

Chapter 6. Fourier Series

6.1 Periodic functions

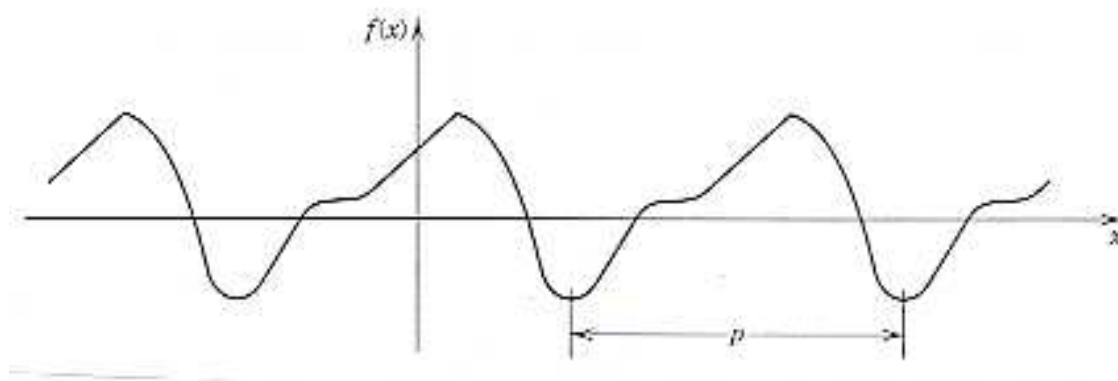
A function $f(x)$ is called *periodic* if it is defined for all real x and if there is some positive number p such that

$$f(x + p) = f(x) \text{ for all } x. \quad (1)$$

The number p is called the *period* of $f(x)$.

6.1.1 Graphs of periodic functions

The graph of such a function can be obtained by periodic repetition of its graph in any interval of length p .



For example, sine and cosine functions are periodic 2π .

$f(x) = c$, c constant, is a periodic function of period p for every positive number p .

x , x^2 , x^3 , \dots , e^x , $\ln x$ are not periodic.

6.1.2 Some algebraic properties of periodic functions

From (1),

$$f(x + 2p) = f((x + p) + p) = f(x + p) = f(x).$$

Thus (by induction) for any positive integer n ,

$$f(x + np) = f(x), \text{ for all } x.$$

Hence $2p, 3p, \dots$ are also periods of f .

Further, if f and g have period p , then the function

$h(x) = a f(x) + b g(x)$ with a, b constants also has period p .

6.1.3 Trigonometric series

Our aim is to represent various periodic functions of period 2π in terms of simple functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx, \dots \quad (2)$$

which have period 2π .

The series that arises in this connection will be of the

form

$$\begin{aligned} a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots \\ = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \end{aligned} \quad (3)$$

where $a_0, a_1, a_2, \cdots, b_1, b_2, \cdots$ are real constants.

Series (3) is called a *trigonometric series*, and a_n and b_n are called *coefficients* of the series.

The set of functions (2) is often called a *trigonometric system*.

We note that each term of the series (3) has period 2π . Hence if the series converges, its sum will be a periodic function of period 2π .

6.2 Fourier Series

Assume that $f(x)$ is a periodic function of period 2π and that it can be represented by a trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \quad (4)$$

That is, we assume that the series on the right converges and has $f(x)$ as its sum.

We say the right hand side of (4) is the Fourier series of $f(x)$.

Given $f(x)$, our task now is to determine the coefficients a_n and b_n .

