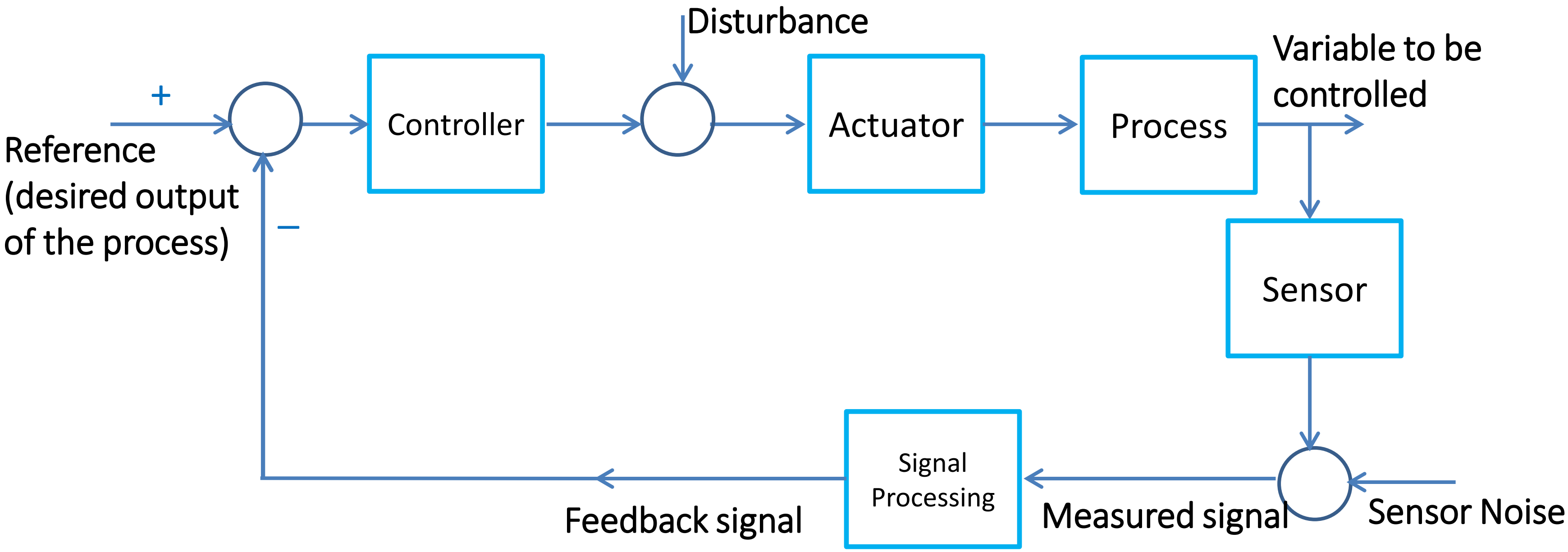
EE3331C/ EE3331E
Feedback Control Systems
Part II: Frequency Response Methods

Chapter 4: Design of Feedback System

Learning Outcome

- At the end of this chapter, students will know
  1) the properties of L(s) to meet steady state error specifications for polynomial type reference
     - System type and Error constants [discussed in the first half]
  2) the properties of L(s) required to meet error specifications for sinusoidal reference
     - Sensitivity function
  3) the properties of L(s) to reduce the effect of sensor noise
     - Complementary sensitivity function
  4) the properties of L(s) to meet transient specifications
  5) what is compensator and how to design
     - Lead compensator and Lag compensator

Design of Feedback System



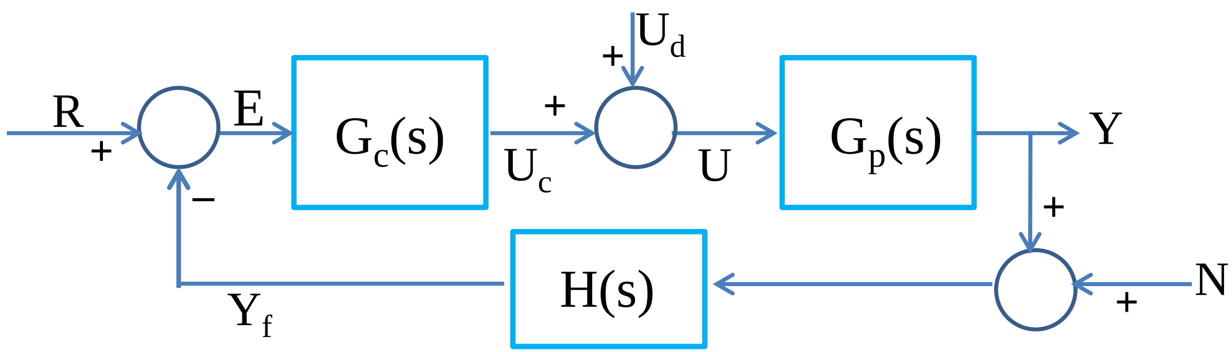
- For a practical case, one should have prior knowledge about
  1) the process
  2) its environment &
  3) the desired output

- Appropriate actuator and sensor
- Characteristics of disturbance and noise
- Signal processing needed

- For this module: Only unknown is the controller that we design

Design of Feedback System

- Assumptions:
  - Known process
  - Known actuation mechanism and sensing mechanism
  - Known properties for reference, noise and disturbances

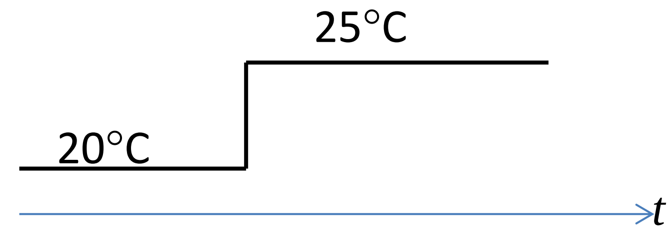


  - Examples of some practical processes and their actuation & sensing mechanisms will be provided as additional materials

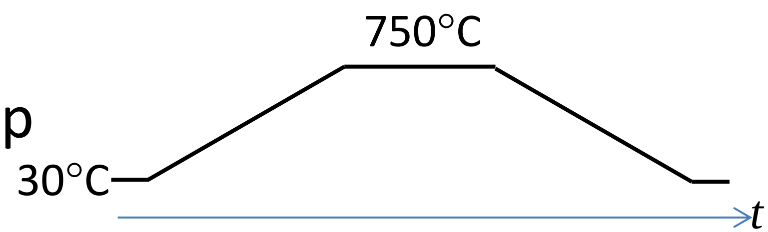
- Design must take into consideration
  a) Properties of all dynamic elements in the process $G_p(s)$
  b) External influences present, and
  c) Desired output

Some of the Signals commonly used as Reference (Desired output)

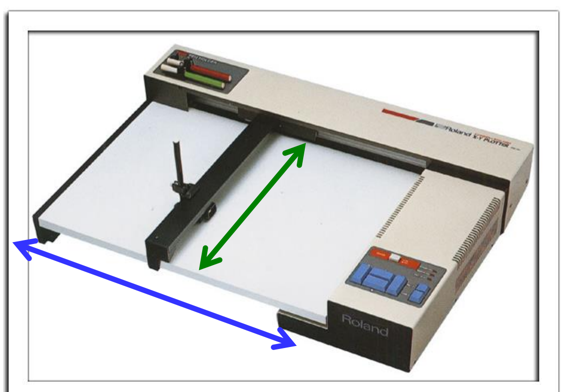
- Set-point control
  - Example: controlling room temp



- Ramp up/down & set-point
  - Example: controlling furnace temp



- Sinusoidal
  - Example: x-position, y-position in X-Y plotter to plot ellipse



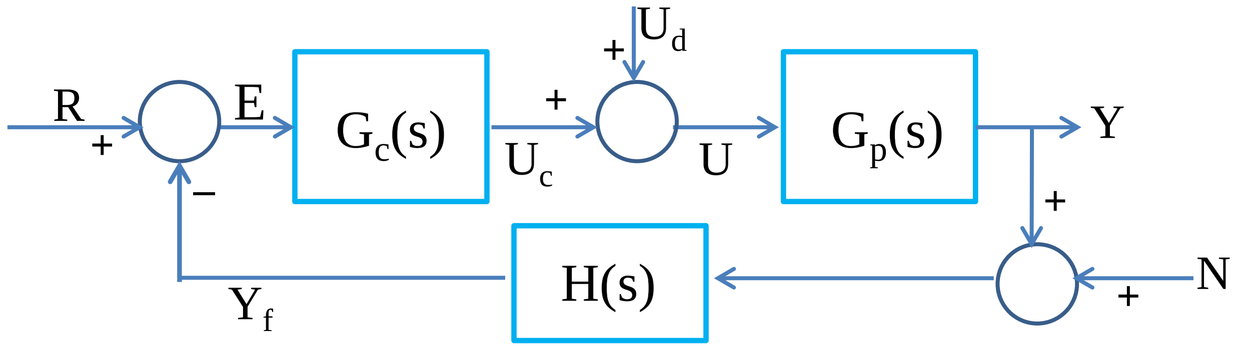
And many others….

Design of Feedback System

Design Targets

- Steady-state error, e_{ss}
  - How closely the output tracks (follows) the reference
- Response time: how quickly
  - the output follows the reference or
  - the system overcomes influence of undesirable disturbances
    - Measured using rise time, settling time etc.
- Properties of transient response
  - How steady-state is reached: exponential or oscillatory
    - Indicated by the damping factor
- Attenuation of sensor noise
  - How well the effect of sensor noise is minimized

Modification of Loop Characteristics by the Controller



- $G_p(s)$ and H(s) are known
- $G_c(s)$ is designed to achieve desired CL properties

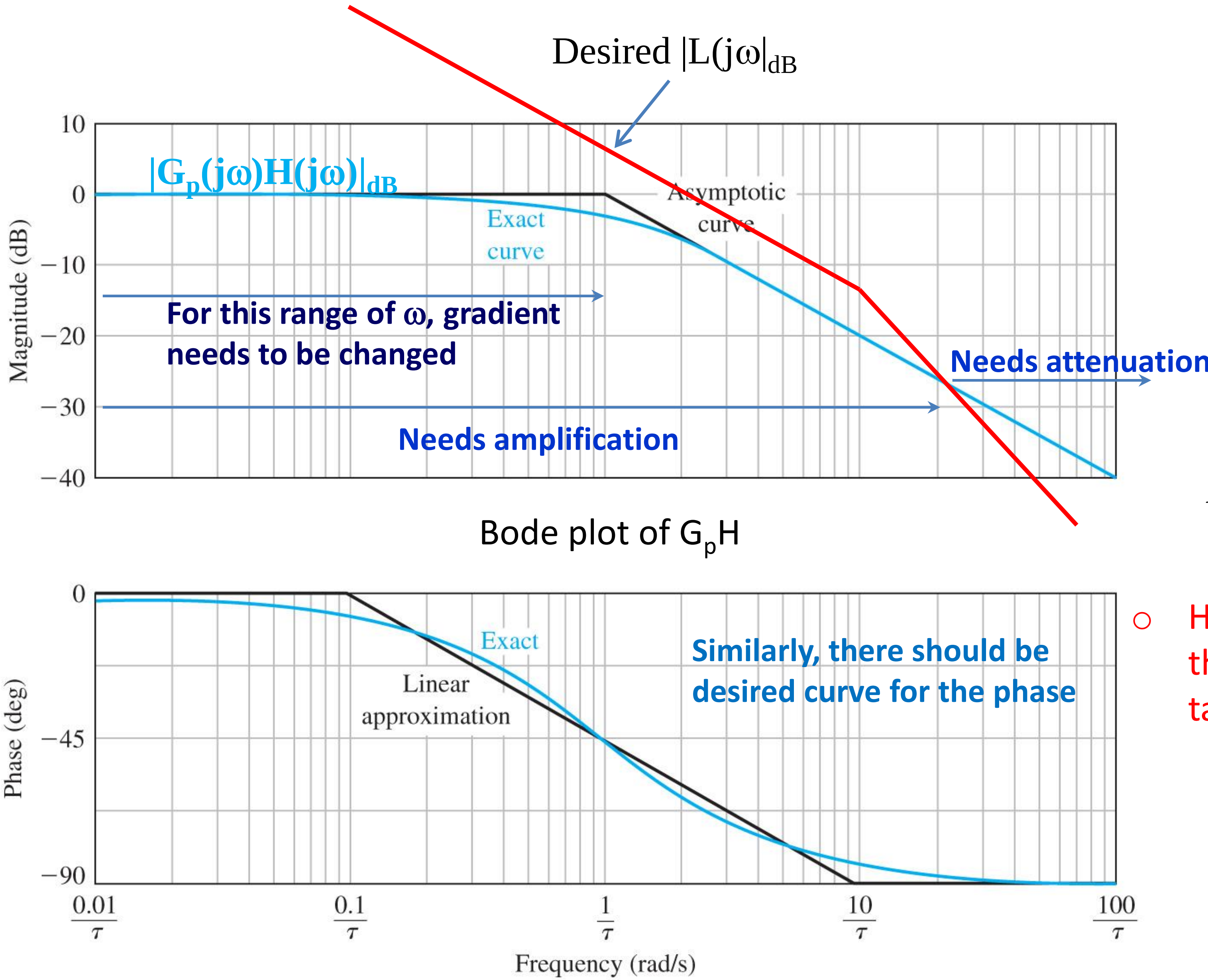
$$G_{cl}(s) = \frac{Y(s)}{R(s)} = \frac{G_c G_p}{1 + G_c G_p H}$$

Closed Loop Transfer Function (CLTF)

- It is easier for analysis if the desired performances can be expressed using properties of loop transfer function

$$L(s) = G_c G_p H$$

Design of Feedback System



$$L = G_c G_p H$$

- How to choose the desired/ target L(s)?

