)$ defined by

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

  for those $s \in \mathbf{C}$ for which the integral make sense.
  - F is a complex-valued function of complex numbers
  - s is called the (complex) frequency variable, with units sec^{-1}; t is called the time variable (in sec); st is unitless
- Common notation: lower case letter denotes signal; capital letter denotes its Laplace transform.

- Example: what is the LT of a unit step? i.e. $f(t) = 1$ for $t \geq 0$

$$F(s) = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}$$

provided we can say $e^{-st} \to 0$ as $t \to \infty$, which is true for $\Re s > 0$ since

$$\left| e^{-st} \right| = \underbrace{\left| e^{-j(\Im s)t} \right|}_{=1} \left| e^{-(\Re s)t} \right| = e^{-(\Re s)t}$$

  - the integral defining F make sense for all s with $\Re s > 0$
  - but the resulting formula for F make sense for all s except $s = 0$
- Gets complicated for other signals, instead we will make use of Laplace Transform properties and common LT pairs to solve more complicated problems.

Common Laplace transform pairs

$f(t)$	$\Leftrightarrow$	$F(s)$	$f(t)$	$\Leftrightarrow$	$F(s)$
$\delta(t)$		1	$\sin(at)$		$\frac{a}{s^2+a^2}$
$U(t)$		$\frac{1}{s}$	$\cos(at)$		$\frac{s}{s^2+a^2}$
t		$\frac{1}{s^2}$	$e^{-at}\sin(bt)$		$\frac{b}{(s+a)^2+b^2}$
t^k		$\frac{k!}{s^{k+1}}$	$e^{-at}\cos(bt)$		$\frac{s+a}{(s+a)^2+b^2}$
e^{-at}		$\frac{1}{s+a}$			
te^{-at}		$\frac{1}{(s+a)^2}$			
$\frac{1}{(k-1)!}t^{k-1}e^{-at}$		$\frac{1}{(s+a)^k}$			

Table 3.1 : Laplace transform pairs

Laplace transform properties

- Linearity: if f and g are any signals and α and β are any scalar, we have

$$\mathcal{L}\left\{\alpha f(t) + \beta g(t)\right\} = \alpha F(s) + \beta G(s)$$

i.e. homogeneity and superposition hold.

  - example:

$$\mathcal{L}(2\delta(t) - 3e^{-t}) = 2\mathcal{L}(\delta(t)) - 3\mathcal{L}(e^{-t}) = 2 - 3\frac{1}{s+1} = \frac{2s-1}{s+1}$$

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- Derivative: if a signal f is continous at $t=0$, then
$$\mathcal{L}(f'(t))=sF(s)-f(0^{-})$$
  - time-domain differentiation becomes multiplication by frequency variable s
  - plus a term that includes initial condition (i.e., $-f(0^{-})$)

- higher-order derivatives

$$\mathcal{L}(f^n(t)) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^k(0^-)$$

- example:
  - $f(t)$ is unit ramp, so $f'(t)$ is unit step

$$\mathcal{L}(f'(t)) = s\left(\frac{1}{s^2}\right) - 0 = \frac{1}{s}$$

  - $f(t) = e^{-at}$, so $f'(t) = -ae^{-at}$

$$\mathcal{L}(f'(t)) = -a\frac{1}{s+a} = \frac{-a}{s+a}$$

    using the formula

$$\mathcal{L}(f'(t)) = sF(s) - f(0^-) = s\frac{1}{s+a} - 1 = \frac{-a}{s+a}$$

- Integral: if

$$g(t) = \int_0^t f(\tau) d\tau$$

then

$$G(s) = \frac{1}{s} F(s)$$

i.e. time-domain integral becomes division by frequency variable s.

  - example: if $f(t)$ is an impulse, $F(s) = 1$; then $g(t)$ is the unit step

$$G(s) = 1/s$$

  - example: if $f(t)$ is unit step function, what is $G(s)$?

