Chapter 4

Sequences and Series

Key Results

- Introductory study of series of real numbers
- Convergence of power series
- Taylor series that give definition of functions as power series
- Approximation of function values using polynomials
- Estimating errors in approximations

Infinite Series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is called an infinite series.

The term a_n is called the nth term of the series.

For example,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$

is an infinite series whose $\boxed{n\text{th term is } \frac{1}{2^n}}$

Partial Sums

Consider the sums

$$s_n = a_1 + a_2 + a_3 \cdots + a_n = \sum_{k=1}^{n} a_k$$

$$\boxed{s_1, s_2, s_3, \ldots, s_n, \ldots}$$

is called the sequence of partial sums of the series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

The number s_n is called the *n*th partial sum.

Convergence and Divergence

If the sequence of partial sums $\{s_n\}$ converges to a limit L, then the series is said to be convergent and that its sum is L.

Write

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots = L.$$

If the sequence of partial sum does not converge, then the series is said to be divergent.

Geometric Series

Fix real numbers a and r. The series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \boxed{\sum_{n=1}^{\infty} ar^{n-1}}$$

is a geometric series

First term is a.

Common ratio is r.

Sum Formulas

Consider the *n*th partial sum

$$\begin{aligned} s_n &= \boxed{a} + \boxed{ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}} \\ rs_n &= \phantom{a + {}} \boxed{ar + ar^2 + ar^3 + \cdots + ar^{n-1}} + \boxed{ar^n} \\ s_n - rs_n &= \boxed{a - ar^n} \end{aligned}$$

$$\boxed{s_n = a\frac{1-r^n}{1-r} \qquad r \neq 1}$$

If $a = 0$, the series is a sum of zeros, giving sum 0.
Therefore, let a be nonzero.

If $r = 1$, then $s_n = na \to \infty \ (\text{or } -\infty)$ series is divergent

Consider $\boxed{s_n = a\frac{1-r^n}{1-r},}$ $\quad r \neq 1$

If $\boxed{|r| < 1,}$ then $r^n \to 0$

$$s_n \to \frac{a}{1-r}$$

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \boxed{\frac{a}{1-r}}$$

If $\boxed{|r| > 1,}$ then $|r|^n \to \infty$

$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ diverges

Example

Consider the series

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$$

First term $a = \frac{1}{9}$ Common ratio $\boxed{r = \frac{1}{3} < 1}$

This is a geometric series.

Series converges to

$$\frac{a}{1-r} = \frac{\frac{1}{9}}{1-\frac{1}{3}} = \boxed{\frac{1}{6}}$$

Example

Consider the series

$$4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} + \cdots$$

First term $a = 4$ Common ratio $r = -\frac{1}{2}$

$$\boxed{|r| = \frac{1}{2} < 1}$$

This is a geometric series.

Series converges to

$$\frac{a}{1-r} = \frac{4}{1-\left(-\frac{1}{2}\right)} = \boxed{\frac{8}{3}}$$

Some Rules

If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then

(1) Sum rule: $\sum_{n=1}^{\infty}(a_n + b_n) = A + B$

(2) Difference rule: $\sum_{n=1}^{\infty}(a_n - b_n) = A - B$

(3) Constant multiple rule: : $\sum_{n=1}^{\infty}(ka_n) = kA$

Example

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = 0 + \boxed{\frac{1}{3}} \xrightarrow{\times\frac{2}{3}} \boxed{\frac{2}{9}} \xrightarrow{\times\frac{13}{24}} \boxed{\frac{13}{108}} + \cdots$

This is not a geometric series,

but a difference of two geometric series:

$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \boxed{\sum_{n=1}^{\infty} \boxed{\frac{1}{2^{n-1}}} - \sum_{n=1}^{\infty} \boxed{\frac{1}{6^{n-1}}}}$$

$$a = 1$$

$$|r| < 1$$

$$r = \frac{1}{2} \qquad r = \frac{1}{6}$$

$$= \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{6}} = \boxed{\frac{4}{5}} \qquad \frac{a}{1-r}$$

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms.

Suppose $$\boxed{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho}$$

Then

Greek letter rho ρ

(1) the series converges if $\rho < 1$

(2) the series diverges if $\rho > 1$

(3) there is no conclusion if $\rho = 1$

Case 2 includes the situation $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = +\infty$$

Example

Consider the series

$$\boxed{\sum_{n=1}^{\infty} a_n = 1 + \frac{1}{3} + \frac{2\cdot 1}{5\cdot 3} + \frac{3\cdot 2\cdot 1}{7\cdot 5\cdot 3} + \cdots}$$

$$a_1 = 1 \qquad \boxed{a_{n+1} = \frac{n}{2n+1}a_n}$$

$$n = 1: \qquad a_2 = \frac{1}{2\cdot 1+1}a_1 = \frac{1}{3}$$

$$n = 2: \qquad a_3 = \frac{2}{2\cdot 2+1}a_2 = \frac{2}{5}\cdot\frac{1}{3}$$

$$n = 3: \qquad a_4 = \frac{3}{2\cdot 3+1}a_3 = \frac{3}{7}\cdot\frac{2}{5}\cdot\frac{1}{3}$$

Example

Consider the series

$$\sum_{n=1}^{\infty} a_n = 1 + \frac{1}{3} + \frac{2\cdot 1}{5\cdot 3} + \frac{3\cdot 2\cdot 1}{7\cdot 5\cdot 3} + \cdots$$

$$a_1 = 1 \qquad \boxed{a_{n+1} = \frac{n}{2n+1} a_n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}} \longrightarrow \frac{1}{2} = \boxed{\rho < 1}$$

$$\text{as } n \longrightarrow \infty$$

By ratio test, the given series converges.