Desired Properties of L(s)

- Characteristics the target L(s) must have to achieve desired performance for the following-
  A. Tracking a polynomial reference input
  B. Tracking a sinusoidal reference input
  C. Achieving good damping
  D. Rejecting input disturbances
  E. Minimizing effect of sensor noise

Desired Properties of L(s)

A. Tracking Polynomial Input as Reference: Unity Feedback

$$r(t) = At^0$$

$$r(t) = At^1$$

$$r(t) = At^2$$

- Steady-state error for such input is characterized using
  1) **System type** and
  2) **Error constant**
- If L(s) a **type N system,** the resulting closed loop is able to achieve zero steady-state error to polynomials of degree less than N
  - N is the number of integrators in L(s)

$$L(s) = \frac{K}{s^N} P(s) \qquad \{P(s) \text{ has no integrator}\}$$

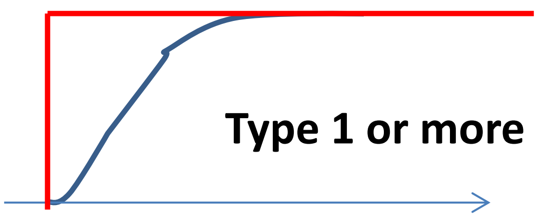
N=0	N=1	N=2
Type 0	Type 1	Type 2
$L(s) = \frac{10}{s^2 + 4s + 5}$	$L(s) = \frac{10}{s(s+4)}$	$L(s) = \frac{10}{s^2(s^2 + 6s + 5)}$

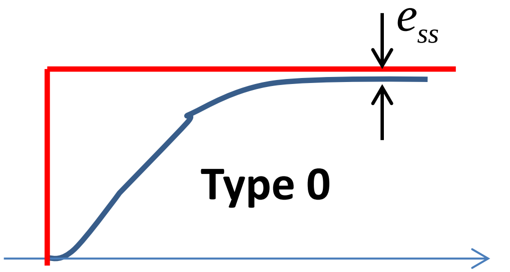
Desired Properties of L(s)

- For a type **N** system, the CL output follows $r(t)$ with zero e_{ss} if

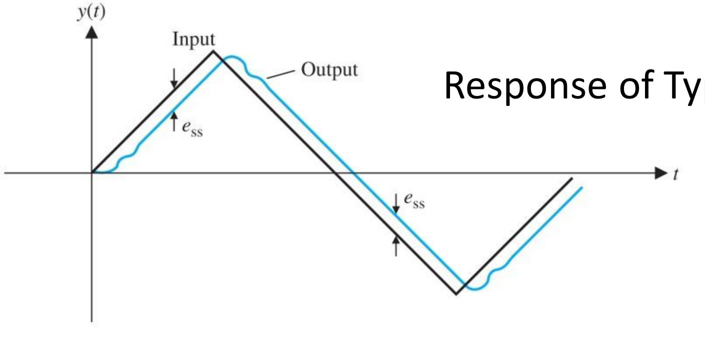
$$r(t) = At^{N-1}$$

- Step input: $r(t) = A$
  - L(s) must be type 1 or more to achieve zero steady-state error





- Similarly, for ramp input, L(s) must be type 2 or more to get e_{ss}=0



Response of Type 1 system to ramp input

Desired Properties of L(s)

	Step	*Ramp*	*Parabolic*
Type 0	Finite e_{ss}	0	0
Type 1	∞	Finite e_{ss}	0
Type 2	∞	∞	Finite e_{ss}

- **Error constant** quantifies the error

Position Error Constant	Velocity Error Constant	Acceleration Error Constant
$k_p = \lim_{s\to 0} L(s)$	$k_v = \lim_{s\to 0} sL(s)$	$k_a = \lim_{s\to 0} s^2 L(s)$

Steady-state error for different polynomial reference inputs

Step	Ramp	Parabolic
$r(t) = A$	$r(t) = At$	$r(t) = At^2$
$e_{ss,s} = \dfrac{A}{1+k_p}$	$e_{ss,r} = \dfrac{A}{k_v}$	$e_{ss,p} = \dfrac{2A}{k_a}$

Desired Properties of L(s)

- *Example* 4-1:
$$G_p(s) = \frac{K}{s^2 + as + b}, \quad H(s) = 1, \quad G_c(s) = K_P$$
$$a > 0, \quad b > 0, \quad K > 0$$
  - What is steady-state error if the reference is (i) unit step and (ii) unit ramp?

Answer:

$$L(s) = \frac{KK_P}{s^2 + as + b}$$

(i) $k_p = \lim_{s \to 0} L(s) = \frac{KK_P}{b}$

$$e_{ss,s} = \frac{A}{1 + k_p}$$
$$= \frac{1}{1 + \frac{KK_P}{b}}$$
$$= \frac{b}{b + KK_P}$$

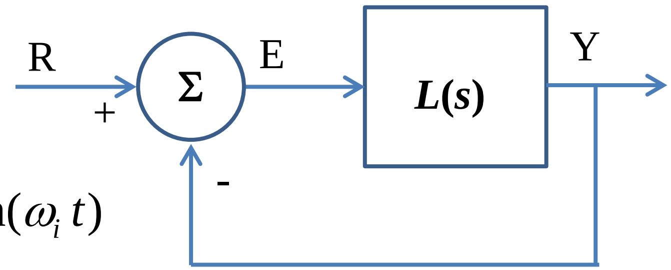
(ii) $k_v = \lim_{s \to 0} sL(s) = 0$

$$e_{ss,s} = \frac{1}{k_v}$$
$$= \infty$$

Desired Properties of L(s)

B. Tracking Sinusoidal Reference: Unity Feedback

- For this case, steady-state error is quantified using gain of error transfer function

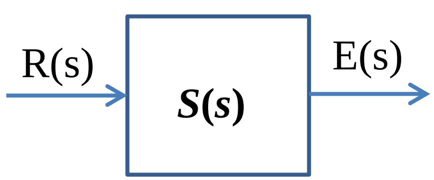


$$r(t) = R_m \sin(\omega_i t)$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+L(s)} = S(s)$$

***S*(*s*):** error transfer function or sensitivity function

Note: For polynomial input, we used Final Value Theorem to find steady-state error and to define the error constants. Final Value Theorem can not be used for sinusoidal signal. Instead, we use frequency response of S.



$$r(t) = R_m \sin(\omega_i t) \quad \Rightarrow \quad e(t) = E_m \sin(\omega_i t + \theta)$$

$$\frac{E_m}{R_m} = |S(j\omega_i)|, \quad \theta = \angle S(j\omega_i)$$

$$\frac{E_m}{R_m} \le \varepsilon_s \quad \Rightarrow \quad |S(j\omega_i)| \le \varepsilon_s$$

Desired Properties of L(s)

- Sensitivity is usually specified for a range of ω

$$\left|S(j\omega_i)\right| \le \varepsilon_s, \quad where \quad \varepsilon_s << 1$$
$$\omega_L \le \omega_i \le \omega_U$$

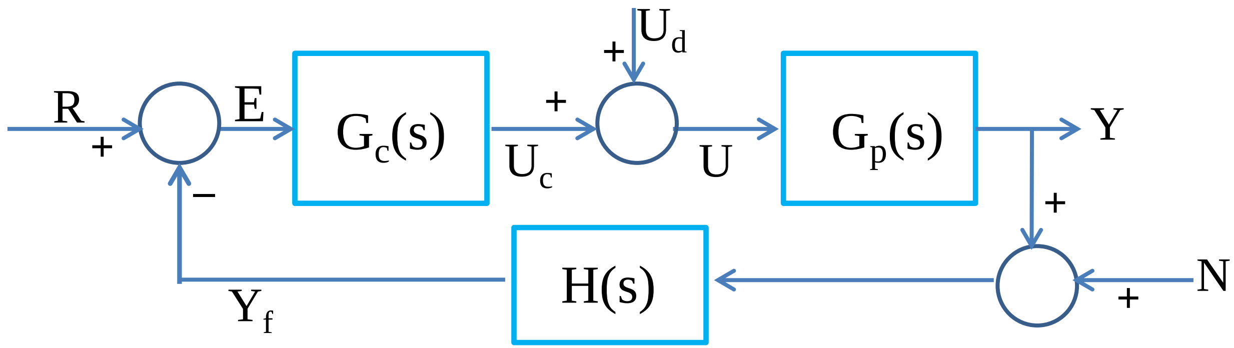
- For ease of design, this specification is transformed into equivalent specification for $L(j\omega)$

$$\left|S(j\omega_i)\right| \le \varepsilon_s \quad \Rightarrow \quad \left|L(j\omega_i)\right| \ge \frac{\varepsilon_s + 1}{\varepsilon_s}, \quad \omega_L \le \omega_i \le \omega_U$$

{For proof, refer to Appendix 4-A at the end of this chapter}

- $|L(j\omega)|$ can be easily modified by changing controller gain K

C. Minimizing Effect of Noise on Output



- Noise is a random signal with spectrum in high frequency range
  - Sum of sinusoids of random amplitude and phase

$$n(t) = N_m \sin(\omega_i t) \quad \Rightarrow \quad y(t) = Y_{m,n} \sin(\omega_i t + \theta)$$

$$\frac{Y(s)}{N(s)} = \frac{-G_c G_p H}{1 + G_c G_p H} = -\frac{L(s)}{1 + L(s)}$$

Let, $\dfrac{L(s)}{1 + L(s)} = T(s)$

$$\frac{Y(s)}{N(s)} = -T(s)$$

***T(s)*:** complementary sensitivity function

$$\frac{Y_{m,n}}{N_m} = \left|T(j\omega_i)\right|$$

$$\frac{Y_{m,n}}{N_m} \le \varepsilon_T \quad \Rightarrow \quad \left|T(j\omega_i)\right| \le \varepsilon_T$$

Desired Properties of L(s)

- Specifications on $|T(j\omega)|$ for a range of frequencies

$$\left|T(j\omega_i)\right| \le \varepsilon_T, \quad where \quad \varepsilon_T << 1$$
$$\omega_1 \le \omega_i \le \omega_2$$

- For ease of design, this specification is transformed into equivalent specification for $L(j\omega)$

$$\left|L(j\omega_i)\right| < \frac{\varepsilon_T}{1+\varepsilon_T}, \quad \varepsilon_T << 1, \quad \omega_1 < \omega_i < \omega_2$$

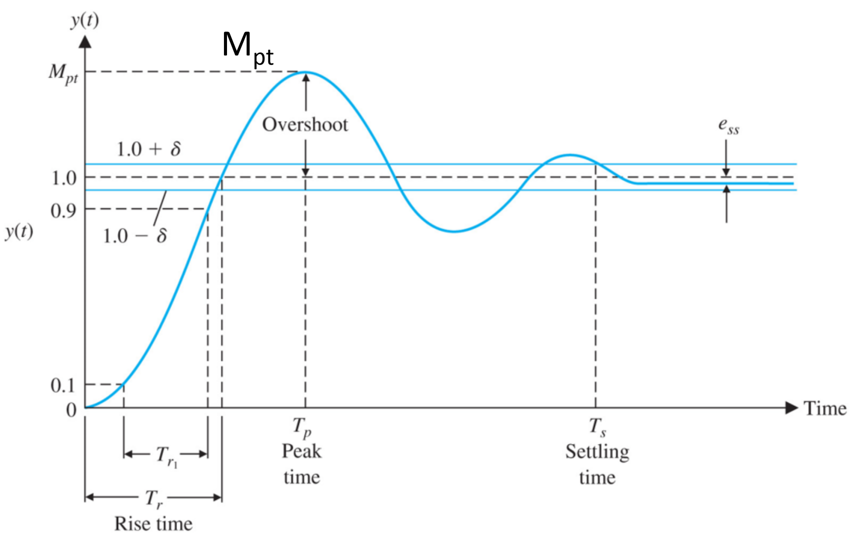
{For proof, refer to Appendix 4-B at the end of this chapter}

D. Transient Properties

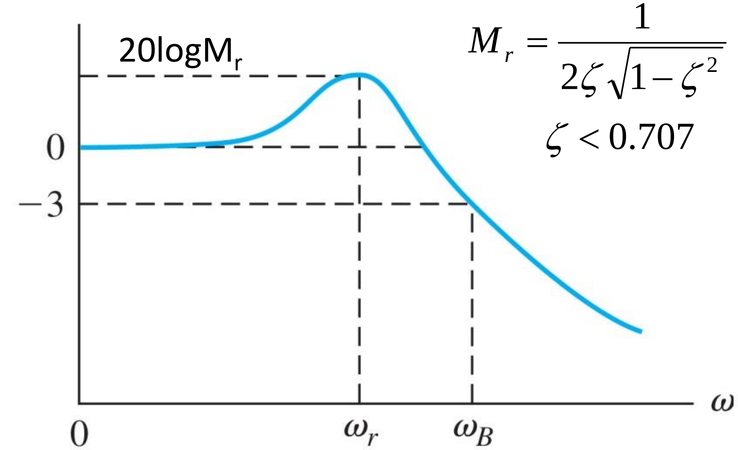
- Dominant pole(s) of the CLTF determines transient behavior
  - For a dominant 2nd order (complex conjugate poles) system,

$$G(s)=\frac{\omega_n^2}{s+2\zeta\omega_n s+\omega_n^2}$$

Step response:



Frequency response:



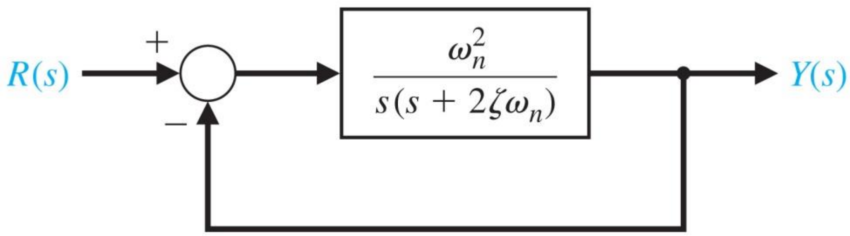
$$M_r=\frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\zeta<0.707$$

$$T_r\cong\frac{1.8}{\omega_n},\quad T_p=\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}=\frac{\pi}{\omega_d},\qquad \%OS=\frac{e^{-\pi\zeta}}{\sqrt{1-\zeta^2}},\quad T_s=\frac{4.6}{\zeta\omega_n}=\frac{4.6}{\sigma},$$

Desired Properties of L(s)

- For the system below, the CLTF is 2nd order



$$L(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

$$G_{cl}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

  - CLTF is underdamped 2nd order for the following as well

$$L(s) = \frac{K}{(s+a)(s+b)}, \quad K > K'$$

  - For simplicity, the first case is used to derive relation between
    - Phase margin and damping factor
    - Gain-crossover frequency and CL bandwidth

  - We use these relations as qualitative guideline for setting design specifications for transient

Desired Properties of L(s)

$$L(s)=\frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

$$L(j\omega)=\frac{\omega_n^2}{(j\omega)(j\omega+2\zeta\omega_n)}$$

$$L(j\omega)=\frac{1}{\dfrac{j\omega}{\omega_n}\left(\dfrac{j\omega+2\zeta\omega_n}{\omega_n}\right)}=\frac{1}{j\dfrac{\omega}{\omega_n}\left(j\dfrac{\omega}{\omega_n}+2\zeta\right)}$$

- Define $\upsilon=\omega/\omega_n$

$$L(j\upsilon)=\frac{1}{j\upsilon\left(j\upsilon+2\zeta\right)}$$

$$\left|L(j\upsilon)\right|=\frac{1}{\upsilon\sqrt{\upsilon^2+4\zeta^2}}$$

- The gain-crossover frequency is obtained by solving

$$\frac{1}{\upsilon_{cg}\sqrt{\upsilon_{cg}^2+4\zeta^2}}=1$$

Desired Properties of L(s)

Gain-crossover frequency (ω_{cg}) of

$$L(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

$$\frac{1}{\upsilon_{cg}\sqrt{\upsilon_{cg}^2 + 4\zeta^2}} = 1 \quad \Rightarrow \quad \upsilon_{cg}\sqrt{\upsilon_{cg}^2 + 4\zeta^2} = 1$$

$$\upsilon_{cg}^4 + 4\zeta^2\upsilon_{cg}^2 - 1 = 0$$

$$\upsilon_{cg}^2 = -2\zeta^2 + \sqrt{4\zeta^4 + 1}$$

$$\upsilon_{cg} = \sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}} \qquad \upsilon_{cg} = \frac{\omega_{cg}}{\omega_n}$$

$$\omega_{cg} = \omega_n\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}$$

Conclusion:

- Higher the gain-crossover frequency of L(s), higher is the natural frequency (ω_n) of the dominant 2nd-order CL poles $\Rightarrow$ shorter rise time

Desired Properties of L(s)

