MA 1505 Mathematics I

Tutorial 7 Solutions

1. (a) $$\int_0^b \int_0^a (x^2+y^2)\,dxdy = \int_0^b \left[\frac{1}{3}x^3 + xy^2\right]_{x=0}^{x=a} dy = \int_0^b \left(\frac{1}{3}a^3 + ay^2\right) dy$$
$$= \left[\frac{1}{3}a^3 y + \frac{1}{3}ay^3\right]_0^b = \frac{1}{3}a^3 b + \frac{1}{3}ab^3.$$

(b) $$\int_1^2 \int_0^1 \frac{xy}{\sqrt{4-x^2}}\,dxdy = \int_1^2 \left[-\frac{1}{2}y\left(2(4-x^2)^{1/2}\right)\right]_{x=0}^{x=1} dy$$
$$= \int_1^2 -y(3^{1/2}-4^{1/2})\,dy$$
$$= (2-\sqrt{3})\left[\frac{1}{2}y^2\right]_{y=1}^{y=2} = 3 - \frac{3}{2}\sqrt{3}.$$

2. (a) The region can be regarded as a Type A region

$$D: \quad 0 \le y \le x, \quad 0 \le x \le 1.$$

$$\int_0^1 \int_0^x e^{x^2}\,dydx = \int_0^1 \left[ye^{x^2}\right]_{y=0}^{y=x} dx = \int_0^1 xe^{x^2}\,dx$$
$$= \frac{1}{2}\left[e^{x^2}\right]_0^1 = \frac{1}{2}(e-1).$$

(b) The region can be regarded as a type A region with bottom boundary $y = x^2$ and top boundary $y = \sqrt{x}$.
Since the two curves intersect at $x = 0$ and $x = 1$, the left and right are bounded by $x = 0$ and $x = 1$ respectively. So

$$D: \quad x^2 \le y \le \sqrt{x}, \quad 0 \le x \le 1.$$

$$\int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)\,dydx = \int_0^1 \left[xy + \frac{1}{2}y^2\right]_{y=x^2}^{y=\sqrt{x}} dx = \int_0^1 (x^{3/2} + \frac{1}{2}x - x^3 - \frac{1}{2}x^4)\,dx$$
$$= \left[\frac{1}{5}2x^{5/2} + \frac{1}{4}x^2 - \frac{1}{4}x^4 - \frac{1}{10}x^5\right]_0^1 = \frac{3}{10}.$$

3. The line joining $(1, 0)$ and $(4, 2)$ has equation

$$\frac{y-0}{x-1} = \frac{2-0}{4-1} = \frac{2}{3} \iff y = \frac{2}{3}x - \frac{2}{3} \iff x = \frac{3}{2}y + 1.$$

The line joining $(1, 0)$ and $(9, -3)$ has equation

$$\frac{y-0}{x-1} = \frac{(-3)-0}{9-1} = -\frac{3}{8} \iff y = -\frac{3}{8}x + \frac{3}{8} \iff x = -\frac{8}{3}y + 1.$$

The region D is is the union of D_1 and D_2, where

$$D_1: \quad y^2 \le x \le \frac{3}{2}y + 1, \quad 0 \le y \le 2,$$

$$D_2: \quad y^2 \le x \le -\frac{8}{3}y + 1, \quad -3 \le y \le 0.$$

Hence the required answer is

$$\begin{aligned}
\iint_D x\, dA &= \iint_{D_1} x\, dA + \iint_{D_2} x\, dA \\
&= \int_0^2 \int_{y^2}^{(3y/2)+1} x\, dxdy + \int_{-3}^0 \int_{y^2}^{-(8y/3)+1} x\, dxdy \\
&= \frac{19}{5} + \frac{106}{5} = 25,
\end{aligned}$$

since

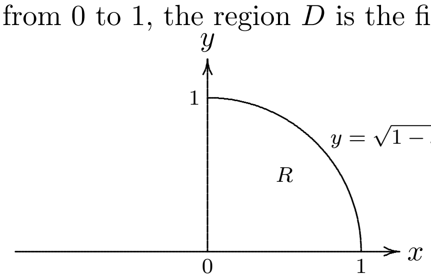
$$\int_0^2 \int_{y^2}^{(3y/2)+1} x\, dxdy = \int_0^2 \frac{1}{8}(9y^2 + 12y + 4 - 4y^4)\, dy = \frac{19}{5},$$

$$\int_{-3}^0 \int_{y^2}^{-(8y/3)+1} x\, dxdy = \int_{-3}^0 \frac{1}{18}(64y^2 - 48y + 9 - 9y^4)\, dy = \frac{106}{5}.$$

4. The region in Cartesian coordinates is given by

$$D: \quad 0 \le y \le \sqrt{1 - x^2}, \quad 0 \le x \le 1$$

This is a type A region with x-axis as the bottom boundary and upper half of the unit circle as the upper boundary.
Since the range of x is from 0 to 1, the region D is the first quadrant of the unit disk.



