EE3331C/EE3331E
Feedback Control Systems
Part II - Frequency Response Methods

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Teaching Materials

Textbook

1. G.F. Franklin, J.D. Powell, A. Emami-Naeini, Feedback Control of Dynamic Systems, Pearson International Edition, Fourth Edition, Prentice Hall

2. R.C. Dorf and R.H. Bishop, Modern Control Systems, Pearson International Edition, Eleventh Edition, Prentice Hall

Lecture Notes, Tutorial Problems, Additional Materials

1. Lecture slides are available in IVLE
2. Tutorial problems will be made available before the tutorial session. Solutions to tutorial problems will be posted after the tutorial session.
3. Additional materials will be posted in IVLE as and when required.

Tests and Assessments

1. No graded CA (e.g. mid-term test) for this half of the module
2. Spot tests & Home works – marked but not counted towards final grade / purpose is to sense students' understanding of the lecture materials

Teaching Materials

Lecturer's Contact

Office: E4-08-18
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Email: eleaam@nus.edu.sg

Consultation

- No fixed time for consultation
- I am available for consultation on
  - Tuesday (3:30 pm – 5:30 pm)
  - Wednesday (9:30 am – 12:00 noon)
  - Friday (3:30 pm – 5:30 pm)
- Prior appointment is required – through email or phone call

Final Exam

- 24 Nov 2015, Tuesday, AM session
- Two (2) questions (total marks 50) from frequency domain methods

Contents

Prior knowledge

Complex numbers

Frequency Domain Methods in Feedback Control

Fundamental topics

- What is Frequency Response?
- How to get frequency response data?
- Presentation of data
  - Bode Plot
  - Polar and Nyquist plot
  - Sketching
  - Estimate transfer function model from Bode plot

Stability Analysis

- Coming from the s-domain (root locus) analysis
  - Condition for Marginal Stability
- Stability Margins
- Nyquist Stability Criterion
  - Contour mapping
  - Principle of Argument
  - Apply PoA to Stability Analysis

Feedback Design

- Open loop properties and steady-state error
  - System Type
  - Error Constants
- Open loop properties and damping
- Open loop properties and error for sinusoidal signal
  - Sensitivity function
  - Complementary sensitivity function
- Compensator
  - Lead Vs Lag compensator
  - Design Formulae
  - Step-by-Step Design

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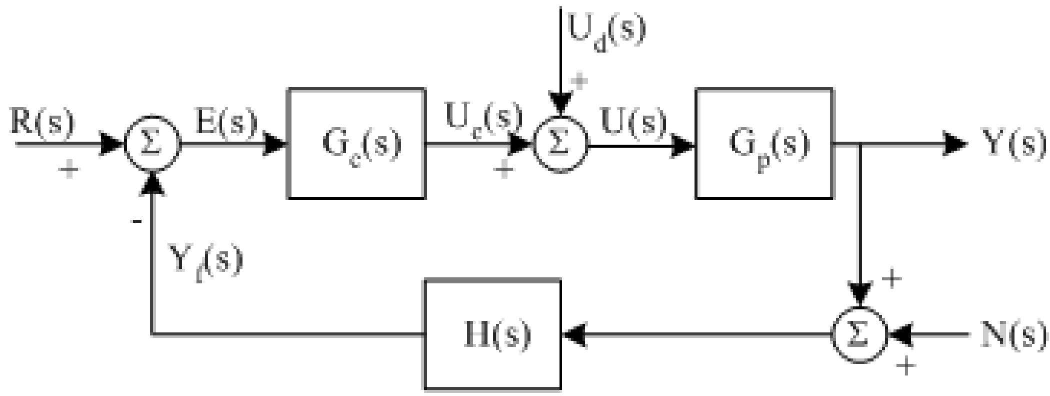
Chapter 1: Introduction

Chapter Learning Objectives

- **At the end of this chapter, you will be able to answer the following …**

1. What is frequency domain method in feedback control?
2. What is frequency response of a linear system?
3. How is frequency response of a linear system related to its transfer function?

