EE3331C/EE3331E
Feedback Control Systems
Part II: Frequency Response Methods

Chapter 3: Stability Analysis

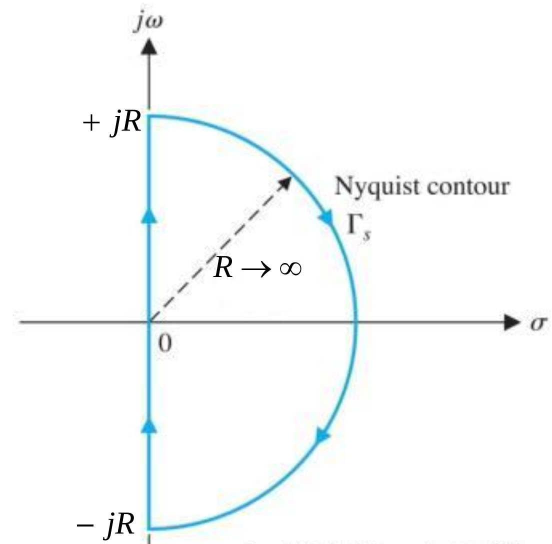
Part 3B – Nyquist Stability Criterion

NSC: The Concept

- Closed loop transfer function

$$G_{CL}(s) = \frac{L(s)}{1 + L(s)} = \frac{L(s)}{F(s)}$$

- Stability check: Is there any pole of G_{CL} in the RHP?



- **Nyquist Contour**:
  - A closed contour that surrounds the entire RHP
  - Radius of the semicircle R→∞

- Stability check objective using the Nyquist contour:

Is there any pole of G_{CL} in the area bounded by the Nyquist contour?

NSC : The Concept

$$G_{CL}(s) = \frac{L(s)}{1+L(s)} = \frac{L(s)}{F(s)}$$

- **A pole of $G_{CL}(s)$ is a zero of $F(s)$**

$$G_{CL}(s_1) = \infty \qquad \Rightarrow \qquad F(s_1) = 0$$

Transfer Function Pole:
if $G(s_1) = \infty$ then $s = s_1$ is a pole of $G(s)$

Transfer Function Zero:
if $G(s_1) = 0$ then $s = s_1$ is a zero of $G(s)$

- Stability check objective using the Nyquist contour and the transfer function F(s):

Is there any zero of $F(s)$ in the area bounded by the Nyquist contour?

$$F(s) = 1 + L(s)$$

- *Example* 3b-1: Closed loop pole is zero of the transfer function $F(s)$

$$L(s)=\frac{10}{s(s+5)}$$

- Then,

$$F(s)=1+L(s)$$
$$=1+\frac{10}{s(s+5)}$$
$$=\frac{s(s+5)+10}{s(s+5)}$$

$$F(s)=\frac{s^2+5s+10}{s(s+5)}$$

$$s^2+5s+10=0$$
$$s_{1,2}=-2.5\pm j1.94$$

[zeros of F(s)]

- The closed loop transfer function,

$$G_{CL}(s)=\frac{L(s)}{1+L(s)}=\frac{L(s)}{F(s)}$$
$$=\frac{\frac{10}{s(s+5)}}{\frac{s^2+5s+10}{s(s+5)}}$$
$$=\frac{10}{s^2+5s+10}$$

$$s^2+5s+10=0$$
$$s_{1,2}=-2.5\pm j1.94$$

[Closed loop poles]

Closed Loop Stability Check

Search for a pole of $G_{CL}(s)$ in the RHP?

⇓

Search for a pole of $G_{CL}(s)$ inside the Nyquist Contour?

⇓

Search for a zero of F(s) inside the Nyquist Contour?

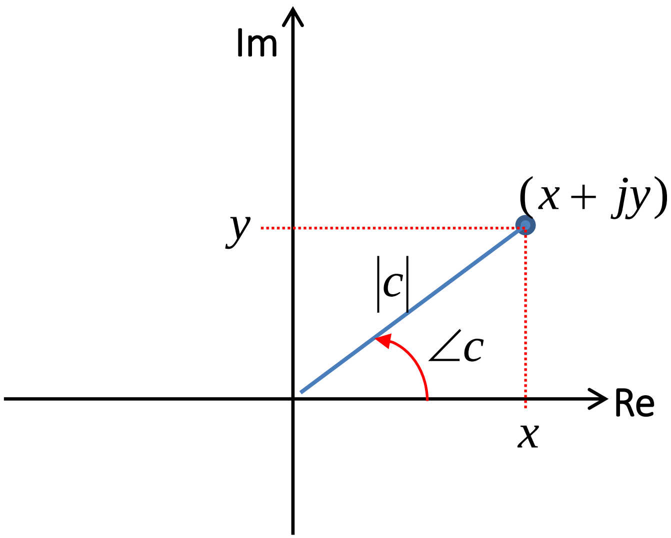
Define a closed contour that surrounds the entire RHP $\Rightarrow$ the Nyquist Contour

A pole of $G_{CL}(s)$ is also a zero of F(s) where F(s) = 1+L(s)

Principle of Argument

Cauchy's Principle of Argument or the Argument Principle

a. Argument of a Complex Number $c = (x + jy)$



- **Modulus** (magnitude) – distance from the origin

$$|c| = \sqrt{x^2 + y^2}$$

- **Argument** – angle between the line drawn from the point to the origin and the positive real-axis

$$\angle c = \tan^{-1} \frac{y}{x}$$

Principle of Argument

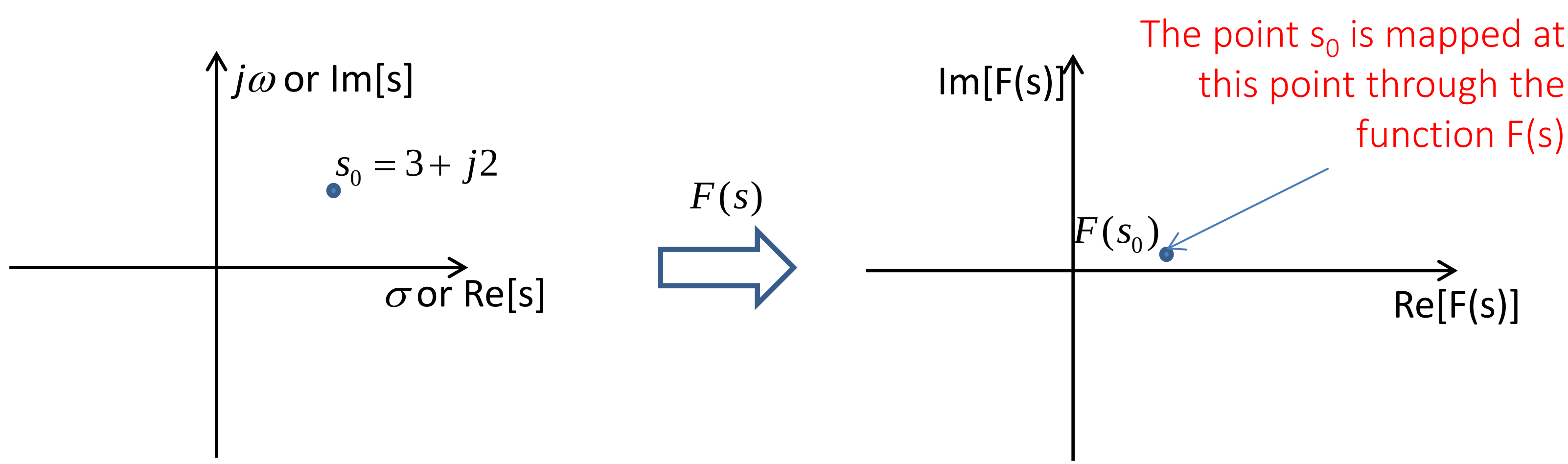
- If a function $F(s)$ of complex variable s is evaluated for $s=s_0$, the result is a complex number $F(s_0)$
- *Example* 3b-2:

$$F(s)=\frac{s+2}{s+3} \qquad s_0=3+j2$$

$$F(s_0)=\frac{5+j2}{6+j2}=\frac{\sqrt{29}\angle 21.8^\circ}{\sqrt{40}\angle 18.4^\circ}$$

$$|F(s_0)|=\frac{\sqrt{29}}{\sqrt{40}}=0.85$$

$$\angle F(s_0)=21.8^\circ-18.4^\circ=3.4^\circ$$



σ-jω plane or s-plane **Re[F]-Im[F] plane or the F(s)-plane**

- The point s_0 is mapped to the point $F(s_0)$
  - Principle of Argument (PoA) deals with the $\angle F(s)$

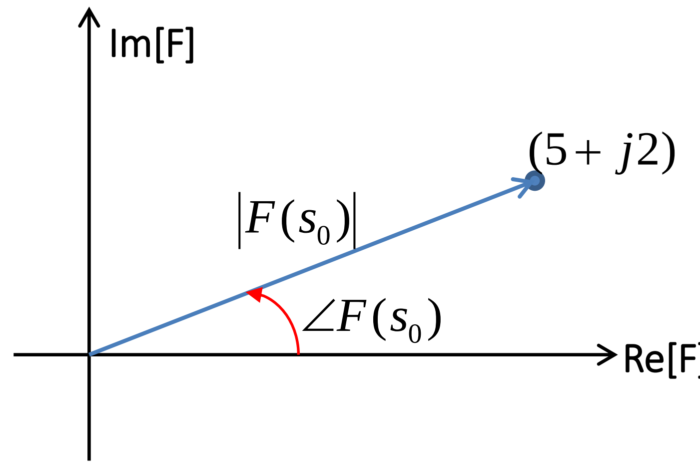
Principle of Argument

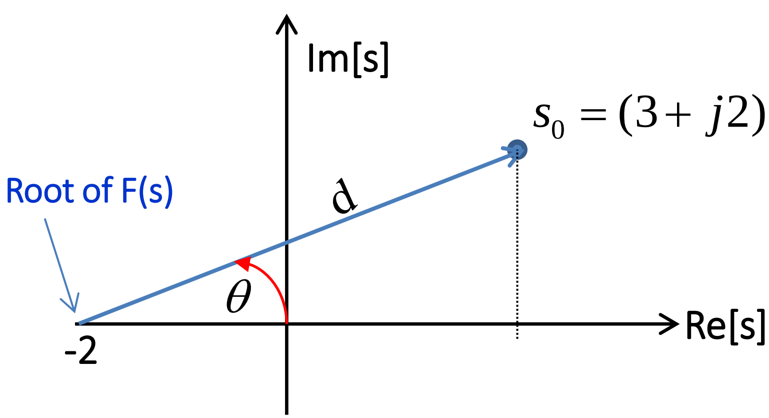
- Consider the point s_0 = 3+j2 mapped to $F(s_0)$ where $F(s)$ has one real-axis zero and no pole, e.g., $F(s)$=(s+2)

$$F(s) = s + 2$$
$$F(s_0) = (3 + j2) + 2$$
$$= 5 + j2$$

$$|F(s_0)| = \sqrt{5^2 + 2^2} = \sqrt{29}$$

$$\angle F(s_0) = \tan^{-1}\frac{2}{5}$$





$|F(s_0)|$ is equal to the distance between s_0 and the root of $F(s)$.

$$d = \sqrt{5^2 + 2^2} = \sqrt{29}$$

$\angle F(s_0)$ is equal to the angle between the line drawn from the root of $F(s)$ to s_0 and the +ve real axis

$$\theta = \tan^{-1}\frac{2}{5}$$

Principle of Argument

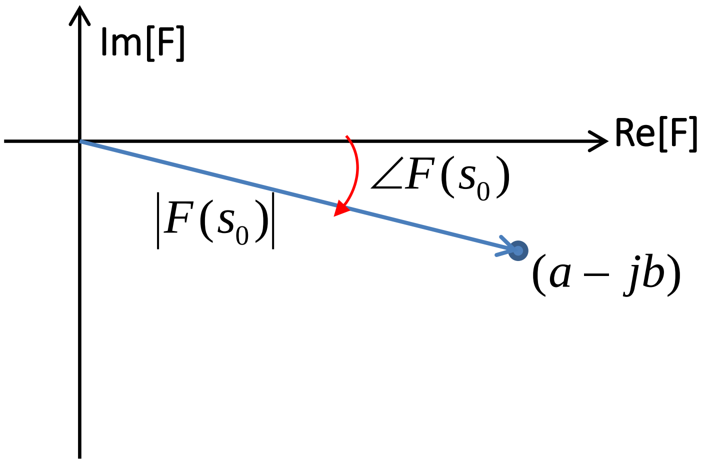
- Now consider the mapping of 3+j2 when $F(s)$ has one real-axis pole and no zero

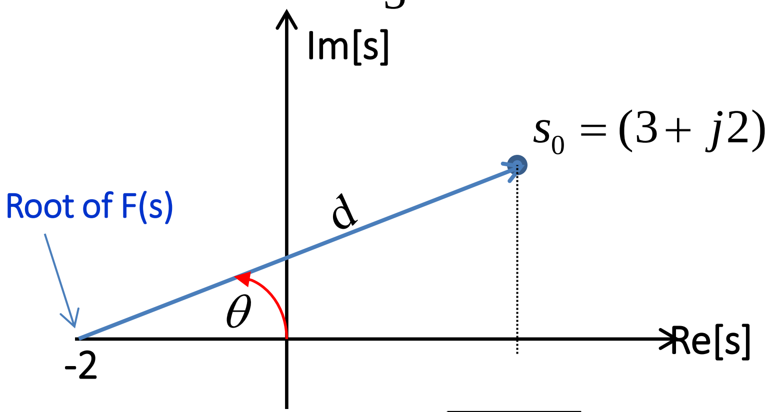
$$F(s)=\frac{1}{s+2}$$

$$F(s_0)=\frac{1}{3+j2+2}=\frac{1}{5+j2}=(a-jb)$$

$$|F(s_0)|=\frac{1}{\sqrt{5^2+2^2}}$$

$$\angle F(s_0)=-\tan^{-1}\frac{2}{5}$$





$$d=\sqrt{5^2+2^2}$$

$$\theta=\tan^{-1}\frac{2}{5}$$

$|F(s)|$ is equal to the reciprocal of d

$$|F(s_0)|=\frac{1}{d}$$

$\angle F(s)$ is equal to -θ

$$\angle F(s_0)=-\theta$$

Principle of Argument

- One more example: $F(s)$ with two complex zeros

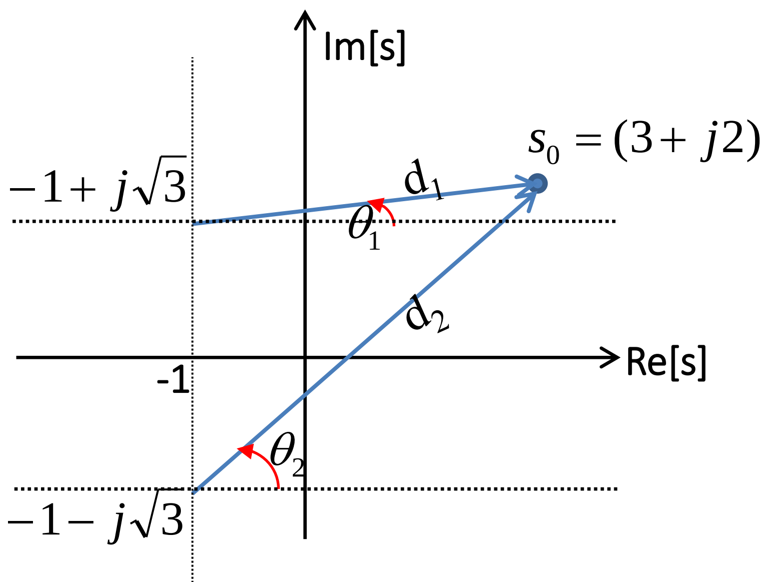
$$F(s) = s^2 + 2s + 4, \quad s_{1,2} = -1 \pm j\sqrt{3}$$

$$s_0 = 3 + j2$$

$$F(s_0) = (3 + j2)^2 + 2(3 + j2) + 4$$
$$= 15 + j16$$

$$|F(s_0)| = \sqrt{15^2 + 16^2}$$
$$= 21.9$$

$$\angle F(s_0) = \tan^{-1}\frac{16}{15}$$
$$= 46.8°$$



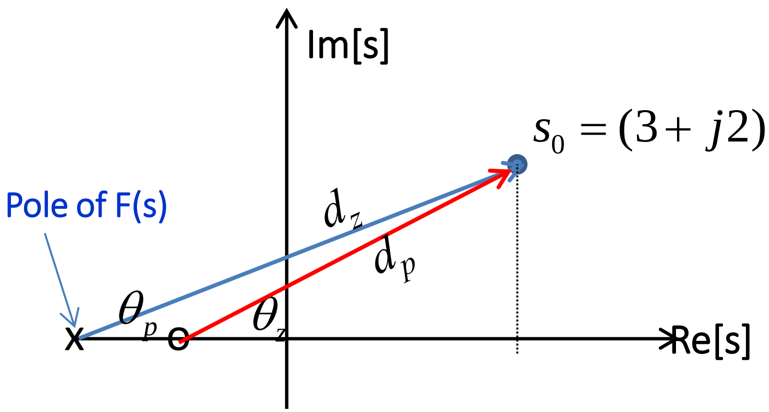
$$d_1 = \sqrt{4^2 + (2 - \sqrt{3})^2} \cong 4$$
$$d_2 = \sqrt{4^2 + (2 + \sqrt{3})^2} \cong 5.47$$
$$d_1 \times d_2 \cong 21.9$$

$$\theta_1 = \tan^{-1}\frac{(2 - \sqrt{3})}{4} = 3.8°$$
$$\theta_2 = \tan^{-1}\frac{2 + \sqrt{3}}{4} = 43.0°$$
$$\theta_1 + \theta_2 = 46.8°$$

Principle of Argument

- Another one: $F(s)$ having one pole and one zero

$$F(s) = \frac{s+1}{s+2}$$



$$F(s_0) = \frac{4 + j2}{5 + j2}$$

$$|F(s_0)| = \frac{\sqrt{4^2 + 2^2}}{\sqrt{5^2 + 2^2}} = \frac{d_z}{d_p}$$

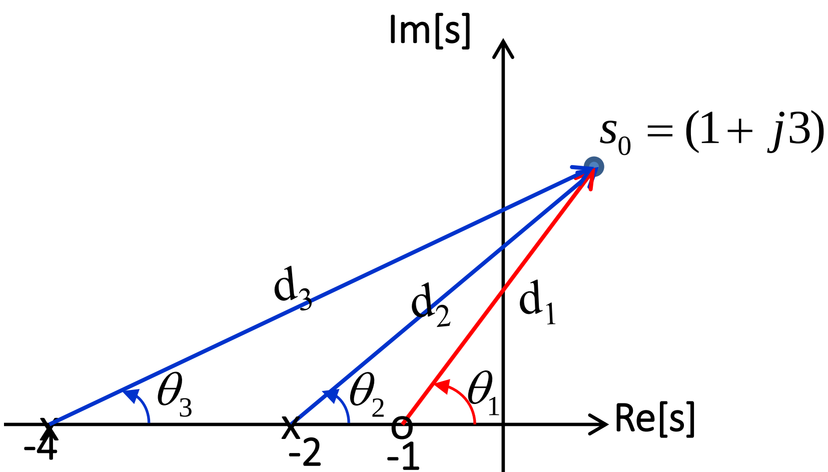
$$\angle F(s_0) = \tan^{-1}\frac{2}{4} - \tan^{-1}\frac{2}{5} = \theta_z - \theta_p$$

- In general, if
  - θ_{z1}, θ_{z2}, θ_{z3}, etc. are the angles formed by the lines drawn from s_0 to the zeros of $F(s)$ and
  - θ_{p1}, θ_{p2}, θ_{p3}, etc. are the angles formed by the lines drawn from s_0 to the poles of $F(s)$, then

$$\angle F(s_0) = (\theta_{z1} + \theta_{z2} + \theta_{z3} + \ldots) - (\theta_{z1} + \theta_{z2} + \theta_{z3} + \ldots)$$

Principle of Argument

$$F(s) = \frac{(s+1)}{(s+2)(s+5)}$$



Red line is used for the ‘zero’

Blue line is used for the ‘pole’

$$F(s_0) = \frac{d_1 \angle \theta_1}{(d_2 \angle \theta_2)(d_2 \angle \theta_2)}$$

$$|F(s_0)| = \frac{d_1}{d_2 \times d_3}$$

$$\angle F(s_0) = \theta_1 - \theta_2 - \theta_3$$

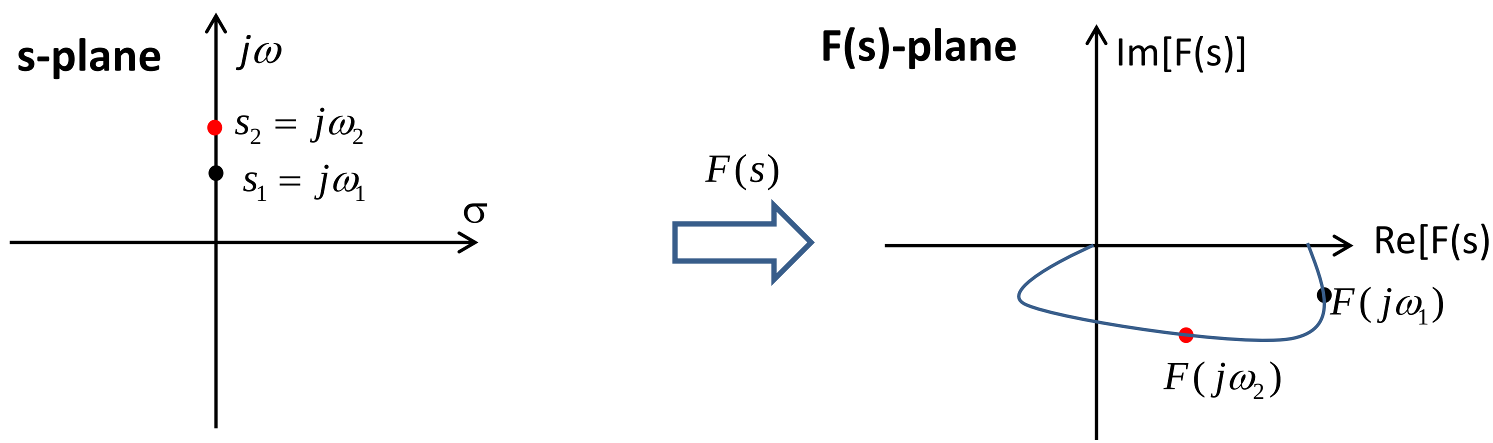
- For the Principle of Argument, we are interested to know the variations in the **argument of $F(s)$** as s_0 is varied around a **closed contour**

