Chapter 9

Line Integrals

Key Results

- Area of region under a graph over a curve in *xy*-plane calculated as a line integral of a scalar function.
- Work done calculated as a line integral of a vector field.
- Fundamental Theorem for Line Integrals.
- Line integrals for conservative fields.
- Green's Theorem.

Motivation

In this chapter, integrals of the form

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

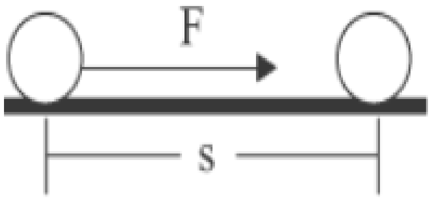
are studied, where **F** is a vector function and *C* is a space curve in three-dimensional space.

What do these integrals calculate?

In physics and engineering, an important application is that these integrals calculate work done by a force **F**.

Work Done I

Consider a constant force **F** applied to an object

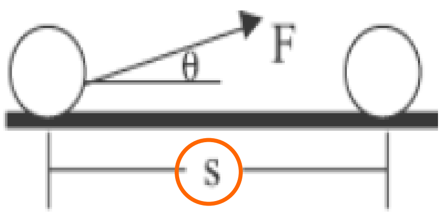


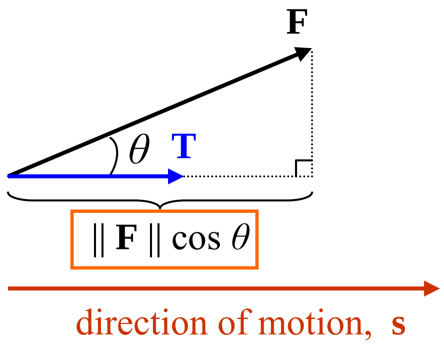
moving the object a distance s in the direction of **F**.

Work done W is given by:

$$\boxed{W = \| \mathbf{F} \| s}$$

Next, consider a constant force **F** moving an object in a direction that is at an angle θ to **F**:





Work done W is given by:

$$W = \|\mathbf{F}\|(\cos\theta)\, s = (\mathbf{F}\cdot\mathbf{T})\, s = \boxed{\mathbf{F}\cdot(s\mathbf{T})}$$

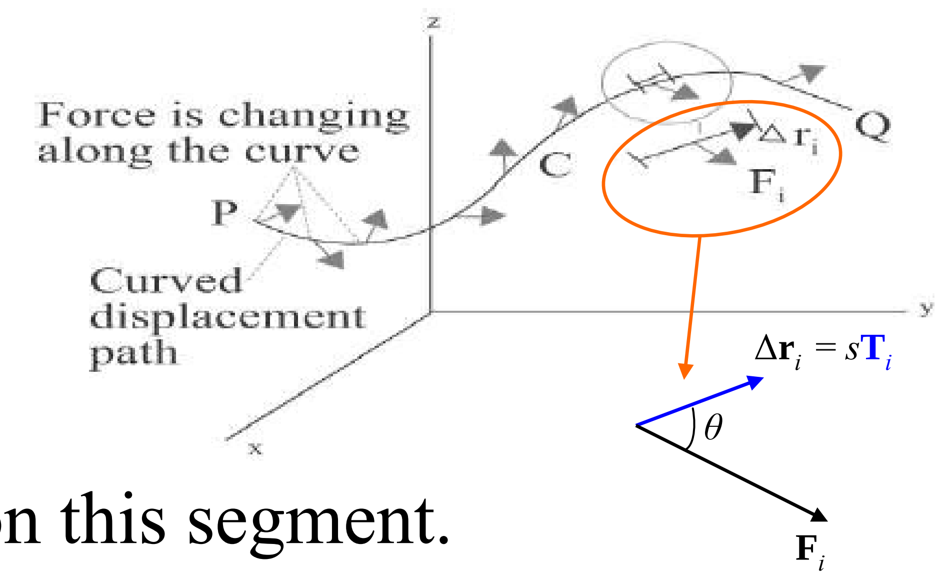
where **T** is a unit vector in the displacement direction.

Work Done II

Let $\mathbf{F}(x, y, z)$ be a variable force acting on an object, moving the object along a space curve C given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

Find work done.



Divide C into n segments.

Each *small segment* $\Delta \mathbf{r}_i$ *treated as a straight line*, $\mathbf{F}$ approximately *constant* $\mathbf{F}_i$ on this segment.

Note that $\Delta \mathbf{r}_i$ may also be written as $s\mathbf{T}_i$, where $\mathbf{T}_i$ is a unit tangent along the ith segment.

previous formula: $W = \mathbf{F} \cdot (s\mathbf{T})$

Work done on ith segment: $W_i \approx \mathbf{F}_i \cdot (s\mathbf{T}_i) = \boxed{\mathbf{F}_i \cdot \Delta\mathbf{r}_i.}$

Total work done is approximately

$$W_{\text{total}} \approx \sum_{1}^{n} \mathbf{F}_i \cdot \Delta\mathbf{r}_i$$

By taking n to infinity, actual work done is

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r}}$$

This is called a line integral of the vector function $\mathbf{F}$ along the space curve C.

§9.3 will show how to calculate these line integrals.

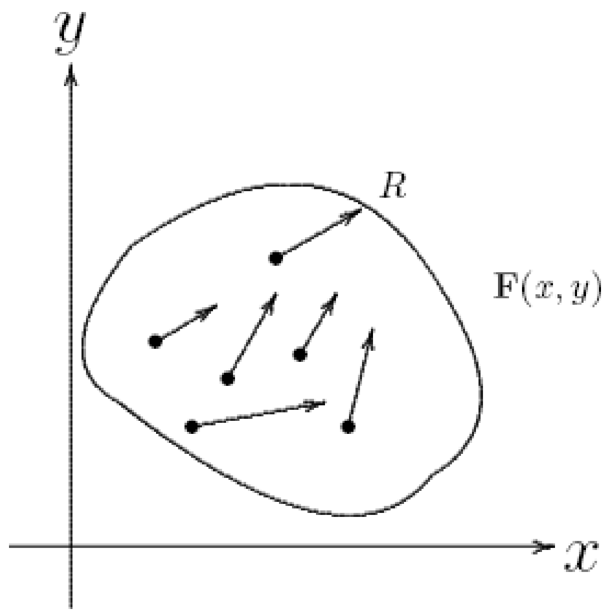
Vector Fields

A vector field is a vector function whose domain is a region in the *xy*-plane (or three-dimensional space) and whose range is also a subset of the *xy*-plane (or three-dimensional space).

Let R be a region in the *xy*-plane and $\mathbf{F}(x, y)$ be a vector field on R.

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

$P(x, y)$ and $Q(x, y)$ are called component functions.



Example

$$\boxed{\mathbf{F}(x, y) = (-y)\,\mathbf{i} + x\,\mathbf{j}.}$$

$\mathbf{F}(1, 1) = -\mathbf{i} + \mathbf{j}$

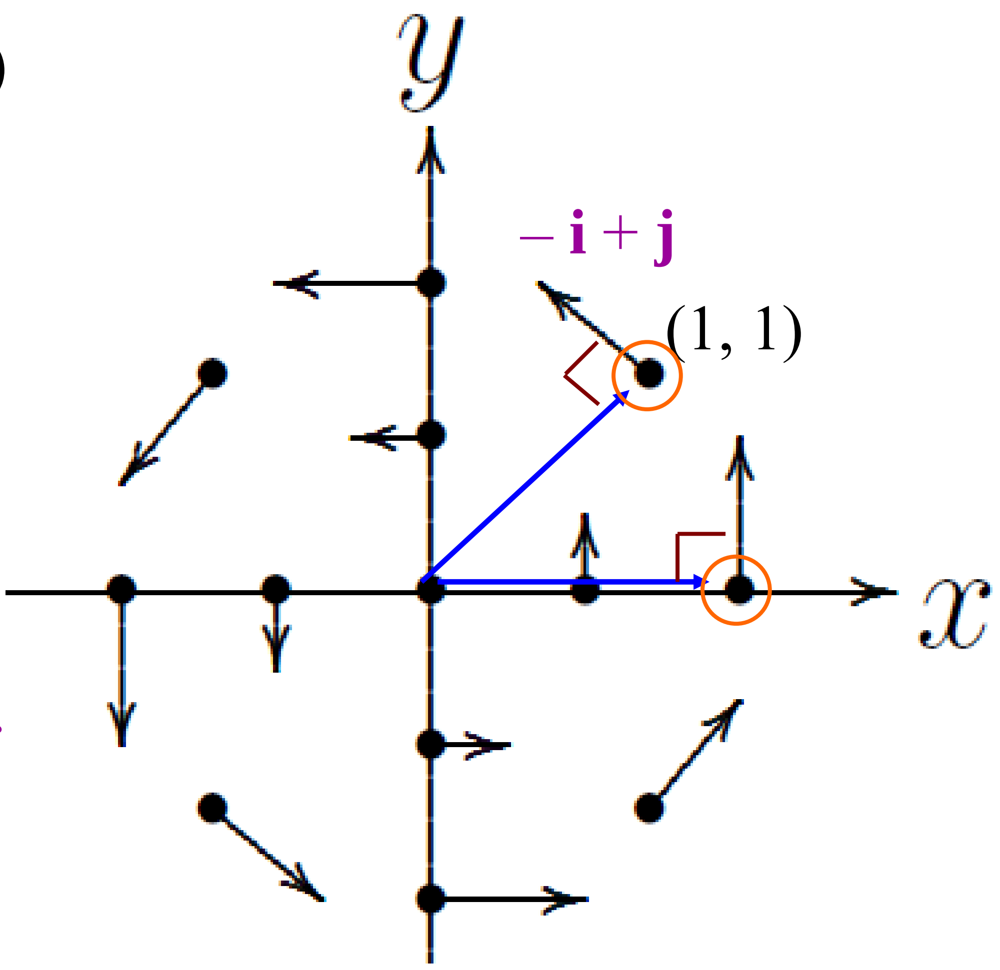
Position vector of (x, y) is $x\,\mathbf{i} + y\,\mathbf{j}$.

$$\mathbf{F}(x, y) \cdot (x\,\mathbf{i} + y\,\mathbf{j})$$

$$= (-y)\,x + xy$$

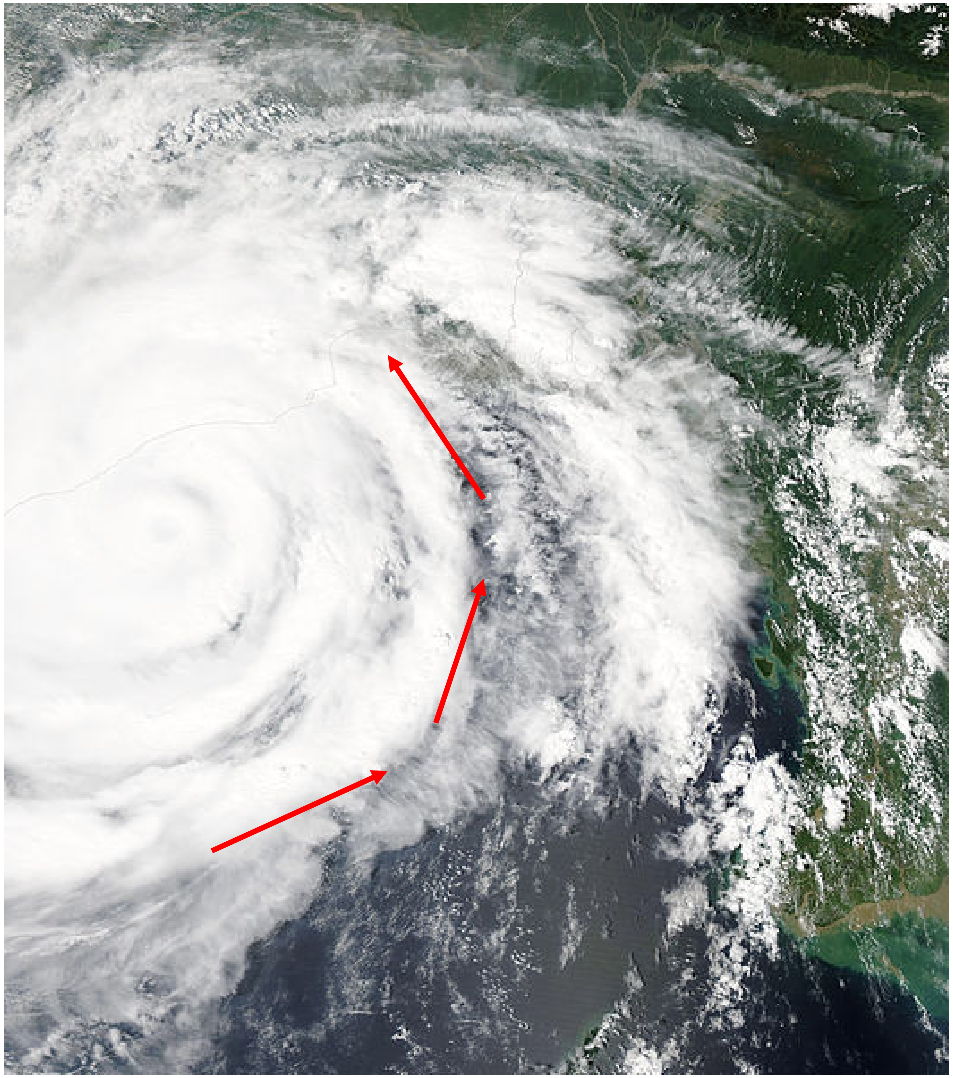
$$= 0$$

$\mathbf{F}(x, y)$ is perpendicular to $x\,\mathbf{i} + y\,\mathbf{j}$.