Why Ratio Test Works

Consider the previous example

$$\boxed{\left|\frac{a_{n+1}}{a_n}\right|} = \boxed{\frac{n}{2n+1}}\boxed{<}\boxed{\frac{n}{2n}} = \boxed{\frac{1}{2}}$$

$$\boxed{a_{n+1} < \frac{1}{2}a_n}$$

set $n = 1$

$a_1 = 1$
(given)

$$a_2 < \frac{1}{2}a_1 = \frac{1}{2}$$

$$a_3 < \frac{1}{2}a_2 < \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2^2}$$

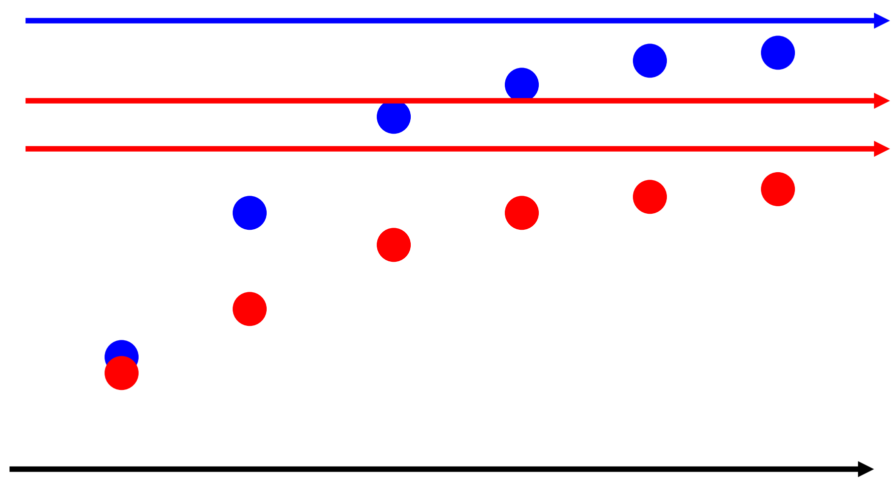
$$a_4 < \frac{1}{2}a_3 < \frac{1}{2}\cdot\frac{1}{2^2} = \frac{1}{2^3}$$

Thus,

$$\boxed{a_1 + a_2 + a_3 + a_4 + \cdots}$$

$$< \boxed{1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots}$$

$$= \frac{1}{1 - \frac{1}{2}} = 2$$



geometric series

$a = 1 \quad r = \frac{1}{2} \quad \frac{a}{1-r}$

$\sum_{n=1}^{\infty} a_n$ converges but may not converge to 2.

Example

Consider $\displaystyle\sum_{n=1}^{\infty} \boxed{\frac{(n!)^2}{(2n)!}}$ *converges by ratio test*

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\boxed{(n+1)!\,(n+1)!}}{\boxed{(2n+2)!}} \cdot \frac{\boxed{(2n)!}}{\boxed{n!\,n!}} = \frac{\boxed{(n+1)(n+1)}}{\boxed{(2n+2)(2n+1)}}$$

$$= \frac{n+1}{2(2n+1)} = \frac{1+\frac{1}{n}}{2\left(2+\frac{1}{n}\right)} \longrightarrow \frac{1}{4} = \boxed{\rho < 1}$$

as $n \longrightarrow \infty$

$$[\,2(n+1)\,]! = (2n+2)!$$

$$\boxed{(2n+2)! = \boxed{(2n+2)\,(2n+1)}\,\underbrace{(2n)\,(2n-1)\,\cdots 3\cdot 2\cdot 1}_{\boxed{(2n)!}}}$$

Example

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n+5}$ *diverges by ratio test*

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{2^{n+1}+5} \cdot \frac{2^n+5}{3^n}$$

$$= \ 3 \cdot \frac{1+\dfrac{5}{2^n}}{2+\dfrac{5}{2^n}} \rightarrow \frac{3}{2} = \boxed{\rho \ > \ 1}$$

as $n \rightarrow \infty$

Power Series

A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

where x is a variable and c_0, c_1, c_2, … are constants.

Thus, if a power series converges, it can be regarded as a function of x.

Generalizing, a power series about $x = a$ is of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

The number a is called the centre of the series.

Example

Consider $\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$

This is a power series about $x = 0$.

It is (also) a geometric series with first term $a = 1$ and common ratio $r = x$.