Feedback System: Block Diagram



Block diagram of a typical feedback control system

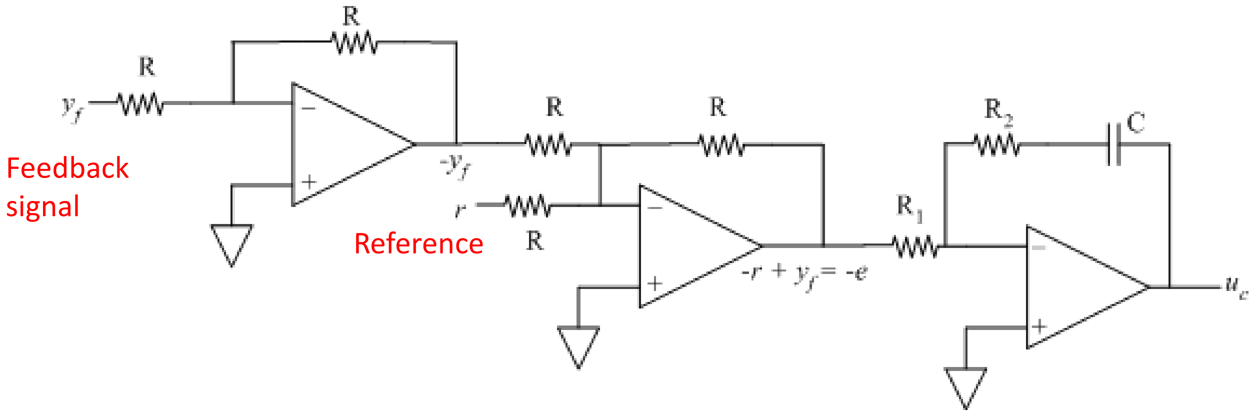
$Y(s)$:	Laplace transform of $y(t)$, the physical variable to be controlled
$G_p(s)$:	Physical process + Actuation mechanism $\Rightarrow$ Plant
$G_c(s)$:	Controller
$H(s)$:	Any dynamics involved in the feedback mechanism
$N(s)$:	Laplace transform of output disturbance or sensor noise
$U(s)$:	Laplace transform of input to the plant
$U_d(s)$:	Laplace transform of input disturbance
$R(s)$:	Laplace transform of reference signal

Everything is given or known except $G_c(s)$, which we need to design

Feedback System: Controller

- In most cases, controller is implemented as an electrical system, for example,
  a) An analog circuit involving Op-Amp, resistor, capacitor etc.
  b) A digital circuit involving a processor, DAC, ADC etc.

a) Controller Implemented using Op-Amp



PI Controller Implemented using Op-Amp Circuit

$$U_c(s) = -\frac{R_2 + \frac{1}{Cs}}{R_1}(-E(s))$$

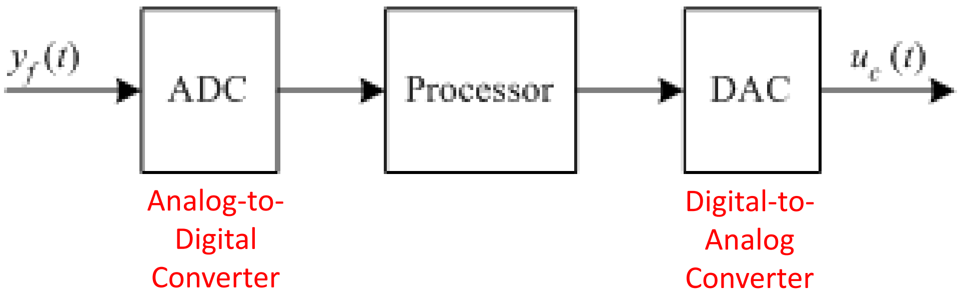
$$= \left(\frac{R_2}{R_1} + \frac{1}{R_1 Cs}\right)E(s)$$

$$= \left(K_P + \frac{K_I}{s}\right)E(s)$$

- Assuming that we know how to physically realize the controller, the objective in the study of feedback control system is to find the values of controller parameters, e.g., K_P and K_I for P+I controller

Feedback System: Controller

b) Controller Implemented in a Processor ⇒ Digital Implementation




```
previous_error = 0
integral = 0
start:
        error = reference - measured_value
        integral = integral + error*dt
        derivative = (error - previous error)/dt
        output = Kp*error + Ki*integral + Kd*derivative
        previous_error = error
        wait (dt)
        goto start
```

