EE3331C/EE3331E Feedback Control Systems
L4: Dynamic Response

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Outline I

Outline II

Response of LTI system

We have introduced previously how to obtain the dynamic model of a system. Our focus now is to understand the dynamic response of the system. Our approach is to use linear analysis techniques to approximate the system response – provide insight into why solution has certain features and how the system might be modified in a desired direction.

Common type of responses

- Impulse response – output response when the input is an impulse.
- Step response – output response when the input is a step signal.
- Sinusoidal response – output response when the input is a sinusoidal signal (will be covered in second-half of the module).

Response of LTI system

Each of these responses is important to LTI systems because they define certain behaviours that can be generalized for such systems.

- Impulse response is related to the transfer function of the system.
- Step response is very commonly encountered in practice and they relate to some physical parameters in LTI systems.
- Sinusoidal response is related to the frequency response of the system.

We have previously dertermined the transfer function, our next step is to understand its poles and zeros which provide a great deal about the system characteristics, responses and stability.

Definition

- A rational transfer function can be described either as a ratio of two polynomials in s,

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

or as a ratio in factored pole-zero form

$$H(s) = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)} \tag{4.1}$$

  - $K = b_m / a_n$ is called the transfer function gain
  - the roots of the numerator, $z_1, z_2, \ldots, z_m$ are called the finite zeros of the system
  - the roots of the denominator, $p_1, p_2, \ldots, p_n$ are called the poles of the system
  - assuming the coefficients of a and b are real, complex poles or zeros come in complex conjugate pairs

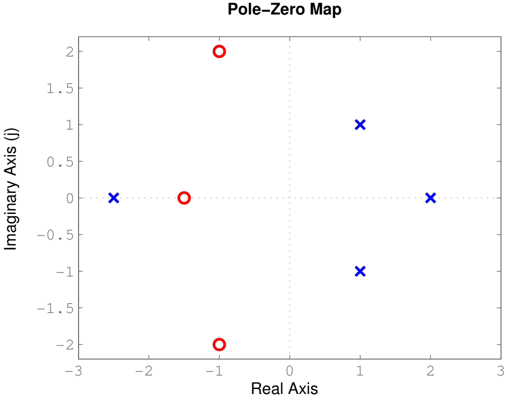
- In addition, if $Y(s) = G(s)U(s)$, then
  - system poles are the roots of the denominator polynomial of the transfer function, $G(s)$
  - input/excitation poles are the roots of the polynomial appearing in the denominator of $U(s)$, where $u(t) = \mathcal{L}^{-1}\{U(s)\}$ is the input/excitation signal

  - The poles of the system determine its stability properties (more on this later)
  - The poles of the system determine the natural behavior of the system, referred to as the modes of the system.

Poles and zeros plots

- Poles and zeros of a rational functions are often shown in a pole-zero plot: poles marked by "×", zeros marked by "○".

$$F(s) = k\frac{(s+1.5)(s^2+2s+5)}{(s+2.5)(s-2)(s^2-2s+2)}$$



Definition

- The impulse response of a LTI system is the system's output signal when the input signal, $u(t)$, is the unit impulse function, $\delta(t)$, with zero initial conditions.
- Suppose the transfer function representation of a LTI system is

$$G(s) = \frac{Y(s)}{U(s)}$$

Its impulse response is given by $u(t) = \delta(t) \quad \Rightarrow \; U(s) = 1$. We then have

$$Y(s) = G(s)$$

Inverse LT gives

$$y(t) = \mathcal{L}^{-1}\{G(s)\} = g(t)$$

$$\boxed{\text{System Transfer Function, } \; G(s) = \mathcal{L}\{\text{Impulse Response}\}}$$

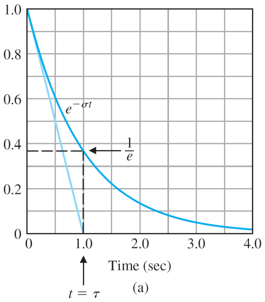
First-order system

- Consider a first-order system

$$G(s) = \frac{1}{s+\sigma}$$

inverse laplace transform (using Table 3.1) gives

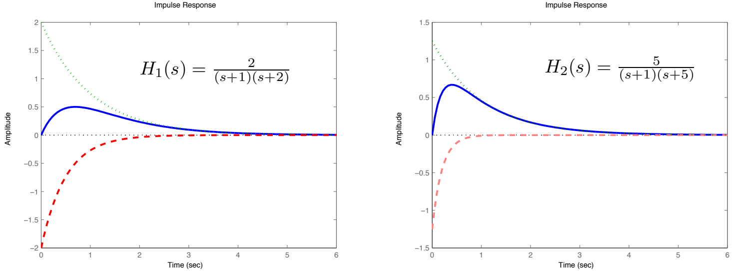
$$g(t) = e^{-\sigma t}$$



- it is clear that when $\sigma > 0$, $g(t) \to 0$ as $t \to \infty$; we say the impulse response is stable
- if $\sigma < 0$, the exponential grows with time and the impulse response is unstable.
- the larger the σ, the faster the exponential expression decay to 0.
- the time constant of the system, τ, is defined as $\tau = 1/\sigma$ as the time when the response is $1/e = 0.368$ times the initial value (a measure of the rate of decay).

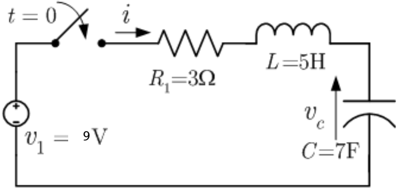
- Another example: Consider the time responses of the following systems:

$$H_1(s) = \frac{2}{(s+1)(s+2)}, \quad H_2(s) = \frac{5}{(s+1)(s+5)}$$



  - notice that the shape of the system response is dominated by the the shape of the smaller pole (in terms of size)
  - in general, poles farther to the left in the s-plane are associated with natural signals that decay faster than those associated with poles closer to the imaginary ($j\omega$) axis.

