Chapter 7

Functions of Several Variables

Key Results

- Partial derivatives of functions of several variables.
- Chain rule.
- Directional derivatives.
- Approximation of changes in function values.
- Second derivative test for extrema.
- Lagrange multipliers

Introduction

- In earlier chapters, functions of one variable $f(x)$ were studied.
- But many physical quantities in real life are described by real-valued functions of several variables.
- The temperature in a room can be described by a function $T(x, y, z)$, where x, y, z are coordinates of a point in the room.

Functions of Two Variables

A function of two variables is a rule that assigns to each ordered pair of real numbers (x, y) a real number denoted by $f(x, y)$.

The set of inputs (x, y) is called the domain of f.

Independent variables are x and y.

Write $z = f(x, y)$. The set of outputs z is called the range of f.

Dependent variable is z.

Examples

1. Polynomial function $f(x, y) = x^2 y^3$.

   Defined for all values x and y.
   Domain consists of all ordered pairs (x, y).

2. Radical function $f(x,y) = \sqrt{1 - x^2 - y^2}$

   function is only defined when $1 - x^2 - y^2 \geq 0$

   equivalently $x^2 + y^2 \leq 1$

   domain of f is the set $D = \{(x,y) : x^2 + y^2 \leq 1\}$

   That is, all points in the *xy*-plane lying within and on the unit circle.

Example

3. Compound functions or piecewise-defined function

$$f(x, y) = \begin{cases} \sqrt{x-y} & \text{if } x > y, \\ \sqrt{y-x} & \text{if } x < y, \\ 1 & \text{if } x = y\,. \end{cases}$$

Graphs

By analogy with the graph of a function of one variable, the graph of a function $f(x, y)$ consists of all the points (x, y, z) in thee-dimensional space such that $z = f(x, y)$.

Graph is a surface in three-dimensional space.

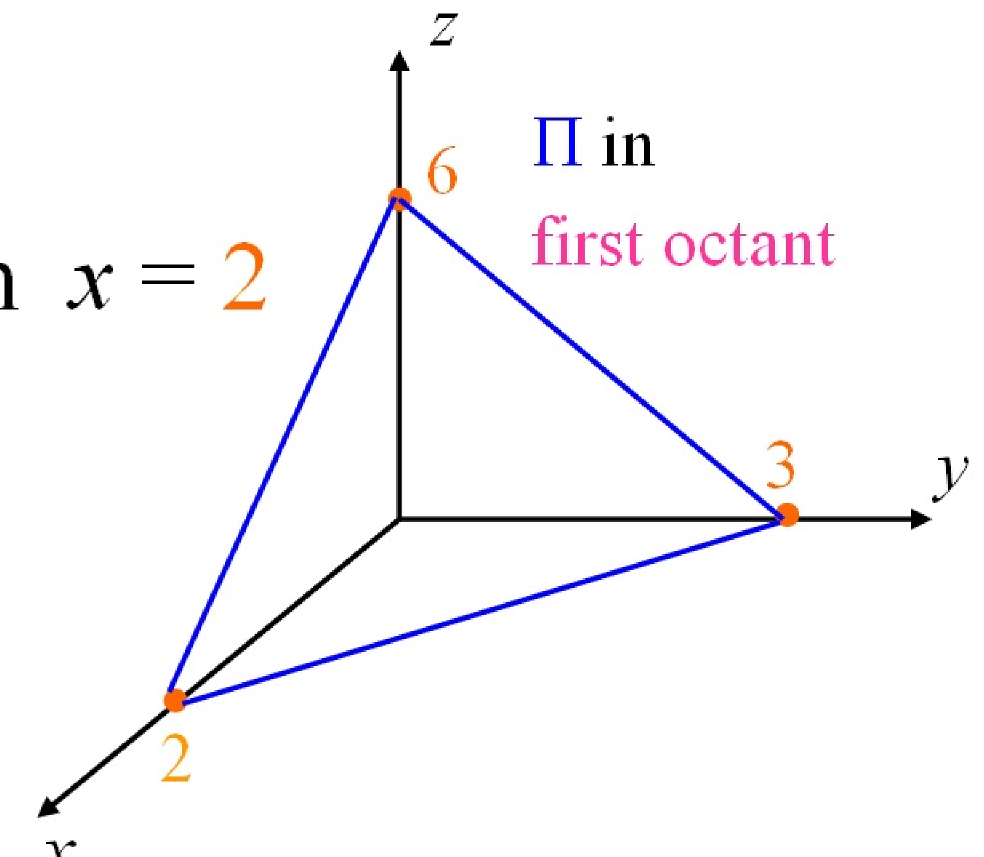
Example (Plane)

The plane $\boxed{\Pi: \ 3x + 2y + z = 6}$ can be expressed as

$$f(x, y) = z = 6 - 3x - 2y$$

The portion of Π lying in the first octant,

where $x \geq 0,\ y \geq 0,\ z \geq 0$,
is sketched:



x-intercept: if $y = 0,\ z = 0$, then $x = 2$

$(2, 0, 0)$

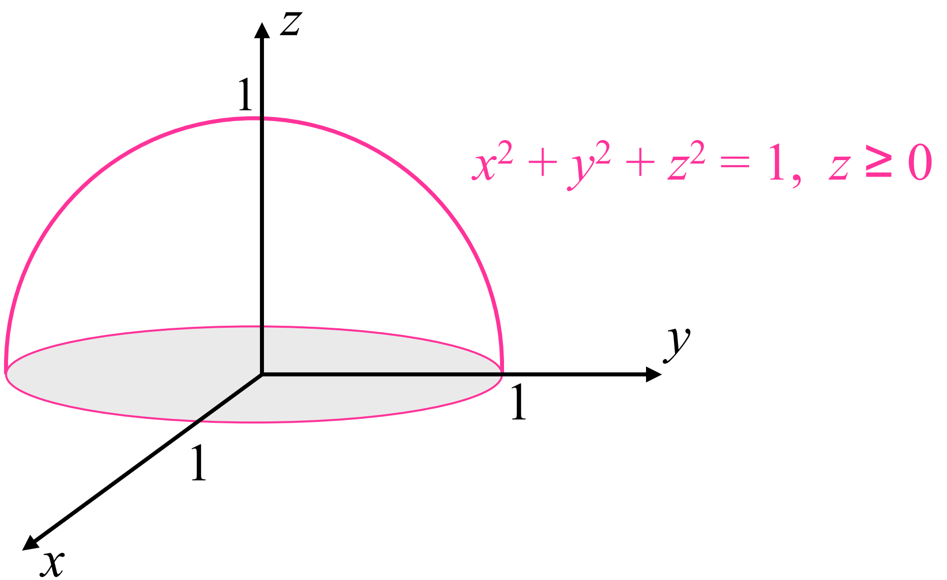
Similarly, $(0, 3, 0)$ and $(0, 0, 6)$ are on Π.

Example (Sphere, Hemisphere)

The surface $f(x, y) = \boxed{z = \sqrt{1 - x^2 - y^2}}$ can be expressed as

$$\boxed{x^2 + y^2 + z^2 = 1,} \quad z \geq 0$$

which is the equation of the upper hemisphere of a sphere of radius 1 and centred at (0, 0, 0).



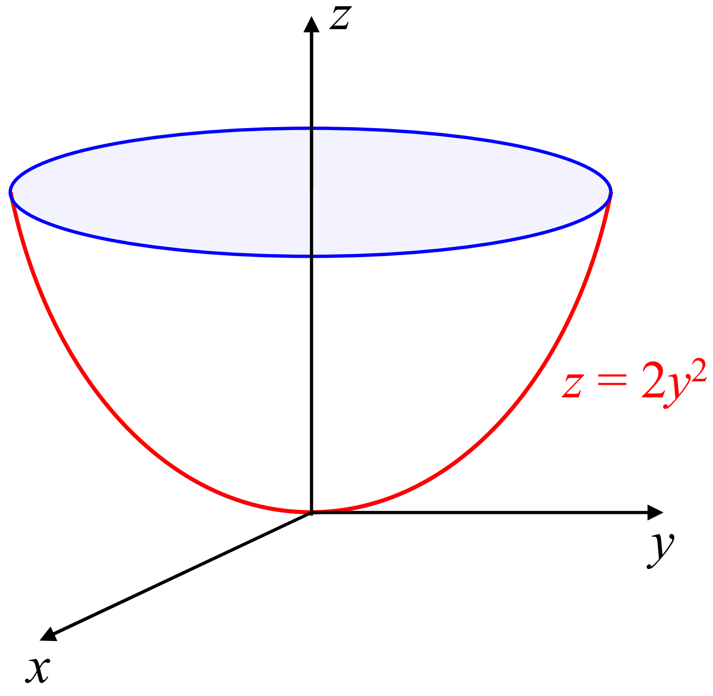
Example (Paraboloid)

$\boxed{z = 8x^2 + 2y^2}$

Set $x = 0$:

Then $z = 2y^2$

Parabola in plane $x = 0$
i.e. yz-plane.



