EE3331C: Modeling Electrical Circuits

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Physical laws

1. Two basic laws:
   - conservation of charge: Kirchhoff's current law
   - conservation of energy: Kirchhoff's voltage law
2. Voltage-current relationship:
   - Resistor: $v(t) = Ri(t)$
   - Capacitor: $v_c(t) = \dfrac{1}{C}\displaystyle\int i(\tau)d\tau$
     the capacitance C of a capacitor is a measure of how much charge can be stored for a given voltage difference across the element, with units of charge per volt (i.e. farad (F)).
   - Inductor: $v_L(t) = L\dfrac{di(t)}{dt}$
     the unit of inductance L is the henry (H) which is one volt-second per ampere.

   The resistor, capacitor and inductor voltage-current relationships together with the conservation of charge and conservation of energy can be used to obtain circuit models.

Circuit anaylsis with impedances

1. Impedance:
   - idea: resistance resists/'*impedes*' current flow: $\frac{v}{i} = R$
   - capacitance and inductance elements also impede the flow of current
   - impedance is defined as the ratio of a voltage transform $V(s)$ to a current transform $I(s)$, i.e. $Z(s) = \dfrac{V(s)}{I(s)}$ (or the transfer function from current, $I(s)$ to voltage, $V(s)$).
   - note that in time domain, v and i are related by convolution, i.e. $v = z * i$.
   - impedance of resistor is thus: $Z_R(s) = R$.

- impedance of capacitor:

$$
\begin{aligned}
v(t) &= \frac{1}{C}\int_0^t i(\tau)d\tau \\
\text{LT gives} \quad V(s) &= \frac{1}{Cs}I(s) \qquad \text{with zero initial voltage} \\
\Rightarrow \quad Z(s) &= \frac{V(s)}{I(s)} = \frac{1}{Cs}
\end{aligned}
$$

- impedance of inductor:

$$
\begin{aligned}
v(t) &= L\frac{di(t)}{dt} \\
\text{LT gives} \quad V(s) &= LsI(s) \qquad \text{with zero initial current} \\
\Rightarrow \quad Z(s) &= \frac{V(s)}{I(s)} = Ls
\end{aligned}
$$

2. Series and parallel impedances:

- concept of impedance useful, individual elements can be combined in series or parallel law;
- key assumption, initial conditions for inductor and capacitor are zero;
- two or more impedances are in series if they have the *same current*, the total impedance is the sum of the individual impedance:

$$Z(s) = Z_1(s) + \ldots + Z_n(s)$$

e.g. the series RLC circuit, the equivalent impedance is given by

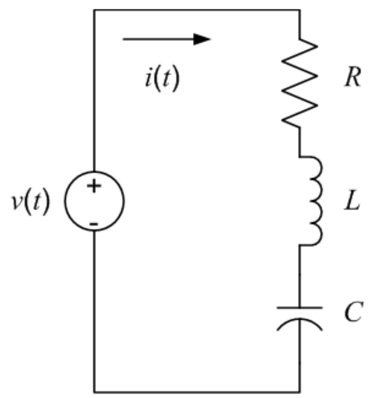
$$Z(s) = R + sL + \frac{1}{sC} = \frac{LCs^2 + RCs + 1}{sC}$$

- two or more impedances are in parallel if they have the *same voltage difference* across them, their impedances combine by the reciprocal rule:

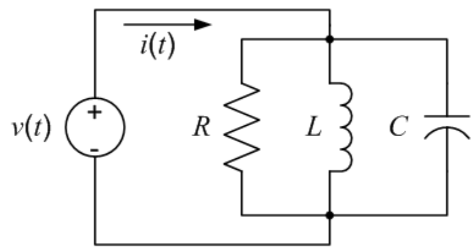
$$\frac{1}{Z(s)} = \frac{1}{Z_1(s)} + \ldots + \frac{1}{Z_n(s)}$$

e.g. the parallel RLC circuit, the equivalent total impedance is

$$
\begin{aligned}
\frac{1}{Z(s)} &= \frac{1}{R} + \frac{1}{sL} + \frac{1}{1/(Cs)} = \frac{RLCs^2 + Ls + R}{RLs} \\
Z(s) &= \frac{RLs}{RLCs^2 + Ls + R}
\end{aligned}
$$



