MA1505

Your Group's Lecturer:

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- **Assessment**
- Mid Term Test: Weightage: 20%
- Final Examination: Weightage: 80%

Course Outline

1. Functions
2. Differentiation
3. Integration
4. Taylor Series
5. Vectors
6. Fourier Series
7. Partial Differentiation
8. Multiple Integration
9. Line Integrals
10. Surface Integrals

Reference Text:

Thomas' Calculus

Author : Giordano, Hass, Thomas, Weir

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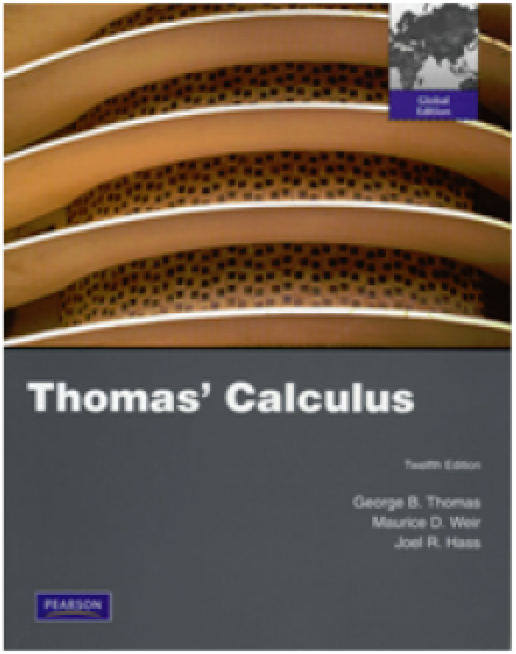


Chapter 1: Functions

1.1 Functions

It is common that the values of one variable depend on the values of another. E.g. the area A of a region on the plane enclosed by a circle depends on the radius r of the circle ($A = \pi r^2$, $r > 0$.) Many years ago, the Swiss mathematician Euler invented the symbol $y = f(x)$ to denote the statement that "y is a function of x".

A function represents a rule that assigns a *unique* value y to each value x.

We refer to x as the *independent variable* and y the *dependent* variable.

One can also think of a function as an input-output system/process: input the value x and output the value $y = f(x)$. (This becomes particularly useful when we combine or composite functions together.)

1.2 Operations on Functions

1.2.1 Arithmetical operations

Let f and g be two functions.

(i) The functions $(f \pm g)(x) = f(x) \pm g(x)$, called the sum or difference of f and g.

(ii) The function $(fg)(x) = f(x)g(x)$, called the product of f and g.

(iii) The function $(f/g)(x) = f(x)/g(x)$, called the quotient of f by g, is defined where $g(x) \neq 0$;

1.2.2 Composition

Let $f : D \longrightarrow \mathbb{R}$ and $g : D' \longrightarrow \mathbb{R}$ be two (real) functions with domains D and D' respectively.

The function

$$(f \circ g)(x) = f(g(x)),$$

called f *composed with* g or f *circle* g, is defined on the subset of D' for which the values $g(x)$ (i.e. the range of g) are in D.

1.2.3 Example

Let $f(x) = x - 7$ and $g(x) = x^2$ (defined on all of $\mathbb{R}$). Then

$$(f \circ g)(2) \ = \ f(g(2)) = f(4) = -3, \quad \text{and}$$

$$(g \circ f)(2) \ = \ g(f(2)) = g(-5) = 25.$$

Note that in general $f \circ g \neq g \circ f$.

1.3 Limits

In this section we are interested in the behaviour of f as x gets closer and closer to a.

