Chapter 2

Differentiation

Key Results

- Definition and geometrical meaning of the derivative
- Chain rule, implicit and parametric differentiation
- Extreme values of functions
- First and second derivative tests
- Optimization
- Indeterminate forms, L'Hopital's rule

Formal Derivative

$f(x)$ is a function.

At a point a ($x = a$), the derivative of f is

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

provided the limit exists.

equivalent formulation (set $x = a + h$)

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

Notation

use y as the dependent variable, i.e., $y = f(x)$.

$$f'(a) \quad = \quad \left.\frac{dy}{dx}\right|_{x=a} = \frac{dy}{dx}(a)$$

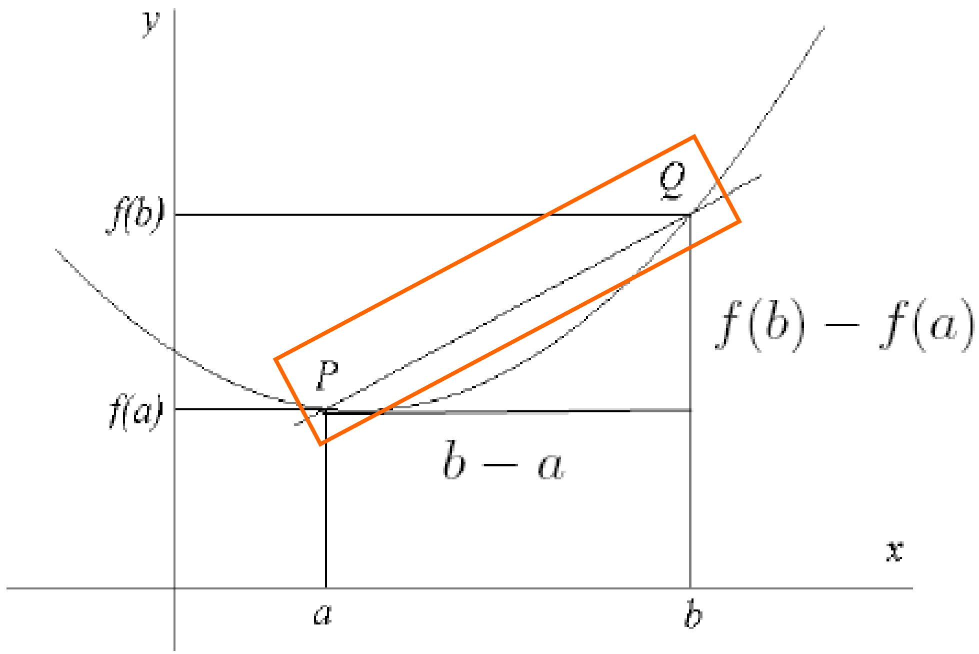
If the derivative $f'(a)$ exists,

f is *differentiable* at the point a.

Geometrical Meaning

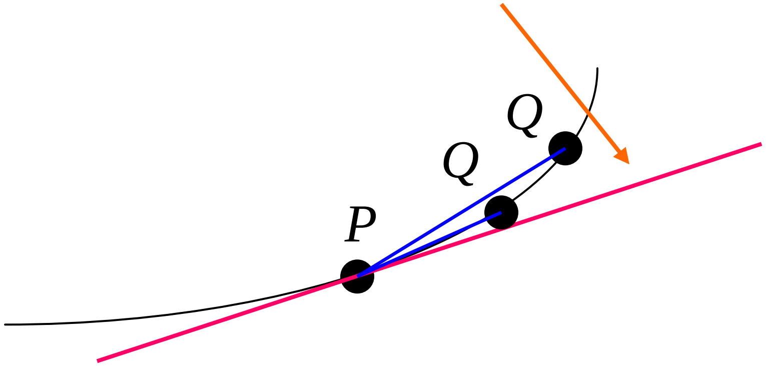
$$\frac{f(b) - f(a)}{b - a} = \text{slope of the straight line joining the two points}$$

$$P = (a, f(a)) \text{ and } Q = (b, f(b))$$



Tangent Line

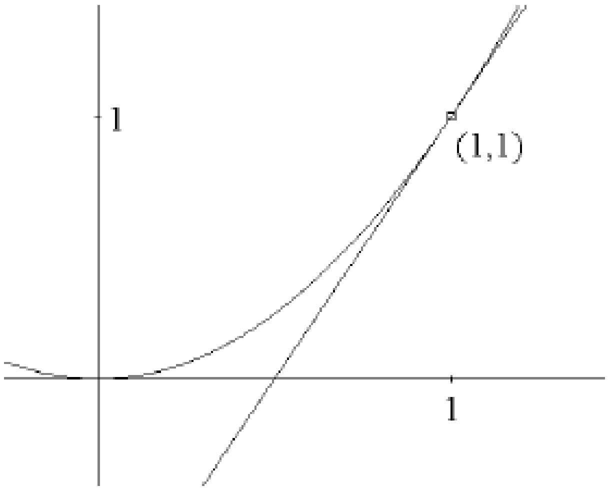
- As Q approaches P, secant lines approach the "tangent line" at P.
- Slopes of the secant lines approach a limit value.
- Derivative is the slope of the tangent at P.



