EE3331C/EE3331E Feedback Control Systems
L2: Review of Signals & Systems

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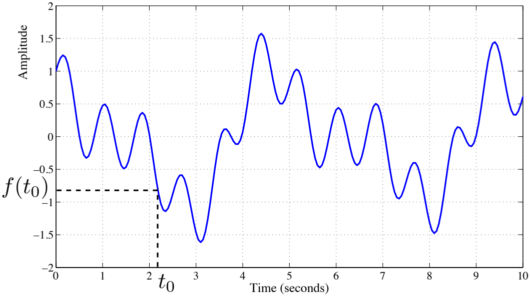
Outline I

Outline II

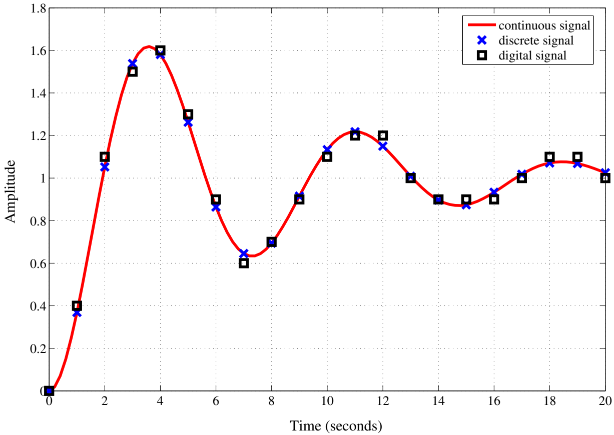
Notation and meaning

- A signal is a function of time, e.g., v_{out} is the circuit output voltage.
- Notations:
  - f or $f(.)$ refer to the whole signal or function
  - $f(t)$, $v_{out}(1.2)$, $p(t+1)$ refer to the value of the signals at times t, 1.2, and $t+1$ respectively.



- Physical units of a signal, e.g., V, mA, m/s
  sometimes the physical units are 1 (i.e. unitless) or unspecified.

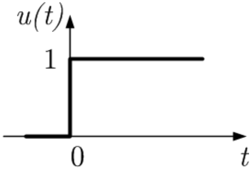
- Signals are classified as continuous-time signals and discrete-time signals based on the continuity of the independent variable.
- Many human-made signals are discrete, e.g., stock index, digital image., which can be processed by modern digital computers.



Elementary Signals

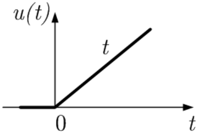
- To increase our understanding of physical processes, signal are described mathematically using basic building blocks known as elementary signals, e.g., step function, impulse function, ramp function, sinusoidal function, exponential function etc.
- All signals can be constructed by combining (add or multiply) elementary signals.
- A constant (or static or DC) signal: $u(t) = k$, where k is some constant.
- The unit step signal (sometimes denoted as $1(t)$ or $U(t)$),

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



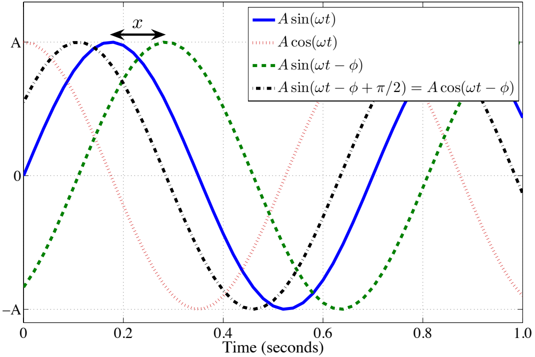
- The unit ramp signal,

$$u(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$$



- A sinusoidal signal,

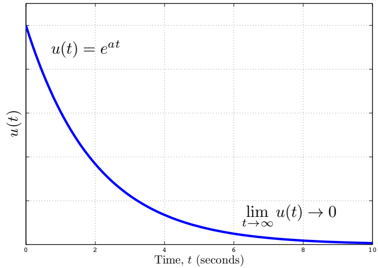
$$\begin{aligned} u(t) &= A\cos(\omega t \pm \phi) \\ \text{or} \quad & A\sin(\omega t \pm \phi) \end{aligned}$$

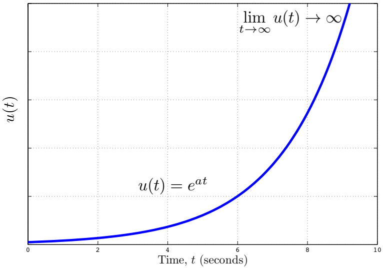


- An exponential signal,

$$u(t) = e^{at}$$

signals decays to zero when $a < 0$ and grow to ∞ when $a > 0$.