- Derivative of Transform (Multiplication by time):

$$\begin{aligned} F'(s) &= \mathcal{L}\{-tf(t)\} \\ \text{and} \quad \frac{d^n}{ds^n}F(s) &= (-1)^n \mathcal{L}\{t^n f(t)\} \end{aligned}$$

  i.e. differentiation in the frequency domain corresponds to multiplication by time

  - if $f(t) = \sin \omega t$, then

$$\mathcal{L}\{t \sin \omega t\} = -\frac{d}{ds}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \frac{2\omega s}{(s^2+\omega^2)^2}$$

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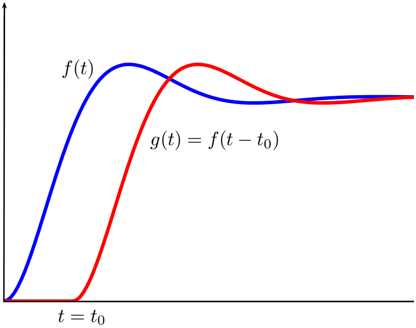
- Time delay (shift in time-domain):
  if a function $f(t)$ is delayed by t_0 unit of time

$$g(t) = \begin{cases} 0, & 0 \leq t < t_0 \\ f(t-t_0), & t \geq t_0 \end{cases}$$

  i.e. g is delayed version of f by t_0 seconds and 'zero-padded' up to t_0.

- Its Laplace transform is then given by

$$G(s) = e^{-st_0}F(s)$$



- Shift in frequency (shift in s-domain):
  multiplication of a signal $f(t)$ by an exponential expression in the time domain corresponds to a shift in frequency

$$\mathcal{L}(e^{-at} f(t)) = F(s+a)$$

  - example: since

$$\mathcal{L}(\cos t) = \frac{s}{s^2+1}$$

    hence

$$\mathcal{L}(e^{-2t}\cos t) = \frac{s+2}{(s+2)^2+1} = \frac{s+2}{s^2+4s+5}$$

- Final value theorem:

$$\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)$$

  - Allows us to compute the constant steady-state value of a time function given its Laplace transform *without having to solve the differential equation or perform inverse laplace transform*.
  - Note: the final value theorem is only applicable to stable system.
  - Example: given

$$Y(s) = \frac{3(s+2)}{s(s^2+2s+10)}$$

applying FVT, we have

$$y(\infty) = sY(s)|_{s=0} = s\frac{3(s+2)}{s(s^2+2s+10)}\bigg|_{s=0} = \frac{3\cdot 2}{10} = 0.6$$

- Convolution:
  - The convolution of two signals $f(t)$ and $g(t)$ (denoted by $h = f * g$) is given by
    $$h(t) = \int_0^t f(\tau)g(t-\tau)d\tau \tag{3.1}$$
    same as
    $$h(t) = \int_0^t f(t-\tau)g(\tau)d\tau \tag{3.2}$$
  - It can be shown that the Laplace transform of Equations (3.1) or (3.2) is given by
    $$H(s) = F(s)G(s)$$
    Laplace transform turns convolution into multiplication!
    More on this later.

Properties of Laplace transform

	Time Function	Laplace Transform	Comments
-	$f(t)$	$F(s)$	Transform pairs
1	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$	Superposition
2	$f(t - t_0)$	$e^{-st_0} F(s)$	Time delay ($t_0 \geq 0$)
3	$f(at)$	$\frac{1}{\lvert a \rvert} F\left(\frac{s}{a}\right)$	Time scaling
4	$e^{-at} f(t)$	$F(s + a)$	Shift in frequency
5	$f^n(t)$	$s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^k(0^-)$	Differentiation
6	$\int_0^t f(\tau) d\tau$	$\frac{1}{s} F(s)$	Integration
7	$f(t) * g(t)$	$F(s)G(s)$	Convolution
8	$tf(t)$	$-\frac{d}{ds} F(s)$	Multiplication by time
9	$f(0^+)$	$\lim_{s \to \infty} sF(s)$	Initial value theorem
10	$\lim_{t \to \infty} f(t)$	$\lim_{s \to 0} sF(s)$	Final value theorem

Inverse Laplace transform

- Used to transform the s-domain solution back to the original domain
- In principle, we can recover $f(t)$ from $F(s)$ via

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

  where σ is large enough that $F(s)$ is defined for $\Re s \geq \sigma$
- As expected, this formula is not really useful!
- Instead, the easiest way to find $f(t)$ from $F(s)$, if $F(s)$ is rational, is to expand $F(s)$ as a sum of simpler terms that can be found in the tables (see Laplace Transform table).
- The basic tool to perform this operation is called **partial-fraction expansion**.

