2013/2014 SEMESTER 1 MID-TERM TEST

MA1505 MATHEMATICS I

1 October 2013

8:30pm to 9:30pm

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **FOURTEEN (14)** printed pages.
2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.
3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1/10).
4. Use **only 2B pencils** for FORM CC1/10.
5. On FORM CC1/10 (section B for matric numbers starting with A, section C for others), **write** your **matriculation number** and **shade** the corresponding numbered circles **completely**. Your FORM CC1/10 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.
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9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.
10. **Do not fold** FORM CC1/10.
11. Submit FORM CC1/10 before you leave the test hall.

Formulae List

1. The **Taylor series** of f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

2.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

3.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

4.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

5.

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

6.

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

7. Let $P_n(x)$ be the nth order Taylor polynomial of $f(x)$ at $x = a$. Then

$$f(x) = P_n(x) + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

8. The **projection** of a vector $\mathbf{b}$ onto a vector $\mathbf{a}$, denoted by $\mathrm{proj}_{\mathbf{a}}\mathbf{b}$ is given by

$$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \frac{(\mathbf{a} \cdot \mathbf{b})}{||\mathbf{a}||^2}\,\mathbf{a}.$$

9. The shortest distance from a point $S\ (x_0,\ y_0,\ z_0)$ to a plane $\Pi\ :\ ax + by + cz = d$, is given by

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

1. Find the slope of the tangent line to the parametric curve

$$x = t^3 - t, \quad y = 4 - t^2$$

at the point corresponding to $t = 3$.

(A) $\frac{5}{11}$

(B) $-\frac{2}{7}$

(C) $\frac{8}{15}$

(D) $-\frac{3}{13}$

(E) None of the above

2. The side OB of a triangle OBC in the first quadrant of the xy-plane joins the origin O to a variable point B with positive x-coordinate on the x-axis, while the side OC lies on the line $y = 3x$. If the side BC passes through the point $(1, 1)$, find the slope of BC such that the area of $\triangle OBC$ is minimum.

(A) -3

(B) $-\dfrac{1}{3}$

(C) $-\dfrac{1}{2}$

(D) -2

(E) None of the above.

3. $\frac{d}{dx}\int_0^x e^{t^2}dt =$

(A) $x^2 e^{x^2-1}$

(B) e^{x^2}

(C) $2xe^{x^2}$

(D) $x^2 e^{x^2}$

(E) None of the above

4. Let $f(x)$ be a differentiable function whose derivative $f'(x)$ is continuous. It is known that

$$f(0) = 1, \quad f(4) = 3, \quad f'(0) = \frac{1}{3}, \quad f'(4) = \frac{1}{2},$$

$$\text{and} \quad \int_0^4 (f(x))^2 \, dx = 10.$$

Find the exact value of the integral

$$\int_0^4 x\,(f(x))\,(f'(x))\,dx.$$

(A) 6

(B) 13

(C) 30

(D) 45

(E) None of the above

5. The ellipse C in the xy-plane is given parametrically as follows:

$$C \ : \quad x = 6\cos\theta \quad \text{and} \quad y = 2\sin\theta.$$

In the first quadrant, find the area of the finite region bounded by C, the y-axis and the line L given by

$$L \ : \quad y = \frac{1}{\sqrt{3}}\, x.$$

(A) π

(B) $\sqrt{3}\,\pi$

(C) $\dfrac{2\pi}{\sqrt{3}}$

(D) 2π

(E) None of the above.

6. Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n(\sqrt{n+2})}\, x^n.$$

(A) $3\sqrt{2}$

(B) $\dfrac{1}{3\sqrt{2}}$

(C) 3

(D) $\dfrac{1}{3}$

(E) None of the above.

7. You want to solve the equation

$$3e^x - 2x - 3\sin x + \cos x - 5 = 0.$$

You know that there is a solution between -1 and 0. To find an approximate value of this solution, you replace e^x by its Taylor polynomial of order 3 at $x = 0$, replace $\sin x$ by its Taylor polynomial of order 3 at $x = 0$, and replace $\cos x$ by its Taylor polynomial of order 2 at $x = 0$. After simplifying the resulting expression, you arrive at a cubic equation. Which cubic equation do you get?