Phase margin (PM) of $L(s)=\dfrac{\omega_n^2}{s(s+2\zeta\omega_n)}$

$$L(j\upsilon)=\frac{1}{j\upsilon\left(j\upsilon+2\zeta\right)},\quad \upsilon=\frac{\omega}{\omega_n}$$

$$\angle L(j\upsilon)=-90^\circ-\tan^{-1}\frac{\upsilon}{2\zeta}$$

$$\angle L(j\upsilon_{cg})=-90^\circ-\tan^{-1}\frac{\upsilon_{cg}}{2\zeta}$$

$$\begin{aligned}PM &=180^\circ+\angle L(j\upsilon_{cg})\\ &=90^\circ-\tan^{-1}\frac{\upsilon_{cg}}{2\zeta}\\ &=\tan^{-1}\frac{2\zeta}{\upsilon_{cg}}\end{aligned}$$

$$PM=\tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{4\zeta^4+1}}}$$

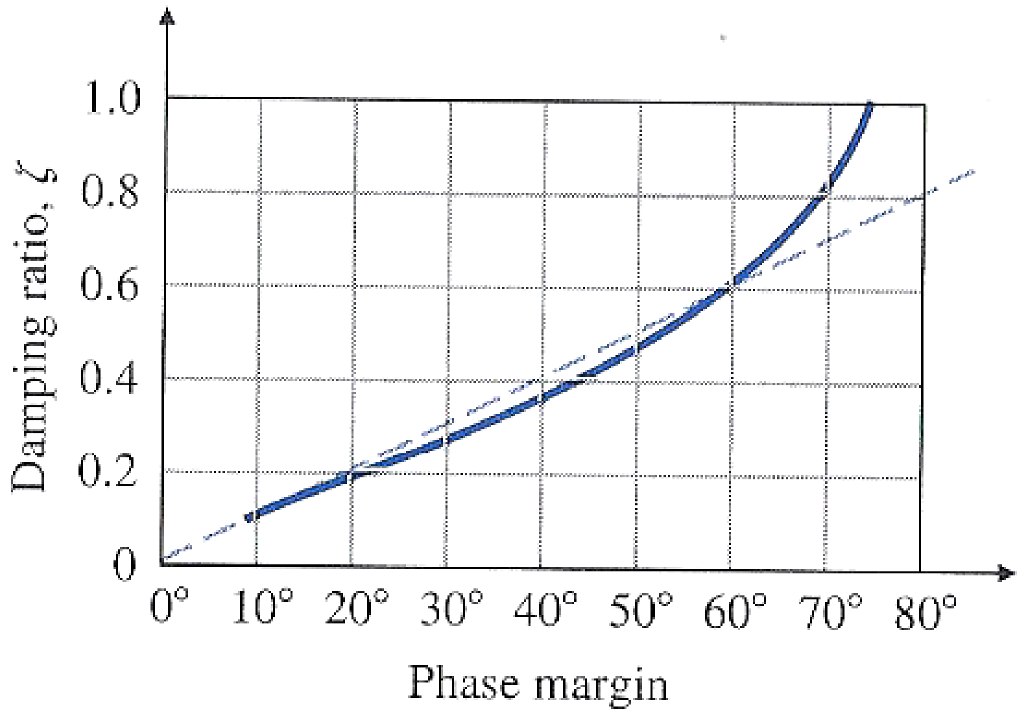
Conclusion:

- PM is related only to the damping factor of the dominant 2nd-order closed loop poles

Desired Properties of L(s)

$$L(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

$$PM = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{4\zeta^4+1}}}$$



- Straight line approximation for PM<60°

$$\zeta \cong \frac{PM}{100}$$

Controller Design by Shaping L(s)

Try Simple Solution First

- If $G_p(s)$ meets the requirement for system type, try proportional control ($G_c(s)=K_P$)

$$L(s) = K_P G_p(s) H(s)$$

  - Check stability by looking at GM and PM

Note: GM and PM are indicators of stability under certain conditions. In this chapter, while we are learning controller design, only plants satisfying those conditions are considered. For more general case, stability check is carried out using the Nyquist Stability Criterion, covered in Chapter 3B.

- *Example* 4-2: Design a controller such that e_{ss} is less than 10% for step input.

$$G_p(s) = \frac{50}{s^2+10s+50}, \quad H(s)=1, \quad G_c(s)=K_P$$

Answer:

$$L(s) = \frac{50K_P}{s^2+10s+50}$$

$$k_p = L(0) = K_P$$

$$e_{ss,step} = \frac{1}{1+k_p}$$

$$k_p = \frac{1}{e_{ss,step}} - 1 = 9$$

$$K_P = 9$$

$$L(s) = \frac{450}{s^2+10s+50}$$

$$G_{cl}(s) = \frac{450}{s^2+10s+500}$$

Controller Design by Shaping L(s)

- If $G_p(s)$ does not meet requirement for system type
  - First, add the required number of integrators
  - Then, try proportional control
  - Check stability by looking at GM and PM

- *Example* 4-3: Design a controller to achieve zero steady-state error for step input and e_{ss} less than 10% for ramp input.

$$G_p(s) = \frac{50}{s^2 + 10s + 50}, \quad H(s) = 1, \quad G_c(s) = K_P$$

Answer: Plant is type 0. An integrator must be added to make it type 1.

$$L(s) = \frac{50K_P}{s(s^2 + 10s + 50)}$$

$$k_p = \infty$$

$$k_v = K_P$$

$$e_{ss,step} = \frac{1}{1 + k_p} = 0$$

$$e_{ss,ramp} = \frac{1}{k_v}$$

$$k_v = \frac{1}{e_{ss,ramp}} = 10$$

$$K_P = 10$$

$$L(s) = \frac{500}{s(s^2 + 10s + 50)}$$

$$G_{cl}(s) = \frac{500}{s^3 + 10s^2 + 50s + 00}$$

Is the CL stable?

Controller Design by Shaping L(s)

- *Example* 4-4: Design a controller to achieve zero steady-state error for step input and maximum 10% error for ramp input.

$$G_p(s) = \frac{10}{s^2 + 6s + 10}, \quad H(s) = 1$$

Answer:

- Plant is **type 0** resulting in finite value for k_p and zero-valued k_v and k_a
- To meet error specifications of $e_{ss,step}$=0 and $e_{ss,ramp}$ < 10%, the open loop must be made type 1

$$L(s) = \frac{K}{s}\frac{10}{(s^2 + 6s + 10)}$$

- Single integrator ensures zero steady-state error for step

$$\begin{aligned} k_v &= \lim_{s\to 0} sL(s) \\ &= \lim_{s\to 0} \frac{10K}{s^2 + 6s + 10} \\ &= K \end{aligned}$$

$$e_{ss,ramp} < 10\% \quad \Rightarrow \quad \frac{1}{k_v} < 0.1$$

$$k_v > 10$$

$$K > 10$$

- Next, we find PM to check stability

Controller Design by Shaping L(s)

$$L(s) = \frac{K}{s}\frac{10}{(s^2+6s+10)}$$

- Let K=10

$$L(j\omega) = \frac{10}{j\omega}\frac{10}{(10-\omega^2 + j6\omega)}$$

Find ω_{cg} by solving $\left|L(j\omega_{cg})\right| = 1$

$$\left|\frac{100}{j\omega_{cg}(10-\omega_{cg}^2 + j6\omega_{cg})}\right| = 1$$

$$\frac{100}{\omega_{cg}\sqrt{(10-\omega_{cg}^2)^2 + 36\omega_{cg}^2}} = 1$$

$$\omega_{cg}^2\left(100+\omega_{cg}^4 - 20\omega_{cg}^2 + 36\omega_{cg}^2\right) = 10000$$

$$\omega_{cg}^6 + 16\omega_{cg}^4 + 100\omega_{cg}^2 - 10000 = 0$$

$$x^3 + 16x^2 + 100x - 10000 = 0$$

Solving numerically, $x = +16.15, \quad -16.07 \pm j18.99$

Taking x = +16.15, $\omega_{cg} \approx 4$

$$\angle L(j\omega_{cg}) = -90° - (180° - \tan^{-1}\frac{24}{6}) = -194°$$

Phase margin is -14°

Negative PM indicates unstable CL

Compensator Design

- A simple controller (proportional or integral)
  - can meet target spec on steady-state error for polynomial input
  - can be tuned to meet sensitivity spec as well

- Only one parameter to tune
  - No guarantee of satisfying multiple target specs

- Points to note:
  - Steady-state error depends on dc-gain of certain transfer function
    - Position error constant is dc-gain of L(s), velocity error constant is dc-gain of sL(s), etc.
  - Phase margin depends on loop property around ω_{cg}

Compensator: A transfer function that modifies the loop in selected range of frequencies

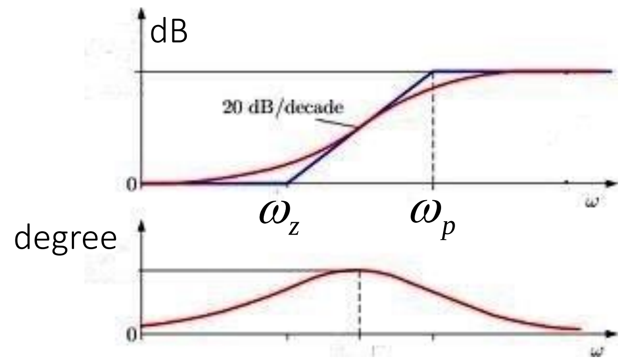
Compensator Design

$L_u(s)$: uncompensated loop transfer function

$C(s)$: compensator transfer function

$L(s) = C(s)L_u(s)$: compensated loop transfer function

Bode plot of a compensator known as lead compensator



- Significant change in phase for $\omega_z<\omega<\omega_p$ but very little change for $\omega<<\omega_z$ or $\omega>>\omega_p$
- No change in dc-gain
  - gain is changed for $\omega>\omega_z$

Compensator Design

General Structure of Compensator Transfer Function

$$C(s)=\frac{(s\tau_{z,1}+1)(s\tau_{z,2}+1)\ldots(s\tau_{z,M}+1)}{(s\tau_{p,1}+1)(s\tau_{p,2}+1)\ldots(s\tau_{p,M}+1)},\quad \tau_{z,i}>0,\ \tau_{p,i}>0$$

- ❑ Same number of poles and zeros
- ❑ 0 phase for $\omega\to 0$
- ❑ 0 phase for $\omega\to\infty$ as pole excess is 0

- The simplest one $\Rightarrow$ 1st-order compensator

$$C(s)=\frac{\tau_z s+1}{\tau_p s+1}=\frac{T\,s+1}{\alpha T\,s+1}$$

  - Ratio between the corner frequencies of zero and pole is α

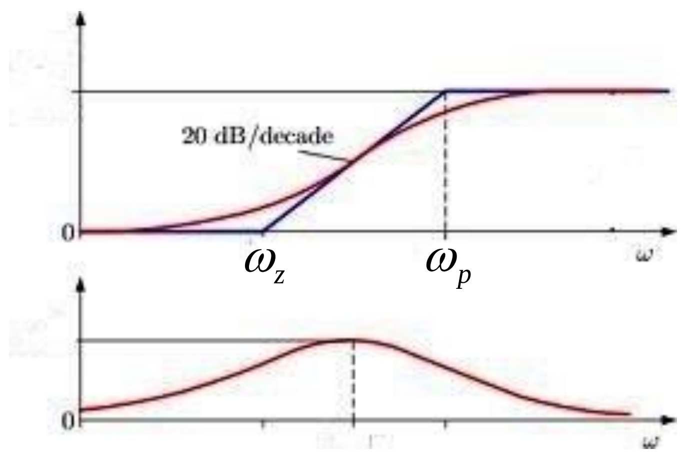
$$\omega_z=\frac{1}{T},\quad \omega_p=\frac{1}{\alpha T}$$

- ❑ Compensator zero at $-1/T$
- ❑ Compensator pole at $-1/(\alpha T)$

Lead Compensator (α<1)

- Corner frequency of zero is lower than corner frequency of pole

$$\omega_z = \frac{1}{T}, \quad \omega_p = \frac{1}{\alpha T} \quad \xrightarrow{\alpha < 1} \quad \omega_z < \omega_p$$

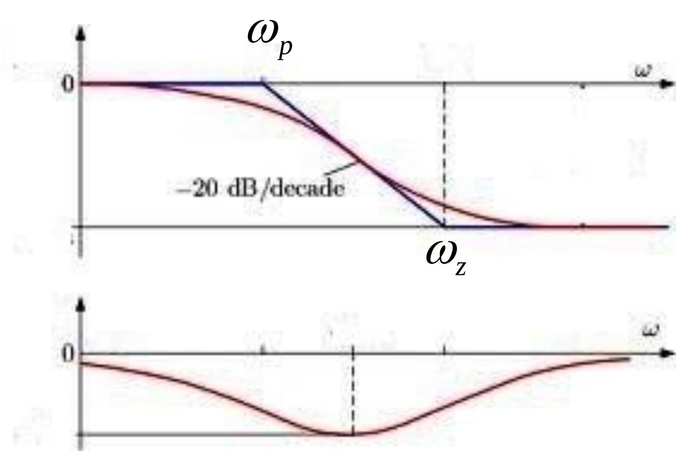


- Phase is **>0** for ∀ω except ω=0 and ω=∞ (**phase lead**)

Lag Compensator (α>1)

- Corner frequency of pole is lower than corner frequency of zero

$$\omega_z = \frac{1}{T}, \quad \omega_p = \frac{1}{\alpha T} \quad \xrightarrow{\alpha > 1} \quad \omega_p < \omega_z$$



- Phase is **<0** for ∀ω except ω=0 and ω=∞ (**phase lag**)

PD and PI are Special Cases of Lead & Lag Compensators

Lead Compensator

$$C_{lead}(s) = \frac{T\,s+1}{\alpha T\,s+1}, \quad \alpha < 1$$

Zero at -1/T
Pole at -1/(αT)

Special case: α=0

$$T\,s+1 = C_{pd}(s)$$

Zero at -1/T
Pole at -∞

Lag Compensator

$$C_{lag}(s) = \frac{T\,s+1}{\alpha T\,s+1}, \quad \alpha > 1$$

Zero at -1/T
Pole at -1/(αT)

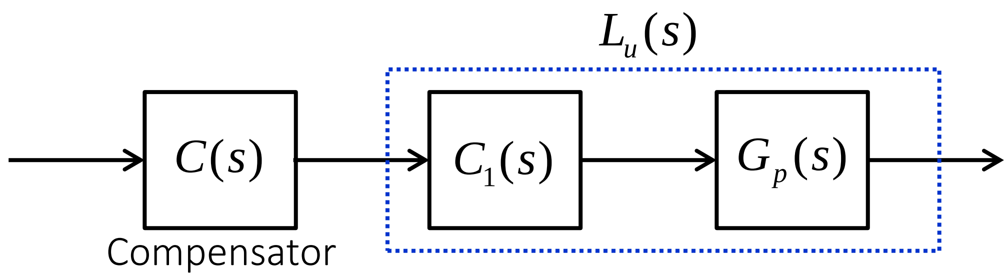
Special case: $\alpha \rightarrow \infty$

$$\frac{Ts+1}{\alpha Ts} = C_{pi}(s)$$

Zero at -1/T
Pole at 0

Guideline for Design: 1st-order Compensator

- DC-gain is one, i.e., 0 dB
  - DC property of uncompensated loop remains same after a cascade compensator is added

