EE3331C/EE3331E Feedback Control Systems
L5: Feedback Control

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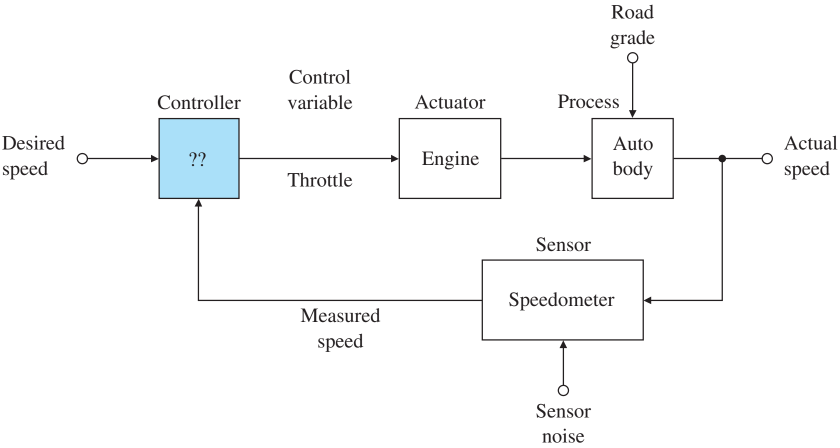
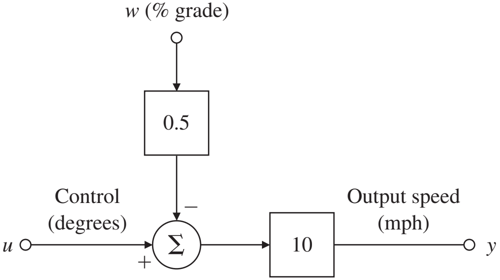
Outline

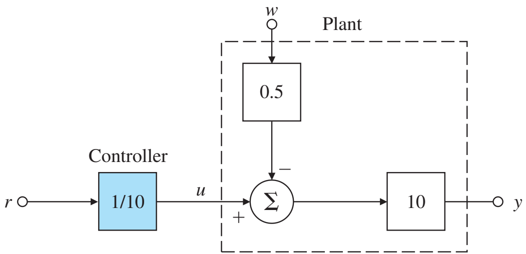
- We have briefly introduce the motivation of feedback earlier, we now demonstrate its advantages and disadvantages via quantitative analysis.
- *Example:* Consider a simplified model of the cruise control of an automobile.



- Assumptions:
  - Ignore dynamics of the system, consider only steady-state behavior.
  - System is linear.
  - Vehicle is moving on **level** road at 65 mph, a 1^o change in the throttle angle (control variable) causes a 10 mph change in speed.
  - When the grade (road conditions) changes by 1%, the speed change by 5 mph.
  - i.e. two things affect the final speed: the throttle angle, u, and the road condition, w.



- Objective: to analyze the effects of a 1% grade on the output speed when the reference is set for 65 mph with and without feedback.



- Open-loop control:
  - controller does not use the speedometer reading
  - the controller is set to $u = r/10$, the output is then

$$\begin{aligned} y_{ol} &= 10(u - 0.5w) \\ &= 10\left(\frac{r}{10} - 0.5w\right) = r - 5w \end{aligned}$$

- Open-loop control:
  - the error in the output speed is

$$e_{ol} = r - y_{ol} = 5w$$

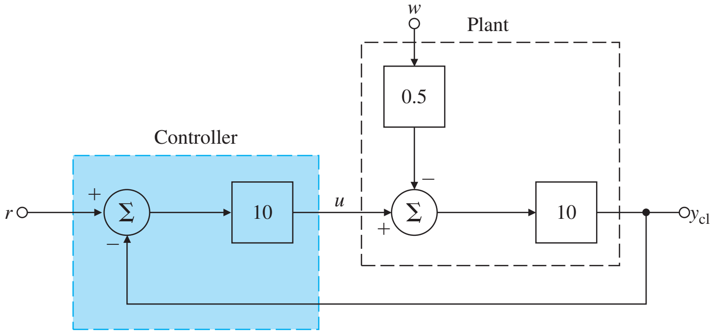
  - for $r = 65$, and if the road is level, i.e. $w = 0$, the speed, y, would be 65 mph with no error
  - if $w = 1$ corresponding to a 1% grade, then the speed, y, will be 60 mph with a 5 mph error
- For open-loop control, to have zero error, the controller needs to be an exact inverse of the plant.
- Not practical as the plant is subjected to changes and errors will be introduced, e.g. if the plant gain is now 11 (a 10% change), the error is now