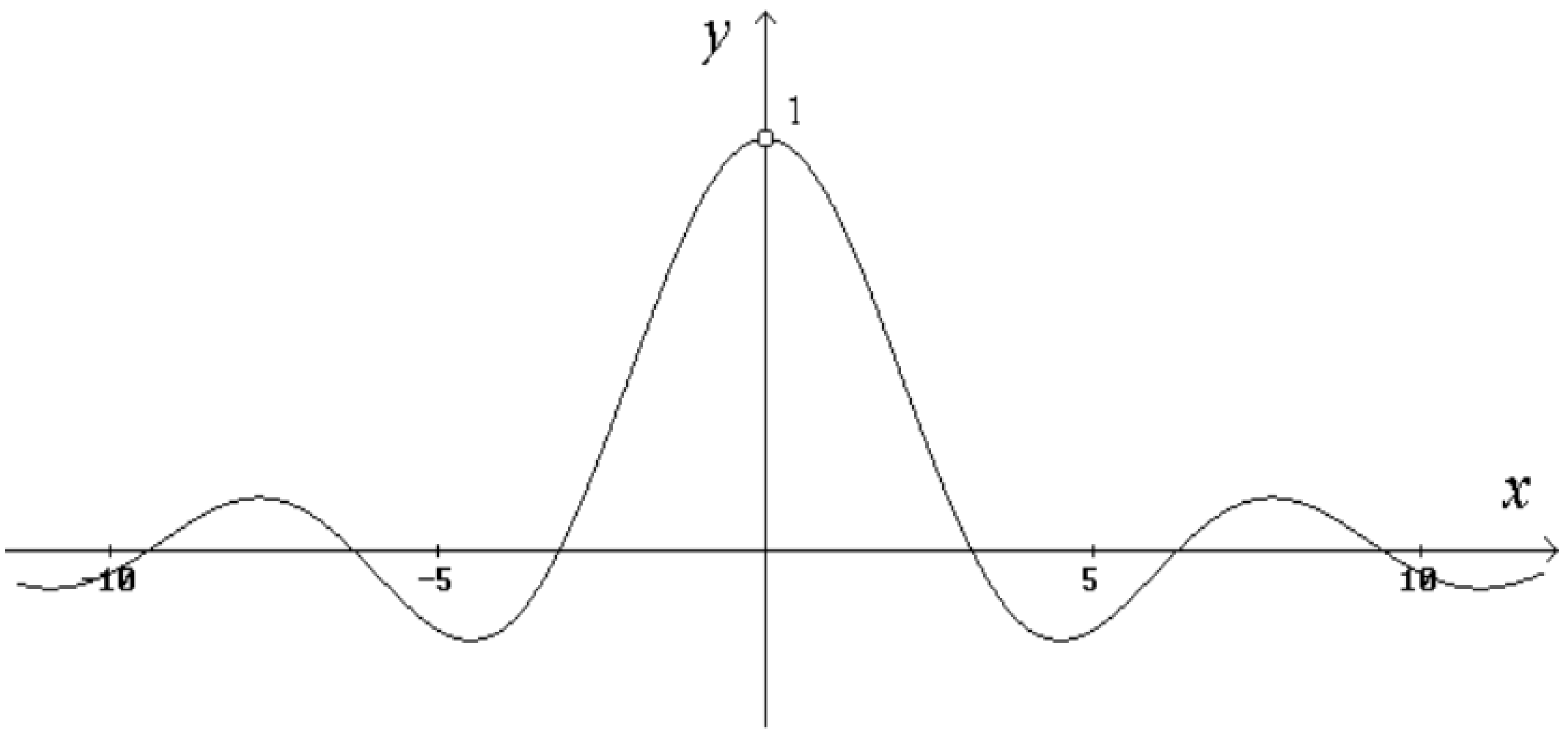
1.3.1 Example

Let $D = \{x \in \mathbb{R} : x \neq 0\}$ and we consider the function $f : D \longrightarrow \mathbb{R}$ given by $f(x) = \dfrac{\sin(x)}{x}$. ($x$ is in radian.) Describe its behaviour as x tends to 0.

Clearly when $x = 0$, $\dfrac{\sin(0)}{0} = \dfrac{0}{0}$ does not make sense. It is defined everywhere except at 0 and thus it makes sense to ask how it behaves as it is evaluated at arguments which are closer and closer to 0.

If we plot the graph of $f(x)$, we see that as x gets closer and closer to 0 from either sides (and not reaching 0 itself), $f(x)$ approaches 1. In this case, we say that "the limit of f as x tends to 0 is equal to 1". We use the following notation:

$$\lim_{x \to 0} f(x) = 1.$$



y
1
x
-10
-5
5
10

1.3.2 Informal Definition

Let $f(x)$ be defined on an open interval I containing x_0, except possibly at x_0 itself. If $f(x)$ gets arbitrary close to L when x is sufficiently close to x_0, then we say that the limit of $f(x)$ as x tends to x_0 is the number L and we write

$$\lim_{x \longrightarrow x_0} f(x) = L.$$

1.3.3 Rules of Limits

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = L'$, then the following statements are easy to verify:

(i) $\lim_{x\to a}(f \pm g)(x) = L \pm L'$;

(ii) $\lim_{x\to a}(fg)(x) = LL'$;

(iii) $\lim_{x\to a} \frac{f}{g}(x) = \frac{L}{L'}$ provided $L' \neq 0$;

(iv) $\lim_{x\to a} kf(x) = kL$ for any real number k.

Chapter 2. Differentiation

2.1 Derivative

2.1.1 Derivative

Let $f(x)$ be a given function. The derivative of f at the point a, denoted by $f'(a)$, is defined to be

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad (*)$$

provided the limit exists.