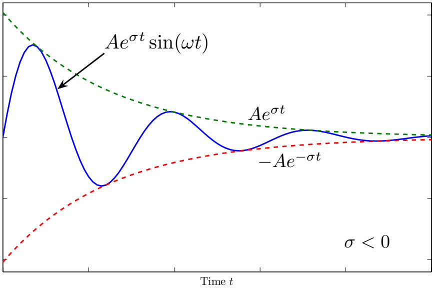
Euler's identity, $e^{j\phi} = \cos\phi + j\sin\phi$

$$A\cos(\omega t + \phi) = \Re\{Ae^{j\omega t}e^{j\phi}\}$$
$$A\sin(\omega t + \phi) = \Im\{Ae^{j\omega t}e^{j\phi}\}$$

- Complex exponential function, Ae^{zt} where $z = \sigma + j\omega$.
  - obtained by multiplying a sinusoidal signal with an exponential signal

$$\begin{aligned} y(t) &= Ae^{(\sigma+j\omega)t} = Ae^{\sigma t}e^{j\omega t} \\ &= Ae^{\sigma t}[\cos(\omega t) + j\sin(\omega t)] \end{aligned}$$

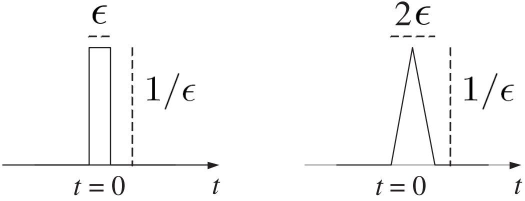
$$\Re\{Ae^{zt}\} = Ae^{\sigma t}\cos(\omega t) \qquad \Im\{Ae^{zt}\} = Ae^{\sigma t}\sin(\omega t)$$



  - if $\sigma < 0$, then signal is exponentially damped

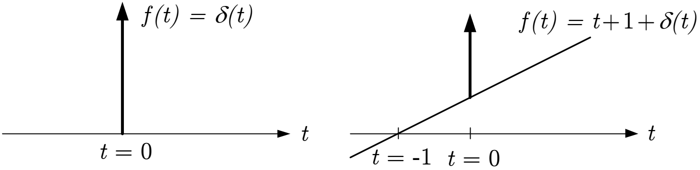
Impulsive Signals

- (Dirac's) delta function or impulse δ is an idealization of a signal that
  - is very large near $t = 0$
  - is very small away from $t = 0$
  - has area under the curve equal 1



  - the exact shape of the function doesn't matter
  - ϵ is small (which depends on context)

- On plots, δ is shown as a solid arrow:



- The δ function is defined with the following property

$$\int_a^b f(t)\delta(t)dt = f(0)$$

provided $a < 0$, $b > 0$, and f is continous at $t = 0$.

- The idea is that δ acts over a time interval very small, over which $f(t) \approx f(0)$
  - $\delta(t) = 0$ for $t \neq 0$
  - $\delta(0)$ isn't really defined

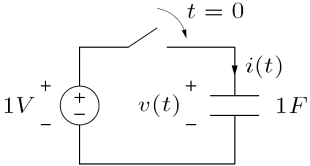
- Scaled impulses & Sifting property:
  - $\alpha\delta(t-T)$ is an impulse at time T, with magnitude α

$$\int_a^b \alpha\delta(t-T)f(t)dt = \alpha f(T)$$

    for $a < T < b$ and f is continuous at T.
  - Example:

$$\begin{aligned}
&\int_{-2}^{2} f(t)(3+\delta(t+1)-2\delta(t-1)+2\delta(t+3))dt \\
&= 3\int_{-2}^{2} f(t)dt + \int_{-2}^{2} f(t)\delta(t+1)dt - 2\int_{-2}^{2} f(t)\delta(t-1)dt \\
&\quad +2\int_{-2}^{2} f(t)\delta(t+3)dt \\
&= 3\int_{-2}^{2} f(t)dt + f(-1) - 2f(1)
\end{aligned}$$

