EE3331C Feedback Control System
Guidelines for Sketching Root Locus

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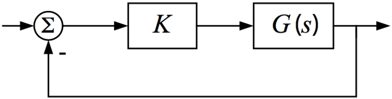
Definition

Consider the following feedback system



The closed-loop poles are given by

$$1 + KG(s) = 0 \tag{A.1}$$

or

$$KG(s) = -1$$

If K is to be real and positive, $G(s)$ must be real and negative. If we arrange $G(s)$ in polar form as magnitude and phase, then the phase of $G(s)$ must be 180^o in order to satisfy the above equation.
Equation A.1 can be rewritten in polar form as

$$|KG(s)|\underline{/KG(s)} = -1 + j0 \tag{A.2}$$

Definition

The root locus is the set of values of s for which $1 + KG(s) = 0$ is satisfied as the real parameter K varies from 0 to ∞. In addition, the root locus of $G(s)$ is the set of points in the s-plane where the phase of $G(s)$ is 180^0.

- Computing the phase of a transfer function is relatively easy.
- If we pick a test point, s_0, it will be a solution of $1 + KG(s) = 0$ if its magnitude and phase satisfy the earlier conditions:

$$\boxed{|KG(s)| = 1 \qquad \text{and} \qquad \angle KG(s) = 180^o + 360^o(l-1)}$$

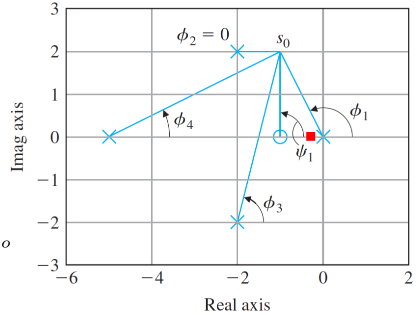
- If we further define the angle to the test point from a zero as ψ_i and the angle to the test point from a pole as ϕ_i, then we have

$$\sum \psi_i - \sum \phi_i = 180^o + 360^o(l-1)$$

Example

- Consider the following transfer function, with poles and zero shown below.

$$G(s) = \frac{s+1}{s(s+5)[(s+2)^2+4]}$$



- If we select the test point $s_0 = -1 + 2j$.
- The phase is given by

$$\begin{aligned}
\angle G &= \angle(s_0+1) - \angle s_0 - \angle(s_0+5) \\
&\quad -\angle[(s_0+2)^2+4] \\
&= \psi_1 - \phi_1 - \phi_2 - \phi_3 - \phi_4 \\
&= 90^o - 116.6^o - 0^o - 76^o - 26.6^o \\
&= -129.2^o
\end{aligned}$$

$\rightarrow$ s_0 is not on the root locus.

- If we now choose the test point, s_1 to be at the red square, the phase is given by

$$\begin{aligned}
\angle G &= \psi_1 - \phi_1 - \phi_2 - \phi_3 - \phi_4 \\
&= 0^o - 180^o - 307^o - 53^o - 0^o = -540^o
\end{aligned}$$

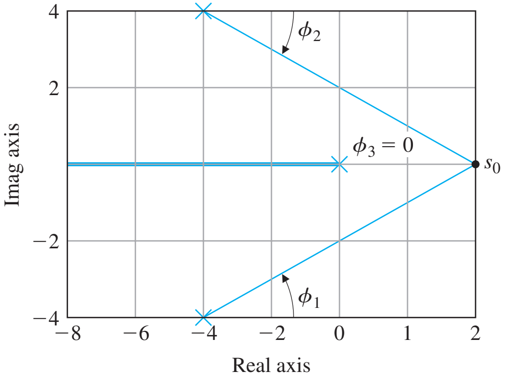
$\rightarrow$ s_1 is on the root locus.

Rules for plotting root locus

Rule 1 Mark n poles and m zeros. The n branches of the locus start at the poles of $G(s)$ and m of these branches end on the zeros of $G(s)$.

Rule 2 Draw the locus on the real-axis to the left of an **odd** number of real poles plus zeros.