Example (tangent)

Tangent to curve $y = x^2$ at $x = 1$.

$$f(x) = x^2$$



$$f'(x) = 2x$$

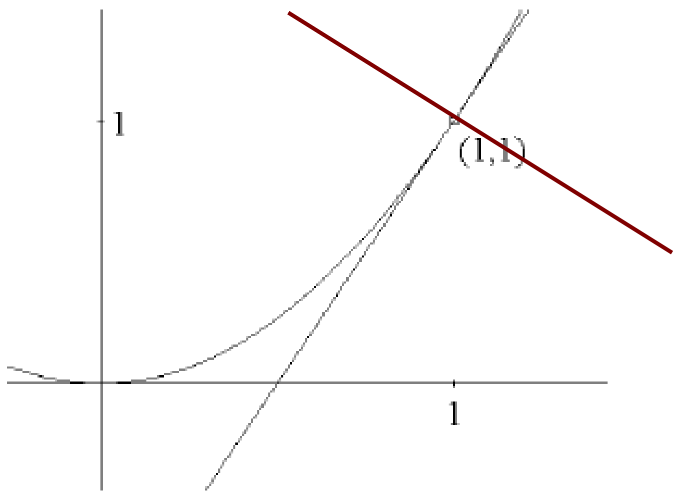
slope of the tangent is $f'(1) = 2$.

Point-slope formula, an equation of the tangent is

$$\frac{y-1}{x-1} = 2$$

$$\boxed{y = 2x - 1.}$$

Example (normal line)



normal line has slope $-\dfrac{1}{2}$

Point-slope formula, an equation of the normal line is

$$\frac{y-1}{x-1} = -\frac{1}{2}$$

$$\boxed{y = -\frac{1}{2}x + \frac{3}{2}.}$$

Rules of Differentiation

f and *g* are differentiable functions, *k* is a constant.

Linear property

$$\boxed{\begin{array}{lrcl} (1) & (kf)'(x) & = & kf'(x) \\ (2) & (f \pm g)'(x) & = & f'(x) \pm g'(x) \end{array}}$$

Product rule

$$\boxed{(fg)'(x) = f'(x)g(x) + f(x)g'(x)}$$

Quotient rule

$$\boxed{\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}}$$

Chain Rule

- Suppose $f \circ g$ and $f' \circ g$ are defined. Then

$$\boxed{(f \circ g)'(x) \;=\; f'(g(x))g'(x)}$$

Example (new)

Let $\boxed{f(x) = x^3}$ $\boxed{g(x) = x^2 - 1}$

Then

$$(f \circ g)'(x) = \boxed{f'(g(x))}\boxed{g'(x)} = \boxed{3\left(x^2-1\right)^2} \cdot \boxed{2x}$$

$$= \boxed{6x\left(x^2-1\right)^2}$$

$f'(x) = 3x^2$ $\boxed{g'(x) =}\boxed{2x}$

$$f'(g(x)) = 3(g(x))^2 = \boxed{3\left(x^2-1\right)^2}$$

Note that $(f \circ g)(x) = \left(x^2-1\right)^3$

Inside function $\boxed{x^2-1}$

Outside function $\boxed{(\quad)^3}$

Chain Rule (another formulation)

Suppose $f \circ g$ and $f' \circ g$ are defined. Then

$$\boxed{(f \circ g)'(x) \ =} \boxed{f'(g(x))} \boxed{g'(x)}$$

Another formulation with a new variable u, write

$$\boxed{y = f(u),} \quad \boxed{u = g(x).}$$

Then $\boxed{y(x) = f(g(x)) = (f \circ g)(x)}$

$$\boxed{\frac{dy}{dx} = \left.\frac{dy}{du}\right|_{u=u(x)} \cdot \frac{du}{dx}}$$

Parametric Differentiation

Suppose x and y are functionally dependent but are expressed in terms of a parameter t.

That is, $x = u(t)$ and $y = v(t)$

Then

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{v'(t)}{u'(t)}}$$

provided the derivatives exist.

Example

Let $\boxed{x = a(t - \sin t)}$ and $\boxed{y = a(1 - \cos t)}$

Then

$$\frac{dy}{dx} = \frac{\boxed{\dfrac{dy}{dt}}}{\boxed{\dfrac{dx}{dt}}} = \frac{a \sin t}{a(1 - \cos t)} \qquad a \neq 0$$

$$= \boxed{\frac{\sin t}{1 - \cos t}} \qquad \text{(reasonable answer)}$$

$$= \frac{2 \sin \frac{t}{2} \cos \frac{t}{2}}{1 - \left(1 - 2 \sin^2 \frac{t}{2}\right)} = \boxed{\cot \frac{t}{2}.}$$

Implicit Differentiation

- An application of chain rule.
- *x* and *y* are functionally dependent, but dependence implicitly expressed as

$$F(x, y) = 0.$$

Not easy to express y in terms of x to find $\dfrac{dy}{dx}$.

- Differentiate both sides of above equation.

Example

Consider $\boxed{x^2} + \boxed{y^2} - a^2 = 0 \qquad y > 0$

Differentiate the equation with respect to *x*:

$$\boxed{2x + 2y\frac{dy}{dx} = 0}$$

Obtain $\boxed{\frac{dy}{dx} = -\frac{x}{y}.}$

Chain rule gives

$$\frac{d}{dx}y^2 = \frac{d}{dy}y^2 \cdot \frac{dy}{dx} = 2y\frac{dy}{dx}.$$

Example

Let $y = x^x$, $x > 0$. Find $\dfrac{dy}{dx}$.

- ***Not*** power rule

$$\frac{d}{dx}x^n = nx^{n-1}, \quad n \text{ is constant}, \; n \neq 0$$

- ***Not*** "simple" exponential

$$\frac{d}{dx}e^x = e^x$$

Example (cont'd)

Let $y = x^x$, $x > 0$. Find $\dfrac{dy}{dx}$.

