CH 7- *Functions* of *Several Variables*

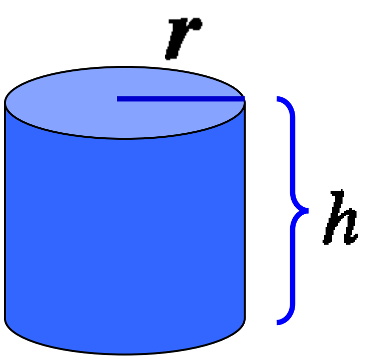
$$f(x,y) \qquad f(x,y,z)$$

7.1 Introduction

In many practical situations, the value of a quantity may depend on more than one variable.

- **Volume** of **right circular cylinder**:

$$V = \pi r^2 h$$



- **The Ideal Gas Law**. The ***pressure***(***P***), ***temperature***(***T***) & ***volume***(***V***) of a gas are related as follows: $PV = kT$, where k is a constant.

- ***Current*** in electrical circuit depends on ***capacitance***, ***electromotive force***, ***impedance*** & ***resistance*** in the circuit.

- ***Output*** of a factory depends on ***amount*** of ***capital invested*** & the ***size*** of ***manpower***

Aim: To ***extend*** some methods of single-variable differential calculus to ***fns*** of **2** variables.

Functions of two variables

- $\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ — a rule that assigns to each pair of real numbers $(\boldsymbol{x},\boldsymbol{y})$, a *real* number $\boldsymbol{z}$ $(= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}))$
  - $\boldsymbol{z}$ is a ***function*** of $\boldsymbol{x}$ & $\boldsymbol{y}$
  - $\boldsymbol{x}$ & $\boldsymbol{y}$ — ***independent*** variables
  - $\boldsymbol{z}$ — ***dependent*** variable

♣ In general,

$\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{x}_1,\boldsymbol{x}_2,\cdots,\boldsymbol{x}_n)$ — ***function*** of $\boldsymbol{n}$ variables

Domain of f = D_f = { (x,y) | $f(x,y)$ is defined }

$f(x, y)$

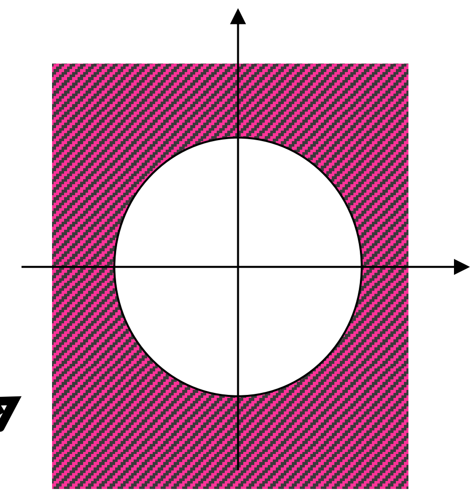
Example

- $f(x, y) = 3 + x\sin y - x^2 y^5$

  $D_f = \{\, (x, y) \mid x, y \text{ are } \mathit{real} \,\}$

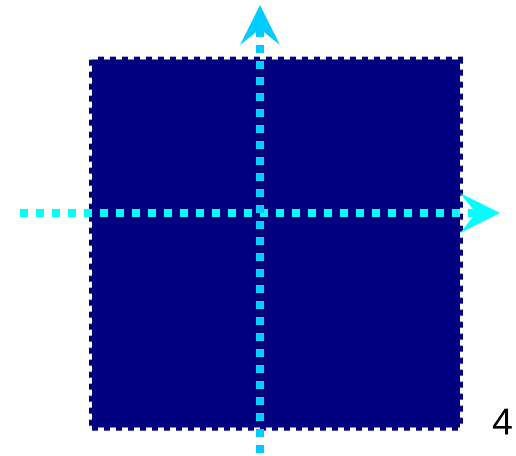
- $f(x, y) = \sqrt{x^2 + y^2 - 1}$

  $D_f = \{\, (x, y) \mid x^2 + y^2 \geq 1 \}$



- $f(x, y) = \dfrac{1}{xy}$

  $D_f = \{\, (x, y) \mid x \neq 0 \;\&\; y \neq 0 \,\}$



7.2 *Geometric Representation*

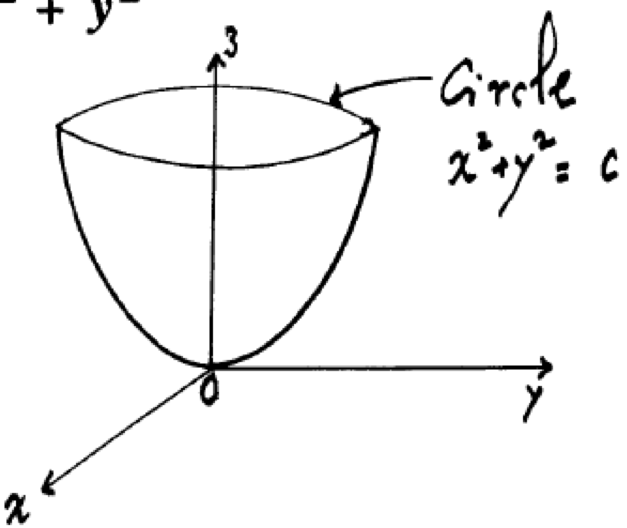
The graph of f(x,y) is a surface

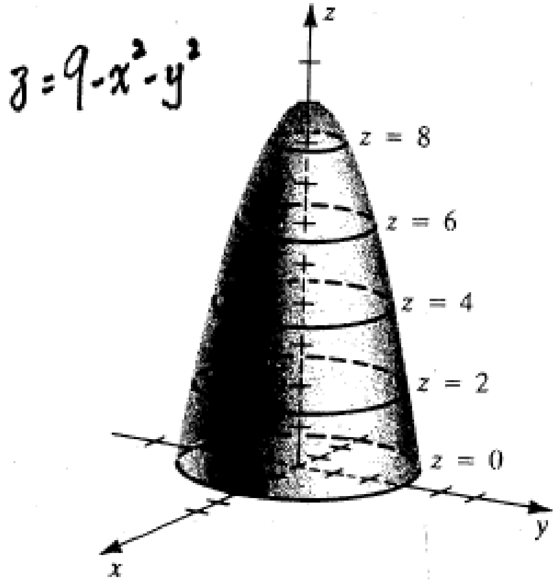
http://www.math.uri.edu/~bkaskosz/flashmo/tools/graph3d/

