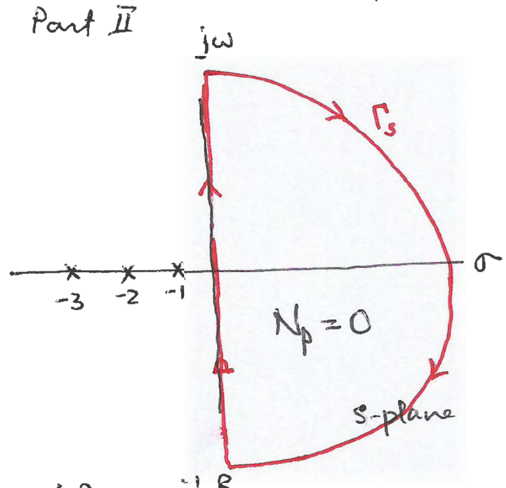
Tutorial 3

EE3331 Feedback Control Systems
Part II



1. i) $$L(s) = \frac{50}{(s+1)(s+2)(s+3)}$$

$$L(0) = \frac{50}{(0+1)(0+2)(0+3)} = \frac{25}{3}\angle 0^\circ$$

$$L(jR)_{R\to\infty} = \frac{50}{(1+jR)(2+jR)(3+jR)}$$

$$= \frac{50}{\sqrt{R^2+1}\sqrt{R^2+4}\sqrt{R^2+9}} \angle -\tan^{-1}\frac{R}{1} - \tan^{-1}\frac{R}{2} - \tan^{-1}\frac{R}{3}$$

$$\approx 0 \angle(-270^\circ + \delta) \quad \text{as } R\to\infty$$

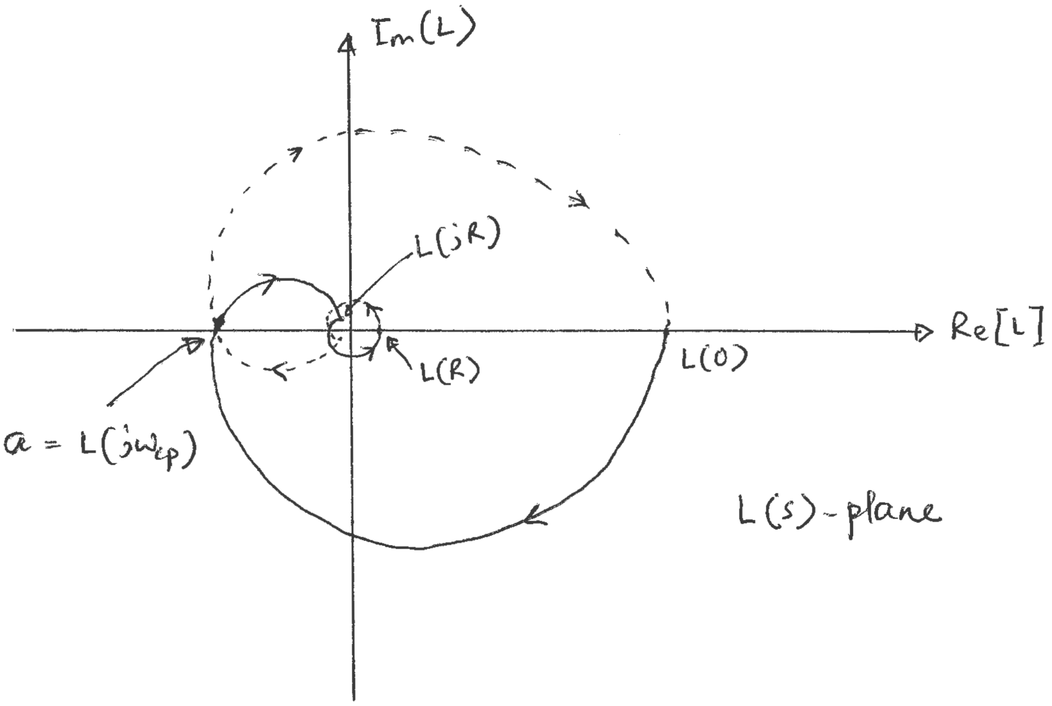
$$L(R) = \frac{50}{(1+R)(2+R)(3+R)} \to 0\angle 0^\circ \quad \text{as } R\to\infty$$

$$\angle L(j\omega) = -\tan^{-1}\frac{\omega}{1} - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{3}$$

