Tutorial 1
Laplace Transform problems:
(e)

$$\ddot{y} - 2\dot{y} + 4y = 0 \quad , y(0) = 1 , \dot{y}(0) = 2$$

L.T., $\left[s^2Y(s) - sy(0) - \dot{y}(0)\right] - 2\left(sY(s) - y(0)\right) + 4Y(s) = 0$

$$\Rightarrow Y(s) = \frac{s}{s^2-2s+4} = \frac{s}{(s-1)^2+3}$$

$$= \frac{(s-1)+1}{(s-1)^2+(\sqrt{3})^2} = \frac{(s-1)}{(s-1)^2+(\sqrt{3})^2} + \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{(s-1)^2+(\sqrt{3})^2}$$

$$y(t) = e^t\cos\sqrt{3}t + \frac{1}{\sqrt{3}}e^t\sin\sqrt{3}t$$

(f)

$$\ddot{y} + y = t \quad , y(0) = 1 , \dot{y}(0) = -1$$

L.T., $s^2Y(s) - sy(0) - \dot{y}(0) + Y(s) = \frac{1}{s^2}$

$$\Rightarrow Y(s) = \frac{s^3-s^2+1}{s^2(s^2+1)} = \frac{1}{s^2} + \frac{s}{s^2+1} - 2\frac{1}{s^2+1}$$

$$y(t) = t + \cos t - 2\sin t$$

(g)

$$\begin{aligned}
\mathcal{L}\{tg(t)\} &= -\frac{d}{ds}G(s) \\
\mathcal{L}\{\sin at\} &= \frac{a}{s^2+a^2} \\
\mathcal{L}\{\cos at\} &= \frac{s}{s^2+a^2} \\
\mathcal{L}\{f(t)\} &= \mathcal{L}\{t\sin 3t\} - 2\mathcal{L}\{t\cos t\} \\
&= -\frac{d}{ds}\frac{3}{s^2+9} - 2\left(-\frac{d}{ds}\frac{s}{s^2+1}\right) \\
&= \frac{-(2s*3)}{(s^2+9)^2} - 2\frac{((s^2+1)-(2s)s)}{(s^2+1)^2} \\
&= \frac{-6s}{(s^2+9)^2} + \frac{2(s^2-1)}{(s^2+1)^2}
\end{aligned}$$

$$\begin{aligned}
f(t) &= \int_0^t \cos(t-\tau)\sin\tau d\tau \\
\mathcal{L}\{f(t)\} &= \mathcal{L}\{\int_0^t \cos(t-\tau)\sin\tau d\tau\} = \mathcal{L}\{\cos(t)*\sin(t)\}
\end{aligned}$$

This is just the definition of the convolution theorem,

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{s}{s^2+1}\frac{1}{s^2+1} \\
&= \frac{s}{s^4+2s^2+1}
\end{aligned}$$

Q1

$$u(t) = t \rightarrow U(s) = \frac{1}{s^2}$$

$$y(t) = e^{-t} - \frac{1}{4}e^{-2t} - \frac{3}{4} + \frac{1}{2}t \Rightarrow \underbrace{\frac{1}{s+1} - \frac{1}{4}\frac{1}{s+2} - \frac{3}{4}\frac{1}{s} + \frac{1}{2}\frac{1}{s^2}}$$

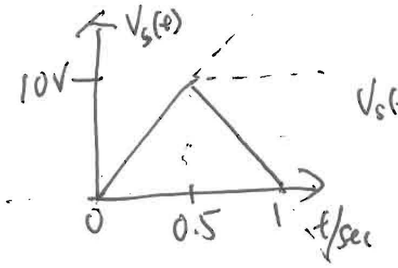
$$Y(s) = \frac{1}{s^2(s+1)(s+2)}$$

$$G(s) = Y(s)/U(s) = \frac{1}{(s+1)(s+2)}$$

Q2

$$\frac{V_o(s)}{V_s(s)} = \frac{R_1 + \frac{1}{sC_1}}{R_1 + \frac{1}{sC_1} + \left(\frac{1}{R_2} + sC_2\right)^{-1}}$$