(A) $x^3 + 2x^2 - 2x - 2 = 0$

(B) $x^3 + 2x^2 - 3x - 1 = 0$

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8. Let $\sum_{n=0}^{\infty} a_n x^n$ denote the Taylor's series of

$$\left(1+x^2\right)\tan^{-1} x$$

at $x = 0$. What is the exact value of a_9 ?

(A) $\frac{2}{27}$

(B) $\frac{2}{45}$

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(D) $-\frac{2}{63}$

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9. Let Π be a given plane. It is known that the perpendicular line from the origin O to Π intersects Π at the point $(3, -6, 2)$. Find the exact value of the distance from the point $(1, 3, -3)$ to the plane Π.

(A) 4

(B) 6

(C) 8

(D) 10

(E) None of the above

10. The three points $A(1,2,2)$, $B(3,3,7)$ and $C(6,3,13)$ lie on the plane Π, while the two points $P(1,0,3)$ and $Q(2,1,5)$ lie on the line L. If the point $R(a,b,c)$ is the intersection of L with Π, find the exact value of the product abc.

(A) 108

(B) 118

(C) 128

(D) 168

(E) None of the above.

END OF PAPER

Additional blank page for you to do your calculations

National University of Singapore
Department of Mathematics

2013-2014 Semester 1   MA1505 Mathematics I   Mid-Term Test Answers

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Question	1	2	3	4	5	6	7	8	9	10
Answer	D	A	B	B	A	C	C	D	D	A

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Then

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1. Find the slope of the tangent line to the parametric curve

$$x = t^3 - t, \quad y = 4 - t^2$$

at the point corresponding to $t = 3$.

(A) $\frac{5}{11}$

(B) $-\frac{2}{7}$

(C) $\frac{8}{15}$

(D) $-\frac{3}{13}$

(E) None of the above

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2t}{3t^2 - 1}$$

When $t = 3$

$$\frac{dy}{dx} = \frac{-2(3)}{3(3)^2 - 1} = \underline{\underline{-\frac{3}{13}}}$$

2. The side OB of a triangle OBC in the first quadrant of the xy-plane joins the origin O to a variable point B with positive x-coordinate on the x-axis, while the side OC lies on the line $y = 3x$. If the side BC passes through the point $(1, 1)$, find the slope of BC such that the area of $\triangle OBC$ is minimum.

(A) -3

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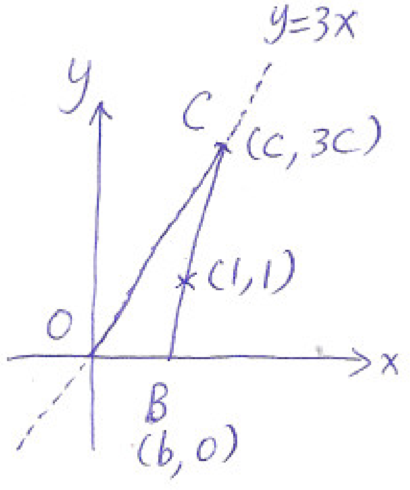
(C) $-\frac{1}{2}$

(D) -2

(E) None of the above.



Let $B=(b,0)$, $b>0$

$C=(c,3c)$

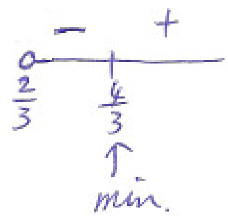
$\because (1,1)$ lies on BC

$\therefore \frac{1-0}{1-b} = \frac{3c-1}{c-1}$

$\therefore c = \frac{b}{3b-2}$ (note that $\because c>0$, $\therefore b>\frac{2}{3}$)

Area of $\triangle OBC = S = \frac{1}{2}b(3c) = \frac{3}{2}bc = \frac{3}{2}\left\{\frac{b^2}{(3b-2)}\right\}$

$\frac{dS}{db} = \frac{3}{2}\left\{\frac{2b(3b-2)-b^2(3)}{(3b-2)^2}\right\} = \frac{3}{2}\left\{\frac{b(3b-4)}{(3b-2)^2}\right\}$



$\therefore S$ attains minimum when $b=\frac{4}{3}$

slope of $BC = \frac{1-0}{1-\frac{4}{3}} = \underline{\underline{-3}}$

3. $\frac{d}{dx}\int_0^x e^{t^2}\,dt =$

(A) $x^2 e^{x^2-1}$

(B) e^{x^2}

(C) $2xe^{x^2}$

(D) $x^2 e^{x^2}$

(E) None of the above

Recall that $\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$.