Principle of Argument

b. Mapping of a line, i.e., when s is varied

- For example, consider s varied along the +jω-axis

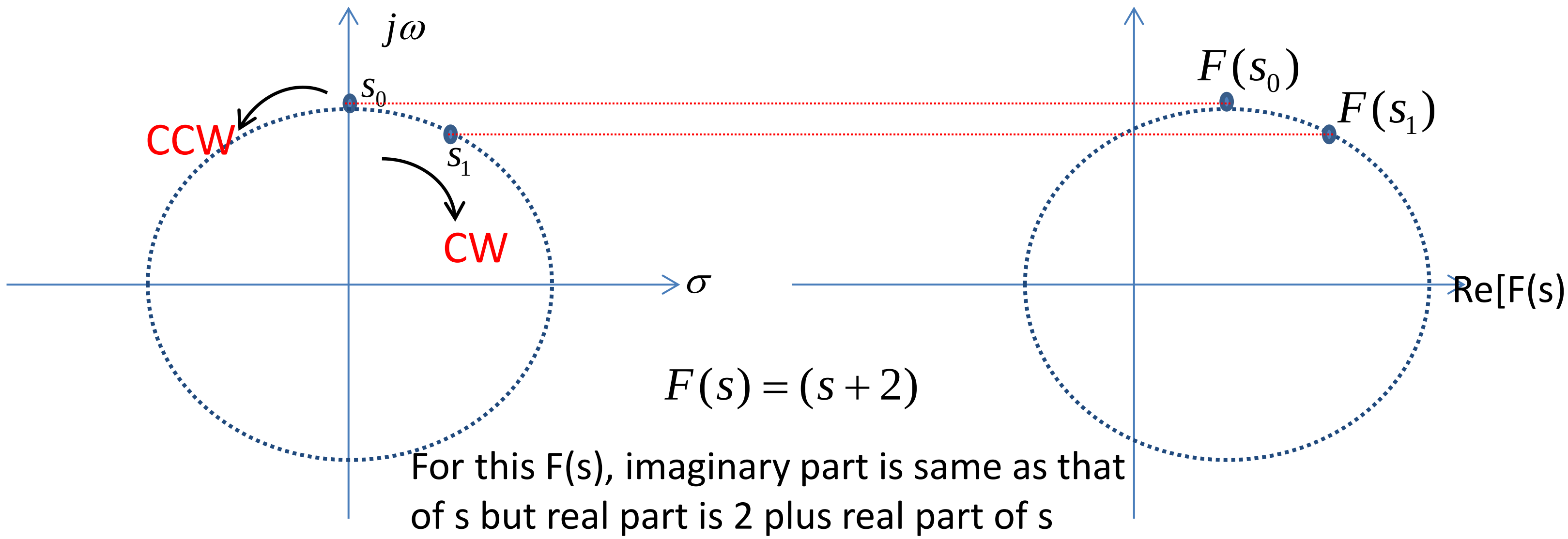
$$F(s)\Big|_{s=+j\omega} = F(j\omega), \quad 0 \le \omega \le \infty$$



- $F(j\omega)$ for ω: 0→∞, is the frequency response of the transfer function $F(s)$
  - Mapping of the $+j\omega$ axis is the frequency response plot on the $F(j\omega)$ plane. It is also called the **polar plot** of $F(j\omega)$.

Principle of Argument

- If s is varied along a closed path, the resulting plot is also closed



- While mapping a closed path, we need to define the direction
  - **Clockwise (CW) or**
  - **Counter-clockwise (CCW)**

- The Nyquist Stability Criterion is based on the properties of the argument of a transfer function while s is varied along the **Nyquist contour** (which is a closed path)

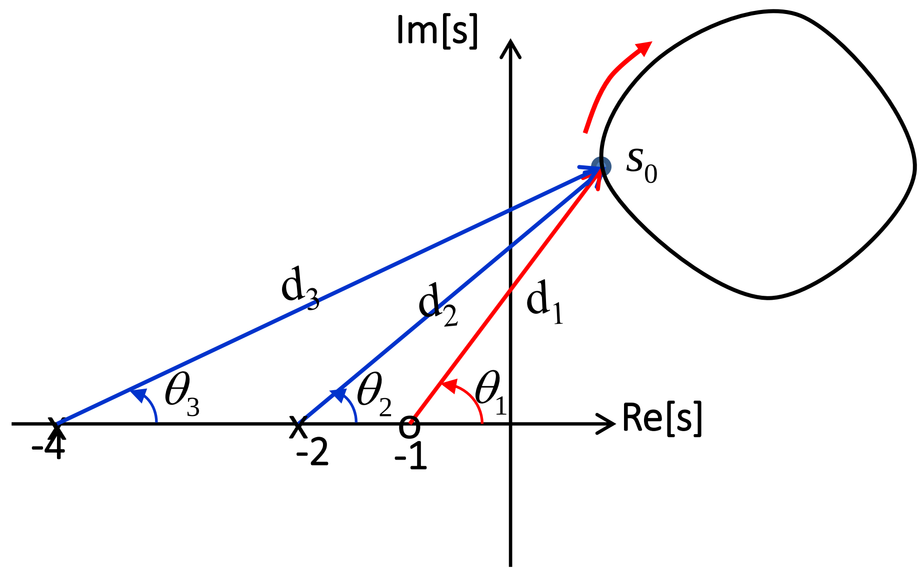
Principle of Argument

- When a closed contour is mapped through a complex function $F(s)$,
  - Variation in $\angle F(s)$ is determined by the variations in angles θ_1, θ_2, θ_3, etc. as the point s_0 is moved around the contour

$$F(s) = \frac{(s+1)}{(s+2)(s+5)}$$

$$|F(s_0)| = \frac{d_1}{d_2 \times d_3}$$

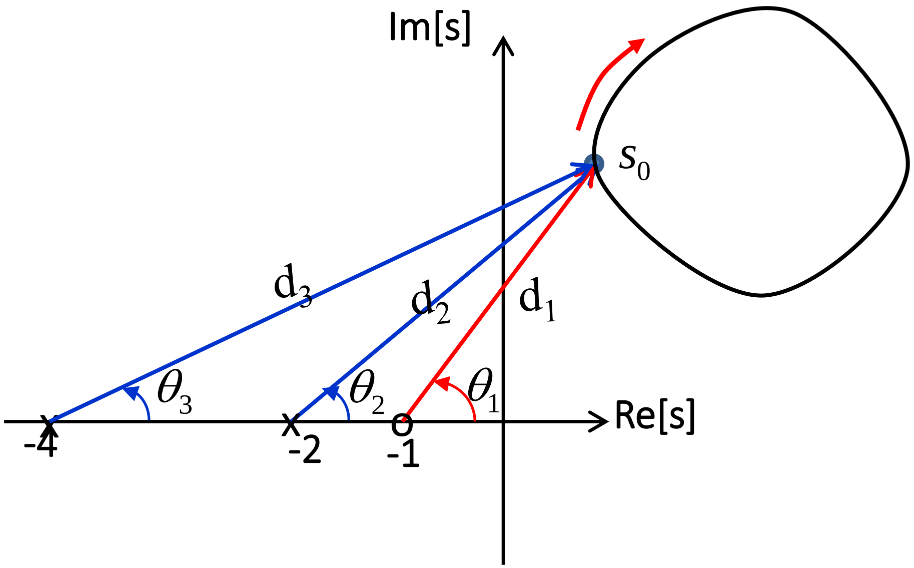
$$\angle F(s_0) = \theta_1 - \theta_2 - \theta_3$$

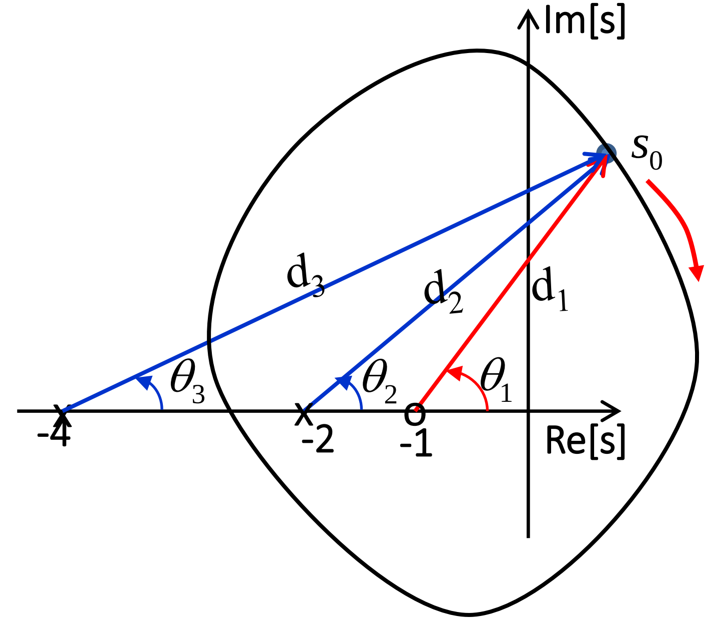


  - How $\angle F(s)$ is varied depends on the positions of the poles and zeros of $F(s)$ with respect to the closed contour
    - Is a pole (or zero) inside the contour or outside?

Principle of Argument

- Assume, the point s_0 is moved around the contour in CW direction





All poles and zeros are outside the closed contour $\Rightarrow$ pole or zero is **not enclosed** by the contour

Variations in all the angles (θ_1, θ_2, and θ_3) are less than 360° as s_0 is moved around the contour

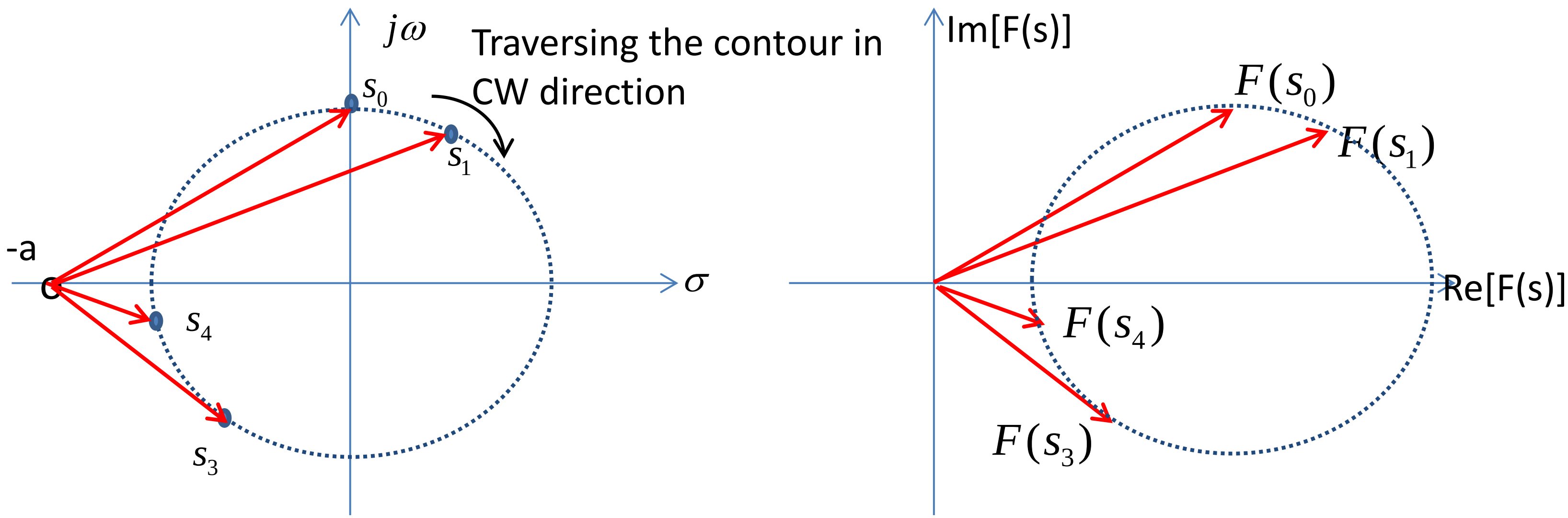
One pole and one zero are inside the closed contour (**enclosed** by the contour)

θ_1 and θ_2 goes through net change of 360° as s_0 is moved around the contour

Principle of Argument

- *Illustration # 1*: $F(s)$ with a single zero outside the contour

$$F(s) = (s + a)$$



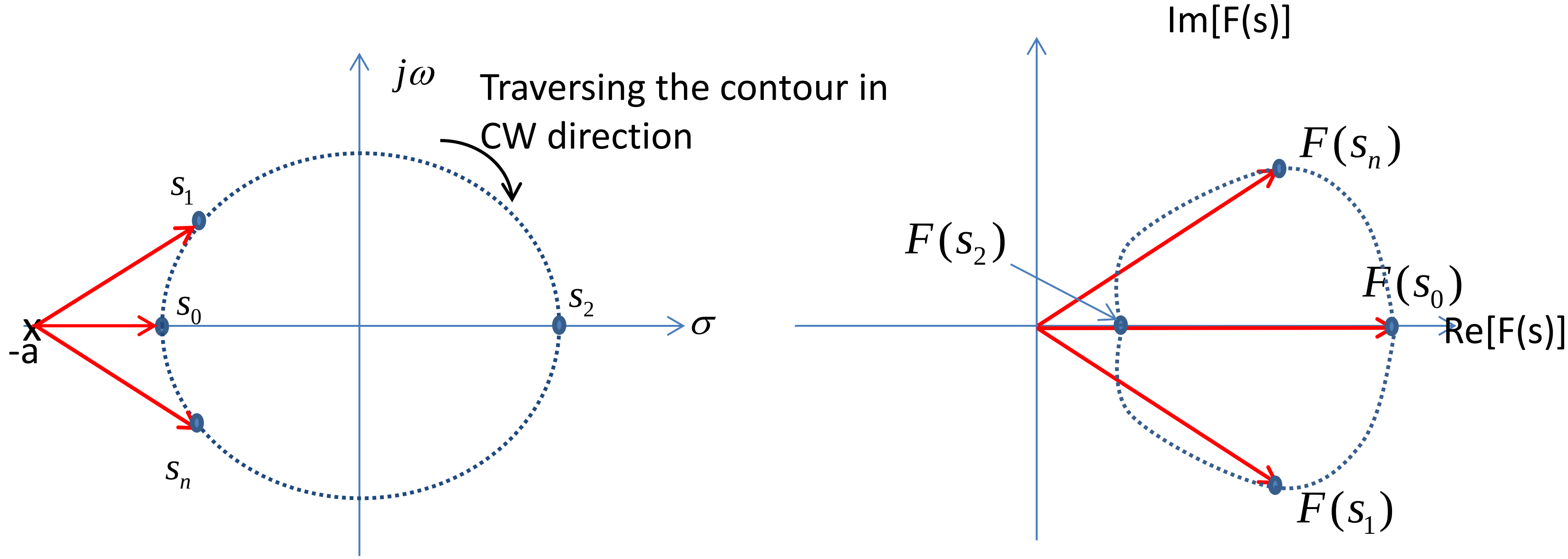
Here and in the examples that follow, the contour is traversed in the CW direction

- Observation:
  - The zero of $F(s)$ is **outside** the contour
  - Net change in $\angle$F(s) is less than 360° $\Rightarrow$ The plot of $F(s)$ **doesn't encircle** the origin

[Only argument, not the modulus, of $F(s)$ is of interest for PoA]

- *Illustration # 2*: $F(s)$ with a single pole outside the contour

$$F(s) = \frac{1}{s+a}$$

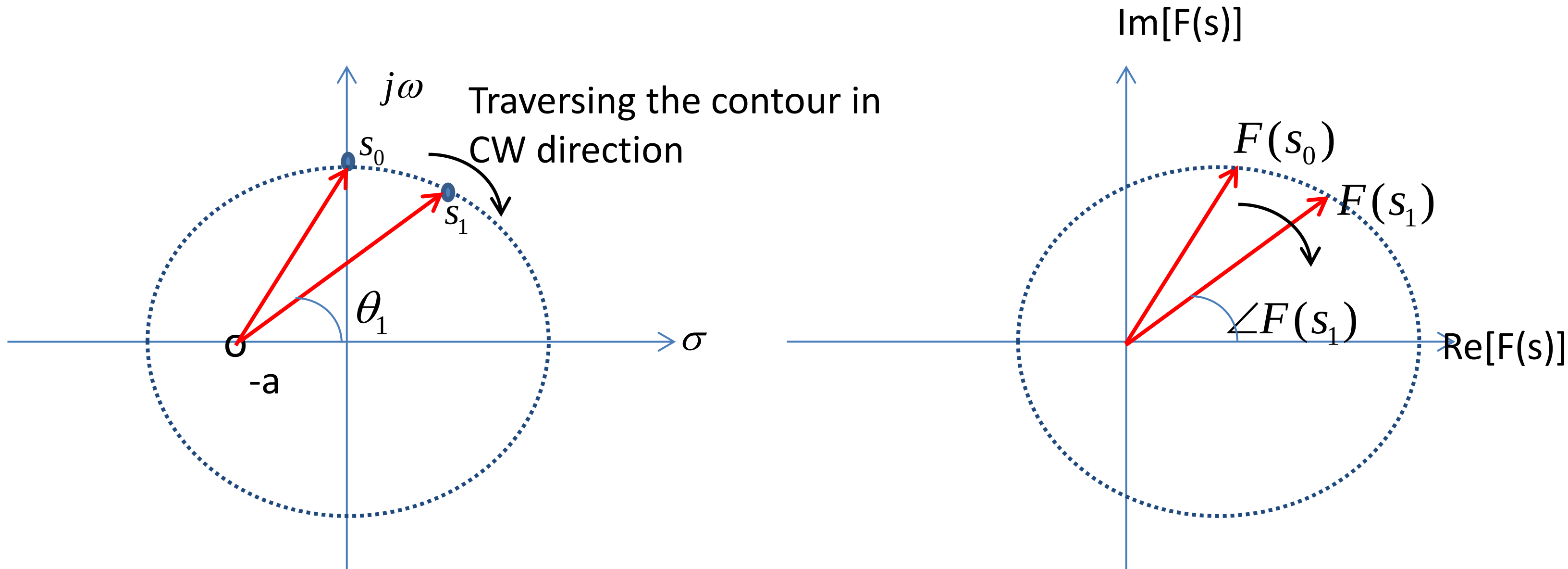


- Observation:
  - The pole of $F(s)$ is **outside** the contour
  - Net change in $\angle$F(s) is less than 360° $\Rightarrow$ The plot of $F(s)$ **doesn't encircle** the origin

Principle of Argument

- *Illustration # 3*: $F(s)$ with a single zero enclosed by the contour

$$F(s) = (s + a)$$



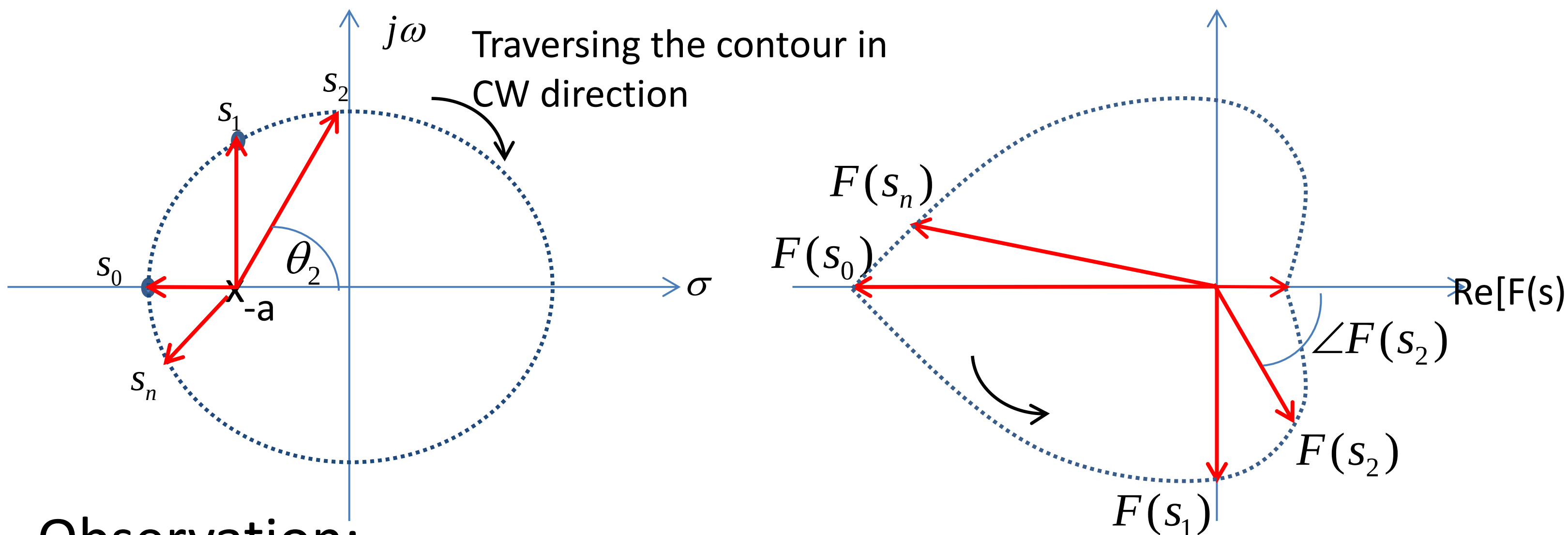
- Observation:
  - The zero of $F(s)$ is **enclosed** the contour
  - $\angle$F(s) goes through a net change of 360° $\Rightarrow$ The plot of $F(s)$ **encircles** the origin in the same direction as traversing the contour in s-plane

$$\angle F(s) = \theta$$

Principle of Argument

- *Illustration # 4*: $F(s)$ with a single pole enclosed by the contour

$$F(s) = \frac{1}{s+a}$$



- Observation:
  - The pole of $F(s)$ is **enclosed** by the contour
  - $\angle$F(s) goes through a net change of 360° $\Rightarrow$ The plot of $F(s)$ **encircles** the origin in the direction opposite to the traversing direction in s-plane

$$\angle F(s) = -\theta$$

Principle of Argument

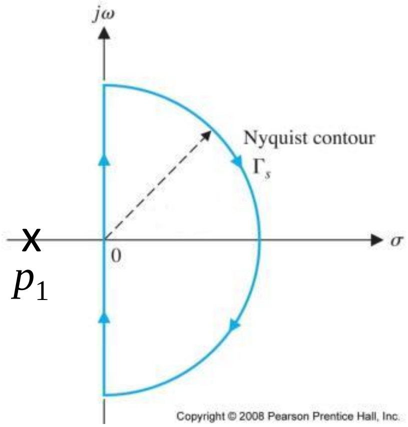
Summary:

- If zeros and poles of $F(s)$ are **not** enclosed by the contour, none of the vectors drawn from the zeros and poles to the CW traversing point on the contour rotate through 360°
  - $\angle F(s)$ is not varied by 360° or more
  - Resulting $F(s)$-plot doesn't encircle the origin of F(s) plane

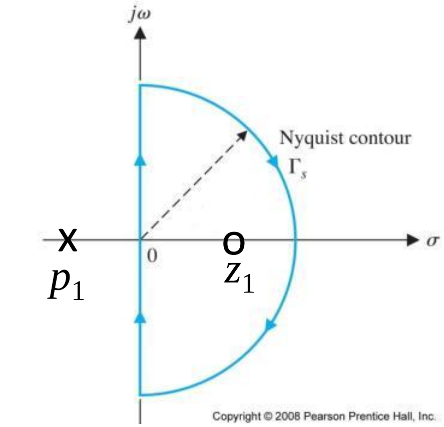
- If a zero (or a pole) is enclosed by the contour, the vector drawn from the zero (or pole) to the CW traversing point rotates through 360°
  - $\angle F(s)$ is changed through 360° for each pole or zero
  - Origin of the $F(s)$-plane is encircled (CW for zero and CCW for pole)

Principle of Argument

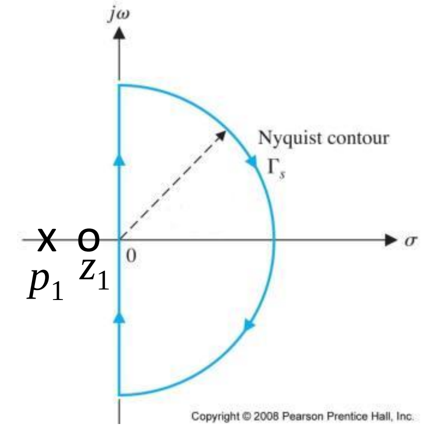
Illustration (enclosure of pole or zero in the s-plane)



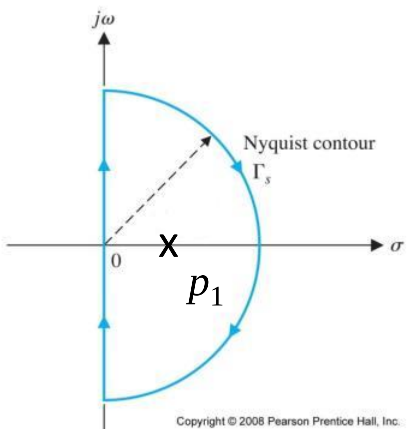
Pole p_1 is not enclosed



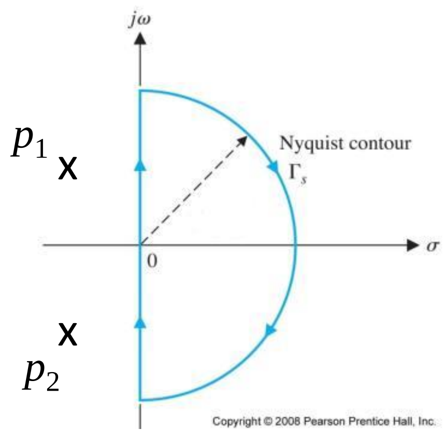
Pole p_1 is not enclosed
Zero z_1 is enclosed



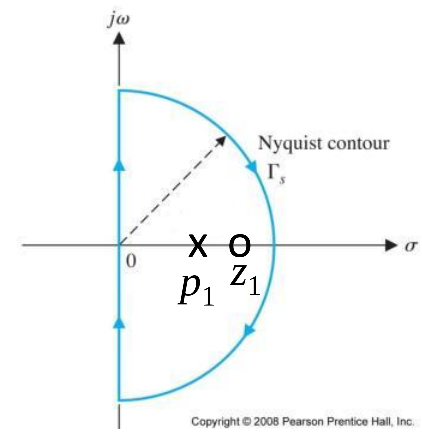
Pole p_1 is not enclosed
Zero z_1 is not enclosed



Pole p_1 is enclosed



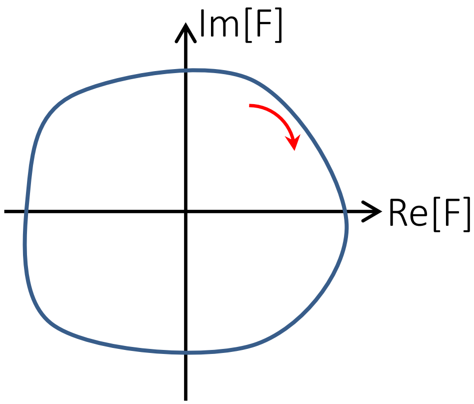
Pole p_1 is not enclosed
Pole p_2 is not enclosed



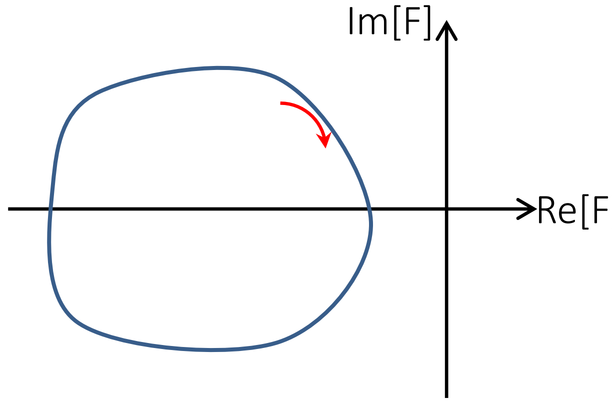
Pole p_1 is enclosed
Zero z_1 is enclosed

Principle of Argument

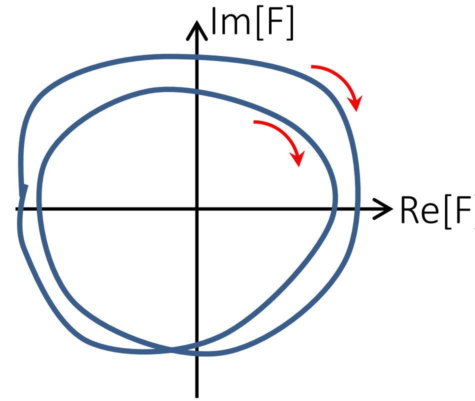
Illustration (encirclement of origin in the F(s)-plane)



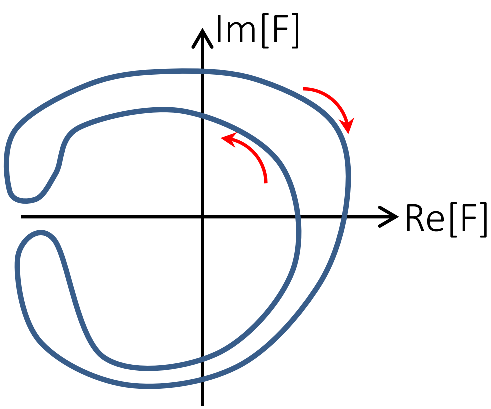
Origin encircled ones CW



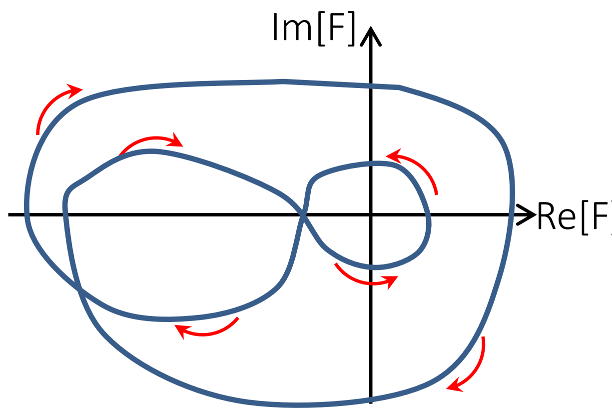
Origin NOT encircled



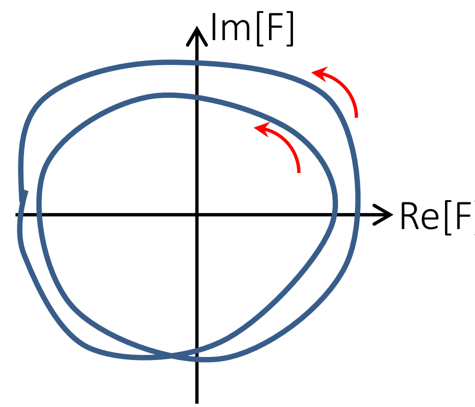
Origin encircled twice CW



Origin NOT encircled



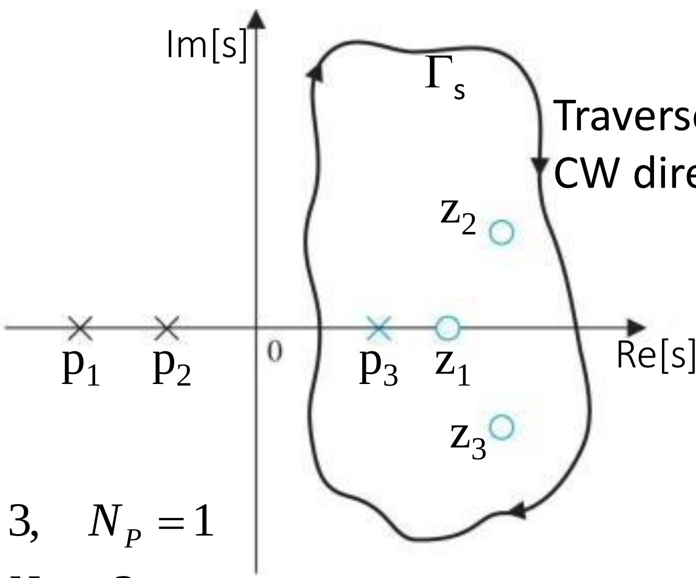
Origin encircled once CW
and once CCW



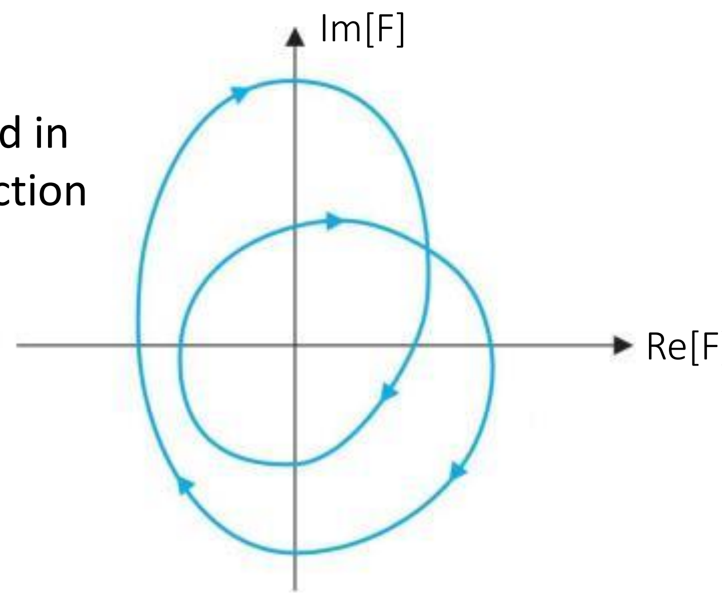
Origin encircled twice CCW

Statement of the Principle of Argument

- If a complex function $F(s)$ has $\mathbf{N_Z}$ number of zeros and $\mathbf{N_P}$ number of poles enclosed by a contour Γ_s then the contour mapping encircles the origin $(N_Z\text{-}N_P)$ times in the same direction as the direction of traversing the contour



$$N_Z = 3, \quad N_P = 1$$
$$N_Z - N_P = 2$$

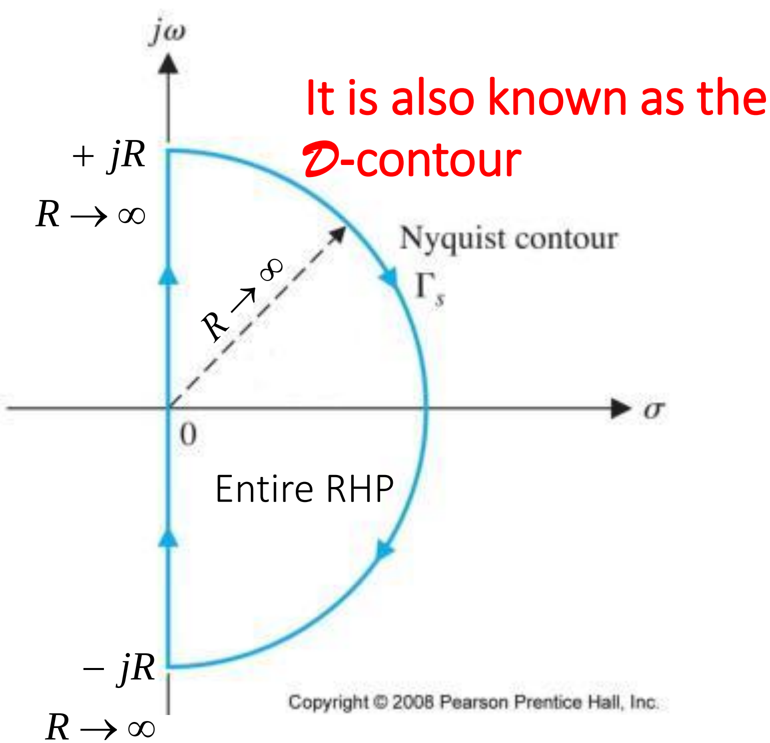


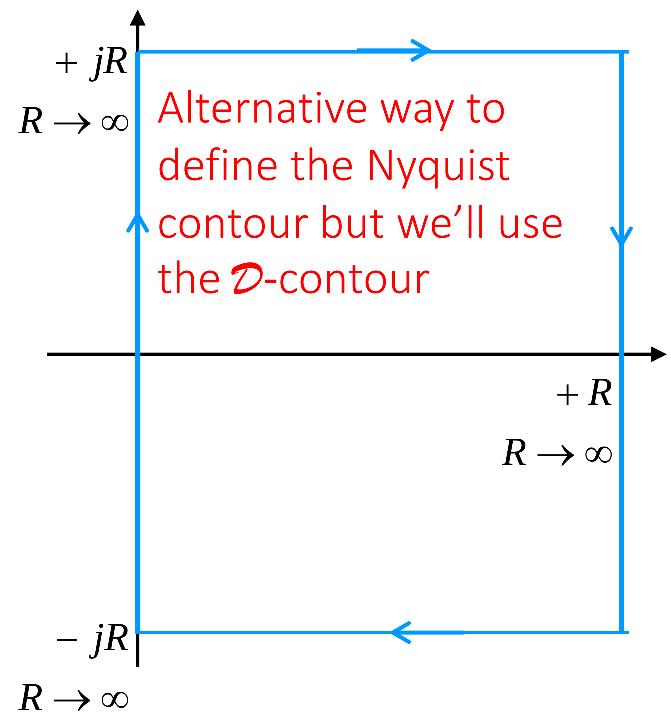
Plot of F(s) encircles the origin of F(s)-plane twice in the CW direction

Application of PoA to check CL Stability

- **Argument Principle** defines the relation between encirclement of origin by $F(s)$-plot and number of poles and zeros enclosed in a predefined contour

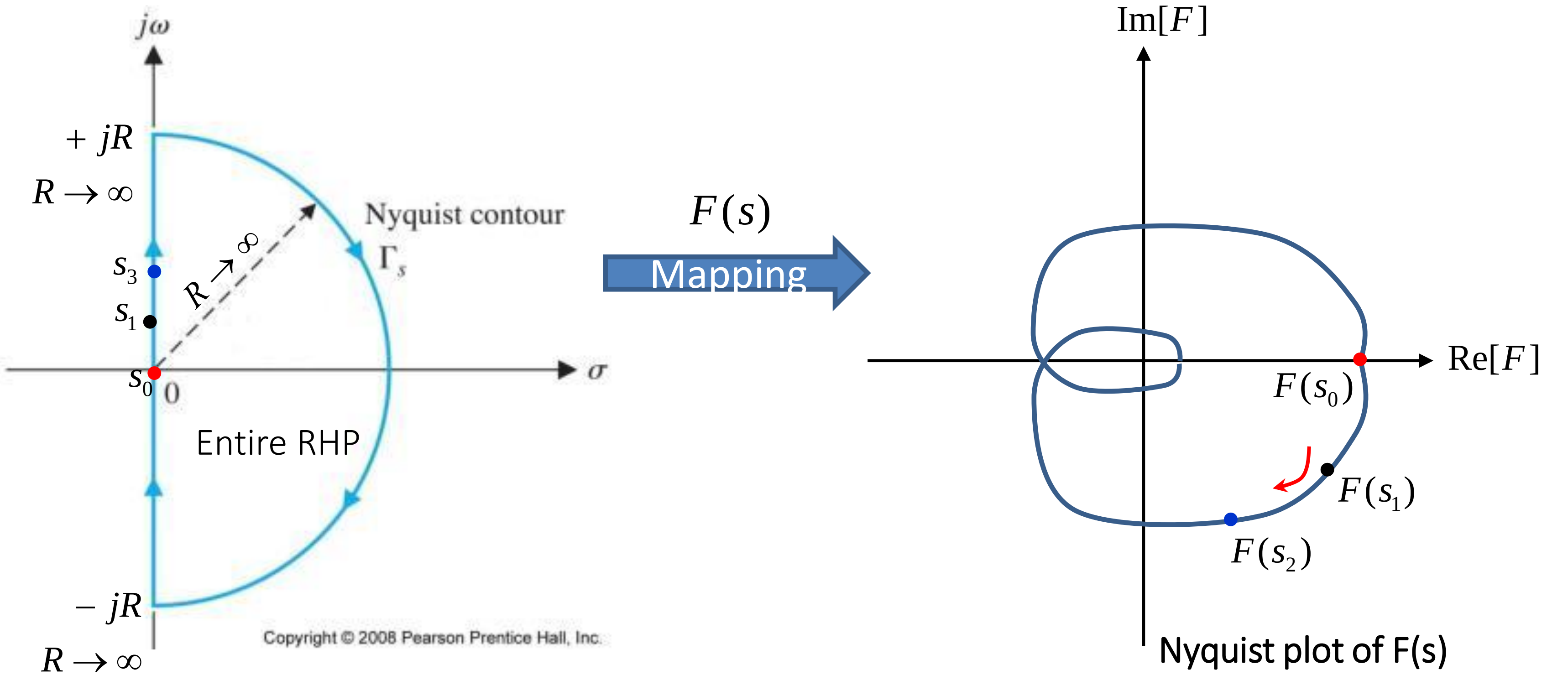
- **Stability Check**: is there any zero of $F(s) = 1+L(s)$ enclosed by the **Nyquist contour**?





Application of PoA to check CL Stability

- Mapping the Nyquist contour is a key step in applying PoA for testing stability
- **Nyquist Plot**: The $F(s)$-plot generated by evaluating $F(s)$ as s is varied in a particular direction (CW) along the Nyquist contour.