- Example: consider the series RLC circuit,



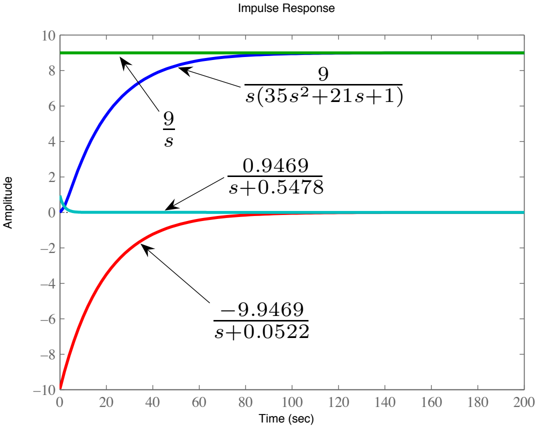
$$\frac{V_c(s)}{V_1(s)} = \frac{1}{35s^2+21s+1}, \quad V_1(s) = \frac{9}{s}$$

  - system poles are roots of $35s^2+21s+1=0$, i.e. $s_1 = -0.052$ and $s_2 = -0.548$
  - input/excitation pole (the pole of input) is $s = 0$

$$\begin{aligned} V_c(s) &= \frac{V_c(s)}{V_1(s)} \times V_1(s) = \frac{1}{35s^2+21s+1} \times \frac{9}{s} \\ v_c(t) &= \underbrace{-9.94e^{-0.052t} + 0.94e^{-0.548t}}_{\text{transient}} + \underbrace{9}_{\text{steady-state}} \end{aligned}$$

System poles appear in the transient response term, while input pole determine the steady-state term.

- again, notice that the shape of the component parts of $v_c(t)$ are determined by the locations of the poles of the transfer function.
- "fast poles" and "slow poles" (dominant poles) refer to the relative rate of signal decay



Second-order system

- Consider the following second-order transfer function with 2 poles:

$$H(s) = \frac{1}{s^2 + s + 1}$$

the poles are given by $s = -0.5 \pm j0.86$ (complex poles!!) (can be obtained in Matlab using roots$([1\ 1\ 1])$)

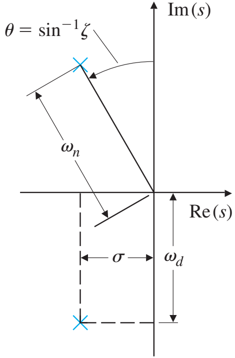
- A pair of complex poles, can be defined in terms of their real and imaginary parts, as follows:

$$s = -\sigma \pm j\omega_d$$

where

$$\sigma = \zeta\omega_n \quad \text{and} \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

ζ is the damping ratio and ω_n is the undamped natural frequency.



- A standard 2nd order transfer function is generally expressed as

$$H(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K\omega_n^2}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

where K is the gain of the system.

- Its impulse response (see LT table) is given by

$$y_i(t) = h(t) = \frac{K\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin(\omega_d t)$$

- Its step response (either via LT table or integrating the impulse response) is given by

$$y_s(t) = K - \frac{K}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin\left(\omega_d t + \cos^{-1}\zeta\right)$$

- The impulse and step responses for different values of ζ is shown in Figure 4.1, the pole locations corresponding to selected value of ζ is shown in Figure 4.2.

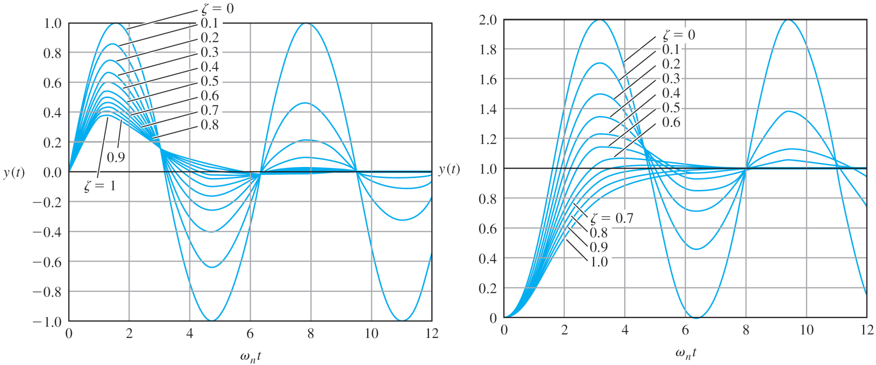


Figure 4.1 : Impulse and step responses of second-order systems

- for very low damping, the response is oscillatory. (i.e. $\zeta \to 0$, poles close to $j\omega$-axis)
- for large damping, the response shows no oscillation. ($\zeta \to 1$)
- main difference between step and impulse, for a stable system, the impulse response will always decay to zero.

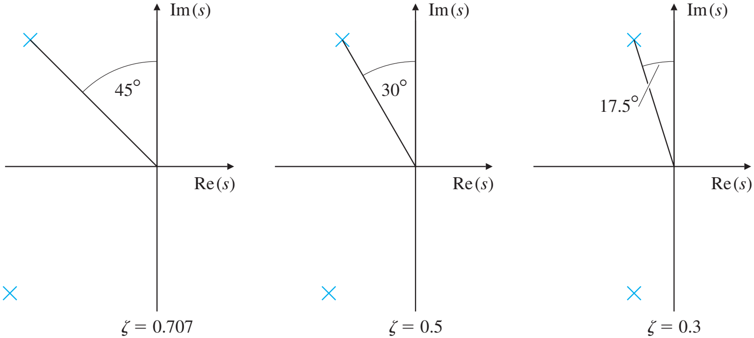


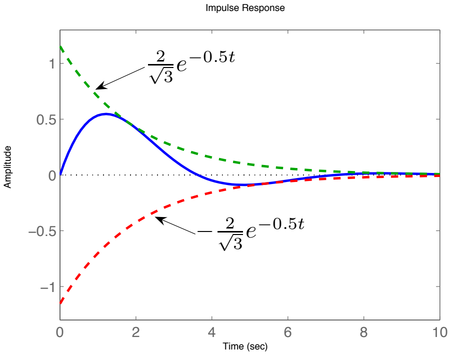
Figure 4.2 : Pole locations corresponding to selected value of ζ

- Examples: what is the impulse response of the following transfer function

$$H(s) = \frac{1}{s^2+s+1} = \frac{2}{\sqrt{3}} \frac{\sqrt{3}/2}{(s+0.5)^2 + (\sqrt{3}/2)^2}$$

Inverse Laplace transform

$$h(t) = \frac{2}{\sqrt{3}} e^{-0.5t} \sin\left(\frac{\sqrt{3}}{2} t\right)$$



Summary of impulse responses of various pole locations:

- real, positive poles correspond to growing exponential terms
- real, negative poles correspond to decaying exponential terms
- a poles at $s = 0$ correspond to a constant term
- complex pole pairs with positive real part correspond to exponentially growing sinusoidal terms
- complex pole pairs with negative real part correspond to exponentially decaying sinusoidal terms
- pure imaginary pole pairs correspond to sinusoidal terms
- repeated poles yield same types of terms, multiplied by powers of t

A sketch of these pole locations and corresponding natural responses are given in Figure 4.3.

