Chapter 10

Surface Integrals

Key Results

- Parametric representations of various common surfaces.
- Surface integrals of scalar functions.
- Volume flow rate of a fluid calculated as a surface integral of a vector field.
- Surface integrals of general vector fields.
- Stokes' Theorem.
- Divergence Theorem of Gauss.

Parametric Surfaces

Just as in line integrals $\int_C \mathbf{F} \cdot d\mathbf{r}$ where the curve C

is described parametrically, in surface integrals the surface S is also described parametrically.

Describing surfaces parametrically also facilitates formulation of theorems:

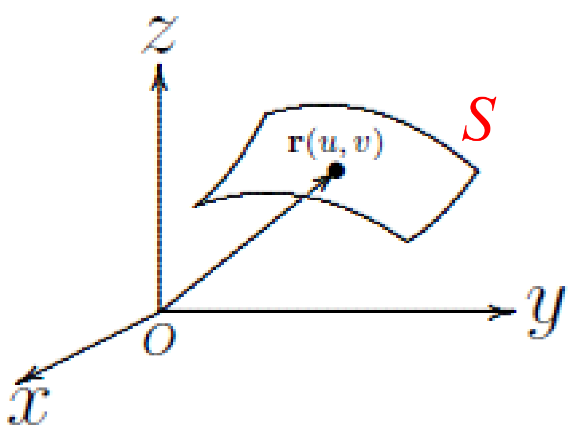
> different coordinate systems - e.g. Cartesian, cylindrical, spherical systems - can be 'standardized'.

A parametric representation of a surface S is given by a two-variable vector function

$$\boxed{\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}}$$

where u and v are two independent parameters.

The equations $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ are the parametric equations of S.



Example (Planes)

$$\Pi : ax + by + cz = d.$$

Let two of the variables be replaced by u and v.

The third variable is then determined in terms of u and v.

Example 1 $\quad \Pi_1 : \boxed{3x + 2y - 4z = 6.}$

Let $x(u,v) = u,\ y(u,v) = v.$

Then

$$z(u,v) = \frac{1}{4}(3u + 2v - 6)$$

Parametric representation of Π_1 is:

$$\mathbf{r}(u,v) = \underset{x}{u}\mathbf{i} + \underset{y}{v}\mathbf{j} + \underset{z}{\left(\frac{1}{4}(3u+2v-6)\right)}\mathbf{k}.$$

Example 2 $\quad \Pi_2 : \boxed{x + 2y = 7.}$

Variable z is absent. Let $z(u, v) = u$ and $\boxed{y(u, v) = v.}$

Then $x = 7 - 2v$.

Parametric representation of Π_2 is:

$$\boxed{\mathbf{r}(u, v) = \underset{x}{(7 - 2v)}\mathbf{i} + \underset{y}{v}\mathbf{j} + \underset{z}{u}\mathbf{k}.}$$

Example 3 $\quad \Pi_3$: xy-plane. $\quad$ That is, $\Pi_3 : z = 0$.

The variables x and y are independent.

Let $x(u, v) = u$ and $y(u, v) = v$.

Parametric representation of Π_3 is:

$$\boxed{\mathbf{r}(u, v) = \underset{x}{u}\mathbf{i} + \underset{y}{v}\mathbf{j} + \underset{z}{0}\mathbf{k}.}$$

Natural Parametric Representation

A surface of the form $z = f(x, y)$ has natural parametric representation

$$\boxed{\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}}$$

That is, $x(u, v) = u$ and $y(u, v) = v$.

Example 1 The paraboloid $z = x^2 + y^2$

has natural parametric representation

$$\boxed{\mathbf{r}(u, v) = \underset{x}{u}\mathbf{i} + \underset{y}{v}\mathbf{j} + (\underset{z}{u^2 + v^2})\mathbf{k}.}$$

Example 2 The upper cone $\boxed{z = \sqrt{x^2 + y^2}.}$

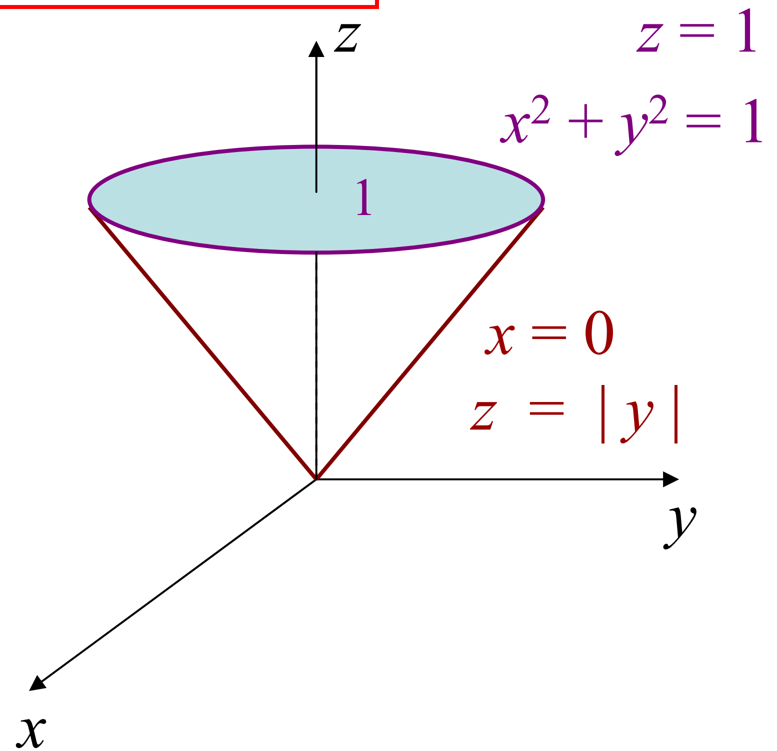
Natural parametric representation:

$$\boxed{\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{u^2 + v^2}\mathbf{k}.}$$

$x = 0$ $\quad z = \sqrt{y^2}$

$z = |y|$

$z = 1$ $\quad x^2 + y^2 = 1$



Spheres

The sphere of radius *a* and centred at the origin has Cartesian equation

$$x^2 + y^2 + z^2 = a^2$$

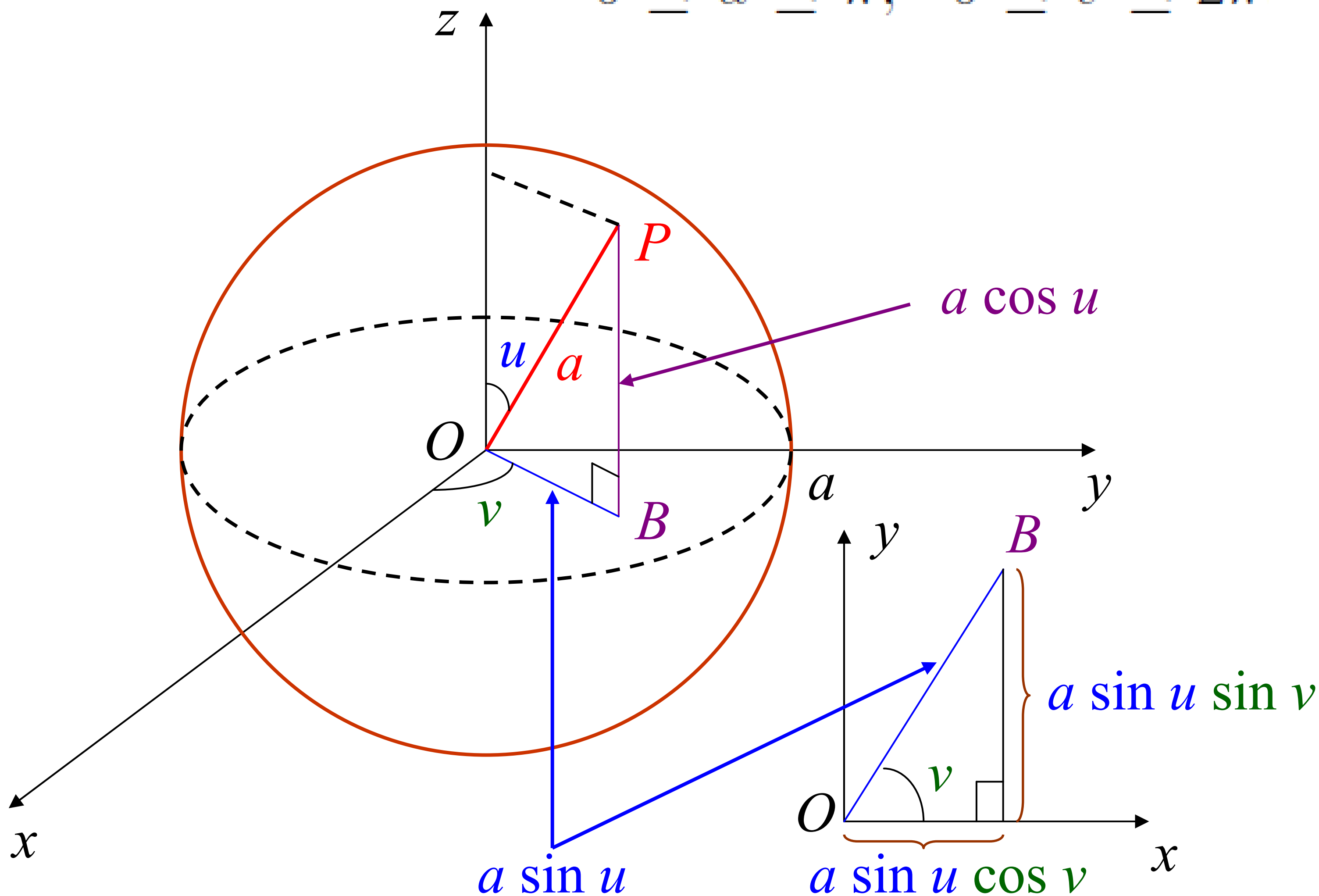
The standard parametric representation is

$$\boxed{\mathbf{r}(u,v) = (a\sin u\cos v)\mathbf{i} + (a\sin u\sin v)\mathbf{j} + (a\cos u)\mathbf{k}.}$$

where $\quad 0 \le u \le \pi, \quad 0 \le v \le 2\pi$

$$\mathbf{r}(u,v) = \underset{x}{\boxed{(a\sin u)\cos v}}\,\mathbf{i} + \underset{y}{\boxed{(a\sin u)\sin v}}\,\mathbf{j} + \underset{z}{\boxed{(a\cos u)}}\,\mathbf{k}.$$

$$0 \leq u \leq \pi, \quad 0 \leq v \leq 2\pi$$



$$\mathbf{r}(u, v) = (a \sin u \cos v)\mathbf{i} + (a \sin u \sin v)\mathbf{j} + (a \cos u)\mathbf{k}.$$

$$0 \le u \le \pi, \quad 0 \le v \le 2\pi$$

Why not 2π ?