$$= \frac{(R_1C_1s+1)(R_2C_2s+1)}{(R_1C_1s+1)(R_2C_2s+1) + R_2C_1s} = \boxed{\frac{s+1}{2s+1}} = G(s)$$




$$V_s(t) = 20t\,u(t) - 40(t-0.5)u(t-0.5) + 20(t-1)u(t-1)$$

$$V_s(s) = \frac{20}{s^2} - \frac{40}{s^2}e^{-0.5s} + \frac{20}{s^2}e^{-s}$$

$$= \frac{20}{s^2}\left(1 - 2e^{-0.5s} + e^{-s}\right)$$

$$V_o(s) = G(s)V_s(s)$$

$$= \frac{20(s+1)}{s^2(2s+1)}\left(1 - 2e^{-0.5s} + e^{-s}\right)$$

$$= \left(\frac{20}{s^2} - \frac{20}{s} + \frac{20}{s+0.5}\right)\left[1 - 2e^{-0.5s} + e^{-s}\right]$$

$$V_o(t) = 20(t - 1 + e^{-0.5t})u(t) - 40\left(t - 0.5 - 1 + e^{-0.5(t-0.5)}\right)u(t-0.5)$$

$$+ 20\left(t - 1 - 1 + e^{-0.5(t-1)}\right)u(t-1)$$

$$V_o(1) = 0.98\,V$$

Q3

$$G(s) = \frac{5}{s^2+9s-10}$$

a) DC gain, $G(0) = -\frac{5}{10} = -0.5$

b) System unstable $\because$ $s^2+9s-10=0$

$\Rightarrow (s+10)(s-1)=0$

$s = -10, 1 \rightarrow$ unstable.

$\Rightarrow$ Final value $= \infty$

Q4

$$G(s) = \frac{w(s)}{V(s)} = \frac{K}{s\tau+1}$$

• DC gain, $G(0) = K = \left.\frac{o/p}{i/p}\right|_{\text{at steady-state}} = \frac{70}{5} = 14$

• $w(s) = G(s)V(s) = \frac{K}{s\tau+1}\cdot\frac{5}{s}$

$= \frac{5K}{s} - \frac{5K\tau}{s\tau+1}$

$\Rightarrow w(t) = 5K - 5K e^{-t/\tau}$

At $t = 2\,\text{sec}$, $w(2) = 5K - 5K e^{-2/\tau} = 30$

$\Rightarrow \tau = 3.57$ sec

Q5

(a) Step response of $G_1(s)$ increases monotonically to the steady-state value i.e. no oscillations and $\frac{dy(t)}{dt}|_{t=0} \neq 0$. Hence, $G_1(s)$ is a first-order plus dead-time system and the step response is

$$\begin{aligned}
y_{step}(t) &= \left[K - Ke^{-\frac{t-t_d}{\tau}}\right] U(t-t_d) \\
Y_{step}(s) &= \mathcal{L}\{y_{step}(t)\} \\
&= \left[\frac{K}{s} - \frac{K}{s+\frac{1}{\tau}}\right] e^{-st_d} \\
&= \frac{K}{s(\tau s+1)} e^{-st_d}
\end{aligned}$$

Since input is a unit step function, $U(s) = \frac{1}{s}$ and $Y_{step}(s) = G(s)U(s)$. Hence,

$$G_1(S) = \frac{K}{\tau s+1} e^{-st_d}$$

Let $t' = t - 0.25$, then $y_{step}(t') = [K - Ke^{-\frac{t'}{\tau}}]U(t')$. When $t' = 0$,

$$\left.\frac{\mathrm{d}y_{step}(t')}{\mathrm{d}t'}\right|_{t'=0} = \frac{K}{\tau}$$

Q5 (continued)

$$\begin{aligned}\lim_{t\to\infty} y_{step}(t) &= K = 3 \\ \left.\frac{\mathrm{d}y_{step}(t')}{\mathrm{d}t'}\right|_{t'=0} &= \frac{K}{\tau} = 6 \qquad \Longrightarrow \tau = 0.5\end{aligned}$$