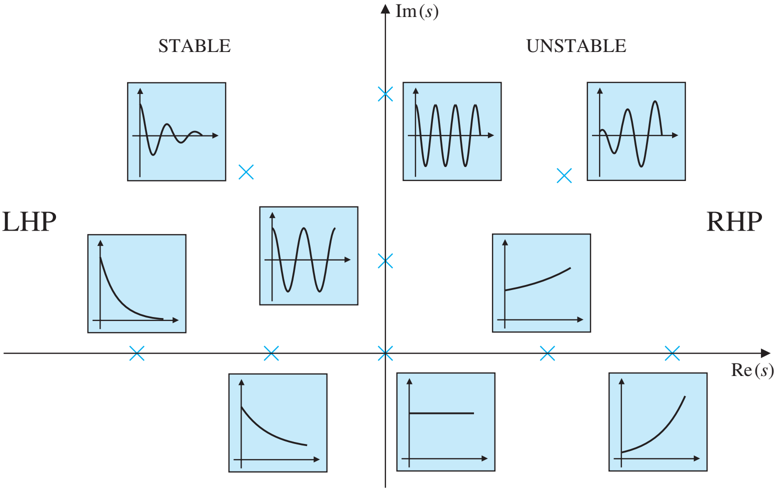
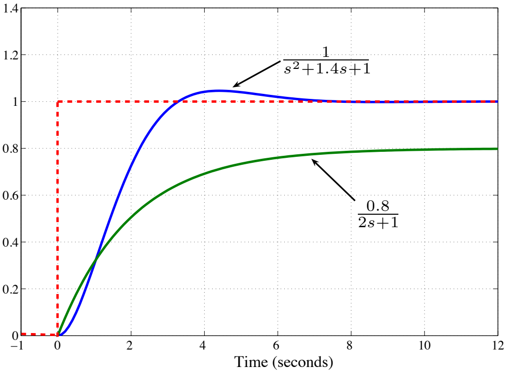


Figure 4.3 : Impulse responses associated with points in the s-plane.

Definition

- The step response is the response of the system $G(s)$ to a unit step input $u(t) = 1(t)$.
- The step response transform is given by $Y_s(s) = G(s)U(s)$. Since $U(s) = 1/s$, we have

$$y_s(t) = \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}$$



- Example: find the unit step response of $G(s) = \frac{1}{(s+1)(s+2)}$.

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} &= \frac{1}{(s+1)(s+2)} \times \frac{1}{s} \\
&= \underbrace{\frac{0.5}{s+2} - \frac{1}{s+1}}_{\text{system poles}} + \underbrace{\frac{0.5}{s}}_{\text{input pole}} \\
&= \underbrace{0.5e^{-2t} - e^{-t}}_{y_{tr}(t)} + \underbrace{0.5}_{y_{ss}(t)}
\end{aligned}
$$

  - the input pole gives rise to the steady-state term, $y_{ss}(t)$
  - if the system is stable, then $\lim_{t\to\infty} y_{tr} = 0$ and

$$
\begin{aligned}
\lim_{t\to\infty} \text{step response} &= \lim_{t\to\infty} (y_{tr}(t) + y_{ss}(t)) \\
&= 0 + y_{ss}(t) \\
&= \text{constant}
\end{aligned}
$$

This constant is the gain of the system $\times$ the step size.

- Static/Steady-state/DC gain: this is the ratio of the output of a system to its input (presumed constant) after all transients have decayed.
  If the magnitude of the step input is A, i.e.
  $u(t) = AU(t) \Rightarrow U(s) = \dfrac{A}{s}$, we then have (via Final Value Theorem)

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG(s)\frac{A}{s} = AG(0)$$

  Hence,

$$\text{DC gain,} \quad K = \frac{\lim\limits_{t\to\infty} y(t)}{\lim\limits_{t\to\infty} U(t)} = \frac{AG(0)}{A} = G(0)$$

Step response of common transfer functions

- Step responses are encountered frequently in practice. Hence, it is useful to derive and analyse the step response of common transfer functions:
  - integrator
  - differentiator
  - transportation delay or dead-time
  - first order system
  - second order system
- They can be viewed as the basic building blocks of a physical system. (We have already seen some of these in the earlier lectures.)

- Integrator:

$$y(t) = \int_0^t K_i u(\tau) d\tau$$

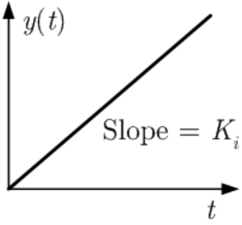
where K_i is known as the integrator gain.
The transfer function is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K_i}{s}$$

- step response of an integrator

$$Y(s) = G(s)U(s) = G(s)\frac{1}{s} = \frac{K_i}{s^2}$$

- inverse LT: $y(t) = K_i t$
- input bounded but output unbounded! What input would have resulted in a bounded output?

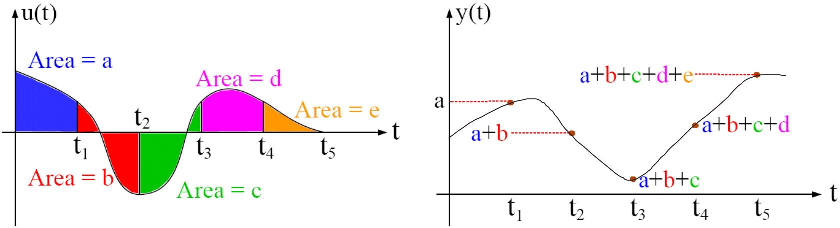


- Integrator is marginally stable as its impulse response is non-decreasing.
- Consistent with conclusion drawn from system pole location. Pole of an integrator is $s = 0$ (the origin), and systems with non-repeated pole on the imaginary axis is marginally stable.
- Since the step response of an integrator is unbounded, the term Steady-state/DC gain is meaningless. We introduce a new term, K_i, to characterize the integrator long term behavior.

$$K_i = \lim_{s \to 0} sG(s)$$

  where K_i is the slope of the integrator's step response and $G(s) = K_i/s$.
- An example of the system whose transfer function contains an integrator is a capacitor fed with a current source.

