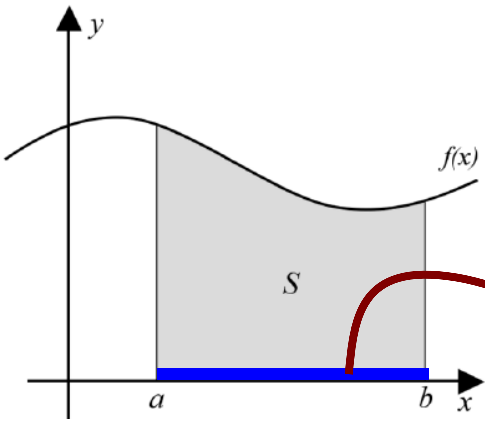
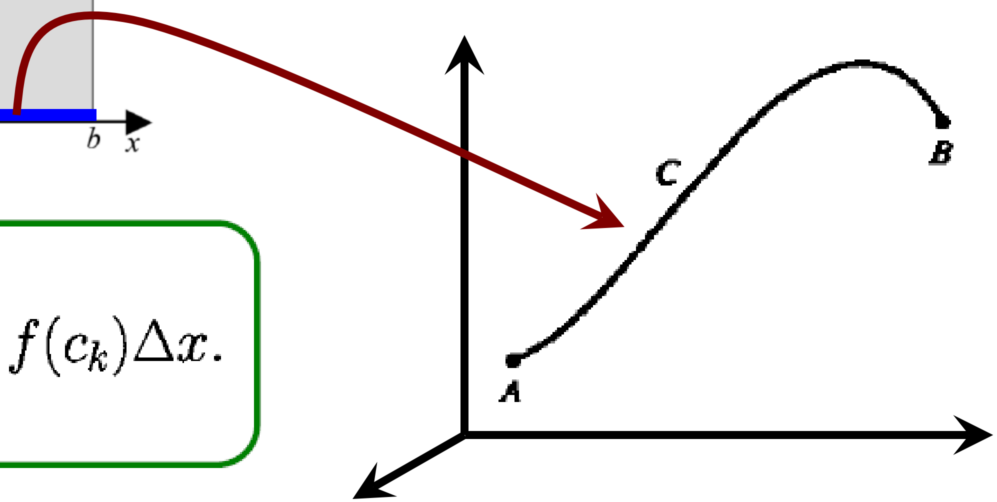
CH 9 Line Integrals

Find the ***area*** of the ***shaded*** region:



$$\int_C$$ ***C*** is a **curve** in **space**



$$\boxed{\int_a^b} f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x.$$

Line Integrals

Type I

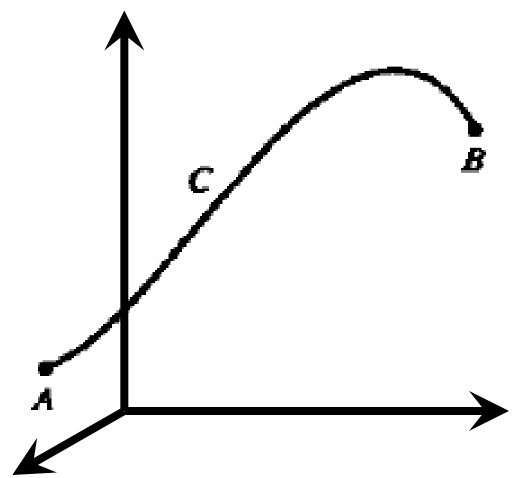
$$\int_C \boxed{f(x,y)}\, ds$$

scalar function

Type II

$$\int_C \boxed{\mathbf{F}} \cdot d\mathbf{r}$$

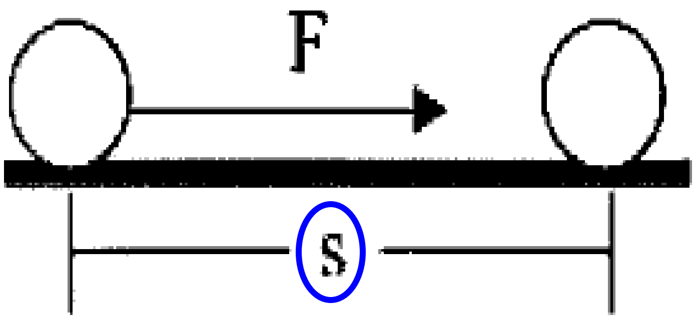
vector field F



We shall give the motivation of type II linear integral first in Sections 9.1,9.2

9.1 Introduction

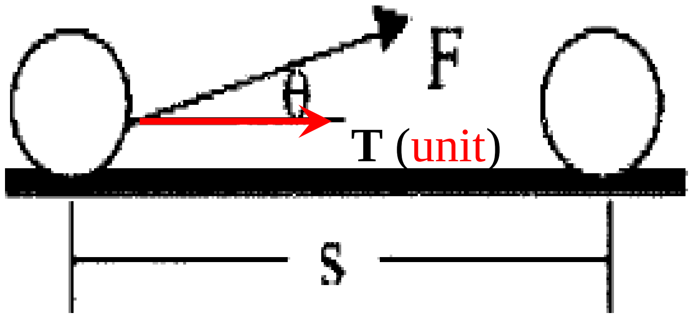
♣ **Work Done I**



(i) Let $\mathbf{F}$ be a **constant** force acting on a particle in the displacement direction as shown. If the distance moved by the particle is s, then the **work done** is:

$$W = \|\mathbf{F}\| \times s.$$

(ii) Let $\mathbf{F}$ be a constant force acting on a particle in the direction which forms an angle θ against the displacement direction



$$\boxed{\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\| \, \|\mathbf{v}_2\|}.}$$

If the distance moved by the particle is **s**, then the **work done** is :

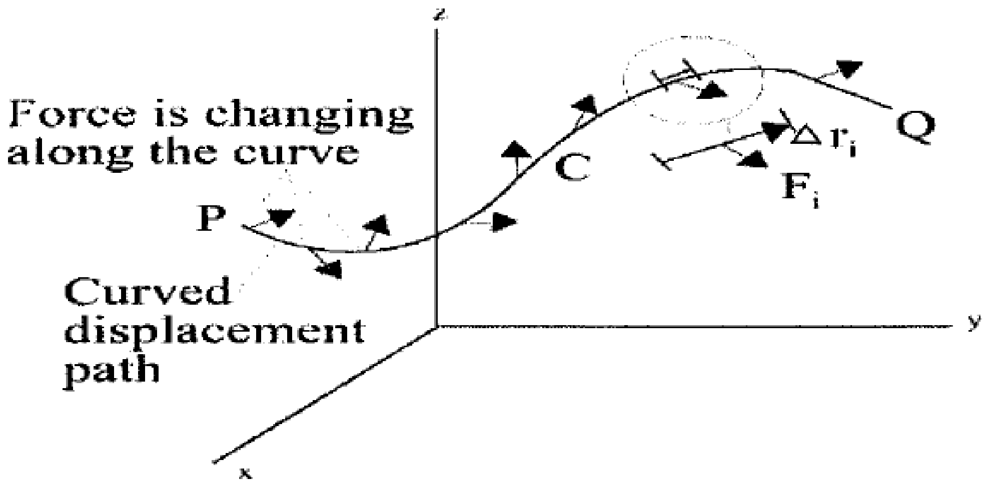
$$W = \|\mathbf{F}\| \cos\theta \times s = (\mathbf{F} \cdot \mathbf{T}) \times s = \mathbf{F} \cdot s\mathbf{T}$$

where $\mathbf{T}$ is the unit vector in the displacement direction.

$$\boxed{W = \mathbf{F} \cdot s\mathbf{T}}$$

♣ Work Done II

Let $\mathbf{F}(x, y, z)$ be a variable force acting on a particle which moves along the curve C with vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

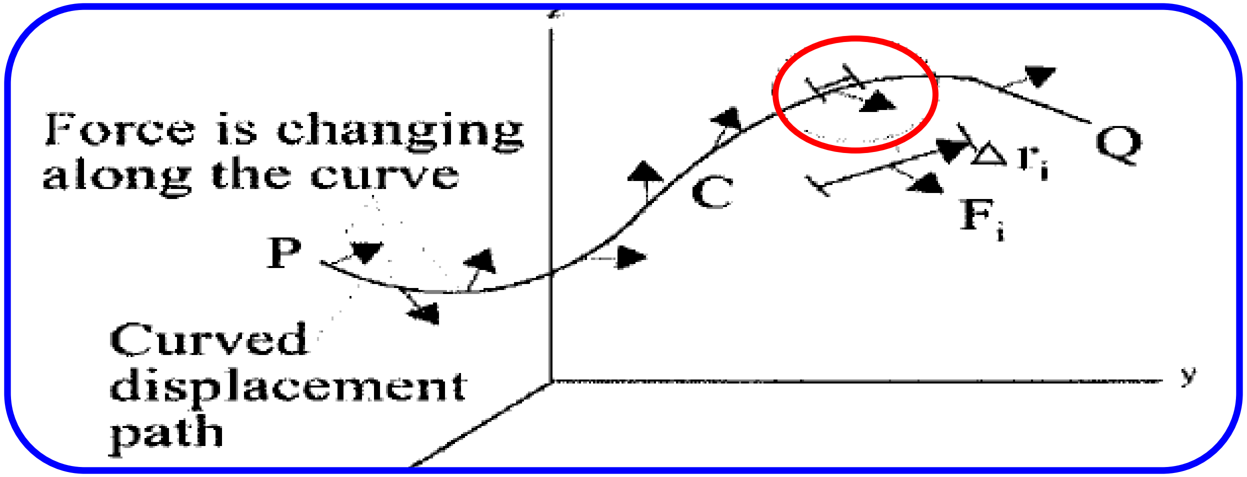


If the particle moves from P to Q, what is the **work done** ?

We divide *C* into *n* segments. As each one is small, each can be treated as a **line segment** & the force within which assumed to be **constant** $\mathbf{F}_i$. Then the **work done** for such a segment is approximately

$$W_i \approx \mathbf{F}_i \cdot \Delta \mathbf{r}_i$$

where $\mathbf{r}_i = s\mathbf{T}_i$ and $\mathbf{T}_i$ is the unit tangent vector along this segment.



- Thus the **total work done** is approximately

$$W_{\text{total}} \approx \sum_{1}^{n} \mathbf{F}_i \cdot \Delta \mathbf{r}_i.$$

As $n \to \infty$, we write this as

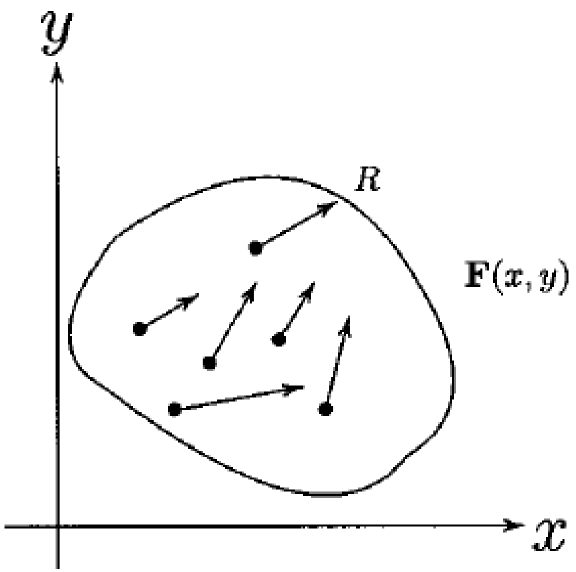
$$(\clubsuit) \qquad \int_C \mathbf{F} \cdot d\mathbf{r}$$

which gives the actual total work done.

