Review of Chapter 6

(1) Let f be a piecewise continuous function defined on [-L,L].

We would like to represent f by

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(n\frac{\pi}{L}x\right) + b_n \sin\left(n\frac{\pi}{L}x\right) \right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(n\frac{\pi}{L}x\right)dx$$

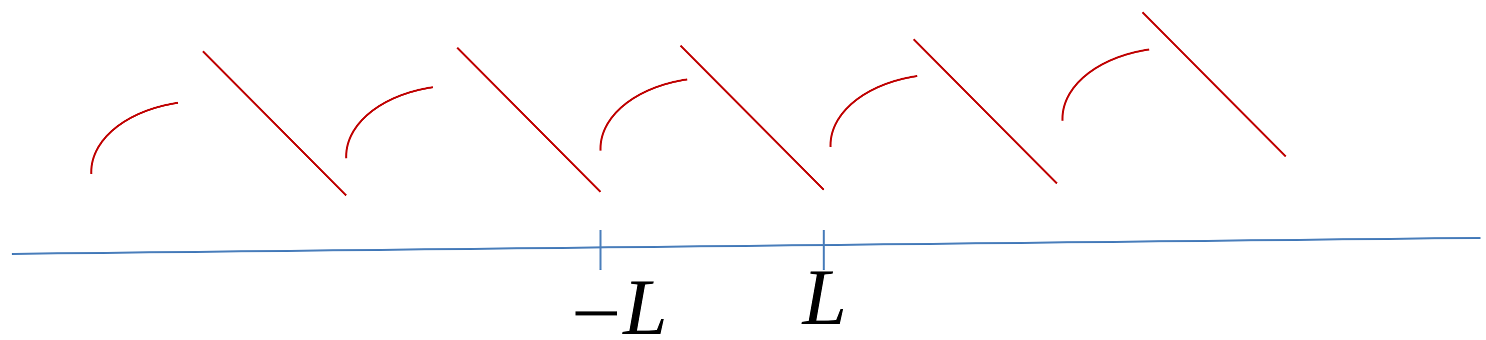
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\frac{\pi}{L}x\right)dx$$

First note that

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\frac{\pi}{L}x) + b_n \sin(n\frac{\pi}{L}x))$$

is 2L periodic function. So for convenience, we extend

f to the whole real line and f is 2L periodic function.



Is $\quad f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\frac{\pi}{L}x) + b_n \sin(n\frac{\pi}{L}x))$

ANSWER

Let ***f*** be a 2L periodic function *such that* ***f*** and ***f´*** are piecewise continuous.

(1) If f is continuous at the point x

then

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(n\frac{\pi}{L}x\right) + b_n \sin\left(n\frac{\pi}{L}x\right)\right)$$

(2) If f is discontinuous at x, then

$$\frac{1}{2}\left(f(x^+) + f(x^-)\right) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(n\frac{\pi}{L}x\right) + b_n \sin\left(n\frac{\pi}{L}x\right)\right)$$

(2) (A) Suppose f is even, then $\boxed{b_n = 0}$

Hence the Fourier Series of f is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}x\right)$$

called Fourier cosine series

(B) Suppose f is odd, then $\boxed{a_n = 0}$

Hence the Fourier Series of f is

$$\sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}x\right)$$

called Fourier sine series

(3)(A) Suppose f is defined on [0,L]

Then we may extend f to be on [-L,0]

such that f is even

Hence the Fourier series of the extended function f on [-L,L] is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(n\frac{\pi}{L}x)$$

called half-range Fourier series or half-range expansion

(3)(B) Suppose f is defined on [0,L]

Then we may extend f to be on [-L,0]

such that f is odd

Hence the Fourier series of the extended function f on [-L,L] is

$$\sum_{n=1}^{\infty} b_n \sin(n\frac{\pi}{L}x)$$

called half-range Fourier series or half-range expansion

(4) Before we compute the Fourier coefficients a_n, b_n

We will check whether the given function is odd, even, or not odd not even

How to check?

$x, x^3, \sin nx$ are odd functions

constant, $x^2, \cos nx, |x|, |\sin x|, |odd|$

are even functions

even even=even   odd odd= even

odd even= odd

(4)(cont)

If you can’t see from the given function,
then you can sketch the graph of the function f.
From the graph, we can see easily and get the answer.