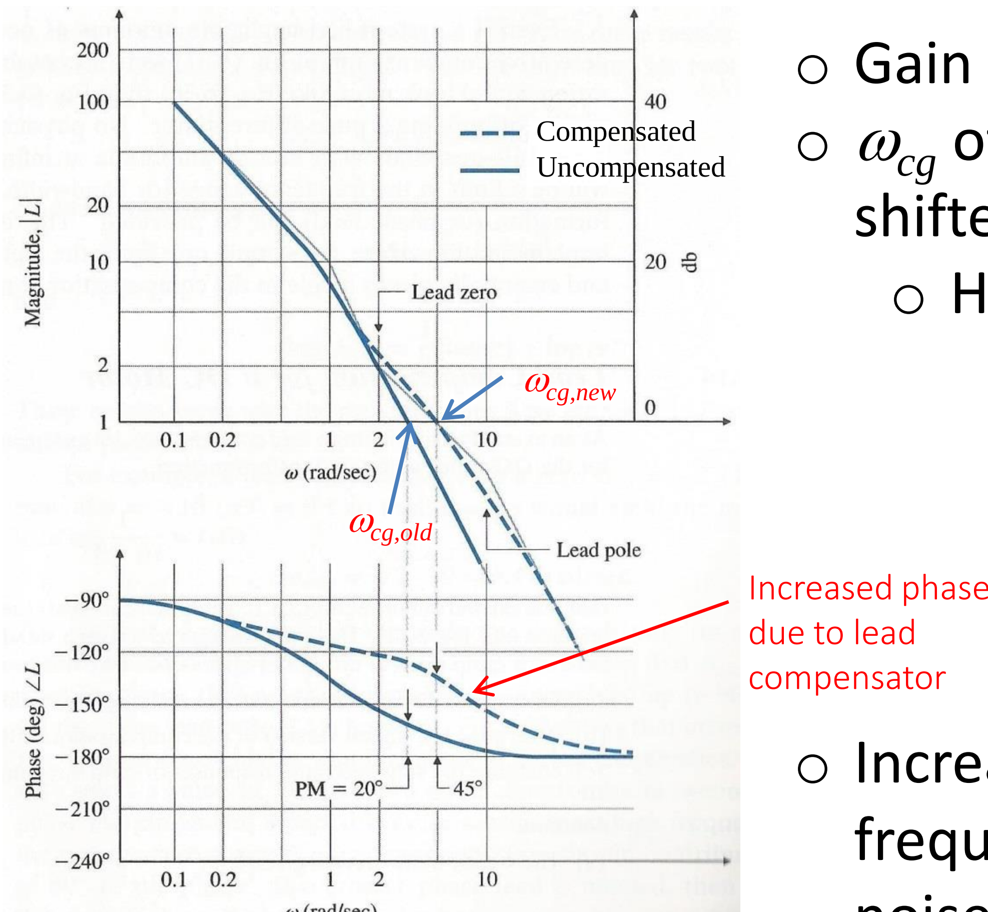


$$L(s) = C(s)L_u(s)$$

- $C_1(s)$ to meet specifications dependent on dc-gain, e.g., to meet system type and error constants
  - $C(s)$, when added, does not alter those conditions

$$L(j\omega) = C(j\omega)L_u(j\omega)$$
$$L(j\omega) = L_u(j\omega), \quad for \quad \omega \to 0$$

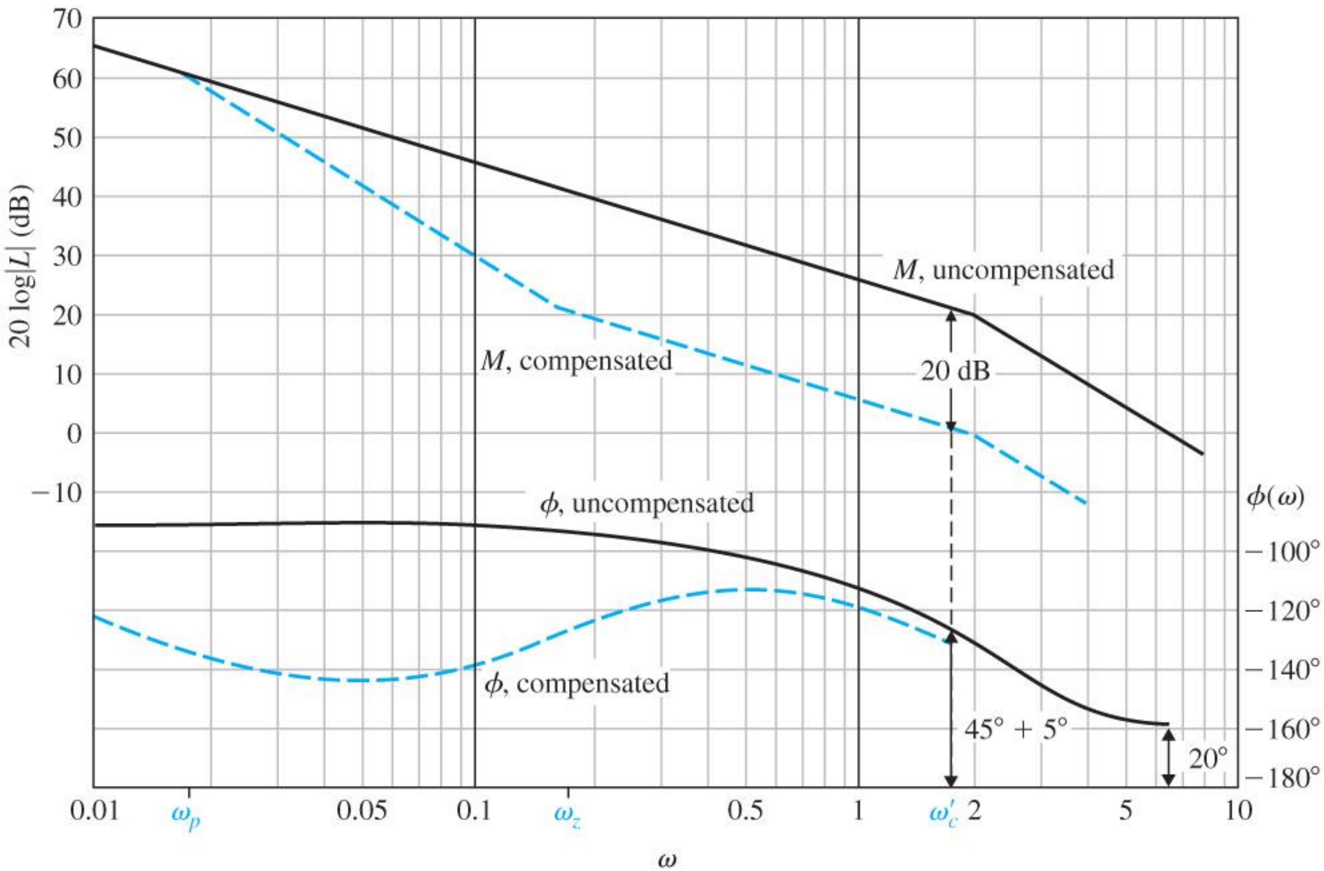
- **Lead compensator**: to increase PM by adding positive phase around the gain-crossover frequency of $L_u(j\omega)$



- Gain is increased for $\omega > \omega_z$
- ω_{cg} of the compensated loop is shifted to a higher frequency
  - Higher CL bandwidth
- Increased gain in high frequency $\Rightarrow$ amplified sensor noise

Compensator Design

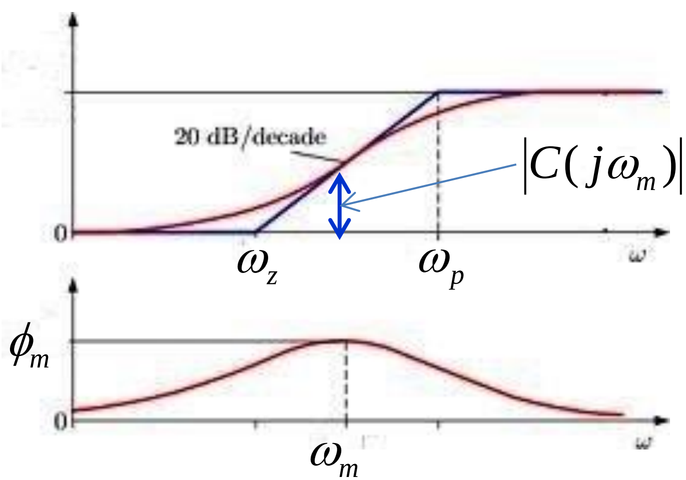
- **Lag compensator**: can improve PM by reducing gain



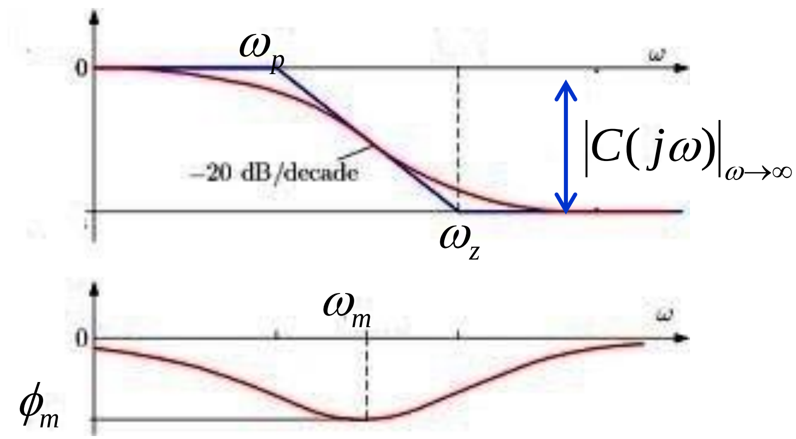
- Helps to reduce the detrimental effect of sensor noise
- Gain crossover frequency shifted to lower frequency ⇒ lower bandwidth of the CL

Design Formulae

Lead compensator



Lag compensator



- How the compensator parameters (α, T) are related to
  - ω_m (frequency at which phase change is maximum)
  - ϕ_m (maximum change of phase)
  - $|C(j\omega)|_\infty$ (maximum change of gain)
  - $|C(j\omega_m)|$ (Gain at $\omega=\omega_m$)

Compensator Design

Find ω_m (the frequency at which ϕ is maximum or minimum)

$$\frac{d\phi(\omega)}{d\omega} = 0$$

- The 1st-order compensator, $C(s) = \dfrac{Ts+1}{\alpha Ts+1} \quad \Rightarrow \quad C(j\omega) = \dfrac{(j\omega T+1)}{(j\alpha\omega T+1)}$

$$\phi = \angle C(j\omega) = \tan^{-1}(\omega T) - \tan^{-1}(\alpha\omega T)$$

$$\frac{d\phi}{d\omega} = \frac{T}{1+(\omega T)^2} - \frac{\alpha T}{1+(\alpha\omega T)^2}$$

$$\frac{T}{1+(\omega_m T)^2} - \frac{\alpha T}{1+(\alpha\omega_m T)^2} = 0$$

$$\frac{T\left(1+(\alpha\omega_m T)^2\right) - \alpha T\left(1+(\omega_m T)^2\right)}{\left(1+(\omega_m T)^2\right)\left(1+(\alpha\omega_m T)^2\right)} = 0$$

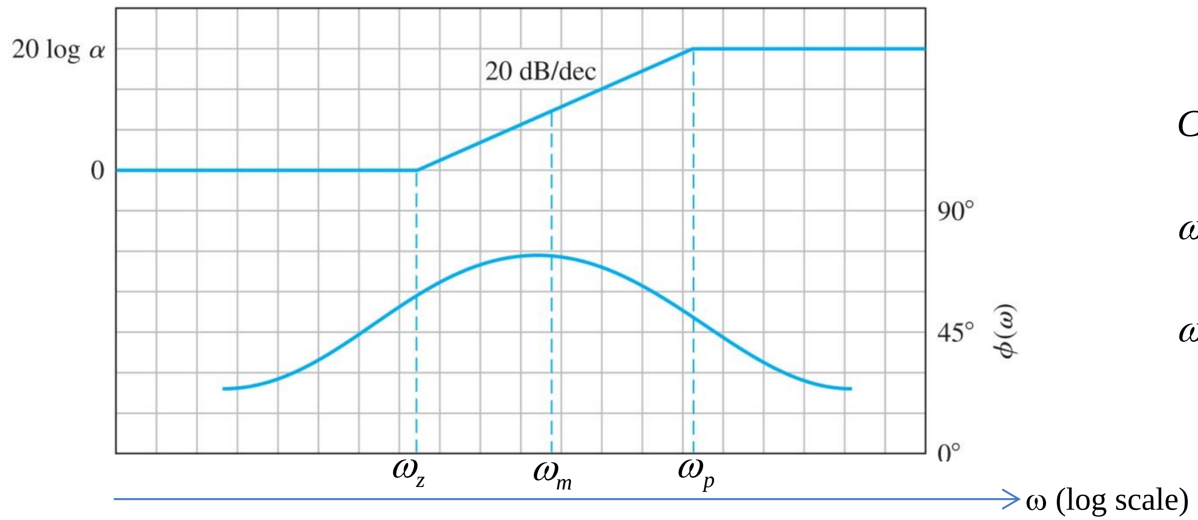
$$(1+\alpha^2\omega_m^2 T^2) - \alpha(1+\omega_m^2 T^2) = 0$$

$$\alpha\omega_m^2 T^2(\alpha-1) = \alpha-1$$

$$\alpha\omega_m^2 T^2 = 1$$

$$\boxed{\omega_m = \frac{1}{T\sqrt{\alpha}}}$$

Alternatively (from Bode plot),



$$C(s) = \frac{Ts+1}{\alpha Ts+1}$$

$$\omega_z = \frac{1}{T}$$

$$\omega_p = \frac{1}{\alpha T}$$

$$\log \omega_m = \frac{1}{2}\left(\log \omega_z + \log \omega_p\right)$$

$$2\log \omega_m = \log \omega_z + \log \omega_p$$
$$= \log\left(\omega_z \omega_p\right)$$

$$\log(\omega_m)^2 = \log\left(\omega_z \omega_p\right)$$

$$\omega_m^2 = \omega_z \omega_p$$

$$\omega_m = \sqrt{\omega_z \omega_p}$$

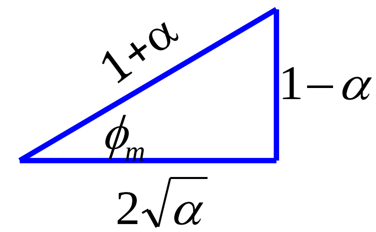
$$\omega_m = \sqrt{\frac{1}{T}\frac{1}{\alpha T}}$$
$$= \frac{1}{T\sqrt{\alpha}}$$

$$\boxed{\omega_m = \frac{1}{T\sqrt{\alpha}}}$$

Find ϕ_m

$$\phi = \tan^{-1}(\omega T) - \tan^{-1}(\alpha\omega T)$$

$$\phi_m = \tan^{-1}(\omega_m T) - \tan^{-1}(\alpha\omega_m T)$$



$$\omega_m = \frac{1}{T\sqrt{\alpha}} \quad \Rightarrow \quad \omega_m T = \frac{1}{\sqrt{\alpha}}$$

$$\phi_m = \tan^{-1}\left(\frac{1}{\sqrt{\alpha}}\right) - \tan^{-1}\left(\sqrt{\alpha}\right)$$

$$\sin(\phi_m) = \frac{1-\alpha}{1+\alpha}$$

$$\tan(A-B) = \frac{\tan(A) - \tan(B)}{1+\tan(A)\tan(B)}$$

$$\tan\phi_m = \frac{\frac{1}{\sqrt{\alpha}} - \sqrt{\alpha}}{1+\left(\frac{1}{\sqrt{\alpha}}\right)\left(\sqrt{\alpha}\right)}$$

$$= \frac{\frac{1}{\sqrt{\alpha}} - \sqrt{\alpha}}{2}$$

$$= \frac{1-\alpha}{2\sqrt{\alpha}}$$

$$\boxed{\alpha = \frac{1-\sin(\phi_m)}{1+\sin(\phi_m)}}$$

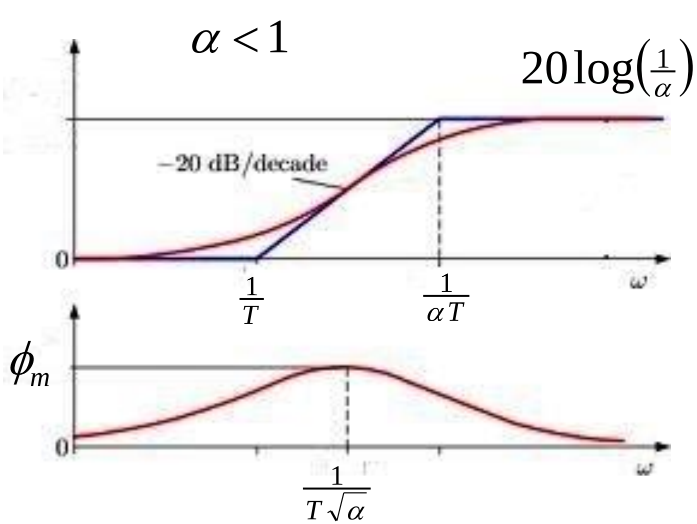
Find maximum change in gain, i.e., |C(jω)| as ω→∞