$$e_{ol} = r - y_{ol} = r - 11(u - 0.5w) = r - 11\left(\frac{r}{10} - 0.5w\right) = -\frac{1}{10}r + 5.5w$$

which results in a 10% error when $w = 0$.

Open-loop control: not robust to changes in plant dynamics.

- Consider the case when we make use of the speedometer readings, i.e. there is some form of feedback.



  - the controller gain is set to 10
  - assume that the feedback sensor (speedometer) has no dynamics with unity gain

- We then have

$$\begin{aligned} y_{cl} &= 10u - 5w \\ u &= 10(r - y_{cl}) \end{aligned}$$

Combining yields

$$\begin{aligned} y_{cl} &= 10(10(r - y_{cl})) - 5w \\ &= 100r - 100y_{cl} - 5w \\ 101y_{cl} &= 100r - 5w \\ y_{cl} &= \frac{100}{101}r - \frac{5}{101}w \\ e_{cl} &= r - y_{cl} = \frac{1}{101}r + \frac{5}{101}w \end{aligned}$$

- Notice that feedback has reduced the sensitivity of the speed error to the grade, w, by a factor of 101 when compared with the open-loop system. When $w = 1$, the error in speed is only $\frac{5}{101} = 0.05$ mph. Feedback can reduce the effect of disturbance.

- Notice that on a level ground, $w = 0$, the final speed is now

$$y_{cl} = \frac{100}{101} r = 0.99r \text{ mph}, \quad e_{cl} = \frac{1}{101} r$$

i.e. there is now a small error.

- However, if the plant gain is now 11, the error is now

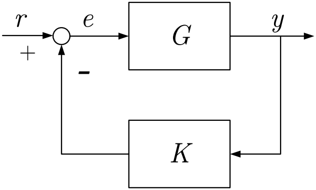
$$\begin{aligned} e_{cl} &= r - y_{cl} = r - 11u \\ &= r - 11(10e_{cl}) \\ &= \frac{1}{111} r \end{aligned}$$

which results in a 0.9% error.

Feedback is robust to plant change.

- Note that the reduction in sensitivity due to disturbances and plant changes is due to the high loop gain (product of plant and controller) in the above example. There is however a practical limit on how high this loop gain can go.

- Consider the following standard feedback block.

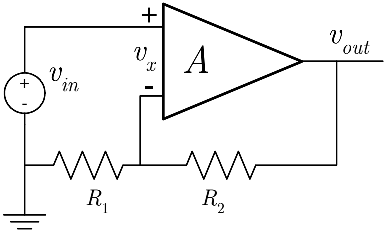


  - r is the reference input signal; y is the output signal; e is called the error signal
  - G is called the forward or open-loop system or plant (combination of controller and plant)
  - K is called the feedback system
  - again, assume static case for now, i.e. r, e, y are all (constant) real numbers
  - suppose also that the forward and feedback systems are linear, i.e. G, K are numbers ('gains')

$$y = Ge, \quad e = r - Ky$$

- Let $y = Hr$ where
$$H = \frac{G}{1 + GK}$$
is called the closed-loop system gain (G is called the open-loop system gain)
- $L = GK$ is called the loop gain – it is the gain around the feedback loop cut at the summing junction.
  - if $L = GK$ is large, then $H \approx 1/K$ and relatively independent of G
  - how close is this approximation? The relative error is given by
$$\frac{1/K - H}{H} = \frac{1}{GK} = \frac{1}{L}$$
  - If $K = 0.1$ and $G = 100$, then we have $H \approx 9$.
  - As G varies from 100 to 1000, $H \approx 9.9$.
  - Large variations in open-loop gain, G, lead to much smaller variations in closed-loop gain, H.
  - Closed-loop gain is much smaller than the original open-loop gain due to feedback.