Series converges to $\dfrac{1}{1-x}$ when $|x| < 1$ $\qquad \dfrac{a}{1-r}$

Stated as $\boxed{\dfrac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots}$

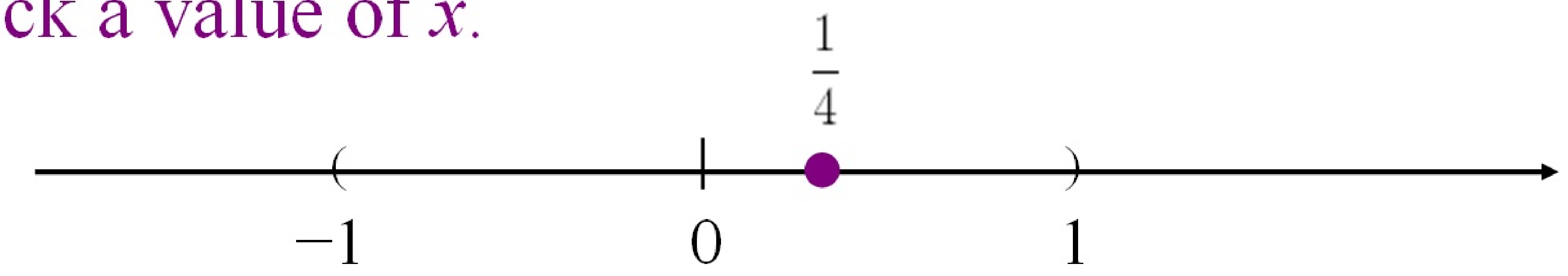
$\boxed{\text{for } -1 < x < 1}$

Consider

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots \quad = \boxed{\frac{1}{1-x}}$$

$$\text{for } \boxed{-1 < x < 1}$$

Pick a value of *x*.



$$x = \frac{1}{4} \qquad \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

converges

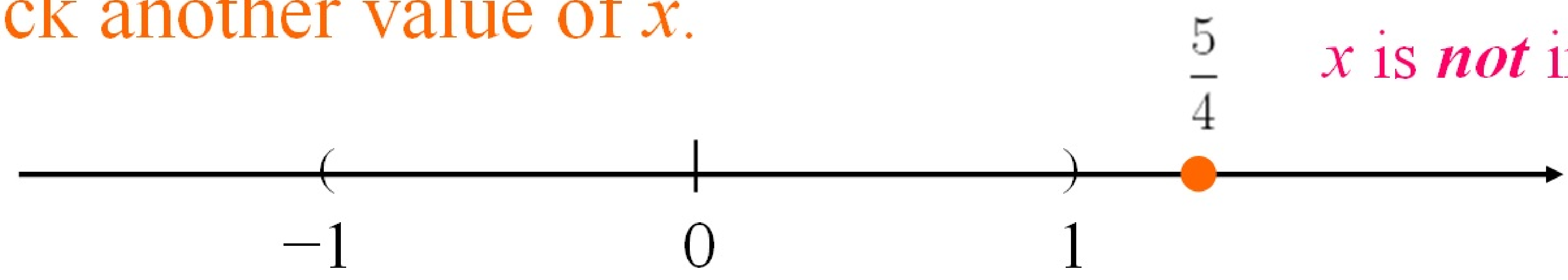
Consider

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots \; = \frac{1}{1-x}$$

$$\text{for } \boxed{-1 < x < 1}$$

Pick another value of *x*.

x is ***not*** in interval.



$$x = \frac{1}{4} \qquad \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

converges

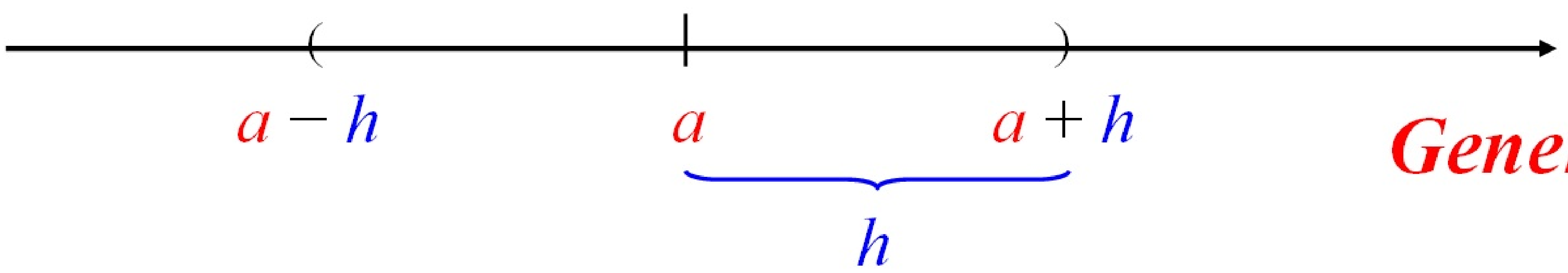
$$x = \frac{5}{4} \qquad \sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n \to \infty$$

diverges

Consider

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots \ = \ \frac{1}{1-x}$$

for $-1 < x < 1$



Generalize!