First apply ln $\boxed{\ln y} = \boxed{x \ln x}$

Differentiate implicitly with respect to *x*:

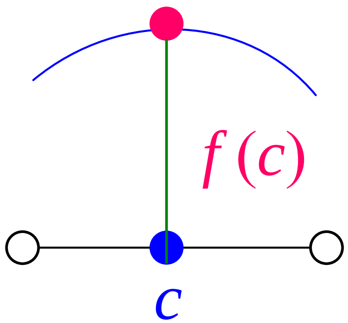
$$\frac{1}{y} \cdot \frac{dy}{dx} = \boxed{1 + \ln x}$$ (use product rule)

Obtain $$\frac{dy}{dx} = y\left(1 + \ln x\right) = \boxed{x^x\left(1 + \ln x\right)}$$

Local Maximum/Minimum

A function f has a local (relative) maximum value at a point c of its domain if

$$\boxed{f(x) \leq f(c)}$$



for all x in a neighbourhood of c.

- 'neighbourhood' refers to a small open interval containing c.
- $f(c)$ is a 'high' value locally.

Similar definition for local (relative) minimum.

Absolute Maximum/Minimum

A function *f* has an absolute maximum value at a point *c* of its domain if

$$\boxed{f(x) \leq f(c)}$$

for all *x* in the domain of *f*.

- *f*(*c*) is the highest value.

Similar definition for absolute minimum.

Finding Extreme Values

Points where *f* may have an extreme value are:

(1) interior points where $\boxed{f'(x) = 0}$; or

(2) interior points where $\boxed{f'(x) \text{ does not exist}}$; or

(3) end-points of the domain of f.

Test these points for extreme values.

Special terminology for (1) and (2):

An interior point c of the domain of a function f where $f'(c)$ is zero or fails to exist is a **critical point** of f.

Critical Points Example

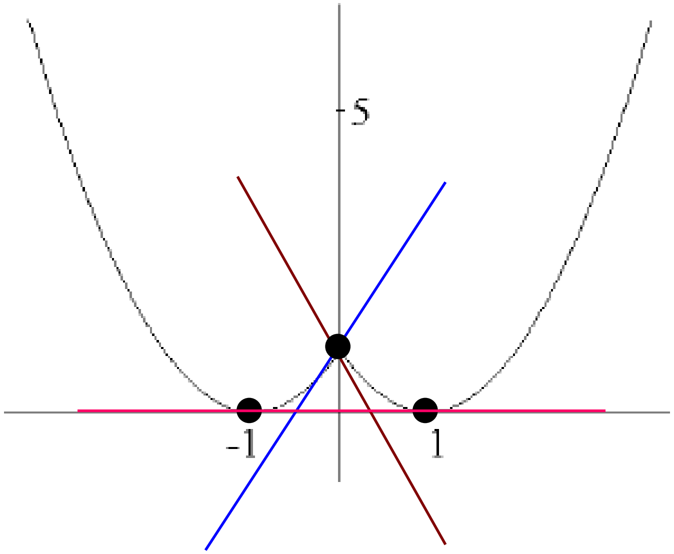
Consider

$$f(x) = \begin{cases} (x-1)^2 & \text{if } x \geq 0 \\ (x+1)^2 & \text{if } x < 0. \end{cases}$$

Critical points at:

$\boxed{x = 0 \quad \text{(local max.)}}$

$f'(x)$ does not exist
no unique tangent



$\boxed{x = -1 \ \text{(local min.)}}$

$\boxed{x = 1 \quad \text{(local min.)}}$

$f'(x) = 0$ slope of tangent is 0

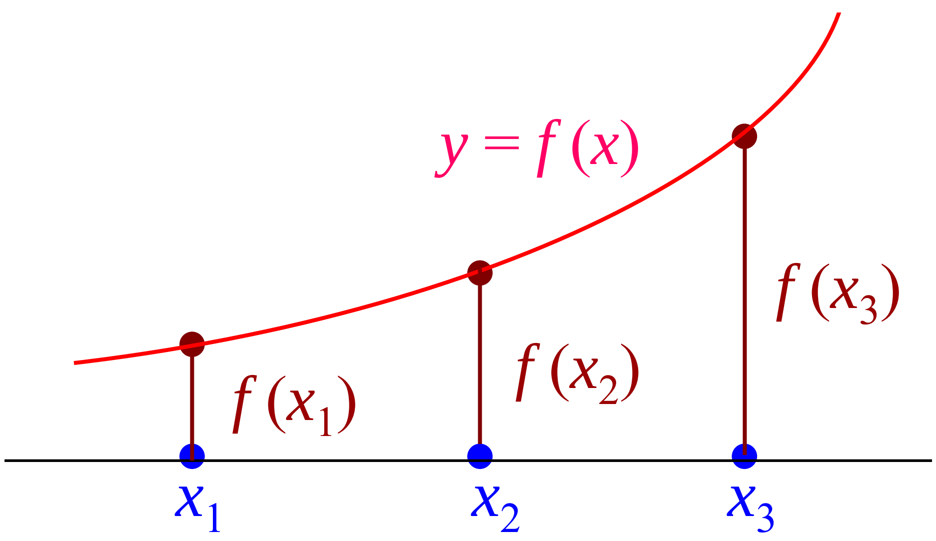
Increasing/Decreasing Functions

A function f defined on an interval I.

- f is increasing on I if

$$\boxed{f(x_2) \; > \; f(x_1)}$$

for any two points x_1 and x_2 in I where $\boxed{x_2 > x_1}$.



Increasing/Decreasing Functions

A function *f* defined on an interval *I*.