Pseudocode of a PID controller

- Digital controller introduces delay in the feedback system
  - Delay is equal to the sum of the time required by the processor to compute control signal, conversion time of ADC, and conversion time of DAC

Feedback System: Things to Know

- **Study of feedback system involves**
  - Analysis of closed loop (CL) stability
    - CL stability is what we are concerned about; open loop may be stable or unstable
  - Steady-state error analysis
  - Transient response analysis
  - Controller design

$\left.\begin{array}{l}\text{Steady-state error analysis}\\ \text{Transient response analysis}\end{array}\right\}$ **Performance of feedback system**

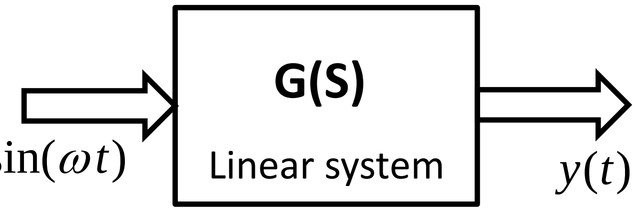
- **Laplace Transform based methods -**
  - Stability: Routh-Hurwitz criterion or root locus
  - Steady-state error (for step, ramp etc): system type and error constant
  - Transient response: rise time, overshoot, settling time etc. for step input
  - Controller: three-term controller (P+I+D)

The 2^{nd}-part of EE3331 explores the frequency-domain methods

Frequency Response

- **What is frequency response?**
  - Steady-state response of a linear system subject to sinusoidal input

$u(t) = A\sin(\omega t)$



$y(t) = y_{tr}(t) + y_{ss}(t)$

- You can use Laplace Transform based method to find $y_{ss}(t)$

- **Step 1**: Take Laplace Transform of $u(t)$ and then find Y(s)

$$u(t) = A\sin(\omega t) \quad \Rightarrow \quad U(s) = A\frac{\omega}{s^2+\omega^2}$$

$$Y(s) = G(s)U(s)$$

$$Y(s) = \frac{N(s)}{(s+p_1)(s+p_2)\ldots(s+p_n)}\left(\frac{A\omega}{s^2+\omega^2}\right)$$

Next step: use partial fraction decomposition

Frequency Response

- **Step 2**: Partial Fraction Decomposition of Y(s)

$$Y(s)=\frac{N(s)}{(s+p_1)(s+p_2)\ldots(s+p_n)}\left(\frac{A\omega}{s^2+\omega^2}\right)$$

$$Y(s)=\frac{k_1}{(s+p_1)}+\frac{k_2}{(s+p_2)}+\ldots+\frac{\alpha s+\beta}{(s^2+\omega^2)}$$

$$Y(s)=\frac{k_1}{(s+p_1)}+\frac{k_2}{(s+p_2)}+\ldots+\alpha\frac{s}{(s^2+\omega^2)}+\frac{\beta}{\omega}\frac{\omega}{(s^2+\omega^2)}$$

  - Note: For each individual fraction, the denominator is either a pole of G(s) or a pole of U(s)

- **Step 3**: Take Inverse Laplace Transform of Y(s) expressed as sum of partial fractions in step 2

$$y(t)=k_1e^{-p_1t}+k_2e^{-p_2t}+\ldots.+\alpha\cos(\omega t)+\frac{\beta}{\omega}\sin(\omega t)$$

$$=y_{tr}(t)+y_{ss}(t)$$

Poles of G(s) contribute to $y_{tr}(t)$ and poles of U(s) contribute to $y_{ss}(t)$

Frequency Response

$$y(t) = k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + ... + \alpha\cos(\omega t) + \frac{\beta}{\omega}\sin(\omega t)$$

$$= y_{tr}(t) + y_{ss}(t)$$

- For stable G(s),

$$e^{-p_i t} \to 0 \quad as \quad t \to \infty \qquad \Rightarrow \qquad y_{tr}\big|_{t\to\infty} = 0$$

- So the steady-state response for sinusoidal input is,

$$y_{ss}(t) = \alpha\cos(\omega t) + \frac{\beta}{\omega}\sin(\omega t)$$

- Compare this with $B\sin\phi\cos(\omega t) + B\cos\phi\sin(\omega t) = B\sin(\omega t + \phi)$

$$B\sin\phi = \alpha, \qquad B\cos\phi = \frac{\beta}{\omega}$$

$$B = \frac{\sqrt{\beta^2 + \alpha^2\omega^2}}{\omega}, \quad \phi = \tan^{-1}\frac{\alpha\omega}{\beta}$$