Parametric representation of a surface S is supposed to assign unique 'coordinates' (u, v) to each point on S.

Exercise: Suppose $0 \le u \le 2\pi$ is allowed.

On the sphere, locate the point(s) P_1 and P_2 with 'coordinates' as follows:

$$P_1(u, v) = P_1\left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \qquad P_2(u, v) = P_2\left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$$

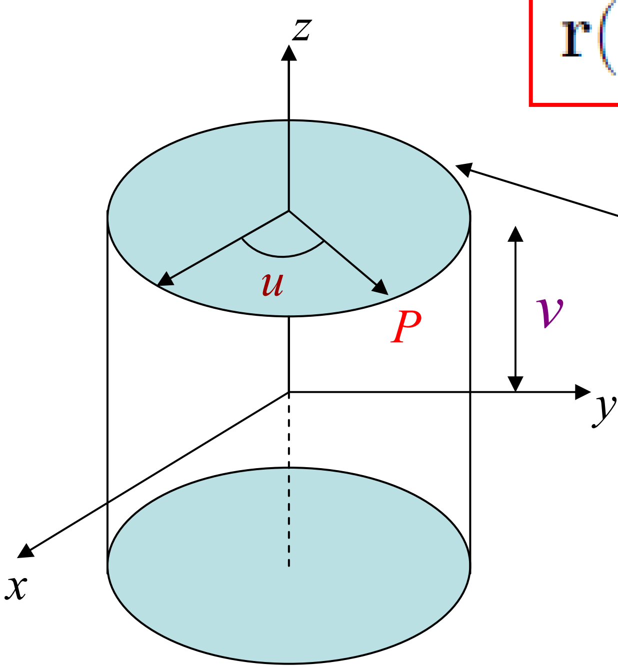
Circular Cylinder

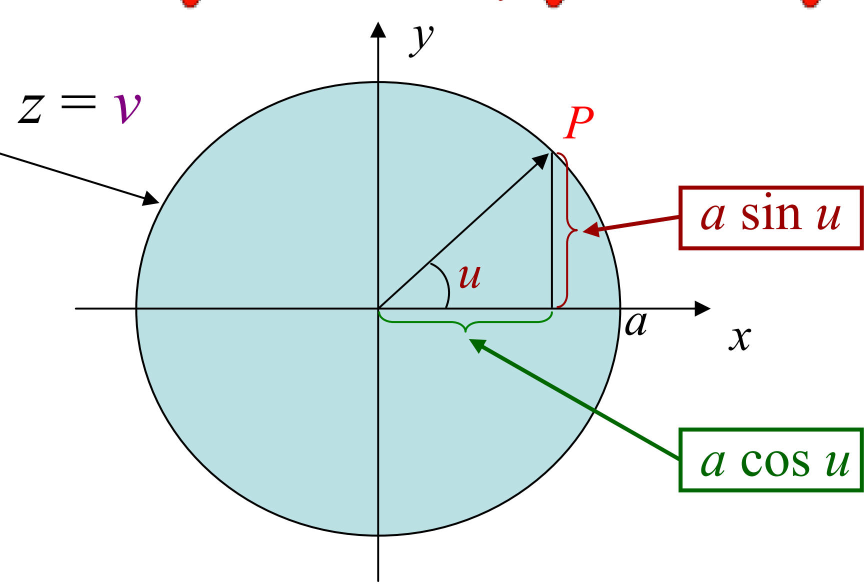
The circular cylinder of radius a about the z-axis has Cartesian equation

$$x^2 + y^2 = a^2$$

The standard parametric representation is

$$\boxed{\mathbf{r}(u,v) = (\underbrace{a\cos u}_{x\ \checkmark})\mathbf{i} + (\underbrace{a\sin u}_{y\ \checkmark})\mathbf{j} + \underbrace{v}_{z\ \checkmark}\mathbf{k}.}$$





Tangent Planes

S is a surface with parametric representation:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

A point P_0 on S has position vector $\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$.

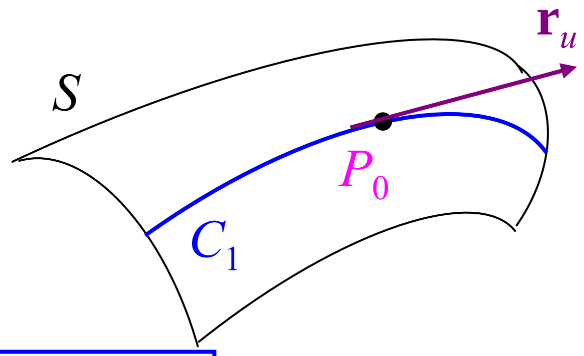
Fix $v = v_0$ in representation above to obtain:

$$\boxed{\mathbf{r}(u, v_0) = x(u, v_0)\mathbf{i} + y(u, v_0)\mathbf{j} + z(u, v_0)\mathbf{k}}$$

This gives a curve C_1 passing through P_0 on S.

The tangent vector to C_1 at P_0 is given by

$$\frac{d}{du}\mathbf{r}(u, v_0)\,\Big|_{u=u_0}$$



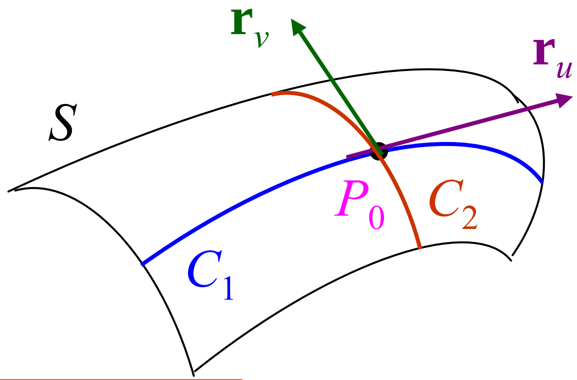
$$\boxed{\boxed{\mathbf{r}_u} \equiv \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.}$$

$S: \quad \mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$

Similarly, fix $u = u_0$ in representation above to obtain:

$$\boxed{\mathbf{r}(u_0,v) = x(u_0,v)\mathbf{i} + y(u_0,v)\mathbf{j} + z(u_0,v)\mathbf{k}.}$$

which is another curve C_2 passing through P_0 on S.



The tangent vector to C_2 at P_0 is given by

$$\boxed{\boxed{\mathbf{r}_v} \equiv \frac{\partial x}{\partial v}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0,v_0)\mathbf{k}.}$$

$S: \quad \mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$

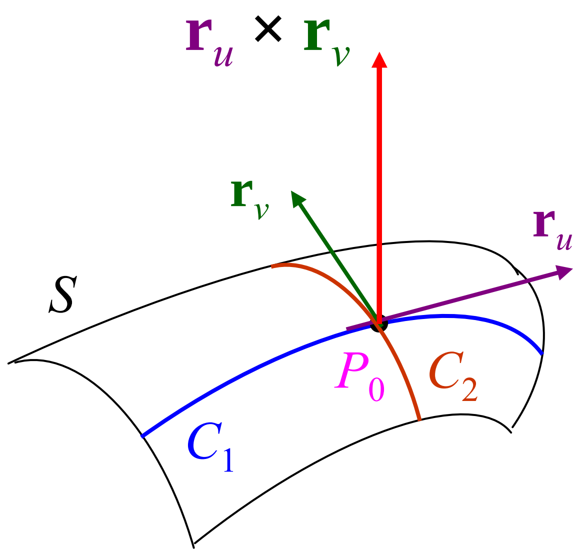
Both vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ lie on the tangent plane Π to S at P_0.

Thus, $\boxed{\mathbf{r}_u \times \mathbf{r}_v}$ (if it is nonzero) is a normal vector to Π.

Let $\mathbf{w} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ denote an arbitrary point on Π.

An equation for Π is:

$$\boxed{(\mathbf{w} - \mathbf{r}_0) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 0.}$$



Example

$S:\ \boxed{\mathbf{r}(u,v) = \underset{x}{u}\mathbf{i} + v^2\mathbf{j} + \underbrace{\boxed{(u^2 - v)}}_{z}\mathbf{k}}$

Point $\boxed{P_0(1, 4, -1)}$. Set $\mathbf{r}_0(u, v) = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$

This gives $\boxed{u = 1.}$ Then $u^2 - v = -1 \Rightarrow \boxed{v = u^2 + 1 = 2.}$

$$\mathbf{r}_u = \mathbf{i} + 0\mathbf{j} + 2u\mathbf{k}$$
$$\mathbf{r}_v = 0\mathbf{i} + 2v\mathbf{j} - \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 2v & -1 \end{vmatrix}$$

$$= \boxed{-4uv\mathbf{i} + \mathbf{j} + 2v\mathbf{k}}$$

$\boxed{\mathbf{r}_u \times \mathbf{r}_v = \boxed{-8\mathbf{i} + \mathbf{j} + 4\mathbf{k}}}$ is a normal vector to the plane Π.