$x = \frac{1}{4}$ $\qquad \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \ = \ \frac{1}{1 - \frac{1}{4}} \ = \ \frac{4}{3}$ $\qquad$ converges

$x = \frac{5}{4}$ $\qquad \sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n \ \to \infty$ $\qquad$ diverges

For most power series $\displaystyle\sum_{n=0}^{\infty} c_n (x-a)^n$,

there is a "radius of convergence" $h > 0$ such that (Case 2)

$$\sum_{n=0}^{\infty} c_n (x-a)^n \begin{cases} \text{converges} & \text{if } a-h < x < a+h \\ \text{diverges} & \text{if } x < a-h \text{ or } x > a+h \end{cases}$$

For "end-points" $x = a - h$ and $x = a + h$, separate calculations are required to determine convergence.

Special cases

- $h = 0$ (Case 1): "interval" is only a single point!
- h is infinite (Case 3): interval is the whole real line, i.e. power series converges for all values of x.

Example

Consider $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - + \cdots$

Write $\boxed{u_n = (-1)^{n-1}\frac{x^n}{n}}$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|(-1)^{(n+1)-1}\frac{x^{n+1}}{n+1}\cdot\frac{n}{(-1)^{n-1}x^n}\right|$$

$$= \frac{n}{n+1}|x| = \frac{1}{1+\frac{1}{n}}|x| \xrightarrow[\text{as } n \to \infty]{} \boxed{|x| = \rho}$$

By ratio test, given series converges if $|x| < 1$ and diverges if $|x| > 1$.

Radius of convergence is 1.

Example

Consider $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to \boxed{0 = \rho < 1}$$

as $n \to \infty$ for any x

By ratio test, given series converges for all values of x,

i.e. the interval of convergence is $(-\infty, \infty)$

The radius of convergence is infinite.

Example

Consider $\sum_{n=0}^{\infty} n!x^n = \boxed{1} + \boxed{x + 2!x^2 + 3!x^3 + \cdots}$

$= 0$ if $x = 0$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right|$$

$$= \boxed{(n+1)|x|} \to \boxed{\infty = \rho > 1} \quad \text{if } x \neq 0$$

as $n \to \infty$

$$\text{series} \begin{cases} \text{converges if } x = 0 \\ \text{diverges if } x \neq 0 \end{cases}$$

interval is a single point

Radius of convergence is 0.

Differentiation of Power Series

Suppose radius of convergence $h > 0$.

The power series defines a function f. Write

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n} \qquad a - h < x < a + h$$

For these values of x, f has derivatives of all orders obtained by term-by-term differentiation:

$$\boxed{f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}$$

Example

Consider the (geometric) power series

$$f(x) = \boxed{\frac{1}{1-x}} = \boxed{1+}\boxed{x+}\boxed{x^2+}\boxed{x^3+}\cdots+\boxed{x^n+}\cdots$$

$$\text{for } -1 < x < 1$$

$$f'(x) = \boxed{\frac{1}{(1-x)^2}} = 0+\boxed{1+2x+3x^2+\cdots+nx^{n-1}+\cdots}$$

$$= \boxed{1+2x+3x^2+\cdots+nx^{n-1}+\cdots}$$

$$f''(x) = \frac{2}{(1-x)^3} = 0+\boxed{2+6x+\cdots+n(n-1)x^{n-2}+\cdots}$$

$$= 2+6x+\cdots+n(n-1)x^{n-2}+\cdots$$

Integration of Power Series

Suppose radius of convergence $h > 0$.

The power series defines a function f. Write

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n} \qquad a - h < x < a + h$$

For these values of x, f has antiderivatives obtained by term-by-term integration:

$$\boxed{\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} + c}$$

Example

Recall the (geometric) power series

$$\frac{1}{1+t} = \boxed{1 - t + t^2 - + \cdots + (-1)^n t^n + \cdots}$$

for $-1 < t < 1$

common ratio $r = -t$

$$\frac{a}{1-r}$$

$$\boxed{\ln(1+x)} = \int_0^x \boxed{\frac{1}{1+t}} dt$$

$$= \int_0^x \left(\boxed{1 -}\boxed{t +}\boxed{t^2 -} + \cdots + \boxed{(-1)^n t^n + \cdots}\right) dt$$

$$= \left[t - \frac{t^2}{2} + \frac{t^3}{3} - + \cdots + (-1)^n \frac{t^{n+1}}{n+1} + \cdots \right]_{t=0}^{t=x}$$

$$\boxed{= x - \frac{x^2}{2} + \frac{x^3}{3} - + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots}$$

Taylor Series

Let f be a function with derivatives of all orders over some interval containing a as an interior point.