So, $0^\circ > \angle L(j\omega) > -270^\circ$ for $0 < \omega < \infty$

$L(j\omega)$ plot lies in the 4^{th}, the 3^{rd} and the 2^{nd} quadrant of the $L(s)$-plane

$|L(j\omega)|$ decreases with increasing value of ω



Encirclement of $(-1,0)$ depends on the value of $a = L(j\omega_{cp})$

$$L(s) = \frac{50}{(s+1)(s+2)(s+3)} = \frac{50}{s^3+6s^2+11s+6}$$

$$L(j\omega) = \frac{50}{(6-6\omega^2)+j\omega(11-\omega^2)}$$

$L(j\omega_{cp})$ is negative real number. Therefore $Im[L(j\omega_{cp})] = 0$

$$\omega_{cp}^2 = 11 \Rightarrow \omega_{cp} = \sqrt{11}$$

$$a = L(j\omega_{cp}) = \frac{50}{6-6\times 11} = \frac{50}{-60} = -\frac{5}{6}$$

as $|a| < 1$ the point $(-1,0)$ is not encircled by the Nyquist plot. So, $N_{cw} = 0$

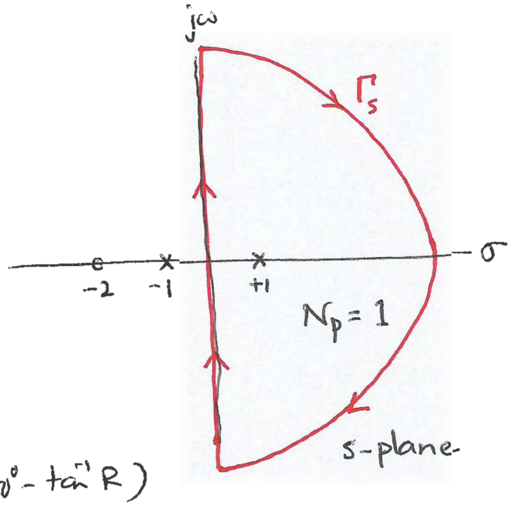
$L(s)$ has no RHP pole. So, $N_p = 0$

$\therefore$ $N_z = 0$ closed loop is stable.

Note: As we are interested to know encirclement of $(-1,0)$, arc of infinitesimally small radius around the origin is insignificant. We can ignore mapping of the big arc of the Nyquist contour.

You can verify that closed loop poles are
-5.77, -0.11+j3.11 and -0.11-j3.11

ii) $$L(s) = \frac{s+2}{(s+1)(s-1)}$$



$$L(0) = \frac{0+2}{(0+1)(0-1)} = -2 = 2\angle \pm 180^\circ$$

$$L(jR)_{R\to\infty} = \frac{2+jR}{(1+jR)(-1+jR)}$$

$$= \frac{\sqrt{4+R^2}\ \angle \tan^{-1}\frac{R}{2}}{\left(\sqrt{1+R^2}\ \angle \tan^{-1}R\right)\left(\sqrt{1+R^2}\ \angle(180^\circ - \tan^{-1}R\right)}$$

$$= \frac{\sqrt{4+R^2}}{1+R^2}\ \angle\left(\tan^{-1}\frac{R}{2} - 180^\circ\right)$$

$$\approx 0\angle -90^\circ \quad \text{as } R\to\infty$$

$$L(R) = \frac{R+2}{(R+1)(R-1)} \to 0\angle 0^\circ \quad \text{as } R\to\infty$$

As we are interested to know encirclement of the point (-1,0), arcs of infinitesimally small radius around the origin is insignificant. We can ignore mapping of the big semicircle.

