STATISTICAL SIGNAL ANALYSIS

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Module objective

Module: Statistical signal processing

Knowledge and understanding

- Understand the fundamentals of statistical signal processing, learning distribution from signal, and statistical classification

Key skills

- Design, build, implement a statistical classification approach that could use features extracted from the wavelet transofrmation

Statistical signal processing

- Learning: For a given set of observations $\boldsymbol{X} = [\boldsymbol{X}_1, \boldsymbol{X}_{2,}, \cdots, \boldsymbol{X}_N]$ from some random variables, learn a model $f(\boldsymbol{X}|\boldsymbol{\theta})$ (defined by the parameter $\boldsymbol{\theta}$) to best describe the data.
- Estimation: Given a noisy observation $\boldsymbol{X}_n = \boldsymbol{Z}_n + \boldsymbol{w}_n$, where $\boldsymbol{Z}_n$ is the unknown clean signal and $\boldsymbol{w}_n$ is noise, and the learned model $f(\boldsymbol{X}|\boldsymbol{\theta})$, estimate the unknown clean signal.
- Prediction: Assume that we have two classes of objects as ω_1 and ω_2, and we already have learned models $f(\boldsymbol{X}|\boldsymbol{\theta_1})$ and $f(\boldsymbol{X}|\boldsymbol{\theta_2})$ for ω_1 and ω_2, respectively. For a new observation $\boldsymbol{X}_n$, decide which class it belongs to, that is $\boldsymbol{f}(\omega_1, \omega_2|\boldsymbol{X}_n, \boldsymbol{\theta_1}, \boldsymbol{\theta_2})$.

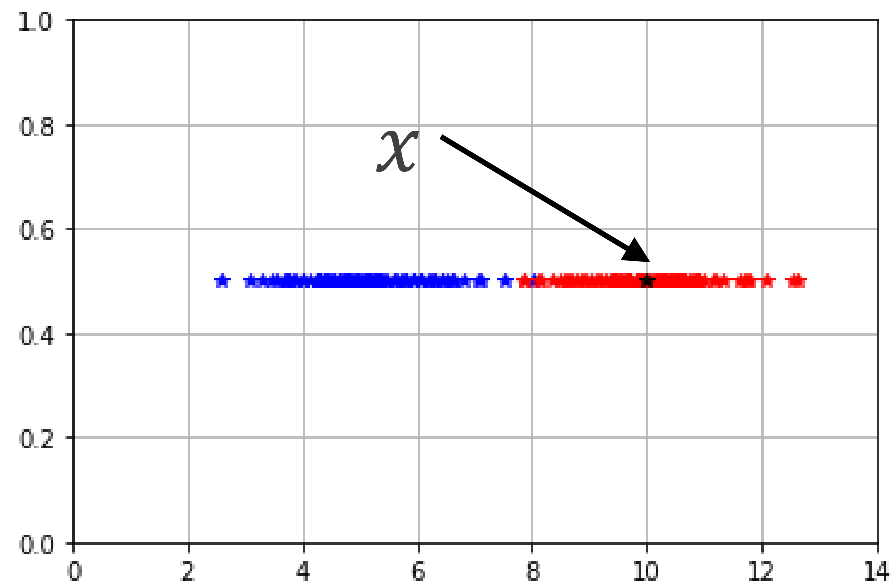
Signal classification

- Given a signal x, two categories of data, say ω_1, ω_2, does the signal x belong to ω_1 or ω_2?
- That means, we need to evaluate $P(\omega_1|x)$ and $P(\omega_2|x)$
- The class posterior probability is

Likelihood Priors

$$\boxed{P(\omega_i|x) = \frac{P(x|\omega_i)P(\omega_i)}{P(x)}}$$

Evidence



Signal classification

- Class priors $P(\omega_i)$
  - How much of each class? $P(\omega_i) \approx N_i/N$
- Class likelihood $P(x|\omega_i)$
  - Requires that we have a model for each ω_i, for example, ω_i can be modelled as Gaussian distribution
- Evidence
  - $P(x) = P(x|\omega_1)P(\omega_1) + P(x|\omega_2)P(\omega_2)$
- Classification rule
  - If $P(\omega_1|x) \geq P(\omega_2|x)$, then x is classified as ω_1
  - If $P(\omega_1|x) < P(\omega_2|x)$, then x is classified as ω_2

Example: Naive Bayes classifier

Given: $\boldsymbol{x}$=(Outlook=*Sunny,* Temperature=*Cool,* Humidity=*High,* Wind=*Strong*)
Predict: PlayTennis Yes or No?

