1. $V = I \times R \implies I = \frac{V}{R}$.

   (i) $\frac{\partial I}{\partial V} = \frac{1}{R}$.

   If $R = 15$, then $\frac{\partial I}{\partial V} = \frac{1}{15} \approx 0.0667$ A/V.

   (ii) $\frac{\partial I}{\partial R} = -\frac{V}{R^2}$. If $V = 120$ and $R = 20$, then $\frac{\partial I}{\partial R} = -\frac{120}{20^2} = -0.3$ A/Ω.

   (iii) By Chain rule,

   $$\frac{dI}{dt} = \frac{\partial I}{\partial V}\frac{dV}{dt} + \frac{\partial I}{\partial R}\frac{dR}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{V}{R^2}\frac{dR}{dt}.$$

   Since $R = 400$, $I = 0.08$, $V = 32$, $\frac{dV}{dt} = -0.01$, $\frac{dR}{dt} = 0.03$, so

   $$\frac{dI}{dt} = \frac{1}{400}(-0.01) - \frac{(0.08)(400)}{400^2}(0.03) = -3.1 \times 10^{-5}.$$

2. $f_x = e^{2y-x} + xe^{2y-x}(-1) = e^{2y-x}(1-x)$ and $f_y = 2xe^{2y-x}$.

   So $f_x(-2,-1) = 3$ and $f_y(-2,-1) = -4$.

   (i) $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ is a unit vector. Thus,

   $$D_{\mathbf{u}}f(-2,-1) = 3 \times \frac{1}{\sqrt{2}} - 4 \times \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

   (ii) The unit vector in the direction of $3\mathbf{i} + 4\mathbf{j}$ is $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$. Thus,

   $$D_{\mathbf{u}}f(-2,-1) = 3 \times \frac{3}{5} - 4 \times \frac{4}{5} = -\frac{7}{5}.$$

   Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ be a unit vector. Then

   $$\begin{aligned} D_{\mathbf{u}}f(-2,-1) &= f_x(-2,-1) \times a + f_y(-2,-1) \times b \\ &= (f_x(-2,-1)\mathbf{i} + f_y(-2,-1)\mathbf{j}) \bullet (a\mathbf{i} + b\mathbf{j}) \\ &= \|f_x(-2,-1)\mathbf{i} + f_y(-2,-1)\mathbf{j}\| \, \|\mathbf{u}\| \cos\theta \quad (*) \end{aligned}$$

   where θ is the angle between $f_x(-2,-1)\mathbf{i} + f_y(-2,-1)\mathbf{j}$ and $\mathbf{u}$.

   Since the largest value of $\cos\theta$ is 1 which occurs when $\theta = 0$, this means that the largest possible value of $D_{\mathbf{u}}f(-2,-1)$ occurs when $\mathbf{u}$ is in the same direction as $f_x(-2,-1)\mathbf{i} + f_y(-2,-1)\mathbf{j} = 3\mathbf{i} - 4\mathbf{j}$.

3. $f_x = yz\cos{(xyz)}, \quad f_y = xz\cos{(xyz)} \quad$ and $\quad f_z = xy\cos{(xyz)}$.

So $f_x(\frac{1}{2}, \frac{1}{3}, \pi) = \dfrac{\sqrt{3}}{6}\pi, \quad f_y(\frac{1}{2}, \frac{1}{3}, \pi) = \dfrac{\sqrt{3}}{4}\pi \quad$ and $\quad f_z(\frac{1}{2}, \frac{1}{3}, \pi) = \dfrac{\sqrt{3}}{12}$.

(i) Let $\mathbf{u} = \dfrac{1}{\sqrt{3}}\mathbf{i} - \dfrac{1}{\sqrt{3}}\mathbf{j} + \dfrac{1}{\sqrt{3}}\mathbf{k}$.

Thus, the rate of change of f at P in the direction $\mathbf{u}$ is given by

$$D_{\mathbf{u}}f\left(\frac{1}{2}, \frac{1}{3}, \pi\right) = \frac{\sqrt{3}}{6}\pi \times \frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{4}\pi \times \left(-\frac{1}{\sqrt{3}}\right) + \frac{\sqrt{3}}{12} \times \frac{1}{\sqrt{3}} = \frac{1}{12}(1 - \pi).$$

(ii) The change in the value of f:

$$df \approx D_{\mathbf{u}}f\left(\frac{1}{2}, \frac{1}{3}, \pi\right) \times dt = \frac{1}{12}(1 - \pi) \times 0.1 \approx -0.01785.$$

So the value of f will *decrease* by about 0.01785 unit.

4. (i) Let $f(x, y) = \ln{(x^2 y)} - xy - 2x = 2\ln x + \ln y - xy - 2x$, where $x > 0$, $y > 0$.

