Chapter 4. Series

4.1 Infinite series

4.1.1 Definition.

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is called an <u>infinite series</u>. The term a_n is the <u>nth term</u> of the series.

For example,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$

is an infinite series whose nth term is $\dfrac{1}{2^n}$.

The sequence $\{s_n\}$ defined by

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

is called the sequence of <u>partial sums</u> of the series.

The number s_n is called the <u>nth partial sum</u>.

If the sequence of partial sums $\{s_n\}$ converges to a

limit L, then we say that the series is <u>convergent</u> and

that its sum is L. We write

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots = L.$$

If the sequence of partial sums does not converge, we

say that the series is <u>divergent</u>.

Note that

$$\sum_{n=1}^{\infty} a_n, \quad \sum_{k=1}^{\infty} a_k, \quad \sum_{r=1}^{\infty} a_r, \quad \sum a_n, \quad \sum a_k, \quad \sum a_r$$

represent the same series.

4.1.2 Geometric series

The series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

is called a <u>geometric series</u>, where a and r are fixed

numbers, a is called the first term, and r is the (com-

mon) ratio.

For this series, the nth partial sum s_n is given by

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n.$$

Thus $s_n - rs_n = a - ar^n$,

$$\implies s_n = a\frac{1 - r^n}{1 - r}, \qquad r \neq 1.$$

If $r = 1$, then clearly $s_n = na \to \infty$ (or $-\infty$), if

$a \neq 0$, and the series is divergent.

If $|r| < 1$, then $r^n \to 0$. Thus

$$s_n \to \frac{a}{1 - r},$$

and the sum of the series is $\dfrac{a}{1 - r}$.

If $|r| > 1$, then $|r|^n \to \infty$, and the series diverges.

We summarize this as follow:

4.1.3 Convergence of Geometric series

The geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

with $a \neq 0$ converges to the sum $\dfrac{a}{1-r}$ if $|r| < 1$ and it diverges if $|r| \geq 1$.

4.1.4 Example

(i) $\dfrac{1}{9} + \dfrac{1}{27} + \dfrac{1}{81} + \cdots$ is a geometric series whose first term is $\frac{1}{9}$ and common ratio $\frac{1}{3}$. It converges to $\frac{1}{6}$.

(ii) $4 - 2 + 1 - \dfrac{1}{2} + \dfrac{1}{4} - \cdots$ converges to $\frac{8}{3}$.

4.1.5 Some rules on series

If $\sum a_n = A$, and $\sum b_n = B$, then

(1) Sum rule. $\quad \sum (a_n + b_n) = A + B.$

(2) Difference rule. $\quad \sum (a_n - b_n) = A - B.$

(3) Constant multiple rule. $\quad \sum (ka_n) = kA.$

4.1.6 Ratio test

Let $\sum a_n$ be a series, and let

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho.$$

Then

(1) the series converges if $\rho < 1$.

(2) the series diverges if $\rho > 1$.

(3) no conclusion if $\rho = 1$.

4.1.7 Example

(i) $a_1 = 1$, $a_{n+1} = \dfrac{n}{2n+1} a_n$ and the series is

$$\sum a_n = 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \cdots .$$

For this series $\left| \dfrac{a_{n+1}}{a_n} \right| = \dfrac{n}{2n+1} \to \dfrac{1}{2}$ as $n \to \infty$. So

the ratio test implies the convergence of the series.

(ii) $\displaystyle\sum \frac{(n!)^2}{(2n)!}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!(n+1)!}{(2n+2)!} \frac{(2n)!}{n!n!}$$

$$= \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{n+1}{2(2n+1)} \to \frac{1}{4}.$$

So the given series is convergent by ratio test.

(iii) $\displaystyle\sum \frac{3^n}{2^n + 5}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{3^{n+1}}{2^{n+1}+5} \frac{2^n+5}{3^n} = 3 \cdot \frac{1 + 5 \cdot 2^{-n}}{2 + 5 \cdot 2^{-n}} \to \frac{3}{2}.$$

By ratio test, $\displaystyle\sum \frac{3^n}{2^n+5}$ is divergent.

4.2 Power Series

4.2.1 Power series about $x = 0$

A **power series** about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

where $c_0, c_1, \ldots, c_n, \ldots$ are constants while x is a variable.

So a power series can be regarded as a function for x where it converges.

4.2.2 Example

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

This power series about $x = 0$ converges to $\dfrac{1}{1-x}$ when $|x| < 1$.

We state this as

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots , \quad -1 < x < 1.$$

4.2.3 Power series about $x = a$

More generally, a **power series** about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \cdots$$
$$+ c_n (x - a)^n + \cdots .$$

The number a is called the centre of the power series.

4.2.4 Convergence of power series

We observe that a power series always behaves in exactly **one** of the following three ways:

Case 1.   The series $\sum c_n(x-a)^n$ converges at $x = a$

and diverges elsewhere.

Case 2.   There is a positive number h such that the

series converges for all x in the interval $(a-h, a+h)$

but diverges for all $x > a + h$ and $x < a - h$.