- Example: feedback amplifier



  - the input, r, is v_{in}; the output, y, is v_{out} and the error signal, e, is v_x; the open-loop gain is the amplifier gain, $G = A$.
  - the output voltage is given by

$$
\begin{aligned}
v_{out} &= Av_x = A\left(v_{in} - \frac{R_1}{R_1 + R_2}v_{out}\right) \\
v_{out} &= \frac{A}{1 + AK}v_{in} = Hv_{in}
\end{aligned}
$$

  where the feedback gain, $K = R_1/(R_1 + R_2)$.

Sensitivity

- Objective: to derive an expression for changes in the closed-loop gain, H due to small changes in the open-loop gain, G.

$$\frac{\partial H}{\partial G} = \frac{\partial}{\partial G}\left(\frac{G}{1+GK}\right) = \frac{1}{(1+GK)^2}$$

For small changes in G, i.e. δG, we have

$$\delta H \approx \frac{1}{(1+GK)^2}\delta G$$

in terms of fractional gain changes:

$$\begin{aligned}\frac{\delta H}{H} &= \frac{1}{(1+GK)^2}\delta G \frac{1}{H} \\ &= \frac{1}{(1+GK)^2}\delta G \frac{1+GK}{G} \\ &= \frac{1}{(1+GK)}\frac{\delta G}{G} = \mathcal{S}\frac{\delta G}{G}\end{aligned}$$

where $\mathcal{S}$ is the sensitivity.

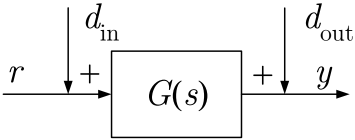
- The sensitivity $\mathcal{S}$ is thus the ratio of the fractional change in closed-loop gain over the fractional change in open-loop gain.

$$\mathcal{S} = \frac{1}{(1+GK)} = \frac{1}{1+L}$$

- For large loop gain, sensitivity $\approx 1/$loop gain; and $|\mathcal{S}| \ll 1$, hence small fractional changes in G yield much smaller fractional changes in H. i.e. the sensitivity of gain w.r.t. G is reduced by about L.
- The closed-loop gain, $H = \dfrac{G}{1+GK}$; for large loop gain, L, the gain is reduced by about L.

Feedback allows us to trade gain for reduced sensitivity.

- Consider the following open-loop system with input and output disturbances.

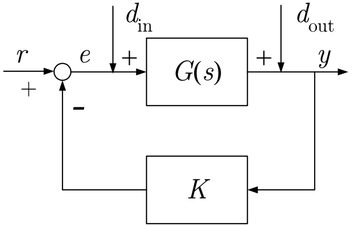


- The output is given by

$$y = Gr + Gd_{\text{in}} + d_{\text{out}}$$

- Effect of disturbances appears at the output.

- Consider the following closed-loop system with input and output disturbances.



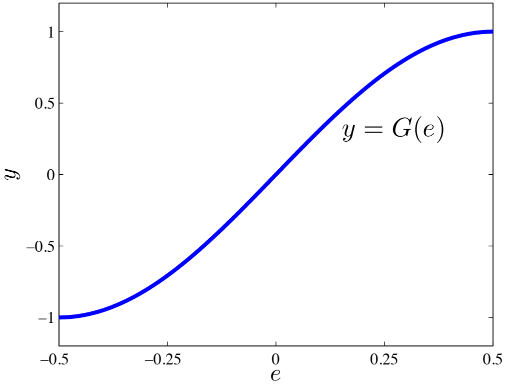
- The output is given by

$$y \quad = \quad \frac{G}{1+GK}r + \frac{G}{1+GK}d_{\text{in}} + \frac{1}{1+GK}d_{\text{out}}$$

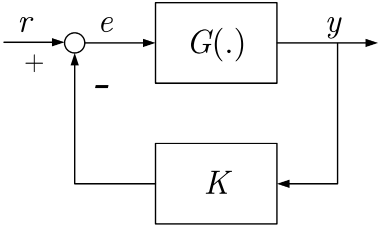
Can you derive the above equation?