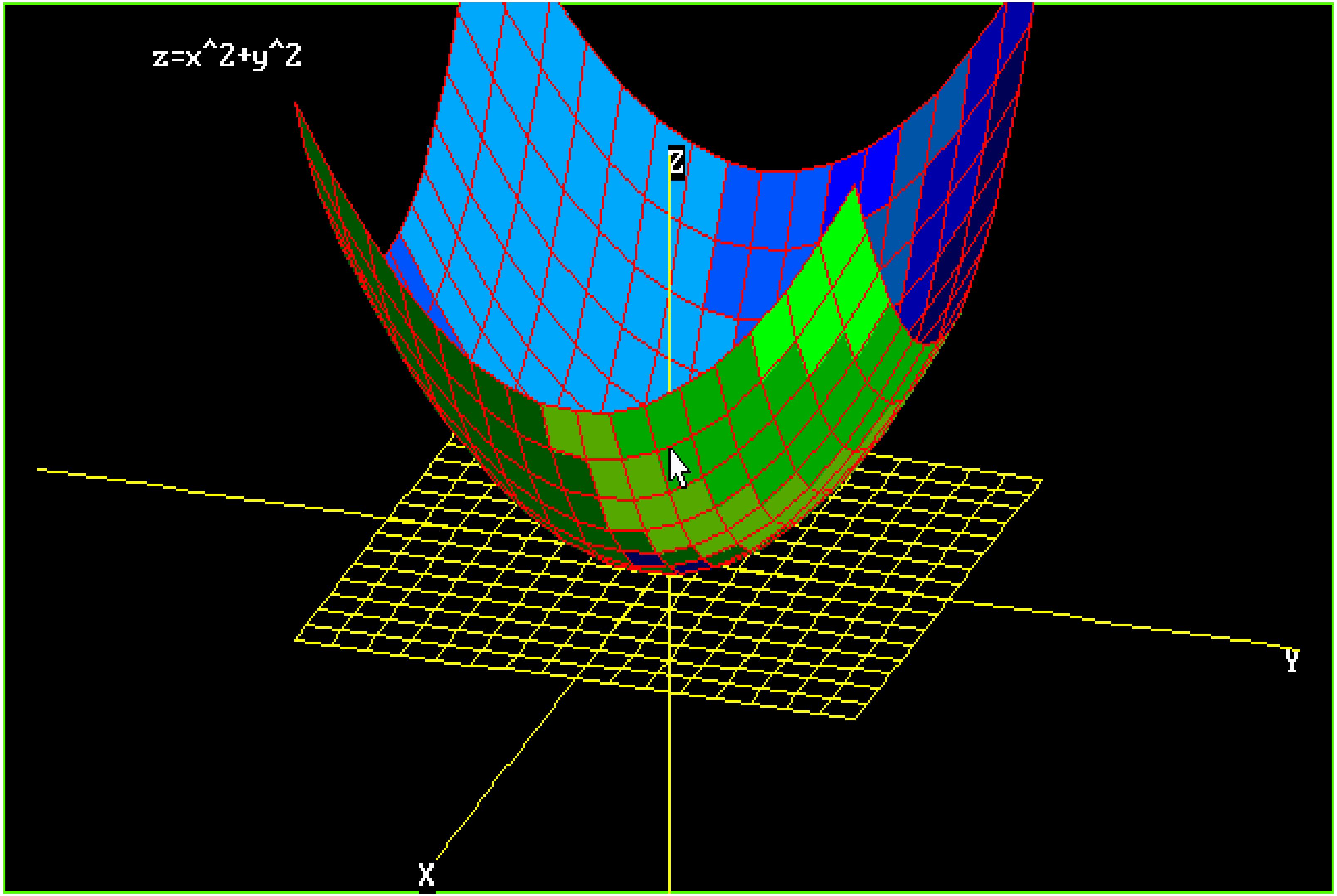
Inverted paraboloid

- $z = x^2 + y^2$

paraboloid






z=x^2+y^2
Z
Y
X

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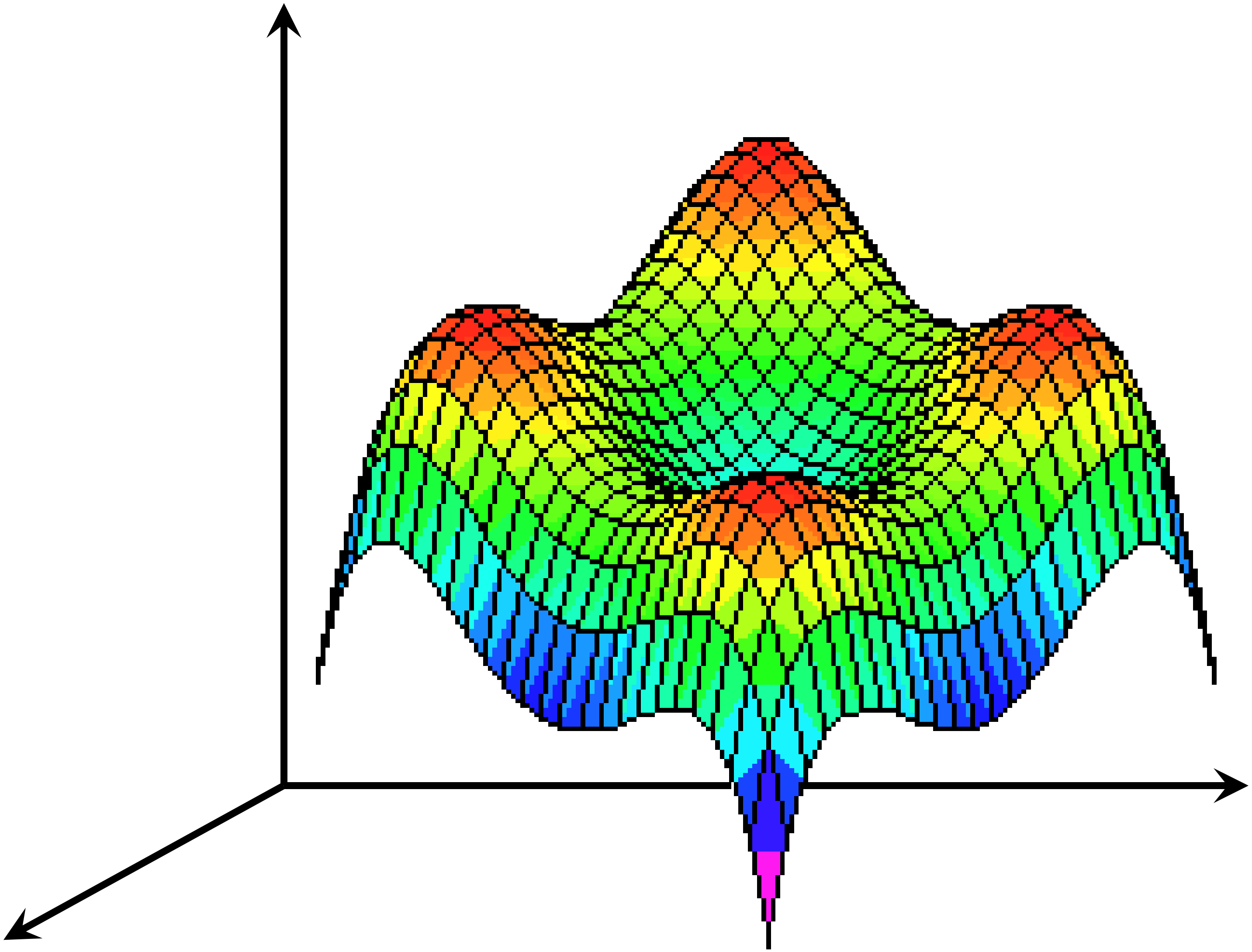
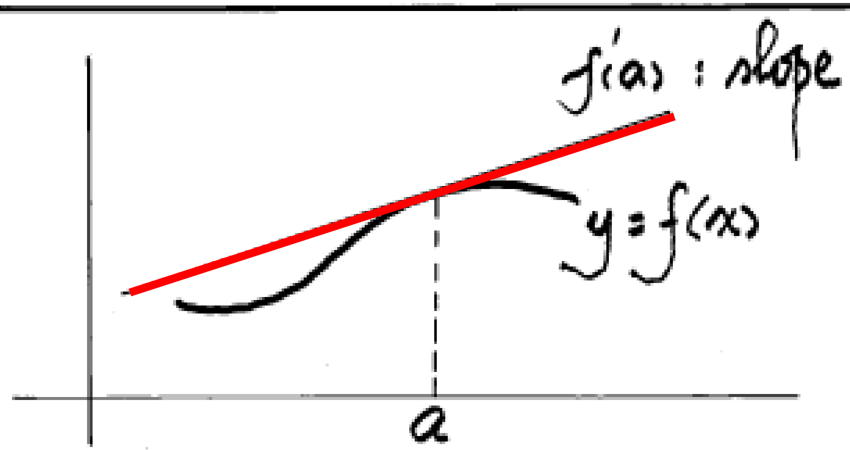


7.3 *Partial Derivatives*

Recall for ***functions*** of ***one*** variable:

$f(x)$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



Objective: Given ***f***(***x***,***y***), to find its ***derivative*** *wrt* one of the 2 variables when the other is held ***constant***. $f(x,b)$ $f(a,y)$

Function of one variable

The ***partial derivative*** of ***f*** *wrt* **x** at (a, b) is

$$\lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

We denote it by $\boxed{\left.\frac{\partial f}{\partial x}\right|_{(a,b)}}$ or $\boxed{f_x(a,b)}$

if the ***limit*** exists.

The ***partial derivative*** of ***f*** *wrt* ***y*** at (a, b) is

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a,b) = \lim_{h\to 0} \frac{f(a, b+h) - f(a,b)}{h}$$

Notation. If $z = f(x, y)$, we also write

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} \quad \& \quad f_y = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}$$

Question. Given $z = f(x, y)$, how to compute

$$\frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} \quad \textbf{?}$$

♣ **Let $z = x^3 \sin(y^2 + x)$. Find $\dfrac{\partial z}{\partial x}$ & $\dfrac{\partial z}{\partial y}$.**

$$\frac{\partial z}{\partial x} = x^3 \cos(y^2 + x) + 3x^2 \sin(y^2 + x)$$ **fix y**