- *f* is decreasing on *I* if

$$f(x_2) < f(x_1)$$

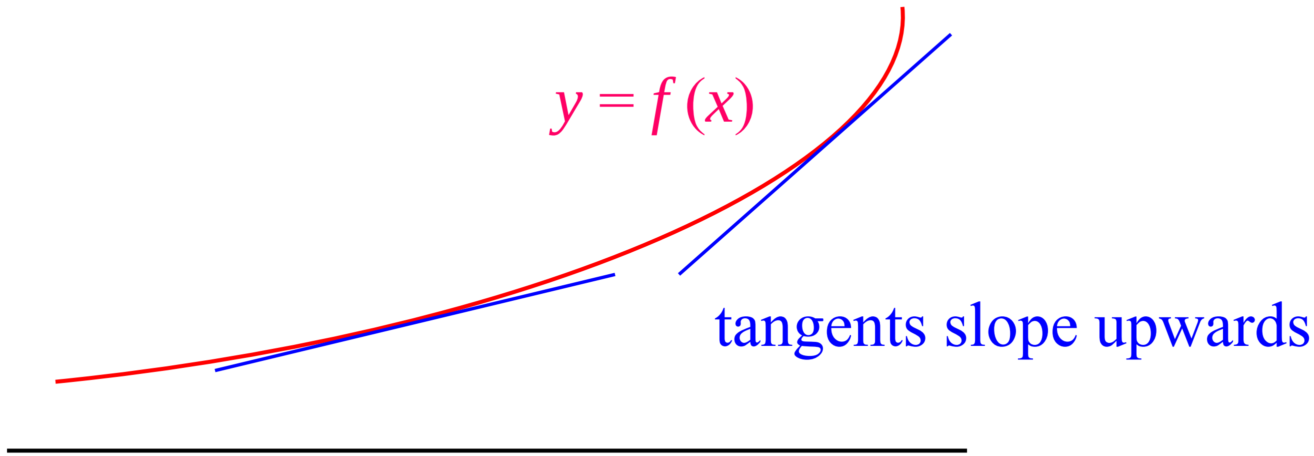
for any two points x_1 and x_2 in I where $x_2 > x_1$.

Use Derivative to Test

A function f is

(1) $\boxed{\text{increasing}}$ on an interval I if $\boxed{f'(x) > 0}$ on I;

(2) $\boxed{\text{decreasing}}$ on I if $\boxed{f'(x) < 0}$ on I.