Wikipedia: Cyclone Phailin, 12 October 2013

Category 5 hurricane



Gradient Fields

If $f(x, y)$ is a scalar (real-valued) function, then

$$\boxed{\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}}$$

is a vector field called the gradient (field) of f.

grad(f)

Example:

The gradient field of $f(x,y) = \boxed{xy^2 + x^3}$ is

$$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$
$$= \boxed{(y^2+3x^2)\mathbf{i} + (2xy)\mathbf{j}.}$$

Conservative Fields

A vector field $\mathbf{F}(x, y)$ is called a conservative field if there is a scalar function $f(x, y)$ such that

$$\boxed{\mathbf{F}(x, y) = \nabla f(x, y)}$$

The scalar function $f(x, y)$ is called a potential function for $\mathbf{F}(x, y)$.

Example: In Example 9.2.5,

The gradient field of $f(x, y) = xy^2 + x^3$ is

$$\nabla f(x, y) = (y^2 + 3x^2)\mathbf{i} + (2xy)\mathbf{j}.$$

$f(x, y) = xy^2 + x^3$ is a potential function for

$$\mathbf{F}(x, y) = (y^2 + 3x^2)\mathbf{i} + (2xy)\mathbf{j}$$

Example

Let $\mathbf{F}(x, y) = \boxed{(3 + 2xy)\,\mathbf{i}} + \boxed{(x^2 - 3y^2)\,\mathbf{j}}.$

Find a potential function f for $\mathbf{F}$.

Require $\mathbf{F}(x, y) = \nabla f(x, y) = \boxed{f_x(x, y)\mathbf{i}} + \boxed{f_y(x, y)\mathbf{j}}$

That is, $\boxed{f_x = 3 + 2xy}$ and $\boxed{f_y = x^2 - 3y^2}$

Integrating f_x w.r.t. x gives

$$\boxed{f(x, y) = 3x + x^2 y + g(y)}$$

Then $f_y(x, y) = 0 + x^2 + g'(y) \overset{\text{set}}{=} x^2 - 3y^2$

to give $\boxed{g'(y) = -3y^2.}$

Integrating $g'(y)$ w.r.t. y gives $\boxed{g(y) = -y^3 + K.}$

Thus, $\boxed{f(x, y) = 3x + x^2 y - y^3 + K.}$

Vector Fields in 3D-space

The concepts of vector fields, gradient fields, and conservative fields generalize to 3D-space:

(page 6, vector field)

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$

(page 7, gradient field)

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

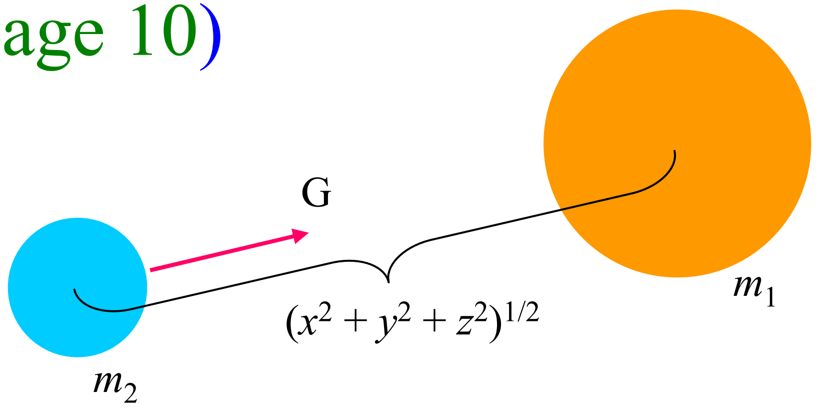
$\mathbf{F}(x, y, z)$ is a conservative field if

$$\boxed{\mathbf{F}(x, y, z) = \nabla f(x, y, z)}$$

for some potential function $f(x, y, z)$.

Example (page 10)

The gravitational field given by



$$\mathbf{G} = \left(\frac{-m_1 m_2 K x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)\mathbf{i} + \left(\frac{-m_1 m_2 K y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)\mathbf{j} + \left(\frac{-m_1 m_2 K z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)\mathbf{k}$$

is conservative because it is the gradient of the gravitational potential function

$$\boxed{g(x,y,z) = \frac{m_1 m_2 K}{\sqrt{x^2+y^2+z^2}},}$$

where K is the gravitational constant, m_1 and m_2 are the masses of two objects.

To verify that $g(x, y, z)$ is a potential function, calculate g_x , g_y , g_z to see that these are the respective **i**, **j**, **k** components of $\mathbf{G}(x, y, z)$.

For example, write

$$g(x,y,z) = \frac{m_1 m_2 K}{\sqrt{x^2+y^2+z^2}} = \boxed{m_1 m_2 K \left(x^2+y^2+z^2\right)^{-1/2}}$$

Then by power rule and chain rule,

$$g_x = m_1 m_2 K \cdot \left[-\frac{1}{2}\right] \left(\boxed{x^2+y^2+z^2}\right)^{-3/2} \cdot 2x$$

$$= \boxed{\frac{-m_1 m_2 K x}{(x^2+y^2+z^2)^{\frac{3}{2}}}}$$

which is the **i** component of $\mathbf{G}(x, y, z)$.

Similar calculations hold for g_y and g_z.

Criteria for Conservative Fields

Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$

F is conservative if and only if $\boxed{\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}}$ ✓

Motivation for criterion

F is conservative means that

$$P\mathbf{i} + Q\mathbf{j} = \mathbf{F} = \nabla f = f_x\mathbf{i} + f_y\mathbf{j}$$

for some potential function f.

$$\Rightarrow \boxed{P = f_x,} \quad \boxed{Q = f_y}$$

Then $$\frac{\partial P}{\partial y} = (f_x)_y = (f_y)_x = \frac{\partial Q}{\partial x}$$

(from Chapter 7) $f_{xy} = f_{yx}$

Example

$$\mathbf{F}(x, y) = \underbrace{\boxed{(3 + 2xy)\mathbf{i}}}_{P} + \underbrace{\boxed{(x^2 - 3y^2)\mathbf{j}}}_{Q}.$$

Check

$$\frac{\partial Q}{\partial x} = \frac{\partial(x^2 - 3y^2)}{\partial x} = \boxed{2x}$$

$$\frac{\partial P}{\partial y} = \frac{\partial(3 + 2xy)}{\partial y} = \boxed{2x}$$

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$ implies that **F**(x, y) is conservative.

Let $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$

F is conservative if and only if

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},} \quad \boxed{\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x},} \quad \boxed{\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}}$$

Motivation for criteria is similar to that for $\mathbf{F}(x, y)$:

need to check 3 equalities.

F is conservative means that

$$\boxed{P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}} = \mathbf{F} = \nabla f = \boxed{f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}}$$

for some potential function f.

$$\Rightarrow \boxed{P = f_x,} \quad Q = f_y, \quad \boxed{R = f_z}$$

Then (for example) (check the other two equalities!)

$$\frac{\partial P}{\partial z} = (f_x)_z \stackrel{\checkmark}{=} (f_z)_x = \frac{\partial R}{\partial x}$$

Example

$\mathbf{F}(x, y, z) = \underbrace{xz}_{P}\mathbf{i} + \underbrace{xyz}_{Q}\mathbf{j} - y^2\mathbf{k}$ is not conservative ✓

because

$$\frac{\partial P}{\partial y} = \frac{\partial(xz)}{\partial y} = \boxed{0}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial(xyz)}{\partial x} = \boxed{yz}$$

$\boxed{\dfrac{\partial P}{\partial y} \text{ and } \dfrac{\partial Q}{\partial x} \text{ are different.}}$

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},} \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

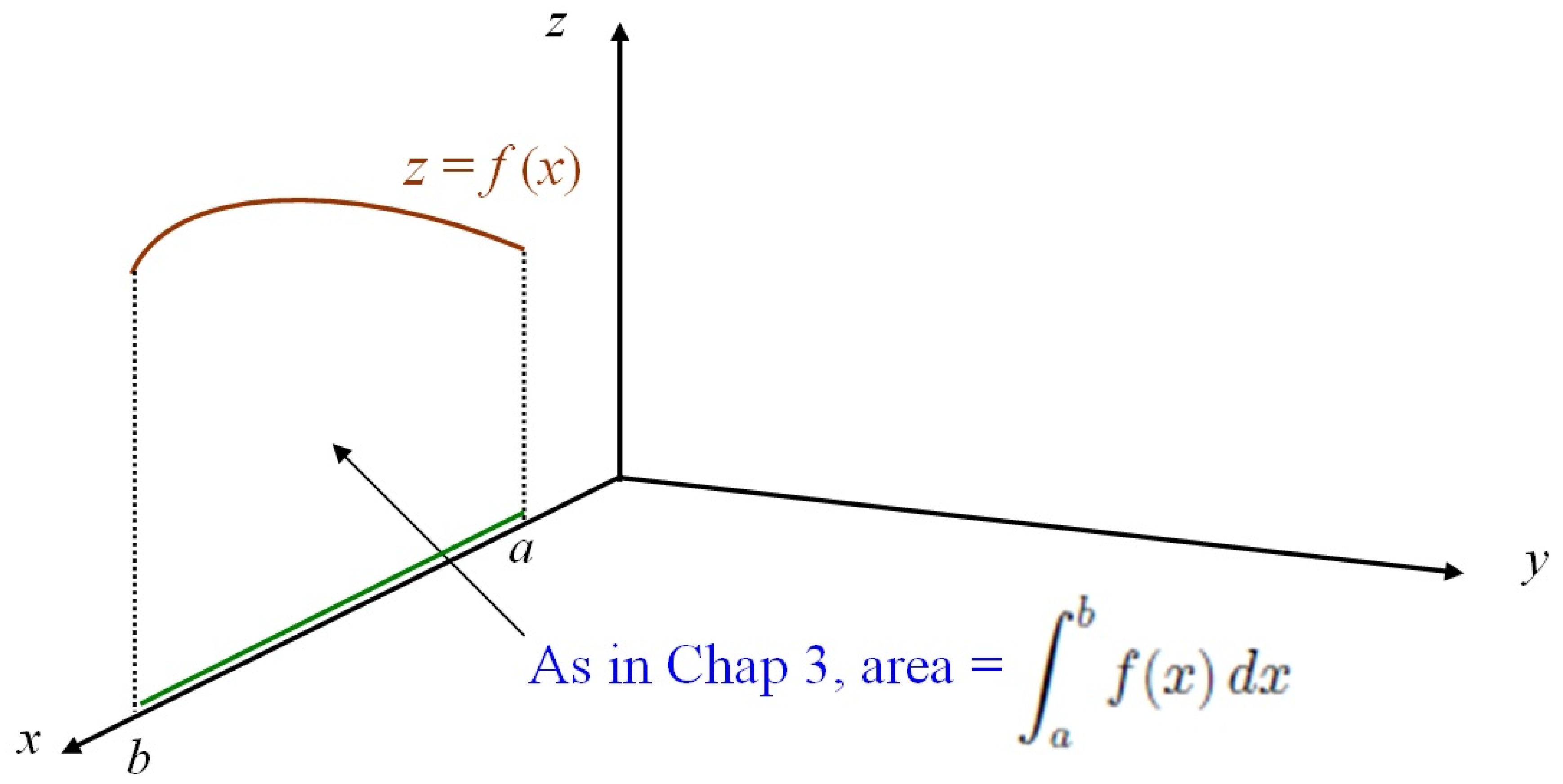
Line Integral of a Scalar Function

In Chapter 3 the area of a region A under the graph of $y = f(x)$ was found as a definite integral.

