Chapter 8

Multiple Integrals

Key Results

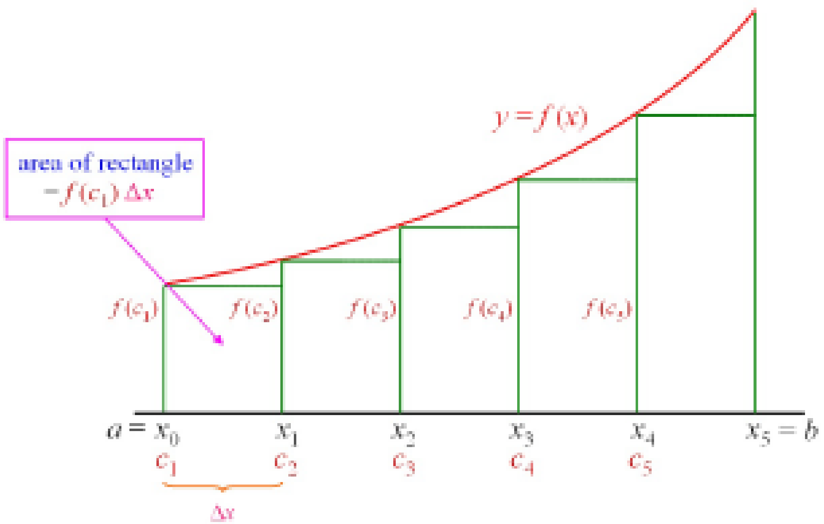
- Double integrals calculated as iterated integrals.
- Cartesian and polar forms of double integrals.
- Calculation of volumes of solid regions under a surface.
- Calculation of surface area.

Review

Recall definite integral from Chapter 3:

$$\sum_{k=1}^{5} f(c_k)\Delta x$$

sum of areas of rectangles approximates area of region under a graph.



Exact area of region is

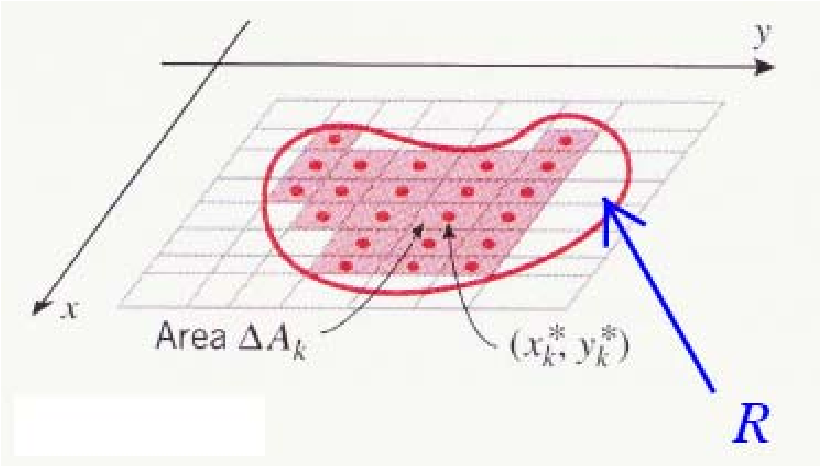
$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x = \int_a^b f(x)\,dx$$

Now generalize this idea to get volume of a region in 3D.

Double Integrals

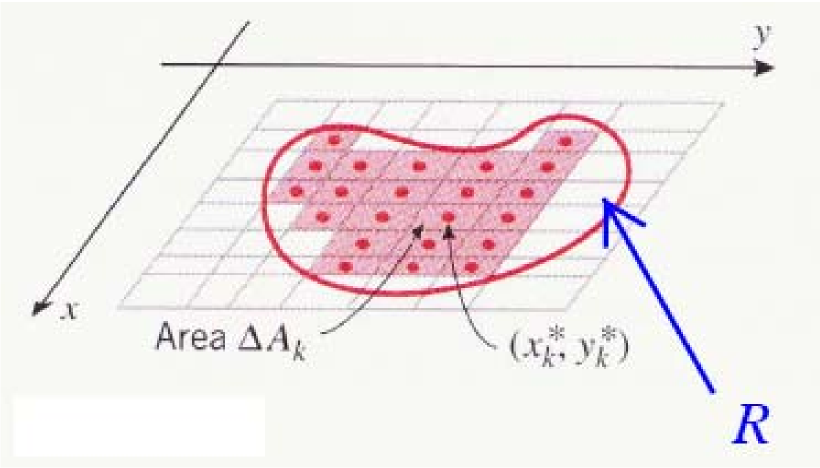
Suppose a surface $z = f(x, y)$ lies over a region R in the xy-plane.

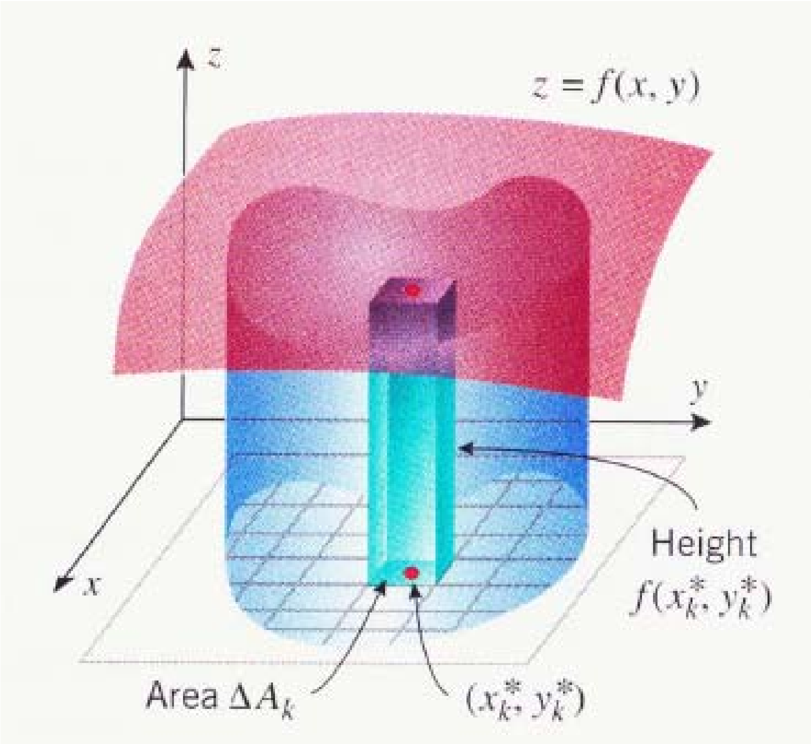
Subdivide R into rectangles R_k :



Each rectangle R_k has area ΔA_k and a chosen point $(x_k{}^*, y_k{}^*)$.

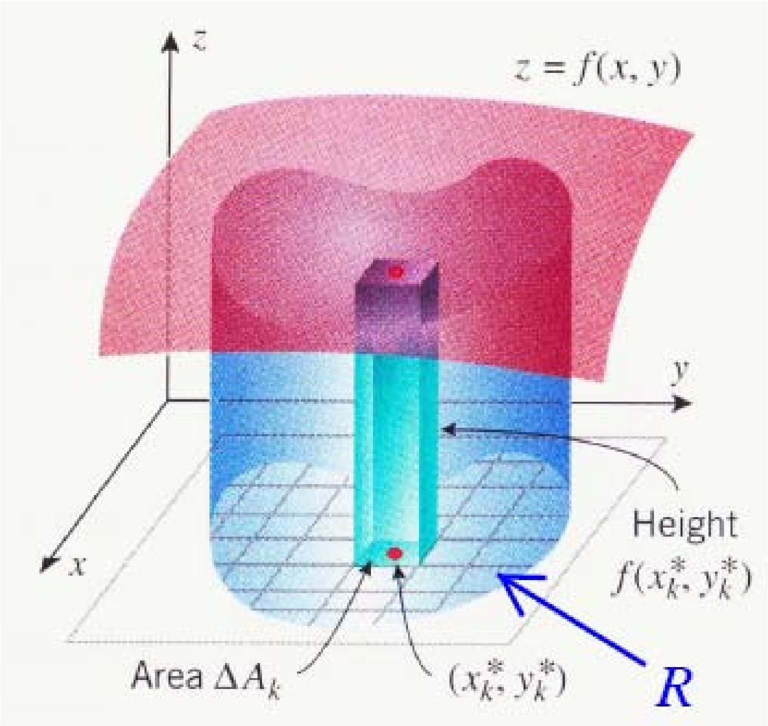
Source: *Calculus* by Anton, Bivens, Davis





Approximate volume of solid region under the surface and over the rectangle R_k is

$$\boxed{f(x_k^*, y_k^*)\Delta A_k}$$



The double integral of *f* over *R* is

$$\boxed{\iint_R f(x,y)\,dA = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k}$$

gives the volume of the solid region under the surface over *R*.

Properties

Similar to properties of functions of one variable and their integrals

(1) $$\iint_R (f(x,y) + g(x,y))\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA.$$

(2) $$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA,$$ where c is a constant.

(3) If $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$,

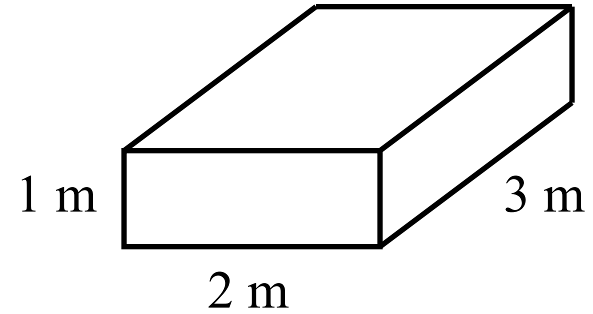
then $$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA.$$

(4) $\iint_R dA \left(= \iint_R 1\, dA \right) = A(R)$, the area of R.

$f(x, y) = 1$ constant height 1

Consider a rectangular box

of height 1.



