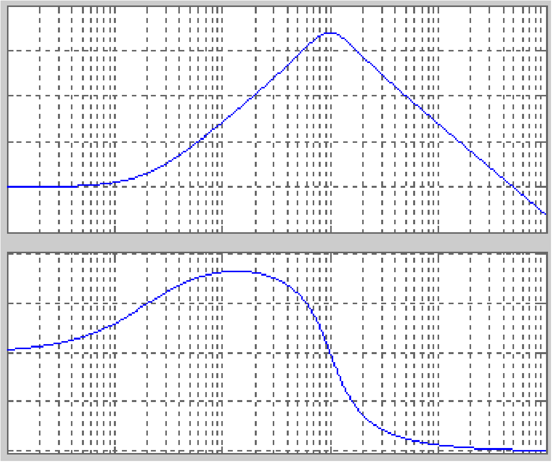
EE3331C/ EE3331E
Feedback Control Systems
Part II: Frequency Response Methods

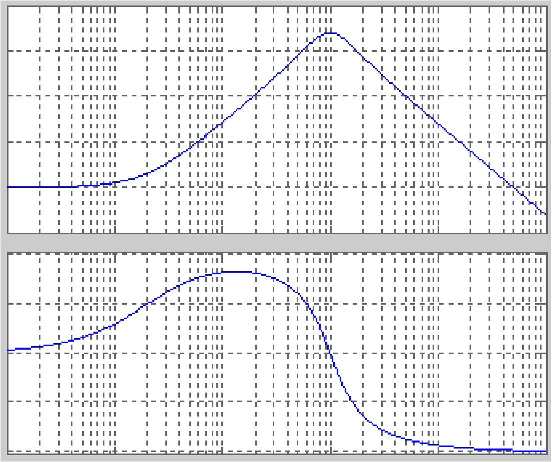
Chapter 2: Frequency Response Plot – Bode plot

Chapter Learning Objectives

1. Given a transfer function G(s), you should be able to sketch the frequency response plot by hand (i.e., not using a software)
2. Given the plot of frequency response of a linear system, you should be able to estimate the transfer function G(s) of the linear system

$G(s)$





$G_{est}(s)$

Frequency Response Plots

- **Why frequency response plot?**
  1) Good way to visualize data in a frequency response

$$G(s) \underset{s=j\omega_i}{\Longrightarrow} G(j\omega_i), \quad 0 \le \omega_i \le \infty$$

[$G(j\omega_i)$ is obtained from transfer function $G(s)$]

$$G(j\omega_i) = M(\omega_i)\angle\phi(\omega_i), \quad 0 \le \omega_i \le \infty$$

[M and ϕ can be measured experimentally]

In practical measurement, ω_i is bounded by finite limits,
$\omega_h \ge \omega_i \ge \omega_l$

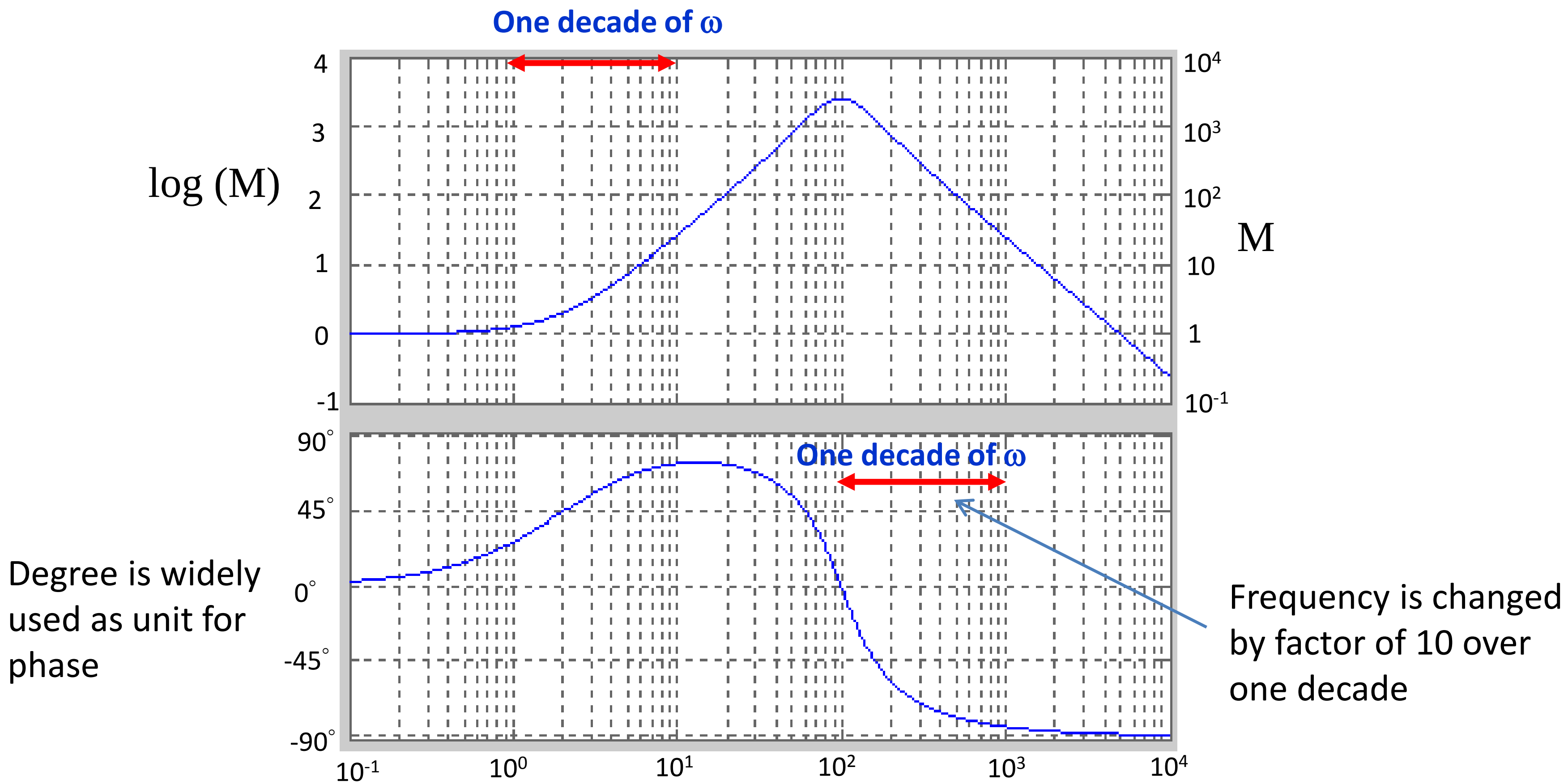
$G(j\omega_i)$ is an array of complex numbers which is not convenient for getting insight

  2) Frequency response plot helps to develop good intuition so that we can identify if there is any error in the data generated

- **Different frequency response plots**
  - Bode plot (in this chapter)
    - Useful for design of controller
  - Polar plot and Nyquist plot
    - To be discussed when we study stability analysis

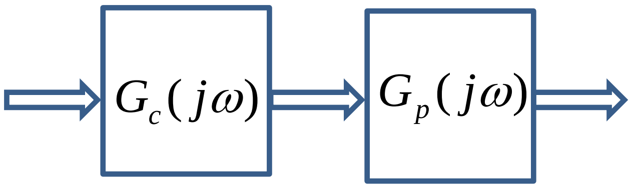
Bode Plot

- It consists of two plots
  - Plot of $\log(M)$ versus $\log \omega$
  - Plot of ϕ versus $\log \omega$
    - M: gain, ϕ: phase, ω: frequency



Bode Plot

- **Why Log Magnitude?**
  - In frequency domain, controller design requires to find how the frequency response is altered if a controller is put in series with the plant



  - Overall frequency response:

$$G(j\omega) = G_c(j\omega)G_p(j\omega)$$

$$|G|\angle G = \left(|G_c|\angle G_c\right)\left(|G_p|\angle G_p\right)$$

  - Overall gain and phase of the inter-connected system:

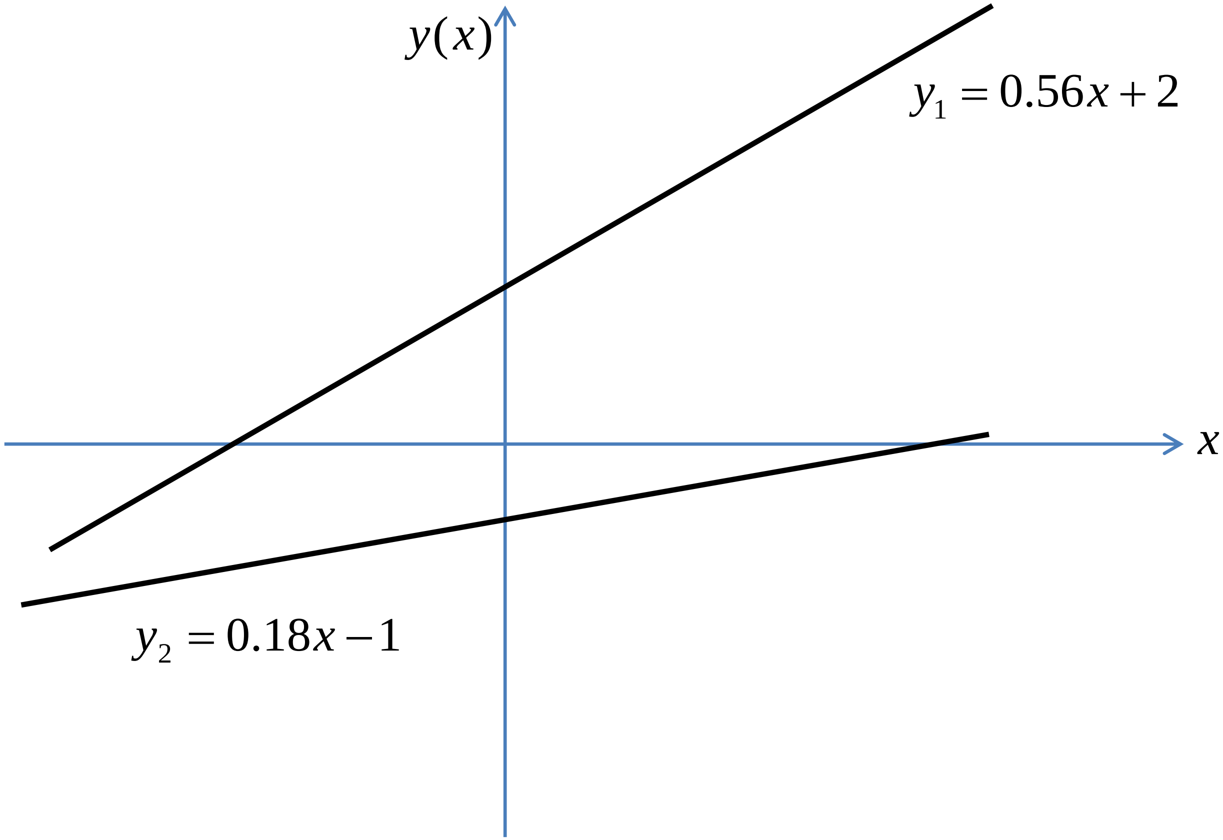
$$|G| = |G_c||G_p|$$

$$\angle G = \angle G_c + \angle G_p$$

    - Overall gain is product of individual gains
    - Overall phase is sum of individual phases

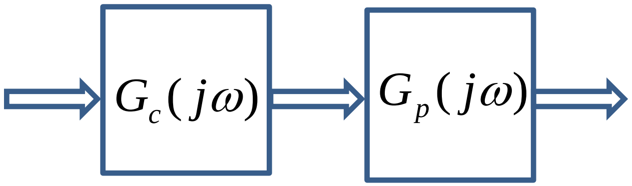
Bode Plot

- **Hand-sketching Sum of functions versus Product of functions**
  - Given the plots of two functions $y_1(x)$ and $y_2(x)$,
    - Hand-sketching the plot of $y_1(x)+y_2(x)$ is easier than sketching the plot of $y_1(x)y_2(x)$



  - Hand-sketch the plots for y_1+y_2 and y_1y_2.

