Tut 3 Q1. $KG(s) = \dfrac{K}{s(s+a)(s+b)}$

(a) $a = 1$, $b = 2$

(b) At 'A', $\zeta = 1$

- Actual ⊄ poles : $1 + KG(s) = 0$

$$s^3 + 3s^2 + 2s + K = 0 \quad — ①$$

- Desired ⊄ poles :-

$(s+\alpha)(s+\beta)^2$ or $(s+\alpha)(s^2 + 2\omega_n s + \omega_n^2)$ $\quad \zeta = 1$ at 'A'

$$s^3 + (\alpha + 2\beta)s^2 + (2\alpha\beta + \beta^2)s + \alpha\beta^2 = 0 \quad — ②$$

① & ②, $s^2 : \ \alpha + 2\beta = 3 \rightarrow \alpha = 3 - 2\beta$

$s^1 : \ 2\alpha\beta + \beta^2 = 2$

$2(3-2\beta)\beta + \beta^2 = 2$

$3\beta^2 - 6\beta + 2 = 0$

$\Rightarrow \beta = 0.4226 \ , \ 1.5774$ (cannot $\because 0 < \beta < 1$).

$\Rightarrow \alpha = 2.1547$

$s^0 : \ K = \alpha\beta^2 = 0.385$ ※.

(c) At 'B', 'C', $\zeta = 0$.

$\Rightarrow$ ⊄ poles :- $s = j\omega$

Sub $s = j\omega$ into ①

$-j\omega^3 - 3\omega^2 + j2\omega + K = 0$

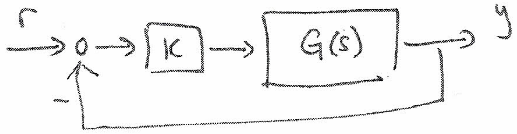
imag : $-\omega^3 + 2\omega = 0$

$\Rightarrow \omega = \sqrt{2}$

real : $-3\omega^2 + K = 0$

$\Rightarrow K = 6$ ※

Q2.



$G(s) = \frac{s^2+s+1}{s^3}$

(a) CL poles, $K=1$, $s=-1, \pm j$

(b) No.

(c). When $K=5$,

CL poles :- $1+GK = 0$

$$1 + K\frac{s^2+s+1}{s^3} = 0$$

$$\Rightarrow s^3+Ks^2+Ks+K = 0$$

$$s^3+5s^2+5s+5 = 0. \quad -①$$

From root locus, CL poles:-

$$(s+\alpha)(s+0.462\pm j1) = 0$$

$$(s+\alpha)(s^2+0.9240s+1.2134) = 0 \quad -②$$

① & ②, s^0 : $1.2134\alpha = 5$

$$\alpha = 4.12 \;\#$$

(d). When $K=5$.