- Physical interpretation:
  Impulse function are used to model physical signals
  - that act over short time intervals
  - whose effect depends on integral of signal
- Example: rapid charging of capacitor[1]

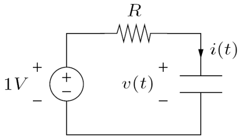


assuming $v(0) = 0$, what is $v(t)$, $i(t)$ for $t > 0$?
  - $i(t)$ is very large, for a very short time
  - a unit charge is transferred to the capacitor 'almost' instantaneously
  - $v(t)$ increase to $v(t) = 1$ 'almost' instantaneously

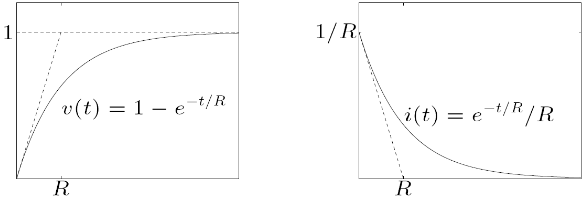
to calculate i and v, we need a more detailed model.

[1] "Linear Systems" by T Kailath, page 14.

- Consider the same circuit with a small resistance added



$$i(t) = \frac{dv(t)}{dt} = \frac{1 - v(t)}{R}, \quad v(0) = 0$$



as $R \to 0$, i approaches an impulse, v approaches a unit step!

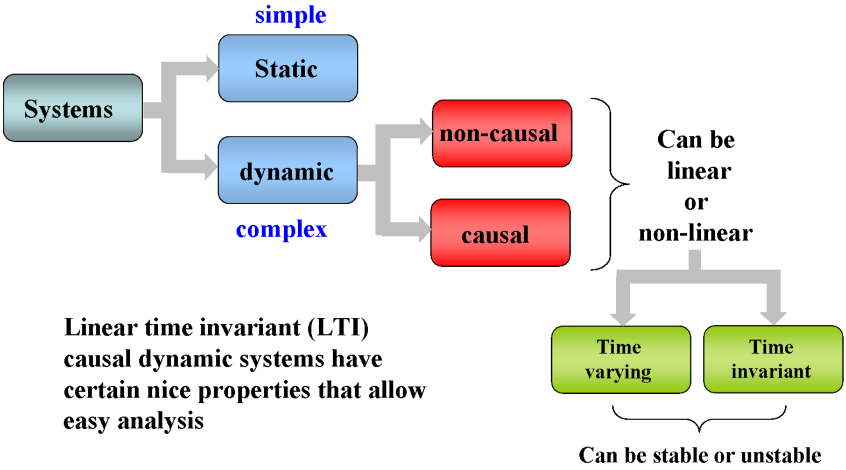
- What is the relationship between the unit step and the unit impulse?

Notation and meaning

- A system can be viewed as a process in which input signals are transformed by the system or cause the system to respond in some way, resulting in other signals as output.
- Examples:
  - Hi-fi system with tone controls, change in tonal quality of the reproduced signal.
  - The previous RC circuit can be viewed as a system with input voltage, $v_i(t)$ and output voltage, $v(t)$.
  - An automobile can be thought of as a system with pressure on accelerator as input and speed as output.
  - An image-enhancement system transforms an input image into an output image that has some desired properties.
- Notation:

  $y = Gu$ or $y = G(u)$ means that the system, G, acts on input signal, u, to produce output signal, y.