6.2.1 Determine a_0

We integrate both sides of (4) from $-\pi$ to π :

$$\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} \left(a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right) dx.$$

Assuming that term by term integration is allowed,
we obtain

$$\begin{aligned} & \int_{-\pi}^{\pi} f(x) dx \\ &= a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \, dx \right) \\ &= 2\pi a_0 + \sum_{n=1}^{\infty} \left(\left[a_n \frac{\sin nx}{n} \right]_{-\pi}^{\pi} + \left[b_n \frac{\cos nx}{-n} \right]_{-\pi}^{\pi} \right) \\ &= 2\pi a_0 \end{aligned}$$

$$\text{So } a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx.$$

6.2.2 Determine a_m , $m > 0$

We multiply both sides of (4) by $\cos mx$ and integrate term by term from $-\pi$ to π :

$$\begin{aligned} & \int_{-\pi}^{\pi} f(x) \cos mx \, dx \\ = & a_0 \int_{-\pi}^{\pi} \cos mx \, dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \cos mx \, dx \right. \\ & \left. + b_n \int_{-\pi}^{\pi} \sin nx \cos mx \, dx \right) \quad (5) \end{aligned}$$

Computing the three integrations on the right hand sides of (5):

$$\begin{aligned} \text{(i)} \quad & \int_{-\pi}^{\pi} \cos mx \, dx = \left[\frac{\sin mx}{m} \right]_{-\pi}^{\pi} = 0. \\ \text{(ii)} \quad & \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0, \end{aligned}$$

since $\sin nx$ is odd and $\cos mx$ is even

and

$$\begin{aligned}
\text{(iii)} \quad & \int_{-\pi}^{\pi} \cos nx \cos mx \, dx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} (\cos(m+n)x + \cos(m-n)x) \, dx \\
&= \begin{cases} \frac{1}{2} \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right]_{-\pi}^{\pi} & m \neq n \\ \frac{1}{2m} [mx + \sin mx \cos mx]_{-\pi}^{\pi} & m = n \end{cases} \\
&= \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}
\end{aligned}$$

Substituting the above results back in (5), we get

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx, \quad m = 1, 2, \dots$$

6.2.3 Determine b_m , $m > 0$

We multiply (4) by $\sin mx$ and integrate from $-\pi$

to π :

$$\begin{aligned}
& \int_{-\pi}^{\pi} f(x) \sin mx \, dx \\
&= a_0 \int_{-\pi}^{\pi} \sin mx \, dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \sin mx \, dx \right. \\
&\quad \left. + b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \right) \quad (6) \\
&= \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx
\end{aligned}$$

as the first two integrands on the right hand side of (6) are odd functions.

$$\begin{aligned}
\text{Now } & \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} (\cos(n-m)x - \cos(n+m)x) \, dx \\
&= \begin{cases} \frac{1}{2} \left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m} \right]_{-\pi}^{\pi} & m \neq n \\ \frac{1}{2m} [mx - \sin mx \cos mx]_{-\pi}^{\pi} & m = n \end{cases} \\
&= \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}
\end{aligned}$$

$$\text{Thus } b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx, \quad m = 1, 2, \dots$$

6.2.4 Euler formulas

Given a periodic function $f(x)$ of period 2π with Fourier series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Its coefficients are known as *Fourier coefficients* and are given by

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad n = 1, 2, \dots \quad (7) \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \dots \end{aligned}$$

(7) are known as Euler formulas.

6.2.5 Representation by a Fourier series

If a periodic function $f(x)$ with period 2π is piecewise continuous in the interval $-\pi \leq x \leq \pi$ and has a left hand derivative and right hand derivative at each point of the interval, then the Fourier series with coefficients (7) is convergent. Its sum is $f(x)$ except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left hand and right hand limits of f at x_0 .