Comparing with $G_s(s)$ with $G_i(s) = \dfrac{K}{as^2 + bs + c}e^{-sL}$,

$$K = 3,\ a = 0,\ b = 0.5,\ c = 1,\ L = 0.25$$

(b) Step response, $y_{step}(t) = 4(t - 0.5)U(t - 0.5)$

$$\begin{aligned}Y_{step}(s) &= \mathcal{L}\{y_{step}(t)\} \\ &= \frac{4}{s^2}e^{-0.5s}\end{aligned}$$

Since $Y_{step}(s) = \frac{G(s)}{s}$. Hence,

$$G_2(s) = \frac{4}{s}e^{-0.5s}$$

Comparing with $G_2(s)$ with $G_i(s) = \dfrac{K}{as^2 + bs + c}e^{-sL}$,

$$K = 4,\ a = 0,\ b = 1,\ c = 0,\ L = 0.5$$

(c) Impulse response, $y_{impulse}(t) = 4(t - 0.5)U(t - 0.5)$

$$\begin{aligned}Y_{impulse}(s) &= \mathcal{L}\{y_{impulse}(t)\} \\ &= \frac{4}{s^2}e^{-0.5s}\end{aligned}$$

When input is a unit impulse function, $U(s) = 1$ and $Y_{impulse}(s) = G_3(s)U(s) = G_3(s)$. Hence,

$$K = 4,\ a = 1,\ b = 0,\ c = 0,\ L = 0.5$$

(d) Impulse response, $y_{impulse}(t) = 1.5U(t - 1)$

$$\begin{aligned}Y_{impulse}(s) &= \mathcal{L}\{y_{impulse}(t)\} \\ &= \frac{1.5}{s}e^{-s}\end{aligned}$$

Since $Y_{impulse}(s) = G_4(s)U(s) = G_4(s)$. Hence,

$$K = 1.5,\ a = 0,\ b = 1,\ c = 0,\ L = 1$$

Q6

$$G(s) = \frac{K}{s^2+bs+c} \quad \text{compare with} \quad \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

- DC gain, $G(0) = \frac{K}{c} = \frac{o/p}{i/p} = \frac{1}{0.5} = 2$
- $M_p = (\text{step magnitude}) \times (\text{DC gain})\, e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.52$

$\Rightarrow 0.5\left(\frac{K}{c}\right) e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.52$

$\Rightarrow \zeta = 0.2038$

$$\boxed{\begin{aligned} & e^{-\pi\zeta/\sqrt{1-\zeta^2}} = \alpha \\ \Rightarrow\ & -\pi\zeta/\sqrt{1-\zeta^2} = \ln(\alpha) \\ & \pi^2\zeta^2/(1-\zeta^2) = (\ln\alpha)^2 \\ \Rightarrow\ & \zeta = \sqrt{\frac{(\ln\alpha)^2}{\pi^2+(\ln\alpha)^2}} \end{aligned}}$$

- Period of signal, $T_p = 1.78 - 1.14 = 0.64$ sec

$\omega_d = 2\pi/T_p = 9.82$

$\because \omega_d = \omega_n\sqrt{1-\zeta^2} \Rightarrow \omega_n = \omega_d/\sqrt{1-\zeta^2}$

$\approx 10$ rad/sec

$\Rightarrow c = \omega_n^2 = 100$

$b = 2\zeta\omega_n = 4.1$

$\frac{K}{c} = 2 \Rightarrow K = 200$

Q7

$\%M_p = 10\%$; $t_s = 0.5$ sec → $s = -8 \pm j10.9$

$\%M_p = 15\%$; $t_p = 0.25$ sec → $s = -7.6 \pm j12.6$

$t_r = 1$ sec; $t_s = 5$ sec → $s = -0.8 \pm j1.6$

Q9

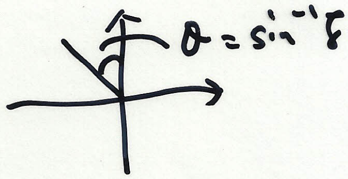
| | DC gain | $\omega_n$ | $\zeta$ | |
|---|---|---|---|---|
| $G_a$ | 0.5 | 2 | 0.2 | → IV |
| $G_b$ | 1 | 1 | 0.2 | → I |
| $G_c$ | 1 | 2 | 0.2 | → II |
| $G_d$ | 0.5 | 1 | 0.2 | → III |
| $G_e$ | 0.5 | 1 | 0.7 | → V |