- Effect of input & output disturbances multiplied by $G\mathcal{S}$ and $\mathcal{S}$ respectively. The disturbances effect will be small if the loop gain is large.

- Consider the system gain, G with some form of nonlinearity, very common for amplifiers, transducers etc, to be at least a bit nonlinear. i.e. G is a function from $\mathbf{R}$ into $\mathbf{R}$.



- Consider again our feedback block diagram with nonlinear element G,



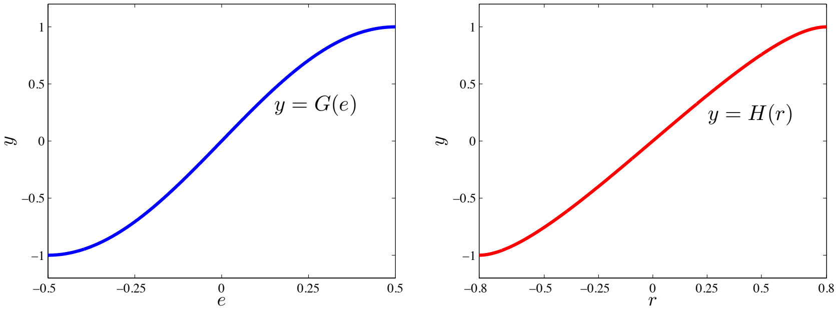
We have

$$y = G(e) \qquad e = r - Ky$$

solving results in $y = H(r)$.

  - coupled nonlinear equations
  - usually not possible to solve analytically
  - can be solved graphically or by computer

- Example: consider the function $y = \sin(e)$



  - closed-loop characteristics obtained using $K = 0.3$
  - note the gain of the closed-loop system is lower
  - characteristics more linear (for $|y| < 1$)

- How do you check that it is more linear?
- Consider the closed-loop characteristic function, G

$$y = H(r) = G(e), \qquad e = r - KH(r)$$

differentiating w.r.t. r gives

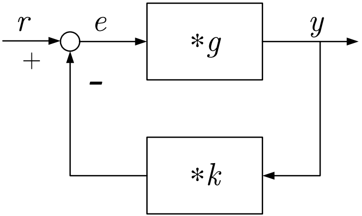
$$H'(r) = G'(e)\frac{de}{dr}, \qquad \frac{de}{dr} = 1 - KH'(r)$$

rearranging, we have

$$\begin{aligned} H'(r) &= G'(e)\big(1 - KH'(r)\big) \\ H'(r) &= \frac{G'(e)}{1 + G'(e)K} \end{aligned}$$

  - for r such that $|G'K| \gg 1$, we have $H' \approx 1/K$ (independent of r), i.e. H is almost linear
  - notice also the slope of H is smaller than the slope of G by a factor of $(1 + G'K)$

- We now consider dynamic systems: assume that all signals (r, y, e) are dynamic, i.e. change with time.
- Open-loop and feedback systems are convolution operators, with impulse responses g and k, respectively



$$y(t) = \int_0^t g(\tau)e(t-\tau)d\tau, \qquad e(t) = r(t) - \int_0^t k(\tau)y(t-\tau)d\tau$$

which are complicated integral equations.
(Consider what happened in frequency domain.)

- Taking Laplace transform, we have

$$Y(s) = G(s)E(s), \qquad E(s) = R(s) - K(s)Y(s)$$

combining gives

$$Y(s) = \frac{G(s)}{1 + G(s)K(s)}U(s) = H(s)U(s)$$

where $H(s)$ is the closed-loop transfer function.

- Exactly, the same formula as in the static case except that G, K, H are transfer functions. $H(s)$ is the closed-loop transfer function.
- Also, the loop transfer function, $L = GK$ and sensitivity transfer function, $\mathcal{S} = \dfrac{1}{1 + GK}$.
- What's new? $L, \mathcal{S}, H$ are now complex-valued, depend on frequency s and can be stable or unstable.
- In addition, $L, \mathcal{S}, H$ can be large for some frequencies and small for others; here large and small means complex magnitude.