- Note the output of an integrator, $y(t)$, at time $t = t_i$ is the total area under the curve defined by the input signal, $u(t)$, from $t = 0$ to t_i:



- The output of an integrator depends on the entire past history of the input, i.e., it has infinite memory.
- Another useful property:

$$\boxed{\int_0^t u(\tau)d\tau = \text{constant} \quad \forall t > t_0 \quad \text{if and only if} \quad u(t) = 0 \quad \forall t > t_0}$$

- Differentiator:

$$y(t) = K_d \frac{du(t)}{dt}$$

  where K_d is the derivative gain.

  The transfer function is $G(s) = \dfrac{Y(s)}{U(s)} = K_d s$

- Step response of a differentiator is

$$Y(s) = G(s)U(s) = \frac{G(s)}{s} = K_d$$

  Inverse LT gives

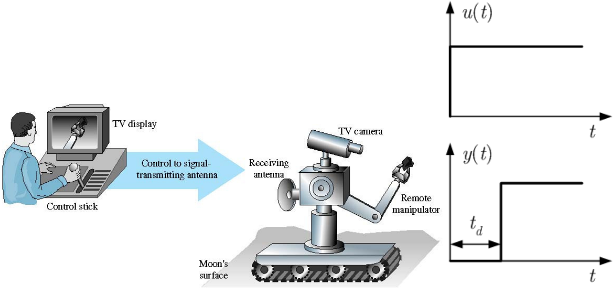
$$y(t) = K_d \delta(t), \quad \text{impulse function}$$

- A differentiator is useful since it provides the ability to "predict ahead in time". An example is the voltage across an inductive coil.

- Transportation delay: also called transport lag or dead-time

$$y(t) = u(t - t_d), \quad t_d = \frac{\text{distance}}{\text{speed}}$$

- Transportation delay is a type of time delay that occurs in systems which require a finite time to move material or transmit signal from one point to another, example:



- The transfer function of a transportation lag is given by

$$\begin{aligned}\mathcal{L}\{y(t)\} &= \mathcal{L}\{u(t-t_d)\} \\ Y(s) &= e^{-st_d}U(s) \\ G(s) &= \frac{Y(s)}{U(s)} = e^{-st_d}\end{aligned}$$

- The transfer function of a lag is non-rational, an approximation in terms of poles and zeros is:

$$e^{-st_d} = \frac{e^{-st_d/2}}{e^{st_d/2}} = \frac{1 - st_d/2 + \frac{(-st_d/2)^2}{2!} + \dots}{1 + st_d/2 + \frac{(st_d/2)^2}{2!} + \dots} \approx \frac{1 - st_d/2}{1 + st_d/2}$$

- In general, it is undesirable to have time delays because it implies the system is slow to react to any changes.

- First-order systems:
  - The d.e. of a linear first-order system is generally written as

$$\tau \frac{dy(t)}{dt} + y(t) = Ku(t)$$

    where K is the steady-state/static gain and τ is called the time constant.
  - its transfer function is given as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1}; \quad y(0) = 0$$

    where the pole is located at $s = -\frac{1}{\tau}$.
  - Common 1st-order systems:

$$\text{Series RC circuit:} \quad G(s) = \frac{V_c(s)}{V(s)} = \frac{1}{RCs + 1}$$

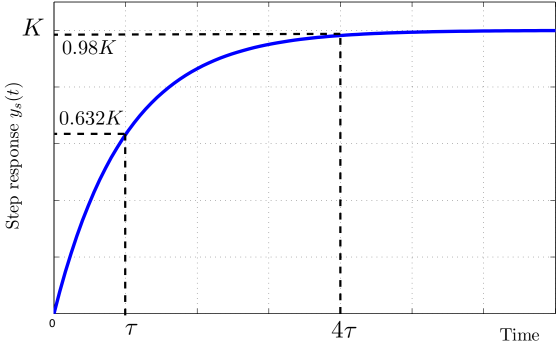
$$\text{Series RL circuit:} \quad G(s) = \frac{I_L(s)}{V(s)} = \frac{1/R}{L/Rs + 1}$$

- Its unit step response is given by

$$Y(s) = G(s)U(s) = \frac{K}{\tau s + 1}\frac{1}{s} = \frac{K}{s} - \frac{K\tau}{\tau s + 1}$$

Inverse LT gives

$$y_s(t) = K - Ke^{-t/\tau}$$



$$\boxed{\text{Steady-state output} = \lim_{s\to 0} G(s) \times \text{ magnitude of step}}$$

- We have previously shown the definition of the time constant for a first-order system from its impulse response (page 4-10). From its step response, it corresponds to the time the system response takes to reach 63.2% of the final value.
- When $t = 4\tau$, $y_s(t) = K(1 - e^{-4}) \approx 0.98K$; i.e. the 2% settling time is 4τ.
- Pole is located at $s = -1/\tau$, a larger time constant corresponds to a pole closer to the imaginary ($j\omega$)-axis. Therefore the farther the pole is to the left of the origin, the faster the rate at which steady-state is reached (transient decays away).
- Show that $\tau = K \bigg/ \left.\dfrac{dy_s(t)}{dt}\right|_{t=0}$.