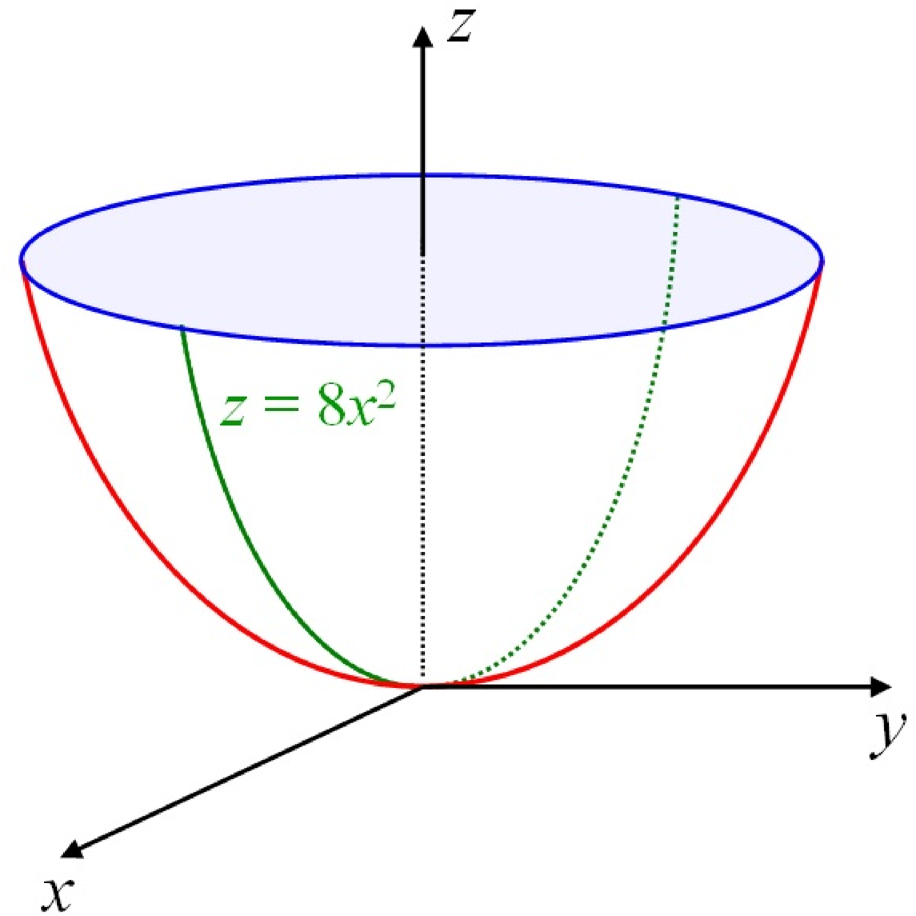
Example (Paraboloid)

$z = 8x^2 + 2y^2$

Set $y = 0$:

Then $z = 8x^2$

Parabola in plane $y = 0$
i.e. xz-plane.



Example (Paraboloid)

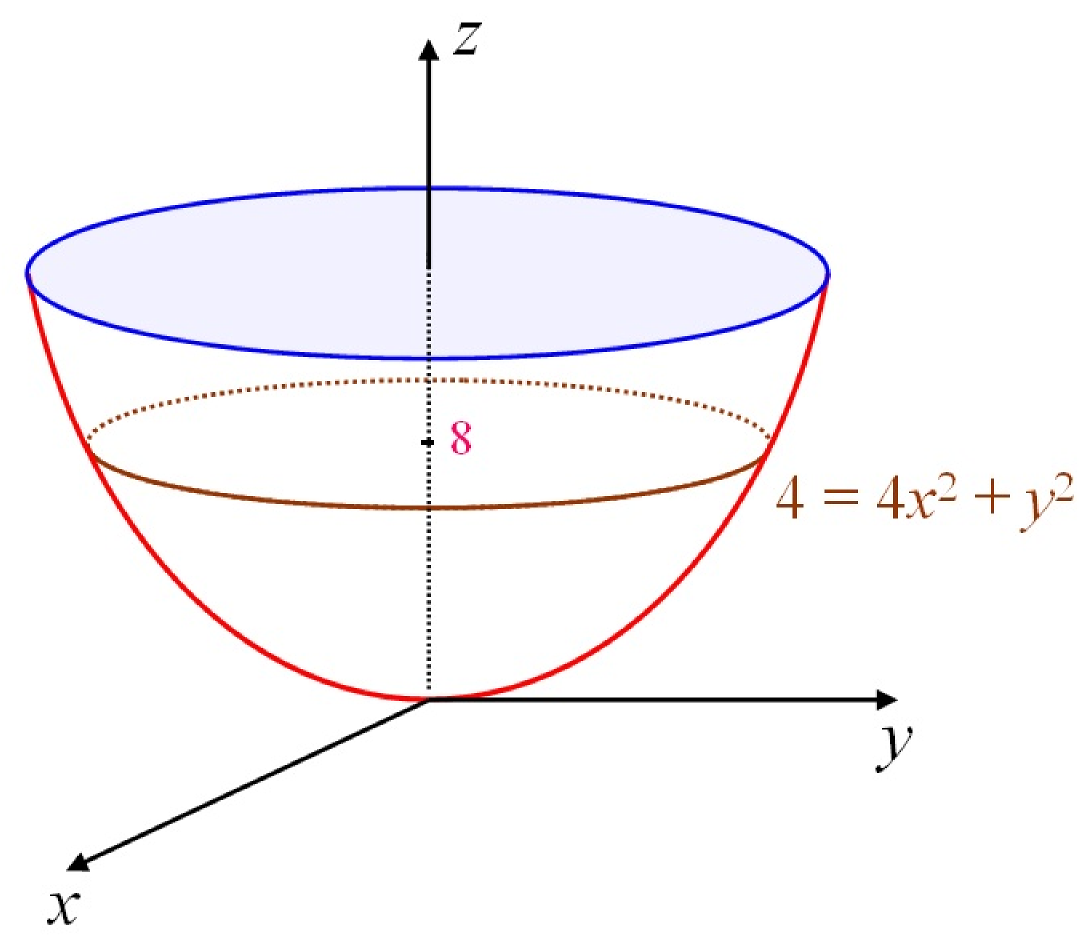
$z = 8x^2 + 2y^2$

Set $z = 8$:

Then $8 = 8x^2 + 2y^2$

$4 = 4x^2 + y^2$

Ellipse in plane $z = 8$.



Partial Derivatives

For $f(x, y)$, a change in x value or y value causes a change in $f(x, y)$.

To study such changes, may begin with fixing x or y in $f(x, y)$.

For example, consider $f(x,y) = x^2 - 2xy + 3y^3$

Fix $y = 2$ to obtain $f(x,2) = x^2 - 4x + 24$

which is a function of x only.

Differentiation with respect to x is now possible.

First Order Partial Derivatives

The first order partial derivative of $z = f(x, y)$ with respect to x at the point (a, b) is

$$\frac{d}{dx} f(x,b)\Big|_{x=a} = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

provided the limit exists.

$$f_x = \frac{\partial z}{\partial x} \quad f_x(a,b) \quad \frac{\partial f}{\partial x}\Big|_{(a,b)}$$

The first order partial derivative of $z = f(x, y)$ with respect to y at the point (a, b) is

$$\frac{d}{dy} f(a,y)\Big|_{y=b} = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

provided the limit exists.

$$f_y = \frac{\partial z}{\partial y}. \quad f_y(a,b) \quad \frac{\partial f}{\partial y}\Big|_{(a,b)}$$

Example

Let $f(x,y) = (x^3 + y)\cos(y^2)$. Find $f_x(2,0)$

Treat y as a constant

$$\frac{d}{dx} f(x,y) = \frac{d}{dx}(x^3 + y)\cos(y^2)$$

$$f_x(x,y) = 3x^2 \cos(y^2)$$

$$f_x(2,0) = 3(2)^2 \cos(0^2) = \boxed{12}$$

Find $f_y(2,0)$. Treat x as a constant

$$\frac{d}{dy} f(x,y) = \frac{d}{dy}(x^3 + y)\cos(y^2) \qquad \text{(product rule)}$$