- Example: consider

$$G(s) = \frac{10}{10s+1}, \quad K = 10$$

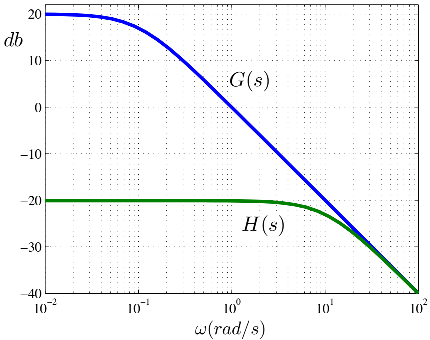
the closed-loop transfer function is

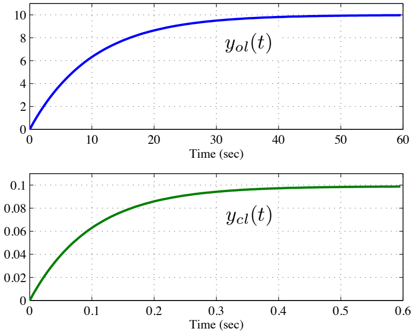
$$H(s) = \frac{G(s)}{1+G(s)K} = \frac{\frac{10}{1+10K}}{\frac{10}{1+10K}s+1} = \frac{0.099}{0.099s+1}$$

  - the open-loop DC gain is large: 10; the closed-loop DC gain is small: $0.1 \approx 1/K$
  - the open-loop and closed-loop bandwidth is around 0.1 rad/s and 10 rad/s respectively
  - the open-loop and closed-loop 2% settling time is around 40 sec and 0.4 sec respectively

- With feedback, the system has lower gain, higher bandwidth (faster response).

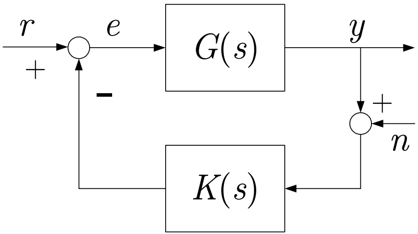
- Bode plots and step responses:





Noise

- Suppose sensor has noise, n,



- The effect of noise on the output is given by

$$\frac{Y(s)}{N(s)} = \frac{-G(s)K(s)}{1 + G(s)K(s)} = -\mathcal{T}$$

Not good! we want a large loop gain $L(s) = G(s)K(s)$, which means that $\mathcal{T}$ is approximately 1. i.e. the sensor noise appear at the output!

Summary

So far... benefits of feedback

- reduce sensitivity to plant variations
- reduce effect of disturbance
- linearize a nonlinear system
- faster response when system is stable (not always!)

benefits determined by $\mathcal{S} = \dfrac{1}{1 + GK}$

both sensitivity and nonlinearity are reduced by $\mathcal{S}$

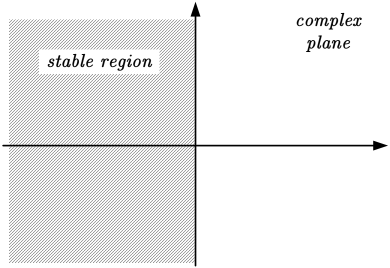
Hence, a large loop gain, $L = GK$, is desirable.

but, output is affected by sensor noise!

Closed-loop stability

- We have previously seen that stability is determined by the pole locations of the transfer function, $G(s)$.
- How about the stability of feedback systems?
- Check the closed-loop poles from the closed-loop transfer function.

1. find the closed-loop transfer function (cltf)
2. find the closed-loop poles by equating the denominator of the cltf to zero



- Consider the following feedback system:

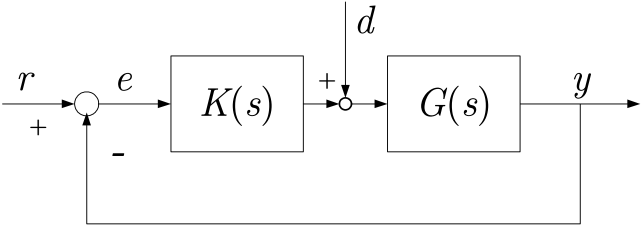


Figure 5.1 : Feedback Control System.

