EE3331C/EE3331E Feedback Control System
L6: Root Locus Analysis

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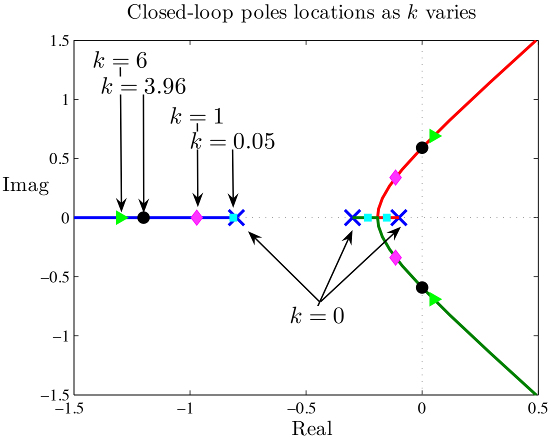
Outline

Review:

- In our earlier heater example, we show a plot of how the closed-loop poles varies by solving the closed-loop c.e. for different values of gain, k.



Also known as the root locus plot.

- Key Idea: Consider the feedback system in Figure 6.1, the closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{G(s)K(s)}{1 + G(s)K(s)G_s(s)}$$

with the closed-loop poles given by

$$1 + G(s)K(s)G_s(s) = 0$$

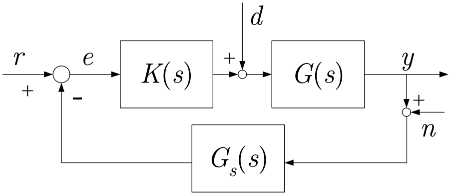


Figure 6.1 : Basic closed-loop block diagram

- Assuming that we are interested in a parameter which we called K here (note: does not need to be the controller gain k). The closed-loop c.e. can be written as

$$1 + KL(s) = 0$$

and if we let $L(s) = \dfrac{b(s)}{a(s)}$, we then have

$$\begin{aligned} 1 + K\frac{b(s)}{a(s)} &= 0 \\ a(s) + Kb(s) &= 0 \qquad (6.1) \\ \frac{a(s)}{K} + b(s) &= 0 \end{aligned}$$

where $a(s)$ and $b(s)$ are of order n and m respectively and $n \geq m$.

  - As $K \to 0$, the poles of the closed-loop system are $a(s) = 0$ or the poles of $L(s)$
  - As $K \to \infty$, the poles of the closed-loop system are $b(s) = 0$ or the zeros of $L(s)$.

- Regardless of the value of K, the closed-loop system must always have n poles:
  - the root locus must have n branches, each branch starts at a pole of $L(s)$ and goes to a zero of $L(s)$
  - if $L(s)$ more poles than zeros (as is often the case), $n > m$, and we say that $L(s)$ has zeros at infinity; the number of zeros at infinity is $n - m$, and is the number of branches of the root locus that go to infinity (asymptotes).
- We have shown previously that by varying the gain, K, we can plot out the locus of all possible roots (closed-loop poles). The resulting plot can then aid us in selecting the best value of K to achieve the desired performance specifications. We next demonstrate the idea via a few examples.

- In Matlab,
  1. Describe $L(s)$ in Matlab as a transfer function.
  2. Plot the root locus with 'rlocus(sys)' where 'sys' is $L(s)$.

Example 1: Root locus of first-order system

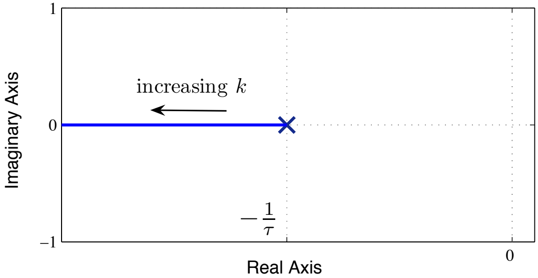
- Consider unity feedback control system with

$$G(s) = \frac{k_p}{s\tau + 1}, \quad K(s) = k$$

Closed-loop poles given by

$$1 + G(s)K(s) = 1 + \frac{kk_p}{s\tau + 1} = 0 \quad \Rightarrow \quad s = -\frac{1 + kk_p}{\tau}$$

for k_p and $\tau \geq 0$, the pole is always on the left-half plane.