- Second-order systems:
  - d.e. of a general second-order system may be expressed as

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = K\omega_n^2 u(t)$$

    where K, ζ and ω_n are the steady-state/static gain, damping ratio and undamped natural frequency respectively.
  - its transfer function is given by (see page 4-15)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad y(0) = y'(0) = 0$$

  - common examples include:

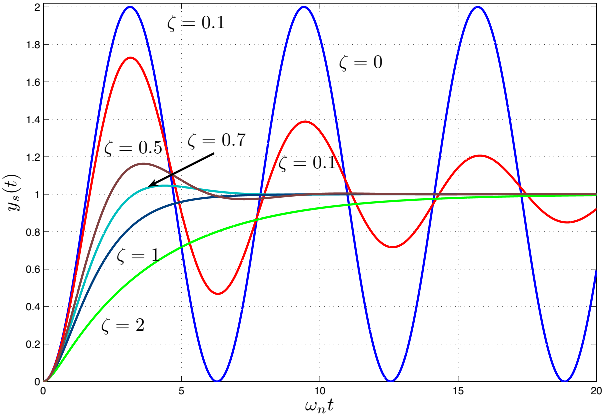
$$\text{mass-spring-damper system:} \quad G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$

$$\text{baking a wafer:} \quad G(s) = \frac{T_w(s)}{U(s)} = \frac{k}{s^2 + \alpha s + \beta}$$

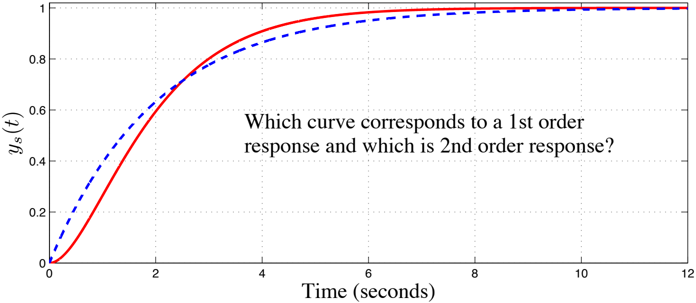
- Poles of a second-order system (see page 4-14):

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad \Rightarrow s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

  - $\zeta > 1$, poles are real and distinct
  - $\zeta = 1$, poles are real and equal (repeated)
  - $\zeta < 1$, poles are complex conjugate



- ζ, provides a measure of the degree of damping in the system.
  - when $0 < \zeta < 1$, system is underdamped (oscillatory response)
  - when $\zeta > 1$, system is overdamped (no osciallations)
  - when $\zeta = 1$, system is critically damped as the step response has the fastest rise time without oscillation and without exceeding the steady-state value (overshoot)
- When $\zeta > 1$, step responses of 1st and 2nd order systems have similar characteristic (no oscillations).



- Poles of a second-order system (see page 4-14)

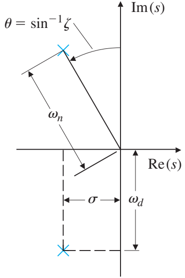
A pair of complex poles, can be defined in terms of their real and imaginary parts, as follows:

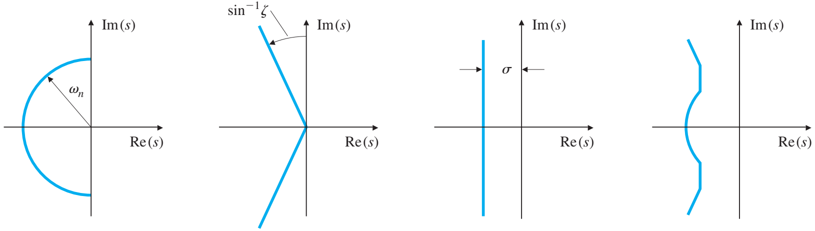
$$s = -\sigma \pm j\omega_d$$

where

$$\sigma = \zeta\omega_n \quad \text{and} \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

ζ is the damping ratio and ω_n is the undamped natural frequency.





(a) (b) (c) (d)

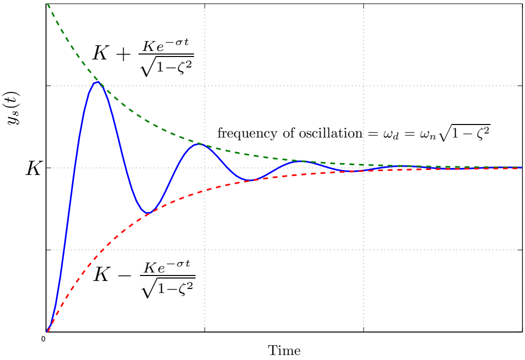
- The step response of a second-order system is given by

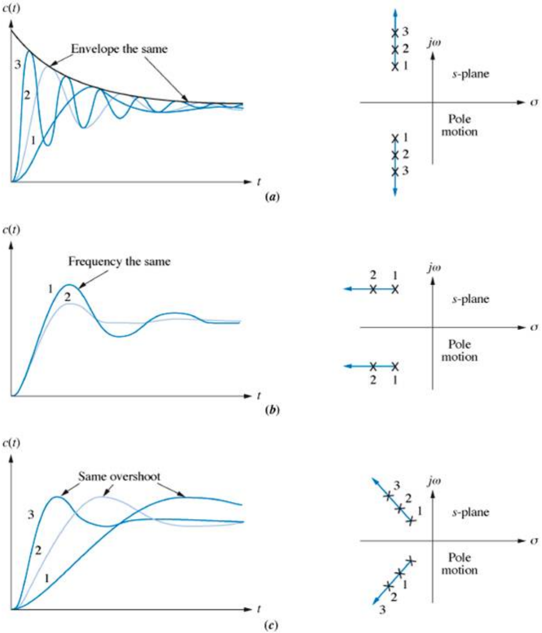
$$
\begin{aligned}
\frac{Y(s)}{U(s)} &= \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
Y(s) &= \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\,\frac{1}{s} \\
&= \frac{K}{s} - \frac{K(s + 2\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \omega_n^2 - \zeta^2\omega_n^2} \\
&= \frac{K}{s} - \frac{K(s + \zeta\omega_n)}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} - \frac{K\zeta}{\sqrt{1-\zeta^2}}\,\frac{\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}
\end{aligned}
$$

Inverse LT gives:

$$
\begin{aligned}
y(t) &= K - Ke^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}\,t\right) - \frac{K\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right) \\
&= K - \frac{Ke^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\,t\right) + \phi\right] \\
&= K\left(1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\sin\left[\omega_d t + \phi\right]\right)
\end{aligned}
$$

