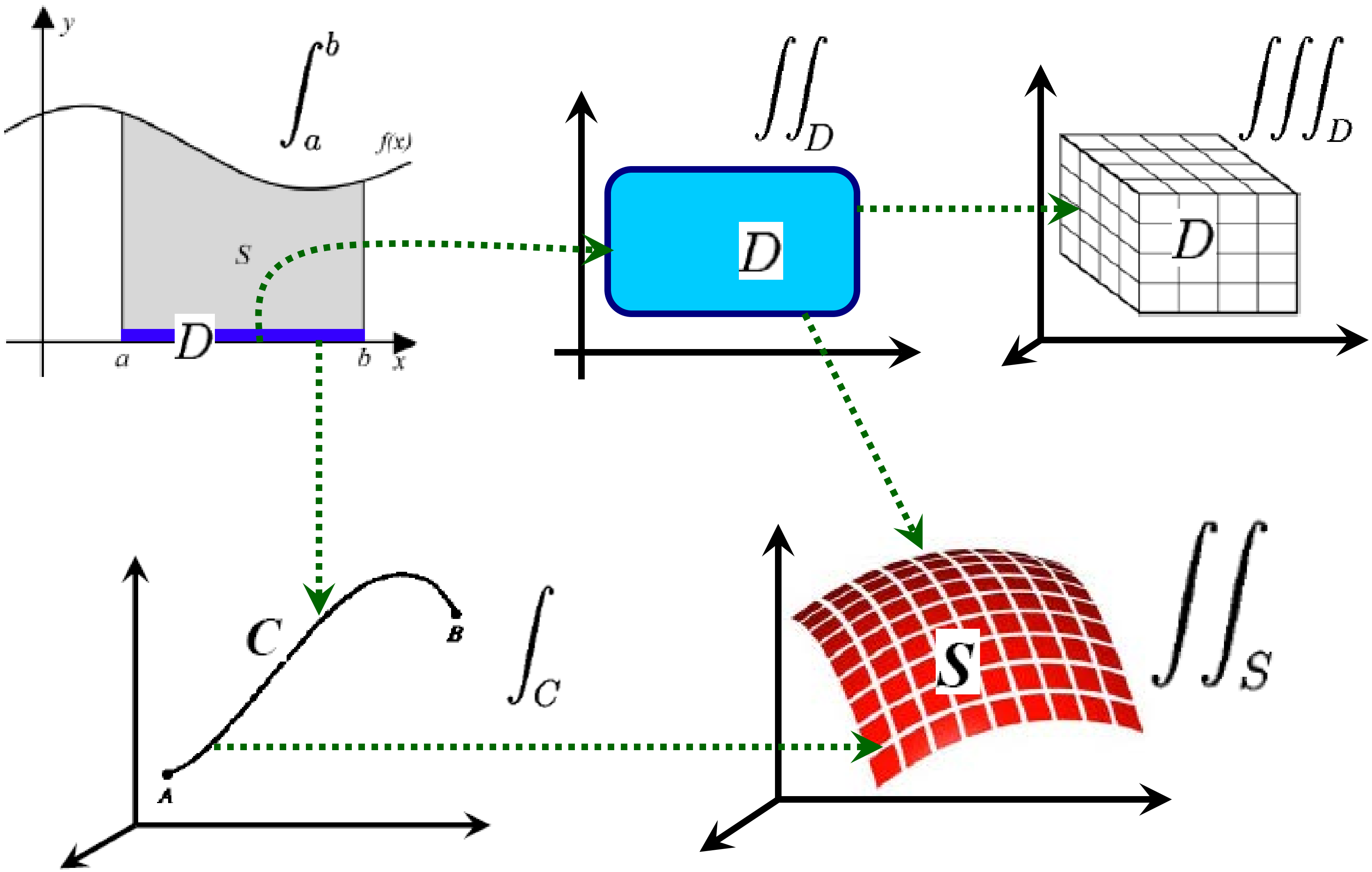
# CH 10 Surface Integrals



# 10.1 *Parametric* Surfaces

- ***Parametric curves*** in space :

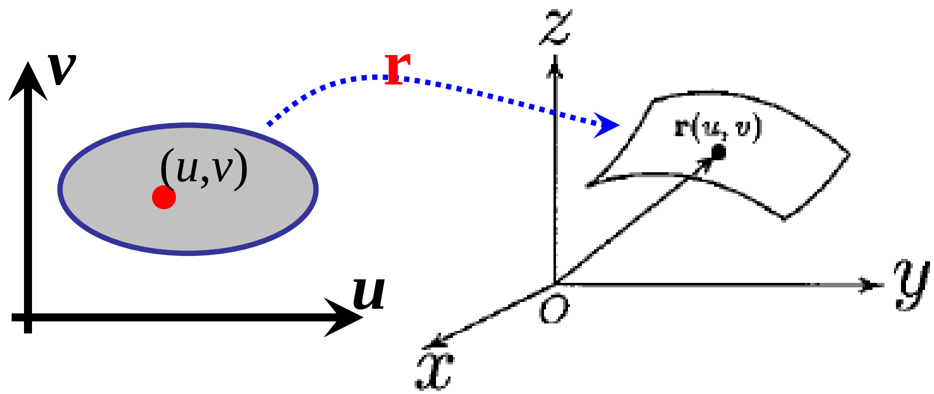
$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}\,, \qquad a \le t \le b$$

- ***Parametric surfaces*** in space :

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad (1)$$

where $u$ and $v$ are two independent parameters.

$$\begin{cases} x = x(u, v), \\ y = y(u, v), \\ z = z(u, v) \end{cases}$$




— the ***parametric*** equations of the surface

# **Why *parametric* ?**

- **It represents the *points* on surfaces explicitly**
- **It describes certain surfaces which *cannot* be expressed as Cartesian equations**
- **It can be used to compute *surface integrals***

**http://www.math.uri.edu/~bkaskosz/flashmo/tools/parsur/**

♣ ***Planes*** : $\boldsymbol{ax + by + cz = d}$

Let 2 of the 3 components be $u$ & $v$, & obtain the remaining component in terms of $u$ & $v$ by the equation.

○ $2x - 5y + 3z = 4$.

Let $x = u$, $y = v$, & so $z = \frac{1}{3}(4 - 2u + 5v)$ .

Thus the ***parametric*** representation of the plane is $\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + \frac{1}{3}(4 - 2u + 5v)\mathbf{k}$ .

- If 1 variable is ***absent from*** the eqn., let it be $u$ or $v$.
  - $3x - y = 5$.

    Let $z = u$. Then $x = v$, $y = 3x - 5 = 3v - 5$, &

$$\mathbf{r}(u,v) = v\mathbf{i} + (3v - 5)\mathbf{j} + u\mathbf{k}\ .$$

- If 2 variables are ***absent from*** the eqn., let them be $u$ & $v$ .
  - The $yz$-plane ($x = 0$) is represented by

$$\mathbf{r}(u,v) = 0\mathbf{i} + u\mathbf{j} + v\mathbf{k}.$$

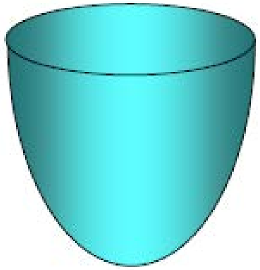
## ♣ **Surfaces** of the form $z = f(x, y)$ !!

$$\text{Let } x(u,v) = u,\ \ y(u,v) = v.$$

$$\text{Then } z(u,v) = z = f(x,y) = f(u,v)$$

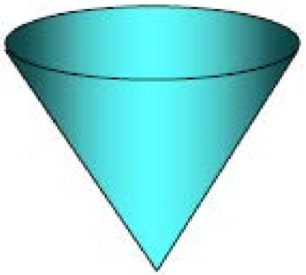
- The paraboloid $z = x^2 + y^2$.

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + (u^2 + v^2)\mathbf{k}.$$



- The upper cone $z = \sqrt{x^2 + y^2}$.

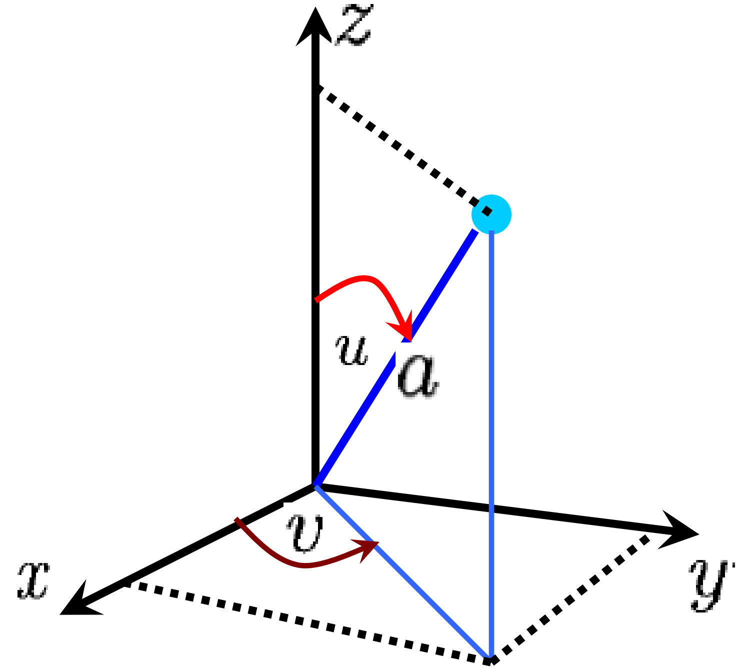
$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + \sqrt{u^2 + v^2}\mathbf{k}.$$



http://www.math.uri.edu/~bkaskosz/flashmo/tools/graph3d/

♣ **Spheres** $(x^2 + y^2 + z^2 = a^2$ with **radius** $a)$

$$\mathbf{r}(u, v) = (a \sin u \cos v)\mathbf{i} + (a \sin u \sin v)\mathbf{j} + (a \cos u)\mathbf{k}.$$




- **Full sphere** :

$$0 \le u \le \boxed{\pi}, \quad 0 \le v \le 2\pi$$

- **Upper hemisphere** :

$$0 \le u \le \boxed{\pi/2,} \quad 0 \le v \le 2\pi$$

http://www.math.uri.edu/~bkaskosz/flashmo/tools/sphcoords/

http://www.math.uri.edu/~bkaskosz/flashmo/tools/sphplot/

## ♣ Circular Cylinder :

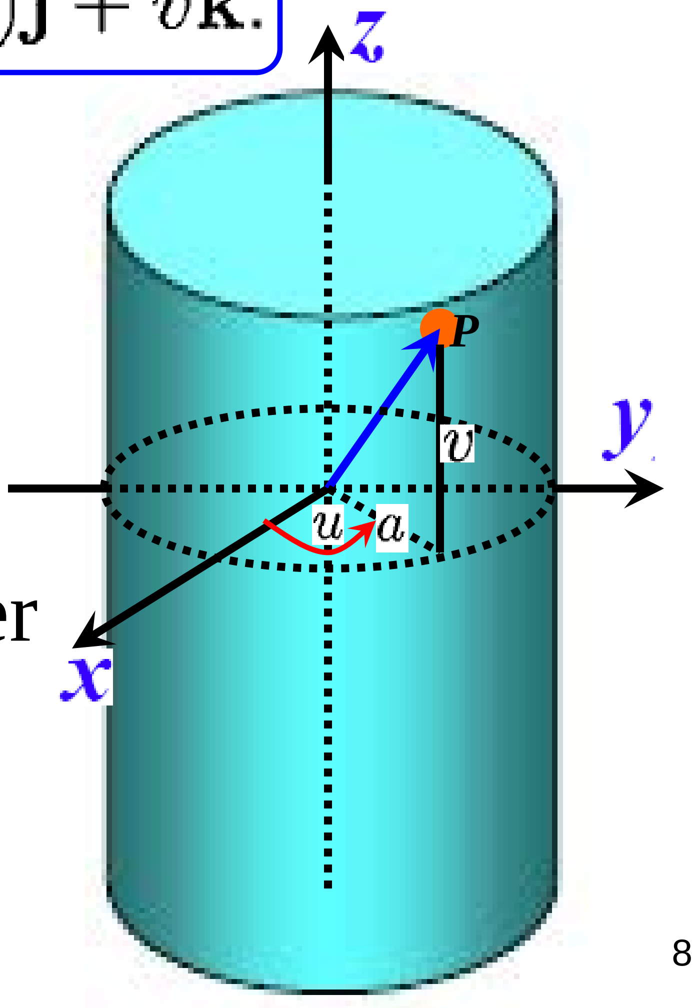
( $x^2 + y^2 = a^2$ **about** the ***z*-axis**)

$$\boxed{\mathbf{r}(u,v) = (a\cos u)\mathbf{i} + (a\sin u)\mathbf{j} + v\mathbf{k}.}$$

