EE3331C/EE3331E Feedback Control Systems
L7: Control System Performance: transient & steady-state

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Outline

Review

- We have so far looked at the advantages and disadvantages of open-loop vs. closed-loop control. In this chapter, we are interested in the performance of closed-loop control systems.
- Performance refers to how well the system responds to inputs. It is usually given in terms of time domain specifications such as rise time, settling time, steady state error, etc.
- Steady-state response: DC gain, $G(0)$.
- Transient response: time constants, τ, damping ratios, ζ, and natural frequencies, ω_n.

- How does DC gain affect the steady state response?
- DC gain is defined as the gain at zero frequency.

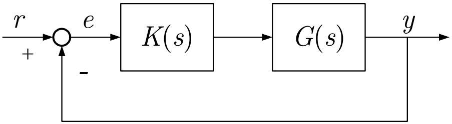


Figure 7.1 : Unity feedback system.

- For the above unity feedback system,
  - Open-loop t.f.: $L(s) = G(s)K(s)$,

    the open-loop DC gain is $G(0)K(0)$.
  - Closed-loop t.f.: $G_{cl}(s) = \dfrac{G(s)K(s)}{1 + G(s)K(s)}$,

    the closed-loop DC gain is $G_{cl}(0) = \dfrac{G(0)K(0)}{1 + G(0)K(0)}$.

- Suppose the setpoint is a step input of magnitude r, the output is given by:

$$Y(s) = \frac{G(s)K(s)}{1 + G(s)K(s)} R(s)$$

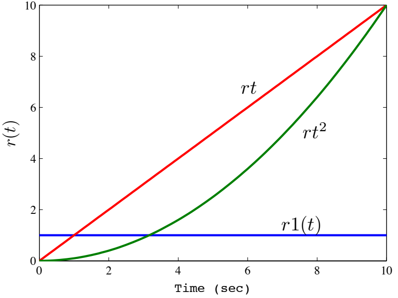
If the closed-loop is stable, then the steady-state output is (via final value theorem):

$$\begin{aligned} y_{ss} &= \lim_{s \to 0} sY(s) \\ &= \lim_{s \to 0} \frac{sG(s)K(s)}{1 + G(s)K(s)} \frac{r}{s} \\ &= \frac{G(0)K(0)}{1 + G(0)K(0)} r \\ &= G_{cl}(0) r \end{aligned}$$

- Hence for zero steady-state error (i.e. $y_{ss} = r$), we require $G_{cl}(0) = 1$.

  Note that the above statement is true only for the step inputs of arbitrary magnitude.

- For the feedback configuration in Figure 7.1, the transfer function from r to error, e is
$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)K(s)}$$
- We will consider steady-state errors in stable systems for general polynomial inputs,
$$r(t) = rt^{n-1}$$
Its Laplace transform is given by $R(s) = \dfrac{C\,r}{s^n}$ where $C = (n-1)!$



- $n = 1$: step input
- $n = 2$: ramp input
- $n = 3$: parabolic input

- For stable system, the steady-state error is given by

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} s\frac{1}{1+G(s)K(s)}\frac{C\,r}{s^n} \\ &= \lim_{s\to 0} \frac{C\,r}{s^{n-1}+s^{n-1}G(s)K(s)} \end{aligned}$$

- Suppose that the open-loop transfer function, $G(s)K(s) = \frac{1}{s^m}P(s)$ where $P(s)$ does not contain any integrators, then

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} \frac{C\,r}{s^{n-1}+s^{n-1}\frac{P(s)}{s^m}} \\ &= \lim_{s\to 0} \frac{C\,rs^m}{s^{n+m-1}+s^{n-1}P(s)} \\ &= \lim_{s\to 0} \frac{C\,rs^{m-n+1}}{s^m+P(s)} \end{aligned}$$

To have $e_{ss} = 0$, we need $m > n-1$.

- For $m = 0$, i.e. $G(s)K(s)$ without integrators.

  The steady-state error can be further simplify to

$$
\begin{aligned}
e_{ss} &= \lim_{s\to 0} \frac{C\,r s^{m-n+1}}{s^m + P(s)} \\
&= \lim_{s\to 0} \frac{C\,r s^{-(n-1)}}{1+p} \\
&= \begin{cases} \dfrac{C\,r}{1+p} & \text{if} \quad n = 1 \quad \text{constant step input} \\ \infty & \text{if} \quad n > 1 \quad \text{ramp and other higher order inputs} \end{cases}
\end{aligned}
$$

where $p = P(s)|_{s=0} = P(0)$.