The Taylor series of f at a ($x = a$) is

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \\
= \quad & f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 \\
& + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots
\end{aligned}$$

Example

At $x = 0$, find the Taylor series of $f(x) = e^x$

$$f(0) = 1$$

$$f^{(n)}(x) = e^x \qquad \boxed{f^{(n)}(0) = 1}$$

Therefore, Taylor series is

$$\boxed{e^x \;=} \sum_{n=0}^{\infty} \frac{\boxed{f^{(n)}(0)}}{n!} x^n = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

Example

At $x = 0$, find the Taylor series of $f(x) = \cos x$

$f(x) = \cos x$	$f(0) = 1$	even integers $2n$
$f'(x) = -\sin x$	$f'(0) = 0$	odd integers $2n+1$
$f^{(2)}(x) = -\cos x$	$f^{(2)}(0) = -1$	$f^{(2n)}(0) = (-1)^n$
$f^{(3)}(x) = \sin x$	$f^{(3)}(0) = 0$	$f^{(2n+1)}(0) = 0$
$f^{(4)}(x) = \cos x$	$f^{(4)}(0) = 1$	
$\cdots$	$\cdots$	

$$\cos x \ =$$

$$f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2$$

$a = 0$

$$+ \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$f(x) = \cos x$ $\quad f(0) = 1$ $\quad$ even integers $2n$

$f'(x) = -\sin x$ $\quad f'(0) = 0$ $\quad$ odd integers $2n+1$

$f^{(2)}(x) = -\cos x$ $\quad f^{(2)}(0) = -1$ $\quad f^{(2n)}(0) = (-1)^n$

$f^{(3)}(x) = \sin x$ $\quad f^{(3)}(0) = 0$ $\quad f^{(2n+1)}(0) = 0$

$f^{(4)}(x) = \cos x$ $\quad f^{(4)}(0) = 1$

$\cdots \quad \cdots$

$$\cos x = 1 + 0 \cdot x - 1 \cdot \frac{x^2}{2!} + 0 \cdot x^3 + 1 \cdot \frac{x^4}{4!} + \cdots$$

Example

At $x = 0$, find the Taylor series of $f(x) = \cos x$

$$f(x) = \cos x \qquad f(0) = 1 \qquad \text{even integers } 2n$$

$$f'(x) = -\sin x \qquad f'(0) = 0 \qquad \text{odd integers } 2n+1$$

$$f^{(2)}(x) = -\cos x \qquad f^{(2)}(0) = -1 \qquad f^{(2n)}(0) = (-1)^n$$

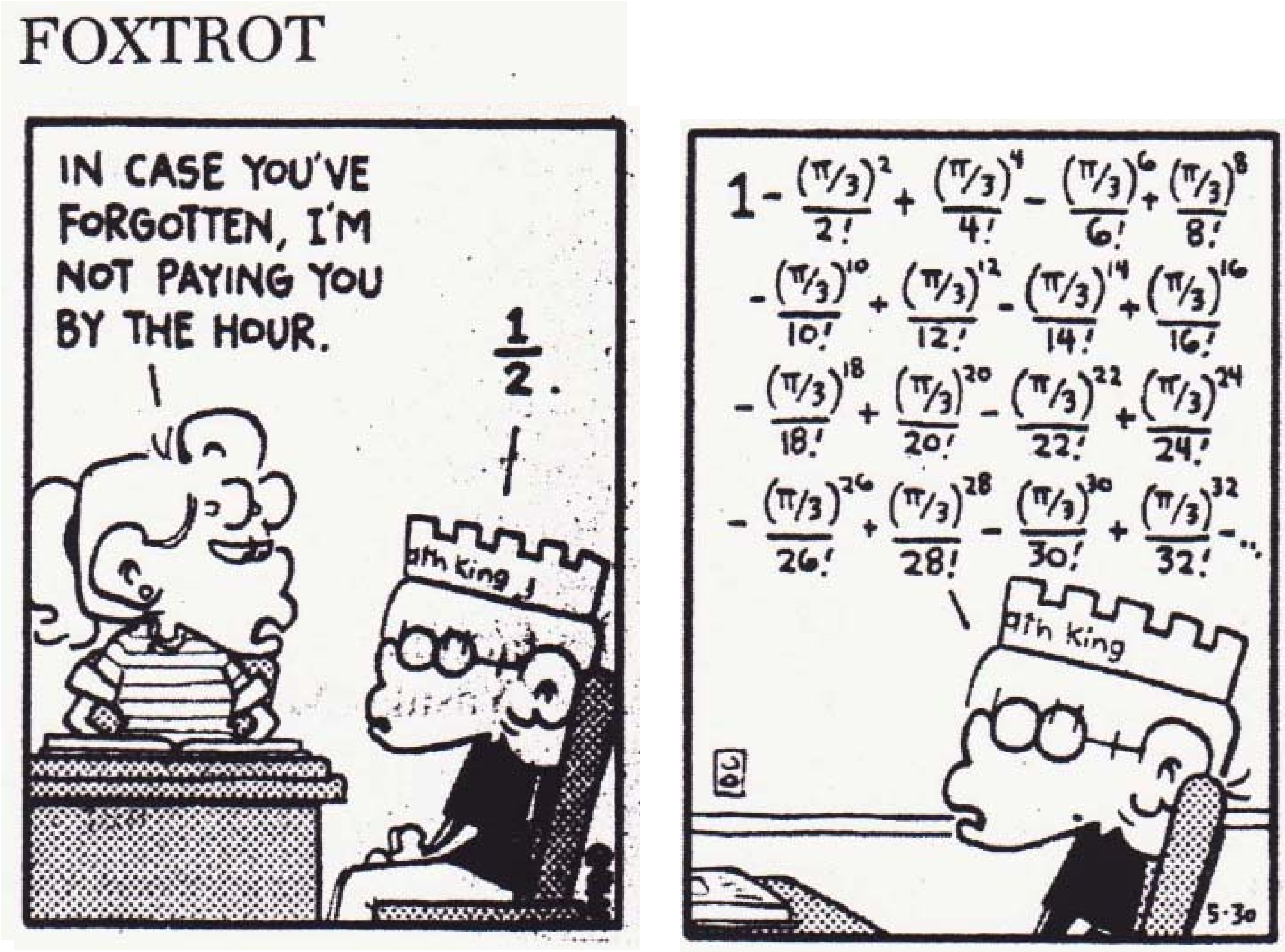
$$f^{(3)}(x) = \sin x \qquad f^{(3)}(0) = 0 \qquad f^{(2n+1)}(0) = 0$$