***u*** : measures the ***angle*** from the positive *x*-axis

***v*** : the ***height*** from the *xy*-plane along the cylinder

***P*** : (***a* cos *u***, ***a* sin *u***, ***v***)



- For **cylinder** about ***y*-axis** ( $x^2+z^2=a^2$ ) :

$$\boxed{\mathbf{r}(u,v)=(a\cos u)\mathbf{i}+v\mathbf{j}+(a\sin u)\mathbf{k}}$$

- For **cylinder** about ***x*-axis** ( $y^2+z^2=a^2$ ) :

$$\boxed{\mathbf{r}(u,v)=v\mathbf{i}+(a\cos u)\mathbf{j}+(a\sin u)\mathbf{k}}$$

**http://www.math.uri.edu/~bkaskosz/flashmo/tools/cylin/**

## • Tangent Planes

**Given** : surface ***S***

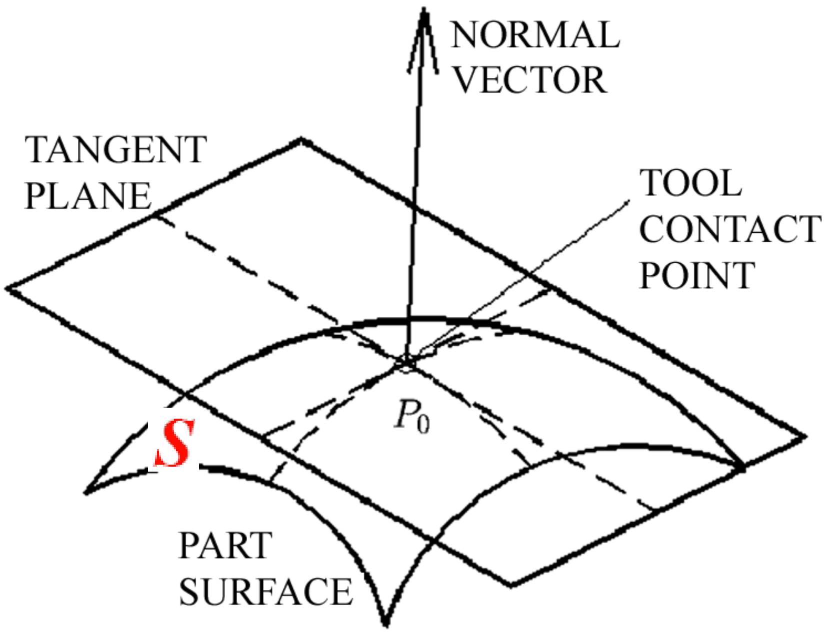
$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

& a point $P_0$ with position vector $\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$

**Find** : the equation

of the **tangent plane**

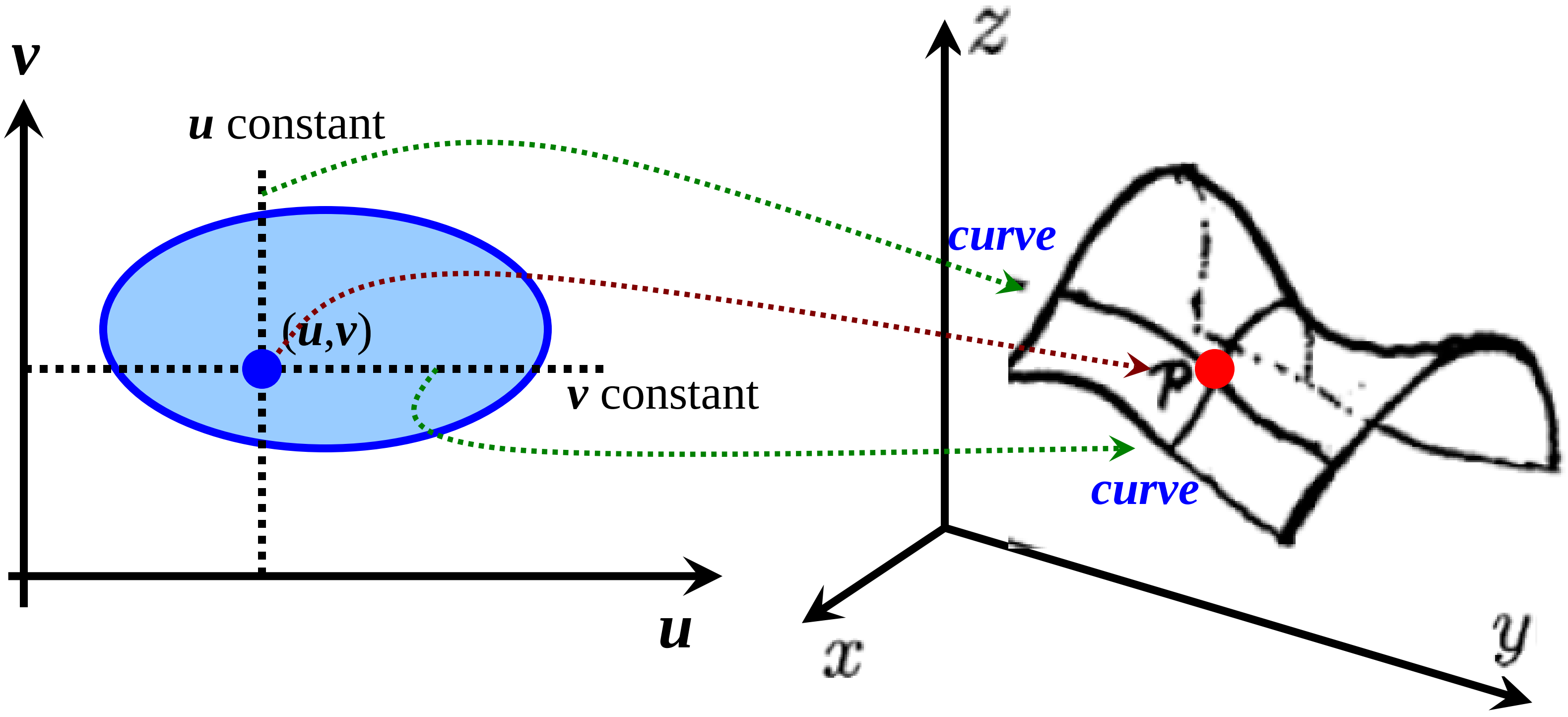
to ***S*** at $P_0$



- ***Parametric surfaces*** in space :

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad (1)$$

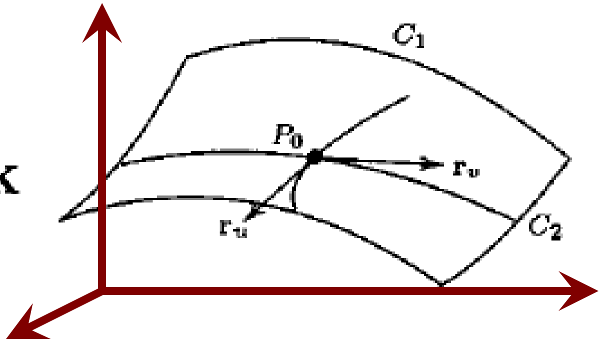
where $u$ and $v$ are two independent parameters.



- **Fix** $v = v_0$ . **Curve** $C_1$

$$\mathbf{r}(u, v_0) = x(u, v_0)\mathbf{i} + y(u, v_0)\mathbf{j} + z(u, v_0)\mathbf{k}$$

Tangent vector to $C_1$ at $P_0$




$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}$$

- **Fix** $u = u_0$ . **Curve** $C_2$

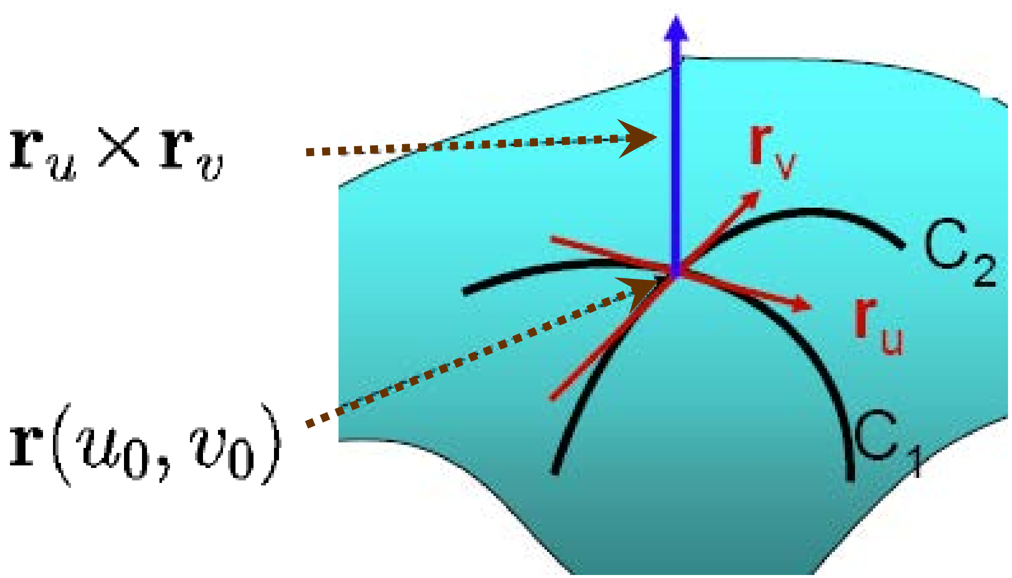
$$\mathbf{r}(u_0, v) = x(u_0, v)\mathbf{i} + y(u_0, v)\mathbf{j} + z(u_0, v)\mathbf{k}$$

Tangent vector to $C_2$ at $P_0$

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}$$

● As $\mathbf{r}_u$ and $\mathbf{r}_u$ lie in the tangent plane to $S$ at $P_0$, the cross product $\mathbf{r}_u \times \mathbf{r}_v$ provides a **normal** vector to the tangent plane. Thus the **equation** of the **tangent plane** is :

$$\boxed{(\mathbf{r} - \mathbf{r}_0) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 0}$$




**See CH5, 5.5 planes in Space**

$(r-r_0)\bullet n = 0$

♣ **Find** the eqn. of the tangent plane to the surface

$$\mathbf{r}(u,v) = u\mathbf{i} + v^2\mathbf{j} + (u^2 - v)\mathbf{k}$$

at (1, 4, −1).

$$\boxed{(\mathbf{r} - \mathbf{r}_0)\cdot(\mathbf{r}_u \times \mathbf{r}_v) = 0}$$

$$\mathbf{r} - \mathbf{r}_0 = (x-1)\mathbf{i} + (y-4)\mathbf{j} + (z+1)\mathbf{k}$$

$$\mathbf{r}_u = \mathbf{i} + 0\mathbf{j} + 2u\mathbf{k} \qquad \mathbf{r}_v = 0\mathbf{i} + 2v\mathbf{j} - \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = -4uv\mathbf{i} + \mathbf{j} + 2v\mathbf{k}$$

$$(1, 4, -1) \Longleftrightarrow (u, v) \quad ?$$

- $\mathbf{r}(u, v) = u\mathbf{i} + v^2\mathbf{j} + (u^2 - v)\mathbf{k}$

**Point** (1, 4, −1): $\mathbf{r}_0 = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$

We have :

$$\begin{cases} u = 1 \\ v^2 = 4 \\ u^2 - v = -1 \end{cases}$$

which imply **(*u*, *v*) = (1, 2)** .