- Magnitude of the real part of pole, $|\mathcal{R}\{s\}| = \sigma = \zeta\omega_n$, determines the exponential envelope.
- Like the case of a first-order system, steady-state is reached more quickly if the pole is farther to the left of the imaginary axis, i.e., $|\mathcal{R}\{s\}| = \sigma$ is large.
- Imaginary part of pole, $\mathcal{I}\{s\} = \omega_d = \omega_n\sqrt{1-\zeta^2}$ determines the frequency of the sinusoidal signal.
- Frequency of sinusoid, and hence the amount of oscillation, is larger if pole is farther away from the real axis, i.e., $|\mathcal{I}\{s\}| = \omega_d$ is large.




c(t)
Envelope the same
t
(a)
jω
s-plane
σ
Pole motion
c(t)
Frequency the same
t
(b)
jω
s-plane
σ
Pole motion
c(t)
Same overshoot
t
(c)
jω
s-plane
σ
Pole motion

Summary on the relationship between the pole positions and step response:

- K determines the steady state output response to a step input.
- 3 types of responses possible: underdamped $\zeta < 1$, overdamped $\zeta > 1$ and critically damped $\zeta = 1$.
- For underdamped response
  - real part of pole ($\sigma = \zeta\omega_n$) determines how quickly the oscillations decay away
  - imaginary part of pole ($\omega_d = \omega_n\sqrt{1-\zeta^2}$) gives you the frequency of the oscillation
- For over- and critically damped systems, they behave more like first order systems, except more sluggish.

Effect of an additional zero

- For transient analysis, the zeros exert their influence by modifying the coefficients of the exponential terms whose shape is decided by the poles.
- Consider the following 2 transfer functions:

$$H_1(s) = \frac{2}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{2}{s+2}$$

$$H_2(s) = \frac{2(s+1.1)}{1.1(s+1)(s+2)} = \frac{0.18}{s+1} + \frac{1.64}{s+2}$$

  - the coefficient of the $(s+1)$ term has been modified from 2 in $H_1(s)$ to 0.18 in $H_2(s)$
  - a zero near a pole reduces the amount of that term in the total response.

- Let $H_1(s)$ be a transfer function with N poles and no zeros. Its step response is given by

$$y_1(t) = \mathcal{L}^{-1}\left\{H_1(s)U(s)\right\}$$

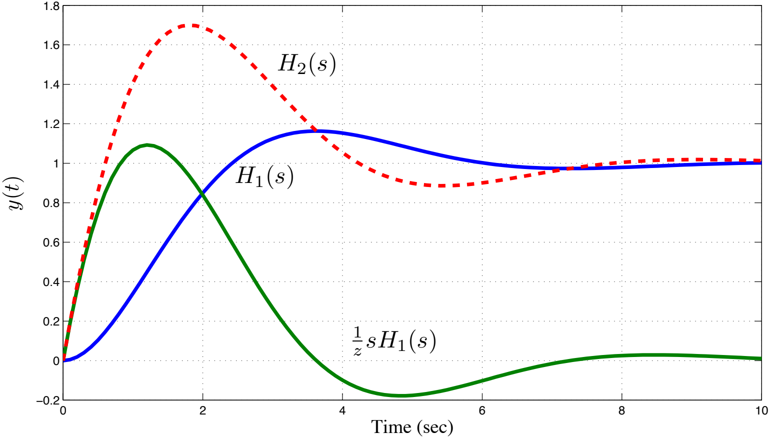
- Suppose $H_2(s)$ is formed by adding a zero to $H_1(s)$, i.e.

$$\begin{aligned} H_2(s) &= \left(\frac{s}{z}+1\right)H_1(s) \\ &= \frac{1}{z}sH_1(s)+H_1(s) \end{aligned}$$

The first term, due to the zero, is a product of a constant $(1/z)$ times s times the original term. Recall that the Laplace transform of df/dt is $sF(s)$, hence we have

$$\begin{aligned} y_2(t) &= \mathcal{L}^{-1}\left\{H_2(s)U(s)\right\} \\ &= \mathcal{L}^{-1}\left\{\left(\frac{1}{z}sH_1(s)+H_1(s)\right)U(s)\right\} \\ &= \frac{1}{z}\frac{dy_1(t)}{dt}+y_1(t) \end{aligned}$$

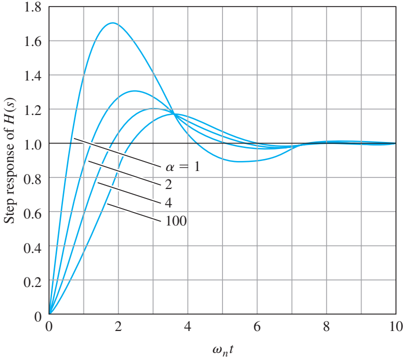
- The derivative has a large hump in the early part of the curve, and adding to the original response lifts up the total response of $H_2(s)$.



- Example: Consider the transfer function with two complex poles and one zero:

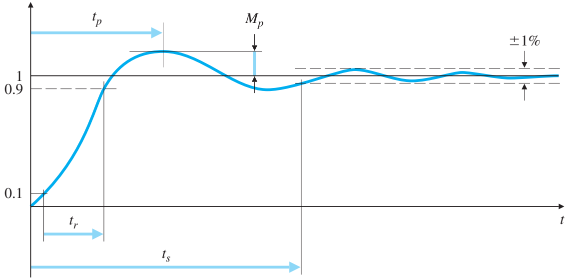
$$H(s) = \frac{(s/\alpha\zeta\omega_n + 1)}{(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1}$$

  - the zero is located at $s = -\alpha\zeta\omega_n = -\alpha\sigma$
  - the effect is shown below, the major effect of the zero is to increase the overshoot M_p, decrease the rise-time, whereas it has very little influence on the settling time.



Overview

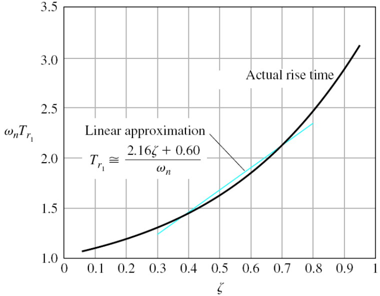
- In control system design, the following time-domain specifications are often used.
  - **rise time**, t_r: the time it takes the system to reach the vicinity of its new set point
  - **settling time**, t_s: the time it takes the system transient to decay
  - **overshoot**, M_p: the maximum amount the system overshoots its final value divided by its final value
  - **peak time**, t_p: the time it takes the system to reach the maximum overshoot point



- Many possible definitions, the most commonly used one is the 10% to 90% rise time, t_r, defined as the time taken for the system response to rise from 10% to 90% of the steady-state value.
- Difficult to derive analytical expressions for rise time.
- For a standard 2nd-order transfer function, from Figure 4.1, it is possible to plot the normalized rise time, $\omega_n t_r$ versus ζ as shown here.