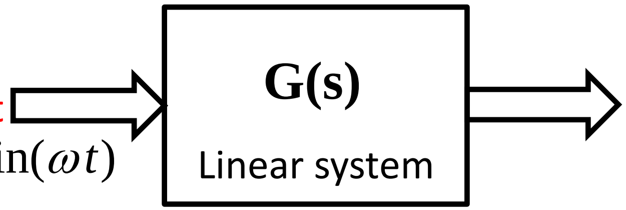
- Therefore,

$$y_{ss}(t) = B\sin(\omega t + \phi)$$

Frequency Response

Gain and Phase

Sinusoidal input
$u(t) = A\sin(\omega t)$



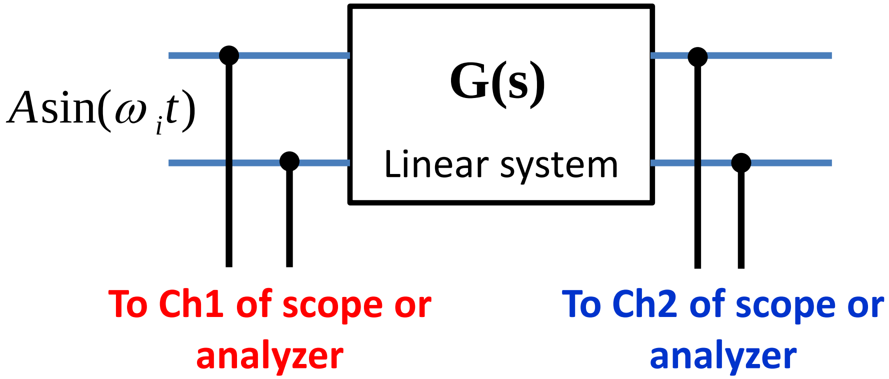
Steady-state response
$y_{ss}(t) = B\sin(\omega t + \phi)$

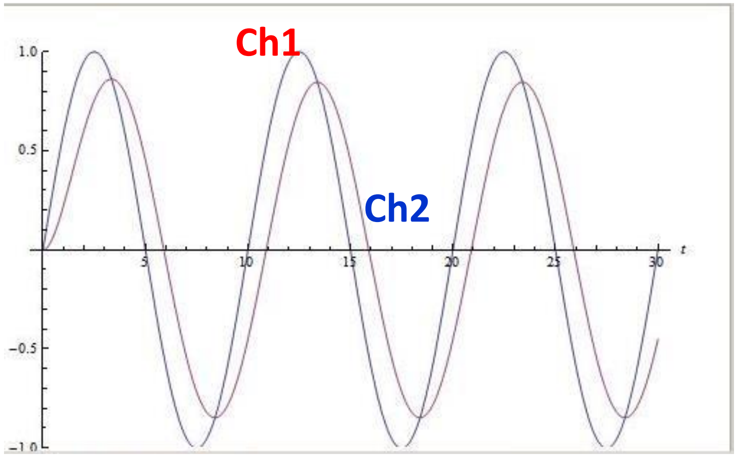
- **Conclusion**:
  - For sinusoidal input, the steady-state response of linear system is also sinusoidal
  - Same frequency (ω) at both input and output
  - Input and output signals differ in amplitude and phase
- **Gain** of the linear system G(s): $M = \dfrac{B}{A}$
- **Phase** of the linear system G(s): ϕ
  - Both gain and phase vary if frequency ω is varied
  - M(ω) and ϕ(ω) define the frequency response of a linear system

$$B = \frac{\sqrt{\beta^2 + \alpha^2\omega^2}}{\omega}, \quad \phi = \tan^{-1}\frac{\alpha\omega}{\beta}$$

Frequency Response

Measurement





1) Apply $A\sin(\omega_i t)$ as the input to the linear system and allow output to reach its steady-state
2) Measure gain and phase for frequency ω_i
3) Repeat steps 1 and 2 for all frequencies of interest

In most frequency response analyzers, measurement is automated. Users need to specify the range of frequencies only