$$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = 1$$

$$\cdots \quad \cdots$$

$$\cos x = 1 + 0 \cdot x - 1 \cdot \frac{x^2}{2!} + 0 \cdot x^3 + 1 \cdot \frac{x^4}{4!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$



$$\cos x \;=\; 1-\frac{x^2}{2!}+\frac{x^4}{4!}-+\;\cdots$$

Short List of Taylor Series

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

for all values of x

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

for $-1 < x < 1$

Algebraic Method

Consider the (geometric) Taylor series

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \quad \text{for } |t| < 1$$

If $t = x - 2$, then

$$\frac{1}{3-x} = \frac{1}{1-(x-2)} = \sum_{n=0}^{\infty} (x-2)^n$$

Thus, the Taylor series of $\frac{1}{3-x}$ at $x = 2$ is $\sum_{n=0}^{\infty} (x-2)^n$

Can write

$$\boxed{\frac{1}{3-x} = \sum_{n=0}^{\infty} (x-2)^n \quad \text{for } |x-2| < 1}$$

Example

Find the Taylor series of $\dfrac{1}{2x+1}$ at $x=-2$

$$\frac{1}{2x+1} = \frac{1}{2(x+2)-4+1} = \frac{1}{-3+2(x+2)}$$

$$= -\frac{1}{3}\left(\frac{1}{1-\left[\frac{2}{3}(x+2)\right]}\right)$$

$$t = \frac{2}{3}(x+2)$$

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \quad \text{for } |t|<1$$

$$\sum_{n=0}^{\infty} c_n (x+2)^n$$

Example

Find the Taylor series of $\dfrac{1}{2x+1}$ at $x=-2$

$$\frac{1}{2x+1} = \frac{1}{2(x+2)-4+1} = \frac{1}{-3+2(x+2)}$$

$$= -\frac{1}{3}\left(\frac{1}{1-\left[\frac{2}{3}(x+2)\right]}\right)$$

$$= -\frac{1}{3}\sum_{n=0}^{\infty}\left[\frac{2}{3}(x+2)\right]^n \qquad t=\frac{2}{3}(x+2)$$

$$= \boxed{\sum_{n=0}^{\infty}\left(-\frac{2^n}{3^{n+1}}\right)(x+2)^n} \qquad \sum_{n=0}^{\infty} c_n(x+2)^n$$

Brook Taylor

(1685 – 1731)

Taylor Polynomials

Let f be a function with derivatives of all orders up to order n over some interval containing a as an interior point.

The nth order Taylor polynomial of f at a ($x = a$) is

$$\boxed{\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 \\ &\quad + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}}$$

provides the best polynomial approximation of degree n.