- **Why Log Magnitude?**



$$|G| = |G_c||G_p|$$
$$\angle G = \angle G_c + \angle G_p$$

  - Given the plots of $\angle G_{\mathrm{p}}(\log\omega)$ and $\angle G_{\mathrm{c}}(\log\omega)$,
    - It is straightforward to hand-sketch the plot of $\angle G(\log\omega)$
  - But it is not so simple to hand-sketch the plot of $|G(\log\omega)|$ from the plots of $|G_{\mathrm{p}}(\log\omega)|$ and $|G_{\mathrm{c}}(\log\omega)|$
  - If Log magnitude is considered then it becomes sum

$$\log|G(j\omega)| = \log|G_c(j\omega)| + \log|G_p(j\omega)|$$

  - So instead of plotting gain (or magnitude), we plot log of gain
  - Sometime, magnitude axis is labeled in **dB** (decibel) scale

$$|G(j\omega)|_{dB} = 20\log|G(j\omega)|$$

Bode Plot

- **Sketching the Bode Plot**

  a) Point by Point sketching
  Data can be obtained either from experiment of by evaluating $G(j\omega)$ for $\omega = \omega_0, \omega_1, \omega_2, \ldots..$, if G(s) is given

  b) Direct hand-sketching if G(s) is given
  We are going to learn this technique next

- Any transfer function can be expressed as product of three types of factors

$$G(s) = K_0 \frac{(s+z_1)(s^2 + 2\zeta_1\omega_1 s + \omega_1^2)\ldots}{s^N (s+p_1)(s^2 + 2\zeta_2\omega_2 s + \omega_2^2)\ldots}$$

1) $\dfrac{K_0}{s^N}$ [pole or zero at the origin]

2) $(s+a)^{\pm 1}$ [real-axis pole or zero giving linear factor]

3) $\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)^{\pm 1}$ [complex pole or zero resulting in quadratic factor]

Bode Plot

- First, we learn to sketch the plots for each of the basic factors
- Bode plot of a composite G(s) can be sketched by adding the Bode plots of these individual factors

$$G(s) = 10\frac{(s+5)}{s(s+50)}$$

  - Bode plot of this transfer function can be sketched by adding Bode plots of the following terms

1) $\dfrac{10}{s}$ [one pole at the origin]

2) $(s+5)$ [real-axis zero]

3) $(s+50)^{-1}$ [real-axis pole]

Pole/zero at the origin

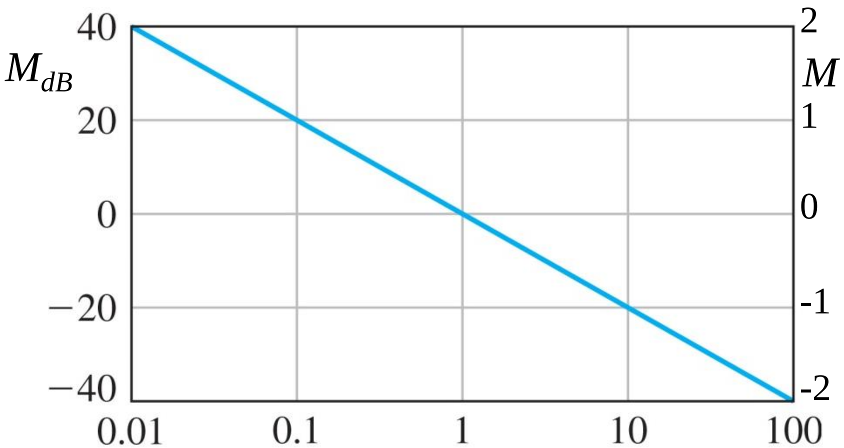
$$\frac{K_0}{s^N} \Rightarrow \frac{K_0}{(j\omega)^N} = \frac{K}{\omega^N \angle N \times 90^\circ}$$

N > 0: **Integrator**
N < 0: **Differentiator**
N = 0: **Pure Gain**

Magnitude/Gain plot: $$M(\omega) = \frac{K_0}{\omega^N} \Rightarrow \log M(\omega) = -N\log\omega + \log K_0$$

- With $\log\omega$ as the independent variable, the equation above represents a straight line with slope of –N
  - In Bode plot terminology, it is called –N/decade slope

- If dB scale is used,
  $$M_{dB} = -20N\log\omega + 20\log K_0$$
- The slope is –20N dB/decade
- The intersection of the gain plot and the horizontal axis happens at $\omega=\omega_0$ where
  $$\frac{K_0}{\omega_0^N} = 1 \Rightarrow \omega_0^N = K_0$$



Bode Plot

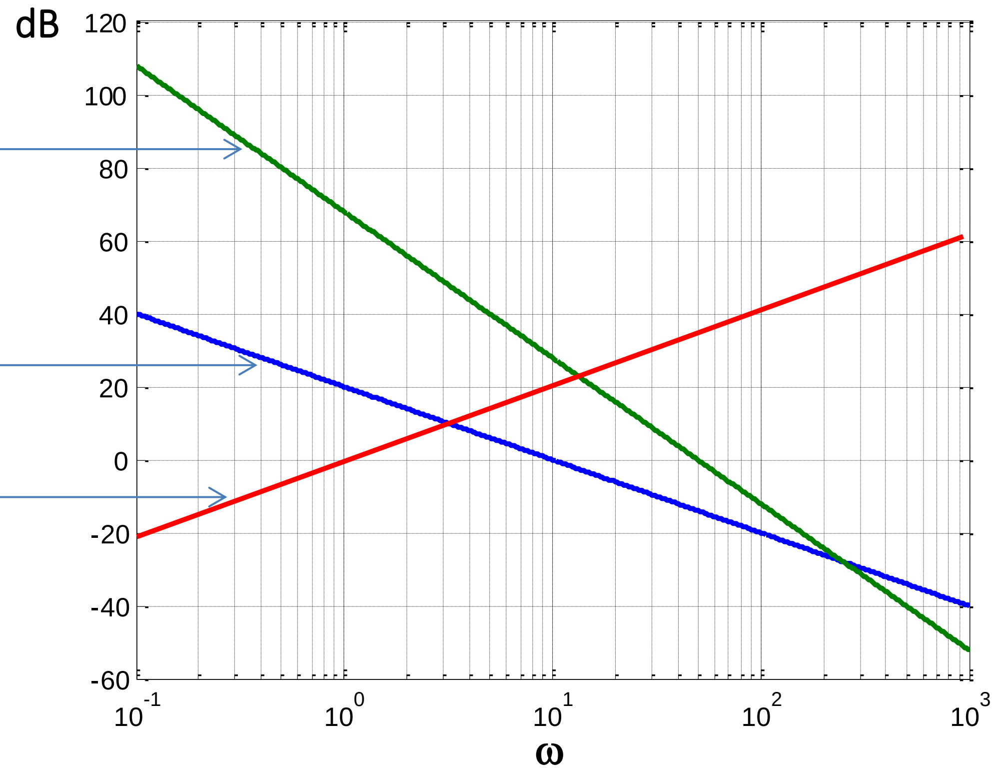
$$\frac{K_0}{s^N} \Rightarrow \frac{K_0}{(j\omega)^N} = \frac{K}{\omega^N \angle N \times 90°}$$

- Slope -20N dB/decade
- Intersects the horizontal axis at ω_0 where $\omega_0^N = K_0$

$$G(s) = \frac{50^2}{s^2}$$

$$G(s) = \frac{10}{s}$$

$$G(s) = s$$

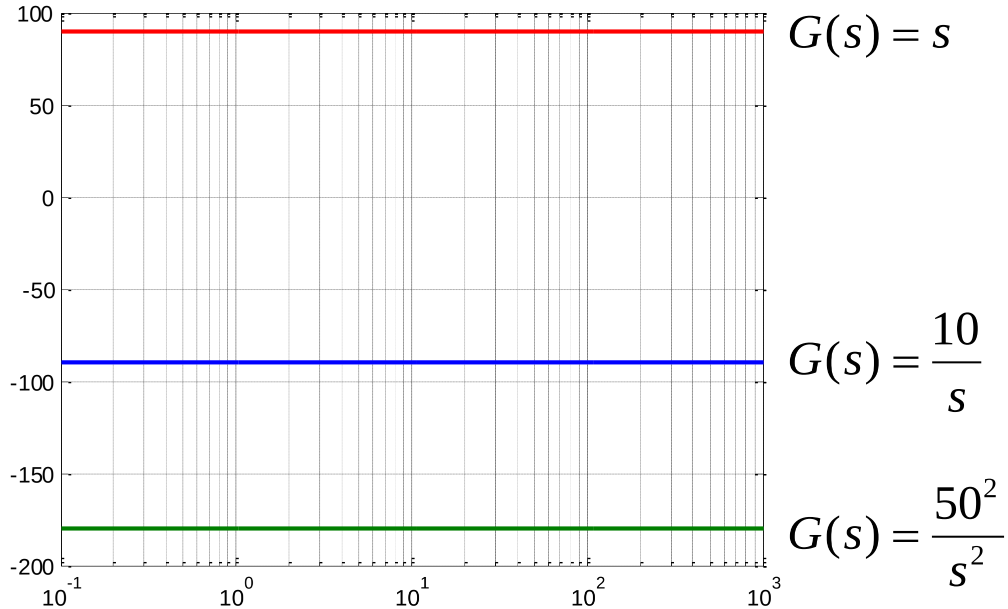


Bode Plot

Phase plot:

$$\frac{K_0}{s^N} \quad \Rightarrow \quad \frac{K_0}{(j\omega)^N} = \frac{K}{\omega^N \angle N \times 90^\circ} \qquad\qquad \phi(\omega) = -N \times 90^\circ$$

- Phase is constant regardless the value of ω
  - Phase plot is a horizontal line with vertical axis value equal to $(-N\times 90^\circ)$



Real-axis pole/zero (magnitude plot)

