MA 1505 Mathematics I
Tutorial 5 Solutions

1. Rewrite the function:

$$f(x) = \frac{1}{2}(x + |x|) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

The Fourier series of $f(x)$ is given by

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

$$a_0 = \frac{1}{2\pi} \int_0^{\pi} x \; dx = \frac{\pi}{4}.$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} x \cos nx \; dx = \frac{1}{\pi} \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi} = \frac{(-1)^n - 1}{\pi n^2}.$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} x \sin nx \; dx = \frac{1}{\pi} \left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi} = \frac{(-1)^{n+1}}{n}.$$

So the Fourier series is

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left\{ \frac{(-1)^n - 1}{\pi n^2} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \right\}.$$

More explicitly, we have

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right) + \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots \right)$$

2. From the graph, the function is given by :

$$f(x) = \begin{cases} 2 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

The Fourier series of $f(x)$ is given by

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{0} 2 \; dx + \frac{1}{2\pi} \int_0^{\pi} 1 \; dx = \frac{3}{2}.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{0} 2 \cos nx \; dx + \frac{1}{\pi} \int_0^{\pi} \cos nx \; dx = 0.$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{0} 2\sin nx\ dx + \frac{1}{\pi}\int_{0}^{\pi} \sin nx\ dx \\
&= \frac{1}{\pi}\left[-\frac{2\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{0}^{\pi} \\
&= \frac{1}{\pi}\left(\frac{-2+2\cos n\pi}{n}\right) + \frac{1}{\pi}\left(\frac{-\cos n\pi + 1}{n}\right) \\
&= \frac{1}{\pi}\left(\frac{\cos n\pi - 1}{n}\right) \\
&= \begin{cases} 0 & \text{if } \ n = 2m \ \text{ even} \\ \frac{-2}{(2m-1)\pi} & \text{if } \ n = 2m-1 \ \text{ odd} \end{cases}.
\end{aligned}
$$

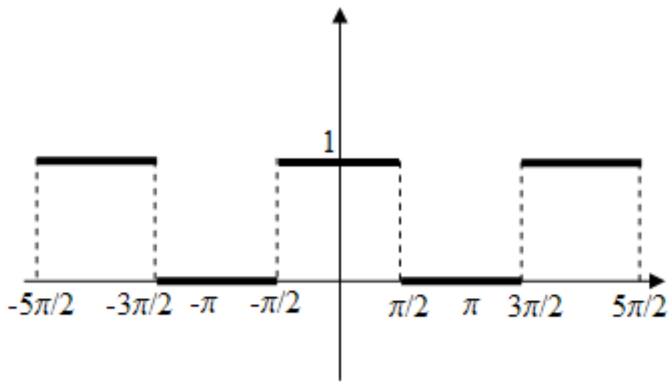
So the Fourier series is

$$f(x) = \frac{3}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}.$$

More explicitly, we have

$$f(x) = \frac{3}{2} - \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).$$

3. The graph of f is given as follow:



Since the graph is symmetrical about y-axis, $f(x)$ is an even function.

So $b_n = 0$ for all n.

The Fourier series of $f(x)$ is given by $a_0 + \sum_{n=1}^{\infty}(a_n \cos nx)$.

$$a_0 = 2\left(\frac{1}{2\pi}\int_0^{\pi/2} 1\ dx\right) = \frac{1}{2}.$$

$$a_n = 2\left(\frac{1}{\pi}\int_0^{\pi/2} \cos nx \ dx\right) = \frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi/2} = \begin{cases} 0 & \text{if } \ n = 2m \ \text{ even} \\ \frac{2}{\pi}\frac{(-1)^{m+1}}{2m-1} & \text{if } \ n = 2m-1 \ \text{ odd} \end{cases}$$

So the Fourier series is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(2n-1)x}{2n-1}.$$