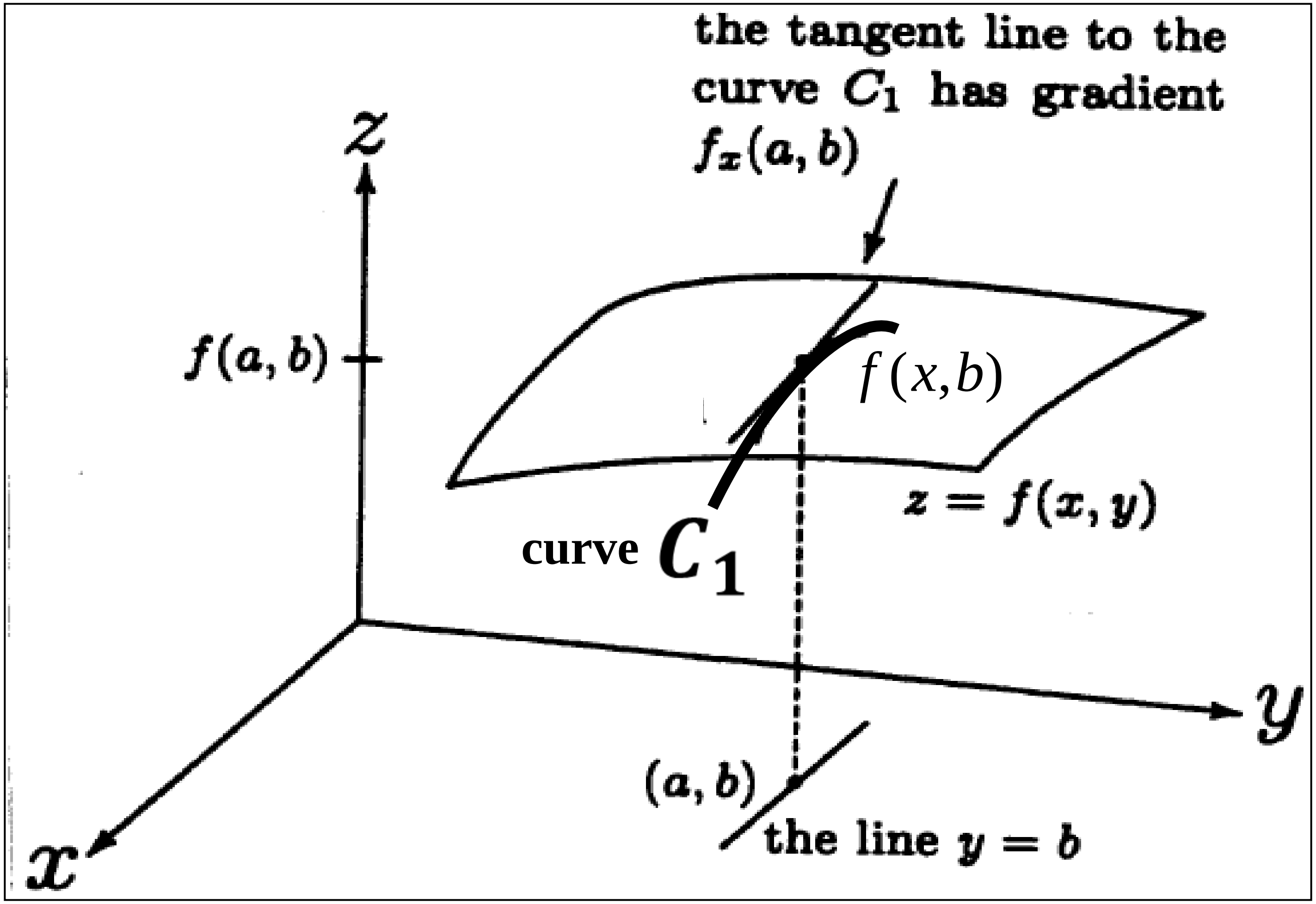
$$\frac{\partial z}{\partial y} = x^3 \cos(y^2 + x) \cdot (2y)$$ **fix x**

♣ **Let $z = e^{xy} \ln y$.**

$$\frac{\partial z}{\partial x} = (\ln y)\, e^{xy} \cdot y$$ **fix y**

$$\frac{\partial z}{\partial y} = e^{xy}\, {}^{1}/_{y} + x\, e^{xy} \ln y$$ **fix x**

● *Geometric interpretation*



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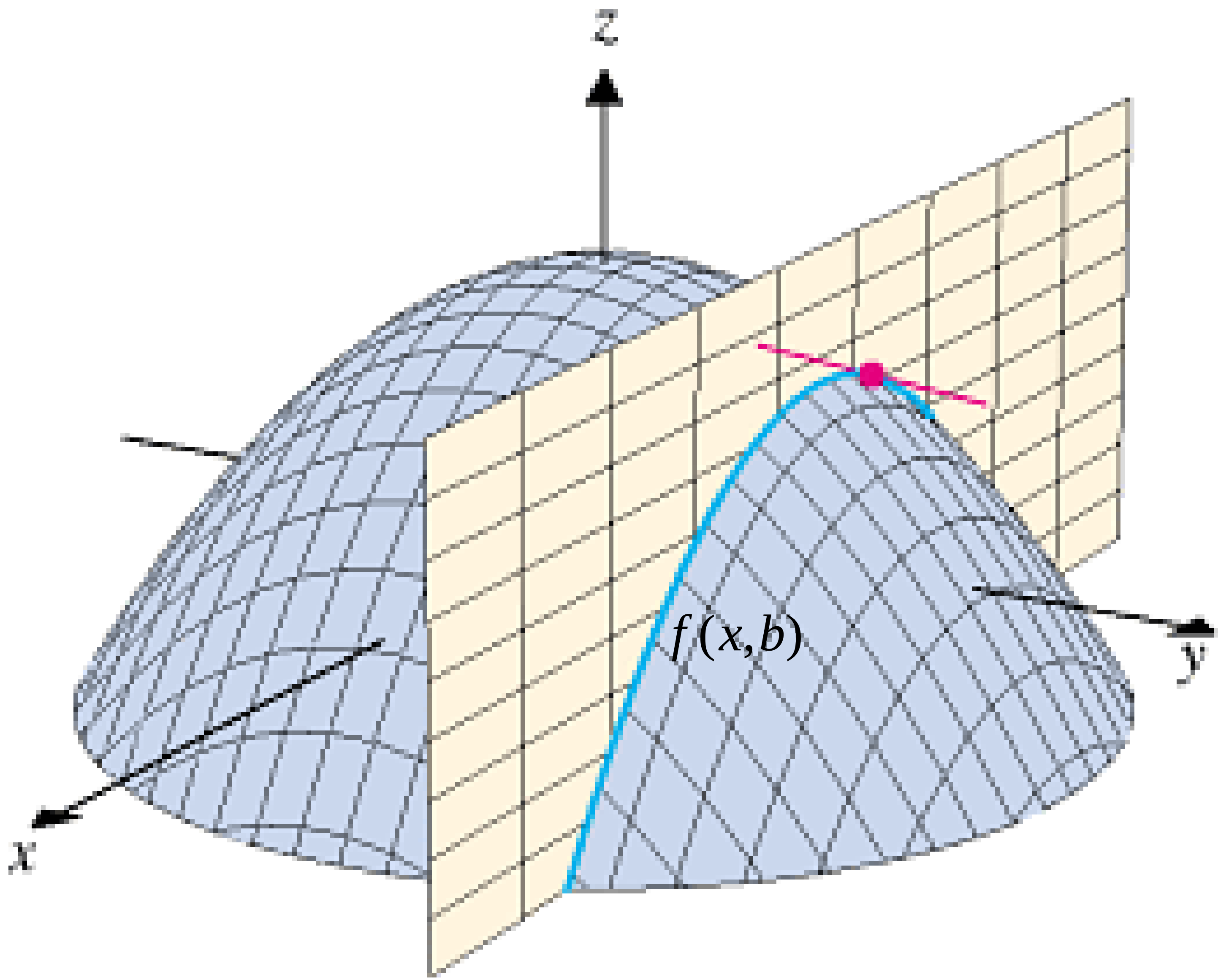
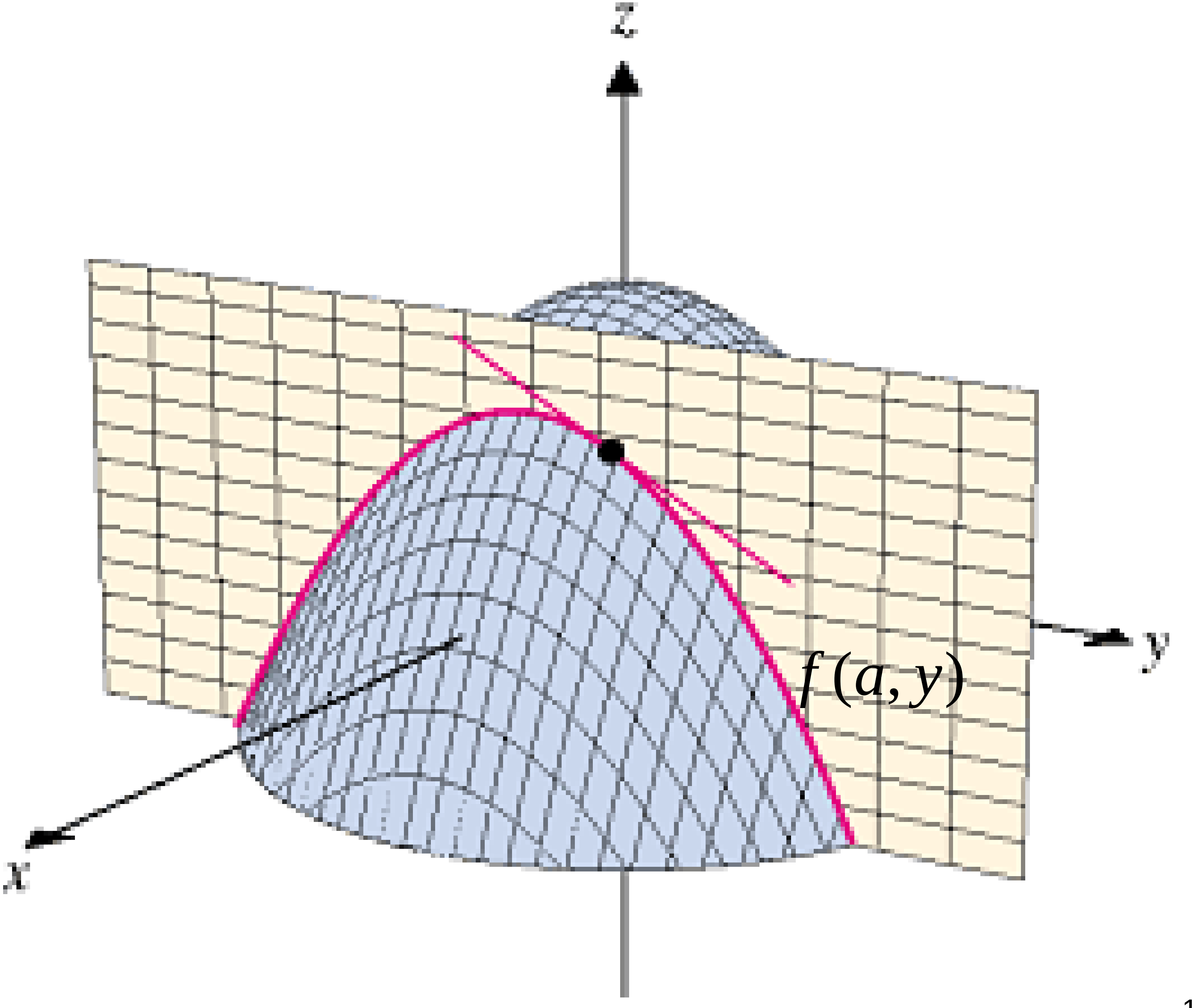