Base area $= 2 \times 3 = \boxed{6 \text{ m}^2.}$

Volume $=$ base area $\times$ height

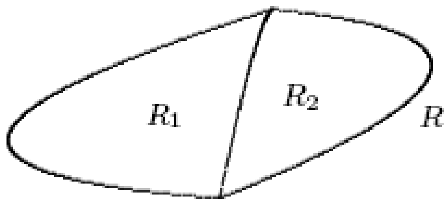
$= 6 \times 1 \text{ m}^3$

$= \boxed{6 \text{ m}^3.}$

Volume is numerically the same value as base area

because the height is 1 m.

Two regions R_1 and R_2 do not overlap except perhaps on their boundary



$$(5) \quad \iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA,$$

where $R = R_1 \cup R_2$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

How to calculate these double integrals efficiently?

Scheme of study

1. First calculate double integrals for the simplest regions R: rectangular regions.

2. Then calculate double integrals for more complicated regions R: Type A or Type B.

Rectangular Regions

A rectangular region *R* in the *xy*-plane can be described in terms of inequalities:

$$a \le x \le b, \qquad c \le y \le d.$$

Then

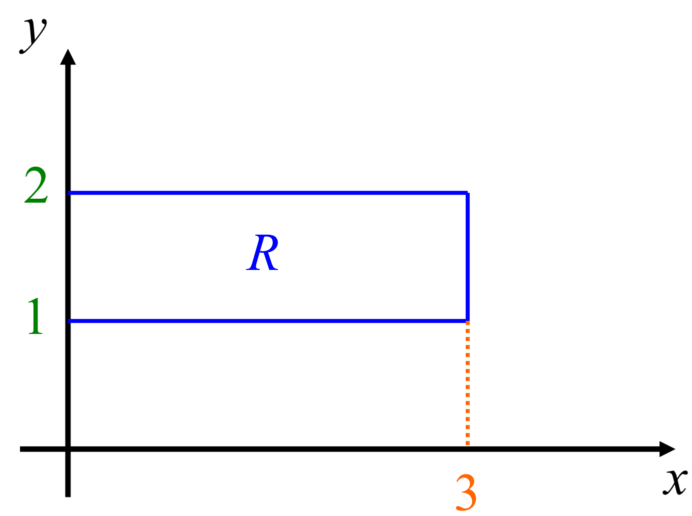
$$\boxed{\iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx\right]\,dy.}$$

The integral expression on the right is called an iterated integral.

Example (page 6, example (b))

Rectangular region *R* described by:

$0 \le x \le 3, \quad 1 \le y \le 2.$



$$\int_1^2 \int_0^3 (x + 2y)\,\boxed{dx}\,\boxed{dy}$$

$$= \int_1^2 \left[\cancelto{0}{\frac{x^2}{2}} + \cancelto{0}{2xy}\right]_{\boxed{x=0}}^{\boxed{x=3}} dy \quad = \quad \int_1^2 \boxed{\left[\frac{9}{2} + 6y\right]} dy$$

$$= \left[\frac{9y}{2} + 3y^2\right]_{y=1}^{y=2} = \boxed{27/2.}$$

For rectangular regions, the order of the variables of integration can be changed easily. (page 5)

$$\boxed{\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\;dy\right]\;dx.}$$

$$\int_1^2 \int_0^3 (x+2y)\,dx\,dy \;=\; \int_0^3 \int_1^2 (x+2y\boxed{)\,dy}dx$$

$$= \int_0^3 \left[xy+y^2\right]_{\boxed{y=1}}^{\boxed{y=2}} dx \;=\; \int_0^3 \left[(2x+4)-(x+1)\right]\,dx$$

$$= \int_0^3 (x+3)\,dx \;=\; \left[\frac{x^2}{2}+3x\right]_{x=0}^{x=3} \;=\boxed{27/2}$$

Special Case

For rectangular regions *R*, where

$$a \le x \le b, \quad c \le y \le d$$ *a*, *b* constant

if the integrand $f(x, y)$ factors as

$$f(x,y) = g(x)h(y)$$

then the double integral also 'factors':

$$\boxed{\iint_R g(x)h(y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)}$$

Example

Let R be the rectangular region $\quad 0 \le x \le 4, \quad 1 \le y \le 2.$

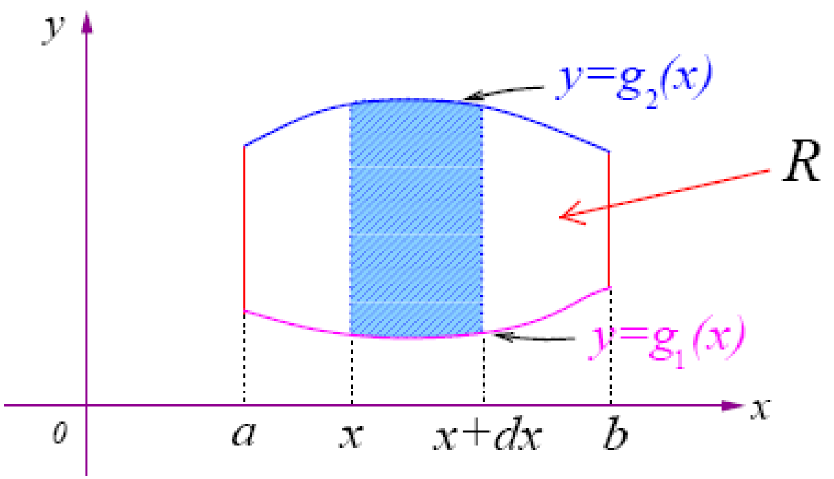
$$\begin{aligned}
\iint_R x^2 y \, dA &= \int_0^4 \int_1^2 x^2 y \, dy dx \\
&= \left(\int_0^4 x^2 \, dx \right) \left(\int_1^2 y \, dy \right) \\
&= \left[\frac{1}{3} x^3 \right]_{x=0}^{x=4} \cdot \left[\frac{1}{2} y^2 \right]_{y=1}^{y=2} \\
&= \frac{64}{3} \times \frac{3}{2} \\
&= \boxed{32}
\end{aligned}$$

General Regions – Type A

To compute a general double integral $\iint_R f(x, y)\, dA$

the region R should be described in a way that limits of integration can be included:

Consider

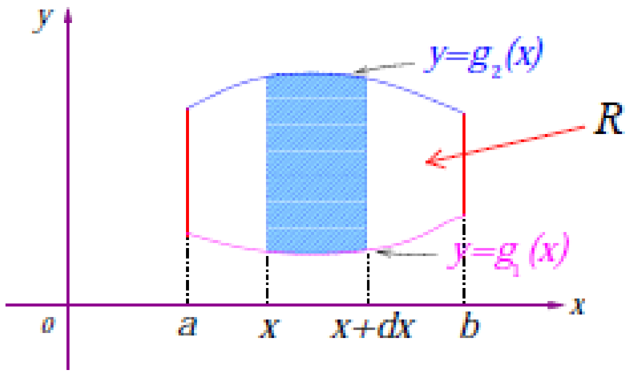


Left boundary: line $x = a$

Right boundary: line $x = b$

Lower boundary: curve $y = g_1(x)$

Upper boundary: curve $y = g_2(x)$



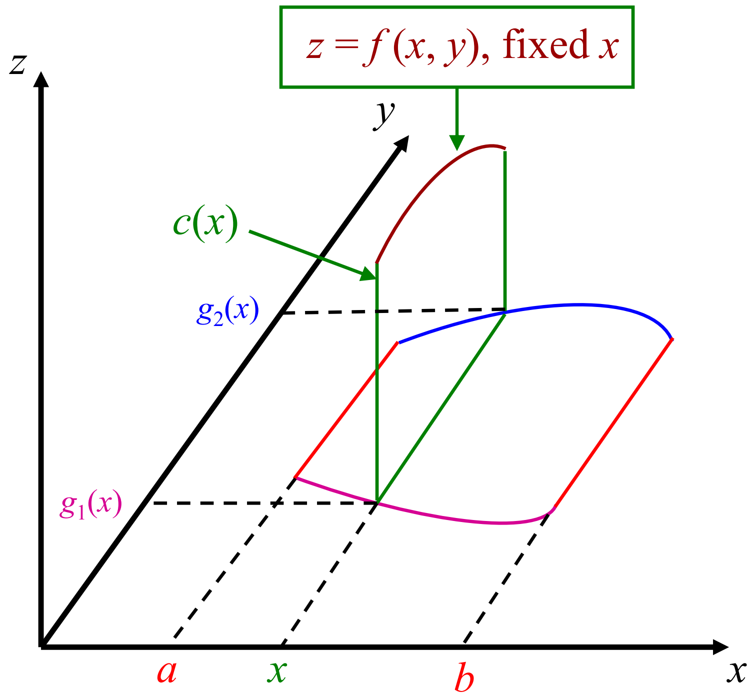
Region R is described as Type A by:

$$R: \quad g_1(x) \leq y \leq g_2(x), \quad a \leq x \leq b.$$

Then

$$\boxed{\iint_R f(x, y)\, dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right] dx}$$

How is this obtained?

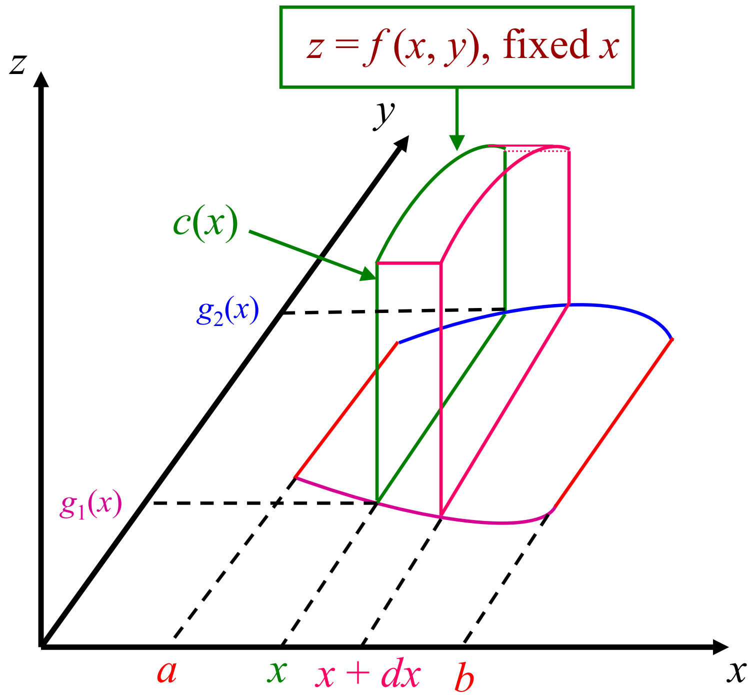


Fix x. Then $z = f(x, y)$ is a function of y only.

