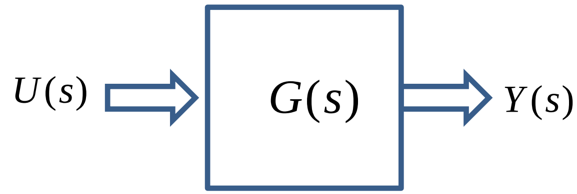
EE3331C/EE3331E
Feedback Control Systems
Part II: Frequency Response Methods

Chapter 3: Stability Analysis

Part 3A – Marginal stability & Stability margin

Closed Loop Stability

- A system is stable if all poles of its transfer function are in the **left half-plane (LHP)** of the complex plane



$$G(s) = \frac{s+2}{s+10}$$ Pole at -10 (Stable)

$$G(s) = \frac{s-2}{(s+1)(s+5)}$$ Poles at -1 & -5 (Stable)

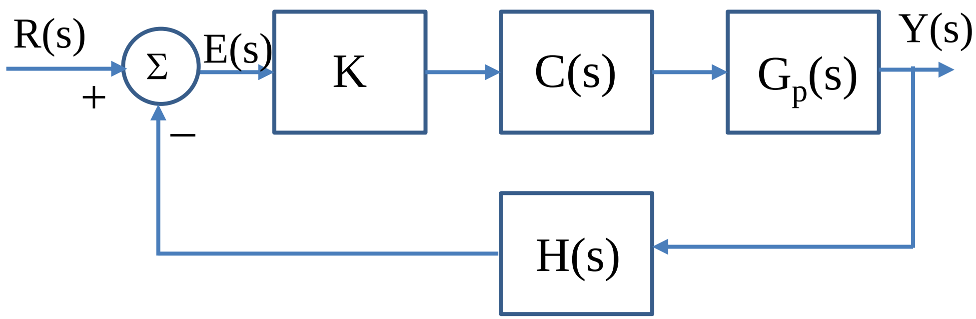
$$G(s) = \frac{s+1}{(s-1)(s+5)}$$ One pole at +1 (Unstable)

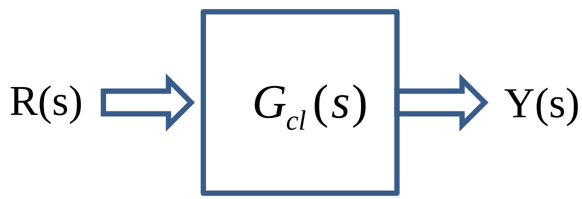
$$G(s) = \frac{8(s+1)}{s^2-4s+8}$$ Poles at +2±j2 (Unstable)

$$G(s) = \frac{8(s+1)}{s^2+4s+8}$$ Poles at -2±j2 (Stable)

- **Stability of Feedback System**
  - A closed loop feedback system is stable if the **closed loop** (CL) transfer function is stable
    - CL stability is determined by CL poles, not by open loop poles

○ Stability of Feedback System





$$G_{cl}(s) = \frac{KC(s)G_p(s)}{1 + KC(s)G_p(s)H(s)}$$

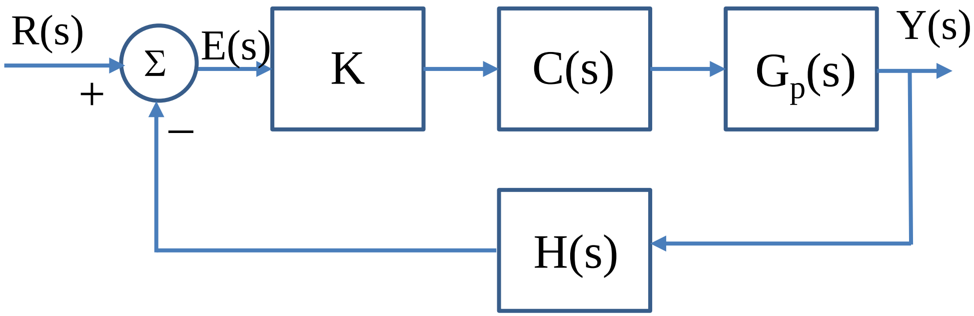
- CL Stability $\Rightarrow$ all poles of $G_{cl}(s)$ must be in the LHP

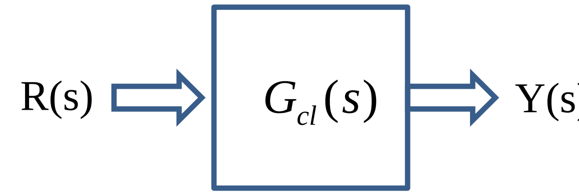
  $\Rightarrow$ all solutions of the following equation must have negative real part

$$1 + KC(s)G_p(s)H(s) = 0$$

Characteristic Equation

Closed Loop Stability





$$G_{cl}(s) = \frac{KC(s)G_p(s)}{1+KC(s)G_p(s)H(s)}$$

- *Example* 3a-1: Check closed loop stability for the following

$$K=10,\ C(s)=1,\ G_p(s)=\frac{1}{(s+2)},\ H(s)=1$$ (Open loop is stable)