z
$f(x,b)$
y
x

•



z
y
x
$f(a, y)$

- ***Higher order partial derivatives***

The ***2nd order* partial derivatives** of ***f*** are:

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = (f_y)_x = \frac{\partial^2 f}{\partial x \partial y} \qquad f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2}.$$

If ***z = f*(*x,y*)**, we also write:

$$f_{xx} = \frac{\partial^2 z}{\partial x^2} \qquad f_{xy} = \frac{\partial^2 z}{\partial y \partial x} \qquad f_{yx} = \frac{\partial^2 z}{\partial x \partial y} \qquad f_{yy} = \frac{\partial^2 z}{\partial y^2}.$$

♣ Let $\boldsymbol{z = f}(x, y) = x^3 \sin(y^2 + x)$. Then

$$f_x = x^3 \cos(y^2 + x) + 3x^2 \sin(y^2 + x)$$

fix y

$$f_y = 2x^3 y \cos(y^2 + x)$$

fix x

$$f_{xx} = 3x^2\cos(y^2+x) + x^3(-\sin(y^2+x)) + 6x\sin(y^2+x) + 3x^2\cos(y^2+x)$$

fix y

$$f_{xx} = (6x - x^3) \sin(y^2 + x) + 6x^2 \cos(y^2 + x)$$

$$f_{xy} = -2x^3 y \sin(y^2 + x) + 6x^2 y \cos(y^2 + x)$$

$$f_{yx} = \quad ?$$

fix x

Note

Let $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ be a function defined on a region D containing (a,b). If

$\boldsymbol{f}_x$, $\boldsymbol{f}_y$, $\boldsymbol{f}_{xy}$ & $\boldsymbol{f}_{yx}$ are all ***continuous*** in D, then

$$\boldsymbol{f}_{xy}(a,b) = \boldsymbol{f}_{yx}(a,b).$$

♣ **Let $f(x, y) = xy^3 + \frac{\ln y}{\sin y}$. Find $f_{yx}(1, 3)$.**

To compute f_{yx} is difficult, we shall compute f_{xy}

$f_x = y^3$ $\qquad$ $f_{xy} = 3y^2$

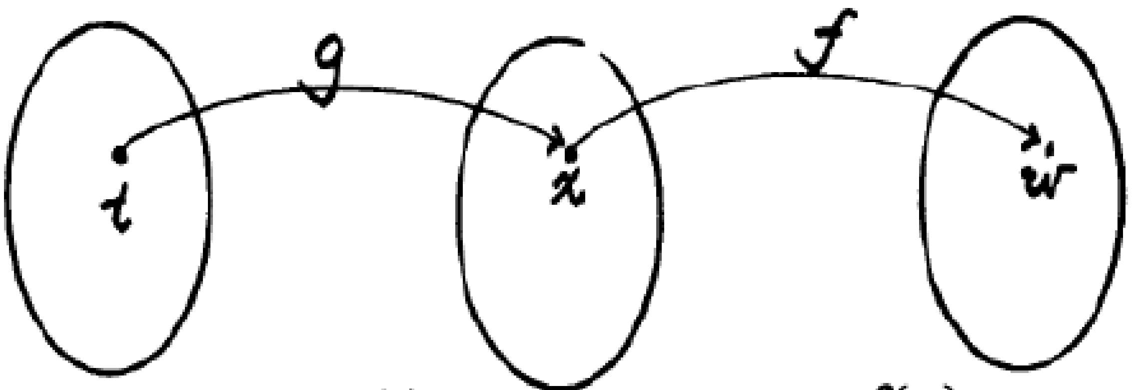
$$f_{yx}(1,3) = f_{xy}(1,3) = 27$$

Remark: For $f(x,y,z)$, we can similarly define

$$f_x\left(= \frac{\partial f}{\partial x}\right),\ f_y\left(= \frac{\partial f}{\partial y}\right)\ \&\ f_z\left(= \frac{\partial f}{\partial z}\right)$$

7.4 The *Chain Rule*

Chain rule for ***functions*** of 1 variable



$x = g(t)$ $w = f(x)$

$$\boxed{\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}}$$

♣ ***Chain rule*** for functions of more than 1 variable - several forms

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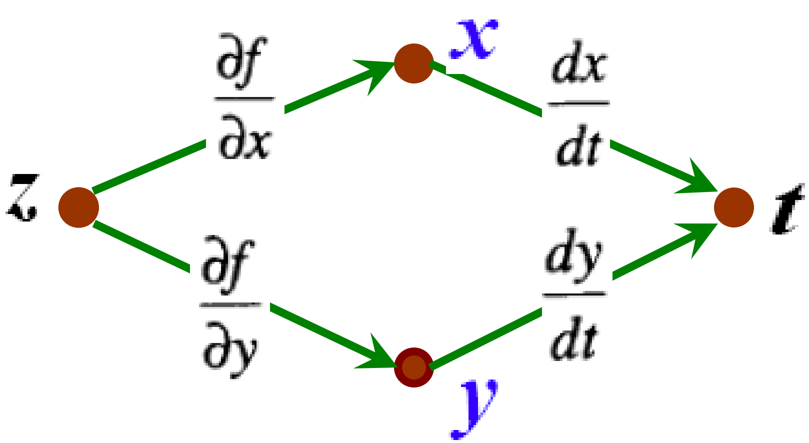
- ***z* = *f*(*x*,*y*)** & ***x* = *x*(*t*), *y* = *y*(*t*)**

  Then ***z* is a *function* of '*t*'**



$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}}$$

♣ $z = x^2 + xy + y^2$,

where $x = \cos t$, $y = \sin t$.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= (2x + y)(-\sin t) + (x + 2y)\cos t$$