- We'll learn soon how to sketch the Nyquist plot for a given transfer function

Application of PoA to check CL Stability

- CL stability check $\Rightarrow$ is there any zero of $F(s)$ inside the Nyquist contour?

$$\begin{aligned} F(s) &= 1 + L(s) \\ &= 1 + KG(s) \end{aligned}$$

- If there are N_Z zeros of $F(s)$ and N_P poles of $F(s)$ inside the Nyquist contour, then mapping of the contour in clockwise (CW) direction results in CW encirclement of the origin N_{CW} times where

$$N_{CW} = N_Z - N_P$$

  - **We want to find the value of N_Z**
  - **We can determine N_{CW} from the Nyquist plot of $F(s)$**
  - **What is N_P?**

Application of PoA to check CL Stability

What is N_P?

$$L(s) = KG(s) = K\frac{N_L(s)}{D_L(s)}$$

- If $s=s_1$ is an open loop pole then, $D_L(s_1) = 0$

$$F(s) = 1 + K\frac{N_L(s)}{D_L(s)} = \frac{D_L(s) + KN_L(s)}{D_L(s)}$$

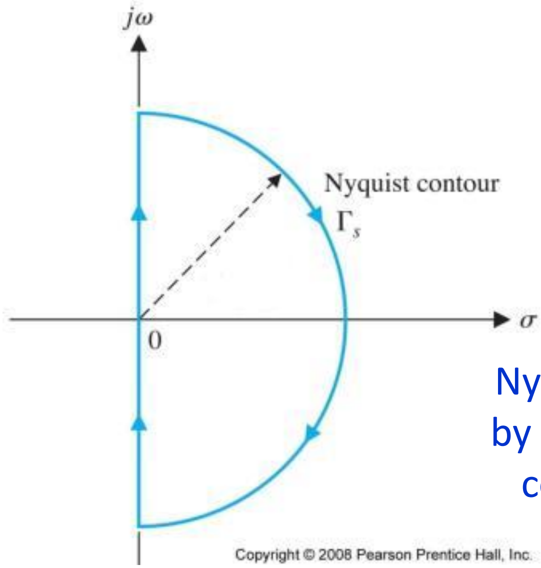
- If $D_L(s_1)=0$ then $s=s_1$ is a pole of $F(s)$
  - **A pole of $F(s)$ is also a pole of $L(s)$, i.e., open loop pole**

- As $L(s)$ is already known, we know how many of its poles are inside the Nyquist contour $\Rightarrow$ we know the value of N_P
  - If OL is stable, no pole of $L(s)$ is in RHP, i.e., $N_P=0$
  - If OL is unstable, N_P is the number of unstable poles of $L(s)$

Application of PoA to check CL Stability

$$L(s) = G_c G_p H$$
$$= KG(s)$$

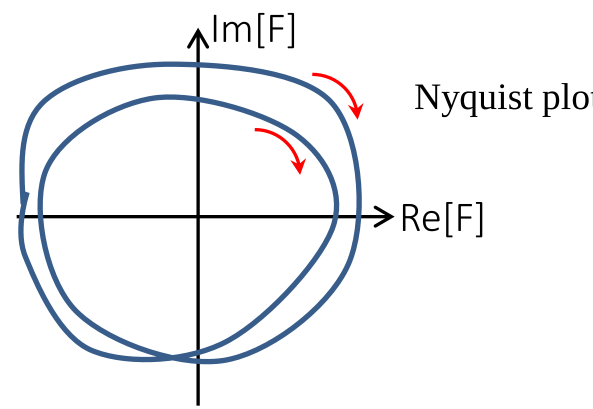
$$G_{CL}(s) = \frac{L(s)}{1+L(s)} = \frac{L(s)}{F(s)}$$



$$F(s) = 1 + L(s)$$

Mapping

Nyquist contour is mapped by evaluating F(s) along the contour in CW direction



- **Count number of CW encirclement (N_{CW}) of origin by the Nyquist plot**

$$N_{CW} = N_Z - N_P$$

- **Count N_{CW} from the Nyquist plot**
- **Get N_P from the open loop transfer function $L(s)$**
- **N_Z is the number of unstable zero of $F(s)$, *i.e.* unstable pole of $G_{CL}(s)$**
  - **For CL to be stable, N_Z must be 0**

Can we use $L(s)$ to Check Stability?

$$F(s) = 1 + L(s) \quad \Rightarrow \quad L(s) = F(s) - 1$$

$$F(s_1) = a + jb \quad \Longrightarrow \quad \begin{aligned} L(s_1) &= F(s_1) - 1 \\ &= (a-1) + jb \end{aligned}$$

- *Example* 3b-3: Consider the following functions evaluated at $s_1 = 2 + j1$

$$F(s) = \frac{s+1}{s+5}$$

$$F(s_1) = \frac{2 + j1 + 1}{2 + j1 + 5} = \frac{3 + j1}{7 + j1}$$

$$F(s_1) = \frac{3 + j1}{7 + j1} \times \frac{7 - j1}{7 - j1} = \frac{21 + j7 - j3 + 1}{49 + 1}$$

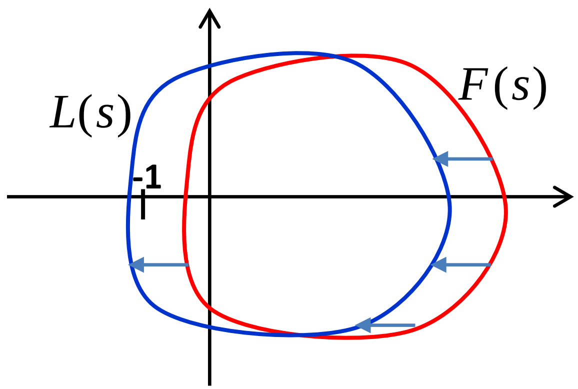
$$F(s_1) = \frac{22 + j4}{50} = 0.44 + j0.08$$

$$\begin{aligned} L(s_1) &= F(s_1) - 1 \\ &= 0.44 + j0.08 - 1 \\ &= -0.56 + j0.08 \end{aligned}$$

Application of PoA to check CL Stability

$$F(s_1) = a + jb \quad \Rightarrow \quad L(s_1) = (a-1) + jb$$

- Plot of $L(s) \Rightarrow$ plot of $F(s)$ shifted to left by 1



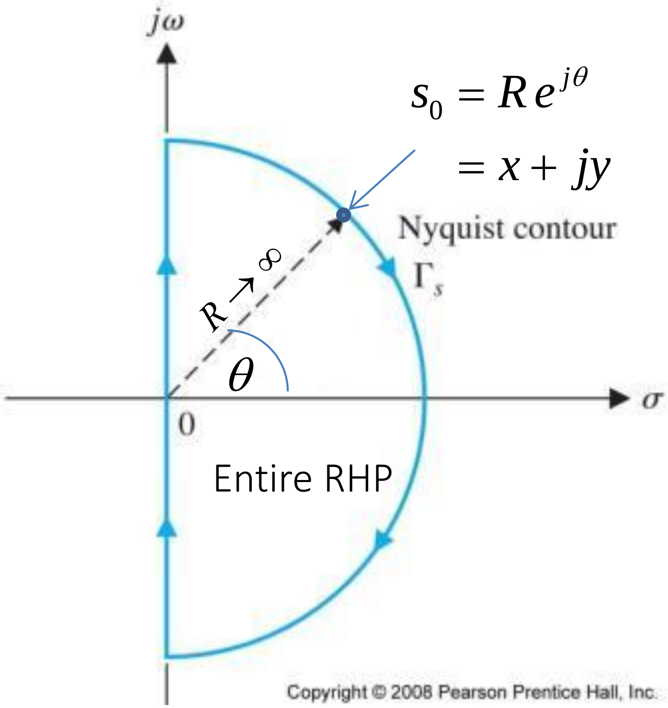
If the $F(s)$ encircles the origin, $L(s)$ will encircle the point (-1,0) or (-1+j0)

- **Stability check**:
  - **Sketch the Nyquist plot of $L(s)$ and find N_{CW}, the number of clockwise encirclement of the point (-1,0)**
  - **From the knowledge of OL, find the value of N_P**
  - **Find number of unstable CL poles (N_Z) by solving**

$$N_{CW} = N_Z - N_P$$

Sketching the Nyquist Plot

- The Nyquist plot of $L(s)$ is the mapping of the Nyquist Contour using the transfer function $L(s)$



Segment by Segment Mapping

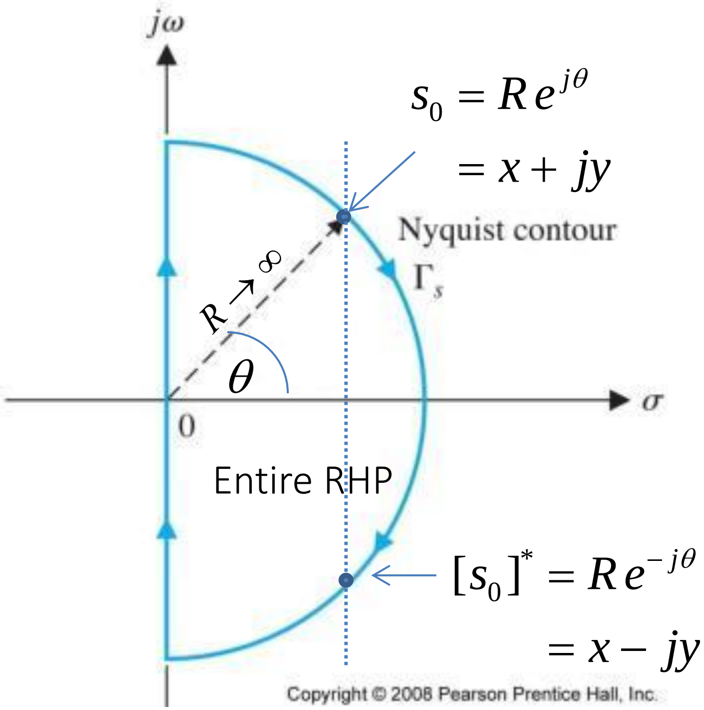
- Segment 1: $s = j\omega, \quad -R \le \omega \le R$
  $R \to \infty$

$$L(s)\big|_{seg1} = L(j\omega), \quad -R \le \omega \le R$$
$$R \to \infty$$

- Segment 2: $s = R\cos\theta + jR\sin\theta$
  $= Re^{j\theta}$
  $R \to \infty, \theta: +90° \to 0° \to -90°$

$$L(s)\big|_{seg2} = L(Re^{j\theta}),$$
$$R \to \infty, \theta: +90° \to 0° \to -90°$$

Symmetry of the Nyquist Contour



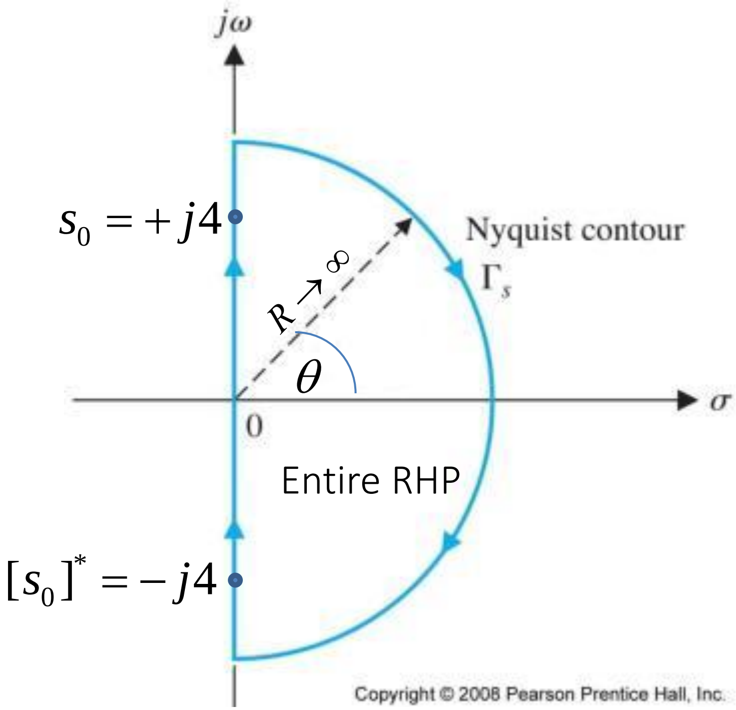
- If $L(s_0) = a + jb$
- Then,

$$L([s_0]^*) = [L(s_0)]^*$$
$$= a - jb$$

- We can map the upper half of the Nyquist contour, i.e., find $L(s_0)$
  - Then $L([s_0]^*)$ is the mirror image of the point $L(s_0)$

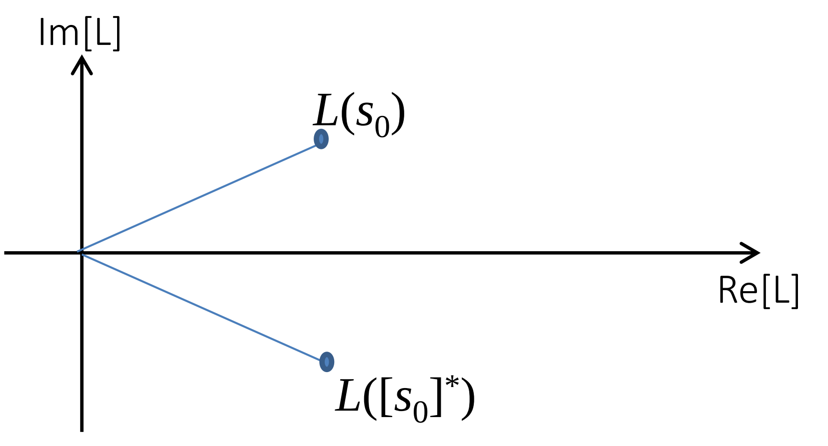
Sketching the Nyquist Plot

- *Example* 3b-4: Evaluate $L(s)$ for s_0=+j4 and $[s_0]^*$=-j4 when $L(s) = \dfrac{s+2}{s+5}$



$$L(s_0) = \frac{2+j4}{5+j4} = \frac{\sqrt{20}\angle 63.4^\circ}{\sqrt{41}\angle 38.7^\circ} = 0.7\angle 24.7^\circ$$

$$L([s_0]^*) = \frac{2-j4}{5-j4} = \frac{\sqrt{20}\angle -63.4^\circ}{\sqrt{41}\angle -38.7^\circ} = 0.7\angle -24.7^\circ$$

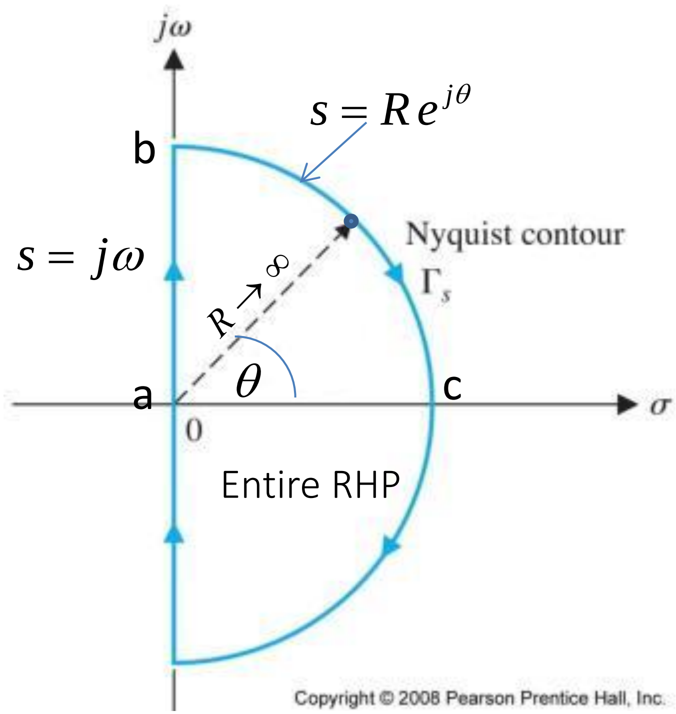


{mirror image of $L(s_0)$}

Complex variable *s* along the upper half of the $\mathcal{D}$-contour

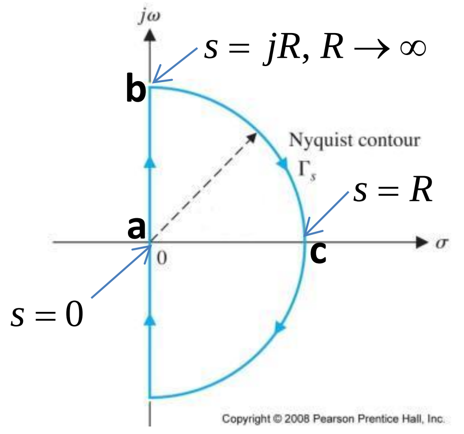
1) Along the +jω axis: $s = j\omega, \quad 0 \le \omega \le R$, $R \to \infty$ [Going upward from point **a** to **b** in the figure below]

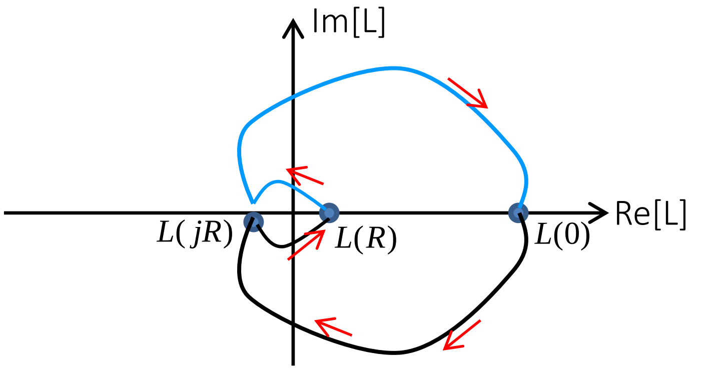
2) Along the top half of the semicircle: $s = Re^{j\theta}$
[Going CW from point **b** to **c** in the figure below] $R \to \infty, \theta : +90° \to 0°$



Sketching the Nyquist Plot: L(s) with no Integrator

$$L(s) = K\frac{(s+b_1)(s+b_2)(s^2+c_1s+d_1)....}{(s+a_1)(s+a_2)(s^2+p_1s+q_1)....}$$





1) **Map the upper half of the Nyquist contour**
   a) Map the points 'a', 'b', and 'c'
      - Determine the points $L(0)$, $L(jR)$ where $R \to \infty$, and $L(R)$
   b) Connect $L(0)$ to $L(jR)$
      - How is $L(j\omega)$ changed as ω is varied from 0 to R
   c) Connect $L(jR)$ to $L(R)$
      - How $L(s)$ is changed as s is varied along the arc from 'b' to 'c'
2) **Complete the Nyquist plot by adding the mirror image of the plot obtained in step 1**

Sketching the Nyquist Plot: L(s) with no Integrator

- *Example* 3b-5: Map the points 'a', 'b' and 'c' of the Nyquist contour for the transfer function

$$L(s)=\frac{(s+10)}{(s+1)(s+100)}$$

- Answer:

'a'

$$s=0,\quad L(0)=\frac{(0+10)}{(0+1)(0+100)}=0.1$$

'b'

$$s=jR,$$

$$L(jR)=\frac{(jR+10)}{(jR+1)(jR+100)}$$

$$=\frac{\sqrt{R^2+100}}{\sqrt{R^2+1}\sqrt{R^2+10000}}\angle\left(\tan^{-1}\tfrac{R}{10}-\tan^{-1}R-\tan^{-1}\tfrac{R}{100}\right)$$

$$as\quad R\to\infty,\quad |L(jR)|\cong\frac{1}{R},\quad \angle L(jR)=-90^\circ$$

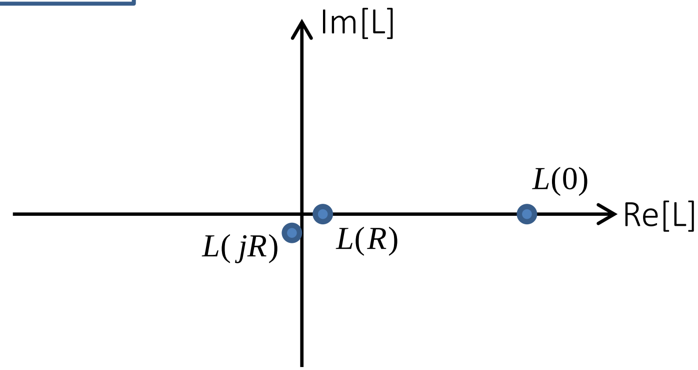
'c'

$$s=R,$$

$$L(R)=\frac{(R+10)}{(R+1)(R+100)}$$

$$=\frac{R+10}{R^2+101R+100}$$

$$as\quad R\to\infty,\quad L(R)\cong\frac{1}{R}$$



How to connect L(0) to L(jR) and L(jR) to L(R)?