The area of the region under the graph of $z = f(x, y)$ is:

$$\boxed{c(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy}$$

(cross-sectional area)



$$c(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\ dy$$

Consider a small thickness dx.

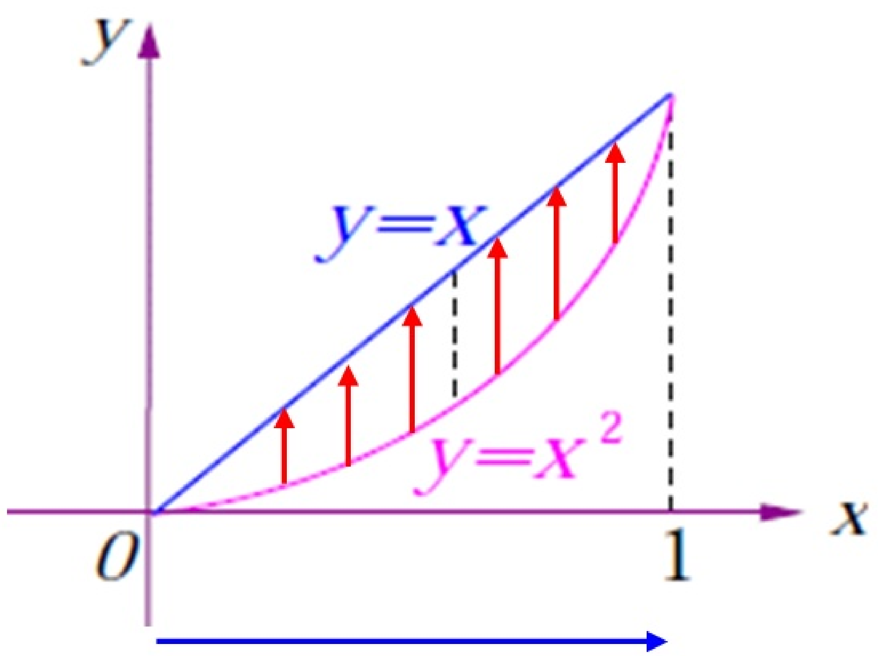
Volume of a slice = $c(x)\ dx$.

Volume of the region under the surface $z = f(x, y)$ is:

$$\iint_R f(x, y)\, dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\ dy \right] dx$$

Example (Type A) (page 11)

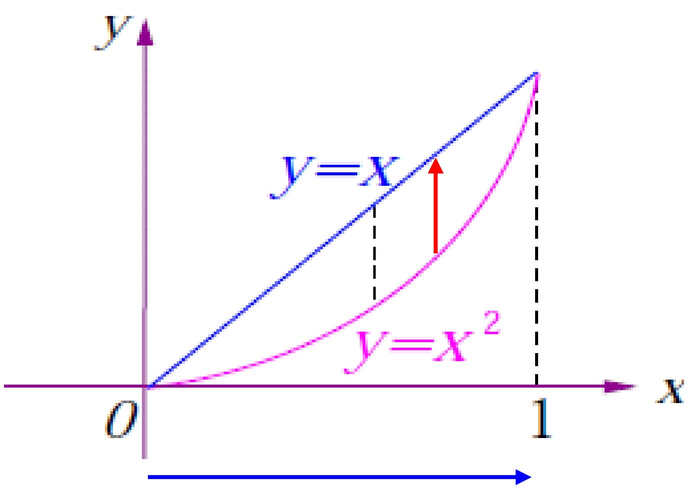
Region R is bounded by $y = x$ and $y = x^2$.



Describe R as Type A:

$$R: \quad x^2 \leq y \leq x, \quad 0 \leq x \leq 1$$

$$
\begin{aligned}
\iint_R 30xy\,dA &= \int_0^1 \boxed{\left[\int_{x^2}^{x} 30xy\,dy\right]} dx \\
&= \int_0^1 \boxed{\left[15xy^2\right]_{y=x^2}^{y=x}} dx \\
&= \int_0^1 15x(x^2 - x^4)\,dx \\
&= \left[\frac{15x^4}{4} - \frac{15x^6}{6}\right]_{x=0}^{x=1} \\
&= \boxed{\frac{5}{4}.}
\end{aligned}
$$



General Regions – Type B

In another situation, to compute the double integral

$$\iint_R f(x,y)\,dA$$

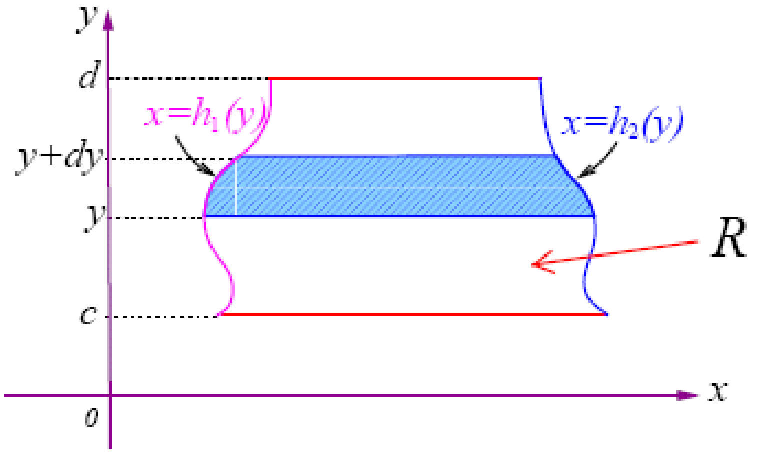
the region R may be described as follows:

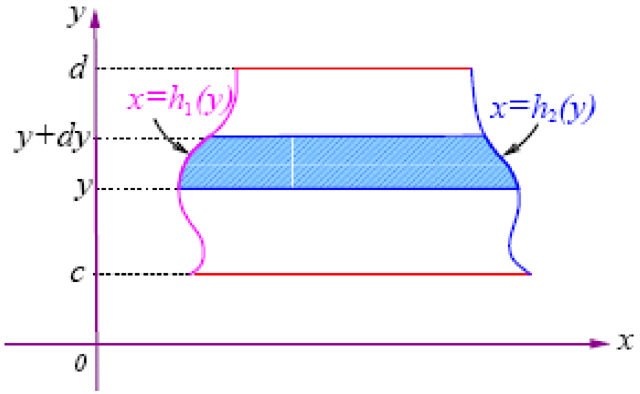
Left boundary: curve $x = h_1(y)$

Right boundary: curve $x = h_2(y)$

Lower boundary: line $y = c$

Upper boundary: line $y = d$





Region R is described as Type B by:

$$R: \quad h_1(y) \leq x \leq h_2(y), \quad c \leq y \leq d.$$

Then

$$\boxed{\iint_R f(x,y)\, dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx \right] dy}$$

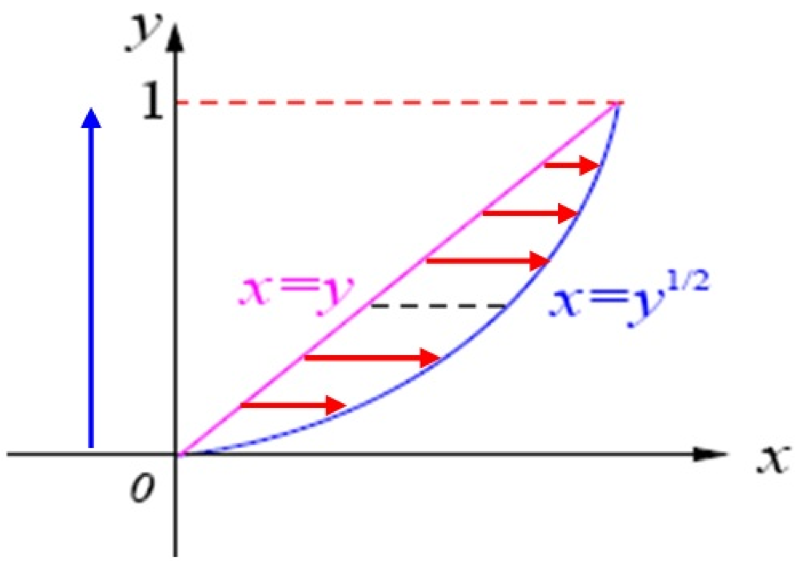
Example (Type B) (page 12)

Region R is bounded by $y = x$ and $y = x^2$.