An equivalent formulation of $(*)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

If we use y as the dependent variable, i.e., $y = f(x)$,

then we also use the notation

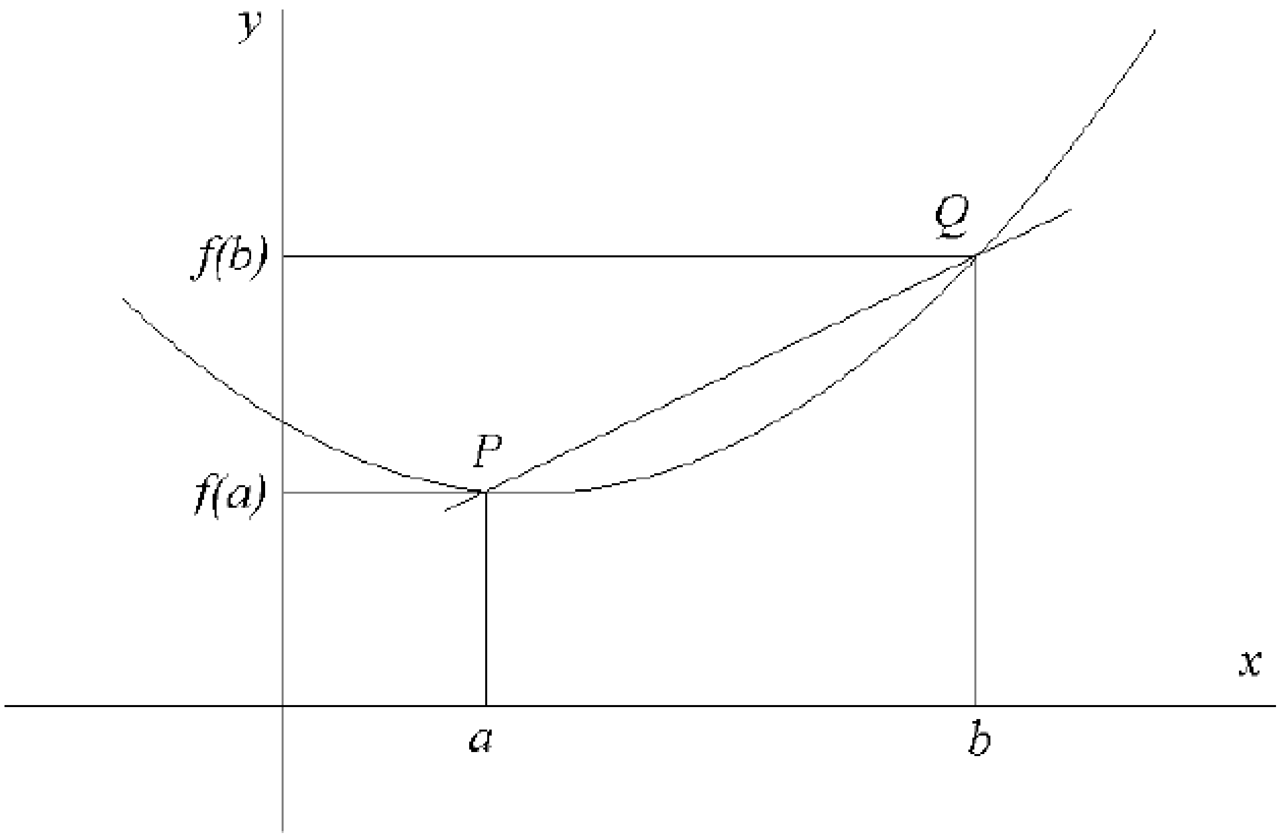
$$\left.\frac{dy}{dx}\right|_{x=a} = \frac{dy}{dx}(a) = f'(a).$$

2.1.2 Differentiable functions

If the derivative $f'(a)$ exists, we say that the function f is *differentiable* at the point a. If a function is differentiable at every point in its domain, we say that the function is differentiable.

2.1.3 Geometrical Meaning

Let us start with the graph of a function f which is differentiable at a (see figure below). Then $\dfrac{f(b) - f(a)}{b - a}$ is the slope of the straight line joining the two points $P = (a, f(a))$ and $Q = (b, f(b))$ (such a line is called a secant to the graph). As b tends to a (so Q approaches P), the secant becomes the tangent, and thus, geometrically, the derivative is just the slope of the tangent to the graph.

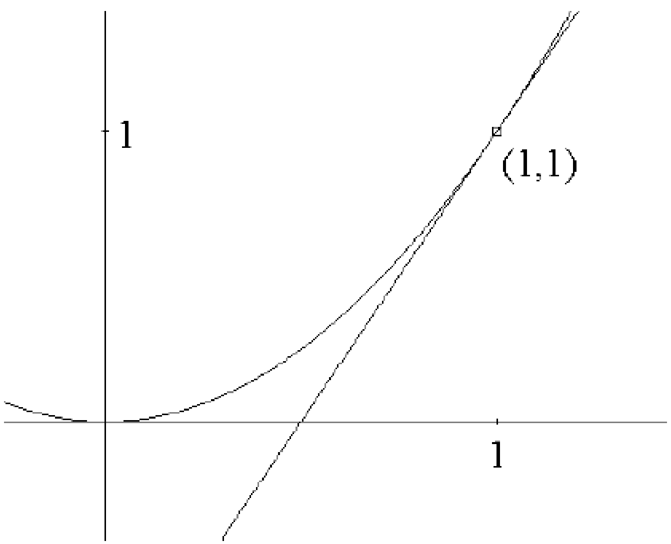


y
Q
f(b)
P
f(a)
x
a
b

Therefore, a function has a derivative at a point a if the slopes of the secant lines through the point $P = (a, f(a))$ and a nearby point Q on the graph approach a limit as Q approaches P. Whenever the secants fail to take up a limiting position or become vertical as Q approaches P, the derivative does not exist.