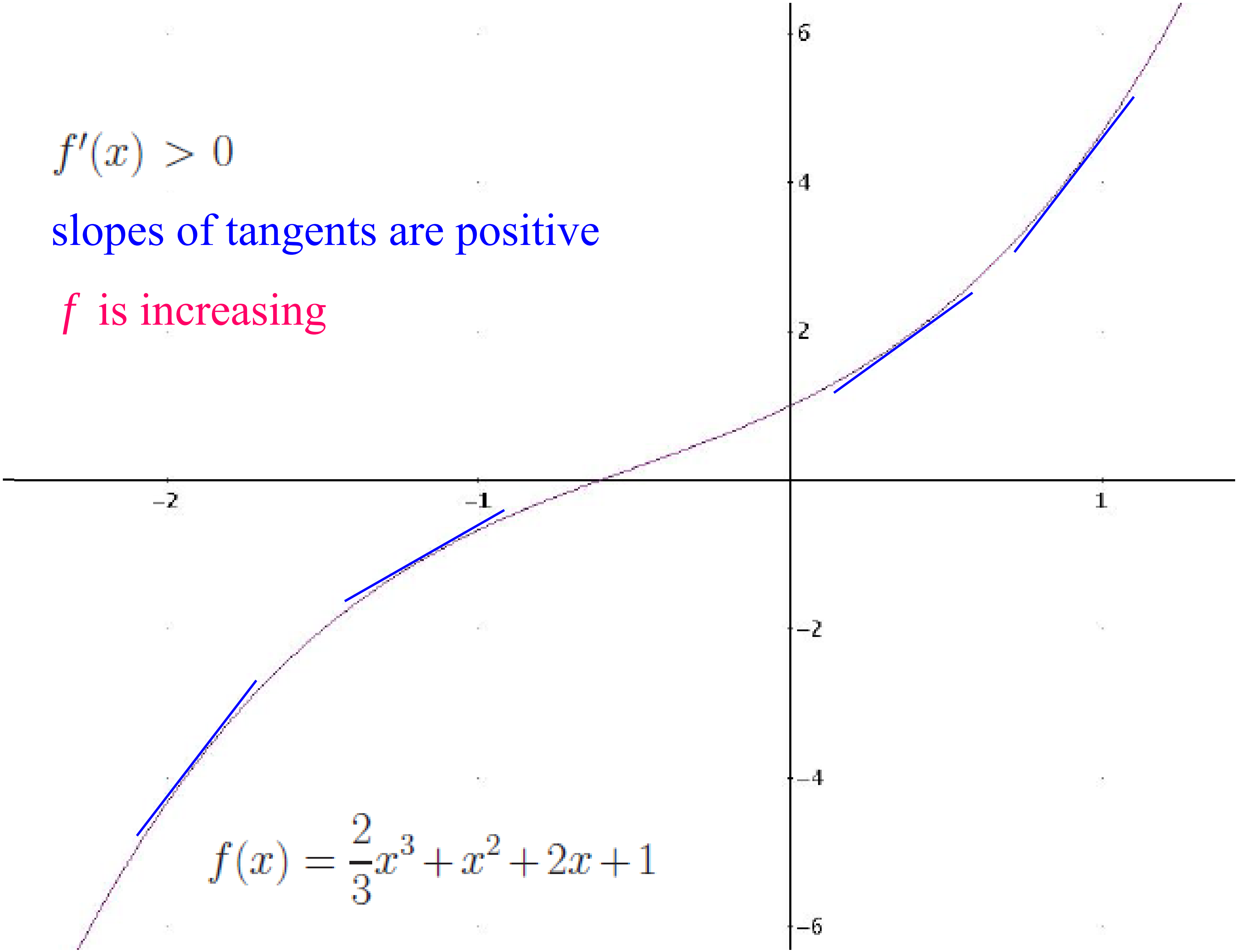


Example

Consider $f(x) = \frac{2}{3}x^3 + x^2 + 2x + 1$

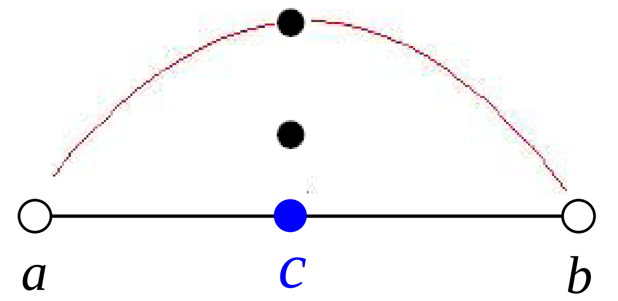
$$\begin{aligned}
\boxed{f'(x)} &= 2x^2 + 2x + 2 \\
&= 2\left[\boxed{x^2 + x + 1}\right] \\
&= 2\left[\boxed{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}\right] \quad \text{(complete the square)} \\
&\boxed{> 0 \text{ for all } x.}
\end{aligned}$$

f is increasing on any interval.



$f'(x) > 0$
slopes of tangents are positive
f is increasing
$f(x) = \frac{2}{3}x^3 + x^2 + 2x + 1$
6
4
2
−2
−4
−6
−2
−1
1

First Derivative Test



graph is in one piece

Let f be a continuous function on an interval (a, b).

$c \in (a, b)$ is a critical point of f

If $f'(x) > 0$ for $x \in (a, c)$, and $f'(x) < 0$ for $x \in (c, b)$, then $f(c)$ is a local maximum.

If $f'(x) < 0$ for $x \in (a, c)$, and $f'(x) > 0$ for $x \in (c, b)$, then $f(c)$ is a local minimum.

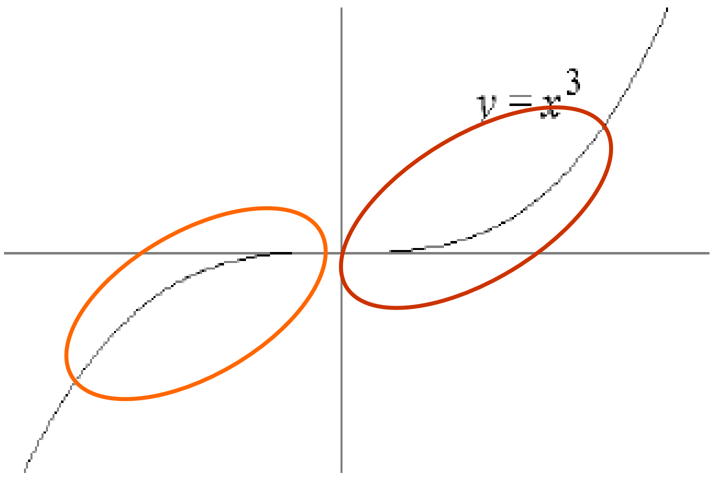
Concavity

Describes how a curve bends.

$f(x) = x^3$.

Curve bends down (concave down) as it approaches the origin from the left.

Curve bends up (concave up) as it moves from the origin into the first quadrant.



Graph is on page 18

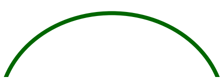
Concavity Test

Use second derivative

The graph of $y = f(x)$ is

(1) concave up on any interval where $\boxed{y'' > 0}$

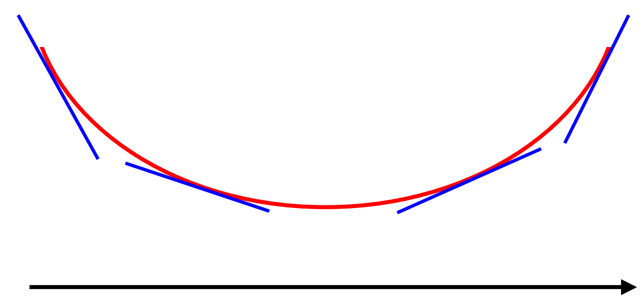
(2) concave down on any interval where $\boxed{y'' < 0}$



Let $s = y' =$ slope of tangent

$$\boxed{s'} = \boxed{y''}\boxed{> \ 0}$$

slope s is increasing



f increases on an interval I when $f'(x) > 0$

Example

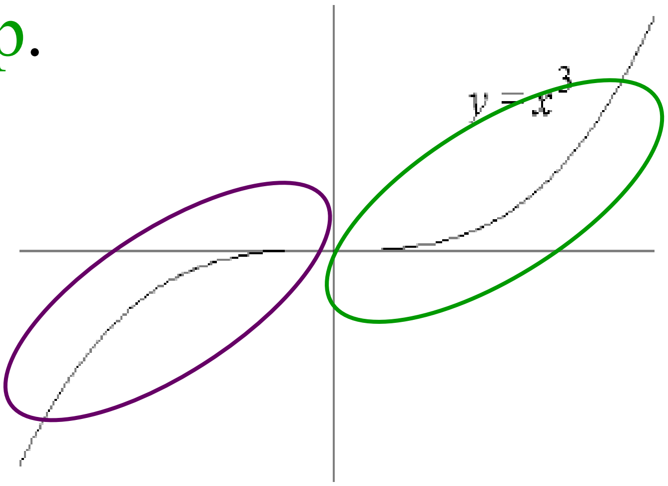
Let $y = x^3$. Then $y' = 3x^2$ and $\boxed{y'' = 6x.}$

If $x < 0$, then $y'' < 0$

and the cubic graph is concave down.

