Summary

CH 9 Line Integrals

(A)

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\|\mathbf{r}'(t)\|dt$$

Equation of curve *C*: $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

$$a \le t \le b$$

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\|r'(t)\|dt$$

C : $\mathbf{x}(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$,

$$a \le t \le b$$

(B)

curve *C* :

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad a \le t \le b$$

F(x,y,z) : vector field defined on ***C***

$$\mathbf{F}(\mathbf{r}(t)) = \mathbf{F}(x(t), y(t), z(t))$$

Then

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt}$$

Also holds for F(x,y)

(C) $\int_C \mathbf{F} \cdot d\mathbf{r}$ in (B) can be computed by the following formulae

If $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$

then

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy.}$$

Similarly, for if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$,

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz.}$$

Suppose $$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

$$C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad t \in [a,b].$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy$$

$$= \int_a^b P(x(t), y(t))\,x'(t)\,dt + \int_a^b \mathrm{Q}(x(t), y(t))\,\mathrm{y}'(t)\,dt$$

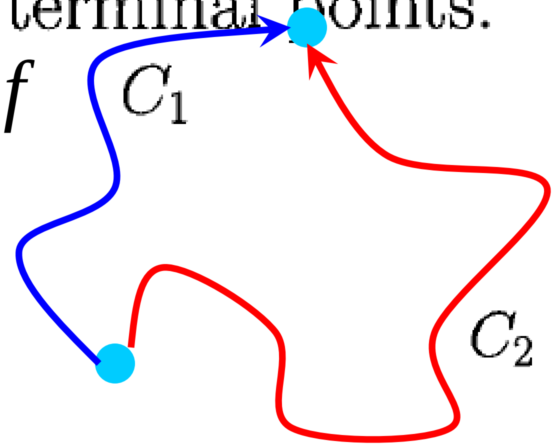
Similarly for F(x,y,z)

Conservative vector field

(I)(a) If $\mathbf{F}$ is a conservative vector field, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is *independent of path*,

i.e. $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any 2 paths C_1 and C_2 that have the same initial and terminal points.

(b) If **F** is conservative and $F = \nabla f$



$$\int_C \mathrm{F} \cdot \mathrm{dr} = \int_C \nabla f \cdot \mathrm{dr}$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Example

Curve C: $\mathbf{r}(t) = \cos t\mathbf{i} + e^t \sin t\mathbf{j}, \;\; t \in [0, \pi]$

$\mathbf{r}(0) = \mathbf{i} = \mathbf{i} + 0\mathbf{j} \rightarrow (1,0)$

$\mathbf{r}(\pi) = -\mathbf{i} = -\mathbf{i} + 0\mathbf{j} \rightarrow (-1,0)$

$(0,0) \longrightarrow (1,0)$

position vector i

$\mathrm{F}=\nabla f\,, where\ f(x,y) = xy^2 + x^3$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(-1,0) - f(1,0) \;\; = -2.$$

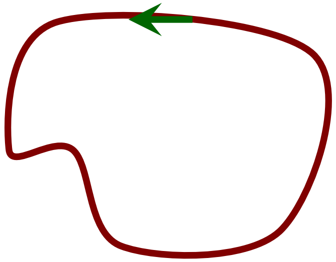
$$\boxed{\int_C \mathrm{F} \cdot \mathrm{dr} = \int_C \nabla f \cdot \mathrm{dr} \;\; = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).}$$

Conservative vector field

(II) If $\mathbf{F}$ is a conservative vector field,

$$\text{then } \oint_{\ell} \mathbf{F} \cdot d\mathbf{r} = 0$$

$\boxed{\text{for any } \textit{closed} \text{ curve } \ell}$

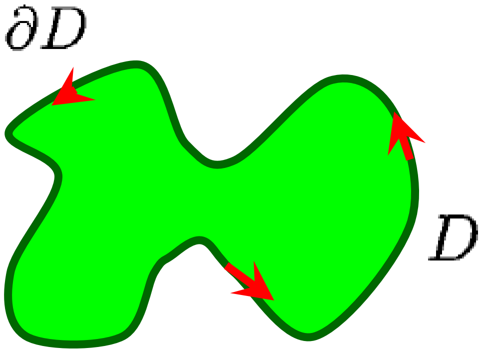


DO YOU KNOW HOW TO CHECK
WHETHER VECTOR FIELD F(x,y) IS CONSERVATIVE?