- At this point,

$$(\mathbf{r}-\mathbf{r}_0)\cdot(\mathbf{r}_u\times\mathbf{r}_v)=0$$

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= -4uv\mathbf{i}+\mathbf{j}+2v\mathbf{k} \\ &= -8\mathbf{i}+\mathbf{j}+4\mathbf{k}\end{aligned}$$

**(*u*, *v*) = (1, 2)**

Thus the **equation** of the **tangent plane** is :

$$[(x-1)\mathbf{i}+(y-4)\mathbf{j}+(z+1)\mathbf{k}]\cdot(-8\mathbf{i}+\mathbf{j}+4\mathbf{k})=0$$

or $$-8x+y+4z+8=0.$$

♣ For the surface $\boxed{S : z = f(x, y),}$ its **parametric** representation is :

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}.$$

Thus, $\mathbf{r}_u = \mathbf{i} + 0\mathbf{j} + f_u\mathbf{k}$

& $\mathbf{r}_v = 0\mathbf{i} + \mathbf{j} + f_v\mathbf{k}$

& so the **normal vector** is :

$$\mathbf{r}_u \times \mathbf{r}_v = -f_u\mathbf{i} - f_v\mathbf{j} + 1\mathbf{k}.$$

# 10.2 *Surface* Integrals (two types)

- **Type I** : **Scalar** function $f(x, y, z)$

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA$$

**Recall line integral**

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\|\mathbf{r}'(t)\|dt$$

- **Type II** : **Vector field** $\mathbf{F}(x, y, z)$

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA$$

**Recall line integral**

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$$

**Electrical charge** distributed over ***S***
***f*(*x*,*y*,*z*)** – **charge density**
To find the ***total charge*** on ***S***

- **Surface integrals** of **scalar** functions

Given: Surface ***S*** –

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

***D*** – the corresponding **domain** for (*u*, *v*)

***f*(*x*,*y*,*z*)** – a (scalar) function defined on ***S***.

The ***surface integral*** of ***f*** over ***S*** is :

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA.$$

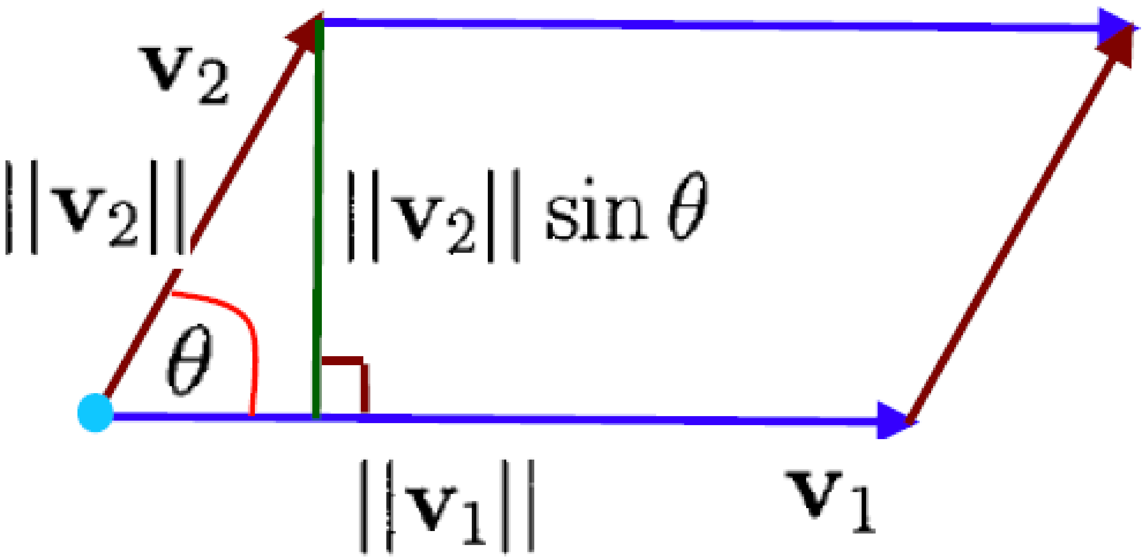
$$dS = \|\mathbf{r}_u \times \mathbf{r}_v\| dA = \|\mathbf{r}_u \times \mathbf{r}_v\| du dv$$

**Why?**

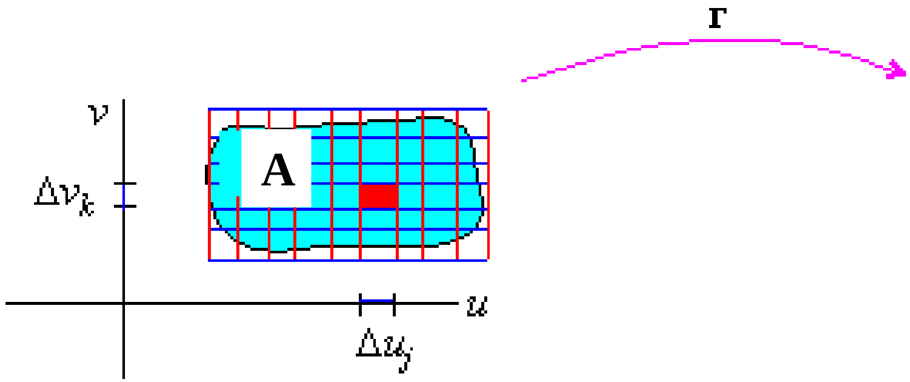
**Recall** that

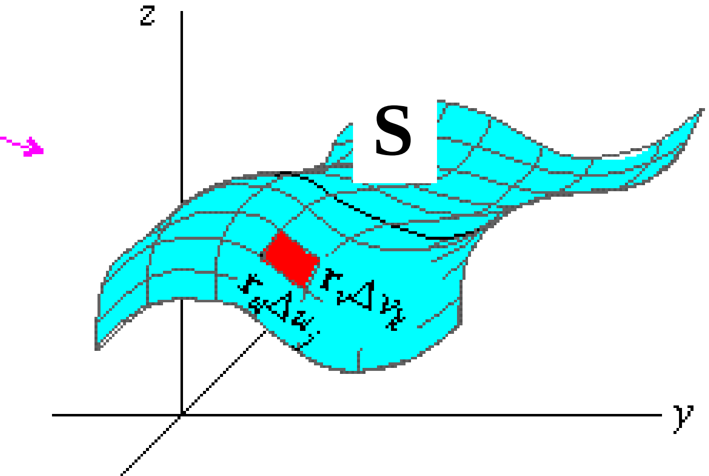
$$||\mathbf{v}_1 \times \mathbf{v}_2||$$

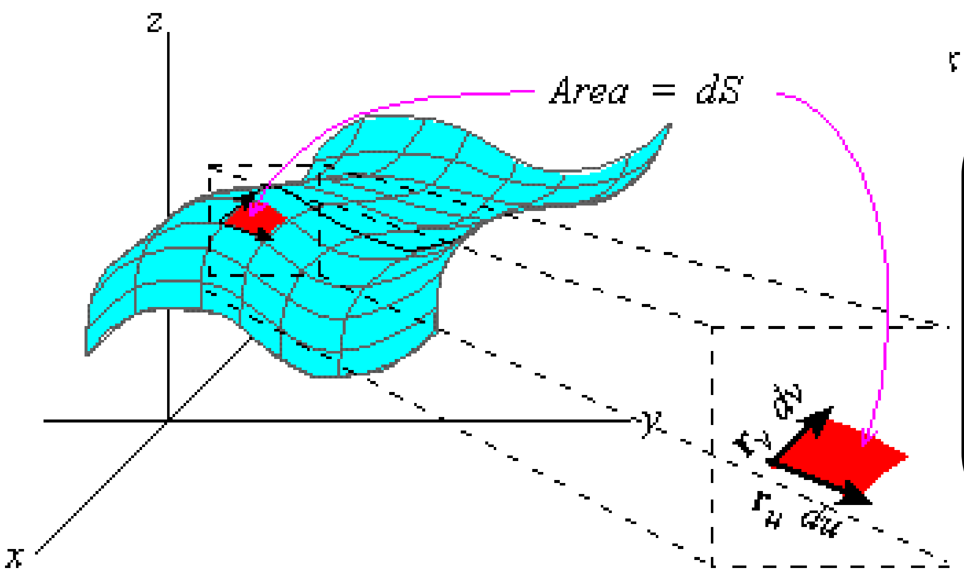
= the ***area*** of the following ***parallelogram***



## ♣ $dS$ ? $dA = (du dv)$



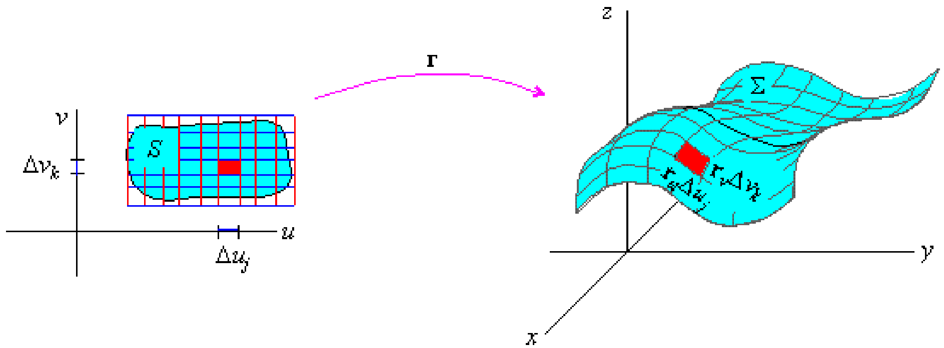



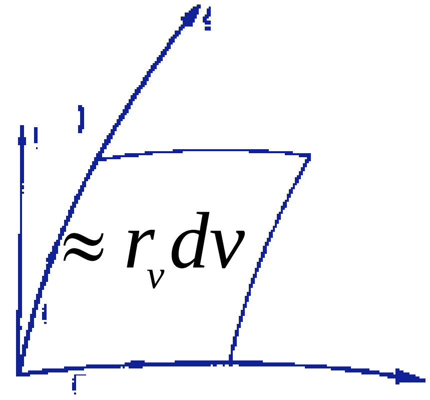




Why? see next slide

$$dS = |\mathbf{r}_u\, du \times \mathbf{r}_v\, dv|$$

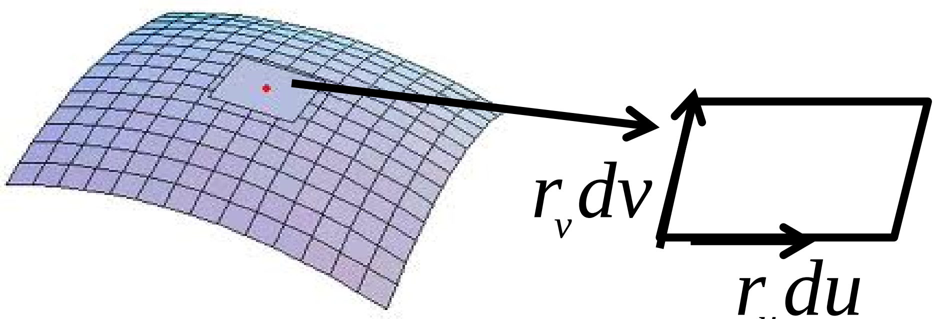
$$= |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$$



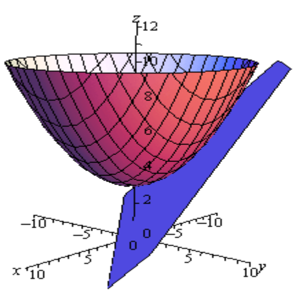




$$\approx r_u du$$

**See line integral of vector fields , CH 9 section 9.3.7**




$$area \text{ of } \square = \|\mathrm{r}_u du \times \mathrm{r}_v dv\| = \|\mathrm{r}_u \times \mathrm{r}_v\| dudv$$



$$dS \approx \|\mathrm{r}_u du \times \mathrm{r}_v dv\| = \|\mathrm{r}_u \times \mathrm{r}_v\| dudv$$

**However we always write**

$$dS = \|\mathrm{r}_u du \times \mathrm{r}_v dv\| = \|\mathrm{r}_u \times \mathrm{r}_v\| dudv$$

$$\iint_S f(x,y,z)\,dS$$

$$= \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA.$$

$$dS = \|\mathbf{r}_u \times \mathbf{r}_v\| dA = \|\mathbf{r}_u \times \mathbf{r}_v\| dudv$$

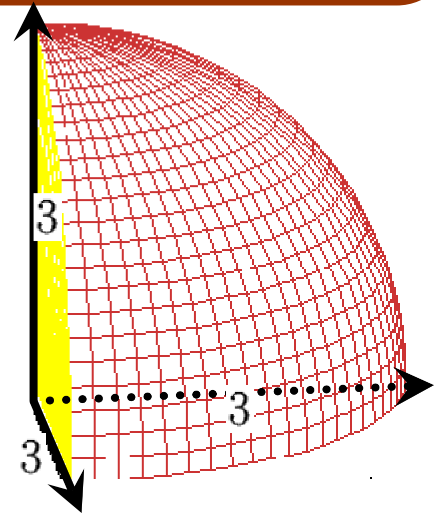
- **Physical Meaning**
  - If $f(x, y, z)$ is a **density** function of a surface $S$, then the **surface integral** gives the **mass** of the **surface**.
  - If $f(x, y, z) = 1$, then the **surface integral** gives the **area** of the **surface**.

♣ Evaluate $\iint_S (xz + yz)\, dS$, where $S$ is part of the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.

$S$ : $\mathbf{r}(u, v) = 3\sin u \cos v\mathbf{i} + 3\sin u \sin v\mathbf{j} + 3\cos u\mathbf{k}$

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\|\mathbf{r}_u \times \mathbf{r}_v\|\, dA$$

$D$ : $0 \le u \le \pi/2$ and $0 \le v \le \pi/2$.