Example

The first few Taylor polynomials of

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

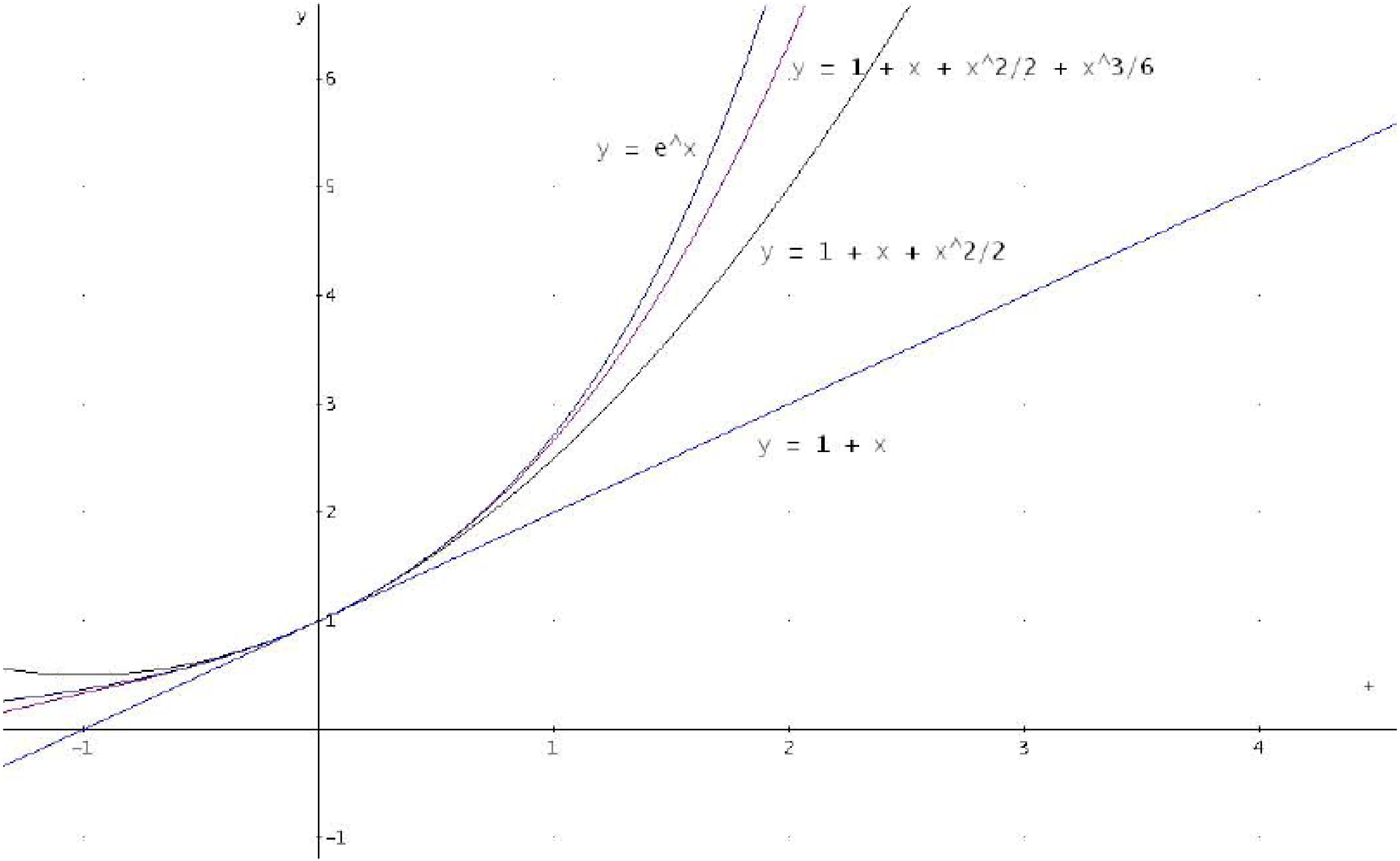
at $x = 0$ are:

$$P_1(x) = 1 + x$$

$$P_2(x) = 1 + x + \frac{x^2}{2}$$

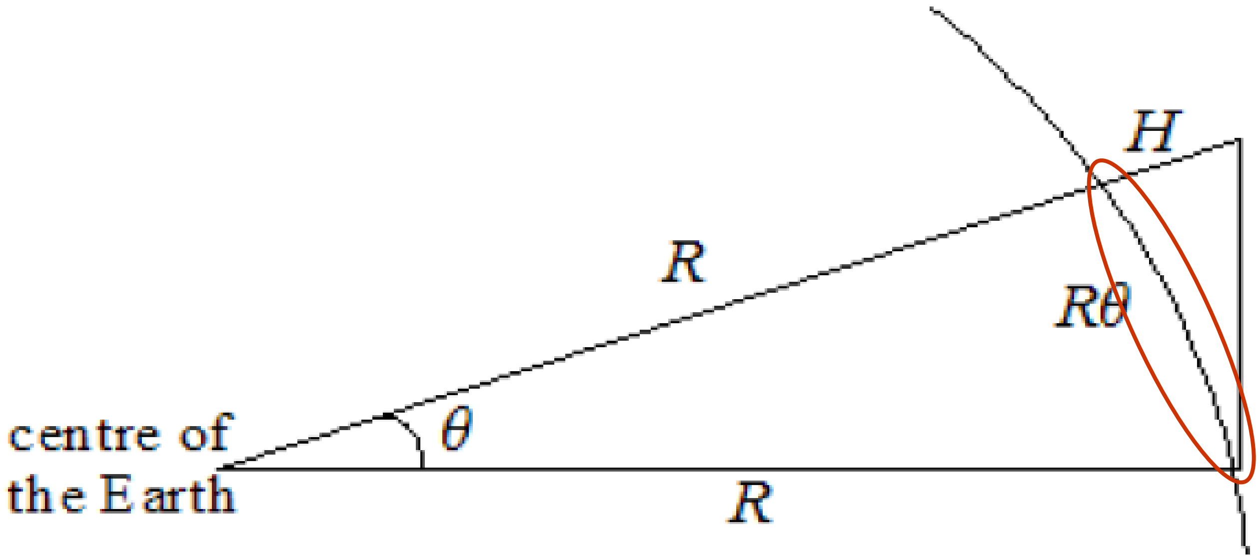
$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Taylor Polynomials



How Far Away is the Horizon?