Cartesian equation of Π:

$$-8x + y + 4z$$
$$= -8\cdot 1 + 1\cdot 4 + 4\cdot(-1)$$

$$\boxed{\Pi : 8x - y - 4z = 8}$$

A Formula for $\mathbf{r}_u \times \mathbf{r}_v$

A surface S with Cartesian equation $z = f(x, y)$ has natural parametric representation:

$$\boxed{\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + f(u,v)\mathbf{k}.}$$

Then

$$\boxed{\mathbf{r}_u = \mathbf{i} + 0\mathbf{j} + f_u\mathbf{k}} \quad \text{and} \quad \boxed{\mathbf{r}_v = 0\mathbf{i} + \mathbf{j} + f_v\mathbf{k}.}$$

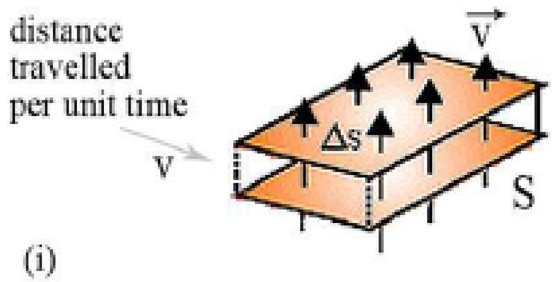
$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = \boxed{-f_u\,\mathbf{i} - f_v\,\mathbf{j} + 1\,\mathbf{k}}$$

Motivation for Surface Integrals

Fluid with velocity **v** flows through a surface *S*.

How to calculate the total volume of fluid flowing out of S per unit time?

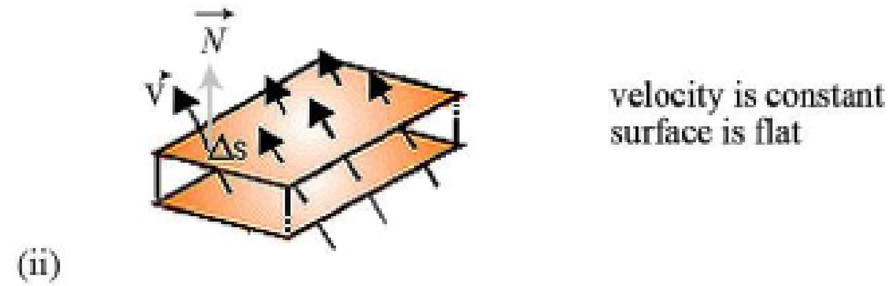
Case (i)



The fluid velocity **v** is constant over flat surface *S* and its direction is perpendicular to *S*.

Volume flow rate is $\boxed{w = \|\mathbf{v}\|\Delta s.}$

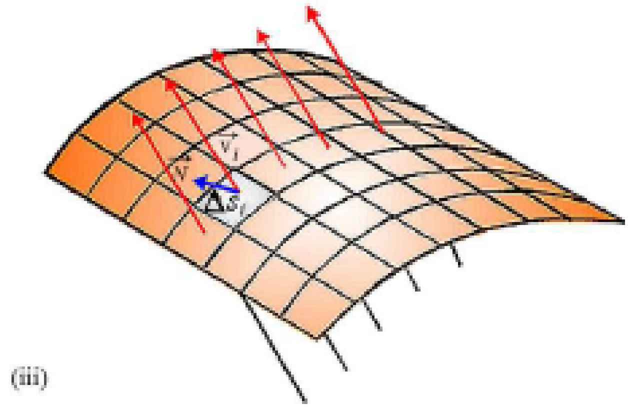
Case (ii)



The fluid velocity **v** is constant over flat surface *S* but its direction is *not* perpendicular to *S*.

Volume flow rate is $\boxed{w = \mathbf{v} \cdot \mathbf{N} \Delta s}$

Case (iii)



General case (velocity is changing on a curved surface)

The fluid velocity **v** is *changing* over *curved* surface *S*.

Divide *S* into small segments.

On a segment volume flow rate is

$$w_i \approx \mathbf{v}(x_i, y_i, z_i) \cdot \mathbf{N}_\mathrm{i} \Delta s_i.$$

Over *S* the total volume flow rate is

$$\boxed{w \approx \sum_{1}^{n} \mathbf{v}(x_i, y_i, z_i) \cdot \mathbf{N}_\mathrm{i} \Delta s_i}$$

By considering smaller but more segments on *S* and the expression

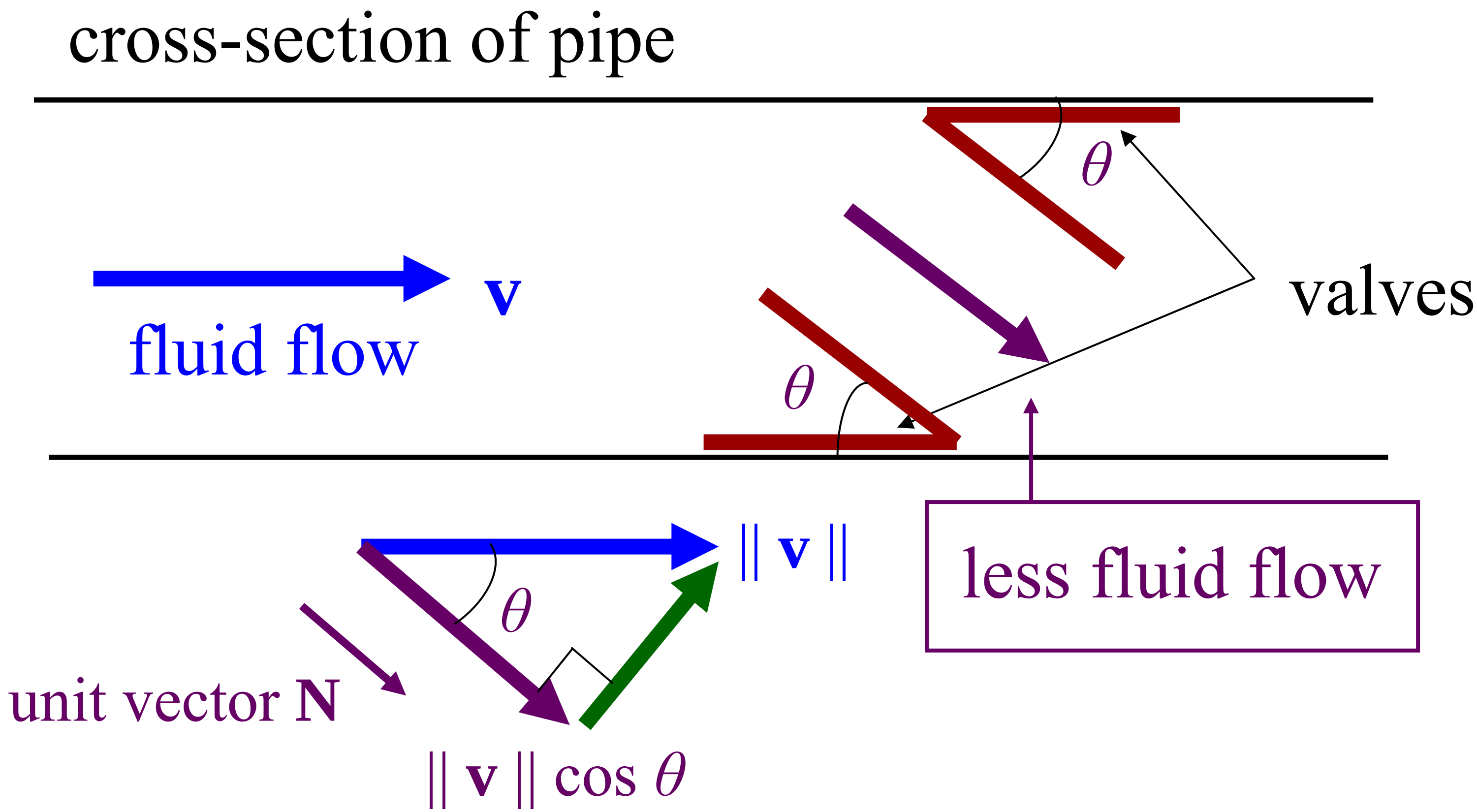
$$w \approx \sum_{1}^{n} \mathbf{v}(x_i, y_i, z_i) \cdot \mathbf{N_i} \Delta s_i$$

the actual total volume flow rate is given by:

$$\boxed{\iint_S \mathbf{v}(x, y, z) \cdot \mathbf{N} ds}$$

Why dot product?

Consider a pipe with valves that may restrict fluid flow:



volume flow rate

$$= \| \mathbf{v} \| \cos \theta \, (\Delta S) = \| \mathbf{v} \| \cos \theta \, \| \mathbf{N} \, \Delta S \| = \boxed{\mathbf{v} \cdot \mathbf{N} \, \Delta S}$$

ΔS is small surface area element

Obtained actual total volume flow rate:

$$\iint_S \mathbf{v}(x, y, z) \cdot \mathbf{N} ds$$

Thus, one concept this chapter studies involves integrals of the form

$$\iint_S \mathbf{F} \cdot \mathbf{N} dS$$ which is also written as $$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S}}$$

This integral is the surface integral of the vector field **F** over the surface *S*.

Similar to line integrals, there are also surface integrals of scalar functions (over a surface *S*):

$$\boxed{\iint_S f(x, y, z)\, dS}$$

Surface Integrals of Scalar Functions

$S: \quad \mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$

where the ordered pairs (u, v) are from some bounded domain D.

Scalar function $f(x, y, z)$ defined on S.

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA.}$$

This is the surface integral of the scalar function f over the surface S.