Example 2: Root locus of motor position control

- Consider the following normalized transfer function of a DC motor

$$\frac{Y(s)}{U(s)} = G(s) = \frac{A}{s(s+c)}$$

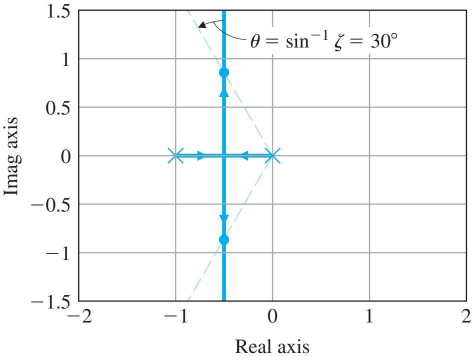
  - we are interested how the closed-loop varies as the system gain A varies
  - referring to Figure 6.1, we have $K(s) = 1, G_s(s) = 1$ and we also assumed $c = 1$
- We then have $L(s) = \frac{1}{s(s+1)}$, and $K = A$

$$\begin{aligned} 1 + KL(s) = 1 + \frac{K}{s(s+1)} &= 0 \\ s^2 + s + K &= 0 \end{aligned}$$

solving, we have

$$r_1, r_2 = -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2}$$

- Two branches of the root locus: the locus starts from the open-loop poles and ends at the open-loop zeros or infinity.
- As K increases, the poles moved towards each other along the real-axis, meeting at $s = -0.5$ before breaking away from the real-axis towards infinity with real-part of the poles at $s = -0.5$.
- By varying the gain K, we can have any closed-loop poles along the locus to meet design specifications. e.g. plot shows $\zeta > 0.5$.




```
>> sys = tf(1,[1 1 0]);
>> rlocus(sys);
```

Example 3: Root locus of motor position control (cont)

- If we are now interested to find out how the closed-loop poles varies as c varies, assuming $A = 1$, we have

$$1 + G(s) = 1 + \frac{1}{s(s+c)}$$

The closed-loop poles is given by

$$s^2 + cs + 1 = 0$$

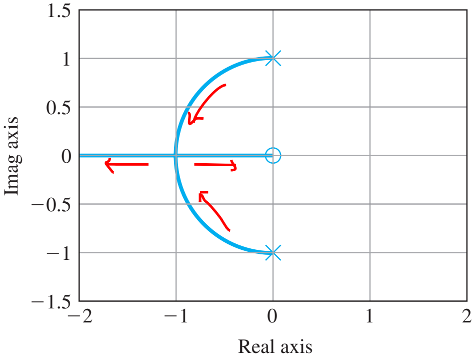
Rearranging, we have

$$1 + c\frac{s}{s^2+1} = 0$$

The roots are given by

$$r_1, r_2 = -\frac{c}{2} \pm \frac{\sqrt{c^2-4}}{2}$$

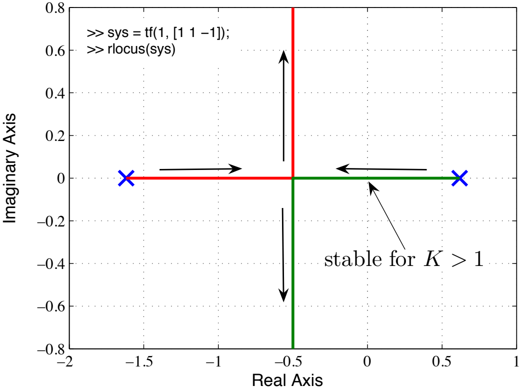
- At $c = 0$, the poles are at the open-loop pole locations.
- At $c = 2$, the two poles are both at $s = -1$.
- For $c > 2$, the two locus segments change direction and move in opposite directions; one towards the open-loop zero and the other towards infinity.




```
>> sys2 = tf([1 0],[1 0 1]);
>> rlocus(sys2);
```

Example 4: Unstable open-loop system.

- Consider $L(s) = \dfrac{1}{s^2 + s - 1}$, the open-loop poles are at $s = \frac{-1 \pm \sqrt{5}}{2}$.

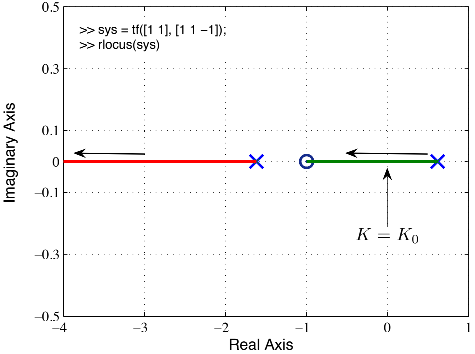


- At $K = 1$, CL poles crossed over to the RHP, giving the CL only marginal stability.