The integral **(♣)** is called the ***line integral*** of **F** along the curve ***C*****.**

9.2 Vector Fields

● **Vector fields** (**2 variables**)



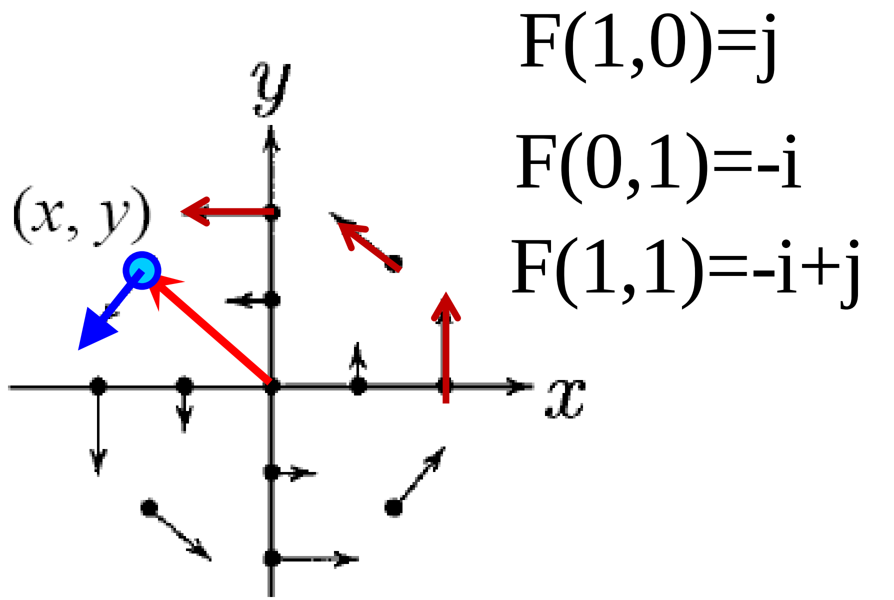
Let ***R*** be a region in *xy*-plane.
A ***vector field*** on ***R*** is a vector fn **F** that assigns to each point (x, y) in ***R*** a 2-*D* vector $\mathbf{F}(x,y)$.
We may write $\mathbf{F}(x,y)$, which is a vector, in component form:

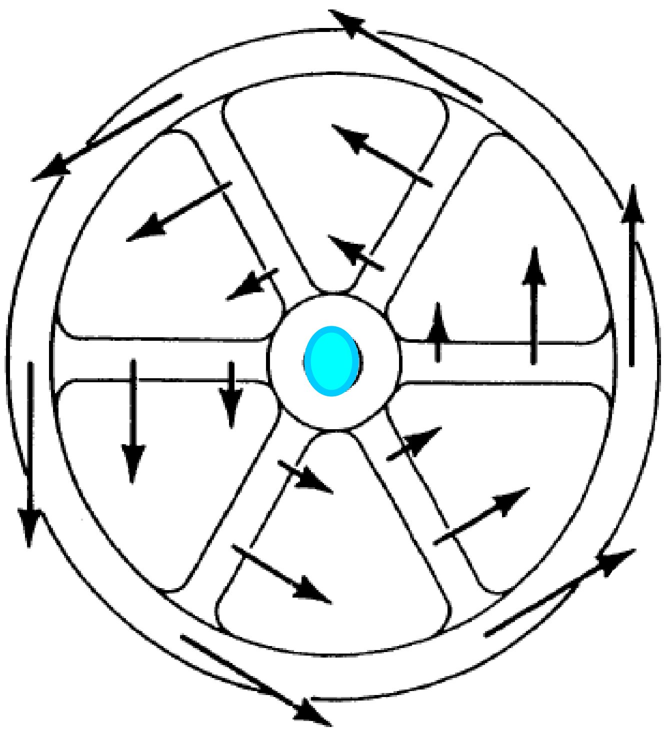
$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

or simply $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$.

♣ $\mathbf{F}(x,y) = (-y)\mathbf{i} + x\mathbf{j}$

$$(-y, x)\bullet(x, y) = 0$$





This Velocity field determined by a **wheel rotating about an axle**

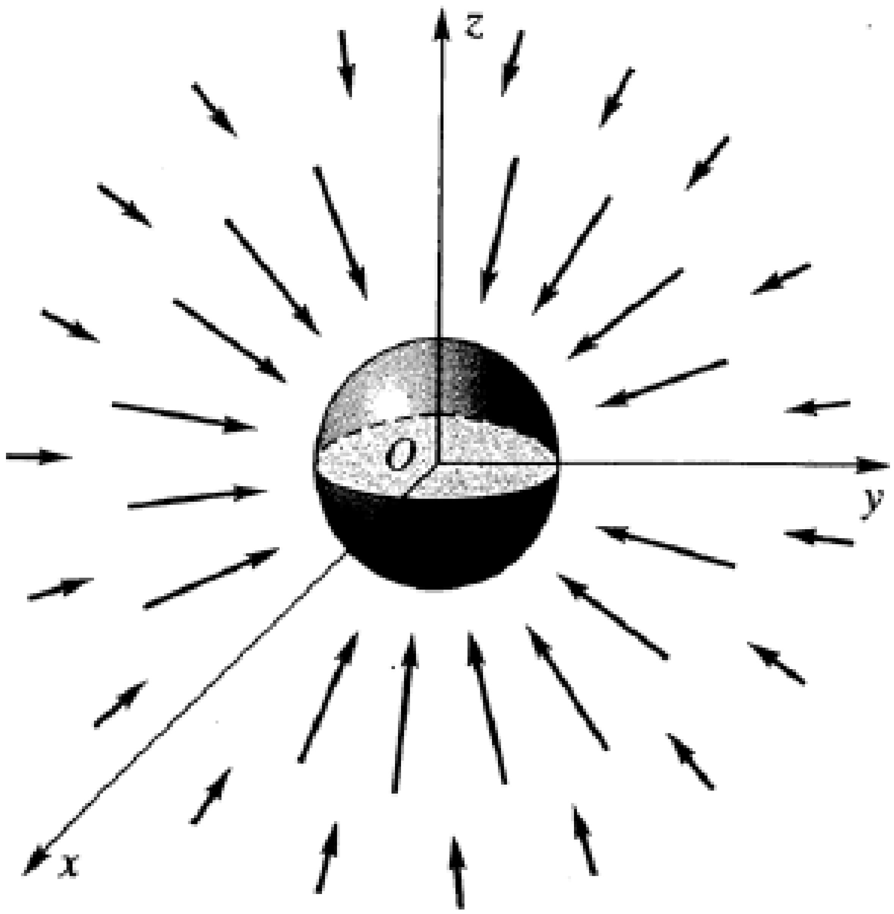
- **Vector field** (**3 variables**)

Let D be a solid region in xyz-space. A $\boxed{\textbf{vector field}}$ on D is a vector function $\mathbf{F}$ that assigns to each point (x, y, z) in D a three-dimensional vector $\mathbf{F}(x, y, z)$.

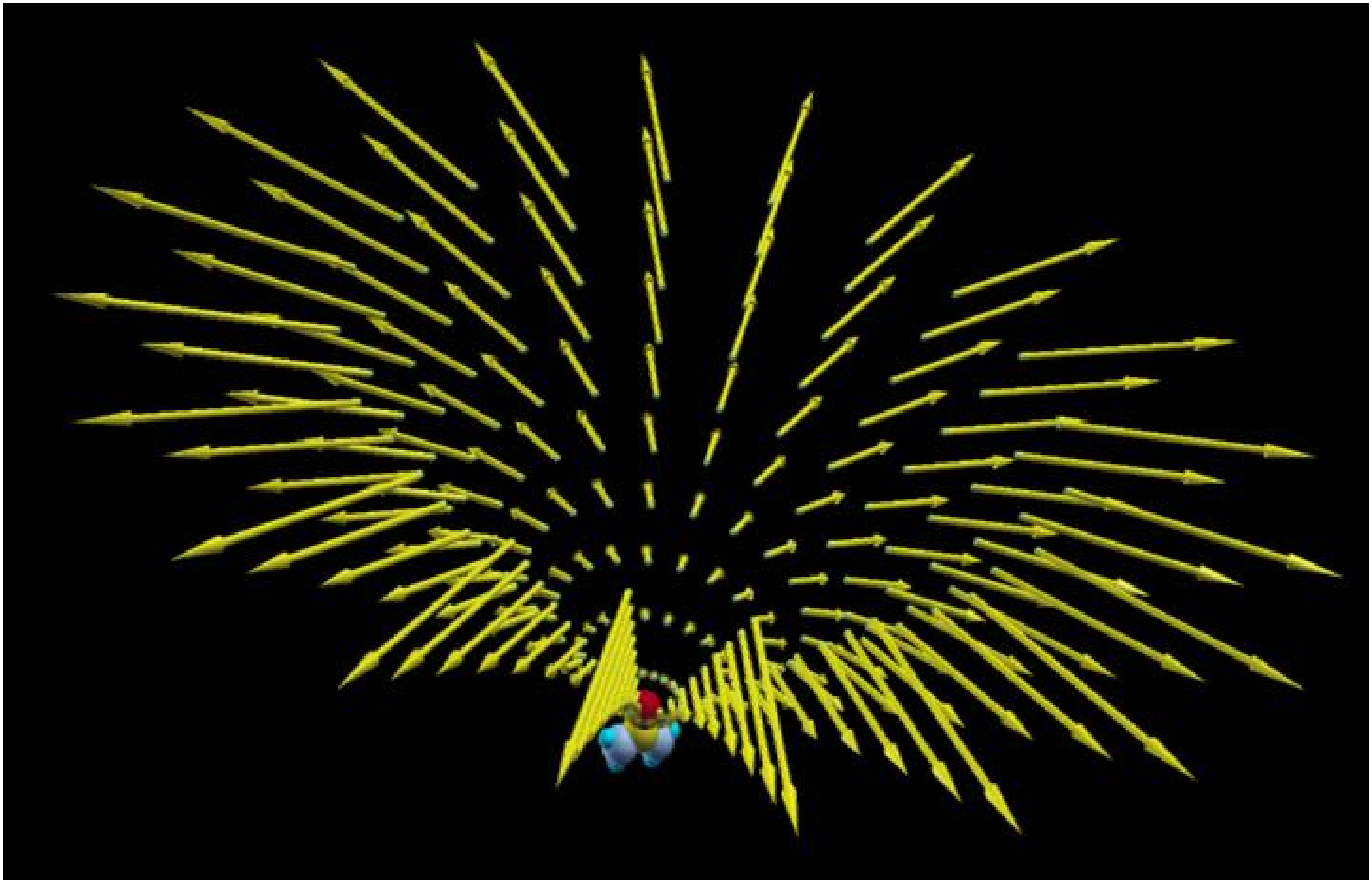
$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$