If $x > 0$, then $y'' > 0$

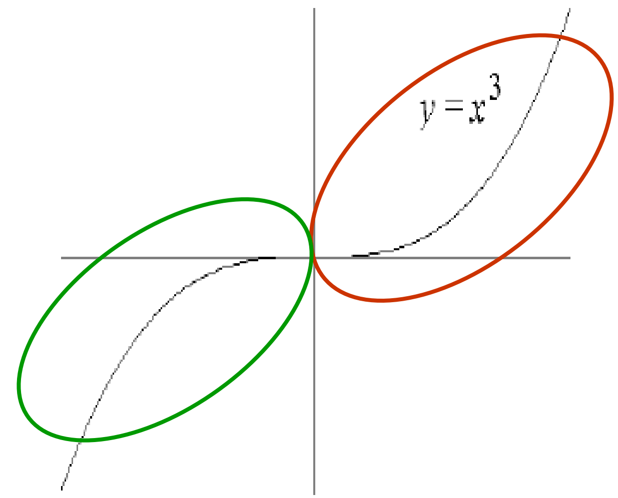
and the cubic graph is concave up.



Concavity and increasing/decreasing function are different concepts

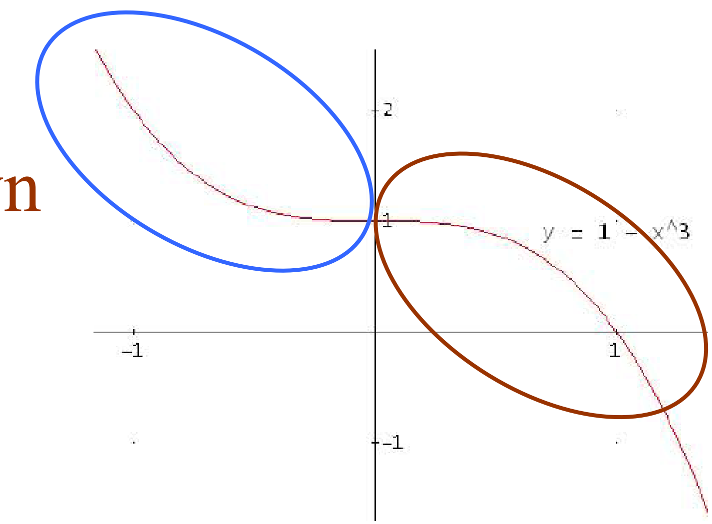
$y = x^3$

- $x > 0$: increasing, concave up
- $x < 0$: increasing, concave down



$y = 1 - x^3$

- $x > 0$: decreasing, concave down
- $x < 0$: decreasing, concave up



Points of Inflection

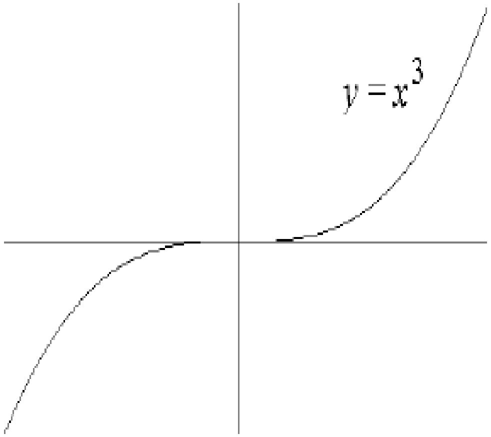
A point c is a point of inflection of the function f if

- f is continuous at c, and
- there is an open interval containing c such that the graph of f changes concavity at c.

Remarks

- f need not be differentiable at c.
- If f is differentiable at c, derivative need not be 0.

Examples



1. $y = x^3$ has a point of inflection at $x = 0$ because concavity changes at $x = 0$.