PlayTennis: training examples

Day	Outlook	Temperature	Humidity	Wind	PlayTennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

Example

Given: $\boldsymbol{x}$=(Outlook=*Sunny,* Temperature=*Cool,* Humidity=*High,* Wind=*Strong*)
Predict: PlayTennis Yes or No?

Bayesian Rule	
P(*Yes*|**x**)	0.0053
[P(*Sunny*|Yes)P(*Cool*| *Yes*)P(*High*|*Yes*)P(*Strong*| *Yes*)]P(Play=*Yes*)	
P(*No*|**x**)	0.0206
[P(*Sunny*|No) P(*Cool*|No)P(*High*|*No*)P(*Strong*|*No*)]P(Play=*No*)	

Decision: Given the fact $P(Yes|\boldsymbol{x}) < P(No|\boldsymbol{x})$, we decide $\boldsymbol{x}$ to be *No*.

P(Outlook=*Sunny*|Play=*Yes*) = 2/9
P(Temperature=*Cool*|Play=*Yes*) = 3/9
P(Huminity=*High*|Play=*Yes*) = 3/9
P(Wind=*Strong*|Play=*Yes*) = 3/9
P(Play=*Yes*) = 9/14

P(Outlook=S*unny*|Play=*No*) = 3/5
P(Temperature=*Cool*|Play==*No*) = 1/5
P(Huminity=*High*|Play=*No*) = 4/5
P(Wind=*Strong*|Play=*No*) = 3/5
P(Play=*No*) = 5/14

Details, refer to next slide

Example

Outlook	Play=*Yes*	Play=*No*
Sunny	2/9	3/5
Overcast	4/9	0/5
Rain	3/9	2/5

Temperature	Play=*Yes*	Play=*No*
Hot	2/9	2/5
Mild	4/9	2/5
Cool	3/9	1/5

Humidity	Play=*Yes*	Play=No
High	3/9	4/5
Normal	6/9	1/5

Wind	Play=*Yes*	Play=*No*
Strong	3/9	3/5
Weak	6/9	2/5

Total: $P(Play = Yes) = 9/14, P(Play = No) = 5/14$

Learning distributions for data

- Problem: Given a collection of examples from some data, estimate its distribution
- Solution:
  - Assign a model to the distribution
  - Learn parameters of model from data

Bernoulli distribution

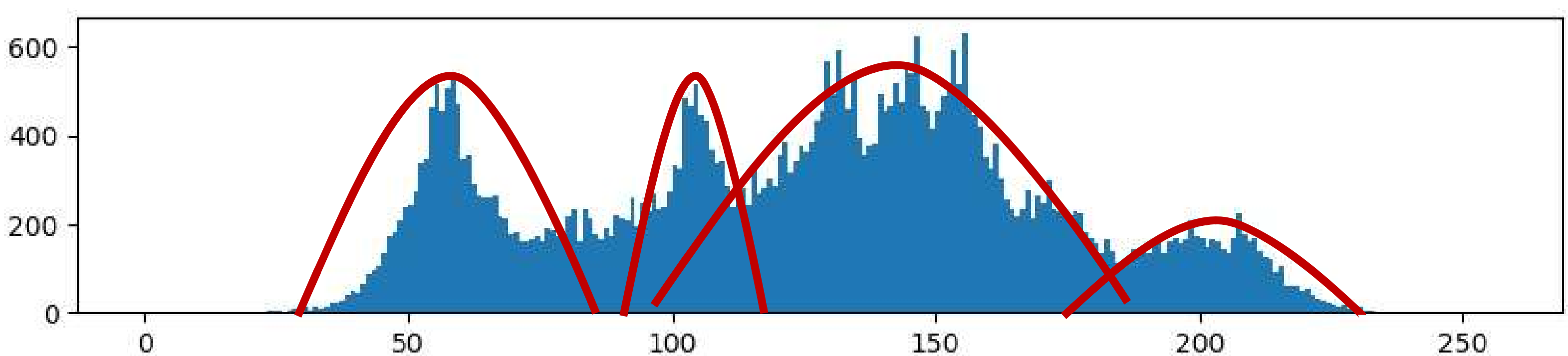
$$p(\mathbf{x}|\boldsymbol{\mu}) = \prod_i \mu_i^{x_i}(1-\mu_i)^{x_i}$$