Green's Theorem

Let ***D*** be a bounded region in the *xy*-plane & ***∂D*** the boundary of ***D***. Then

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\, dA.$$

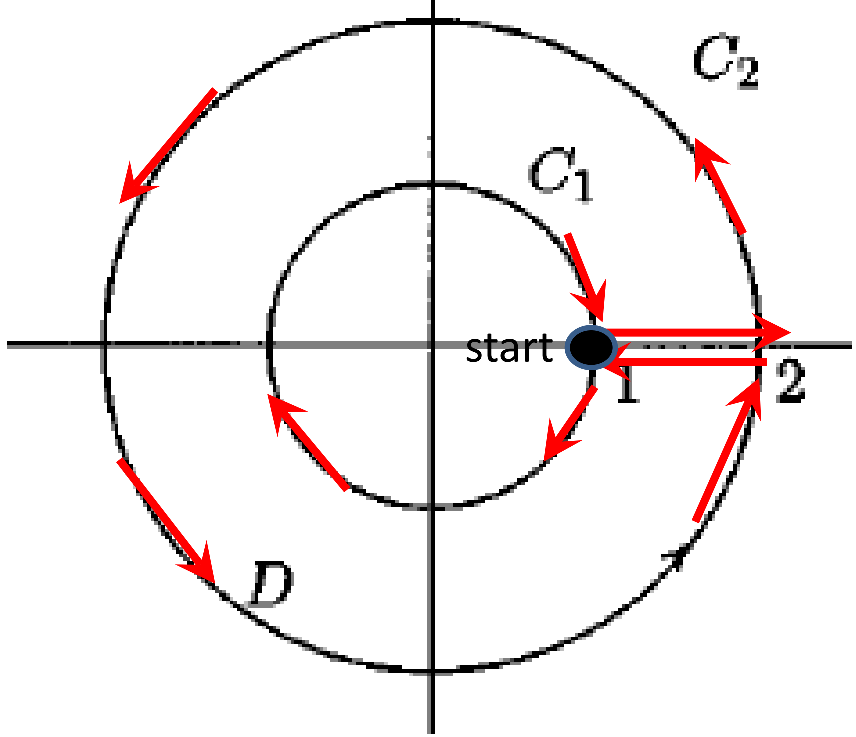


We move along *∂D* in the direction that the region ***D*** always on the **left. This direction is called positive direction.**

Positive directions, for many cases, always anticlockwise. However for some special cases, they are clockwise, see next example

Region enclosed by

C_1 and C_2. How to choose positive direction



$$\partial D = -C_1 + l + C_2 - l = C_2 - C_1$$

C_1, C_2 anticlockwise by given

l bridge form left to right

Remarks of Green's theorem

(A) Green's Theorem only applied to vector fields F(x,y) of two variables: F(x,y)=P(x,y)i+Q(x,y)j

$$\oint_{\partial D} \mathrm{F} \cdot \mathrm{dr} = \oint_{\partial D} P dx + Q dy = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dA$$

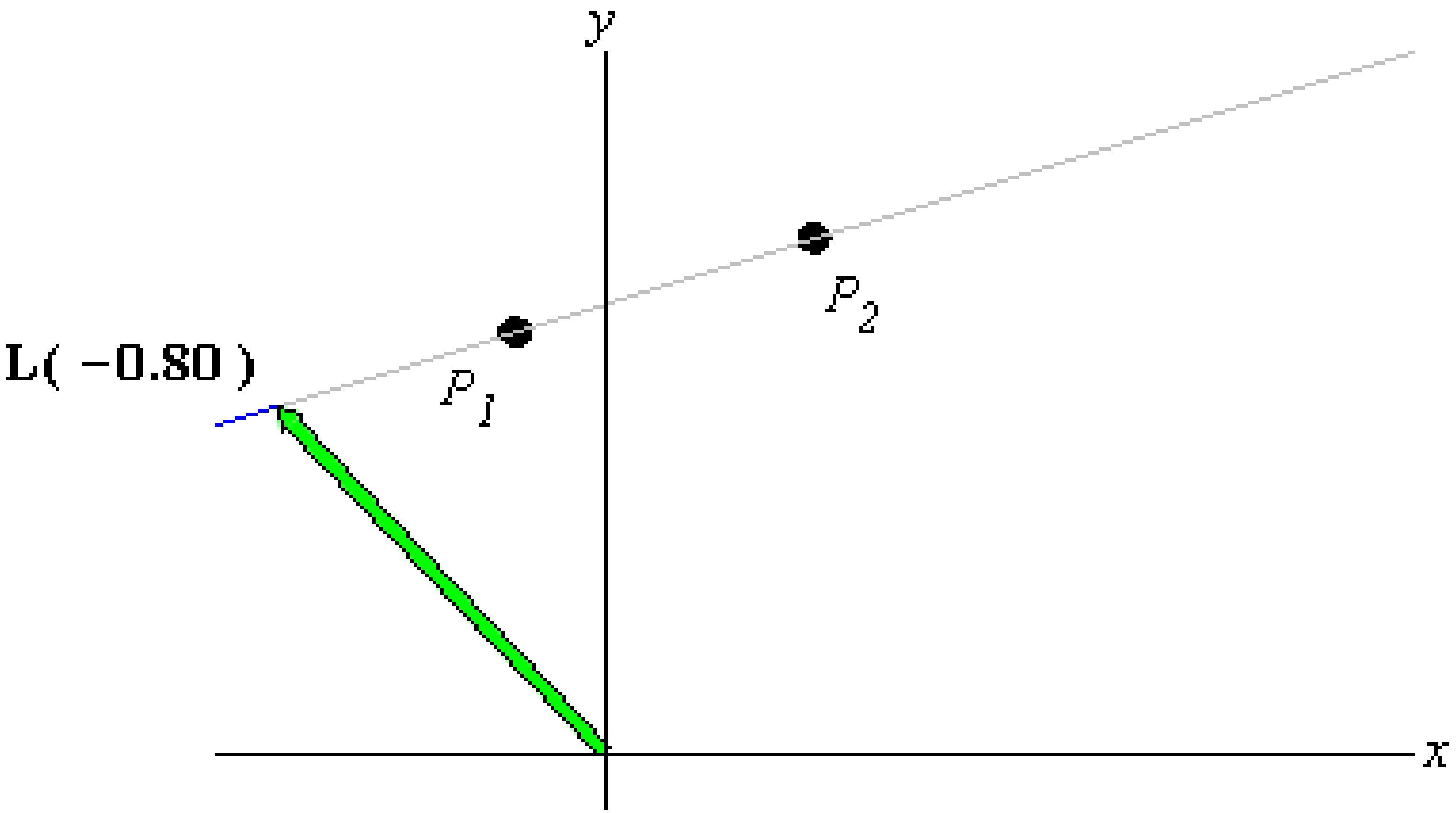
(B) Green's Theorem only applied to closed curves

