Complex Numbers

1. Definition:
   A complex number, z, is expressed in the form

$$z = a + jb$$

   where a and b are real numbers, j (sometimes we use i) is the imaginary unit that satisfies the relation

$$j^2 = -1$$

   - a is called the real part of z, $\mathrm{Re}(z) = a$,
   - b is called the imaginary part of z, $\mathrm{Im}(z) = b$.

2. Properties:

   (a) Two complex numbers, $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$ are equal if and only if $a_1 = a_2$ and $b_1 = b_2$.

   (b) Arithmetic operations on complex numbers. Consider two complex numbers, $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$,

   - Addition/Subtraction:

$$z_1 \pm z_2 = (a_1 + jb_1) \pm (a_2 + jb_2) = (a_1 \pm a_2) + j(b_1 \pm b_2)$$

   - Multiplication:

$$\begin{aligned} z_1 z_2 &= (a_1 + jb_1)(a_2 + jb_2) \\ &= a_1 a_2 + j a_1 b_2 + j a_2 b_1 - b_1 b_2 \\ &= (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_2 b_1) \end{aligned}$$

   - Division:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a_1 + jb_1}{a_2 + jb_2} \\ &= \frac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)} \\ &= \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + j\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} \end{aligned}$$

   Exercise: Solve $\dfrac{(-1 + 5j)^2(3 - 4j)}{1 + 3j} + \dfrac{10 + 7j}{5j}$.
   Ans: $10 + 38.2j$

- Calculation with complex numbers are reduced to calculation with real numbers.
- Addition and multiplication are commutative and associative, i.e.

$$z_1 + z_2 = z_2 + z_1$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

- Distributive law also holds

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

(c) Argand Diagram:
Complex numbers can be represented as points in the $x - y$ plane (the complex plane) as shown in Figure 1. The x-axis is called the real axis (Re) and the y-axis the imaginary axis (Im).

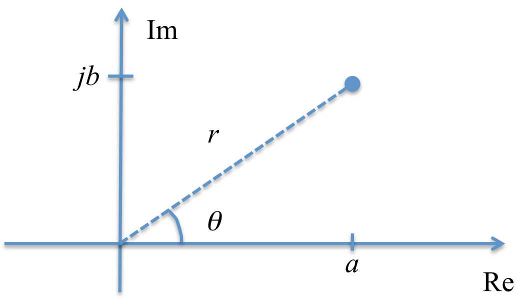


Figure 1: Argand diagram

<u>Exercise</u>: Given the $\arg(a + jb) = \theta$, where $a > 0, b > 0$, find in terms of θ and π, the value of (a) $\arg(a - jb)$, (b)$\arg(-a + jb)$ and (c) $\arg(-a - jb)$.

From Figure 1, we can also represent the complex number in polar notation where r is the radius (magnitude) and angle (phase or argument) of the complex number in the form: $r\angle\theta$. The two representations are related by

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ \theta &= \tan^{-1}\left(\frac{b}{a}\right) \\ a &= r\cos\theta \\ b &= r\sin\theta \end{aligned}$$

3. Euler's formula
   Euler's formula relates complex exponentials and trigonometric functions:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

   - $e^{j\pi} = -1$, $e^{j\pi/2} = j$, $e^{j3\pi/2} = e^{j\pi/2} = -j$. (see Figure 1)
   - If we multiply Euler's formula by a constant, $r > 0$, we get the two forms of complex numbers:

$$z = re^{j\theta} = r\cos\theta + jr\sin\theta$$

   - Examples:

     Convert the following complex numbers from rectangular to polar form:

     (a) $1 + j$
     (b) $-1 - j$
     (c) $-5 + j12$

     Convert the following complex numbers from polar to rectangular form:

     (a) $5e^{j\pi/4}$
     (b) $e^{-3\pi/2}$
     (c) $10e^{j2.618}$

   - More examples on arithmetic operations:

     Addition and subtraction is easy in rectangular form.

     Multiplication and division is easier in polar form.

$$r_1e^{j\theta_1}r_2e^{j\theta_2} = (r_1r_2)e^{j(\theta_1+\theta_2)}$$

     Note that the magnitude of a product is the product of the magnitudes. Argument of the product of complex numbers is the sum of the arguments.

