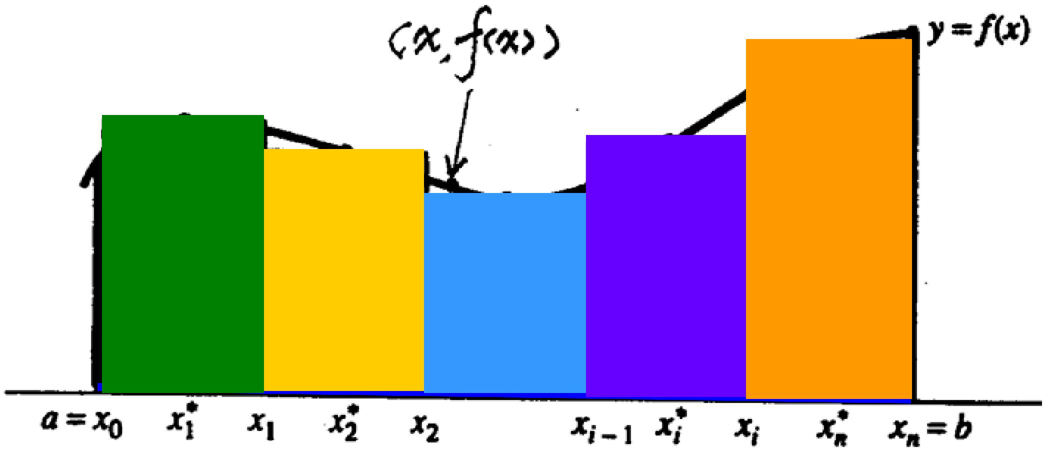
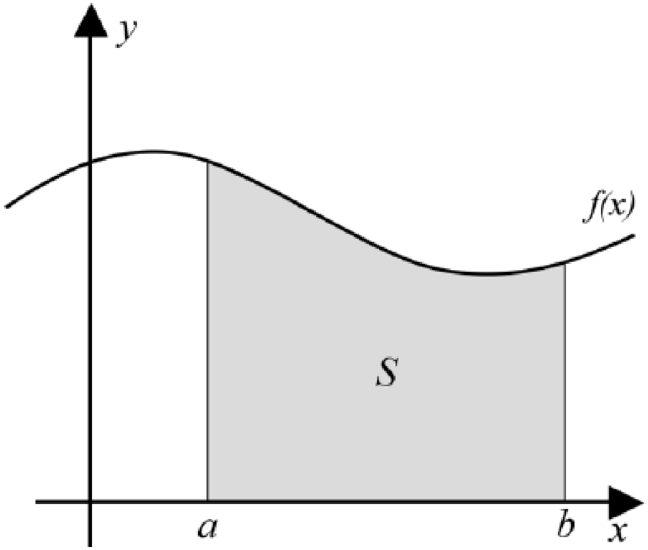
CH 8- *Multiple Integrals*

- **Recall (definite integral)**



Find the *area* of the *shaded* region:

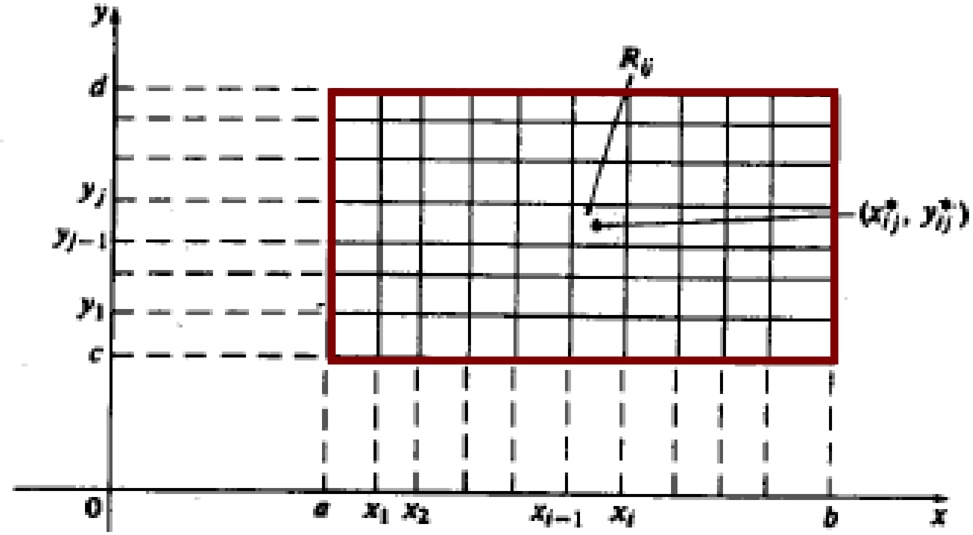


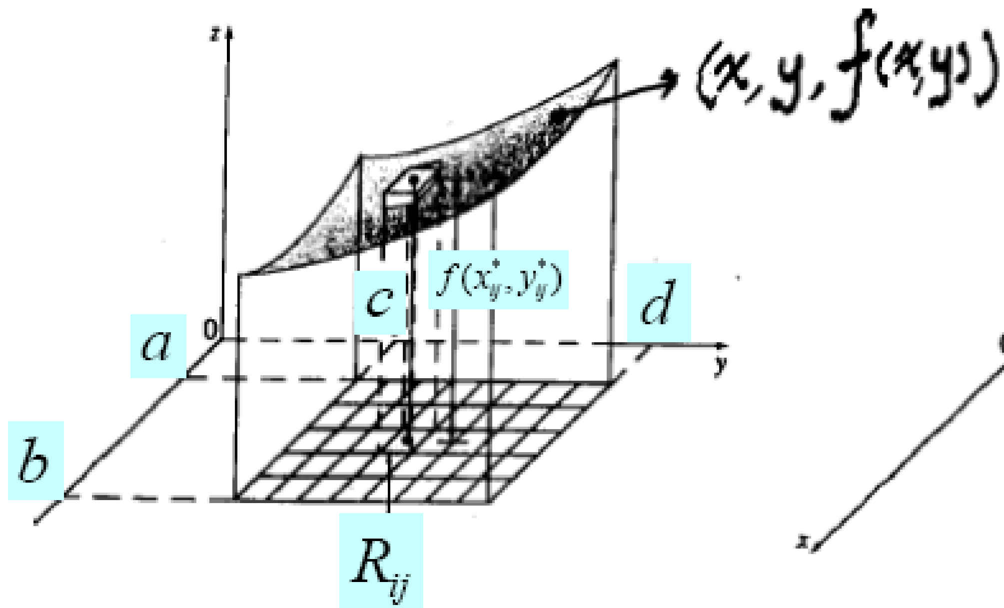
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

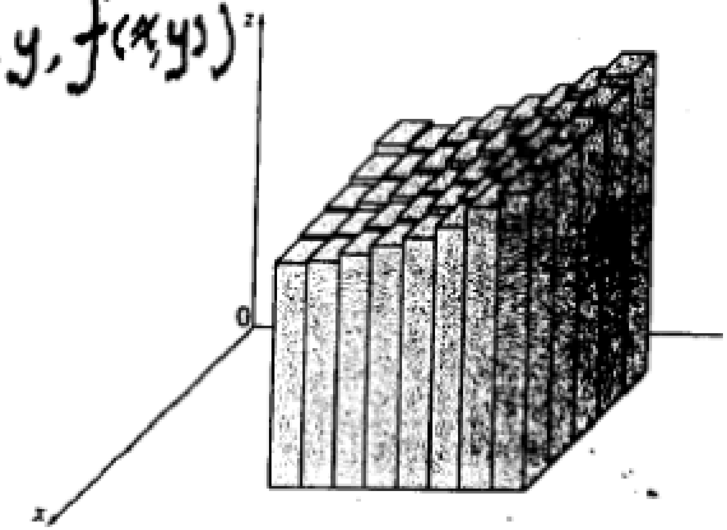
8.1 Double Integrals

8.1.1 Definition

$z = f(x, y)$

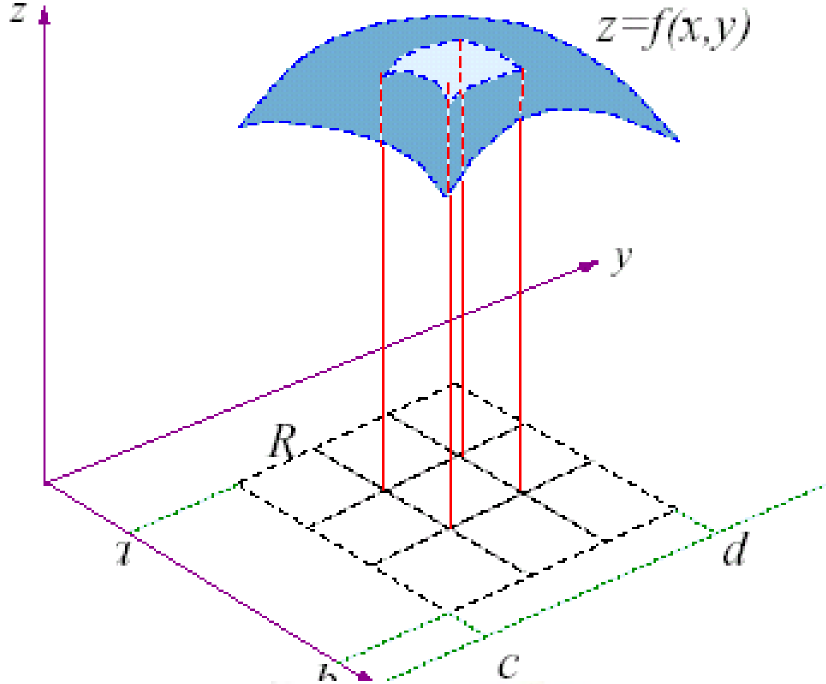




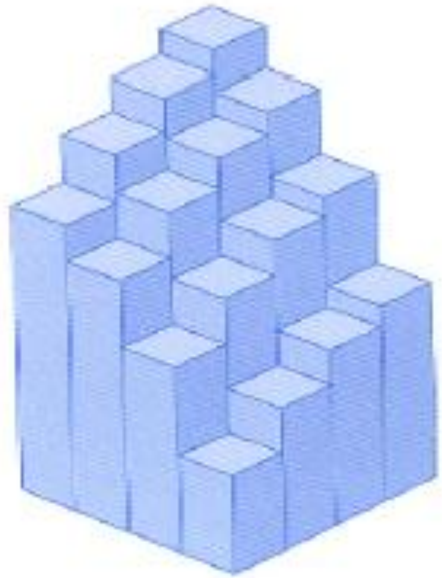


(b)

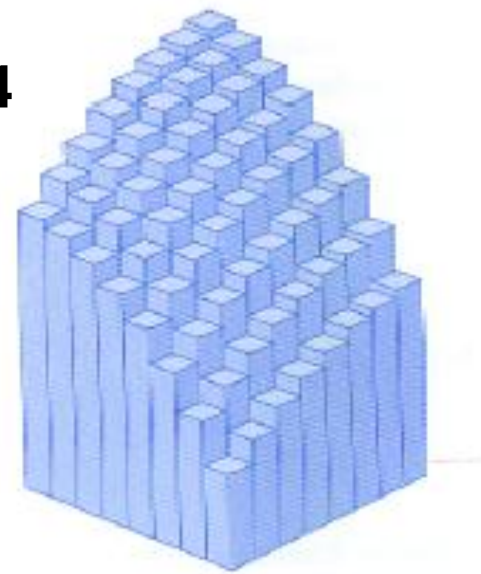
$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i{}^*, y_i{}^*)\Delta A_i$$