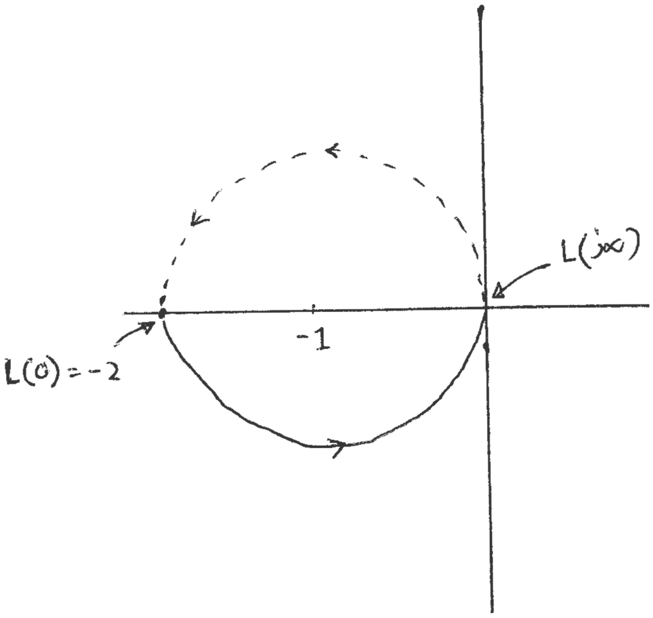
$$L(j\omega) = \frac{2+j\omega}{(1+j\omega)(-1+j\omega)}$$

$$= \frac{\sqrt{4+\omega^2}\ \angle \tan^{-1}\frac{\omega}{2}}{\left(\sqrt{1+\omega^2}\ \angle \tan^{-1}\omega\right)\left(\sqrt{1+\omega^2}\ \angle \cancel{\tan}(180^\circ - \tan^{-1}\omega\right)}$$

$$= \frac{\sqrt{4+\omega^2}}{1+\omega^2}\ \angle\left(\tan^{-1}\frac{\omega}{2} - 180^\circ\right)$$

With increasing values of ω, $\tan^{-1}\frac{\omega}{2}$ approaches 90°.

So, $-180^\circ < \angle L(j\omega) < -90^\circ$. So, the plot of $L(j\omega)$ lies in the 3rd Quadrant of the $L(s)$ plane.



$N_{cw} = -1$ One ccw encirclement of $(-1, 0)$

$N_p = 1$ There is a pole at +1 for $L(s)$

$N_{cw} = N_z - N_p$

$N_z = 0$

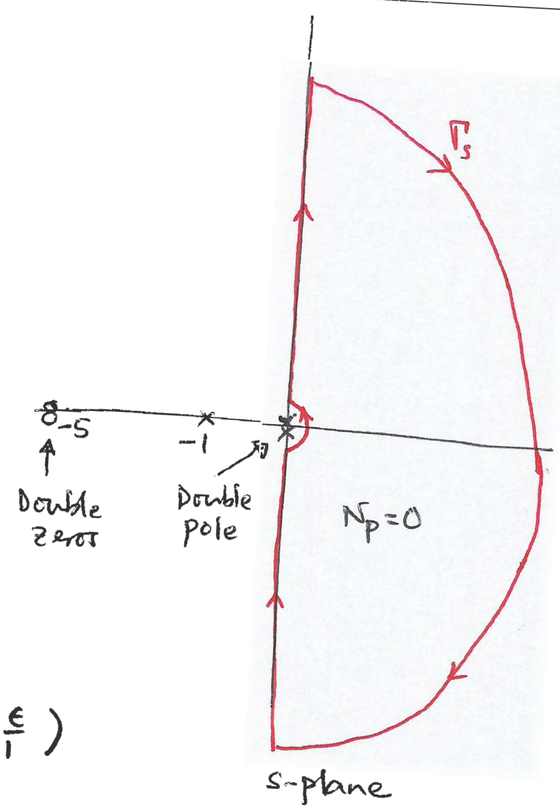
Closed loop is stable.

You can verify that closed loop poles are
-0.5+j0.87 and -0.5-j0.87

Q.1 (iii)

$$L(s) = \frac{(s+5)^2}{s^2(s+1)}$$

$$\underset{\epsilon \approx 0^+}{L(\epsilon)} = \frac{(5+\epsilon)^2}{\epsilon^2(1+\epsilon)} \approx \infty \angle 0^\circ \quad \text{as } \epsilon \to 0$$



$$L(j\epsilon) = \frac{(5+j\epsilon)^2}{(j\epsilon)^2(1+j\epsilon)}$$

$$= \frac{(25+\epsilon^2)\angle 2\times \tan^{-1}\frac{\epsilon}{5}}{(\epsilon^2\angle +180^\circ)(\sqrt{1+\epsilon^2}\angle \tan^{-1}\frac{\epsilon}{1})}$$