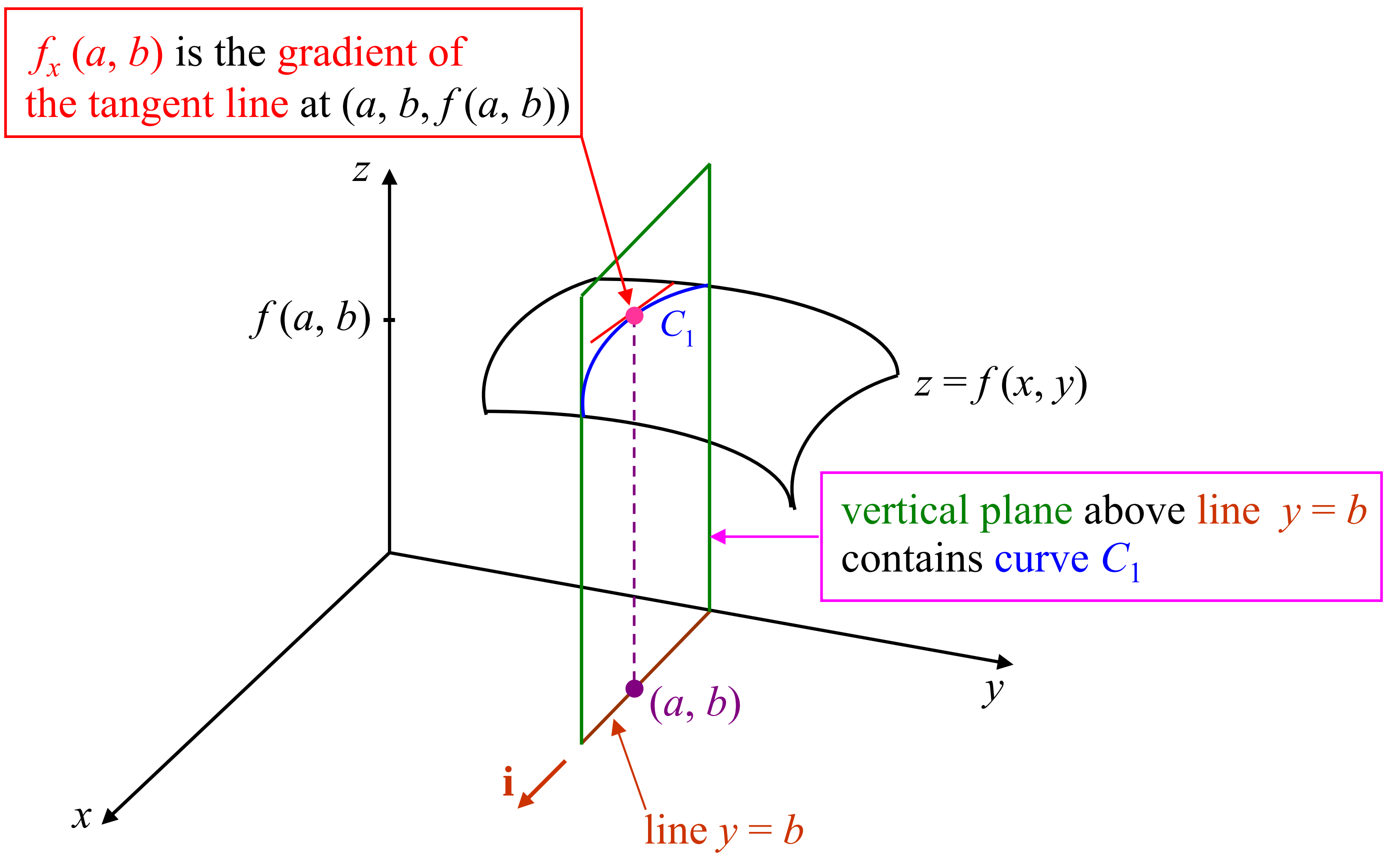
$$f_y(x,y) = \cos(y^2) - (x^3 + y)\sin(y^2)\, 2y.$$

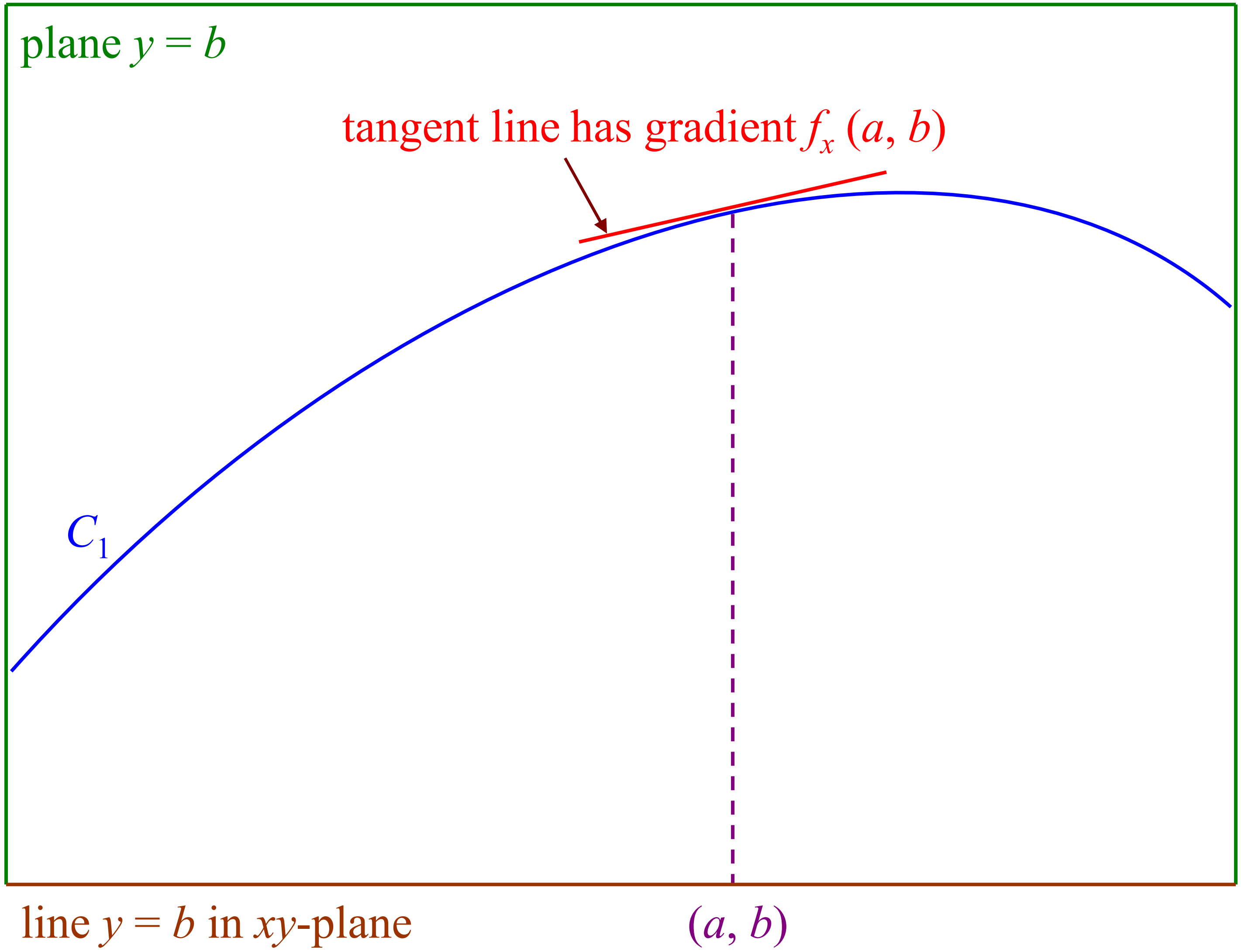
$$f_y(2,0) = \cos(0^2) - (2^3 + 0)\sin(0^2)\, 2(0) = \boxed{1}$$

Geometric Interpretation

Geometrically, $f_x(a, b)$ is the rate of change of f in the direction of vector $\mathbf{i}$ at the point (a, b) on the xy-plane.

Vertical plane $y = b$ intersects surface $z = f(x, y)$ in a curve C_1.