Frequency Response

Relation between $G(s)$ and its Frequency Response

- $G(s)$: Transfer function of a linear system
- $M(\omega_i)$: Gain of that system for frequency ω_i rad/s
- $\phi(\omega_i)$: Phase of that system for frequency ω_i rad/s
- Then,

$$M(\omega_i) = |G(j\omega_i)|, \quad \phi(\omega_i) = \angle G(j\omega_i)$$

{Refer to the Appendix 1-A for proof}

- *Example* 1-1: Find gain and phase of the following transfer function for $\omega = 4$ rad/s

$$G(s) = \frac{10}{s^2 + 3s + 9}$$

$$G(j\omega) = \frac{10}{(j\omega)^2 + j3\omega + 9} = \frac{10}{9 - \omega^2 + j3\omega}$$

$$G(j4) = \frac{10}{9 - 4^2 + j3 \times 4} = \frac{10}{-7 + j12}$$

$$G(j4) = \frac{10}{13.9\angle 120.3°} = \frac{10}{13.9}\angle(0° - 120.3°)$$

$$\boxed{M = 0.07, \quad \phi = -120.3°}$$

(A brief review is given in Appendix 1B on basic calculations using complex numbers)

Frequency Response

- For sinusoidal input, the steady-state output of a linear system G(s) can be found without using the Laplace transform approach
  - Instead, the gain and phase of G(s) for the frequency of the input sinusoid is used

- *Example* 1-2: Find the output signal in the steady-state of the following transfer function if the input is $u(t) = 2\sin(2t)$

$$G(s) = \frac{10}{s^2 + 3s + 9}$$

  - Frequency of the input is ω=2 rad/s. Find gain and phase of G(s) for ω=2.

$$G(j\omega) = \frac{10}{(j\omega)^2 + j3\omega + 9} = \frac{10}{9 - \omega^2 + j3\omega}$$

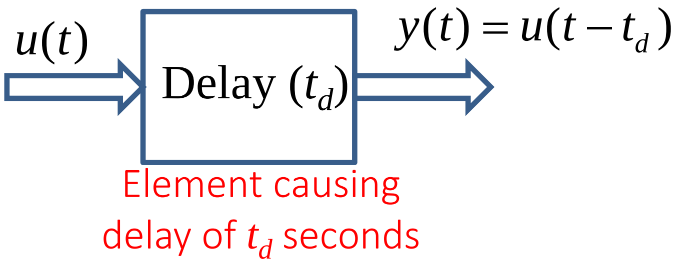
$$G(j2) = \frac{10}{9 - 2^2 + j3\times 2} = \frac{10}{5 + j6}$$

$$G(j2) = \frac{10}{7.8\angle 50.2^\circ} = 1.28\angle -50.2^\circ$$

$$M = 1.28, \quad \phi = -50.2^\circ$$

$$y_{ss}(t) = 1.28 \times 2\sin(2t - 50.2^\circ)$$
$$= 2.56\sin(2t - 50.2^\circ)$$

Frequency Response of Delay



$$u(t) \underset{L.T.}{\Rightarrow} U(s)$$

$$u(t - t_d) \underset{L.T.}{\Rightarrow} e^{-st_d}U(s)$$

- Transfer function of the delay element, $G_{delay}(s) = \dfrac{Y(s)}{U(s)} = \dfrac{e^{-st_d}U(s)}{U(s)} = e^{-st_d}$

- Frequency response, $G_{delay}(j\omega) = e^{-j\omega t_d}$

$$G_{delay}(j\omega) = \cos(\omega t_d) - j\sin(\omega t_d)$$

$$M = \left|G_{delay}(j\omega)\right| = 1$$

$$\phi = \angle G_{delay}(j\omega) = -\omega t_d$$

- Delay operation does not change the amplitude but the output sine wave is phase shifted by -ωt_d rad/s with respect to the input sine wave

Frequency Response: Exercise

1. Find gain and phase of the following transfer functions for ω = 5 rad/s

$$a.\quad \frac{1}{s+2},\qquad b.\quad \frac{1}{s^2+7s+10},\qquad c.\quad \frac{s+1}{s+10},\qquad d.\quad \frac{3}{s(s+3)}e^{-2s}$$