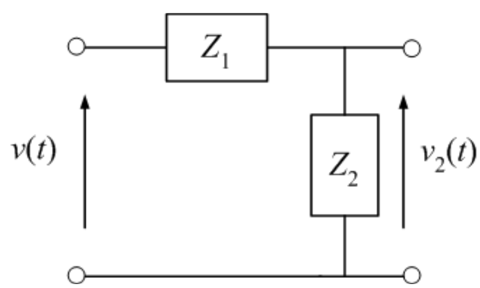
(a) Series RLC circuit



(b) Parallel RLC circuit

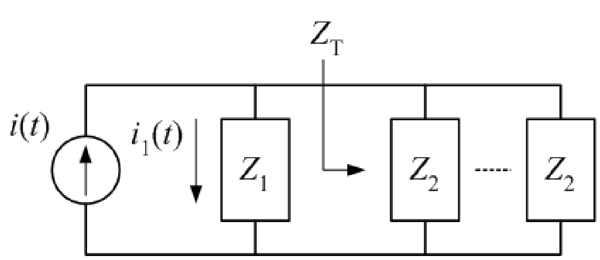
3. Voltage divider rule:

$$V_2(s) = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} V(s)$$



4. Current divider rule:

$$I_1(s) = \frac{Z_T(s)}{Z_1(s) + Z_T(s)} I(s)$$



Practice Problems

1. Consider the following circuit:

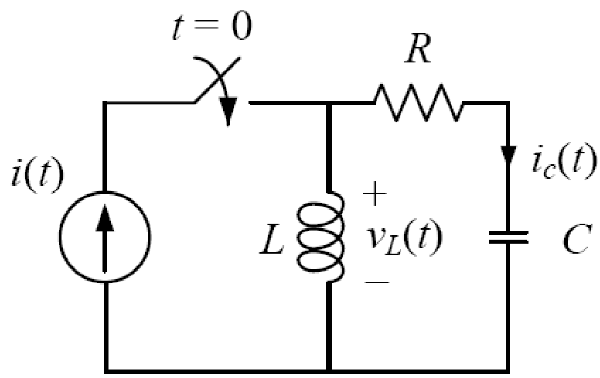


Figure 1: Q1

   (i) Find the transfer function relating the input current, $i(t)$, to capacitor current, $i_c(t)$.

   (ii) Find the transfer function relating the input current, $i(t)$, to the inductor voltage, $v_L(t)$.

2. Consider the following parallel RLC circuit:

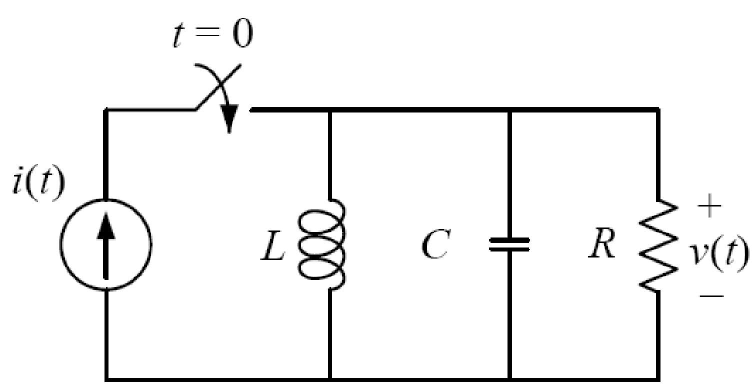


Figure 2: Q2

   (i) Find the differential equation that relates the input current, $i(t)$, to the voltage, $v(t)$.

   (ii) Find the transfer function, $G(s) = \frac{V(s)}{I(s)}$.