$\therefore$ Answer is $\underline{\underline{e^{x^2}}}$

4. Let $f(x)$ be a differentiable function whose derivative $f'(x)$ is continuous. It is known that

$$f(0)=1,\quad f(4)=3,\quad f'(0)=\frac{1}{3},\quad f'(4)=\frac{1}{2},$$

$$\text{and}\quad \int_0^4 (f(x))^2\,dx = 10.$$

Find the exact value of the integral

$$\int_0^4 x\,(f(x))\,(f'(x))\,dx.$$

(A) 6

(B) 13

(C) 30

(D) 45

(E) None of the above

$$\int_0^4 x\,f(x)f'(x)\,dx = \int_0^4 \underbrace{x f(x)}_{u}\, d(\underbrace{f(x)}_{v})$$

$$= \left[x(f(x))^2\right]_0^4 - \int_0^4 f(x)\underbrace{\{f(x)+xf'(x)\}dx}_{du}$$

$$2\int_0^4 x f(x) f'(x)\,dx = 4(f(4))^2 - \int_0^4 (f(x))^2\,dx = 26$$

$$\therefore \int_0^4 x f(x) f'(x)\,dx = \underline{\underline{13}}$$

Second Solution:

$$\int_0^4 x f(x) f'(x)\,dx = \int_0^4 x\, d\left\{\frac{1}{2}(f(x))^2\right\}$$

$$= \frac{1}{2}x(f(x))^2\Big|_0^4 - \frac{1}{2}\int_0^4 (f(x))^2\,dx$$

$$= 2(f(4))^2 - 5$$

$$= 18 - 5 = \underline{\underline{13}}$$

5. The ellipse C in the xy-plane is given parametrically as follows:

$$C \;:\; x = 6\cos\theta \quad \text{and} \quad y = 2\sin\theta.$$

In the first quadrant, find the area of the finite region bounded by C, the y-axis and the line L given by

$$L \;:\; y = \frac{1}{\sqrt{3}}x.$$

(A) π

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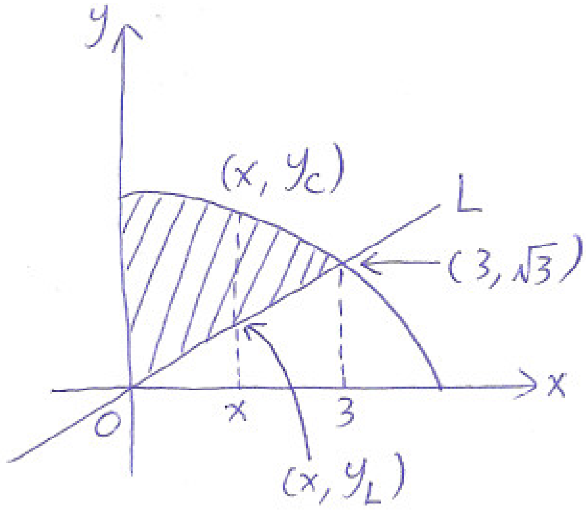

Solving for point of intersection

$y = \frac{1}{\sqrt{3}}x \Rightarrow \frac{2\sin\theta}{6\cos\theta} = \frac{1}{\sqrt{3}} \Rightarrow \tan\theta = \sqrt{3}$

$\Rightarrow \theta = \frac{\pi}{3} \Rightarrow x = 3, y = \sqrt{3}$

Observe that when $x = 0$, we have $6\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

$$\text{Area} = \int_0^3 (y_C - y_L)\,dx$$

$$= \int_0^3 y_C\,dx - \int_0^3 y_L\,dx$$

$$= \int_{\pi/2}^{\pi/3} 2\sin\theta(-6\sin\theta)\,d\theta - \frac{1}{2}3(\sqrt{3})$$

$$= 6\int_{\pi/3}^{\pi/2} (1 - \cos 2\theta)\,d\theta - \frac{3\sqrt{3}}{2} = \underline{\underline{\pi}}$$

6. Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n(\sqrt{n+2})} x^n.$$

(A) $3\sqrt{2}$

(B) $\frac{1}{3\sqrt{2}}$

(C) 3

(D) $\frac{1}{3}$

(E) None of the above.