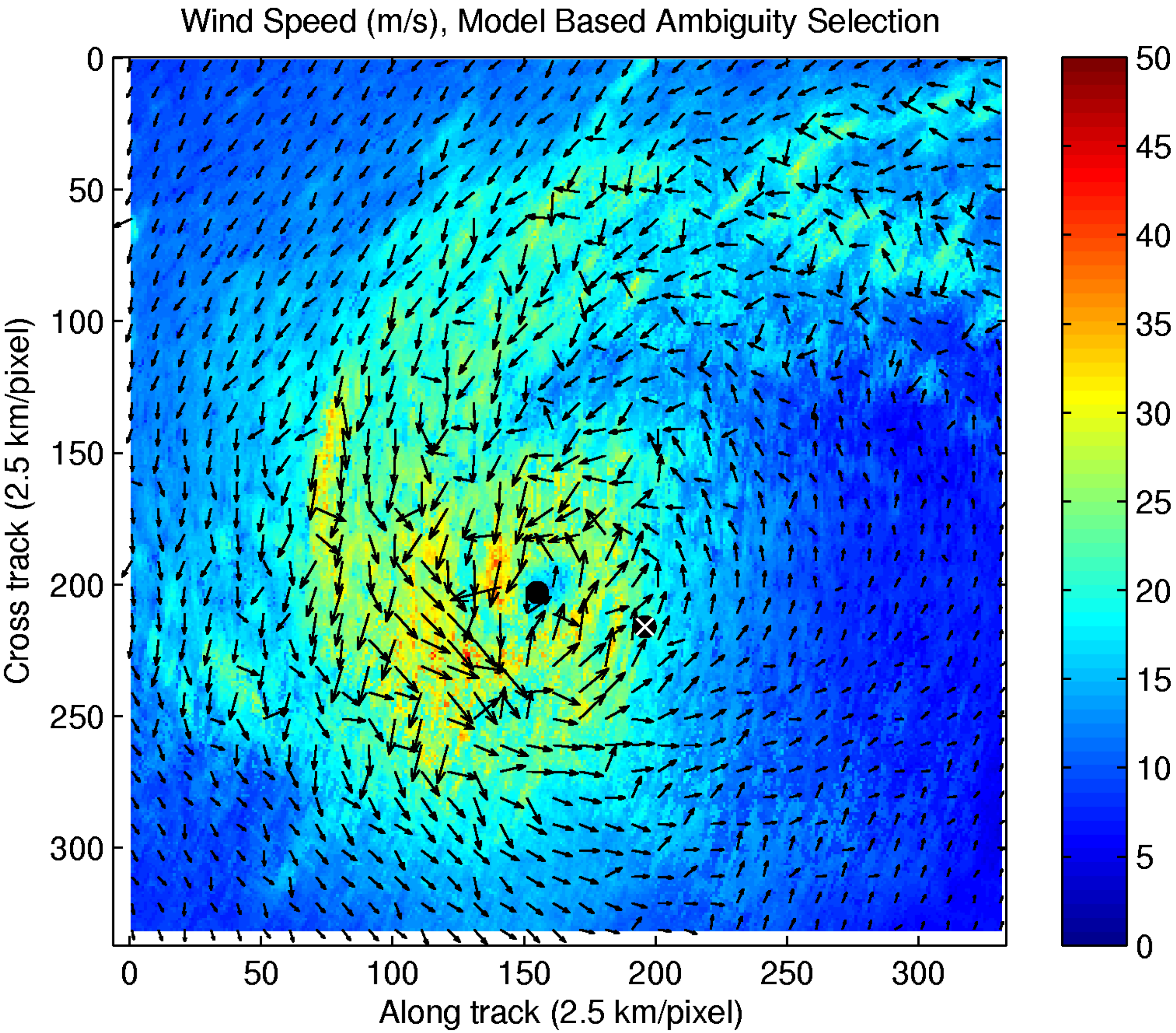
- **VF**s are used to model the strength & direction of certain ***force*** (magnetic or gravitational) or the speed & direction of a moving ***fluid*** in space.

♣ **Gravitational field** of **force**



♣




Wind Speed (m/s), Model Based Ambiguity Selection
Cross track (2.5 km/pixel)
Along track (2.5 km/pixel)
0
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100
150
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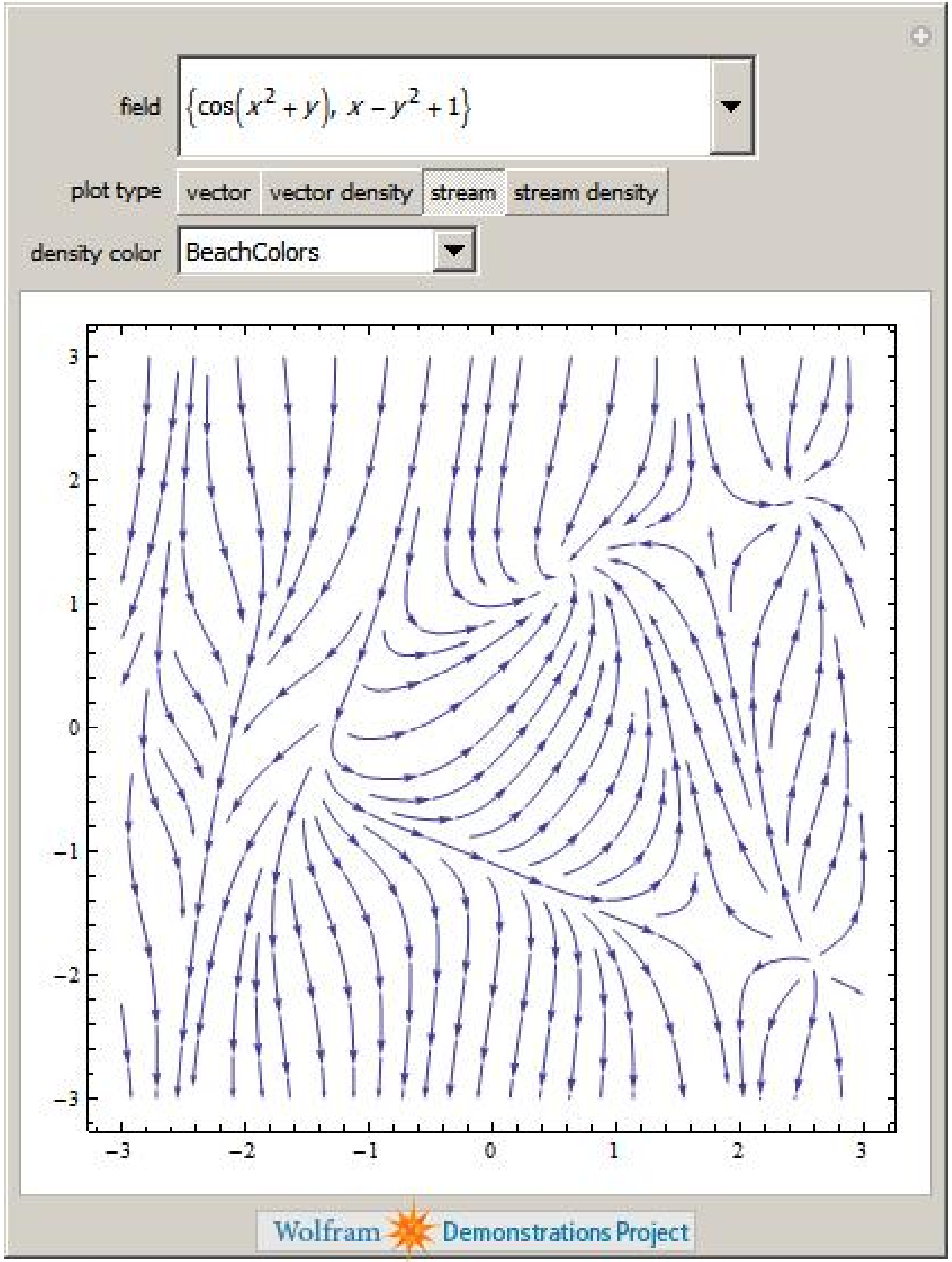
50