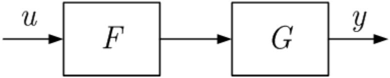
Classification of Systems



Linear time invariant (LTI) causal dynamic systems have certain nice properties that allow easy analysis

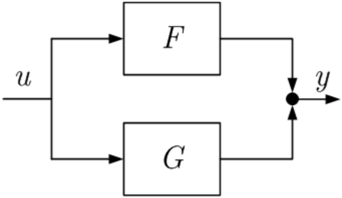
Interconnection of systems

- We can connect systems to form new systems, e.g..
  - cascade (or series): $y = G(Fu) = GFu$

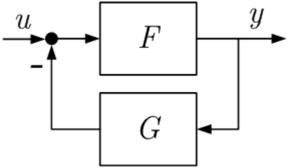


Note: block diagram and algebra are reversed!!

  - sum (or parallel): $y = Fu + Gu$

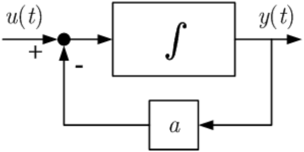


- feedback: $y = F(u - Gy)$



  - the minus sign is just a tradition, it is often not there
  - in general, block diagram are just a symbolic way to describe a connection of systems; we can just as well write out the equations relating the signals

- Example:



  - input to the integrator is $u - ay$, we then have

$$\int^t (u(\tau) - ay(\tau))d\tau = y(t)$$

    we will be able to give an explicit expression for y in terms of u in the next lecture (via Laplace transform)
  - alternatively, the input to the integrator is the derivative of its output, so we have

$$u - ay = y'$$

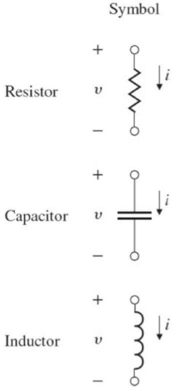
System model

When we interact with a *system*, we need some concept of how its variables (input, output, etc.) relate to each other $\rightarrow$ *Model*.

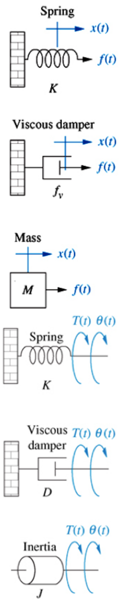
- various shapes
- phrased with varying degrees of mathematical formalism
- degree of sophistication depend on the intended use
- acceptance of model: **"usefulness"** rather than **"truth"**
- why 'model'?
  <u>easier, cheaper, safer, or faster than working with a real system</u>

Differential Equations Model of Physical Systems

- As discuss earlier, the type of systems that we are interested in this module are LTI and can be represented by linear d.e. with constant coefficients.
- Electrical Circuits:
  - Resistor: $v_R(t) = Ri_R(t)$
  - Capacitor: $v_C(t) = \dfrac{1}{C}\displaystyle\int_0^t i_C(\tau)d\tau$ or $i_C(t) = C\dfrac{dv_C(t)}{dt}$
  - Inductor: $v_L(t) = L\dfrac{di_L(t)}{dt}$



- Linear motions (force-displacement relationship)
  - Mass: $f(t) = M\dfrac{d^2x(t)}{dt}$
  - Spring: $f(t) = Kx(t)$
  - Damper: $f(t) = f_v\dfrac{dx(t)}{dt}$
- Angular motions (torque-angular displacement relationship)
  - Inertia: $T(t) = J\dfrac{d^2\theta(t)}{dt}$
  - Spring: $T(t) = K\theta(t)$
  - Damper: $T(t) = D\dfrac{d\theta(t)}{dt}$



Figures from *Nise, Control Systems Engineering, Wiley.*

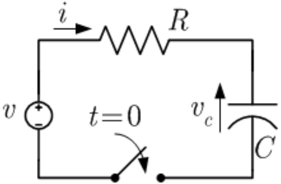
Example 1: RC circuit

- Consider the following series RC circuit:
  - input is the applied voltage, $v(t)$
  - output is the voltage across the capacitor, $v_c(t)$

- initial voltage: $v_c(0)$

- KVL yield:

$$
\begin{aligned}
v_R(t) + v_c(t) &= v(t) \\
Ri(t) + v_c(t) &= v(t)
\end{aligned}
$$