$$= \frac{25+\epsilon^2}{\epsilon^2\sqrt{1+\epsilon^2}} \angle\left(2\times\tan^{-1}\frac{\epsilon}{5} - 180^\circ - \tan^{-1}\frac{\epsilon}{1}\right)$$

$$\approx \infty \angle -180^\circ - \delta$$

$$L(jR) = \frac{(5+jR)^2}{(jR)^2(1+jR)}$$

$$= \frac{25+R^2}{R^2\sqrt{1+R^2}} \angle\left(2\times\tan^{-1}\frac{R}{5} - 180^\circ - \tan^{-1}R\right)$$

as $R \to \infty$ $\quad L(jR) \approx 0\angle -90^\circ$

$$L(j\omega) = \frac{(5+j\omega)^2}{(j\omega)^2(1+j\omega)}$$

$$= \frac{(25-\omega^2)+j10\omega}{-\omega^2(1+j\omega)} \times \frac{(1-j\omega)}{(1-j\omega)}$$

$$= \frac{(25+9\omega^2) - j\omega(15-\omega^2)}{-\omega^2(1+\omega^2)}$$

$$= -\frac{25+9\omega^2}{\omega^2(1+\omega^2)} + j\frac{(15-\omega^2)}{\omega(1+\omega^2)}$$

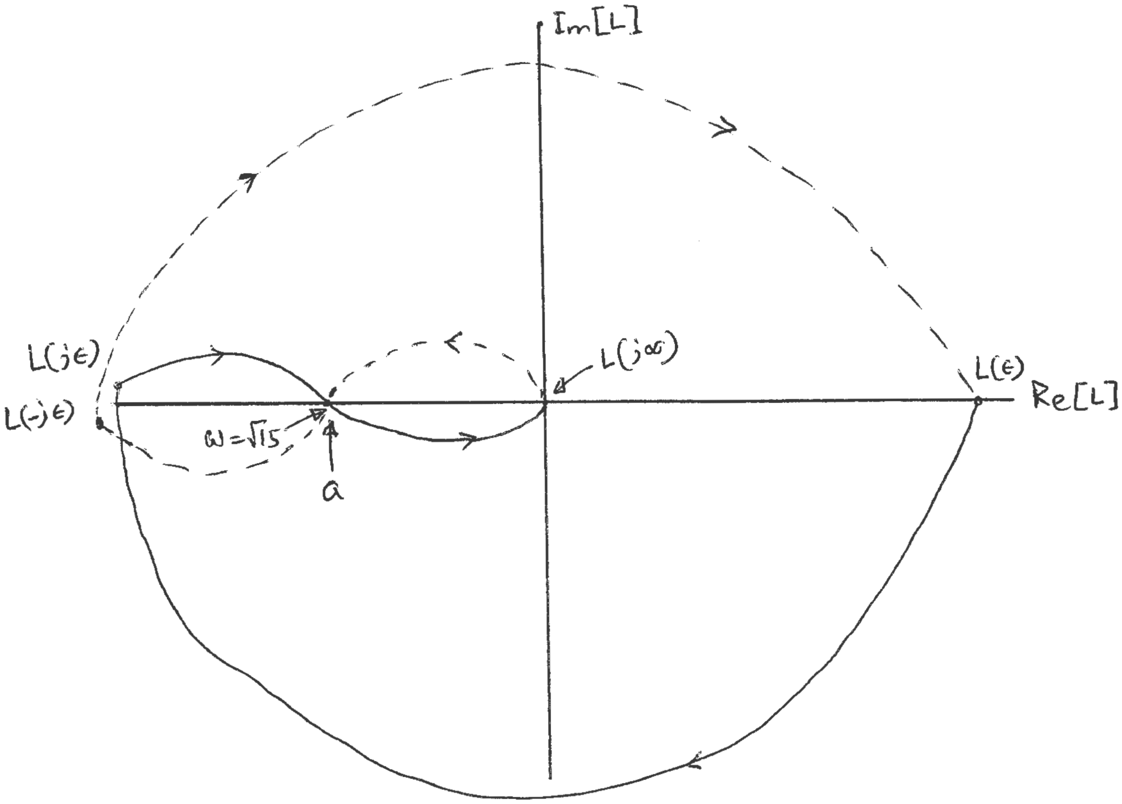
$\mathrm{Im}[L] = 0 \Rightarrow 15-\omega^2 = 0 \quad$ i.e. $\omega = \sqrt{15}$