2.1.4 Example

Find equations of the lines which are tangent and normal to the curve $y = x^2$ at $x = 1$ respectively.



Solution. The slope of the tangent is given by $f'(1) = 2$. The point of contact between the slope and the curve is $(1, 1)$. Thus an equation of the slope, by the point-slope form, is $y - 1 = 2(x - 1)$.

As for the normal, the slope of the normal is $-1/2$ and it contains the same point $(1, 1)$. So an equation of the normal is $y - 1 = (-1/2)(x - 1)$.

2.1.5 Rules of Differentiation

Let k be a constant and let f and g be differentiable.

Linearity

(i) $(kf)'(x) = kf'(x)$, and

(ii) $(f \pm g)'(x) = f'(x) \pm g'(x)$.

Product Rule

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Quotient Rule

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Chain Rule

Assume that the compositions $f \circ g$ and $f' \circ g$ are defined. Then

$$(f \circ g)'(x) = f'(g(x))g'(x) \equiv (f' \circ g)(x)g'(x)$$

2.1.6 Remark

The Chain Rule is often phrased in the following way:

We start with a function $y = f(u)$. Then we make a change of variable $u = u(x)$ (i.e., we write the *old* variable u in terms of a *new* variable x). Substituting the change of variable $u(x)$ into the function, we get a function $y = \tilde{f}(x) = f(u(x))$ of the new variable x. Now we take the derivative of the function $\tilde{f}$ in terms of the new variable x. The Chain Rule is:

$$\frac{dy}{dx} = \left.\frac{dy}{du}\right|_{u=u(x)} \frac{du}{dx}, \qquad (**)$$

where

$$\frac{dy}{dx} = \tilde{f}'(x) = \frac{d}{dx} f(u(x)),$$

$$\frac{dy}{du} = f'(u) = \frac{d}{du} f(u), \qquad \left.\frac{dy}{du}\right|_{u=u(x)} = f'(u)|_{u=u(x)} .$$

General Formulas

Power:

$$\frac{d}{dx}x^n = nx^{n-1}$$

Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Exponential and Logarithmic Functions

$$\frac{d}{dx} e^x = e^x \qquad \frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} a^x = a^x \ln a \qquad \frac{d}{dx} (\log_a x) = \frac{1}{x \ln a}$$

Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1 + x^2} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2 - 1}}$$

Note: For inverse trigonometric functions:

(1) $\sec^{-1} x = \cos^{-1} \frac{1}{x}$

(2) $\csc^{-1} x = \sin^{-1} \frac{1}{x}$

(3) $\cot^{-1} x = \tan^{-1} \frac{1}{x}$

Proof: e.g. (1)

$$\cos\{\sec^{-1}x\} = \frac{1}{\sec\{\sec^{-1}x\}}$$

$$= \frac{1}{x}$$

$$\therefore \sec^{-1}x = \cos^{-1}\frac{1}{x} \quad //$$

e.g. To find $\frac{d}{dx}\cos^{-1}x$:

Let $y = \cos^{-1}x$

$\therefore \cos y = x$

$\therefore \frac{d}{dx}\cos y = \frac{dx}{dx} = 1$

$-\sin y \frac{dy}{dx} = 1$ (use Chain Rule $\frac{d}{dx}\cos y = \left(\frac{d}{dy}\cos y\right)\left(\frac{dy}{dx}\right)$)

$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}} \quad //$$

e.g. To find $\frac{d}{dx}\sec^{-1}x$:

$$\frac{d}{dx}\sec^{-1}x = \frac{d}{dx}\cos^{-1}\frac{1}{x}$$

$$= -\frac{1}{\sqrt{1-\frac{1}{x^2}}}\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$$

$$= \frac{1}{\frac{x^2}{\sqrt{x^2}}\sqrt{x^2-1}}$$

$$= \frac{1}{\frac{x^2}{|x|}\sqrt{x^2-1}} \quad \left(\text{note: } \sqrt{x^2} = |x|\right)$$

$$= \frac{1}{\frac{|x|^2}{|x|}\sqrt{x^2-1}} \quad \left(\text{note: } x^2 = |x|^2\right)$$

$$= \frac{1}{|x|\sqrt{x^2-1}} \quad //$$

2.2 Other Types of Differentiation

2.2.1 Parametric Differentiation

Suppose x and y are functionally dependent but are both expressed in terms of a parameter t. Then we can differentiate y with respect to x (provided it exists) as follows:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

In other words, suppose the function $y = f(x)$ is determined by the following equations

$$\begin{cases} y = u(t), \\ x = v(t). \end{cases}$$

Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{u'(t)}{v'(t)}.$$

2.2.2 Example

Let $x = a(t - \sin t)$ and $y = a(1 - \cos t)$. Then

$$\frac{dy}{dx} = \frac{a \sin t}{a(1 - \cos t)} = \cot \frac{t}{2}.$$

2.2.3 Implicit Differentiation

This is an application of the chain rule. This method is used when x and y are functionally dependent but this dependence is given implicitly by means of the equation

$$F(x, y) = 0.$$

In other words, the function $y = y(x)$ is determined by the above equation. To compute $\frac{dy}{dx}$ we may differentiate both sides of the above equation with respect to x, and solve $\frac{dy}{dx}$.