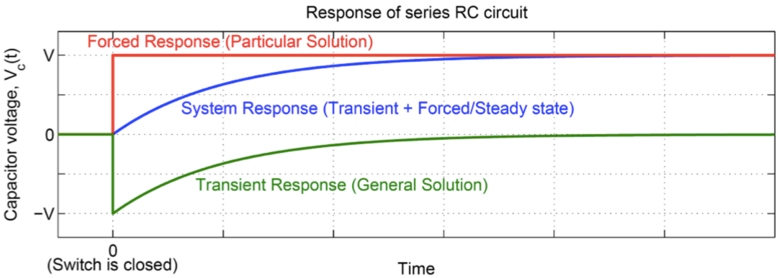
Since $v_c(t) = \frac{1}{C}\int i(\tau)d\tau$

or $i(t) = C\frac{dv_c(t)}{dt}$, we have

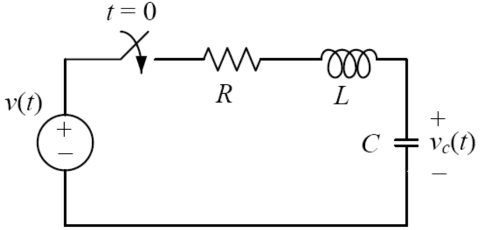
$$RC\frac{dv_c(t)}{dt} + v_c(t) = v(t)$$

- Solving the first-order differential equation and given that the input voltage $v(t) = V$, we have

$$v_c(t) = \underbrace{(v_c(0) - V)e^{-\frac{t}{RC}}}_{\text{transient}} + \underbrace{V}_{\text{Forced/Steady-state}}$$

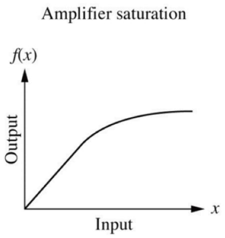


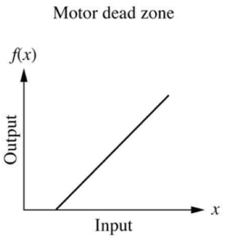
- Next, consider the following series RLC circuit, find the differential equation describing the input, $v(t)$ and output, $v_c(t)$.



Linearization

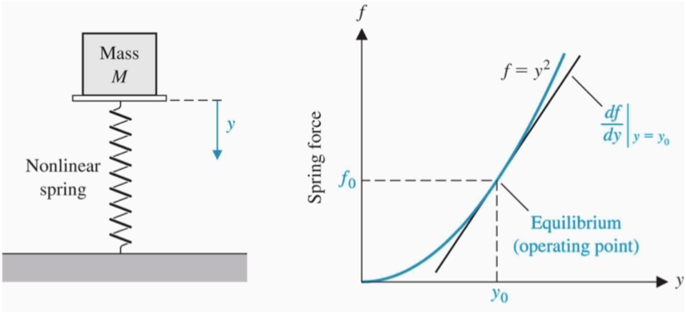
- Many physical systems are nonlinear in nature. To continue to use the tools that you have learnt previously such as Laplace Transform, we can linearize the nonlinear system around an operating point.
- The approach presented here is based on the expansion of the nonlinear function into a Taylor series about the operating point and the retention of only the linear term.
- Because we ignore the higher order terms of the Taylor series expansion, these neglected terms must be small, i.e., the variables must deviate only slightly from the operating condition.





Linearization

- Example: consider the case of a mass-spring system shown below,



- The normal operating point is the equilibrium position that occurs when the spring force balances the gravitational force, Mg, where g is the gravitational constant, i.e., $f_0 = Mg$.