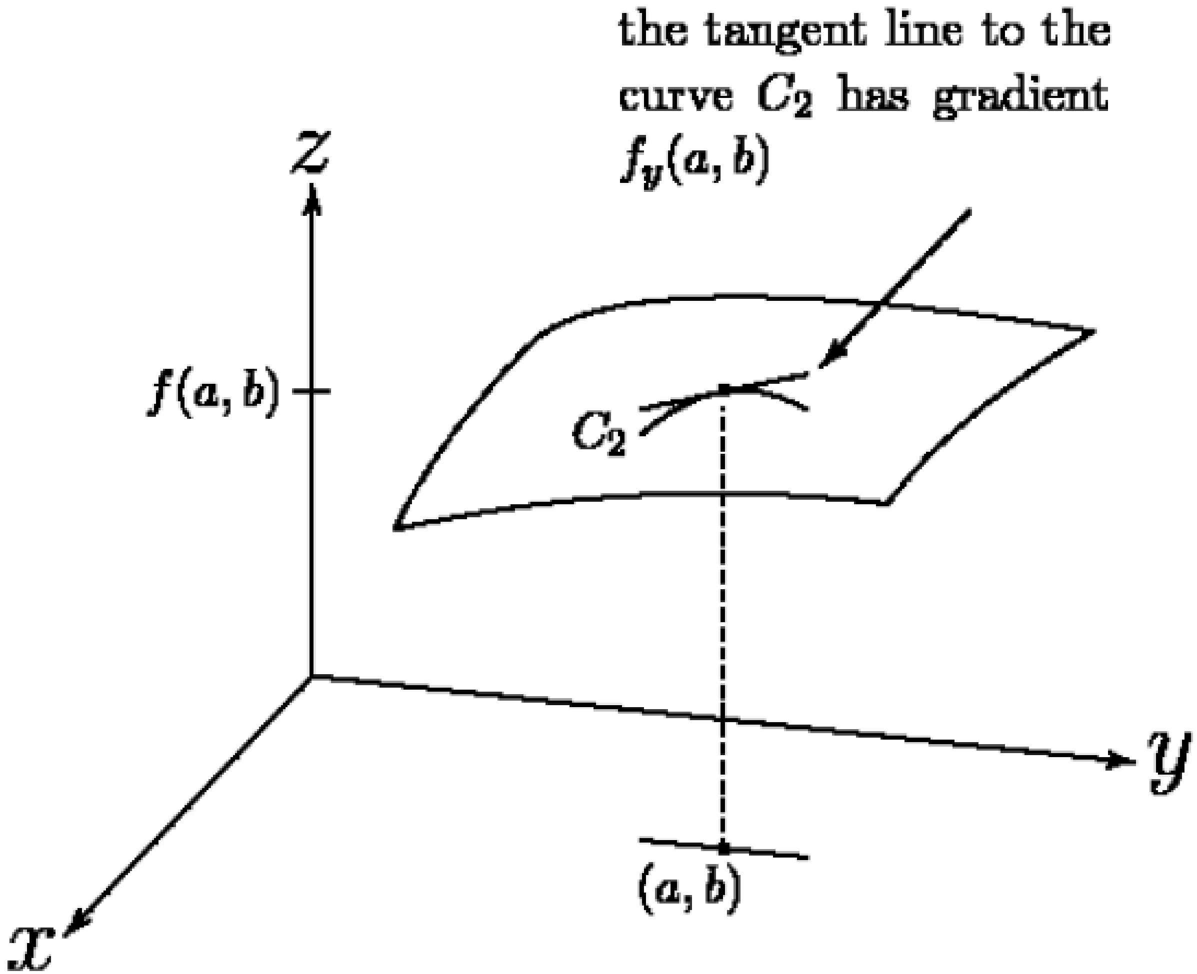




plane $y = b$
tangent line has gradient $f_x\,(a, b)$
C_1
line $y = b$ in xy-plane
(a, b)

A similar gradient description holds for $f_y\,(a, b)$.

The curve C_2 lies in the vertical plane $x = a$.



Higher Order Partial Derivatives

For $z = f(x, y)$, the partial derivatives f_x and f_y are also functions of x and y.

The functions f_x and f_y may also have partial derivatives with respect to x and y.

The second order partial derivatives of $f(x, y)$ are:

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Example

$$f(x, y) = 4x^3 + x^2y^3 - 6y^2$$

$$f_x = 12x^2 + 2xy^3 \qquad f_y = 3x^2y^2 - 12y$$

$$f_{xx} = 24x + 2y^3 \qquad \boxed{f_{xy} = 6xy^2}$$

$$f_{yy} = 6x^2y - 12 \qquad \boxed{f_{yx} = 6xy^2}$$

Mixed Derivatives

Suppose $f(x, y)$ is defined on a disk D that contains the point (a, b). If f_{xy} and f_{yx} are both continuous on D, then

For most functions in practice,

$$\boxed{f_{xy}(a, b) = f_{yx}(a, b)}$$

Chain Rule (page 16)

For $z = f(x, y)$, suppose that $x = x(t)$ and $y = y(t)$ are functions of t.

Then z is a function of t: $z(t) = f(x(t), y(t))$.

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

or

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$z(x(t))$

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt}$$

Example

Let $z = 3xy^2 + x^4y$, where $x = \sin 2t$, $y = \cos t$.

$$\frac{dz}{dt} = \boxed{\frac{\partial z}{\partial x} \cdot \frac{dx}{dt}} + \boxed{\frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

$$= \boxed{(3y^2 + 4x^3y)(2\cos 2t) + (6xy + x^4)(-\sin t)}$$

Chain Rule for $f(x, y, z)$

Suppose $w = f(x, y, z)$, where $x = x(t)$, $y = y(t)$, $z = z(t)$.

Then w is a function of t: $w(t) = f(x(t), y(t), z(t))$.

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}}$$

Example (page 14)

Suppose the length ℓ, width w and height h of a box change with time. At time t_0, the dimensions of the box are $\ell = 2$ m, $w = 3$ m, $h = 4$ m, and ℓ and w are increasing at a rate of 5 ms^{-1} while h is decreasing at a rate of 6 ms^{-1}. What is the rate of change of the volume of the box at time t_0?

Volume of box is a function of length, width and height,

$$V = V(\ell, w, h)$$

which are functions of time:

$$\ell = \ell(t) \qquad w = w(t) \qquad h = h(t)$$

Thus, volume is a function of time:

$$V(t) = V(\ell(t),\ w(t),\ h(t))$$

(page 19) By chain rule,

$$\frac{dV}{dt} = \frac{\partial V}{\partial \ell} \cdot \frac{d\ell}{dt} + \frac{\partial V}{\partial w} \cdot \frac{dw}{dt} + \frac{\partial V}{\partial h} \cdot \frac{dh}{dt}$$

$$\text{Volume} \quad \boxed{V(\ell, w, h) \;=\; \ell w h}$$

$$\begin{aligned}
\frac{dV}{dt} &= \boxed{\frac{\partial V}{\partial \ell}} \cdot \frac{d\ell}{dt} + \boxed{\frac{\partial V}{\partial w}} \cdot \frac{dw}{dt} + \boxed{\frac{\partial V}{\partial h}} \cdot \frac{dh}{dt} \\
&= wh\frac{d\ell}{dt} + \ell h\frac{dw}{dt} + \ell w\frac{dh}{dt} \\
&= 3 \cdot 4 \cdot 5 + 2 \cdot 4 \cdot 5 + 2 \cdot 3 \cdot (-6) \\
&= \boxed{64 \ \ \mathrm{m}^3\mathrm{s}^{-1}}
\end{aligned}$$

Given: $\ell = 2\text{ m},\ w = 3\text{ m},\ h = 4\text{ m}.$

$$\frac{d\ell}{dt} = 5\,\mathrm{ms}^{-1},\ \frac{dw}{dt} = 5\,\mathrm{ms}^{-1} \text{ and } \frac{dh}{dt} = -6\,\mathrm{ms}^{-1}$$

Chain Rule Generalizations

The chain rule generalizes. Generalizations depend on the function f and the input variables to f which may also be functions of other variables.

Two situations are described on pages 17 and 20.

Observe that the equations are similar:

sums of products of various derivatives.

Directional Derivatives

Partial derivatives of $f(x, y)$ give the rates of change with respect to x and y, i.e. along the directions of the x-axis and y-axis.

What about the rate of change along an arbitrary direction?

This leads to the notion of directional derivatives.

Ingredients: $f(x, y)$ at a point (a, b).

A unit vector $\boxed{\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}}$ in the xy-plane.

The directional derivative of $f(x, y)$ at (a, b) in the direction of a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is

$$\boxed{D_{\mathbf{u}}f(a,b) = \lim_{h\to 0}\frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}}$$

if this limit exists.

$u_1 = 1$ $u_2 = 0$

$$D_{\mathbf{i}}f(a,b) = \lim_{h\to 0}\frac{f(a+h, b) - f(a,b)}{h} = f_x(a,b)$$

Ingredients: $f(x, y)$ at a point (a, b).

A unit vector $\boxed{\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}}$ in the xy-plane.

The directional derivative of $f(x, y)$ at (a, b) in the direction of a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is

$$\boxed{D_{\mathbf{u}}f(a,b) = \lim_{h\to 0}\frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}}$$

if this limit exists.

$u_1 = 0$ $\quad$ $u_2 = 1$

$$D_{\mathbf{i}}f(a,b) = \lim_{h\to 0}\frac{f(a+h,b) - f(a,b)}{h} = f_x(a,b)$$

$$D_{\mathbf{j}}f(a,b) = \lim_{h\to 0}\frac{f(a,b+h) - f(a,b)}{h} = f_y(a,b)$$

Ingredients: $f(x, y)$ at a point (a, b).

A unit vector $\boxed{\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}}$ in the xy-plane.