CLTF : $H(s) = \frac{GK}{1+GK} = \frac{5s^2+5s+5}{s^3+5s^2+5s+5}$

From ②, $s = -0.462\pm j, -4.12$

$|0.462| << |4.12|$

$\Rightarrow$ 2nd order

$s^2+2\zeta\omega_n s+\omega_n^2 = 0 \quad -③$

$s^2+0.9240s+1.2134=0 \quad -④$

③ & ④, $\omega_n = \sqrt{1.2134}$

$\zeta = 0.42 \;\#$

(e). $t_s = 4/\zeta\omega_n = 8.65s$

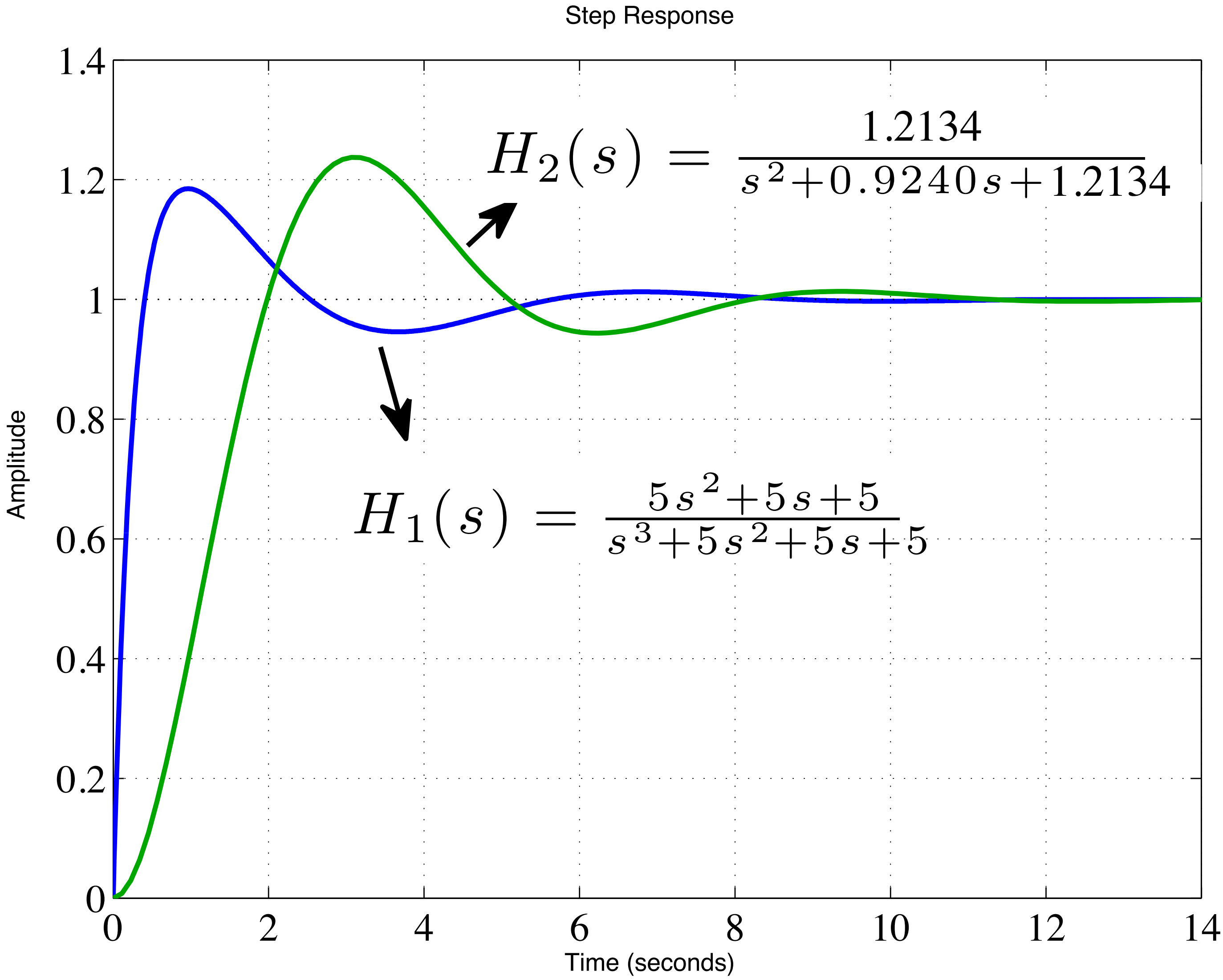
$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.234$

2nd order, $H_2(s) = \frac{\bar{K}}{s^2+2\zeta\omega_n s+\omega_n^2}$

Compare DC gain, $H_2(0) = H(0)$

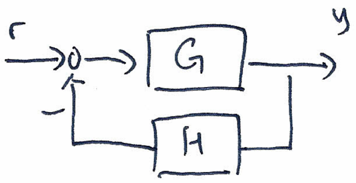
$$\Rightarrow \frac{\bar{K}}{\omega_n^2} = \frac{5}{5} = 1$$

$$\Rightarrow \bar{K} = \omega_n^2 \;\#$$



Step Response
Amplitude
Time (seconds)
$H_2(s) = \frac{1.2134}{s^2+0.9240s+1.2134}$
$H_1(s) = \frac{5s^2+5s+5}{s^3+5s^2+5s+5}$
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0
2
4
6
8
10
12
14

Q3



$$G(s) = \frac{K}{s(\tau_t s+1)(\tau_e s+1)}$$

$$H(s) = \frac{1}{\tau_m s+1}$$

CLTF : $\frac{y}{r} = \frac{G}{1+GH}$.