Linearization

- For the nonlinear spring with $f = y^2$, the equilibrium position is

$$y_0 = \sqrt{Mg} = (Mg)^{1/2}$$

- Linear model for a small deviation is obtained by taking the truncated (1st order) Taylor Series Expansion about the operating point, y_0,

$$\begin{aligned} f(y) - f(y_0) &\approx \left.\frac{df}{dy}\right|_{y=y_0} (y - y_0) \\ \Delta f(y) &= \left.\frac{df}{dy}\right|_{y=y_0} \Delta y = m\Delta y \end{aligned}$$

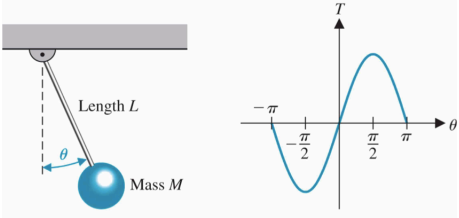
  This equation shows a linear relationship between $\Delta f(y)$ and Δy for small excursions away from y_0. In this example, $m = 2y_0$.

- A linear approximation is as accurate as the assumption of small signals is applicable to the specific problem.

Example 2: Pendulum oscillator model

- Consider the pendulum oscillator, the torque on the mass is given by

$$T = MgL\sin\theta$$



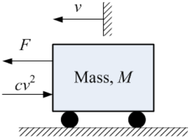
- The equilibrium condition for the mass is $\theta_0 = 0^o$ and $T_0 = 0$, the first derivative evaluated at equilibrium provides the linear approximation,

$$\begin{aligned} T - T_0 &= MgL \left.\frac{d\sin\theta}{d\theta}\right|_{\theta=\theta_0} (\theta - \theta_0) \\ T &= MgL(\cos 0^o)(\theta - 0^o) \\ T &= MgL\theta \end{aligned}$$

- reasonably accurate for $-\pi/4 \leq \theta \leq \pi/4$.

Example 3: Nonlinear drag force

- Consider the following dynamic system with nonlinear drag force



where

- v is the velocity of the car moving on a horizontal surface
- F is the force created by cars engine to propel it forward
- c is a coefficient that relates velocity-dependent wind resistance force to the velocity. Its value depends on cars coefficient of drag, frontal area as well as density of air the car is travelling through
- M is the mass of the car

The nonlinear differential equation model is given by

$$M\frac{dv}{dt} = F - cv^2$$

Example 3: Nonlinear drag force

- Assume that the car is moving at a constant velocity of $v_0 = 90$ km/h and $M = 1000$ kg, $c = 0.4$ N/(m/s)^2.
  - we now linearize the nonlinear ODE around the operating point, $v_0 = 90$ km/h = 25 m/s

$$M\frac{dv}{dt} = F - cv^2 \Rightarrow 1000\frac{dv}{dt} = F - 0.4v^2$$

  - at the operating point, the car velocity is constant, i.e., acceleration is zero

$$\begin{aligned} 1000 \times 0 &= F_0 - 0.4v^2 \\ F_0 &= 0.4v^2 = 0.4 \times 25^2 = 250 \text{ N} \end{aligned}$$

- We next define two new variables δv and δF, which represent small deviation in motor velocity and applied force at the operating point,

$$v = v_0 + \delta v, \quad F = F_0 + \delta F \quad \text{and} \quad \frac{dv}{dt} = \frac{d(v_0 + \delta v)}{dt} = \frac{d(\delta v)}{dt}$$

Example 3: Nonlinear drag force

- Substituting v, F and $\frac{dv}{dt}$ into the nonlinear ODE, and applying Taylor series expansion of v^2.

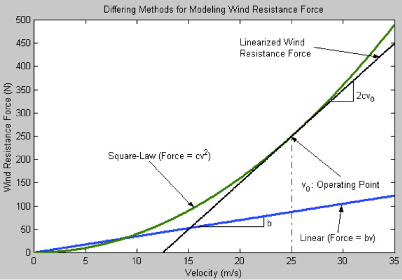
$$1000\frac{d(\delta v)}{dt} + 0.4v^2 = F_0 + \delta F$$

$$1000\frac{d(\delta v)}{dt} + 0.4\left(v_0^2 + \left.\frac{d(v^2)}{dv}\right|_{v=v_0}\delta v\right) = F_0 + \delta F$$

$$1000\frac{d(\delta v)}{dt} + 0.8v_0\delta v = F_0 - 0.4v_0^2 + \delta F$$

$$1000\frac{d(\delta v)}{dt} + 0.8v_0\delta v = \delta F$$

$$1000\frac{d(\delta v)}{dt} + 20\delta v = \delta F$$



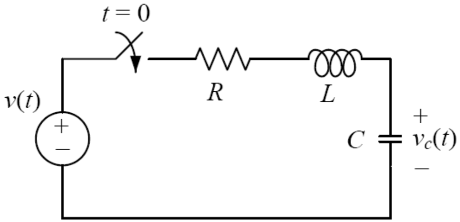
$\rightarrow$ linear ODE in terms of variables δv and δF!