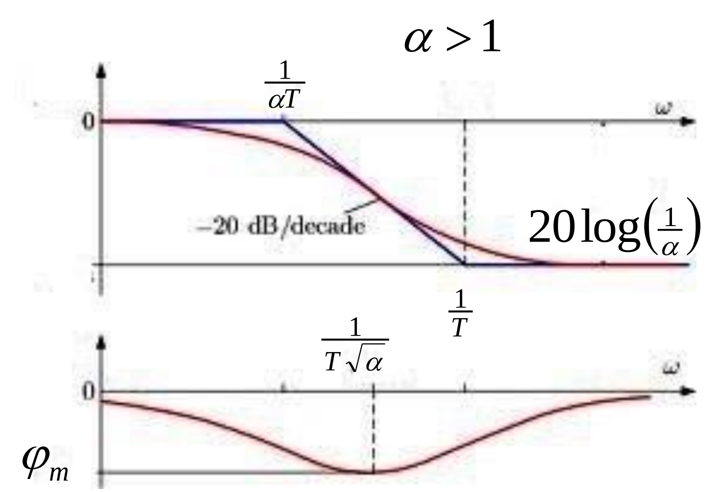
$$C(j\omega) = \frac{(j\omega T + 1)}{(j\alpha\omega T + 1)}$$

$$C(j\omega)\big|_{\omega\to\infty} = \frac{j\omega T}{j\alpha\omega T} = \frac{1}{\alpha}$$

$$\boxed{\left|C(j\omega)\right|_{\omega\to\infty} = \frac{1}{\alpha}}$$

- For lead compensator, $\alpha<1$ and hence high frequency gain is greater than 1 or positive dB
- For lag compensator, $\alpha>1$ and hence high frequency gain is less than 1 or negative dB





Find compensator gain at $\omega = \omega_m$

$$C(j\omega) = \frac{(j\omega T + 1)}{(j\alpha\omega T + 1)} \quad \Rightarrow \quad C(j\omega_m) = \frac{(j\omega_m T + 1)}{(j\alpha\omega_m T + 1)}$$

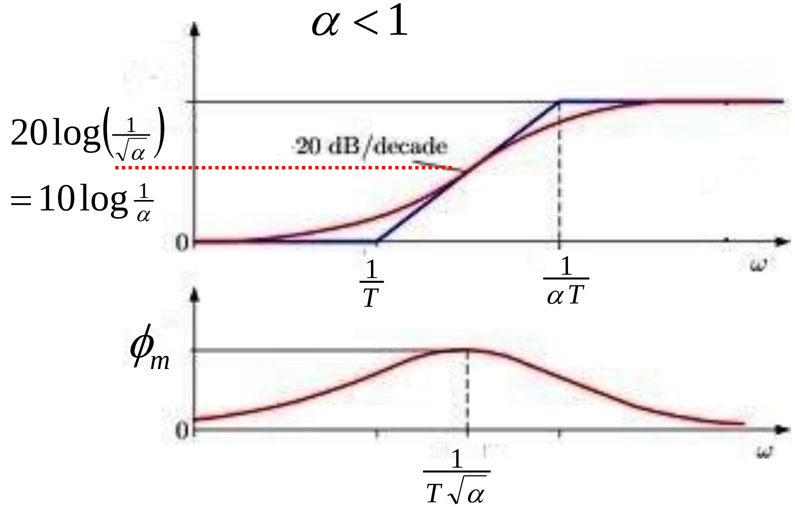
$$\omega_m = \frac{1}{T\sqrt{\alpha}} \quad \Rightarrow \quad \omega_m T = \frac{1}{\sqrt{\alpha}}$$

$$C(j\omega_m) = \frac{\left(j\frac{1}{\sqrt{\alpha}} + 1\right)}{\left(j\sqrt{\alpha} + 1\right)} = \frac{\sqrt{\alpha} + j1}{\sqrt{\alpha} + j\alpha}$$

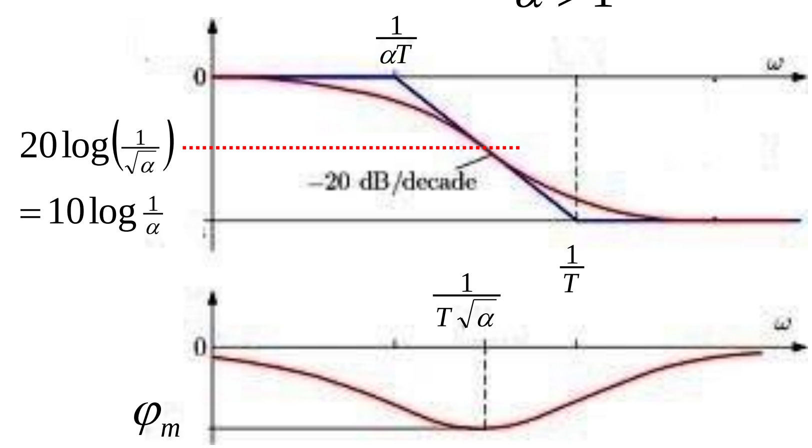
$$|C(j\omega_m)| = \sqrt{\frac{\alpha + 1}{\alpha + \alpha^2}} = \sqrt{\frac{\alpha + 1}{\alpha(1+\alpha)}} = \frac{1}{\sqrt{\alpha}}$$

$$\boxed{|C(j\omega_m)| = \frac{1}{\sqrt{\alpha}}}$$

$\alpha < 1$



$\alpha > 1$



Compensator Design

- *Example* 4-5: Write the transfer function of a 1st-order lead compensator that adds +40° phase at 100 rad/s

$$\omega_m = 100, \quad \phi_m = +40°$$

$$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m} = \frac{1-0.64}{1+0.64} = 0.22$$

$$\omega_m = \frac{1}{T\sqrt{\alpha}} \quad \Rightarrow \quad T = \frac{1}{100\times\sqrt{0.22}} = 0.02$$

$$C(s) = \frac{0.02s+1}{0.22\times 0.02s+1}$$

- *Example* 4-6: Write the transfer function of a 1st-order lag compensator that gives -10dB gain but negligible phase change at 100 rad/s

$$C(s) = \frac{Ts+1}{\alpha Ts+1}, \quad \alpha > 1$$

$$C(j\omega)\big|_{\omega\to\infty} = \frac{1}{\alpha}, \quad \alpha > 1$$

$$20\log\frac{1}{\alpha} = -10, \quad \Rightarrow \quad \alpha = 3.16$$

100 rad/s >> ω_z

$$100 >> \frac{1}{T}$$

Let, $_z$

$$100 = 20\frac{1}{T}, \quad \Rightarrow \quad T = 0.2$$

$$C(s) = \frac{0.2s+1}{3.16\times 0.2s+1}$$

To verify

$$|C(j100)| = 0.32 \quad or \quad -9.9\,dB$$

$$\angle C(j100) = -1.9°$$

Design Example - Improvement of PM using Lag Compensator

- *Example* 4-6: Consider a 1st-order process $G_p(s) = \dfrac{1}{0.5s+1}$
  - Design a controller to meet the following target specifications
    a) Zero steady-state error for constant reference input
    b) 5% error for ramp input
    c) Phase margin of 45°

- Answer:
  - The process is type 0 (no integrator in the process model)
  - An integrator must be added in the loop to meet the 1st target

$$L_u(s) = \frac{K}{s}\left(\frac{1}{0.5s+1}\right)$$

[Uncompensated loop transfer function]

  - To meet the second specification, the velocity error constant must be 20

$$e_{ss,ramp} = \frac{1}{k_v} \quad \Rightarrow \quad k_v = \frac{1}{0.05} = 20$$

$$k_v = \lim_{s\to 0} sL_u(s) = \lim_{s\to 0} K\left(\frac{1}{0.5s+1}\right) = K$$

$$k_v = 20 \quad \Rightarrow \quad K = 20$$

Compensator Design

- The first two specifications (regarding steady-state error) can be met using

$$C_1(s) = \frac{20}{s} \quad \Rightarrow \quad L_u(s) = \frac{20}{s(0.5s+1)}$$

- Check PM (third specification)

$$L_u(j\omega) = \frac{20}{j\omega(j0.5\omega+1)}$$

$$\left|L_u(j\omega_{cg})\right| = 1$$

$$\frac{20}{\omega_{cg}\sqrt{0.25\omega_{cg}^2+1}} = 1$$

$$\omega_{cg}\sqrt{0.25\omega_{cg}^2+1} = 20$$

$$\omega_{cg}^2\left(0.25\omega_{cg}^2+1\right) = 400$$

$$0.25\omega_{cg}^4 + \omega_{cg}^2 - 400 = 0$$

$$\omega_{cg}^4 + 4\omega_{cg}^2 - 1600 = 0$$

$$\omega_{cg}^2 \cong 38$$

$$\omega_{cg} \cong 6.2$$

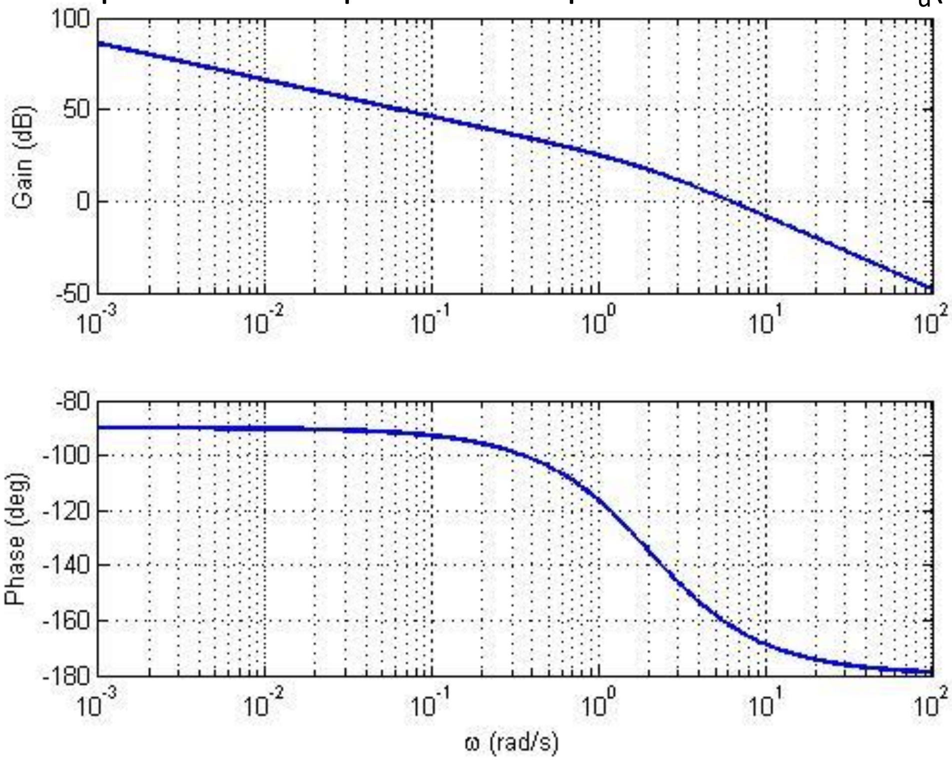
$$\angle L_u(j\omega_{cg}) = -90^\circ - \tan^{-1}\frac{0.5\times 6.2}{1}$$

$$= -162^\circ$$

$$PM_u = 18^\circ$$

Compensator Design

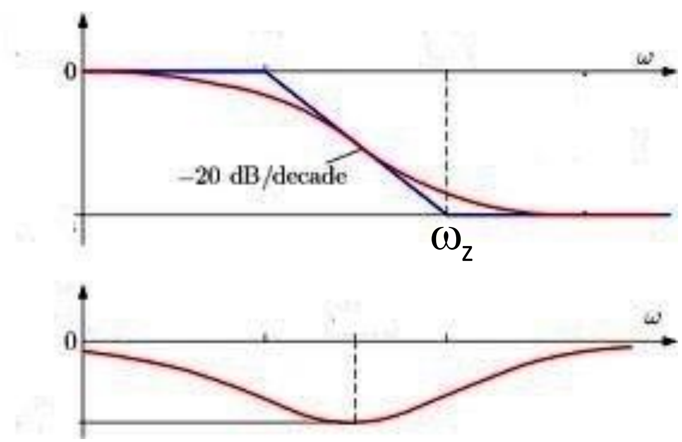
Bode plot of uncompensated loop transfer function $L_u(s)$



Phase margin is 18°

Target PM is 45° so required

$$\angle L(j\omega_{cg}) = -135°$$



- First find ω_x so that $\angle L_u(j\omega_x) = -135°$
- Then reduce the gain at that frequency to unity or 0 dB
  - Lag compensator helps to reduce gain at frequencies $>\omega_z$
  - However, it also adds negative phase