- For $m > 0$, i.e. $G(s)K(s)$ has at least one integrator.
  The steady-state error can be further simplify to

$$
\begin{aligned}
e_{ss} &= \lim_{s \to 0} \frac{C\, r s^{m-n+1}}{s^m + P(s)} \\
&= \lim_{s \to 0} \frac{C\, r s^{m-(n-1)}}{p} \\
&= \begin{cases} 0 & \text{if } \; m > n-1 \\ \dfrac{C\, r}{p} & \text{if } \; m = n-1 \\ \infty & \text{if } \; m < n-1 \end{cases}
\end{aligned}
$$

- In conclusion, if $G(s)K(s)$ contains n or more integrators, then the closed loop system will track an input given by $r(t) = rt^{(n-1)}$ without any steady state error.

- For the unity feedback system in Figure 7.1, the steady-state error is summarized in the following table:

$r(t) = rt^{(n-1)}$	Constant input $r(t) = r$ $R(s) = \frac{r}{s}$ $n = 1$	Ramp input $r(t) = rt$ $R(s) = \frac{r}{s^2}$ $n = 2$	Parabolic input $r(t) = rt^2$ $R(s) = \frac{2r}{s^3}$ $n = 3$
0 integrator in the loop i.e. $m = 0$	$\frac{r}{1+p}$	∞	∞
1 integrator in the loop i.e. $m = 1$	0	$\frac{r}{p}$	∞
2 integrators in the loop i.e. $m = 2$	0	0	$\frac{2r}{p}$
3 or more integrators in the loop i.e. $m \geq 3$	0	0	0

Table 7.1 : Steady-state error for different input and system types.

- A stable system can be classified as a **system type**. A system of **type** k indicates the ability of the system to achieve zero steady-state error to polynomials of degree less than but not equal to k, i.e. k is the number of integrators in the system $(G(s)K(s))$.
- We can further define the following error constants:
  - Position Error Constant:

$$K_p = \lim_{s \to 0} G(s)K(s)$$

  - Velocity Error Constant:

$$K_v = \lim_{s \to 0} sG(s)K(s)$$

  - Acceleration Error Constant:

$$K_a = \lim_{s \to 0} s^2 G(s)K(s)$$

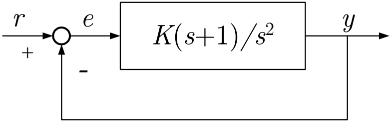
Control systems are often described in terms of their system type and error constants, K_p, K_v, and K_a. *How are these error constants related to the steady-state errors?*

- For the unity feedback system in Figure 7.1, the error constants and system types is summarized in the following table (compare with Table 7.1):

No. of integrators in the loop (System Type)	Constant input $r(t) = r$ $R(s) = \frac{r}{s}$	Ramp input $r(t) = rt$ $R(s) = \frac{r}{s^2}$	Parabolic input $r(t) = rt^2$ $R(s) = \frac{2r}{s^3}$
Type 0	$\frac{r}{1+K_p}$	∞	∞
Type 1	0	$\frac{r}{K_v}$	∞
Type 2	0	0	$\frac{2r}{K_a}$
Type $k \geq 3$	0	0	0

Table 7.2 : Steady-state error as a function of system types.

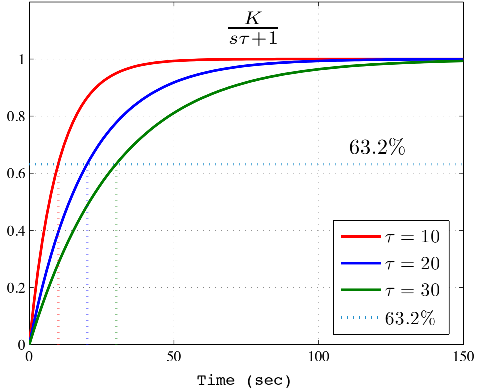
- Example: A laser beam can be used to weld, drill, etch, cut and mark metals. Suppose we require an accurate laser beam to mark a parabolic path with a closed loop control system as shown below. Calculate the necessary gain to result in a steady state error of 5 mm for $r(t) = t^2$ cm.



- Do you expect to have a steady state error for $r(t) = t^2$?
- How many integrators do you need to have zero steady state error to this $r(t)$?