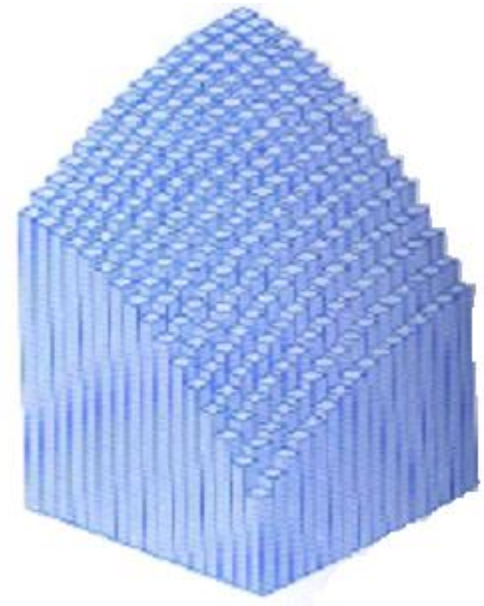
$n = 16$



$n = 64$



$n = 256$



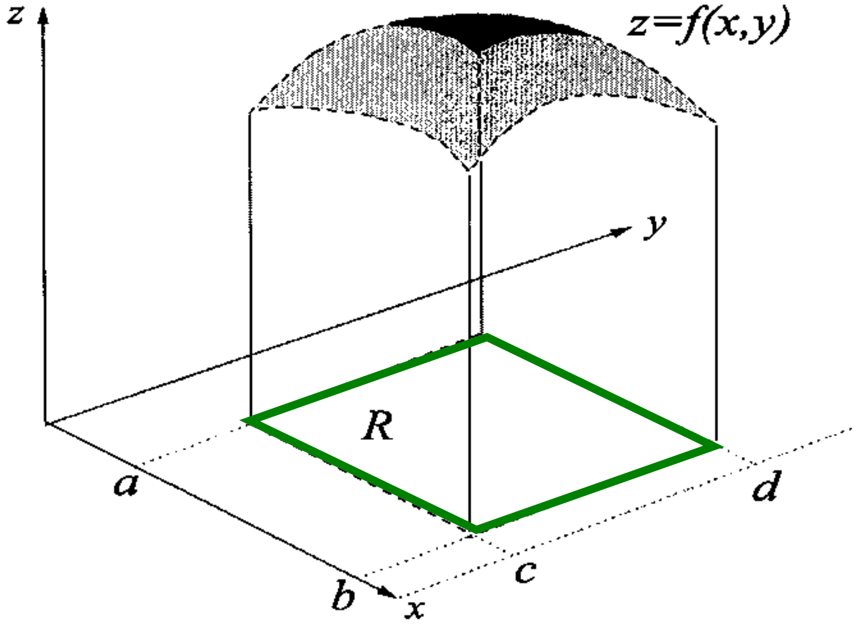
http://calculus7.com/id34.html

8.1.2 *Geometrical Meaning*

♣ If $\boldsymbol{f}(x,y) \geq 0$, then

$$\boxed{\iint_R f(x,y)\,dA}$$

= the ***volume*** of the ***solid*** as shown.



8.1.3 *Properties of Double Integrals*

(1) $\displaystyle\iint_R (f(x,y)+g(x,y))\;dA$

$$= \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA.$$

(2) $\displaystyle\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$, where c is a constant.

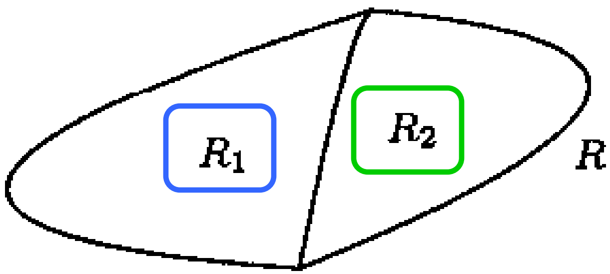
(3) If $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$,

then $\displaystyle\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA.$

(4) $$\iint_R dA \left(= \iint_R 1\, dA \right) = A(R), \text{ the area of } R.$$

(5) $$\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA,$$

where $R = R_1 \cup R_2$



(6) If $m \leq f(x,y) \leq M$ for all $(x,y) \in R$, then

$$mA(R) \leq \iint_R f(x,y)\, dA \leq M\, A(R).$$

How to evaluate

$$\iint_R f(x, y)\,dA$$

efficiently ?

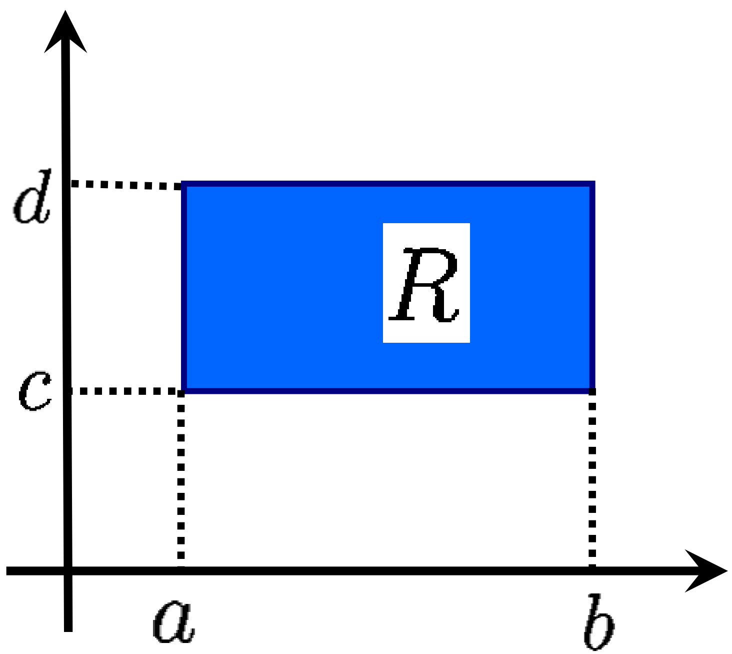
http://calculus7.com/id34.html

8.2 *Evaluation* of * $\iint_R f(x,y)\,dA$ *

8.2.1 Rectangular regions

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_c^d \left[\int_a^b f(x,y)\;dx\right]\;dy\\ &= \int_a^b \left[\int_c^d f(x,y)\;dy\right]\;dx\end{aligned}$$

(**iterated integral**)



$a \leq x \leq b,$

$c \leq y \leq d.$

♣ The above result was proved by Italian mathematician **Fubini** (1907) under the condition that $\boldsymbol{f}(x,y)$ is ***continuous*** throughout the region $\boldsymbol{R}$.

Guido Fubini
(1879-1943)

8.2.2

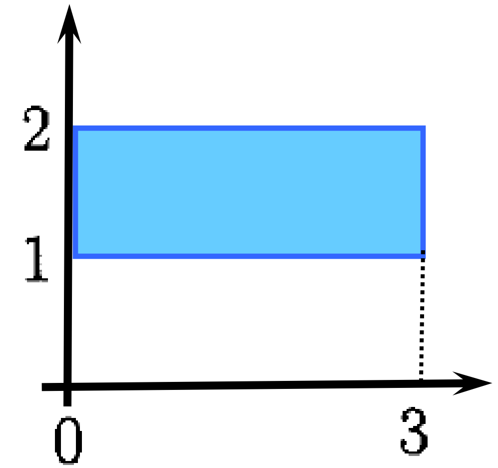
♣

(a) $$\int_0^3 \int_1^2 (x+2y)\,dy\,dx$$

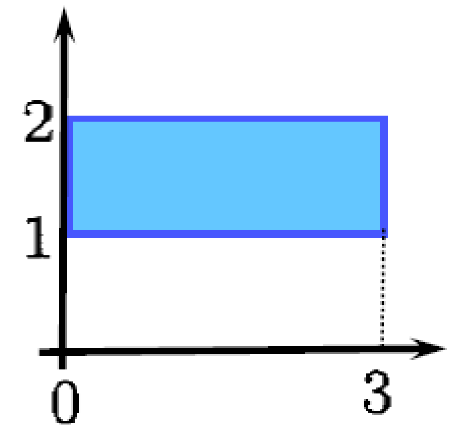
$$= \int_0^3 \left[xy + y^2\right]_{y=1}^{y=2} dx$$

$$= \int_0^3 (x+3)\,dx$$

$$= \left[\frac{x^2}{2} + 3x\right]_{x=0}^{x=3} = 27/2.$$



♣ (b) $\int_1^2 \int_0^3 (x+2y)\,dx\,dy$



$$= \int_1^2 \left[\frac{x^2}{2} + 2xy\right]_{x=0}^{x=3} dy$$

$$= \int_1^2 \left[\frac{9}{2} + 6y\right] dy$$

$$= \left[\frac{9y}{2} + 3y^2\right]_{y=1}^{y=2} = 27/2.$$

8.2.3

Let R be the rectangular region

$$0 \le x \le 4, \quad 1 \le y \le 2.$$

Evaluate $\iint_R x^2 y \, dA.$ **(*)**

Solution:

$$(*) = \int_0^4 \int_1^2 x^2 y \, dy dx$$

$$= \left(\int_0^4 x^2 \, dx\right)\left(\int_1^2 y \, dy\right)$$

$$= \frac{64}{3} \times \frac{3}{2} = 32.$$

8.2.4 Remark If ***f*(*x*,*y*) = *g*(*x*) *h*(*y*),** then

$$\iint_R g(x)h(y) \, dA = \left(\int_a^b g(x) \, dx\right)\left(\int_c^d h(y) \, dy\right)$$

where R is the rectangular region :