The directional derivative of $f(x, y)$ at (a, b) in the direction of a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is

$$\boxed{D_{\mathbf{u}}f(a,b) = \lim_{h\to 0} \frac{f(a + hu_1, b + hu_2) - f(a,b)}{h}}$$

if this limit exists.

$$\boxed{D_{\mathbf{i}}f(a,b) =} \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} = \boxed{f_x(a,b)}$$

$$\boxed{D_{\mathbf{j}}f(a,b) =} \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h} = \boxed{f_y(a,b)}$$

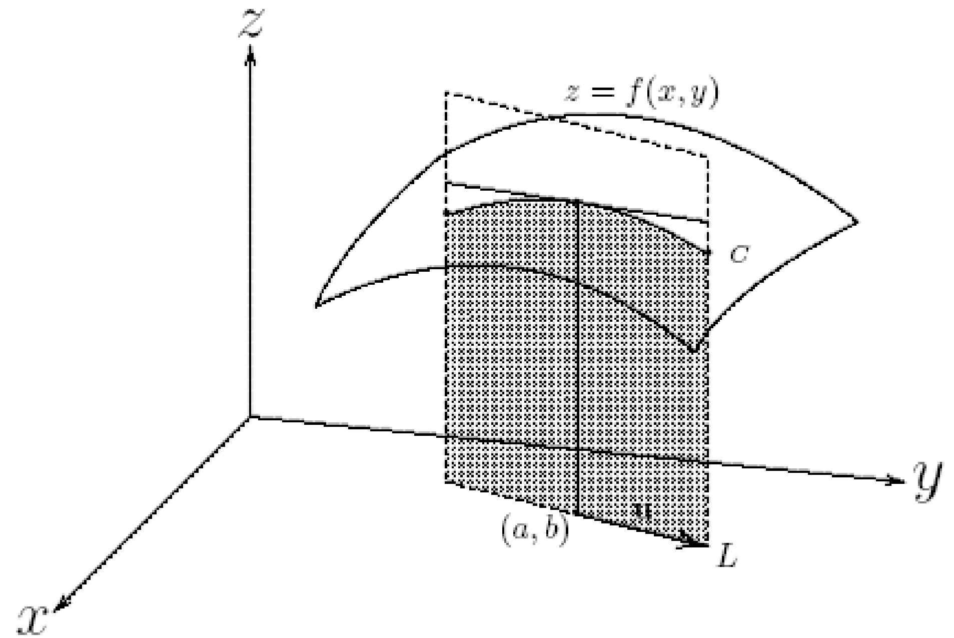
Geometric Interpretation

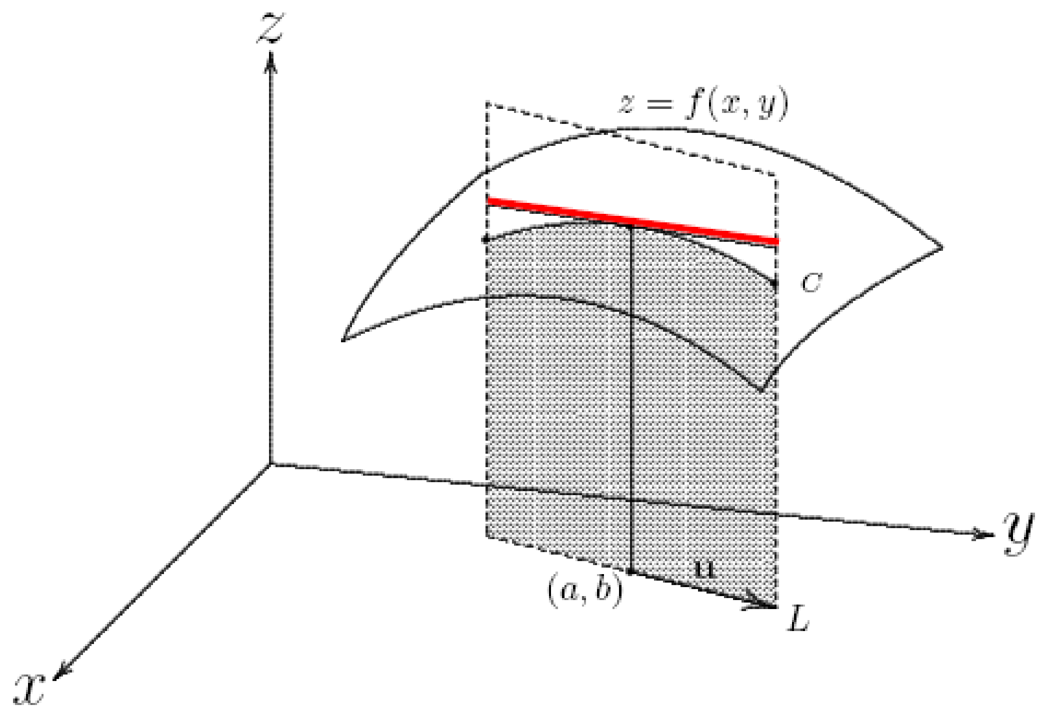
The line L with parametric equations

$$x = a + u_1 t, \quad y = b + u_2 t, \quad z = 0,$$

lies in the xy-plane, passes through the point $(a, b, 0)$, and is parallel to the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$.

A vertical plane over L intersects the surface $z = f(x, y)$ in a curve C.





$D_{\mathbf{u}} f(a, b)$ gives the gradient of the tangent line to C at $(a, b, f(a, b))$ in the direction of $\mathbf{u}$.

$$\boxed{D_{\mathbf{u}} f(a, b) = f_x(a, b) \cdot u_1 + f_y(a, b) \cdot u_2}$$

The formula is derived easily from the $\boxed{\text{chain rule.}}$

First note that the line L has parametric equations

$$\boxed{x = a + u_1 t,} \quad \boxed{y = b + u_2 t,} \quad z = 0,$$

with parameter t.

$D_{\mathbf{u}} f(a, b)$ is the rate of change of $f(x, y)$ in the direction of $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$ at (a, b) on L.

Treating f as a function of t, it follows that

$$\boxed{D_{\mathbf{u}} f(a, b) =} \frac{df}{dt} = \boxed{\frac{\partial f}{\partial x} \cdot \boxed{\frac{dx}{dt}}^{= u_1} + \frac{\partial f}{\partial y} \cdot \boxed{\frac{dy}{dt}}^{= u_2}}$$

$$= \boxed{f_x(a, b) \cdot u_1 + f_y(a, b) \cdot u_2}$$

Example

Let $\boxed{f(x, y) = x^2 - 3xy^2 + 2y^3.}$ Find $D_{\mathbf{u}} f(2, 1)$, where

$$\mathbf{u} = \boxed{\frac{\sqrt{3}}{2}}\mathbf{i} + \boxed{\frac{1}{2}}\mathbf{j}.$$

$$f_x(x, y) = 2x - 3y^2. \qquad f_y(x, y) = -6xy + 6y^2$$

$$f_x(2, 1) = 2\cdot 2 - 3\cdot 1^2 = \boxed{1} \qquad f_y(2, 1) = -6\cdot 2\cdot 1 + 6\cdot 1^2 = \boxed{-6.}$$

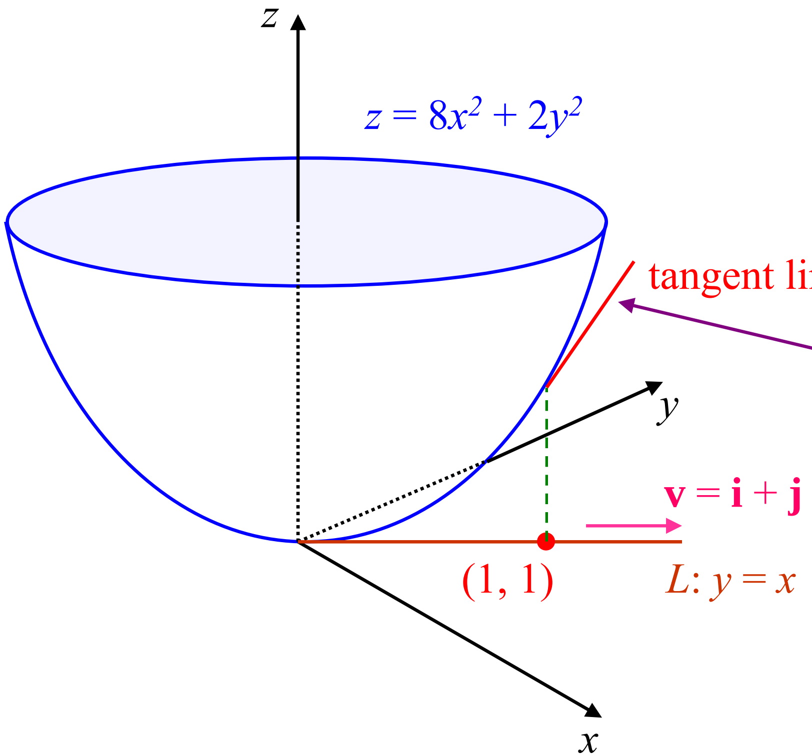
$$\boxed{D_{\mathbf{u}} f(2, 1) \ =} \ 1 \cdot \frac{\sqrt{3}}{2} + (-6) \cdot \frac{1}{2} = \boxed{\frac{\sqrt{3} - 6}{2}}$$

$$D_{\mathbf{u}} f(a, b) = \boxed{f_x(a, b)} \cdot \boxed{u_1} + \boxed{f_y(a, b)} \cdot \boxed{u_2.}$$

Example

Consider the paraboloid $z = f(x, y) = 8x^2 + 2y^2$.

Line L in the xy-plane with equation $y = x$.



Point (1, 1) on L.

Vector $\mathbf{v} = \mathbf{i} + \mathbf{j}$

is in the direction of L with increasing x.