  - The characteristics equation,

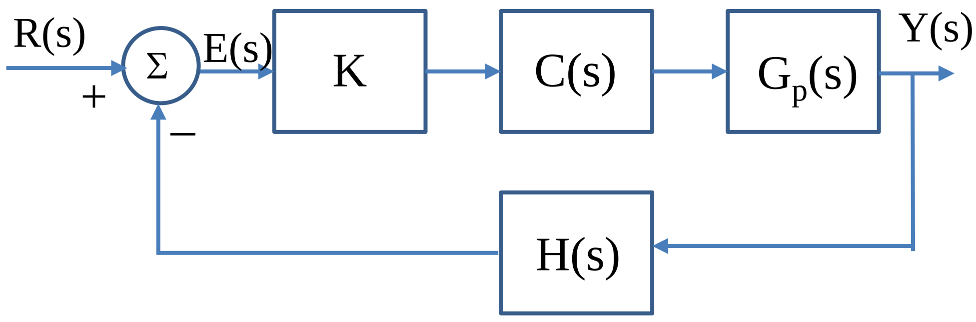
$$1+KCG_pH=0$$

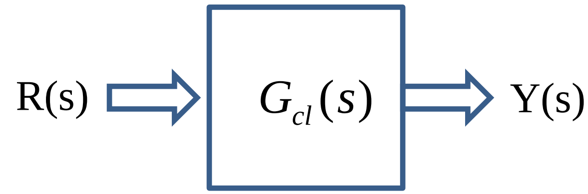
$$1+10\frac{1}{s+2}=0 \quad \Rightarrow \quad \frac{s+2+10}{s+2}=0 \quad \Rightarrow \quad s+12=0$$

Closed loop pole is at s=-12

Closed loop is stable

Closed Loop Stability





$$G_{cl}(s) = \frac{KC(s)G_p(s)}{1 + KC(s)G_p(s)H(s)}$$

- *Example* 3a-2: Check closed loop stability for the following

$$K = 10, C(s) = (s+1), G_p(s) = \frac{1}{(s-2)}, H(s) = 1$$

(Open loop is unstable)

- The characteristics equation,

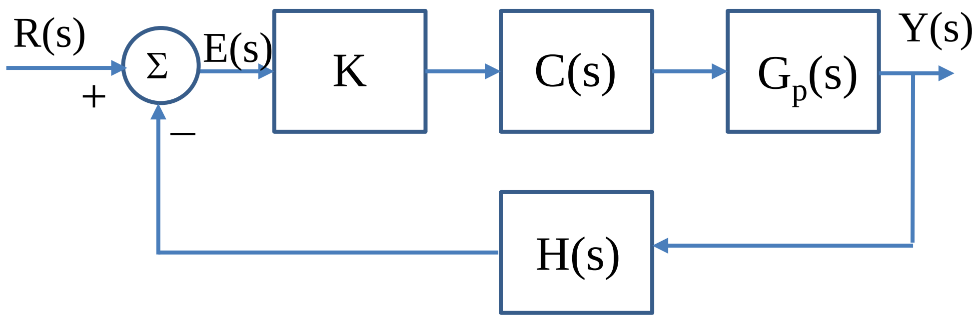
$$1 + KCG_pH = 0$$

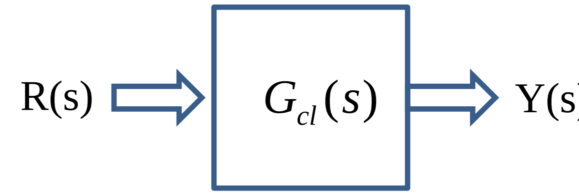
$$1 + 10\frac{s+1}{s-2} = 0 \quad \Rightarrow \quad \frac{s-2+10(s+1)}{s-2} = 0 \quad \Rightarrow \quad 11s + 8 = 0$$

Closed loop pole is at s=-8/11

Closed loop is stable

Closed Loop Stability





$$G_{cl}(s) = \frac{KC(s)G_p(s)}{1 + KC(s)G_p(s)H(s)}$$

- *Example* 3a-3: Check closed loop stability for the following,

$$K = 100,\ C(s) = 1,\ G_p(s) = \frac{1}{(s+1)(s+2)(s+3)},\ H(s) = 1$$

(Open loop is stable)

  - The characteristics equation, $1 + \frac{100}{(s+1)(s+2)(s+3)} = 0$

$$\frac{(s+1)(s+2)(s+3)+100}{(s+1)(s+2)(s+3)} = 0 \quad \Rightarrow \quad s^3 + 6s^2 + 11s + 106 = 0$$

  - Closed loop pole are at s_1 = -6.71, $s_{2,3}$ = +0.36±j3.96

(Closed loop is unstable)