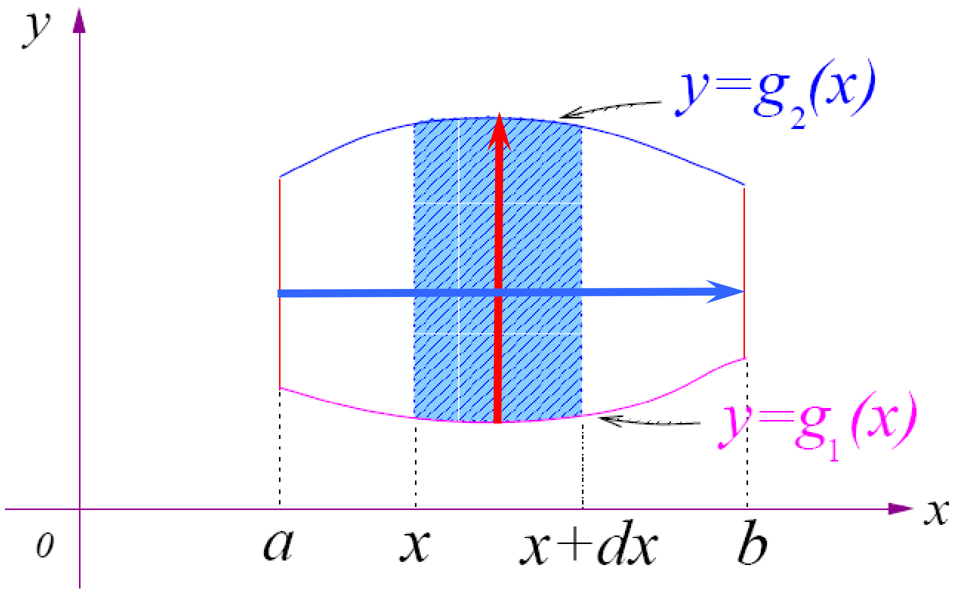
$$a \le x \le b, \quad c \le y \le d.$$

8.2.5 General regions — *Type* A

♣ The **region** ***R*** **:**

$$g_1(x) \leq y \leq g_2(x),$$

$$a \leq x \leq b.$$

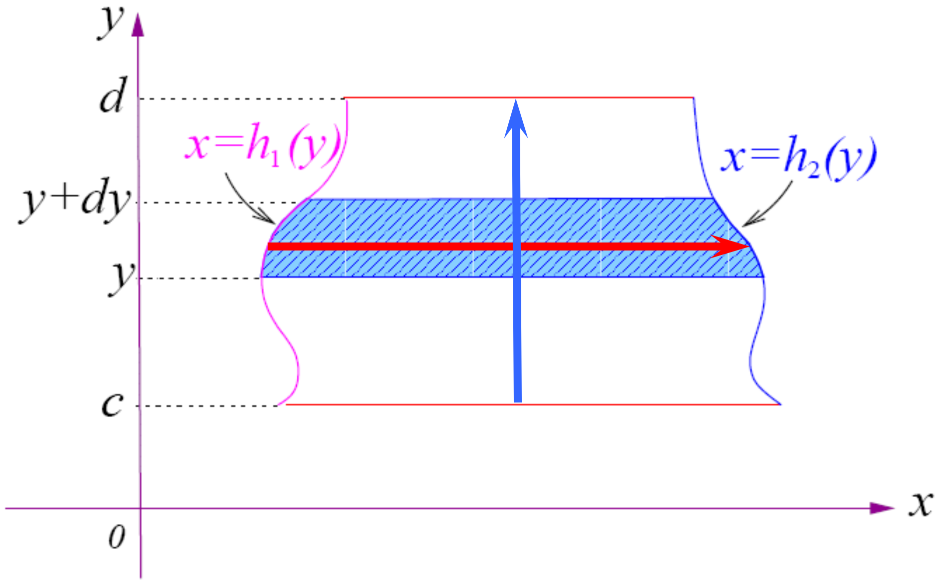


$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx$$

8.2.6 General regions — *Type* B

♣ The **region** ***R*** **:**

$$h_1(y) \le x \le h_2(y),$$
$$c \le y \le d.$$



$$\iint_R f(x,y)\, dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx \right] dy$$

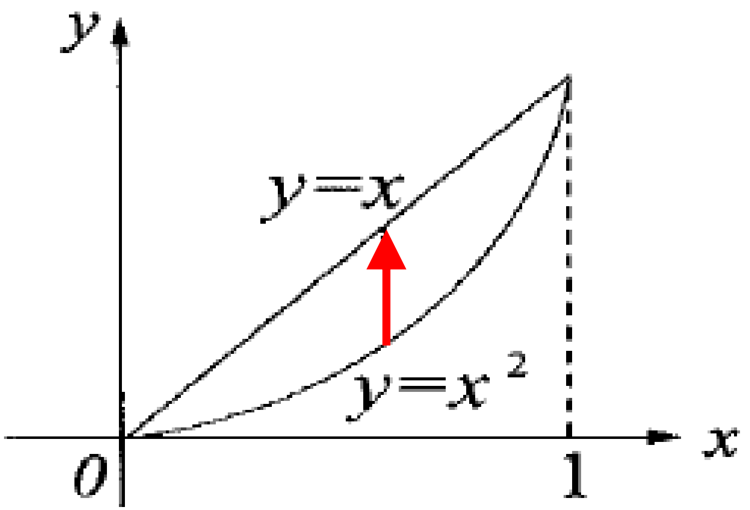
8.2.7

(♣) Find $\iint_R xy\,dA$, $R: x^2 \le y \le x,\ 0 \le x \le 1.$

(Type A) 1. Sketch (*R*)

2. *y*-limits

3. *x*-limits



$$(\clubsuit) = \int_0^1 \left[\int_{x^2}^{x} xy\,dy\right] dx$$

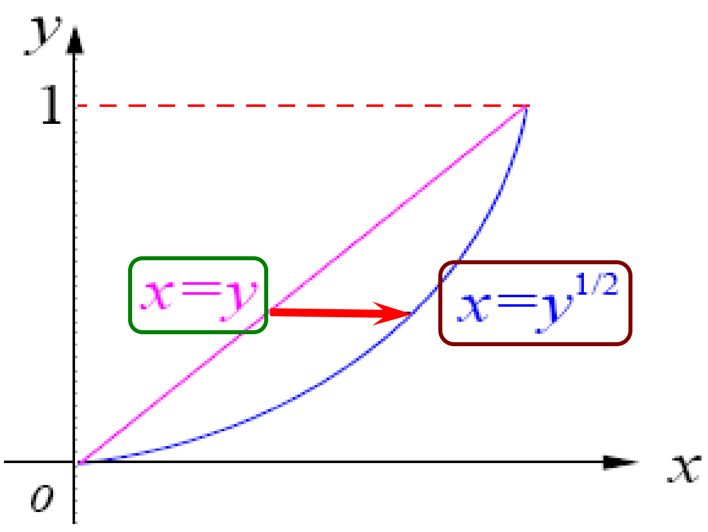
$$= \tfrac{1}{2}\int_0^1 xy^2\Big]_{y=x^2}^{y=x} dx = \tfrac{1}{2}\int_0^1 x(x^2 - x^4)\,dx = 1/24.$$

$(\clubsuit)$ Find $\iint_R xy\, dA$, $R: \underline{x^2 \leq y} \leq x,\ 0 \leq x \leq 1.$

(Type B) 1. Sketch (R)

2. x-limits

3. y-limits



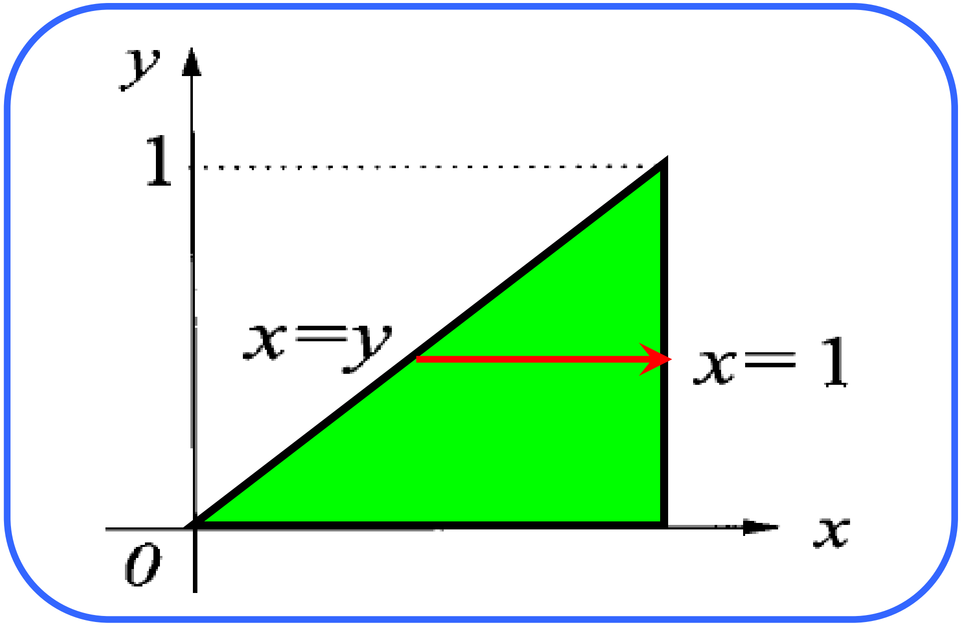
$$(\clubsuit) = \int_0^1 \left[\int_y^{\sqrt{y}} xy\ dx \right]\ dy$$

$$= ½ \int_0^1 x^2 y \Big]_{x=y}^{x=\sqrt{y}}\ dy\ = ½ \int_0^1 (y^2 - y^3)dy\ = 1/24$$

8.2.8

(♣) **Evaluate** $\displaystyle\iint_R \frac{\sin\ x}{x}\, dA$ where $\boldsymbol{R}$ **:**

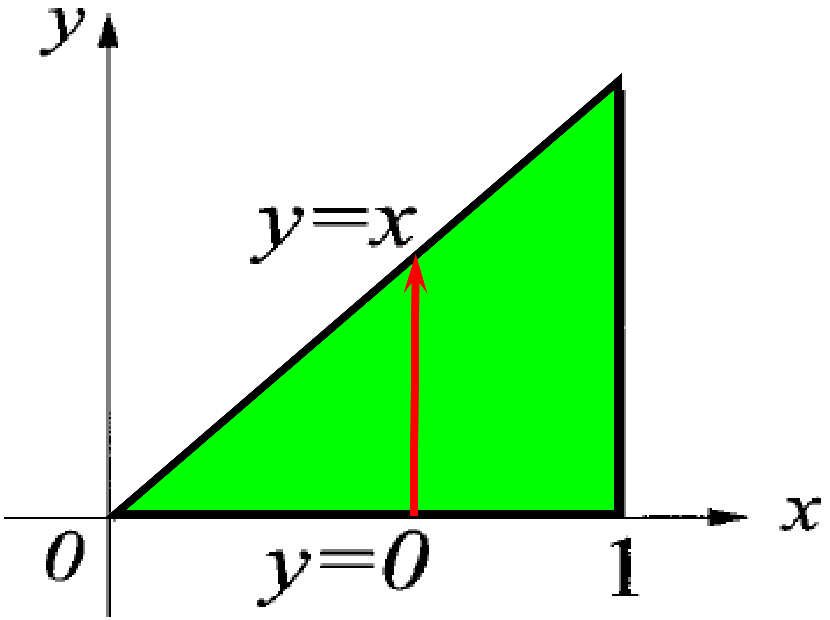
(Type B)



(♣) = $\displaystyle\int_0^1 \left[\int_y^1 \frac{\sin\ x}{x}\, dx\right] dy$ **???**

(♣) **Evaluate** $\displaystyle\iint_R \frac{\sin\ x}{x}\, dA$ where ***R*** **:**

(Type A)