3. Consider the following series RLC circuit:

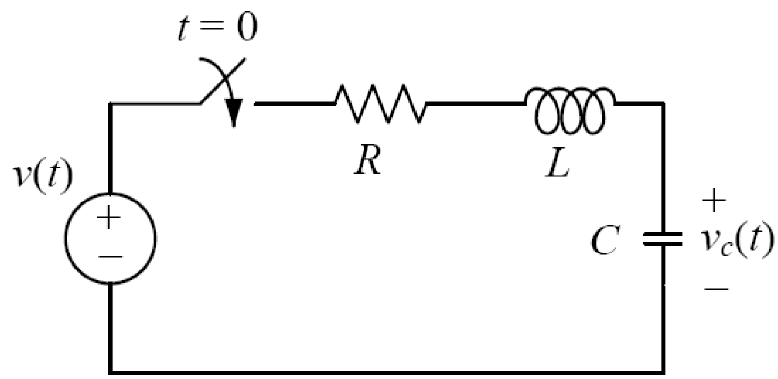


Figure 3: Q3

(i) Find the differential equation relating the source voltage, $v(t)$, to capacitor voltage, $v_c(t)$.

(ii) Find the transfer function, $G(s) = \frac{V_c(s)}{V(s)}$.

(iii) Given $R = 4\omega$, $L = 1$H and $C = 0.24$F, find $v_c(t)$ when $v(t) = U(t)$.

4. Consider the following circuit:

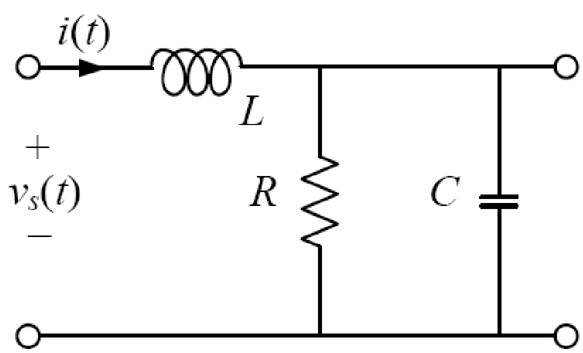


Figure 4: Q4

(i) Find the differential equation relating the source voltage, $v(t)$, to the current, $i(t)$.

(ii) Find the transfer function, $G(s) = \frac{I(s)}{V(s)}$.

5. Consider the following coupled RC circuit:

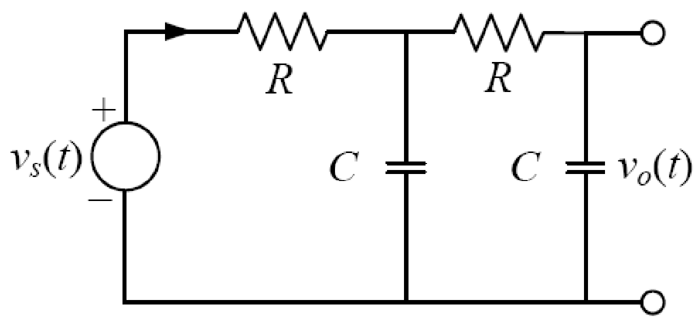


Figure 5: Q5

(i) Find the transfer function, $G(s) = \frac{V_o(s)}{V_s(s)}$.

6. Consider the following electrical circuit:

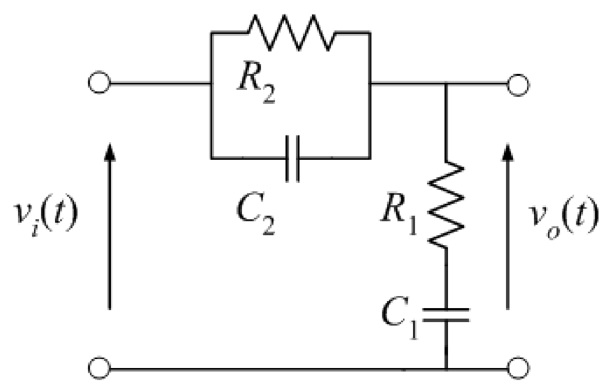


Figure 6: Q6

(i) Find the transfer function, $G(s) = \frac{V_o(s)}{V_i(s)}$.