♣


field
$\{\cos(x^2+y),\ x-y^2+1\}$
plot type
vector
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stream
stream density
density color
BeachColors
3
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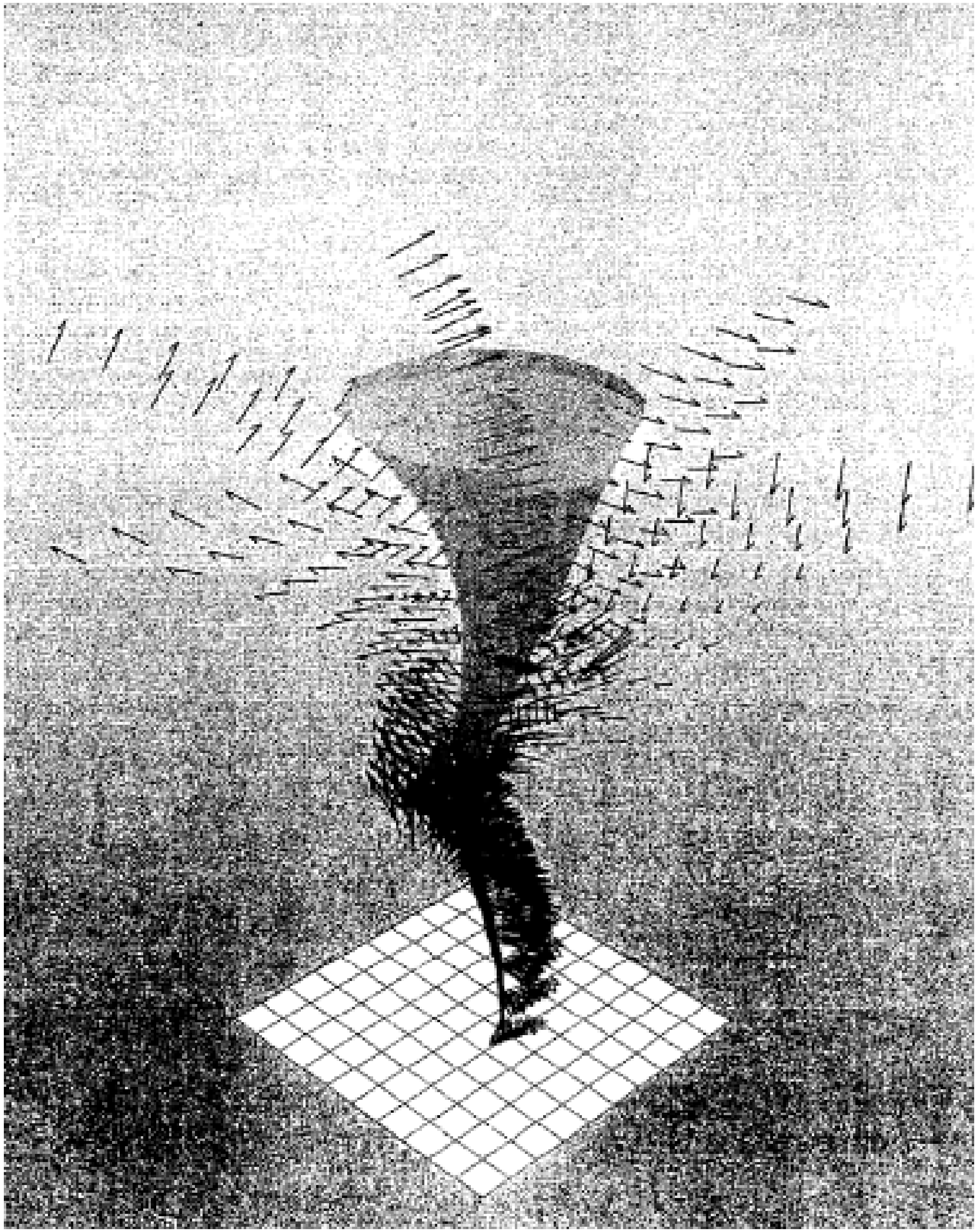
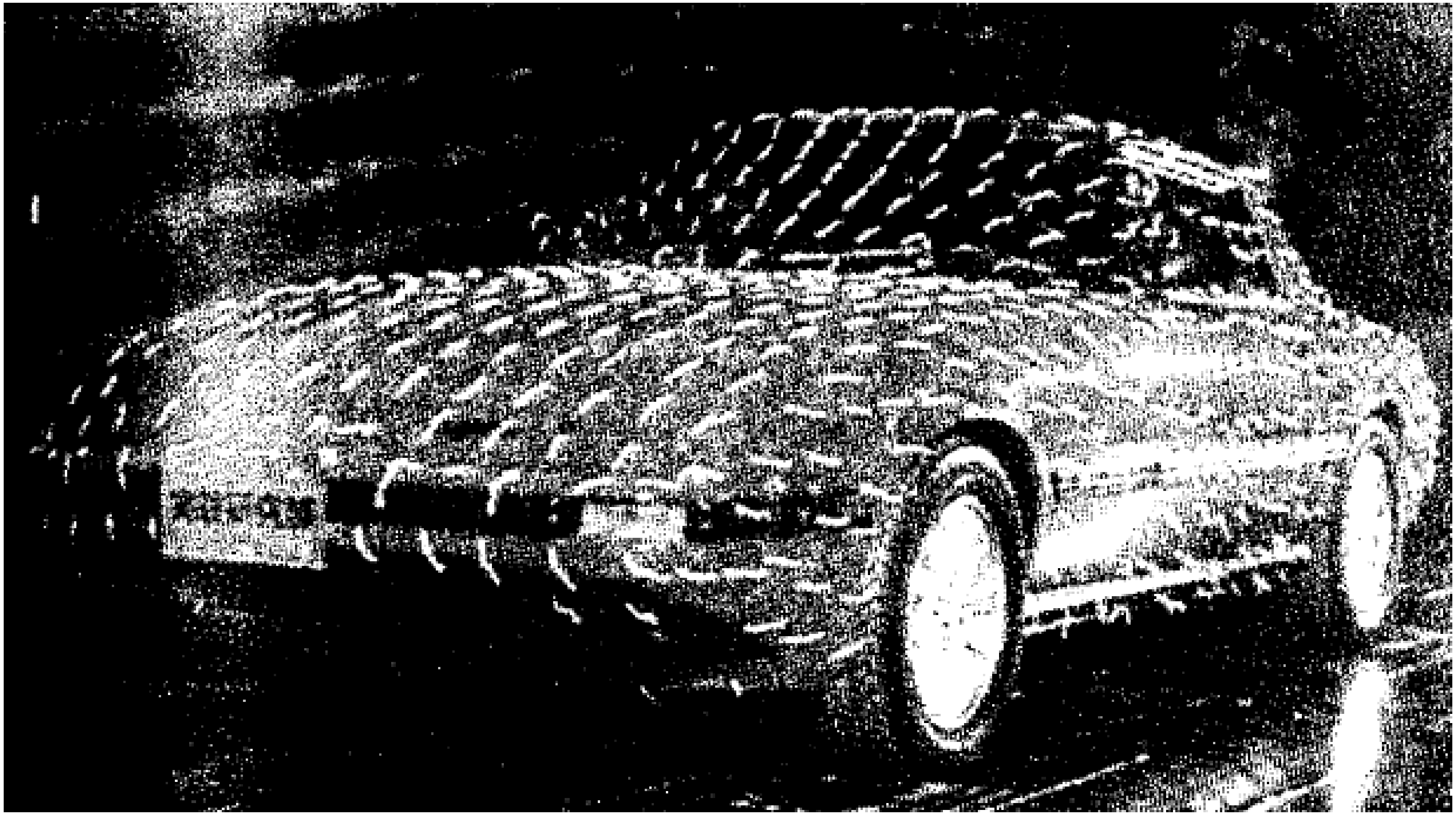
Wolfram Demonstrations Project

♣ Tornado



♣ **Velocity vector field** for **airflow** around a car



• Gradient fields (see CH 7 7.5.4)

The ***gradient*** (***field***) of ***f***(*x*, *y*) is

$$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

a **vector field** in *xy*-plane

The ***gradient*** (***field***) of ***f***(*x*, *y*, *z*) is

$$\nabla f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}$$

a **vector field** in space

♣ The gradient field of $f(x,y) = xy^2 + x^3$ is

$$\nabla f(x,y) = (y^2 + 3x^2)\mathbf{i} + (2xy)\mathbf{j}.$$

♣ A **relation** between $D_{\underset{\sim}{u}} f(a,b)$ & $\nabla f(a,b)$ where $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a ***unit*** vector

$$D_{\underset{\sim}{u}} f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2$$

$$= \nabla f(a,b) \cdot (u_1 \underset{\sim}{i} + u_2 \underset{\sim}{j})$$

$$= \nabla f(a,b) \cdot \underset{\sim}{u}$$

$$\boxed{D_{\underset{\sim}{u}} f(a,b) = \nabla f(a,b) \cdot \underset{\sim}{u}}$$

• Conservative fields

$$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

A vector field **F** is ***conservative*** if it is the **gradient** of some (scalar) function. That is, there is a fn ***f*** s.t.

Conservative $\rightarrow$ $\boxed{\mathbf{F}} = \nabla f$ $\leftarrow$ ***Potential function*** for **F**

♣ The **vector field** $\mathbf{F}(x,y) = (y^2+3x^2)\mathbf{i} + (2xy)\mathbf{j}$ is ***conservative*** as it has a ***potential fn*** ***f***$(x, y) = xy^2 + x^3$.

$$f(x,y) = xy^2 + x^3$$
$$\nabla f(x,y) = (y^2+3x^2)\mathbf{i} + (2xy)\mathbf{j}$$

$$\boxed{\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}}$$

♣ Let $\mathbf{F}(x,y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$. Find a potential function f for $\mathbf{F}$.

$\nabla f = \mathbf{F}$ $\Rightarrow f_x(x,y) = 3 + 2xy$

$\Rightarrow f(x,y) = \int f_x(x,y)dx = 3x + x^2y + g(y)$

$\Rightarrow f_y(x,y) = x^2 + g'(y)$

$\Rightarrow x^2 - 3y^2 = x^2 + g'(y)$

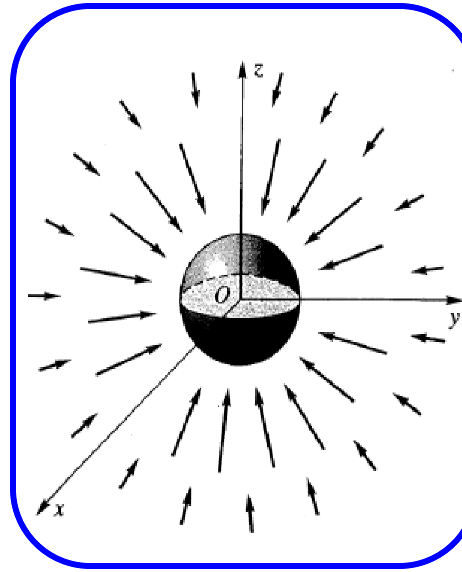
$\Rightarrow g'(y) = -3y^2$

$\Rightarrow g(y) = -y^3 + K$

Thus, $f(x,y) = 3x + x^2y - y^3 + K$

The $\boxed{\text{gravitational field}}$ given by

$$\mathbf{G} = \left(\frac{-m_1 m_2 K x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) \mathbf{i} + \left(\frac{-m_1 m_2 K y}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) \mathbf{j} + \left(\frac{-m_1 m_2 K z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right) \mathbf{k}$$



is conservative because it is the gradient of the gravitational potential function

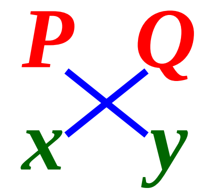
$$g(x, y, z) = \frac{m_1 m_2 K}{\sqrt{x^2+y^2+z^2}},$$

where K is the gravitational constant, m_1 and m_2 are the masses of two objects.