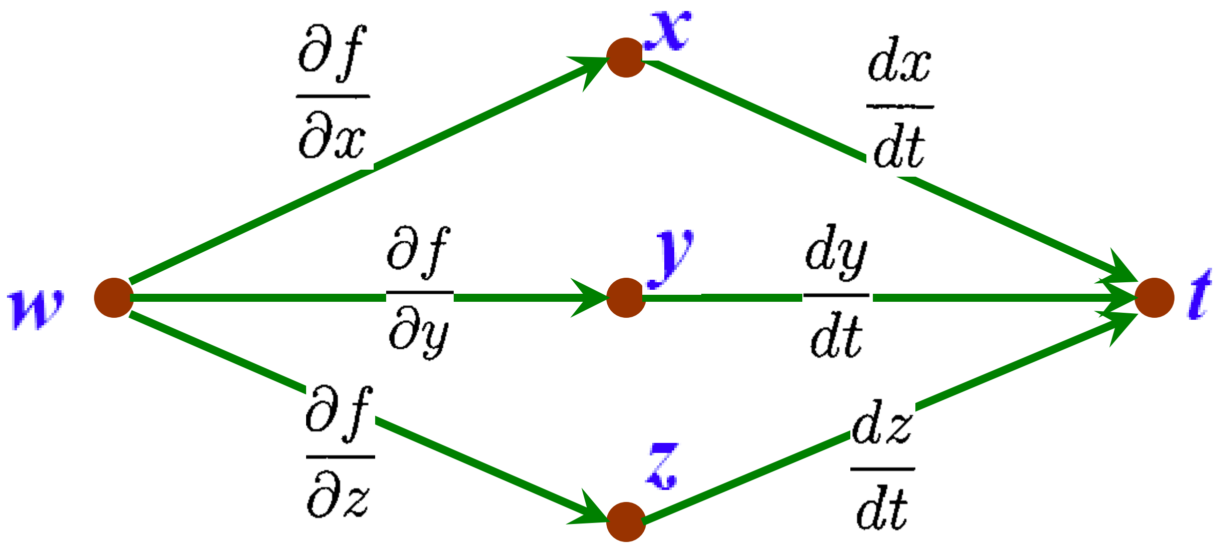
Note we may do:

$$z = (\cos t)^2 + (\cos t)(\sin t) + (\sin t)^2$$

$$\frac{dz}{dt}$$

- $w = f(x,y,z)$ & $x = x(t)$, $y = y(t)$, $z = z(t)$

Then ***w*** is a ***function*** of '***t***'



$$\boxed{\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}}$$

♣ $w = z - \sin xy$,

where $x = t,\ y = \ln t,\ z = e^{t-1}$

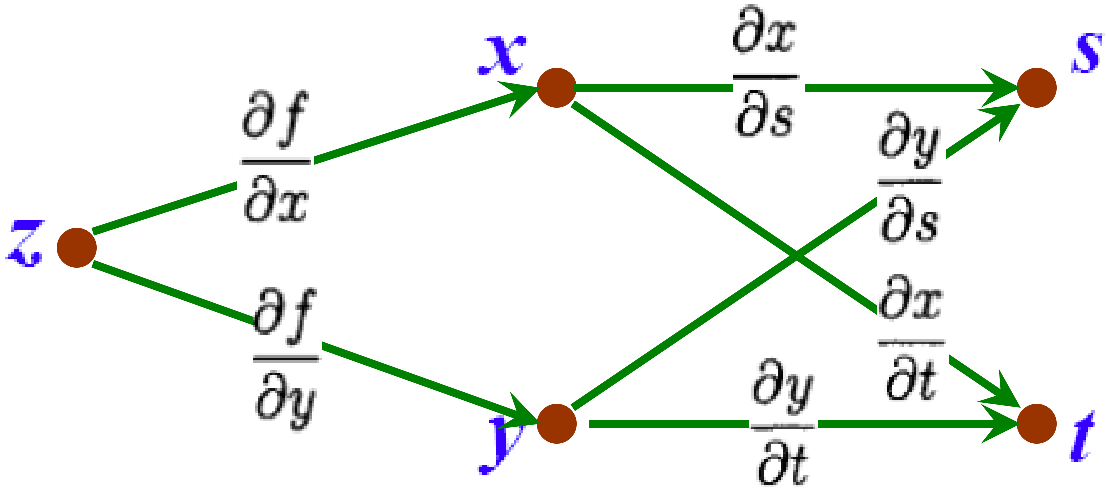
$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= (-\cos xy)\, y \cdot 1 + (-\cos xy)\, x \cdot {}^{1}\!/_{t}$$

$$+ 1 \cdot e^{t-1}$$

- $z = f(x,y)$ & $x = x(s,t)$, $y = y(s,t)$

Then z is a *function* of 's' & 't'

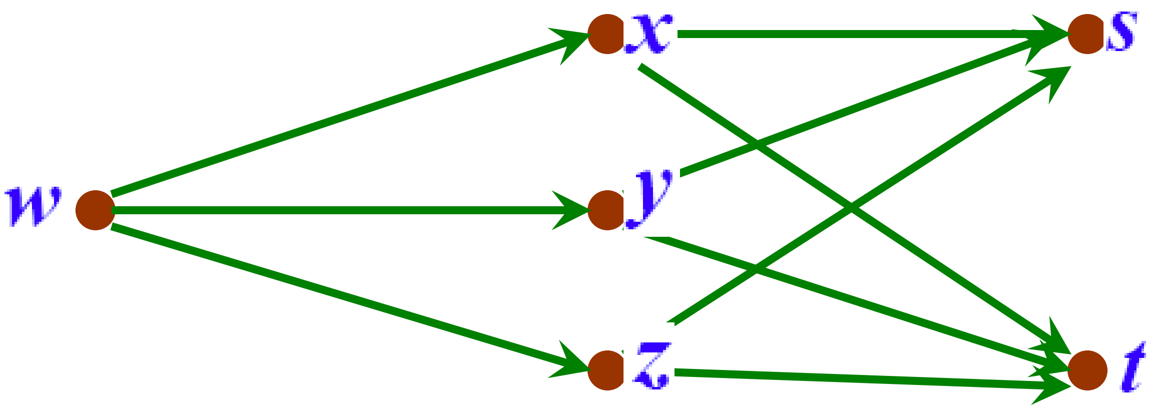


$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}}$$

$$\boxed{\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}}$$

$w = f(x,y,z)$ & $x = x(s,t)$, $y = y(s,t)$, $z = z(s,t)$

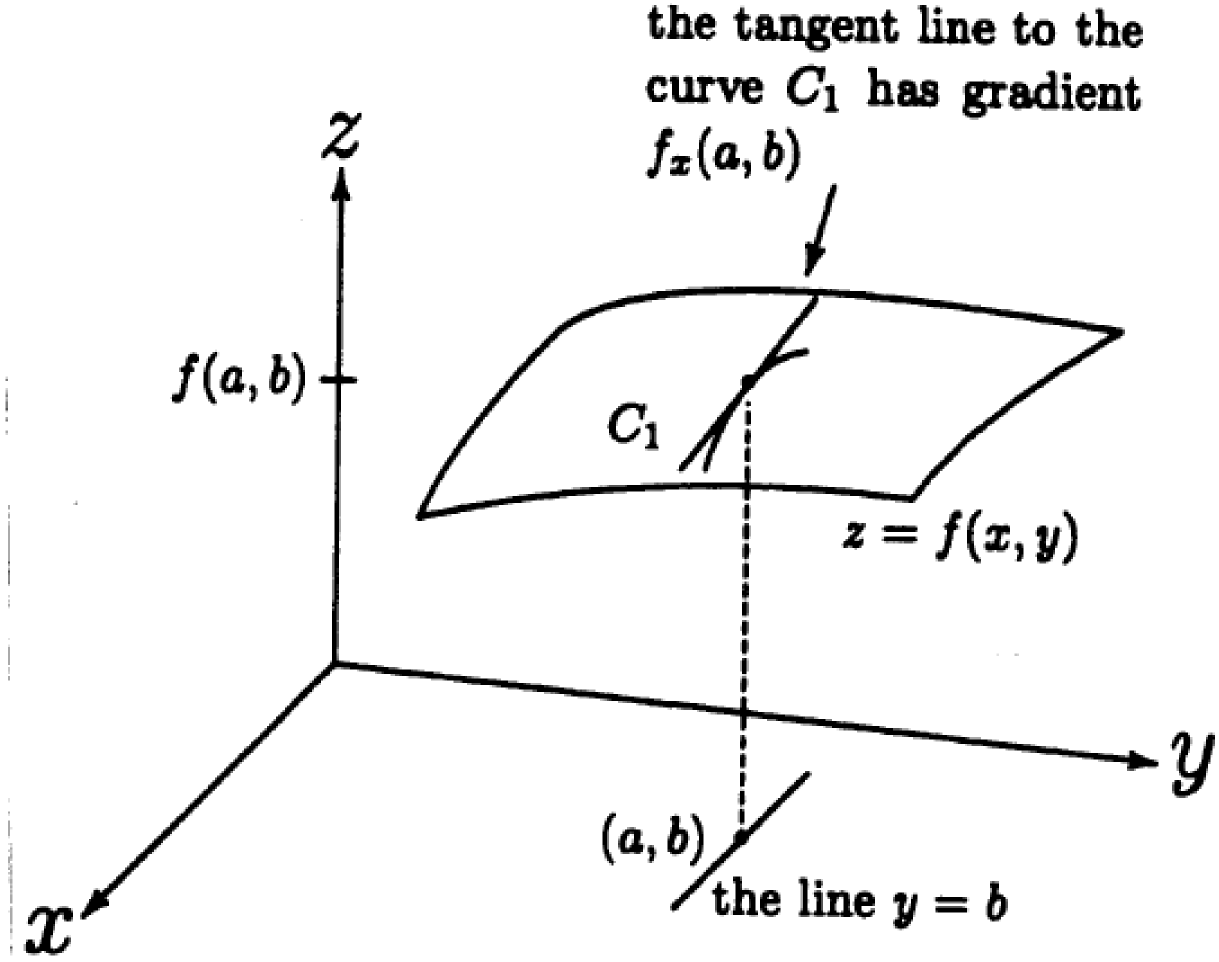
Then w is a *function* of 's' & 't'



$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

Recall: ***Geometric interpretation***



7.5 *Directional Derivatives*

Given ***z* = *f*(*x*,*y*)**,

$f_x(a, b)$ — ***rate of change*** of ***f*** along ***direction*** of ***x*-axis** at (a, b)

$f_y(a, b)$ — ***rate of change*** of ***f*** along ***direction*** of ***y*-axis** at (a, b)

Question: How about the ***rate of change*** of ***f*** along an ***arbitrary direction***?