     Solve (a) $5e^{j0.927} + e^{j\pi/4}$, (b) (4+3j)(1-j) and (c) (5+j12)/(1+j)

   - De Moivre's Theorem
     Let n be a real rational number, then

$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta$$

     This is equivalent to

$$(e^{j\theta})^n = e^{jn\theta}$$

     Exercise: Use De Moivre's Theorem to evaluate $\dfrac{(1+j)^{11}}{(1-j)^9}$.

     Ans: -2.

4. Complex Conjugates

Let $z = a + jb$, the complex conjugate of z, denoted by z^*, is defined as

$$z^* = a - jb$$

Note that z^* is the reflection of z on the x-axis, and, $|z^*| = |z|$, $\arg(z^*) = -\arg(z) = -\theta$.

- Properties of conjugates:
  (a) $z + z^* = 2Re(z) = 2a$
  (b) $z - z^* = j2Im(z) = j2b$
  (c) $zz^* = |z|^2 = a^2 + b^2$
  (d) $(z^*)^* = z$
  (e) $(z_1 + z_2)^* = z_1^* + z_2^*$
  (f) $(z_1 z_2)^* = z_1^* z_2^*$

5. Complex numbers as vectors

In an Argand diagram, let the points Z and W represent the complex numbers z and w respectively. Then $\overrightarrow{OZ}$ and $\overrightarrow{OW}$ are the corresponding vectors representing z and w.

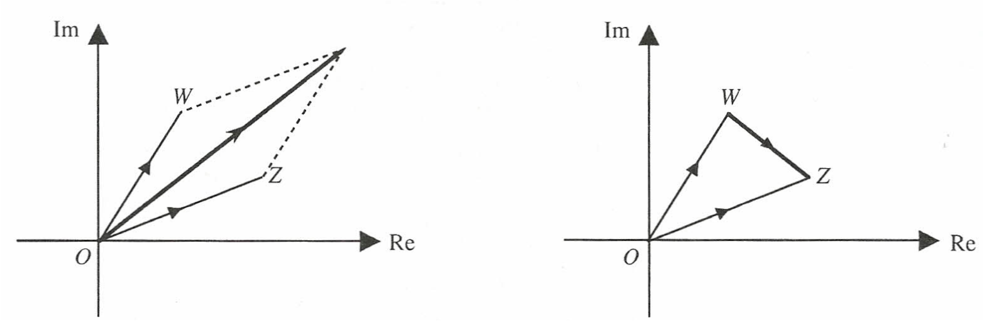


Figure 2: Complex numbers and Vectors

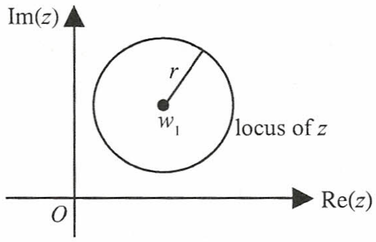
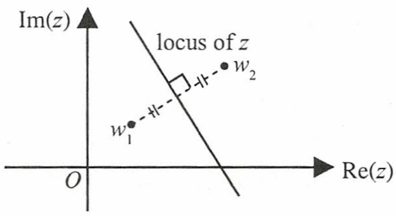
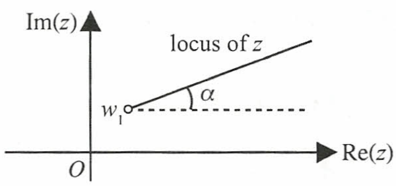
The sum of two complex numbers, $z + w$, is equivalent to the vector sum $\overrightarrow{OZ} + \overrightarrow{OW}$, which is given by the parallelogram of vectors addition.

The difference, $z - w$, is the vector $\overrightarrow{WZ}$ according to the triangle law of vector addition.

<u>Exercise</u>: The complex number $z = e^{j\theta}$ where $-\pi < \theta \leq -\pi/2$. The complex number, w_1 is such that $w_1 = -j2z$.