Q10

- $\% M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$

  5% $\rightarrow \xi = 0.69 \rightarrow 43.6°$

  10% $\rightarrow \xi = 0.59 \rightarrow 36.1°$

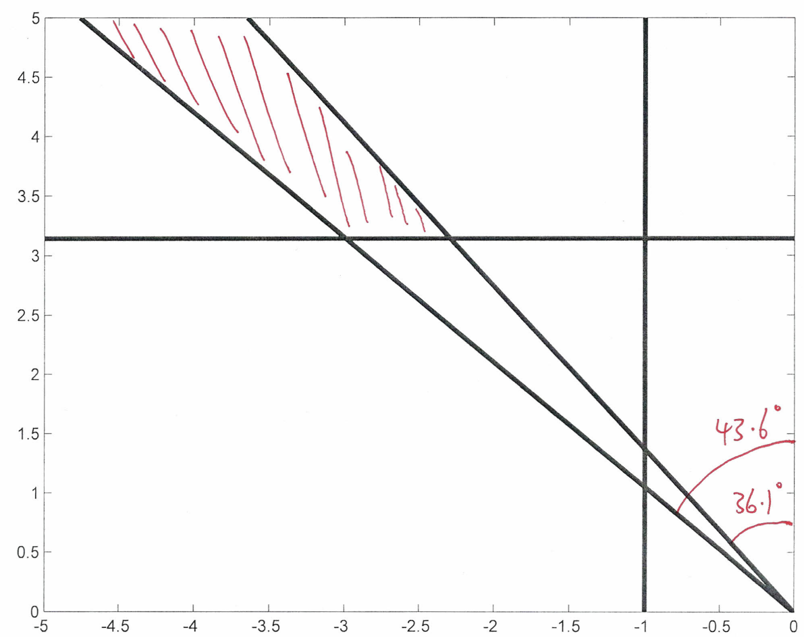



- 2% $t_s < 4$ sec

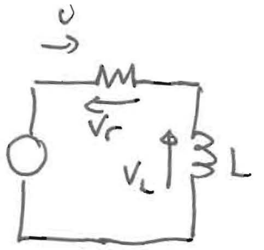
  $\Rightarrow \frac{4}{\xi\omega_n} < 4 \Rightarrow \xi\omega_n > 1$

- $t_p < 1$ sec

  $\frac{\pi}{\omega_d} < 1 \Rightarrow \omega_d > \pi$



Q11: Linearization

a) 


KVL, $v_r(t) + v_L(t) = v(t)$

$$v_r(t) + L\frac{di(t)}{dt} = v(t) \quad \text{— (A)}$$

(1) At the operating point, at s.s.

$$v_r = v$$

$$\underline{10\ln\left(\frac{i}{2}\right) = v_0 = 20} \Rightarrow \underline{i_0 = 14.78\,A}$$

(2) Define $\delta i$ & $\delta v$

$$i = i_0 + \delta i, \quad v = v_0 + \delta v$$

(A)

$$L\frac{d(i_0+\delta i)}{dt} + 10\ln\left(\frac{i}{2}\right) = v_0 + \delta v \quad \text{— (B)}$$

$$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$$

(3) linearize $\ln\left(\frac{i}{2}\right)$

$$\ln\left(\frac{i}{2}\right) = \ln\left(\frac{i_0}{2}\right) + \underbrace{\frac{d(\ln\frac{i}{2})}{di}\Big|_{i=i_0}}_{\frac{1/2}{i/2}\big|_{i=i_0}} \delta i$$

$$\frac{1/2}{i/2}\Big|_{i=i_0}\delta i = \frac{\delta i}{i_0}$$

$$\Rightarrow \ln\left(\frac{i}{2}\right) = \ln\left(\frac{i_0}{2}\right) + \frac{\delta i}{i_0}$$