Suppose the y-axis is replaced by the z-axis.

Clearly, Chapter 3 techniques hold for finding the area of a region A under the graph of $z = f(x)$.

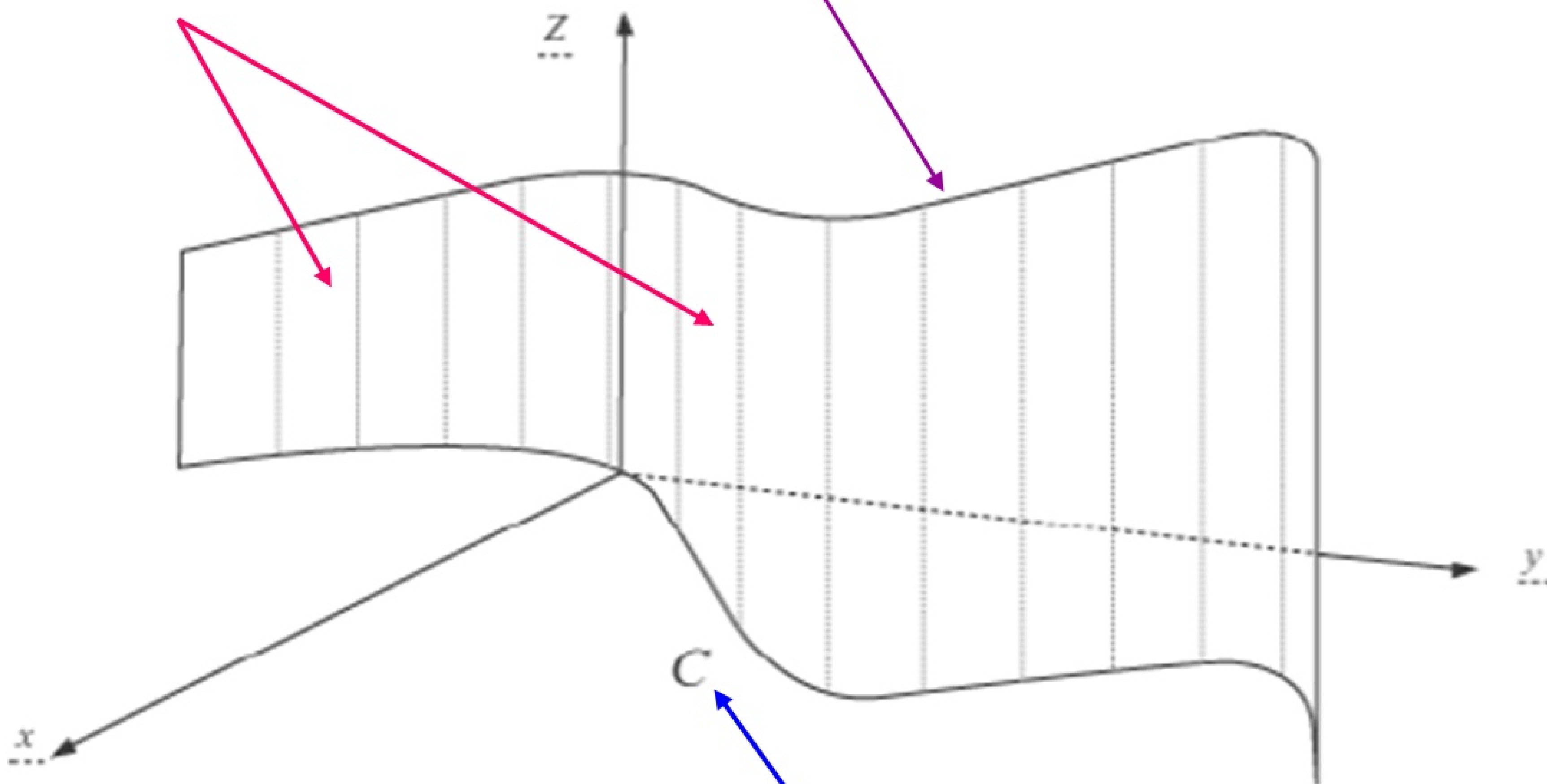
In 3D the line integral of a scalar function $f(x, y)$ generalizes this concept of area of a region A under a graph, where the base of A is not on the x-axis but on some curve C in the xy-plane.



Generalize: interval $[a, b]$ becomes a curve C in xy-plane

How to calculate area of region?

‘graph’ lies on surface $z = f(x, y)$

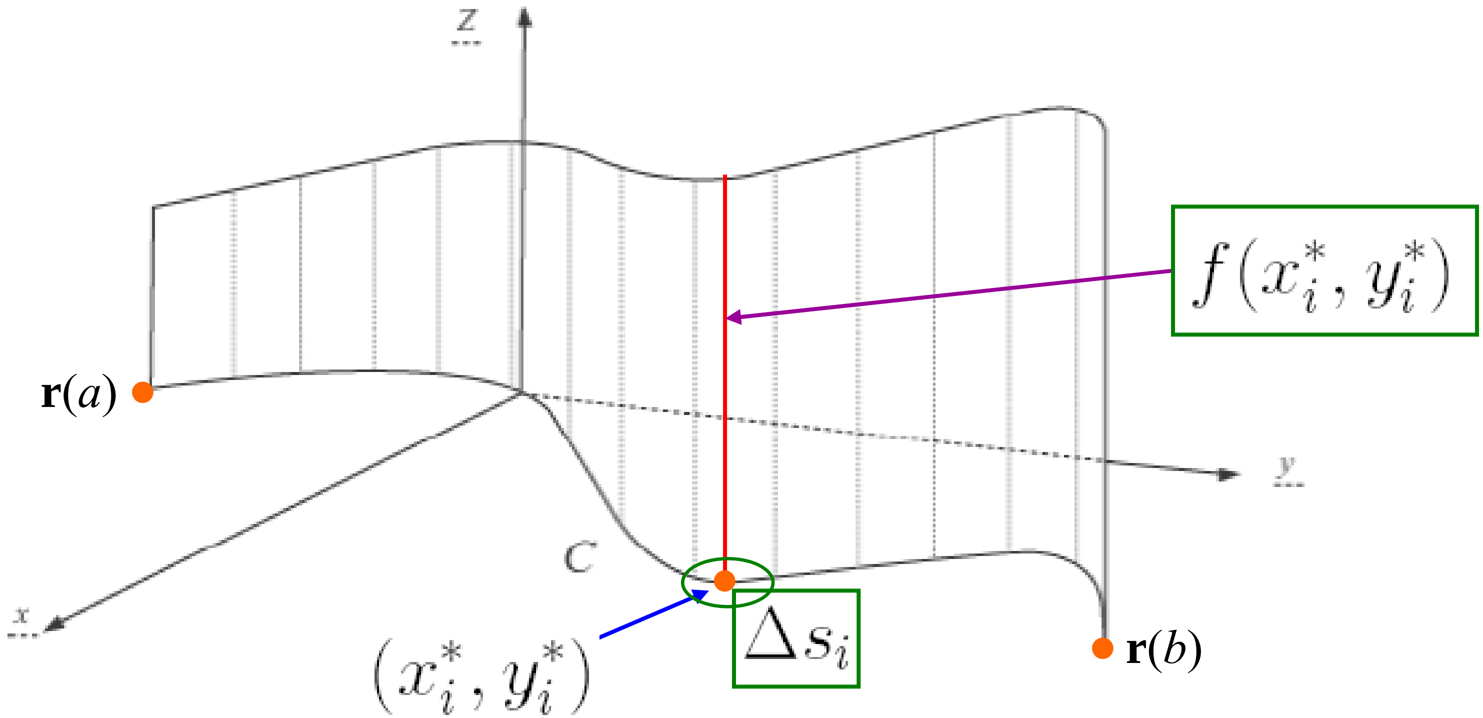


C is a smooth curve.

$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$

$\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t)$ is never zero

Calculating area of region:



Subdivide the curve from $\mathbf{r}(a)$ to $\mathbf{r}(b)$ into n small arcs of length Δs_i, $i = 1, 2, 3, \ldots, n$.

Pick an arbitrary point (x_i^*, y_i^*) inside the ith small arc and form the sum $\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$.

The surface area is given by

$$\int_C f(x,y)\,\boxed{ds} = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

If the limit exists, then it is called the line integral of the scalar function $f(x, y)$ along the plane curve C.

s denotes the arc length of C. As a function of t,

$$s(t) = \int_a^{t} \|\mathbf{r}'(u)\|\,du$$

By FTC, $$\frac{ds}{dt} = \|\mathbf{r}'(t)\|$$

$$ds = \boxed{\|\mathbf{r}'(t)\|\,dt}$$

$$s = \int_a^{b} \|\mathbf{r}'(t)\|\,dt.$$

A Line Integral Formula

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\|\mathbf{r}'(t)\|dt}$$

$$= \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

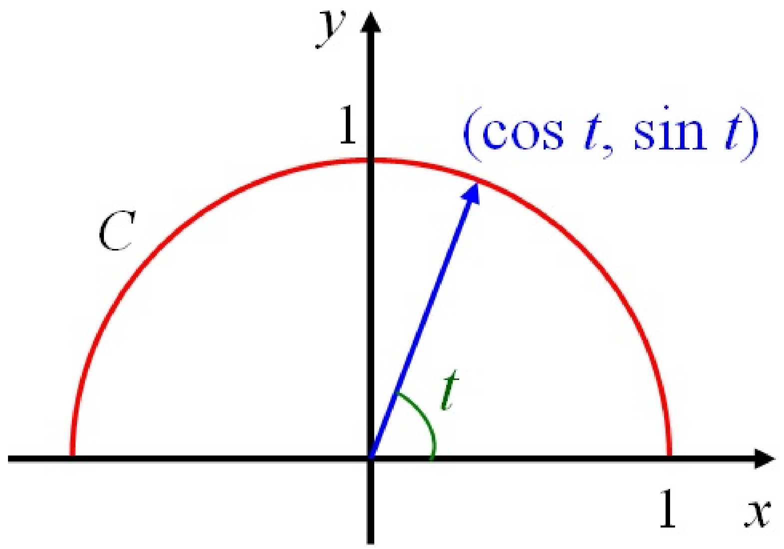
where $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ is the vector equation of the plane curve C.

Example

$$\int_C (2y + x^2 y)\,ds$$ C is the upper half of the unit circle.

$$C:\ \mathbf{r}(t) = \underbrace{\cos t}_{x}\,\mathbf{i} + \underbrace{\sin t}_{y}\,\mathbf{j} \qquad 0 \le t \le \pi$$

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$



$$\int_C (2y + x^2 y)\,ds = \int_0^\pi (2\sin t + \cos^2 t \sin t)\sqrt{\sin^2 t + \cos^2 t}\; dt$$

$$= \int_0^\pi (2\sin t + \cos^2 t \sin t)\; dt$$

$$= \left[-2\cos t - \frac{1}{3}\cos^3 t\right]_0^\pi = \boxed{\frac{14}{3}}$$

Line Integral for 3D-Space

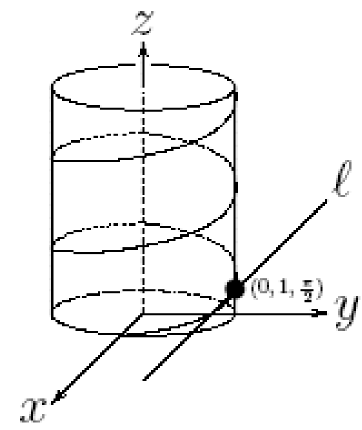
Scalar function $f(x, y, z)$

Space curve C : $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$

$$\boxed{\begin{aligned} &\int_C f(x, y, z)\, ds \\ &= \int_a^b f(x(t), y(t), z(t))\, \|\mathbf{r}'(t)\|\, dt \end{aligned}}$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example



$$\int_C xy\sin z\,ds$$

$$C:\ \boxed{\mathbf{r}(t) = \underset{x}{\cos t}\,\mathbf{i} + \underset{y}{\sin t}\,\mathbf{j} + \underset{z}{t}\,\mathbf{k},}\ \boxed{0 \le t \le \frac{\pi}{2}}$$

$$\boxed{\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}}$$

$$\int_C \boxed{xy\sin z}\,\boxed{ds}$$

$$= \boxed{\int_0^{\pi/2}} (\cos t)(\sin t)(\sin t)\sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1} + 1}\ \boxed{dt}$$

$$= \sqrt{2}\int_0^{\pi/2} \cos t\sin^2 t\,dt = \frac{\sqrt{2}}{3}\left[\sin^3 t\right]_0^{\pi/2} = \boxed{\frac{\sqrt{2}}{3}}$$

substitution $u = \sin t$

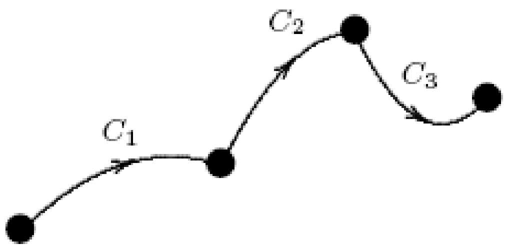
$$\int_a f(x(t), y(t), z(t))\,\boxed{\|\mathbf{r}'(t)\|\,dt}$$

Line Integral over Joint Curves

Union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$:

$$C = C_1 + C_2 + \cdots + C_n$$

C is a piecewise-smooth curve.