Example 5: Unstable open-loop system with open-loop zero.

$$L(s) = \frac{s+1}{s^2+s-1}$$

same open-loop poles as in previous example.

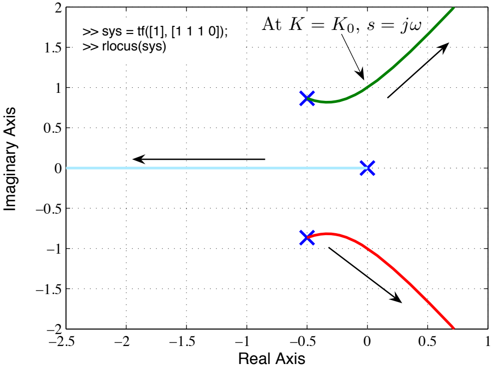


- Closed-loop is only stable when $K > K_0$, the closed-loop c.e. is

$$s^2 + s(K+1) + K - 1 = 0$$

Example 6: loop transfer function with three poles, unity feedback

$$G(s) = \frac{1}{s^2 + s + 1}, \qquad K(s) = \frac{K}{s}$$

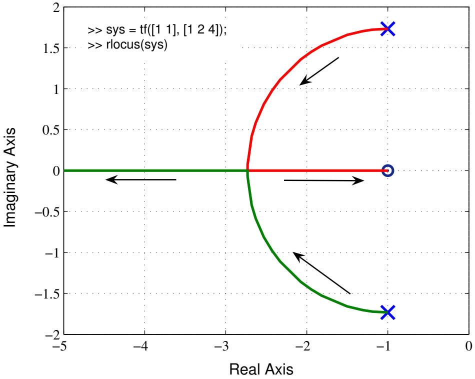


- 3 branches of root locus, the locus starting from $s = 0$ is always stable, the other 2 locus becomes unstable for $K > K_0$.
- Closed-loop c.e.: $s^3 + s^2 + s + K = 0$. How to find K_0?

Example 7: loop transfer function with two complex poles, one zeros

$$L(s) = \frac{s+1}{s^2+2s+4}$$

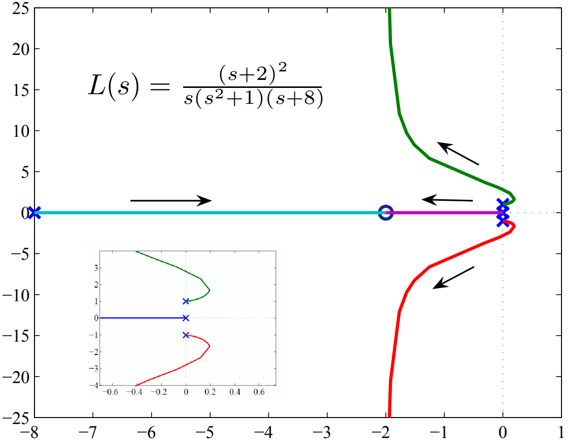
open-loop poles at $s = -1 \pm j\sqrt{3}$, zero at $s = -1$.



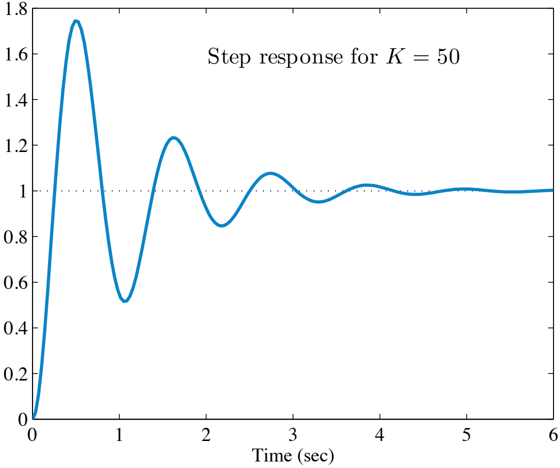
- Closed-loop c.e.: $s^2 + (2+K)s + 4 + K = 0$. Closed-loop always stable.