Closed Loop Stability

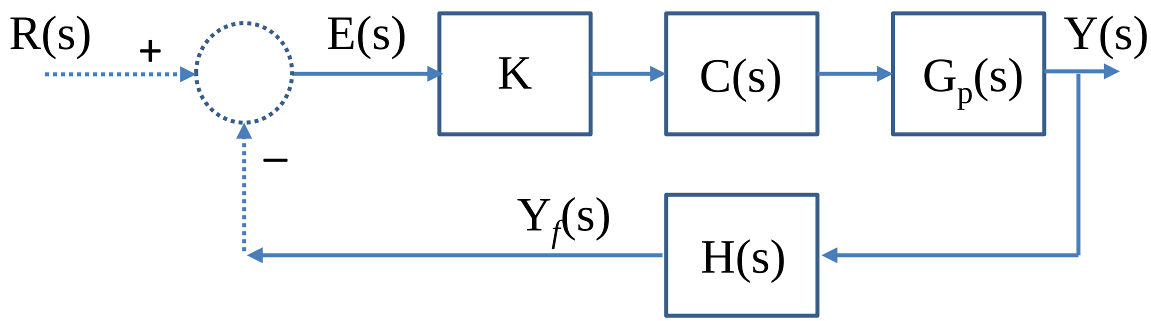
- **In summary,**
  a. A feedback system is stable if the closed loop transfer function has all poles in the LHP
  b. An open loop stable system may become unstable if the feedback is not designed properly
  c. It is possible to stabilize an open loop unstable system by appropriate choice of controller

- In examples shown so far, the stability is tested by finding the CL poles
  - It is inconvenient, especially, for large transfer function

- **Can we test CL stability using OL transfer function?**

Checking CL Stability using OL Transfer Function (OLTF)

- **Objectives:**
  1. To tell whether the closed loop is stable using OLTF
  2. To find the range of feedback gain for which the CL is stable