(a) 4 poles.

(b) (i) $K = 0.5$

(ii) system is marginally stable,

$K = 0.5$, $\mathbb{C}$ poles : $-0.92 \pm j0.26$, $\pm j\omega$

$\mathbb{C}$ poles :- $1+GH = 0$

$$1 + \frac{K}{s(s+1)(2s+1)(3s+1)} = 0$$

$$\Rightarrow 6s^4 + 11s^3 + 6s^2 + s + K = 0 \quad — (1)$$

$$\hookrightarrow s^4 + \frac{11}{6}s^3 + s^2 + \frac{1}{6}s + \frac{K}{6} = 0 \quad - (2)$$

sub $s = j\omega$ into (1)

$$\Rightarrow 6\omega^4 - j11\omega^3 - 6\omega^2 + j\omega + K = 0$$

imag : $-11\omega^3 + \omega = 0$

$$\omega = 1/\sqrt{11} = 0.3$$

(iii) When $K = \alpha$, $s = -0.97, -0.62, -0.14, -0.097$

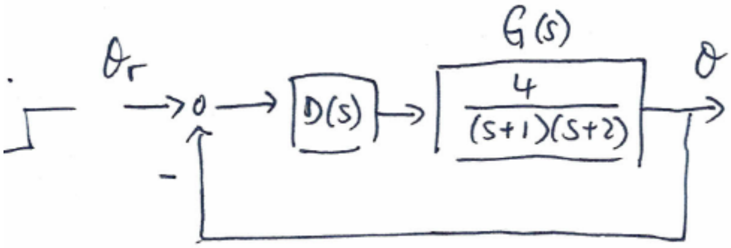
c.e.

$$\Rightarrow (s+0.97)(s+0.62)(s+0.14)(s+0.097) = 0$$

$$s^4 + \ldots \underbrace{(0.97)(0.62)(0.14)(0.097)}_{\frac{\alpha}{6}} = 0 \quad — (3)$$

$\Rightarrow \alpha = 0.05$ #

Q4



$$e = \theta_r - \theta$$
$$= \theta_r - GDe$$
$$\frac{e}{\theta_r} = \frac{1}{1+GD}$$

(i) $D(s) = 1$, find e_{ss}

FVT, $e_{ss} = \lim_{s \to 0} sE(s)$

$$= \lim_{s \to 0} s \frac{1}{1+\frac{4}{(s+1)(s+2)}} \cdot \frac{1}{s}$$

$$= \frac{1}{3}$$

ξ poles:- $1+GD = 0$

$$1 + \frac{4}{(s+1)(s+2)} = 0$$
$$s^2+3s+6 = 0$$
$$s = -1.5 \pm j1.94$$

(ii) $D(s) = 1 + \frac{0.1}{s} = \frac{s+0.1}{s}$

FVT, $e_{ss} = \lim_{s \to 0} sE(s)$

$$= \lim_{s \to 0} s \frac{1}{1+\frac{s+0.1}{s}\frac{4}{(s+1)(s+2)}} \cdot \frac{1}{s}$$

$$= \lim_{s \to 0} s \frac{s(s+1)(s+2)}{s(s+1)(s+2)+4(s+0.1)} \cdot \frac{1}{s}$$

