MA 1505 Mathematics I
Tutorial 10 Solutions

1. Use fundamental Theorem of line integral:

$$\int_C \nabla f \bullet d\mathbf{r} = f(\text{terminal point}) - f(\text{initial point}).$$

(a) C has parametric equation $\mathbf{r}(t) = (t^2+1)\mathbf{i} + (t^3+t)\mathbf{j}, \quad 0 \le t \le 1$.

So initial point is $\mathbf{r}(0) = (1,0)$ and terminal point is $\mathbf{r}(1) = (2,2)$.

So $\displaystyle\int_C \nabla f \bullet d\mathbf{r} = f(2,2) - f(1,0) = 9 - 3 \quad \text{(from table)} = 6$.

(b) The unit circle is a closed curve and ∇f is conservative. So $\displaystyle\oint_C \nabla f \bullet d\mathbf{r} = 0$.

2. The work done is given by the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ where

$\mathbf{F}$ is the gravitational force contributed by the weight of the man + the pail of water;

C is the path traced out (which is a helix) by the man as he climbed up the staircase.

C has vector equation given by

$$\mathbf{r}(t) = 6\cos t\mathbf{i} + 6\sin(t)\mathbf{j} + \lambda t\mathbf{k}, \quad 0 \le t \le 6\pi$$

where λ is some constant.

As t increases from 0 to 6π (3 revolutions), the $\mathbf{k}$ component λt (representing the height) of C increases from 0 to 30.

i.e. $6\pi\lambda = 30$ or $\lambda = 5/\pi$. Hence

$$\mathbf{r}(t) = 6\cos t\mathbf{i} + 6\sin(t)\mathbf{j} + \frac{5}{\pi}t\mathbf{k} \quad 0 \le t \le 6\pi.$$

On the other hand, since $\mathbf{F}$ is given by gravitation, the vector field is of the form

$$\mathbf{F}(x,y,z) = 0\mathbf{i} + 0\mathbf{j} - (W_m + W_p)g\mathbf{k}$$

where W_m is the mass of the man and W_p is the mass of the pail of water.

Since the pail leaks 2kg of water throughout the ascent, W_p varies (linearly) according to z:

As z increases from 0 to 30, W_p decreases from 10 to 8. i.e.

$$\frac{z-0}{W_p - 10} = \frac{30-0}{8-10} \Longrightarrow z = -15(W_p - 10) \Longrightarrow W_p = 10 - \frac{z}{15}.$$

Hence

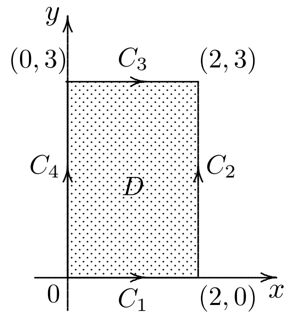
$$\mathbf{F}(x,y,z) = 0\mathbf{i} + 0\mathbf{j} - (90 - \frac{z}{15})g\mathbf{k}.$$

$$\begin{aligned}
\int_C \mathbf{F} \bullet d\mathbf{r} &= \int_0^{6\pi} (0\mathbf{i} + 0\mathbf{j} - (90 - \frac{z}{15})g\mathbf{k}) \bullet (-6\sin t\mathbf{i} + 6\cos(t)\mathbf{j} + \frac{5}{\pi}\mathbf{k})\ dt \\
&= -\frac{5}{\pi} g \int_0^{6\pi} (90 - \frac{t}{3\pi})\ dt \\
&= -\frac{5}{\pi} g \left[90t - \frac{t^2}{6\pi} \right]_0^{6\pi} \\
&= -2670g
\end{aligned}$$

So the work done is $2670g$ kg-m^{2}s^{-2} (against the gravity).

3. C is a piecewise smooth curves made up of 4 straight lines C_1, C_2, C_3, C_4.

Along C_1: $x = t$, $y = 0$
$\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 0$, for $0 \le t \le 2$.
Along C_2: $x = 2$, $y = t$
$\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 1$, for $0 \le t \le 3$.
Along C_3: $x = t$, $y = 3$
$\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 0$, for $0 \le t \le 2$.
Along C_4: $x = 0$, $y = t$
$\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 1$, for $0 \le t \le 3$.