6.2.6 Example

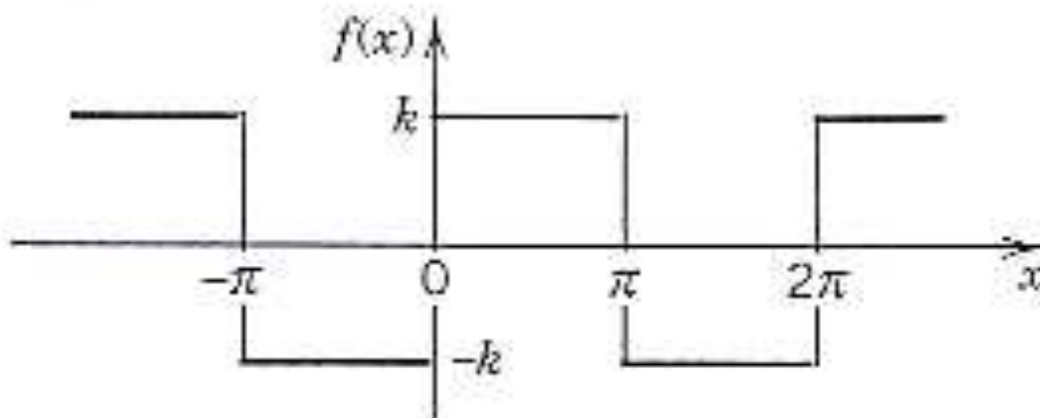
Find the Fourier series of $f(x)$ given by

$$f(x) = \begin{cases} -k, & \text{if } -\pi < x < 0 \\ k, & \text{if } 0 < x < \pi \end{cases}$$

and $f(x) = f(x + 2\pi)$.

Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc.

$f(x)$ is piecewise continuous and the value of f at a single point does not affect the integral. We can therefore leave f undefined at $x = 0, x = \pm\pi$.



Solution. We observe that over the interval $(-\pi, \pi)$, f is an odd function. Thus $f(x) \cos nx$ is also an odd

function. Thus by (7), $a_n = 0$ for $n = 0, 1, 2, \dots$,
and

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} k \sin nx \, dx = \frac{2k}{n\pi} (1 - \cos n\pi) \\ &= \frac{2k}{n\pi} (1 - (-1)^n). \\ b_1 &= \frac{4k}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{4k}{3\pi}, \\ b_4 &= 0, \quad b_5 = \frac{4k}{5\pi}, \dots \end{aligned}$$

The Fourier series for the square wave is, therefore,

$$\frac{4k}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right).$$

6.2.7 An approximation for π

From the previous section, the series converges to
 $f(x)$ in $(0, \pi)$.

Setting $x = \frac{\pi}{2}$, we get

$$k = \frac{4k}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots \right)$$

i.e. $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$ (Leibniz).

Note that at all the points of discontinuity ($0, \pi, etc$) of f , the sum of the series is equal to 0, which is the average of the left hand and right hand limits of f (e.g. they are $-k$ and k respectively at $x = 0$).

6.2.8 Periodic functions of period $p = 2L$

Let $f(x)$ be a periodic function of period $p = 2L$.

We set $v = \frac{\pi x}{L}$. Then $x = \frac{vL}{\pi}$ and at $x = \pm L$, $v = \pm\pi$.

We now view f as a function of v and put $f(x) =$

$g(v)$. Then g becomes a periodic function of period 2π . If $f(x)$ has a fourier series, then so has $g(v)$. We have

$$g(v) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv)$$

with

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(v) dv = \frac{1}{2\pi} \int_{-L}^L g(v) \frac{\pi}{L} dx \\ &= \frac{1}{2L} \int_{-L}^L f(x) dx \end{aligned}$$

and for $n = 1, 2, 3, \dots$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \cos nv dv \\ &= \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \\ b_n &= \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx. \end{aligned}$$

Since $g(v) = f(x)$, we get

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x \right)$$

with a_0, a_n and b_n as given above.

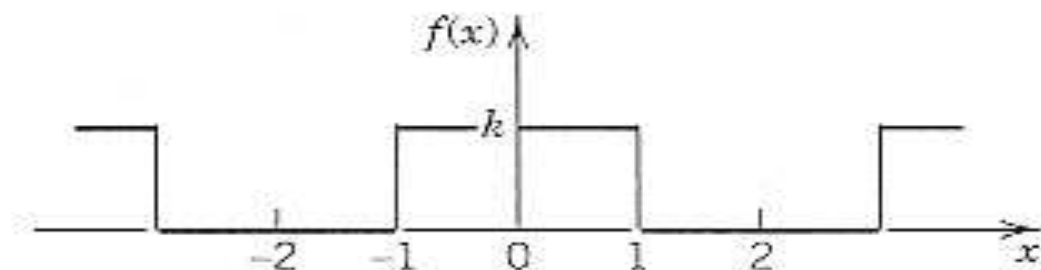
The interval of integration in the above formula can be replaced by any interval of length $p = 2L$, for example, by $0 \leq x \leq 2L$ or $L \leq x \leq 3L$.