The closed-loop transfer function from r to y and d to y are given by

$$\frac{Y(s)}{R(s)} = \frac{G(s)K(s)}{1+G(s)K(s)}, \qquad \frac{Y(s)}{D(s)} = \frac{G(s)}{1+G(s)K(s)}.$$

- In general, closed-loop transfer function for single-input single-output system with negative feedback may be obtained from the rule [1]

$$\text{Output} = \frac{\text{“direct”}}{1 + \text{“loop”}} \cdot \text{Input}$$

  where “direct” represents the transfer function from the direct effect of the input on the output (with feedback path open) and “loop” is the transfer function around the loop (i.e. our loop gain, $L(s)$)
- For the closed-loop system in Figure 5.1, our closed-loop poles are given by

$$\begin{aligned} 1 + L(s) &= 0 \qquad (5.1) \\ 1 + G(s)K(s) &= 0 \end{aligned}$$

- Equation 5.1 also known as the closed-loop characteristic equation (c.e.)

[1] Multivariable Feedback Control, by Skogestad and Postlethwaite, pg 23.

- Consider again our closed-loop c.e., let

$$G(s) = \frac{N_g(s)}{D_g(s)}, \quad \text{and} \quad K(s) = \frac{N_k(s)}{D_k(s)}$$

- The closed-loop transfer function from r to y is then given by

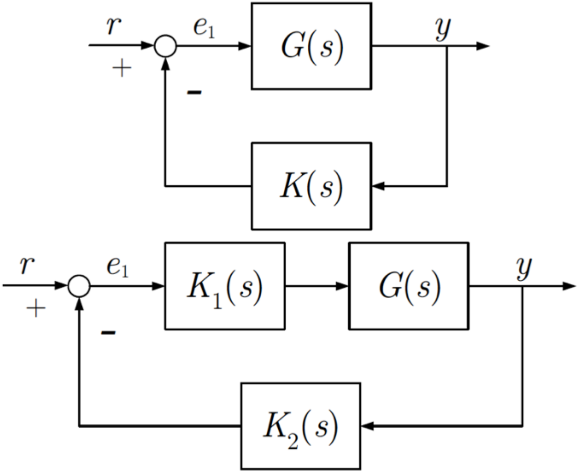
$$\begin{aligned} H(s) &= \frac{G(s)K(s)}{1 + G(s)K(s)} \\ &= \frac{N_g(s)N_k(s)}{D_g(s)D_k(s) + N_g(s)N_k(s)} \end{aligned}$$

The closed-loop c.e. is thus given by

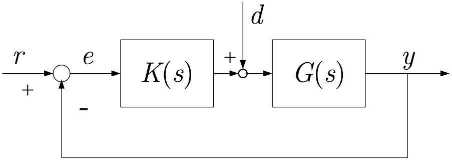
$$D_g(s)D_k(s) + N_g(s)N_k(s) = 0$$

Note that the open-loop poles are given by $D_g(s) = 0$. The closed-loop poles are much more complicated, its stability depends on the controller $K(s)$.

- What are the closed-loop poles for the following feedback configurations?



- Example: Control of heating plate.



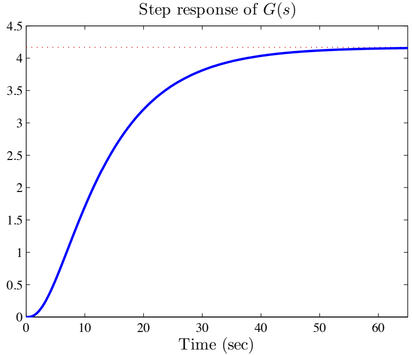
- Objective is to control the temperature of the heating plate, $y = T$ to its desired temperature, $r = T_{ref}$.

