MA 1505 Mathematics I

Tutorial 4 Solutions

1. (a) Let $u_n = (-1)^n \frac{(x+2)^n}{n}$.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{n+1} \cdot \frac{n}{(x+2)^n}\right| = |x+2|.$$

By ratio test, the power series is convergence in $|x+2|<1$.

So the radius of convergence is 1.

(b) Let $u_n = \frac{(3x-2)^n}{n}$.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(3x-2)^{n+1}}{n+1} \cdot \frac{n}{(3x-2)^n}\right| = |3x-2|.$$

By ratio test, the power series is convergence in $|3x-2|<1 \Rightarrow |x-\frac{2}{3}|<\frac{1}{3}$.

So the radius of convergence is $\dfrac{1}{3}$.

(c) Let $u_n = (-1)^n(4x+1)^n$.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(4x+1)^{n+1}}{(4x+1)^n}\right| = |4x+1|.$$

By ratio test, the power series is convergence in $|4x+1|<1 \Rightarrow |x+\frac{1}{4}|<\frac{1}{4}$.

So the radius of convergence is $\dfrac{1}{4}$.

(d) Let $u_n = \frac{(3x)^n}{n!}$.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(n+1)!} \cdot \frac{n!}{(3x)^n}\right| = \lim_{n\to\infty}\left|\frac{3x}{n+1}\right| = 0.$$

Since the limit is less than 1 for any x, so the radius of convergence is ∞.

(e) Let $u_n = (nx)^n$.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{((n+1)x)^{n+1}}{(nx)^n}\right| = \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^n (n+1)x\right| = \infty$$

for all $x \neq 0$.

So the radius of convergence is 0.

(f) Let $u_n = \frac{(4x-5)^{2n+1}}{n^{3/2}}$.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(4x-5)^{2n+3}}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{(4x-5)^{2n+1}}\right| = |4x-5|^2.$$

By ratio test, the power series is convergence in $|4x-5|^2<1 \Rightarrow |4x-5|<1 \Rightarrow |x-\frac{5}{4}|<\frac{1}{4}$.

So the radius of convergence is $\dfrac{1}{4}$.

2. The first term of the geometric series is $a = 1$ and the common ratio is $r = -\dfrac{(x-3)}{2}$.

   So the sum of the series is

$$\frac{a}{1-r} = \frac{1}{1+(x-3)/2} = \frac{2}{x-1}$$

   provided $\left|\dfrac{(x-3)}{2}\right| < 1 \Rightarrow 1 < x < 5$.

3. (a)

$$\frac{x}{1-x} = x\left(\frac{1}{1-x}\right) = x\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}$$

   (b) Let $f(x) = \frac{1}{x^2}$.

   Then $f'(x) = -\dfrac{2}{x^3}$, $f''(x) = \dfrac{3!}{x^4}$, ... and in general $f^{(n)}(x) = (-1)^n \dfrac{(n+1)!}{x^{n+2}}$.

   So $f^{(n)}(1) = (-1)^n (n+1)!$.

   The Taylor series of f at $x = 1$ is thus:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n.$$

   (c)

$$\begin{aligned}\frac{x}{1+x} &= \frac{(1+x)-1}{1+x} = 1 - \frac{1}{1+x} = 1 - \frac{1}{1+(x+2)-2} \\ &= 1 + \frac{1}{1-(x+2)} = 1 + \sum_{n=0}^{\infty}(x+2)^n = 2 + \sum_{n=1}^{\infty}(x+2)^n\end{aligned}$$

4. Given $f(x) = \sin x$, compute the following:

$$\begin{aligned} f(x) &= \sin x & f(0) &= \sin 0 = 0 \\ f^{(1)}(x) &= \cos x & f^{(1)}(0) &= \cos 0 = 1 \\ f^{(2)}(x) &= -\sin x & f^{(2)}(0) &= -\sin 0 = 0 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -\cos 0 = -1 \\ f^{(4)}(x) &= \sin x & & \end{aligned}$$