Sketching the Nyquist Plot: L(s) with no Integrator

Connecting L(0) to L(jR), R→∞

- For this section,

$$s = j\omega, \quad 0 \le \omega \le R$$

$$L(s) = K\frac{(s+b_1)(s+b_2)(s^2+c_1s+d_1)....}{(s+a_1)(s+a_2)(s^2+p_1s+q_1)....}$$

$$L(j\omega) = K\frac{(j\omega+b_1)(j\omega+b_2)((j\omega)^2+jc_1\omega+d_1)....}{(j\omega+a_1)(j\omega+a_2)((j\omega)^2+jp_1\omega+q_1)....}, \quad 0 \le \omega \le R, \quad R \to \infty$$

- There are different ways to find the variations in $L(j\omega)$

    1) Using trigonometric expression of the argument
    2) Finding points of intersections with the axes
    3) Sketching the Bode (phase) plot

Sketching the Nyquist Plot: L(s) with no Integrator

1) Using Trigonometric Expression for ∠L(jω)

- *Example* 3b-6: For the following transfer function, determine how ∠L is changed as ω is varied from 0 to R where R→∞

$$L(s)=\frac{K}{(s+2)(s+10)}$$

- Answer:

$$L(j\omega)=\frac{K}{(j\omega+2)(j\omega+10)}$$

$$L(0)=\frac{K}{(0+2)(0+10)}=\frac{K}{20}$$

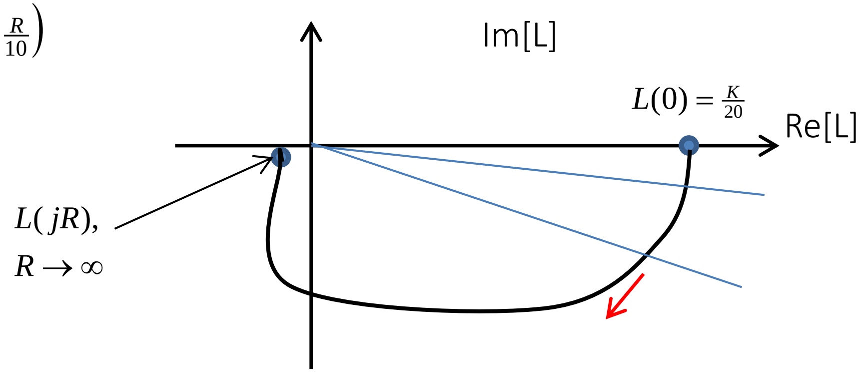
$$L(jR)=\frac{K}{(jR+2)(jR+10)}$$

$$=\frac{K}{\sqrt{R^2+100}\sqrt{R^2+4}}\angle\left(-\tan^{-1}\tfrac{R}{2}-\tan^{-1}\tfrac{R}{10}\right)$$

$$\cong\frac{K}{R^2}\angle-180^\circ$$

$$\angle L=-\tan^{-1}\frac{\omega}{2}-\tan^{-1}\frac{\omega}{10}$$

1. $\angle L$ is negative for all $\omega>0$
2. $\angle L\rightarrow$ (-180°) as $\omega\rightarrow\infty$
3. Plot must be in the 4th and the 3rd quadrant



Sketching the Nyquist Plot: L(s) with no Integrator

2) Finding points of intersections with the axes

- Intersection with real-axis:

$$\mathrm{Im}[L(j\omega)] = 0$$
$$\angle L(j\omega) = 0^\circ \; or \; \pm 180^\circ$$

- Intersection with imaginary-axis:

$$\mathrm{Re}[L(j\omega)] = 0$$
$$\angle L(j\omega) = \pm 90^\circ$$

- *Example* 3b-7: For the following transfer function, determine how $\angle$L is changed as ω is varied from 0 to R where R$\to\infty$

$$L(s) = \frac{K}{(s+2)(s+10)}$$

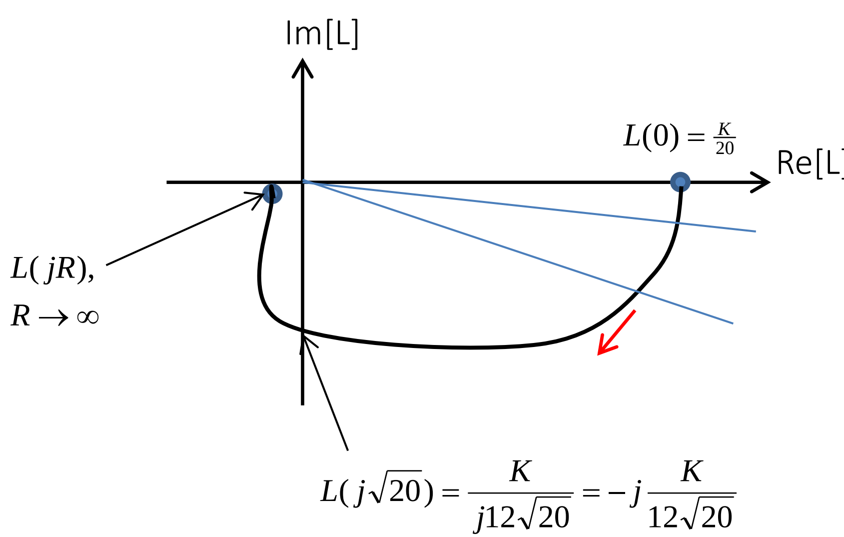
- Answer:

$$L(j\omega) = \frac{K}{(j\omega+2)(j\omega+10)}$$
$$= \frac{K}{(20-\omega^2) + j12\omega}$$

Im[L] = 0 when ω=0.
This point is effectively the dc-gain, L(j0)

Re[L] = 0 when,

$$(20-\omega^2) = 0 \quad \Rightarrow \quad \omega = \sqrt{20}$$



Sketching the Nyquist Plot: L(s) with no Integrator

- *Example* 3b-8: For the following transfer function, find points of intersection of the frequency response curve with the real-axis and the imaginary-axis

$$L(s)=\frac{K}{(s+2)(s^2+2s+5)}$$

- Answer:

$$L(s)=\frac{K}{(s+2)(s^2+2s+5)}=\frac{K}{s^3+4s^2+9s+10}$$

$$L(j\omega)=\frac{K}{-j\omega^3-4\omega^2+j9\omega+10}=\frac{K}{(10-4\omega^2)+j\omega(9-\omega^2)}$$

  - The curve intersects with real-axis at two points

$$\omega=0 \quad\Rightarrow\quad L(j0)=\frac{K}{10}, \qquad \omega=3 \quad\Rightarrow\quad L(j3)=-\frac{K}{26}$$

  - And it intersects with imaginary-axis at one point,

$$\omega=\frac{\sqrt{10}}{2} \quad\Rightarrow\quad L(j\tfrac{\sqrt{10}}{2})=\frac{K}{j\frac{\sqrt{10}}{2}\left(9-\frac{10}{4}\right)}=-j\frac{4K}{13\sqrt{10}}$$

Sketching the Nyquist Plot: L(s) with no Integrator

- *Example* 3b-9: For the following transfer function, find points of intersection of the frequency response curve with real-axis and imaginary-axis

$$L(s)=\frac{K(s+2)}{(s^2+2s+5)}$$

- Answer:

$$L(j\omega)=\frac{K(j\omega+2)}{-\omega^2+j2\omega+5}=\frac{K(j\omega+2)}{(5-\omega^2)+j2\omega}$$

[multiply both numerator and denominator by the conjugate of the denominator]

$$L(j\omega)=\frac{K(j\omega+2)}{(5-\omega^2)+j2\omega}\times\frac{(5-\omega^2)-j2\omega}{(5-\omega^2)-j2\omega}$$

$$L(j\omega)=K\frac{10+j\omega(1-\omega^2)}{(5-\omega^2)^2+4\omega^2}$$

  - The imaginary part is zero at two frequencies,

$$\omega=0 \quad\Rightarrow\quad L(j0)=\frac{2K}{5}, \qquad \omega=1 \quad\Rightarrow\quad L(j1)=+\frac{K}{2}$$

  - The real part is not zero for any $\infty \geq \omega \geq 0$.

Sketching the Nyquist Plot: L(s) with no Integrator

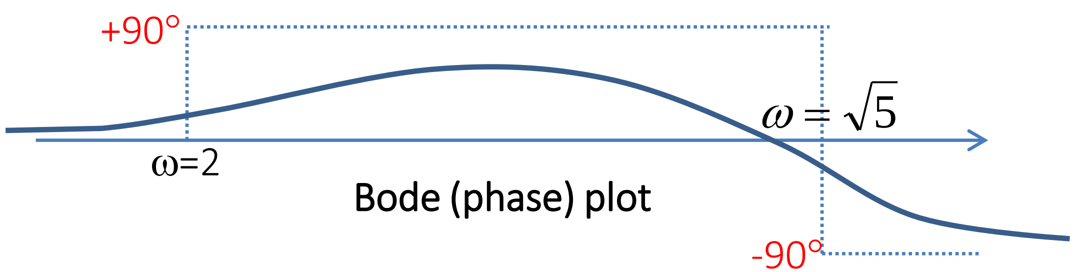
3) Using Bode (Phase) Plot

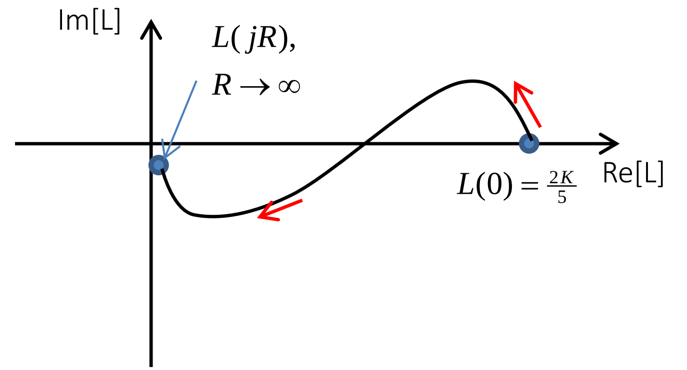
- Hand-sketch the Bode (phase) plot of L(jω) to find how ∠L(jω) changes as ω is varied from 0 to ∞

- *Example* 3b-10: For the following transfer function, find the trend in ∠L(jω) as ω is varied from 0 to ∞

$$L(s) = \frac{K(s+2)}{(s^2+2s+5)}$$

  - Answer: The corner frequency of the zero is 2 and natural frequency of the complex poles is $\omega_n = \sqrt{5}$





- Phase of the transfer function is bounded between +90° and -90°. $L(j\omega)$-plot must be on the right side of the $\mathrm{Im}[L(j\omega)]$ axis.

Sketching the Nyquist Plot: L(s) with no Integrator

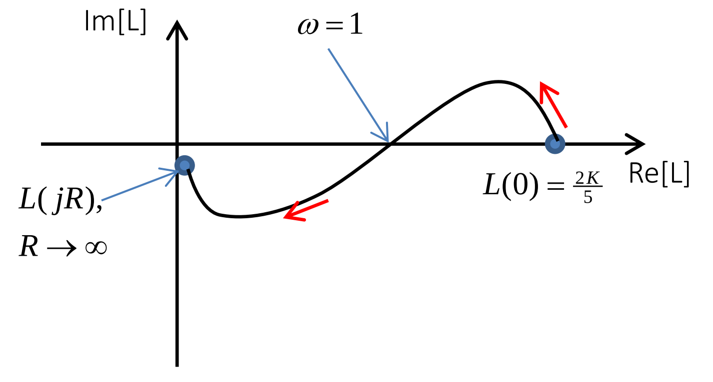
- Keeping the same example, let's verify the result using the method of finding the points of inter-section
- *Example* 3b-11: $L(s)=\dfrac{K(s+2)}{(s^2+2s+5)}$
- Answer:

$$L(j\omega)=K\frac{2+j\omega}{(5-\omega^2)+j2\omega}$$

$$L(j\omega)=K\frac{2+j\omega}{(5-\omega^2)+j2\omega}\times\frac{(5-\omega^2)-j2\omega}{(5-\omega^2)-j2\omega}$$

$$L(j\omega)=K\frac{10+j\omega(1-\omega^2)}{(5-\omega^2)^2+4\omega^2},$$

$$\because (a+jb)(a-jb)=a^2+b^2$$

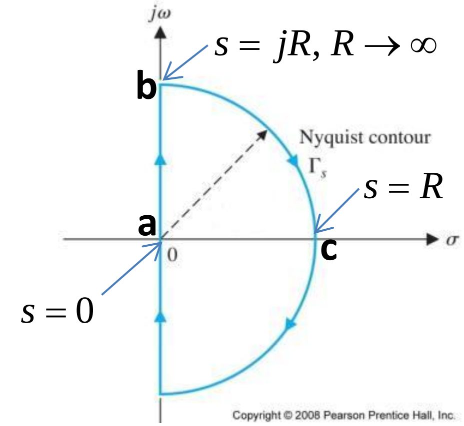


Real part of L is >0 for all ω
Imaginary part is zero for ω=0 and ω=1
Imaginary part is positive for ω<1 and negative for ω>1

Connecting L(jR) to L(R), R→∞

- For this segment (upper half of the semicircle),

$$s = Re^{j\theta}, \quad R \to \infty, \theta : 90° \to 0°$$



- Consider $L(s)$ having m poles and n zeros with $m \geq n$

$$L(s) = K\frac{(s+b_1)(s+b_2)(s^2+c_1s+d_1)....}{(s+a_1)(s+a_2)(s^2+p_1s+q_1)....}$$

$$s = Re^{j\theta} \quad \Rightarrow \quad s^2 = R^2e^{j2\theta}$$

$$L(s) = K\frac{(Re^{j\theta}+b_1)(Re^{j\theta}+b_2)(R^2e^{j2\theta}+c_1Re^{j\theta}+d_1)....}{(Re^{j\theta}+a_1)(Re^{j\theta}+a_2)(R^2e^{j2\theta}+p_1Re^{j\theta}+q_1)....}$$

Linear factor

$$(Re^{j\theta}+b_1) = R(e^{j\theta}+b_1R^{-1})$$
$$= R(e^{j\theta}), \quad \because R^{-1} \approx 0$$

Quadratic factor

$$(R^2e^{j2\theta}+c_1Re^{j\theta}+d_1) = R^2(e^{j2\theta}+c_1R^{-1}e^{j\theta}+d_1R^{-2})$$
$$= R^2(e^{j2\theta}), \quad \because R^{-1} \approx 0$$

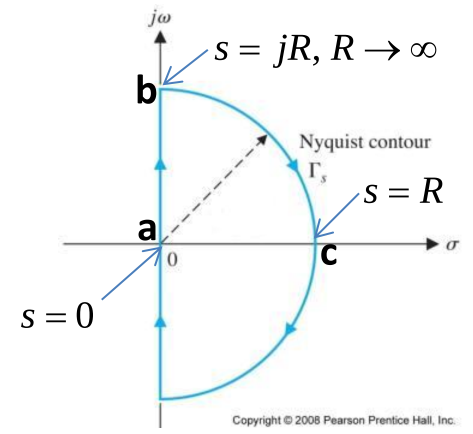
Sketching the Nyquist Plot: L(s) with no Integrator

$$s = Re^{j\theta}, \quad R \to \infty, \theta : 90^\circ \to 0^\circ$$

$$L(s) = K\frac{(Re^{j\theta}+b_1)\ldots(R^2e^{j2\theta}+c_1Re^{j\theta}+d_1)\ldots.}{(Re^{j\theta}+a_1)\ldots(R^2e^{j2\theta}+p_1Re^{j\theta}+q_1)\ldots.}$$

$$= K\frac{(Re^{j\theta})\ldots(R^2e^{j2\theta})}{(Re^{j\theta})\ldots(R^2e^{j2\theta})}$$

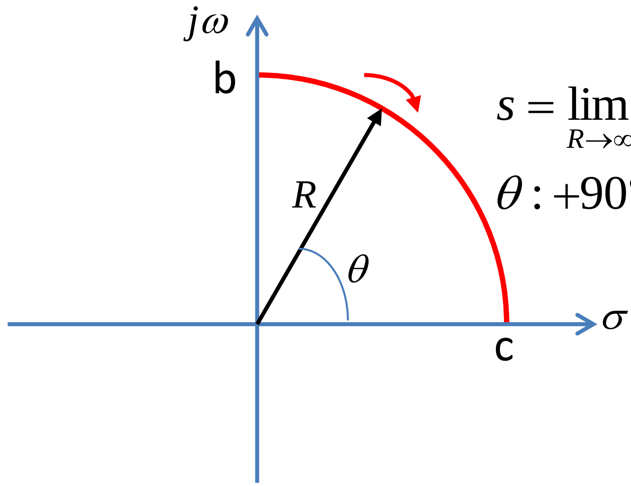
$$= K\frac{R^m e^{jm\theta}}{R^n e^{jn\theta}}$$



$$|L(s)| = \frac{K}{R^{n-m}}, \quad \angle L(s) = -(n-m)\theta \quad \Rightarrow \quad \left\{ \begin{array}{l} \theta = 90^\circ, \quad \angle L(s) = -(n-m)\times 90^\circ \\ \Downarrow \\ \theta = 0^\circ, \quad \angle L(s) = -(n-m)\times 0^\circ = 0^\circ \end{array} \right.$$

- Plot of L(s) is an arc of infinitesimally small radius
- Argument of L(s) is changed through $-(n\text{-}m)\theta$ as θ is varied
  - As θ is varied from 90° to 0° clockwise while going from 'b' to 'c', $\angle$L(s) is sees a change of (n-m)×90° counterclockwise

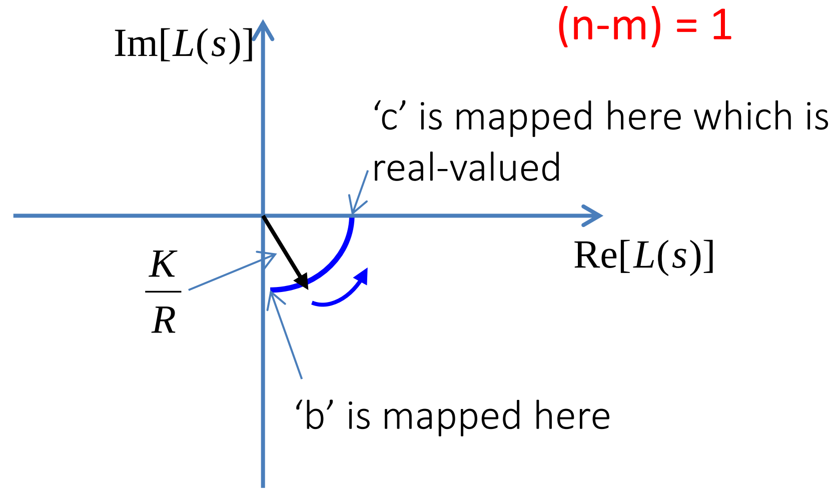
Sketching the Nyquist Plot: L(s) with no Integrator



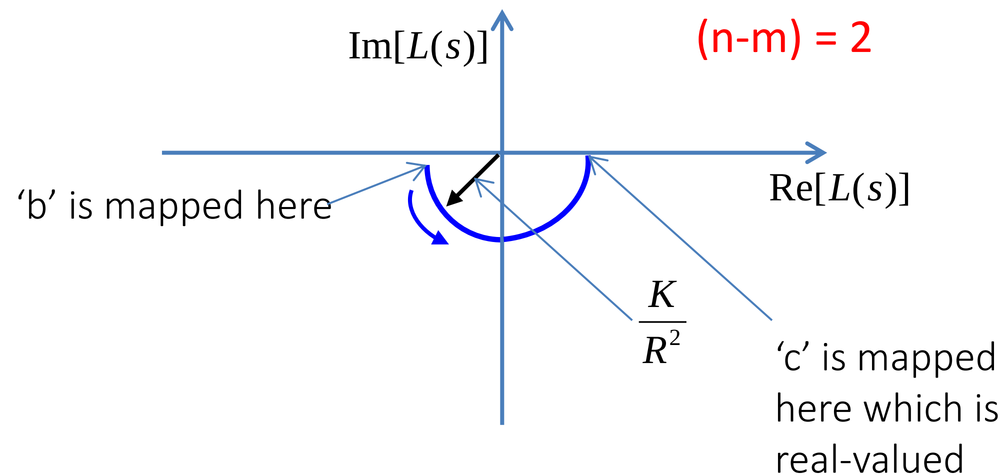
$$s = \lim_{R\to\infty} R e^{j\theta}$$

$$\theta : +90^\circ \to 0^\circ$$

σ-$j\omega$ plane or the s-plane



$L(s)$-plane



Sketching the Nyquist Plot: L(s) with no Integrator

- *Example* 3b-12: Sketch the Nyquist plot for

$$L(s)=\frac{K}{(s+2)(s+10)}$$

- Answer:

$$L(0)=\frac{K}{(0+2)(0+10)}=\frac{K}{20}$$

$$L(jR)=\frac{K}{(jR+2)(jR+10)}$$

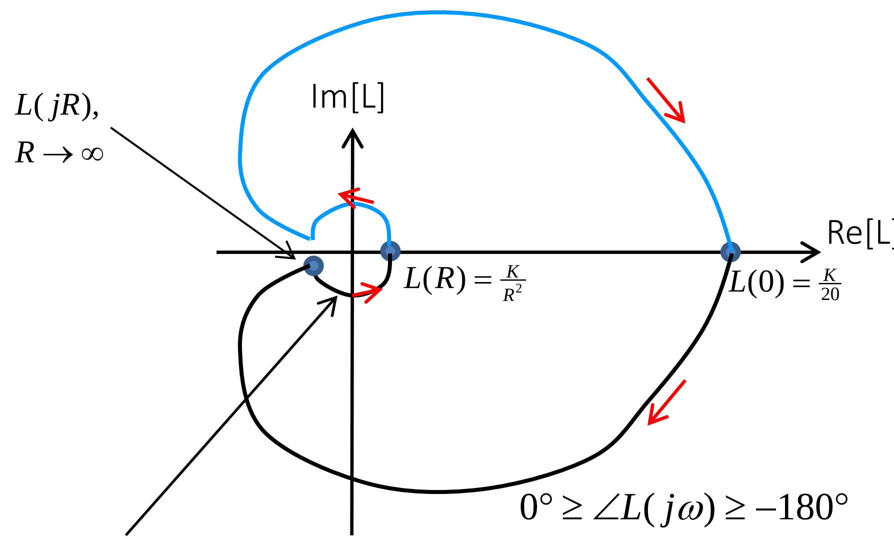
$$|L(jR)|\cong\frac{K}{R^2},\quad \angle L(jR)\cong -180^\circ$$

$$L(R)=\frac{K}{(R+2)(R+10)}\cong\frac{K}{R^2}$$

$$L(j\omega)=\frac{K}{(j\omega+2)(j\omega+10)}$$

$$\angle L(j\omega)=\left(-\tan^{-1}\tfrac{\omega}{2}-\tan^{-1}\tfrac{\omega}{10}\right)$$

$$0^\circ\geq\angle L(j\omega)\geq -180^\circ$$



(n-m)=2 means CCW rotation through 180°

Sketching the Nyquist Plot: L(s) with no Integrator

- *Example* 3b-13: Sketch the Nyquist Plot of

$$L(s)=\frac{1}{s-3}$$

- Answer:

$$L(0)=\frac{1}{0-3}=-\frac{1}{3}$$

$$L(jR)=\frac{1}{jR-3}=\frac{1}{\sqrt{R^2+9}\angle(180^\circ-\tan^{-1}\frac{R}{3})}$$

$$|L(jR)|\cong\frac{1}{R},\quad \angle L(jR)=-90^\circ,\quad \because R\to\infty$$

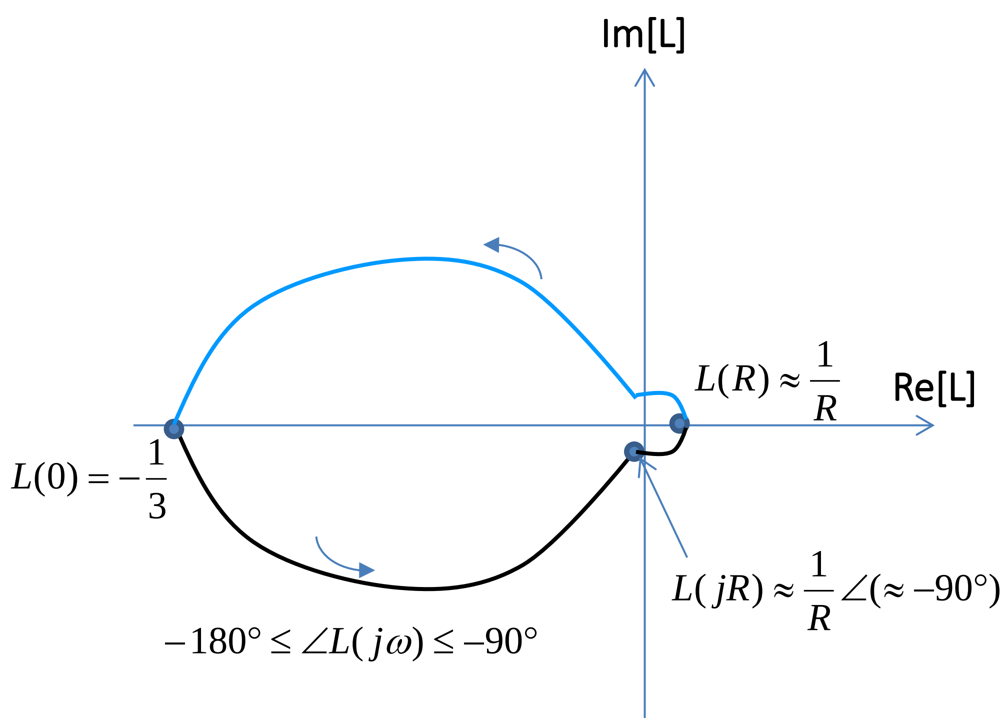
$$L(R)=\frac{1}{R-3}\cong\frac{1}{R}$$

$$L(j\omega)=\frac{1}{j\omega-3},\quad 0\le\omega\le R,\quad R\to\infty$$

$$\angle L(j\omega)=\frac{1}{\angle(180^\circ-\tan^{-1}\frac{\omega}{3})}$$

$$=\angle(-180^\circ+\tan^{-1}\frac{\omega}{3})$$

$$as\quad 0^\circ\le\tan^{-1}\frac{\omega}{3}<90^\circ,\quad -180^\circ\le\angle L(j\omega)\le-90^\circ$$