Answer: a. 0.19, -68.2°, b. 0.03, -113.2°, c. 0.46, 52.1°, d. 0.12, -722°

2. Transfer function model of a dynamic system is $G(s) = \dfrac{s+10}{s+2}$.

Find the steady-state output if the input signal is

$$a.\quad u(t) = 5\sin(5t),\quad b.\quad u(t) = 10\sin(t),\quad c.\quad u(t) = 5\sin(3t+30^\circ)$$

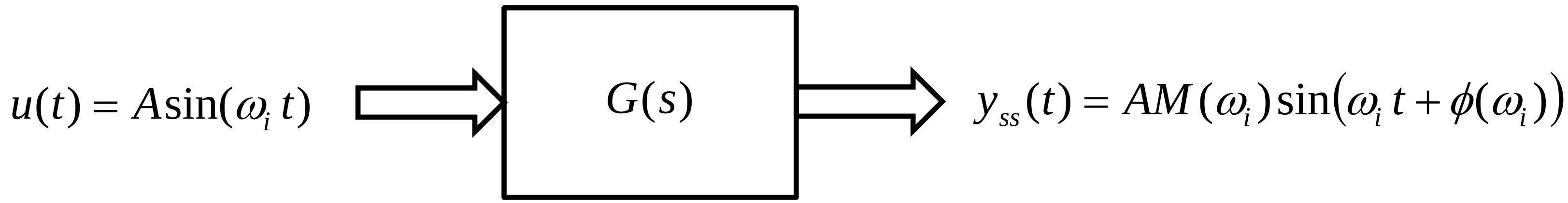
Answer:

$$a)\quad 10.4\sin(5t-41.6^\circ)$$

$$b)\quad 44.9\sin(t-20.9^\circ)$$

$$c)\quad 14.5\sin(3t-9.6^\circ)$$

Frequency Response of G(s) is G(jω)



- $M(\omega_i)$ and $\phi(\omega_i)$ are gain and phase of G(s) for input frequency ω_i

- Find $y_{ss}(t)$ for G(s) using Laplace transform method

$$U(s) = \frac{A\omega_i}{s^2 + \omega_i^2} = \frac{A\omega_i}{(s + j\omega_i)(s - j\omega_i)}$$

$$Y(s) = G(s)\frac{A\omega_i}{(s + j\omega_i)(s - j\omega_i)}$$

- If $$G(s) = \frac{m(s)}{n(s)} = \frac{m(s)}{(s+p_1)(s+p_2)....(s+p_n)} = \frac{m(s)}{\prod_{i=1}^{n}(s+p_i)}$$

then $$Y(s) = \frac{m(s)}{\prod_{i=1}^{n}(s+p_i)}\frac{A\omega_i}{(s + j\omega_i)(s - j\omega_i)}$$

Frequency Response of G(s) is G(jω)

- After Partial Fraction decomposition,

$$Y(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \ldots.. + \frac{k_n}{s+p_n} + \frac{\alpha}{s+j\omega_i} + \frac{\beta}{s-j\omega_i}$$

- Inverse transform gives

$$y(t) = k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \ldots. + k_n e^{-p_n t} + \alpha\, e^{-j\omega_i t} + \beta e^{+j\omega_i t}$$

  - If $\mathrm{G(s)}$ is stable, all terms except the last two approach zero in the steady state

$$k_i e^{-p_i t}\Big|_{t\to\infty} = 0$$

$$y_{ss}(t) = y_{t\to\infty} = \alpha\, e^{-j\omega_i t} + \beta e^{+j\omega_i t}$$

  - Next, we see how the parameters α and β are related to the transfer function $\mathrm{G(s)}$

Frequency Response of G(s) is G(jω)

$$G(s)\frac{A\omega_i}{(s+j\omega_i)(s-j\omega_i)} = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \ldots + \frac{k_n}{s+p_n} + \frac{\alpha}{s+j\omega_i} + \frac{\beta}{s-j\omega_i}$$

- Multiply both sides by $(s+j\omega_i)(s-j\omega_i)$

$$G(s)A\omega_i = (s+j\omega_i)(s-j\omega_i)\left[\frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \ldots.. + \frac{k_n}{s+p_n}\right] + \alpha(s-j\omega_i) + \beta(s+j\omega_i)$$

- We can evaluate α by substituting $s=-j\omega_i$ in the above equation

$$G(-j\omega_i)A\omega_i = \alpha(-j\omega_i - j\omega_i)$$

$$\alpha = \frac{G(-j\omega_i)A\omega_i}{-2j\omega_i} = -\frac{AG(-j\omega_i)}{2j}$$