- We have previously seen the effects of the time constant, damping ratio and natural frequencies on the transient response.
- Time constant, τ: Consider the first-order system

$$G(s) = \frac{K}{s\tau + 1}$$

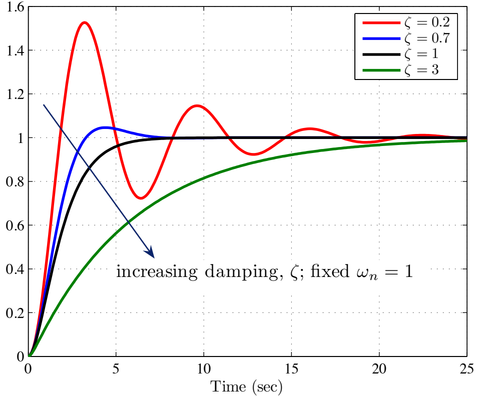


- the larger the τ, the closer the pole to the origin $\rightarrow$ the slower the response
- thus poles location determine the transient response

- Damping ratio, ζ: Consider the standard second-order system

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

with poles at $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$.

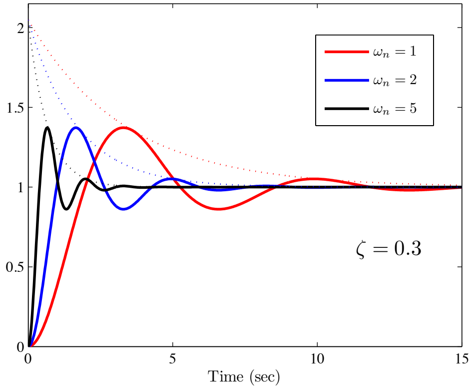


- $\zeta < 1$, underdamped system, complex poles: as $\zeta \uparrow$, the response is more slugglish and less oscillations.
- $\zeta = 1$, critically-damped system, repeated real poles with no oscillations.
- $\zeta > 1$, overdamped system, distinct real poles.

- Note: for $\zeta \leq 1$, it is called the damping ratio and defined as $\zeta = \cos\beta$.

- **Natural frequency, ω_n**: Consider the standard second-order system

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



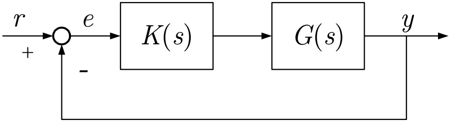
- the smaller the ω_n, the more sluggish is the response
- since ζ is the same, the overshoot is the same:

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

- performance also define by: rise-time t_r, peak-time t_p, settling-time t_s.

Example 1

- Consider the feedback system with $G(s) = \dfrac{2}{3s+4}$



Design a gain controller, K, such that the closed loop system is stable and has a time constant less than 0.25 sec and steady state error of less than 1%.

- Closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = G(s) = \frac{G(s)K(s)}{1+G(s)K(s)}$$

$$= \frac{2K}{3s+4+2K} = \frac{\frac{K}{2+K}}{\frac{3}{4+2K}s+1}$$

where the closed-loop time constant and steady-state gain are given by $\tau_c = \dfrac{3}{4+2K}$ and $K_c = \dfrac{K}{2+K}$

- The specification requires the following:

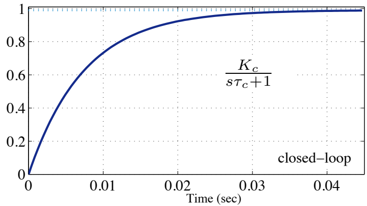
$$
\begin{aligned}
\tau_c = \frac{3}{4+2K} &< 0.25 \\
\Rightarrow K &> 4
\end{aligned}
$$

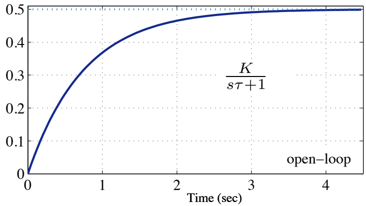
and $0.99 < K_c < 1.01$ which gives

$$
\begin{aligned}
0.99 < \frac{K}{2+K} \quad &\text{and} \quad \frac{K}{2+K} < 1.01 \\
1.98 + 0.99K < K \quad &\quad K < 2.02 + 1.01K \\
K > 198 \quad &\quad K > -2.02
\end{aligned}
$$

combining the two specifications gives $K > 198$.