Sketching the Nyquist Plot: L(s) with Integrator

- If a pole of $L(s)$ lies on the D-contour, for example, an integrator of $L(s)$, then that point can't be mapped

$$L(s) = \frac{K}{(s)^N} \frac{(s+b_1)(s+b_2)(s^2+c_1 s+d_1)....}{(s+a_1)(s+a_2)(s^2+p_1 s+q_1)....}$$

$$L(j\omega) = \frac{K}{(j\omega)^N} \frac{(j\omega+b_1)(j\omega+b_2)((j\omega)^2+jc_1\varepsilon+d_1)....}{(j\omega+a_1)(j\omega+a_2)((j\omega)^2+jp_1\varepsilon+q_1)....}, \quad 0 \le \omega \le R, \quad R \to \infty$$

  - While mapping the jω-axis, we can't evaluate $L(j0)$,

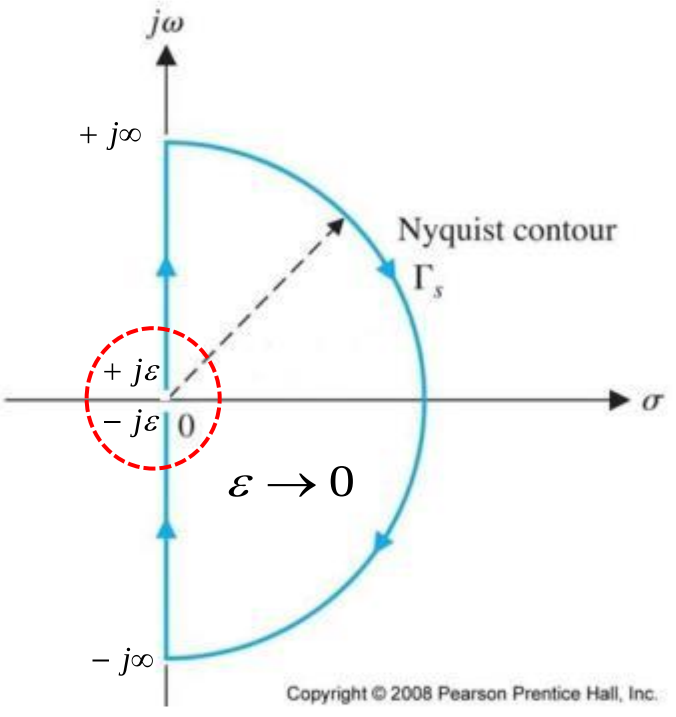
$$|L(j0)| = \infty, \quad \angle L(j0) = ?$$

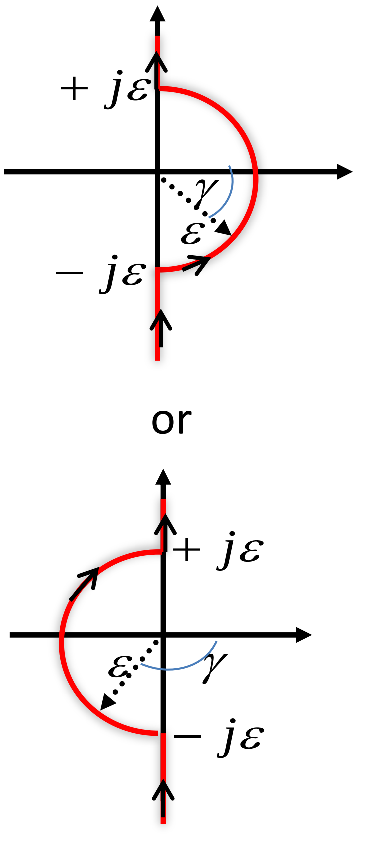
  - Instead of finding $L(j0)$, we find the point for infinitesimally small value of ω

$$L(j\varepsilon) \quad where \quad \varepsilon \approx 0$$

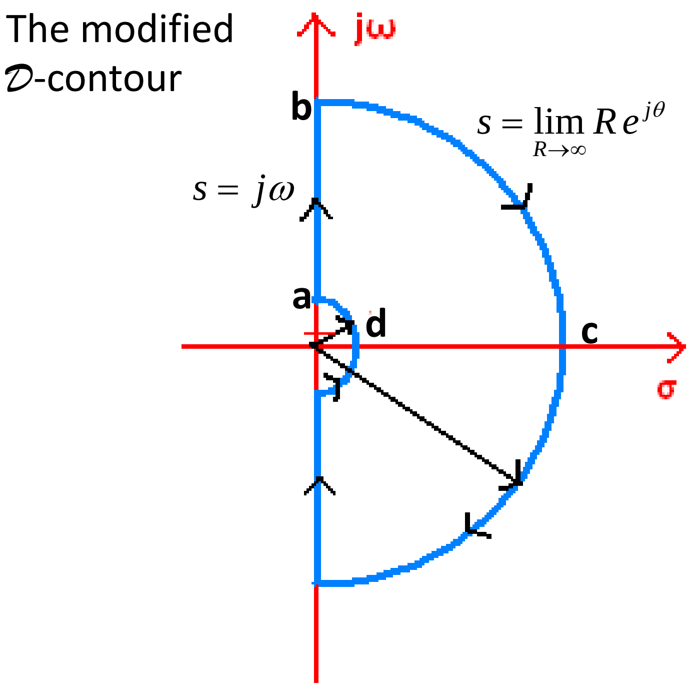
Sketching the Nyquist Plot: L(s) with Integrator

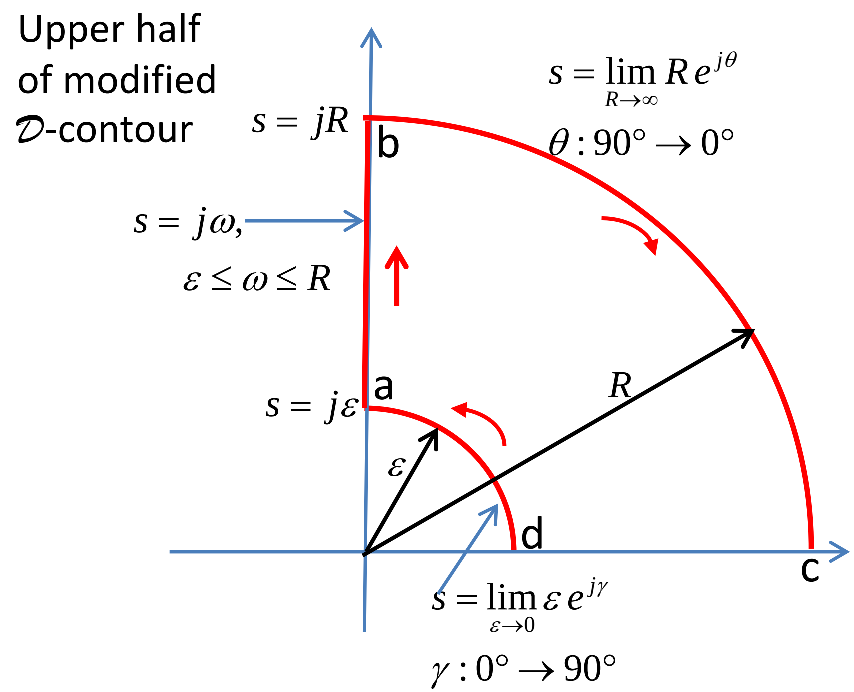
- The Nyquist Contour is modified to exclude the point of singularity





- Modified contour is same as the $\mathcal{D}$-contour except for the small semicircle around the pole





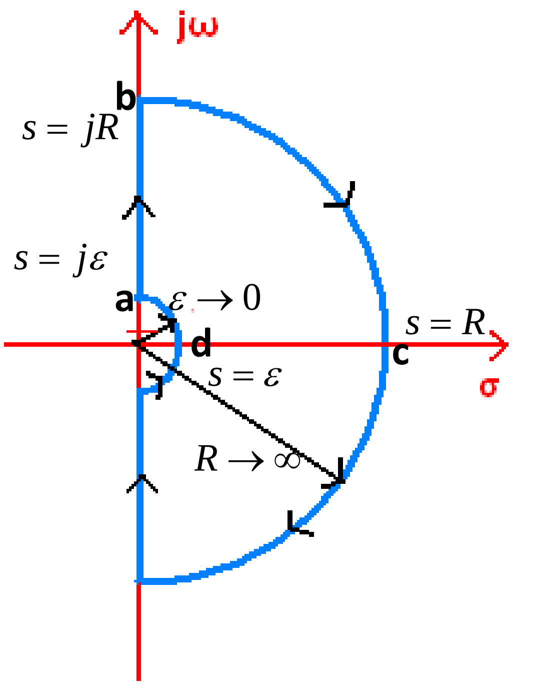
- The modified $\mathcal{D}$-contour is also symmetric about the real axis
  - We consider only the upper half

Sketching the Nyquist Plot: L(s) with Integrator

$$L(s) = \frac{K}{(s)^N} \frac{(s+b_1)(s+b_2)(s^2+c_1 s+d_1)....}{(s+a_1)(s+a_2)(s^2+p_1 s+q_1)....}$$

1) **Map the upper half of the Nyquist contour**
   a) Map the points 'd', 'a', 'b', and 'c'
      - Determine the points $L(\varepsilon)$, $L(j\varepsilon)$, $L(j$
        where $R \rightarrow \infty$, and $L(R)$
   b) Connect $L(\varepsilon)$ to $L(j\varepsilon)$
      - How is $L(s)$ changed as s is varied al
        the arc from 'd' to 'a'
   c) Connect $L(j\varepsilon)$ to $L(jR)$
   d) Connect $L(jR)$ to $L(R)$
      - Steps c and d are quite similar to wh
        was done for L(s) without integrator

2) **Complete the Nyquist plot by adding the mi**
   image of the plot obtained in step 1



Mapping of the points ‘d’ [L(ε)]

$$L(s) = \frac{K}{(s)^N} \frac{(s+b_1)\ldots(s^2+c_1s+d_1)\ldots}{(s+a_1)\ldots(s^2+p_1s+q_1)\ldots}$$

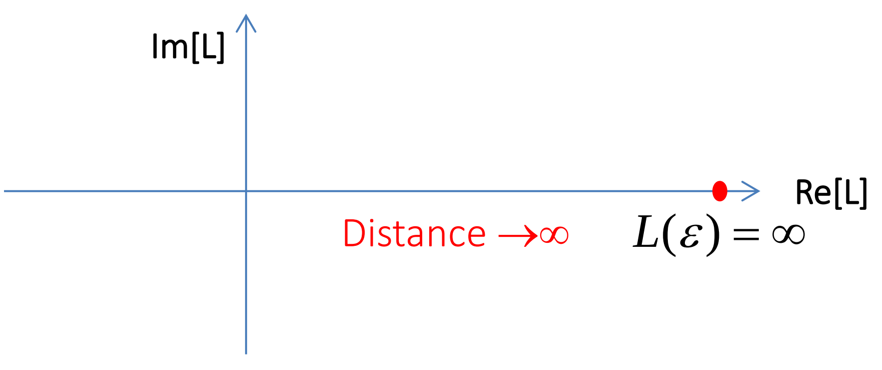
$$L(\varepsilon) = \frac{K}{(\varepsilon)^N} \frac{(\varepsilon+b_1)\ldots(\varepsilon^2+c_1\varepsilon+d_1)\ldots}{(\varepsilon+a_1)\ldots(\varepsilon^2+p_1\varepsilon+q_1)\ldots}$$

[real-valued]

As ε → 0,

$$L(\varepsilon) = \frac{K}{(\varepsilon)^N} \frac{(b_1)\ldots(d_1)\ldots}{(a_1)\ldots(q_1)\ldots} \rightarrow \infty$$

The point $L(\varepsilon)$ lies on the real-axis of the $L(s)$-plane and it is at infinite distance from the origin



Mapping of the points 'a' [L(jε)]

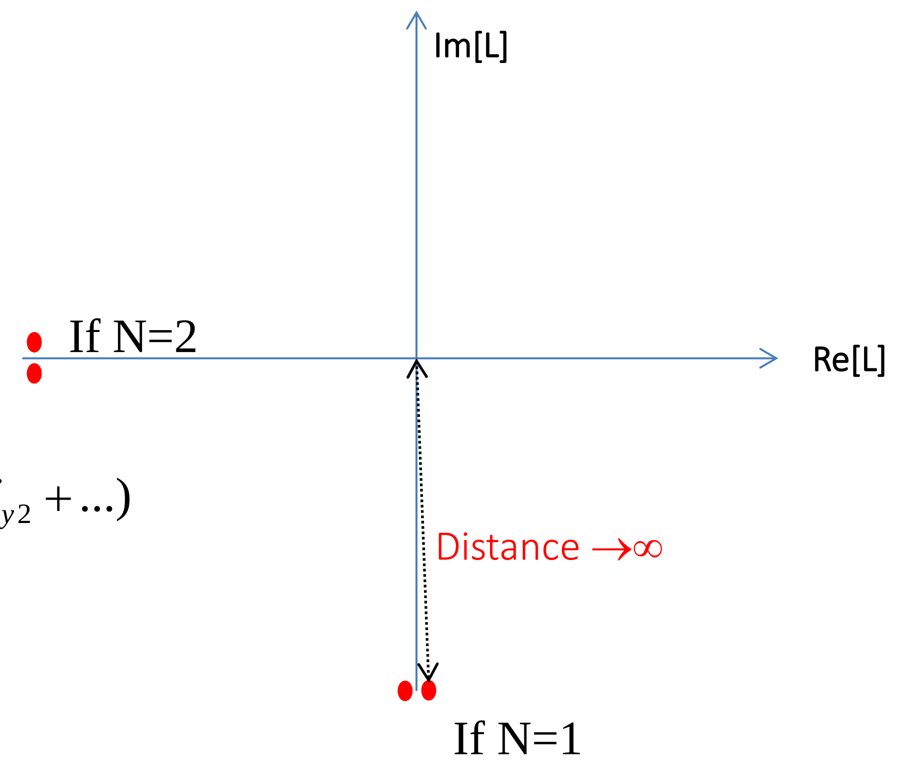
$$L(s)=\frac{K}{(s)^N}\frac{(s+b_1)\ldots(s^2+c_1s+d_1)\ldots}{(s+a_1)\ldots(s^2+p_1s+q_1)\ldots} \Rightarrow L(j\varepsilon)=\frac{K}{(j\varepsilon)^N}\frac{(j\varepsilon+b_1)\ldots(-\varepsilon^2+jc_1\varepsilon+d_1)\ldots}{(j\varepsilon+a_1)\ldots(-\varepsilon^2+jp_1\varepsilon+q_1)\ldots}$$

$$L(j\varepsilon)\Big|_{\varepsilon\to 0}=\frac{K}{(\varepsilon\angle 90^\circ)^N}\frac{(r_{x1}\angle\delta_{x1})(r_{x2}\angle\delta_{x2})\ldots}{(r_{y1}\angle\delta_{y1})(r_{y2}\angle\delta_{y2})\ldots}$$

[$\angle\delta_{x1}$, $\angle\delta_{x2}$, etc. are infinitesimally small angle]

$$|L(j\varepsilon)|=\frac{K}{(\varepsilon)^N}\frac{(r_{x1})(r_{x2})\ldots}{(r_{y1})(r_{y2})\ldots}\cong\infty$$

$$\angle L(j\varepsilon)=-N\times 90^\circ+(\angle\delta_{x1}+\delta_{x2}+\ldots)-(\angle\delta_{y1}+\delta_{y2}+\ldots)$$
$$=-N\times 90^\circ\pm\delta$$



Sketching the Nyquist Plot: L(s) with Integrator

- *Example* 3b-14: Map points ($s=\varepsilon$) and ($s=j\varepsilon$) for the transfer function

$$L(s)=\frac{K(s+5)}{s(s^2+2s+5)}$$

- Answer:

$$s=\varepsilon,\quad L(\varepsilon)=\frac{K}{\varepsilon}\,\frac{(\varepsilon+5)}{(\varepsilon^2+2\varepsilon+5)}$$

$$\varepsilon\to 0,\quad L(\varepsilon)=\frac{K}{\varepsilon}\to\infty$$

$$s=j\varepsilon,\quad s^2=-\varepsilon^2$$

$$L(j\varepsilon)=\frac{(j\varepsilon+5)}{(j\varepsilon)(5-\varepsilon^2+j2\varepsilon)}$$

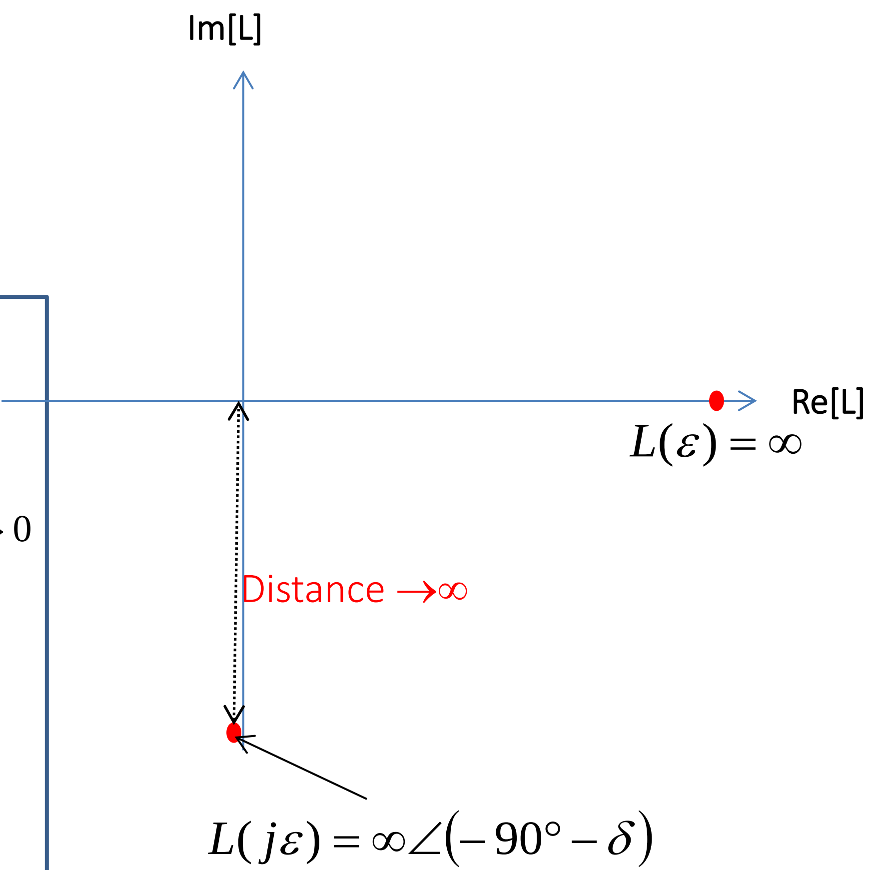
$$\left|L(j\varepsilon)\right|=\frac{K\sqrt{\varepsilon^2+25}}{\varepsilon\sqrt{(5-\varepsilon^2)^2+4\varepsilon^2}}\cong\frac{5K}{5\varepsilon}=\infty,\quad \because\varepsilon\to 0$$

$$\angle L(j\varepsilon)=\tan^{-1}\tfrac{\varepsilon}{5}-90^\circ-\tan^{-1}\tfrac{2\varepsilon}{5-\varepsilon^2}$$

$$\tfrac{\varepsilon}{5-\varepsilon^2}>\tfrac{\varepsilon}{5}\quad\Rightarrow\quad\tfrac{2\varepsilon}{5-\varepsilon^2}>\tfrac{\varepsilon}{5}$$

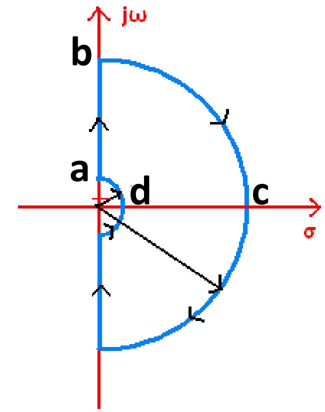
$$\tan^{-1}\tfrac{2\varepsilon}{5-\varepsilon^2}>\tan^{-1}\tfrac{\varepsilon}{5}$$

$$\angle L(j\varepsilon)=-90^\circ-\delta$$



Connecting L(ε) to L(jε), ε→0

- For this segment (upper half of the small semicircle),



$$s = \varepsilon\, e^{j\gamma}, \quad \varepsilon \to 0, \gamma : 0^\circ \to 90^\circ$$

$$L(s) = \frac{K}{(s)^N} \frac{(s+b_1)\ldots(s^2+c_1 s+d_1)\ldots.}{(s+a_1)\ldots(s^2+p_1 s+q_1)\ldots.}$$

$$s = \varepsilon\, e^{j\gamma} \quad \Rightarrow \quad s^2 = \varepsilon^2 e^{j2\gamma}$$

$$L(s) = \frac{K}{(\varepsilon\, e^{j\gamma})^N} \frac{(\varepsilon\, e^{j\gamma}+b_1)\ldots(\varepsilon^2\, e^{j2\gamma}+c_1\varepsilon\, e^{j2\gamma}+d_1)\ldots.}{(\varepsilon\, e^{j\gamma}+a_1)\ldots(\varepsilon^2\, e^{j2\gamma}+p_1\varepsilon\, e^{j\gamma}+q_1)\ldots.}$$

$$L(s) = \frac{K}{(\varepsilon^N \angle N\times\gamma)} \frac{(b_1)\ldots(d_1)\ldots.}{(a_1)\ldots(q_1)\ldots.} = \infty\angle(-N\times\gamma)$$