We have $f_x = \dfrac{2}{x} - y - 2$ and $f_y = \dfrac{1}{y} - x$. Also, $f_{xx} = -\dfrac{2}{x^2}, f_{xy} = -1$ and $f_{yy} = -\dfrac{1}{y^2}$. Then $f_y = 0$ implies that $x = \dfrac{1}{y}$, and substitution into $f_x = 0$ gives $2y - y - 2 = 0$, i.e. $y = 2$. So $x = 1/2$. Thus, the only critical point is $(1/2, 2)$.

Now $D(1/2, 2) = (-8)(-1/4) - 1^2 = 1 > 0$, $f_{xx}(1/2, 2) = -8 < 0$ implies that $f(1/2, 2) = -\ln 2 - 2$ is a local maximum.

(ii) Let $g(x, y) = xy(1 - x - y)$.

We have $g_x = y - 2xy - y^2, g_y = x - x^2 - 2xy, g_{xx} = -2y, g_{yy} = -2x$ and $g_{xy} = 1 - 2x - 2y$.

Then $g_x = 0$ implies $y = 0$ or $y = 1 - 2x$. Substituting $y = 0$ into $g_y = 0$ gives $x = 0$ or $x = 1$. Substituting $y = 1 - 2x$ into $g_y = 0$ gives $x = 0$ or $x = 1/3$.

Thus the critical points are $(0, 0), (1, 0), (0, 1)$ and $(1/3, 1/3)$.

Now $D(0, 0) = D(1, 0) = D(0, 1) = -1$ while $D(1/3, 1/3) = 1/3$ and $g_{xx}(1/3, 1/3) = -2/3 < 0$.

Thus $(0, 0), (1, 0)$ and $(0, 1)$ are saddle points, and $g(1/3, 1/3) = 1/27$ is a local maximum.

(iii) Let $h(x, y) = x^2 + y^2 + x^{-2}y^{-2}$.

We have $h_x = 2x - 2x^{-3}y^{-2}$, $h_y = 2y - 2x^{-2}y^{-3}$, $h_{xx} = 2 + 6x^{-4}y^{-2}$, $h_{yy} = 2 + 6x^{-2}y^{-4}$ and $h_{xy} = 4x^{-3}y^{-3}$.

Then $h_x = 0$ implies that $2x^4y^2 - 2 = 0$ or $y^2 = x^{-4}$. Note that neither x nor y can be zero. Now $h_y = 0$ implies that $2x^2y^4 - 2 = 0$, and with $y^2 = x^{-4}$ this implies $2x^{-6} - 2 = 0$ or $x^6 = 1$, that is $x = \pm 1$. If $x = 1$, then $y = \pm 1$. If $x = -1$, then $y = \pm 1$.

So the critical points are $(1, 1), (1, -1), (-1, 1)$ and $(-1, -1)$.

Now $D(\pm 1, \pm 1) = D(\pm 1, \mp 1) = 64 - 16 > 0$ and h_{xx} is always greater than zero. Thus $h(\pm 1, \pm 1) = h(\pm 1, \mp 1) = 3$ are local minima.

5. Let $v = \dfrac{\partial u}{\partial y}$, then the given condition $u\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}$ implies that $u\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial x}v$.
Therefore we have
$$\frac{u\dfrac{\partial v}{\partial x} - v\dfrac{\partial u}{\partial x}}{u^2} = 0,$$

and hence $\dfrac{\partial}{\partial x}\left(\dfrac{v}{u}\right) = 0$. This means that $\frac{v}{u}$ is a function of one variable y only. Now the result follows by observing that $\frac{v}{u} = \frac{1}{u}\dfrac{\partial u}{\partial y} = \dfrac{\partial(\ln u)}{\partial y}$.

6. The function $P(L, K)$ is maximized, subject to the constraint $L + K = 150$.

Let
$$\tilde{P}(L, K, \lambda) = 50L^{2/5}K^{3/5} - \lambda(L + K - 150).$$

Then set

$$\begin{aligned}
\frac{\partial \tilde{P}}{\partial L} &= 20L^{-3/5}K^{3/5} - \lambda = 0 \\
\frac{\partial \tilde{P}}{\partial K} &= 30L^{2/5}K^{-2/5} - \lambda = 0 \\
\frac{\partial \tilde{P}}{\partial \lambda} &= -L - K + 150 = 0.
\end{aligned}$$

Eliminate λ from the first two equations to get

$$20L^{-3/5}K^{3/5} = 30L^{2/5}K^{-2/5}$$
$$K = \frac{3}{2}L.$$

Substituting this into the third equation yields

$$L = 60 \quad \text{and} \quad K = 90.$$

Thus, $P(60, 90) = 50(60)^{2/5}(90)^{3/5} \approx 3826$.

Answer: Allocate $60,000 to labour and $90,000 to capital investment to generate a maximum output of approximately 3,826 units.

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