- Procedure for computing $f(t)$
  1. Simplify complicated functions using partial fractions expansion. 3 basic cases,
     - distinct linear factors
       $$\frac{k(s)}{(s+\alpha_1)\cdots(s+\alpha_n)} = \frac{A_1}{s+\alpha_1} + \cdots + \frac{A_n}{s+\alpha_n}$$
     - repeated linear factors
       $$\frac{k(s)}{(s+\alpha)^n} = \frac{A_1}{s+\alpha} + \cdots + \frac{A_n}{(s+\alpha)^n}$$
     - quadratic factor
       $$\begin{aligned}\frac{k(s)}{(\alpha_1 s^2+\beta_1 s+\gamma_1)(\alpha_2 s^2+\beta_2 s+\gamma_2)} &= \frac{As+B}{(\alpha_1 s^2+\beta_1 s+\gamma_1)} \\ &\quad + \frac{Cs+D}{(\alpha_2 s^2+\beta_2 s+\gamma_2)}\end{aligned}$$
  2. Obtain the inverse Laplace transform, $f(t)$ using the LT table.

- Example1: distinct real roots

$$Y(s) = \frac{(s+2)(s+4)}{s(s+1)(s+3)}$$

find $y(t)$.

- Example 2: given

$$Y(s) = \frac{3}{s(s-2)}$$

applying FVT, we have

$$y(\infty) = sY(s)|_{s=0} = -\frac{3}{2}$$

is this right? Inverse laplace transform (from table), we have

$$y(t) = \left(-\frac{3}{2} + \frac{3}{2}e^{2t}\right)$$

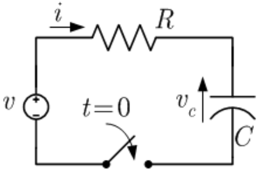
as $t \to \infty$, $y(\infty) \to \infty$, it is unbounded!!

Finding the Laplace transform

- You should **know** the Laplace Transform of some basic signals, e.g.
  - unit step and impulse
  - exponential
  - sinusoidal

  these combined with a table of Laplace transforms and the properties given above (linearity, time delay, ...) will get you pretty far; and of course you can always integrate, using the formula.
- example: circuit analysis
  - initial voltage: $v_c(0)$
  - KVL yield:

$$RC\frac{dv_c(t)}{dt} + v_c(t) = V \cdot U(t)$$



- take Laplace transform, we have

$$RC\left(sV_c(s) - v_c(0)\right) + V_c(s) = \frac{V}{s}$$

solve for V_c

$$V_c(s) = \frac{RCv_c(0)}{sRC+1} + \frac{V}{s(sRC+1)}$$

in the time domain (applying inverse Laplace transform):

$$\begin{aligned} v_c(t) &= \mathcal{L}^{-1}\left\{\frac{RCv_c(0)}{sRC+1} + \frac{V}{s(sRC+1)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{v_c(0)}{s+1/RC}\right\} + \mathcal{L}^{-1}\left\{\frac{V}{s} - \frac{V}{s+1/RC}\right\} \\ &= (v_c(0) - V)e^{-\frac{t}{RC}} + V \end{aligned}$$

- as $t \to \infty$, $v_c(\infty) = V$, show that FVT gives the same answer.

A final note

- Some interesting patterns between
  - time domain (i.e. signals) and
  - frequency domain (i.e. their Laplace transforms)
- differentiation in one domain corresponds to multiplication by the variable in the other domain
- multiplication by an exponential in one domain corresponds to a shift (or delay) in the other domain
  we'll see these patterns (and others) throughout the course

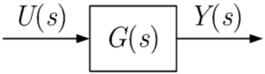
Definition

- The transfer function, $G(s)$, of a system is the transfer gain from input, $U(s)$, to output, $Y(s)$.
- It is the ratio of the Laplace transform of the output to the Laplace transform of the input,

$$G(s) = \frac{Y(s)}{U(s)}$$

Key assumption: all initial conditions on the system are zero.

- if $u(t)$ is an unit impulse, then $U(s) = 1$, we have $Y(s) = G(s)$, i.e. the transfer function $G(s)$ is the Laplace transform of the unit impulse response $h(t)$.