$$= 0$$

ξ poles: $1+GD = 0$

$$s^3+3s^2+6s+0.4 = 0$$
$$s = -0.069$$
$$-1.46 \pm j1.91$$

(iii) $D(s) = 1+0.3s$

FVT, $e_{ss} = \lim_{s \to 0} sE(s)$

$$= \lim_{s \to 0} s \frac{1}{1+\frac{4(1+0.3s)}{(s+1)(s+2)}} \cdot \frac{1}{s}$$

$$= \frac{1}{3} \text{ ※.}$$

$$1+GD = 0$$
$$s^2+4.2s+6 = 0$$
$$s = -2.1 \pm j1.26$$

Q5

$$
\begin{aligned}
G_c(s) &= K\left(1+\frac{1}{sT_i}+sT_d\right) && (1)\\
G_c'(s) &= K'\left(1+\frac{1}{sT_i'}\right)(1+sT_d') \\
&= K'\left[\left(1+\frac{T_d'}{T_i'}\right)+\frac{1}{sT_i'}+sT_d'\right] && (2)
\end{aligned}
$$

Compare coefficient of Equations (1) and (2) gives

$$
\begin{aligned}
K &= K'\left(1+\frac{T_d'}{T_i'}\right) && (3)\\
\frac{K}{T_i} &= \frac{K'}{T_i'} && (4)\\
KT_d &= K'T_d' && (5)
\end{aligned}
$$

From Equations (4) and (5) we obtain

$$
\begin{aligned}
K'^2\frac{T_d'}{T_i'} &= K^2\frac{T_d}{T_i} \\
\frac{T_d'}{T_i'} &= \frac{K^2}{K'^2}\frac{T_d}{T_i} && (6)
\end{aligned}
$$

Substitute (6) into (3) gives

$$
\begin{aligned}
K &= K'\left(1+\frac{K^2}{K'^2}\frac{T_d}{T_i}\right)\\
0 &= K'^2 - K'K + K^2\frac{T_d}{T_i}\\
K' &= \frac{K}{2}\left[1\pm\sqrt{1-4\frac{T_d}{T_i}}\right]
\end{aligned}
$$

Choose

$$
K' = \frac{K}{2}\left[1+\sqrt{1-4\frac{T_d}{T_i}}\right]
$$

gives

$$
\begin{aligned}
T_i' &= \frac{T_i}{2}\left[1+\sqrt{1-4\frac{T_d}{T_i}}\right]\\
T_d' &= \frac{2T_d}{1+\sqrt{1-4\frac{T_d}{T_i}}}
\end{aligned}
$$

Choose

$$
K' = \frac{K}{2}\left[1-\sqrt{1-4\frac{T_d}{T_i}}\right]
$$

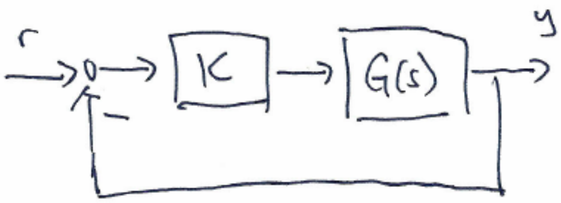
gives

$$
\begin{aligned}
T_i' &= \frac{T_i}{2}\left[1-\sqrt{1-4\frac{T_d}{T_i}}\right]\\
T_d' &= \frac{2T_d}{1-\sqrt{1-4\frac{T_d}{T_i}}}
\end{aligned}
$$

Since K, T_i, T_d, K', T_i', T_d' are real numbers, $G_c'(s)$ can only give real zeros whereas $G_c(s)$ can give complex zeros as well. Hence $G_c(s)$ is more general.