We apply the Ratio Test:

$$\left| \frac{\frac{(-1)^{n+1}}{3^{n+1}\sqrt{n+3}} x^{n+1}}{\frac{(-1)^n}{3^n\sqrt{n+2}} x^n} \right|$$

$$= \left(\sqrt{\frac{n+2}{n+3}}\right) \frac{1}{3}|x|$$

$$= \left(\sqrt{\frac{1+\frac{2}{n}}{1+\frac{3}{n}}}\right) \frac{1}{3}|x| \longrightarrow \frac{1}{3}|x|$$

$$\therefore \frac{1}{3}|x| < 1 \Rightarrow |x| < \underline{\underline{3}}$$

7. You want to solve the equation

$$3e^x - 2x - 3\sin x + \cos x - 5 = 0.$$

You know that there is a solution between -1 and 0. To find an approximate value of this solution, you replace e^x by its Taylor polynomial of order 3 at $x = 0$, replace $\sin x$ by its Taylor polynomial of order 3 at $x = 0$, and replace $\cos x$ by its Taylor polynomial of order 2 at $x = 0$. After simplifying the resulting expression, you arrive at a cubic equation. Which cubic equation do you get?

(A) $x^3 + 2x^2 - 2x - 2 = 0$

(B) $x^3 + 2x^2 - 3x - 1 = 0$

(C) $x^3 + x^2 - 2x - 1 = 0$

(D) $x^3 + x^2 - x - 2 = 0$

(E) None of the above.

$$3\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right) - 2x - 3\left(x-\frac{x^3}{6}\right) + \left(1-\frac{x^2}{2}\right) - 5 = 0$$

$$\underline{\underline{x^3+x^2-2x-1=0}}$$

8. Let $\sum_{n=0}^{\infty} a_n x^n$ denote the Taylor's series of

$$(1+x^2)\tan^{-1} x$$

at $x = 0$. What is the exact value of a_9 ?

(A) $\frac{2}{27}$

(B) $\frac{2}{45}$

(C) $-\frac{2}{81}$

(D) $-\frac{2}{63}$

(E) None of the above.

$$(1+x^2)\tan^{-1} x = (1+x^2)\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{2n+1}$$

$$= x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{2n+1}$$

$$= x + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+3}}{2n+3} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{2n+1}$$

$$= x + \sum_{n=0}^{\infty} \left\{ \frac{(-1)^{n+1}}{2n+3} + \frac{(-1)^n}{2n+1} \right\} x^{2n+3}$$

Put $n = 3 \Rightarrow a_9 = \frac{(-1)^4}{9} + \frac{(-1)^3}{7} = \underline{\underline{-\frac{2}{63}}}$

9. Let Π be a given plane. It is known that the perpendicular line from the origin O to Π intersects Π at the point $(3, -6, 2)$. Find the exact value of the distance from the point $(1, 3, -3)$ to the plane Π.

(A) 4

(B) 6

(C) 8

(D) 10

(E) None of the above

The vector $\begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$ is normal to π

$\therefore$ equation of π is $3x - 6y + 2z = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix} = 49$

$$\text{distance} = \frac{|3(1) - 6(3) + 2(-3) - 49|}{\sqrt{49}}$$

$$= \frac{|-70|}{7} = \underline{\underline{10}}$$

10. The three points $A(1,2,2)$, $B(3,3,7)$ and $C(6,3,13)$ lie on the plane Π, while the two points $P(1,0,3)$ and $Q(2,1,5)$ lie on the line L. If the point $R(a,b,c)$ is the intersection of L with Π, find the exact value of the product abc.

(A) 108

(B) 118

(C) 128

(D) 168

(E) None of the above.

END OF PAPER

$\vec{AB} = (2,1,5)$; $\vec{BC} = (3,0,6)$

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 5 \\ 3 & 0 & 6 \end{vmatrix} = 6\vec{i} + 3\vec{j} - 3\vec{k}$$

$\therefore \Pi : 6x + 3y - 3z = 6$

$L : (x, y, z) = (1,0,3) + t(1,1,2) = (1+t, t, 3+2t)$

$6(1+t) + 3(t) - 3(3+2t) = 6 \Rightarrow t = 3$

$\therefore (a, b, c) = (4, 3, 9)$

$abc = \underline{\underline{108}}$

Additional blank page for you to do your calculations