To describe R as Type B, need to express graphs as functions of y:

$$y = x \quad \Rightarrow \quad x = y \qquad\qquad y = x^2 \quad \Rightarrow \quad x = \sqrt{y}$$

$$R: \quad y \le x \le \sqrt{y}, \quad 0 \le y \le 1.$$



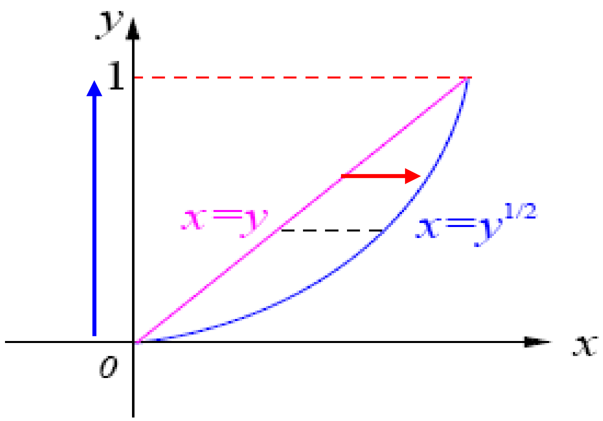
$$\iint_R 30xy\ dA = \int_0^1 \boxed{\left[\int_y^{\sqrt{y}} 30xy\ dx\right]}\ dy$$

$$= \int_0^1 \boxed{\left[15x^2y\right]_{x=y}^{x=\sqrt{y}}}\ dy$$

$$= \int_0^1 \left(15y^2 - 15y^3\right)\ dy$$

$$= \left[\frac{15y^3}{3} - \frac{15y^4}{4}\right]_{y=0}^{y=1}$$

$$= \boxed{\frac{5}{4}}$$



The iterated integrals formulas, e.g Type A formula below

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\;dy\right]\;dx$$

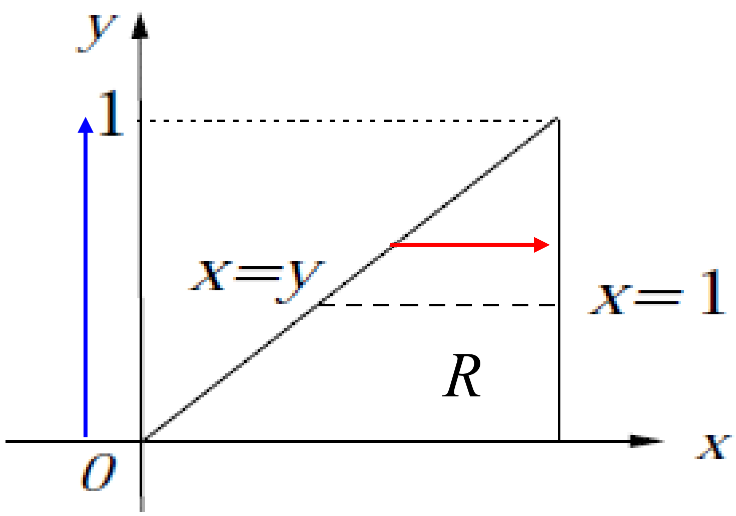
are part of Fubini's Theorem.

Guido Fubini
(1879 – 1943)

Example (Where Only One Type Works)

Region R is the triangle in the xy-plane described as Type B:

$$R: \quad y \le x \le 1, \quad 0 \le y \le 1$$



Then

$$\iint_R \frac{\sin x}{x}\, dA$$

$$= \int_0^1 \left[\int_y^1 \frac{\sin x}{x}\, dx \right] dy$$

which cannot be evaluated by elementary means.

But region *R* can be described as Type A:

$$R: \quad 0 \le y \le x, \quad 0 \le x \le 1.$$

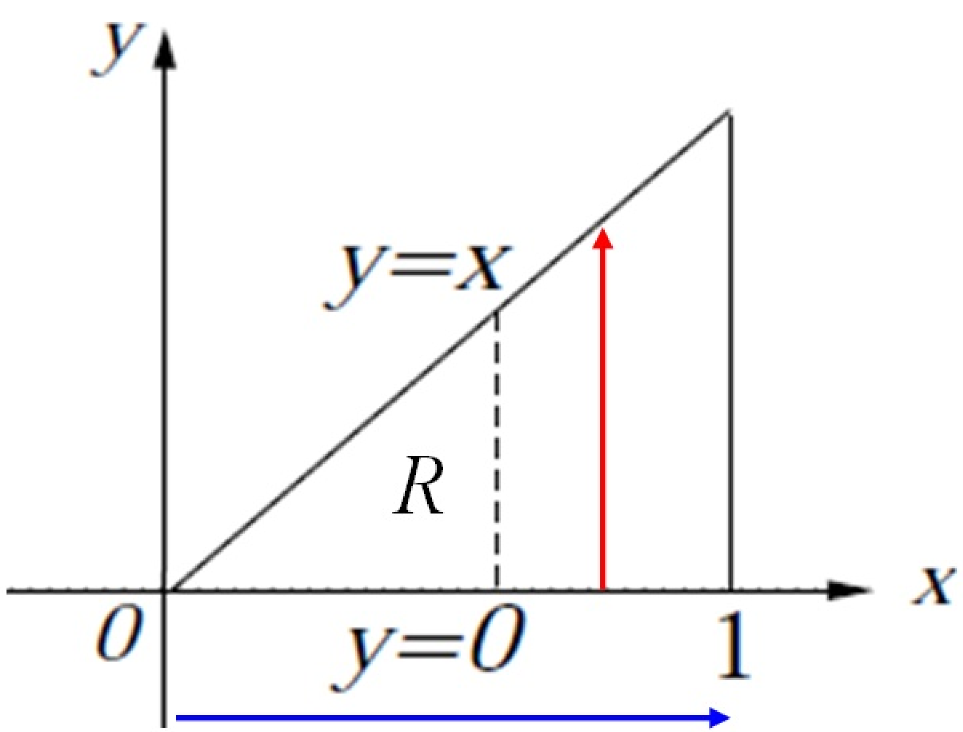
no y-term

$$\iint_R \frac{\sin x}{x}\, dA = \int_0^1 \left[\int_0^x \boxed{\frac{\sin x}{x}}\, \boxed{dy}\right] dx$$

$$= \int_0^1 \left[\boxed{y}\frac{\sin x}{x}\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 (\sin x - 0)\, dx$$

$$= \Big[-\cos x\Big]_{x=0}^{x=1} = \boxed{1 - \cos 1.}$$



Switching/Interchanging Integrals

Sometimes, a given double integral cannot be evaluated directly, but an interchange of the integrals allows the double integral to be evaluated.

This involves recognizing the region R as Type A (or Type B) and re-describing R as Type B (or Type A).

Example

Find $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\ dx$

Clearly, it is difficult to integrate e^{y^3} directly.

Perhaps a switch of the integrals may facilitate the integration.

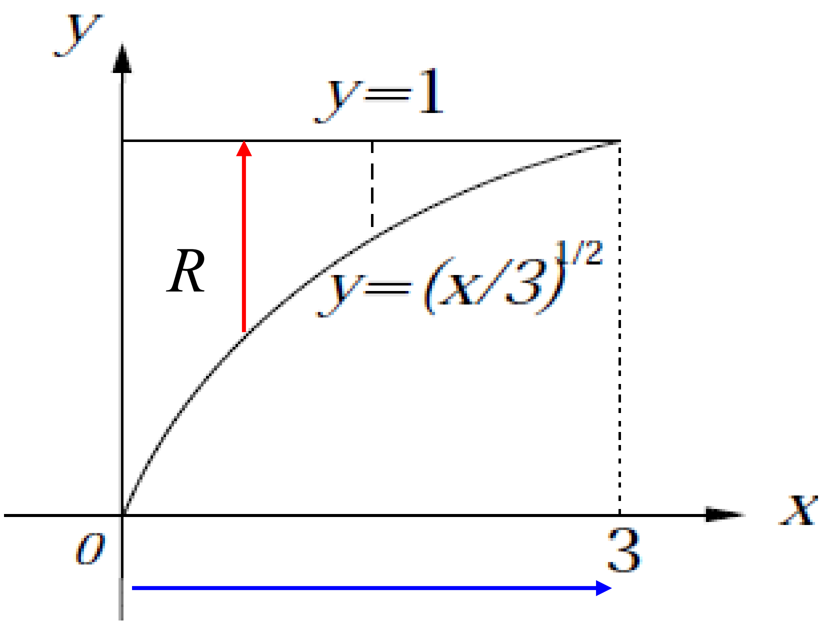
First identify the region *R* of integration.

$$\boxed{\int_0^3 \int_{\sqrt{x/3}}^1} e^{y^3} \boxed{dy \quad dx}$$

Region R is described as Type A:

$$R: \quad \boxed{\sqrt{\frac{x}{3}}} \le y \le \boxed{1,} \quad 0 \le x \le 3.$$

To sketch R,
identify the curves



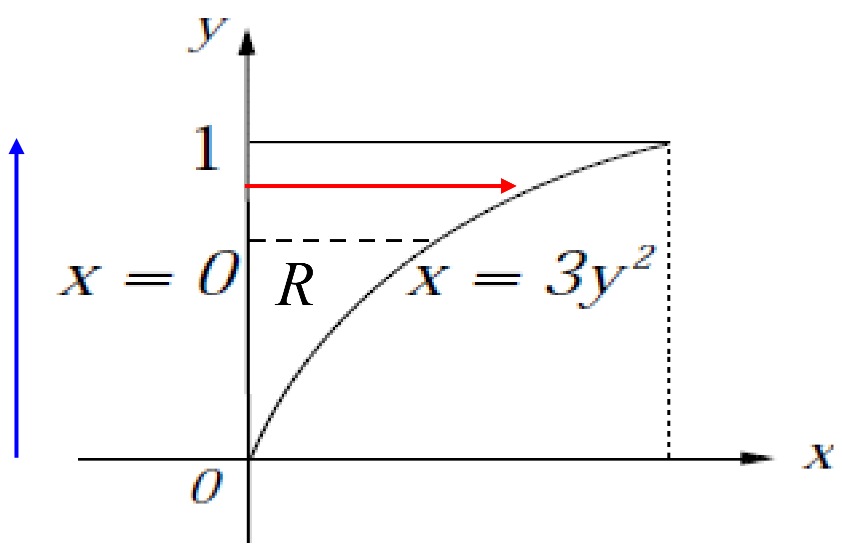
Curve 1: $y = \sqrt{\dfrac{x}{3}}$

Curve 2: $y = 1$

Now describe R as Type B:

Note that curve 1 can be described as follows:

$$y = \sqrt{\frac{x}{3}} \quad \Rightarrow \quad x = 3y^2$$



$$R: \quad 0 \le x \le 3y^2, \qquad 0 \le y \le 1.$$

Thus, $$\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy \;\; dx = \int_0^1 \boxed{\left[\int_0^{3y^2} e^{y^3} \;\; dx\right]} \; dy$$

$$= \int_0^1 \boxed{\left[x e^{y^3}\right]_{x=0}^{x=3y^2}} \; dy$$

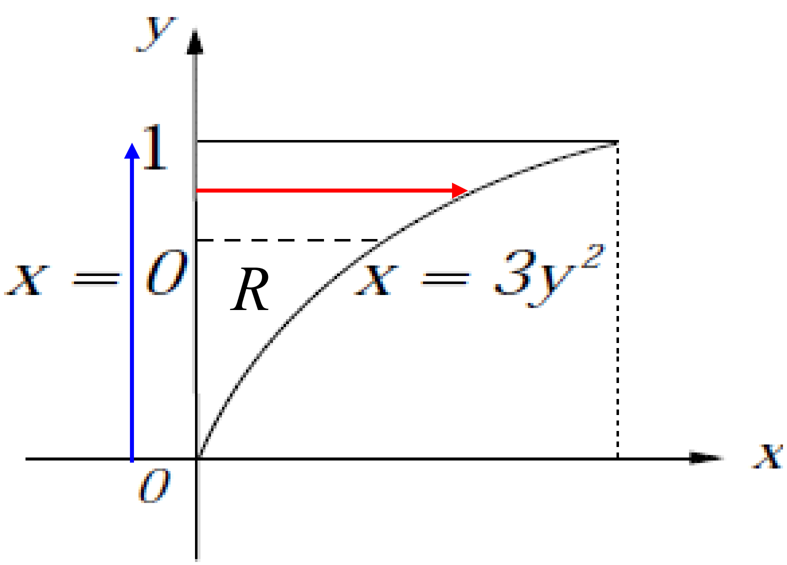
$$= \int_0^1 3y^2 e^{y^3} \; dy$$

substitution $u = y^3$

$$= \int_0^1 e^u du$$

$$= \boxed{e - 1}$$

R as Type B



Double Integrals (Polar Form)

Certain regions are better described using polar coordinates, which may facilitate double integration.

Polar coordinates (r, θ).

r is the distance from origin to a point in the region.

θ is the angle of elevation of a point from the x-axis.

Cartesian/Polar Coordinates

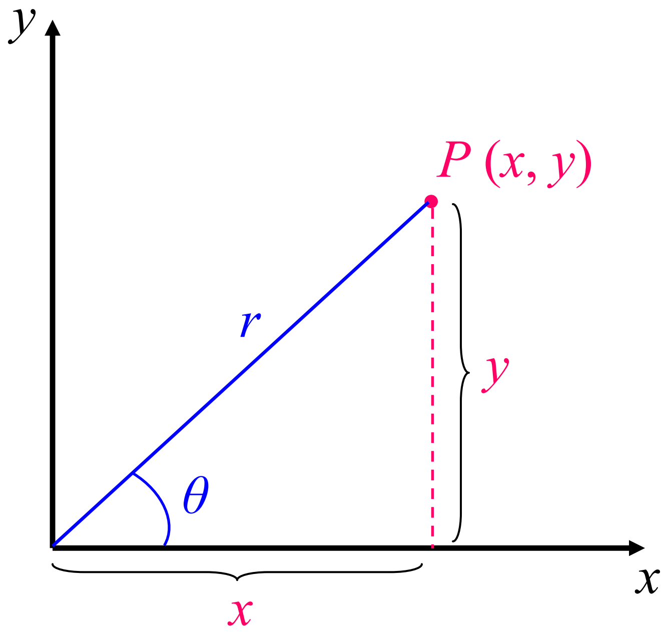
Cartesian coordinates $P(x, y)$

Polar coordinates (r, θ)

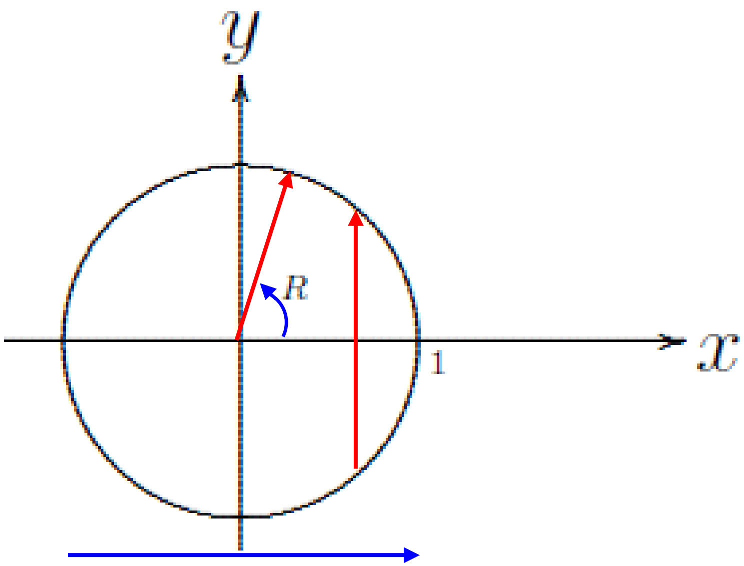
Conversion formulas:

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$$\boxed{\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}}$$



Circle



In polar coordinates, *R* may be described easily as:

$$R: \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$$

In Cartesian coordinates, the circle may be described as Type A:

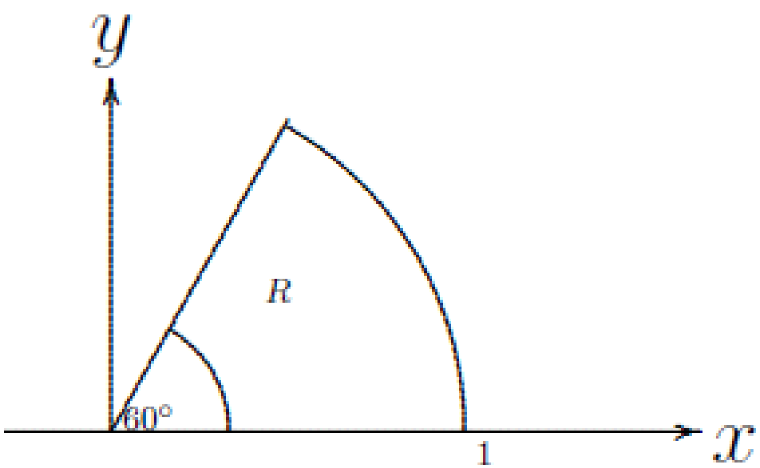
$$R: \quad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, \quad -1 \le x \le 1.$$

Sector of a Circle

In Cartesian coordinates, the sector *R* may be described as a Type B region:

$$R: \quad \frac{1}{\sqrt{3}}y \le x \le \sqrt{1-y^2}, \quad 0 \le y \le \frac{\sqrt{3}}{2}.$$

Complicated to use for integration.



In polar coordinates, the sector *R* is described easily by:

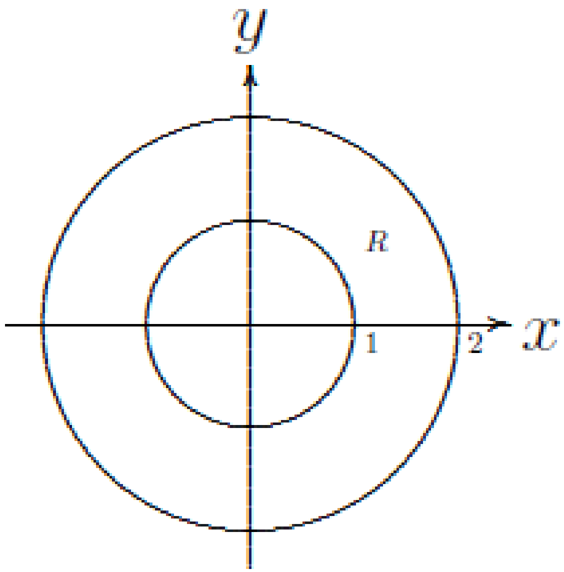
$$\boxed{R: \quad 0 \le r \le 1, \quad 0 \le \theta \le \pi/3}$$

Ring

In Cartesian coordinates, the ring R is not easily described as a Type A or Type B region.

In polar coordinates, the ring R is described easily by:

$$\boxed{R: \quad 1 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi}$$

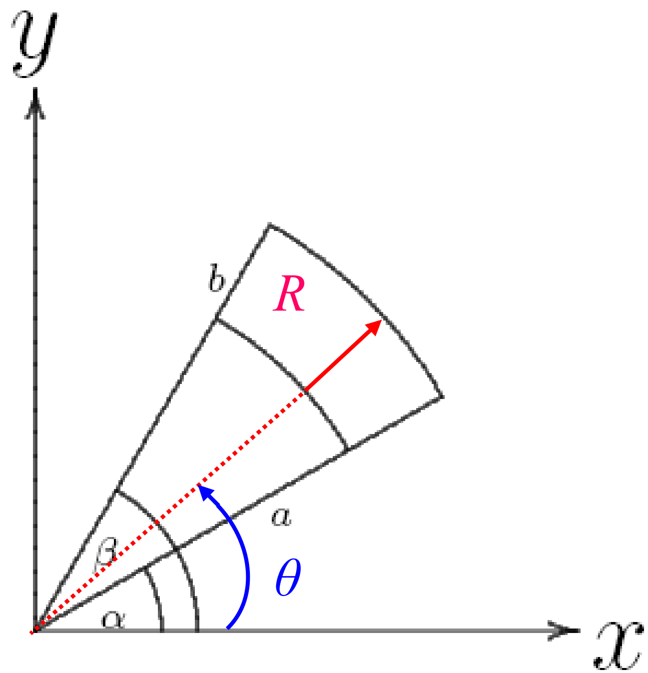


Polar Rectangle

In polar coordinates, a region R described by:

$$\boxed{R: \quad a \le r \le b, \quad \alpha \le \theta \le \beta}$$

is called a polar rectangle.



Note that the boundary values of r, namely a and b, are constants independent of θ.

Exercise

Sketch the region R, which is given in polar coordinates as:

$$R \ : \ 0 \ \leq \ r \ \leq \ 1 + \cos\theta, \quad 0 \ \leq \ \theta \ \leq \ 2\pi.$$

The region R is ‘heart-shaped’ and its boundary is called a cardioid.