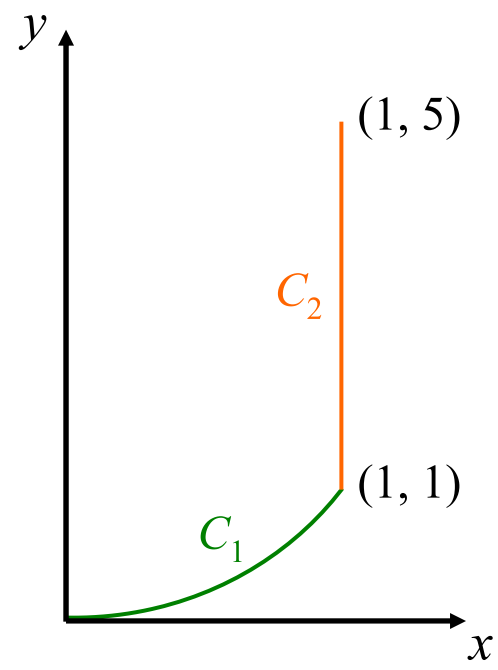
$$\boxed{\int_C f(x, y)\, ds = \int_{C_1} f(x, y)\, ds + \cdots + \int_{C_n} f(x, y)\, ds.}$$

Example (page 20)

Find $\int_C 9y\, ds$ with $C = C_1 + C_2$,

where C_1 is the arc of $y = x^3$ from (0, 0) to (1, 1),

C_2 is the line segment from (1, 1) to (1, 5).



Example (page 20)

C_1 : $y = x^3$ from (0, 0) to (1, 1)

$$\mathbf{r}_1(t) = \underset{x}{t}\mathbf{i} + \underset{y}{t^3}\mathbf{j} \qquad 0 \le t \le 1$$

$$\mathbf{r}_1'(t) = \mathbf{i} + 3t^2\mathbf{j} \qquad \|\mathbf{r}'_1(t)\| = \sqrt{1 + (3t^2)^2}$$

$$\int_{C_1} 9y\,ds = \int_0^1 9t^3\sqrt{1+9t^4}\,dt$$

$$= 9 \cdot \frac{1}{4}\int_0^1 (1+9u)^{1/2}\,du$$

check limits

substitution

$$u = t^4$$

$$\frac{du}{dt} = 4t^3$$

$$\frac{1}{4}\,du = t^3\,dt$$

Example (page 20)

$C_1 : y = x^3$ from (0, 0) to (1, 1)

$$\mathbf{r}_1(t) = \underset{x}{t}\mathbf{i} + \underset{y}{t^3}\mathbf{j} \qquad 0 \le t \le 1$$

$$\mathbf{r}_1'(t) = \mathbf{i} + 3t^2\mathbf{j} \qquad \|\mathbf{r}'_1(t)\| = \sqrt{1 + (3t^2)^2}$$

$$\int_{C_1} 9y\,ds = \int_0^1 9t^3\sqrt{1+9t^4}\,dt$$

$$= 9 \cdot \frac{1}{4}\int_0^1 (1+9u)^{1/2}\,du$$

$$= \frac{9}{4}\left[\frac{1}{9} \cdot \frac{2}{3}(1+9u)^{3/2}\right]_0^1$$

$$= \boxed{\frac{1}{6}(10\sqrt{10} - 1)}$$

substitution

$$u = t^4$$

$$\frac{du}{dt} = 4t^3$$

$$\frac{1}{4}du = t^3\,dt$$

C_2 : vertical line segment from (1, 1) to (1, 5)

$$\mathbf{r}_2(t) = \underset{x}{\mathbf{i}} + \underset{y}{t\mathbf{j}} \qquad 1 \le t \le 5$$

$$\mathbf{r}_2'(t) = \mathbf{j} \qquad \|\mathbf{r}'_2(t)\| = 1$$

$$\int_{C_2} 9y\,ds = \int_1^5 9t\,dt = \boxed{108.}$$

Thus,

$$\int_C 9y\,ds = \int_{C_1} 9y\,ds + \int_{C_2} 9y\,ds$$

$$= \frac{1}{6}(10\sqrt{10} - 1) + 108$$

$$= \boxed{\frac{1}{6}(10\sqrt{10} + 647)}$$

Try different description of C_1 : $x = t^3 \qquad y = x^3 = t^9$

$$\mathbf{r}(t) = \underset{x}{t^3}\mathbf{i} + \underset{y}{t^9}\mathbf{j} \qquad 0 \le t \le 1$$

$$\mathbf{r}'(t) = 3t^2\mathbf{i} + 9t^8\mathbf{j} \qquad ||\mathbf{r}'(t)|| = \sqrt{(3t^2)^2 + (9t^8)^2}$$

$$= 3t^2\sqrt{1 + 9t^{12}}$$

$$\int_{C_1} 9y\, ds = \int_0^1 9t^9 \cdot 3t^2\sqrt{1 + 9t^{12}}\ dt$$

$$= 27\int_0^1 t^{11}\sqrt{1 + 9t^{12}}\, dt$$

$$= 27 \cdot \frac{1}{12}\int_0^1 (1 + 9u)^{1/2}\ du$$

$$= \frac{9}{4}\left[\frac{1}{9} \cdot \frac{2}{3}\ (1 + 9u)^{3/2}\right]_0^1 = \boxed{\frac{1}{6}\left[10\sqrt{10} - 1\right]}$$

substitution

$$u = t^{12}$$

$$\frac{du}{dt} = 12t^{11}$$

$$\frac{1}{12}du = t^{11}\, dt$$

There are two types of line integrals:

1. for scalar function $f(x, y)$ or $f(x, y, z)$

   $$\int_C f(x, y)\, ds$$ calculates area of region

   under a graph over a curve C.

2. for vector field $\mathbf{F}(x, y)$ or $\mathbf{F}(x, y, z)$

   $$\int_C \mathbf{F} \cdot d\mathbf{r}$$ calculates, for example, work done

   by force $\mathbf{F}$ in moving an object over a curve C.

Line Integrals of Vector Fields

In the *xy*-plane or 3D-space,

(1) **F** is a vector field;

(2) C is a smooth curve given by vector function $\mathbf{r}(t)$.

The line integral of **F** along C is given by

$$\boxed{\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt}$$

If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then

$$\mathbf{F}(\mathbf{r}(t)) = \mathbf{F}(x(t), y(t), z(t))$$

refers to the vector field along the curve.

Example

$$\boxed{\mathbf{F}(x,y,z) = x\mathbf{i} + xy\mathbf{j} + xyz\mathbf{k}}$$

$$C: \boxed{\mathbf{r}(t) = \underset{x}{t}\mathbf{i} + \underset{y}{t^2}\mathbf{j} + \underset{z}{t^3}\mathbf{k},} \quad \boxed{0 \le t \le 2}$$

$$\boxed{\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}}$$

$$\boxed{\mathbf{F}(\mathbf{r}(t))} \cdot \boxed{\mathbf{r}'(t)} = (t\mathbf{i} + t \cdot t^2\mathbf{j} + t \cdot t^2 \cdot t^3\mathbf{k}) \cdot (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})$$

$$= \boxed{t + 2t^4 + 3t^8}$$

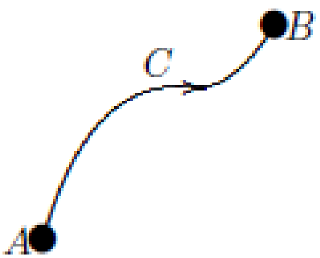
Thus,

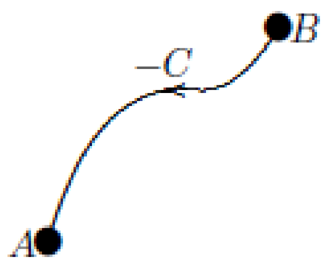
$$\int_C \mathbf{F} \cdot \boxed{d\mathbf{r}} = \boxed{\int_0^2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \boxed{dt} = \int_0^2 (t + 2t^4 + 3t^8)\, dt$$

$$= \boxed{2782/15}.$$

Orientation of Curves

The vector equation of a curve C determines an **orientation** (direction) of C. The same curve with the opposite orientation of C is denoted by $-C$.





$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_{C} \mathbf{F} \cdot d\mathbf{r}$$

But for scalar functions $f(x, y, z)$,

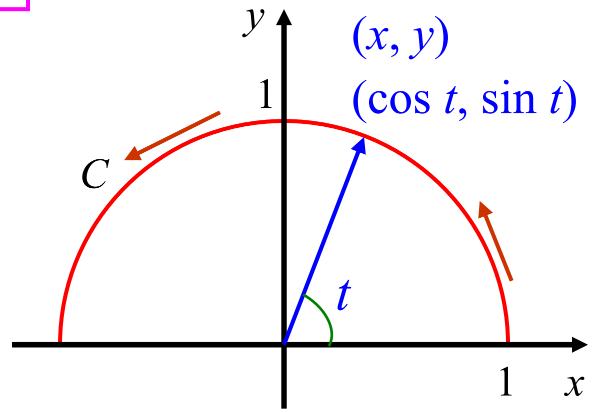
$$\int_{-C} f(x, y, z)\, ds = \int_{C} f(x, y, z)\, ds$$

Example

$$\boxed{\mathbf{F}(x,y) = (-y)\mathbf{i} + x\mathbf{j}}$$

$$C : \boxed{\mathbf{r}_1(t) = \underbrace{\cos t}_{x}\,\mathbf{i} + \underbrace{\sin t}_{y}\,\mathbf{j}} \qquad \boxed{0 \le t \le \pi}$$

$$\boxed{\mathbf{r}_1'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}}$$



$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \boxed{\int_0^{\pi}} (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j}) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\boxed{dt} \\
&= \int_0^{\pi} \underbrace{\left(\sin^2 t + \cos^2 t\right)}_{1}\, dt \\
&= \boxed{\pi}
\end{aligned}$$

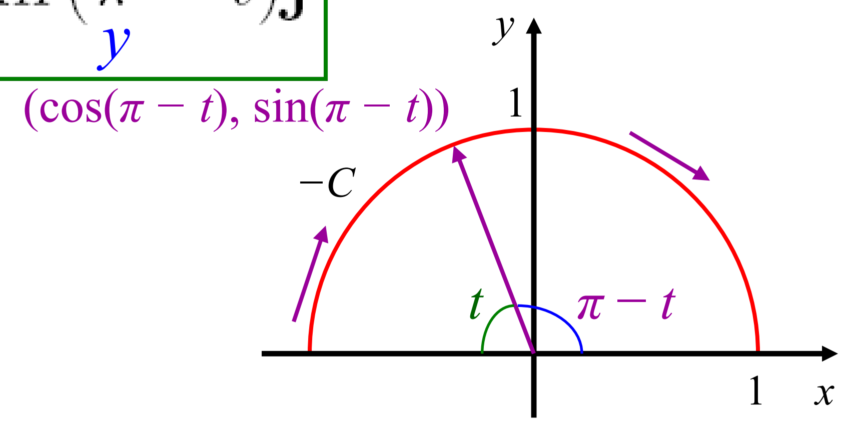
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \boxed{\mathbf{F}(\mathbf{r}(t))} \cdot \boxed{\mathbf{r}'(t)}\, dt.$$