4. The period $2L = \frac{2\pi}{w} \Rightarrow L = \frac{\pi}{w}$.

The Fourier series of $u(t)$ is given by

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nwt + b_n \sin nwt).$$

$$a_0 = \frac{w}{2\pi}\int_0^{\pi/w} \sin wt \ dt = \frac{1}{\pi}.$$

$$\begin{aligned} a_n &= \frac{w}{\pi}\int_0^{\pi/w} \sin wt \cos nwt \ dt \\ &= \frac{w}{2\pi}\int_0^{\pi/w} [\sin(1+n)wt + \sin(1-n)wt] \ dt \\ &= \frac{w}{2\pi}\left[-\frac{\cos(1+n)wt}{(1+n)w} - \frac{\cos(1-n)wt}{(1-n)w}\right]_0^{\pi/w} \\ &= \frac{1}{2\pi}\left(\frac{-\cos(1+n)\pi+1}{1+n} + \frac{-\cos(1-n)\pi+1}{1-n}\right) \qquad (*) \\ &= \begin{cases} 0 & \text{if } \ n \text{ is odd and } \neq 1 \\ \frac{-2}{(n-1)(n+1)\pi} & \text{if } \ n \text{ is even} \end{cases} \end{aligned}$$

Note that the second term in $(*)$ is not defined at $n = 1$.

$$\begin{aligned} a_1 &= \frac{w}{\pi}\int_0^{\pi/w} \sin wt \cos wt \ dt \\ &= \frac{w}{2\pi}\int_0^{\pi/w} \sin 2wt \ dt \\ &= \frac{w}{2\pi}\left[-\frac{\cos 2wt}{2w}\right]_0^{\pi/w} \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{w}{\pi}\int_0^{\pi/w} \sin wt \sin nwt \ dt \\
&= \frac{w}{2\pi}\int_0^{\pi/w} [-\cos(1+n)wt + \cos(1-n)wt] \ dt \\
&= \frac{w}{2\pi}\left[-\frac{\sin(1+n)wt}{(1+n)w} + \frac{\sin(1-n)wt}{(1-n)w}\right]_0^{\pi/w} \\
&= \frac{1}{2\pi}\left(\frac{-\sin(1+n)\pi}{1+n} + \frac{\sin(1-n)\pi}{1-n}\right) \qquad (*) \\
&= 0 \text{ if } \ n \geq 2
\end{aligned}
$$

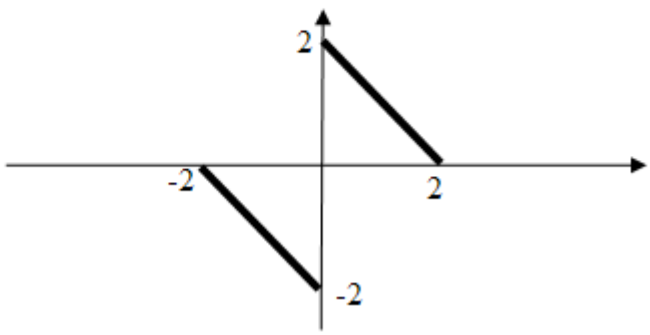
Note that the second term in $(*)$ is not defined at $n = 1$.

$$
\begin{aligned}
b_1 &= \frac{w}{\pi}\int_0^{\pi/w} \sin^2 wt \ dt \\
&= \frac{w}{2\pi}\int_0^{\pi/w} 1 - \cos 2wt \ dt \\
&= \frac{w}{2\pi}\left[t - \frac{\sin 2wt}{2w}\right]_0^{\pi/w} \\
&= \frac{1}{2}
\end{aligned}
$$

So the Fourier series is

$$
f(x) = \frac{1}{\pi} + \frac{1}{2}\sin wt - \frac{2}{\pi}\left(\frac{1}{1\cdot 3}\cos 2wt + \frac{1}{3\cdot 5}\cos 4wt + \cdots\right)
$$

5. Note that this function is an odd function with period $2L = 4$:



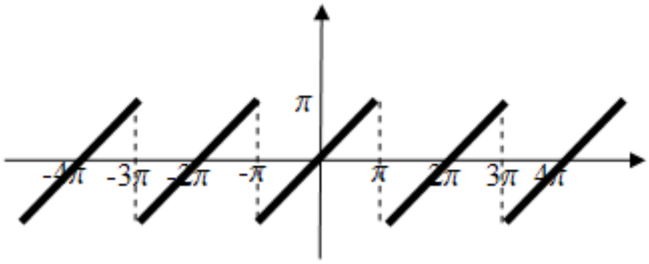
$a_n = 0$ for all n.