$$\begin{aligned} L(s) &= KC(s)G_p(s)H(s) \\ &= KG(s) \end{aligned}$$

Loop Transfer Function

- Can we check **CL stability** using **L(s)?**

Checking CL Stability using OLTF

s-Domain method using root locus

- Consider the OL transfer function

$$L(s)=\frac{K}{s(s+2)}$$

- Characteristic Equation

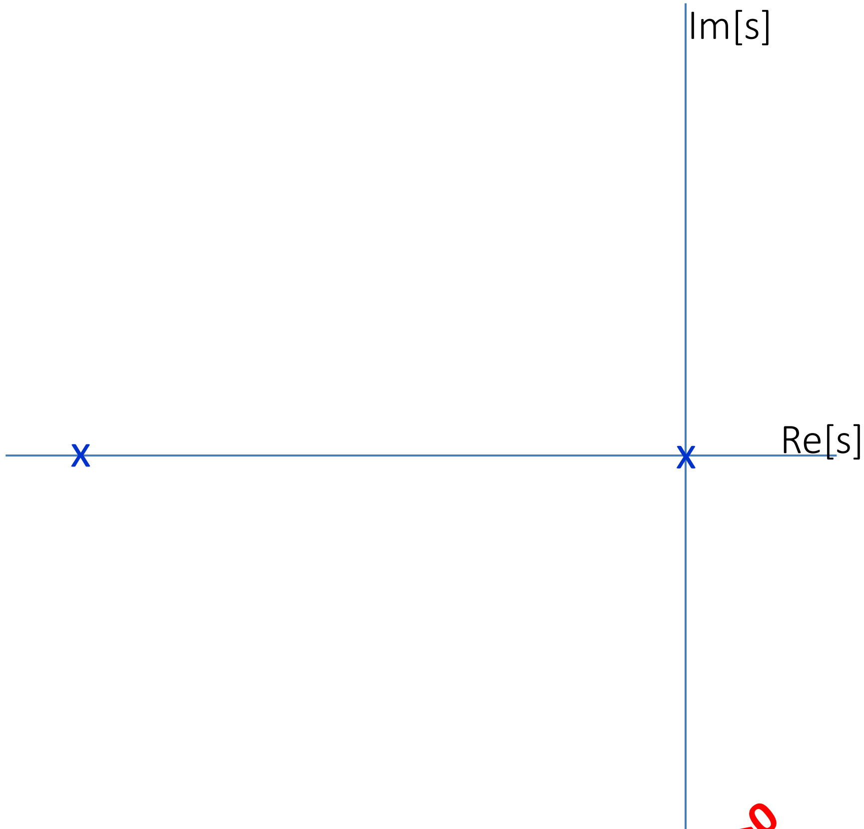
$$1+\frac{K}{s(s+2)}=0$$

$$s(s+2)+K=0$$

$$s^2+2s+K=0$$

- CL Poles $s_1=-\frac{2}{2}+\frac{\sqrt{4-4K}}{2}$

$$s_2=-\frac{2}{2}-\frac{\sqrt{4-4K}}{2}$$



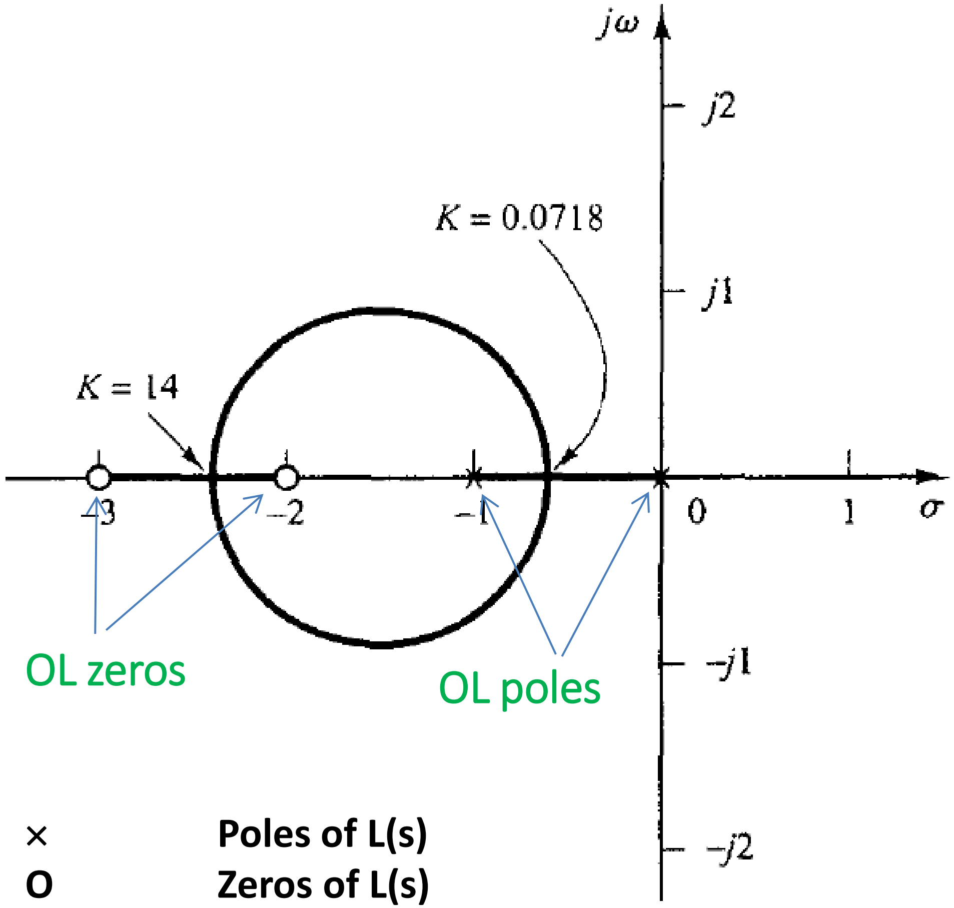
CL is stable for all K>0

- CL poles are real and distinct if 0<K<1
- CL poles are real and identical if K = 1
- CL poles are complex conjugate if K>1
  - Real part of these complex conjugate pole is -1

Checking CL Stability using OLTF

- **s-Domain method using root locus: Example**

$$L(s) = K\frac{(s+2)(s+3)}{s(s+1)}$$

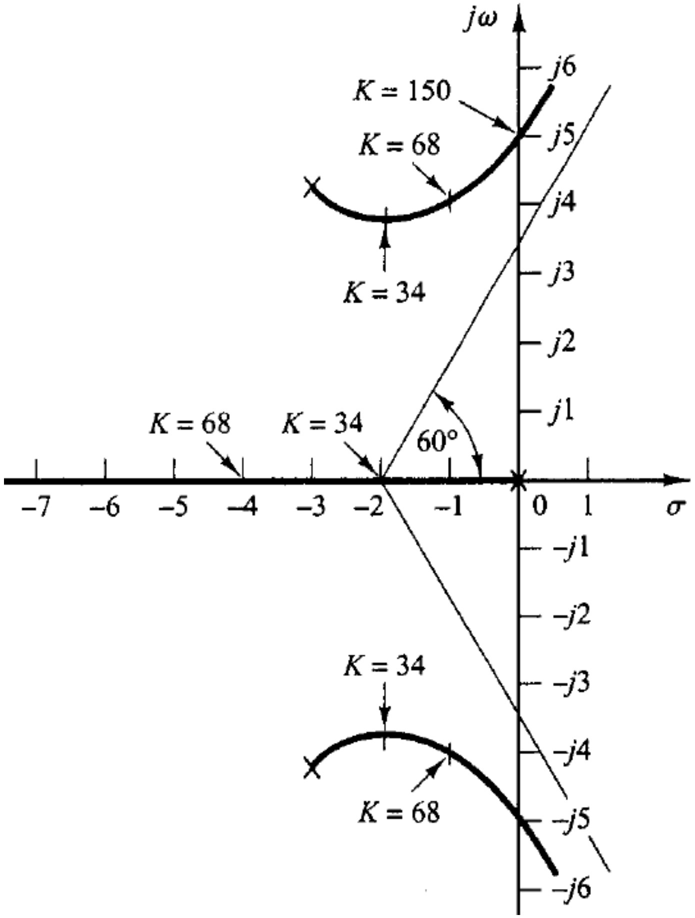


× Poles of L(s)
O Zeros of L(s)

- Solid lines are the loci of CL poles as K is varied from 0 to ∞
- The CL is stable for all K>0

Checking CL Stability using OLTF

- **s-Domain method using root locus: Example**



$$L(s) = K\frac{1}{s(s^2+6s+25)}$$

- For small values of K>0, the loci are in the LHP $\Rightarrow$ stable CL
- For K>150, the CL is unstable
- CL is marginally stable for K = 150
  - Let's designate this gain as K_m

× **Poles of L(s)**
O **Zeros of L(s)**

Frequency Domain Condition for Marginal Stability of CL

- The loop transfer function $L(s) = KC(s)G_p(s)H(s)$
$= KG(s)$
  - The characteristic equation, $1 + L(s) = 0$
  - The point $s = s_1$ is a CL pole if $1 + L(s_1) = 0$
  - The CL is marginally stable if it has poles at $s = j\omega_1$

$$1 + L(j\omega_1) = 0$$
$$L(j\omega_1) = -1 \quad \Longrightarrow \quad |L(j\omega_1)| = 1, \quad \angle L(j\omega_1) = \pm 180°$$

  - $L(j\omega)$ is the frequency response of $L(s)$

❑ Condition for marginal stability of CL:

If there exists a frequency at which the frequency response of L(s) has unity gain and ±180° phase, then the CL is marginally stable.