• ***Criteria*** of **conservative fields**

(a) Let $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be a vector field on the xy-plane. Then

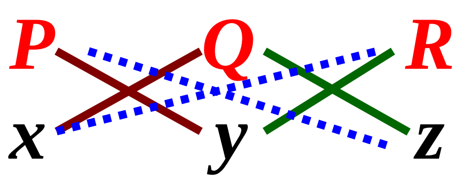
$$\mathbf{F} \text{ is conservative} \quad \Leftrightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$



(b) Let $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ be a vector field on the xyz-space. Then

$$\mathbf{F} \text{ is conservative} \quad \Leftrightarrow$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$



♣ The vector field

$$\mathbf{F}(x,y) = (3+2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$$

is ***conservative*** as

$$\frac{\partial(x^2-3y^2)}{\partial x} = 2x = \frac{\partial(3+2xy)}{\partial y}$$

♣ $\mathbf{F}(x,y,z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$

is ***not conservative***

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}}$$

as

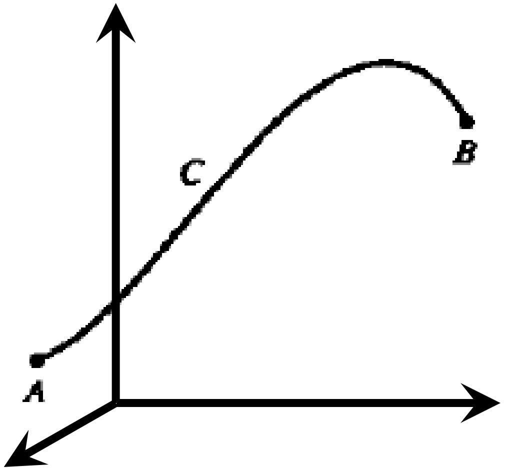
$$\frac{\partial P}{\partial y} = 0 \neq yz = \frac{\partial Q}{\partial x}$$

9.3 *Line Integrals*

Type I

$$\int_C \boxed{f(x,y)}\, ds$$

scalar function



Type II

$$\int_C \boxed{\mathbf{F}} \cdot d\mathbf{r}$$

vector field **F**

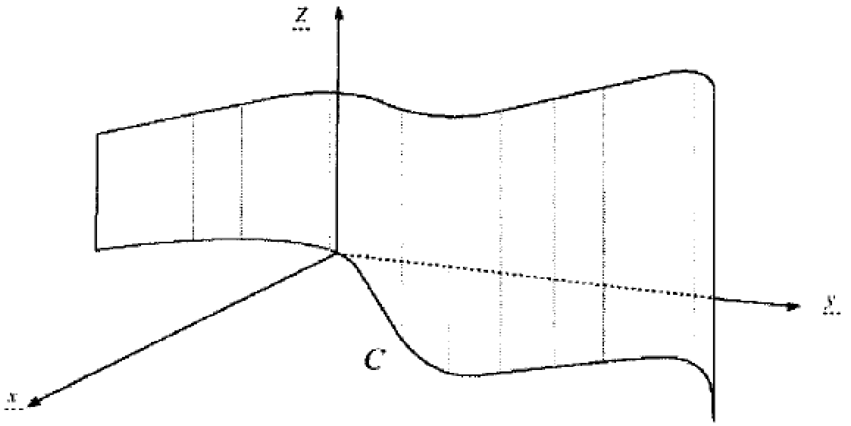
♣ **Line integrals of scalar fns** (2 variables)

Problem Find the area of the surface with

base: a smooth curve C

on xy-plane

top: described by $f(x,y)$

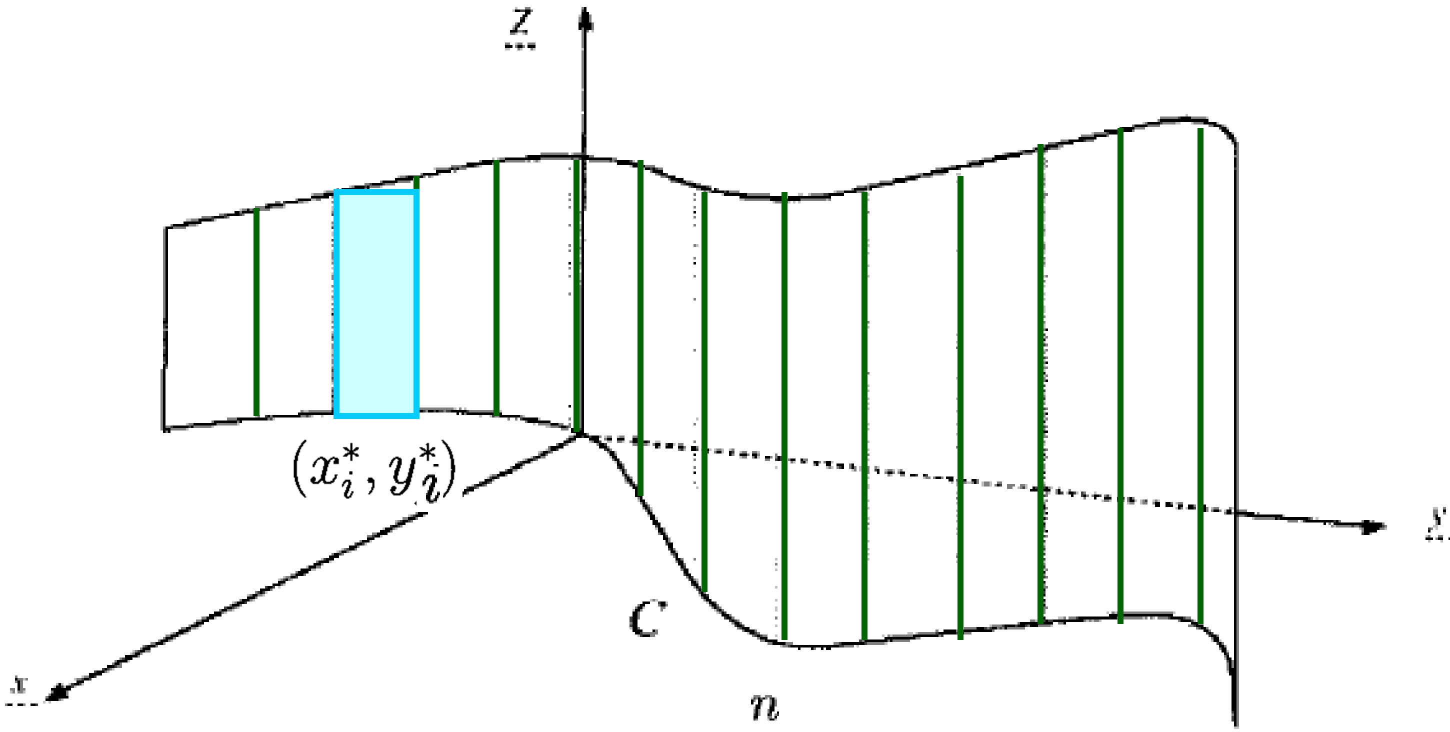


C : $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \ a \le t \le b$

The ***arc length*** of C is (see **Ch 6**):

$$s = \int_a^b \|\mathbf{r}'(t)\| dt.$$

♣



The surface area is $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$

— the ***line integral*** of the **scalar** fn ***f***,

& denoted by $\int_C f(x,y)\,ds$

Here, s denotes the arc length of C.

- $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$: vector fn of a curve ***C***, $a \le t \le b$.

The **arc length** of ***C*** :

$$s = \int_a^b \|\mathbf{r}'(t)\| dt.$$

If we replace the endpoint b by a variable t, then

$$s(t) = \int_a^t \|\mathbf{r}'(u)\| du.$$

By **FTC**,

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|.$$

Thus

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \|\mathbf{r}'(t)\| dt$$

$$= \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Evaluation of $\int_C f(x,y)\,ds$

(1) Find a **parametrization** of ***C*** :

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad a \le t \le b$$

(2) Compute $\|\mathbf{r}'(t)\|$

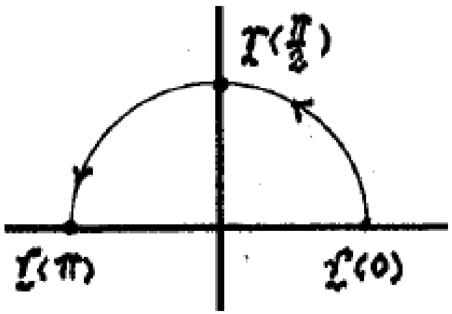
(3) Apply the formula

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\|\mathbf{r}'(t)\|dt}$$

♣ **Evaluate** $\int_C (2y + x^2 y)\,ds$, where C is the upper half of the unit circle centered at the origin.

(1) $C:\ \mathbf{r}(t) = \underbrace{\cos t}_{x}\,\mathbf{i} + \underbrace{\sin t}_{y}\,\mathbf{j}$

with $0 \le t \le \pi$.



(2) Then

$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = \mathbf{1}$$

(3) Thus

$$\boxed{\int_a^b f(x(t), y(t))\|\mathbf{r}'(t)\|\,dt}$$

$$\int_C (2y + x^2 y)\,ds = \int_0^{\pi} \left(2\sin t + \cos^2 t \sin t\right)\,dt$$

$$= \left[-2\cos t - \frac{1}{3}\cos^3 t\right]_0^{\pi} = \frac{14}{3}$$

Line integral: $\int_C f(x,y,z)\,ds.$

- For line integral of a function $f(x,y,z)$ along a space curve C, we have

$$C : x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}\ ,\quad a \le t \le b$$

$$\int_C f(x,y,z)\,ds =$$

$$\int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

♣ Evaluate $\int_C xy \sin z \, ds$, where C is the circular helix

$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $t \in [0, \pi/2]$.

$$\int_C xy \sin z \, ds$$

$$\boxed{\int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt}$$

$$= \int_0^{\pi/2} (\cos t)(\sin t)(\sin t)\sqrt{\sin^2 t + \cos^2 t + 1} \, dt$$

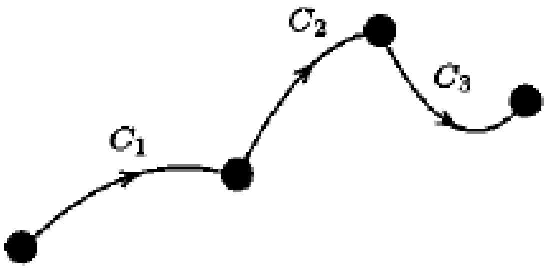
$$= \sqrt{2} \int_0^{\pi/2} \cos t \sin^2 t dt \quad = \frac{\sqrt{2}}{3} \left[\sin^3 t\right]_0^{\pi/2}$$

$$= \frac{\sqrt{2}}{3}$$

Piecewise smooth **curves**

- If ***C*** is a ***piecewise smooth*** curve obtained by joining *n* ***smooth*** curves

$$C_1, C_2, \cdots, C_n$$

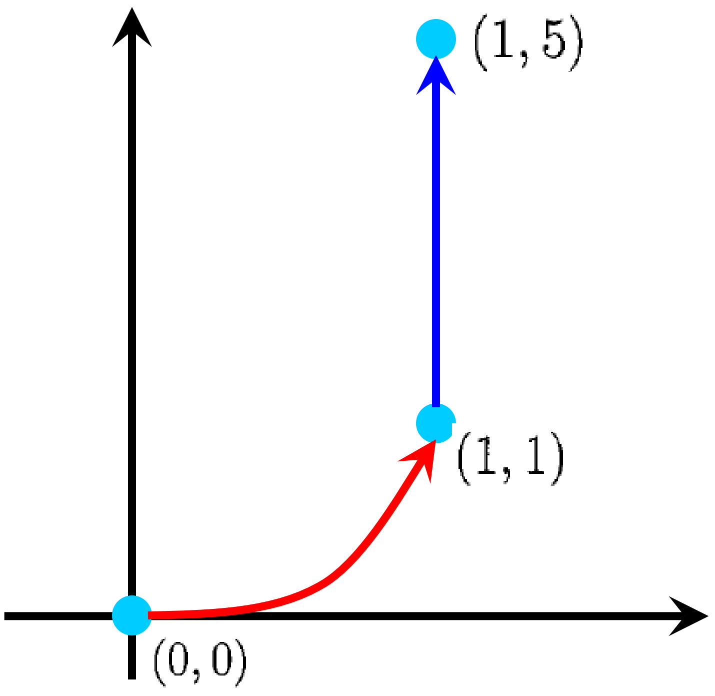


we denote it by $C = C_1 + C_2 + \cdots + C_n.$

The ***line integral*** of ***f*** along ***C*** is :

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds.$$

♣ Evaluate $\int_C 9y\,ds$, where C consists of the arc C_1 of the cubic $y = x^3$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,5)$.