Real-axis Pole:

$$G(s)=\frac{1}{s+a} \quad \Rightarrow \quad G(j\omega)=\frac{1}{(j\omega+a)}$$

- Gain of the transfer function as function of ω

$$M(\omega)=\frac{1}{\sqrt{\omega^2+a^2}}=(\omega^2+a^2)^{-1/2}$$

$$\log(M)=-\tfrac{1}{2}\log(\omega^2+a^2)$$

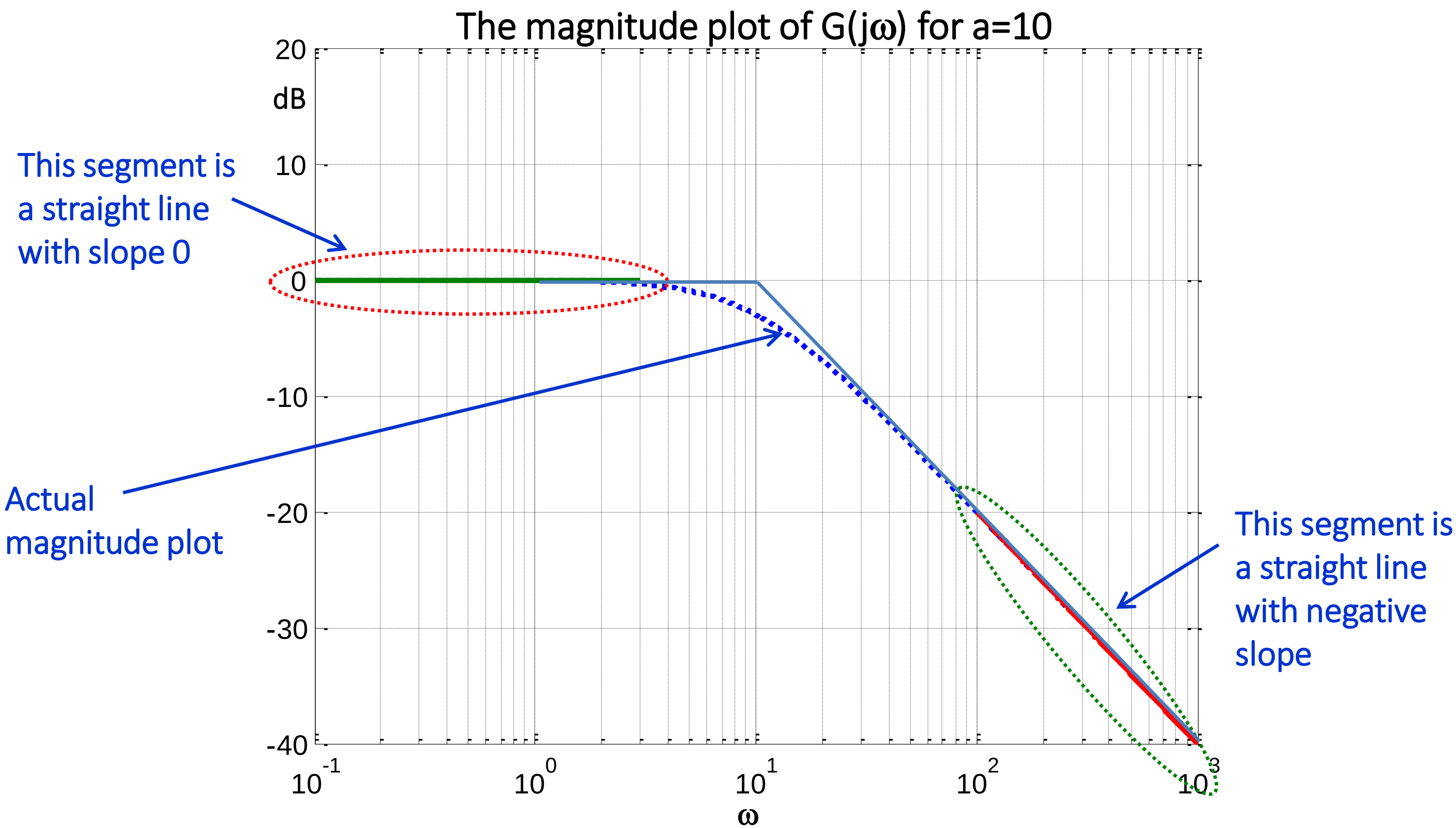
$$M_{dB}=-10\log(\omega^2+a^2)$$

  - Unlike the case of integrator or differentiator, this is not a straight line's equation

- However, the plot of log(M) or M_{dB} versus logω can be approximated by straight lines for ω<<a and for ω>>a
  - **Asymptotic Bode plot**

Bode Plot

$$G(s) = \frac{1}{s+a} \quad \Rightarrow \quad G(j\omega) = \frac{1}{(j\omega + a)}$$



Bode Plot

$$G(s)=\frac{1}{s+a} \quad \Rightarrow \quad G(j\omega)=\frac{1}{(j\omega+a)}$$

- Now, we find the equations of straight lines that approximate the actual plot

i. Low frequency, $\omega << a$

$$a+j\omega \cong a, \quad as \quad \omega << a$$

$$G(j\omega)=\frac{1}{a+j\omega}\cong\frac{1}{a}$$

$$\log M(\omega)=0-\log a$$

$$M_{dB}=-20\log a$$

- The approximation of magnitude plot is a horizontal straight line for $\omega << a$
  - The value is same as the gain for ω=0 or **DC-gain**

$$G(j\omega)\big|_{\omega=0}=\frac{1}{a} \quad \Rightarrow \quad M_{dB}=-20\log a$$

Bode Plot

$$G(s)=\frac{1}{s+a} \quad \Rightarrow \quad G(j\omega)=\frac{1}{(j\omega+a)}$$

ii. High frequency, $\omega >> a$

$$a+j\omega \cong j\omega, \quad as \quad \omega >> a$$

$$G(j\omega)=\frac{1}{a+j\omega}\cong\frac{1}{j\omega}$$

$$M(\omega)=\frac{1}{\omega}$$

$$\log M(\omega) = -\log \omega$$

$$M_{dB} = -20\log\omega$$

- Magnitude plot approximation is a straight line with slope of -20 dB/decade
  - We can find the point of intersection of the two straight lines

$$M_{dB} = -20\log a, \quad for \quad \omega << a$$

$$M_{dB} = -20\log \omega, \quad for \quad \omega >> a$$

  - Two lines intersect at a point $\omega=\omega_c$ where, $\omega=\omega_c=a$

This frequency is known as the **corner frequency**

Bode Plot

Real-axis Zero:

$$G(s) = (s+b) \quad \Rightarrow \quad G(j\omega) = (j\omega + b)$$

$$M = \sqrt{\omega^2 + b^2}$$

$$\log(M) = +\tfrac{1}{2}\log(\omega^2 + b^2)$$

- As we did for the real-axis pole, we are going to approximate the plot of log(M) or M_{dB} versus logω using two straight lines: one for ω<<b and the other for ω>>b

i. Low frequency range, $\omega << b$

$$b + j\omega \cong b, \quad \because \quad \omega << b$$

$$G(j\omega) = (b + j\omega) \cong b$$

$$M = b$$

$$\log M = +\log b$$

$$M_{dB} = +20\log b$$

- Magnitude plot approximation is a horizontal straight line with vertical-axis value equal to the DC-gain, $20\log b$

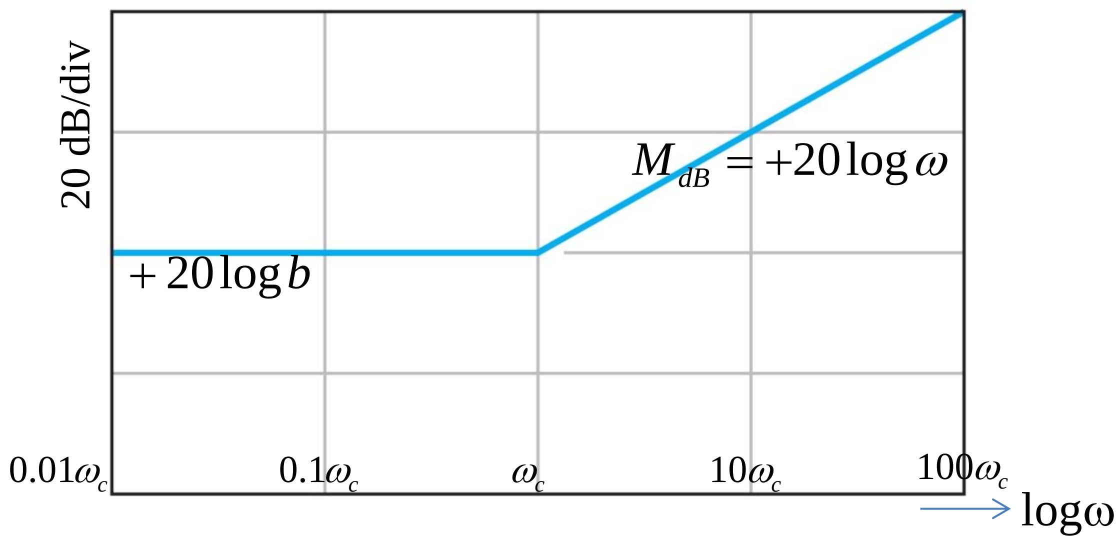
Bode Plot

ii. High frequency range, $\omega >> b$

$$b + j\omega \cong j\omega, \quad as \quad \omega >> b \qquad \log M(\omega) = +\log\omega$$

$$G(j\omega) = (b + j\omega) \cong j\omega \qquad M_{dB} = +20\log\omega$$

- Magnitude plot approximation is a straight line with slope of +20 dB/decade



- **Two lines intersect at a point ω=ω$_c$ where,** $\omega = \omega_c = b$

This frequency is known as the **corner frequency**

Sketching Bode (magnitude) Plot of Composite Transfer Function

- Now we know how to sketch Bode (magnitude) plot of integrator, differentiator, real-axis pole and real-axis zero.
  - How do we combine them to sketch Bode plot of a composite G(s)

Bode (magnitude) plot of integrator/ differentiator/ real pole/ real zero: summary

Integrator/Differentiator	Real pole	Real zero
$G(s)=\frac{K_0}{s^N}$	$G(s)=(s+a)^{-1}$	$G(s)=(s+b)$
	$\omega_c=a$	$\omega_c=b$
Line with -20N dB/decade slope for all ω	Line with slope 0 for $\omega<\omega_c$	
	Line with -20 dB/decade slope for $\omega>\omega_c$	Line with +20 dB/decade slope for $\omega>\omega_c$
Line intersects 0 dB line at ω_0 where $(\omega_0)^N=K_0$	Two lines intersects at $\omega=\omega_c$	
Infinite dc-gain; can’t be shown on the plot	DC-gain is finite; vertical-axis value for the low frequency is equal to the dc-gain	

Bode Plot

- *Example 2-1*: Sketch Bode (magnitude) plot of

$$G(s) = 25\frac{(s+10)}{(s+1)(s+100)}$$

- There is no integrator, the dc-gain is finite.

$$G(j\omega) = 25\frac{(j\omega+10)}{(j\omega+1)(j\omega+100)} \quad \Rightarrow \quad G(j0) = 25\frac{(10)}{(1)(100)} = 2.5$$

- The vertical-axis value of Bode (magnitude) for frequency much smaller than the smallest corner frequency is the dc-gain