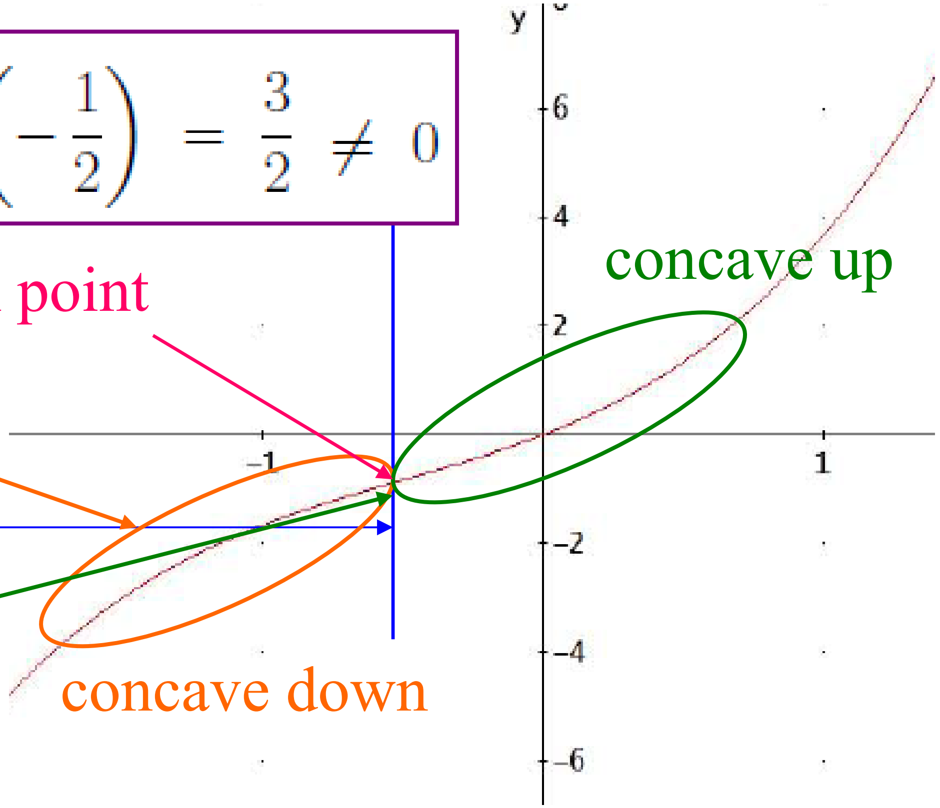
2. $$f(x) = \frac{2}{3}x^3 + x^2 + 2x$$

$$f'(x) = 2x^2 + 2x + 2$$

$$\boxed{f'\left(-\frac{1}{2}\right) = \frac{3}{2} \neq 0}$$

$$f''(x) = 4x + 2 \begin{cases} \boxed{< 0 \quad \text{if} \quad x < -\frac{1}{2}} \\ = 0 \quad \text{if} \quad x = -\frac{1}{2} \\ \boxed{> 0 \quad \text{if} \quad x > -\frac{1}{2}} \end{cases}$$

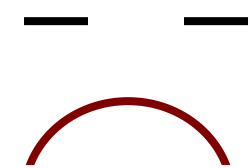
inflection point



Second Derivative Test

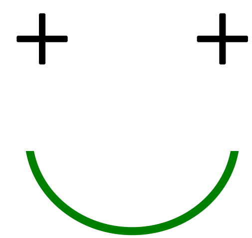
(1) If $f'(c) = 0$ and $f''(c) < 0$,

f has a local maximum at $x = c$.



(2) If $f'(c) = 0$ and $f''(c) > 0$,

f has a local minimum at $x = c$.



Example

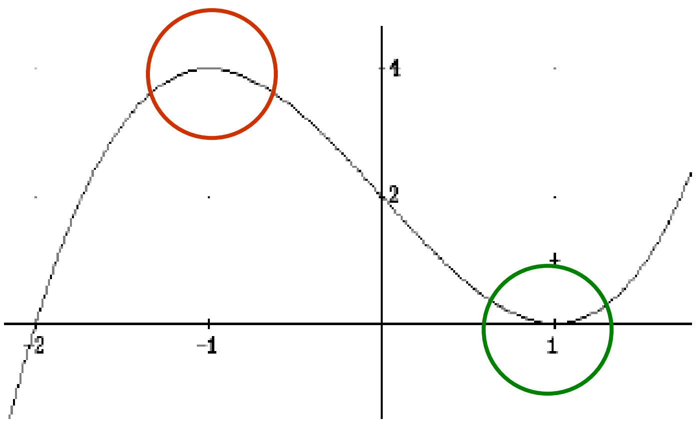
Find all local maxima and minima of

$\boxed{y = x^3 - 3x + 2}$ on the interval $(-\infty, \infty)$.

Local extrema can only occur where

$$\boxed{y' = 3x^2 - 3} = 0, \text{ i.e. at } \boxed{x = \pm 1.}$$

$$y'' = \boxed{6x} = \begin{cases} \boxed{< 0 \quad \text{if} \quad x = -1 \ \text{(local max.)}} \\ \boxed{> 0 \quad \text{if} \quad x = 1 \ \text{(local min.)}} \end{cases}$$



$y(-1) = 4$ is a local max value

$y(1) = 0$ is a local min value

Optimization Problems

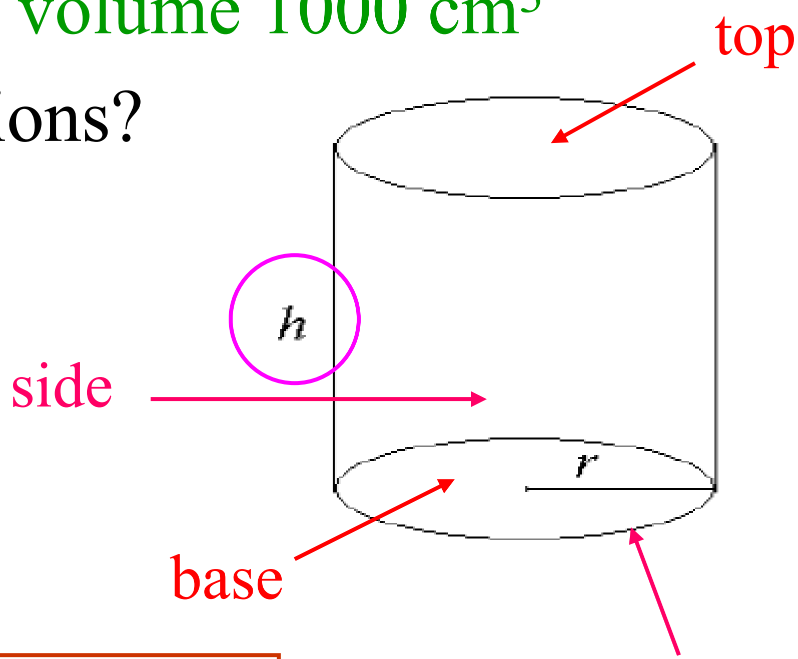
Optimization refers to finding the absolute maximum or minimum value of a function over a closed interval.