Change of Variables

When a region R is first described using Cartesian coordinates and then re-described using polar coordinates, there is a change of variables from (x, y) to (r, θ), where

$$x = r\cos\theta, \quad y = r\sin\theta.$$

In this case, the variable of integration dA will change from $dy\,dx$ (or $dx\,dy$) to

$$\boxed{r\,dr\,d\theta}$$

If the region *R* in the *xy*-plane is re-described in polar form as:

$$R: \quad a \leq r \leq b, \quad \alpha \leq \theta \leq \beta,$$

then

$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.}$$

Example

The region R is the semi-circular ring in the upper half-plane between the semi-circles

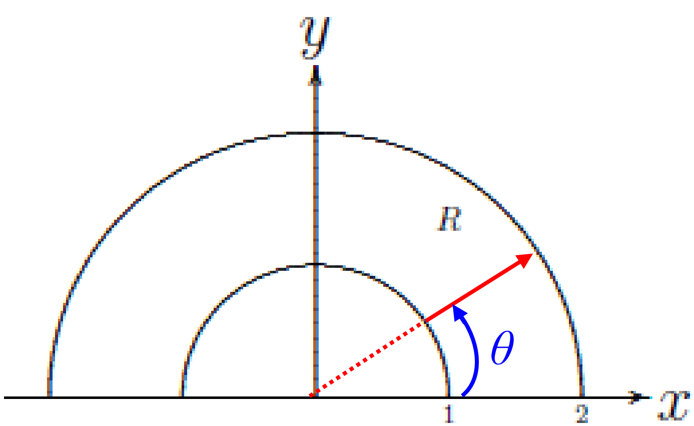
$$x^2 + y^2 = 1 \quad \text{and} \quad x^2 + y^2 = 4.$$

$$\boxed{R: \quad 1 \le r \le 2, \quad 0 \le \theta \le \pi.}$$

$$\iint_R (3x + 4y^2)\, dA = \int_0^{\pi} \boxed{\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\,dr}\, d\theta$$

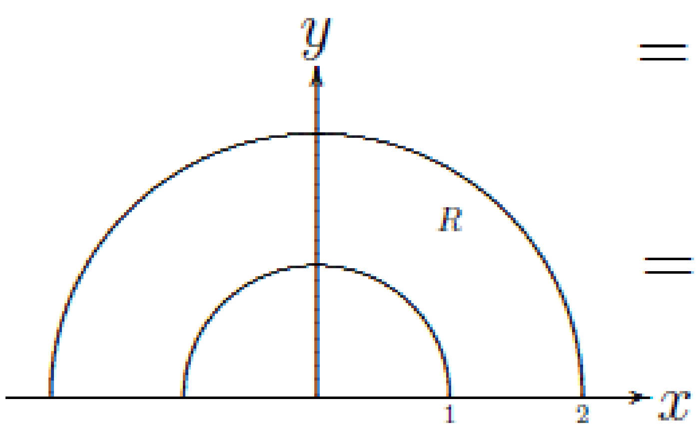
$$= \int_0^{\pi} \boxed{\left[r^3\cos\theta + r^4\sin^2\theta\right]_{r=1}^{r=2}} d\theta$$

$$= \int_0^{\pi} \left(7\cos\theta + 15\sin^2\theta\right) d\theta$$



$$x = r\cos\theta, \qquad y = r\sin\theta.$$

$$
\begin{aligned}
\iint_R (3x+4y^2)\,dA &= \int_0^{\pi}\int_1^{2} \left(3r\cos\theta + 4r^2\sin^2\theta\right)\, r\,dr\,d\theta \\
&= \int_0^{\pi} \left[r^3\cos\theta + r^4\sin^2\theta)\right]_{r=1}^{r=2} d\theta \\
&= \int_0^{\pi} \left(7\cos\theta + 15\sin^2\theta\right) d\theta \\
&= \int_0^{\pi} \left(7\cos\theta + \frac{15}{2}(1-\cos 2\theta)\right) d\theta \\
&= \Big[7\underbrace{\sin\theta}_{0} + \boxed{\tfrac{15}{2}\big(\theta} - \underbrace{\frac{\sin 2\theta}{2}}_{0}\big)\Big]_{\theta=0}^{\boxed{\theta=\pi}} \\
&= \boxed{\frac{15\pi}{2}}
\end{aligned}
$$



Why is there an 'r' in $r\,dr\,d\theta$?

Consider

Interval $[0, a]$, $0 \le r \le a$

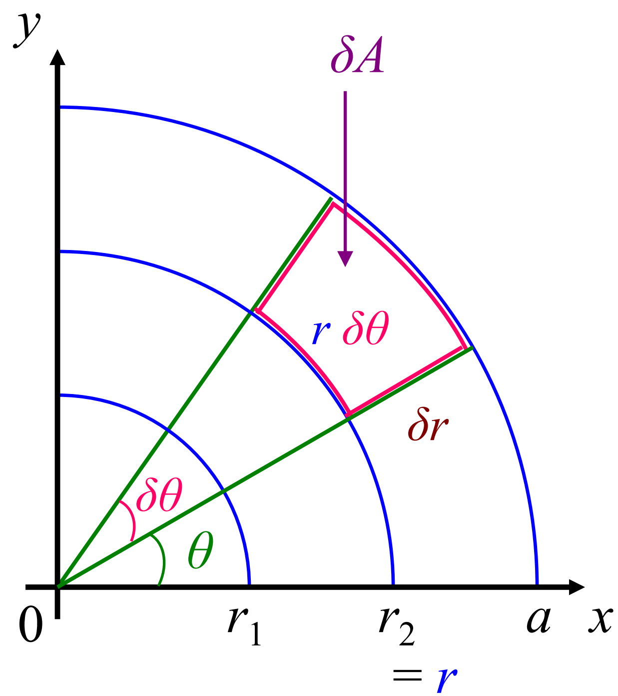
Partition $[0, a]$

$0 = r_0 < r_1 < r_2 < r_3 = a$

Interval $[0, \pi/2]$, $0 \le \theta \le \pi/2$

Partition $[0, \pi/2]$

$0 = \theta_0 < \theta_1 < \theta_2 < \theta_3 = \pi/2$



Sub-region is approximately rectangular

$\delta A \approx \delta r \cdot r\,\delta\theta$ That is, $\boxed{dA = r\,dr\,d\theta}$

Application Example (page 24)

Paraboloid $x^2 + 2y^2 + z = 16$
can be expressed as

$$z = f(x, y) = 16 - x^2 - 2y^2$$

It is an elliptic paraboloid because horizontal cross-sections are ellipses.

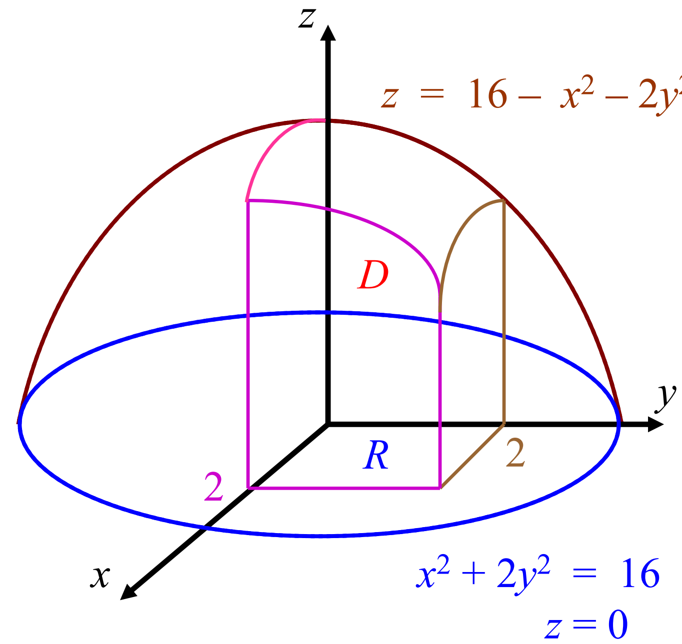
For example, $z = 0$ gives

$$x^2 + 2y^2 = 16$$

In first octant, put in

plane $x = 2$

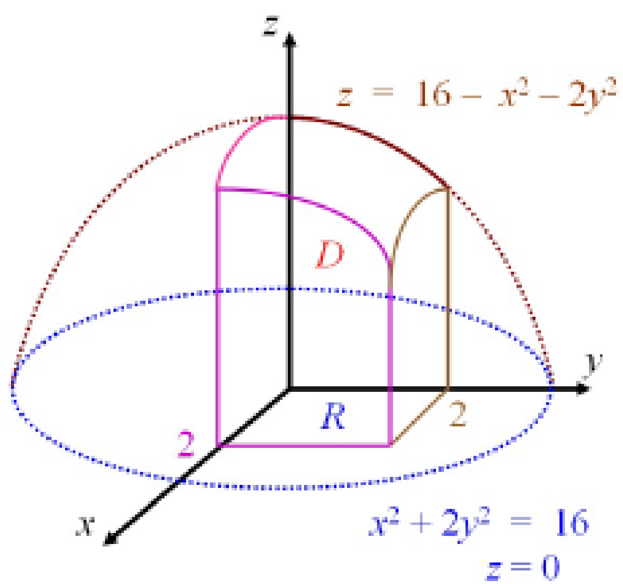
plane $y = 2$



The solid region D is bounded by the paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$, $y = 2$, and the three coordinate planes ($x = 0$, $y = 0$, $z = 0$).

Region R is described as Type B: (Type A also works.)

$$\boxed{R: \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 2}$$



Volume of D is

$$\iint_R (16 - x^2 - 2y^2)\,dA$$

$$= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\,dx\,dy$$

$$= \boxed{48.}$$

(Details left as exercise.)