$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\cos u \cos v & 3\cos u \sin v & -3\sin u \\ -3\sin u \sin v & 3\sin u \cos v & 0 \end{vmatrix}$$

$$= 9\sin^2 u \cos v\mathbf{i} + 9\sin^2 u \sin v\mathbf{j} + 9\sin u \cos u\mathbf{k}$$

Therefore, $\|\mathbf{r}_u \times \mathbf{r}_v\| = 9\sin u$.

- Hence $\iint_S (xz+yz)\,dS$

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA$$

$$\mathbf{r}(u,v) = \underbrace{3\sin u\cos v}_{x}\,\mathbf{i} + \underbrace{3\sin u\sin v}_{y}\,\mathbf{j} + \underbrace{3\cos u}_{z}\,\mathbf{k}$$

$$= \iint_D (9\sin u\cos u\cos v + 9\sin u\cos u\sin v)\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \underline{81\sin^2 u\cos u}(\underline{\cos v + \sin v})\,du\,dv$$

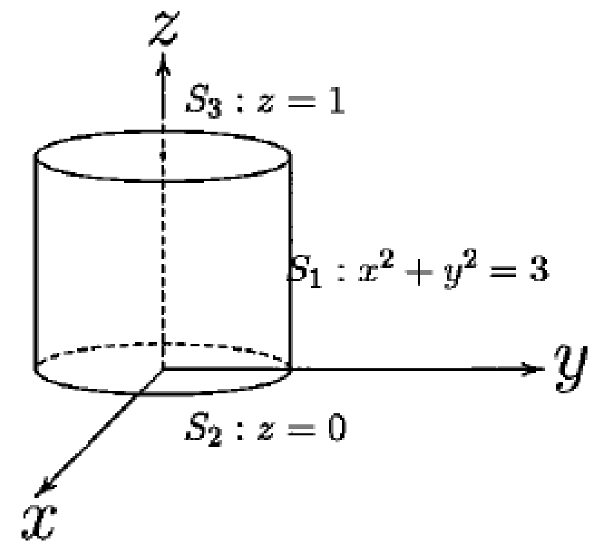
$$0 \le u \le \pi/2$$
$$0 \le v \le \pi/2$$

$$= 81\int_0^{\pi/2} \sin^2 u\cos u\left[\int_0^{\pi/2}(\cos v+\sin v)\,dv\right]du$$

$$= 81\int^{\pi/2} \sin^2 u\cos u\;du\int_0^{\pi/2}(\cos v+\sin v)\;dv$$

$$= 81\left(\left[\frac{1}{3}\sin^3 u\right]_0^{\pi/2}\right)(2) = 54.$$

♣ Evaluate $\iint_S z\,dS$, where $S$ is the closed surface bounded laterally by $S_1$: the cylinder $x^2+y^2=3$; bounded below by $S_2$: the $xy$-plane and bounded on top by $S_3$: the horizontal plane $z=1$.



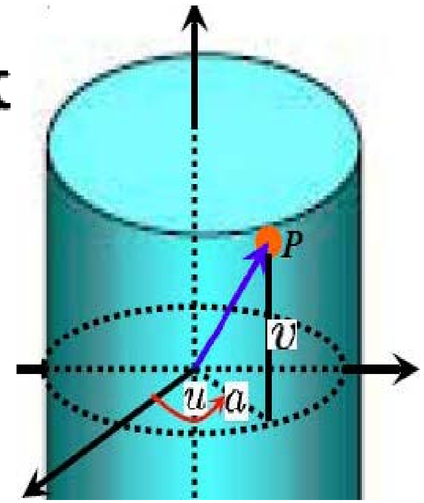

**Note** that $\boxed{f(x,y,z) = z}$

$$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS.$$

$$\mathbf{r}(u,v) = (a\cos u)\mathbf{i} + (a\sin u)\mathbf{j} + v\mathbf{k}.$$

- For $S_1$ :

$$\mathbf{r}(u,v) = \underbrace{\sqrt{3}\cos u}_{x}\mathbf{i} + \underbrace{\sqrt{3}\sin u}_{y}\mathbf{j} + \underbrace{v}_{z}\mathbf{k}$$




**Check:**

$$\mathbf{r}_u \times \mathbf{r}_v = \sqrt{3}\cos u\mathbf{i} + \sqrt{3}\sin u\mathbf{j} + 0\mathbf{k}$$

and $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{3}.$

Then

$$\iint_{S_1} z\,dS = \iint_D v\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{3}v\,dv\,du = \sqrt{3}\pi.$$

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA$$

$$D\left\{\begin{matrix} 0 \le u \le 2\pi \\ 0 \le v \le 1 \end{matrix}\right\}$$

- $S_2$ is on the $xy$-plane, so we have $z = 0$.

Thus $$\iint_{S_2} z\,dS = 0.$$

$$\boxed{\iint_S z\,dS}$$

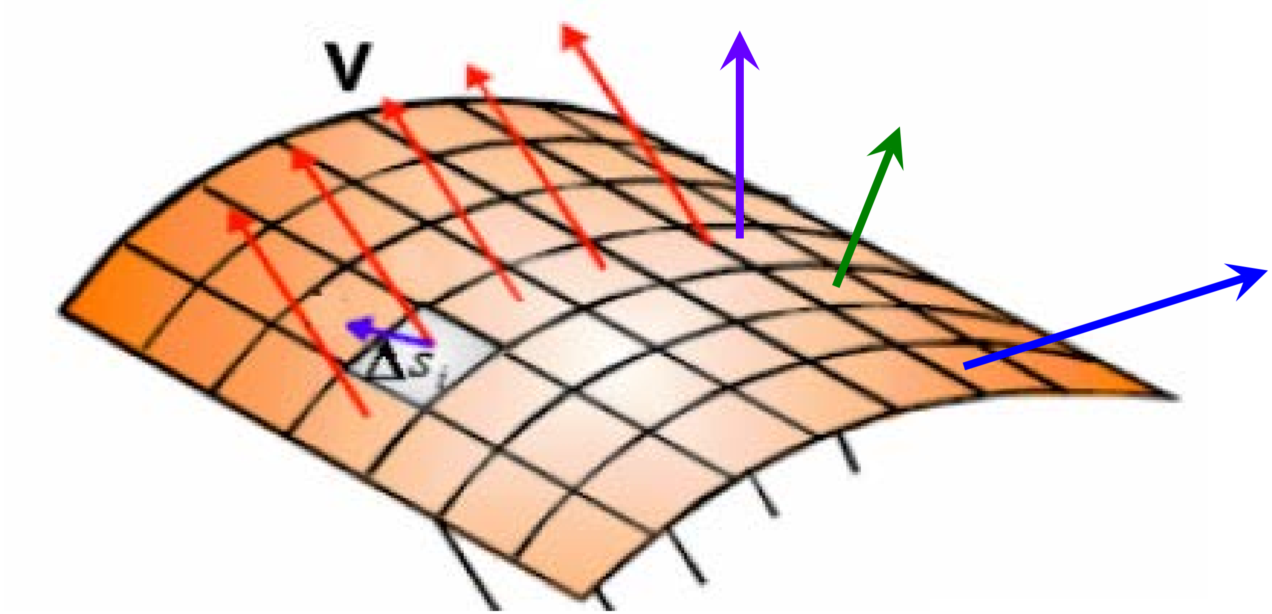
- $S_3$ is on the horizontal plane $z = 1$.

Thus $$\iint_{S_3} z\,dS = \iint_{S_3} dS = \text{area of } S_3 = \pi(\sqrt{3})^2 = 3\pi.$$

☺ Hence

$$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z dS + \iint_{S_3} z dS = (3+\sqrt{3})\pi.$$

- **Objective** : To calculate the total volume of fluid flowing out of ***S*** per unit time.

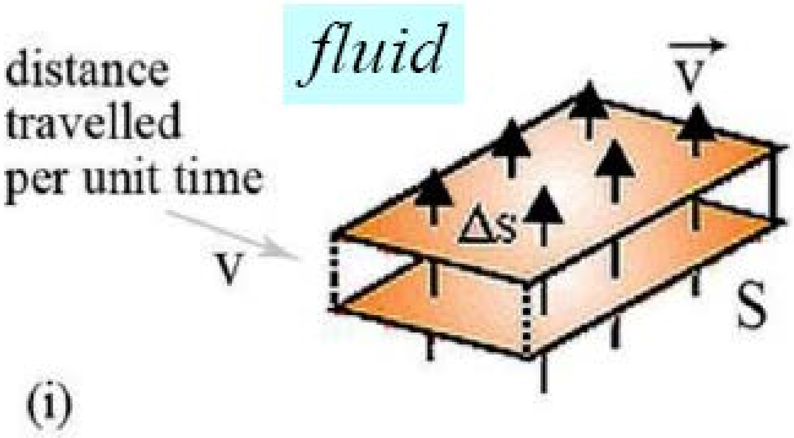



A fluid with velocity **v** flows through ***S***

## ♣ Surface integrals *of* vector fields

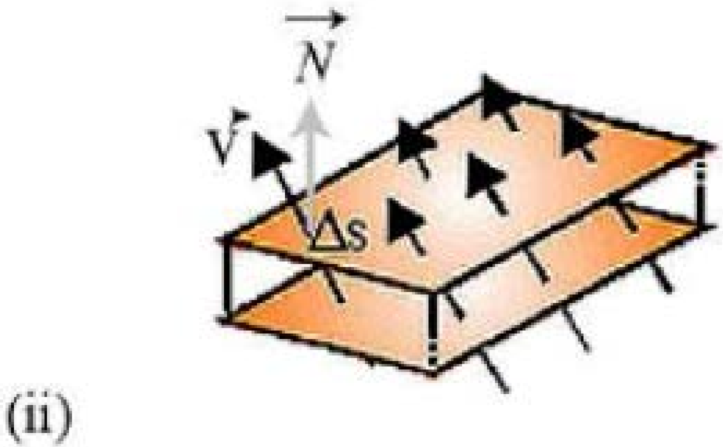
Volume flow rate through flat surface

constant velocity field v




The volume flow rate

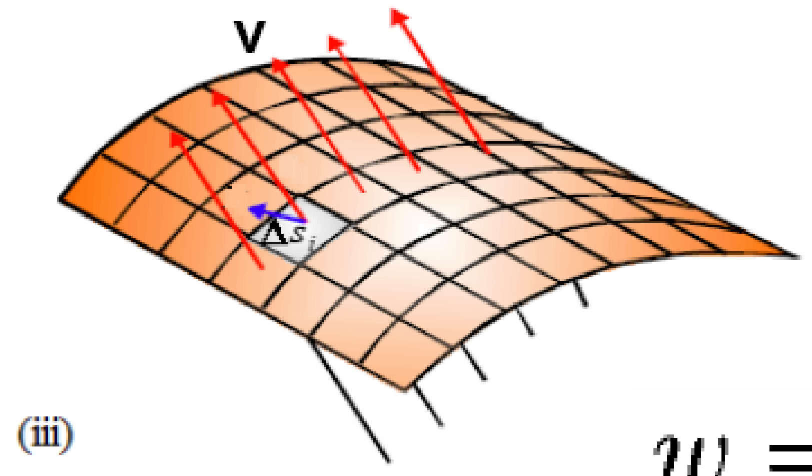
$$w = \|\mathbf{v}\| \Delta s$$




The volume flow rate

$$w = \mathbf{v} \cdot \mathbf{N} \Delta s$$

- Volume flow rate through non-flat surface non-constant velocity field v(x,y,z)




$$w = \mathbf{v} \cdot \mathbf{N} \Delta s$$

In a particular segment,

$$w_i \approx \mathbf{v}(x_i, y_i, z_i) \cdot \mathbf{N_i} \Delta s_i.$$

- Thus the **total flow rate** is approximately

$$w \approx \sum_{1}^{n} \mathbf{v}(x_i, y_i, z_i) \cdot \mathbf{N_i} \Delta s_i$$

If n goes to infinity, the above RHS becomes

$$\iint_S \mathbf{v}(x, y, z) \cdot \mathbf{N} ds$$

which represents the **actual total volume flow rate**. This integral is called a **surface integral** (Flux) of the **vector field v**.

# ***Surface integrals*** of ***vector fields***

**Given** : ***S*** − surface with a unit normal vector **n**,

**F** − continuous **vf** defined on ***S***.

The **surface integral** of **F** over ***S*** is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS \quad \text{or} \quad \iint_S \mathbf{F}\cdot d\mathbf{S}$$

**Bold font**

This integral is also called the **flux** of **F** over ***S*** as it is related to the ***volume flow rate*** of ***fluid***.