2.2.4 Example

Consider the function $y = y(x)$ which is determined by the equation

$$x^2 + y^2 - a^2 = 0.$$

To compute $\frac{dy}{dx}$, we differentiate the equation with respect to x:

$$2x + 2y\frac{dy}{dx} = 0,$$

from which we get $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

Note that we used the Chain rule to get

$$\frac{d}{dx}y^2 = \frac{d}{dy}y^2 \cdot \frac{dy}{dx} = 2y\frac{dy}{dx}.$$

2.2.5 Example

Find $\dfrac{dy}{dx}$ if $2y = x^2 + \sin y$.

Solution. Differentiate both sides with respect to x,

$$2\frac{dy}{dx} = 2x + \cos\; y\frac{dy}{dx}.$$

So

$$(2-\cos\ y)\frac{dy}{dx}=2x\Rightarrow\frac{dy}{dx}=\frac{2x}{2-\cos y}.$$

2.2.6 Example

Let $y = x^x$, $x > 0$. Find $\dfrac{dy}{dx}$.

Solution. $y = x^x$. Then $\ln y = x \ln x$. Differentiate

both sides with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x.$$

So

$$\frac{dy}{dx} = y(1 + \ln x) = x^x(1 + \ln x).$$

2.2.7 Higher Order Derivatives

Higher order derivatives are obtained when we differentiate repeatedly. Let $y = f(x)$, then the following notation is used:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x), \qquad \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x).$$

In general, the nth derivative is denoted by

$$\frac{d^n y}{dx^n} \quad \text{or} \quad f^{(n)}(x).$$

2.2.8 Example

Let $f(x) = \sqrt{x}$. Compute $f'''(x)$.

Solution

$$f'(x) = \frac{1}{2}x^{-1/2}, \quad f''(x) = -\frac{1}{4}x^{-3/2}, \quad f'''(x) = \frac{3}{8}x^{-5/2}.$$

2.3 Maxima and Minima

2.3.1 Local and absolute extremes

A function f has a *local (relative) maximum* value at a point c of its domain if $f(x) \leq f(c)$ for all x in a neighborhood of c. The function has an *absolute maximum* value at c if $f(x) \leq f(c)$ for all x in the domain.

Similarly a function f has a *local (relative) minimum* value at a point c of its domain if $f(x) \geq f(c)$ for all x in a neighborhood of c. The function has an *absolute minimum* value at c if $f(x) \geq f(c)$ for all x in the domain.

Local (respectively, absolute) minimum and maximum values are called local (respectively, absolute) *extremes.*

2.3.2 Finding extreme values

Points where f can have an extreme value are

(1) Interior points where $f'(x) = 0$.

(2) Interior points where $f'(x)$ does not exist.

(3) End points of the domain of f.

2.3.3 Critical points

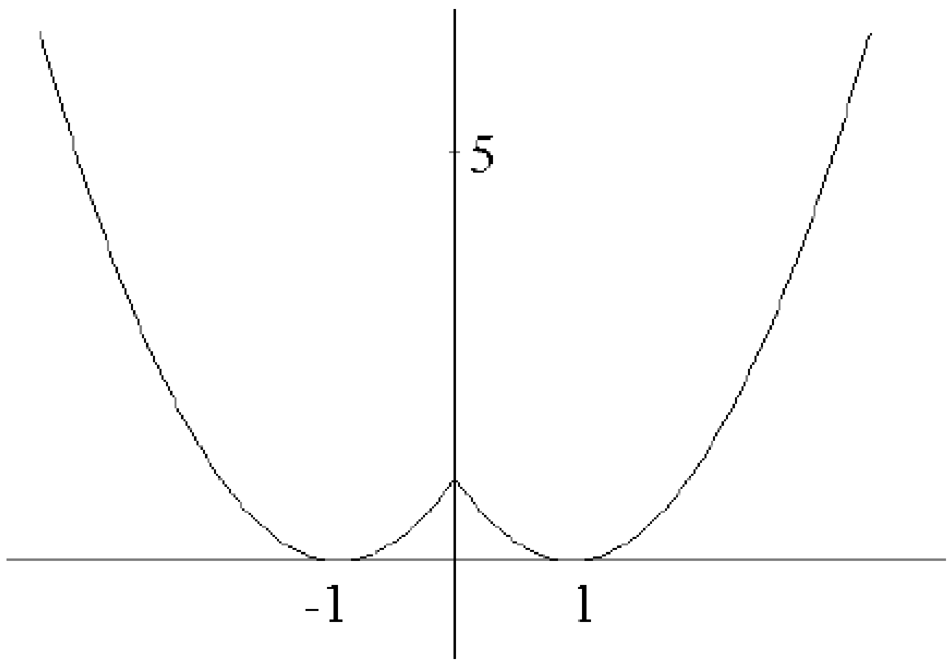
An interior point of the domain of a function f where f' is zero or fails to exist is a *critical point* of f.