Example

Circular cylinder $\boxed{x^2 + y^2 = 9.}$

Paraboloid $\boxed{z = x^2 + y^2.}$

Paraboloid meets cylinder when $z = 9$

Plane $x + z = 20.$

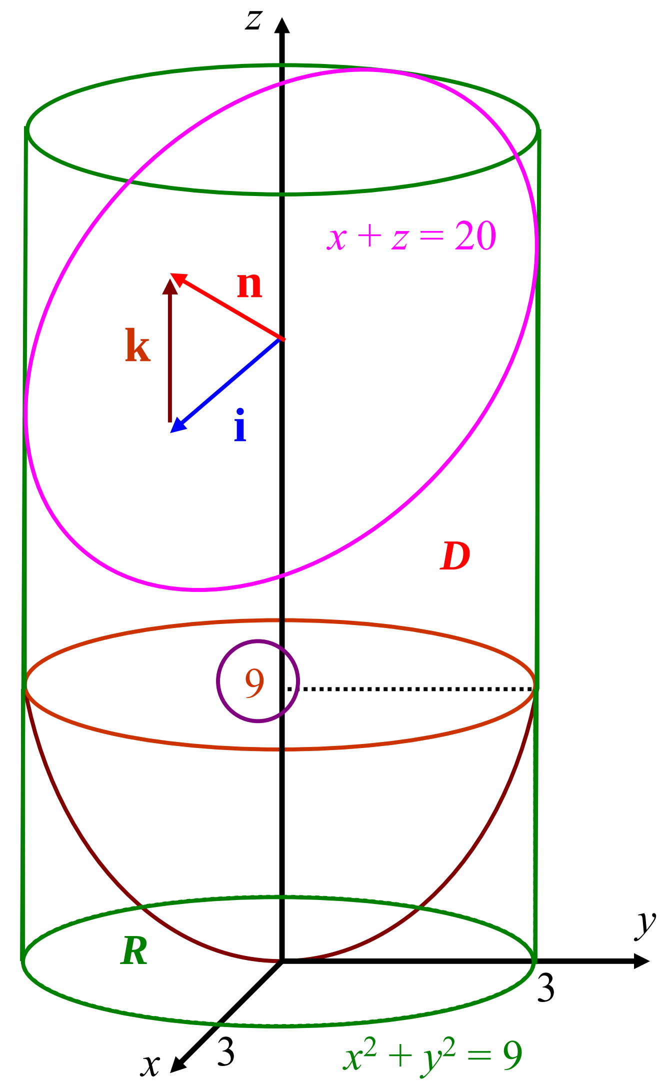
A normal vector is $\mathbf{n} = \mathbf{i} + \mathbf{k}$

Within cylinder, largest value of x is 3.

Smallest value of z for plane is

$$z = 20 - x = 17 > \mathbf{9}.$$

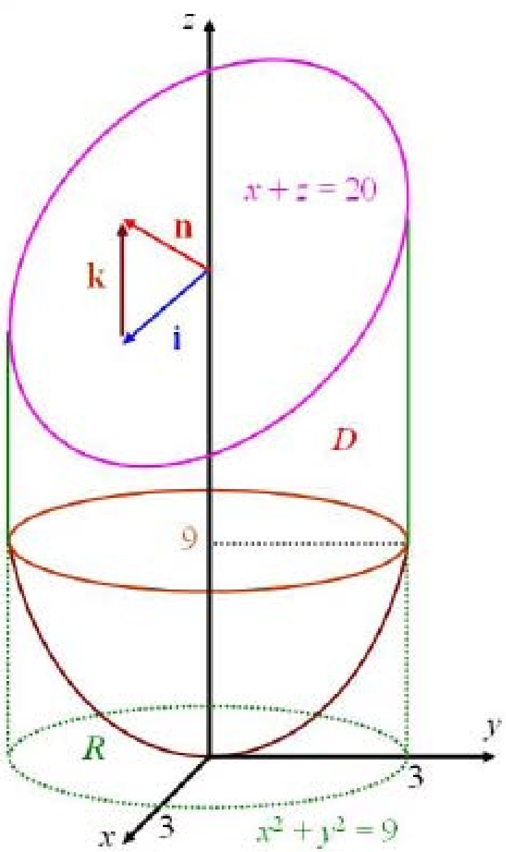
Plane will not intersect paraboloid.



The region *R* is described using polar coordinates:

$$0 \le r \le 3, \quad 0 \le \theta \le 2\pi$$

Volume of solid region *D* is



$$= \boxed{\iint_R (20 - x)\,dA} - \boxed{\iint_R (x^2 + y^2)\,dA}$$

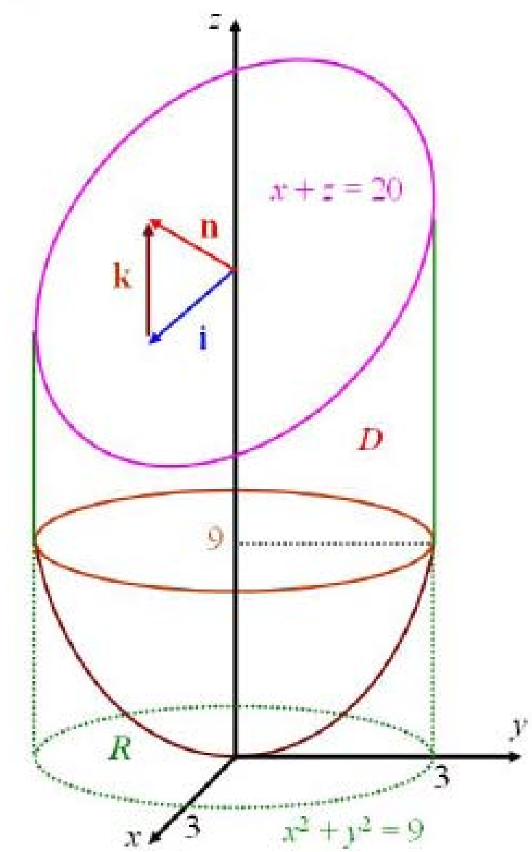
Volume under plane   Volume under paraboloid

$$= \int_0^{2\pi} \int_0^3 (20 - r\cos\theta) r \ dr \ d\theta$$

$$- \int_0^{2\pi} \int_0^3 (r^2) r \ dr \ d\theta$$

$$x = r\cos\theta \quad y = r\sin\theta \quad r = \sqrt{x^2 + y^2}$$

$$
\begin{aligned}
V &= \int_0^{2\pi} \boxed{\int_0^3 20r - r^2\cos\theta - r^3 \; dr} \; d\theta \\
&= \int_0^{2\pi} \boxed{\left[10r^2 - \frac{r^3}{3}\cos\theta - \frac{r^4}{4}\right]_0^3} \; d\theta \\
&= \int_0^{2\pi} 90 - 9\cos\theta - \frac{81}{4} \; d\theta \\
&= \left[\boxed{\frac{279}{4}\theta} - \underbrace{9\sin\theta}_{0}\right]_0^{\boxed{2\pi}} \\
&= \boxed{\frac{279}{2}\pi}
\end{aligned}
$$



Surface Area

If $z = f(x, y)$ has continuous first order partial derivatives over a region *R* in the *xy*-plane, then the area of the surface over *R* is given by

$$\boxed{S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA.}$$

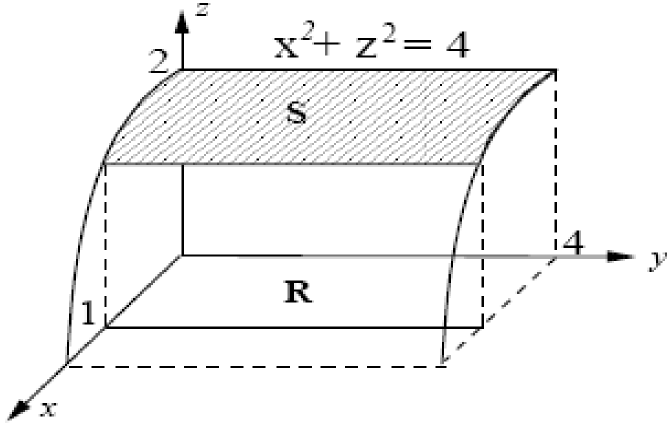
The above formula is easily obtained from results in Chapter 10 on surface integrals.

Example

A rectangle R in the xy-plane is described as Type B by:

$$R: \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 4.$$

Find area of portion of cylinder $x^2 + z^2 = 4$ lying above R.



For the portion of the cylinder above the xy-plane, write

$$z = \sqrt{4 - x^2}$$

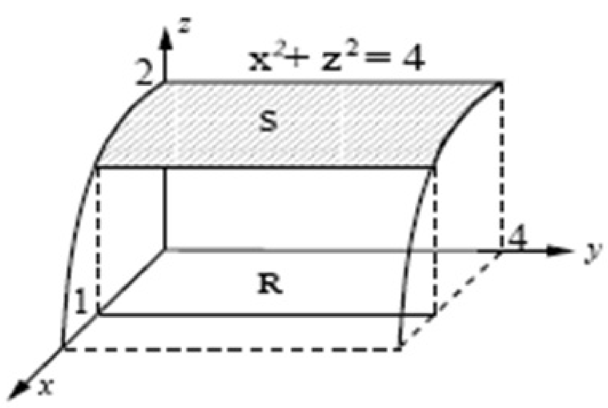
$$\text{Surface area } S = \iint_R \sqrt{\left(-\frac{x}{\sqrt{4-x^2}}\right)^2 + 0^2 + 1}\, dA$$

$$= \int_0^4 \left[\int_0^1 \frac{2}{\sqrt{4-x^2}}\, dx\right] dy$$

$$= 2\int_0^4 \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{x=0}^{x=1} dy$$