  The heating-plate dynamics is given by

$$G(s) = \frac{0.1}{(s+0.1)(s+0.3)(s+0.8)}$$

  The controller is $K(s) = k$.

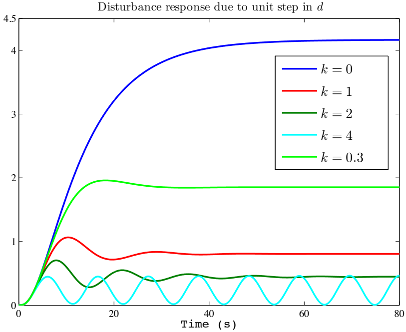
- Step response of the heating-plate is slow due to the large thermal mass of the system.

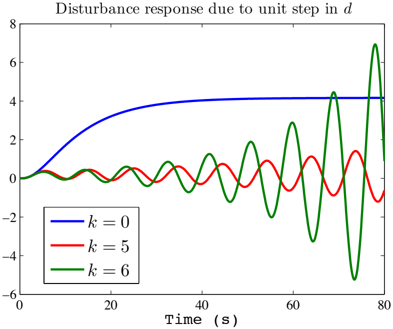


- Assume system operating in steady-state condition, a unit step disturbance of 1 W is introduced into the system. The transfer function from d to y is given by (show it!)

$$\frac{Y(s)}{D(s)} = \frac{G(s)}{1 + G(s)K(s)} = \frac{0.1}{(s+0.1)(s+0.3)(s+0.8)+0.1k}$$

- The output response for different gains, k:





  - closed-loop system exhibit oscillatory response
  - the system is stable for $k \leq 4$ (approximately); for $k > 4$ (approximately) it is unstable
  - when stable, the larger the gain, the smaller the effect of disturbance, however, the system also more oscillatory
  - stability requirement limits the size of the gain, k (hence loop gain)

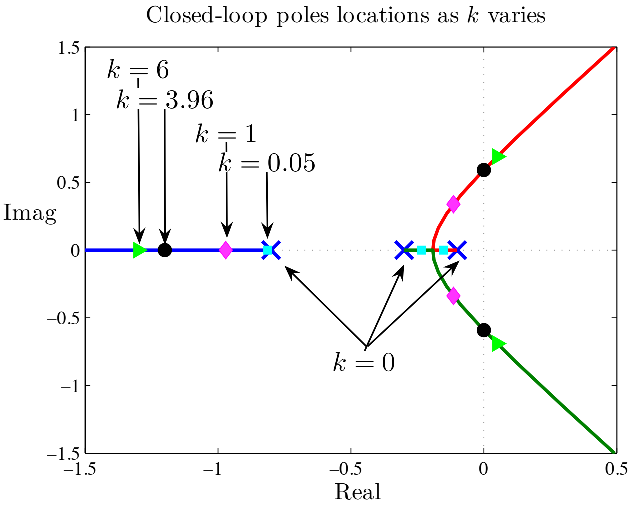
these are general phenomena

- For the heating system in Figure 5.1, the closed-loop pole is located at:

$$
\begin{aligned}
1 + G(s)K(s) &= 0 \\
1 + \frac{0.1k}{(s+0.1)(s+0.3)(s+0.8)} &= 0 \\
s^3 + 1.2s^2 + 0.35s + 0.024 + 0.1k &= 0
\end{aligned}
$$

k	poles
0	$-0.1,\ -0.3,\ -0.8$
1	$-0.9711,\ -0.1145 \pm 0.3385j$
2	$-1.0686,\ -0.0657 \pm 0.4531j$
3.96	$-1.2,\ \pm 0.5916j$
4	$-1.2022,\ 0.0011 \pm 0.5939j$
5	$-1.2541,\ 0.0270 \pm 0.6458j$
6	$-1.3000,\ 0.0500 \pm 0.6910j$

- Plot shows how the closed-loop poles varies as the gain k is increased.