(♣)

$$= \int_0^1 \left[\int_0^x \frac{\sin\ x}{x}\, dy\right]\, dx$$

$$= \int_0^1 \left[y\frac{\sin\ x}{x}\right]_{y=0}^{y=x}\, dx$$

$$= \int_0^1 (\sin\ x - 0)\, dx \qquad = 1 - \cos\ 1.$$

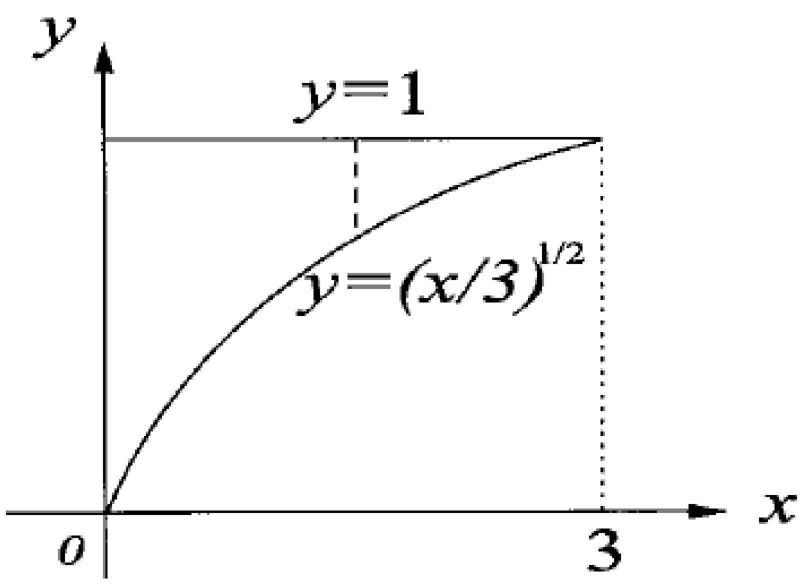
8.2.9

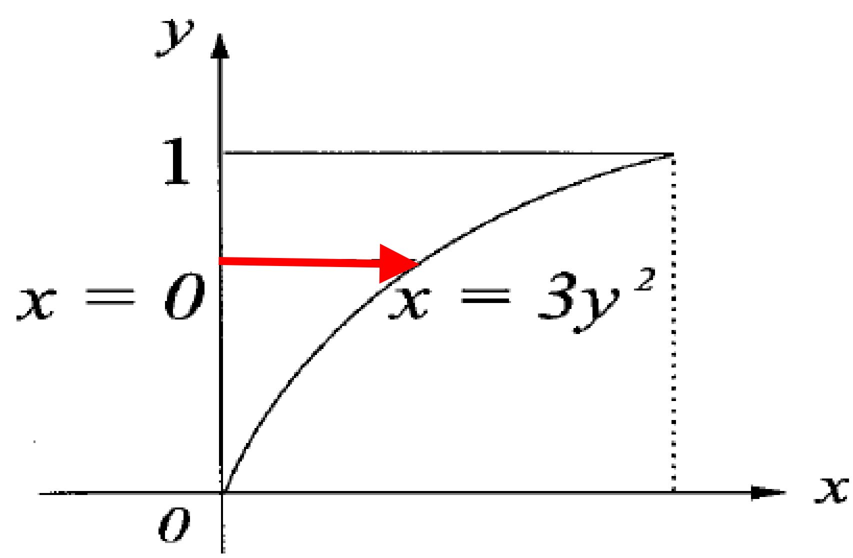
(♣) **Evaluate** $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy \; dx.$

(1) Type A or B ? **(B)**

(2) Identify *R*

$$y = \sqrt{\frac{x}{3}} \Longrightarrow x = 3y^2$$





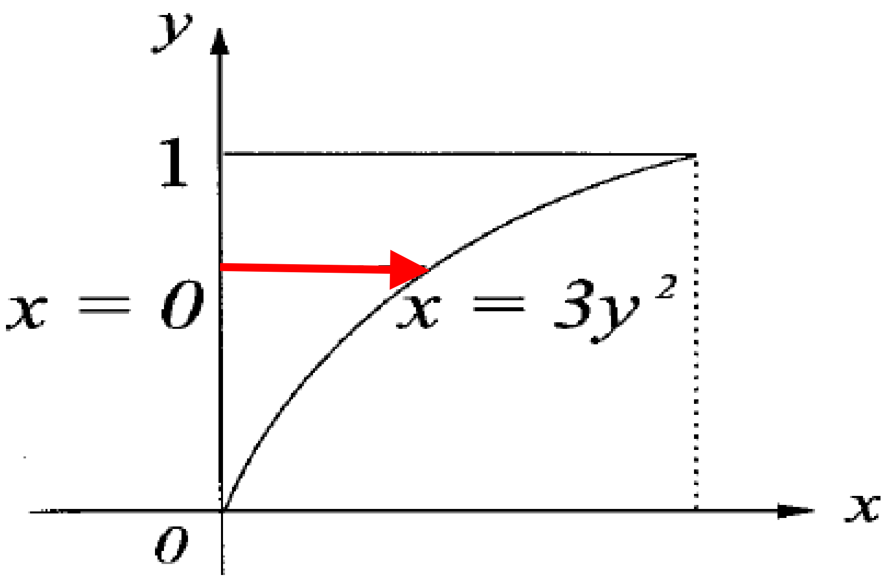
$$(\clubsuit) = \int_0^1 \left[\int_0^{3y^2} e^{y^3}\, dx \right] dy$$

$$= \int_0^1 \left[x e^{y^3} \right]_{x=0}^{x=3y^2} dy$$

$$= \int_0^1 3y^2 e^{y^3}\, dy$$

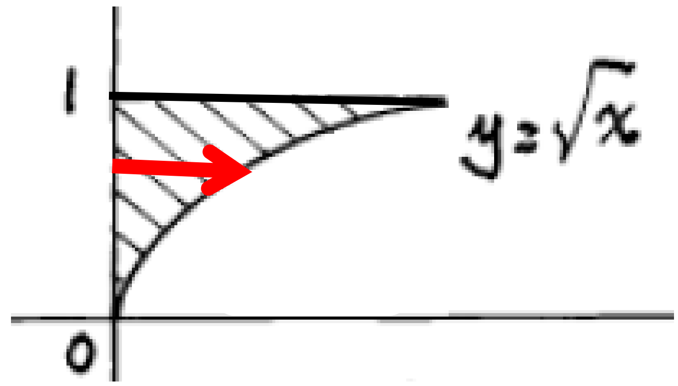
$$= \int_0^1 e^u du \qquad \left(u = y^3 \right)$$

$$= e - 1.$$



♣ **Evaluate** $\displaystyle\int_0^1 \left[\int_{\sqrt{x}}^1 \sin\left(\frac{y^3+1}{2}\right) dy\right] dx.$

$$\int_0^1 \left[\int_0^{y^2} \sin\left(\frac{y^3+1}{2}\right) dx \right] dy$$



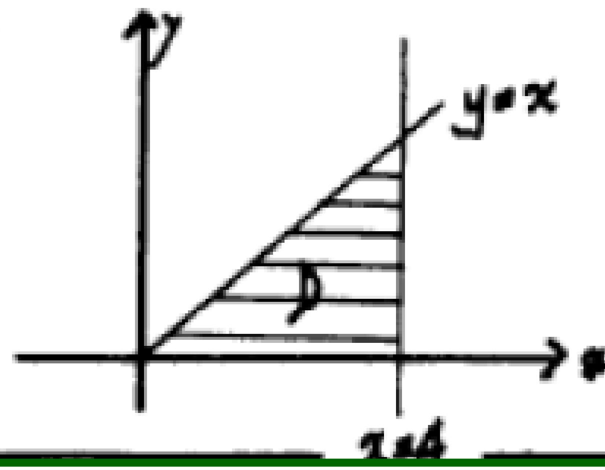
$$\int_0^1 y^2 \sin\left(\frac{y^3+1}{2}\right) dy \longrightarrow \frac{2}{3}\left(\cos\frac{1}{2} - \cos 1\right)$$

Question 5 (b) [5 marks]

Evaluate

$$\int\int_D \left(4e^{x^2} - 5\sin y\right) dx\,dy$$

where D is the region in the first quadrant bounded by the graphs of $y = x$, $y = 0$, and $x = 4$.



$$\iint_D \left(4e^{x^2} - 5\sin y\right) dx\,dy$$

$$= \int_0^4 \int_0^x \left(4e^{x^2} - 5\sin y\right) dy\,dx$$

Question 5 (b) [5 marks]

Let k be a positive constant. Evaluate

$$\iint_D x^2 e^{xy}\,dx\,dy$$

where D is the plane region given by

$$D: \quad 0 \le x \le 2k \text{ and } 0 \le y \le \frac{1}{2k}.$$

$$\iint_D x^2 e^{xy}\,dx\,dy = \int_0^{2k}\left\{\int_0^{\frac{1}{2k}} x^2 e^{xy}\,dy\right\}dx$$

$$= \int_0^{2k}\left[x e^{xy}\right]_{y=0}^{y=\frac{1}{2k}} dx$$

$$= \int_0^{2k}\left[x e^{\frac{x}{2k}} - x\right] dx$$

$$= 2k\int_0^{2k} x\, d\left(e^{\frac{x}{2k}}\right) - \int_0^{2k} x\,dx$$

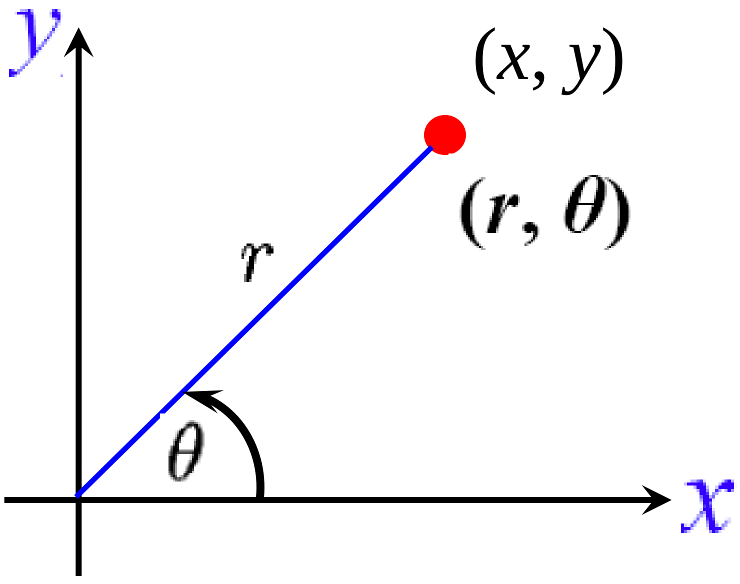
$$= 2k\left\{\left[x e^{\frac{x}{2k}}\right]_0^{2k} - \int_0^{2k} e^{\frac{x}{2k}}dx\right\} - \left[\frac{1}{2}x^2\right]_0^{2k}$$