$$= 2\int_0^4 \frac{\pi}{6}\, dy$$

$$= \boxed{\frac{4\pi}{3}}$$



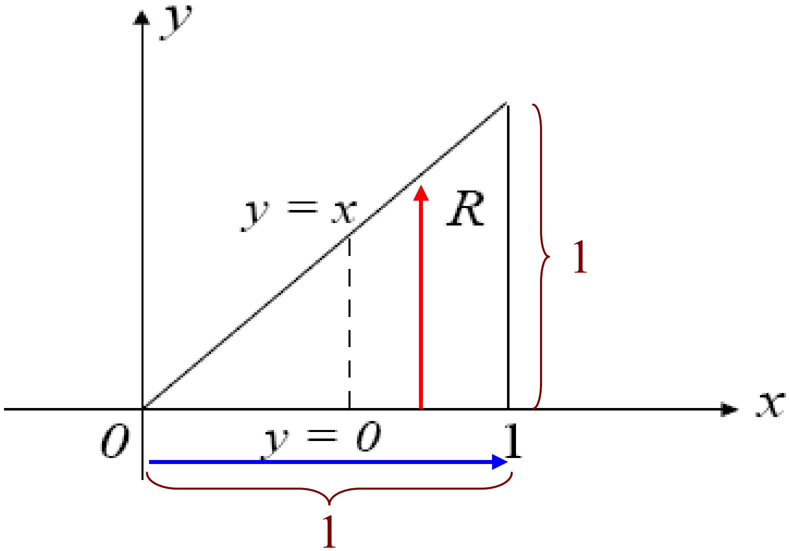
$$z = \sqrt{4-x^2}$$

Average Value of a Function

The average value of $f(x, y)$ over a region R is given by

$$\boxed{\frac{1}{\text{Area of } R}\iint_R f(x,y)\,dA.}$$

Example



Area of triangular region R

$$= \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

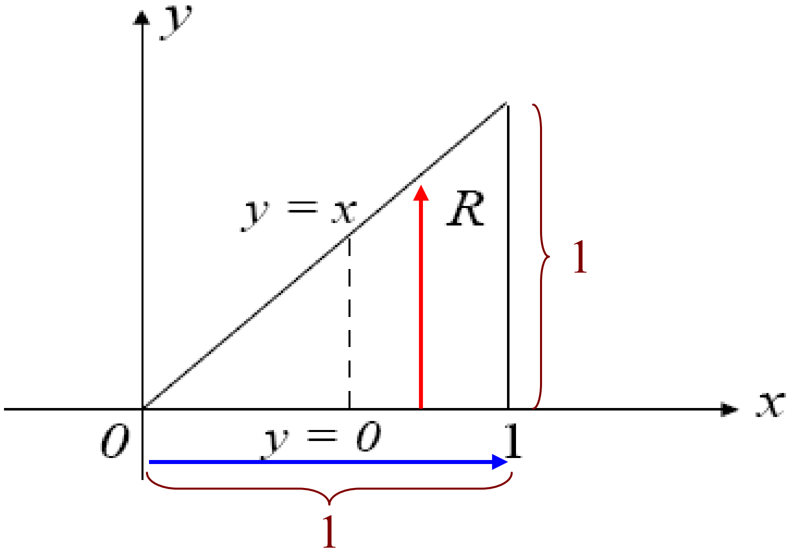
Region R described as Type A

$$R: \; 0 \le y \le x, \quad 0 \le x \le 1.$$

Find the average value of the function $f(x, y) = xe^y$

on the triangular region with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$.

Example



Area of triangular region R

$$= \frac{1}{2}\cdot 1\cdot 1 = \boxed{\frac{1}{2}}$$

Region R described as Type A

$$R:\ \boxed{0 \le y \le x,\quad 0 \le x \le 1.}$$

Average value $$= \frac{1}{\boxed{1/2}} \int_0^1 \left[\int_0^x x e^y \, dy\right] dx$$

$$= 2\int_0^1 \boxed{\left[x e^y\right]_{y=0}^{y=x}} \, dx = 2\int_0^1 \left(\boxed{x e^x} - x\right) dx$$ (integrate by parts)

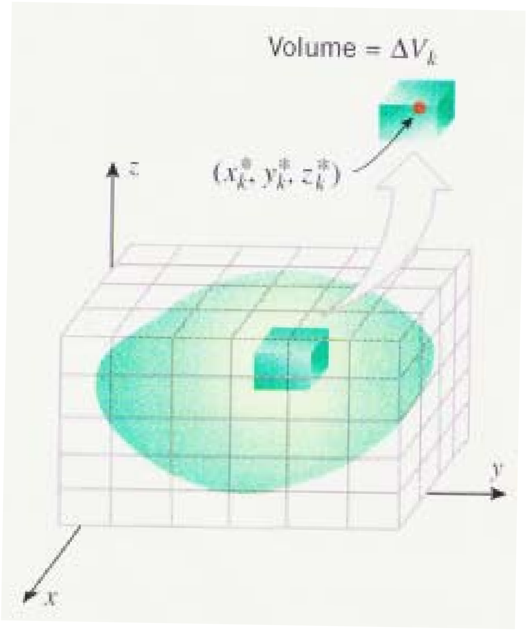
$$= 2\left[\boxed{x e^x - e^x} - \frac{1}{2}x^2\right]_{x=0}^{x=1} = 2\left[\left(e - e - \frac{1}{2}\right) - (-1)\right] = \boxed{1.}$$

Triple Integrals

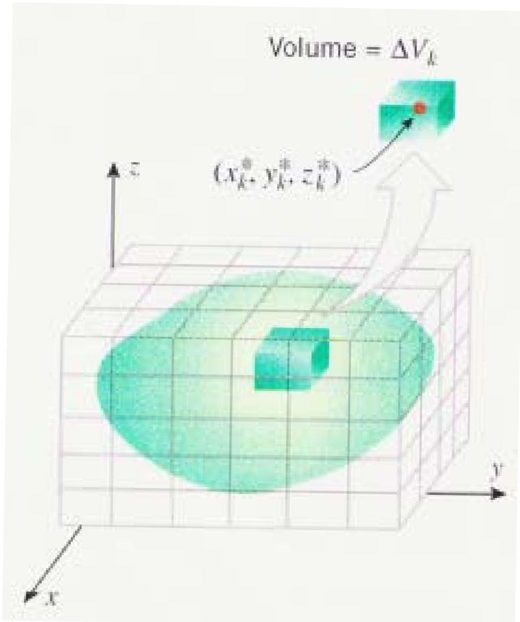
Triple integrals can be defined for functions $f(x, y, z)$ over a solid region D in three-dimensional space.

Use planes parallel to the coordinate planes to subdivide D into n smaller cubic regions D_k :

Each cubic region D_k has volume ΔV_k and a chosen point (x_k^*, y_k^*, z_k^*).



Source: *Calculus* by Anton, Bivens, Davis



The triple integral of *f* over *D* is:

$$\boxed{\iiint_D f(x,y,z)\,dV = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\,\Delta V_k\,.}$$

$$\boxed{\iiint_D f(x, y, z)\, dV = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\, \Delta V_k\,.}$$

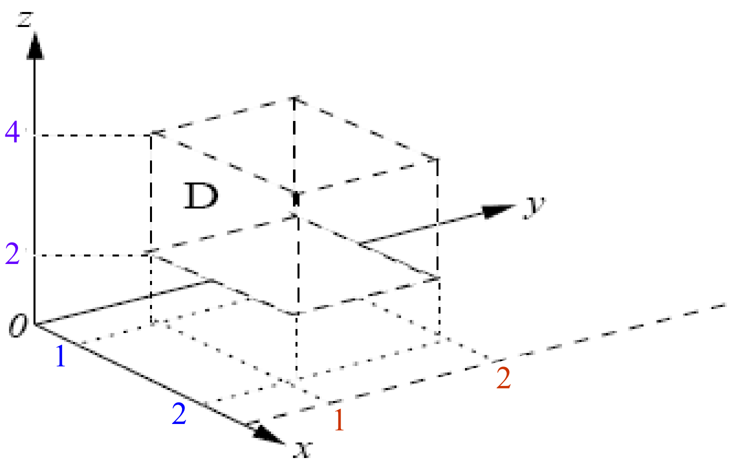
Compared to double integrals to find volumes, there is *no direct geometrical meaning* for triple integrals.

In special cases when f represents some physical quantity, e.g. density, the triple integral may have some physical meaning.

Example

Suppose the points (x, y, z) in a rectangular solid D are bounded as follows:

$$D\ :\ \boxed{1 \le x \le 2,}\ \boxed{1 \le y \le 2,}\ \boxed{2 \le z \le 4.}$$



D is made up of various compounds such that its density δ is given by:

$$\boxed{\delta(x, y, z) = (x + y)z.}$$

Find the mass of D.

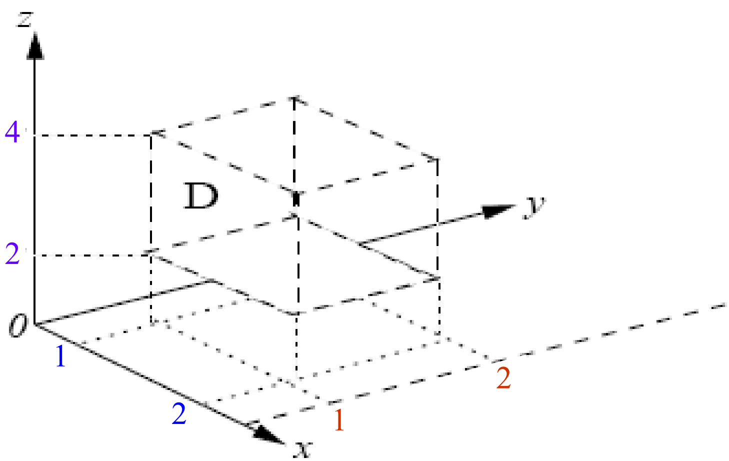
$$\text{Mass of } D = \iiint_D \delta(x,y,z)\,dV$$

D : $1 \le x \le 2$,

$1 \le y \le 2$,

$2 \le z \le 4$.

$\delta(x, y, z) = (x + y)z$.



$$= \int_1^2 \int_1^2 \boxed{\int_2^4 (x+y)z\,dz}\,dy\,dx$$

$$= \int_1^2 \boxed{\int_1^2 \boxed{(x+y)\left[\frac{z^2}{2}\right]_{z=2}^{z=4}}\,dy}\,dx$$

$$= \int_1^2 \boxed{6\left[xy + \frac{y^2}{2}\right]_{y=1}^{y=2}}\,dx$$

$$= \int_1^2 6\left[x + \frac{3}{2}\right]\,dx$$

$$= 6\left[\frac{x^2}{2} + \frac{3}{2}x\right]_1^2 = \boxed{18}$$

End of Chapter 8