Vector equations of line segments in **THREE** dim space

Line and line segment

Recall vector equation of line in 2-dim space

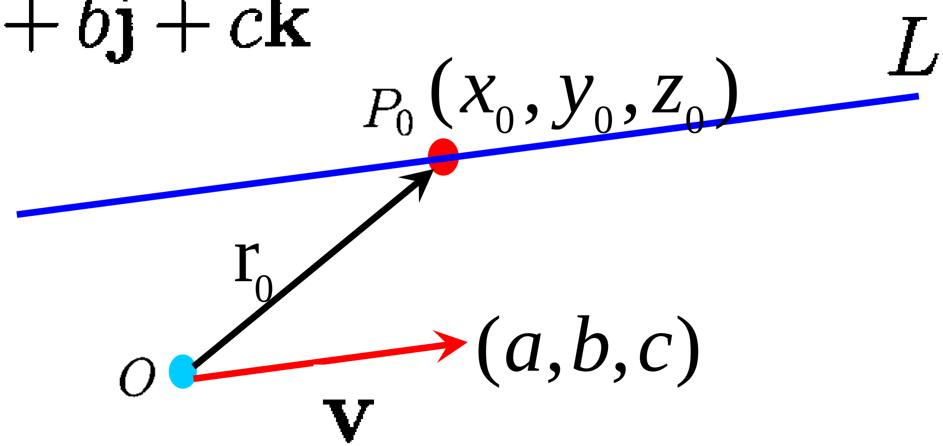
We can draw a line by vectors



Problem Given point P_0 with position vector

$$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$$

& vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$



find the ***equation*** of the ***line*** ***L*** passing through P_0

& parallel to $\mathbf{v}$

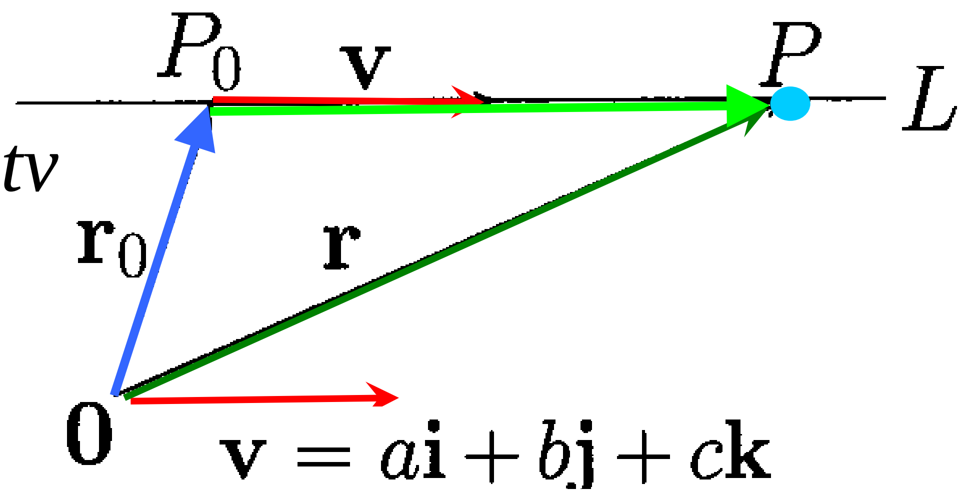
Let ***P*** be a point on **line** *L* with ***position*** vector