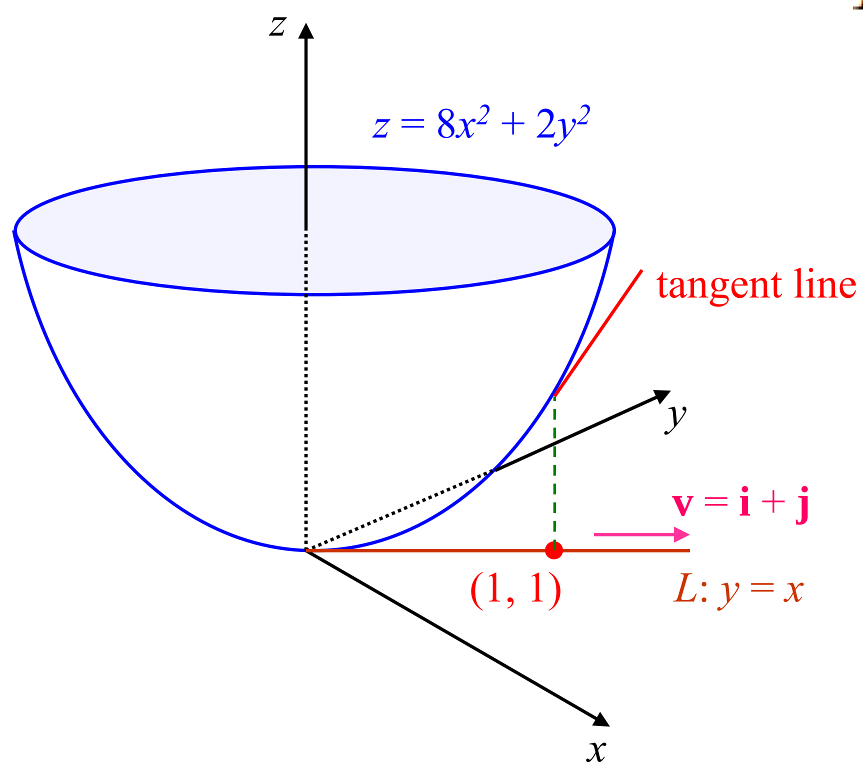
$D_{\mathbf{u}} f(1, 1)$ is the slope of the tangent line.

What is $\mathbf{u}$?

$$\mathbf{v} = \mathbf{i} + \mathbf{j}$$

$$u_1\mathbf{i} + u_2\mathbf{j} = \mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \boxed{\frac{1}{\sqrt{2}}}\mathbf{i} + \boxed{\frac{1}{\sqrt{2}}}\mathbf{j}$$

$$\boxed{f(x, y) = 8x^2 + 2y^2} \quad \boxed{f_x(x, y) = 16x} \quad \boxed{f_y(x, y) = 4y}$$



$$\begin{aligned} D_{\mathbf{u}}f(1, 1) &= \boxed{f_x(1, 1)} \cdot \boxed{u_1} + \boxed{f_y(1, 1)} \cdot \boxed{u_2} \\ &= \boxed{16 \cdot \frac{1}{\sqrt{2}} + 4 \cdot \frac{1}{\sqrt{2}}} \\ &= \frac{20}{\sqrt{2}} \\ &= \boxed{10\sqrt{2}} \end{aligned}$$

Gradient Vector

The directional derivative can be expressed as a dot product to obtain additional results.

First define a vector using partial derivatives:

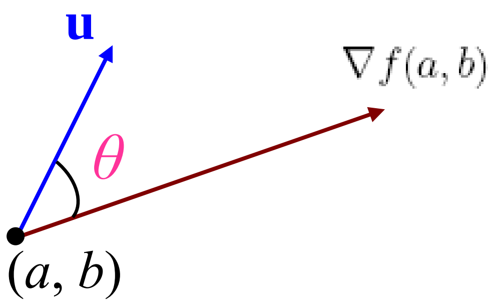
$$\boxed{\nabla f = f_x \mathbf{i} + f_y \mathbf{j}.}$$

called gradient of f. 'grad f' 'del f'

For a given unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, consider

$$\begin{aligned} \boxed{\nabla f(a,b) \cdot \mathbf{u} \quad =} & \; (f_x(a,b)\,\mathbf{i} + f_y(a,b)\,\mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j}) \\ = & \; f_x(a,b) \cdot u_1 + f_y(a,b) \cdot u_2 \\ = & \; \boxed{D_{\mathbf{u}} f(a,b).} \end{aligned}$$

Let θ be the angle between $\mathbf{u}$ and $\nabla f(a,b)$.



$$\boxed{D_{\mathbf{u}} f(a,b)}$$

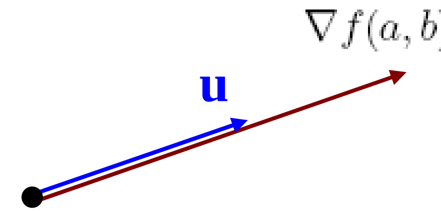
$$= \nabla f(a,b) \cdot \mathbf{u}$$

$$= \|\nabla f(a,b)\| \underbrace{\boxed{\|\mathbf{u}\|}}_{} \overset{=1}{\cos\theta}$$

$$= \boxed{\|\nabla f(a,b)\| \cos\theta}$$

Assume

$\nabla f(a,b) \neq \mathbf{0}$



The largest value of $\cos\theta$ is 1 when $\theta = 0$.

Thus, the largest value of $D_{\mathbf{u}} f(a,b)$ is $\boxed{\|\nabla f(a,b)\|}$

(f increases most rapidly)

in the direction of $\boxed{\nabla f(a,b)}$.

Similarly, the smallest value of $D_{\mathbf{u}} f(a,b)$ is $\boxed{-\|\nabla f(a,b)\|}$

(f decreases most rapidly)

in the direction of $\boxed{-\nabla f(a,b)}$.

Example

$f(x, y) = \boxed{\sqrt{9 - x^2 - y^2}}$. Find the largest value of $D_{\mathbf{u}} f(2, 1)$ and the corresponding direction given by **u**.

$$\boxed{f_x = \frac{-x}{\sqrt{9 - x^2 - y^2}}} \qquad \boxed{f_y = \frac{-y}{\sqrt{9 - x^2 - y^2}}}$$

The largest value of $D_{\mathbf{u}} f(2, 1)$ is obtained when **u** is in the direction of

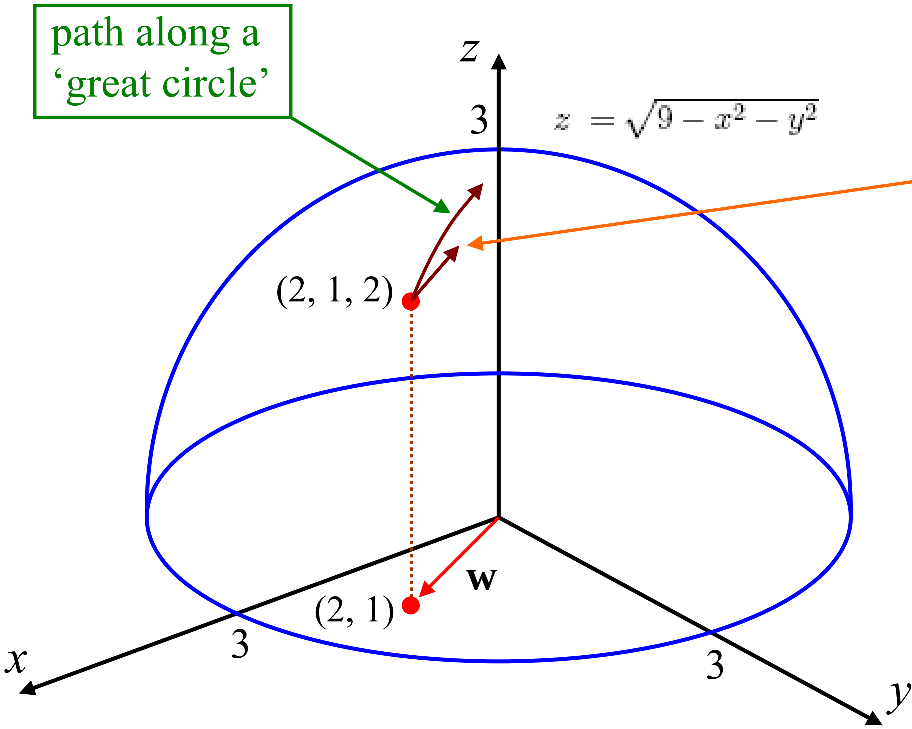
$$\boxed{\nabla f(2, 1) =} \boxed{f_x(2, 1)\mathbf{i}} + \boxed{f_y(2, 1)\mathbf{j}}$$

$$= \boxed{-\mathbf{i} - \frac{1}{2}\mathbf{j}.}$$

Largest value of $D_{\mathbf{u}} f(2, 1)$ is

$$\boxed{\|\nabla f(2, 1)\| =} \sqrt{(-1)^2 + \left(-\frac{1}{2}\right)^2} = \boxed{\frac{\sqrt{5}}{2}.}$$

The surface $z = \sqrt{9 - x^2 - y^2}$ is the upper hemisphere centred at (0, 0, 0) and of radius 3.