6.2.9 Example

Let f be a periodic square wave of period $p = 2L = 4$

defined as follows :

$$f(x) = \begin{cases} 0, & \text{if } -2 < x < -1 \\ k, & \text{if } -1 < x < 1 \\ 0, & \text{if } 1 < x < 2 \end{cases}$$



To find the Fourier series of f , we compute

$$a_0 = \frac{1}{4} \int_{-2}^2 f(x) dx$$

$$a_n = \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi x}{2} dx$$

and since f is even,

$$b_n = 0 \quad \text{for } n = 1, 2, \dots.$$

$$a_0 = \frac{1}{4} \int_{-2}^2 f(x) dx = \frac{1}{4} \int_{-1}^1 k dx = \frac{k}{2}$$

$$a_n = \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi x}{2} dx = \frac{1}{2} \int_{-1}^1 k \cos \frac{n\pi x}{2} dx$$

$$= \frac{2k}{n\pi} \sin \frac{n\pi}{2}$$

Hence $a_n = 0$ if n is even and

$$a_n = \begin{cases} \frac{2k}{n\pi} & \text{if } n = 1, 5, 9, \dots \\ -\frac{2k}{n\pi} & \text{if } n = 3, 7, 11, \dots \end{cases}$$

Hence

$$f(x) = \frac{k}{2} + \frac{2k}{\pi} \left(\cos \frac{\pi}{2}x - \frac{1}{3} \cos \frac{3\pi}{2}x + \frac{1}{5} \cos \frac{5\pi}{2}x - \dots \right).$$

6.2.10 Fourier cosine and sine series

Using

$$\int_{-L}^L f(x)dx = \begin{cases} 0 & \text{if } f \text{ is odd} \\ 2 \int_0^L f(x)dx & \text{if } f \text{ is even.} \end{cases}$$

we obtain the following two series.

The Fourier series of an even function $f(x)$ of period

$2L$ is the *Fourier cosine series*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

with

$$\begin{aligned} a_0 &= \frac{1}{L} \int_0^L f(x) dx, \\ a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 1, 2, \dots \end{aligned}$$

The Fourier series of an odd function $f(x)$ of period $2L$ is a *Fourier sine series*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

with $b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$

6.2.11 Sum and Scalar multiplication

The Fourier coefficients of $f_1 + f_2$ are the sums of corresponding Fourier coefficients of f_1 and f_2 .

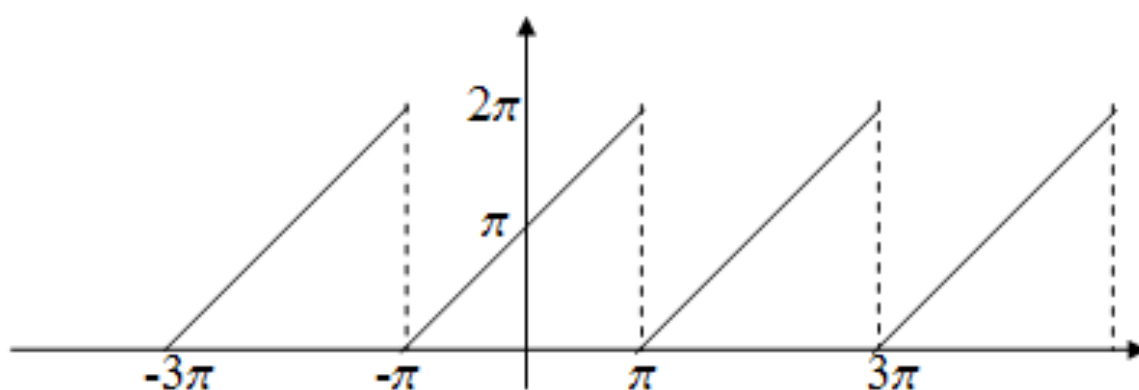
The Fourier coefficients of cf (c a constant) are c times the corresponding Fourier coefficients of f .