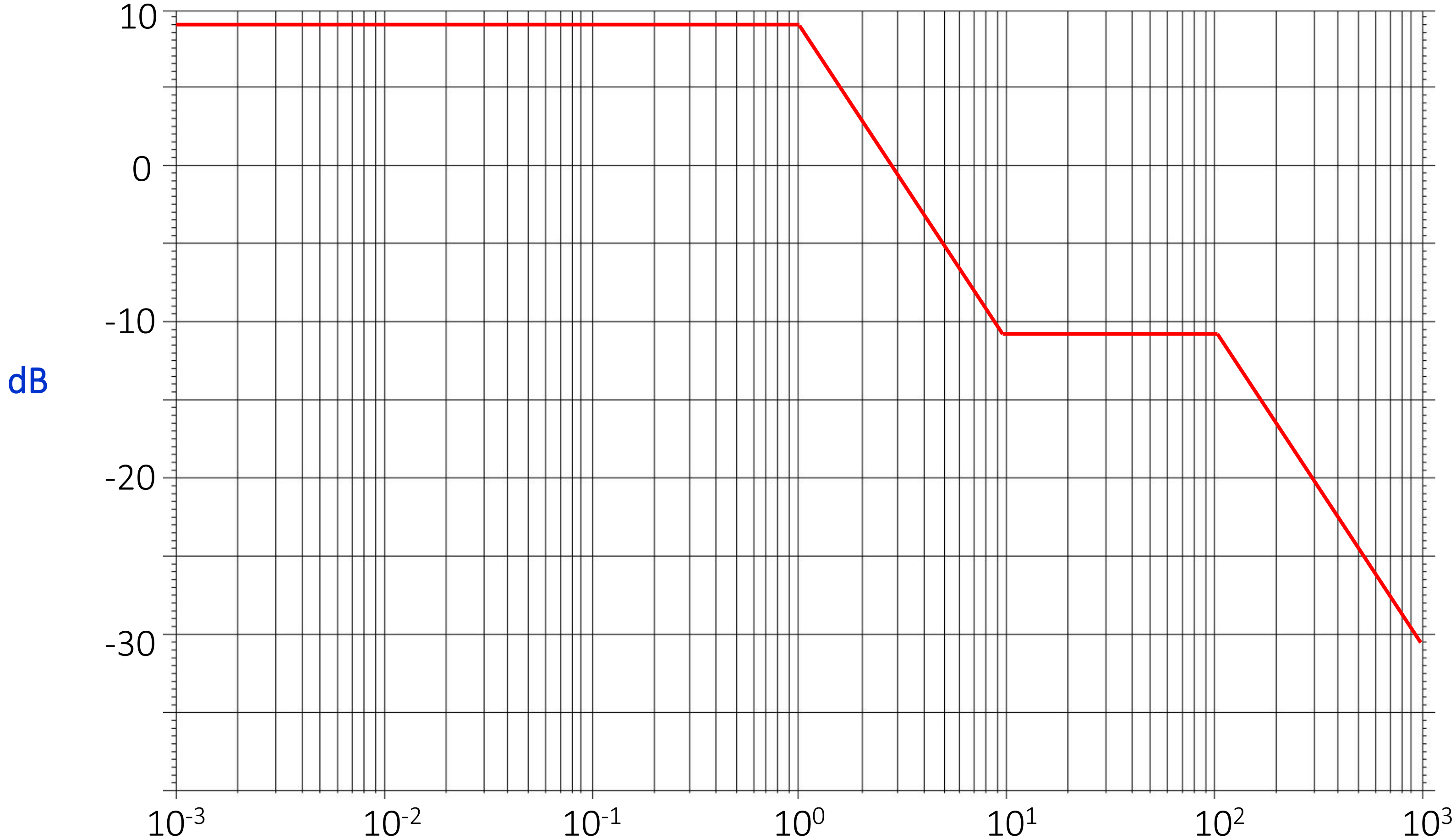
$$20\log(2.5) = +7.95\,dB$$

  - Starting at this value for low frequency, the magnitude plot is horizontal line up to the smallest of the corner frequency, i.e., ω=1 where the slope is changed to -20 dB/decade

  - As we move towards higher frequency, at every corner frequency the slope is changed by +20 dB/decade for a zero and -20 dB/decade for a pole

Bode Plot

$$G(s) = 25\frac{(s+10)}{(s+1)(s+100)} \qquad dc\ gain = +7.9\ dB$$



Bode Plot

- *Example 2-2*: Sketch Bode (magnitude) plot of the transfer function

$$G(s) = 25\frac{(s+10)}{s(s+100)}$$

- The dc-gain is infinite. How to choose the leftmost point of the magnitude plot?

- Option 1: Choose a frequency much smaller than the smallest of the corner frequencies of real-axis poles and real-axis zeros
  - Determine the gain at that frequency. For the given G(s), let's choose ω=1

$$G(j\omega) = 25\frac{(j\omega+10)}{(j\omega)(j\omega+100)} \quad \Rightarrow \quad G(j1) = 25\frac{(10+j1)}{(j1)(100+j1)}$$

$$|G(j1)| = 25\frac{\sqrt{100+1}}{1\sqrt{10000+1}} = 2.5$$

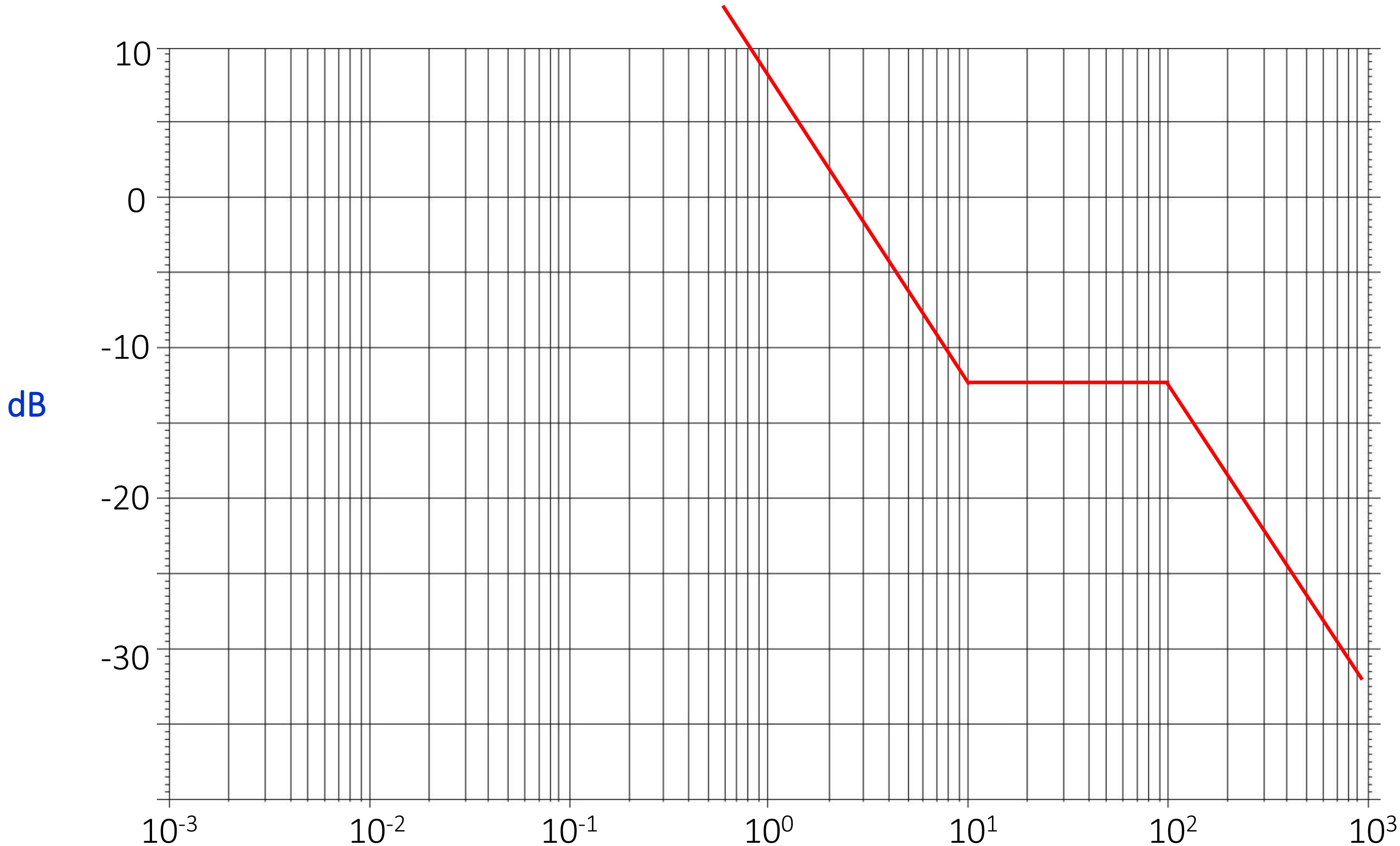
$$20\log(2.5) = +7.95\,dB$$

  - Starting at this value, draw a line with -20 dB/decade slope
  - As we move towards higher frequency, at every corner frequency the slope is changed by +20 dB/decade for a zero and -20 dB/decade for a pole

Bode Plot

$$G(s) = 25\frac{(s+10)}{s(s+100)}$$

$$|G(j1)| = +7.95\,dB$$



Bode Plot

- *Example* 2-3: Sketch the Bode (magnitude) plot of the transfer function

$$G(s) = 25\frac{(s+10)}{s(s+100)}$$

- The dc-gain is infinite. How to choose the leftmost point of the magnitude plot?
- Option 2: Use Bode form of the transfer function

- **Bode Form**:
  - All factors of the transfer function, except pole/zero at the origin, are expressed in such a way that their constant term is 1. For example,

$$G(s) = 25\frac{(s+10)}{s(s+100)} \quad \Rightarrow \quad G(s) = 25\frac{10(0.1s+1)}{s100(0.01s+1)} = \frac{2.5}{s}\frac{(0.1s+1)}{(0.01s+1)}$$

  - The corner frequencies of the linear factors remain unchanged
  - All factors (except pole/zero at origin) now have dc-gain of 1