In polar coordinates, this is given by

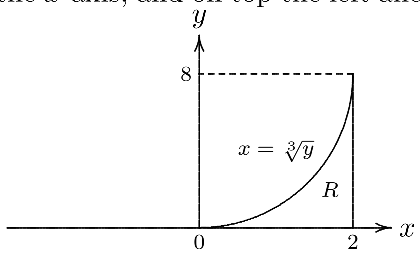
$$D: \quad 0 \le r \le 1, \quad 0 \le \theta \le \pi/2.$$

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\, dydx &= \int_0^{\pi/2} \int_0^1 e^{r^2} r\, drd\theta = \int_0^{\pi/2} d\theta \int_0^1 e^{r^2} r\, dr \\
&= \frac{\pi}{2}\left[\frac{1}{2}e^{r^2}\right]_0^1 = \tfrac{1}{4}\pi(e-1).
\end{aligned}$$

5. (a) The type B region R is given by

$$\sqrt[3]{y} \leq x \leq 2, \quad 0 \leq y \leq 8.$$

It is bounded on the left by the cubic curve $\sqrt[3]{y} = x$ and on the right by the vertical line $x = 2$.
Below it is bounded by the x-axis, and on top the left and right boundaries intersect at $y = 8$.



Converting to type A region, the lower boundary is $y = 0$, the top boundary is the cubic curve $y = x^3$.
On the left, these two boundaries intersect at $x = 0$ and on the right, it is bounded by $x = 2$.
So the region is given by

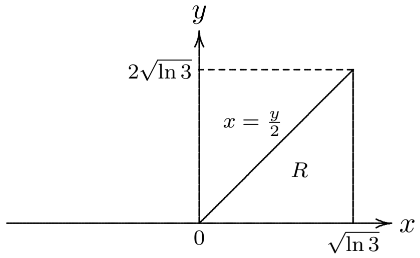
$$0 \leq y \leq x^3, \quad 0 \leq x \leq 2.$$

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dxdy = \int_0^2 \int_0^{x^3} e^{x^4}\, dydx = \int_0^2 e^{x^4} [y]_{y=0}^{y=x^3}\, dx = \int_0^2 x^3 e^{x^4}\, dx$$
$$= \left[\frac{1}{4} e^{x^4}\right]_0^2 = \frac{1}{4}(e^{16} - 1).$$

(b) The type B region R is given by

$$y/2 \leq x \leq \sqrt{\ln 3}, \quad 0 \leq y \leq 2\sqrt{\ln 3}.$$

It is bounded on the left by the straight line $x = y/2$ and on the right by the vertical line $x = \sqrt{\ln 3}$.
Below it is bounded by the x-axis, and on top the left and right boundaries intersect at $y = 2\sqrt{\ln 3}$.



Converting to type A region, the lower boundary is $y = 0$, the top boundary is the line $y = 2x$.
On the left, these two boundaries intersect at $x = 0$ and on the right, it is bounded by $x = \sqrt{\ln 3}$.
So the region is given by

$$0 \leq y \leq 2x, \quad 0 \leq x \leq \sqrt{\ln 3}.$$

$$\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\, dxdy = \int_0^{\sqrt{\ln 3}} \int_0^{2x} e^{x^2}\, dydx = \int_0^{\sqrt{\ln 3}} e^{x^2} [y]_{y=0}^{y=2x}\, dx = \int_0^{\sqrt{\ln 3}} 2xe^{x^2}\, dx$$
$$= \left[e^{x^2}\right]_0^{\sqrt{\ln 3}} = e^{\ln 3} - 1 = 2.$$

6. We change to polar coordinates:

$$\int\int_D \sqrt{4 - x^2 - y^2}dxdy = \int_{\frac{\pi}{2}}^{\pi} \left(\int_1^2 (\sqrt{4 - r^2} \quad)rdr \right) d\theta$$

$$\int_{\frac{\pi}{2}}^{\pi} \left(\int_1^2 (\sqrt{4 - r^2} \quad)rdr \right) d\theta = \frac{1}{2}\sqrt{3}\pi.$$

7. Interchange the order of integration and then apply L'Hopital's Rule and the Fundamental Theorem of Calculus, we have
$\lim\limits_{t\to 0^+} \frac{1}{t^4} \int_0^t \int_x^t \sin y^2 dydx$
$= \lim\limits_{t\to 0^+} \frac{1}{t^4} \int_0^t \int_0^y \sin y^2 dxdy$
$= \lim\limits_{t\to 0^+} \frac{\int_0^t y \sin y^2 dy}{t^4}$
$= \lim\limits_{t\to 0^+} \frac{t \sin t^2}{4t^3}$
$= \frac{1}{4} \lim\limits_{\theta\to 0^+} \frac{\sin\theta}{\theta}$, where $\theta = t^2$
$= \frac{1}{4}$