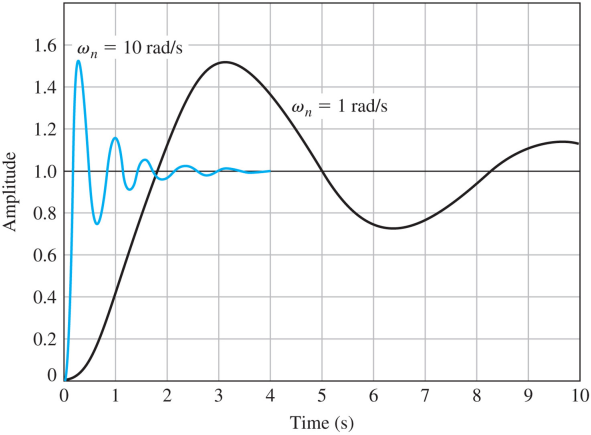
As a rough estimate, we have
$t_r = \dfrac{2.16\zeta + 0.60}{\omega_n}$ for $0.3 \leq \zeta \leq 0.8$.
Taking average (for $\zeta = 0.55$), we have

$$t_r \approx \frac{1.8}{\omega_n} \qquad (4.2)$$

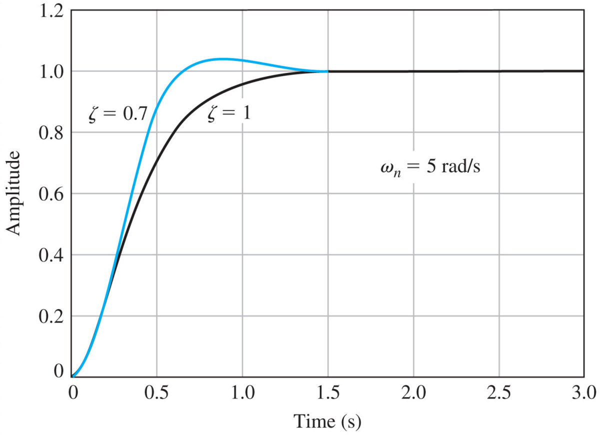


Note: formula for 2nd order system with no zeros, for all other systems only a very rough approximation.

- For a given damping ratio, ζ, the response is faster for larger ω_n.
- Notice that the overshoot is independent of ω_n.



- For a given undamped natural frequency, ω_n, the response is slightly faster for smaller ζ.



- The overshoot, M_p, occurs when the derivative of the signal, $y(t)$ is zero.
- Previously shown that the unit step response of 2nd-order transfer function, $H(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, is given by the inverse Laplace transfer of $H(s)/s$

$$
\begin{aligned}
y(t) &= K - Ke^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right) \\
&= K - Ke^{-\sigma t}\sqrt{1 + \frac{\sigma^2}{\omega_d^2}}\cos(\omega_d t + \beta)
\end{aligned}
$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$, $\sigma = \zeta\omega_n$ and $\beta = \tan^{-1}(\sigma/\omega_d)$. K is the gain of the system.

- When the output response $y(t)$ reaches its maximum value, its derivative is zero:

$$\begin{aligned}\dot{y}(t) &= K\sigma e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right) \\ &\quad -Ke^{-\sigma t}\left(-\omega_d\sin\omega_d t + \sigma\cos\omega_d t\right) = 0 \\ &= Ke^{-\sigma t}\left(\frac{\sigma^2}{\omega_d}\sin\omega_d t + \omega_d\sin\omega_d t\right) = 0\end{aligned}$$

 this will occurs when $\sin\omega_d t = 0$, hence $\omega_d t_p = \pi$ and the peak time, t_p is given by $t_p = \dfrac{\pi}{\omega_d}$

- Substituting t_p in the expression for $y(t)$, we have

$$\begin{aligned}y(t_p) &\triangleq K + M_p = K - Ke^{-\sigma\pi/\omega_d}\left(\cos\pi + K\frac{\sigma}{\omega_d}\sin\pi\right) \\ &= K + Ke^{-\sigma\pi/\omega_d}\end{aligned}$$

 the overshoot formula is thus given by

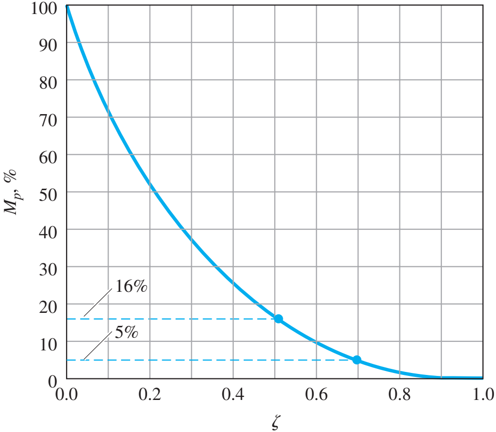
$$M_p = Ke^{-\pi\zeta/\sqrt{1-\zeta^2}}, \quad 0 \le \zeta < 1. \tag{4.3}$$

- The percentage overshoot is given as

$$\%M_p = \frac{M_p}{y_{ss}} \times 100\% = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\%$$

which is independent of the system gain K and the step size.