Compensator Design

Find ω_x

- As compensator adds small negative phase (not zero), find this frequency with some allowance in phase
  - Instead of -135°, find ω_x such that $\angle L_u$ is -130°

$$L_u(j\omega) = \frac{20}{j\omega(j0.5\omega+1)} \quad \Rightarrow \quad \angle L_u = -90^\circ - \tan^{-1}(0.5\omega)$$

$$-130^\circ = -90^\circ - \tan^{-1}(0.5\omega_x)$$

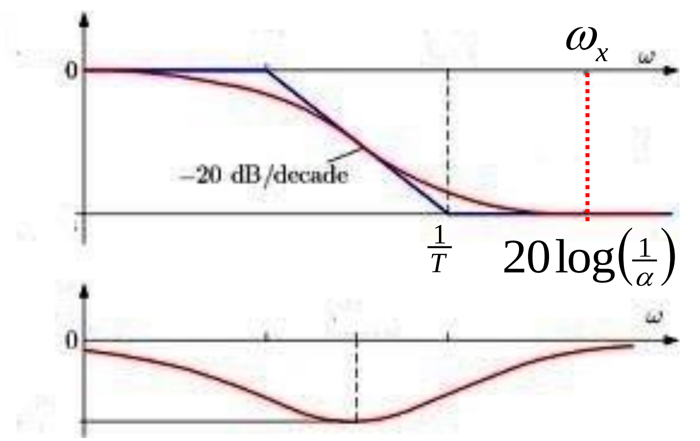
$$\tan^{-1}(0.5\omega_x) = 40^\circ$$

$$\omega_x = 1.7$$

Select T

- Choose $\omega_z << \omega_x$, for example, $\omega_z = \frac{\omega_x}{10} \quad \Rightarrow \quad \frac{1}{T} = \frac{1.7}{10} \quad \Rightarrow \quad T \cong 6$

Find α



- Gain of uncompensated loop at ω_x

$$|L_u(j\omega_x)| = \frac{20}{\omega_x\sqrt{(0.5\omega_x)^2+1}} = 9$$

- Compensator gain at ω_x should be such that

$$|C(j\omega_x)||L_u(j\omega_x)| = 1 \quad \Rightarrow \quad |C(j\omega_x)| = \frac{1}{9}$$

$$\frac{1}{\alpha} = \frac{1}{9} \quad \Rightarrow \quad \alpha = 9$$

Compensator Transfer Function

$$C(s) = \frac{Ts+1}{\alpha Ts+1} = \frac{6s+1}{54s+1}$$

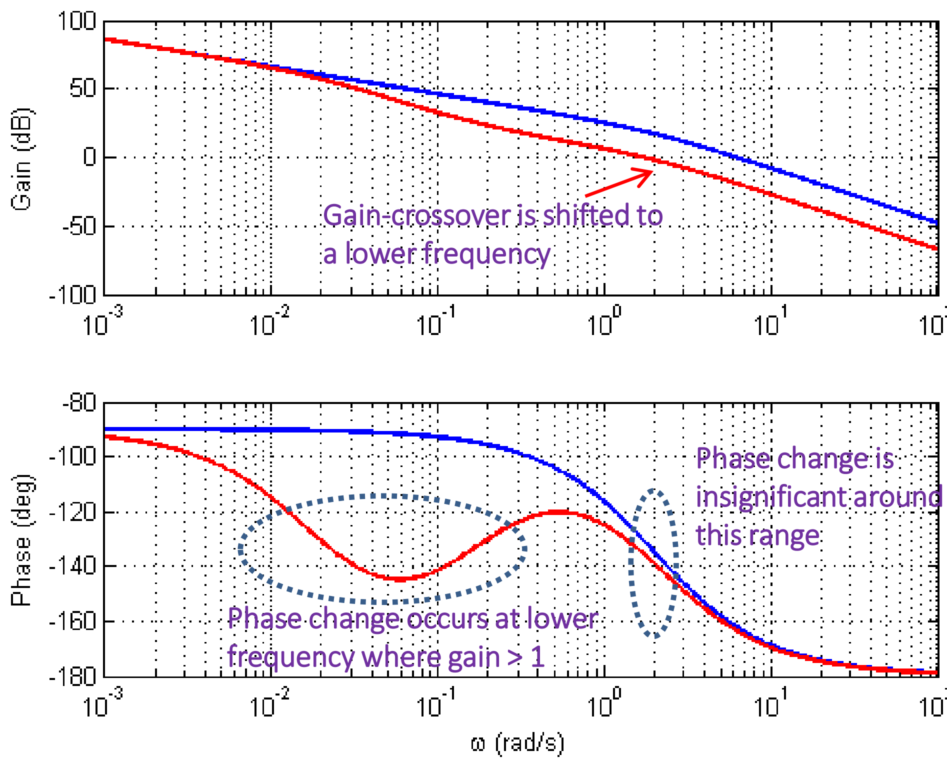
- Compensated loop transfer function

$$\begin{aligned} L(s) &= C(s)L_u(s) \\ &= \frac{(6s+1)}{(54s+1)}\frac{20}{s(0.5s+1)} \end{aligned}$$

- Steady-state error performance is not affected by appending C(s)
    - Integrator ensures zero steady-state error for step input
    - Velocity error constant is not changed

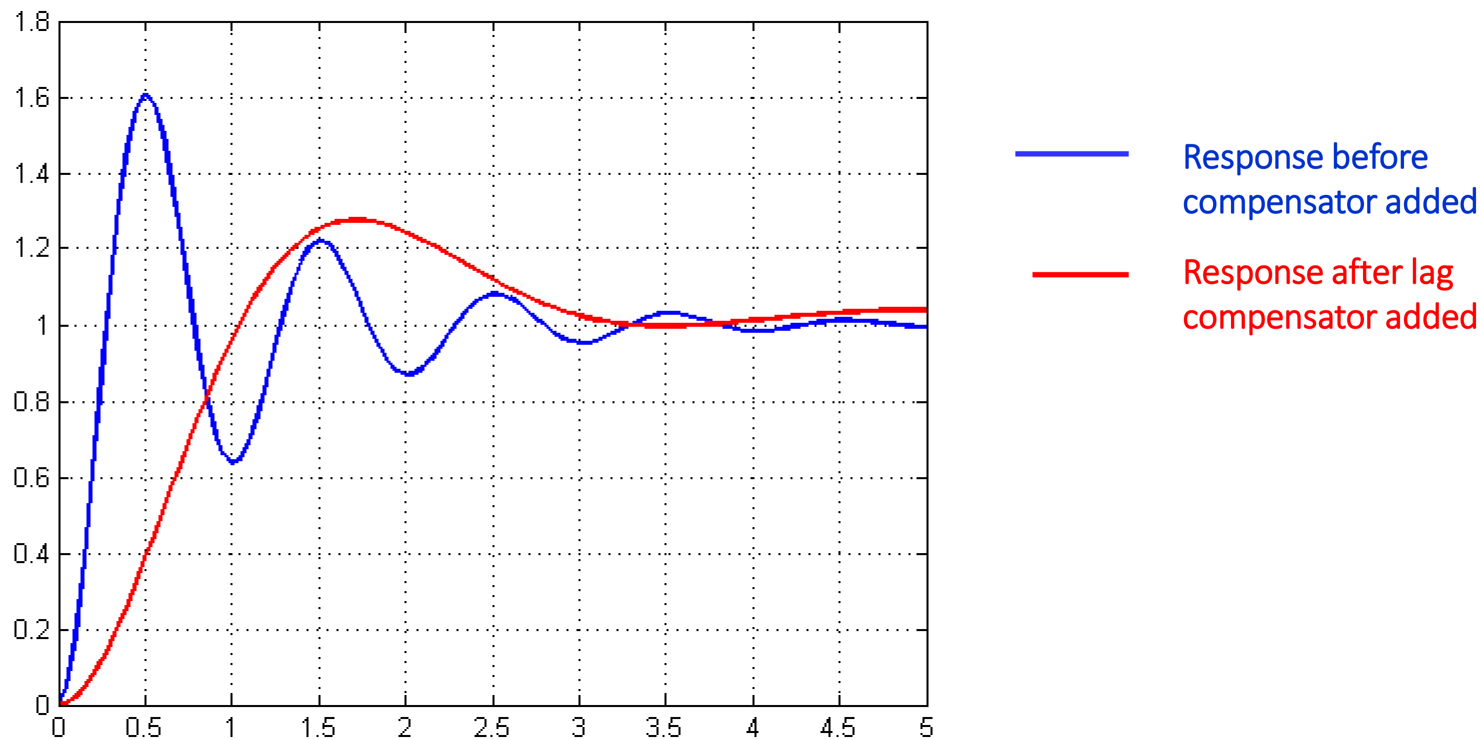
$$\begin{aligned} k_v &= \lim_{s\to 0} sL(s) \\ &= \lim_{s\to 0} s\frac{(6s+1)}{(54s+1)}\frac{20}{s(0.5s+1)} \\ &= 20 \end{aligned}$$

Effects of Adding Lag Compensator



- Increased PM means better damping of the CL transfer function
  - Less oscillatory transient response
- Lower gain-crossover frequency means smaller ω_n of the CL poles
  - Longer rise time after adding lag compensator

Effects of Adding Lag Compensator



Design Example - Improvement of PM using Lead Compensator

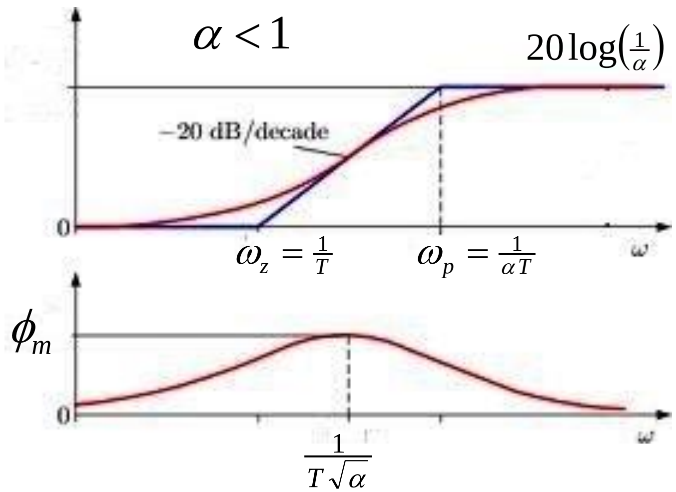
- *Example* 4-7: Consider a 1st-order process $G_p(s) = \frac{1}{0.5s+1}$
  - Design a controller to meet the following target specifications
    a) Zero steady-state error for constant reference input
    b) 5% error when the reference is unit ramp
    c) Phase margin of 45°

- Answer:
  - Following modification of the loop meets the 1st two specifications (example 4-6)

$$L_u(s) = \frac{20}{s}\left(\frac{1}{0.5s+1}\right)$$ [Uncompensated loop]

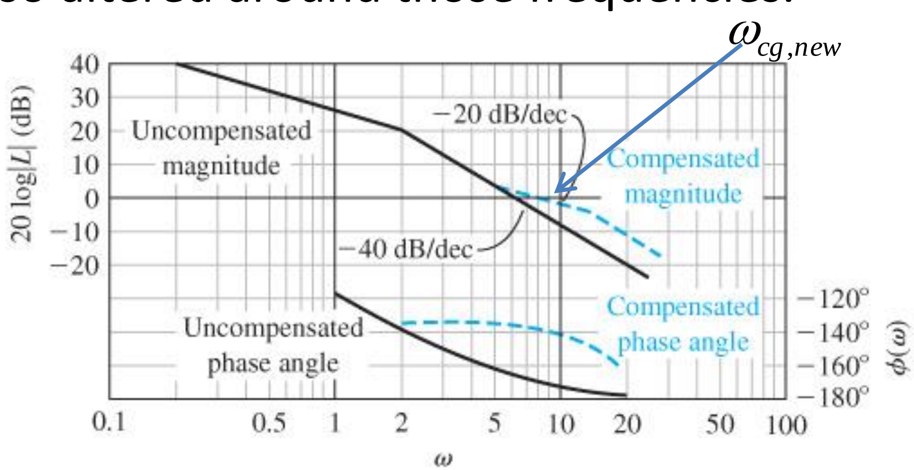
  - Gain-crossover frequency and PM of the uncompensated loop are 6.2 rad/s and 18° (example 4-6)
  - Apparently, if we add a lead compensator with ϕ_m=27° and ω_m=6.2 then the PM is increased to 45°
    - But lead compensator alters gain around ω_m

- Before we start using the design formulae, let's examine the effects of lead compensator on the frequency response



- Phase is increased for $\omega_z < \omega < \omega_p$
  - Phase is also affected in frequencies outside this band
  - Effect is insignificant for $\omega << \omega_z$ and $\omega >> \omega_p$
- Gain is increased for $\omega > \omega_z$.

- If a lead compensator is used to increase phase in some range of frequencies, the gain is also altered around those frequencies.



Gain-crossover is shifted to a higher frequency

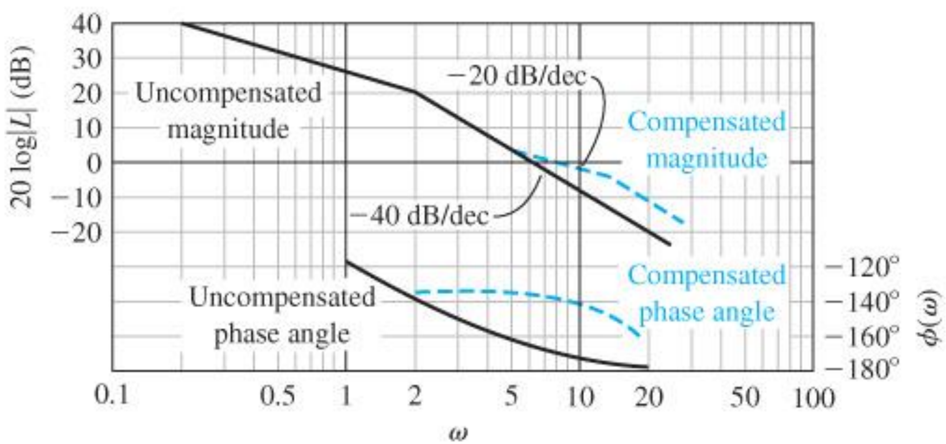
$$\omega_{cg,new} > \omega_{cg,u}$$

Compensator Design

- For the current example, the required phase lead of 27° is calculated assuming ω_{cg} = 6.2 rad/s
  - How much phase lead is required at $\omega_{cg,new}$?

- If the phase of the uncompensated loop transfer function doesn't change abruptly, it can be assume to be slightly more negative at $\omega_{cg,new}$

$$\angle L_u(j\omega_{cg,new}) = \angle L_u(j\omega_{cg,u}) - \delta$$



- Design a compensator to give $\phi_m = PM_t - PM_u + \delta$
  - Typical value of the allowance is 10% of (PM_t – PM_u)

Compensator Design

- Continuing with Example 4-7,

$$PM_t = 45°, \quad PM_u = 18°, \quad \Rightarrow \quad \delta = 0.1 \times 27° \cong 3°$$

$$\begin{aligned}\phi_m &= 45° - 18° + 3° \\ &= 30°\end{aligned}$$

- The parameter α of a compensator depends solely on the value of ϕ_m

$$\alpha = \frac{1 - \sin\phi_m}{1 + \sin\phi_m}$$

$$\phi_m = 30° \quad \Rightarrow \quad \alpha = 0.33$$

- The second parameter T of the compensator can be found using the formula

$$\omega_m = \frac{1}{T\sqrt{\alpha}}$$

[The value of α is already decided but what is ω_m?]