Bode Plot

$$G(s) = 25\frac{(s+10)}{s(s+100)}$$

- Express the transfer function in the Bode form

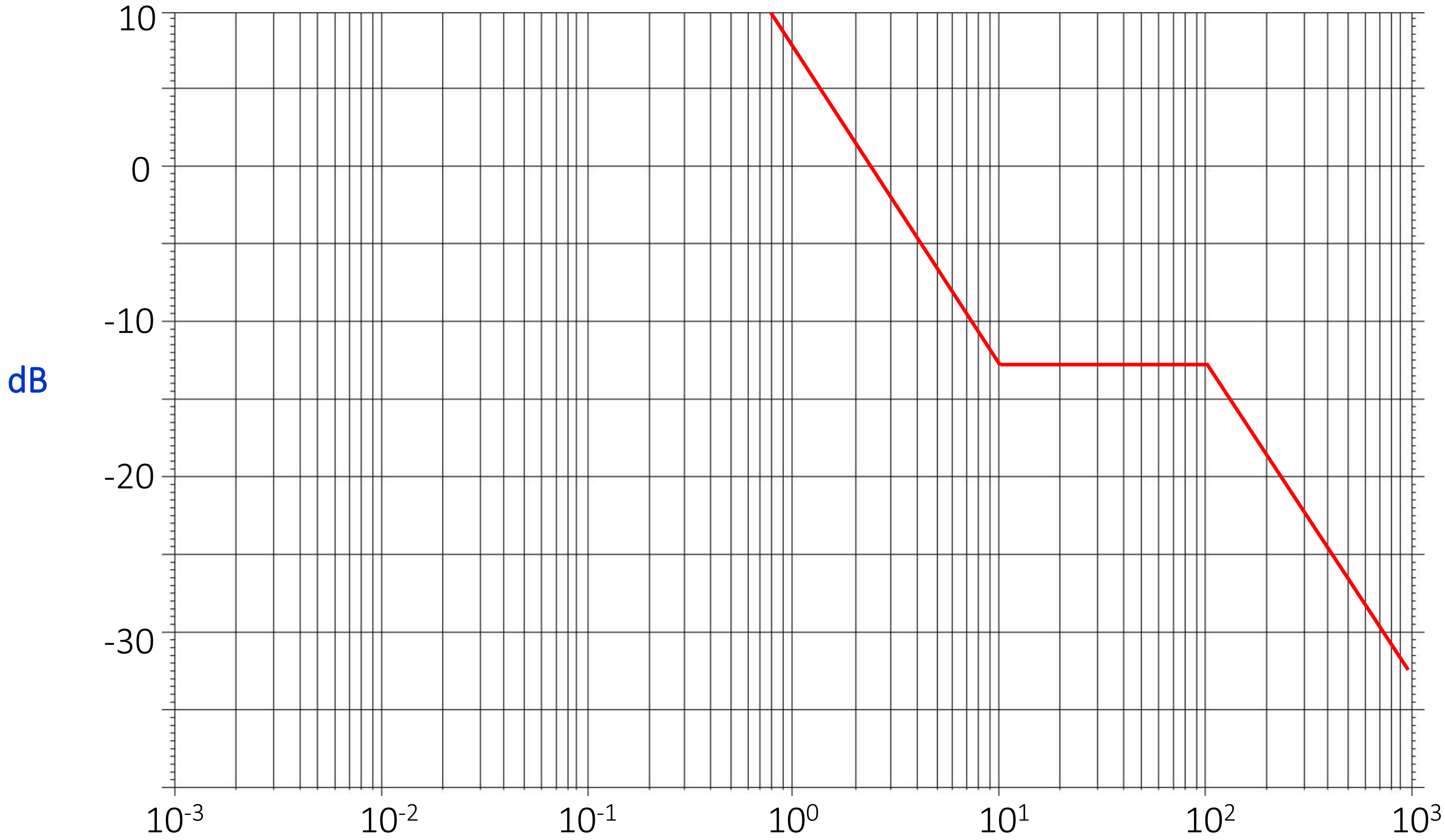
$$G(s) = \frac{2.5}{s}\frac{(0.1s+1)}{(0.01s+1)}$$

  - Sketch the magnitude plot with the integrator term, i.e., a line with -20 dB/decade slope intersecting the 0 dB line at ω that satisfies the condition

$$\omega^N = K_0$$

  - For the given transfer function, N=1 and K_0=2.5
  - As we move towards higher frequency, at every corner frequency the slope is changed by +20 dB/decade for a zero and -20 dB/decade for a pole

Bode Plot



Real-axis pole/zero (phase plot)

Real-axis Pole: $G(s) = \frac{1}{s+a} \quad \Rightarrow \quad G(j\omega) = \frac{1}{(j\omega + a)}$

- Phase of the transfer function

$$\phi(\omega) = -\tan^{-1}\frac{\omega}{a}$$

  - This is also a nonlinear function and we consider piecewise linear approximation

i. Low frequency: $\omega << a$

$$\phi(\omega) = \quad 0°, \quad for \quad \omega << a$$

ii. High frequency: $\omega >> a$

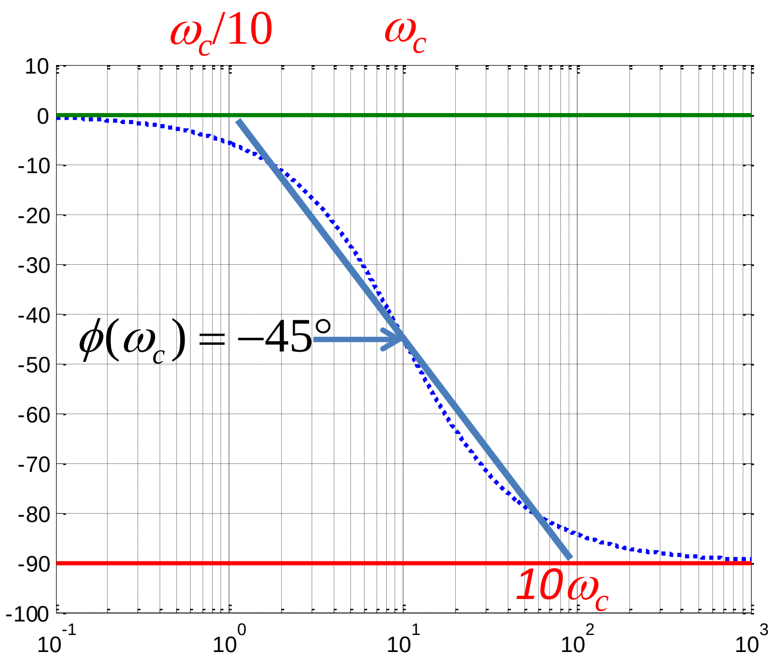
$$\phi(\omega) = \quad -90°, \quad for \quad \omega >> a$$

iii. At the corner frequency, $\omega = \omega_c = a$,

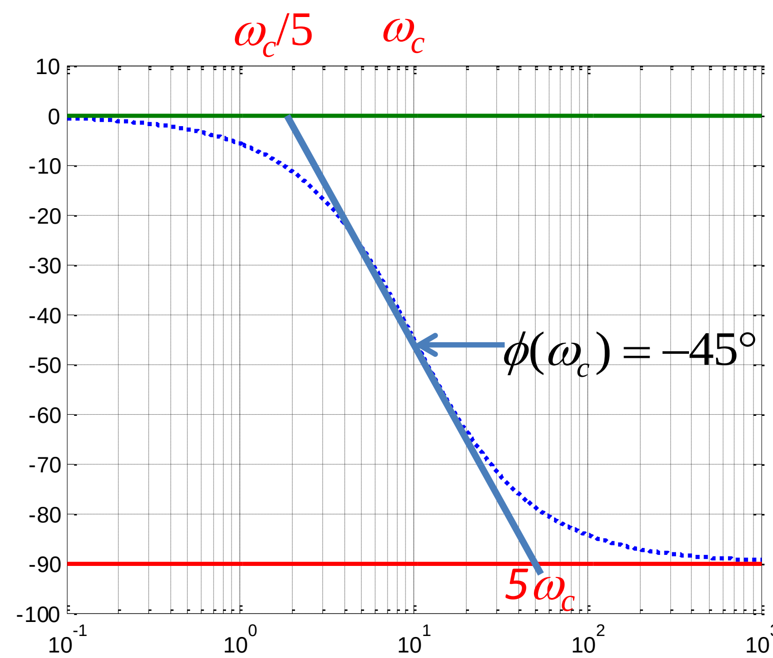
$$\phi(\omega) = \quad -45°, \quad for \quad \omega = a$$

Bode Plot

- Phase plot for real-axis plot with corner frequency at 10 rad/s



Approximation suggested in the text by Franklin & Powell



Approximation suggested in the text by Bishop & Dorf

Bode Plot

Real-axis Zero: $G(s) = (s+b) \quad \Rightarrow \quad G(j\omega) = (j\omega + b)$

- Phase of the transfer function as function of ω

$$\phi(\omega) = +\tan^{-1}\frac{\omega}{b}$$

i. Low frequency: $\phi(\omega) = \quad 0^\circ, \quad for \quad \omega << b$

ii. High frequency: $\phi(\omega) = \quad +90^\circ, \quad for \quad \omega >> b$

iii. At the corner frequency, $\omega = \omega_c = b$, $\phi(\omega) = \quad +45^\circ, \quad for \quad \omega = b$

- Straight-line approximation similar to the one shown for real-axis pole is also applicable here
  - However, for composite transfer function phase is sketched in a simpler way (illustrated in the next slide)

Phase plot of composite transfer function

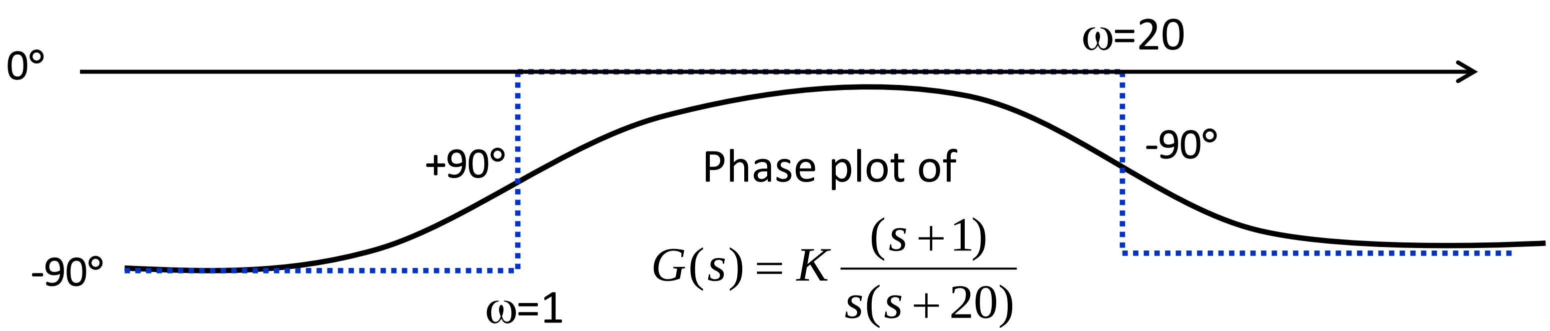
$$G(s) = K\frac{(s+a)\ldots}{s^N(s+b)\ldots}$$

- Phase as $\omega \to 0$ depends on the number of integrator/ differentiator in G(s)

$$\phi(\omega)\big|_{\omega\to 0} = N\times(-90^\circ)$$

1) Start sketching with this value
2) Keep it same with increasing values of ω until a corner frequency is reached
   - Add 90° if the corner frequency is of a zero or
   - Subtract 90° if the corner frequency is of a pole

- This produces a $\phi(\omega)$ plot showing step changes at the corner frequencies
  - Hand-sketch a smooth curve through these steps



Bode Plot

Quadratic Factor

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

ω_n: natural frequency

ζ: damping coefficient ($0<\zeta<1$)

- Modify this into Bode form so that the dc-gain is 1 (0 dB)

$$\omega_n^2\left(\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1\right)$$

