EE3331C/EE3331E Feedback Control Systems
Part II, Tutorial 2

Section 1

1. For the transfer function below, find
   a) the frequency at which the phase is -123.7° and
   b) the frequency at which gain is 1.

$$L(s)=4\frac{(s+5)}{s(s+1)}$$

Solution: $L(j\omega)=4\frac{(5+j\omega)}{j\omega(1+j\omega)}$

(a)

$$\phi(\omega)=\tan^{-1}(\omega/5)-90^\circ-\tan^{-1}(\omega)$$

$$\tan^{-1}(\omega/5)-90^\circ-\tan^{-1}(\omega)=-123.7^\circ$$
$$\tan^{-1}(\omega/5)-\tan^{-1}(\omega)=-33.7^\circ$$
$$\tan\left[\tan^{-1}(\omega/5)-\tan^{-1}(\omega)\right]=\tan[-33.7^\circ]$$
$$\frac{\frac{\omega}{5}-\omega}{1+\frac{\omega^2}{5}}=-0.67$$
$$\frac{4\omega}{\omega^2+5}=0.67$$
$$0.67\omega^2-4\omega+3.35=0$$

$$\boxed{\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp(\tan A)(\tan B)}}$$

Solving this quadratic equation, we get ω= 4.96 and ω=1.

(b)

$$|L(j\omega)|=4\frac{\sqrt{25+\omega^2}}{\omega\sqrt{1+\omega^2}}$$

$$4\frac{\sqrt{25+\omega^2}}{\omega\sqrt{1+\omega^2}}=1$$
$$16\frac{(25+\omega^2)}{\omega^2(1+\omega^2)}=1$$
$$\omega^4+\omega^2=16(25+\omega^2)$$
$$\omega^4-15\omega^2-400=0$$

Solving this quadratic equation, we get $\omega^2=28.86$, $\omega=5.37$

2. Find gain margin if the loop transfer function is

$$L(s)=\frac{0.5}{s(2s+1)(3s+1)}.$$

Solution:

$$L(j\omega)=\frac{0.5}{j\omega(1+j2\omega)(1+j3\omega)}$$

$$\phi(\omega)=-90^\circ-\tan^{-1}(2\omega)-\tan^{-1}(3\omega)$$

$$-90^\circ-\tan^{-1}(2\omega_{cp})-\tan^{-1}(3\omega_{cp})=-180^\circ$$
$$\tan^{-1}(2\omega_{cp})+\tan^{-1}(3\omega_{cp})=+90^\circ$$
$$\tan\left[\tan^{-1}(2\omega_{cp})+\tan^{-1}(3\omega_{cp})\right]=\tan[90^\circ]$$
$$\frac{2\omega_{cp}+3\omega_{cp}}{1-6\omega_{cp}^2}=\infty$$
$$1-6\omega_{cp}^2=0$$
$$\omega_{cp}=\frac{1}{\sqrt{6}}\ rad/s$$

$$|L(j\omega_{cp})|=\frac{0.5}{\omega_{cp}\sqrt{1+4\omega_{cp}^2}\sqrt{1+9\omega_{cp}^2}}$$
$$=\frac{0.5}{\sqrt{\frac{1}{6}}\sqrt{1+\frac{4}{6}}\sqrt{1+\frac{9}{6}}}$$
$$=\frac{0.5}{\sqrt{\frac{1}{6}\times\frac{10}{6}\times\frac{15}{6}}}=0.6$$

Gain margin, $GM=-20\log(0.6)$
$=4.44\ dB$

3. Consider the loop transfer function given below and determine the closed loop performance in terms of
   a) Steady-state error for unit step, unit ramp and unit parabolic input
   b) Amplitude of error if the reference input is sine wave of frequency 1 radian per sec and amplitude of 5 unit.
   c) Phase margin and hence approximate damping of the dominant closed loop poles.

$$L(s) = 4\frac{(s+10)}{s(s+1)}$$

Solution:

(a) It is type 1 system; steady-state error is zero for step input and infinite for parabolic input. For ramp input, e_{ss} is calculated below.

$$k_v = \lim_{s\to 0} sL(s)$$
$$= \lim_{s\to 0} \frac{4(s+10)}{(s+1)} = 40 \qquad e_{ss,ramp} = \frac{1}{k_v} = 0.025$$

Error in tracking a ramp input will be 2.5% .

(b)
$$S(s) = \frac{1}{1+L(s)} = \frac{1}{1+\dfrac{4(s+10)}{s(s+1)}} = \frac{s^2+s}{s^2+5s+40}$$

$$S(j\omega) = \frac{-\omega^2 + j\omega}{40-\omega^2 + j5\omega}$$

$$S(j1) = \frac{-1+j1}{39+j5} \qquad |S(j1)| = \frac{\sqrt{1+1}}{\sqrt{39^2+5^2}} = 0.036$$

Amplitude of the error is 5×0.036 = 0.18 unit.