If C is given positive orientation, then $C = C_1 + C_2 - C_3 - C_4$. Thus

$$\oint_C xy^2\,dx + x^3\,dy = \oint_{C_1+C_2-C_3-C_4} xy^2\,dx + x^3\,dy$$

$$= \int_0^2 0\,dt + \int_0^3 8\,dt - \int_0^2 9(t)\,dt - \int_0^3 0\,dt = 0+24-18-0 = 6.$$

We may also use Green's theorem to evaluate the line integral.

$$\oint_C xy^2\,dx + x^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial y}(xy^2) \right] dA$$

$$= \int_0^2 \int_0^3 (3x^2 - 2xy)\,dydx = \int_0^2 (9x^2 - 9x)\,dx = 6.$$

4. Let D be the ring region enclosed by two concentric circle of radius a and b respectively.

   In polar coordinates, $D: \quad a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi$.

   By Green's Theorem

$$\oint_C (x^5 - y^5)\,dx + (x^5 + y^5)\,dy = \int\int_D \left[\frac{\partial}{\partial x}(x^5 + y^5) - \frac{\partial}{\partial y}(x^5 - y^5)\right] dA$$

$$= \int\int_D (5x^4 + 5y^4)\,dA = 5\int\int_D [(x^2+y^2)^2 - 2x^2y^2]\,dA$$

$$= 5\int_0^{2\pi}\int_a^b [(r^2)^2 - 2(r\cos\theta)^2(r\sin\theta)^2]r\,drd\theta = 5\int_0^{2\pi}\int_a^b r^5(1 - 2(\cos\theta)^2(\sin\theta)^2)\,drd\theta$$

$$= 5\left[\int_a^b r^5\ dr\right]\left[\int_0^{2\pi}(1 - 2(\cos\theta)^2(\sin\theta)^2)\ d\theta\right] = 5\left[\int_a^b r^5\ dr\right]\left[\int_0^{2\pi}(1 - \frac{1}{2}\sin^2 2\theta)\ d\theta\right]$$

$$= 5\left[\int_a^b r^5\ dr\right]\left[\int_0^{2\pi}\left(\frac{3}{4} + \frac{1}{4}\cos 4\theta\right)\ d\theta\right] = \left[\frac{5}{6}(b^6 - a^6)\right]\left[\frac{3}{2}\pi\right] = \frac{5}{4}\pi(b^6 - a^6).$$

5. Applying Green's Theorem, we have

$$\begin{aligned}
&\oint_C \left(xy - \tan\left(y^2\right)\right)\ dx\ +\ \left(x^2 - 2xy\sec^2\left(y^2\right)\right)\ dy \\
=\ &\iint_D \left(\frac{\partial\left(x^2 - 2xy\sec^2\left(y^2\right)\right)}{\partial x} - \frac{\partial\left(xy - \tan\left(y^2\right)\right)}{\partial y}\right) dA \\
=\ &\iint_D x\,dA,
\end{aligned}$$

where D is the triangle with vertices at $(0,0)$, $(1,0)$, $(0,2)$. We can write D as a Type A domain:

$$D = \left\{(x,y) \in R^2 :\ 0 \leq y \leq 2 - 2x,\quad 0 \leq x \leq 1.\right\}$$

and then

$$\oint_C \left(xy - \tan\left(y^2\right)\right) dx + \left(x^2 - 2xy\sec^2\left(y^2\right)\right) dy = \int_0^1 \left(\int_0^{2-2x} xdy\right) dx = \frac{1}{3}.$$

6. Let L denote the straight line segment that joins $(-1,0)$ to $(1,0)$.

   Let D denote the domain bounded by the closed curve $L-C$.

   Apply Green's Theorem to D, we have

   $\oint_{L-C} \left\{y\left(x^2+e^x\right)+x^2\right\} dx + \left(e^x - xy^2\right) dy$
   $= \int\int_D \left(e^x - y^2 - x^2 - e^x\right) dxdy$
   $= \int_0^\pi \int_0^1 -r^2 r dr d\theta$
   $= -\frac{\pi}{4}$

   Therefore, using the parametrization: $x=t$, $y=0$, $-1 \le t \le 1$ for L, we have

   $\int_C \left\{y\left(x^2+e^x\right)+x^2\right\} dx + \left(e^x - xy^2\right) dy$
   $= \frac{\pi}{4} + \int_L \left\{y\left(x^2+e^x\right)+x^2\right\} dx + \left(e^x - xy^2\right) dy$
   $= \frac{\pi}{4} + \int_{-1}^1 t^2 dt$
   $= \frac{\pi}{4} + \frac{2}{3}$