- The term ω_n^2 outside the bracket is to be multiplied to the integrator term of the composite transfer function (example below)

$$G(s) = 10\frac{(s+5)}{s(s^2+6s+15)} = \frac{10/3}{s}\frac{\left(\frac{s}{5}+1\right)}{\left(\frac{s^2}{15}+\frac{6}{15}s+1\right)}$$

- We use the following term for understanding the nature of its Bode plot

$$\left(\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1\right)$$

Bode Plot

$$\left(\frac{s^2}{\omega_n^2}+2\zeta\frac{s}{\omega_n}+1\right)$$

- Corresponding frequency response -

$$\left(\frac{(j\omega)^2}{\omega_n^2}+2\zeta\frac{j\omega}{\omega_n}+1\right)$$

- Depending on its placement in the transfer function, it represents either two poles (denominator) or two zeros (numerator)

- Intuitively,
  - Magnitude plot should be
    - Low frequency slope of 0
    - High frequency slope of 2×(-20) dB/decade for poles and 2×(+20) dB/decade slope for zeros
  - Phase plot should be
    - 0° for low frequency
    - 2×(-90°) for poles and 2×(+90°) for zeros in the high frequency

Bode Plot

$$\left(\frac{(j\omega)^2}{\omega_n^2} + 2\zeta \frac{j\omega}{\omega_n} + 1 \right)$$

- Let's define a normalized frequency $\upsilon = \dfrac{\omega}{\omega_n}$

$$(j\upsilon)^2 + j2\zeta\upsilon + 1 \quad \Rightarrow \quad (1-\upsilon^2) + j2\zeta\upsilon$$

- $\omega \ll \omega_n$ implies $\upsilon \ll 1$ and $\omega \gg \omega_n$ implies that $\upsilon \gg 1$

Sketching Bode (magnitude) Plot for Complex Poles

i. Low frequency range,

$$G(j\upsilon) = \frac{1}{(1-\upsilon^2) + j2\zeta\upsilon}, \qquad as \quad \upsilon \ll 1, \quad G(j\upsilon) \cong 1 \qquad \left|G(j\upsilon)\right| \approx 1$$

$$G(j\upsilon)\big|_{dB} \approx 0$$

- For $\omega \ll \omega_n$, the Bode (magnitude) plot is a horizontal line at 0 dB

Bode Plot

$$G(j\upsilon) = \frac{1}{(1-\upsilon^2) + j2\zeta\upsilon}$$

ii. High frequency range,

$$as \quad \upsilon >> 1, \quad G(j\upsilon) \cong \frac{1}{-\upsilon^2 + j2\zeta\upsilon}$$

$$|G(j\upsilon)| = \frac{1}{\sqrt{\upsilon^4 + 4\zeta^2\upsilon^2}}$$

$$= \frac{1}{\sqrt{\upsilon^2(\upsilon^2 + 4\zeta^2)}}$$

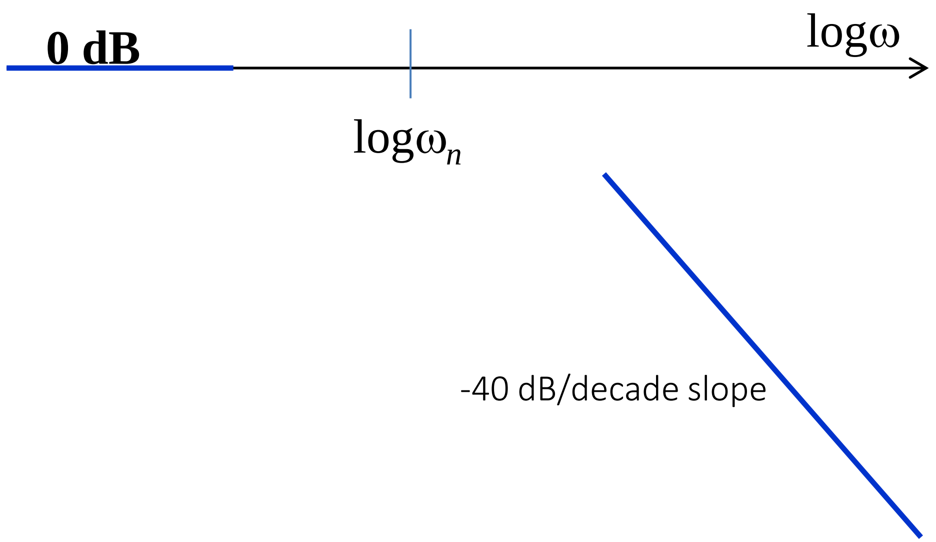
$$\approx \frac{1}{\sqrt{\upsilon^2(\upsilon^2)}}$$

$$= \frac{1}{\upsilon^2}$$

$$|G(j\upsilon)|_{dB} = 20\log(\upsilon^{-2})$$

$$= -40\log\upsilon$$

-40 dB/decade slope



Bode Plot

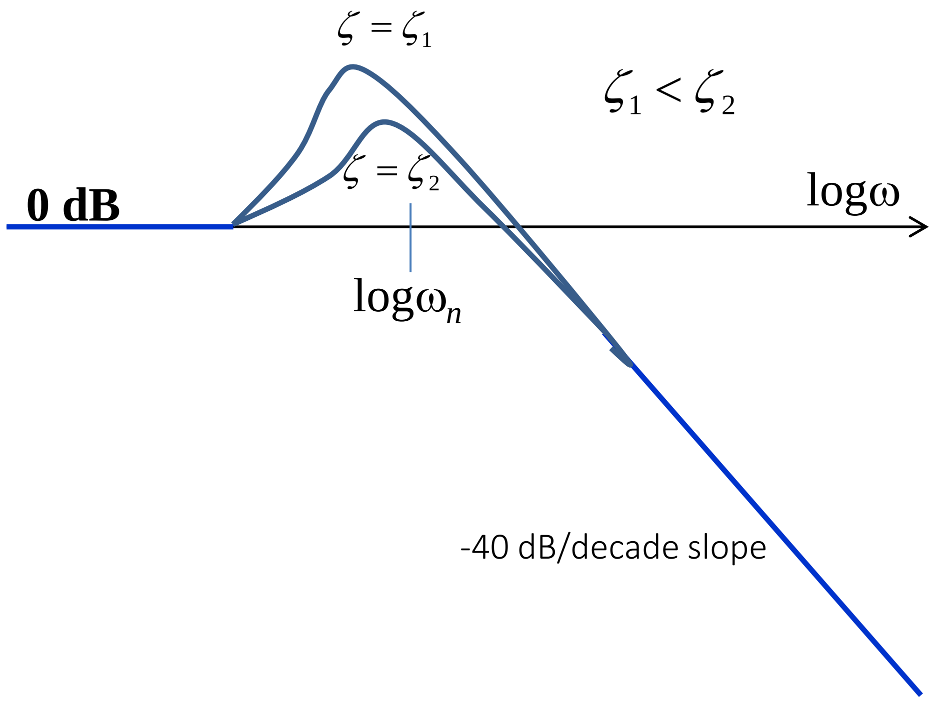
$$G(j\upsilon) = \frac{1}{(1-\upsilon^2) + j2\zeta\upsilon}$$

iii. For $\omega=\omega_n$,

$$as \quad \upsilon = 1, \quad G(j\upsilon) \cong \frac{1}{j2\zeta}$$

$$\left|G(j\upsilon)\right|_{\upsilon=1} = \frac{1}{2\zeta}$$

- For $\omega=\omega_n$, the gain depends on the damping coefficient ζ
  - Magnitude can be >1, i.e., >0 dB
- Smaller the value of ζ, larger is the gain at natural frequency



Bode Plot

Sketching Bode (magnitude) Plot for Complex Zeros

$$G(j\upsilon) = ((j\upsilon)^2 + j2\zeta\upsilon + 1)$$
$$= (1-\upsilon^2) + j2\zeta\upsilon$$

i. **Low frequency range,**

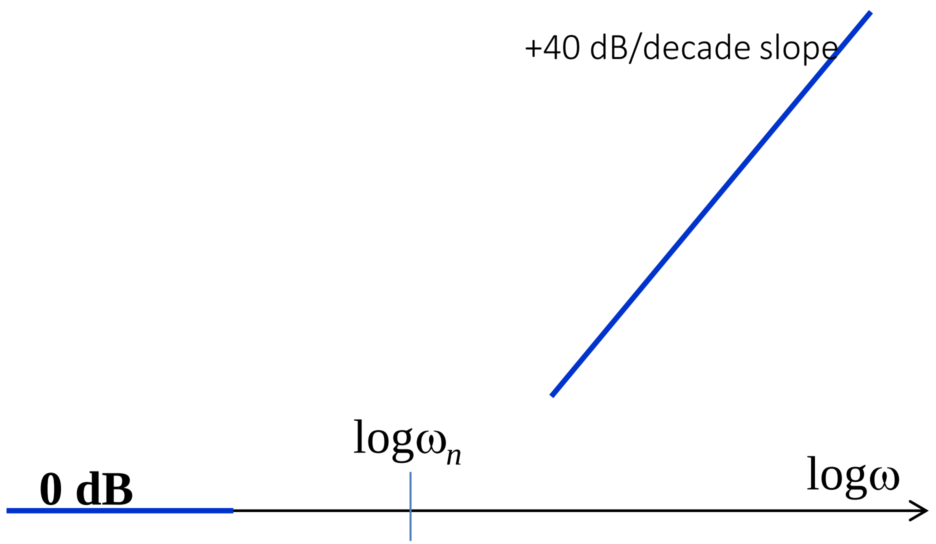
$$as \quad \upsilon << 1, \quad G(j\upsilon) \approx 1 \qquad |G(j\upsilon)| \approx 1$$
$$G(j\upsilon)|_{dB} \approx 0$$

ii. **High frequency range,**

$$as \quad \upsilon >> 1, \quad G(j\upsilon) \approx -\upsilon^2 + j2\zeta\upsilon$$

$$|G(j\upsilon)| = \sqrt{\upsilon^4 + 4\zeta^2\upsilon^2}$$
$$\approx \sqrt{\upsilon^2(\upsilon^2)}$$
$$= \upsilon^2$$

$$|G(j\upsilon)|_{dB} = +40\log\upsilon$$ +40 dB/decade slope



Bode Plot

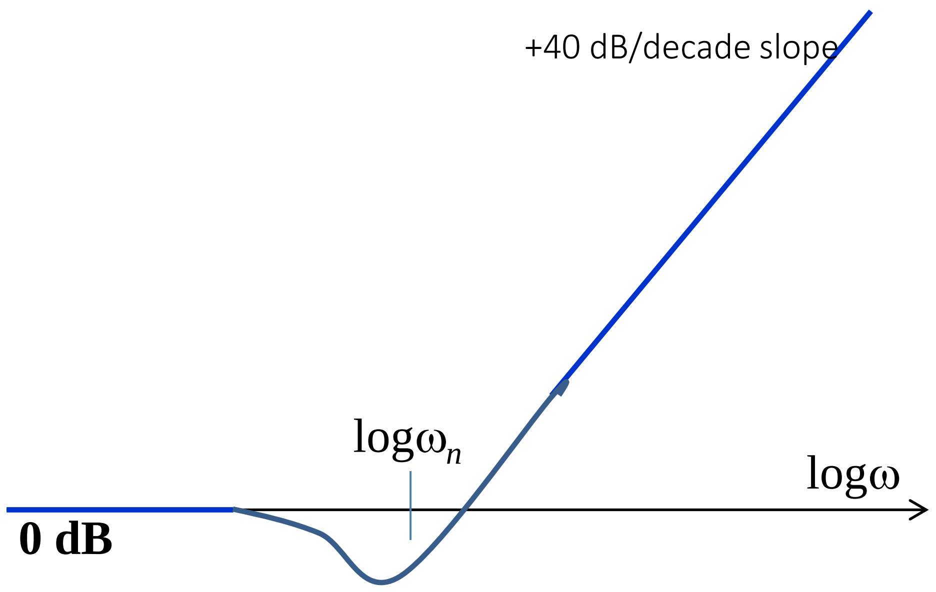
$$G(j\upsilon) = ((j\upsilon)^2 + j2\zeta\upsilon + 1)$$
$$= (1-\upsilon^2) + j2\zeta\upsilon$$

iii. For ω=ω$_n$,

$$as \quad \upsilon = 1, \quad G(j\upsilon) = j2\zeta$$

$$\left|G(j\upsilon)\right|_{\upsilon=1} = 2\zeta$$

- For $\omega=\omega_n$, the gain depends on the damping coefficient ζ
  - Magnitude can be <1, i.e., <0 dB
- Smaller the value of ζ, smaller is the gain at natural frequency



