EE3331C/EE3331E Feedback Control Systems, Part II
Solution to Tutorial 2

1. Plant transfer function of a unity feedback system is $G_p(s) = \dfrac{K}{s(s+2)(s+20)}$.
   - i. Select amplifier gain K so that steady-state error to ramp input is less than or equal to 5%.
   - ii. Design a lag compensator to make phase margin greater than or equal to 45°.

Solution:

(i) $e_{ss,ramp} \le 5\% \quad \Rightarrow \quad k_v \ge 20$ For the transfer function given, $k_v = \dfrac{K}{40}$

$$\frac{K}{40} \ge 20 \quad \Rightarrow \quad K \ge 800.$$

Let K=800. Then uncompensated loop transfer function: $L_u(s) = \dfrac{800}{s(s+2)(s+20)}$.

(ii)

Find the frequency, ω_x, at which uncompensated phase is -130° $\angle L_u(j\omega_x) = -130°$ $-90° - \tan^{-1}\frac{\omega_x}{2} - \tan^{-1}\frac{\omega_x}{20} = -130°$ $\tan^{-1}\frac{\omega_x}{2} + \tan^{-1}\frac{\omega_x}{20} = 40°$ $\dfrac{\frac{\omega_x}{2}+\frac{\omega_x}{20}}{1-(\frac{\omega_x}{2})(\frac{\omega_x}{20})} = \tan 40°$ $\dfrac{22\omega_x}{40-\omega_x^2} = 0.8$ $0.8\omega_x^2 + 22\omega_x - 32 = 0$ Solving the quadratic equation, $\omega_x \cong 1.38$	Comment: To achieve 45° PM, phase of the compensated loop should be -135°at the gain-crossover frequency. But lag compensator gives small negative phase. We take an allowance of 5° and design for 50° PM. Trigonometric identity used: $\tan(A+B) = \dfrac{\tan A + \tan B}{1-(\tan A)(\tan B)}$
Choose the corner frequency of the compensator zero to be one tenth of this frequency $\dfrac{1}{T} = \dfrac{\omega_x}{10} \quad \Rightarrow \quad T = \dfrac{10}{\omega_x} \cong 7.2$	
Find uncompensated gain at this frequency, $\lvert L_u(j\omega_x)\rvert = \left\lvert \dfrac{800}{(j1.38)(2+j1.38)(20+j1.38)} \right\rvert$ $= \dfrac{800}{1.38 \times \sqrt{4+1.9}\sqrt{400+1.9}}$ $= 11.9$	As ω_x is large compared to the corner frequency of the compensator zero, the gain of compensator at this frequency can be approximated as $\lvert C(j\omega_x)\rvert = \dfrac{1}{\alpha}$
Choose $\alpha = \lvert L_u(j\omega_x)\rvert$ so that $\lvert C\rvert \times \lvert L_u\rvert = 1$ at ω_x. $\alpha = 11.9$ $C(s) = \dfrac{7.2s+1}{85.7s+1}, \quad L(s) = \dfrac{800(7.2s+1)}{s(s+2)(s+20)(85.7s+1)}$	

2. Plant transfer function of a unity feedback system is $G_p(s) = \dfrac{30}{s^2+s}$.

   i. What are the steady-state errors for (a) unit step input, (b) unit ramp input, and (c) unit parabolic input?
   ii. What is the phase margin?
   iii. Design a lead compensator to make phase margin greater than or equal to 40°.

Solution:

(i) It is type 1. So steady-state error is zero for step input and infinity for parabolic input.

$$k_v = 30$$

Steady-state error for ramp input is, $e_{ss,ramp} = \dfrac{1}{k_v} \times 100\% \quad = \dfrac{100}{30}\% = 3.3\%$

(ii)