$\omega_{cp} = \sqrt{15}$ rad/s

For all $\omega > 0$, $\mathrm{Re}[L] < 0. \Rightarrow L(j\omega)$ lies on the left of Imaginary axis

For $0 < \omega < \sqrt{15}$, $\mathrm{Im}[L] > 0$

For $\omega > \sqrt{15}$, $\mathrm{Im}[L] < 0$



$$a = L(j\sqrt{15}) = -\frac{25+9\times 15}{15(1+15)} \approx -0.67$$

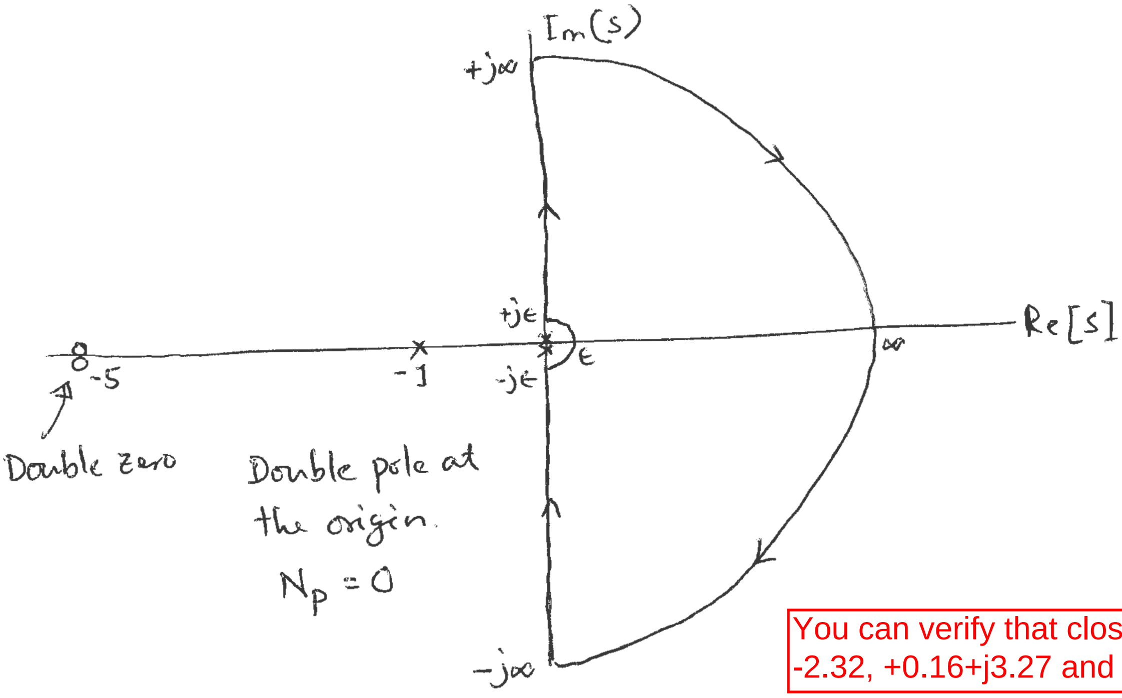
The point (-1,0) lies on the left of the point 'a'.

$N_{cw} = 2$

$N_p = 0$ ⟶ The modified D-contour used is shown below.

$N_z = 2$

Unstable CL



You can verify that closed loop poles are -2.32, +0.16+j3.27 and +0.16-j3.27

Q.2 (i) Find the intersection of $L(s)$ plot with negative real axis.

$a = L(j\omega_{cp})$. $\qquad L(s) = \dfrac{K}{s^3 + 4s^2 + 9s + 10}$

$\angle L(j\omega_{cp}) = -180^\circ \qquad L(j\omega) = \dfrac{K}{(10 - 4\omega^2) + j\omega(9 - \omega^2)}$

$\Downarrow$

$Im[L(j\omega_{cp})] = 0$

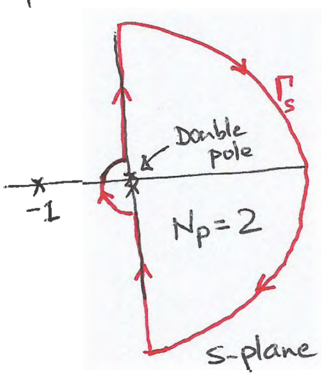
$\omega_{cp} = \sqrt{9} = 3$ rad/s. $\qquad L(j\omega_{cp}) = \dfrac{K}{10 - 4\times 9} = \dfrac{K}{-26}$

If $K > 26$, $\quad N_{cw} = +2, \quad N_p = 0, \quad N_z = 2 \quad \rightarrow$ Unstable with two RHP poles.