If the given function is piecewise continuous, then
we need to sketch the graph piece by piece

From the graph, we also know the discontinuous points,
so we will know where f(x)=F S of f at x.

(5) Functions f in Chapter six may be constant,

$$x, x^2, \sin \alpha x, \cos \beta x, \ |x|, |\sin \alpha x|, |\cos \beta x|$$

or combination of functions just mentioned,

and ...

so formulae of integrals given
in my lecture slide "some useful facts" may be useful

(6) The answers of coefficients of Fourier series always have the following terms

$$\cos n\pi = (-1)^n \qquad n = 1, 2, 3, \ldots$$

$$\cos(2n-1)\frac{\pi}{2} = 0 \qquad n = 1, 2, 3, \ldots$$

$$\cos(n\frac{\pi}{2}) = \begin{cases} \cos(2m\pi / 2) = \cos(m\pi) = (-1)^m & \text{if n=2m} \\ \cos[(2m-1)\pi / 2] = 0 & \text{if n=2m-1} \end{cases}$$

$$\sin(2n-1)\frac{\pi}{2} = (-1)^{n+1} \;\; or \;\; = (-1)^{n-1} \qquad n = 1, 2, 3, \ldots$$

$$\sin n\pi = 0 \qquad n = 1, 2, 3, \ldots$$

$$\sin(n\frac{\pi}{2}) = \begin{cases} \sin(2m\pi / 2) = \sin(m\pi) = 0 & \text{if n=2m} \\ \sin[(2m-1)\pi / 2] = (-1)^{m+1} & \text{if n=2m-1} \end{cases}$$

(6)(cont) For example,

$$b_n = \frac{2k}{n\pi}(1-\cos n\pi) = \frac{2k}{n\pi}(1-(-1)^n)$$

$$b_9 = \frac{2k}{9\pi}(1-(-1)^9) = \frac{2k}{9\pi}(1-(-1)) = \frac{2k}{9\pi}(2)$$

$$b_{100} = \frac{2k}{100\pi}(1-(-1)^{100}) = \frac{2k}{100\pi}(1-1) = 0$$

$$a_n = \frac{4k}{n^2\pi^2}(2\cos\frac{n\pi}{2} - \cos n\pi - 1)$$

$$a_3 = \frac{4k}{n^2\pi^2}(2\cos\frac{n\pi}{2} - (-1)^n - 1)$$

$$= \frac{4k}{3^2\pi^2}(2\cos\frac{3\pi}{2} - (-1)^3 - 1)$$

$$= \frac{4k}{3^2\pi^2}(0 - (-1) - 1) = 0$$

$$a_6 = \frac{4k}{6^2\pi^2}(2\cos\frac{6\pi}{2} - (-1)^6 - 1)$$

$$= \frac{4k}{6^2\pi^2}(2\cos 3\pi - (-1)^6 - 1)$$

$$= \frac{4k}{6^2\pi^2}(2(-1)^3 - (-1)^6 - 1)$$

$$= \frac{4k}{6^2\pi^2}(2(-1) - 1 - 1)$$

♣ **(2005)**

Let $f(x) = 2x + 1$ for all $x \in (-\pi, \pi)$ and $f(x) = f(x + 2\pi)$. Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier Series which represents $f(x)$. Find the value of $a_0 + a_5 + b_5$.

Let $g(x) = x$. **(odd)** Thus $g(x) = \sum_{n=1}^{\infty} c_n \sin nx$,

where $c_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx\, dx = (-1)^{n+1} \frac{2}{n}$.

Hence F S of f(x)=2x+1 is

$$2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nx + 1 = 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{n} \sin nx$$

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

and so $a_0 + a_5 + b_5 = 1 + 0 + (-1)^6 \frac{4}{5} = \boxed{\frac{9}{5}}$.

Let

$$f(x) = x^2\sqrt{\pi^2 - x^2}, \quad -\pi \le x \le \pi,$$

and $f(x + 2\pi) = f(x)$ for all x. Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

(2008)

be the Fourier Series which represents $f(x)$. Find the **exact value** of $b_2 + b_3 + \sum_{n=1}^{\infty} a_n$.