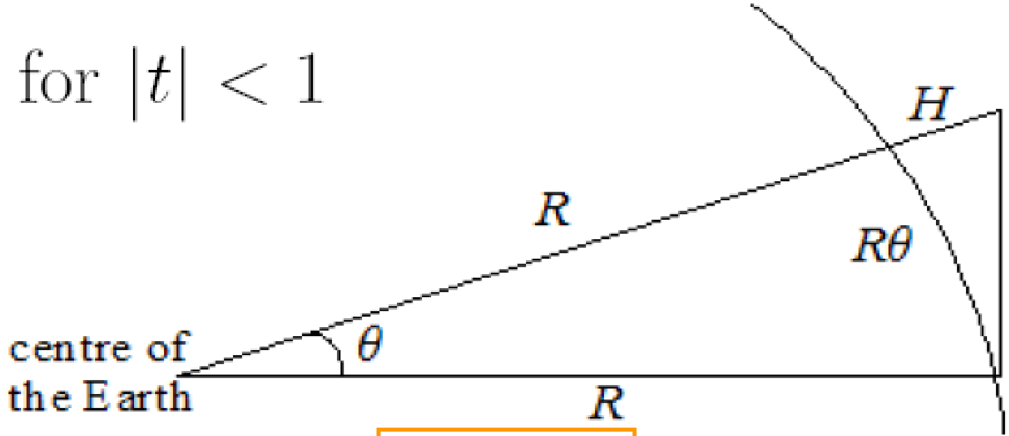
Stand at top of lighthouse, height H = 100 m = 0.1 km.



Estimate the distance from the foot of lighthouse to the horizon, i.e. the value of $R\theta$

$$\frac{1}{1\boxed{-t}} = \boxed{\sum_{n=0}^{\infty} t^n} \quad \text{for } |t| < 1$$

$$t = -\frac{H}{R}$$



Simple trigonometry gives $\boxed{\dfrac{R}{R+H}} = \boxed{\cos\theta}$

$$\boxed{\frac{R}{R+H}} = \frac{1}{1\boxed{+\dfrac{H}{R}}} = 1 - \frac{H}{R} + \left(\frac{H}{R}\right)^2 - + \cdots$$

$$\boxed{\approx 1 - \frac{H}{R}}$$

$$\boxed{\cos\theta} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - + \cdots \boxed{\approx 1 - \frac{\theta^2}{2}}$$

$R\,\theta$

Thus,

$$1 - \frac{H}{R} \approx 1 - \frac{\theta^2}{2}$$

$$R^2\,\theta^2 = R \cdot R\,\theta^2 \approx R \cdot 2H = 2RH$$

$$R\,\theta \approx \sqrt{2R}\,\sqrt{H}$$

$R \approx 6370$

$$\approx 113\sqrt{H}$$

$H = 0.1$

$$\boxed{\approx 35.7(\mathrm{km})}$$

Example

Use the Taylor polynomial of e^x of order 5 at $x = 0$ to approximate e.

First observe that e can now be defined using series, namely

$$e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

$$\boxed{P_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}}$$

$$\boxed{e \approx} \; P_5(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$$

$$\boxed{\approx 2.7167}$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots}$$

Approximations

Consider

$$e \approx P_5(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \approx 2.7167$$

- *e* is approximated by a finite series of terms
- only arithmetic operations +, –, ×, ÷ are used in calculations
- hardwired circuitry can perform arithmetic operations in binary form very fast.

How good is the approximation?

Taylor's Theorem

Let $P_n(x)$ be the nth order Taylor polynomial of $f(x)$ at $x = a$.

Then

$$\boxed{f(x) = P_n(x) + R_n(x)}$$

where

$$\boxed{R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}$$

for some c between a and x.

The function $R_n(x)$ is called the remainder of order n.

$$\frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\boxed{f(x) = P_n(x)} + \boxed{R_n(x)}$$

Using $P_n(x)$ to approximate $f(x)$ incurs an error given by $R_n(x)$.

$R_n(x)$ is thus called the error term for the approximation of $f(x)$ by $P_n(x)$.

Continuation of Example

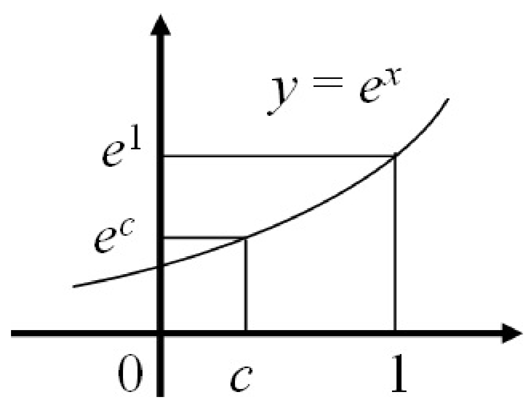
$$\boxed{e} \approx P_5(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \; \boxed{\approx 2.7167}$$

$$\boxed{R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}$$

$$f(x) = e^x \qquad a = 0,\; x = 1 \qquad n = 5 \qquad f^{(6)}(c) = e^c$$

$$\boxed{R_5(1)} = \frac{e^c}{6!}1^6 < \frac{e^1}{6!} < \frac{3}{6!} \; \boxed{\approx 4.167 \times 10^{-3}}$$

2.7167 approximates e with an error less than 0.005.



e^x is increasing function

$e^1 = e < 3$ (justify!)

Exercise

Compare the series for e against the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{2^n}$$

to obtain the 'crude' estimate $e < 3$.

End of Chapter 4