Gaussian distribution

$$p(\mathbf{x}|\boldsymbol{\mu},\Sigma) = \frac{1}{\sqrt{(2\pi)^N|\Sigma|}} exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

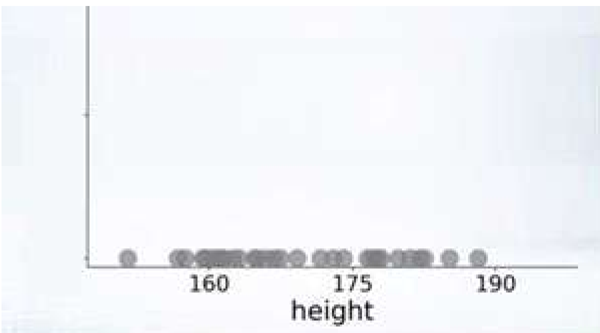
How about complicated data

- Example: The dataset (illustrated in figure below) is complicated so that we can not use a single function (e.g., single Gaussian function) to fit data.
- We can fit the following dataset using several (say, 4) Gaussian functions. How do we know which data is used to fit which Gaussian function?
- We need to perform data clustering (data labelling) and distribution estimation together.

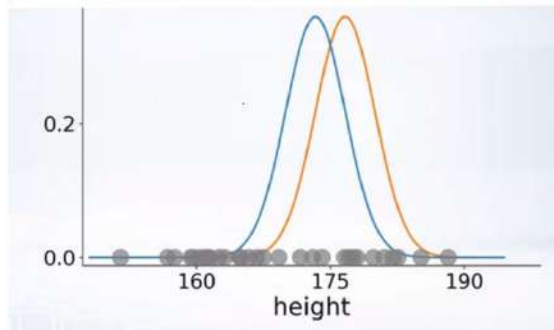


Example: Clustering

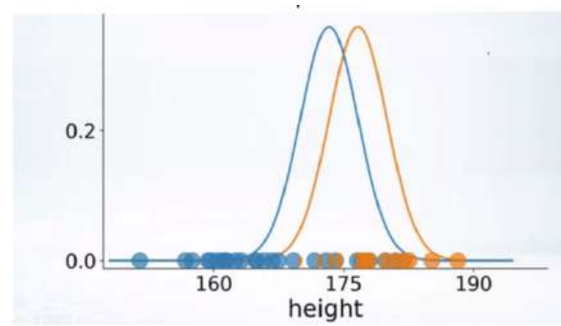
Dataset: Students' height



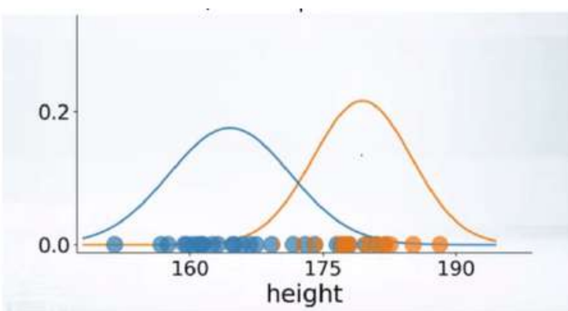
Step1: (randomly) Initialize distribution