Crossover Frequency

- **Gain-Crossover Frequency (ω_{cg})**
  - The frequency at which the loop transfer gain or open loop gain is 1 (one)

$$\left|L(j\omega_{cg})\right| = 1$$

$$\left|L(j\omega_{cg})\right|_{dB} = 20\log 1 = 0$$

  - Bode (magnitude) plot L(s) crosses the 0 dB line at ω_{cg}

- **Phase-Crossover Frequency (ω_{cp})**
  - The frequency at which the phase of the loop transfer function is $\pm 180°$

$$\angle L(j\omega_{cp}) = \pm 180°$$

  - Bode (phase) plot crosses the $\pm 180°$ line at ω_{cp}

Marginal Stability and Crossover Frequency

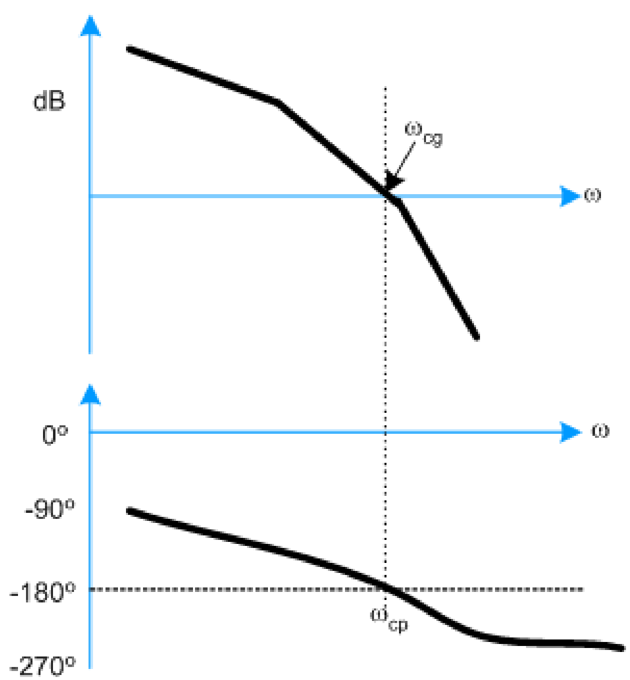
- Condition for marginal stability is $L(j\omega_1) = -1$

$$|L(j\omega_1)| = 1, \quad \angle L(j\omega_1) = \pm 180°$$

$$|L(j\omega_1)|_{dB} = 0, \quad \angle L(j\omega_1) = \pm 180°$$

[ω_1 is ω_{cg}] [ω_1 is ω_{cp}]

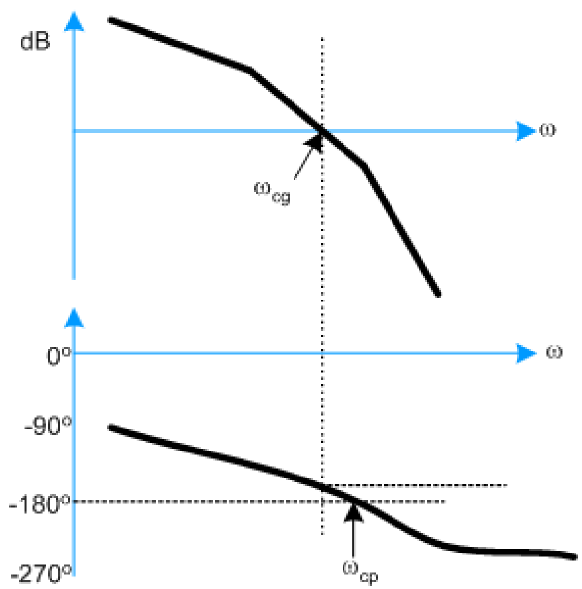
- The closed loop is marginally stable if $\omega_{cg} = \omega_{cp}$



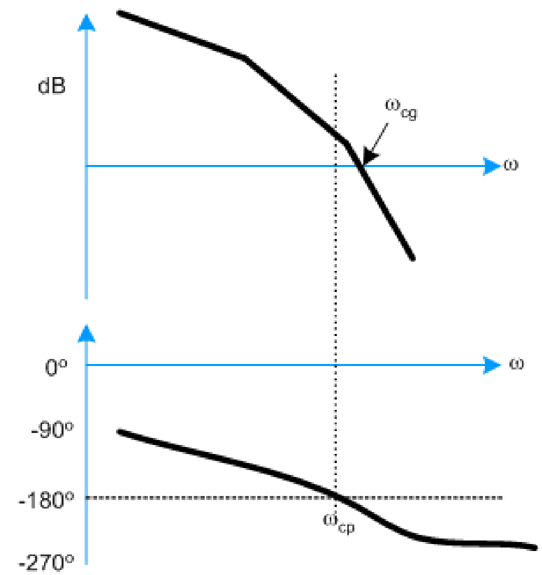
Bode plot of a loop transfer function that will result in a marginally stable closed loop

Stability & Crossover Frequency

- If $\omega_{cg} \neq \omega_{cp}$, then CL is either stable or unstable
  - How are these conditions, either $\omega_{cg} < \omega_{cp}$ or $\omega_{cg} > \omega_{cp}$, related to stability?