Since f is even, $b_n = 0$ for each $n = 1, 2, 3, \cdots$.

f is continuous everywhere, just look at the values of f at endpts $\pi, -\pi$

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = f(x) = x^2\sqrt{\pi^2 - x^2}$$

Putting $x = 0$, we have $a_0 + \sum_{n=1}^{\infty} a_n = f(0) = 0$.

That is, $\sum_{n=1}^{\infty} a_n = -a_0$.

Now,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \quad = \frac{1}{\pi}\int_0^{\pi} x^2\sqrt{\pi^2 - x^2}\,dx$$

$$(\text{let} \quad x = \pi\sin\theta)$$

$$= \frac{1}{\pi}\int_0^{\pi/2} (\pi^2\sin^2\theta)(\pi\cos\theta)(\pi\cos\theta d\theta)$$

This part is optional

$$= \frac{\pi^3}{4}\int_0^{\pi/2} \sin^2 2\theta d\theta \quad = \frac{\pi^3}{8}\int_0^{\pi/2}(1 - \cos 4\theta)d\theta \quad = \frac{\pi^4}{16}\ .$$

Thus, $\quad b_2 + b_3 + \sum_{n=1}^{\infty} a_n = -\frac{\pi^4}{16}$

♣ **(2007)**

Let $f(x) = |\sin x|$ for all $x \in (-\pi, \pi)$, and $f(x + 2\pi) = f(x)$ for all x. Let

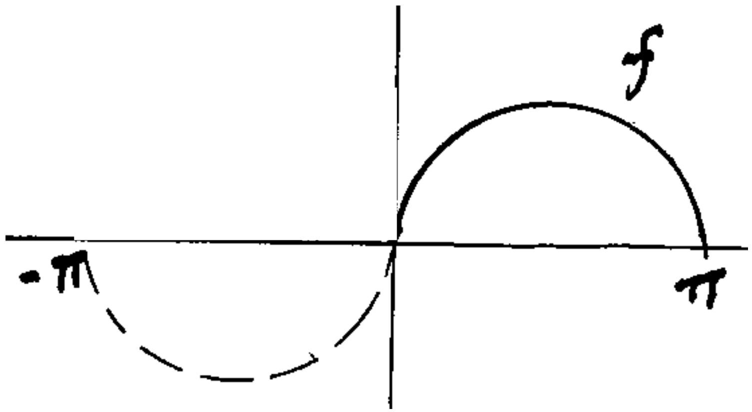
$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier Series which represents $f(x)$. Let m denote a fixed positive integer. Find $a_0 + a_2 + a_{2m+1} + b_m$.

As f is even,

$$b_n = 0 \quad \forall\, n = 1, 2, \ldots$$

Observe that



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{2}{\pi},$$

- $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$$
$(n \geq 2)$

$$\boxed{\begin{aligned}&2\cos A \sin B\\ &= \sin(A+B) - \sin(A-B)\end{aligned}}$$

$$= \frac{1}{\pi}\int_0^{\pi}\left\{\sin(n+1)x - \sin(n-1)x\right\}dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\frac{(-1)^{n+2}}{n+1} + \frac{1}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n-1}\right\}$$

thus $a_{2m+1} = 0 \quad \forall\, m = 1, 2, \cdots$

$$a_2 = \frac{1}{\pi}\left(\frac{2}{3} - 2\right) = -\frac{4}{3\pi}$$

Hence $a_0 + a_2 + a_{2m+1} + b_m = \dfrac{2}{\pi} - \dfrac{4}{3\pi} = \dfrac{2}{3\pi}$

(7) Suppose f is 2L periodic function

However the function is given on [0,2L]

Can we still apply the formula

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\frac{\pi}{L}x)dx$$

ANS:YES

$$a_n = \frac{1}{L}\int_{-L}^{L} \overbrace{f(x)\cos(n\frac{\pi}{L}x)}^{\text{2L periodic}}dx$$

$$= \frac{1}{L}\int_{0}^{2L} f(x)\cos(n\frac{\pi}{L}x)dx$$

(8) Fourier series of $f(x) = x^2$

is $$\frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} \cos kx$$

Therefore

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} \cos kx$$ for all x

Since f is continuous at every pt x

Now we shall look at some special points to get some interesting equalities. Try $x = 0, \pi / 2, \pi$

Let $x = \pi$ get

$$\pi^2 = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} \cos k\pi$$

So

$$\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos k\pi = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} (-1)^k = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$$

Let x=0, subst into

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} \cos kx$$

get

$$\frac{\pi^2}{12} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2}$$

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$$

(9) (A)

Let $f(x) = 2 + \sin 10x + 8\cos 50x$

which is 2π periodic function

What is the F S of f?