- Many systems can be described by a linear constant coefficient ordinary differential equation:

$$\begin{aligned} a_n y^n + \cdots + a_1 y' + a_0 y &= b_m u^m + \cdots + b_2 u'' + b_1 u' + b_0 u \\ \sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} &= \sum_{m=0}^{M} b_m \frac{d^m u(t)}{dt^m} \end{aligned}$$

  - n is called the order of the system, the largest power of s in the denominator
  - $b_0, \ldots, b_m, a_0, \ldots, a_n$ are the coefficients of the system

Notice that the above formula gives an implicit description of a system, using Laplace transform, we can explicitly express y in terms of u.

- Example: find the transfer function of

$$a_2\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_0 u(t)$$

Taking LT on both side,

$$a_2[s^2Y(s) - sy(0^-) - \dot{y}(0^-)] + a_1[sY(s) - y(0^-)] + a_0Y(s) = b_0U(s)$$

Assuming zero initial conditions, i.e., $y(0^-) = \dot{y}(0^-) = 0$,

$$\begin{aligned} \left[a_2s^2 + a_1s + a_0\right] Y(s) &= b_0U(s) \\ \frac{Y(s)}{U(s)} &= \frac{b_0}{a_2s^2 + a_1s + a_0} \end{aligned}$$

- For a general N^{th}-order system with zero initial conditions:

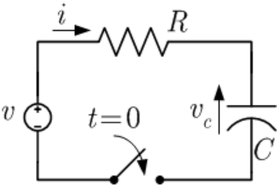
$$\begin{aligned} \sum_{n=0}^{N} a_n \mathcal{L}\left\{\frac{d^n y(t)}{dt^n}\right\} &= \sum_{m=0}^{M} b_m \mathcal{L}\left\{\frac{d^m u(t)}{dt^m}\right\} \\ \frac{Y(s)}{U(s)} &= \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \end{aligned}$$

Examples

- Example 1:
  - Resistor: $v(t) = Ri(t) \Rightarrow G(s) = \dfrac{V(s)}{I(s)} = R$
  - Capacitor: $v_c(t) = \dfrac{1}{C}\displaystyle\int_0^t i(\tau)d\tau$
    Applying Integral rule,
    $$G(s) = \frac{V_c(s)}{I(s)} = \frac{1}{Cs}; \quad v_c(0) = 0$$
  - Inductor: $v_L(t) = L\dfrac{di(t)}{dt}$
    Applying Derivative rule, what is $G(s)$?

- Example 2: RC circuit examples, previously we have
  - initial voltage: $v_c(0)$
  - KVL yield:

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v(t)$$



Laplace transform gives

$$\begin{aligned} RC(sV_c(s) - v_c(0)) + V_c(s) &= V(s) \\ (sRC+1)V_c(s) &= V(s) + RCv_c(0) \end{aligned}$$

Assuming that the initial voltage, $v_c(0)$ is zero, we have

$$G(s) = \frac{V_c(s)}{V(s)} = \frac{1}{sRC+1}$$

- Given $v(t) = VU(t)$, find $v_c(t)$

$$V_c(s) = \frac{1}{sRC+1}\frac{V}{s}$$

Partial fraction expansion gives

$$V_c(s) = \frac{1}{s} - \frac{RC}{sRC+1}$$

Taking inverse LT, we have

$$\begin{aligned} v_c(t) &= \mathcal{L}^{-1}\left\{\frac{V}{s} - \frac{RCV}{sRC+1}\right\} \\ &= V - Ve^{-\frac{t}{RC}} \end{aligned}$$

Non-zero initial conditions

- Most real-world problems do not satisfy the zero-initial conditions assumption which is key to derivation of the transfer function.
  - example, consider building a model for controlling our room temperature, $y(t)$; since the ambient room temperature is not zero, $y(0^-) \neq 0$.
- It is possible to relax the zero initial condition assumption. Output signal may be derived from transfer functions as long as system is initially at rest, i.e., $\left(y(0) \neq 0; \quad \dfrac{d^n y}{dt^n} = 0 \right)$
- For systems with non-zero initial conditions, we introduce the following dummy output and input variables:

$$\begin{aligned} \tilde{y}(t) &= y(t) - y(0) \\ \tilde{u}(t) &= u(t) - u(0) \end{aligned}$$

where $y(0)$ and $u(0)$ are the output and input signals initial conditions. We now have $\tilde{y}(0) = \tilde{u}(0) = 0$.