$\omega_{cg} < \omega_{cp}$

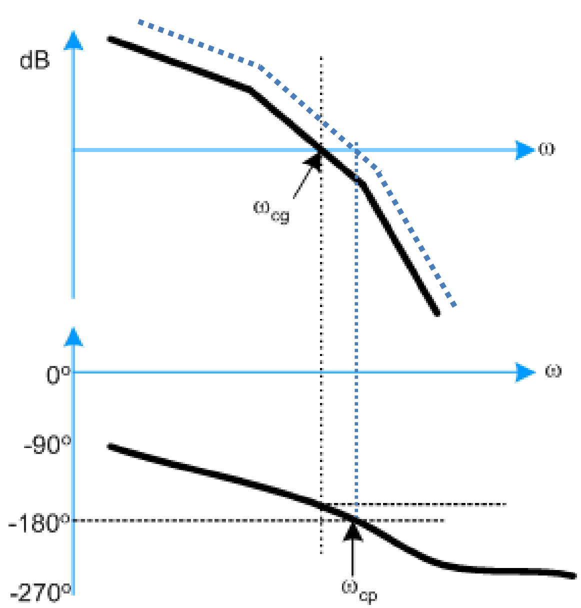


$\omega_{cg} > \omega_{cp}$

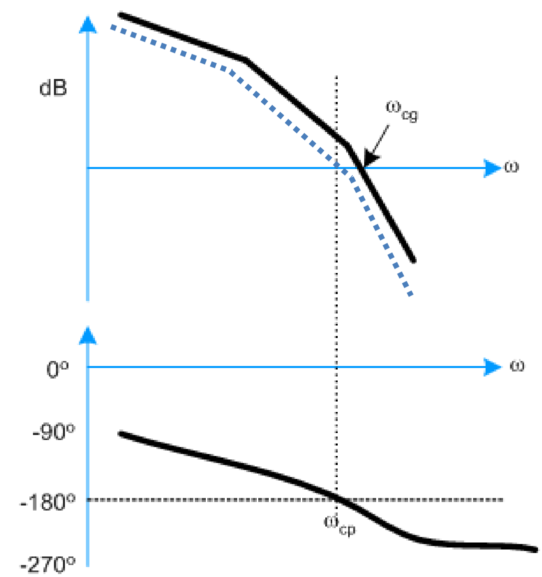
Stability & Crossover Frequency

$$L(s) = KG(s) \quad \Rightarrow \quad L(j\omega) = KG(j\omega) \quad \Longrightarrow \quad |L(j\omega)| = K|G(j\omega)|, \quad \angle L(j\omega) = \angle G(j\omega)$$

- Let $K=K_m$ makes the CL marginally stable



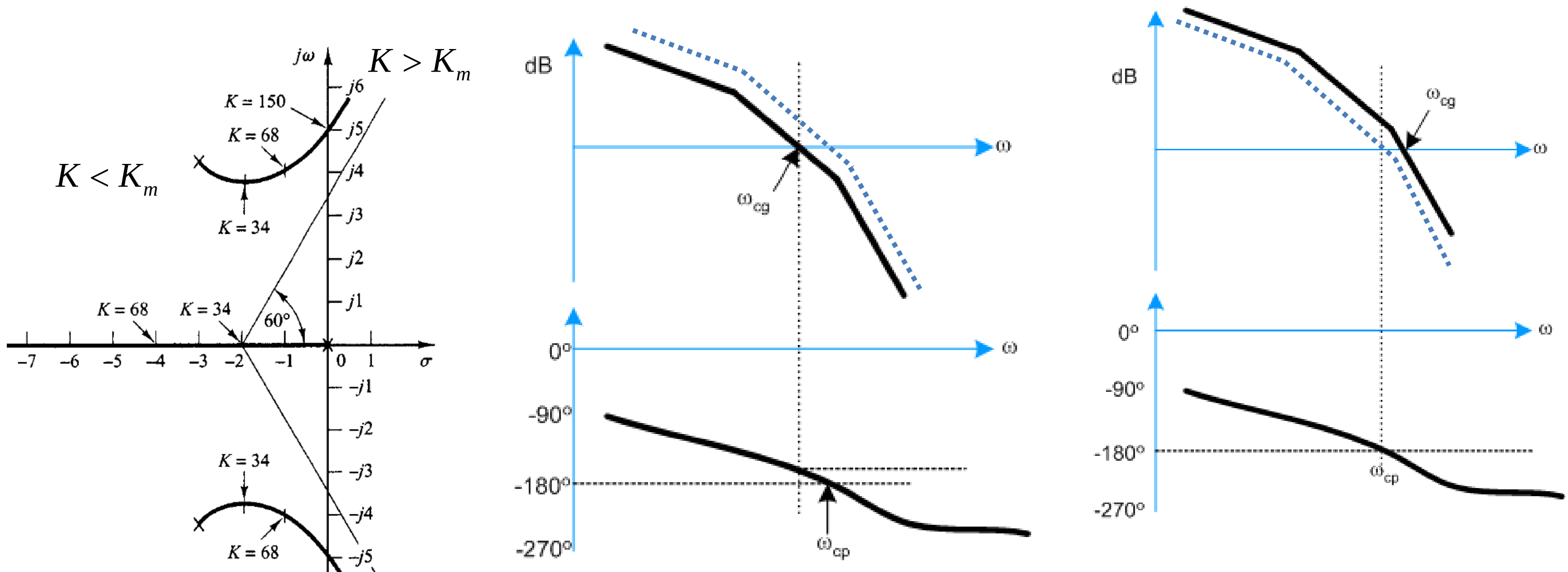
In both figures, the magnitude plot shown in dotted line is for $K=K_m$