$$= 2k\left\{2ke - 2k\left[e^{\frac{x}{2k}}\right]_0^{2k}\right\} - 2k^2$$

$$= 4k^2 - 2k^2$$

$$= \underline{\underline{2k^2}}$$

Polar *Coordinates*

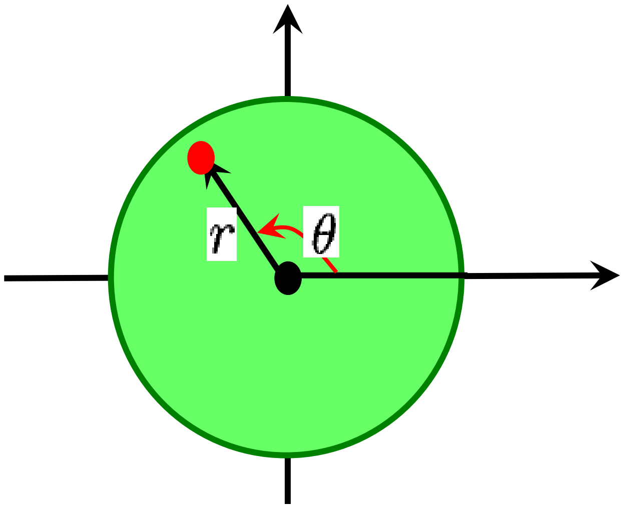


8.3 Double Integral in *Polar* Coordinates

- **Circle (unit)**

R **:** $0 \le r \le 1,$

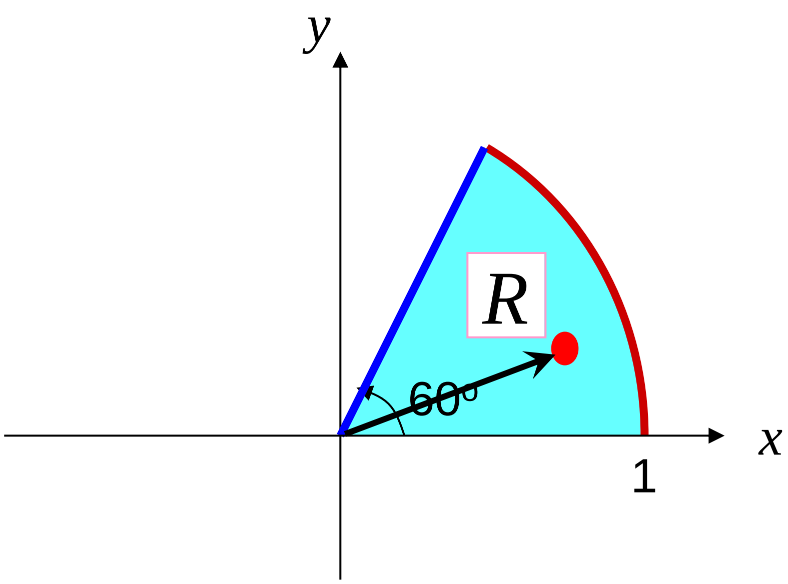
$0 \le \theta \le 2\pi$



- **Sector of a circle**

R **:** $0 \le r \le 1,$

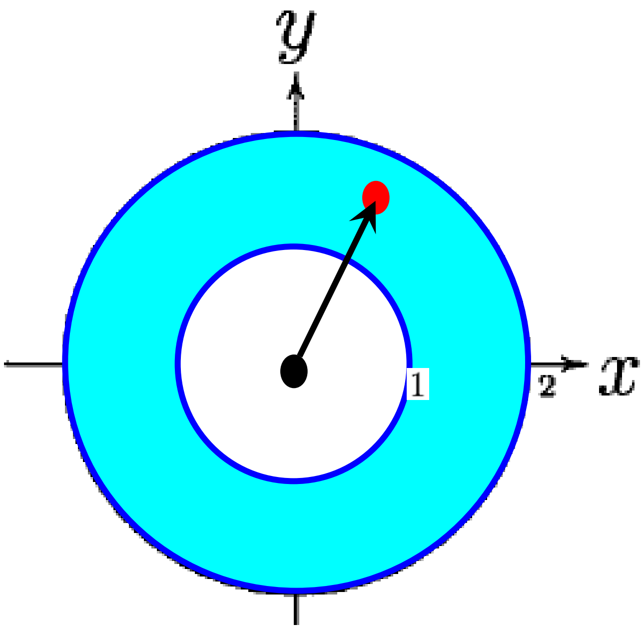
$0 \le \theta \le \pi/3$



- **Ring**

R **:** $1 \le r \le 2,$

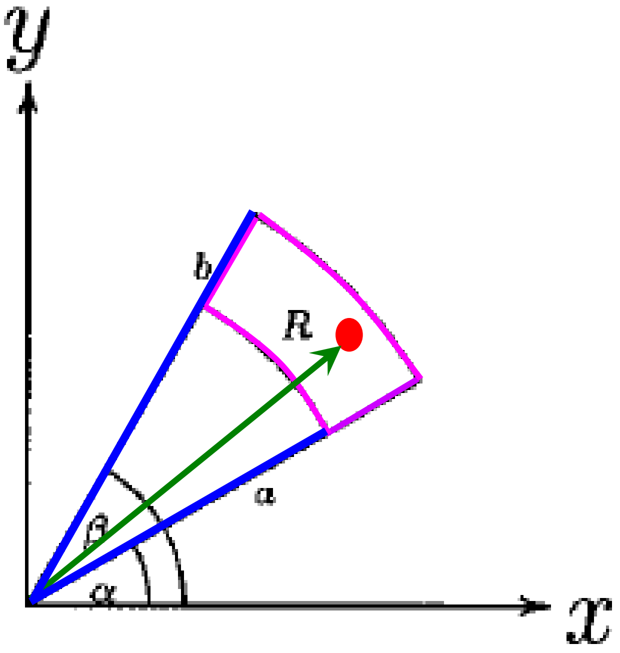
$0 \le \theta \le 2\pi$



- **Polar rectangle**

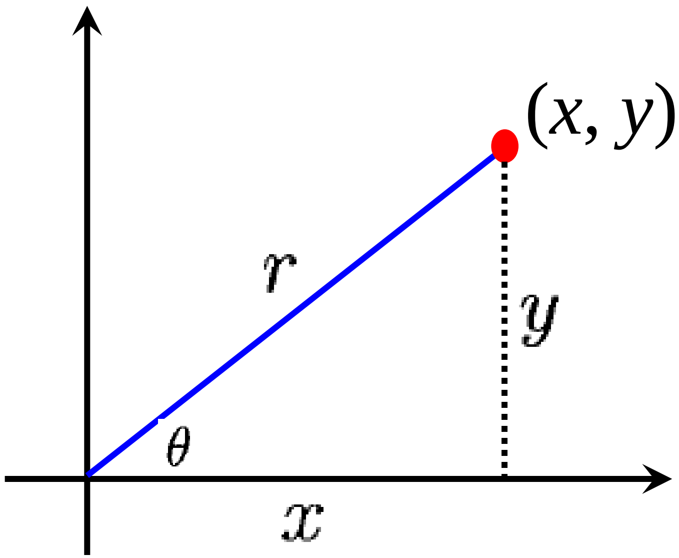
R **:** $a \le r \le b,$

$\alpha \le \theta \le \beta$



Change of *variables*

♣ $(x, y) \rightarrow (r, \theta)$, $\quad \iint_R f(x,y)\,dA \quad \rightarrow$?

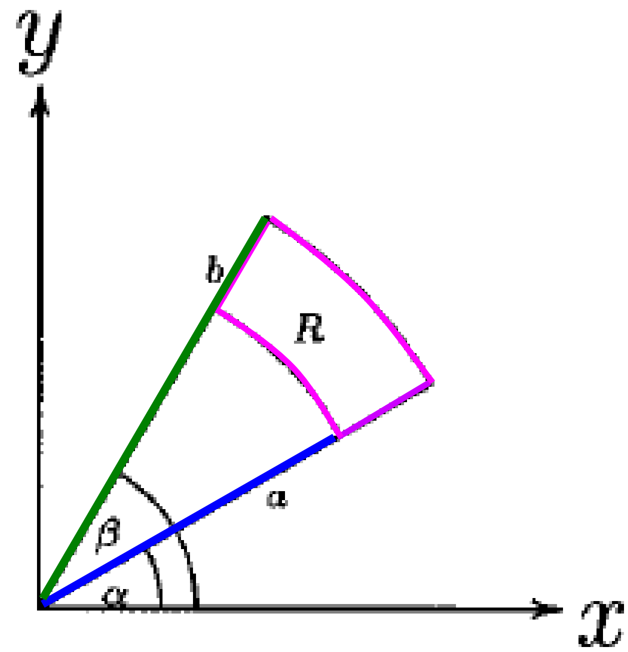


$$x = r\cos\theta,$$
$$y = r\sin\theta$$

$$dA \quad \rightarrow \quad r\,dr\;d\theta$$

♣ If $\boldsymbol{R}$: $a \le r \le b,$

$\alpha \le \theta \le \beta,$



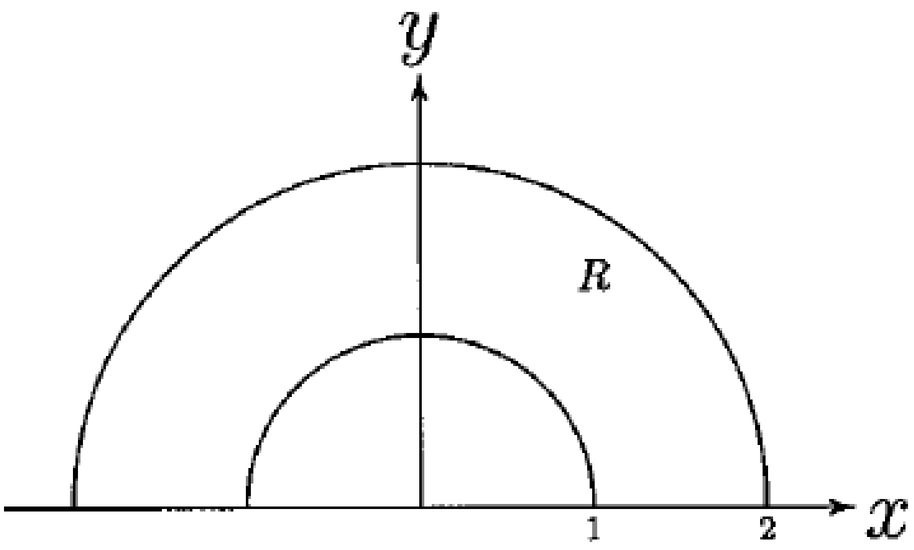
then

$$\iint_R f(x,y)\,dA$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

♣ **Evaluate**

$$\iint_R (3x + 4y^2)\, dA$$



$\boldsymbol{R}: 1 \le r \le 2, \quad 0 \le \theta \le \pi$

$$x = r\cos\theta,$$

$$y = r\sin\theta$$

$$dA \quad \longrightarrow \quad r\ dr\ d\theta$$

$$= \int_0^{\pi} \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\ r dr d\theta$$

$$\int_0^{\pi}\int_1^{2}(3r\cos\theta + 4r^2\sin^2\theta)\ r\,dr\,d\theta$$

$$= \int_0^{\pi}\left[r^3\cos\theta + r^4\sin^2\theta)\right]_{r=1}^{r=2} d\theta$$

$$= \int_0^{\pi}\left(7\cos\theta + 15\,\underline{\sin^2\theta}\right) d\theta$$

$$= \int_0^{\pi}\left(7\cos\theta + \frac{15}{2}\underline{(1-\cos 2\theta)}\right) d\theta$$