Example

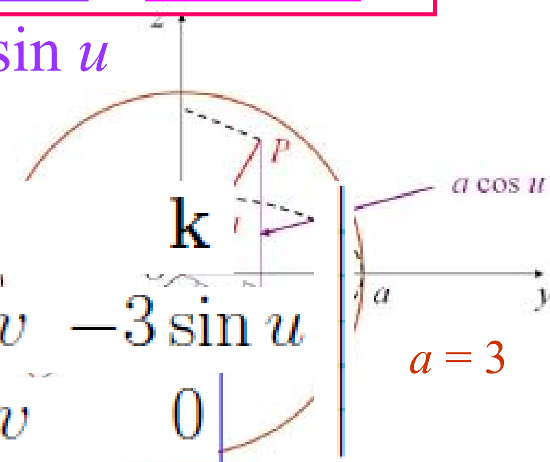
S : $\boxed{\text{First octant}}$ of sphere $x^2 + y^2 + z^2 = 9$.

Standard parametric representation (see Example 10.1.3) of sphere:

$$\boxed{\mathbf{r}(u, v) = \boxed{3\sin u\cos v\mathbf{i} + 3\sin u\sin v\mathbf{j}} + \boxed{3\cos u}\mathbf{k}.}$$

$$\boxed{D: \quad 0 \le u \le \pi/2 \text{ and } 0 \le v \le \pi/2.}$$

3sin u



$a = 3$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\cos u\cos v & 3\cos u\sin v & -3\sin u \\ -3\sin u\sin v & 3\sin u\cos v & 0 \end{vmatrix}$$

$$= \boxed{9\sin^2 u\cos v\mathbf{i} + 9\sin^2 u\sin v\mathbf{j}} + \boxed{9\sin u\cos u}\mathbf{k}.$$

$$= \boxed{3\sin u}\,\mathbf{r}(u, v)$$

$$\iint f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\|\mathbf{r}_u \times \mathbf{r}_v\|\, dA.$$

S : First octant of sphere $x^2 + y^2 + z^2 = 9$.

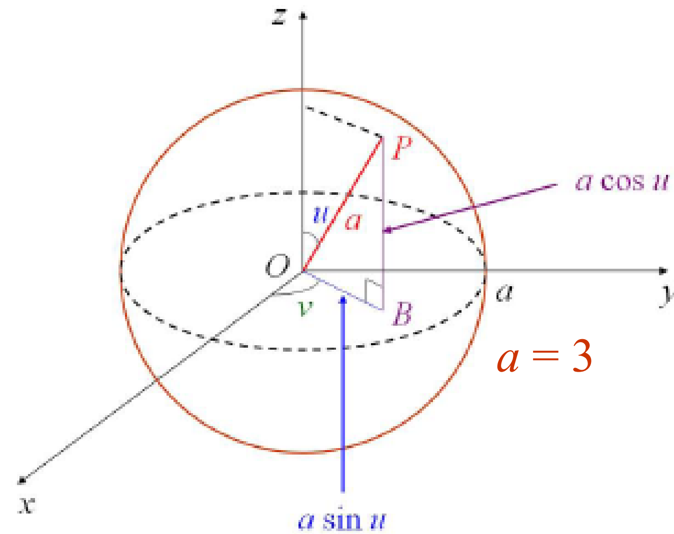
$$\mathbf{r}(u, v) = \underset{x}{3\sin u\cos v}\,\mathbf{i} + \underset{y}{3\sin u\sin v}\,\mathbf{j} + \underset{z}{3\cos u}\,\mathbf{k}.$$

D : $\boxed{0 \le u \le \pi/2}$ and $\boxed{0 \le v \le \pi/2.}$

$$\mathbf{r}_u \times \mathbf{r}_v = 3\sin u\,\boxed{\mathbf{r}(u, v)}$$

$\times\ 3$ Vector OP

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = 9\sin u$$



$$\iint_S (xz + yz)\,dS$$

$$= \iint_D (9\sin u\cos u\cos v + 9\sin u\cos u\sin v)\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} 81\sin^2 u\,\cos u\,(\cos v + \sin v)\,du\,dv$$

$\iint_S \qquad \iint_D \qquad \mathbf{r}(u, v))\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA.$

$$\iint_S (xz + yz)\, dS$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} 81 \boxed{\sin^2 u \ \cos u} \boxed{(\cos v + \sin v}) \ du \ dv$$

$$= 81 \int_0^{\pi/2} \sin^2 u \ \cos u\, du \ \int_0^{\pi/2} (\cos v + \sin v)\, dv$$

$$= 81 \cdot \left[\frac{1}{3} \sin^3 u\right]_0^{\pi/2} \cdot 2$$

$$= \boxed{54.}$$

Chap 8 page 7: $\iint_R \boxed{g(x)} \boxed{h(y)} dA = \left(\int_a^b g(x)\, dx\right) \left(\int_c^d h(y)\, dy\right)$

An interpretation of the surface integral calculations

Let $T(x, y, z) = xz + yz$ denote the temperature at the point (x, y, z) in the first octant.

Then $$\iint_S (xz + yz)\, dS = 54$$

gives the '*total temperature*' of all the points on the surface of the first octant of the sphere of radius 3.

$$\boxed{\text{Surface area} = \frac{1}{8} \cdot 4\pi \cdot 3^2 = \frac{9}{2}\pi}$$

Thus, 'average temperature' $= \dfrac{\text{total temperature}}{\boxed{\text{surface area}}} = \dfrac{54}{\frac{9}{2}\pi}$

$$= \boxed{\frac{12}{\pi}}$$

Exercise

Given the standard parametric representation of the sphere:

$$\mathbf{r}(u,v) = (a\sin u\cos v)\mathbf{i} + (a\sin u\sin v)\mathbf{j} + (a\cos u)\mathbf{k}.$$

show that

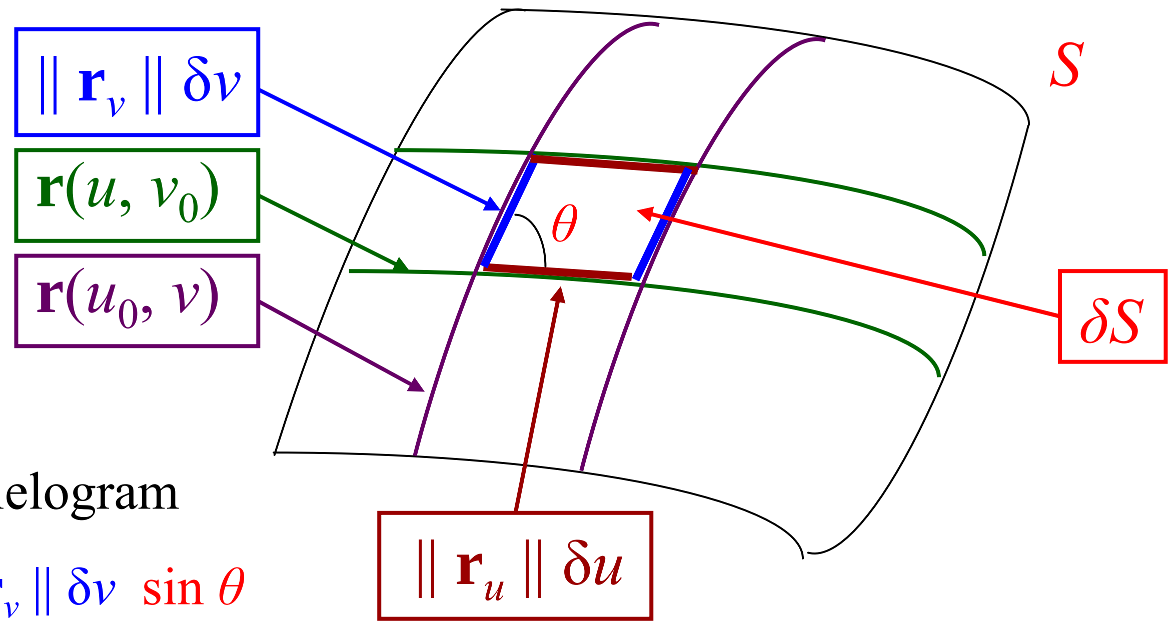
$$\boxed{\mathbf{r}_u \times \mathbf{r}_v \;=\; a\sin u\,\mathbf{r}(u,v)}$$

Interpret this result geometrically:

(1) Why is $\mathbf{r}_u \times \mathbf{r}_v$ parallel to $\mathbf{r}(u, v)$?

(2) How is $\| \mathbf{r}_v \|$ related to $a\sin u$?

$$\iint_S f(x,y,z)\,\boxed{dS} = \iint_D f(\mathbf{r}(u,v))\,\boxed{\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA.}$$ **?** ✓



δS

$\approx$ area of parallelogram

$\approx\ \|\mathbf{r}_u\|\,\delta u\ \|\mathbf{r}_v\|\,\delta v\ \sin\theta$

$=\ \boxed{\|\mathbf{r}_u\|\,\|\mathbf{r}_v\|\ \sin\theta}\ \boxed{\delta u\ \delta v}$

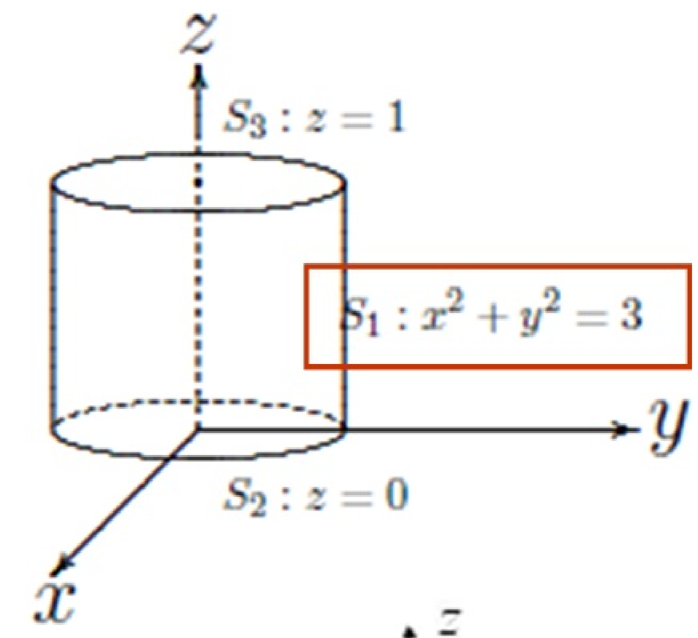
$=\ \|\mathbf{r}_u \times \mathbf{r}_v\|\ \ \delta A$