If $\boldsymbol{S}: \mathbf{r} = \mathbf{r}(u, v)$ with domain $\boldsymbol{D}$,

then
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

$$= \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}\, dS$$

$$\boxed{dS = \|\mathbf{r}_u \times \mathbf{r}_v\| dA}$$

$$= \iint_D \left[\mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}\right] \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA$$

$$= \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.$$

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA}$$

- **Type I** : **Scalar** function $f(x, y, z)$

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA}$$

- **Type II** : **Vector field** $\mathbf{F}(x, y, z)$

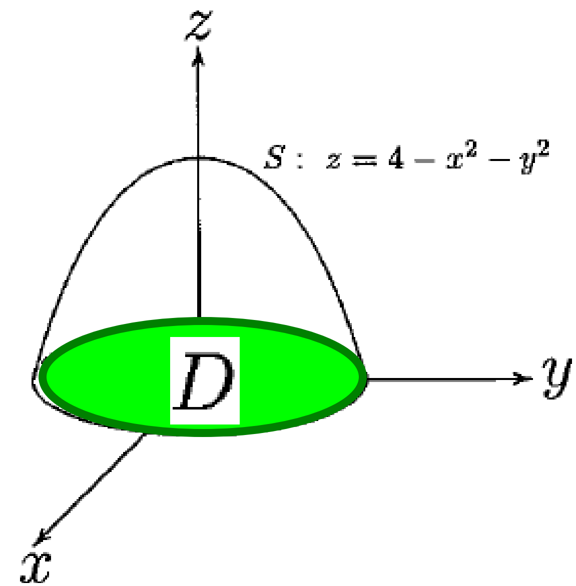
$$\boxed{\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA}$$

♣ **Evaluate** $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + xy\mathbf{k},$$

***S*** : $z = 4 - x^2 - y^2$

above the $xy$-plane.




The **parametric representation** of ***S***:

$\boxed{z = f(x, y)}$

$$\mathbf{r}(u,v) = \underline{u}\mathbf{i} + \underline{v}\mathbf{j} + (\underline{4 - u^2 - v^2})\mathbf{k}.$$

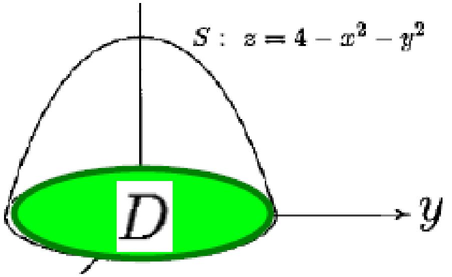
The **domain *D*** is then the **projection** onto ***xy*-plane** (a **circle** of **radius 2**)

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + (4 - u^2 - v^2)\mathbf{k}.$$
$$\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + xy\mathbf{k}$$

- **Check** :

$$\mathbf{r}_u = \mathbf{i} + 0\mathbf{j} - 2u\mathbf{k},$$
$$\mathbf{r}_v = 0\mathbf{i} + \mathbf{j} - 2v\mathbf{k}$$
$$\mathbf{r}_u \times \mathbf{r}_v = 2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k}.$$




Thus, $$\boxed{\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA}$$

$$= \iint_D (u\mathbf{i} + v\mathbf{j} + uv\mathbf{k})\cdot(2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k})\,dA$$

$$= \iint_D (2u^2 + 2v^2 + uv)\,dA$$

$u = r\cos\theta$
$v = r\sin\theta$
$dA = r\,dr\,d\theta$

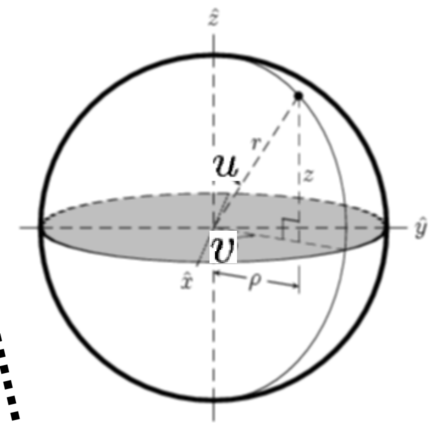
$$= \int_0^{2\pi}\int_0^2 (2r^2 + r^2\cos\theta\sin\theta)\,r\,dr\,d\theta \quad = 16\pi.$$

♣ Let $\mathbf{F}(x,y,z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$. Evaluate $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $S$ is the sphere $x^2+y^2+z^2=1$.

---

$S$ : $\mathbf{r}(u,v) = \underbrace{\sin u\cos v}_{x}\mathbf{i} + \underbrace{\sin u\sin v}_{y}\mathbf{j} + \underbrace{\cos u}_{z}\mathbf{k}$,

with $D$ given by $0 \le u \le \pi$ and $0 \le v \le 2\pi$.




**Check** :

**(1)** $\mathbf{r}_u \times \mathbf{r}_v = \sin^2 u\cos v\mathbf{i} + \sin^2 u\sin v\mathbf{j} + \sin u\cos u\mathbf{k}$

**(2)** $\mathbf{F}(\mathbf{r}(u,v)) = \sin u\sin v\mathbf{i} + \sin u\cos v\mathbf{j} + \cos u\mathbf{k}$

**(3)** $\mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v) = 2\sin^3 u\sin v\cos v + \sin u\cos^2 u.$

We thus have :

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

$$= \int_0^{2\pi} \int_0^{\pi} (2 \underline{\sin^3 u}\, \underline{\sin v \cos v} + \underline{\sin u \cos^2 u})\, du dv$$

$$= \int_0^{\pi} \boxed{\sin^3 u}\, du \int_0^{2\pi} \sin 2v\, dv + \int_0^{\pi} \sin u \cos^2 u\, du \int_0^{2\pi} dv$$

$$= 4\pi/3.$$

- **Type I** : **Scalar** function $f(x, y, z)$

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA$$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\|\mathbf{r}'(t)\|dt$$

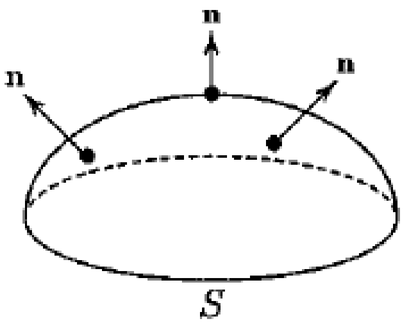
- **Type II** : **Vector field** $\mathbf{F}(x, y, z)$

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$$

# ***Orientation*** of **surfaces**

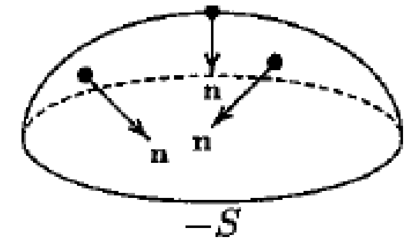
If ***S*** is a surface given by $\mathbf{r} = \mathbf{r}(u, v)$, then the ***normal*** vector $\mathbf{r}_u \times \mathbf{r}_v$ automatically supplies an **orientation** to ***S***.




The **opposite** orientation is given by

$$\mathbf{r}_v \times \mathbf{r}_u = -\mathbf{r}_u \times \mathbf{r}_v$$

& the corresponding oriented surface is denoted by ***−S***, & we have :

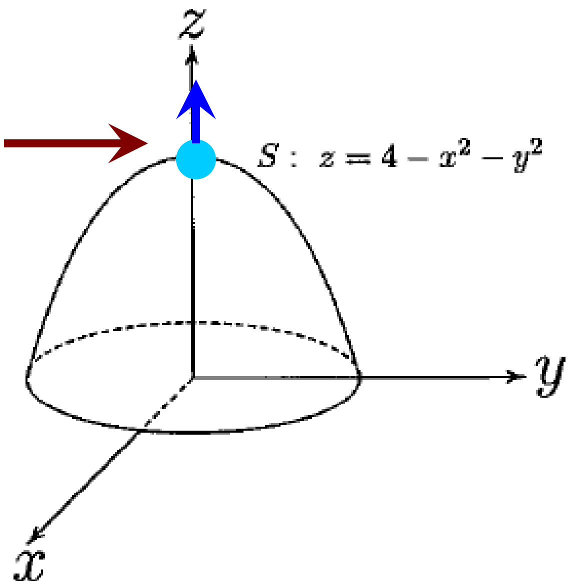



$$\iint_{-S} \mathbf{F} \cdot d\mathbf{S} = -\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

## ♣ (Example 10.2.5)

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + (4 - u^2 - v^2)\mathbf{k}.$$

$u = 0, v = 0 \iff$ point $(0, 0, 4)$




$$\mathbf{r}_u \times \mathbf{r}_v = 2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k}.$$

At this point,

$$\mathbf{r}_u \times \mathbf{r}_v = \mathbf{k},$$

which is pointing "**upwards**".

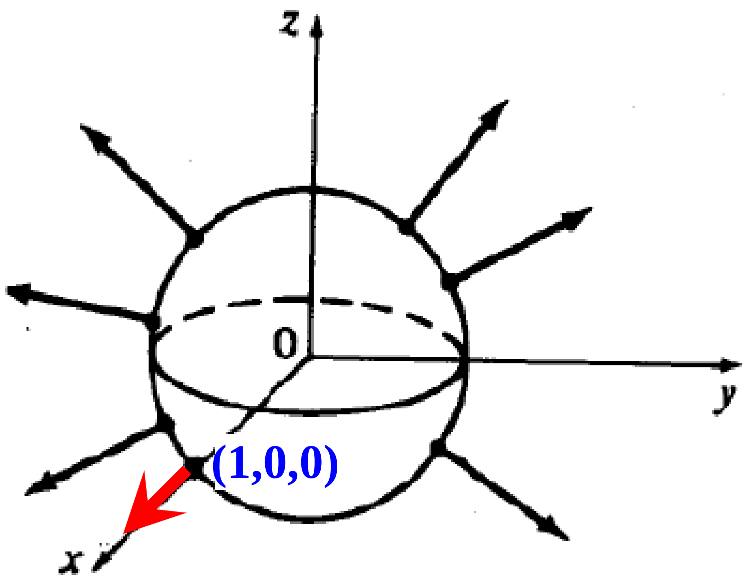
Hence the orientation of the paraboloid in this example is given by the **upward normal vector**.

## ♣ (Example 10.2.6)

$$\mathbf{r}_u \times \mathbf{r}_v = \sin^2 u \cos v \mathbf{i} + \sin^2 u \sin v \mathbf{j} + \sin u \cos u \mathbf{k}$$

$u = \pi/2, v = 0$

⟺ point $(1, 0, 0)$




At this point,

$$\mathbf{r}_u \times \mathbf{r}_v = \mathbf{i},$$

which is pointing "outwards" away from the sphere.

Hence the orientation of the sphere in this example is the "**outward normal vector**".

# 10.3 Curl & Divergence

## • Curl

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field in the space.

The **curl** of **F** is defined by

$$\textbf{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

**Note** that **curl F** is a ***vector field*.**

$\mathbf{F}$ is conservative $\quad \Leftrightarrow$

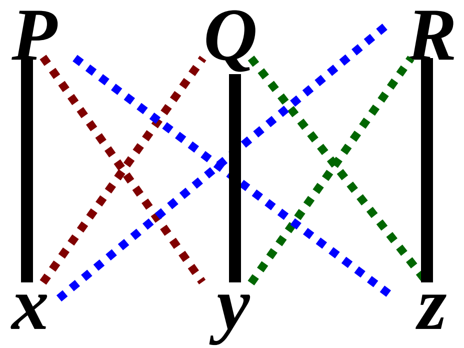
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

# • Divergence

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field in the space.

The **divergence** of **F** is defined by

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$




Note that div **F** is a ***scalar function.***

**Recall curl F** $= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$

## • Del operator

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a **vf**.

Write

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Then

(1) $$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\boxed{\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

& so

$$\boxed{\nabla \times \mathbf{F} = \text{curl } \mathbf{F}}$$

(ii) $$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k})$$
$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

i.e., $$\boxed{\nabla \cdot \mathbf{F} = \mathrm{div}\ \mathbf{F}.}$$

---

Let $\mathbf{F}(x, y, z) = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}.$

Then

**(i) curl F** $= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$

$$= (xz^2 - xy^2)\mathbf{i} + (x^2y - yz^2)\mathbf{j} + (y^2z - x^2z)\mathbf{k}.$$

**(ii)** div **F** $= \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2)$
$$= 6xyz.$$

♣ **Show** that **curl**$(\nabla f) = \mathbf{0}$.