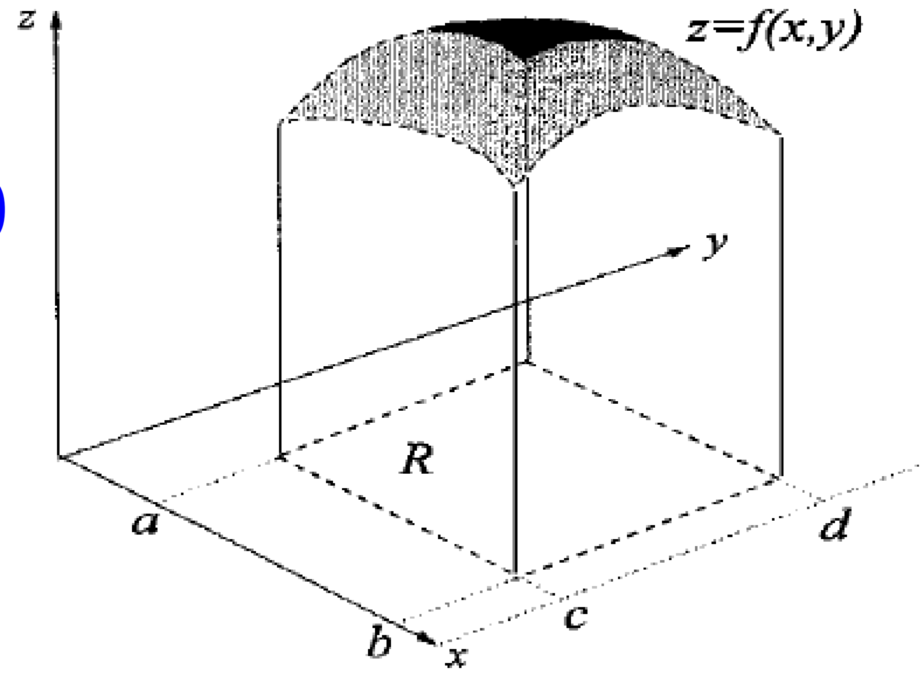
$$= \left[7\sin\theta + \frac{15}{2}(\theta - \frac{\sin 2\theta}{2})\right]_{\theta=0}^{\theta=\pi} \quad = \frac{15\pi}{2}$$

8.4 *Applications*

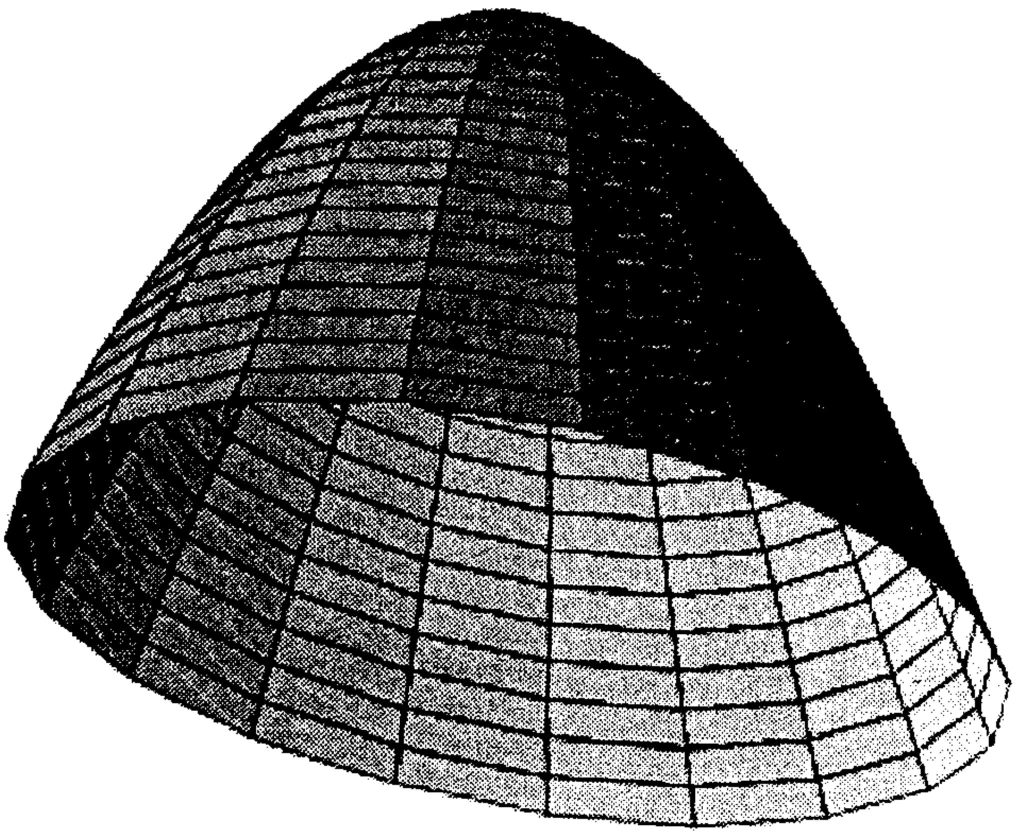
♣ Volume

Suppose ***D*** is a **solid** under the **surface *f*(*x*,*y*)** over a plane region ***R***. Then the volume of ***D*** is given by

$$\iint_R f(x,y)\,dA.$$



Elliptic paraboloid



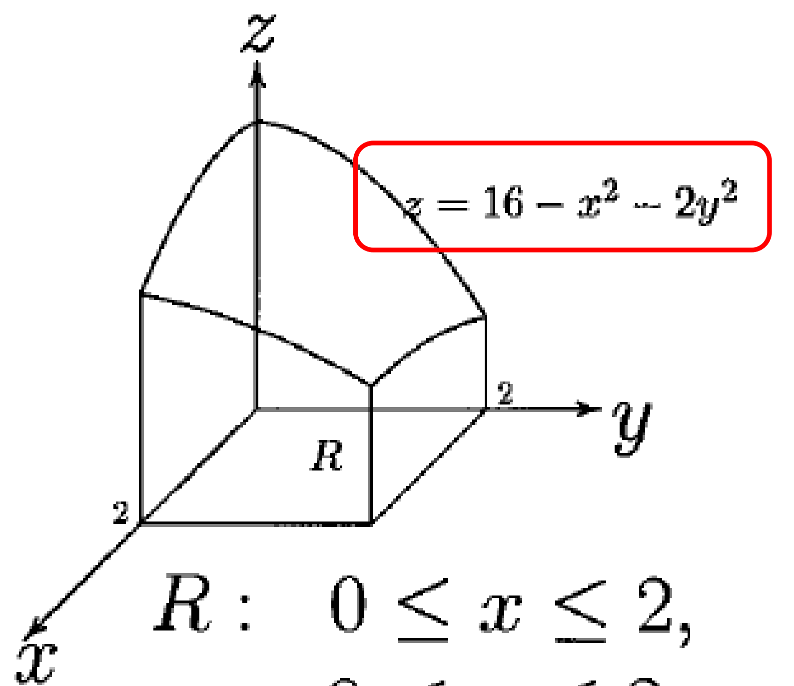
♣ Find the volume of the solid D that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$, $y = 2$, and the 3-coordinate planes.

The volume of D is

$$\iint_R (16 - x^2 - 2y^2)\,dA$$

$$= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\,dx\,dy$$

$$= 48.$$



$R: \quad 0 \le x \le 2,$
$\qquad 0 \le y \le 2.$

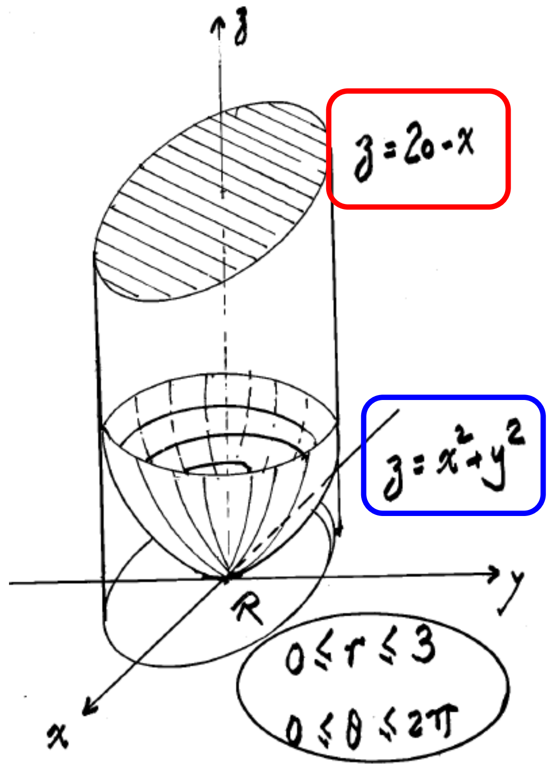
♣ Find the volume of the solid enclosed laterally by the circular cylinder about z-axis of radius 3 and bounded on top by the plane $x + z = 20$ and below by the paraboloid $z = x^2 + y^2$.

The volume can be computed as

$$V = \iint_R f_1(x, y)dA - \iint_R f_2(x, y)dA$$

where
$$f_1(x, y) = 20 - x$$
$$f_2(x, y) = x^2 + y^2$$

and $\boxed{R : 0 \le r \le 3, \quad 0 \le \theta \le 2\pi.}$



- So the volume of the solid is

$$V = \iint_R (20 - x)\,dA - \iint_R (x^2 + y^2)\,dA$$

$$= \int_0^{2\pi} \int_0^3 (20 - r\cos\theta) r \; dr \; d\theta - \int_0^{2\pi} \int_0^3 (r^2) r \; dr \; d\theta$$

$$= \int_0^{2\pi} \int_0^3 \left[20r - r^2 \cos\theta - r^3 \right] dr \; d\theta$$

$$x = r\cos\theta,$$
$$y = r\sin\theta$$
$$dA \;\; \longrightarrow \;\; r \; dr \; d\theta$$

$$= \int_0^{2\pi} \left[10r^2 - \frac{r^3}{3}\cos\theta - \frac{r^4}{4} \right]_0^3 d\theta$$

$$= \int_0^{2\pi} \left[90 - 9\cos\theta - \frac{81}{4} \right] d\theta$$

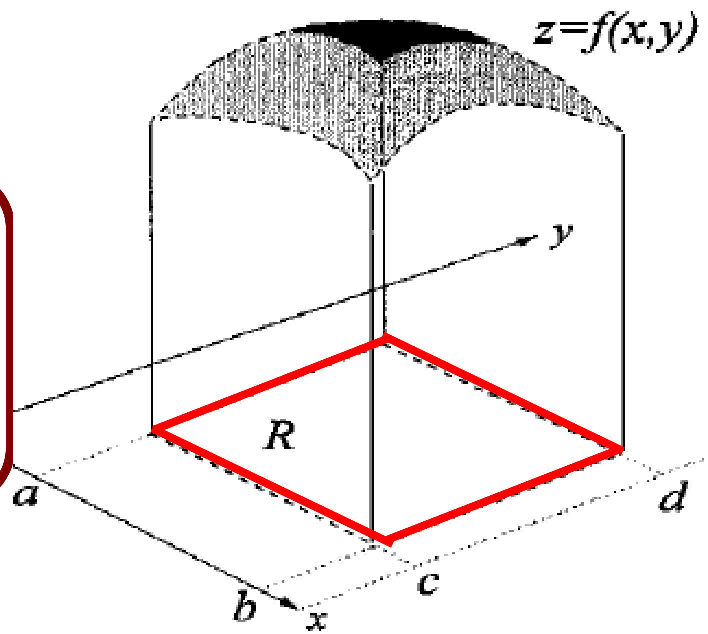
$$= \left[\frac{279}{4}\theta - 9\sin\theta \right]_0^{2\pi} = \frac{279}{2}\pi$$