Example (clockwise)

$$\boxed{\mathbf{F}(x,y) = (-y)\mathbf{i} + x\mathbf{j}}$$

$$-C : \boxed{\mathbf{r}_2(t) = \cos(\underbrace{\pi - t}_{x})\mathbf{i} + \sin(\underbrace{\pi - t}_{y})\mathbf{j}}$$

$$\boxed{0 \le t \le \pi}$$



$$\boxed{\mathbf{r}_2'(t) = \sin(\pi - t)\mathbf{i} - \cos(\pi - t)\mathbf{j}}$$

$$\boxed{\mathbf{r}_1'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}}$$

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = \boxed{\int_0^{\pi}} \boxed{(-\sin(\pi - t)\mathbf{i} + \cos(\pi - t)\mathbf{j})} \boxed{\cdot (\sin(\pi - t)\mathbf{i} - \cos(\pi - t)\mathbf{j})} \boxed{dt}$$

$$= \int_0^{\pi} (-1)\,dt = \boxed{-\pi} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a \boxed{\mathbf{F}(\mathbf{r}(t))} \cdot \boxed{\mathbf{r}'(t)}\,dt.$$

Line Integrals in Component Form

Another popular notation for line integrals of vector fields uses vector components:

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

$$C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \qquad a \leq t \leq b$$

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy.}$$

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

$$C \;:\; \mathbf{r}(t) \;=\; x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad a \leq t \leq b$$

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz.}$$

Derivation

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

$$C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j},$$

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

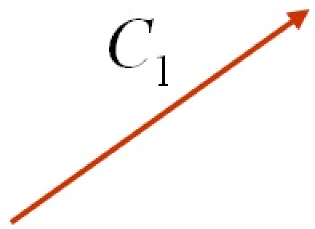
$$= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_a^b [P(\mathbf{r}(t))\mathbf{i} + Q(\mathbf{r}(t))\mathbf{j}] \cdot \left[\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right] dt$$

$$= \int_a^b \left[P(\mathbf{r}(t))\frac{dx}{dt} + Q(\mathbf{r}(t))\frac{dy}{dt}\right] dt$$

$$= \int_C P dx + Q dy.$$

Example (a)



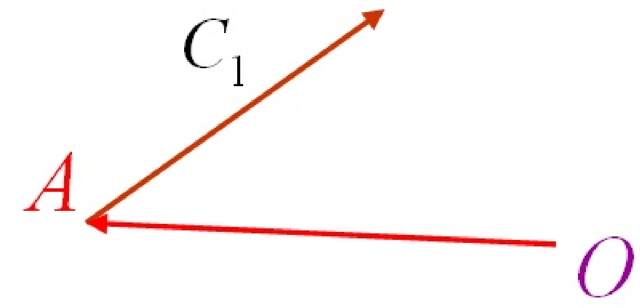
$C = C_1$: line joining $A(-5, -3)$ to $B(0, 2)$.

C_1 is parallel to $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\mathbf{j}) - (-5\mathbf{i} - 3\mathbf{j})$

$\overrightarrow{OA} \quad + \quad t\,\overrightarrow{AB}$ $\qquad = 5\mathbf{i} + 5\mathbf{j}.$

$$\mathbf{r}(t) = (-5\mathbf{i} - 3\mathbf{j}) + t(5\mathbf{i} + 5\mathbf{j}) \qquad 0 \leq t \leq 1.$$

Example (a)



$C = C_1$: line joining $A(-5, -3)$ to $B(0, 2)$.

C_1 is parallel to $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\mathbf{j}) - (-5\mathbf{i} - 3\mathbf{j})$

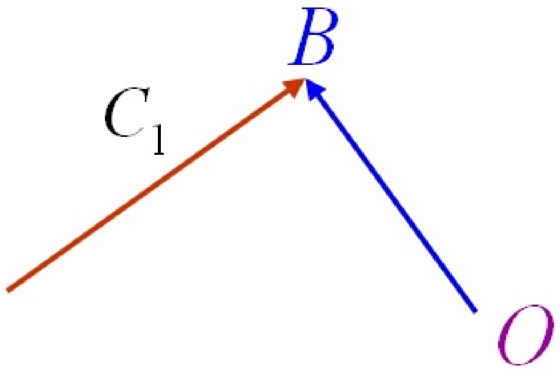
$= 5\mathbf{i} + 5\mathbf{j}.$

$\overrightarrow{OA}$

$\mathbf{r}(0) = (-5\mathbf{i} - 3\mathbf{j})$

$0 \le t \le 1.$

Example (a)



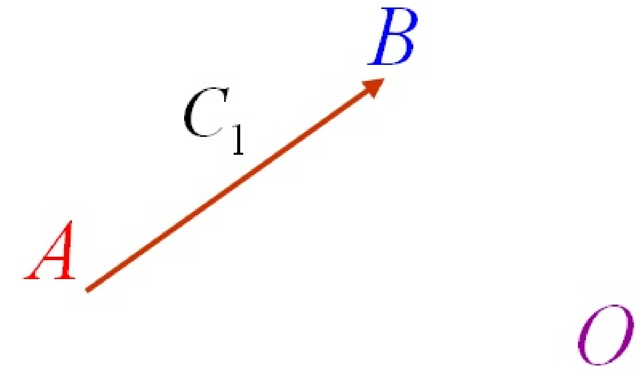
$C = C_1$: line joining $A(-5, -3)$ to $B(0, 2)$.

C_1 is parallel to $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\mathbf{j}) - (-5\mathbf{i} - 3\mathbf{j})$

$$\boxed{\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}} \qquad = 5\mathbf{i} + 5\mathbf{j}.$$

$\mathbf{r}(1) = (-5\mathbf{i} - 3\mathbf{j}) + (5\mathbf{i} + 5\mathbf{j}) \qquad 0 \leq t \leq 1.$

Example (a)



$C = C_1$: line joining $A(-5, -3)$ to $B(0, 2)$.

C_1 is parallel to $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\mathbf{j}) - (-5\mathbf{i} - 3\mathbf{j})$

$= 5\mathbf{i} + 5\mathbf{j}.$

$\overrightarrow{OA} \quad + \quad t\,\overrightarrow{AB}$

$$\mathbf{r}(t) = (-5\mathbf{i} - 3\mathbf{j}) + t(5\mathbf{i} + 5\mathbf{j}) \qquad 0 \le t \le 1.$$

$$= (5t - 5)\mathbf{i} + (5t - 3)\mathbf{j}$$

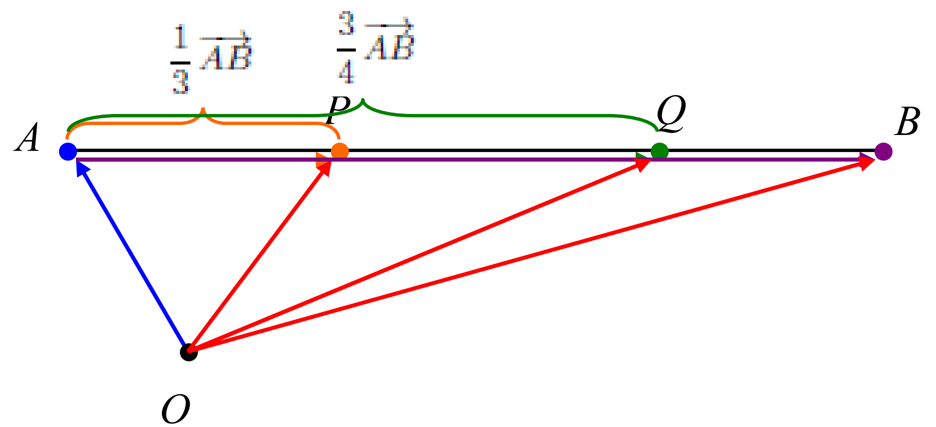
$x \quad \frac{dx}{dt} = 5 \qquad y \quad \frac{dy}{dt} = 5$

$$\int_{C_1} y^2 dx + x dy = \int_0^1 (5t-3)^2 \frac{dx}{dt} dt + \int_0^1 (5t-5) \frac{dy}{dt} dt$$

$$= \boxed{-5/6.} \qquad = \int_0^1 (5t-3)^2 5dt + \int_0^1 (5t-5)5dt$$

Line Segment: Why is $0 \leq t \leq 1$ for $\mathbf{r}(t)$?

Consider line segment joining points A and B.



$$\mathbf{r}(t) = \overrightarrow{OA} + t\,\overrightarrow{AB}$$

$t = 0$: $\boxed{\mathbf{r}(0)} = \overrightarrow{OA}$ $\boxed{\text{point } A}$

$t = \frac{1}{3}$: $\boxed{\mathbf{r}\left(\frac{1}{3}\right)} = \overrightarrow{OA} + \boxed{\frac{1}{3}\overrightarrow{AB}}$ $\boxed{\text{point } P}$

$t = \frac{3}{4}$: $\boxed{\mathbf{r}\left(\frac{3}{4}\right)} = \overrightarrow{OA} + \boxed{\frac{3}{4}\overrightarrow{AB}}$ $\boxed{\text{point } Q}$

$t = 1$: $\boxed{\mathbf{r}(1)} = \overrightarrow{OA} + \overrightarrow{AB}$ $\boxed{\text{point } B}$

Example (b)

$$\int_C y^2\,dx + x\,dy$$

$C = C_2$: arc of parabola $x = 4 - y^2$ joining $A(-5, \boxed{-3})$ to $B(0, \boxed{2})$.

Set $\boxed{y = t}$: $\quad \mathbf{r}(t) = \boxed{\underbrace{(4 - t^2)}_{x}\mathbf{i}} + \boxed{\underbrace{t}_{y}\mathbf{j}}\ \boxed{-3 \le t \le 2.}$

$$\frac{dx}{dt} = -2t \qquad \frac{dy}{dt} = 1$$

$$\int_{C_2} \boxed{y^2\,dx} + \boxed{x\,dy} = \int_{-3}^{2} t^2 \boxed{\frac{dx}{dt}}\,dt + \int_{-3}^{2} (4 - t^2)\boxed{\frac{dy}{dt}}\,dt$$