Chap 6 page 17 $\quad ||\mathbf{v}_1 \times \mathbf{v}_2|| = ||\mathbf{v}_1||\,||\mathbf{v}_2||\sin\theta.\quad$ arc length $ds = \|\mathbf{r}'(t)\|\,dt$

Example

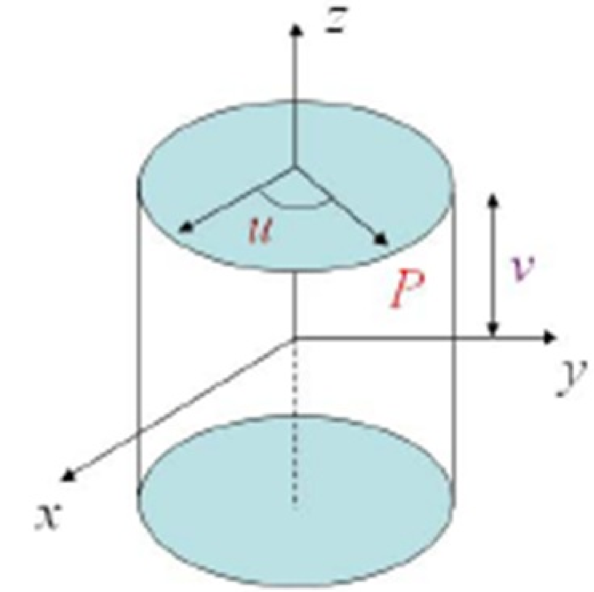
S is a surface consisting of three parts: S_1, S_2, S_3.

S_1 is a circular cylinder with parametric representation:



$$\mathbf{r}(u,v) = \underset{x}{\sqrt{3}\cos u}\,\mathbf{i} + \underset{y}{\sqrt{3}\sin u}\,\mathbf{j} + \underset{z}{v}\,\mathbf{k}.$$

$D:\ 0 \le u \le 2\pi,\ 0 \le v \le 1.$



$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sqrt{3}\sin u & \sqrt{3}\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

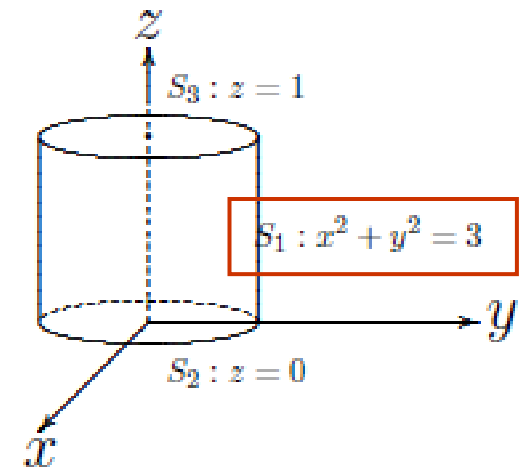
$$= \sqrt{3}\cos u\,\mathbf{i} + \sqrt{3}\sin u\,\mathbf{j} + 0\mathbf{k}$$

$$S_1: \quad \mathbf{r}(u,v) = \underset{x}{\sqrt{3}\cos u}\,\mathbf{i} + \underset{y}{\sqrt{3}\sin u}\,\mathbf{j} + \boxed{\underset{z}{v}}\,\mathbf{k}.$$

$$\boxed{D:\ 0 \le u \le 2\pi,\ \ 0 \le v \le 1.}$$

$$\boxed{\mathbf{r}_u \times \mathbf{r}_v = \sqrt{3}\cos u\,\mathbf{i} + \sqrt{3}\sin u\,\mathbf{j} + 0\mathbf{k}}$$

$$\boxed{\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{3}.}$$



Thus,

$$\iint_{S_1} z\,dS = \iint_D \boxed{v\|\mathbf{r}_u \times \mathbf{r}_v\|}\,dA = \int_0^{2\pi}\int_0^1 \boxed{\sqrt{3}v}\,dv\,du$$

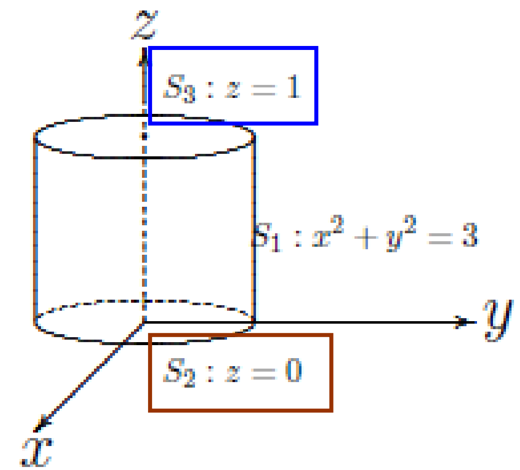
$$= \int_0^{2\pi} \frac{\sqrt{3}}{2}\,du = \boxed{\sqrt{3}\pi.}$$

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA.$$

S_2 lies on the xy-plane : $z = 0$

Thus, $\displaystyle\iint_{S_2} z\,dS = \boxed{0.}$

S_3 lies on the plane $z = 1$.



Thus, $\displaystyle\iint_{S_3} z\,dS = \iint_{S_3} dS$

$= \text{area of } S_3 = \pi(\sqrt{3})^2 = \boxed{3\pi.}$

Finally, $$\iint_S z\,dS = \underbrace{\iint_{S_1} z\,dS}_{\sqrt{3}\pi} + \underbrace{\iint_{S_2} z\,dS}_{0} + \underbrace{\iint_{S_3} z\,dS}_{3\pi}.$$

$$= \boxed{\left(3+\sqrt{3}\right)\pi}$$

Surface Area Formula of Chapter 8

Recall: $\displaystyle\iint_R 1\,dA$ = area of region R in the xy-plane.

Similarly, $\displaystyle\iint_S g(x,y,z)\,dS \;\xrightarrow{\text{set } g \text{ to } 1}\; \iint_S 1\,dS$ = surface area of S.

Surface area of S $\displaystyle = \iint_D 1\cdot||\mathbf{r}_u\times\mathbf{r}_v||\,dA$

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u\times\mathbf{r}_v\|\,dA.$$

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set g to 1

Similarly, $\displaystyle\iint_S g(x,y,z)\,dS = \iint_S 1\,dS$ = surface area of S.

Surface area of S $= \displaystyle\iint_D 1\cdot||\mathbf{r}_u \times \mathbf{r}_v||\,dA$

use natural parametric rep. $z = f(u, v)$

$\mathbf{r}_u \times \mathbf{r}_v = -f_u\,\mathbf{i} - f_v\,\mathbf{j} + \mathbf{k}$

$$= \iint_D \sqrt{(-f_u)^2 + (-f_v)^2 + 1^2}\,dA$$

$$\boxed{\underset{v=y}{\overset{u=x}{=}} \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA}$$

There are two types of surface integrals:

1. for scalar function $f(x, y, z)$

$$\iint_S f(x, y, z)\, dS$$

no general geometric interpretation,

but can be used to calculate surface area and average 'values', e.g. average temperature

2. for vector field $\mathbf{F}(x, y, z)$

$$\iint_S \mathbf{F}(x, y, z) \cdot d\mathbf{S}$$

calculates, for example,

volume flow rate of a fluid with velocity $\mathbf{F}$ moving through a surface S.

Surface Integrals of Vector Fields

Recall that volume flow rate of fluid through a surface *S* may be calculated as a surface integral of the form

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

This integral is also called the 'flux' of **F** over *S*.

$$S: \ \mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

Let *D* be the domain of $\mathbf{r}(u, v)$.

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA}$$

Derivation of formula:

Recall that $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{N}\,dS$

where **N** is a normal unit vector (function) on *S*.

But note that $\mathbf{r}_u \times \mathbf{r}_v$ is normal to *S*.

scalar function

Thus, $$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \boxed{\mathbf{F}\cdot\frac{\mathbf{r}_u\times\mathbf{r}_v}{\|\mathbf{r}_u\times\mathbf{r}_v\|}}\,dS$$

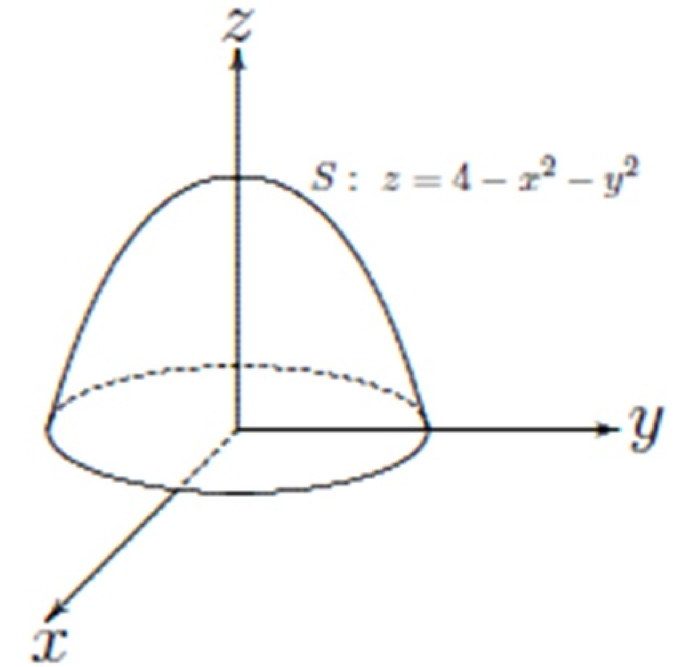
$$= \iint_D \left[\mathbf{F}(\mathbf{r}(u,v))\cdot\frac{\mathbf{r}_u\times\mathbf{r}_v}{\|\mathbf{r}_u\times\mathbf{r}_v\|}\right]\|\mathbf{r}_u\times\mathbf{r}_v\|\,dA$$

$$= \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA.$$

$$\iint_S f(x,y,z)\,dS = \iint_D \boxed{f(\mathbf{r}(u,v))}\,\|\mathbf{r}_u\times\mathbf{r}_v\|\,dA.$$

Example

S is part of paraboloid $z = 4 - x^2 - y^2$ above xy-plane.