- State the modulus of w_1 and express the argument of w_1 in terms of θ. (Ans: $\arg(w) = 1.5\pi - \theta$)
- In an Argand diagram with origin O, the points P, Q, and R are represented by z, w_1 and $w_1 - z$ respectively. Show these points on a diagram. State the shape of OPQR.
- The point S is represented by $z + w_1$. Show this point on the same Argand diagram and find its area. (Ans: 2)

Let Z represent the **variable** complex number z and W, W_1 and W_2 represent **fixed** complex numbers w, w_1 and w_2 respectively.

Equation	Type of locus	Diagram
$\lvert z - w_1 \rvert = r$ Or $z = w_1 + re^{i\theta}, 0 < \theta < 2\pi$	Circle with radius r, centre w_1	
$\lvert z - w_1 \rvert = \lvert z - w_2 \rvert$	Perpendicular bisector of the line $W_1 W_2$	
$\arg(z - w_1) = \alpha$ Or $z = w_1 + re^{i\alpha}, \quad r \in \mathbb{R}^+$	Half-line with end-point W, inclined at an angle α to the Re-axis	

Some examples:

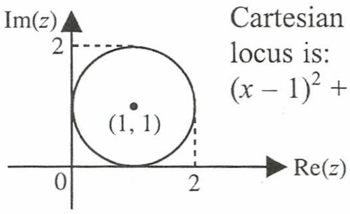
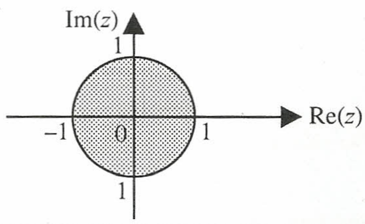
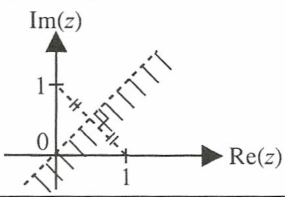
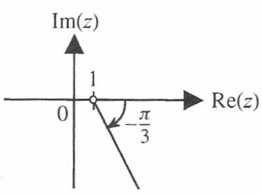
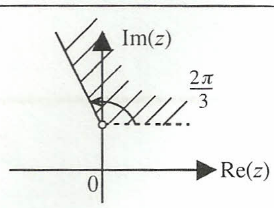
$\lvert z - 1 - i \rvert = 1 \Leftrightarrow \lvert z - (1 + i) \rvert = 1$  Cartesian equation of locus is: $(x - 1)^2 + (y - 1)^2 = 1.$	$\lvert z \rvert \leqslant 1 \Leftrightarrow \lvert z - 0 \rvert \leqslant 1$ 
$\lvert z - 1 \rvert < \lvert z - i \rvert$ 	Indicate that the 2 lines are at right angles to each other and that the perpendicular bisector is equidistant from 1 and i. In fact, it is the line $y = x$.$\lvert z - 1 \rvert < \lvert z - i \rvert$ means that z is nearer to 1 than to i.
$\arg(z - 1) = -\frac{\pi}{3}$ 	$0 < \arg(z - i) \leqslant \frac{2\pi}{3}$ 

Figure 3: Locus

6. Solving Polynomial Equations

   Note that for a polynomial equation with all real coefficients, complex roots occur in conjugate pairs, i.e., if z is a complex root of the polynomial equation, then z^* is also a complex root of the polynomial equation

   - Solve $z^2 + 2z + 3$.

     This is a second-order polynomial, the roots are given by the formula $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, hence we have

$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 12}}{2} \\ &= -1 \pm \sqrt{-2} \\ &= -1 \pm j\sqrt{2} \end{aligned}$$

   - Verify that $z = j2$ is a root of the equation $z^4 - 2z^3 + 7z^2 - 8z + 12 = 0$. Hence, determine the other roots.

7. MA1301 Proficiency test questions

- July 2012

  Express the complex number $z = \dfrac{(1-\sqrt{3}i)^4}{(1+i)^2}$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$.

  On an Argand diagram, the point P represents the complex number z and the point Q represents the complex number w. Given the triangle OPQ oriented in the clockwise sense is equilateral, find the modulus and argument of w.

- July 2011

  Let $z_1 = 2 + i$, $z_2 = 12 - 4i$ and $z_3 = \frac{z_2}{z_1}$. On the Argand diagram, z_1, z_2 and z_3 are represented by the points P, Q and R respectively.

  - Express z_3 in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$.
  - Calculate the exact area of triangle PQR.