6.2.12 Example

Saw tooth function

$$f(x) = x + \pi, \quad -\pi < x < \pi,$$

$$f(x) = f(x + 2\pi).$$



We note that $f = f_1 + f_2$, where $f_1 = x$, $f_2 = \pi$.

The Fourier coefficients for f_2 are $a_0 = \pi$ and

$$a_n = 0 = b_n, \quad n \geq 1.$$

The function $f_1 = x$ is odd.

Thus $a_n = 0$ for all n , and

$$\begin{aligned}
 b_n &= \frac{2}{\pi} \int_0^\pi x \sin nx \, dx \\
 &= \frac{2}{\pi} \left\{ \left[x \left(\frac{-\cos nx}{n} \right) \right]_0^\pi - \int_0^\pi \frac{-\cos nx}{n} dx \right\} \\
 &= \frac{2}{\pi} \left\{ \frac{-(-1)^n \pi}{n} - \left[\frac{-\sin nx}{n^2} \right]_0^\pi \right\} \\
 &= \frac{(-1)^{n+1} 2}{n}
 \end{aligned}$$

So

$$\begin{aligned}
 f(x) &= f_1(x) + f_2(x) \\
 &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2}{n} \sin \frac{n\pi x}{\pi} + \pi \\
 &= \pi + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx
 \end{aligned}$$

6.3 Half-range Expansions

In various applications there is a practical need to use Fourier series in connection with functions f that are given on some interval only, say, $0 \leq x \leq L$.

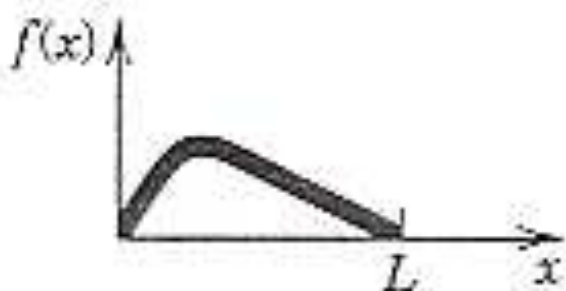


Figure (a)

6.3.1 Extension of $f(x)$

We could extend $f(x)$ as a periodic function with period L and then represent the extended function by a Fourier series, which in general would involve both sine and cosine terms. We can do better and

always get a cosine series by first extending $f(x)$ from $0 \leq x \leq L$ as an even function on the interval $-L \leq x \leq L$ as in figure (b) and then extend this new function as a periodic function of period $2L$, and since it is even, we can represent it by a Fourier cosine series.

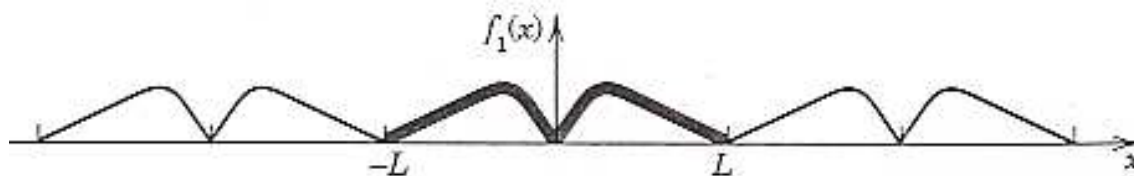


Figure (b)

Also, we can extend $f(x)$ from $0 \leq x \leq L$ as an odd function on $-L \leq x \leq L$ as in figure (c) and then extend this new function as a periodic function of period $2L$, and since it is odd, it is represented by a Fourier sine series.