2.3.4 Example

Let

$$f(x) = \begin{cases} (x-1)^2 & \text{if } \ x \geq 0, \\ (x+1)^2 & \text{if } \ x < 0. \end{cases}$$

We first plot its graph:



The critical points of f are at $x = -1,\ 0,\ \text{and}\ 1$ as can be seen from the graph. Thus local or absolute extrema of f may be attained at these points.

2.4 Increasing and Decreasing Functions

2.4.1 Definition

Let f be a function defined on an interval I. For any

two points x_1 and x_2 in I,

if $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$,

we say f is *increasing* on I;

if $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$,

we say f is *decreasing* on I.

2.4.2 Test for Increasing/Decreasing Functions

f increases on an interval I when $f'(x) > 0$ for all x on I.

f decreases on I when $f'(x) < 0$ for all x on I.

2.4.3 Example

(i) $f(x) = x^2$.

$f'(x) = 2x$ so $f'(x) > 0$ if and only if $x > 0$.

Therefore $f(x)$ is increasing on $x > 0$ and decreasing on $x < 0$.

(ii) $f(x) = \frac{2}{3}x^3 + x^2 + 2x + 1$ is increasing on any interval, since

$$f'(x) = 2x^2+2x+2 = 2\left((x+\frac{1}{2})^2 + \frac{3}{4}\right) > 0 \text{ for all } x.$$

2.4.4 First Derivative Test for Local Extremes

Suppose that $c \in (a, b)$ is a critical point of f. If

(i) $f'(x) > 0$ for $x \in (a, c)$, and $f'(x) < 0$ for $x \in (c, b)$, then $f(c)$ is a local maximum.

(ii) $f'(x) < 0$ for $x \in (a, c)$, and $f'(x) > 0$ for $x \in (c, b)$, then $f(c)$ is a local minimum.

2.5 Concavity

2.5.1 Definition

The graph of a differentiable function is *concave down* on an interval if its shape looks like the graph of $y = -x^2$. It is *concave up* on an interval if its shape looks like the graph of $y = x^2$.

2.5.2 Concavity Test

The graph of $y = f(x)$ is concave down on any interval where $y'' < 0$, and concave up on any interval where $y'' > 0$.

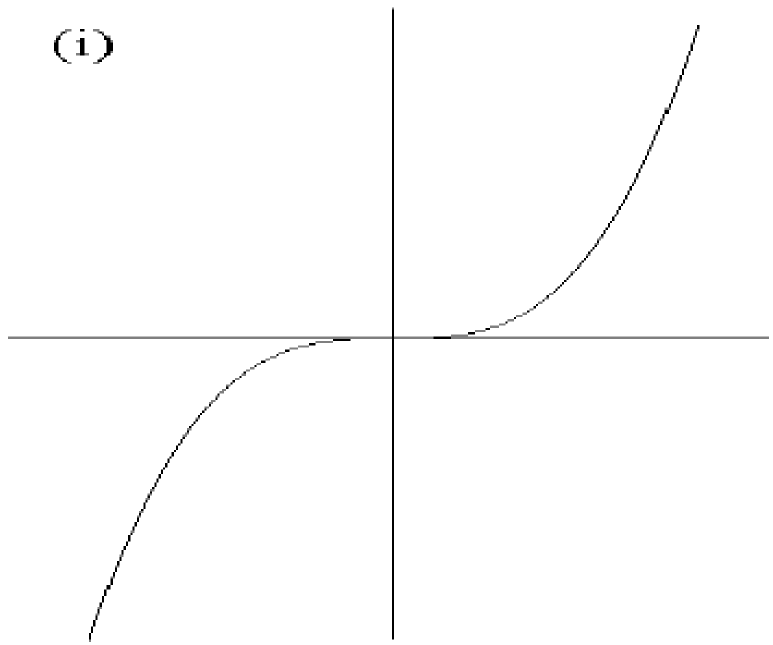
2.5.3 Example

(i) $y = x^3$. Then $y' = 3x^2$, $y'' = 6x$.

When $x < 0$, $y'' < 0$, the curve $y = x^3$ is concave down.