$$= \int_{-3}^{2} t^2(-2t)\,dt + \int_{-3}^{2} (4 - t^2)\,dt$$

$$= \boxed{245/6.}$$

$x = t = -3$

one value of y

acceptable

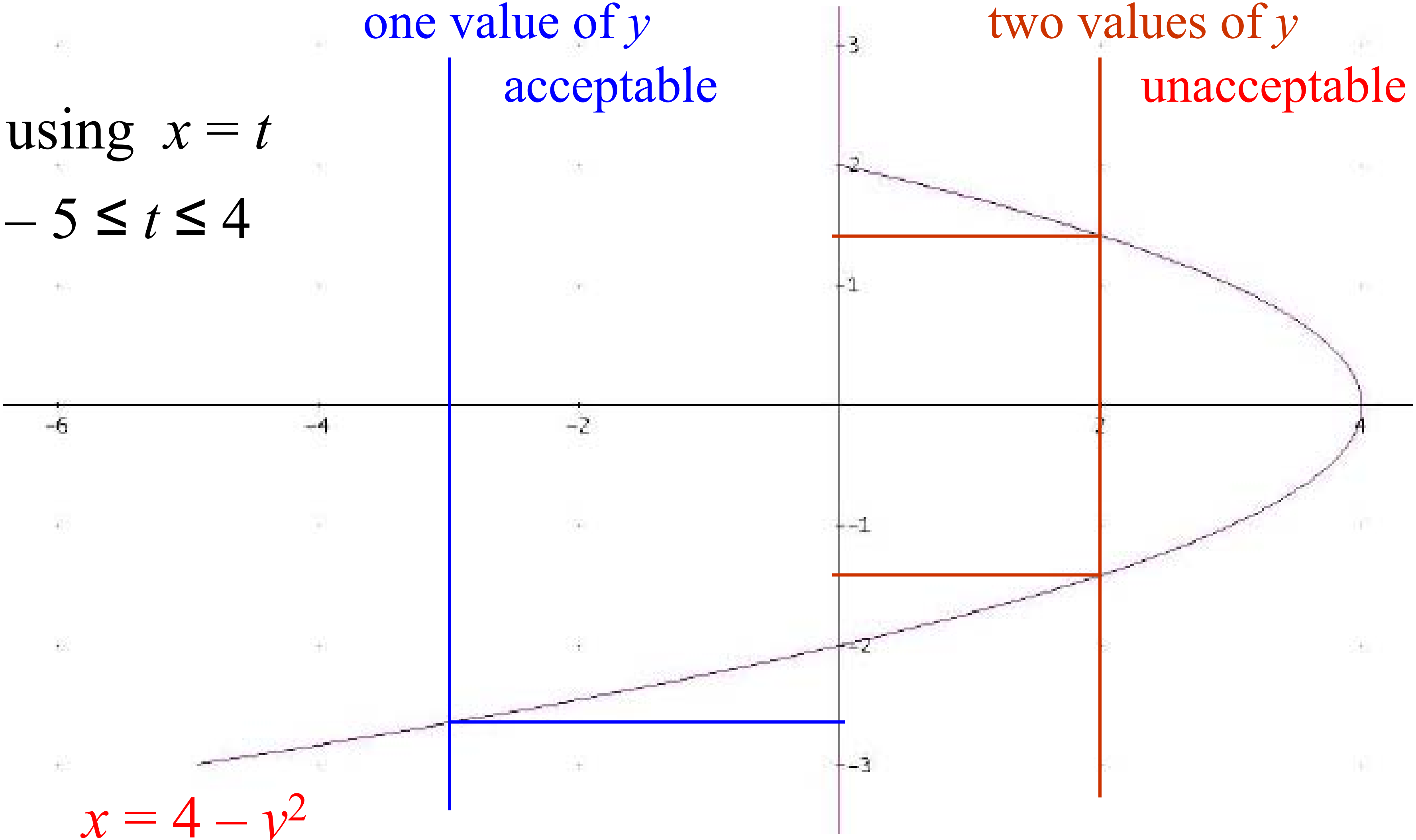
$x = t = 2$

two values of y

unacceptable

using $x = t$

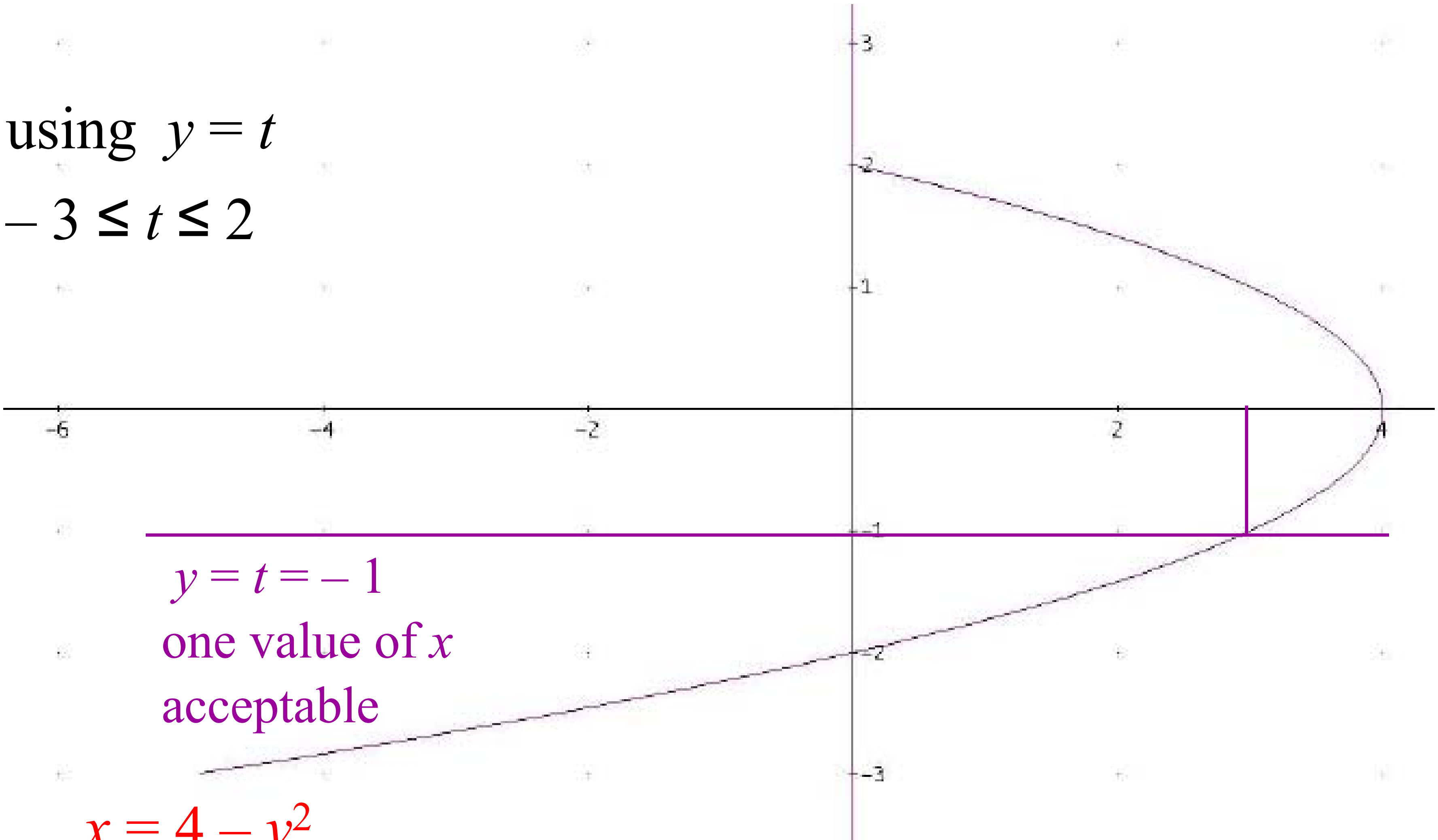
$-5 \le t \le 4$



$x = 4 - y^2$

using $y = t$

$-3 \le t \le 2$



$y = t = -1$

one value of x

acceptable

$x = 4 - y^2$

Each value of t gives one value of x

(and one value of y).

Example (c)

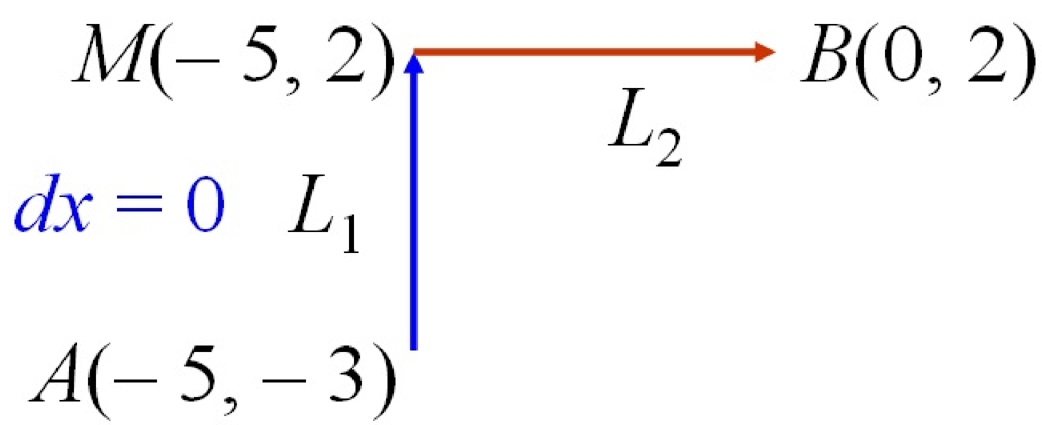
The component form is efficient if the curve is a vertical line segment or horizontal line segment.

Consider the point $M(-5, 2)$.

L_1 : vertical line joining $A(-5, -3)$ to $M(-5, 2)$.

L_2 : horizontal line joining $M(-5, 2)$ to $B(0, 2)$.

$C = L_1 + L_2$



$$\int_C \overset{0}{\boxed{y^2 dx}} + x\,dy$$

$$= \int_{L_1} x\,dy \; +$$

Example (c)

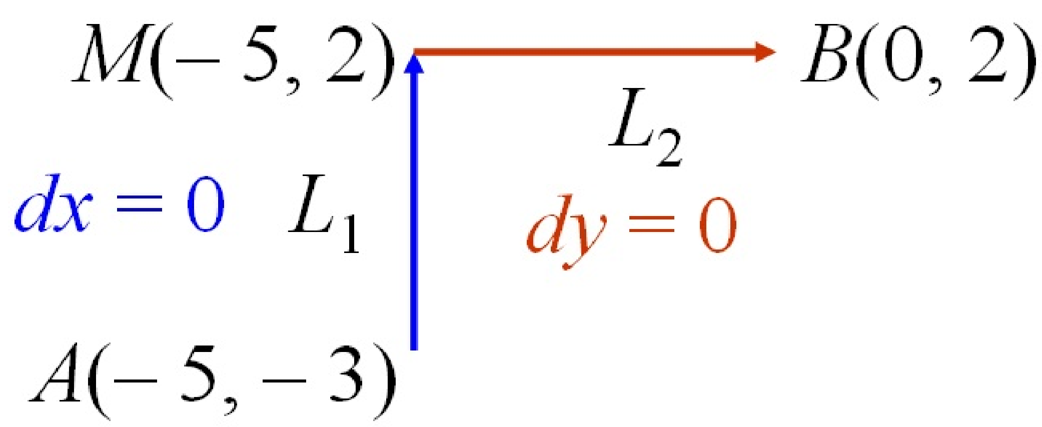
The component form is efficient if the curve is a vertical line segment or horizontal line segment.

Consider the point $M(-5, 2)$.

L_1 : vertical line joining $A(-5, -3)$ to $M(-5, 2)$.

L_2 : horizontal line joining $M(-5, 2)$ to $B(0, 2)$.

$C = L_1 + L_2$



$$\int_C y^2 dx + \boxed{x dy}^{\,0}$$

$$= \int_{L_1} x\, dy + \int_{L_2} y^2\, dx$$

Example (c)

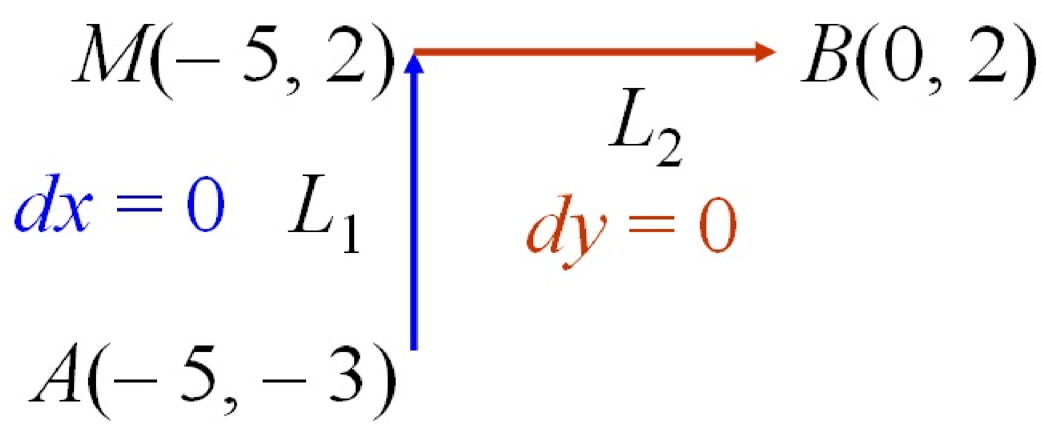
The component form is efficient if the curve is a vertical line segment or horizontal line segment.

Consider the point $M(-5, 2)$.

L_1 : vertical line joining $A(-5, -3)$ to $M(-5, 2)$.

L_2 : horizontal line joining $M(-5, 2)$ to $B(0, 2)$.