Linear factor

$$(\varepsilon\, e^{j\gamma}+b_1) \cong b_1$$

Quadratic factor

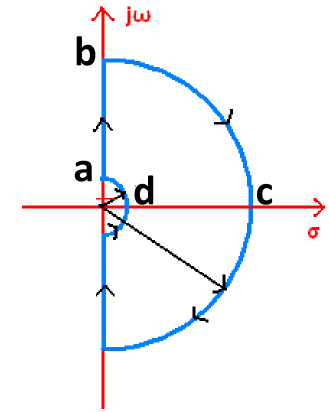
$$(\varepsilon^2 e^{j2\gamma}+c_1\varepsilon\, e^{j\gamma}+d_1) \cong d_1$$

Integrator

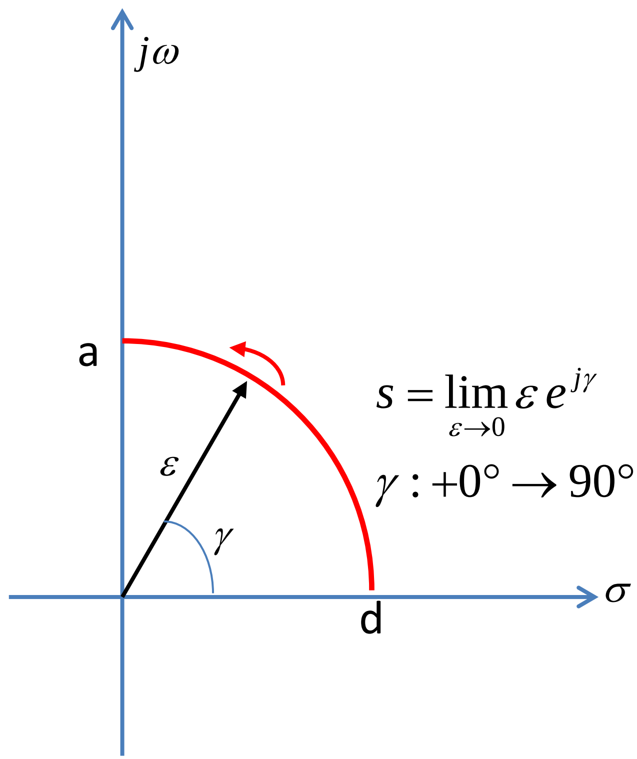
$$\frac{K}{(\varepsilon\, e^{j\gamma})^N} = \frac{K}{\varepsilon^N \angle N\times\gamma}$$

Sketching the Nyquist Plot: L(s) with Integrator

$$L(s) = \frac{K}{(\varepsilon^N \angle N \times \gamma)} \frac{(b_1)\ldots(d_1)\ldots.}{(a_1)\ldots(q_1)\ldots.} = \infty\angle(-N \times \gamma)$$

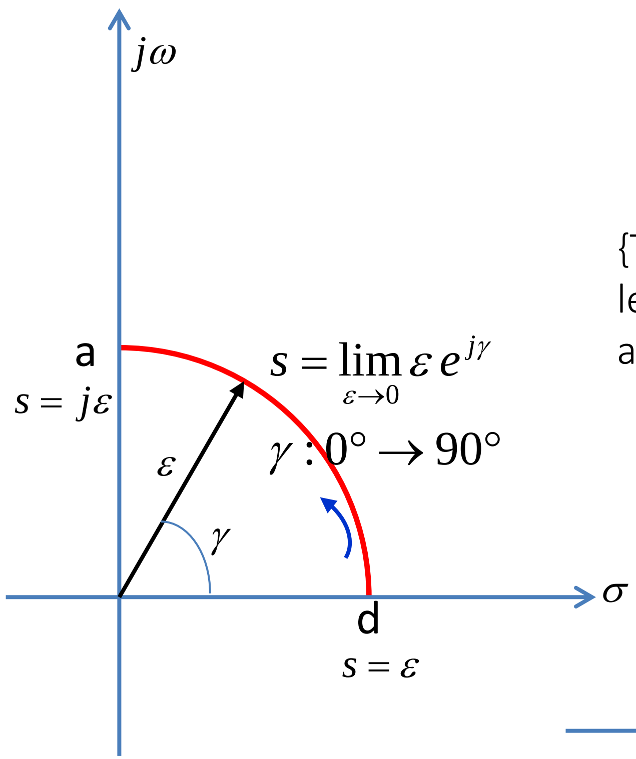


- Plot of L(s) is an arc of infinitely large radius
- Argument of L(s) is changed through $-N\gamma$ as γ is varied

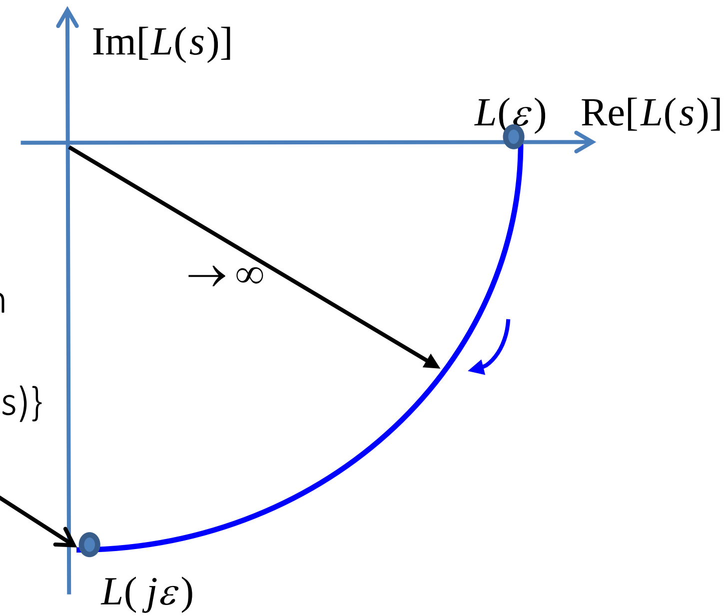


- γ is varied from 0° to 90° CCW while going from ‘d’ to ‘a’
  - **$\angle$L(s) goes through a net clockwise change of N×90°**

Sketching the Nyquist Plot: L(s) with Integrator

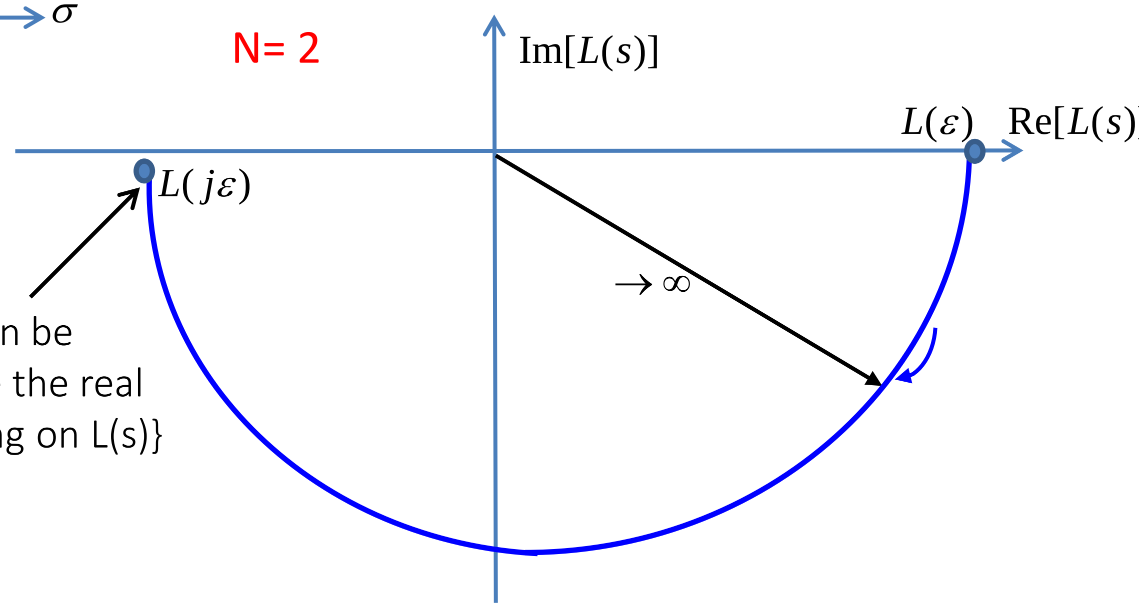


N= 1



{This point can be on left of the imaginary axis depending on L(s)}

N= 2



{This point can be slightly above the real axis depending on L(s)}

Sketching the Nyquist Plot: L(s) with Integrator

- *Example* 3b-15: For the following transfer function, sketch L(s) when $s = \varepsilon e^{j\gamma}$

$$\varepsilon \approx 0, \gamma : +0° \to 90°$$

$$L(s) = \frac{K}{s^2}\frac{(s+2)}{(s+5)}$$

- Answer:

$$L(s) = \frac{K}{(\varepsilon e^{j\gamma})^2}\frac{(\varepsilon e^{j\gamma}+2)}{(\varepsilon e^{j\gamma}+5)}$$

$$\cong \frac{K}{(\varepsilon e^{j\gamma})^2}\frac{(2)}{(5)}$$

$$|L(s)| = \frac{2K}{5(\varepsilon)^2} \to \infty,$$

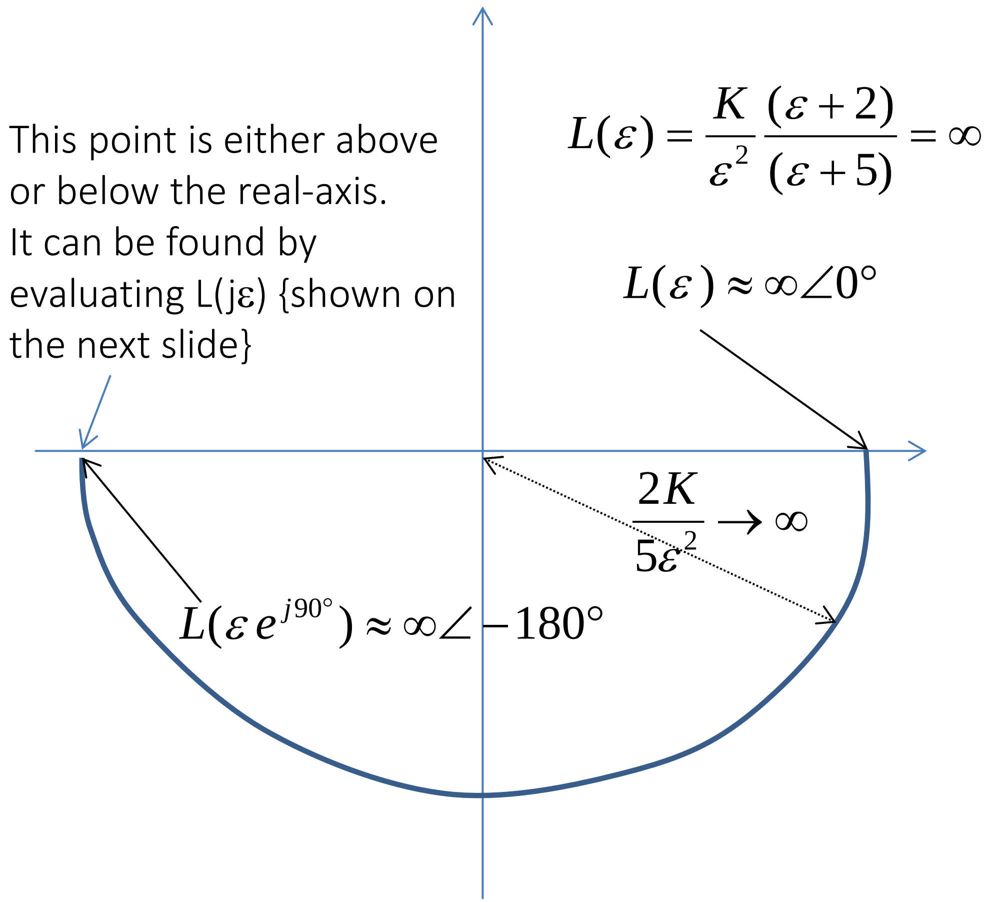
$$\angle L(s) \cong -2 \times \gamma$$

$\gamma = 0° :$

$$|L(s)| \to \infty, \quad \angle L(s) = 0°$$

$\gamma = 90° :$

$$|L(s)| \to \infty, \quad \angle L(s) \cong -180°$$



Sketching the Nyquist Plot: L(s) with Integrator

$$L(s) = \frac{K}{s^2}\frac{(s+2)}{(s+5)}$$

$$s = \varepsilon e^{j\gamma}$$

$$\varepsilon \approx 0, \gamma : +0^\circ \to 90^\circ$$

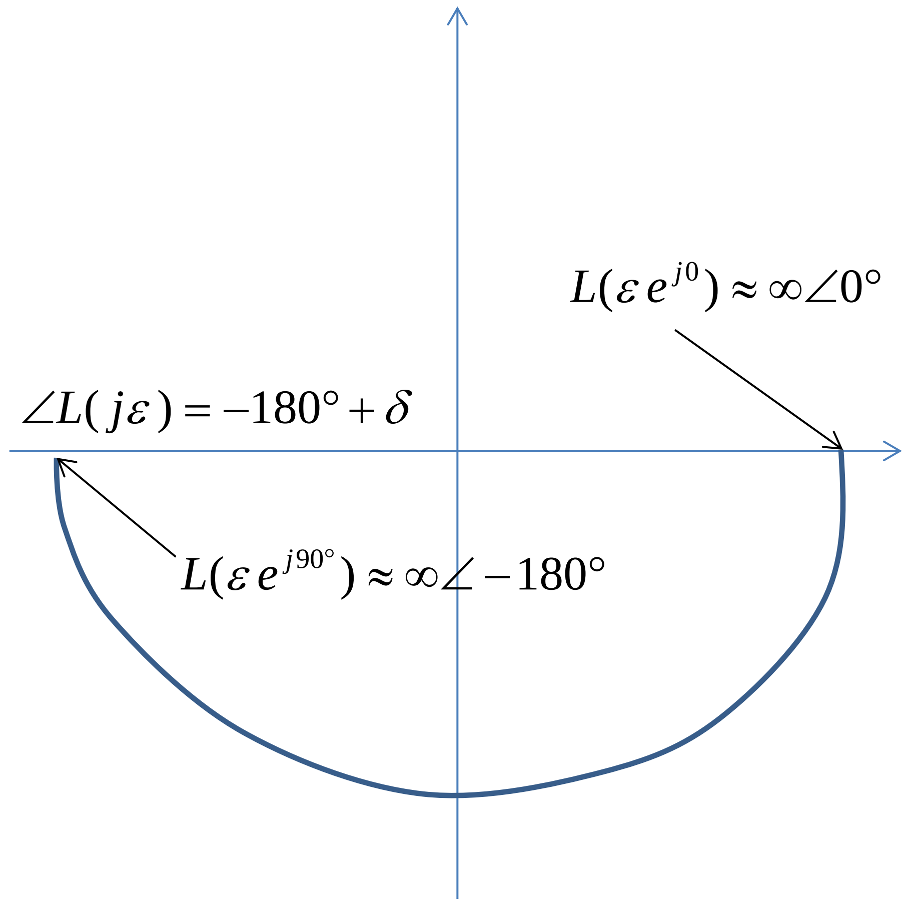
$$\gamma = 90^\circ : \quad s = j\varepsilon$$

$$L(s) = \frac{K}{(j\varepsilon)^2}\frac{(2+j\varepsilon)}{(5+j\varepsilon)}$$

$$\angle L(s) \cong -2\times 90^\circ + \tan^{-1}\frac{\varepsilon}{2} - \tan^{-1}\frac{\varepsilon}{5}$$

$$\tan^{-1}\frac{\varepsilon}{2} > \tan^{-1}\frac{\varepsilon}{5}, \quad for\ \forall \varepsilon$$

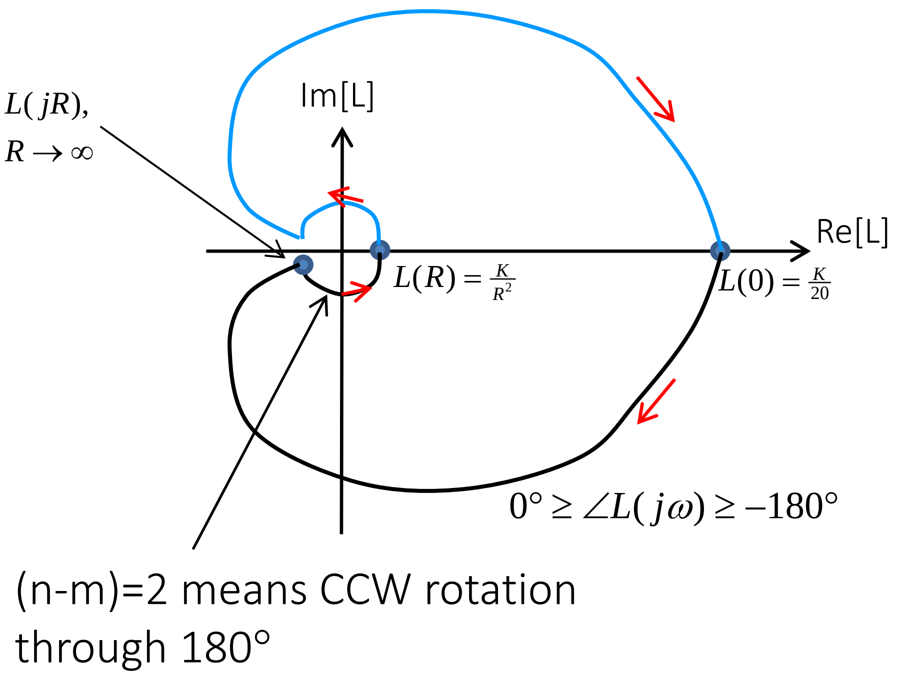
$$\angle L(j\varepsilon) = -180^\circ + \delta, \quad \delta > 0$$

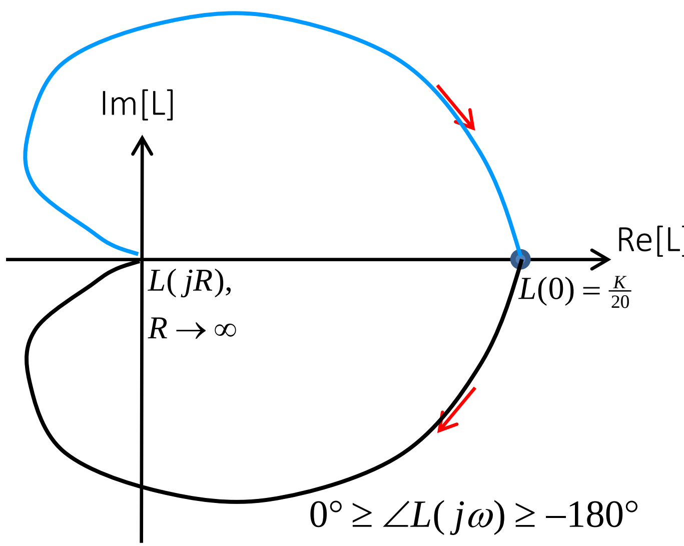


Sketching the Nyquist Plot: Mapping of $Re^{j\theta}$ Ignored

- When the big semicircle is mapped, it gives an arc of infinitesimally small radius $\frac{K}{R^{(n-m)}}$
  - This is too small compared to other parts of the Nyquist plot and hence is often drawn as a point

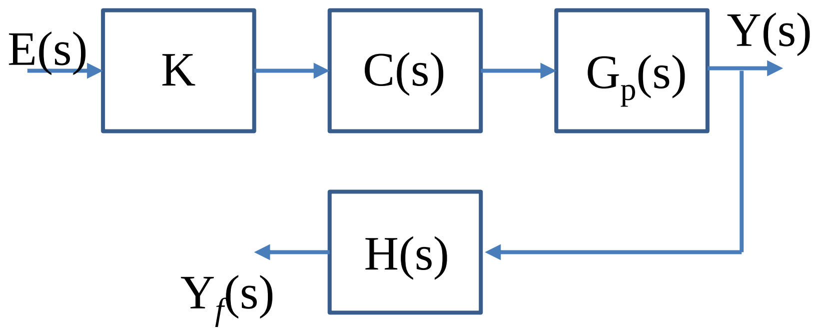
- *Example* 3b-16: $L(s) = \dfrac{K}{(s+2)(s+10)}$





Stability Check using the Nyquist Plot of L(s)

1. Sketch the Nyquist plot of the loop transfer function $L(s)$ by following the Nyquist contour in CW direction



$$L(s) = KC(s)G_p(s)H(s)$$
$$= KG(s)$$

2. Count N_{CW}, the number of CW encirclement of the point (-1,0)

3. Determine N_P, the number of OL poles inside the Nyquist contour

4. Find N_Z from the equation $N_{CW} = N_Z - N_P$

   - N_Z is the number of zeros of F(s) in RHP, i.e., the number of closed loop poles in RHP
   - For CL stability, N_Z must be 0

Stability Check using the Nyquist Plot of L(s)