If $K < 26$, $\quad N_{cw} = 0, \quad N_p = 0, \quad N_z = 0 \longrightarrow$ Stable CL

2 (ii) The modified D-Contour encloses the double pole of $L(s)$.

So, $N_p = 2$



From the Nyquist plot, $\quad N_{cw} = \cancel{2}\ 0$

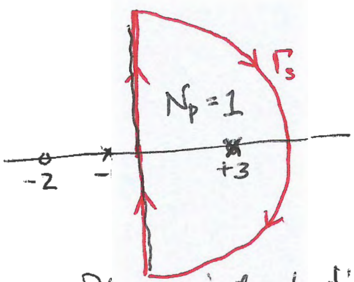
$N_{cw} = N_z - N_p \Rightarrow N_z = N_{cw} + N_p$

$= \cancel{2} + 2$

$= \cancel{4}\ 2$

~~CL is unstable.~~ CL is unstable for all $K > 0$.

2 (iii) $\qquad L(s) = K \dfrac{s+2}{(s+1)(s-3)}$



$L(0) = -\dfrac{2K}{3}$

Plot intersects the negative real axis at another point. Let's find it out.

$L(j\omega) = K \dfrac{2 + j\omega}{(1 + j\omega)(-3 + j\omega)}$

$\angle L(j\omega) = \tan^{-1}\frac{\omega}{2} - \tan^{-1}\omega - [180^\circ - \tan^{-1}\frac{\omega}{3}]$

$= -180^\circ + \tan^{-1}\frac{\omega}{2} + \tan^{-1}\frac{\omega}{3} - \tan^{-1}\omega$

$\angle L(j\omega_{cp}) = -180^\circ \Rightarrow \tan^{-1}\frac{\omega_{cp}}{2} + \tan^{-1}\frac{\omega_{cp}}{3} = \tan^{-1}\omega_{cp}$

$\dfrac{\frac{\omega_{cp}}{2} + \frac{\omega_{cp}}{3}}{1 - \frac{\omega_{cp}^2}{6}} = \omega_{cp} \Rightarrow \dfrac{5\omega_{cp}}{6 - \omega_{cp}^2} = \omega_{cp}$

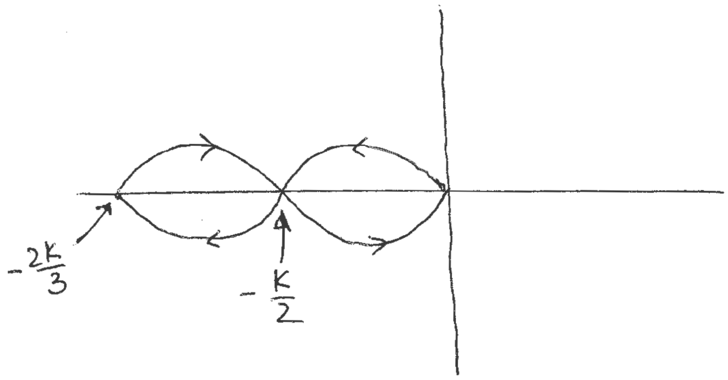
$$\frac{5}{6-\omega_{cp}^2} = 1.$$

$$\omega_{cp}^2 = 1. \qquad \omega_{cp} = 1.\ \text{rad/s}$$

$$|L(j\omega_{cp})| = K\frac{\sqrt{4+\omega_{cp}^2}}{\sqrt{1+\omega_{cp}^2}\sqrt{9+\omega_{cp}^2}}$$

$$= K\frac{\sqrt{4+1}}{\sqrt{1+1}\ \sqrt{9+1}}$$

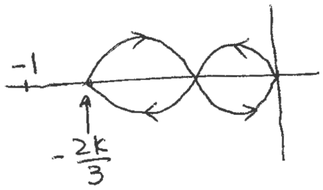
$$= K\frac{\sqrt{5}}{\sqrt{20}} = \frac{K}{2}$$



$$L(s) = K\frac{s+2}{(s+1)(s-3)}$$

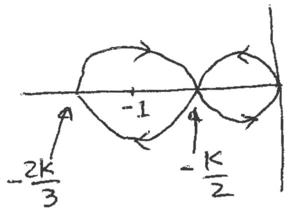
$N_p = 1.$ (pole at $s=+3$)