If $K<K_m$, the magnitude plot is shifted downward making $\omega_{cg} < \omega_{cp}$

If $K>K_m$, the magnitude plot is shifted upward making $\omega_{cg} > \omega_{cp}$

Stability & Crossover Frequency



$K < K_m \quad \Rightarrow \quad \omega_{cg} < \omega_{cp}$ $\qquad$ $K > K_m \quad \Rightarrow \quad \omega_{cg} > \omega_{cp}$

- Assumption: increasing gain leads to instability (as seen in the root locus). Then we can conclude
  - $\omega_{cg} < \omega_{cp}$ implies stable closed loop
  - $\omega_{cg} > \omega_{cp}$ implies unstable closed loop

Stability Margins

- Condition for marginal stability:

$$|L(j\omega_1)| = 1, \quad \angle L(j\omega_1) = \pm 180^\circ$$

$$K_m |G(j\omega_1)| = 1, \quad \angle G(j\omega_1) = \pm 180^\circ$$

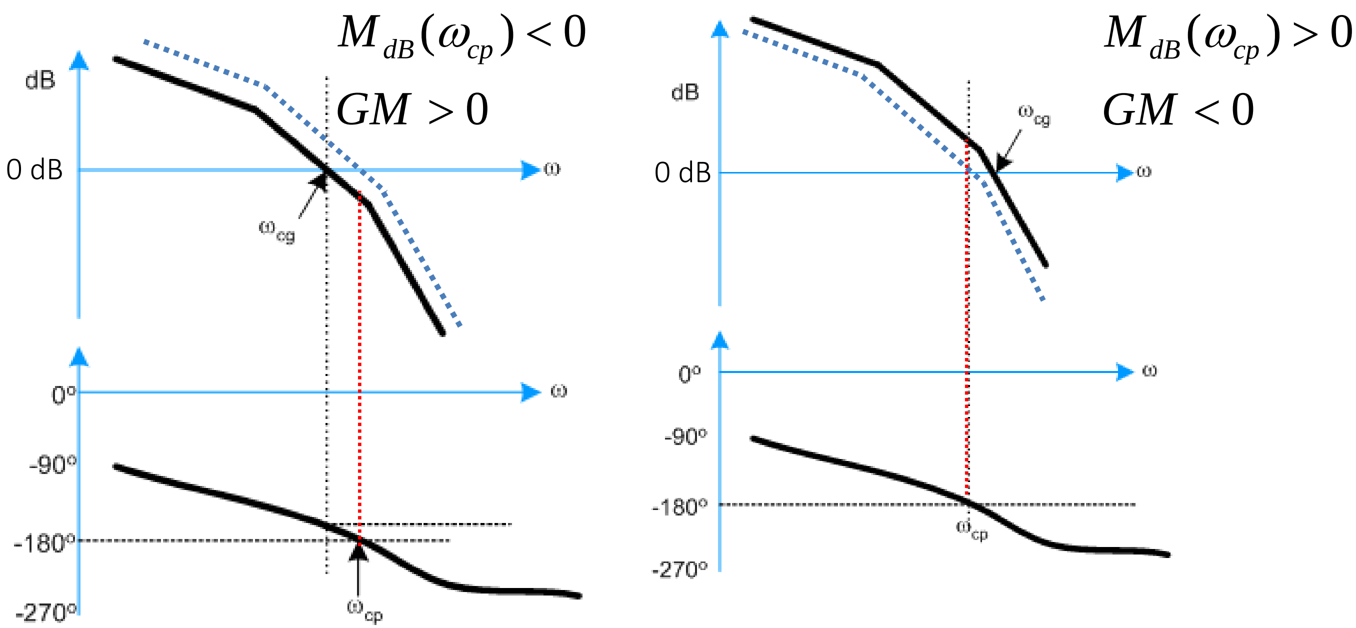
  - For $K \neq K_\mathrm{m}$, closeness to marginal stability is defined in two ways:
    1) $\angle L(j\omega_{cp})$=$\pm 180^\circ$, how much change of gain is required to make the magnitude $|L(j\omega_{cp})|$ equal to 1

    2) $|L(j\omega_{cg})|$ =1, how much phase change is required to make the phase $\angle L(j\omega_{cg})$=$\pm 180^\circ$

  - Accordingly, two stability margins are defined -
    - **Gain margin (GM)** and
    - **Phase margin (PM)**

