

MA 1505 Mathematics I
Tutorial 8 Solutions

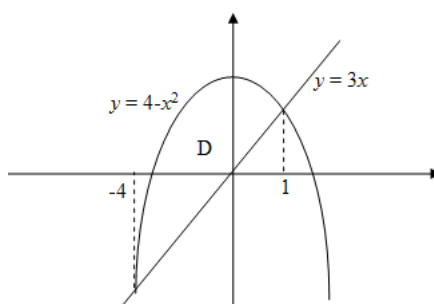
1. The volume is given by the double integral

$$V = \iint_D f(x, y) dA$$

where D is the region bounded by the parabola $y = 4 - x^2$ and straight line $y = 3x$ and $f(x, y)$ is the function whose graph is the plane $x - z + 4 = 0$.

Writing the equation of the plane as $z = x + 4$, we get the function $f(x, y) = x + 4$.

A rough sketch of the region D is shown below:



D can be regarded as type A region

$$D : \quad 3x \leq y \leq 4 - x^2, \quad -4 \leq x \leq 1.$$

(The two limits -4 and 1 of x are obtained by solving the two equation $y = 3x$ and $y = 4 - x^2$.)

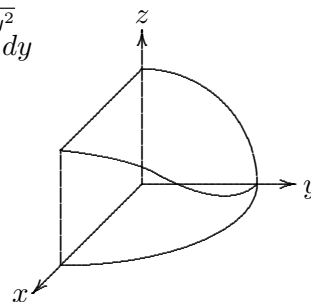
Hence

$$V = \int_{-4}^1 \int_{3x}^{4-x^2} (x+4) dy dx = \int_{-4}^1 (x+4)(4-x^2-3x) dx = \left[16x - 4x^2 - \frac{7}{3}x^3 - \frac{1}{4}x^4 \right]_{-4}^1 = \frac{625}{12}$$

2. By symmetry, the desired volume V is 8 times the volume V_1 in the first octant. Now,

$$\begin{aligned} V_1 &= \int_0^r \int_0^{\sqrt{r^2-y^2}} \sqrt{r^2-y^2} dx dy = \int_0^r \left[x\sqrt{r^2-y^2} \right]_{x=0}^{x=\sqrt{r^2-y^2}} dy \\ &= \int_0^r (r^2-y^2) dy = \left[r^2y - \frac{1}{3}y^3 \right]_0^r = \frac{2}{3}r^3. \end{aligned}$$

Therefore, $V = \frac{16}{3}r^3$.



3. Let $z = \sqrt{2^2 - x^2 - y^2}$. Then
 $z_x = -x(4 - x^2 - y^2)^{-1/2}$ and $z_y = -y(4 - x^2 - y^2)^{-1/2}$.

Substitute $z = 1$ into $x^2 + y^2 + z^2 = 4$ gives

$$x^2 + y^2 + 1 = 4 \Rightarrow x^2 + y^2 = 3$$

which is the equation of a circle of radius $\sqrt{3}$.

This means the plane $z = 1$ intersects the sphere at a circle of radius $\sqrt{3}$.

Hence the projected region R of the part of the sphere is a disk of radius $\sqrt{3}$.

In polar coordinates, this is given by

$$0 \leq r \leq \sqrt{3}, \quad 0 \leq \theta \leq 2\pi.$$

Thus,

$$\begin{aligned} A(S) &= \iint_R \sqrt{\frac{x^2 + y^2}{4 - x^2 - y^2} + 1} dA = \int_0^{2\pi} \int_0^{\sqrt{3}} \left(\frac{r^2}{4 - r^2} + 1 \right)^{\frac{1}{2}} r dr d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} 2r(4 - r^2)^{-\frac{1}{2}} dr d\theta = \int_0^{2\pi} d\theta \left[-2(4 - r^2)^{\frac{1}{2}} \right]_{r=0}^{r=\sqrt{3}} \\ &= (2\pi)[-2[(4 - 3)^{\frac{1}{2}} + 2[(4)^{\frac{1}{2}}] = 4\pi. \end{aligned}$$

4. $\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$, $\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$. Therefore

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} + 1} = \sqrt{2}.$$

Note that D is given as a Type A domain. The surface area is then given by

$$\iint_D \sqrt{2} dx dy = \int_{-1}^2 \left(\int_{x^2}^{x+2} \sqrt{2} dy \right) dx = \frac{9}{2} \sqrt{2}.$$

5. D is given by

$$-\frac{1}{2} \leq x \leq \frac{1}{2}, \quad -\frac{1}{2} \leq y \leq \frac{1}{2}, \quad -\frac{1}{2} \leq z \leq \frac{1}{2}$$

So

$$\begin{aligned} \iiint_D (x^2 + 2z) dV &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} (x^2 + 2z) dx \, dz \, dy \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{1}{3} x^3 + 2xz \right]_{-\frac{1}{2}}^{\frac{1}{2}} dz \, dy \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{12} + 2z \right) dz \, dy \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{z}{12} + z^2 \right]_{-\frac{1}{2}}^{\frac{1}{2}} dy \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{12} dy = \frac{1}{12} \end{aligned}$$

6. Write the equation of the saddle surface as $z = \frac{1}{a}x^2 - \frac{1}{a}y^2$, we have

$$z_x = \frac{2x}{a} \text{ and } z_y = \frac{-2y}{a}.$$

Let D denote the bounded circular region on the xy -plane bounded by the circle $x^2 + y^2 = a^2$.

Then the required surface area is given by

$$\begin{aligned} S &= \iint_D \sqrt{1 + z_x^2 + z_y^2} dx dy \\ &= \frac{1}{a} \iint_D \sqrt{a^2 + 4x^2 + 4y^2} dx dy \\ &= \frac{1}{a} \int_0^{2\pi} \int_0^a \sqrt{a^2 + 4r^2} r dr d\theta \\ &= \frac{2\pi}{a} \int_0^a \sqrt{a^2 + 4r^2} d\left(\frac{a^2 + 4r^2}{8}\right) \\ &= \frac{\pi}{6a} \left[(a^2 + 4r^2)^{\frac{3}{2}} \right]_0^a \\ &= \frac{\pi a^2}{6} \left(5^{\frac{3}{2}} - 1 \right) \end{aligned}$$