The ***directional derivative*** of $\boldsymbol{f}$ at (a,b) in the ***direction*** of ***unit* vector** $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is

$$\boxed{D_{\mathbf{u}}f(a,b) = \lim_{h\to 0}\frac{f(a+hu_1,b+hu_2)-f(a,b)}{h}}$$

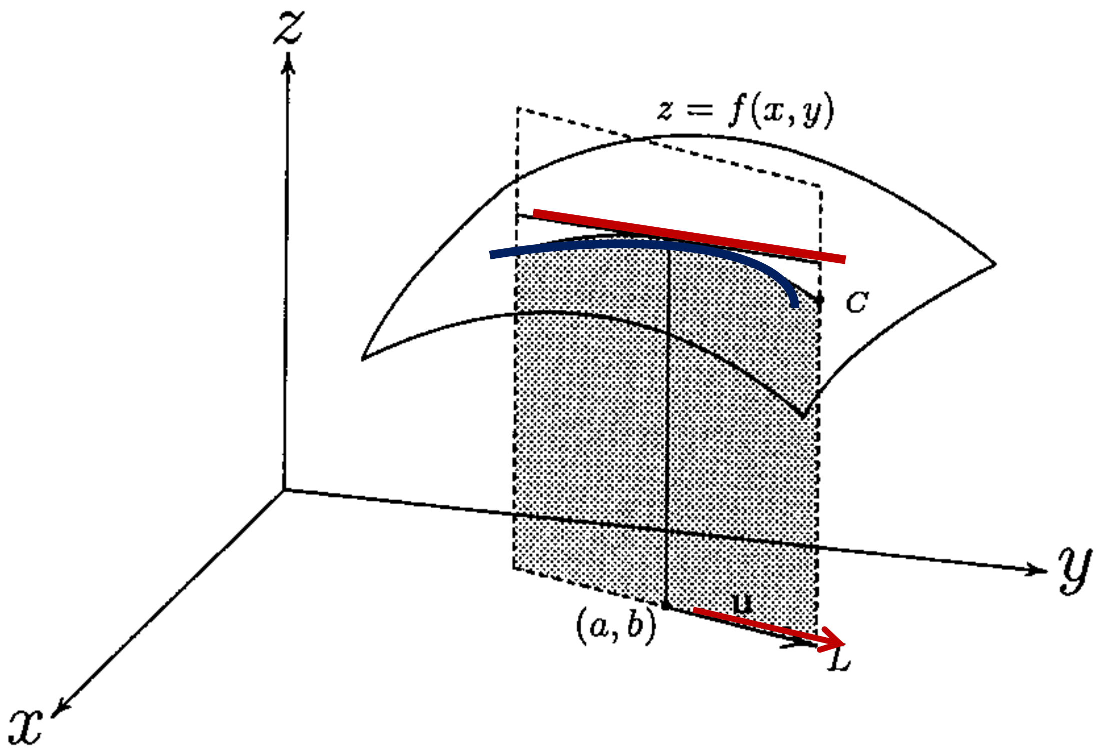
if the limit exists.

$h\mathrm{u} = hu_1\mathrm{i} + hu_2\mathrm{j}$

Note: $\boldsymbol{D_{\mathbf{i}}f}(a,b) = \boldsymbol{f_x}(a,b)$

$\boldsymbol{D_{\mathbf{j}}f}(a,b) = \boldsymbol{f_y}(a,b)$

Geometrical meaning



$D_{\mathbf{u}} f(a, b)$ $\Leftrightarrow$ gradient of the tangent line to the curve C at the point (a, b)

Question: How to compute $D_u f(a,b)$?

$$D_u f(a,b) = f_x(a,b)\, u_1 + f_y(a,b)\, u_2$$

where $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a ***unit*** vector

without proof

♣ Find $D_{\mathbf{u}} f(2,1)$, where
$f(x,y) = x^2 - 3xy^2 + 2y^3$ and $\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$.

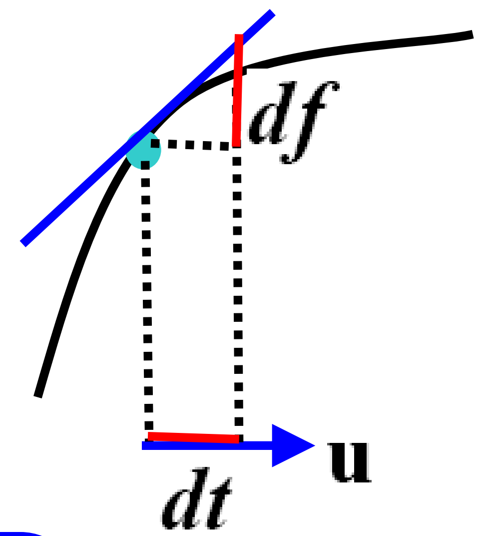
First, $f_x = 2x - 3y^2$, $f_y = -6xy + 6y^2$.
Thus $f_x(2,1) = 1$ and $f_y(2,1) = -6$.

Hence

$$D_{\mathbf{u}} f(2,1) = (1)\left(\frac{\sqrt{3}}{2}\right) + (-6)\left(\frac{1}{2}\right) = \frac{\sqrt{3}-6}{2}.$$

Physical meaning

The ***directional derivative*** $D_u f(a,b)$ measures the ***change*** in the value Δf of a function f when we move a small distance dt from the point (a,b) in the direction of the vector **u**:



$$\Delta f \approx df \text{ where}$$

$$df = D_{\underline{u}} f(a,b) \cdot dt.$$

Usual multiplication

♣ Let $f(x, y) = x^2y^3 + 1$.

$$\boxed{df = D_{\mathbf{u}}f(a,b) \cdot dt.}$$

Estimate how much the value of f will change if a point Q moves 0.1 unit from $(2, 1)$ towards $(3, 0)$.

Q moves in the direction: $(3\,\mathbf{i} + 0\,\mathbf{j}) - (2\,\mathbf{i} + \mathbf{j}) = \mathbf{i} - \mathbf{j}$.

The unit vector $\mathbf{u}$ along this direction is $\dfrac{1}{\sqrt{2}}\mathbf{i} - \dfrac{1}{\sqrt{2}}\mathbf{j}$.

$\boxed{f_x = 2xy^3,\ f_y = 3x^2y^2} \Rightarrow \boxed{f_x(2,1) = 4 \text{ and } f_y(2,1) = 12.}$

Thus $\boxed{D_{\mathbf{u}}f(2,1)} = (4)(\dfrac{1}{\sqrt{2}}) + (12)(-\dfrac{1}{\sqrt{2}}) = -\dfrac{8}{\sqrt{2}}$.

$$\Delta f \approx df = D_{\mathbf{u}}f(2,1) \cdot dt = (-\frac{8}{\sqrt{2}})(0.1) \approx -0.57.$$

So the value of f decreases by approximately 0.57 unit.