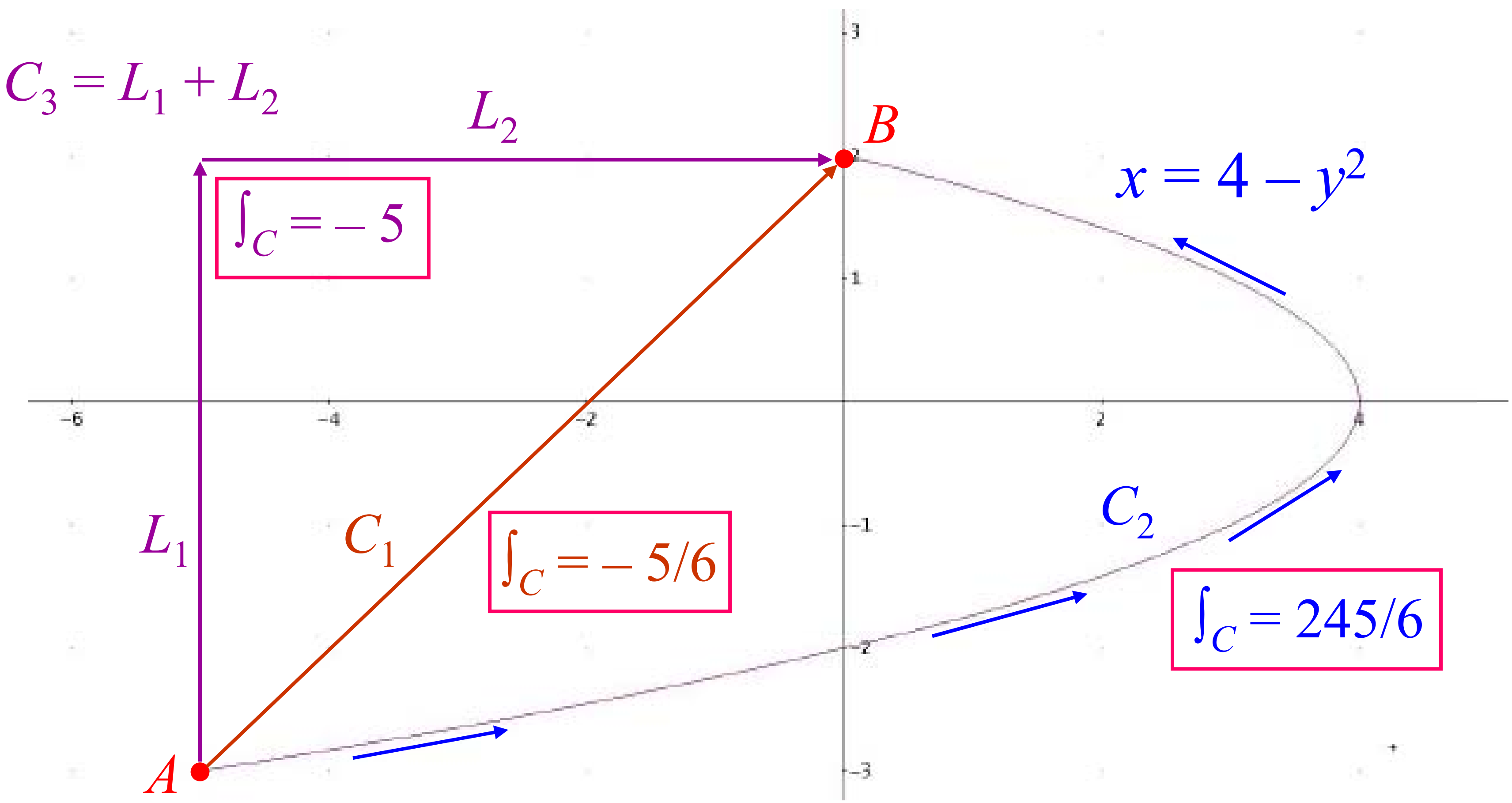
$C = L_1 + L_2$



$$\int_C y^2 dx + x dy$$

$$= \int_{L_1} x\, dy + \int_{L_2} y^2\, dx$$

$$= \int_{-3}^{2} (-5)\, dy + \int_{-5}^{0} 2^2\, dx = -25 + 20 = \boxed{-5}$$



$$\boxed{\int_C y^2 dx + x dy} = \boxed{\int_C \mathbf{F} \cdot d\mathbf{r}}$$

$$\mathbf{F}(x, y) = y^2\mathbf{i} + x\mathbf{j}$$

Is there a vector field **G** for which line integral values are the same for given *A* and *B* ?

Fundamental Theorem

Suppose f is a function of two or three variables such that $\boxed{\nabla f \text{ is continuous,}}$ and $\boxed{C \text{ is a smooth curve}}$ with vector function $\mathbf{r}(t)$, where $a \leq t \leq b$.

Then

$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))}$$

Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Example

In Example 9.2.9 on gravitational field **G**, a potential function $g(x, y, z)$ was given, i.e.

$$\mathbf{G} = \nabla g \quad \text{where} \quad \boxed{g(x, y, z) = \frac{mMK}{\sqrt{x^2 + y^2 + z^2}}}$$

Moving an object of mass m from point $\boxed{A(3, 4, 12)}$ to point $\boxed{B(1, 0, 0),}$ work done by gravity is

$$\int_C \mathbf{G} \cdot d\mathbf{r} = \int_C \nabla g \cdot d\mathbf{r} = \boxed{g(1, 0, 0)} - \boxed{g(3, 4, 12)}$$

$$= \frac{mMK}{\sqrt{1^2 + 0^2 + 0^2}} - \frac{mMK}{\sqrt{3^2 + 4^2 + 12^2}} = \boxed{\frac{12}{13} mMK}$$

Line Integrals in Conservative Fields

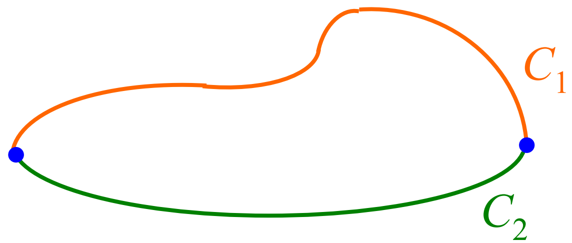
$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

In this formula, only the end-points of C, given by $\mathbf{r}(a)$ and $\mathbf{r}(b)$, are used in calculation.

The interior points on C do not come in the calculations.

The line integral $\boxed{\int_C \nabla f \cdot d\mathbf{r}}$ is independent of path:

If C_1 and C_2 have the same initial points and the same terminal points, then

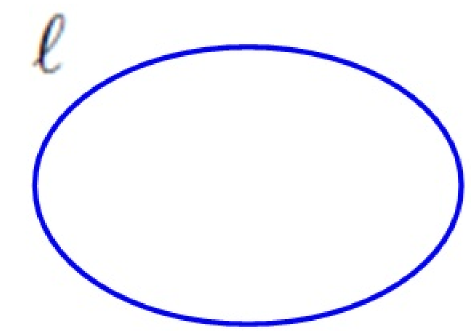


$$\boxed{\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}}$$

For a conservative field $\mathbf{F} = \nabla f$

(1) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path

(2) $\displaystyle\oint_\ell \mathbf{F} \cdot d\mathbf{r} = 0$ if $C = \ell$ is a loop.



A loop is a closed curve, i.e. a curve where the terminal point coincides with the initial point.

Example

From Example 9.2.5, $\mathbf{F}(x, y) = (y^2 + 3x^2)\mathbf{i} + (2xy)\mathbf{j}$

is conservative because **F** has a potential function

$\boxed{f(x, y) = xy^2 + x^3}$

$\boxed{\mathbf{F} = \nabla f}$

(i) $\boxed{\mathbf{r}(t) = \cos t\mathbf{i} + e^t \sin t\mathbf{j},}$ $0 \leq t \leq \pi$

Initial point:	$\mathbf{r}(0) = \mathbf{i}$	$A(x, y) = A(1, 0)$
Terminal point:	$\mathbf{r}(\pi) = -\mathbf{i}$	$B(x, y) = B(-1, 0)$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \boxed{f(-1, 0) - f(1, 0)} = -1 \ -1 \ = \boxed{-2}$$

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Example (cont'd)

From Example 9.2.5, $\mathbf{F}(x,y) = (y^2+3x^2)\mathbf{i} + (2xy)\mathbf{j}$

is conservative because **F** has a potential function

$f(x,y) = xy^2 + x^3$

$$\boxed{\mathbf{F} = \nabla f}$$

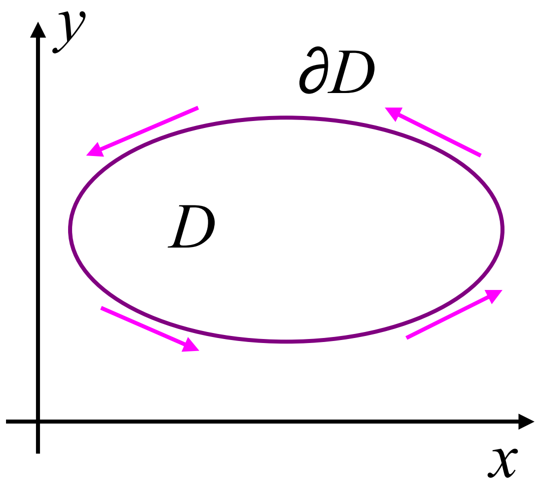
(ii) *C* is the unit circle:

Clearly, *C* is a loop. **F** is conservative now gives

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \boxed{0}.$$

Green's Theorem

D is a bounded region in the xy-plane with boundary ∂D.



Positive orientation:
∂D is traversed with D on the left side.

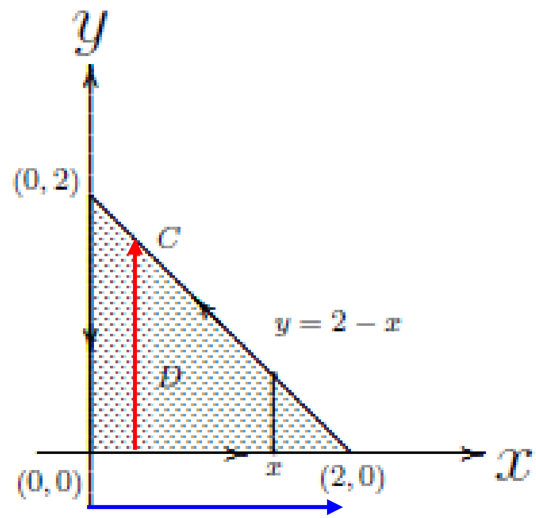
$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$

$P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on D.

$$\boxed{\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} \;=\; \oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.}$$

Example

D as Type A: $\quad 0 \le y \le 2 - x, \;\; 0 \le x \le 2.$



By Green's Theorem,

$$\oint_C \underset{P}{2xy}\, dx + \underset{Q}{xy^2} dy$$

$$= \iint_D \boxed{\left[\frac{\partial(xy^2)}{\partial x} - \frac{\partial 2xy}{\partial y}\right]} \, dA$$

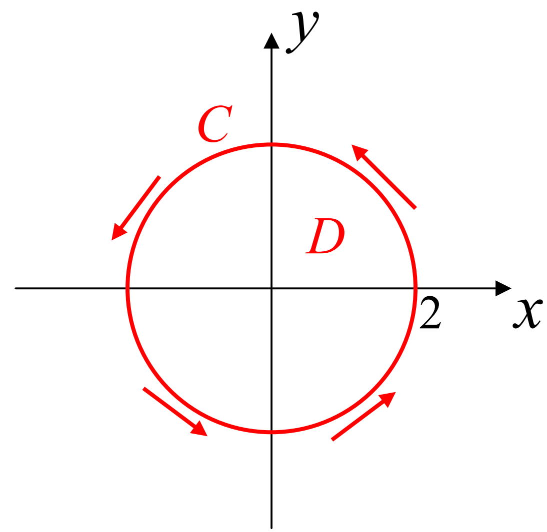
$$= \iint_D \boxed{(y^2 - 2x)}\, dy dx \;=\; \int_0^2 \int_0^{2-x} (y^2 - 2x)\, dy\, dx$$

$$= \boxed{-\frac{4}{3}.}$$

$$\oint_{\partial D} P dx + Q dy = \iint_D \boxed{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)} dA.$$

Example

C is the circle $x^2 + y^2 = 4$ with positive orientation: traversed anti-clockwise.