- $C_1 : y = x^3$

  $(0,0)$ to $(1,1)$

  $x = t$
  $y = t^3$ $\quad 0 \le t \le 1$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\|\mathbf{r}'(t)\|dt$$

$$\mathbf{r}_1(t) = t\mathbf{i} + t^3\mathbf{j}$$

$$\|\mathbf{r}'_1(t)\| = \sqrt{1+(3t^2)^2}$$

$$\int_{C_1} 9y\,ds = \int_0^1 9t^3\sqrt{1+9t^4}\,dt$$

$$= \frac{1}{6}\left[(1+9t^4)^{3/2}\right]_0^1$$

$$= \frac{1}{6}(10\sqrt{10}-1).$$

- $C_2 : x = 1$
  $(1,1)$ to $(1,5)$

$$x = 1 \qquad y = t \qquad 1 \le t \le 5$$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\|\mathbf{r}'(t)\|dt$$

$$\mathbf{r}_2(t) = \mathbf{i} + t\mathbf{j}$$

$$\|\mathbf{r}'_2(t)\| = \sqrt{0+1}$$

$$\int_{C_2} 9y ds = \int_1^5 9t\,dt = 108$$

Hence $\displaystyle \int_C 9y\,ds = \int_{C_1} 9y\,ds + \int_{C_2} 9y\,ds$

$$= \frac{1}{6}(10\sqrt{10} + 647).$$

Evaluation of $\int_C f(x,y)\,ds$

(1) Find a **parametrization** of ***C*** :

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad a \le t \le b$$

(2) Compute $\|\mathbf{r}'(t)\|$

(3) Apply the formula

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\|\mathbf{r}'(t)\|dt}$$

Line integral: $\int_C f(x, y, z)\, ds.$

- For line integral of a function $f(x, y, z)$ along a space curve C, we have

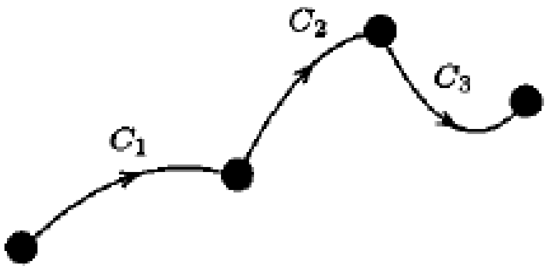
$$C : \mathbf{x}(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}\ ,\quad a \leq t \leq b$$

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Piecewise smooth curves

- If $\boldsymbol{C}$ is a ***piecewise smooth*** curve obtained by joining n ***smooth*** curves

  $C_1, C_2, \cdots, C_n$



we denote it by $C = C_1 + C_2 + \cdots + C_n.$

The ***line integral*** of ***f*** along ***C*** is :

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds.$$

Line integrals of *vector fields*

- **Recall**

  - Thus the total work done is approximately

$$W_{\text{total}} \approx \sum_{1}^{n} \mathbf{F}_i \cdot \Delta \mathbf{r}_i.$$

As $n \to \infty$, we write this as

$$(\clubsuit) \qquad \int_C \mathbf{F} \cdot d\mathbf{r}$$

which gives the actual total work done.

- **Smooth curve C** :

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad a \le t \le b$$

$\mathbf{F}(x,y,z)$: vector field (**vf**) defined on C

$$\mathbf{F}(\mathbf{r}(t)) = \mathbf{F}(x(t), y(t), z(t))$$

Then

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \underline{\mathbf{F}(\mathbf{r}(t))} \cdot \underline{\mathbf{r}'(t)}\, dt}$$

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = x\mathbf{i} + xy\mathbf{j} + xyz\mathbf{k}$$

and C is the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $t \in [0, 2]$

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

$$\mathrm{r}(t) = t\mathrm{i} + t^2\mathrm{j} + t^3\mathrm{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (t\mathbf{i} + t \cdot t^2\mathbf{j} + t \cdot t^2 \cdot t^3\mathbf{k}) \cdot (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})$$

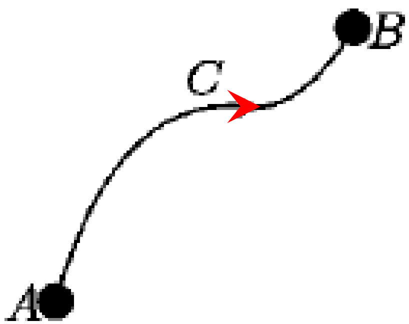
$$= t + 2t^4 + 3t^8$$

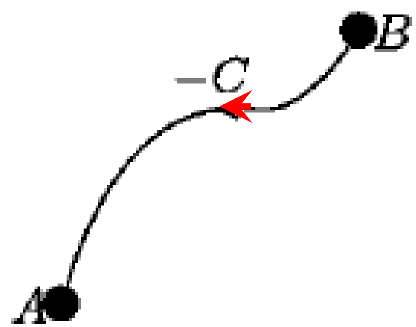
So $\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt}$

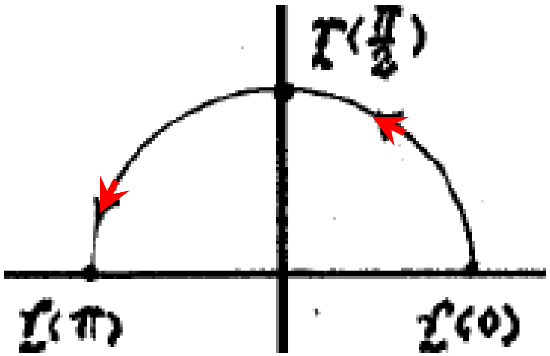
$$= \int_0^2 (t + 2t^4 + 3t^8)\, dt$$

Orientation of *curves*

- The vector equation of a curve C determines an **orientation** (direction) of C. The same curve with the opposite orientation of C is denoted by $-C$.







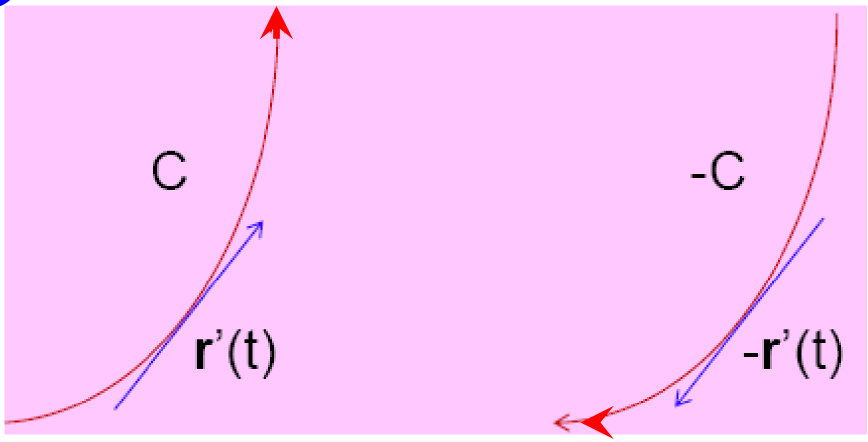
$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$

with $0 \leq t \leq \pi$.

- $$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt}$$

We have:

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$



as $\mathbf{r}'(t)$ changes sign in $-C$.

Note that for ***scalar*** functions,

$$\int_{-C} f(x,y,z)\, ds = \int_C f(x,y,z)\, ds$$

since the arc length is always positive.

☺ $$\int_C (x + 4xy)\, ds$$

$$\int_C xy \sin z \, ds$$

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

☻ $$\int_C y^2 dx + x dy$$ **??**

$$\int_C 2xy\, dx + (x^2 + z)\, dy + y\, dz$$ **???**

Line integrals in ***component form***

- **Suppose** $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$

$$C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad t \in [a, b].$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_a^b [P(\mathbf{r}(t))\mathbf{i} + Q(\mathbf{r}(t))\mathbf{j}] \cdot \left[\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right] dt$$

$$= \int_a^b \left[P(\mathbf{r}(t))\frac{dx}{dt} + Q(\mathbf{r}(t))\frac{dy}{dt}\right] dt$$

$$= \int_C P dx + Q dy.$$

denoted by

- Thus for $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$

$$\boxed{\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy.}$$

Similarly, for **vf**

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k},$$

$$\boxed{\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz.}$$

☺ $$\int_C (x + 4xy)\,ds$$

$$\int_C xy \sin z\,ds$$

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

☻ $$\int_C y^2 dx + x dy$$ **??**

$$\int_C 2xy\,dx + (x^2 + z)\,dy + y\,dz$$ **???**

How to ***compute***, for instance,

$$\int_C y^2 dx + x dy \qquad \textbf{?}$$

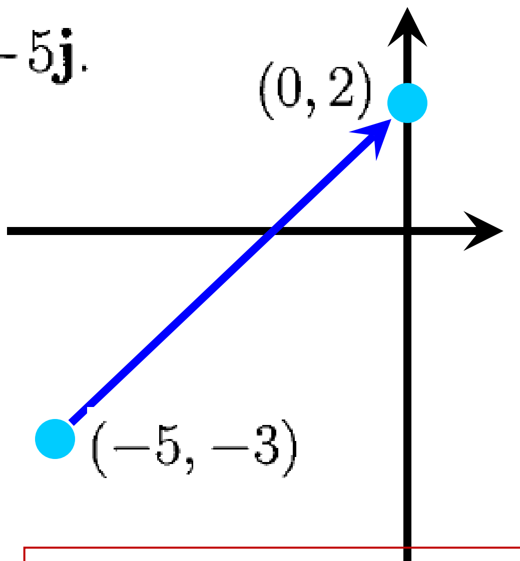
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \boxed{\begin{aligned} &\int_a^b \left[P(\mathbf{r}(t))\frac{dx}{dt} + Q(\mathbf{r}(t))\frac{dy}{dt} \right] dt \\ &= \int_C P dx + Q dy. \end{aligned}}$$

♣ **Evaluate** $\int_C y^2 dx + x dy$, where

$C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$

C_1 is a line passing through the point $(-5,-3)$
and parallel to the vector $(2\mathbf{j}) - (-5\mathbf{i} - 3\mathbf{j}) = 5\mathbf{i} + 5\mathbf{j}$.



$$C_1: \quad \mathbf{r}(t) = (-5\mathbf{i} - 3\mathbf{j}) + t(5\mathbf{i} + 5\mathbf{j})$$

$$= \underline{(5t-5)}\mathbf{i} + \underline{(5t-3)}\mathbf{j} \quad \text{with } 0 \le t \le 1$$