Since F S of f is of the form

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$$

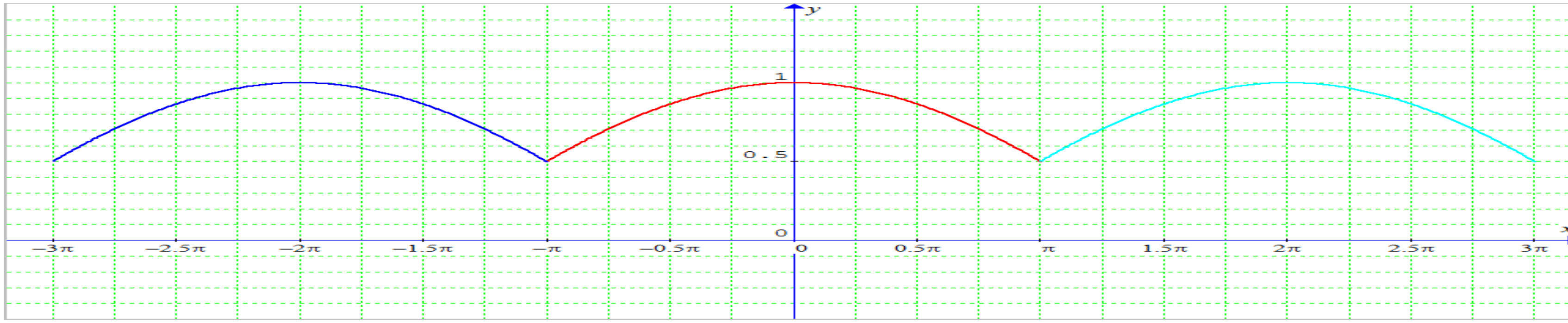
i.e.,

$$f(x) = 2 + \sin 10x + 8\cos 50x = a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$$

Hence $a_0 = 2, a_{50} = 8, b_{10} = 1$

So the F S of f is the function f itself $2 + \sin 10x + 8\cos 50x$

(B) Let $f(x) = \cos\left(\dfrac{x}{3}\right)$ on $[-\pi, \pi]$ *and* $f(x+2\pi) = f(x)$



Is the F S of f is the function f itself? ANS: No, No

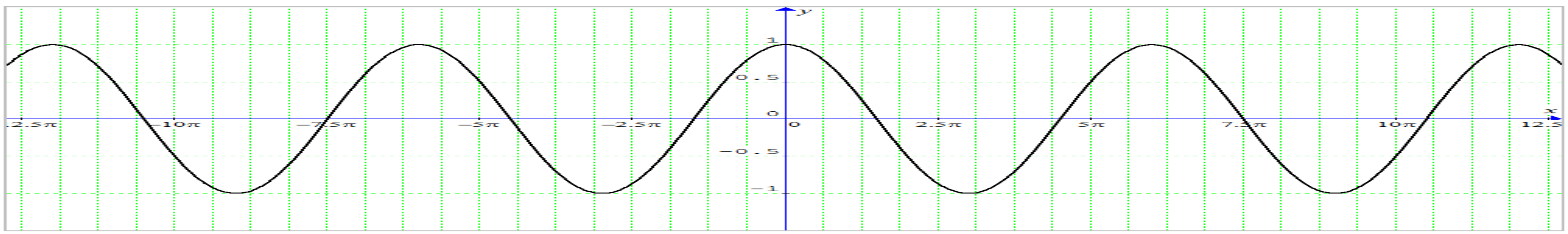
Since the F S of 2π periodic function is of the form

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$\cos\left(\dfrac{x}{3}\right)$ is not of the form cos nx

(B)(cont.)