  - We want the ω_m to be at the new gain-crossover frequency, i.e.,

$$|L(j\omega_m)| = 1$$

$$|L_u(j\omega_m)| \times |C(j\omega_m)| = 1$$

Compensator Design

- Compensator gain at the frequency ω_m depends on the value of α alone

$$|C(j\omega_m)| = \frac{1}{\sqrt{\alpha}}$$

  - In order to have ω_m at the new gain-crossover frequency,

$$|L_u(j\omega_m)| \times |C(j\omega_m)| = 1$$

$$|L_u(j\omega_m)| = \frac{1}{|C(j\omega_m)|}$$

$$|L_u(j\omega_m)| = \sqrt{\alpha}$$

- As the value of α has already been decided, we can find ω_m by solving

$$|L_u(j\omega_m)| = \sqrt{\alpha}$$

$$L_u(j\omega) = \frac{20}{j\omega}\left(\frac{1}{j0.5\omega+1}\right) \Rightarrow \frac{20}{\omega_m\sqrt{(0.5\omega_m)^2+1}} = \sqrt{0.33} \Rightarrow \frac{400}{\omega_m^2(0.25\omega_m^2+1)} = 0.33$$

$$\omega_m^2\left(0.25\omega_m^2+1\right) = 1212 \qquad \omega_m^2 = +67.66 \quad or \quad -71.66$$

$$\omega_m^4 + 4\omega_m^2 - 4848 = 0 \qquad \omega_m = 8.2$$

- Find compensator parameter T by solving $T = \frac{1}{\omega_m\sqrt{\alpha}} = \frac{1}{8.2\sqrt{0.33}} \cong 0.2$

Compensator Design

- Compensator transfer function,

$$C(s) = \frac{0.2s+1}{0.33 \times 0.2s+1}$$

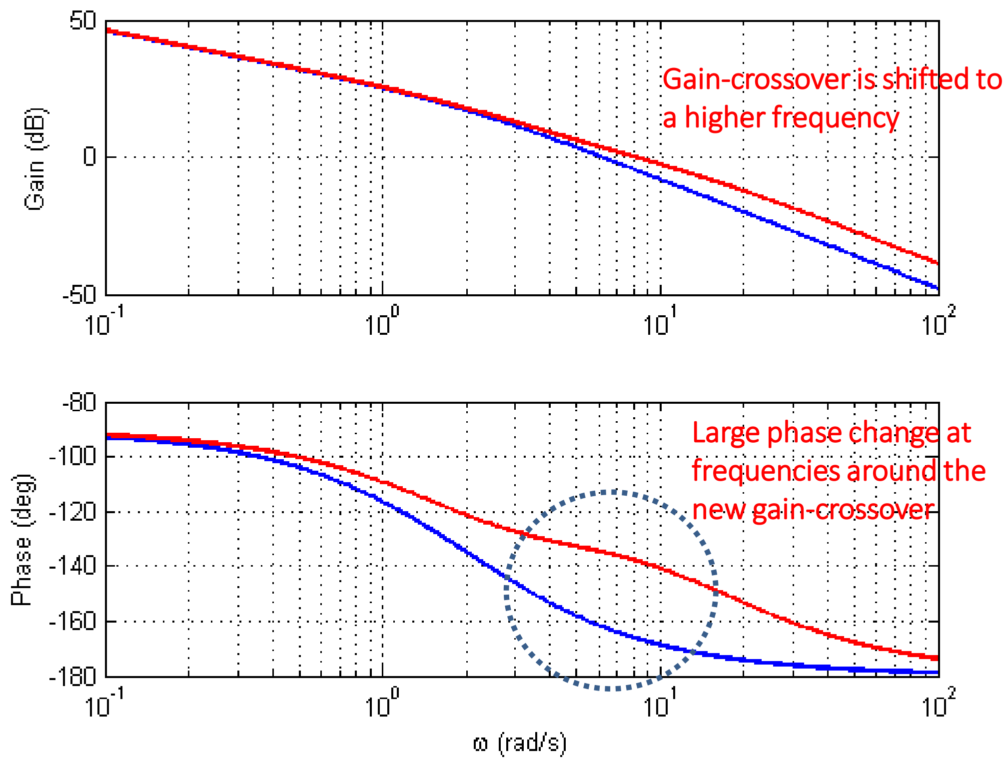
- Compensated loop transfer function

$$\begin{aligned} L(s) &= C(s)L_u(s) \\ &= \frac{(0.2s+1)}{(0.07s+1)} \frac{20}{s(0.5s+1)} \end{aligned}$$

- Steady-state error performance is not affected by appending C(s)
  - Integrator ensures zero steady-state error for step input
  - Velocity error constant is not changed

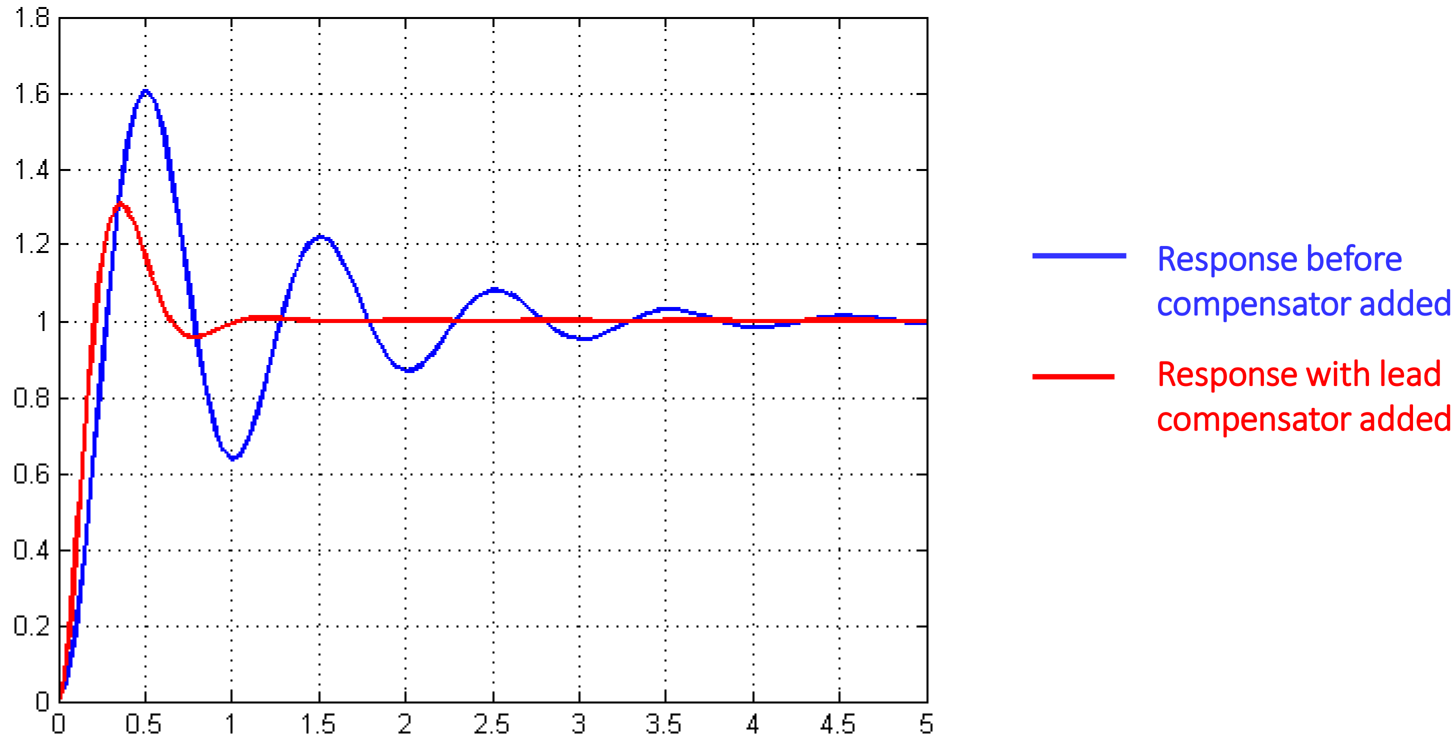
$$\begin{aligned} k_v &= \lim_{s \to 0} sL(s) \\ &= \lim_{s \to 0} s \frac{(0.2s+1)}{(0.07s+1)} \frac{20}{s(0.5s+1)} \\ &= 20 \end{aligned}$$

Effects of Adding Lead Compensator

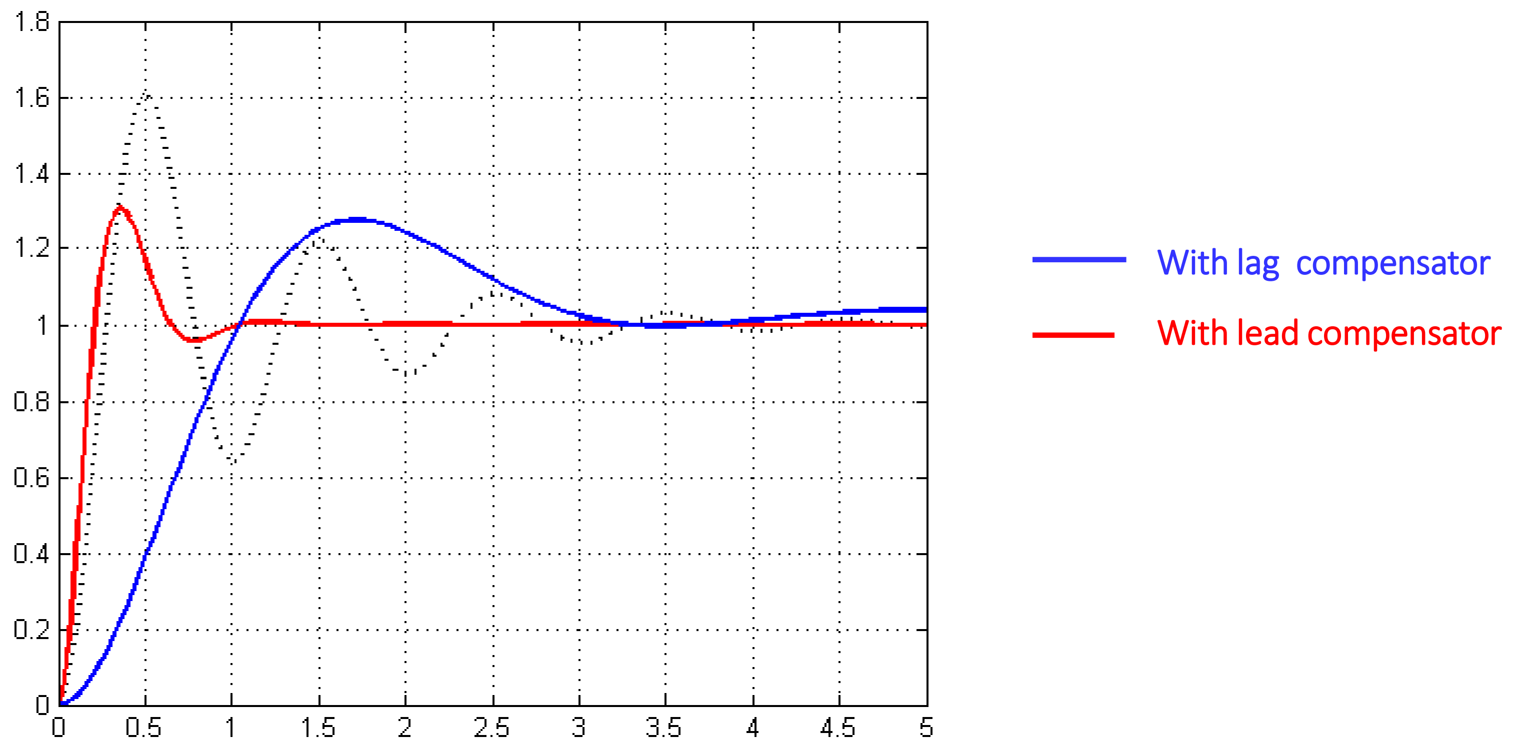


- Increased PM means higher damping of the CL transfer function
  - Less oscillatory transient response
- Higher gain-crossover frequency means larger ω_n of the CL poles
  - Rise time is shorter with lead compensator added

Effects of Adding Lead Compensator



Comparison between Transient Responses with Lag and Lead Compensator



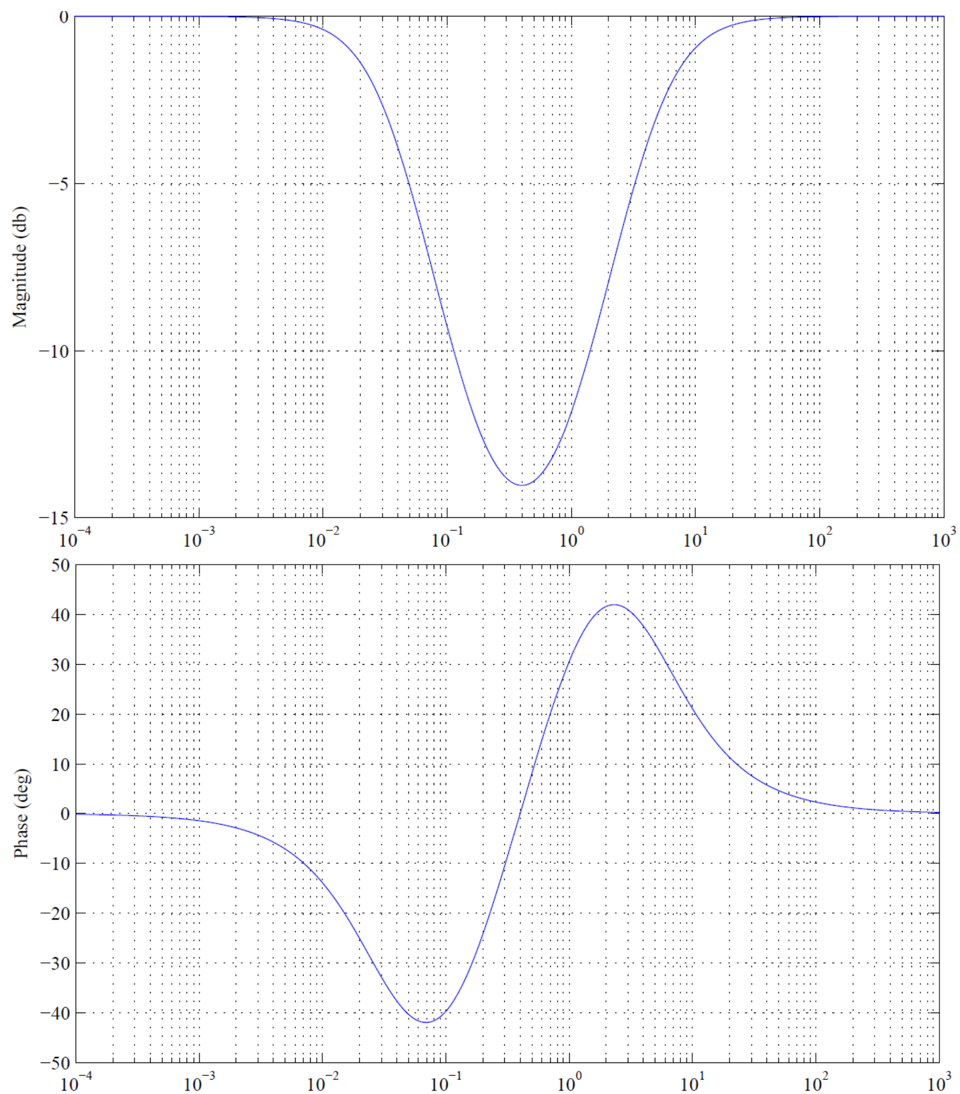
Higher-Order Compensator

$$C(s) = \underbrace{\frac{(T_1 s + 1)}{(\beta T_1 s + 1)}}_{\text{Lag}} \underbrace{\frac{(T_2 s + 1)}{(\alpha T_2 s + 1)}}_{\text{Lead}}, \quad \begin{cases} 0 < \alpha < 1 \\ \beta > 1 \end{cases}$$