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<td>Find the gain-crossover frequency, ω_{cg}, by solving $\left|L_u(j\omega_{cg})\right| = 1$
$$L_u(j\omega) = \frac{30}{-\omega^2 + j\omega}$$
$$\frac{30}{\sqrt{\omega_{cg}^4 + \omega_{cg}^2}} = 1$$
$$\omega_{cg}^4 + \omega_{cg}^2 - 900 = 0$$
Solving the quadratic equation, $\omega_{cg}^2 = 29.5$
[only the positive solution is considered]
$$\omega_{cg} = 5.4$$
Phase of uncompensated loop at gain-crossover:
$$\angle L(j5.4) = -169.5°$$
Phase margin: $PM = 10.5°$</td>
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<td>Lead compensator design:
Apparent phase lead required, $PM_t - PM_u = 40° - 10.5° = 29.5° \cong 30°$
Design for phase lead of, $\phi_m = (1+0.1)\times 30° = 33°$
Find compensator parameter α using
$$\sin\phi_m = \frac{1-\alpha}{1+\alpha} \quad \Rightarrow \quad \alpha = 0.29 \approx 0.3$$</td>
<td>Extra phase lead is chosen to make up for the change in uncompensated phase as the gain-crossover is shifted to higher frequency due to the compensator's gain.</td>
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<td>Choose ω_m such that $\left|L_u(j\omega_m)\right| \times \left|C(j\omega_m)\right| = 1$ $\left|L_u(j\omega_m)\right| \times \dfrac{1}{\sqrt{\alpha}} = 1$
$$\left|L_u(j\omega_m)\right| = \sqrt{\alpha}$$
For the uncompensated loop, $\dfrac{30}{\sqrt{\omega_m^4 + \omega_m^2}} = \sqrt{0.3}$
$$\omega_m^4 + \omega_m^2 = \frac{900}{0.3} = 3000$$
$$\omega_m^4 + \omega_m^2 - 3000 = 0$$</td>
<td>This means ω_m is the gain-crossover frequency after compensation.
Whatever be the value of ωm, compensator gain at this frequency is
$$\left|C(j\omega_m)\right| = \frac{1}{\sqrt{\alpha}}$$</td>
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Solution of the quadratic equation gives,

$$\omega_m^2 \cong 54.3$$
$$\omega_m = 7.36$$

Find compensator parameter T using, $\omega_m = \dfrac{1}{T\sqrt{\alpha}}$ $\qquad T = \dfrac{1}{7.36\sqrt{0.3}} = 0.25$

Compensator transfer function, $C(s) = \dfrac{0.25s+1}{0.075s+1}$

3. A DC motor with negligible armature inductance is to be used in a position control system. It's transfer function is

$$G_p(s) = \frac{250}{s(s+5)}.$$

Assuming unity feedback (i.e., H(s)=1), design a controller to meet the following specifications:

i. The steady-state error to a unit ramp input is less than or equal to 1/200.
ii. Overshoot to unit step response is less than 20%.

Solution:

Spec (i) requires a type 1 system. Plant itself is type 1. We find the required gain to meet the required error constant.

$$k_v = \lim_{s\to 0} s\frac{250K}{s(s+5)} = 50K \qquad e_{ss,ramp} \le \frac{1}{200}, \quad k_v \ge 200 \qquad 50K \ge 200 \;\Rightarrow\; K \ge 4$$

Let, K=4 $\qquad L_u(s) = \dfrac{1000}{s(s+5)}$

(ii)

$$\%OS < 20\% \;\Rightarrow\; e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} < 0.2 \qquad \frac{\pi^2\zeta^2}{1-\zeta^2} = 2.56$$

$$\pi^2\zeta^2 + 2.56\zeta^2 - 2.56 = 0$$

$$\zeta^2 = \frac{2.56}{\pi^2 + 2.56} = 0.21$$

$$\zeta = 0.46$$

Required PM is, $PM_t = 46°$

We design a lead compensator to achieve this PM. Find gain-crossover of the uncompensated loop by solving $|L_u(j\omega_{cg})| = 1$

$$\frac{1000}{\omega_{cg}\sqrt{\omega_{cg}^2 + 5^2}} = 1 \;\Rightarrow\; \omega_{cg} \cong 31\, rad/s$$

$$PM_u = 180° - 90° - \tan^{-1}\frac{31}{5} = 9.2°$$

$\zeta \cong (PM/100)$ is an approximation assuming the loop transfer function is an integrator followed by a real pole. However, after we add a compensator, the loop transfer function has three poles and one zero. If a lag compensator is used, we may end up putting the compensator pole and zero between integrator and plant pole. In that case, the resulting closed loop may become a dominant 1st order transfer function. For this type of situations, lag compensator is not a good option.