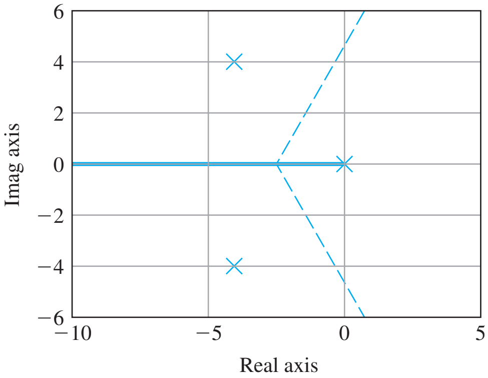
$$G(s) = \frac{1}{s[(s+4)^2 + 16]}$$



Rules for plotting root locus

Rule 3   Draw $n - m$ radial asymptotes centered at α, with angles ϕ_l

$$
\begin{aligned}
\alpha &= \frac{\sum p_i - \sum z_i}{n-m}, \\
\phi_l &= \frac{180^0 + 360^0(l-1)}{n-m}, \qquad l = 1, 2, \ldots, n-m.
\end{aligned}
$$



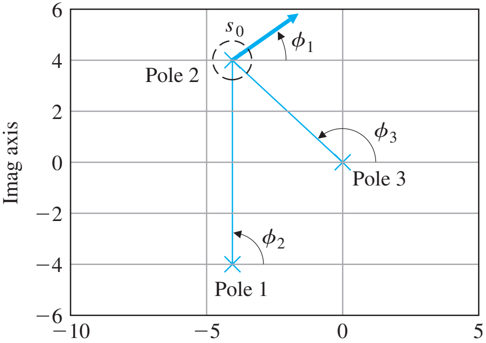
Rules for plotting root locus

Rule 4 Compute departure angles from poles and arrival angles to zeros

$$q\phi_{l,dep} = \sum \psi_i - \sum_{i \neq l} \phi_i - 180^0 - 360^0(l-1),$$

$$q\psi_{l,arr} = \sum \phi_i - \sum_{i \neq l} \psi_i + 180^0 + 360^0(l-1).$$

where q: order of poles/zeros



Rules for plotting root locus

Rule 5 Let $s = j\omega_0$, compute points where locus crosses the imaginary axis.

$$\begin{aligned} 1 + \frac{K}{s[(s+4)^2 + 16]} &= 0 \\ s^3 + 8s^2 + 32s + K &= 0 \end{aligned}$$

Substitute $s = j\omega_0$,

$$(j\omega_0)^3 + 8(j\omega_0)^2 + 32(j\omega_0) + K = 0$$

Solving for the real and imaginary parts,

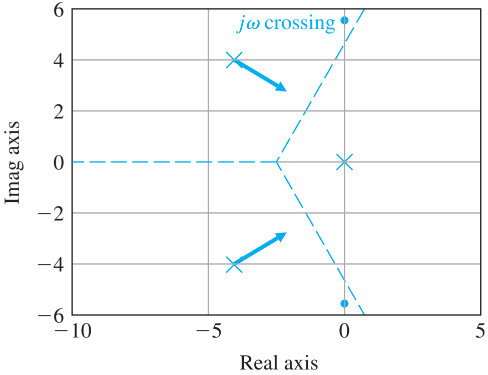
$$-\omega_0^3 + 32\omega = 0 \quad \text{and} \quad -8\omega_0^2 + K = 0$$

we have

$$\omega_0 = \sqrt{32} = 5.66 \quad \text{and} \quad K = 256$$

Rules for plotting root locus

Rule 5 Let $s = j\omega_0$, compute points where locus crosses the imaginary axis.



Rules for plotting root locus

Rule 6 Points along real-axis where the locus breakaway

$$\frac{dK}{ds} = 0.$$

Note that this is a necessary but not sufficient condition to indicate multiple root situation.

In our example, there is no breakaway point, the closed loop poles are

$$\begin{aligned} 1 + \frac{K}{s[(s+4)^2 + 16]} &= 0 \\ K &= -s^3 - 8s^2 - 32s \\ \frac{dK}{ds} &= -3s^2 - 16s - 32 = 0 \end{aligned}$$

Solving, we have $s = -2.67 \pm 1.89j$, however, these points are not on the root locus.

Final plot:

Matlab command:
numG = [1];
denG = [1 8 32 0];
sysG =
tf(numG,denG);
rlocus(sysL)