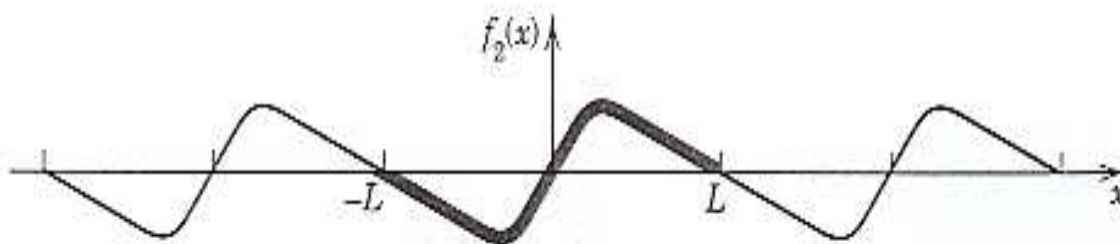


Figure (c)

6.3.2 Half range expansion

Such two series are called the two ‘half range expansions’ of the function f which is given only on ‘half the range’.

The cosine half range expansion is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

with

$$a_0 = \frac{1}{L} \int_0^L f(x) dx,$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 1, 2, \dots$$

The sine half range expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

with

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, \dots$$

6.3.3 Example

Find the two half range expansions for

$$f(x) = \begin{cases} 0, & 0 < x < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < x < \pi. \end{cases}$$

For the cosine half range expansion, we have

$$a_0 = \frac{1}{\pi} \left\{ \int_0^{\pi/2} 0 dx + \int_{\pi/2}^{\pi} 1 dx \right\} = \frac{1}{2}$$

and

$$\begin{aligned}
 & a_n \\
 &= \frac{2}{\pi} \left\{ \int_0^{\pi/2} 0 \cos nx dx + \int_{\pi/2}^{\pi} 1 \cos nx dx \right\} \\
 &= \frac{2}{\pi} \left[\frac{\sin \pi n - \sin \frac{1}{2}\pi n}{n} \right]
 \end{aligned}$$

Thus a_n simplifies to

$$a_n = \frac{2}{n\pi} \left(-\sin \frac{n\pi}{2} \right)$$

Indeed,

$$a_1 = \frac{-2}{\pi}, \quad a_3 = \frac{2}{3\pi}, \quad a_5 = \frac{-2}{5\pi}, \quad , \dots$$

and $a_n = 0$ if $n \geq 1$ and n is even.

The cosine half range expansion is

$$f(x) = \frac{1}{2} + 2 \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m-1)\pi} \cos(2m-1)x$$

For the sine half range expansion, we have

$$\begin{aligned}
 b_n &= \frac{2}{\pi} \left\{ \int_0^{\pi/2} 0 \sin nx dx + \int_{\pi/2}^{\pi} 1 \sin nx dx \right\} \\
 &= \frac{2}{\pi} \left[\frac{-\cos \pi n + \cos \frac{1}{2}\pi n}{n} \right]
 \end{aligned}$$

Thus b_n simplifies to

$$b_n = \frac{2}{n\pi} \left[(-1)^{n+1} + \cos \frac{n}{2}\pi \right]$$

Indeed,

$$b_1 = \frac{2}{\pi}, \quad b_2 = \frac{-4}{2\pi}, \quad b_3 = \frac{2}{3\pi}, \quad b_4 = 0,$$

$$b_5 = \frac{2}{5\pi}, \quad b_6 = \frac{-4}{6\pi}, \quad b_7 = \frac{2}{7\pi}, \quad b_8 = 0, \dots$$

The sine half range expansion is

$$f(x) =$$

$$\frac{2}{\pi} \sum_{m=1}^{\infty} \left\{ \frac{\sin(4m-3)x}{4m-3} - \frac{2 \sin(4m-2)x}{4m-2} + \frac{\sin(4m-1)x}{4m-1} \right\}$$