Let $\mathbf{w} = 2\mathbf{i} + \mathbf{j}$

$$\nabla f(2,1) = -\mathbf{i} - \frac{1}{2}\mathbf{j}$$

$$= -\frac{1}{2}\mathbf{w}$$

is parallel to **w** but points towards origin.

[at (2, 1, 2) points towards *z*-axis]

$D_{\mathbf{u}} f(2, 1)$ gives the largest rate of change of f at point (2, 1) and is in the direction of $\nabla f(2,1)$.

Climbing up a 'great circle' is the steepest climb up the hemisphere.

Physical Meaning

For functions $y(x)$ of one variable x,

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx} \quad \Rightarrow \quad \delta y \approx \frac{dy}{dx} \cdot \delta x$$

(convenient notation) $dy \approx \dfrac{dy}{dx} \cdot dx$

For functions $f(x, y)$ of two variables x and y,

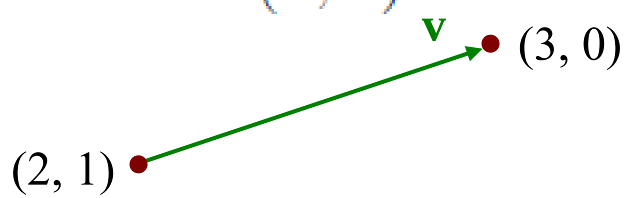
$$\boxed{df \approx D_{\mathbf{u}} f(a, b) \cdot dt}$$

$D_{\mathbf{u}} f(a, b)$ measures the change df in value of f in moving a small distance dt from the point (a, b) in the direction of the unit vector $\mathbf{u}$.

Example

Let $f(x, y) = x^2 y^3 + 1$.

Estimate how much the value of f will change if a point Q moves 0.1 unit from $(2, 1)$ towards $(3, 0)$.



Q moves in the direction

$$\mathbf{v} = (3\mathbf{i} + 0\mathbf{j}) - (2\mathbf{i} + 1\mathbf{j}) = \mathbf{i} - \mathbf{j}$$

The unit vector $\mathbf{u}$ along this direction is

$$u_1\mathbf{i} + u_2\mathbf{j} = \mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$$

$$u_1\mathbf{i}+u_2\mathbf{j} \;=\; \mathbf{u} \;=\; \frac{\mathbf{v}}{||\mathbf{v}||} \;=\; \boxed{\frac{1}{\sqrt{2}}\mathbf{i}}\boxed{-\frac{1}{\sqrt{2}}\mathbf{j}}$$

$$f(x,y)=x^2y^3+1 \qquad f_x(x,y)=\boxed{2xy^3} \qquad f_y(x,y)=\boxed{3x^2y^2}$$

$$D_{\mathbf{u}}f(2,1) \;=\; \boxed{f_x(2,1)\cdot u_1 \;+}\boxed{f_y(2,1)\cdot u_2}$$

$$\boxed{=\; 4\cdot\frac{1}{\sqrt{2}} \;+\; 12\cdot\left(-\frac{1}{\sqrt{2}}\right)}$$

$$=\; -\frac{8}{\sqrt{2}} \;=\; -4\sqrt{2}$$

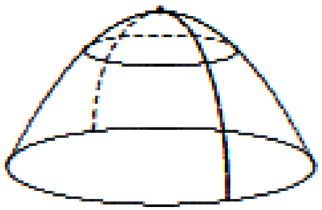
$$df \;\approx\; D_{\mathbf{u}}f(2,1)\cdot dt \;=\; \left(-4\sqrt{2}\right)\cdot 0.1 \;\approx\; -0.57$$

Value of *f* *decreases* by approximately 0.57 unit.

Max./Min. Values

$f(x, y)$ has a local maximum at (a, b) if

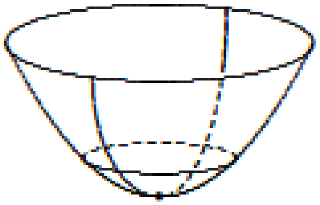
$$f(x, y) \leq f(a, b)$$



for all points (x, y) near (a, b).

The number $f(a, b)$ is called a local maximum value.

Similar definitions hold for local minimum at (a, b) and local minimum value.



Critical Points

A function f may have a local maximum or minimum at (a, b) if

(i) $f_x(a, b) = 0$ and $f_y(a, b) = 0$; or

(ii) $f_x(a, b)$ or $f_y(a, b)$ does not exist.

A point of f that satisfies (i) or (ii) above is called a critical point.

Example

Let $f(x,y) = x^2 + y^2 + 4x - 8y + 24.$

$$f_x(x,y) = 2x + 4 \overset{\text{set}}{=} 0 \;\Rightarrow\; x = -2$$

$$f_y(x,y) = 2y - 8 \overset{\text{set}}{=} 0 \;\Rightarrow\; y = 4$$

critical point

$(x,y) = (-2,4)$

'Inspection method' for simple polynomials: complete the square

$$\boxed{f(x,y)} = 24 + \boxed{x^2 + 4x} + \boxed{y^2 - 8y}$$

$$= 4 + \underbrace{\boxed{(x+2)^2}}_{\geq 0} + \underbrace{\boxed{(y-4)^2}}_{\geq 0}$$

$$\boxed{\geq 4}$$

Point (–2, 4) is a local minimum of f with a minimum value of 4.

Saddle Points

Suppose $\boxed{f_x(a, b) = 0 \text{ and } f_y(a, b) = 0.}$

If there are some directions along which f has a local maximum at (a, b) and some directions along which f has a local minimum at (a, b), then (a, b) is called a saddle point.

Example

Consider $z = f(x, y) = \boxed{2y^2 - 3x^2}$

Find critical points:

$$f_x(x, y) = -6x \overset{\text{set}}{=} 0$$

$$f_y(x, y) = 4y \overset{\text{set}}{=} 0$$

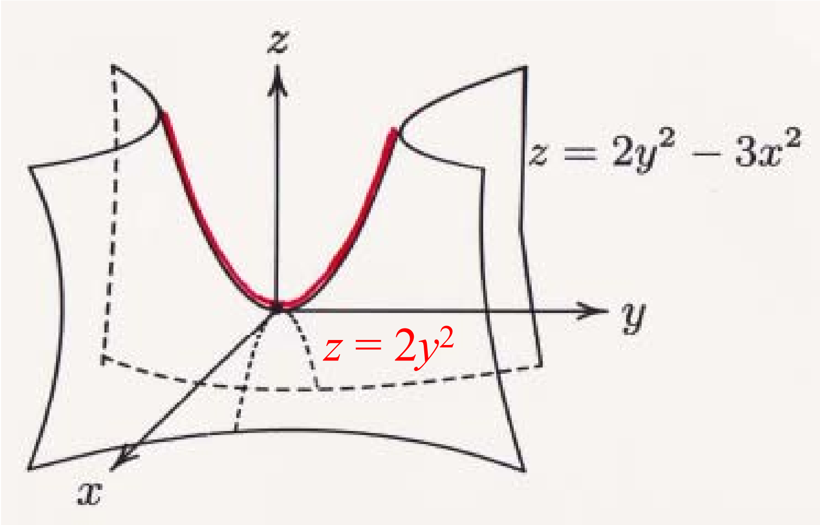
Only one critical point:
$(x, y) = (0, 0)$

It turns out that the point (0, 0) is not a local maximum or a local minimum.

Look at two curves on the surface $z = 2y^2 - 3x^2$.

$$z = f(x,y) = \boxed{2y^2 - 3x^2}$$

(0, 0, 0) is a saddle point.



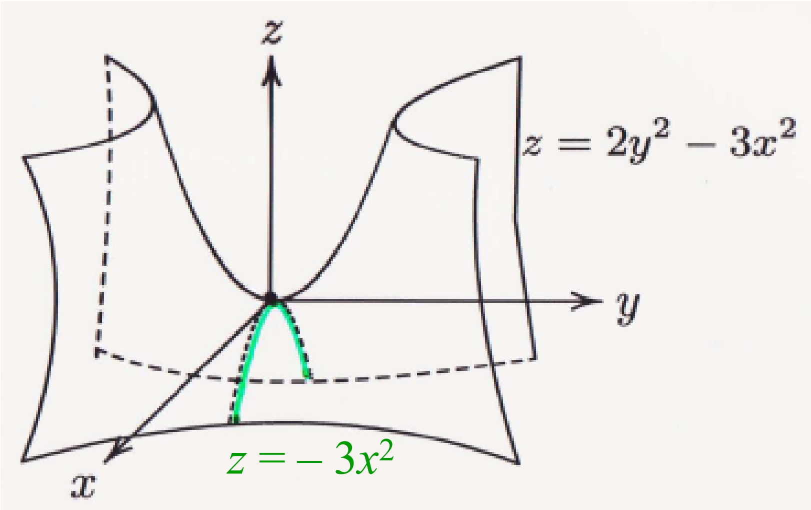
Set $x = 0$:

$$z = f(0,y) = 2y^2$$

Cup-shaped parabola

in the plane $x = 0$.