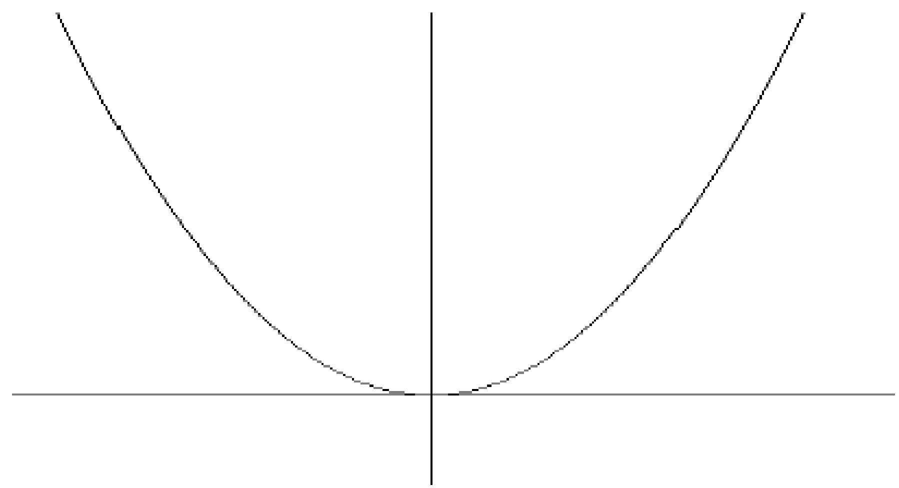
When $x > 0$, $y'' > 0$, the curve $y = x^3$ is concave up.

(i)



(ii) $y = x^2$. Then $y' = 2x$, and $y'' = 2$ is always positive. So the curve $y = x^2$ is concave up on $(-\infty, \infty)$.

(ii)



2.5.4 Points of Inflection

A point c is a *point of inflection* of the function f if f is continuous at c and there is an open interval containing c such that the graph of f changes from concave up (or down) before c to concave down (or up) after c.

Note that the definition does not require that the function be differentiable at a point of inflection.

2.5.5 Examples.

$y = x^3$ has a point of inflection at $x = 0$.

2.5.6 Second Derivative Test for Local Extreme Values

If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

2.5.7 Example

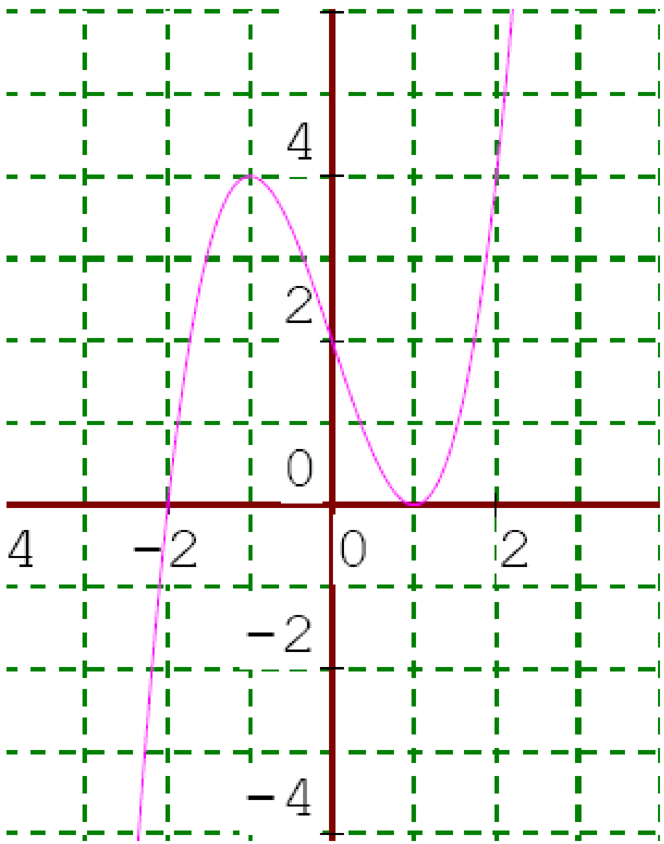
Find all local maxima and minima of the function

$y = x^3 - 3x + 2$ on the interval $(-\infty, \infty)$.

The domain has no endpoints and f is differentiable everywhere. Therefore local extrema can occur only where $y' = 3x^2 - 3 = 0$, which means at $x = 1$ and $x = -1$.

We have $y'' = 6x$, so it is positive at $x = 1$ and negative at $x = -1$.

Hence $y(1) = 0$ is a local minimum value and $y(-1) = 4$ is a local maximum value.



4
2
0
−2
−4
4
−2
0
2

2.6 Optimization Problems

To optimize something means to maximize or minimize some aspect of it. In the mathematical models in which functions are used to describe the things (variables) involved, we are usually required to find the absolute maximum or minimum value of a continuous function over a closed interval.

2.6.1 Finding Absolute Extreme Values

Step 1: Find all the critical points of the function in the interior.