If $K < \frac{3}{2}$



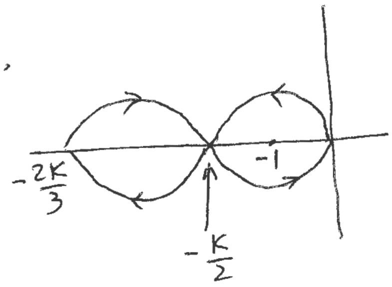
$N_{cw} = 0$
$N_p = 1.$ $\quad N_z = 1$ $\quad$ Unstable with one RHP pole

If $\frac{3}{2} < K < 2,$



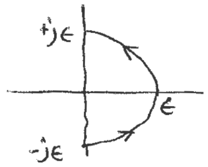
$N_{cw} = +1.$
$N_p = 1.$ $\quad N_z = 2,$ $\quad$ Unstable with two RHP poles

If $K > 2,$



$N_{cw} = -1.$
$N_p = 1,$ $\quad N_z = 0$ $\quad$ Stable CL

3. $L(s) = K \dfrac{(s+1)^2}{s^3}$



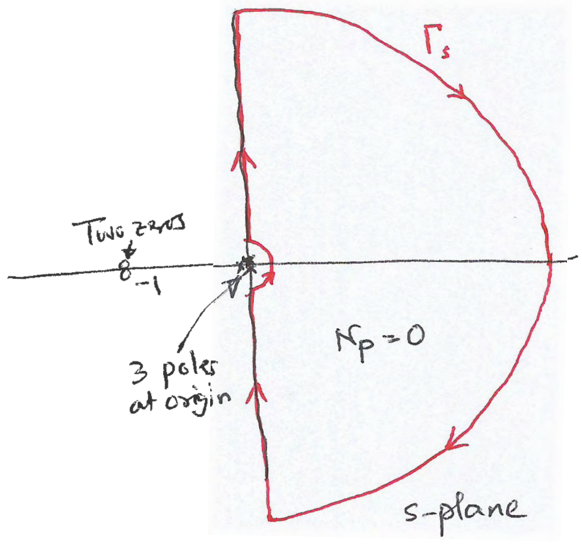
$$L(\epsilon) = K \frac{(1+\epsilon)^2}{\epsilon^3} = \infty \angle 0^\circ$$

$$L(j\epsilon) = K \frac{(1+j\epsilon)^2}{(j\epsilon)^3}$$

$$= K \frac{1+\epsilon^2}{\epsilon^3} \angle -270^\circ + 2 \times \tan^{-1}\epsilon$$