(0, 0, 0) is a local minimum on this parabola.



Set $y = 0$:

$$z = f(x,0) = -3x^2$$

Cap-shaped parabola

in the plane $y = 0$.

(0, 0, 0) is a local maximum on this parabola.

Second Derivative Test

For a function $f(x, y)$, suppose that

$$f_x(a,b) = 0 \quad \text{and} \quad f_y(a,b) = 0.$$ (critical point)

Define

$$\boxed{D \;=\; f_{xx}(a,b) f_{yy}(a,b) - f_{xy}(a,b)^2}$$

(1) If $D > 0$ and $f_{xx}(a,b) < 0$,
then (a, b) is a local maximum of f.

(2) If $D > 0$ and $f_{xx}(a,b) > 0$,
then (a, b) is a local minimum of f.

(3) If $D < 0$, then (a, b) is a saddle point of f.

(4) If $D = 0$, then no conclusion can be drawn.

Example

$$f(x,y) = \boxed{x^3 + y^3 + 3x^2 - 3y^2 - 8}$$

Obtain critical points:

$$f_x(x,y) = 3x^2 + 6x = 3x(x+2) \overset{\text{set}}{=} 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = -2$$

$$f_y(x,y) = 3y^2 - 6y = 3y(y-2) \overset{\text{set}}{=} 0$$

$$\Rightarrow y = 0 \quad \text{or} \quad y = 2$$

$$(\boxed{x = 0} \text{ or } \boxed{x = -2}) \text{ and } (\boxed{y = 0} \text{ or } \boxed{y = 2})$$

Obtain 4 critical points: (0, 0), (0, 2), (– 2, 0), (– 2, 2).

$$f_x(x,y) = 3x^2 + 6x$$

$$\Rightarrow \quad f_{xx}(x,y) = \boxed{6x+6} \quad \text{and} \quad f_{xy}(x,y) = \boxed{0}$$

$$f_y(x,y) = 3y^2 - 6y$$

$$\Rightarrow \quad f_{yy}(x,y) = \boxed{6y-6}$$

	f_{xx}	f_{yy}	f_{xy}	$\boxed{D = f_{xx}f_{yy} - (f_{xy})^2}$
(x, y)	$6x + 6$	$6y - 6$	0	
$(0, 0)$	$\boxed{6}$	$\boxed{-6}$	$\boxed{0}$	$\boxed{-36 < 0}$ saddle
$(0, 2)$	$\boxed{6}$	6	0	$36 > 0$ $f_{xx} > 0$ local min.
$(-2, 0)$	$\boxed{-6}$	-6	0	$36 > 0$ $f_{xx} < 0$ local max.
$(-2, 2)$	-6	6	0	$-36 < 0$ saddle

Lagrange Multipliers

Many optimization models are subject to certain constraints.

For example, production levels depend on labour input and capital expenditure. With a given budget (constraint), how to maximize production?

The method of Lagrange multipliers is used.

Example

Find the absolute extrema of

$$z = f(x, y) = 12x - 16y + 50$$

subject to the constraint $\boxed{x^2 + y^2 = 25.}$

Among all the points on the plane $12x - 16y - z = -50$ *that lie over the circle* $x^2 + y^2 = 25$*, find the highest point and the lowest point.*

Method: write constraint as

$$g(x, y) = x^2 + y^2 - 25.$$

Construct the function

$$\boxed{F(x, y, \lambda) = f(x, y) - \lambda g(x, y)}$$

Lagrange multiplier λ

$$= 12x - 16y + 50 - \lambda(x^2 + y^2 - 25).$$

$$F(x, y, \lambda) = \boxed{12x - 16y + 50 - \lambda(x^2 + y^2 - 25).}$$

Calculate:

set

$$\left.\begin{array}{rl|l} F_x = & 12 - 2\lambda x & = 0 \\ F_y = & -16 - 2\lambda y & = 0 \\ F_\lambda = & -x^2 - y^2 + 25 & = 0 \end{array}\right\}$$ Solve for x and y.

First two equations give: $x = \dfrac{6}{\lambda}, \quad y = \dfrac{-8}{\lambda}$

Substitute these into the third equation:

$$-\frac{36}{\lambda^2} - \frac{64}{\lambda^2} + 25 = 0$$

$$\lambda^2 = \frac{100}{25} = 4$$

$$\boxed{\lambda = \pm 2}$$

$\boxed{x = \frac{6}{\lambda},}$ $\boxed{y = \frac{-8}{\lambda}}$ $z = f(x, y) = \boxed{12x - 16y + 50}$

$$\text{If } \lambda = 2, \text{ then } x = \frac{6}{2} = 3, \quad y = \frac{-8}{2} = -4$$

$$z = f(3, -4) = 12(3) - 16(-4) + 50$$

$$= \boxed{150.}$$

(local max/min value)

max at (3, −4)

$$\text{If } \lambda = -2, \text{ then } x = \frac{6}{-2} = -3, \quad y = \frac{-8}{-2} = 4$$

$$z = f(-3, 4) = 12(-3) - 16(4) + 50$$

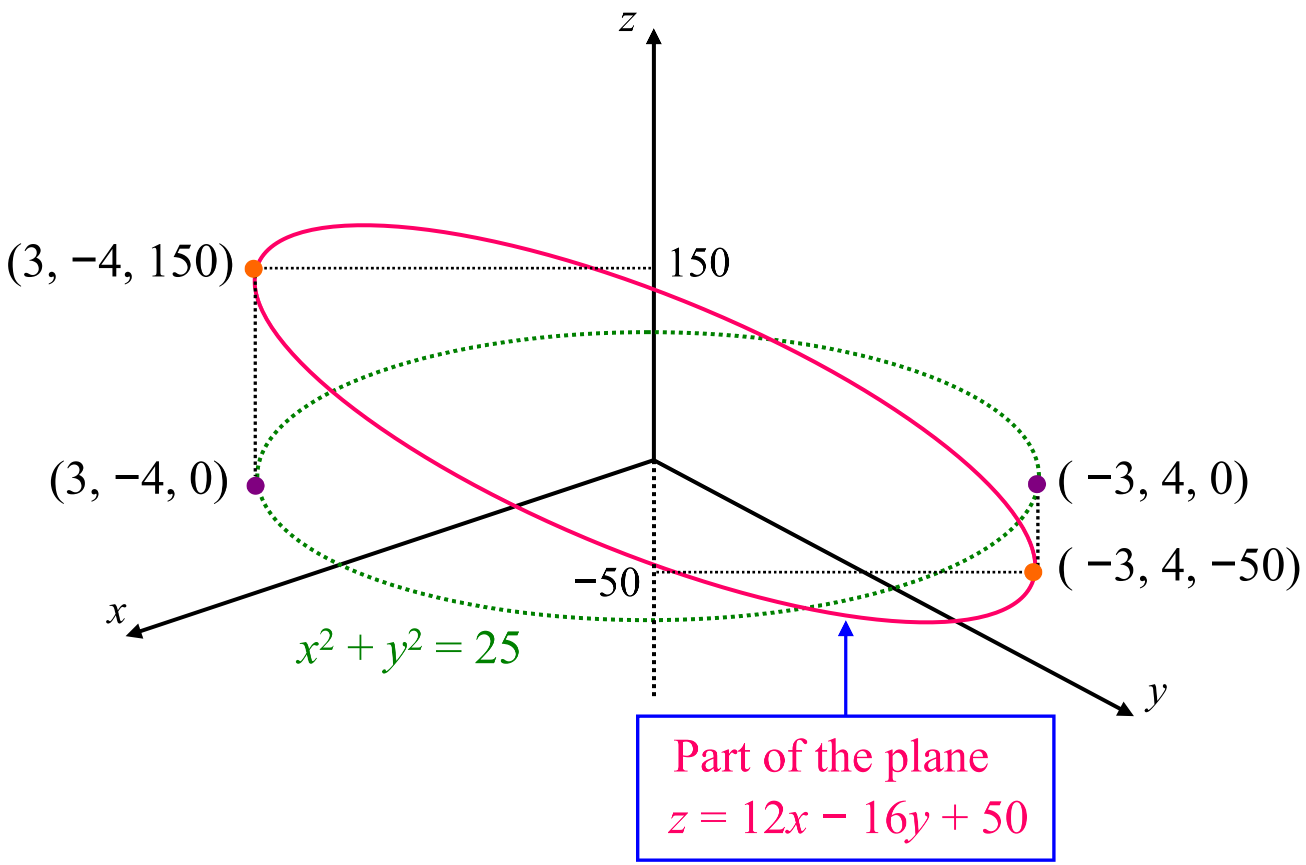
$$= \boxed{-50.}$$

(local max/min value)

min at (−3, 4)

Max of 150 at $(3, -4)$

Min of -50 at $(-3, 4)$



Among all the points on the plane $12x - 16y - z = -50$ *that lie over the circle* $x^2 + y^2 = 25$*, find the highest point and the lowest point.*

End of Chapter 7