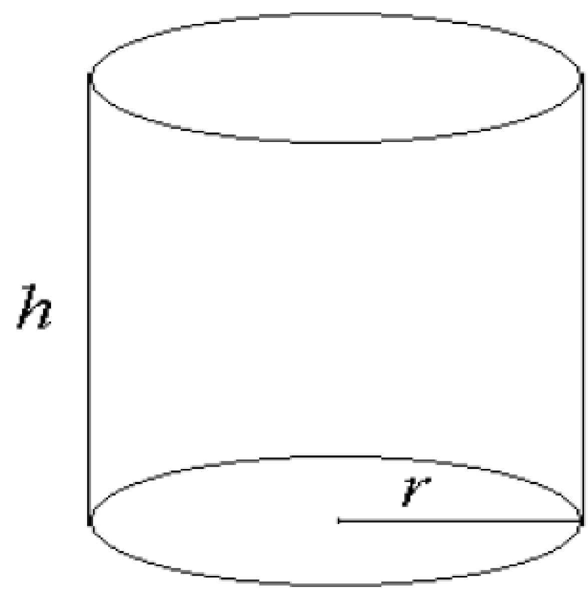
Step 2: Evaluate the functions at its critical points and at the end points of its domain.

Step 3: The largest and smallest of these values will be the absolute maximum and minimum values respectively.

2.6.2 Example.

We are asked to design a 1000cm^3 can shaped like a right circular cylinder. What dimensions will use the least material? Ignore the thickness of the material and waste in manufacturing.

Solution Let r be the radius of the circular base and h the height of the can.



We have volume

$$V = \pi r^2 h = 1000,$$

and so $h = \frac{1000}{\pi r^2}$.

The surface area

$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{2000}{r}, \qquad r > 0.$$

Our aim is to find minimum value of A on $r > 0$.

Now $A' = 4\pi r - \frac{2000}{r^2}$. Setting $A' = 0$, we get $r = (\frac{500}{\pi})^{\frac{1}{3}}$.

$$A'' = 4\pi + \frac{4000}{r^3} > 0, \qquad \text{for } r > 0.$$

Thus $r = (\frac{500}{\pi})^{\frac{1}{3}}$ leads to minimum of A. This value of r gives $h = 2r$.

Thus the dimensions of the can are $r = 5.42$cm and $h = 10.84$cm.

2.7 Indeterminate Forms

If the functions f and g are continuous at $x = a$, but $f(a) = g(a) = 0$, then the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be evaluated by substituting $x = a$. To describe such a situation, we shall symbolically use the expression $\frac{0}{0}$, known as an *indeterminate form*.

2.7.1 L'Hospital's Rule

Suppose that

(1) f and g are differentiable in a neighborhood of x_0;

(2) $f(x_0) = g(x_0) = 0$;

(3) $g'(x) \neq 0$ except possibly at x_0.

Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

In particular,

Suppose $f(a) = g(a) = 0$, $f'(a)$ and $g'(a)$ exist, and $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

2.7.2 Example.

(i) $$\lim_{x\to 0}\frac{3x-\sin x}{x}=\frac{3-\cos x}{1}\bigg|_{x=0}=2.$$

(ii) $$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}}{1}\Bigg|_{x=0}=\frac{1}{2}.$$

(iii)

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}=\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\lim_{x\to 0}\frac{\sin x}{6x}=\left.\frac{\cos x}{6}\right|_{x=0}=\frac{1}{6}.$$

(iv) $$\lim_{x\to 0}\frac{1-\cos x}{x+x^2}=\lim_{x\to 0}\frac{\sin x}{1+2x}=0.$$

2.7.3 Other Indeterminate Forms

If $f(x)$ and $g(x)$ both approach ∞ as $x \to a$, and $f(x)$ and $g(x)$ are differentiable, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists. Here a may be finite or infinite.

2.7.4 Remark.

For all the other indeterminate forms (for example $\infty \cdot 0$, $\infty - \infty$), one needs to change them to either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form and then apply L'Hopital's rule.

2.7.5 Example.

(i) (of form $\frac{\infty}{\infty}$)

$$\lim_{x \to \frac{\pi}{2}-} \frac{\tan x}{1 + \tan x} = \lim_{x \to \frac{\pi}{2}-} \frac{\sec^2 x}{\sec^2 x} = 1.$$

(ii) (of form $\frac{\infty}{\infty}$)

$$\lim_{x\to\infty} \frac{x - 2x^2}{3x^2 + 5} = \lim_{x\to\infty} \frac{1 - 4x}{6x} = \lim_{x\to\infty} \frac{-4}{6} = -\frac{2}{3}.$$

(iii) (of form $0 \cdot \infty$)

$$\lim_{x \to 0^+} x \cot x = \lim_{x \to 0^+} \frac{x}{\tan x} = \lim_{x \to 0^+} \frac{1}{\sec^2 x} = 1.$$

Note that we have changed it to $\frac{0}{0}$ form before we apply L'Hopital's rule.