- *Example* 3b-17: Check stability of the closed loop if the loop transfer function is

$$L(s) = \frac{100}{(s+1)(0.1s+1)}$$

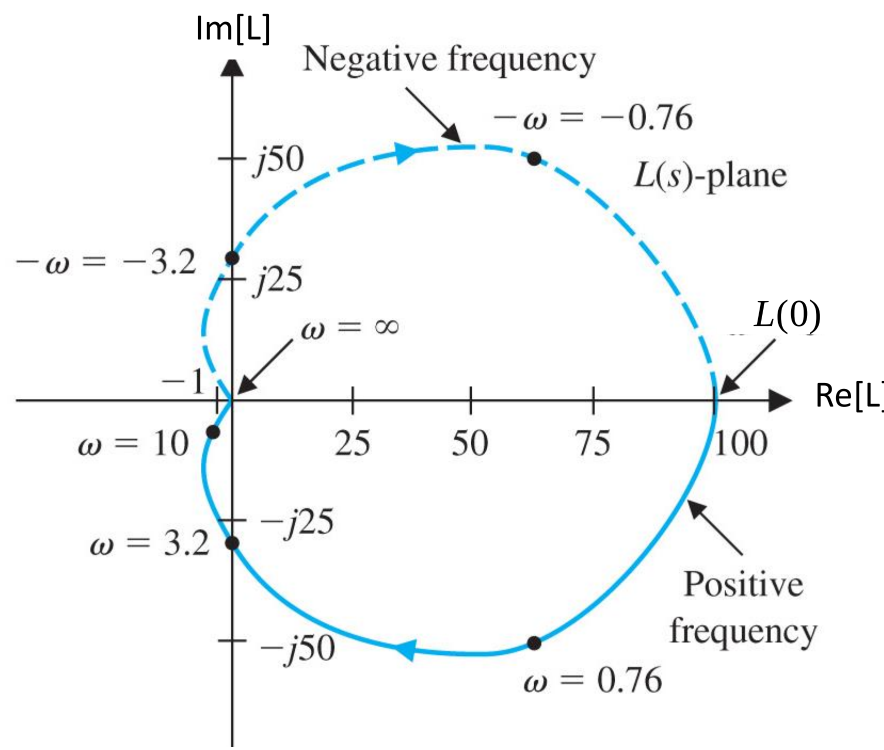
- Answer:

$$L(0) = \frac{100}{(0+1)(0+1)} = 100$$

$$L(jR) = \frac{100}{(jR+1)(j0.1R+1)}$$

$$|L(jR)| = \frac{100}{\sqrt{R^2+1}\sqrt{0.01R^2+1}} \cong 0, \quad \because R \to \infty$$

$$\begin{aligned}\angle L(jR) &= -\tan^{-1} R - \tan^{-1}(0.1R) \\ &\cong -90° - 90°, \quad \because R \to \infty \\ &\cong -180°\end{aligned}$$



- From the Nyquist plot, $N_{CW} = 0$
- $L(s)$ is open loop stable, i.e., $N_P = 0$

$$N_{CW} = N_Z - N_P \quad \Rightarrow \quad N_Z = 0$$

CL is stable

Stability Check using the Nyquist Plot of L(s)

- *Example* 3b-18: Check CL stability if

$$L(s)=\frac{K}{s(\tau s+1)}$$

- Answer:

$$L(\varepsilon)=\frac{K}{\varepsilon(\tau\varepsilon+1)}\to\infty$$

$$L(j\varepsilon)=\frac{K}{(j\varepsilon)(j\varepsilon\tau+1)}$$

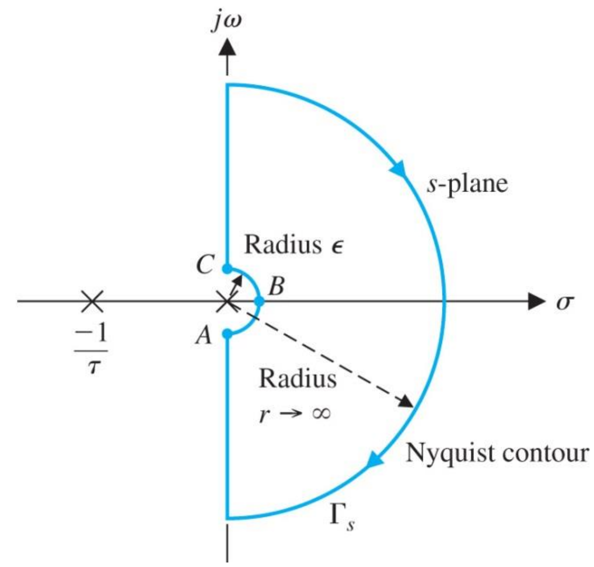
$$|L(j\varepsilon)|\to\infty,\quad \because \varepsilon\to 0$$

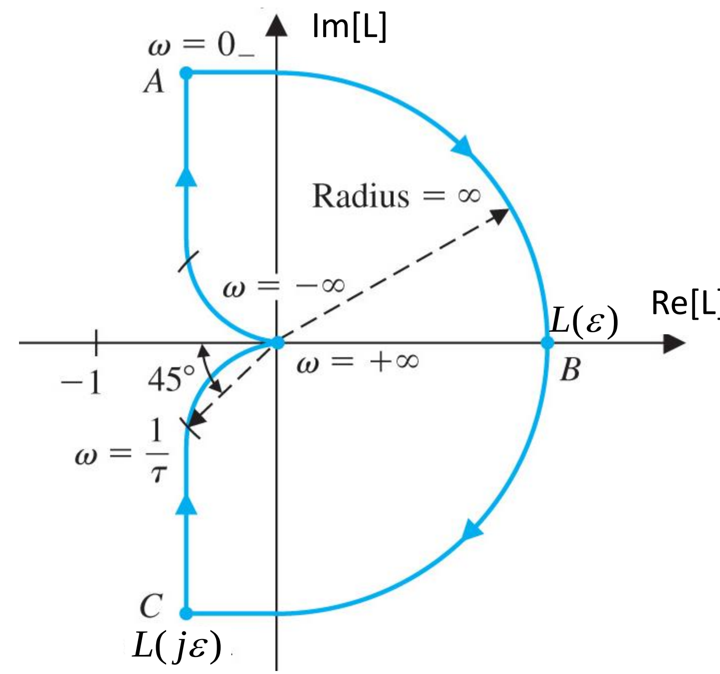
$$\angle L(j\varepsilon)=-90^\circ-\tan^{-1}(\varepsilon\tau)$$
$$\cong -90^\circ-\delta,$$
$$\delta\approx 0,\quad \because \varepsilon\to\infty$$

$$L(jR)=\frac{K}{(jR)(jR\tau+1)}$$

$$|L(jR)|=\frac{100}{R\sqrt{\tau^2R^2+1}}\cong 0$$

$$\angle L(jR)=-90^\circ-\tan^{-1}(\tau R)$$
$$\cong -180^\circ,\quad \because R\to\infty$$





- From the Nyquist plot, $N_{CW}=0$
- $L(s)$ has no pole inside Γ_s, i.e., $N_P=0$

$$N_{CW}=N_Z-N_P \quad\Rightarrow\quad N_Z=0$$

CL is stable

Stability Check using the Nyquist Plot of L(s)

- *Example* 3b-19: For the following L(s), find the range of K such that CL is stable

$$L(s)=\frac{K}{s(s+1)^2}$$

- Answer:

$$L(\varepsilon)=\frac{K}{\varepsilon(\tau_1\varepsilon+1)^2}\rightarrow\infty \qquad L(j\omega)=\frac{K}{(j\omega)(j\omega+1)^2}$$

$$\angle L(j\omega)=-90^\circ-2\tan^{-1}(\omega)$$
$$-90^\circ\geq\angle L(j\omega)\geq-270^\circ$$

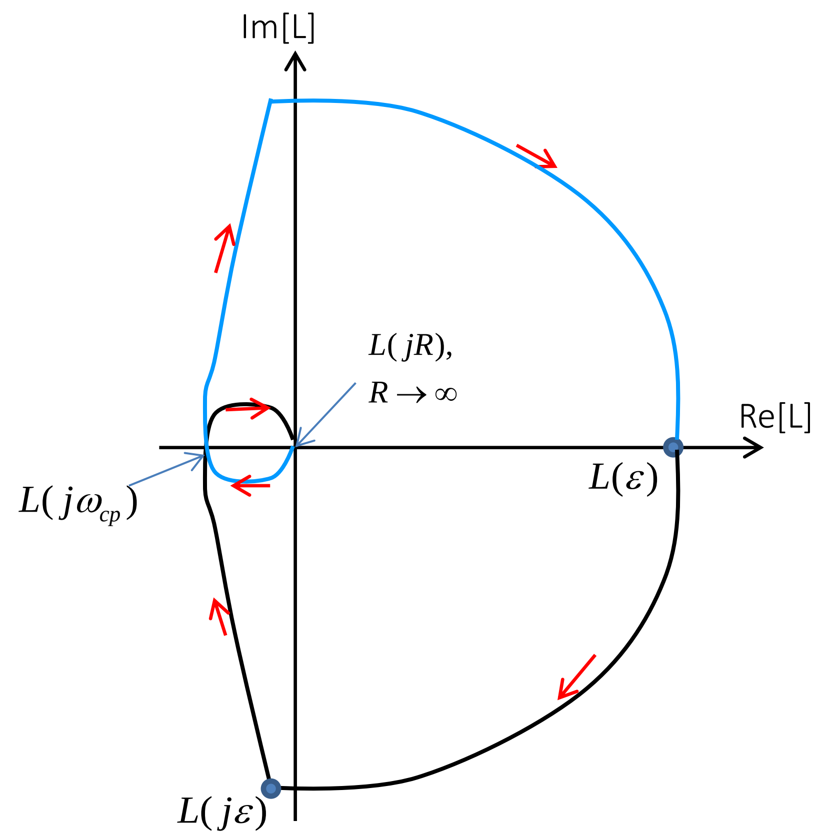
[3rd and 2nd quadrant]

$$|L(j\varepsilon)|\rightarrow\infty,\quad \because \varepsilon\rightarrow 0$$

$$\angle L(j\varepsilon)=-90^\circ-2\tan^{-1}(\varepsilon)$$
$$=-90^\circ-\delta,$$

$$|L(jR)|=\frac{K}{R(R^2+1)}\cong 0$$

$$\angle L(jR)=-90^\circ-2\tan^{-1}(R)$$
$$\cong-270^\circ,\quad \because R\rightarrow\infty$$



- The value of $L(j\omega_{cp})$ determines whether (-1,0) is encircled or not
  - We need to find this point

Stability Check using the Nyquist Plot of L(s)

$$L(j\omega) = \frac{K}{(j\omega)(j\omega+1)^2}$$

$$L(j\omega) = \frac{K}{j\omega(1-\omega^2 + j2\omega)}$$

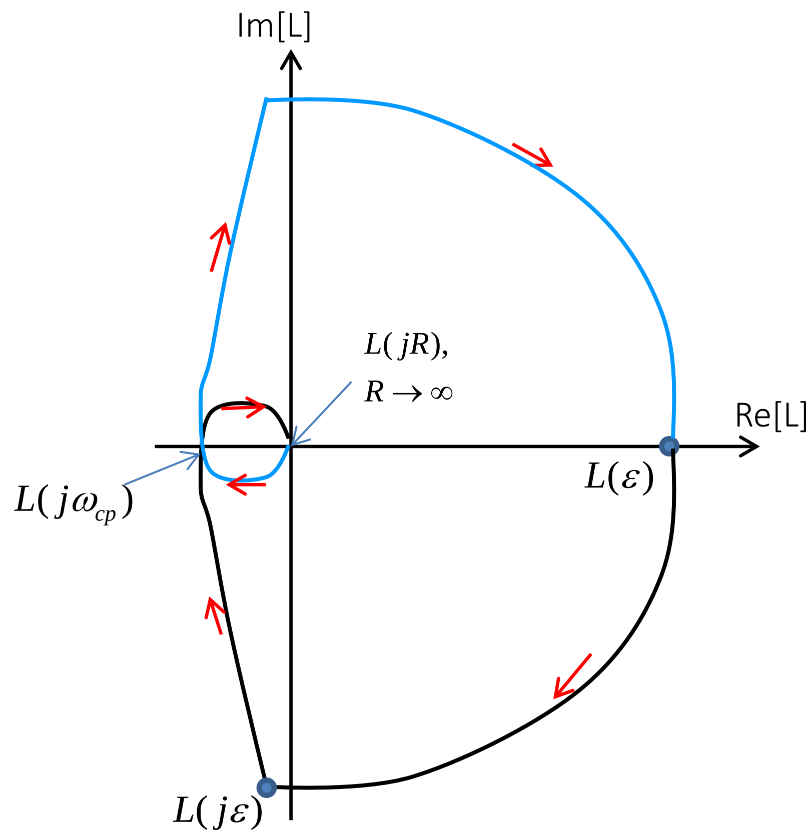
$$= \frac{K}{-2\omega^2 + j\omega(1-\omega^2)}$$

For $L(j\omega_{cp})$, imaginary part is equal to 0

$$\omega_{cp} = 1$$

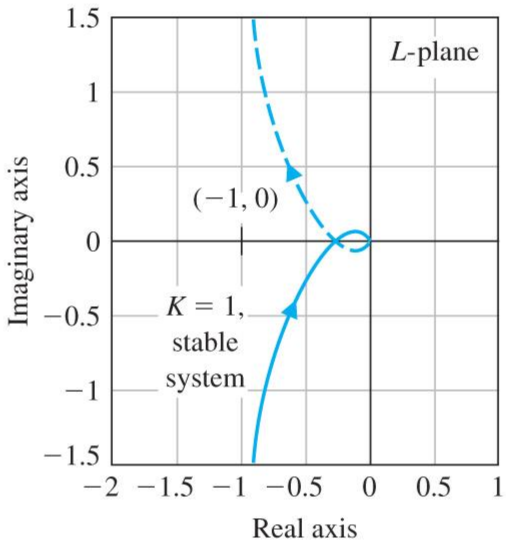
$$L(j\omega_{cp}) = \frac{K}{-2\omega_{cp}^2} = -\frac{K}{2}$$

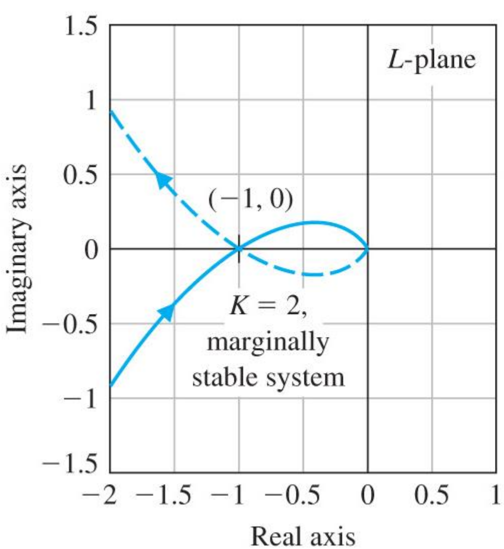
- If $K<2$, $N_{CW}=0$ and the CL is stable
- Otherwise, CL is unstable

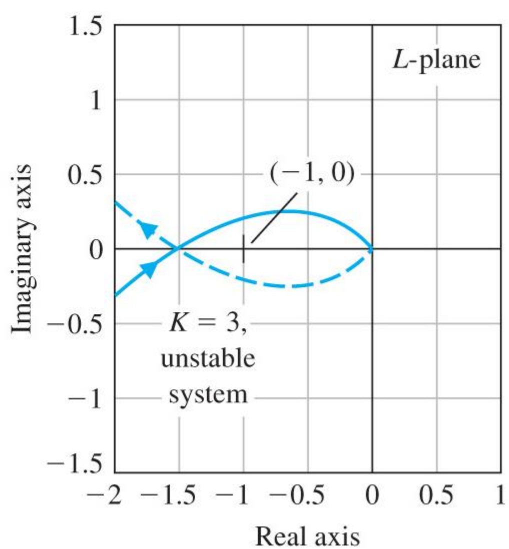


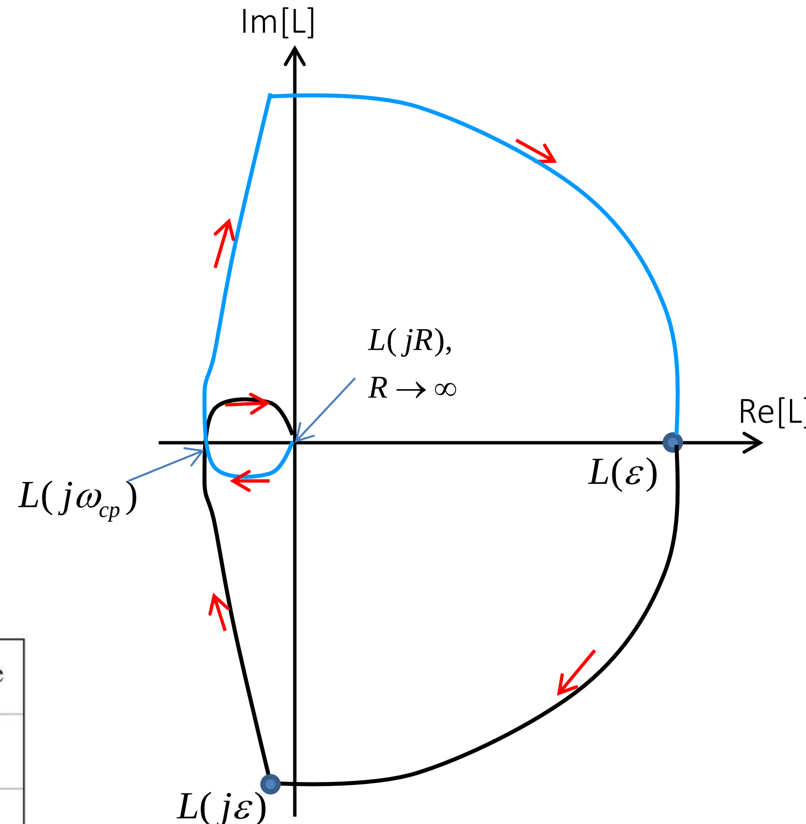
- The next slide shows the situations for three different values of K

Stability Check using the Nyquist Plot of L(s)









Stability Check using the Nyquist Plot of L(s)

- *Example* 3b-20: Check CL stability if $L(s) = \dfrac{K}{s^2(\tau s+1)}$

- Answer:

$$L(\varepsilon) = \frac{K}{\varepsilon^2(\tau\varepsilon+1)} \to \infty$$

$$L(j\omega) = \frac{K}{(j\omega)^2(j\omega\tau+1)}$$

$$\angle L(j\omega) = -180^\circ - \tan^{-1}(\tau\omega)$$

$$-180^\circ \geq \angle L(j\omega) \geq -270^\circ$$

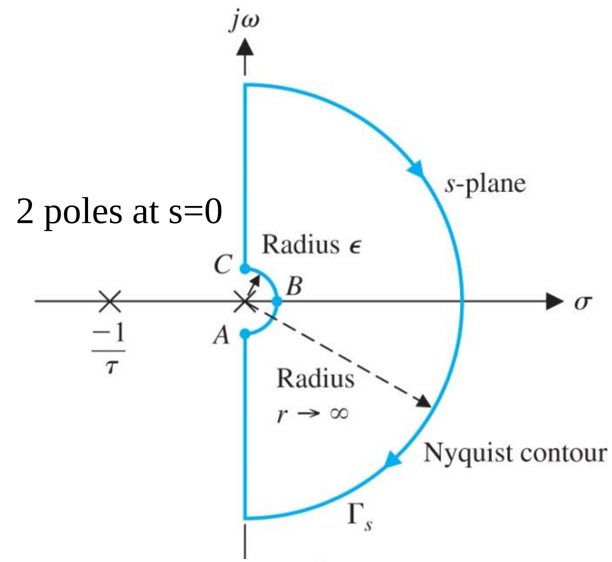
[2nd quadrant]

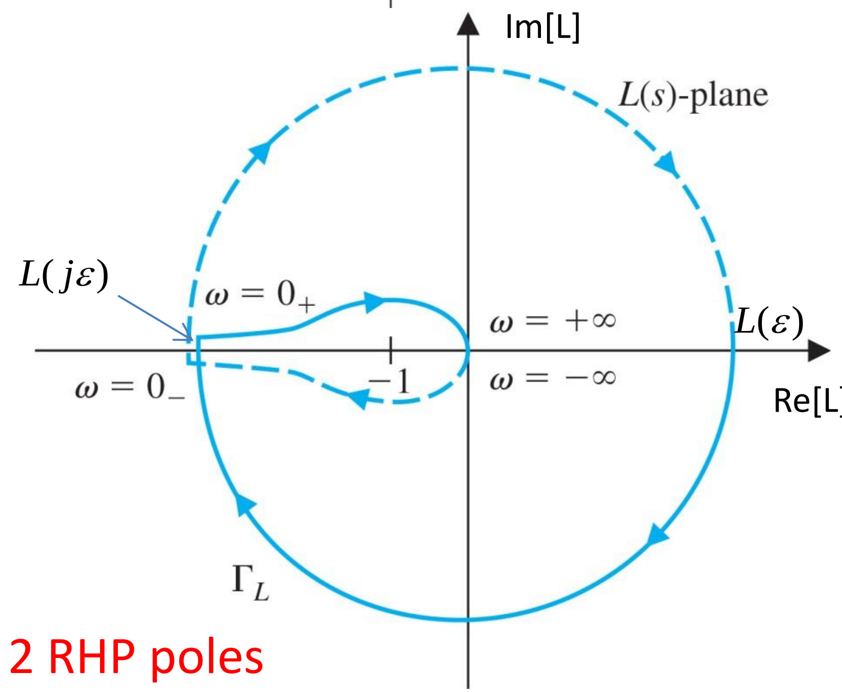
$$|L(j\varepsilon)| \to \infty, \quad \because \varepsilon \to 0$$

$$\angle L(j\varepsilon) = -180^\circ - \tan^{-1}(\varepsilon\tau) \cong -180^\circ - \delta$$

$$|L(jR)| = \frac{K}{R^2\sqrt{\tau^2R^2+1}} \cong 0 \quad \because R \to \infty$$

$$\angle L(jR) = -180^\circ - \tan^{-1}(\tau R) \cong -270^\circ, \quad R \to \infty$$





- From the Nyquist plot, N_{CW} = 2
- $L(s)$ has no pole inside Γ_s, i.e., N_P = 0

$$N_{CW} = N_Z - N_P \quad \Rightarrow \quad N_Z = 2$$

CL is unstable with 2 RHP poles