Question 4 (b) [5 marks]

Let $f(x,y)$ be a differentiable function of two variables such that $f(2,1) = 1506$ and $\frac{\partial f}{\partial x}(2,1) = 4$. It was found that if the point Q moved from $(2,1)$ a distance 0.1 unit towards $(3,0)$, the value of f became 1505. Estimate the value of $\frac{\partial f}{\partial y}(2,1)$.

$$Let\ \frac{\partial f}{\partial y}(2,1) = a$$

$$\vec{u} = unit\ vector\ from\ (2,1)\text{ to }(3,0)$$

$$= \frac{(3,0)-(2,1)}{\|(3,0)-(2,1)\|} = \frac{1}{\sqrt{2}}(1,-1)$$

$$D_u f(2,1) = (4)\frac{1}{\sqrt{2}} + a\frac{-1}{\sqrt{2}} = \frac{4-a}{\sqrt{2}}$$

$$\Delta f = 1505 - 1506 \approx df = \frac{4-a}{\sqrt{2}}(0.1)$$

$$a \approx 4 + 10\sqrt{2} \approx 18.14$$

Functions of **three variables**

- Given ***f***(***x***, ***y***, ***z***), the ***directional derivative*** of ***f*** at (a, b, c) in the direction of a ***unit*** vector : $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k}$ is

$$\boxed{D_{\mathbf{u}} f(a,b,c) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2, c+hu_3) - f(a,b,c)}{h}}$$

if this limit exists.

$$\boxed{D_{\mathbf{u}} f(a,b,c) = f_x(a,b,c)u_1 + f_y(a,b,c)u_2 + f_z(a,b,c)u_3}$$

$$\boxed{df = D_u f(a,b,c) \cdot dt}$$

08/09(Sem 1)

Question 5 (a) [5 marks]

Let $f(x, y, z)$ be a differentiable function of three variables, P be a point in space and $f(P) = 1$. It is known that the values of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ at P are equal to $-\sqrt{3}$, $-\frac{\sqrt{3}}{4}$, $-\frac{1}{\sqrt{12}}$ respectively. Suppose P moves 0.1 unit in the direction of the vector $\boldsymbol{\nu} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ to the point Q. Estimate the value of $f(Q)$.

Self study

Solution. $$df = D_u f(a,b,c)\cdot dt$$

$$f(Q) - f(P) \approx df = D_u f(a,b,c)\cdot dt$$

Direction vector : $v = \mathbf{i} - \mathbf{j} - \mathbf{k}$

Unit direction vector : $u = \dfrac{v}{|v|} = \dfrac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} - \mathbf{k})$

$$\begin{aligned} D_u f(P) &= f_x(P)u_1 + f_y(P)u_2 + f_z(P)u_3 \\ &= (-\sqrt{3})\cdot\frac{1}{\sqrt{3}} + \left(-\frac{\sqrt{3}}{4}\right)\cdot\left(-\frac{1}{\sqrt{3}}\right) + \left(-\frac{1}{\sqrt{12}}\right)\cdot\left(-\frac{1}{\sqrt{3}}\right) \\ &= -1 + \frac{1}{4} + \frac{1}{6} = -\frac{7}{12}. \end{aligned}$$

Thus, $$f(Q) - 1 \approx D_u f(P)\cdot(0.1) = -\frac{7}{120}$$

$$f(Q) \approx \frac{113}{120}.$$

Gradient Vector

The gradient of $f(x, y)$

is the vector (function) $\nabla f = f_x \mathrm{i} + f_y \mathrm{j}$

For a given unit vector $\mathrm{u} = u_1 \mathrm{i} + u_2 \mathrm{j}$

$$\nabla f(a,b) \bullet \mathrm{u} = (f_x(\mathrm{a,b})\mathrm{i} + f_y(\mathrm{a,b})\mathrm{j}) \bullet (u_1 \mathrm{i} + u_2 \mathrm{j})$$

$$= f_x(\mathrm{a,b}) u_1 + f_y(\mathrm{a,b}) u_2$$

$$= D_{\mathrm{u}} f(a,b)$$

Let θ be the angle between two vectors

$\nabla f(a,b)$ and u, where $0 \le \theta \le \pi$

$$D_{\mathrm{u}} f(a,b) = \nabla f(a,b) \bullet \mathrm{u} = \left\|\nabla f(a,b)\right\| \left\|u\right\| \cos\theta = \left\|\nabla f(a,b)\right\| \cos\theta$$

$$- \le \cos\theta \le 1$$

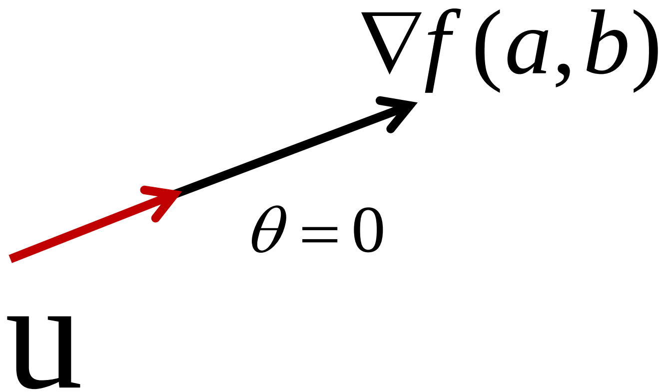
Hence we have

$D_u f(a,b)$ is positive and maxi when $\cos\theta = 1$ i.e., $\theta = 0$

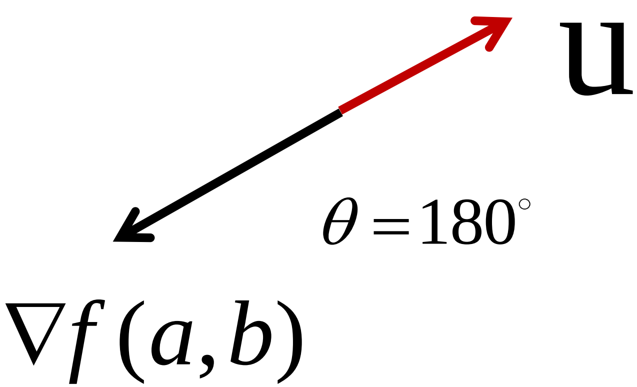
$D_u f(a,b)$ is negative and mini when $\cos\theta = -1$

i.e., $\theta = 180^\circ$

How to choose the direction u such that the following two cases occur



At (a,b), the function f increases most rapidly when u is in the direction $\nabla f(a,b)$



At (a,b), the function f decreases most rapidly when u is in the direction $-\nabla f(a,b)$

Example

Let $f(x,y)=\sqrt{9-x^2-y^2}$

Find the largest possible value of $D_u f(2,1)$

Solution

$D_u f(2,1)$ is maxi when $\theta = 0$

$$D_{\mathrm{u}} f(2,1)=\nabla f(2,1)\bullet \mathrm{u}=\|\nabla f(2,1)\|\|u\|\cos\theta$$

$$=\|\nabla f(2,1)\|\cos\theta=\|\nabla f(2,1)\|$$

$$\nabla f(2,1)= f_x(2,1)\mathrm{i}+f_y(2,1)\,\mathrm{j}$$

$$\boxed{f_x=\frac{-x}{\sqrt{9-x^2-y^2}},\ f_y=\frac{-y}{\sqrt{9-x^2-y^2}}}$$

ANS: maxi value of $D_u f(2,1)=\|\nabla f(2,1)\|=\dfrac{\sqrt{5}}{2}$

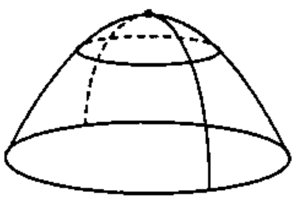
7.6 *Maximum* & *Minimum* Values

- **Local max & min**

(1) $f(x,y)$ has a ***local maximum*** at (a,b) if

$f(x,y) \le f(a,b)$ for ***all*** points (x,y) near (a,b).

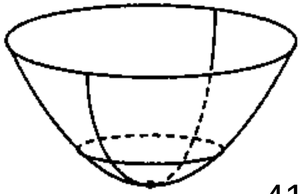
$f(a,b)$ — a ***local maximum value***.



(2) $f(x,y)$ has a ***local minimum*** at (a,b) if

$f(x,y) \ge f(a,b)$ for ***all*** points (x,y) near (a,b).