8.4.4

♣ Surface area

If f has continuous first partial derivatives on a closed region R of the xy-plane, then the area S of that portion of the surface $z = f(x, y)$ that projects onto $\boxed{R}$ is

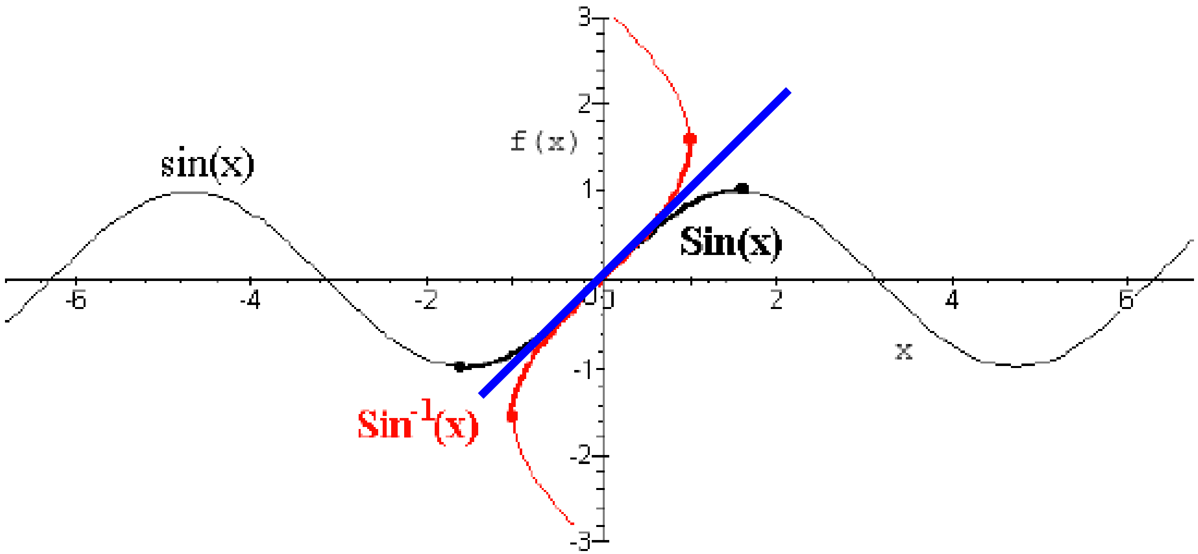
$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA.$$



Recall

Arcsine

$y = \arcsin x \iff x = \sin y$



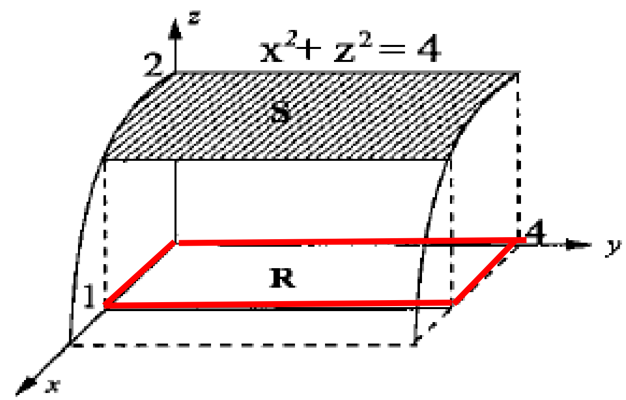
Domain of arcsinx : [−1, 1]

Range of arcsinx: [−½π, ½ π]

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\boxed{\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}(x/a) + C}$$

- Find the surface area of the portion of the cylinder $x^2 + z^2 = 4$ above the rectangle $R: \quad 0 \le x \le 1, \quad 0 \le y \le 4.$



$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

$$= \iint_R \sqrt{\left(-\frac{x}{\sqrt{4-x^2}}\right)^2 + 0^2 + 1}\, dA$$

$$= \int_0^4 \left[\int_0^1 \boxed{\frac{2}{\sqrt{4-x^2}}}\, dx\right] dy$$

$$= 2\int_0^4 \left[\sin^{-1}(x/2)\right]_{x=0}^{x=1} dy$$

$$= 2\int_0^4 \frac{\pi}{6}\, dy = \frac{4\pi}{3}.$$

$$f(x, y) = z = \sqrt{4-x^2}$$

$$\boxed{\frac{x^2 + 4 - x^2}{4 - x^2}}$$

$$\boxed{\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}(x/a) + C}$$

$$\boxed{\sin\frac{\pi}{6} = \frac{1}{2}}$$

8.4.6 *Average value of a function*

average value of the function $f(x,y)$

over a region R is defined to be

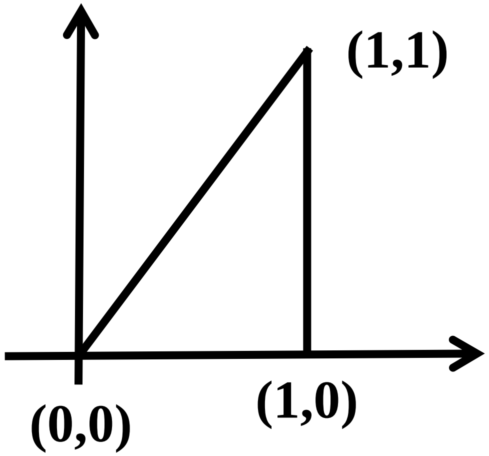
$$\frac{1}{\text{Area of R}}\iint_R f(x,y)\,dA$$

8.4.7 *Example*

Find the average value of $f(x,y) = xe^y$

on the triangular region with vertices

$(0,0),\ (1,0),\ (1,1)$



The area of R is 1/2

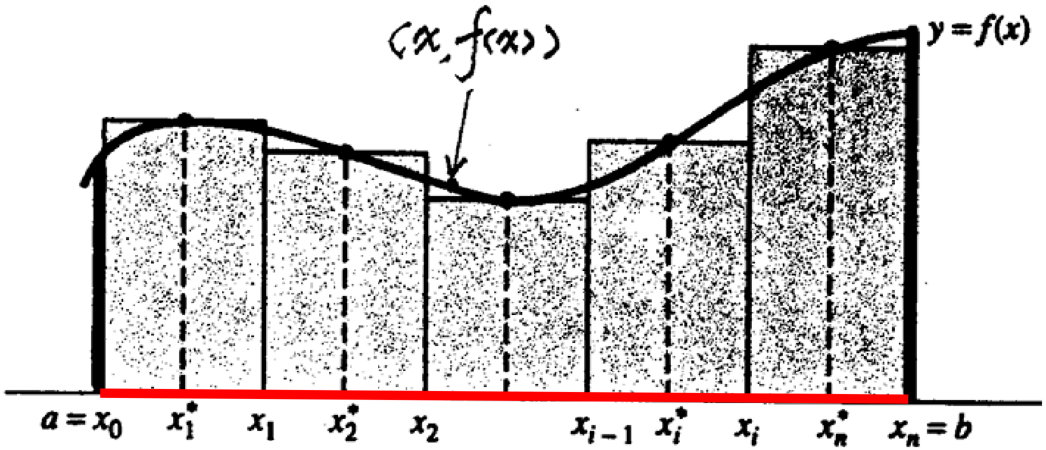
Average value

$$= \frac{1}{1/2}\iint_R xe^y\,dy = 2\int_0^1 \left[\int_0^x xe^y\,dy\right]dx =$$

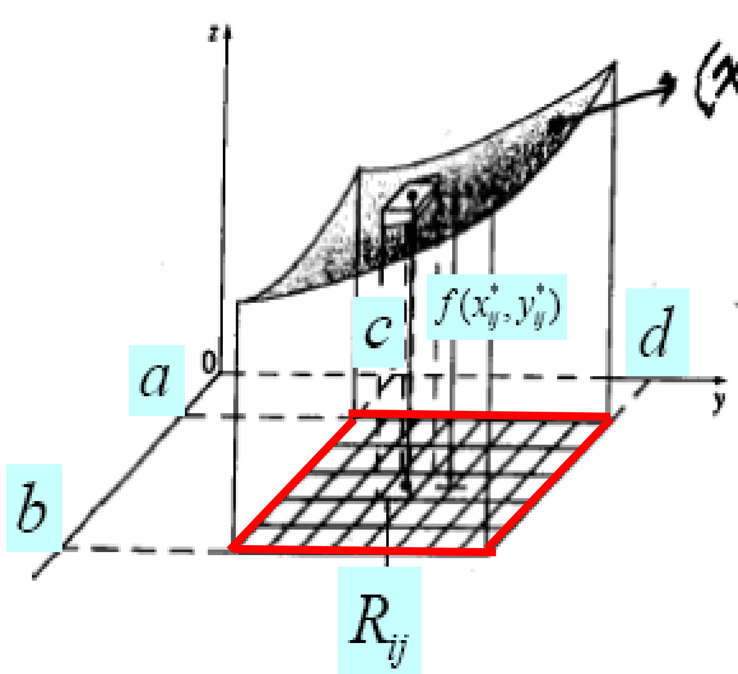
ANS:1

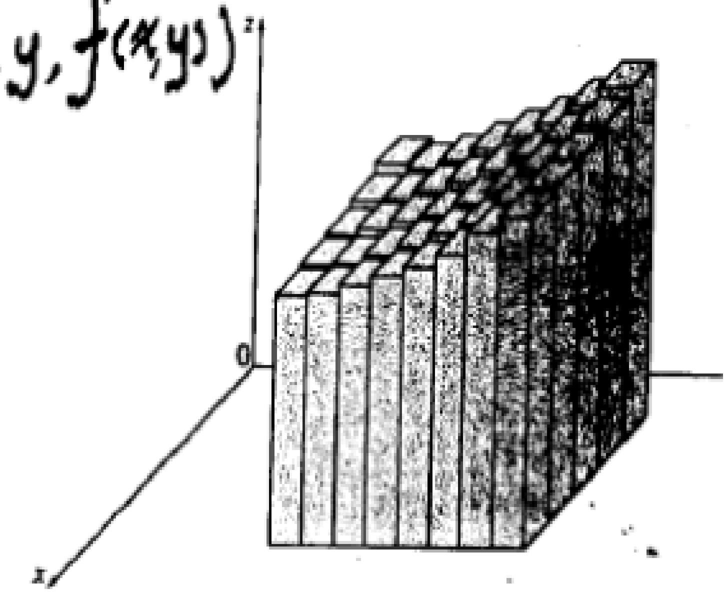
5. *Triple* Integral

Recall :



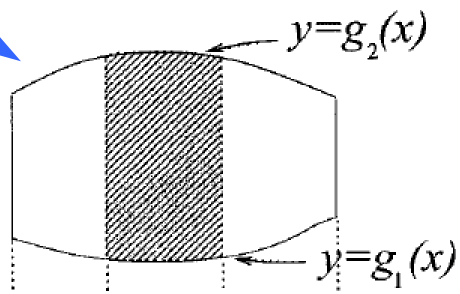
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$





(b)

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i$$



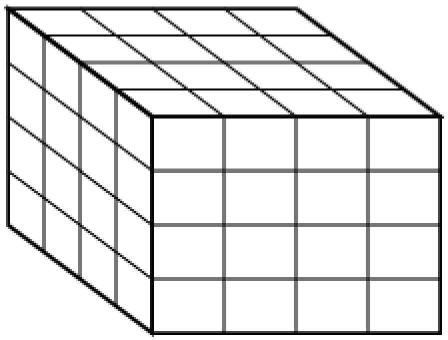
- Let D be a solid region in the xyz space. Subdivide D into smaller cubic region $D_i \quad (i = 1, \ldots, n)$.