Stability Margins

- **Gain Margin (GM)**
  - Find the phase-crossover frequency by solving $\angle L(j\omega_{cp}) = \pm 180°$
  - Then, gain margin: $GM = 0 - M_{dB}(\omega_{cp}) = -20\log|L(j\omega_{cp})|$



Stable CL

Unstable CL

  - Positive GM results in stable closed loop while negative GM means unstable closed loop

Stability Margins

- Gain margin can also be expressed in term of absolute gain (not in dB)

$$GM = -M_{dB}(\omega_{cp}) = -20\log\left|L(j\omega_{cp})\right|$$

$$GM = 20\log\frac{1}{\left|L(j\omega_{cp})\right|}$$

$$gm = \frac{1}{\left|L(j\omega_{cp})\right|}$$

- For the closed loop to be stable, GM > 0, i.e.,

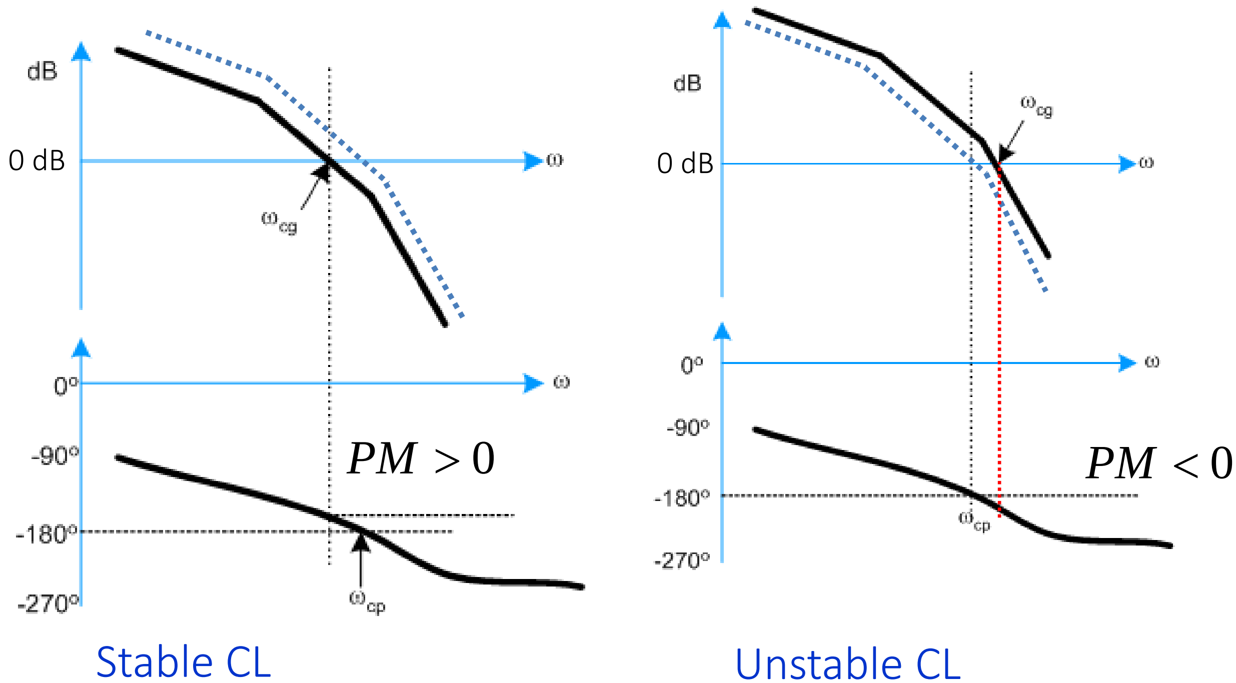
$$\left|L(j\omega_{cp})\right| < 1$$

- The closed loop is unstable if GM < 0, i.e., if

$$\left|L(j\omega_{cp})\right| > 1$$

Stability Margins

- **Phase Margin (PM)**
  - Find the gain-crossover frequency by solving $|L(j\omega_{cg})| = 1$
  - Then, phase margin: $PM = \angle L(j\omega_{cg}) - (-180°)$



  - Positive PM results in stable closed loop while negative PM means unstable closed loop

Stability, Crossover Frequencies and Stability Margins

- In Summary, for loop transfer function $L(s) = KG(s)$

Stable Closed loop	Marginally Stable	Unstable Closed loop
$K < K_m$	$K = K_m$	$K > K_m$
$\omega_{cg} < \omega_{cp}$	$\omega_{cg} = \omega_{cp}$	$\omega_{cg} > \omega_{cp}$
$GM > 0$	$GM = 0$	$GM < 0$
$\lvert L(j\omega_{cp})\rvert < 1$	$\lvert L(j\omega_{cp})\rvert = 1$	$\lvert L(j\omega_{cp})\rvert > 1$
$PM > 0$	$PM = 0$	$PM < 0$
$\angle L(j\omega_{cg})\lvert > -180°$	$\angle L(j\omega_{cg})\lvert = -180°$	$\angle L(j\omega_{cg})\lvert < -180°$

- These conditions are based on the assumption that increased gain leads to instability
  - This assumption holds for most of the practical systems
  - However, there may be cases when it is violated
    - The **Nyquist Stability Criterion (NSC)** is used to test CL stability for any loop transfer function