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$\overrightarrow{P_0P}$ and v are parallel

$\overrightarrow{OP} = r = r_0 + \overrightarrow{P_0P} = r_0 + tv$

for some $t \in \mathbf{R}$.



$$= (x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}) + t(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$$

$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ $\boxed{t \in \mathbf{R}}$ **vector equation** of **line L**

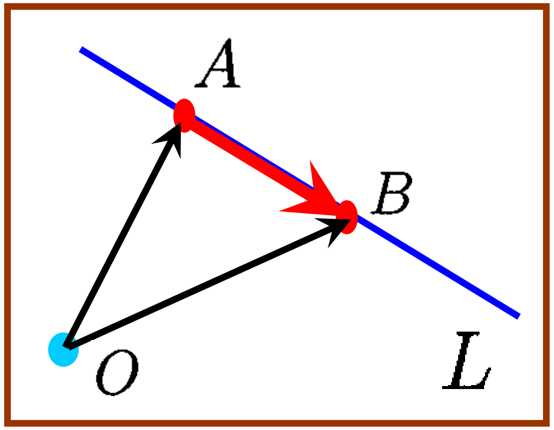
$r = r_0 + tv,\ 0 \le t \le 1$ for line segment P_0P

Example

A and B have position vectors

$$-3\mathbf{i}+2\mathbf{j}-3\mathbf{k} \quad \text{and} \quad \mathbf{i}-\mathbf{j}+4\mathbf{k}$$

Write down vector equation of line segment from A to B



$$\overrightarrow{AB} = (\mathbf{i}-\mathbf{j}+4\mathbf{k})-(-3\mathbf{i}+2\mathbf{j}-3\mathbf{k})$$
$$= 4\mathbf{i}-3\mathbf{j}+7\mathbf{k}.$$

Vector equation of ***line segment AB is***

$$r=(-3i+2j-3k)+t(4i-3j+7k)$$

$$0 \le t \le 1$$

Example

- ♣ The ***vector*** eqn.

$$\mathbf{r}(t) = (1+t)\mathbf{i} + (2+t)\mathbf{j} + (3+t)\mathbf{k}$$

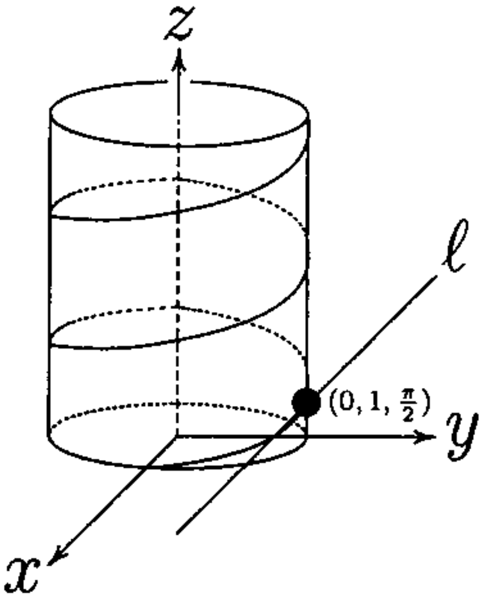
$$= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t(\mathbf{i} + \mathbf{j} + \mathbf{k}) \qquad 0 \le t \le 1$$

represents a line segment from (1, 2, 3) to (2,3,4)

Vector equations of other curves in three dim space are always given, for example,

The ***circular helix***

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$



Vector equations of curves in **TWO** dim space

We are familiar with most of the curves in two dim space

For example

$y = 2x + 3$ $\quad y = 3x^2$ $\quad y = 4 - x^2$ $\quad y = x^3$

$x = y^2$ $\quad x = 5 + y^2$ $\quad x = y^3$ $\quad x^2 + y^2 = 9$

$x = 4$ $\quad y = 3$ $\quad 4x^2 + 9y^2 = 16$

Why in Ch 9 and Ch 10, we use vector equations of the above curves instead of the equations just mentioned?

ANS: Because curves in the formulae in Ch 9 and Ch 10 are in vector equation forms

CAN YOU WRITE DOWN VECTOR EQUATIONS OF THE ABOVE CURVES?