Sketching Bode (phase) Plot for Complex Poles

$$G(j\upsilon) = \frac{1}{(1-\upsilon^2) + j2\zeta\upsilon}, \qquad \angle G(j\upsilon) = -\tan^{-1}\frac{2\zeta\upsilon}{1-\upsilon^2}$$

i. Low frequency range,

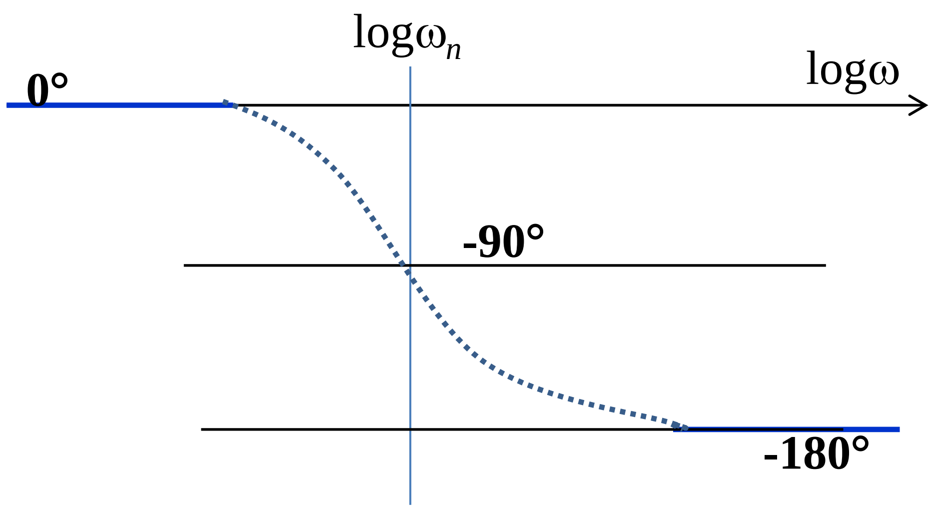
$$as \quad \upsilon << 1, \quad \angle G(j\upsilon) \approx 0°$$

ii. High frequency range,

$$as \quad \upsilon >> 1, \quad \angle G(j\upsilon) \approx -\tan^{-1}\frac{2\zeta\upsilon}{-\upsilon^2} = -\tan^{-1}\frac{2\zeta}{-\upsilon}$$

$$\angle G(j\upsilon) = -\left[180° - \tan^{-1}\frac{2\zeta}{-\upsilon}\right]$$

$$\cong -180°$$



iii. $\omega = \omega_n$,

$$as \quad \upsilon = 1, \quad \angle G(j\upsilon) = -\tan^{-1}\frac{2\upsilon}{0} = -90°$$

Bode Plot

Sketching Bode (phase) Plot for Complex Zeros

$$G(j\upsilon) = (1-\upsilon^2) + j2\zeta\upsilon, \qquad \angle G(j\upsilon) = +\tan^{-1}\frac{2\zeta\upsilon}{1-\upsilon^2}$$

i. Low frequency range,

$$as \quad \upsilon << 1, \quad \angle G(j\upsilon) \approx 0°$$

ii. High frequency range,

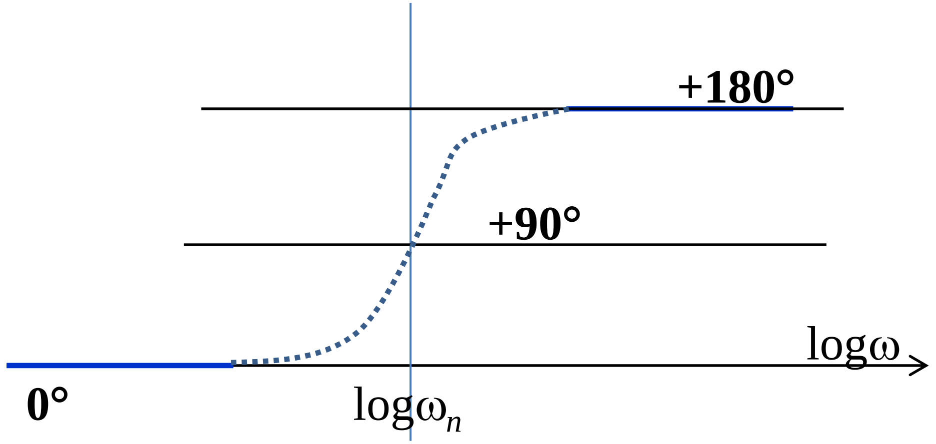
$$as \quad \upsilon >> 1, \quad \angle G(j\upsilon) \approx +\tan^{-1}\frac{2\zeta\upsilon}{-\upsilon^2} = +\tan^{-1}\frac{2\zeta}{-\upsilon}$$

$$\angle G(j\upsilon) = +\left[180° - \tan^{-1}\frac{2\zeta}{-\upsilon}\right]$$

$$\cong +180°$$

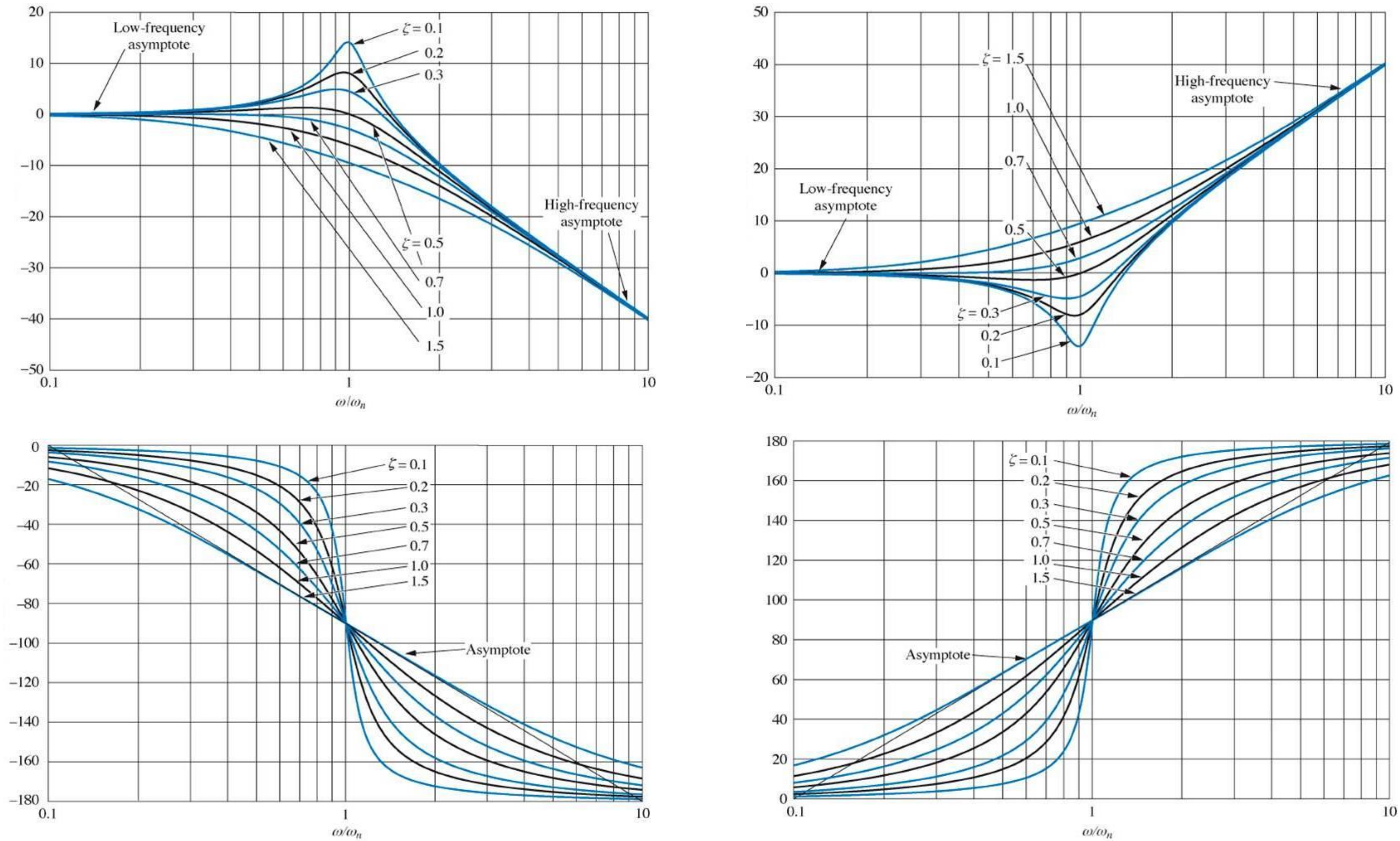
iii. $\omega = \omega_n$,

$$as \quad \upsilon = 1, \quad \angle G(j\upsilon) = +\tan^{-1}\frac{2\upsilon}{0} = +90°$$



Bode Plot

Bode Plot of Complex Poles (left) and Zeros (right) for different values of ζ



Estimating G(s) from Bode Plot

- Given the Bode plot, can we estimate the corresponding G(s)?