- Similarly, we evaluate β by substituting $s=+j\omega_i$ in that equation

$$G(+j\omega_i)A\omega_i = \beta(+j\omega_i + j\omega_i)$$

$$\beta = \frac{G(+j\omega_i)A\omega_i}{+2j\omega_i} = +\frac{AG(-j\omega_i)}{2j}$$

Frequency Response of G(s) is G(jω)

- Substitute values of α and β obtained in the expression for steady-state output,

$$y_{ss}(t) = -\frac{G(-j\omega_i)A}{2j}e^{-j\omega_i t} + \frac{G(+j\omega_i)A}{2j}e^{+j\omega_i t}$$

- $G(j\omega_i)$ is a complex number. $G(+j\omega_i) = X + jY = \sqrt{X^2+Y^2}\angle\tan^{-1}\tfrac{Y}{X}$

$$G(+j\omega_i) = R\angle\theta = Re^{+j\theta}$$

- $G(-j\omega_i)$ is is its conjugate. $G(-j\omega_i) = X - jY = \sqrt{X^2+Y^2}\angle-\tan^{-1}\tfrac{Y}{X}$

$$G(-j\omega_i) = R\angle-\theta = Re^{-j\theta}$$

- Therefore,

$$\begin{aligned} y_{ss}(t) &= -\frac{ARe^{-j\theta}}{2j}e^{-j\omega_i t} + \frac{ARe^{+j\theta}}{2j}e^{+j\omega_i t} \\ &= \frac{AR}{2j}\left[-e^{-j(\omega_i t+\theta)} + e^{+j(\omega_i t+\theta)}\right] \\ &= AR\sin(\omega_i t+\theta) \end{aligned}$$

where $R = \left|G(+j\omega_i)\right|, \quad \theta = \angle G(+j\omega_i)$

Complex Number: Review

- Imaginary Number (j)

$$j=\sqrt{-1},\quad j^2=-1,$$
$$j^3=j\times j^2=-j$$
$$j^4=j^2\times j^2=+1$$

- Complex Number:

$$z=x+jy$$ (Both x and y are real numbers)

- Rectilinear Coordinate & Polar Coordinate

$$z=x+jy$$
$$z=M\angle\theta$$

x: real part of z
y: imaginary part of z
M: modulus of z
ϕ: argument of z

$$M=\sqrt{x^2+y^2},\quad \theta=\tan^{-1}\frac{y}{x}$$
$$x=M\cos\theta,\quad y=M\sin\theta$$

- Addition/ subtraction of Complex Numbers

$$\begin{aligned} z_1&=x_1+jy_1\\ z_2&=x_2+jy_2\end{aligned}\quad\Rightarrow\quad\begin{aligned} z_1+z_2&=(x_1+x_2)+j(y_1+y_2)\\ z_1-z_2&=(x_1-x_2)+j(y_1-y_2)\end{aligned}$$

- Multiplication/ Division of Complex Numbers

$$\begin{aligned} z_1&=M_1\angle\phi_1\\ z_2&=M_2\angle\phi_2\end{aligned}\quad\Rightarrow\quad\begin{aligned} z_1z_2&=(M_1M_2)\angle(\phi_1+\phi_2)\\ z_1/z_2&=(M_1/M_2)\angle(\phi_1-\phi_2)\end{aligned}$$

Complex Number: Review

- Complex Conjugate

$$z = xz + jy \quad \Rightarrow \quad z^* = x - jy$$

$$(z^* \text{ is conjugate of } z)$$

- Product of a complex number and its conjugate is a real number
  - This product is equal to square of modulus

$$\begin{aligned} zz^* &= (x + jy)(x - jy) \\ &= x^2 + jxy - jxy + y^2 \\ &= x^2 + y^2 \\ &= M^2 \end{aligned}$$

- Conjugate in polar coordinate

$$z = M\angle\phi \quad \Rightarrow \quad z^* = M\angle(-\phi)$$

$$\begin{aligned} zz^* &= (M\angle\phi)(M\angle(-\phi)) \\ &= (M^2)\angle(\phi - \phi) \\ &= M^2 \end{aligned}$$