**Proof.**

$$\text{curl}\,(\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \qquad \nabla \times \nabla f$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k}$$

$$= \mathbf{0} \quad [\text{since } f_{xy} = f_{yx} \text{ etc.}]$$

(♥) **Curl** & **conservative fields**

Let **F** be a **vf** in space. Then

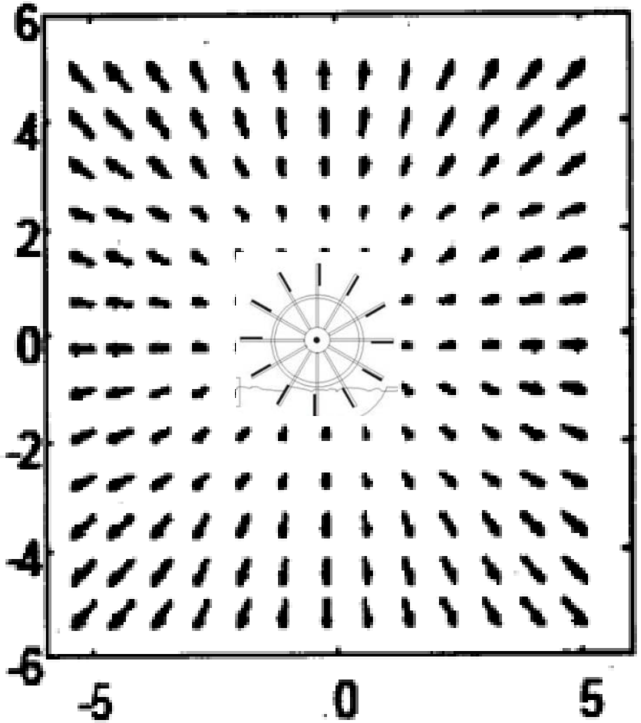
**curl F** = **0** ⟺ **F** is **conservative**

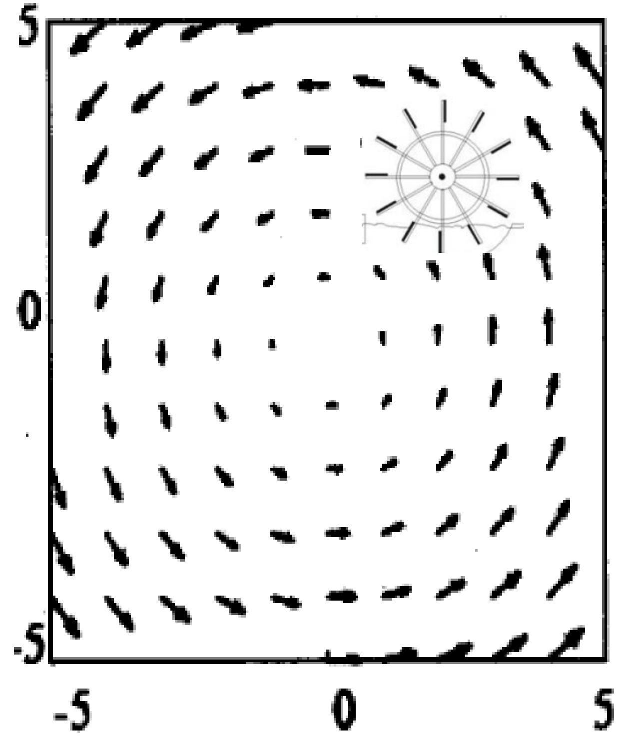
♣ For the **vf** (a) $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$,

$$\text{curl } \mathbf{F_1} = \mathbf{0}, \ \text{div } \mathbf{F}_1 = 2 .$$

(b) $\mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$,

$$\text{curl } \mathbf{F}_2 = 2\mathbf{k}, \ \text{div } \mathbf{F}_2 = 0 .$$



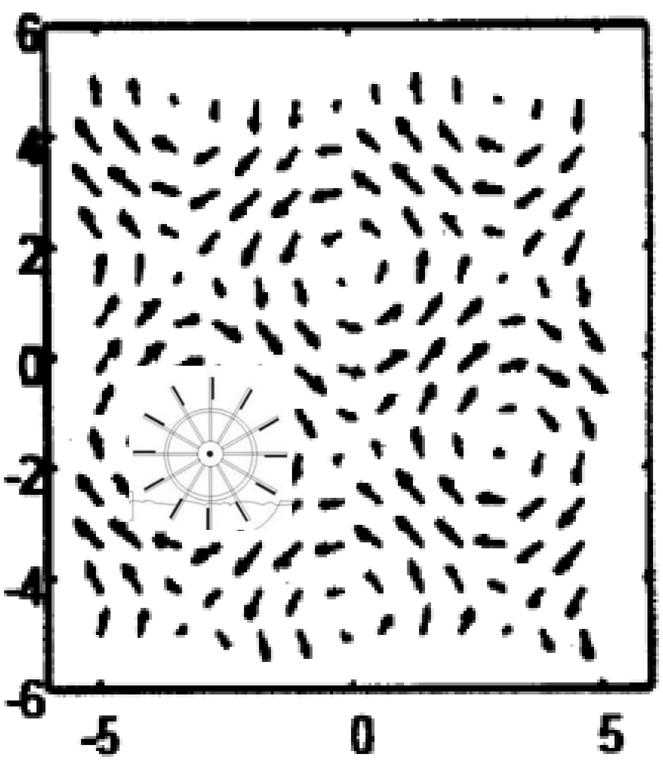



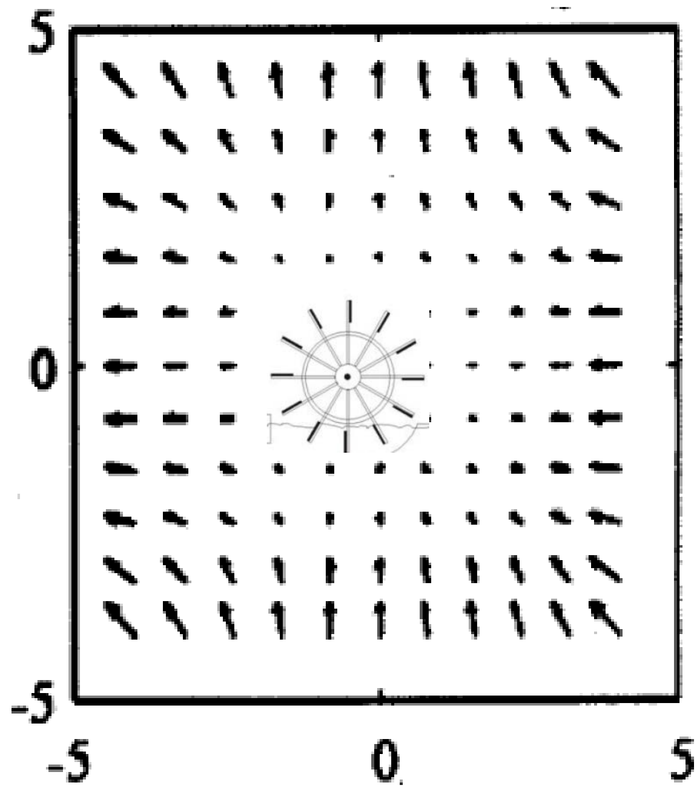
- (c) $\mathbf{F}_3 = \cos y\mathbf{i} + \sin x\mathbf{j}$

  $\text{curl } \mathbf{F}_3 = (\cos x + \sin y)\mathbf{k}$, $\text{div } \mathbf{F}_3 = 0$.

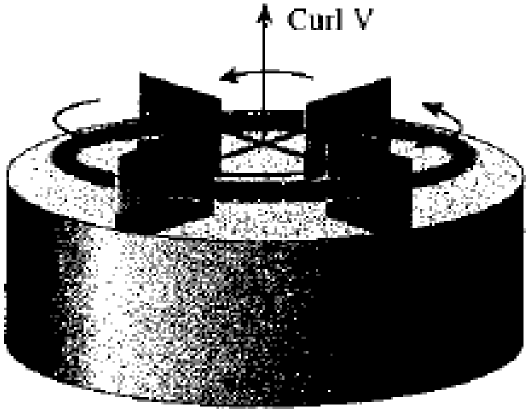
(d) $\mathbf{F} = -x^2\mathbf{i} + y^2\mathbf{j}$

$$\text{curl } \mathbf{F} = \mathbf{0}, \quad \text{div } \mathbf{F} = 2(y - x).$$



- The **curl** of a **vf** measures the **degree** of **swirling** or **rotation** about a given direction.



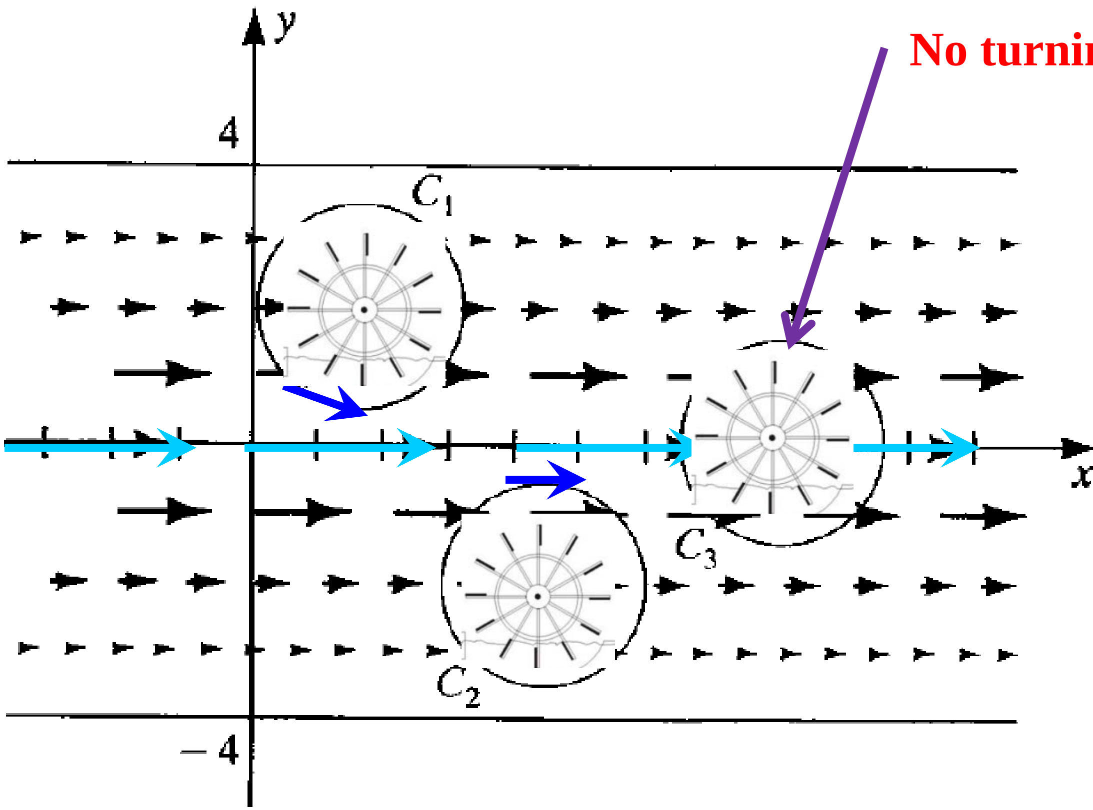

**The direction of curl F** is the **direction** of the **axis** about which the fluid rotates **most rapidly** & $|$**curl F**$|$ is a measure of the **speed** of this rotation. The direction follows the RH-rule.

