Chapter 1. Some Basics

1.1 Functions

It is common that the values of one variable depend on the values of another. E.g. the area A of a region on the plane enclosed by a circle depends on the radius r of the circle ($A = \pi r^2, \;\; r > 0\,.$) Many years ago, the Swiss mathematician Euler invented the symbol $y = f(x)$ to denote the statement that "y is a function of x".

A function represents a rule that assigns a *unique* value y to each value x .

We refer to x as the *independent variable* and y the *dependent* variable.

One can also think of a function as an input-output system/process: input the value x and output the value $y = f(x)$. (This becomes particularly useful when we combine or composite functions together.)

1.2 Operations on Functions

1.2.1 Arithmetical operations

Let f and g be two functions.

(i) The functions $(f \pm g)(x) = f(x) \pm g(x)$, called the sum or difference of f and g.

(ii) The function $(fg)(x) = f(x)g(x)$, called the product of f and g.

(iii) The function $(f/g)(x) = f(x)/g(x)$, called the quotient of f by g, is defined where $g(x) \neq 0$;

1.2.2 Composition

Let $f : D \to \mathbb{R}$ and $g : D' \to \mathbb{R}$ be two (real) functions with domains D and D' respectively.

The function

$$(f \circ g)(x) = f(g(x)),$$

called f *composed with* g or f *circle* g, is defined on the subset of D' for which the values $g(x)$ (i.e. the range of g) are in D.

1.2.3 Example

Let $f(x) = x - 7$ and $g(x) = x^2$ (defined on all of $\mathbb{R}$). Then

$$(f \circ g)(2) \;=\; f(g(2)) = f(4) = -3, \quad \text{and}$$

$$(g \circ f)(2) \;=\; g(f(2)) = g(-5) = 25.$$

Note that in general $f \circ g \neq g \circ f$.

1.3 Limits

In this section we are interested in the behaviour of f as x gets closer and closer to a.

1.3.1 Example

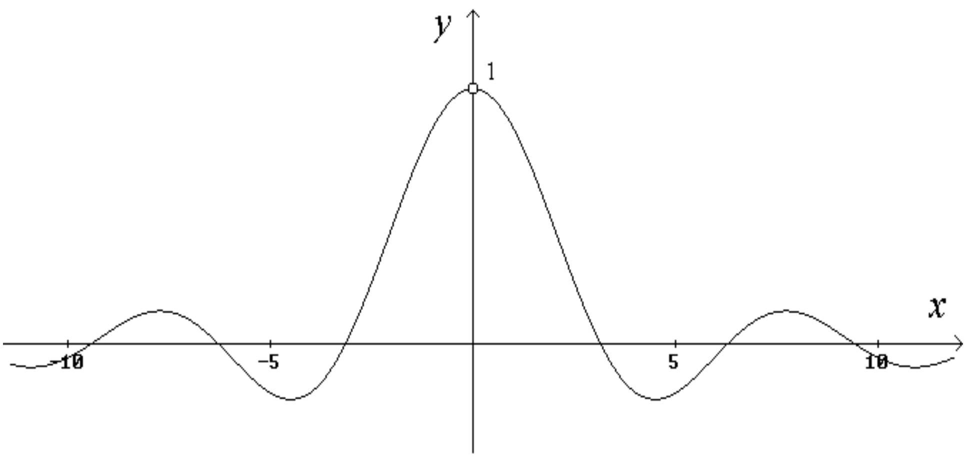
Let $D = \{x \in \mathbb{R} : x \neq 0\}$ and we consider the function $f : D \to \mathbb{R}$ given by $f(x) = \dfrac{\sin(x)}{x}$. ($x$ is in radian.) Describe its behaviour as x tends to 0.

Clearly when $x = 0$, $\dfrac{\sin(0)}{0} = \dfrac{0}{0}$ does not make sense. It is defined everywhere except at 0 and thus it makes sense to ask how it behaves as it is evaluated at arguments which are closer and closer to 0.

If we plot the graph of $\frac{\sin(x)}{x}$, we see that as x gets closer and closer to 0 from either sides (and not reaching 0 itself), $\frac{\sin(x)}{x}$ approaches 1. In this case, we say that "the limit of $\frac{\sin(x)}{x}$ as x tends to 0 is equal to 1".

We use the following notation:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$



1.3.2 Informal Definition

Let $f(x)$ be defined on an open interval I containing x_0, except possibly at x_0 itself. If $f(x)$ gets arbitrary close to L when x is sufficiently close to x_0, then we say that the limit of $f(x)$ as x tends to x_0 is the number L and we write

$$\lim_{x \to x_0} f(x) = L.$$

1.3.3 Rules of Limits

Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = L'$, then the following statements are easy to verify:

(i) $\lim_{x \to a} (f \pm g)(x) = L \pm L'$;

(ii) $\lim_{x \to a} (fg)(x) = LL'$;

(iii) $\lim_{x \to a} \dfrac{f}{g}(x) = \dfrac{L}{L'}$ provided $L' \neq 0$;

(iv) $\lim_{x \to a} kf(x) = kL$ for any real number k.