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\ dx \\
&= \int_0^2 (2-x)\sin\frac{n\pi x}{2}\ dx \\
&= \left[2\left(\frac{-2}{n\pi}\right)\cos\frac{n\pi x}{2}\right]_0^2 + \left[x\left(\frac{2}{n\pi}\right)\cos\frac{n\pi x}{2}\right]_0^2 - \int_0^2\left(\frac{2}{n\pi}\right)\cos\frac{n\pi x}{2}\ dx \\
&= \left[-\frac{4}{n\pi}((-1)^n-1)\right] + \left[\frac{4}{n\pi}((-1)^n-0)\right] - \left[\left(\frac{2}{n\pi}\right)^2\sin\frac{n\pi x}{2}\right]_0^2 \\
&= \frac{4}{n\pi}.
\end{aligned}$$

So the Fourier series is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}.$$

6. Fourier sine half range expansion:

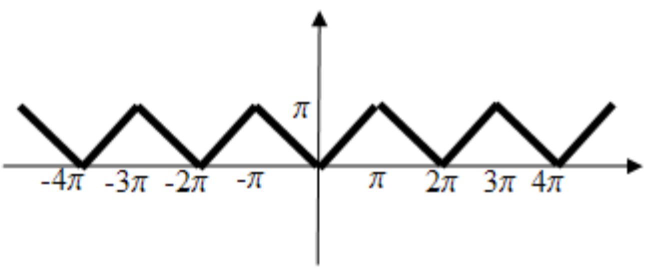


$b_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi x\sin nx\ dx = \dfrac{2}{\pi}\left[-\dfrac{x\cos nx}{n} + \dfrac{\sin nx}{n^2}\right]_0^\pi = \dfrac{(-1)^{n+1}2}{n}.$

So the Fourier sine half range expansion is

$$f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx.$$

Fourier cosine half range expansion:



$a_0 = \dfrac{1}{\pi}\displaystyle\int_0^\pi x\ dx = \dfrac{\pi}{2}.$

$a_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi x\cos nx\ dx = \dfrac{2}{\pi}\left[\dfrac{x\sin nx}{n} + \dfrac{\cos nx}{n^2}\right]_0^\pi = 2\dfrac{(-1)^n - 1}{\pi n^2}.$

So the Fourier cosine half range expansion is

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2((-1)^n - 1)}{\pi n^2}\cos nx.$$

7. From $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, we square both sides to obtain

$(f(x))^2 = \left(\sum_{n=1}^{\infty} b_n \sin nx\right)^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_n b_m \sin nx \sin mx.$

Now integrate both sides from $-\pi$ to π and assume that term by term integration is valid for the right hand side, we have

$\int_{-\pi}^{\pi} (f(x))^2\, dx = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_n b_m \int_{-\pi}^{\pi} \sin nx \sin mx dx.$

Recall from the lecture notes that $\int_{-\pi}^{\pi} \sin nx \sin mx dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$, we then have

$\int_{-\pi}^{\pi} (f(x))^2\, dx = \sum_{n=1}^{\infty} (b_n)^2 \pi.$

That is $\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\, dx = \sum_{n=1}^{\infty} (b_n)^2.$

Apply this formula to $f(x) = x$, we obtain

$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \sum_{n=1}^{\infty} \left(\frac{2(-1)^{n+1}}{n}\right)^2 = \sum_{n=1}^{\infty} \frac{4}{n^2}.$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$