(c) First, find gain-crossover frequency by solving $|L(j\omega_{cg})| = 1$

$$\left|\frac{4(10+j\omega_{cg})}{j\omega_{cg}(1+j\omega_{cg})}\right| = 1$$

$$\frac{4\sqrt{100+\omega_{cg}^2}}{\omega_{cg}\sqrt{1+\omega_{cg}^2}} = 1$$

$$16(100+\omega_{cg}^2) = \omega_{cg}^2(1+\omega_{cg}^2)$$
$$\omega_{cg}^4 - 15\omega_{cg}^2 - 1600 = 0$$

Solving this quadratic equation, $\omega_{cg}^2 = 48.2$

$$\omega_{cg} = 6.9 \quad rad/s$$

Phase margin,
$$PM = \angle L(j\omega_{cg}) - (-180°)$$
$$= -90° - \tan^{-1}(\omega_{cg}) + \tan^{-1}(\omega_{cg}/10) + 180°$$
$$= 90° - \tan^{-1}(6.9) + \tan^{-1}(0.69)$$
$$= 90° - 81.8° + 34.6°$$
$$= 42.8°$$

Closed loop poles' damping coefficient is approximately 42.8/100 = 0.43.

4. For each of the compensators given below, answer the following questions.
   a) What is the type of compensator – lead or lag?
   b) What is ω_m?
   c) What is ϕ_m?
   d) What is the gain at $\omega >> \max\{\omega_z, \omega_p\}$, where ω_z and ω_p are corner frequencies of compensator zero and pole, respectively?
   e) What is the gain at $\omega=\omega_m$?

$$L_1(s) = \frac{(0.5s+1)}{(0.3s+1)}, \quad L_2(s) = 0.1\frac{(s+10)}{(s+1)}$$

Solution:

(a) $L_1(s)$ is a lead compensator and $L_2(s)$ is a lag compensator.

(b)

$$L_1(s) = \frac{0.5s+1}{0.6\times 0.5s+1}, \quad \alpha = 0.6, \quad T = 0.5, \quad \omega_m = \frac{1}{T\sqrt{\alpha}} = 2.58\ rad/s$$

$$L_2(s) = \frac{0.1s+1}{10\times 0.1s+1}, \quad \alpha = 10, \quad T = 0.1, \quad \omega_m = \frac{1}{T\sqrt{\alpha}} = 3.16\ rad/s$$

(c)

$$L_1(s) \quad \Rightarrow \quad \sin\phi_m = \frac{1-0.6}{1+0.6} = 0.25 \quad \Rightarrow \quad \phi_m = 14.5°$$

$$L_2(s) \quad \Rightarrow \quad \sin\phi_m = \frac{1-10}{1+10} = -0.82 \quad \Rightarrow \quad \phi_m = -55°$$

(d)

$$L_1(j\omega) = \frac{1+j0.5\omega}{1+j0.3\omega}, \quad as\ \omega >> \max\{2, 3.33\}, \quad L_1(j\omega) \cong \frac{j0.5\omega}{j0.3\omega} = 1.67$$

$$\left|L_1(j\omega)\right|_{\omega>>\max\{\omega_z,\omega_p\}} = 1.67 \quad or \quad +4.45dB$$

$$L_2(j\omega) = \frac{1+j0.1\omega}{1+j\omega}, \quad as\ \omega >> \max\{10, 1\}, \quad L_2(j\omega) \cong \frac{j0.1\omega}{j\omega} = 0.1$$

$$\left|L_2(j\omega)\right|_{\omega>>\max\{\omega_z,\omega_p\}} = 0.1 \quad or \quad -20dB$$

(e)

$$L_1(j\omega_m) = \frac{1+j0.5\omega_m}{1+j0.3\omega_m}$$

$$\left|L_1(j\omega_m)\right| = \left|\frac{1+j0.5\times 2.58}{1+j0.3\times 2.58}\right| = 1.29 \quad or \quad +2.22dB$$

$$L_2(j\omega_m) = \frac{1+j0.1\omega_m}{1+j\omega_m}$$

$$\left|L_2(j\omega_m)\right| = \left|\frac{1+j0.1\times 3.16}{1+j3.16}\right| = 0.32 \quad or \quad -9.89dB$$

4. A lag compensator is to be designed to provide gain reduction by a factor 0.3 at ω_x = 10 rad/s with negligible phase at that frequency. Choose compensator parameters α and T. Find gain and phase of the compensator at ω_x.