Use natural parametric representation

$$\mathbf{r}(u, v) = \underbrace{u}_{x}\mathbf{i} + \underbrace{v}_{y}\mathbf{j} + \boxed{\underbrace{(4 - u^2 - v^2)}_{z = f(x, y)}}\mathbf{k}.$$

$$\mathbf{r}_u \times \mathbf{r}_v = \boxed{2u\mathbf{i} + \quad}$$

$$\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + xy\,\mathbf{k}$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\mathbf{r}_u \times \mathbf{r}_v = \boxed{-f_u\,\mathbf{i}} - f_v\,\mathbf{j} + \mathbf{k}$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

Example

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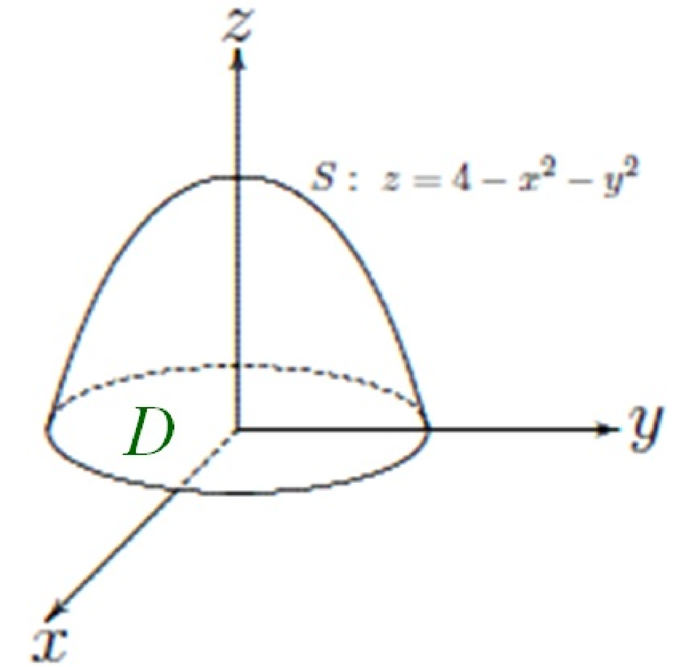


Use natural parametric representation

$$\mathbf{r}(u,v) = \underset{x}{u\mathbf{i}} + \underset{y}{v\mathbf{j}} + \underset{z=f(x,y)}{(4-u^2-v^2)\mathbf{k}}.$$

$$\mathbf{r}_u \times \mathbf{r}_v = 2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k} \qquad \mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + xy\,\mathbf{k}$$

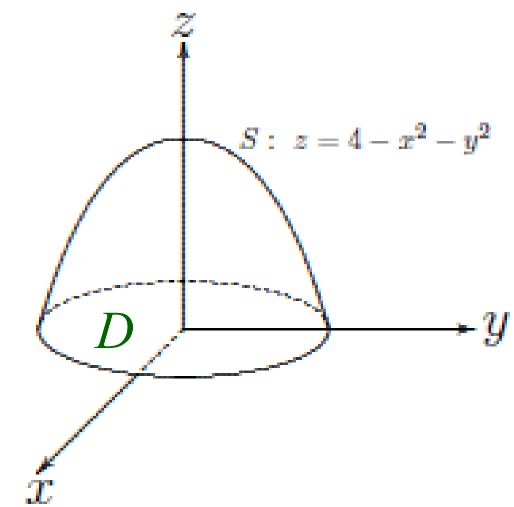
D : disk of radius 2 centred at the origin.

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D (u\mathbf{i} + v\mathbf{j} + uv\mathbf{k})\cdot(2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k})\,dA$$

$$= \iint_D (2u^2 + 2v^2 + uv)\,dA$$

D : disk of radius 2 centred at the origin.

$$0 \le r \le 2, \quad 0 \le \theta \le 2\pi$$



$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

$$= \iint_D \left(\boxed{2u^2 + 2v^2} + \boxed{uv} \right) dA$$

$$= \int_0^{2\pi} \int_0^2 \left(2r^2 + r^2 \cos\theta \sin\theta \right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(2 + \cos\theta \sin\theta \right) d\theta \cdot \int_0^2 r^3\,dr$$

0

$$= 4\pi \cdot 4$$

$$= \boxed{16\pi}$$

$$u^2 + v^2 = x^2 + y^2 = r^2 \qquad uv = xy = r\cos\theta \cdot r\sin\theta$$

Example

Sphere S : $x^2 + y^2 + z^2 = 1$

$$\mathbf{r}(u,v) = \underbrace{\sin u \cos v}_{x}\mathbf{i} + \underbrace{\sin u \sin v}_{y}\mathbf{j} + \underbrace{\cos u}_{z}\mathbf{k},$$

$D:\ 0 \le u \le \pi \text{ and } 0 \le v \le 2\pi.$

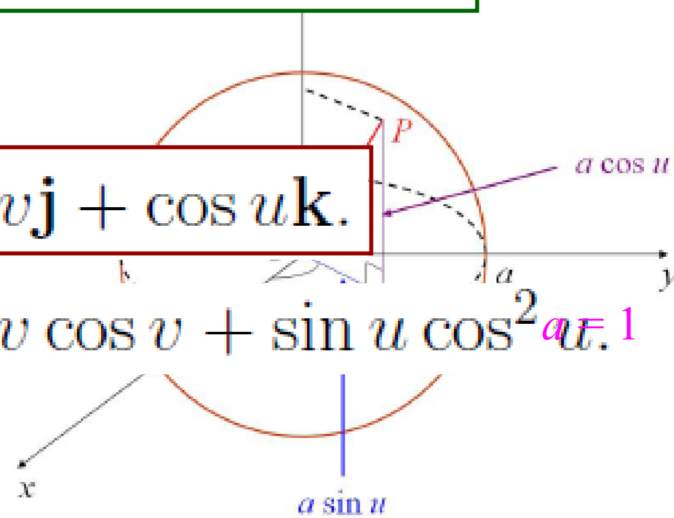
$$\mathbf{r}_u \times \mathbf{r}_v = \sin u\, \boxed{\mathbf{r}(u,v)}$$
$$= \boxed{\sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \sin u \cos u\mathbf{k}}$$

$$\boxed{\mathbf{F}(x,y,z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}.}$$

$$\mathbf{F}(\mathbf{r}(u,v)) = \boxed{\sin u \sin v\mathbf{i} + \sin\ u \cos v\mathbf{j} + \cos u\mathbf{k}.}$$

$$\mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 2\sin^3 u \sin v \cos v + \sin u \cos^2 u.$$

$a = 1$



$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

$$D: \boxed{0 \le u \le \pi \text{ and } 0 \le v \le 2\pi.}$$

$$\mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 2\sin^3 u \sin v \cos v + \sin u \cos^2 u.$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

$$= \boxed{\int_0^{2\pi} \int_0^{\pi}} \boxed{(2\sin^3 u \sin v \cos v} + \boxed{\sin u \cos^2 u)} \boxed{du\,dv}$$

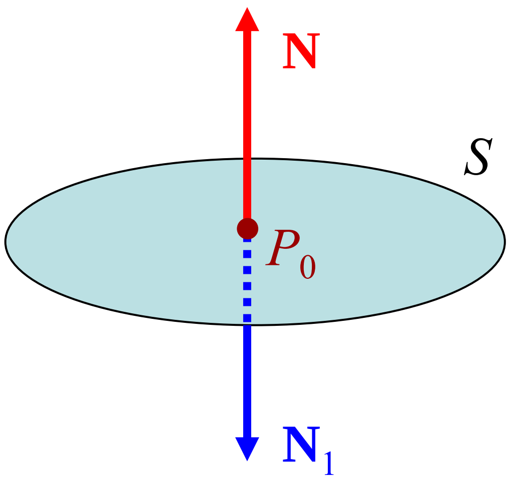
$$= \int_0^{\pi} \sin^3 u \, du \underbrace{\boxed{\int_0^{2\pi} \sin 2v \, dv}}_{=0} + \underbrace{\boxed{\int_0^{\pi} \sin u \cos^2 u \, du}}_{\boxed{\left[-\frac{1}{3}\cos^3 u\right]_0^{\pi}}} \underbrace{\boxed{\int_0^{2\pi} dv}}_{\boxed{2\pi}}$$

$$= \boxed{\frac{4}{3}\pi}$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \boxed{\iint_D} \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

Orientation of Surfaces

At a point P_0 on a surface S, suppose there is a unit normal vector $\mathbf{N}$.



Then $\mathbf{N}_1 = -\mathbf{N}$ is also a unit normal vector at P_0, but in the opposite direction.

This gives a relation between surface integrals:

$$\boxed{\iint_S \mathbf{F}\cdot\mathbf{N}_1\, dS = -\iint_S \mathbf{F}\cdot\mathbf{N}\, dS}$$

Thus, for surface integrals of a vector field, the value depends on the choice of the normal vector.

$$\iint_S \mathbf{F} \cdot \mathbf{N}_1 \, dS = -\iint_S \mathbf{F} \cdot \mathbf{N} \, dS$$

The choice of the normal vector is known as the orientation of the surface S.

If surface S is given an orientation, then the surface with the opposite orientation is denoted by $-S$.