- Choosing $K = 200$ gives $\tau_c = 0.0074$ sec and $K_c = 0.9901$





- Can we have zero steady-state error to a unit step input for the above design?
  In order to have $e_{ss} = 0$, require at least $m = n$ integrators in $G(s)K(s)$ in order to track $r(t) = rt^{(n-1)}$.
- Hence, for a step input given by $r(t) = r$, we therefore have $n = 1$ and hence $G(s)K(s)$ needs to have $m = n = 1$ integrator in order to track $r(t) = r$.
  Since $G(s)$ does not contain any integrator, we should design $K(s)$ to contain an integrator. For example, consider the following controller

$$K(s) = \frac{K}{s}$$

  The new closed-loop transfer function is

$$\begin{aligned} G_{cl2}(s) &= \frac{2K}{s(3s+4)+2K} \\ &= \frac{2K/3}{s^2+(4/3)s+2K/3} \end{aligned}$$

  The closed-loop system is now second-order compared to previous case.

- Choose the controller gain, K, such that the maximum overshoot $M_p < 0.1$

$$M_p = e^{\frac{\pi \zeta}{\sqrt{1-\zeta^2}}} \quad < \quad 0.1$$
$$\Rightarrow \quad \zeta \quad > \quad \sqrt{\frac{(\ln(0.1))^2}{\pi^2 + (\ln(0.1))^2}} = 0.59$$

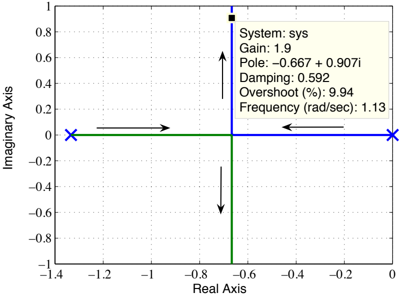
- Comparing the closed-loop transfer function with standard second-order system,

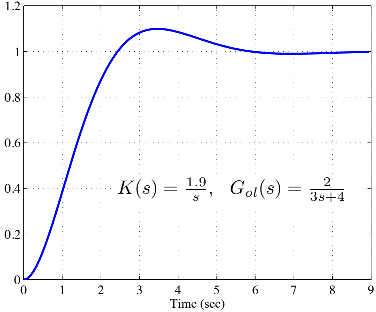
$$G_{cl2}(s) = \frac{2K/3}{s^2 + (4/3)s + 2K/3} \quad \text{c.f.} \quad \frac{\bar{K}\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

we have

$$2\zeta\omega_n = \frac{4}{3}, \quad \omega_n^2 = \frac{2K}{3}, \quad \bar{K} = 1,$$
$$\zeta = \frac{2}{3\omega_n} = \sqrt{\frac{2}{3K}} > 0.59$$
$$\Rightarrow \quad K < 1.915$$

- Verify via Root locus design, consider the open-loop transfer function $G(s)K(s) = \dfrac{K}{s}\dfrac{2}{3s+4}$.





- As shown in the insert in the root locus plot and closed-loop step response, all specifications are met. The response is however much slower compared to the first design with $K(s) = K$.

Example 2

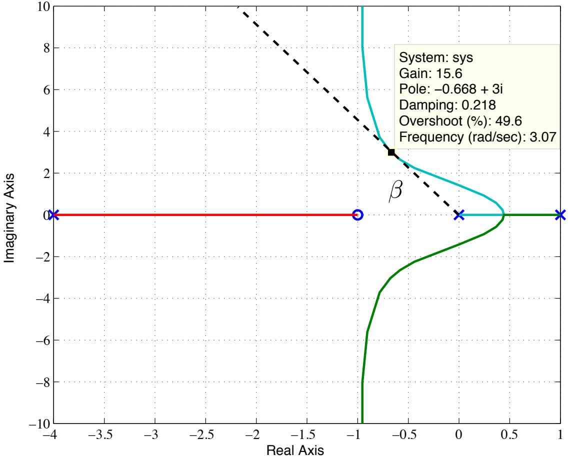
- A unity feedback system has plant: $G(s) = \dfrac{K(s+1)}{s(s-1)(s+4)}$
  1. Determine the range of K for stability
  2. The maximum ζ of the stable complex roots can be estimated from the root locus plots using Matlab.

- Closed-loop c.e. given by $1 + G(s)K(s) = 0$, to find the maximum K before the system becomes unstable, subsititute $s = j\omega$ (i.e. at the stability boundary)

$$
\begin{aligned}
1 + \frac{K(s+1)}{s(s-1)(s+4)} &= 0 \\
s^3 + 3s^2 + (K-4)s + K &= 0 \\
-j\omega^3 - 3\omega^2 + j(K-4)\omega + K &= 0 \\
(K - 3\omega^2) + j\omega(K - 4 - \omega^2) &= 0
\end{aligned}
$$

Equating the real and imaginary parts to zero, we have $K = 6, \omega = \sqrt{2}$.