However if $g(x) = \cos\left(\frac{x}{3}\right)$ *for* all x



Is the F S of this 2(3π) periodic function g is the function g itself?

ANS: YES, YES

Since the F S of 2(3π) periodic function is of the form

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\frac{\pi}{L}x) + b_n \sin(n\frac{\pi}{L}x)) \qquad where\ L = 3\pi$$

2012

Question 1 (a) [5 marks] (Multiple Choice Question)

Let $f(x)$ be a function defined by

$$f(x) = 1505 + 1506x + 1507x^2 + 1508x^3 \quad \text{if} \quad -\pi < x < \pi,$$

and $f(x + 2\pi) = f(x)$.

Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series for $f(x)$.

Find the **exact value** of

$$a_0 + \sum_{n=1}^{\infty} a_n.$$

(A) 1505 (B) 1506 (C) 1507 (D) 1508

Answer 1(a)	(A) or 1505

(Show your working below and on the next page.)

$\because$ f is continuous at $x=0$

$\therefore$ $a_0 + \sum_{n=1}^{\infty} a_n = f(0) = \underline{\underline{1505}}$

2012

Question 1 (b) [5 marks]

Find the first two non-zero terms of the Fourier series of the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = x^2 \quad \text{if } -1 \leq x < 1$$

and $f(x) = f(x+2)$ for all $x \in \mathbb{R}$. Give **exact values** for your answer.

Answer 1(b)	$\frac{1}{3} - \frac{4}{\pi^2} \cos \pi x$

(Show your working below and on the next page.)

f is even $\Rightarrow b_n = 0$ for $n = 1, 2, 3, \ldots$

$$a_0 = \int_0^1 x^2 dx = \tfrac{1}{3} x^3 \Big|_0^1 = \tfrac{1}{3}$$

$$\begin{aligned} a_1 &= 2 \int_0^1 x^2 \cos \pi x \, dx \\ &= \tfrac{2}{\pi} \int_0^1 x^2 \, d(\sin \pi x) \\ &= \tfrac{2}{\pi} \left\{ \left[x^2 \sin \pi x \right]_0^1 - \int_0^1 2x \sin \pi x \, dx \right\} \\ &= \tfrac{4}{\pi^2} \int_0^1 x \, d(\cos \pi x) \\ &= \tfrac{4}{\pi^2} \left\{ \left[x \cos \pi x \right]_0^1 - \int_0^1 \cos \pi x \, dx \right\} \\ &= -\tfrac{4}{\pi^2} \end{aligned}$$

$$\therefore f(x) \sim \underline{\underline{\tfrac{1}{3} - \tfrac{4}{\pi^2} \cos \pi x}} + \ldots$$

2011

Question 3 (a) [5 marks]

Let $f(x)$ be a function defined by

$$f(x) = x^{2010} + x \quad \text{if } -\pi < x < \pi\ ,$$

and $f(x + 2\pi) = f(x)$.
Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series for $f(x)$.
(i) Find the **exact value** of a_0.
(ii) Find the **exact value** of b_{2011}.

Answer 3(a)(i)	$\frac{\pi^{2010}}{2011}$	Answer 3(a)(ii)	$\frac{2}{2011}$

(Show your working below and on the next page.)

Note that x^{2010} is the even part
x is the odd part

$$\therefore a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^{2010}\,dx = \frac{1}{\pi}\int_0^{\pi} x^{2010}\,dx = \underline{\underline{\frac{\pi^{2010}}{2011}}}$$

$$b_{2011} = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin 2011x\,dx = \frac{2}{\pi}\int_0^{\pi} x \sin 2011x\,dx$$

$$= -\frac{2}{2011\pi}\int_0^{\pi} x\, d(\cos 2011x)$$

$$= -\frac{2}{2011\pi}\left\{ x\cos 2011x \Big|_0^{\pi} - \int_0^{\pi} \cos 2011x\,dx \right\}$$

$$= \underline{\underline{\frac{2}{2011}}}$$

2011

Question 3 (b) [5 marks]

Let

$$f(x) = \cos x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2},$$

and $f(x+\pi) = f(x)$. Write down the Fourier series expansion for $f(x)$ up to and including the first two non-zero terms. Give **exact values** in terms of π in the simplest form for your answer.