In particular, the above relation between surface integrals can be expressed as:

$$\boxed{\iint_{-S} \mathbf{F} \cdot d\mathbf{S} = -\iint_S \mathbf{F} \cdot d\mathbf{S}.}$$

If the surface S is described parametrically by $\mathbf{r}(u, v)$, then the normal vector (function) $\boxed{\mathbf{r}_u \times \mathbf{r}_v}$ provides an orientation of S.

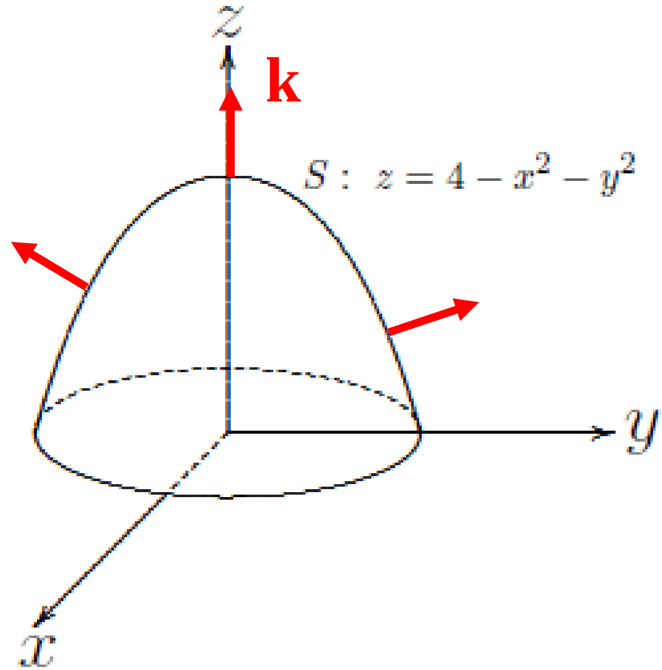
Example 1 (10.2.5)

$$\mathbf{r}(u,v) = \underbrace{u}_{x}\mathbf{i} + \underbrace{v}_{y}\mathbf{j} + \underbrace{(4 - u^2 - v^2)}_{z}\mathbf{k}.$$

$$\boxed{\mathbf{r}_u \times \mathbf{r}_v = 2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k}.}$$

At $u = x = 0$, $v = y = 0$,

note that $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{k}$.



In this example, the orientation of S is given by an upward normal vector.

Example 2 (10.2.6)

$$\boxed{\mathbf{r}(u,v) = \underbrace{\sin u \cos v}_{x}\,\mathbf{i} + \underbrace{\sin u \sin v}_{y}\,\mathbf{j} + \underbrace{\cos u}_{z}\,\mathbf{k},}$$

$$\boxed{\mathbf{r}_u \times \mathbf{r}_v = \sin u\, \mathbf{r}(u,v)}$$

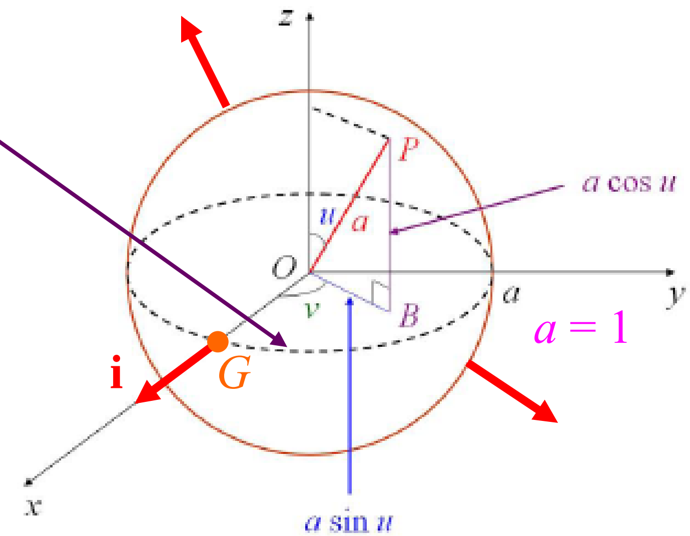
$$= \sin^2 u \cos v\,\mathbf{i} + \sin^2 u \sin v\,\mathbf{j} + \sin u \cos u\,\mathbf{k}$$

At $\boxed{u = \dfrac{\pi}{2}}$ ('equator') and $\boxed{v = 0}$

$\mathbf{r}(u, v) = \mathbf{i}$: point $G(1, 0, 0)$

$\mathbf{r}_u \times \mathbf{r}_v = \mathbf{i}$.

In this example, the orientation of S is given by an outward normal vector.



Terminology

upward normal vector

downward normal vector

outward normal vector outer normal vector

inward normal vector inner normal vector

Curl

Given a vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, the curl of $\mathbf{F}$ is defined as:

$$\boxed{\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}}$$

(page 27)

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Example 10.3.4 (page 28)

$$\mathbf{F}(x, y, z) = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}.$$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy^2z & xyz^2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ x^2yz & xyz^2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ x^2yz & xy^2z \end{vmatrix} \mathbf{k}$$

$$= \boxed{(xz^2 - xy^2)\,\mathbf{i}}$$

$$\frac{\partial}{\partial y}(xyz^2) - \boxed{\frac{\partial}{\partial z}(xy^2z)}$$

Example 10.3.4 (page 28)

$$\mathbf{F}(x,y,z) = \boxed{x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}.}$$

$$\operatorname{curl}\ \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy^2z & xyz^2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ x^2yz & xyz^2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ x^2yz & xy^2z \end{vmatrix} \mathbf{k}$$

$$= \boxed{(xz^2 - xy^2)\,\mathbf{i} - (yz^2 - x^2y)\,\mathbf{j} + (y^2z - x^2z)\,\mathbf{k}}$$

Stokes' Theorem

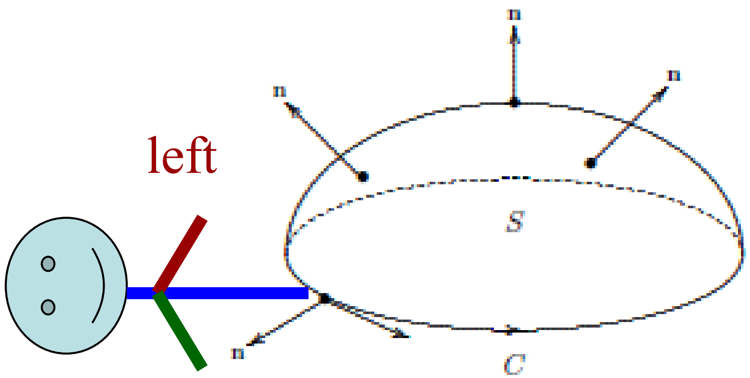
Let S be an oriented piecewise-smooth surface that is bounded by a closed, piecewise-smooth boundary curve C.

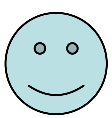
Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on S. Then

$$\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S}}$$

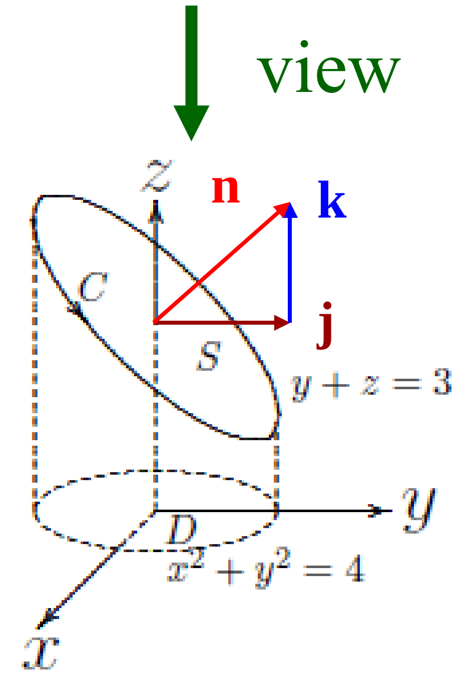
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S}$$

The orientation of C must be consistent with the orientation of S, as follows:



A person  walks on the curve C in the direction of the orientation of C with his head pointing in the direction of the normal vector of S. Surface S must be on his $\boxed{\text{left.}}$

Example



Cylinder $\Sigma: \; x^2 + y^2 = 4.$

Plane $\Pi: \; y + z = 3.$

$\mathbf{n} = \mathbf{j} + \mathbf{k}$ is a normal vector

Curve C is the intersection of Π with Σ.

Orientation of C is anti-clockwise when viewed from above.

S is surface enclosed by C on Π.

Assign orientation $\mathbf{n}$ to S.

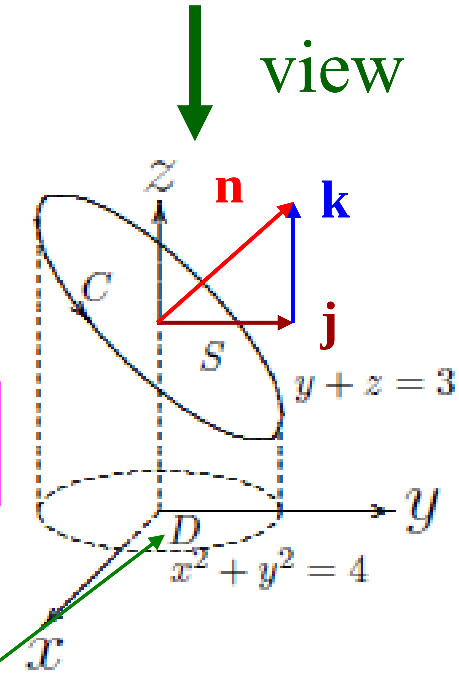
This orientation of S agrees with the orientation of C.

Plane Π : $y+z=3$.