Example 8: conditionally stable system

- Given that $L(s) = \dfrac{(s+2)^2}{s(s^2+1)(s+8)}$,
  - using Matlab, sketch the root locus for $K > 0$
  - determine the range of K for which the closed loop system is stable
  - for what values of $K > 0$ do purely imaginary roots exist? what are the values of these roots?
  - would the use of the dominant roots approximation for an estimate of settling time be justified in this case if K is large ($K > 50$)?



- Closed-loop c.e.: $s^4 + 8s^3 + (K+1)s^2 + (4K+8)s + 4K = 0$
- When $K = 50$, the roots are $s = -1.02 \pm j5.63; -4.66; -1.31$

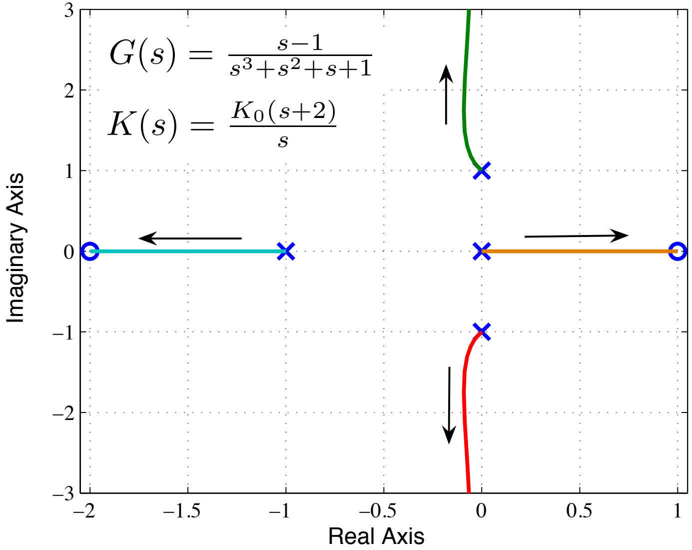

Step response for $K = 50$
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
Time (sec)

Example 9: Unstable system

- Consider an OL plant, $G(s)$, being controlled in a unity negative feedback configuration by a controller, $K(s)$,

$$G(s) = \frac{s-1}{s^3+s^2+s+1}, \qquad K(s) = \frac{K_0(s+2)}{s}$$

- Using Matlab, plot the root locus and determine if the CL system is stable for any values of $K_0 > 0$.


$G(s) = \frac{s-1}{s^3+s^2+s+1}$
$K(s) = \frac{K_0(s+2)}{s}$
Imaginary Axis
Real Axis
3
2
1
0
−1
−2
−3
−2
−1.5
−1
−0.5
0
0.5
1

A final note:

- What we have covered: the use of Matlab for root locus analysis, very powerful tool for controller design.
- What we have not covered:
  - Without Matlab, how to sketch the root locus?
  - In control system design, we are interested to know how to modify the dynamics in such a way as to achieve the desired performance specifications

    $\rightarrow$ useful to know how to sketch root locus for controller design

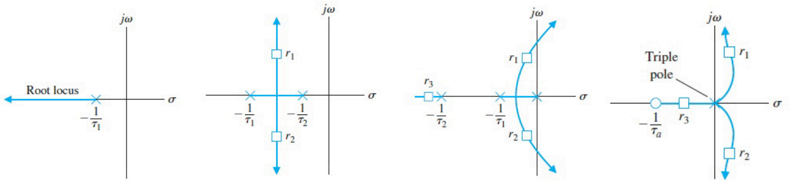
    $\rightarrow$ help to check if computer program is right
  - For the interested reader, please check Section 5.2 of Franklin's book!

Summary

- A root locus is a graph of the values of s that are solutions to the equation $1 + KL(s) = 0$ with respect to a real parameter K.
- A powerful method for controller design. Useful to be able to sketch simple systems.

Review Questions

- How do you find the maximum k before a closed-loop system becomes unstable?
- Discuss the stability of the following systems.



Reading: FPE: section 5.1

Practice Problems

Figure shows the root locus for $1 + k\frac{1}{s(s+2)(s^2+4s+5)}$.

1. What is the maximum k before the system becomes unstable?
2. At $k = 6.5$, the roots are $s_{1,2} = -2.65 \pm j1.23$ and $s_{3,4} = -0.35 \pm j0.8$. What is the settling time and percentage overshoot of its step response? Justify your answer.