- Plot of the maximum overshoot M_p vs. the damping ratio ζ



- This is the time required for the transient to decay to a small value so that $y(t)$ is almost in the steady-state.
- Measure of smallness: $1\%, 2\%$ or 5% have been used.
- Notice that the deviation of y from K is enclosed by the envelop of the exponential function

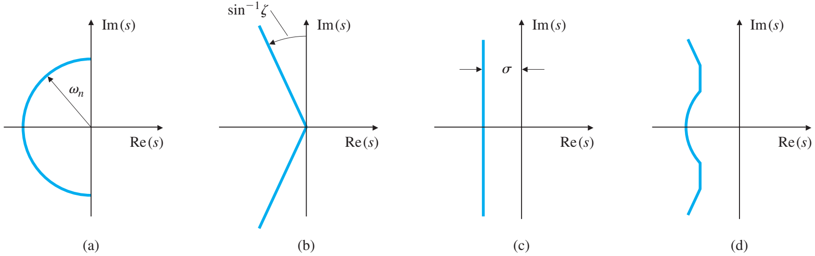
$$K\left(1 \pm \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)$$

The settling time corresponding to a 2% tolerance band may be estimated by the time the exponential curve takes to decay to 0.02 i.e.

$$\begin{aligned} \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} &= 0.02 \\ e^{-\zeta\omega_n t} &\approx 0.02 \\ \zeta\omega_n t_s &\approx 4 \end{aligned}$$

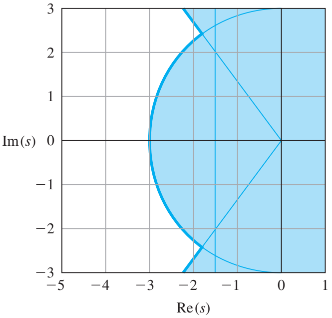
$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma} \tag{4.4}$$

- Equations (4.2)–(4.4) characterize the transient response of a system having no finite zeros and two complex poles and with undamped natural frequency ω_n, and damping ratio ζ.
- Analysis: these parameters are used to estimate rise-time, overshoot and settling time for just about any system (not restricted to 2nd order system – more on this in a while).
- Design Synthesis: selection of pole and zero locations to meet these time-domain specifications for dynamic response.



- Example: find the allowable region in the s-plane for the poles of a transfer function of a system if the system response requirements are $t_r \le 0.6$ sec, $M_p \le 10\%$, and $t_s \le 3$ sec.
- Assuming that the system can be approximated by a second-order system (with no zero), we have

$$
\begin{aligned}
t_r &= \frac{1.8}{\omega_n} \le 0.6 \\
&\Rightarrow \quad \omega_n \ge \frac{1.8}{0.6} = 3.0 \text{ rad/s} \\
M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}} \le 0.1 \\
&\Rightarrow \quad \zeta \ge 0.6 \\
1\% \quad t_s &= \frac{4.6}{\sigma} \le 3 \\
&\Rightarrow \quad \sigma = \zeta\omega_n \ge \frac{4.6}{3} = 1.5 \text{ sec}
\end{aligned}
$$



The unshaded region is the allowable region based on the specifications.

Stability

- A linear time-invariant system is said to be stable if all the roots of the transfer function denominator polynomial have negative real-parts (i.e. all in the left half plane, $\sigma < 0$) and is unstable otherwise ($\sigma > 0$). (see Figure 4.3)
- If the pole is on the $j\omega$ axis, oscillatory response will persist; if the pole is at the origin, small initial conditions will persist.
- Example: for second-order poles:

$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin(\omega_d t)$$

Hence, a system is stable if its transient response decay and unstable if it does not.

- The solution of a LTI system whose transfer function is given by equation 4.1 may be written using partial fraction expansion as

$$y(t) = \sum_{i=1}^{n} K_i e^{p_i t} \tag{4.5}$$

  where p_i are the roots of the transfer function and K_i depend on the initial conditions and zero locations.

- system is stable iff (necessary and sufficient condition) every term in equation (4.5) goes to zero as $t \to \infty$:

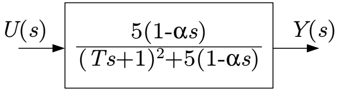
$$e^{p_i t} \to 0 \text{ for all } p_i$$

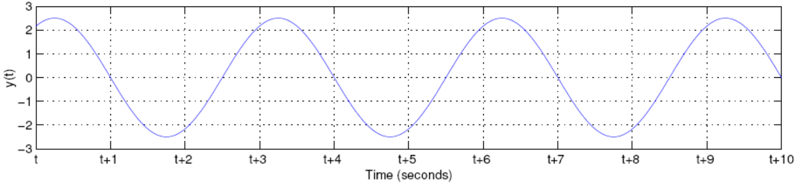
  this will only happen if all the poles of the system are strictly in the LHP

$$\text{Re}\{p_i\} < 0$$

- if the system has any poles in the RHP, it is unstable
- if the system has repeated poles, the response from equation (4.5) must be changed to include a polynomial in t in place of K_i, but the conclusion is the same.
    - example: is $\frac{1}{(s+1)^2}$ stable? inverse LT gives te^{-t} which $\to 0$ as $t \to \infty$.
- $j\omega$ is the stability boundary
- if the system has nonrepeated $j\omega$-axis poles, it is marginally stable
- how about if the system has repeated $j\omega$-axis poles?
- alternative way to finding the roots of the characteristic equation – Routh's stability criterion (back to this later)

- Example: consider the following system





Suppose $\lim_{t\to\infty} u(t) = 0$ and the output signal, $\lim_{t\to\infty} y(t)$ is shown in the figure.
Using the relationship between pole location and stability condition, determine α and T.

Summary

- The locations of poles in the s-plane determine the character of the response as shown in the various impulse and step responses plots.(see Figure 4.3).
- For a standard second-order system, the transient response parameters can be characterized by its rise-time, settling time and overshoot; and are related to the pole locations. A zero on the left-half plane will increase the overshoot.
- The real-part of the pole determine its stability; for a stable system, all poles must be in the LHP.

Review Questions

- What is the effect of zero on the transient response?
- Which is the dominant pole in the following system?

$$H(s) = \frac{100}{s+2} + \frac{1}{s+1}$$

- Is this stable? $1/(s^2+1)^2$

Reading: FPE: sections 3.3, 3.5 and 3.7.1

Practice Problems

1. First-order systems. Find the steady-state gain and time constants of the following systems with transfer function given below:

$$\frac{2}{s+1}, \quad \frac{5}{s+2}, \quad \frac{10}{2s+1}, \quad \frac{1}{0.1s+2}$$

2. Second-order systems. The percentage overshoot and 2% settling time of three second-order system are given below. Find the corresponding closed-poles of the systems.

$$(10\%, 4s), \quad (10\%, 8s), \quad (1\%, 4s)$$

3. Time domain specifications. Suppose you desire the peak time of a given second-order system to be less than $t_p^{'}$. Draw the region in the s-plane that corresponds to values of the poles that meet the specification $t_p < t_p^{'}$.