   Taylor polynomial at $x = 0$ of order 3 is

$$P_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!} x^k = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 = x - \frac{1}{6}x^3.$$

$P_3(0.1)$ approximates $\sin 0.1$ as follows:

$$P_3(0.1) \;=\; 0.1 - \frac{1}{6}\,(0.1)^3 \;\approx\; 0.09983.$$

In using $P_3(x)$ to approximate $\sin x$ the error incurred is

$$R_3(x) \;=\; \frac{f^{(4)}(c)}{4!}\,x^4 \qquad \text{for some } c \text{ between } 0 \text{ and } x.$$

Thus, magnitude of error incurred is

$$\begin{aligned} |R_3(0.1)| \;&=\; \left|\frac{\sin c}{4!}\,(0.1)^4\right| \;\le\; \frac{(0.1)^4}{4!} \qquad \text{since} \quad |\sin\theta| \le 1 \;\text{ for any } \theta \\ &<\; 10^{-5}. \end{aligned}$$

5. (i) We have

$$xe^x = x\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}$$

Note that the radius of convergence of this power series is infinite. Therefore,we can integrate both sides from 0 to 1.

$$\int_0^1 xe^x\;dx = \sum_{n=0}^{\infty}\int_0^1 \frac{x^{n+1}}{n!}\;dx = \sum_{n=0}^{\infty}\frac{1}{n!(n+2)}.$$

On the other hand,

$$\int_0^1 xe^x\;dx = [xe^x]_0^1 - \int_0^1 e^x\;dx = e-(e-1) = 1.$$

So we conclude $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!(n+2)} = 1.$

(ii) We have

$$\frac{e^x-1}{x} = \frac{\sum_{n=1}^{\infty}\frac{x^n}{n!}}{x} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty}\frac{x^n}{(n+1)!}.$$

Note that the radius of convergence of this power series is infinite.
Differentiate both sides with respect to x, we have

$$\frac{xe^x-(e^x-1)}{x^2} = \sum_{n=1}^{\infty}\frac{nx^{n-1}}{(n+1)!} = \sum_{n=0}^{\infty}\frac{(n+1)x^n}{(n+2)!}.$$

The result now follows by setting $x=1$.

6. First we calculate

$\frac{1}{2}\int_0^1 t^{n-1}\left(1-t\right)^2 dt$
$= \frac{1}{2}\int_0^1 t^{n-1}\left(1-2t+t^2\right) dt$
$= \frac{1}{2}\left[\frac{1}{n}t^n - \frac{2}{n+1}t^{n+1} + \frac{1}{n+2}t^{n+2}\right]_0^1$
$= \frac{1}{n(n+1)(n+2)}$

Next we apply this result to $n = 1, 3, 5, 7, \cdots$, we find that

$\frac{1}{1\cdot2\cdot3} + \frac{1}{3\cdot4\cdot5} + \frac{1}{5\cdot6\cdot7} + \frac{1}{7\cdot8\cdot9} + \cdots$
$= \frac{1}{2}\int_0^1 t^{1-1}\left(1-t\right)^2 dt + \frac{1}{2}\int_0^1 t^{3-1}\left(1-t\right)^2 dt + \frac{1}{2}\int_0^1 t^{5-1}\left(1-t\right)^2 dt + \frac{1}{2}\int_0^1 t^{7-1}\left(1-t\right)^2 dt + \cdots$
$= \frac{1}{2}\int_0^1 \left(1-t\right)^2\left(1+t^2+t^4+t^6+\cdots\right) dt$
$= \frac{1}{2}\int_0^1 \left(1-t\right)^2\left(\frac{1}{1-t^2}\right) dt$
$= \frac{1}{2}\int_0^1 \frac{1-t}{1+t} dt$
$= -\frac{1}{2}\int_0^1 \frac{1+t-2}{1+t} dt$
$= -\frac{1}{2}\int_0^1 \left(1-\frac{2}{1+t}\right) dt$
$= \ln 2 - \frac{1}{2}$