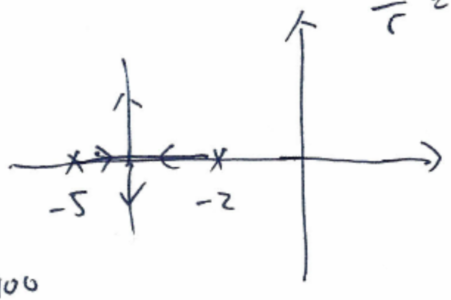
For educational purposes, it is better to give the more general $G_c(s)$ (textbook version). In practice, it is found in the industry that $G_c'(s)$ is good enough. Furthermore, it is easier to use (tune) $G_c'(s)$ because you cannot get complex zeros. Complex zeros are not necessary for most common PID applications.

Q7



$$G(s) = \frac{100}{(s+5)(s+2)}$$

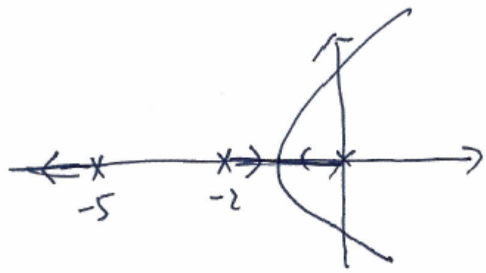
$$\frac{y}{r} = \frac{GK}{1+GK}$$



$r = 100$ km/h

$$y_{ss} = \lim_{s\to 0} sY(s)$$
$$= \lim_{s\to 0} s \frac{GK}{1+GK} R(s)$$
$$= \lim_{s\to 0} s \frac{\frac{100K}{(s+5)(s+2)}}{1+\frac{100K}{(s+5)(s+2)}} \cdot \frac{100}{s}$$
$$= \lim_{s\to 0} s \frac{100K}{(s+5)(s+2)+100K} \cdot \frac{100}{s} \quad , K=1$$
$$= \frac{100}{110} \times 100$$
$$= 90.9 \text{ km/h}$$

Add integrator, $K(s) = \frac{K}{s}$



$$y_{ss} = \lim_{s\to 0} sY(s)$$
$$= \lim_{s\to 0} s \frac{GK}{1+GK} R(s)$$
$$= \lim_{s\to 0} s \frac{\frac{100K}{s(s+5)(s+2)}}{1+\frac{100K}{s(s+5)(s+2)}} \cdot \frac{100}{s}$$
$$= \lim_{s\to 0} s \frac{100K}{s(s+5)(s+2)+100K} \cdot \frac{100}{s}$$
$$= 100 \text{ km/h}$$

Q8

Min time with no overshoot $\Rightarrow \zeta = 1$

& poles :- $1 + GK = 0$

$$1 + \frac{10K}{s(s+10)} = 0$$

$$\Rightarrow s^2 + 10s + 10K = 0 \quad — (1)$$

2nd-order,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s^2 + 2\omega_n s + \omega_n^2 = 0 \quad — (2)$$

(1) & (2), $s^1: \; 2\omega_n = 10 \Rightarrow \omega_n = 5$

$s^0: \; \omega_n^2 = 10K \Rightarrow K = 2.5$ #

$\theta_r(t) = 2 + 0.1t$

$\theta_r(s) = \frac{2}{s} + \frac{0.1}{s^2}$

$$e_{ss} = \lim_{s\to 0} s\,E(s).$$

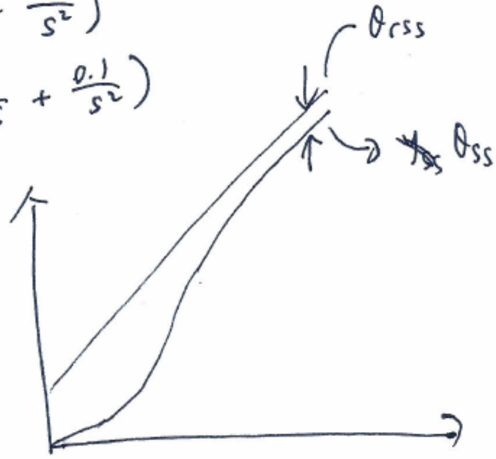
$$= \lim_{s\to 0} s \frac{1}{1+GK} \theta_r(s)$$

$$= \lim_{s\to 0} s \frac{1}{1 + \frac{10K}{s(s+10)}} \left(\frac{2}{s} + \frac{0.1}{s^2}\right)$$