Apparent phase lead required, $PM_t - PM_u = 46° - 9.2° = 36.8°$

Desired phase lead from the compensator, $\phi_m = (1+0.1)\times 36.8° = 40.5°$

Find compensator parameter α using $\sin\phi_m = \dfrac{1-\alpha}{1+\alpha} \;\Rightarrow\; \alpha \approx 0.2$

Choose ω_m such that $|L_u(j\omega_m)|\times|C(j\omega_m)| = 1 \qquad |L_u(j\omega_m)|\times\dfrac{1}{\sqrt{\alpha}} = 1 \;\Rightarrow\; |L_u(j\omega_m)| = \sqrt{\alpha}$

$$\frac{1000}{\omega_m\sqrt{\omega_m^2 + 25}} = \sqrt{0.2} \qquad \omega_m^2 \cong 2224 \;\Rightarrow\; \omega_m = 47$$

$$\omega_m^4 + 25\omega_m^2 - 5000000 = 0 \qquad T = \frac{1}{\omega_m\sqrt{\alpha}} = \frac{1}{47\times\sqrt{0.2}} \cong 0.05 \qquad C(s) = \frac{0.05s+1}{0.01s+1} = 5\frac{(s+20)}{(s+100)}$$

4. Plant transfer function of a unity feedback system is $G_p(s) = \dfrac{100}{s(s+10)}$.

We wish to design a controller to meet the following specifications:

i. Velocity error constant k_v = 100.
ii. Phase margin greater than or equal to 45°
iii. Sinusoidal inputs of up to 1 rad/sec to be reproduced with less than or equal to 2% error.
iv. Sinusoidal inputs with frequency greater than 100 rad/sec to be attenuated at the output to less then 5% of their input amplitudes.

Explain why a single 1st-order compensator cannot meet all the specifications.

Solution:

Spec (i) requires a type 1 system. Plant itself is type 1. We find the required gain to meet the required error constant.

$$k_v = \lim_{s\to 0} s\frac{100K}{s(s+10)} = 10K \qquad 10K = 100 \quad \Rightarrow \quad K = 10 \qquad L_u(s) = \frac{1000}{s(s+10)}$$

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(iii) Check sensitivity specification

$$S_u(s) = \frac{1}{1+L_u(s)} = \frac{s^2+10s}{s^2+10s+1000}$$

$$S_u(j\omega) = \frac{-\omega^2 + j10\omega}{-\omega^2 + j10\omega + 1000}$$

$$S_u(j1) = \frac{-1+j10}{999+j10} \quad \Rightarrow \quad |S_u(j1)| = \frac{\sqrt{1+10^2}}{\sqrt{999^2+10^2}} = 0.01$$

Error for sinusoidal input of up to 1 rad/s will be at the most 1%. Required specification is met.

We design a lead compensator to achieve this PM. Find gain-crossover of the uncompensated loop by solving

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Find the loop gain requirement for specification (iv):
ε_T should be less than or equal to 0.05 for frequency 100 rad/s and above. This requires,

$$|L(j\omega)| \le \frac{\varepsilon_T}{1+\varepsilon_T} \quad \Rightarrow \quad |L(j\omega)| \le 0.05, \quad for \quad \omega \ge 100$$

Find gain of the uncompensated loop at 100 rad/s,

$$|L_u(j100)| = \frac{1000}{100\sqrt{100^2+10^2}} = 0.099$$

This is greater than the gain required (0.05) for ω=100 and higher. If a lead compensator is used to improve PM, the gain around these frequencies will be further increased. So a lead compensator will violate the specification (iv).

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Refer to slide #17 of Chapter 4

Lead compensator amplifies in frequency above compensator zero

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Can we use a lag compensator to improve phase margin? To achieve 45° PM, we first find ω_x such that $\angle L_u(j\omega_x)$ = -135°. An allowance of 5° is included while finding this value. $-90° - \tan^{-1}\frac{\omega_x}{10} = -130° \quad \Rightarrow \quad \omega_x = 8.4$ The corner frequency of compensator zero should be chosen at $\omega_x/10$ or lower. $\frac{1}{T} = 0.84 \quad \Rightarrow \quad T \cong 1.2$ Uncompensated gain at ω=0.84 is $\lvert L_u(j8.4)\rvert = \frac{1000}{8.4\sqrt{(8.4)^2+10^2}} = 9.1$ Compensator parameter α=9.1. Compensator, $C(s) = \frac{1.2s+1}{10.9s+1}$ Compensator gain at ω = 1 rad/s, $\lvert C(j1)\rvert = \frac{\sqrt{(1.2)^2+1}}{\sqrt{(10.9)^2+1}} = 0.14$	Allowance is required as lag compensator adds small negative phase at the frequency where gain reduction is targeted.
As a result, gain of the compensated loop at 1 rad/s will be lower by almost a factor of 10 causing increase in sensitivity at that frequency.	You can find the sensitivity gain after compensation by taking L(s) = L_u(s)C(s). The sensitivity gain at ω=1 is 0.07, much higher than 0.02 required by spec (iii).