Procedure

1. Find all critical points.
2. Evaluate the function at its critical points and at the end points of its domain.
3. The largest and smallest of these values will be the absolute maximum and minimum values respectively.

Cylindrical Can

- Closed right cylindrical can, volume 1000 cm^3
- Least material used, dimensions?



Volume

$$V = \pi r^2 h = 1000$$

$$\boxed{h = \frac{1000}{\pi r^2}}$$

Surface area

$$A = 2\pi r^2 + 2\pi r h = \boxed{2\pi r^2 + \frac{2000}{r},} \quad r > 0$$

circumference $2\pi r$

$$A' = 4\pi r - \frac{2000}{r^2}$$

Cylindrical Can (cont'd)

$$\boxed{A' = 4\pi r - \frac{2000}{r^2}}$$

Setting $A' = 0$, get $\boxed{r = \left(\frac{500}{\pi}\right)^{\frac{1}{3}}}$

leads to minimum value of *A*.

$$A'' = 4\pi + \frac{4000}{r^3} > 0, \quad \text{for} \quad r > 0$$

$$h = \frac{1000}{\pi r^2}. \qquad \boxed{h = 2\left(\frac{500}{\pi}\right)^{\frac{1}{3}}}$$

Dimensions of can are $r \approx 5.42$ cm

$h \approx 10.84$ cm

Indeterminate Forms

Consider

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}$$

Limit cannot be evaluated by substitution $x = 0$ because this will lead to

$$\frac{0}{0}$$

which is undefined. This symbol is known as an indeterminate form.

$\frac{0}{0}$ refers to situations like the following:

$$\frac{10^{-8}}{10^{-3}} = 10^{-5}$$ (small value)

$$\frac{6 \times 10^{-8}}{2 \times 10^{-12}} = 30,000$$ (large value)

Cannot guess the final value of quotient, unless sizes of numerator and denominator are known.

indeterminate!

L'Hopital's Rule

Suppose that

(1) f and g are differentiable on an open interval (a, b) containing x_0;

(2) $f(x_0) = g(x_0) = 0$;

(3) $g'(x) \neq 0$ for every point x in (a, b) except possibly at x_0.

$$\boxed{\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}}$$

provided that the limit on the right exists.

Example

Consider

$$\lim_{x\to 0}\frac{\boxed{\sqrt{1+x}-1}}{\boxed{x}} \qquad \frac{0}{0}$$

L'H = L'Hopital's rule

$$\overset{\text{L'H}}{=} \boxed{\lim_{x\to 0}} \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}}{1}$$

differentiate numerator

differentiate denominator

$$= \boxed{\frac{1}{2}}$$

substitute $x = 0$

Example

Consider $\displaystyle\lim_{x\to 0}\frac{\boxed{x-\sin x}}{\boxed{x^3}}$ $\qquad \dfrac{0}{0}$

$\overset{\text{L'H}}{=} \displaystyle\lim_{x\to 0}\frac{\boxed{1-\cos x}}{\boxed{3x^2}}$ $\qquad \dfrac{0}{0}$

$\overset{\text{L'H}}{=} \displaystyle\lim_{x\to 0}\frac{\sin x}{6x}$ $\qquad \dfrac{0}{0}$

$\overset{\text{L'H}}{=} \displaystyle\boxed{\lim_{x\to 0}}\frac{\boxed{\cos x}}{6}$ $\qquad \cos 0 = 1$

$= \boxed{\dfrac{1}{6}}$

Other Indeterminate Forms

Theorem. If $f(x)$ and $g(x)$ both approach ∞ as $x \longrightarrow a$, and $f(x)$ and $g(x)$ are differentiable, then

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

provided that the limit on the right exists. Here the value a may be finite or infinite.

∞

$-\infty$

Remark. For all the other indeterminate forms (e.g. $\infty \cdot 0$, $\infty - \infty$, 0^0), one needs to change them to either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form and then apply L'Hopital's rule.

Example

(of form $\frac{\infty}{\infty}$)

Consider $\displaystyle\lim_{x\to\infty} \frac{x - 2x^2}{3x^2 + 5}$ $\qquad \frac{-\infty}{\infty}$

L'H

$$= \lim_{x\to\infty} \frac{1 - 4x}{6x} \qquad \frac{-\infty}{\infty}$$

L'H

$$= \lim_{x\to\infty} \frac{-4}{6}$$

$$= \boxed{-\frac{2}{3}}$$

Example

(of form $0 \cdot \infty$)

Consider $\lim_{x \to 0^+} x \cot x$ $\qquad 0 \cdot \infty$

$$= \lim_{x \to 0^+} \frac{x}{\tan x} \qquad \frac{0}{0}$$

L'H

$$= \boxed{\lim_{x \to 0^+}} \frac{1}{\sec^2 x}$$

$\cos 0 = 1$

$\sec 0 = 1$

$$= \boxed{1}$$

Example

$\left(\text{of form } 0^0\right)$

Consider $\lim\limits_{x \to 0^+} x^x$ $\quad 0^0$

Let $y = x^x.$

$$\ln y = x \ln x = \frac{\boxed{\ln x}}{\boxed{\frac{1}{x}}} \qquad \frac{-\infty}{\infty}$$

L'H

$$\lim_{x \to 0^+} \boxed{\ln y} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} -x = \boxed{0}$$

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} y = e^0 = \boxed{1}$$

End of Chapter 2