Summary

- Signals: definition of basic signals (step, impulse, etc), physical interpretations, qualitative properties.
- Systems: types of model, system properties (linearity, time-invariance, causal, etc.)
- Focus will be on linear time-invariant systems.

Review Questions

- Find the differential equation describing the following RC circuit.



Reading: introductory chapters in any signal and systems books

Practice Problems

1. Consider the step signal and impulse signal introduced. Show that you get one from the other via integration and differentiation.
2. Complex exponential signals plays an important role in control systems, discuss the role of σ and ω in the signal on page 2-9. Sketch different values of σ and ω (Time to learn Matlab!).
3. Small signals. Consider the following signals described by

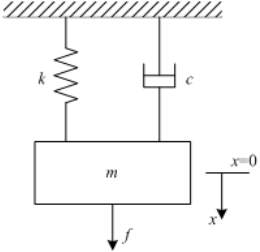
$$u(t) = \begin{cases} 1/\sqrt{d}, & 0 \leq t \leq d \\ 0, & d < t < 1 \end{cases}$$

for $0 \leq t < 1$, and periodic with period 1 (i.e., the signal repeats every second). Sketch the signal for a few values of d. As $d \to 0$, is the signal getting smaller or larger?

Practice Problems

4. Consider the following mass-spring-damper system. Assume that the mass is moving downward from zero position (shown by the arrow) due to the force applied. It makes the spring stretched so that spring force opposes the acceleration of the mass. Damper also opposes acceleration.
   - spring force is proportional to the displacement, i.e., $F_{\text{spring}} = kx$
   - damper force is proportional to the velocity, i.e., $F_{\text{damper}} = c\dot{x}$

   Derive the differential equation model with f as input and x as output.



Static vs. Dynamic

- System is static (memoryless) if the output signal $y(t)$ at time t depends only on the value of the input at time t. i.e.

$$y(t) = Ku(t)$$

example is a resistor, $V(t) = i(t)R$.

- The response of a static system is instantaneous and does not depend on previous inputs.
- Output of a dynamic system (non-zero memory) at time t depends on the past or future values of the input $u(t)$ in addition to the present time, i.e.,

$$y(t) = f\left\{\ldots, u(t-1), u(t), u(t+1), \ldots\right\}$$

examples include

Capacitor, $v(t) = \dfrac{1}{C}\displaystyle\int_0^t i(\tau)d\tau$ Inductor, $v(t) = L\dfrac{di(t)}{dt}$

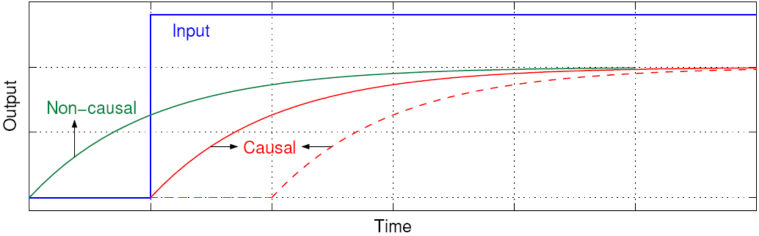
Causal vs. non-causal systems

- A system is causal (non-anticipative) if the output at any time depends only on values of the input at the present time and in the past.

$$y(t) = f\left\{u(t), u(t-1), \ldots, u(t-k)\right\}$$

  - e.g. a RC circuit is causal, the capacitor voltage responds only to the present and past values of the source voltage
  - which is causal? $y_1(t) = u^2(t-1)$, $y_2(t) = u(t+1)$.
  - all real-time physical systems are causal, because time only moves forward; effect occurs after cause

- Causality implies that system does not respond to an input event until that event actually occurs i.e. the response to an event beginning at $t = t_0$ is non-zero only for $t \geq t_0$.