- Characteristic features of the magnitude plot:
  a) At the lowest frequency, the slope of the gain plot is determined by the pole/zero at the origin; low frequency slope is zero for 1st-order terms or quadratic terms

$$slope_{low\_freq} = N \times (-20) \quad dB/decade$$

N: number of integrator in G(s)
negative value of N implies differentiator

  b) At the lowest frequency, the phase is zero for 1st-order term or quadratic term; so low frequency phase is determined by the pole/zero at the origin

$$\phi_{low\_freq} = N \times (-90^\circ)$$

Estimating G(s) from Bode Plot

c) At the highest frequency, a poles gives -20 dB/decade slope and a zero gives +20 dB/decade slope

- If there are ***n*** poles (including pole at the origin) and ***m*** zeros (including zero at the origin) in a transfer function then the high frequency slope of the magnitude plot is

$$
\begin{aligned}
slope_{high\_freq} &= n \times (-20) + m \times (+20) \quad dB/decade \\
&= (n-m) \times (-20)\ dB/decade
\end{aligned}
$$

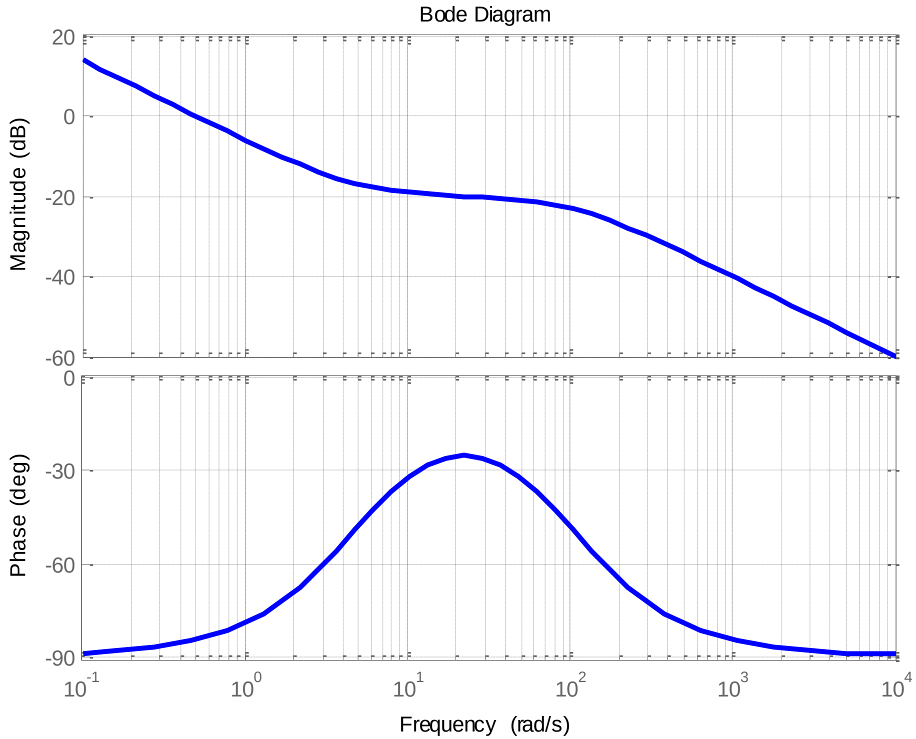
d) At the corner frequency of a zero, the slope of the gain plot is altered by +20 dB/decade

e) At the corner frequency of a pole, the slope is changed by -20 dB/decade

- Change in the slope of the magnitude plot indicates the presence of corner frequency or natural frequency
- Bending upward indicates presence of zero
- Bending downward means pole

Estimating G(s) from Bode Plot

- *Example* 2-4:



-20 dB/dec slope in low frequency $\Rightarrow$ one integrator

-20 dB/dec slope in high frequency $\Rightarrow$ (n-m) = 1

One upward bend (ω≈5) and one downward bend (ω≈100)

Candidate model:

$$G_{est}(s) = \frac{K}{s}\frac{(\tau_z s + 1)}{(\tau_p s + 1)}$$

$$\omega_0 \cong 0.5 \quad \Rightarrow \quad K = 0.5$$

$$\tau_z = 0.2$$

$$\tau_p = 0.01$$

$$\Rightarrow \quad G_{est}(s) = \frac{0.5}{s}\frac{(0.2s+1)}{(0.01s+1)} = \frac{10(s+5)}{s(s+100)}$$

Estimating G(s) from Bode Plot

- Estimating magnitude plot from the magnitude plot alone may lead to erroneous result because of two reasons

1) Transportation delay modifies phase plot but not the magnitude plot. $G_1(s)$ and $G_2(s)$ below have identical magnitude but different phase.

$$G_1(s) = \frac{P(s)}{Q(s)}, \quad G_2(s) = \frac{P(s)}{Q(s)} e^{-t_d s}$$

2) $(s+a)$ and $(s-a)$ have identical magnitude plot but different phase plot

$$(s+a) \quad \Rightarrow \quad (j\omega + a) = \sqrt{\omega^2 + a^2} \angle \tan^{-1} \tfrac{\omega}{a}$$

$$(s-a) \quad \Rightarrow \quad (j\omega - a) = \sqrt{\omega^2 + a^2} \angle (180^\circ - \tan^{-1} \tfrac{\omega}{a})$$

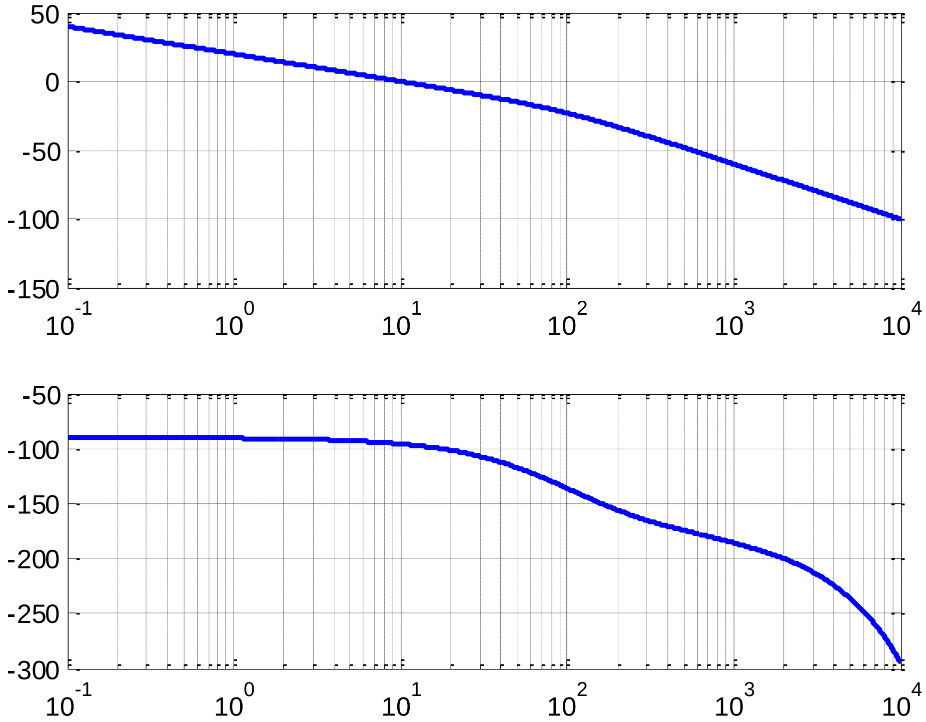
- Following transfer functions have identical magnitude but different phase

$$G_1(s) = \frac{(s+1)}{s(s+10)}, \quad G_2(s) = \frac{(s-1)}{s(s+10)}$$

Non-minimum Phase System: Transfer function has RHS zero or pole

Estimating G(s) from Bode Plot

- *Example* 2-5:



-20 dB/decade slope in low frequency $\Rightarrow$ one integrator

-40 dB/decade slope in high frequency $\Rightarrow$ (n-m) = 2

One downward bend ($\omega \approx 50$)

Candidate model (considering gain alone):

$$G_{est}(s) = \frac{K}{s(\tau_p s + 1)}$$

$$\omega_0 \cong 10 \quad \Rightarrow \quad K = 10$$

$$\tau_p \approx 0.02$$

$$G_{est}(s) = \frac{10}{s(0.02s+1)} = \frac{500}{s(s+50)}$$

Estimating G(s) from Bode Plot

$$G_{est}(s) = \frac{10}{s(0.02s+1)} = \frac{500}{s(s+50)}$$

[this transfer function has two poles and no zero. However, in the given phase plot, high frequency phase doesn't approach -180° as expected.]

- Phase decreases continuously with increasing ω indicating the presence of delay. Modify estimated model to

$$G_{est}(s) = \frac{500}{s(s+50)} e^{-t_d s}$$

- To determine t_d, find from the Bode plot the phase at some high frequency point. Let's choose ω=6000 for this example.

$$\phi\big|_{\omega=6000} \approx -250^\circ$$ {from the phase plot given}

- Phase of the model for ω=6000: $\phi = -90^\circ - \tan^{-1}\frac{6000}{50} - 6000 \times t_d \times \frac{180^\circ}{\pi}$
- Equate this to -250° and solve for t_d

$$t_d \approx 0.2 \times 10^{-3}$$

Complex Poles: Resonance

- We found that dc-gain is 1 and high frequency gain <1 for

$$G(j\omega)=\frac{1}{(j\upsilon)^2+j2\zeta\upsilon+1}=\frac{1}{(1-\upsilon^2)+j2\zeta\upsilon}$$

  - Under what condition the gain is >1?
  - At which frequency, the gain is maximum?

$$M(\upsilon)=\frac{1}{\sqrt{(1-\upsilon^2)^2+(2\zeta\upsilon)^2}},\quad \phi(\upsilon)=-\tan^{-1}\frac{2\zeta\upsilon}{1-\upsilon^2}$$

- When the following function is minimum, the gain is maximum

$$g(\upsilon)=(1-\upsilon^2)^2+(2\zeta\upsilon)^2$$

- The minimum of this function must satisfy

$$\frac{d}{d\upsilon}g(\upsilon)=0,\quad \frac{d^2}{d\upsilon^2}g(\upsilon)>0$$

Complex Poles: Resonance

- We want to find the minimum of the function

$$g(\upsilon)=(1-\upsilon^2)^2+(2\zeta\upsilon)^2$$

- That requires $\dfrac{d}{d\upsilon}g(\upsilon)=0,\quad \dfrac{d^2}{d\upsilon^2}g(\upsilon)>0$

$$\frac{dg}{d\upsilon}=2\left(1-\upsilon^2\right)(-2\upsilon)+4\zeta^2(2\upsilon)$$
$$=4\upsilon\left(\upsilon^2-1+2\zeta^2\right)$$

$$\frac{dg}{d\upsilon}=0 \quad\Rightarrow\quad \upsilon^2-1+2\zeta^2=0$$

$$\upsilon=\sqrt{1-2\zeta^2}$$

$$\frac{\omega}{\omega_n}=\sqrt{1-2\zeta^2}$$

$$\omega_r=\omega_n\sqrt{1-2\zeta^2}$$ (Frequency of resonance)

$$\frac{d^2g}{d\upsilon^2}=12\upsilon^2+4(2\zeta^2-1)$$

$$as\ \upsilon=\sqrt{1-2\zeta^2},$$

$$\frac{d^2g}{d\upsilon^2}=12(1-2\zeta^2)+4(2\zeta^2-1)$$

$$\frac{d^2g}{d\upsilon^2}=8-16\zeta^2$$

$$\frac{d^2g}{d\upsilon^2}>0 \quad if \quad \zeta<\frac{1}{\sqrt{2}}=0.707$$