Thus, $\int_{C_1} y^2 dx + x dy$

$$= \int_0^1 (5t-3)^2 \boxed{\frac{dx}{dt}dt} + \int_0^1 (5t-5) \boxed{\frac{dy}{dt}dt}$$

$$= \int_0^1 (5t-3)^2 5dt + \int_0^1 (5t-5)5dt = -5/6.$$

The equation of the st line is

$y = x + 2$

You may use

$r(s) = si + (s+2)j$

$-5 \le s \le 0$

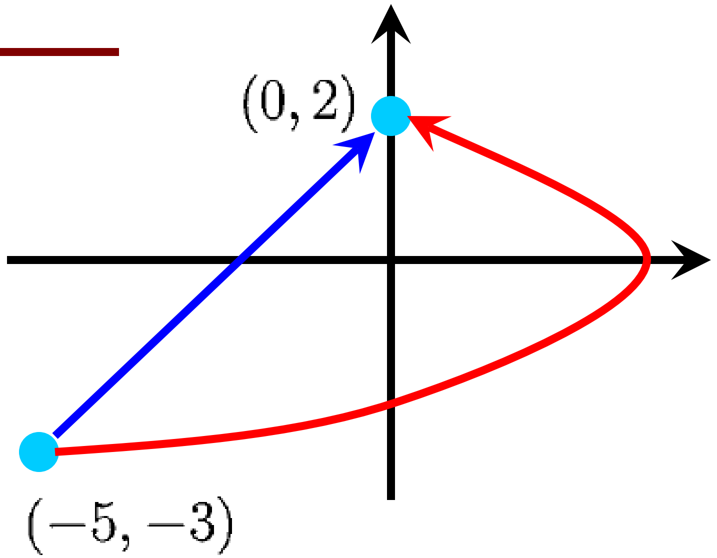
♣ **Evaluate** $\displaystyle\int_C y^2 dx + x dy$, where

$C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$

Let $y = t$.

$$C_2 : \mathbf{r}(t) = \underline{(4-t^2)}\mathbf{i} + \underline{t}\mathbf{j}$$

$$-3 \le t \le 2$$



$$\int_{C_2} y^2 dx + x dy$$

$$= \int_{-3}^{2} t^2 \boxed{\frac{dx}{dt} dt} + \int_{-3}^{2} (4-t^2) \boxed{\frac{dy}{dt} dt}$$

$$= \int_{-3}^{2} t^2(-2t)dt + \int_{-3}^{2} (4-t^2)dt \quad = \; 245/6.$$

Evaluation of *Line integrals* in *component form*

Summary

- **Suppose** $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$

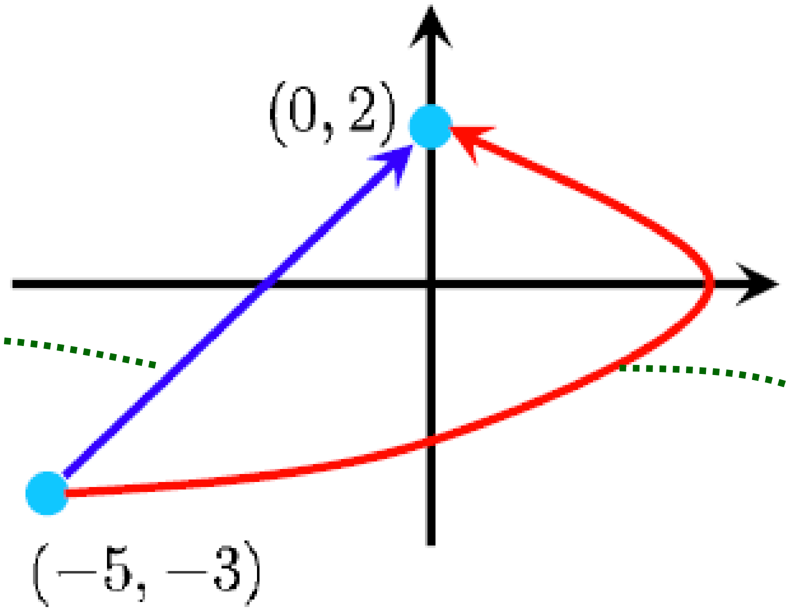
$$C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad t \in [a,b].$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy$$

$$\boxed{= \int_a^b \left[P(\mathbf{r}(t))\frac{dx}{dt} + Q(\mathbf{r}(t))\frac{dy}{dt} \right] dt}$$

♣



$$\int_C y^2 dx + x dy$$

$$= -5/6$$

$$\int_C y^2 dx + x dy$$

$$= 245/6$$

The Fundamental Theorem
for *line integrals*

- **FTC** $$\int_a^b F'(x)\,dx = F(b) - F(a).$$
- **Generalization** for **line integrals**

Let ***f*** be a function defined on a smooth curve ***C*** : $\mathbf{r}(t),\ a \le t \le b$ *s.t.* ∇f is continuous. Then

$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).}$$

♣ Find the work done by the (earth) gravitational field (see Example 9.2.9) in moving a particle of mass m from the point $(3, 4, 12)$ to the point $(1, 0, 0)$ along a curve C.

$$W \equiv \int_C \mathbf{G} \cdot d\mathbf{r} \quad = \int_C \nabla g \cdot d\mathbf{r}$$

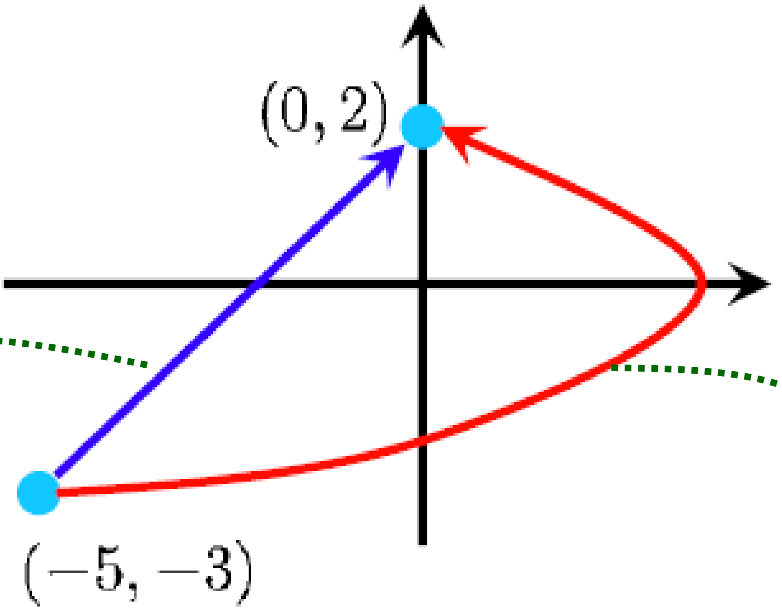
$$= g(1, 0, 0) - g(3, 4, 12).$$

Recall that

$$g(x, y, z) = \frac{mMK}{\sqrt{x^2 + y^2 + z^2}}$$

♣

Recall



$$\int_C y^2dx + xdy$$

$$= -5/6$$

$$\int_C y^2dx + xdy$$

$$= 245/6$$

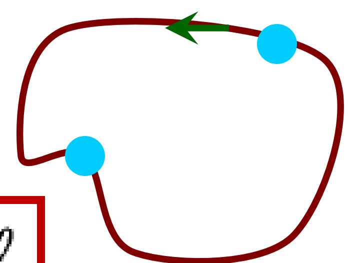
(I) If $\mathbf{F}$ is a conservative vector field, then $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is *independent of path*,

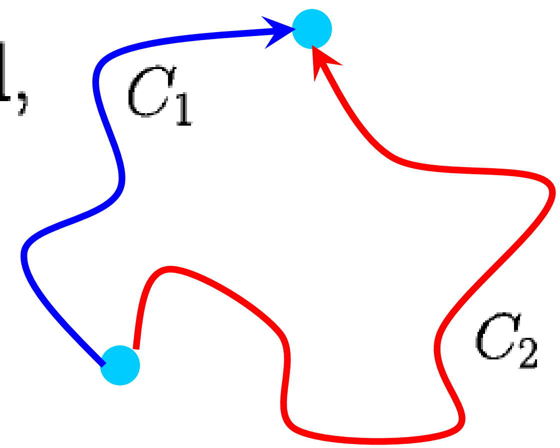
i.e. $\displaystyle\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ for any 2 paths C_1 and C_2 that have the same initial and terminal points.

(II) If $\mathbf{F}$ is a conservative vector field,

then $\displaystyle\oint_{\ell} \mathbf{F}\cdot d\mathbf{r} = 0$

for any *closed* curve ℓ





The **work done** by a **conservative** force field as it moves a particle around a **closed curve** is '**0**'. **Gravitational** fields & **electric** fields are **conservative**.

♣ Let $\mathbf{F}(x,y) = (y^2+3x^2)\mathbf{i}+(2xy)\mathbf{j}$. Show that the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path and evaluate this integral over the curve C where C is

(i) given by $\mathbf{r}(t) = \cos t\,\mathbf{i} + e^t \sin t\,\mathbf{j}, \quad t \in [0,\pi]$;

(ii) the unit circle.

Note that (see Ex 9.2.5)

$$\nabla f = \mathbf{F}$$

where $f(x,y) = xy^2 + x^3$.

So $\mathbf{F}$ is conservative, and

$\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path.

$\mathbf{F}$ is conservative $\Leftrightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

(i) $C : \mathbf{r}(t) = \cos t\mathbf{i} + e^t \sin t\mathbf{j}, \quad t \in [0, \pi]$

$\mathbf{r}(0) = \mathbf{i} = \mathbf{i} + 0\mathbf{j} \rightarrow (1,0)$

$\mathbf{r}(\pi) = -\mathbf{i} = -\mathbf{i} + 0\mathbf{j} \rightarrow (-1,0)$

$$f(x, y) = xy^2 + x^3$$

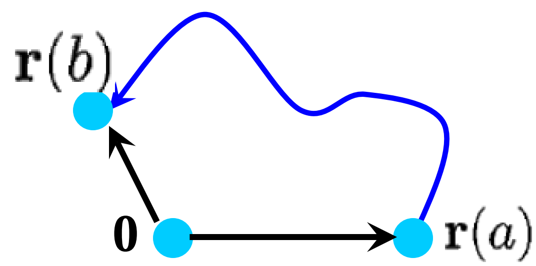
By **FT**,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(-1, 0) - f(1, 0)$$
$$= -2.$$

(ii) As the unit circle is ***closed***

& **F** is ***conservative***,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

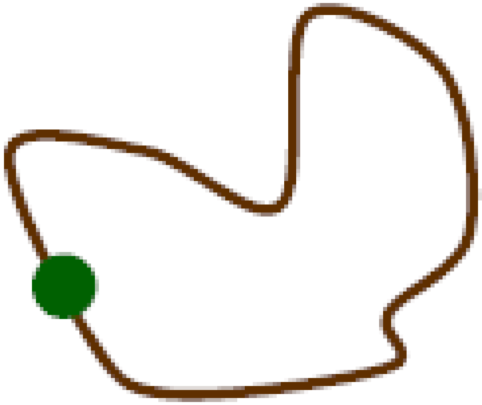


$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$
$$a \le t \le b$$

$$\int_C \nabla f \cdot d\mathbf{r}$$
$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Note. If the curve ℓ is **closed**, the line integral is denoted by

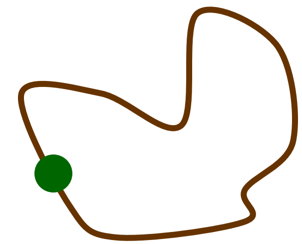
$$\oint_{\ell} \mathbf{F} \cdot d\mathbf{r}.$$



- Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy.$$

Let ℓ be a closed curve in the plane.