- It is called **Lag-Lead** if $\frac{1}{T_1} < \frac{1}{T_2}$
- & **Lead-Lag** if $\frac{1}{T_2} < \frac{1}{T_1}$

- Bode plot shown is for Lag-Lead with

$$\frac{1}{T_1} = 0.2 \qquad \frac{1}{T_2} = 0.8$$
$$\beta = 6.25 \qquad \alpha = 0.16$$



Appendix

Chapter 4: Design of Feedback System

From bound on |S(jω)| to bound on |L(jω)|

- Let M and ϕ be the gain and phase of $L(j\omega)$ at $\omega=\omega_i$.

$$M = |L(j\omega_i)|, \quad \phi = \angle L(j\omega_i)$$

$$L(j\omega_i) = M\cos\phi + jM\sin\phi$$

$$1 + L(j\omega_i) = 1 + M\cos\phi + jM\sin\phi$$

- Then

$$\begin{aligned} |1+L(j\omega_i)| &= \sqrt{(1+M\cos\phi)^2 + (M\sin\phi)^2} \\ &= \sqrt{1 + 2M\cos\phi + M^2\cos^2\phi + M^2\sin^2\phi} \\ &= \sqrt{(1+M^2) + 2M\cos\phi} \end{aligned}$$

- If $M > 1$,

$$(M-1) \le |1+L(j\omega_i)| \le (M+1) \qquad \because \quad -1 \le \cos\phi \le +1$$

$$\min|1+L(j\omega_i)| = (M-1)$$

$$\frac{1}{M-1} \ge \frac{1}{|1+L(j\omega_i)|} \ge \frac{1}{M+1}$$

$$\frac{1}{M-1} \ge |S(j\omega_i)| \ge \frac{1}{M+1}$$

From bound on |S(jω)| to bound on |L(jω)|

$$\frac{1}{M-1} \geq |S(j\omega_i)| \geq \frac{1}{M+1}$$

- Depending on the value of ϕ, the magnitude of the sensitivity function is bounded between two limits
- If maximum magnitude of sensitivity function is less than ε_s, then the bound on sensitivity is guaranteed to be satisfied

$$\text{if } \max|S(j\omega_i)| < \varepsilon_s \quad \text{then} \quad |S(j\omega_i)| < \varepsilon_s$$

$$\frac{1}{M-1} < \varepsilon_s \Rightarrow (M-1) > \frac{1}{\varepsilon_s} \quad \Rightarrow \quad M > 1 + \frac{1}{\varepsilon_s}$$

$$|L(j\omega_i)| > 1 + \frac{1}{\varepsilon_s} \Rightarrow |L(j\omega_i)| > \frac{\varepsilon_s + 1}{\varepsilon_s}$$

From bound on |T(jω)| to bound on |L(jω)|

- Let M and ϕ be the gain and phase of $L(j\omega)$ at $\omega = \omega_i$.

$$M = |L(j\omega_i)|, \quad \phi = \angle L(j\omega_i)$$

$$L(j\omega_i) = M\cos\phi + jM\sin\phi$$

$$1 + L(j\omega_i) = 1 + M\cos\phi + jM\sin\phi$$

- Then

$$\begin{aligned} |1 + L(j\omega_i)| &= \sqrt{(1 + M\cos\phi)^2 + (M\sin\phi)^2} \\ &= \sqrt{1 + 2M\cos\phi + M^2\cos^2\phi + M^2\sin^2\phi} \\ &= \sqrt{(1 + M^2) + 2M\cos\phi} \end{aligned}$$

- If $M < 1$,

$$(1 - M) \le |1 + L(j\omega_i)| \le (1 + M) \qquad \because \quad -1 \le \cos\phi \le +1$$

$$\min|1 + L(j\omega_i)| = (1 - M)$$

$$\frac{1}{1 - M} \ge \frac{1}{|1 + L(j\omega_i)|} \ge \frac{1}{1 + M}$$

$$\frac{L(j\omega_i)}{1 - M} \ge \frac{L(j\omega_i)}{|1 + L(j\omega_i)|} \ge \frac{L(j\omega_i)}{1 + M}$$

From bound on |T(jω)| to bound on |L(jω)|

$$\frac{L(j\omega_i)}{1-M} \geq \frac{L(j\omega_i)}{\left|1+L(j\omega_i)\right|} \geq \frac{L(j\omega_i)}{1+M}$$

$$\frac{M}{1-M} \geq \left|T(j\omega_i)\right| \geq \frac{M}{1+M}$$

- The magnitude of the complementary sensitivity function is bounded between these two limits

$$if \ \max\left|T(j\omega_i)\right| < \varepsilon_T \quad then \quad \left|T(j\omega_i)\right| < \varepsilon_T$$

$$\frac{M}{1-M} < \varepsilon_T \quad \Rightarrow \quad M < \varepsilon_T(1-M)$$

$$M(1+\varepsilon_T) < \varepsilon_T$$

$$M < \frac{\varepsilon_T}{1+\varepsilon_T}$$

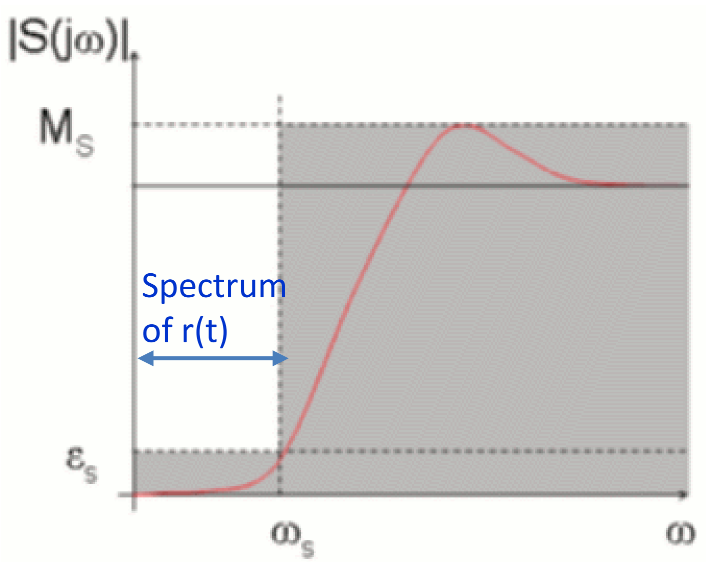
$$\left|L(j\omega_i)\right| < \frac{\varepsilon_T}{1+\varepsilon_T}$$

Trade-Off between |S(jω)| and |T(jω)|

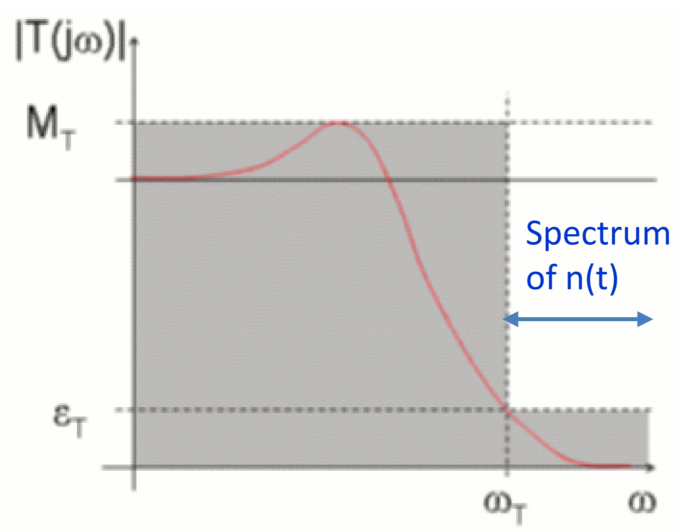
- Good tracking of sinusoidal reference requires $|S(j\omega_i)| << 1$
- Good attenuation of noise requires $|T(j\omega_i)| << 1$
- Both S and T cannot be very small at the same frequency because

$$S(j\omega) + T(j\omega) = 1$$

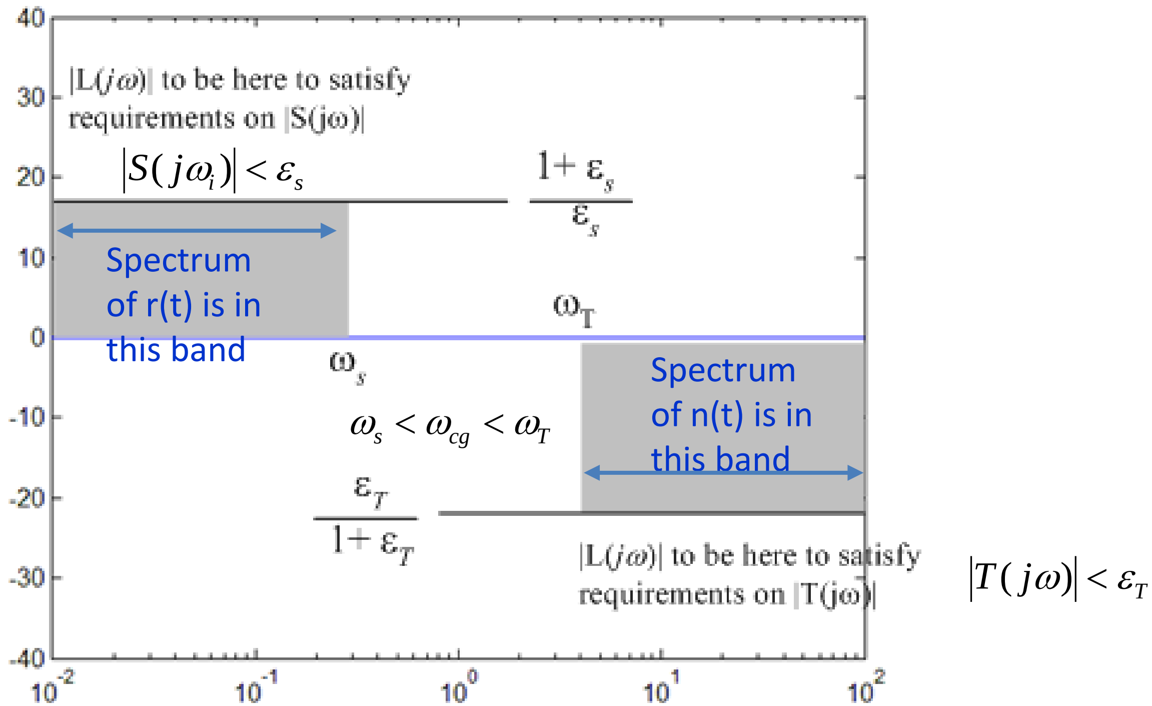
In most practical systems, reference signals have energy contained in low frequency band



Sensor noise usually lies in the high frequency band



Specification for Target |L(jω)|



- For given ω_s and ω_T, smaller values of ε_s and ε_T result in steeper drop in open loop gain
  - Slope of the OL gain at ω_{cg} defines the phase margin and hence affects the damping of the closed loop poles

Specification for Target $|L(j\omega)|$

- For frequencies less than ω_s and frequencies greater than ω_T, the specification involved OL gain only, i.e. $|L(j\omega)|$
  - For ($\omega_s<\omega<\omega_T$), both gain and phase are to be considered
  - Is there a way to relate $|L(j\omega)|$ to $\angle L(j\omega)$?

Some observations from the Bode plot:

	$G(s)=s$	$G(s)=s^{-1}$	$G(s)=(s+a)$	$G(s)=(s+a)^{-1}$
Slope of the Gain plot	+20 db/dec	-20 db/dec	0 for low frequency; +20 db/decade for high frequency	0 for low frequency; -20 db/decade for high frequency
Phase	+90°	-90°	0°for low freq; +90° for high freq	0°for low freq; -90° for high freq

Specification for Target |L(jω)|

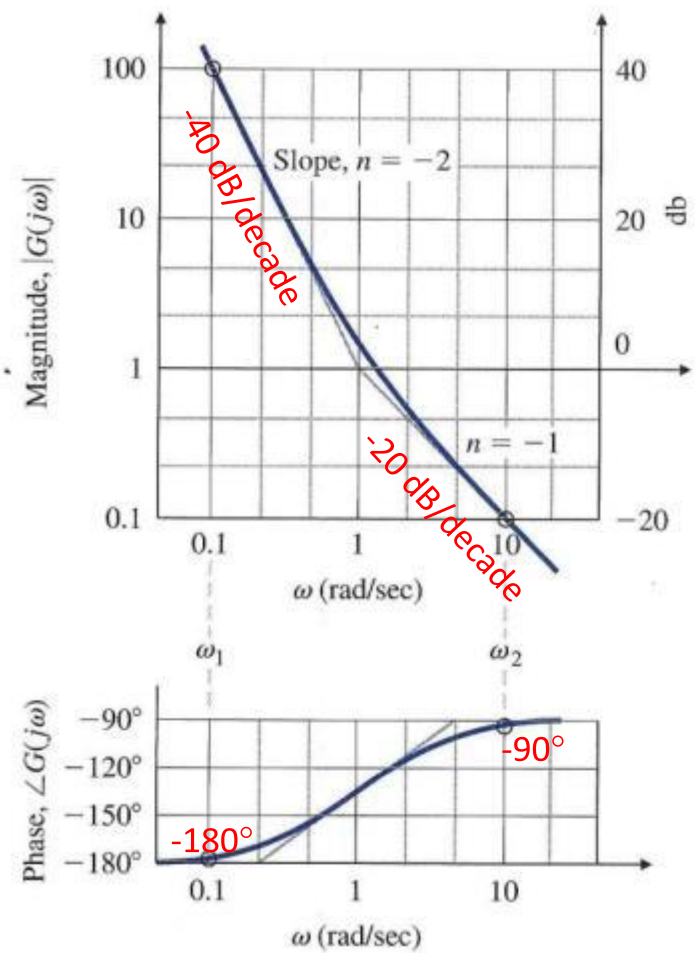
Bode's Gain-Phase Relation

- If slope of the gain curve persists at a constant value of (20n dB/decade) for nearly a decade or more then,

$$\angle G(j\omega_0) \cong n \times 90^\circ$$

- In the figure shown,
  - When the gain slope is 40 dB/decade, phase is -180°
  - When the gain slope is -20 dB/decade, phase is -90°

- This is a simplified approximation of a more accurate formula given by Bode



Specification for Target |L(jω)|

- Let the slope of the Bode (magnitude) plot of $L(j\omega)$ over a range of frequencies around ω_{cg} be **20n** dB/decade. Then

$$\angle L(j\omega_{cg}) = n \times 90^\circ$$

  - The phase margin is:

$$PM = 180^\circ + \angle L(j\omega_{cg})$$
$$\cong 180^\circ + n \times 90^\circ$$

  - Around ω_{cg}, the slope of the Bode (magnitude) plot is negative. Therefore, to have positive PM,

$$|n| < 2, \quad \Rightarrow \quad n = -1$$

  - The slope of the Bode (magnitude) plot of $L(j\omega)$ should be **-20 dB/decade** at the gain-crossover frequency

Specification for Target |L(jω)|