Use natural parametric representation:

$$\mathbf{r}(u,v) = \underset{x}{u}\mathbf{i} + \underset{y}{v}\mathbf{j} + \boxed{\underset{z=f(x,y)}{(3-v)\mathbf{k}}}$$

$$\boxed{\mathbf{r}_u \times \mathbf{r}_v} = -0\,\mathbf{i} - (-1)\,\mathbf{j} + \mathbf{k} = \boxed{\mathbf{j}+\mathbf{k}} = \mathbf{n}$$



D is the disk of radius 2.

$$\mathbf{F}(x,y,z) = \boxed{yz\mathbf{i} - xz\mathbf{j} + xy\mathbf{k}}$$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot d\mathbf{S}$$

$$= \iint_D \boxed{(2u\mathbf{i} - 2(3-v)\mathbf{k})}\cdot\boxed{(\mathbf{j}+\mathbf{k})}\,dA$$

change to curl **F**

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA$$

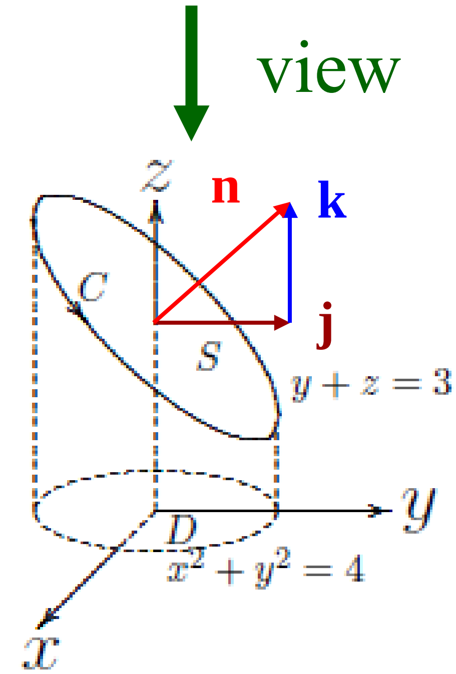
$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & xy \end{vmatrix} = \boxed{2x\mathbf{i} - 2z\mathbf{k}.}$$

$$\mathbf{r}_u \times \mathbf{r}_v = -f_u\,\mathbf{i} - f_v\,\mathbf{j} + \mathbf{k}$$

D is the disk of radius 2.

$D: \; 0 \le r \le 2, \; 0 \le \theta \le 2\pi$

$u = x = r\cos\theta, \; v = y = r\sin\theta$



$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot d\mathbf{S}$$

$$= \iint_D (2u\mathbf{i} - 2(3-v)\mathbf{k})\cdot(\mathbf{j}+\mathbf{k})\, dA$$

$$= \iint_D (-6+2v)\, dA = \int_0^{2\pi}\int_0^2 (-6+2r\sin\theta) r\, dr\, d\theta$$

$$(-6r + 2r^2 \sin\theta)$$

$$= \int_0^{2\pi} \left(-12 + \cancel{\frac{16}{3}\sin\theta}\right) d\theta$$

0

$$= \boxed{-24\pi.}$$

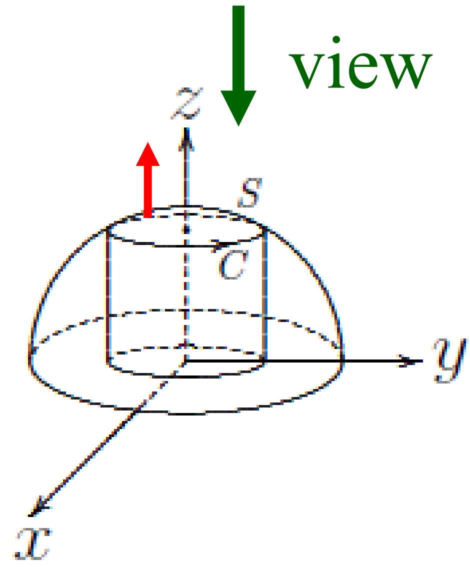
Example

Cylinder $\boxed{x^2 + y^2 = 5}$

Upper hemisphere $\boxed{z = \sqrt{9 - x^2 - y^2}}$

Intersection is a circle C at:

$$z = \sqrt{9 - \boxed{(x^2 + y^2)}} = \boxed{2}$$



$$C: \quad \underbrace{\mathbf{r}(t) = \sqrt{5}\cos t}_{x}\,\mathbf{i} + \underbrace{\sqrt{5}\sin t}_{y}\,\mathbf{j} + \underbrace{2}_{z}\mathbf{k} \qquad \boxed{0 \le t \le 2\pi}$$

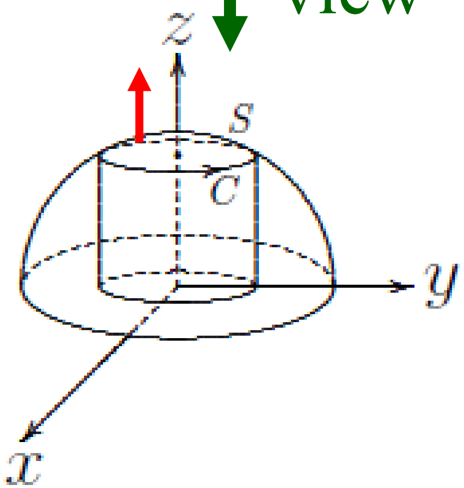
C is traversed in anti-clockwise direction when viewed from the top.

Given orientation of S is the upward normal vector

Orientations of C and S agree.

$$C: \boxed{\begin{array}{l} \mathbf{r}(t) = \boxed{\underset{x}{\sqrt{5}\cos t\,\mathbf{i}} + \underset{y}{\sqrt{5}\sin t\,\mathbf{j}} + \underset{z}{2\mathbf{k}}} \\ \mathbf{r}'(t) = \boxed{-\sqrt{5}\sin t\,\mathbf{i} + \sqrt{5}\cos t\,\mathbf{j} + 0\mathbf{k}} \\ \qquad\qquad 0 \le t \le 2\pi \end{array}}$$

view



$$\mathbf{F}(x, y, z) = \boxed{y^2 z\,\mathbf{i} + x\,\mathbf{j} + (x+y)\mathbf{k}}$$

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \boxed{\mathbf{F}} \cdot \boxed{d\mathbf{r}}$$

Stokes' Theorem

$$= \int_0^{2\pi} \left(10\sin^2 t\,\mathbf{i} + \sqrt{5}\cos t\,\mathbf{j} + \sqrt{5}(\cos t + \sin t)\mathbf{k}\right) \cdot \left(-\sqrt{5}\sin t\,\mathbf{i} + \sqrt{5}\cos t\,\mathbf{j}\right)\, dt$$

$$= \int_0^{2\pi} (\underbrace{-10\sqrt{5}\sin^3 t}_{0} + \underbrace{5\cos^2 t}_{5\pi})\, dt = \boxed{5\pi.}$$

Divergence (page 26)

Given a vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, the divergence of $\mathbf{F}$ is defined as:

$$\boxed{\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

(page 28)

Example (page 28): $\mathbf{F}(x, y, z) = \underbrace{x^2yz}_{P}\mathbf{i} + \underbrace{xy^2z}_{Q}\mathbf{j} + \underbrace{xyz^2}_{R}\mathbf{k}.$

$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2)$$

$$= 2xyz + 2xyz + 2xyz$$

$$= \boxed{6xyz}$$

Divergence Theorem (page 36)

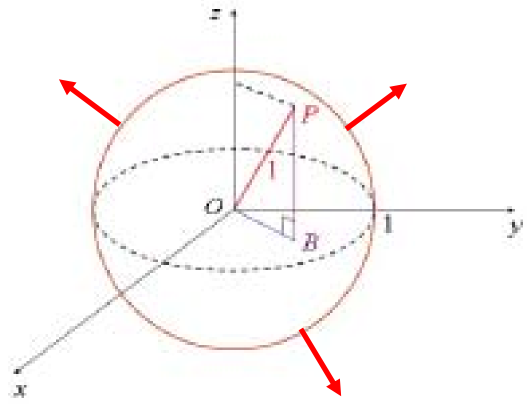
Let E be a solid region with boundary S that is given the outward orientation.

Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field whose component functions P, Q, R have continuous partial derivatives in E.

Then

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV.}$$

Example



Unit sphere $S: x^2 + y^2 + z^2 = 1$,

oriented with outward normal vector.

$$\mathbf{F}(x,y,z) = \underbrace{(x+y)}_{P}\mathbf{i} + \underbrace{(y+z)}_{Q}\mathbf{j} + \underbrace{(z+x)}_{R}\mathbf{k}.$$

$$\boxed{\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 1 + 1 + 1 = 3}$$

By the divergence theorem, with E = solid unit sphere,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E 3\, dV = 3 \boxed{\iiint_E 1\, dV}$$

$$= 3 \times \text{volume of solid unit sphere } E = 3 \cdot \frac{4}{3}\pi \cdot 1^3$$

$$= \boxed{4\pi}$$

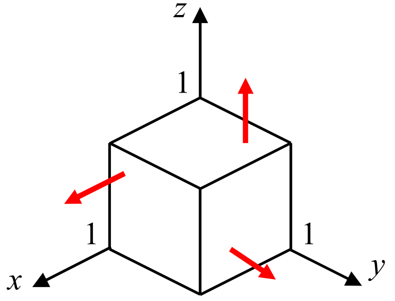
$$\iiint_D 1\, dV = \text{volume of } D.$$

Example

Solid cubic region E : $\boxed{0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1.}$

S = surface of E,

oriented with outward normal vector.



$$\mathbf{F}(x,y,z) = \underbrace{x^2}_{P}\mathbf{i} + \underbrace{(xy + x\cos z)}_{Q}\mathbf{j} + \underbrace{e^{xy}}_{R}\mathbf{k}$$

$$\boxed{\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 2x + x + 0 + 0 = 3x}$$

By the divergence theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F}\, dV = \iiint_E 3x\, dV$$

$$= 3\int_0^1 \int_0^1 \underbrace{\boxed{\int_0^1 x\, dx}}_{= 1/2} dy\, dz = \ldots = \boxed{\frac{3}{2}}$$

Game Over