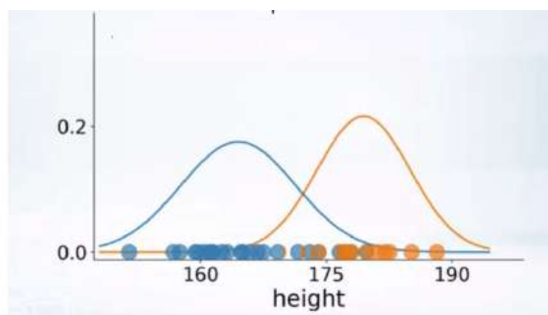
Step2: Assign data label



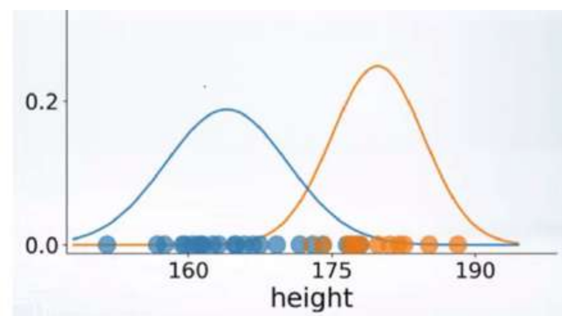
Step3: Update distribution



Step4: Update data label



Step5: Update distribution



Photos: https://towardsdatascience.com/gaussian-mixture-models-d13a5e915c8e

Learn model parameters: A toy task

Example: **6 3 1 5 4 1 2 4 …**

- A person shoots a loaded dice repeatedly
- You observe the series of outcomes
- You can form a good idea of how the dice is loaded, by finding what the probabilities of the various numbers are for dice
- $\mathrm{Probability}(number) = \dfrac{\mathrm{Count}(number)}{\mathrm{Sum}(rolls)}$

This is a *maximum likelihood* estimation. Estimate that makes the observed sequence of numbers most probable.

A toy task with additional rules

- Additional rules
  - Two dice (say, **yellow dice** and **blue dice**) are available.
  - The dice are differently loaded for the two of them.
  - We observe the series of outcome.
  - There is a "caller" who randomly calls out the outcomes.
  - At any time, you do not know which of the two dice you are calling out,
- <u>How do you determine the probability distributions for the two dice?</u> If you do not even know what fraction of time the blue numbers are called, and what fraction are yellow.

Use case: You are given a set of GPA scores for students in TWO classes (AI and DS). You are asked to build the distribution of GPA scores of each class. But you don't know the label (the student belongs to AI or DS class) of individual GPA score.

Key idea: Introduce a hidden label

Objective: We need to formulate following distributions by filling these two tables.

X	1	2	3	4	5	6
$P(X\|\text{Blue})$						
$P(X\|\text{Yellow})$						

$P(Z=\text{Blue})$	
$P(Z=\text{Yellow})$	

- The caller will call out a number X IF
  - He selects "Yellow", and the Yellow dice rolls the number X

  OR
  - He selects "Blue" and the Blue dice rolls the number X
- $P(X) = P(\text{Yellow})P(X|\text{Yellow}) + P(\text{Blue})P(X|\text{Blue})$
  - E.g. $P(6) = P(\text{Yellow})P(6|\text{Yellow}) + P(\text{Blue})P(6|\text{Blue})$

He selects Yellow   Dice rolls number 6   He selects Blue   Dice rolls number 6

Solution: Expectation maximization

- Iterative solution
- Get some initial estimates for all parameters
  - Dice shooter example: This includes probability distributions for dice AND the probability with which the caller selects the dice
- Two steps that are iterated:
  - Expectation Step: Estimate the values of unseen variables
  - Maximization Step: Using the estimated values of the unseen variables as truth, estimates of the model parameters

Expectation maximization

Step 1: Initialization

- We (guess) obtain an initial estimate for the probability distribution of the two sets of dice:

X	1	2	3	4	5	6
$P(X\|\text{Blue})$	0.3	0.3	0.1	0.1	0.1	0.1
$P(X\|\text{Yellow})$	0.4	0.05	0.05	0.05	0.05	0.4

- We (guess) obtain an initial estimate for the probability with which the caller calls out the two shooters

$P(Z = \text{Blue})$	0.5
$P(Z = \text{Yellow})$	0.5

Expectation maximization

Step 2: Estimate hidden labels

- Every observed roll of the dice contributes to both "Yellow" and "Blue"

6 4 5 1 2 3 4 5 2 2 1 4 3 4 6 2 1 6

6 (0.8), 4 (0.33), 5 (0.33), 1 (0.57), 2 (0.14), 3 (0.33), 4 (0.33), 5 (0.33), 2 (0.14), 2 (0.14), 1 (0.57), 4 (0.33), 3 (0.33), 4 (0.33), 6 (0.8), 2 (0.14), 1 (0.57), 6 (0.8)	6 (0.2), 4 (0.67), 5 (0.67), 1 (0.43), 2 (0.86), 3 (0.67), 4 (0.67), 5 (0.67), 2 (0.86), 2 (0.86), 1 (0.43), 4 (0.67), 3 (0.67), 4 (0.67), 6 (0.2), 2 (0.86), 1 (0.43), 6 (0.2)

Probability to be Yellow

Probability to be Blue

Recall that we randomly guess in Step 1

$$P(Z = \text{Blue}) = P(Z = \text{Yellow}) = 0.5$$
$$P(6|\text{Blue}) = 0.1, P(6|\text{Yellow}) = 0.4$$

For roll number 6, we estimate its label

$$P(\text{Yellow}|X = 6) = P(X = 6|Z = \text{Yellow})P(Z = \text{Yellow}) = 0.4 \times 0.5 = 0.2$$
$$P(\text{Blue}|X = 6) = P(X = 6|Z = \text{Blue})P(Z = \text{Blue}) = 0.1 \times 0.5 = 0.05$$

After normalization, we have $P(\text{Yellow}|X = 6) = 0.8, P(\text{Blue}|X = 6) = 0.2$

Expectation maximization

Step 2: Estimate hidden labels

- Every observed roll of the dice contributes to both "Yellow" and "Blue"
- Total count for "Yellow" is the sum of all the posterior probabilities in the Yellow column: 7.31
- Total count for "Blue" is the sum of all the posterior probabilities in the Blue column: 10.69

Called	$P(\text{Yellow}\|X)$	$P(\text{Blue}\|X)$
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	SUM=7.31	SUM=10.69

Expectation maximization

Step 3: Update $P(X|\text{Yellow})$

- Total count for "Yellow": 7.31
  - Total count for 1: 1.71
  - Total count for 2: 0.56
  - Total count for 3: 0.66
  - Total count for 4: 1.32
  - Total count for 5: 0.66
  - Total count for 6: 2.4
- Updated probability of Yellow dice:
  - $P(1 \mid \text{Yellow}) = 1.71/7.31 = 0.234$
  - $P(2 \mid \text{Yellow}) = 0.56/7.31 = 0.077$
  - $P(3 \mid \text{Yellow}) = 0.66/7.31 = 0.090$
  - $P(4 \mid \text{Yellow}) = 1.32/7.31 = 0.181$
  - $P(5 \mid \text{Yellow}) = 0.66/7.31 = 0.090$
  - $P(6 \mid \text{Yellow}) = 2.40/7.31 = 0.328$

Called	$P(\text{Yellow}\|X)$	$P(\text{Blue}\|X)$
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	SUM=7.31	SUM=10.69