- Let **F** be the **vf** of a fluid or gas. Then div **F** at a point A measures the **tendency** of the fluid to diverge away from A or accumulate toward A.
- If div **F** > 0 at A, then, overall, the tendency is for the fluid to diverge away from A, & there is a **source** at A.
- If div **F** < 0 at A, then the fluid is tending to accumulate toward A, & there is a **sink** at A.
- If div **F** = 0 at A, then there is neither a source nor a sink at A.

$$\mathbf{F}(x,y,z)=3(y^2+1)^{-1}\,\mathbf{i}+0\mathbf{j}+0\mathbf{k}$$

$$-4 \le y \le 4$$

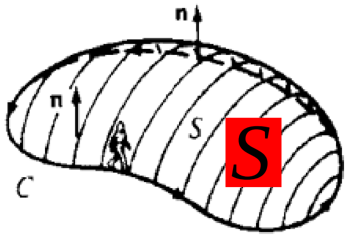
- In the following,the same pattern is repeated in any plane parallel to the ***xy***-plane.




$$\operatorname{div} \mathbf{F} = 0$$

$$\textbf{Curl } \mathbf{F} = 6y/(y^2+1)^2\,\mathbf{k}$$

# 10.4 Stokes' Theorem



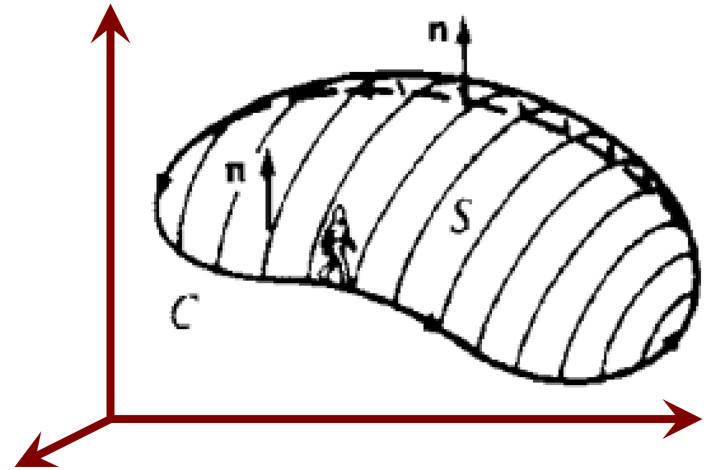

S C

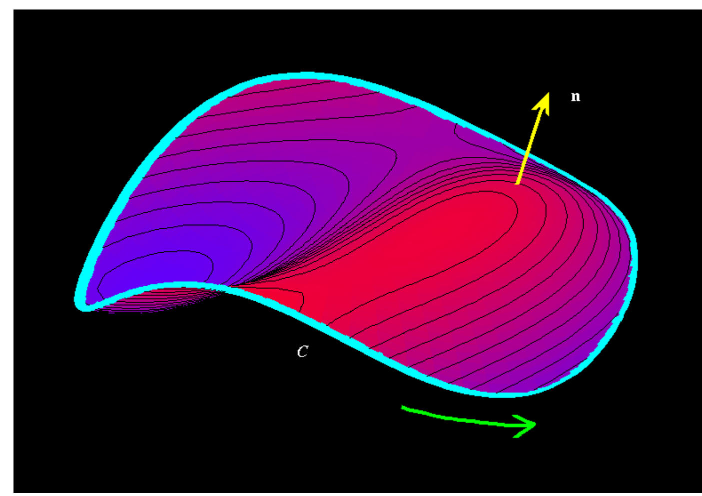
George Gabriel Stokes
(1819-1903)

Let ***S*** be an oriented smooth surface that is bounded by a closed, smooth curve ***C***.
Let **F** be a **vf** whose has continuous partial derivatives on ***S***. Then

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S}.}$$

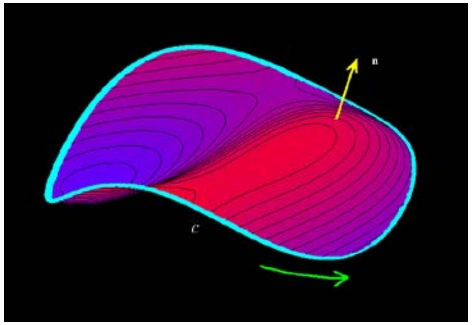
**Note**. In the above equality, the orientation of $C$ must be consistent with that of $S$: when you walk in the orientation around $C$ with your head pointing in the direction of the normal vector of $S$, the corresponding surface $S$ is on your **left**.



- If **F** is a **force field**, the thm says that the **work done** by **F** along ***C*** **equals** the **flux (integral)** of **curl F** over ***S***.
- If **F** is a **velocity field** of a fluid flow, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl }\mathbf{F})\cdot d\mathbf{S}$$

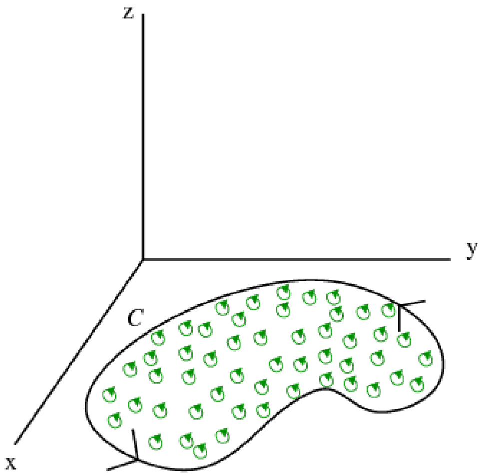


The **circulation** of the fluid around the boundary curve ***C***

The **cumulative tendency** of the fluid to swirl across the surface ***S***

**(Bradley & Smith)**

- **Stokes**' theorem is the "3-variable" version of **Green**'s theorem.



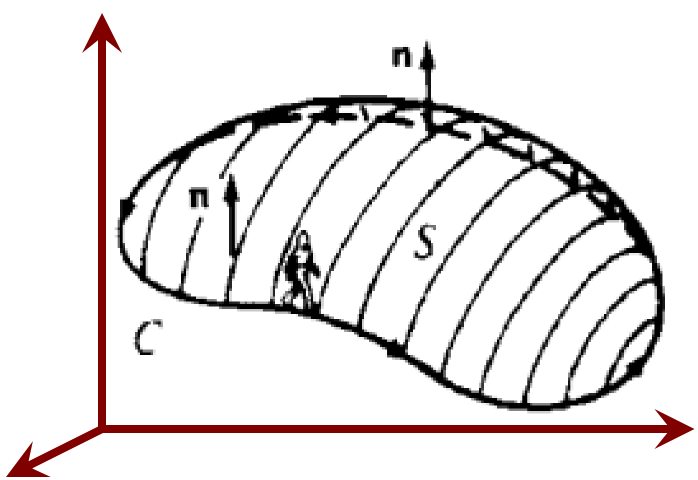

**G** Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. Then

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

**S** Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Then

$$\int_C P\,dx + Q\,dy + R\,dz = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S}$$

$$\left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\mathbf{k}$$



- **Computation**

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

$$\boxed{\int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt}$$

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S}}$$

$$\boxed{\iint_D [\text{curl } \mathbf{F}](\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA}$$

♣ Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + xy\mathbf{k}$ and $C$ is the curve of intersection of the plane $y + z = 3$ and the cylinder $x^2 + y^2 = 4$. ($C$ is oriented in the counterclockwise sense when viewed from above.)

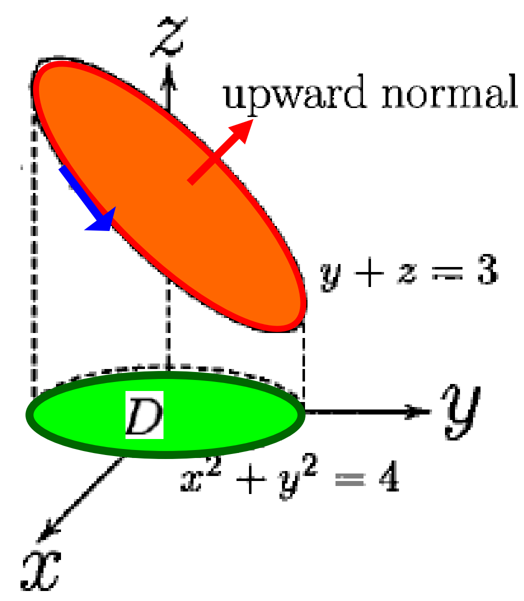
---

Let ***S*** be the surface enclosed by ***C*** on the ***plane*** $z = 3 - y$.

Then ***S*** : $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (3 - v)\mathbf{k}$

where ***D*** is the circle with center 0 & radius 2.

Also, $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{j} + \mathbf{k}$




(**orientations** of ***C*** & ***S*** are **consistent**)

- By **Stokes**' thm, $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$

$$= \iint_D [\text{curl } \mathbf{F}](\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & xy \end{vmatrix}$$

$$= 2x\mathbf{i} - 2z\mathbf{k}$$

$$\boldsymbol{S} : \mathbf{r}(u, v)$$

$$= \underline{u}\mathbf{i} + v\mathbf{j} + \underline{(3 - v)}\mathbf{k}$$

$$[\text{curl } \mathbf{F}](\mathbf{r}(u, v)) = 2u\mathbf{i} - 2(3 - v)\mathbf{k}$$

$$= \iint_D (2u\mathbf{i} - 2(3 - v)\mathbf{k}) \cdot (\mathbf{j} + \mathbf{k})\, dA = \iint_D (-6 + 2v)\, dA$$

- As $\boldsymbol{D}$ is a **circle**, we may use **polar** coordinates:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (-6 + 2v)\, dA$$

$$u = r\cos\theta,$$
$$v = r\sin\theta$$
$$dA \longrightarrow r\ dr\ d\theta$$

$$= \int_0^{2\pi} \int_0^2 (-6 + 2r\sin\theta) r\,dr\,d\theta$$

$\boldsymbol{D}$: $0 \le \boldsymbol{r} \le 2$

$0 \le \boldsymbol{\theta} \le 2\boldsymbol{\pi}$

$$= \int_0^{2\pi} \left(-12 + \frac{16}{3}\sin\theta\right) d\theta = -24\pi.$$

- **Computation**

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

$$\boxed{\int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt}$$

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S}}$$

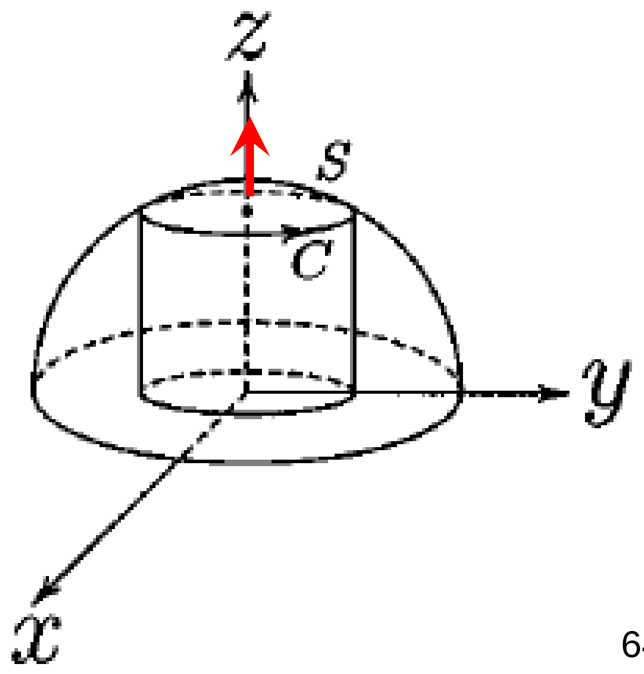
$$\boxed{\iint_D [\text{curl } \mathbf{F}](\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA}$$

♣ Use Stokes' Theorem to compute $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y^2 z\mathbf{i} + x\mathbf{j} + (x + y)\mathbf{k}$ and $S$ is the part of the upper hemisphere $z = \sqrt{9 - x^2 - y^2}$ that lies within the cylinder $x^2 + y^2 = 5$ and the orientation of $S$ is given by the upward normal vector.

By Stokes' Theorem,

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

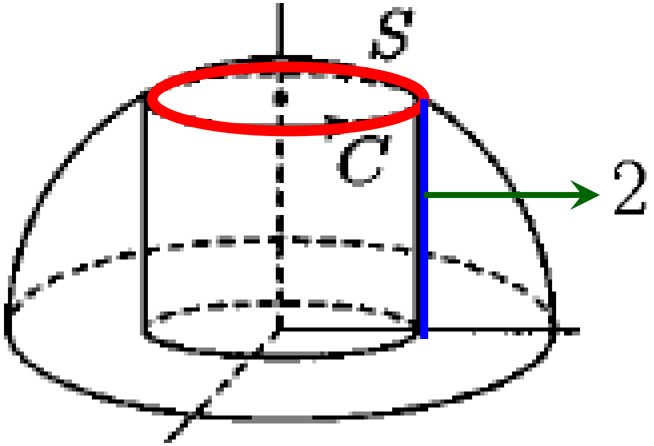


- How to find the parametric rep. $\mathbf{r}(t)$ for $C$ ?

Solving $z = \sqrt{9 - x^2 - y^2}$

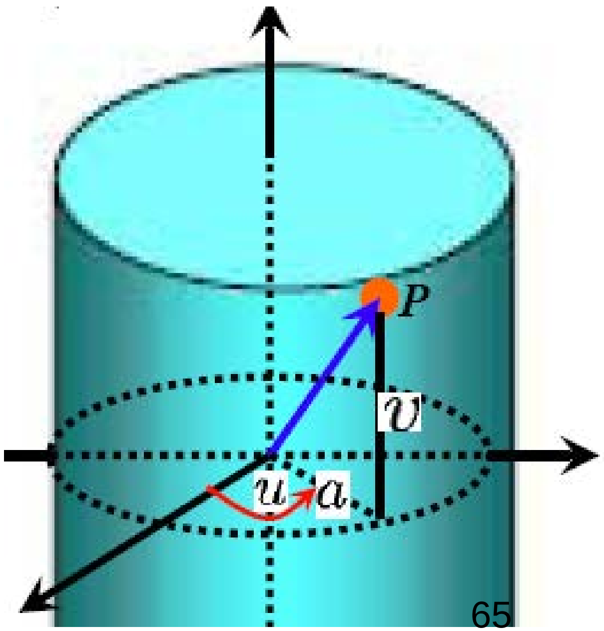
$$x^2 + y^2 = 5$$

yields **$z = 2$**.