(B)

(4)

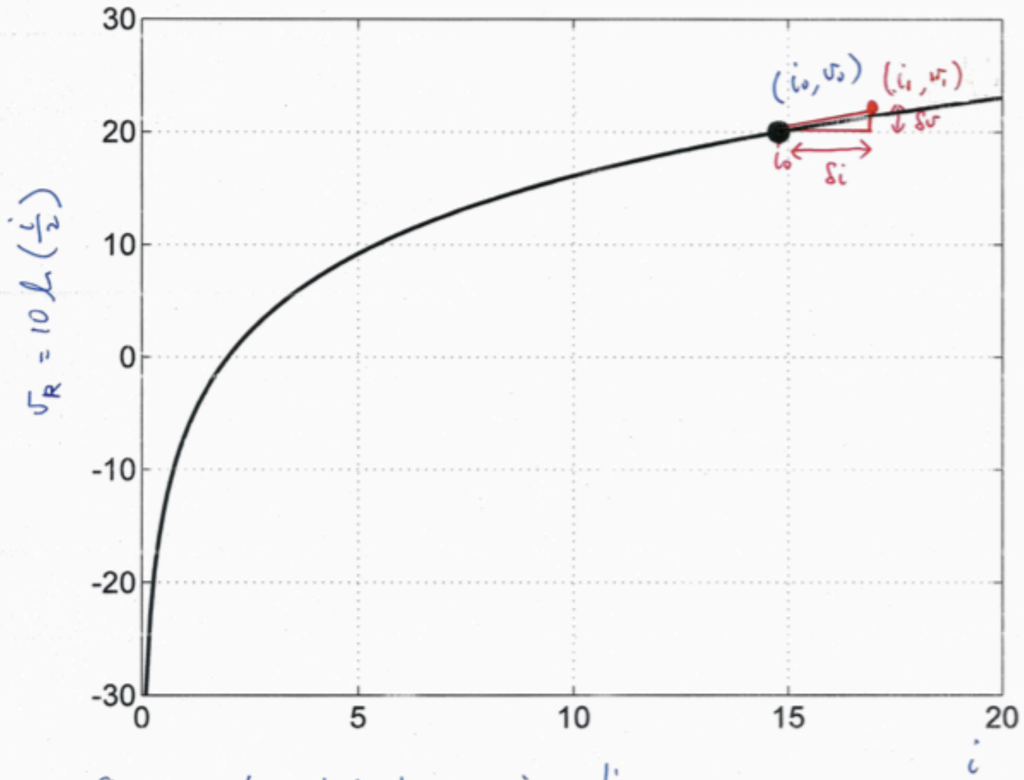
$$L\frac{d(i_0+\delta i)}{dt} + \cancel{10\ln\left(\frac{i_0}{2}\right)} + 10\frac{\delta i}{i_0} = \cancel{v_0} + \delta v$$

$$L\frac{d\delta i}{dt} + \frac{10}{i_0}\delta i = \delta v$$

Take L.T.,

$$\frac{\delta I(s)}{\delta V(s)} = \frac{1}{Ls + 10/i_0}$$

*i* – *v* curve for nonlinear resistor




① It is clear that the curve is nonlinear.

② Our objective is to find a linear equation to describe the curve around the operating point $(i_0, v_0)$.

$v_0 = 10\ln\left(\frac{i_0}{2}\right)$

③ To write an equation, we need at least 2 points: $(i_0, v_0)$ and $(i_1, v_1)$

$v_1 = 10\ln\left(\frac{i_0+\delta i}{2}\right)$

$i_1 = i_0 + \delta i$

⇒ The slope of the red line is given by:

$$\frac{d\left(10\ln\left(\frac{i}{2}\right)\right)}{di} = \frac{v_1 - v_0}{i_1 - i_0}$$

$$= \frac{10\ln\left(\frac{i_0+\delta i}{2}\right) - 10\ln\left(\frac{i_0}{2}\right)}{\delta i}$$

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