Answer 3(b)	$\frac{2}{\pi} + \frac{4}{3\pi}\cos 2x + \cdots$

(Show your working below and on the next page.)

Note that $\cos x$ is even, $L = \frac{\pi}{2}$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos x\,dx = \frac{2}{\pi}\int_0^{\pi/2}\cos x\,dx$$

$$= \frac{2}{\pi}\sin x\Big|_0^{\pi/2} = \frac{2}{\pi}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\cos x\cos 2nx\,dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2}\left[\frac{1}{2}\cos(2n+1)x + \frac{1}{2}\cos(2n-1)x\right]dx$$

$$= \frac{2}{\pi}\left[\frac{\sin(2n+1)x}{2n+1} + \frac{\sin(2n-1)x}{2n-1}\right]_0^{\pi/2}$$

$$a_1 = \frac{2}{\pi}\left[\frac{\sin\frac{3}{2}\pi}{3} + \frac{\sin\frac{\pi}{2}}{1}\right] = \frac{2}{\pi}\left(-\frac{1}{3}+1\right) = \frac{4}{3\pi}$$

$$\therefore f(x) \sim \frac{2}{\pi} + \frac{4}{3\pi}\cos 2x + \cdots$$

Question 3 (a) [5 marks]

2010

Let $f(x)$ be a function defined by

$$f(x) = \cos\frac{x}{2} \quad \text{if } -\pi < x < \pi\,,$$

and $f(x + 2\pi) = f(x)$.
Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series for $f(x)$.
(i) Find the **exact value** of a_0.
(ii) Find the **exact value** of $\pi(a_9)$. Give your answer as a fraction in its simplest form.

Answer 3(a)(i)	$\frac{2}{\pi}$	Answer 3(a)(ii)	$\frac{4}{323}$

(Show your working below and on the next page.)

(i) $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\frac{x}{2}\,dx = \frac{1}{\pi}\int_0^{\pi}\cos\frac{x}{2}\,dx = \frac{2}{\pi}\sin\frac{x}{2}\Big|_0^{\pi} = \underline{\underline{\frac{2}{\pi}}}$

(ii) $a_9 = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos\frac{x}{2}\cos 9x\,dx =$

$= \frac{1}{\pi}\int_0^{\pi}\left(\cos\frac{19}{2}x + \cos\frac{17}{2}x\right)dx$

$= \frac{1}{\pi}\left[\frac{2}{19}\sin\frac{19}{2}x + \frac{2}{17}\sin\frac{17}{2}x\right]_0^{\pi} = \frac{1}{\pi}\left(-\frac{2}{19} + \frac{2}{17}\right) = \frac{4}{323\pi}$

$\therefore \pi a_9 = \underline{\underline{\frac{4}{323}}}$

2010

Question 3 (b) [5 marks]

Let

$$f(x) = x + 1, \quad 0 \le x \le 2.$$

Write down the sine Fourier half range expansion for $f(x)$ up to and including the first two non-zero terms. Give **exact values** in terms of π in the simplest form for your answer.

Answer 3(b)	$\frac{8}{\pi}\sin\frac{\pi x}{2} - \frac{2}{\pi}\sin \pi x$

(Show your working below and on the next page.)

$$b_n = \frac{2}{2}\int_0^2 (x+1)\sin\frac{n\pi x}{2}\,dx$$

$$= -\frac{2}{n\pi}\int_0^2 (x+1)\, d\left(\cos\frac{n\pi x}{2}\right)$$

$$= \left[-\frac{2}{n\pi}(x+1)\cos\frac{n\pi x}{2}\right]_0^2 + \frac{2}{n\pi}\int_0^2 \cos\frac{n\pi x}{2}\,dx$$

$$= -\frac{6}{n\pi}\cos n\pi + \frac{2}{n\pi} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\Big|_0^2$$

$$= -\frac{6}{n\pi}(-1)^n + \frac{2}{n\pi}$$

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} = \underline{\underline{\frac{8}{\pi}\sin\frac{\pi x}{2} - \frac{2}{\pi}\sin \pi x + \cdots}}$$