If $\mathbf{F}$ is ***conservative***, then

$$\oint_\ell \mathbf{F} \cdot d\mathbf{r} = 0$$

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

What can be said of

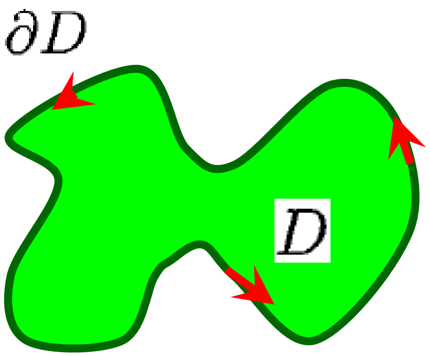
$$\oint_\ell P dx + Q dy$$

if $\mathbf{F}$ is **not** ***conservative*** ?

4. Green's Theorem

Let ***D*** be a bounded region in the *xy*-plane & ***∂D*** the boundary of ***D***. Assume both $P(x, y)$ & $Q(x, y)$ have continuous partial derivatives on ***D***.

Then $$\boxed{\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.}$$

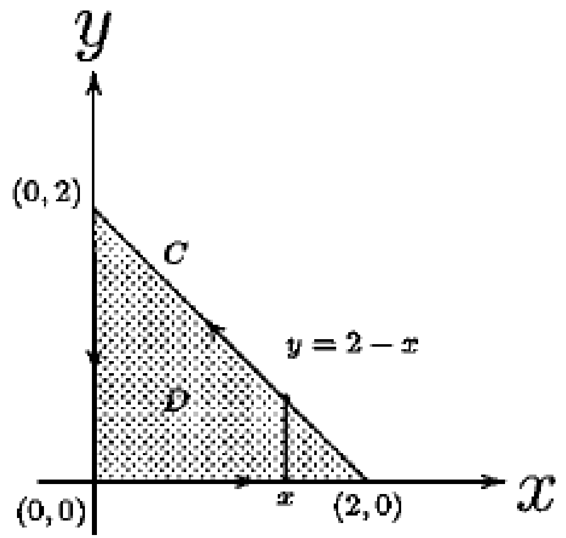


We move along *∂D* in the direction that the region ***D*** always on the **left. This direction is called positive direction.**

Positive directions, for many cases, always anticlockwise. However for some special cases, they are clockwise, see last example

♣ **Evaluate** $\oint_C 2xy\,dx + xy^2 dy,$

where ***C*** is the triangular curve consisting of (0, 0)→(2, 0), (2, 0)→(0, 2) & (0, 2)→(0, 0).



$\boxed{D}$ $0 \le y \le 2-x,\ 0 \le x \le 2.$

$$\oint_C 2xy\,dx + xy^2 dy = \oint_{\partial D} P dx + Q dy$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (y^2 - 2x)\,dydx$$

$$= \int_0^2 \int_0^{2-x} (y^2 - 2x)\,dydx = -\frac{4}{3}.$$

♣ **Evaluate (*)** $\oint_C (4y - e^{x^2})dx + (9x + \sin(y^2 - 1))dy,$

where ***C*** is the circle $x^2 + y^2 = 4$

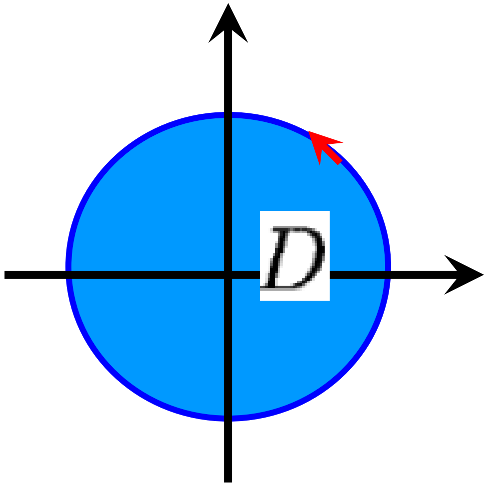
with positive orientation.

$$(*) = \iint_D \left[\frac{\partial(9x + \sin y^2 - 1)}{\partial x} - \frac{\partial(4y - e^{x^2})}{\partial y} \right] dA$$

$$= \iint_D 5\, dA = 5 \iint_D dA$$

$$= 5 \times (\text{area of } D)$$

$$= 5(\pi 2^2) = 20\pi.$$



Problems

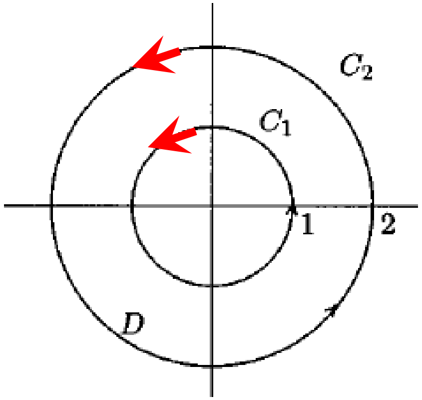
- Evaluate by Green's Theorem

$$\oint_C e^{-x} \sin y \, dx + e^{-x} \cos y \, dy$$

where C is the rectangle with vertices at $(0,0)$, $(\pi, 0)$, $(\pi, \pi/2)$, $(0, \pi/2)$. [Answer: $2(e^{-\pi} - 1)$]

♣ Let $\mathbf{F}(x,y) = y\mathbf{i} + y\mathbf{j}$ and D a region in xy-plane bounded by the two circles centered at the origin with radius 1 and 2. Verify Green's Theorem.

$$\boxed{\int_{\partial D} \mathbf{F}\cdot d\mathbf{r} = \oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$



(i) Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ directly:

$$C_1 : \quad \mathbf{r}_1 = \cos t\mathbf{i} + \sin t\mathbf{j}$$

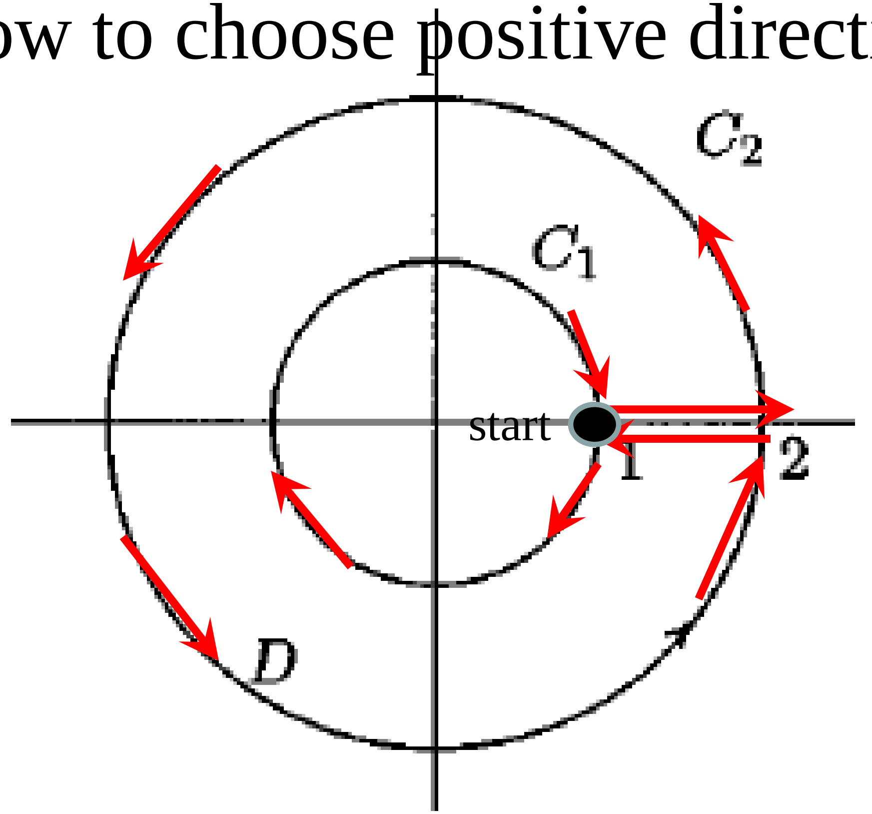
$$C_2 : \quad \mathbf{r}_2 = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$$

$$t \in [0, 2\pi]$$

Note that $\partial D = C_2 - C_1.$ WHY? See next slide

Region enclosed by

C_1 and C_2. How to choose positive direction



$$\partial D = -C_1 + l + C_2 - l = C_2 - C_1$$

C_1, C_2 anticlockwise by given

l bridge form left to right

- $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$\mathbf{F}(x,y) = y\mathbf{i} + y\mathbf{j} \qquad C_1: \quad \mathbf{r}_1 = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$$

$$t \in [0, 2\pi]$$

$$\begin{aligned}
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (\sin t\,\mathbf{i} + \sin t\,\mathbf{j}) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\,dt \\
&= \int_0^{2\pi} (-\sin^2 t + \sin t \cos t)\,dt \\
&= \int_0^{2\pi} \frac{1}{2}(\cos 2t - 1 + \sin 2t)\,dt \\
&= \frac{1}{2}\left[\frac{\sin 2t}{2} - t - \frac{\cos 2t}{2}\right]_0^{2\pi} = -\pi
\end{aligned}$$

- **Likewise**, $$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -4\pi$$

& so

$$\int_{\partial D} \mathrm{F} \bullet d\mathrm{r} = \int_{C_2 - C_1} \mathrm{F} \bullet d\mathrm{r} = \int_{C_2} \mathrm{F} \bullet d\mathrm{r} + \int_{-C_1} \mathrm{F} \bullet d\mathrm{r} = \int_{C_2} \mathrm{F} \bullet d\mathrm{r} - \int_{C_1} \mathrm{F} \bullet d\mathrm{r}$$

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} - \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -3\pi.$$

(ii) Using Green's Theorem, we have

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D \left(\frac{\partial y}{\partial x} - \frac{\partial y}{\partial y} \right) dA$$

$$\boxed{\mathbf{F}(x, y) = y\mathbf{i} + y\mathbf{j}}$$

$$= \iint_D (-1)\, dA.$$

$= (-1) \cdot$ **Area** of $D = -\pi(2^2 - 1^2) = -3\pi$.