- Example: the input-output relationship of a thermometer can be modelled by the following transfer function:

$$5\frac{dy(t)}{dt} + y(t) = 0.99u(t)$$

where $u(t)$ is the temperature of the environment in which the thermometer is placed, $y(t)$ is the measured temperature. Given that the measured temperature is 24.75^oC at time, $t = 0$. Find the transfer function of the thermometer.

Given that $y(0) = 24.75^o$C, we can see that the actual temperature is given by $u(0) = y(0)/0.99 = 25^o$C. It is also clear that for zero initial condition $u(0) = y(0) = 0$, the transfer function is given by

$$\frac{Y(s)}{U(s)} = \frac{0.99}{5s + 1}$$

- In the problem, the i.c. are $y(0) = 24.75^o$C and $u(0) = 25^o$C, so the transfer function cannot be used directly. To match the zero i.c. assumption, define the following new variables:

$$\tilde{y}(t) = y(t) - y(0)$$

and

$$\tilde{u}(t) = u(t) - u(0).$$

- We then have

$$\tilde{y}(0) = y(0) - y(0) = 0$$

and

$$\tilde{u}(0) = u(0) - u(0) = 0$$

- Substituting the new variables, the new differential equation is

$$\begin{aligned} 5\frac{d\tilde{y}(t)}{dt} + \tilde{y}(t) + y(0) &= 0.99(\tilde{u}(t) + u(0)) \\ \Rightarrow \quad 5\frac{d\tilde{y}(t)}{dt} + \tilde{y}(t) &= 0.99\tilde{u}(t) \end{aligned}$$

- Taking Laplace transform,

$$\begin{aligned} (5s+1)\tilde{Y}(s) &= 0.99\tilde{U}(s) \\ \Rightarrow \quad \frac{\tilde{Y}(s)}{\tilde{U}(s)} &= \frac{0.99}{5s+1} \end{aligned}$$

- If the actual temperature is $u(t) = (25+t)1(t)$, we then have $\tilde{u}(t) = u(t) - u(0) = t$ for $t \geq 0$. Laplace transform gives $U\tilde{(}s) = \frac{1}{s^2}$ and we have

$$\tilde{Y}(s) = \frac{0.99}{5s+1}\tilde{U}(s) = \frac{0.99}{s^2(5s+1)}$$

- We then have (after partial fraction expansion)

$$\tilde{Y}(s) = -\frac{4.95}{s} + \frac{0.99}{s^2} + \frac{24.75}{5s+1}$$

inverse Laplace transform gives

$$\tilde{y}(t) = -4.95 + 0.99t + 4.95e^{-t/5}, \qquad t \geq 0$$

change of variable gives

$$y(t) = \tilde{y}(t) + y(0) = 19.8 + 0.99t + 4.95e^{-t/5}, \qquad t \geq 0$$

- Using Matlab for partial fraction expansion,

```
num = 0.99;                    % numerator
den = [5 1 0 0];               % denominator
[r,p,k] = residue(num,den);    % compute the residues
```

which results in the desired answer as hand calculation:

r = [4.9500 -4.9500 0.9900],

p = [-0.2 0 0] and

k =[].

Convolution systems

- Convolution system with input $u \ \ (u(t) = 0, t < 0)$ and output y:

$$y(t) = \int_0^t h(\tau)u(t-\tau)d\tau = \int_0^t h(t-\tau)u(\tau)d\tau \qquad (3.3)$$

  abbreviated as: $y = h * u$.

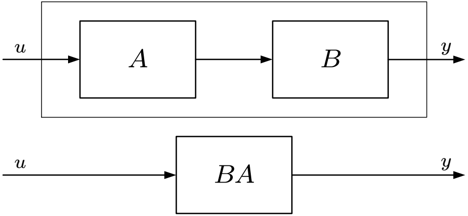
  In the frequency domain, we have $Y(s) = H(s)U(s)$.