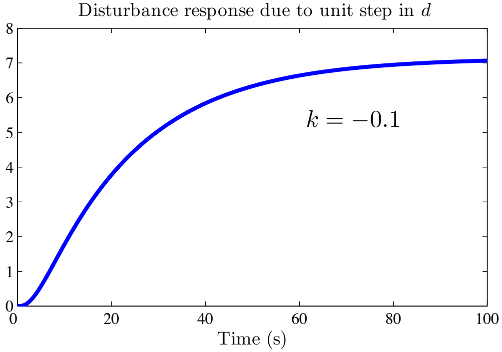


Also known as the root locus plot of $1 + G(s)K(s)$.

Questions:

- Can you find the pole locations and maximum k before it becomes unstable using the closed-loop poles equation $1 + G(s)K(s) = 0$?
- What is the maximum k before the poles are complex?

- For what values k is the system stable?



Stability Test

- We are interested to find the gains k such that the roots of the closed-loop c.e. $(a(s) = 1 + G(s)K(s))$ have negative real-parts.

$$a(s) = a_0 + a_1 s + a_2 s^2 + \ldots + a_n s^n$$

Such polynomials $a(s)$ with all roots with negative real-parts are call Hurwitz.

- if the a_i's are specific numbers, we can easily solved numerically (using a computer), then check ...
but what if the coefficients involve parameters, as in our heater problem?

$$s^3 + 1.2s^2 + 0.35s + 0.024 + 0.1k = 0$$

can we get the roots p_i in terms of the coefficients a_i? ... a classical problem.

- Analytical formulas available for polynomials of degrees ≤ 4.

- It turns out that we can express the Hurwitz condition as a set of algebraic inequalities involving the coefficients, using Routh's stability criterion [2]
- Table shows the conditions for stability for different degrees of polynomials using Routh's stability conditions (note: polynomial highest power of s is 1, if not just divide throughout by the highest coefficient, does not affect stability)

Degree	Hurwitz polynomial	Conditions
1	$a_0 + s$	$a_0 > 0$
2	$a_0 + a_1 s + s^2$	$a_0 > 0, a_1 > 0$
3	$a_0 + a_1 s + a_2 s^2 + s^3$	$a_0 > 0, a_1 > 0, a_2 > 0,$ $a_2 a_1 > a_0$
4	$a_0 + a_1 s + a_2 s^2 + a_3 s^3 + s^4$	$a_0 > 0, a_1 > 0, a_2 > 0, a_3 > 0,$ $a_3 a_2 > a_1, a_1 a_2 a_3 - a_3^2 a_0 > a_1^2$

[2] Feedback Control of Dynamic System by Franklin et al., pg 109, sec. 3.7.2

- Back to our heater example, the closed-loop system is stable for the following k:

$$s^3 + 1.2s^2 + 0.35s + 0.024 + 0.1k = 0$$

Hurwitz conditions:

$$\begin{aligned} a_0 &= 0.024 + 0.1k > 0 \\ &\Rightarrow k > -0.24 \\ a_1 &= 0.35 > 0 \\ a_2 &= 1.2 > 0 \\ a_2 a_1 &= 0.42 > 0.024 + 0.1k \\ &\Rightarrow k < 3.96 \end{aligned}$$

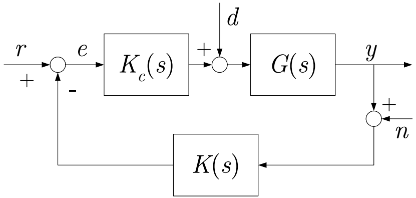
- For stability, we have $-0.24 < k < 3.96$.

Summary

- Why feedback?
  - reduce steady-state errors to disturbances
  - reduce the system's transfer function sensitivity to parameter variations
  - speed up transient response
  - linearize a nonlinear system
- stability requirement often limits the amount of feedback that can be used.

Review Questions

- For the feedback block diagram, derive the closed-loop transfer functions due to r, d and n.



Reading: FPE: sections 3.7.2, 4.1

Practice Problems

1. Consider Figure 5.1 and $G(s) = \dfrac{1}{s^2+s+1}$, for what values of k is the system stable?
2. Compute the transfer function from W to Y as a function of k_p and k_i where

$$v_a = k_p e + k_i \int_0^t e\,dt$$