Let ΔV_i be the volume of D_i and (x_i, y_i, z_i) be a point in D_i.

Let $f(x, y, z)$ be a function of three variables.

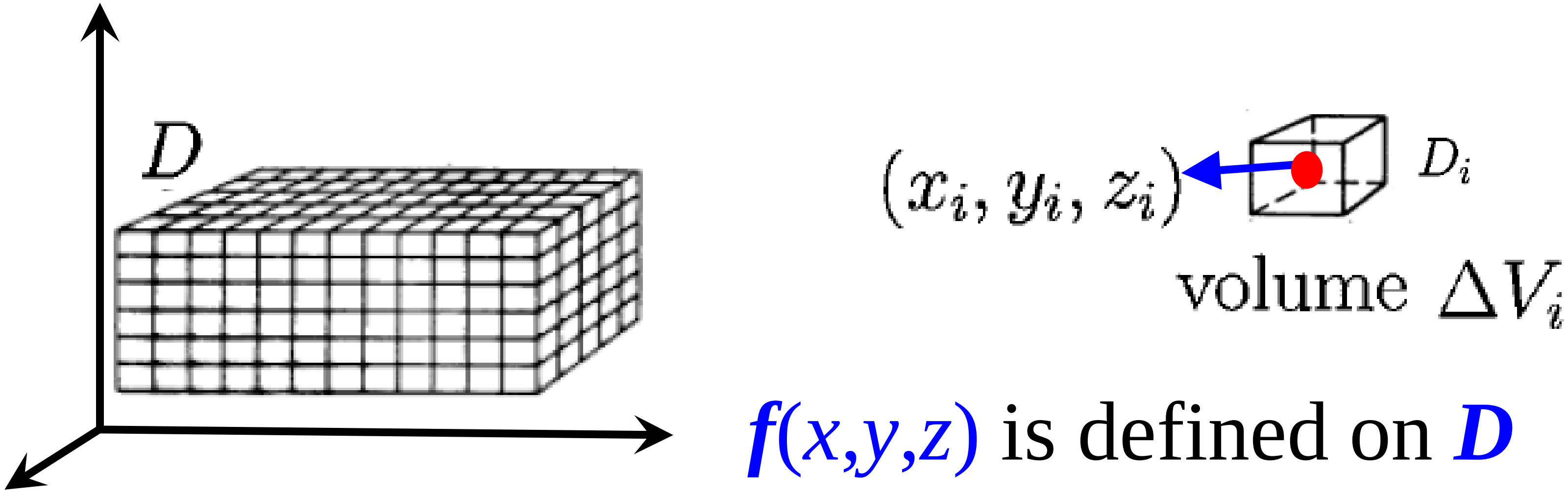
Then the **triple integral** of f over D is

$$\iiint_D f(x, y, z)\, dV = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta V_i.$$



$$\boxed{f(x_k^*)\Delta x}$$

$$\boxed{f(x_i^*, y_i^*)\Delta A_i}$$

- 

f(*x*,*y*,*z*) is defined on ***D***

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$$

$$\boxed{\iiint_D f(x, y, z)\, dV = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i}$$

Physical Meaning of $\iiint_D f(x,y,z)\,dV$

- **No** direct geometrical meaning for $\iiint_D$
- **If** f is the constant function 1, then
$$\iiint_D 1\,dV = \text{ volume of } D.$$
- **If** ***f*** represents certain **physical** quantity, then

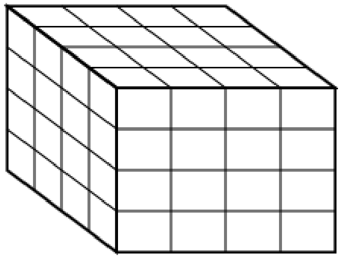
$\iiint_D f(x,y,z)\,dV$ may have some **physical** meaning.

♣ Given a solid object D with volume V and uniform density δ, the mass M of D is given by

$$M = \delta \times V.$$

Now, suppose the ***density*** is a fn $\delta(x,y,z)$ defined on ***D***.

Divide D into subregions D_i as before and let M_i be the mass of the subregion D_i. Then



$$M_i \approx \delta(x_i, y_i, z_i) \times \Delta V_i$$

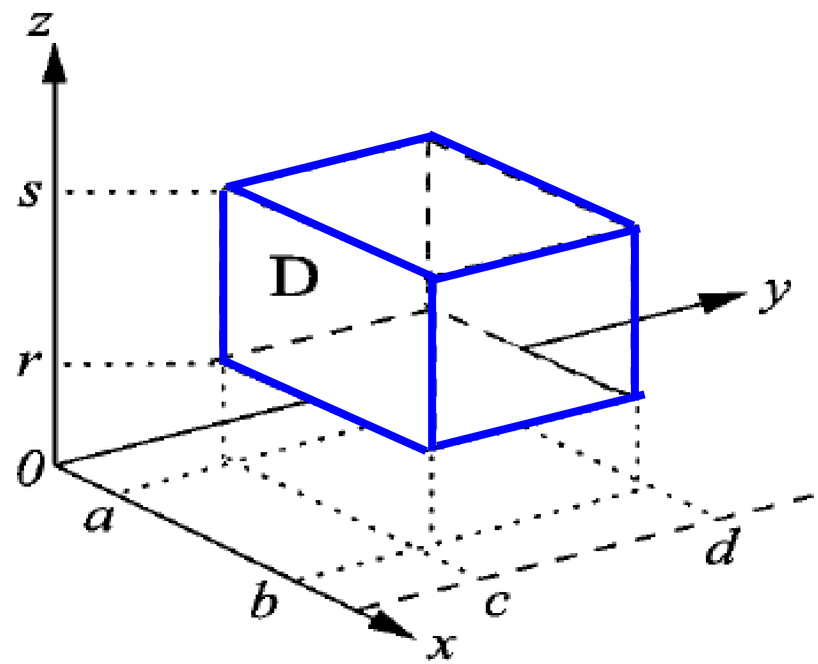
& so

$$M = \lim_{n\to\infty} \sum_{i=1}^{n} \delta(x_i, y_i, z_i)\Delta V_i = \boxed{\iiint_D \delta(x, y, z)\, dV.}$$

♣ Evaluation of triple integral

Suppose D is the rectangular box consisting of points (x, y, z) such that

$$a \leq x \leq b,$$
$$c \leq y \leq d,$$
$$r \leq z \leq s.$$

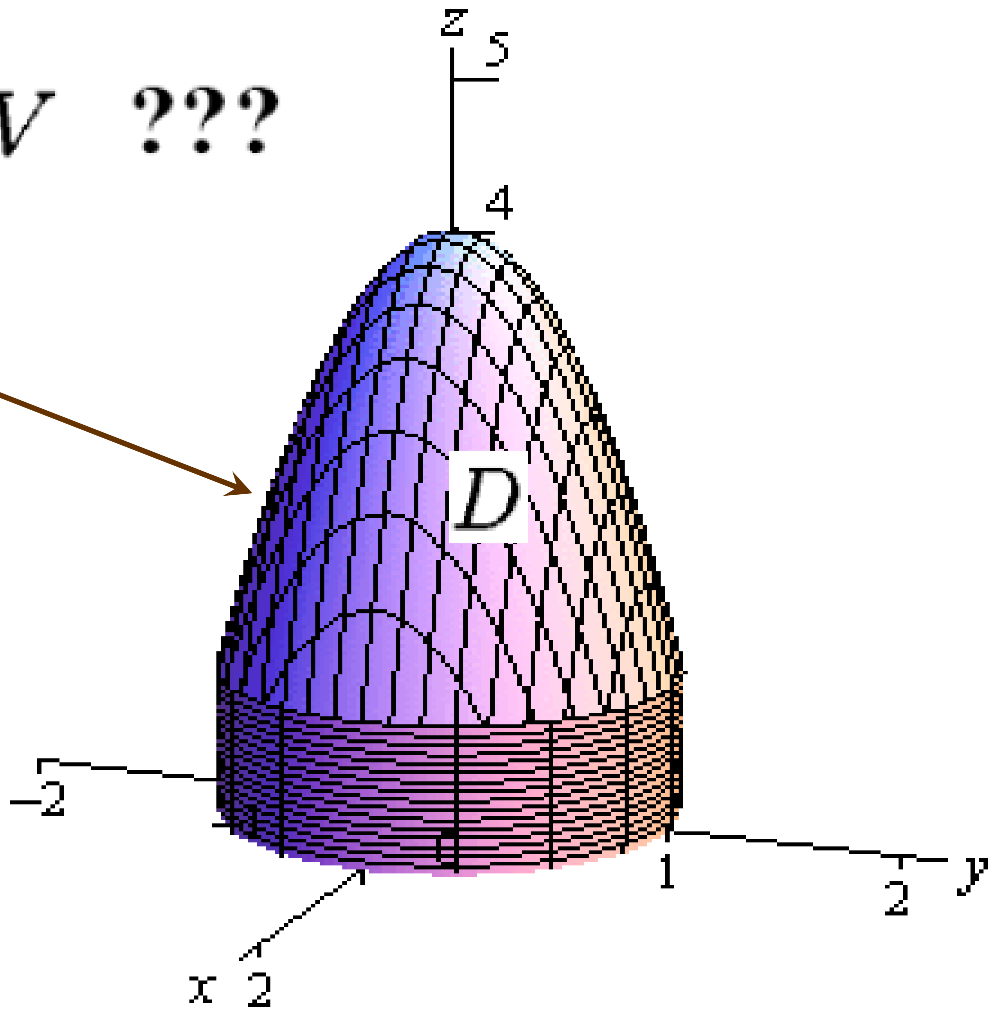


Then

$$\boxed{\iiint_D f(x, y, z)\, dV = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx.}$$

☻ **Solid region** for **triple integral**

$$\iiint_D f(x, y, z)\, dV \quad ???$$



(♣) **Evaluate** $\iiint_D \frac{1}{xyz}\,dV$

$$(♣) = \int_1^e \int_1^e \int_1^e \frac{1}{xyz}\,dz\,dy\,dx$$

$$= \int_1^e \int_1^e \left.\frac{\ln z}{xy}\right|_{z=1}^{z=e} dy\,dx$$

$$= \int_1^e \int_1^e \frac{1}{xy}\,dy\,dx$$

$$= \cdots\cdots = 1$$