Solution:

If $\omega >> \omega_z$, the gain of lag compensator can be approximated by the following.

$$\left|C_{lag}(j\omega)\right|_{\omega>>\omega_z} = \left|\frac{1+j\omega T}{1+j\alpha\omega T}\right|_{\omega>>\frac{1}{T}} = \left|\frac{1+j\omega T}{1+j\alpha\omega T}\right|_{\omega T>>1} \cong \frac{1}{\alpha}$$

$$\left|C_{lag}(j\omega_x)\right| = 0.3 \quad \Rightarrow \quad \frac{1}{\alpha} = 0.3 \quad if \quad \omega_x >> \frac{1}{T}$$

$$\alpha = \frac{1}{0.3} = 3.3$$

$$\omega_x >> \frac{1}{T} \quad let \quad \frac{1}{T} = \frac{\omega_x}{10} = 1\, rad/s$$

$$T = 1$$

Compensator Transfer Function:

$$C_{lag} = \frac{Ts+1}{\alpha Ts+1} = \frac{s+1}{3.3s+1}$$

Gain and phase of this compensator at ω_x:

$$C_{lag}(j\omega) = \frac{1+j\omega}{1+j3.3\omega}, \qquad \left|C_{lag}(j\omega)\right| = \frac{\sqrt{1+\omega^2}}{\sqrt{1+(3.3\omega)^2}}, \qquad \angle C_{lag}(j\omega) = \tan^{-1}\omega - \tan^{-1}(3.3\omega)$$

$$\left|C_{lag}(j10)\right| = \frac{\sqrt{1+(10)^2}}{\sqrt{1+(33)^2}} = 0.3, \qquad \angle C_{lag}(j10) = \tan^{-1}(10) - \tan^{-1}(33) = -3.97°$$

Let's find the solution with (1/T) = (ω_x/20)

$$\omega_x >> \frac{1}{T} \quad let \quad \frac{1}{T} = \frac{\omega_x}{20} = 0.5\, rad/s$$

$$T = 2$$

Compensator Transfer Function:

$$C_{lag} = \frac{Ts+1}{\alpha Ts+1} = \frac{2s+1}{6.6s+1}$$

Gain and phase of this compensator at ω_x:

$$C_{lag}(j\omega) = \frac{1+j2\omega}{1+j6.6\omega}, \qquad \left|C_{lag}(j\omega)\right| = \frac{\sqrt{1+(2\omega)^2}}{\sqrt{1+(6.6\omega)^2}}, \qquad \angle C_{lag}(j\omega) = \tan^{-1}(2\omega) - \tan^{-1}(6.6\omega)$$

$$\left|C_{lag}(j10)\right| = \frac{\sqrt{1+(20)^2}}{\sqrt{1+(66)^2}} = 0.3, \qquad \angle C_{lag}(j10) = \tan^{-1}(20) - \tan^{-1}(66) = -1.99°$$

Note: Further away from ω_x we place the compensator zero, smaller is the compensator phase at ω_x.

Section 2

1. Plant transfer function of a unity feedback system is $G_p(s) = \dfrac{K}{s(s+2)(s+20)}$.
   i. Select amplifier gain K so that steady-state error is less than or equal to 5%.
   ii. Design a lag compensator to make phase margin greater than or equal to 45°.

2. Plant transfer function of a unity feedback system is $G_p(s) = \dfrac{30}{s^2+s}$.
   i. What are the steady-state errors for (a) unit step input, (b) unit ramp input, and (c) unit parabolic input?
   ii. What is the phase margin?
   iii. Design a lead compensator to make phase margin greater than or equal to 40°.

Final Exam, AY14-15, Sem II

3. A DC motor with negligible armature inductance is to be used in a position control system. It's transfer function is

$$G_p(s) = \frac{250}{s(s+5)}.$$

   Assuming unity feedback (i.e., H(s)=1), design a controller to meet the following specifications:

   i. The steady-state error to a unit ramp input is less than or equal to 1/200.
   ii. Overshoot to unit step response is less than 20%.

   Note: The 2nd specification defines a limit on the value of damping factor of the dominant poles. This can be converted into an approximate limit on phase margin.

4. Plant transfer function of a unity feedback system is $G_p(s) = \dfrac{100}{s(s+10)}$.

   We wish to design a controller to meet the following specifications:

   i. Velocity error constant k_v = 100.
   ii. Phase margin greater than or equal to 45°
   iii. Sinusoidal inputs of up to 1 rad/sec to be reproduced with less than or equal to 2% error.
   iv. Sinusoidal inputs with frequency greater than 100 rad/sec to be attenuated at the output to less then 5% of their input amplitudes.

   Explain why a single 1st-order compensator cannot meet all the specifications.

Section 3

1. Find gain margin for the loop transfer function,

$$L(s)=\frac{8.25}{s(s^2+2s+5)}$$

Answer: GM = 1.67 dB and ω_{cg} = 1.8 rad/s

2. For the loop transfer function given below find
   a) The value of K to achieve gain margin of 7.96 dB and
   b) The value of K to achieve phase margin of 45°.

$$L(s)=\frac{K}{s(s^2+2s+5)}$$

Answer: (a) K = 4, (b) K = 5.94

3. Plant transfer function of a unity feedback system is $G_p(s)=\frac{1}{(s+4)^2}$.

Design a controller to achieve approximately 5% error to a step input and phase margin of approximately 45°

Answer: $G_c(s)=304\frac{1.1s+1}{3.7s+1}$ [with ω_x/10 chosen as zero of lag compensator. Allowance of 5° is used while finding ω_x]