Expectation maximization

Step 3: Update $P(X|\mathrm{Blue})$

- Total count for “Blue”: 10.69
  - Total count for 1: 1.29
  - Total count for 2: 3.44
  - Total count for 3: 1.34
  - Total count for 4: 2.68
  - Total count for 5: 1.34
  - Total count for 6: 0.6
- Updated probability of Blue dice:
  - $P(1\,|\,\mathrm{Blue}) = 1.29/11.69 = 0.122$
  - $P(2\,|\,\mathrm{Blue}) = 0.56/11.69 = 0.322$
  - $P(3\,|\,\mathrm{Blue}) = 0.66/11.69 = 0.125$
  - $P(4\,|\,\mathrm{Blue}) = 1.32/11.69 = 0.250$
  - $P(5\,|\,\mathrm{Blue}) = 0.66/11.69 = 0.125$
  - $P(6\,|\,\mathrm{Blue}) = 2.40/11.69 = 0.0563$

Called	$P(\mathrm{Yellow}\|X)$	$P(\mathrm{Blue}\|X)$
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	SUM=7.31	SUM=10.69

Expectation maximization

Step 4: Update $P(Z = \text{Blue})$ and $P(Z = \text{Yellow})$

- Total count for "Yellow": 7.31
- Total count for "Blue": 10.69
- Total instances: $7.31 + 10.69 = 18$
- We also normalize our estimate for the probability that the caller calls out Yellow or Blue
- $P(Z = \text{Yellow}) = 7.31/18 = 0.41$
- $P(Z = \text{Blue}) = 10.69/18 = 0.59$

Called	$P(\text{Yellow}\|X)$	$P(\text{Blue}\|X)$
6	.8	.2
4	.33	.67
5	.33	.67
1	.57	.43
2	.14	.86
3	.33	.67
4	.33	.67
5	.33	.67
2	.14	.86
2	.14	.86
1	.57	.43
4	.33	.67
3	.33	.67
4	.33	.67
6	.8	.2
2	.14	.86
1	.57	.43
6	.8	.2
	SUM=7.31	SUM=10.69

Expectation maximization: Summary

X	1	2	3	4	5	6
Initially, we randomly guess						
$P(X\|\text{Blue})$	0.3	0.3	0.1	0.1	0.1	0.1
$P(X\|\text{Yellow})$	0.4	0.05	0.05	0.05	0.05	0.4
Currently, we have updated results after one iteration of update						
$P(X\|\text{Blue})$	0.122	0.322	0.125	0.250	0.125	0.056
$P(X\|\text{Yellow})$	0.234	0.077	0.090	0.181	0.090	0.328

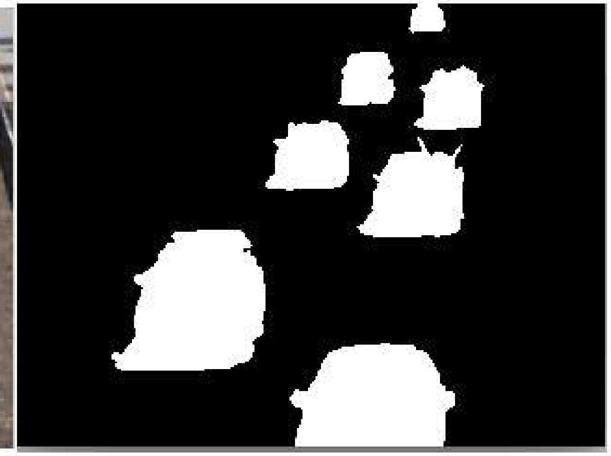
	Previous random guess	Updated results
$P(Z = \text{Blue})$	0.5	0.59
$P(Z = \text{Yellow})$	0.5	0.41

The *Expectation Maximization* (EM) algorithm will continue until the stopping criterion (e.g., # of iterations, etc).

Application: Background modeling in video surveillance

Input image (test)

Background image



Foreground mask

Use case: Given a CCTV video clip captured by the fixed camera, detect the foreground object (e.g., car, human, etc).

- At every pixel location, continuously monitor its color intensity (say, one day).
- Build the statistical model for the color intensity using two-component model (one component for background, one component for foreground)
- Estimate background image (scene modeling)
- Detect the foreground object (mask) from the input image

What we have learnt

- Bayesian signal classification
- Statistical signal modelling including parameter estimation

Thank you!

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