$$|L(j\epsilon)| = \infty \quad \text{as } \epsilon \to 0$$

$$\angle L(j\epsilon) = -270^\circ + \delta \quad \text{as } \epsilon \to 0$$



$$L(jR) = K \frac{(1+jR)^2}{(jR)^3}$$

$$= K \frac{1+R^2}{R^3} \angle -270^\circ + 2 \times \tan^{-1} R$$

$$|L(jR)| = 0 \quad \text{as } R \to \infty$$

$$\angle L(jR) = -270^\circ + 2 \times \tan^{-1} R$$

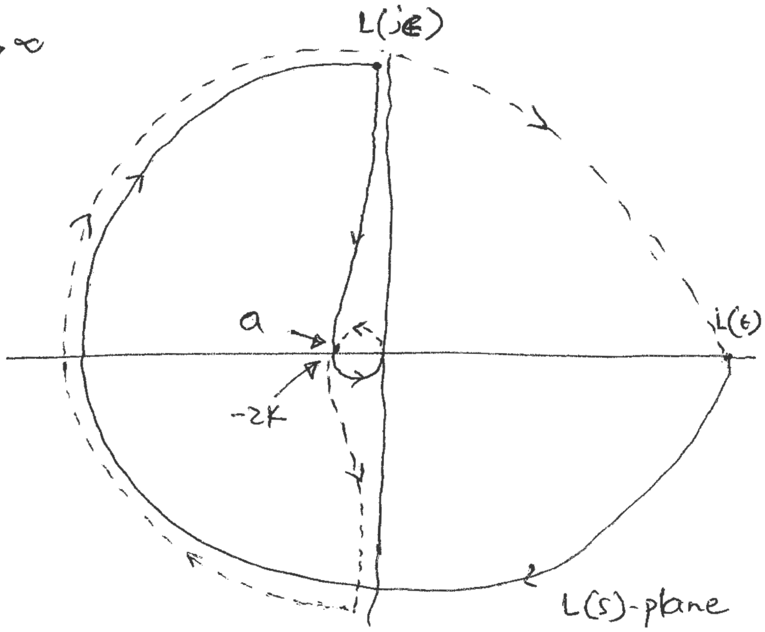
$$= -270 + 180^\circ \quad \text{as } R \to \infty$$

$$= -90^\circ$$

$$L(j\omega) = K \frac{(1+j\omega)^2}{(j\omega)^3}$$

$$\angle L(j\omega) = -270^\circ + 2\tan^{-1}\omega$$

$$-270^\circ \leq \angle L(j\omega) < -90^\circ$$



Find a

$$-270^\circ + 2\tan^{-1}\omega_{cp} = -180^\circ$$

$$2\tan^{-1}\omega_{cp} = +90^\circ$$

$$\tan^{-1}\omega_{cp} = 45^\circ$$

$$\omega_{cp} = 1.$$

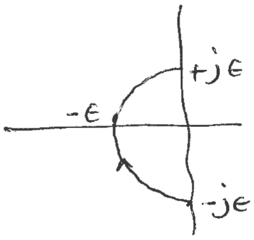
$$L(j\omega_{cp}) = K \frac{(1+j1)^2}{(j1)^3}$$

$$= K \frac{1+j2-1}{-j} = -2K$$

If, $K < 0.5$, $N_{cw} = 2$
$N_z = 2$

If, $K > 0.5$, $N_{cw} = 0$
$N_z = 0$

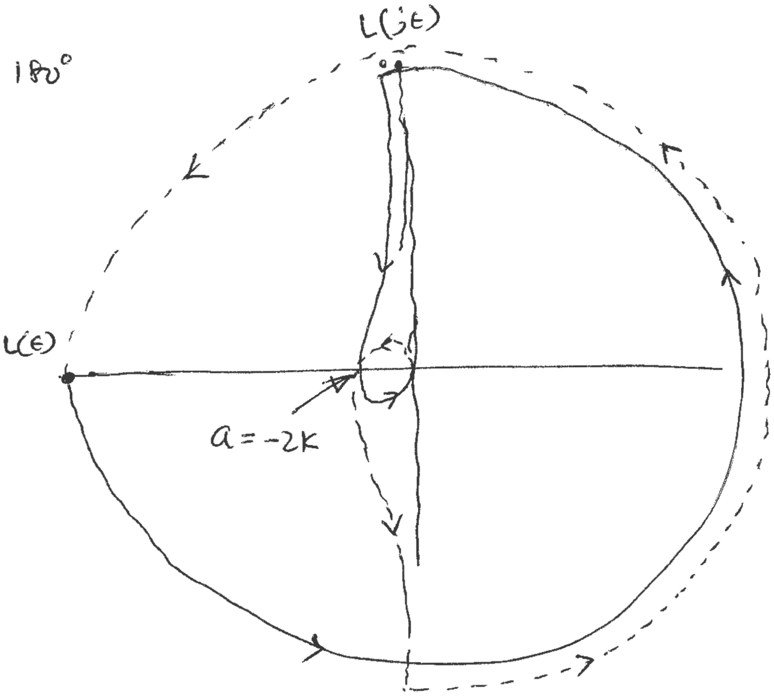
Alternative Γ_s



$$L(s) = K \frac{(1+s)^2}{s^3}$$

$$L(-\epsilon) = K \frac{(1-\epsilon)^2}{-\epsilon^3} = -\infty$$
$$= \infty \angle 180^\circ$$

$$L(j\epsilon) = \infty \angle -270^\circ + \delta$$



$K < 0.5$,

$N_{cw} = -1.$

$N_p = 3$

$N_z = 2$

$K > 0.5$,

$N_{cw} = -3$

$N_p = 3$

$N_z = 0$