By Green's Theorem,

$$\oint_C \underbrace{(4y - e^{x^2})}_{P}dx + \underbrace{(9x + \sin y^2 - 1)}_{Q}dy$$

$$= \iint_D \boxed{\left[\frac{\partial(9x + \sin y^2 - 1)}{\partial x} - \frac{\partial(4y - e^{x^2})}{\partial y}\right]} dA$$

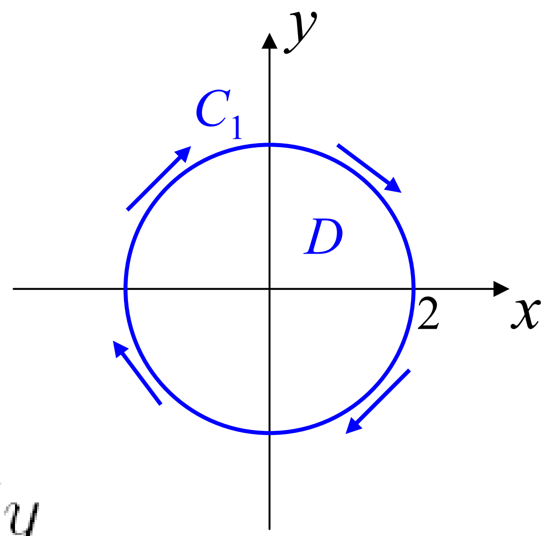
$$= \iint_D 5\, dA = 5 \iint_D dA = 5 \times (\text{area of } D) = 5(\pi 2^2)$$

$$= \boxed{20\pi}$$

Chap 8, page 3 (4) $\iint_R dA \left(= \iint_R 1\, dA \right) = A(R)$, the area of R.

C is the circle $x^2 + y^2 = 4$.

Suppose that C_1 is C with the given negative orientation: traverse C clockwise to get C_1.



$$\oint_{\boxed{C_1}} \left(4y - e^{x^2}\right) dx + \left(9x + \sin y^2 - 1\right) dy$$

$$= - \oint_C (4y - e^{x^2})dx + (9x + \sin y^2 - 1)dy$$

$$= \boxed{-20\pi}$$

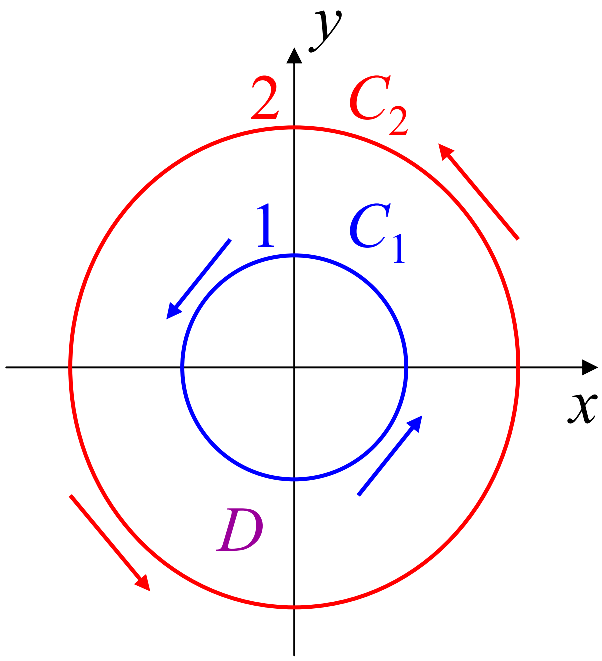
$C_1 = -C$ $\quad \int_{-C} \mathbf{F} \cdot d\mathbf{r} = - \int_C \mathbf{F} \cdot d\mathbf{r}$

Example

$$C_1: \quad \mathbf{r}_1(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$$

$$C_2: \quad \mathbf{r}_2(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$$

$$\boxed{0 \leq t \leq 2\pi}$$



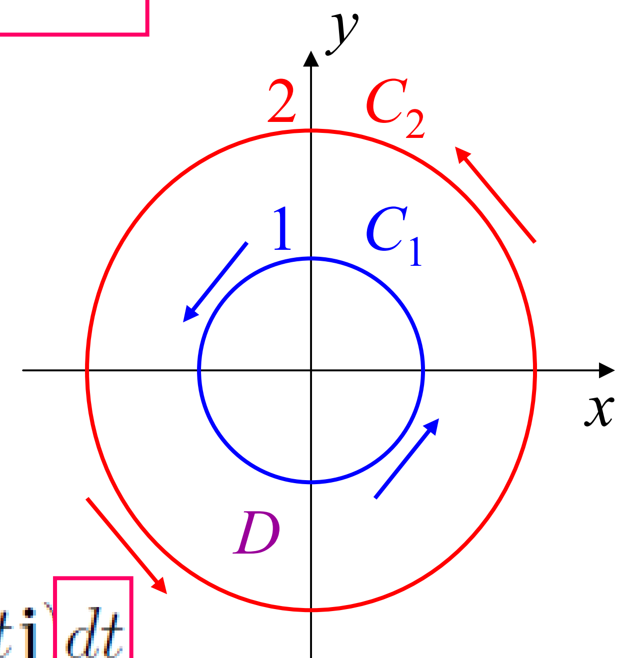
These equations give anti-clockwise orientation to C_1 and C_2.

To get positive orientation for ∂D,

C_2 should be traversed anti-clockwise, while C_1 should be traversed clockwise.

$$\boxed{\partial D = C_2 - C_1}$$

$C_1:$ $\boxed{\begin{aligned} \mathbf{r}_1(t) &= \underbrace{\cos t}_{x}\,\mathbf{i} + \underbrace{\sin t}_{y}\,\mathbf{j} \\ \mathbf{r}_1'(t) &= -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} \end{aligned}}$ $\boxed{0 \le t \le 2\pi}$



$\boxed{\mathbf{F}(x, y) = y\mathbf{i} + y\mathbf{j}}$

$$\int_{C_1} \mathbf{F} \cdot \boxed{d\mathbf{r}}$$

$$= \boxed{\int_0^{2\pi}} (\sin t\,\mathbf{i} + \sin t\,\mathbf{j}) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\,\boxed{dt}$$

$\boxed{\partial D = C_2 - C_1}$

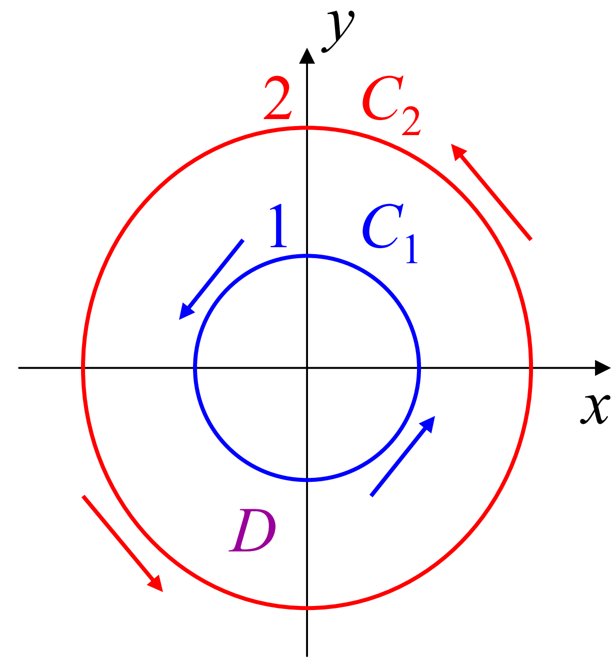
$$= \int_0^{2\pi} (-\sin^2 t + \sin t \cos t)\,dt$$

$$= \int_0^{2\pi} \frac{1}{2}(\cos 2t - 1 + \sin 2t)\,dt = \frac{1}{2}\left[\frac{\sin 2t}{2} - t - \frac{\cos 2t}{2}\right]_0^{2\pi}$$

$$= \boxed{-\pi}$$

$$C_2 : \boxed{\begin{aligned} \mathbf{r}_2(t) &= 2\underset{x}{\cos t}\,\mathbf{i} + 2\underset{y}{\sin t}\,\mathbf{j} \\ \mathbf{r}_2'(t) &= -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} \end{aligned}} \quad \boxed{0 \le t \le 2\pi}$$

$$\boxed{\mathbf{F}(x, y) = y\mathbf{i} + y\mathbf{j}}$$



$$\boxed{\partial D = C_2 - C_1}$$

$$\int_{C_2} \mathbf{F} \cdot \boxed{d\mathbf{r}}$$

$$= \boxed{\int_0^{2\pi}} (2\sin t\,\mathbf{i} + 2\sin t\,\mathbf{j}) \cdot (-2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}) \boxed{dt}$$

$$= 4 \boxed{\int_0^{2\pi} (\sin t\,\mathbf{i} + \sin t\,\mathbf{j}) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\, dt}$$

$$= 4\,(-\pi) = \boxed{-4\pi}$$

refer to $\int_{C_1}$

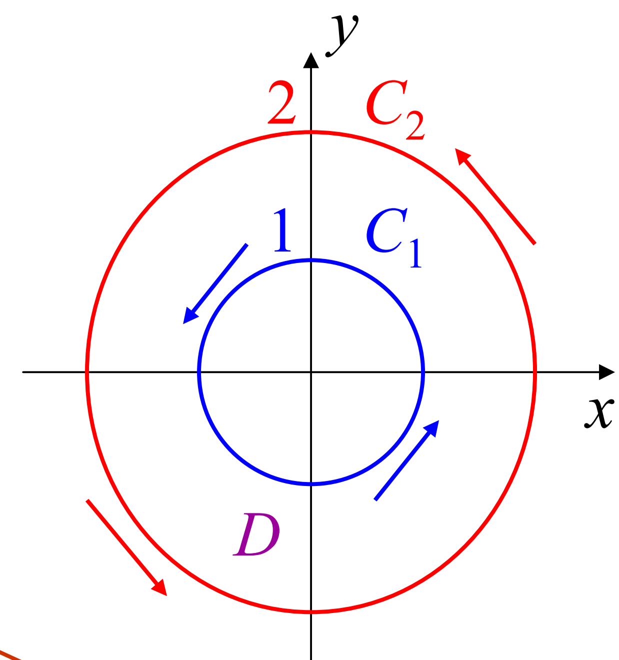
$$\mathbf{F}(x, y) = y\mathbf{i} + y\mathbf{j}$$

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

$$= \boxed{\int_{C_2} \mathbf{F} \cdot d\mathbf{r}} - \boxed{\int_{C_1} \mathbf{F} \cdot d\mathbf{r}}$$

$$= -4\pi - (-\pi)$$

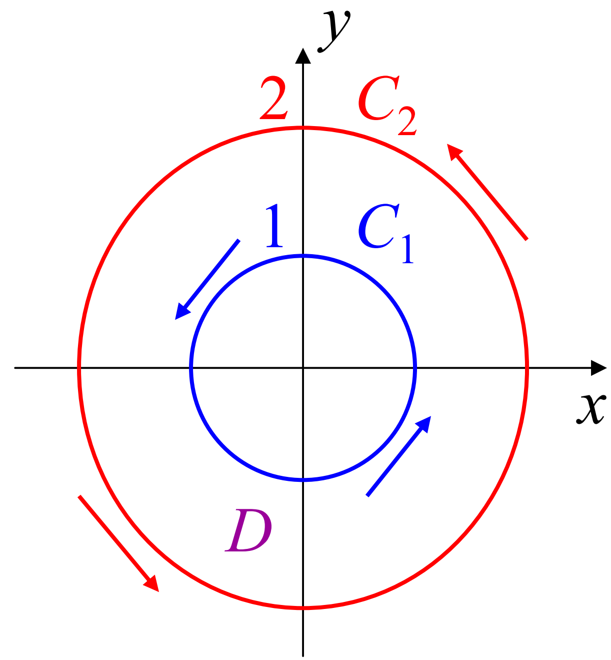
$$= \boxed{-3\pi}$$



$$\boxed{\partial D = C_2 - C_1}$$

$$\mathbf{F}(x,y) = \boxed{\underbrace{y}_{P}\mathbf{i} + \underbrace{y}_{Q}\mathbf{j}}$$

Using Green's Theorem, first note that D is a ring.



$$\boxed{\partial D = C_2 - C_1}$$

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \oint_{\partial D} P\,dx + Q\,dy$$

$$= \iint_D \boxed{\left(\frac{\partial y}{\partial x} - \frac{\partial y}{\partial y}\right)}\, dA$$

$$= \iint_D (-1)\,dA = -\boxed{\iint_D 1\,dA}$$

Area of D

$$= -\boxed{\left(\pi \cdot 2^2 - \pi \cdot 1^2\right)} = \boxed{-3\pi}$$

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \boxed{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}\, dA.$$

End of Chapter 9