- Root locus plot for $G(s) = \dfrac{K(s+1)}{s(s-1)(s+4)}$.



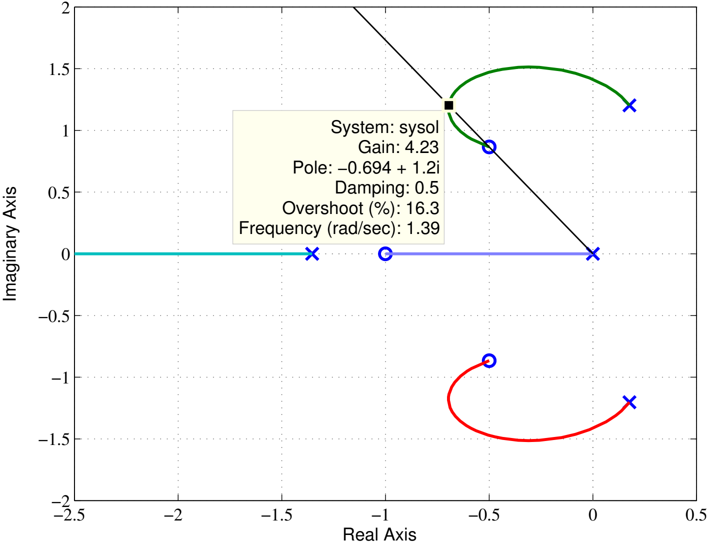
- the maximum ζ occurs when the angle β is smallest in the root locus plot.

Example 3

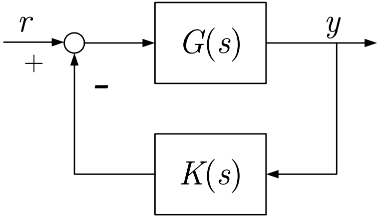
- Consider an OL plant, $G(s)$, being controlled in a unity negative feedback configuration by a controller, $K(s)$:

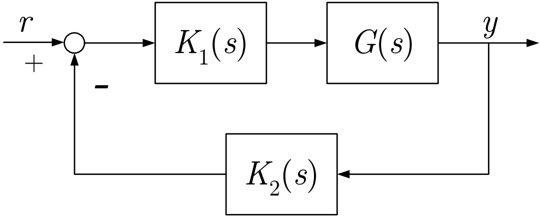
$$G(s) = \frac{s^2 + s + 1}{s^3 + s^2 + s + 2}, \qquad K(s) = \frac{K_0(s+1)}{s}$$

1. Determine if the system is always stable for all values of K_0 using Matlab ('rlocus')
2. Determine $K_0 > 0$ such that the set of complex poles has a damping ratio of 0.5
3. What is the steady-state error to unit step input in the setpoint?
4. What is the steady state error to a unit ramp input in the setpoint?


System: sysol
Gain: 4.23
Pole: −0.694 + 1.2i
Damping: 0.5
Overshoot (%): 16.3
Frequency (rad/sec): 1.39
Imaginary Axis
Real Axis
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−2
−1.5
−1
−0.5
0
0.5

- Derive the steady-state error, e_{ss}, for the following different closed-loop configurations. Note that $E(s) = R(s) - Y(s)$.





Summary

- Steady-state errors in response to polynomial inputs depends on the number of integrators in the loop transfer function.
- Controller design specifications can be in-terms of transient $(t_r, t_s, M_p, t_p,$ etc.$)$ as well as steady-state (e_{ss}) performances.

Review Questions

- How does time constant, τ, damping ratio, ζ and natural frequency, ω_n affect transient performances in first and second-order systems?
- How do you compute the closed-loop steady-state error for a system due to different reference input? Discuss the effect of having integrators in the system?

Reading: FPE: section 4.2.1-4.2.2

Practice Problems

1. For the feedback configuration in page 7-28, $G(s) = \dfrac{1}{s(\tau s + 1)}$, $K_1(s) = k_p$ and $K_2(s) = 1 + k_t s$. Compute the steady-state error due reference input, $R(s) = 1/s^n$ for $n = 1, 2, 3$.

2. Consider a second order plant with transfer function, $G(s) = \dfrac{1}{(s+1)(5s+1)}$ and in a unity feedback structure. Determine the system type and error constant with respect to tracking polynomial reference inputs of the system for the following controllers: (a) $K(s) = k_p$, (b) $K(s) = k_p + k_d s$, and (c) $K(s) = k_p + \frac{k_i}{s} + k_d s$. Let $k_p = 19$, $k_i = 0.5$ and $k_d = 4/19$.