**Circular Cylinder :**

$$\mathbf{r}(u, v) = (a \cos u)\mathbf{i} + (a \sin u)\mathbf{j} + v\mathbf{k}$$




**Note**. $a = \sqrt{5}$, $v = 2$; let $u = t$.

Thus, $C$ :

$$\mathbf{r}(t) = \sqrt{5} \cos t\mathbf{i} + \sqrt{5} \sin t\mathbf{j} + 2\mathbf{k}.$$

**($0 \leq t \leq 2\pi$; orientations** of $C$ & $S$ are **consistent)**

- $$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$\mathbf{r}(t) = \underline{\sqrt{5}\cos t\mathbf{i}} + \underline{\sqrt{5}\sin t\mathbf{j}} + \underline{2\mathbf{k}}.$$

$$\mathbf{r}'(t) = -\sqrt{5}\sin t\mathbf{i} + \sqrt{5}\cos t\mathbf{j} + 0\mathbf{k}$$

$$\mathbf{F}(x, y, z) = y^2 z\mathbf{i} + x\mathbf{j} + (x + y)\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = 10\sin^2 t\mathbf{i} + \sqrt{5}\cos t\mathbf{j} + \sqrt{5}(\cos t + \sin t)\mathbf{k}.$$

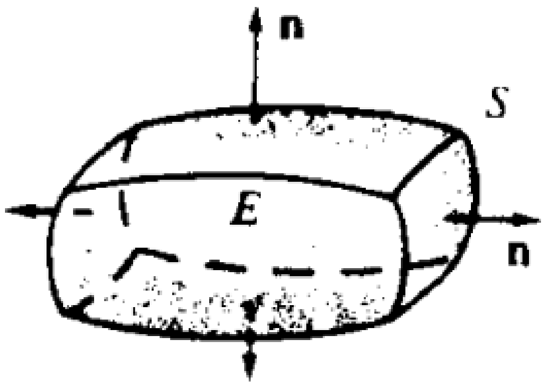
$$= \int_0^{2\pi} (-10\sqrt{5}\sin^3 t + 5\cos^2 t)\, dt = 5\pi.$$

# 10.5 Divergence Theorem (Gauss)

Let ***E*** be a solid & ***S*** the boundary of ***E*** with the **outward** orientation ( the normal vector points outward from ***E*** ).

Let **F** be a **vf** whose component functions have continuous partial derivatives in ***E***. Then

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV.}$$



- The **Divergence Thm** states that the outward flux of a **vf** **F** through ***S*** equals the volume integral of the div **F** over ***E***.

This thm is important in engineering (electrostatics & fluid dynamics).

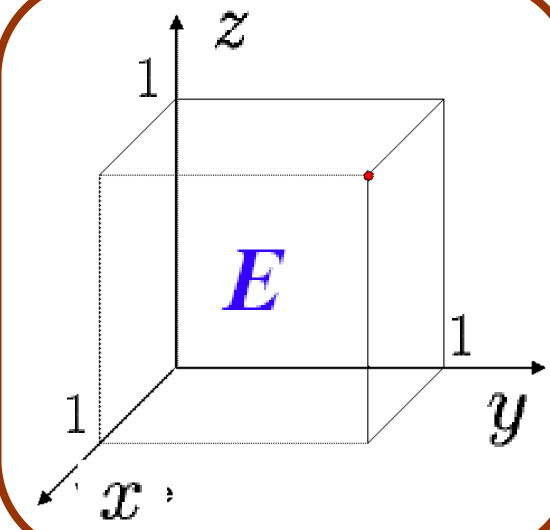
$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV.}$$

**Gauss** (1777-1855) − the Prince of mathematicians

referred to **mathematics** as "the **Queen** of the **Sciences**".

♣ Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F} = x^2\mathbf{i} + (xy + x\cos z)\mathbf{j} + e^{xy}\mathbf{k}$$




**Note** that

$E: \quad 0 \le x \le 1, \;\; 0 \le y \le 1, \;\; 0 \le z \le 1$

By the **Divergence** Thm,

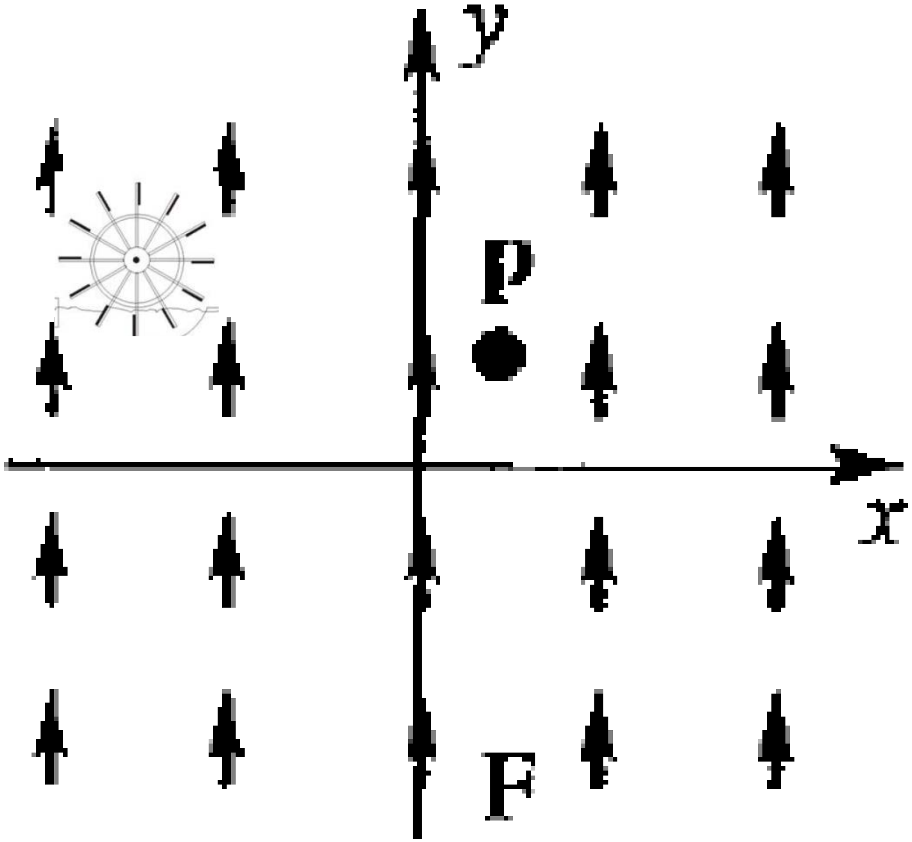
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= \iiint_E 3x\, dV = 3\int_0^1 \int_0^1 \int_0^1 x\, dx dy dz = \frac{3}{2}.$$

Appendix

- $\mathbf{F} = c\mathbf{j}$



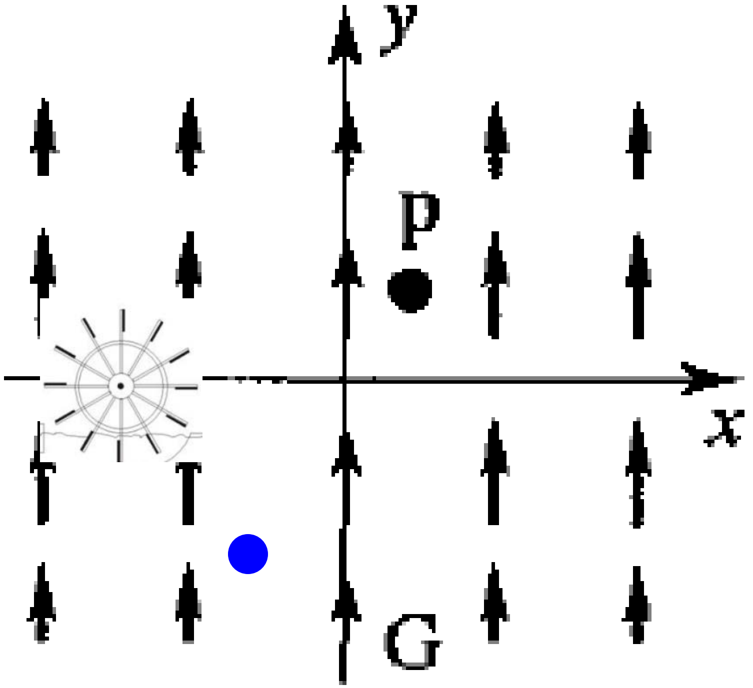

div $\mathbf{F} = 0$

**Curl $\mathbf{F} = \mathbf{0}$**

- $\mathbf{G} = e^{-y^2}\mathbf{j}$

div $\mathbf{G} = -2y\,e^{-y^2}$

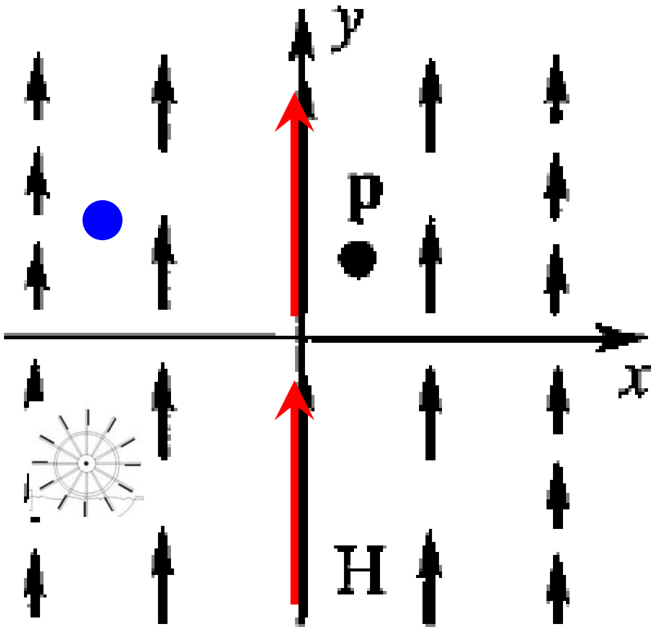
**Curl $\mathbf{G}$ = 0**



- $\mathbf{H} = e^{-x^2}\mathbf{j}$

div $\mathbf{H} = 0$

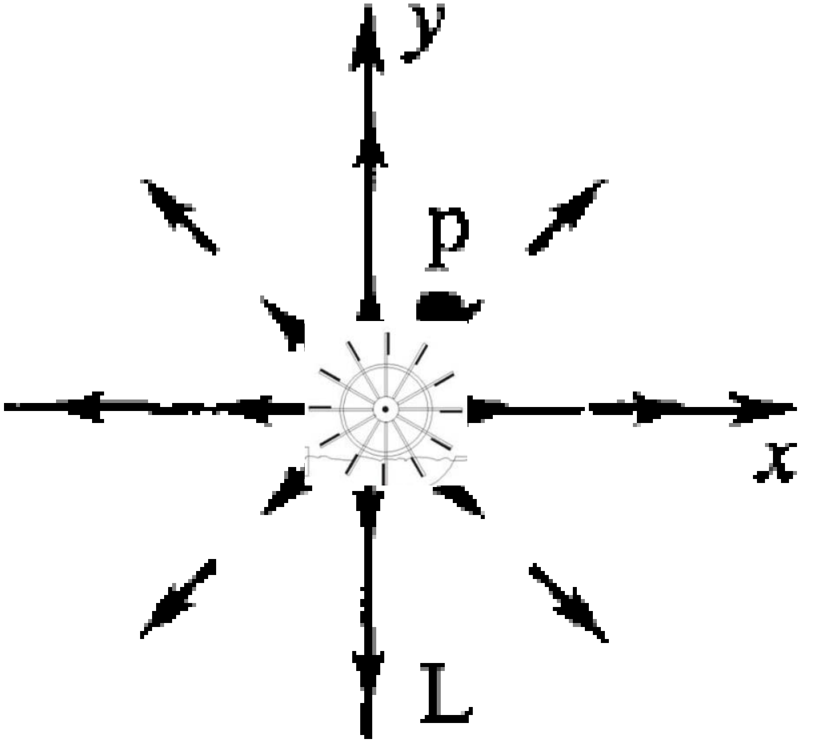
**Curl** $\mathbf{H} = -2xe^{-x^2}\,\mathbf{k}$



- **L** $= (x\mathbf{i} + y\mathbf{j}) / \sqrt{x^2 + y^2}$

div **L** $= 1 / \sqrt{x^2 + y^2}$

**Curl L = 0**




http://www.math.umn.edu/~nykamp/m2374/readings/divcurl/