$f(a,b)$ — a ***local minimum value*.**



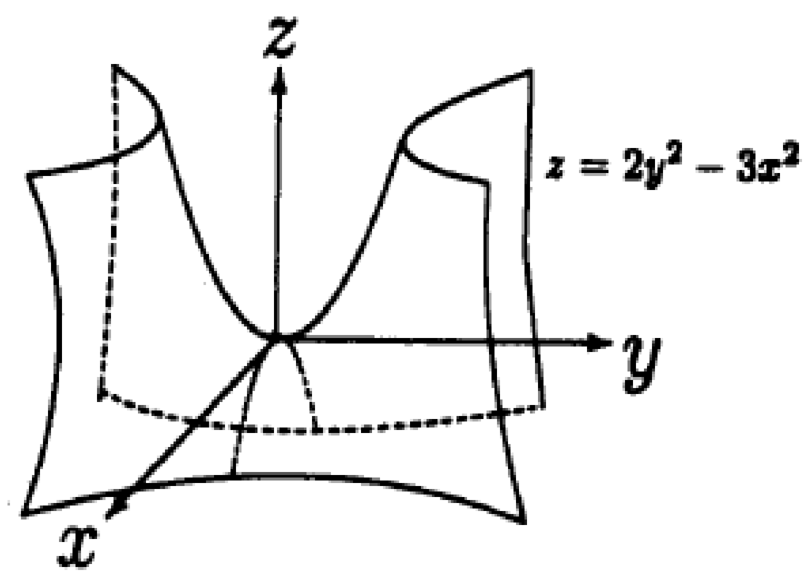
• Critical Points

(First Derivative Test) If $\boldsymbol{f}$ has a local ***maximum*** or ***minimum*** at (a,b), & both $\boldsymbol{f}_x(a,b)$ & $\boldsymbol{f}_y(a,b)$ exist, then

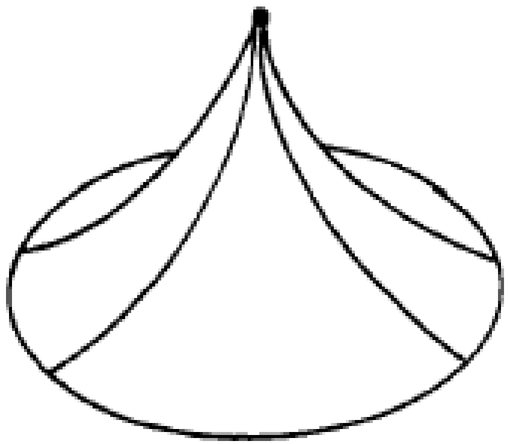
$$\boldsymbol{f}_x(a,b) = 0 \;\&\; \boldsymbol{f}_y(a,b) = 0.$$

The ***converse*** is not true. For example

♣ (***saddle point***)



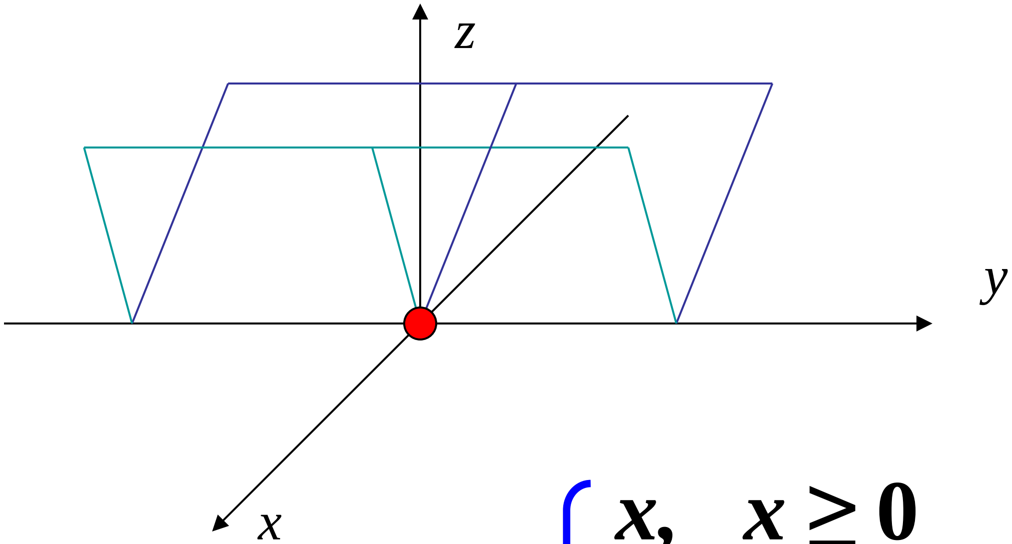
♣ ***f*** may have a **local maximum** or **minimum** at (a,b), where $\boldsymbol{f}_x(a,b)$ or $\boldsymbol{f}_y(a,b)$ does not exist.



A point (a,b) is called a ***critical point*** of $\boldsymbol{f}$ if

(i) $\boldsymbol{f}_x(a,b) = 0$ & $\boldsymbol{f}_y(a,b) = 0$; or

(ii) $\boldsymbol{f}_x(a,b)$ or $\boldsymbol{f}_y(a,b)$ does not exist



$$z = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

z is ***minimum*** at (0,0), but

$${}^{\partial z}/_{\partial x}\,\Big|_{(0,0)}$$

does not exist.

http://www.math.uri.edu/~bkaskosz/flashmo/tools/graph3d/

Second Derivative Test

Assume that f & its 1*st* & 2*nd* partial derivatives are ***continuous*** in a region containing (a,b) *s.t.*

$$f_x(a,b) = 0 \;\&\; f_y(a,b) = 0.$$

Let

$$\boxed{D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2.}$$

Discriminant of f

(a) If $D > 0$ and $f_{xx}(a,b) > 0$, then f has a **local minimum** at (a,b).

(b) If $D > 0$ and $f_{xx}(a,b) < 0$, then f has a **local maximum** at (a,b).

(c) If $D < 0$, then f has a **saddle point** at (a,b).

(d) If $D = 0$, then **no conclusion** can be drawn.

$f(x,y)$, (a,b)

$$f_x(a,b) = 0 = f_y(a,b)$$

$$D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$$

$D > 0$

$f_{xx} > 0$ → Min

$f_{xx} < 0$ → Max

$D < 0$ → Saddle Point

$D = 0$ → *No Conclusion*

♣ $f(x,y) = y^3 + 3x^2y - 3x^2 - 3y^2 + 2$

$f_x = 6x(y - 1)$

$f_y = 3(y^2 + x^2 - 2y)$

Solving $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$

yields 4 ***critical*** points:

(0,0), (0,2), (1,1), (-1,1).

$$f_{xx} = 6(y - 1)$$
$$f_{yy} = 6(y - 1)$$
$$f_{xy} = 6x$$

$$D = f_{xx}f_{yy} - f_{xy}^2$$

	(0,0)	(0,2)	(1,1)	(-1,1)
f_{xx}	-6	6	0	0
f_{yy}	-6	6	0	0
f_{xy}	0	0	6	-6
D	36	36	-36	-36
	Max	**min**	**saddle pts**	

08/09(Sem 2)

Question 5 (b) [5 marks]

Find the local maximum, local minimum and saddle points (if any) of

$$f(x,y) = x^3 - 3xy - y^3.$$

Ans. (0, 0) – saddle point; (-1, 1) – local max.

08/09 (Sem 1)

Question 5 (b) [5 marks]

Let n be a fixed positive integer and $n \geq 2$. Find, if any, the local maximum points, the local minimum points and the saddle points, of the function

$$f(x, y) = \ln(x^n y) - xy - (n-1)x,$$

which is defined in the domain $x > 0$ and $y > 0$.

Self study

Solution. $$f(x,y) = \ln(x^n y) - xy - (n-1)x$$

$$f_x = \frac{nx^{n-1}y}{x^n y} - y - (n-1) = \frac{n}{x} - y - (n-1) = 0 \qquad (1)$$

$$f_y = \frac{x^n}{x^n y} - x = \frac{1}{y} - x = 0 \qquad (2)$$

From (2): $\frac{1}{x} = y \qquad (3)$

Substituting (3) in (1) gives $y(n-1) = n-1$

and so $y = 1$ (as $n \geq 2$).

From (3): $x = 1$.

Thus $(1, 1)$ is the only critical point.

- Observe that

$$f_{xx} = -\frac{n}{x^2}, \qquad f_{xy} = -1, \qquad f_{yy} = -\frac{1}{y^2}.$$

Thus, at (1, 1),

$$D = f_{xx} f_{yy} - f_{xy}{}^2$$

$$= (-n)\,(-1) - (-1)^2 = n - 1 > 0$$

and $$f_{xx} = -n < 0\,.$$

We therefore conclude that (1, 1) is a local maximum point.

7.6.9 Lagrange Multipliers

We shall use one example to illustrate the method of Lagrange multipliers without proof

Example

Find relative extrema of

$$z = f(x, y) = 12x - 16y + 50$$

subject to the constraint $x^2 + y^2 = 25$

Solution

$$\boxed{x^2 + y^2 = 25}$$

constraint

Let $g(x, y) = x^2 + y^2 - 25$

and $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$

$$= 12x - 16y + 50 - \lambda(x^2 + y^2 - 25)$$

Let $F_x = 0,\ F_y = 0,\ F_\lambda = 0$ we get

$$12 - 2\lambda x = 0$$

$$-16 - 2\lambda y = 0$$

$$-x^2 - y^2 + 25 = 0$$

Solve these three equations, get

$$\lambda = 2,\ x = 3,\ y = -4\ \&\ \lambda = -2,\ x = -3,\ y = 4$$

Subject to the constraint $x^2 + y^2 = 25$

$z = f(3,-4) = 150$ Local maxi

$z = f(-3,4) = -50$ Local mini

http://www.math.uri.edu/~bkaskosz/flashmo/tools/graph3d