- Example: given that $u(t) = \delta(t)$, we have $U(s) = 1$, and $Y(s) = H(s)$! or

$$y(t) = \int_0^t \delta(\tau)h(t-\tau)d\tau = h(t)$$

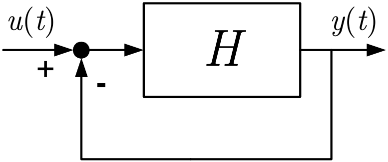
- h is called the **impulse response** of the system
- H is the **transfer function** of the system

- Any convolution system is *linear-time-invariant* (LTI) and causal; any LTI causal system can also be represented by a convolution system.
- Convolution/transfer function representation gives universal description for LTI causal system.
- Composition of convolution systems corresponds to multiplication of transfer functions (note algebra reverse)



  - can manipulate block diagrams with transfer functions as if they were simple gains
  - convolution systems commute with each other

- Example: feedback connection

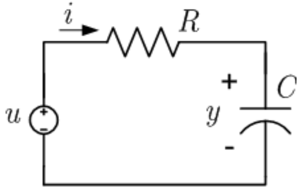


in time domain, we have the convolution integral equation

$$y(t) = \int_0^t h(t-\tau)(u(\tau) - y(\tau))d\tau$$

not easy to understand and solve...
in frequency domain, we have $Y(s) = H(s)(U(s) - Y(s))$, solving for $Y(s)$ gives $Y(s) = \mathcal{H}(s)U(s)$, where $\quad \mathcal{H}(s) = \dfrac{H(s)}{1 + H(s)}$

- More examples:
  - RC circuit: assume zero initial condition for capacitor



  - its transfer function and impulse response

$$H(s) = \frac{1}{sRC + 1}$$

$$h(t) = \mathcal{L}^{-1}(H) = \frac{1}{RC} e^{-t/(RC)}$$

$$y(t) = \frac{1}{RC} \int_0^t e^{-\tau/(RC)} u(t - \tau) d\tau$$

Summary

- Laplace transform is the primary tool to determine the behavior of linear systems.
- The derivative rule is key to finding the transfer function of a system.
- Inverse transform can be found from Laplace transform table.
- Final value theorem is useful in computing the steady-state errors of stable system.

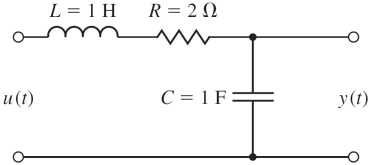
Review Questions

- How do you find the transfer function of a linear ODE?
- What are the key steps in finding the output response $y(t)$ if the input $u(t)$ is sent to the system $G(s)$?

Reading: FPE section 3.1

Practice Problems

1. Given the Laplace transform of $f(t)$ is $F(s)$, find the Laplace transform of (a). $g(t) = f(t)\cos t$ and (b). $g(t) = \int_0^t \int_0^{t_1} f(\tau)d\tau dt_1$.
2. Solve the following ode using Laplace transform:
   - $\ddot{y}(t) + \dot{y}(t) + 3y(t) = 0; y(0) = 1, \dot{y}(0) = 2$
   - $\ddot{y}(t) + y(t) = t; y(0) = 1, \dot{y}(0) = -1$
3. Find $f(t)$ using partial fraction for
   - $F(s) = \frac{3s+2}{s^2+4s+20}$,
   - $F(s) = \frac{1}{s(s+2)^2}$.
4. Write the dyanmic equations describing the circuit below. Assuming a zero input, solve the differential equation for $y(t)$ using Laplace transform given the following initial conditions: $y(0) = 1V, \dot{y}(0) = 0$.



Practice Problems

5. Convolution and Laplace transform.
   - Evaluate $h(t) = e^{-t} * e^{-2t}$ using direct integration for $t \geq 0$ (i.e. from the definition of convolution).
   - Find H, using the expression for h found above.
   - Verify that H is the product of the Laplace transforms of e^{-t} and e^{-2t}.
6. DC motor. (more on this in the lab)
   - A simplified electrical model of the motor can be described by an inductor L in series with a resistance R, so the motor current $i(t)$ satisfies $L\frac{di}{dt} + Ri = v$ where $v(t)$ is the voltage applied to the motor.
   - The motor shaft angle, $\theta(t)$, and the shaft angular velocity, $\omega(t)$, are related by $\omega = \frac{d\theta}{dt}$.
   - The motor current puts a torque on the shaft equal to $ki(t)$, where k is the motor constant. The shaft rotational inertia is J and the damping coefficient is b. Newton's equation then gives $J\frac{d\omega}{dt} = ki - b\omega$.
   - Assuming that $i(0) = 0, \theta(0) = 0$, and $\omega(0) = 0$, find the transfer function relating θ in terms of v.