MA 1505 Mathematics I
Tutorial 9 Solutions

1. We use the criteria of conservative field:

$$\begin{aligned}\frac{\partial}{\partial y}(2xy) &= 2x = \frac{\partial}{\partial x}(x^2+2yz),\\ \frac{\partial}{\partial z}(2xy) &= 0 = \frac{\partial}{\partial x}(y^2),\\ \frac{\partial}{\partial z}(x^2+2yz) &= 2y = \frac{\partial}{\partial y}(y^2).\end{aligned}$$

So $\mathbf{F}$ is conservative, and hence there exists a function f such that $\nabla f = \mathbf{F}$.

To find f, first we know that, by $\mathbf{i}$-component of $\mathbf{F}$, we have $f_x(x,y,z) = 2xy$ so that

$$f(x,y,z) = x^2y + g(y,z) \quad (*).$$

Differentiate $(*)$ w.r.t. y, we have

$$f_y(x,y,z) = x^2 + g_y(y,z).$$

Now by $\mathbf{j}$-component of $\mathbf{F}$, we have $f_y(x,y,z) = x^2 + 2yz$, so

$$g_y(y,z) = 2yz \Rightarrow g(y,z) = y^2z + h(z).$$

Hence $(*)$ becomes

$$f(x,y,z) = x^2y + y^2z + h(z) - -(**)$$

Differentiate $(**)$ w.r.t. z, we have $f_z(x,y,z) = y^2 + h'(z)$.
Now by $\mathbf{k}$-component of $\mathbf{F}$, $f_z(x,y,z) = y^2$ so

$$h'(z) = 0 \Rightarrow h(z) = K.$$

So $f(x,y,z) = x^2y + y^2z + K$, where K is a constant.

2. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + 3t\mathbf{k}$, where $0 \le t \le 1$. Then

$$\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad ||\mathbf{r}'(t)|| = \sqrt{14}, \quad g(x(t),y(t),z(t)) = t^2 - (2t)(3t) + (3t)^2 = 4t^2.$$

Therefore

$$\int_C g(x,y,z)\,ds = \int_0^1 (4t^2)(\sqrt{14})\,dt = \frac{4}{3}\sqrt{14}.$$

3. $\mathbf{F}(\mathbf{r}(t)) = t^5\mathbf{i} + 2t^2\mathbf{j} - t^2\mathbf{k}$. $\mathbf{r}'(t) = 1\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$.

$$\begin{aligned}\int_C \mathbf{F} \bullet d\mathbf{r} &= \int_0^1 (t^5\mathbf{i} + 2t^2\mathbf{j} - t^2\mathbf{k}) \bullet (1\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})\,dt\\ &= \int_0^1 (t^5 - 3t^4 + 4t^3)\,dt = \frac{17}{30}.\end{aligned}$$

4. To parametrize line segment from (a, b, c) to (d, e, f), we can use

$$\mathbf{r}(t) = [a\mathbf{i} + b\mathbf{j} + c\mathbf{k}] + t[(d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) - (a\mathbf{i} + b\mathbf{j} + c\mathbf{k})], \quad 0 \le t \le 1.$$

So we have
the line segment C_1 joining $(0, 0, 0)$ to $(1, 0, 2)$

$$\mathbf{r}(t) = t\mathbf{i} + 0\mathbf{j} + 2t\mathbf{k}, \quad 0 \le t \le 1$$

the line segment C_2 joining $(1, 0, 2)$ to $(3, 4, 1)$

$$\mathbf{r}(t) = (2t + 1)\mathbf{i} + 4t\mathbf{j} + (-t + 2)\mathbf{k}, \quad 0 \le t \le 1$$

Then

$$\int_{C_1} 2xy\ dx + (x^2 + z)\ dy + y\ dz = 0,$$

$$\int_{C_2} 2xy\ dx + (x^2 + z)\ dy + y\ dz$$

$$= \int_0^1 2(2t + 1)(4t)(2dt) + ((2t + 1)^2 + (-t + 2))(4dt) + (4t)(-dt)$$

$$= \int_0^1 (48t^2 + 24t + 12)dt = 40.$$

So $\displaystyle\int_C 2xy\ dx + (x^2 + z)\ dy + y\ dz = 0 + 40 = 40.$

5. Let C be the base of the fence which has vector equation

$$\mathbf{r}(t) = 10\cos t\mathbf{i} + 10\sin t\mathbf{j}, \quad 0 \le t \le 2\pi.$$

So $\|\mathbf{r}'(t)\| = 10$.

The area of the fence is then given by

$$\begin{aligned}
\int_C h(x, y)ds &= \int_0^{2\pi} \left[4 + 0.01[(10\cos t)^2 - (10\sin t)^2]\right] \cdot 10\ dt \\
&= \int_0^{2\pi} 40 + 10\cos 2t\ dt \\
&= [40t + 5\sin 2t]_0^{2\pi} \\
&= 80\pi
\end{aligned}$$

Both sides of the fence will give a total area of $160\pi = 503\ m^2$.

So the amount of paint used is about 5 litre.

6. We can parametrize C_a by $x = t$, $y = a \sin t$, $0 \le t \le \pi$.

   Then

   $I(a) = \int_0^\pi \left(1 + a^3 \sin^3 t\right) dt + (2t + a \sin t)\, a \cos t dt$

   $= \pi - a^3 \int_0^\pi \left(1 - \cos^2 t\right) d(\cos t) + 2a \int_0^\pi t d(\sin t) + \frac{1}{2} a \int_0^\pi \sin 2t dt$

   $= \pi - a^3 \left[\cos t - \frac{1}{3}\cos^3 t\right]_0^\pi + 2a\,[t \sin t]_0^\pi - 2a \int_0^\pi \sin t dt - \frac{1}{4} a^2 [\cos 2t]_0^\pi$

   $= \pi + \frac{4}{3} a^3 - 4a.$

   Therefore, $I'(a) = 4a^2 - 4 = 0$ implies $a = 1$ in the domain $a > 0$.

   Since $I''(a) = 8a > 0$ in the domain $a > 0$,

   we conclude that the minimum value of $I(a)$ in the domain $a > 0$ is $I(1) = \pi - \frac{8}{3}$.