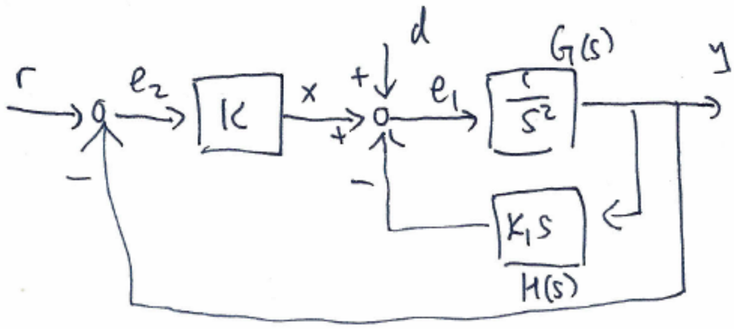
$$= \lim_{s\to 0} s \frac{s(s+10)}{s(s+10) + 10K} \left(\frac{2}{s} + \frac{0.1}{s^2}\right)$$

$$= \frac{10}{10K} \times 0.1$$

$$= 0.04 \; \#.$$



Q9



① CLTF, $y = Ge_1 = G(d + x - Hy)$

$$= G(Ke_2 - Hy)$$

$$= G(K(r-y) - Hy)$$

$$y = GKr - GKy - GHy$$

$$\frac{y}{r} = \frac{GK}{1 + GK + GH}$$

$$= \frac{K/s^2}{1 + K/s^2 + K_1/s}$$

$$= \frac{K}{s^2 + K_1 s + K} \quad \text{✱} \quad \longrightarrow \text{poles:-} \quad s^2 + K_1 s + K = 0 \quad \text{— ①}$$

% overshoot, $M_p < 10\%$

$$\Rightarrow e^{-\pi\zeta/\sqrt{1-\zeta^2}} < 0.1 \quad \Rightarrow \zeta > 0.59$$

2nd order $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$ — ②

① & ②, s^0: $K = \omega_n^2 \Rightarrow \omega_n = \sqrt{K}$

s^1: $2\zeta\omega_n = K_1$

$$\Rightarrow \zeta = K_1/2\omega_n = \frac{K_1}{2\sqrt{K}} > 0.59$$

$$\Rightarrow \frac{K_1}{\sqrt{K}} > 1.18 \quad \#$$

if $K = 100$, $K_1 > 11.8$

— e_{ss} to ramp command.

$$e = r - y = r - \frac{GK}{1+GK+GH} r$$

$$= \frac{1+GH}{1+GK+GH} r.$$

$$e_{ss} = \lim_{s\to 0} s E(s) = \lim_{s\to 0} s \frac{1+GH}{1+GK+GH} R(s).$$

$$= \lim_{s\to 0} s \frac{1+K_1/s}{1+K/s^2+K_1/s} \cdot \frac{1}{s^2}$$

$$= \lim_{s\to 0} s \frac{s^2+K_1 s}{s^2+K_1 s+K} \cdot \frac{1}{s^2}$$

$$= \frac{K_1}{K} \quad \text{— to minimize.}$$

— reduce step disturbance.

$$y = Ge_1 = G(d + x - Hy).$$

$$= G(d + Ke_2 - Hy)$$

$$= G(d + K(-y) - Hy)$$

$$\frac{y}{d} = \frac{G}{1+GK+GH}$$

$$y_{ss} = \lim_{s\to 0} s Y(s)$$

$$= \lim_{s\to 0} s \frac{1/s^2}{1+K/s^2+K_1/s} \cdot \frac{1}{s}$$

$$= \lim_{s\to 0} s \frac{1}{s^2+K_1 s+K} \cdot \frac{1}{s}$$

$$= \frac{1}{K}$$