- All memoryless systems are causal, since the output responds only to the current value of the input.

Linearity

- Most physical systems are nonlinear. e.g. many circuit elements (e.g., diodes), dynamics of aircraft, robotics, manufacturing system etc.
- Our focus in this course is linear system, why?
  - linear models provide accurate representations of behavior of many systems (e.g., linear resistors, capacitors, etc.)
  - can often linearize models to examine "small signal" perturbations around "operating points"
  - many important tools based on linear systems
- Examples: $y_1(t) = 2x(t)$, $y_2(t) = x^2(t)$.
- Differential equations examples:

$$y_1(t) + a\frac{dy_1(t)}{dt} = u(t)$$

$$y_2(t) + a\frac{dy_2(t)}{dt} = \sin(u(t))$$

- A system G is **linear** if the following two properties hold:
  1. **homogeneity**: if u is any signal and a is any scalar,

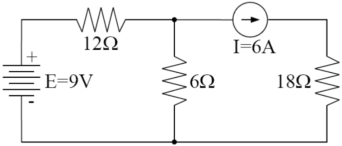
     $$G(au) = aG(u)$$

  2. **superposition**: if u_1 and u_2 are any two signals,

     $$G(u_1 + u_2) = Gu_1 + Gu_2$$

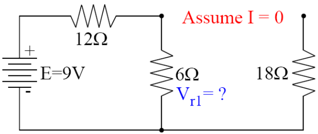
  (watch out – just a few symbols here express a very complex meaning), in words, linearity means:
  - scaling before or after the system is the same
  - summing before or after the system is the same

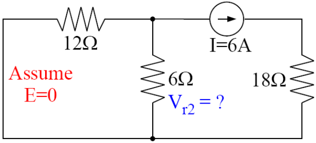
- Example: Recall how circuits with multiple independent current/voltage sources are analysed.



Two independent sources, E and I. What is voltage across 6Ω resistor, V_r?

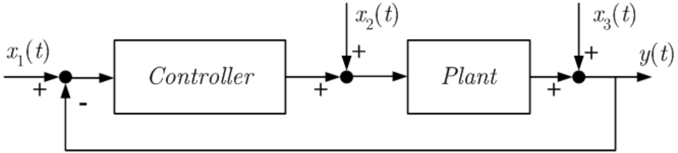
1. "kill" the current source (let $I = 0$), find V_{r1}





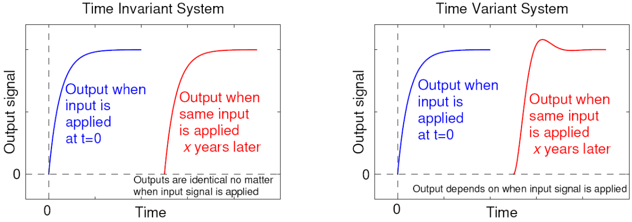
2. "kill" the voltage source (set $E = 0$), find V_{r2}
3. sum solutions i.e. $V_r = V_{r1} + V_{r2}$

- Suppose a linear system is perturbed by n independent input signals, $x_i(t)(i = 1, \ldots, n)$. System output, $y(t)$, is the algebraic sum of the n outputs $y_i(t)(i = 1, \ldots, n)$ produced by the inputs acting separately $(x_i(t) \neq 0$ & $x_{j\neq i}(t) = 0)$.
  - example:



Time Invariance

- System is time invariant if a time delay/advance of the input signal leads to an identical time shift in the output signal i.e. output remains the same regardless of when input is applied.



i.e. if $y(t)$ correspond to input $u(t)$, then a time invariant system will have $y(t-\tau)$ as the output when the input is $u(t-\tau)$. E.g. Consider $y_1(t) = 2u^2(t-1)$ and $y_2(t) = \cos(t)u(t)$.

- Invariant processes can be described by differential equation with constant coeffcients.