The series may or may not converge at either of the

end points $x = a - h$ and $x = a + h$.

Case 3.   The series converges for every x.

4.2.5 Radius of Convergence

The number h in case 2 of the previous section is

called the *radius of convergence* of the series.

Note that we can describe all the points in the inter-

val in this case as $|x - a| < h$. Here a is at the center

of the interval.

If the power series converges for all x, we say that

the radius of convergence is infinite ∞.

If it converges only at a, we say that the radius of

convergence is zero.

4.2.6 **Example**

(i) Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots .$$

<u>Solution.</u> We apply ratio test to the series with nth

term $u_n = (-1)^{n-1}\dfrac{x^n}{n}$.

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{n}{n+1}|x| \to |x| \ \text{ as } \ n \to \infty.$$

Therefore, the series converges for $|x| < 1$. It di-

verges if $|x| > 1$.

The radius of convergence is therefore equal to 1.

(ii)   For the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \to 0 \ \text{ as } \ n \to \infty.$$

Therefore, the series converges for all x.

The radius of convergence is therefore equal to ∞.

(iii) For the series $\displaystyle\sum_{n=0}^{\infty} n! x^n = 1 + x + 2! x^2 + 3! x^3 + \cdots .$

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = (n+1)|x| \to \infty$$

as $n \to \infty$ unless $x = 0$.

Therefore, the series diverges for all x except $x = 0$.

The radius of convergence is zero.

4.2.7 Differentiation and Integration of power series

If $\sum c_n(x-a)^n$ has radius of convergence h,

it defines a function f:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad a - h < x < a + h.$$

(i)   The function f has derivatives of all orders in

$(a - h, a + h)$. The derivatives can be obtained by

differentiating the power series term-by-term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$
$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2}, \ldots.$$

The differentiated series converges for $a - h < x <$

$a + h.$

(ii)   The function f has anti-derivatives in $(a -$

$h, a + h)$. The anti-derivatives can be obtained by

integrating the power series term-by-term:

$$\int f(x)dx = \sum_{0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1} + c.$$

The integrated series converges for $a - h < x < a + h$.

4.2.8 Example

$$f(x) \ = \ \frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^n + \cdots,$$

$$-1 < x < 1$$

$$f'(x) \ = \ \frac{1}{(1 - x)^2} = 1 + 2x + \cdots + nx^{n-1} + \cdots,$$

$$-1 < x < 1$$

$$f''(x) \ = \ \frac{2}{(1 - x)^3} = 2 + 6x + \cdots + n(n - 1)x^{n-2}$$

$$+ \cdots, \ -1 < x < 1$$

4.2.9 Example

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots, \quad -1 < t < 1$$

So

$$
\begin{aligned}
\ln(1+x) &= \int_0^x \frac{dt}{1+t} \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots, \\
&\quad -1 < x < 1.
\end{aligned}
$$

4.3 Taylor Series

4.3.1 Definition

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point.

The **Taylor series** of f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots$$

$$+\frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \quad (1)$$

4.3.2 Example

The Taylor series of e^x at $x = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

This follows from the formula (1) and the fact that

$$\frac{d}{dx}e^x = e^x.$$

The radius of convergence of this series is ∞.

4.3.3 Example

The Taylor series of $\sin x$ and $\cos x$ at $x = 0$ are

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

This follows from the formula (1) and the fact that

$$\frac{d}{dx}\sin x = \cos x$$

and

$$\frac{d}{dx}\cos x = -\sin x$$

The radius of convergence of these two series is ∞.

4.3.4 Example

The Taylor series of $\ln(1+x)$ at $x = 0$ is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$$

This follows from integrating the following geometric

series from 0 to x:

$$\frac{1}{1+t} = \sum_{n=0}^{\infty}(-1)^n t^n$$

The radius of convergence of this series is 1.

4.3.5 Example

The Taylor series of $\tan^{-1} x$ at $x = 0$ is

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

This follows from integrating the following geometric

series from 0 to x:

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}$$

The radius of convergence of this series is 1.

4.3.6 Example

Find the Taylor series of $\dfrac{1}{2x+1}$ at $x = -2$.

Solution.

$$
\begin{aligned}
\frac{1}{2x+1} &= \frac{1}{2(x+2) - 4 + 1} = \frac{1}{-3 + 2(x+2)} \\
&= -\frac{1}{3}\left(\frac{1}{1 - (\frac{2}{3}(x+2))}\right) \\
&= -\frac{1}{3}\sum_{0}^{\infty}\left(\frac{2}{3}(x+2)\right)^n \\
&= \sum_{0}^{\infty}\left(-\frac{2^n}{3^{n+1}}\right)(x+2)^n
\end{aligned}
$$

The radius of convergence of this series is $3/2$.

4.3.7 Taylor polynomials

The nth order **Taylor polynomial** of f at a is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots$$

$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n$$

It provides the best polynomial approximation of degree n.
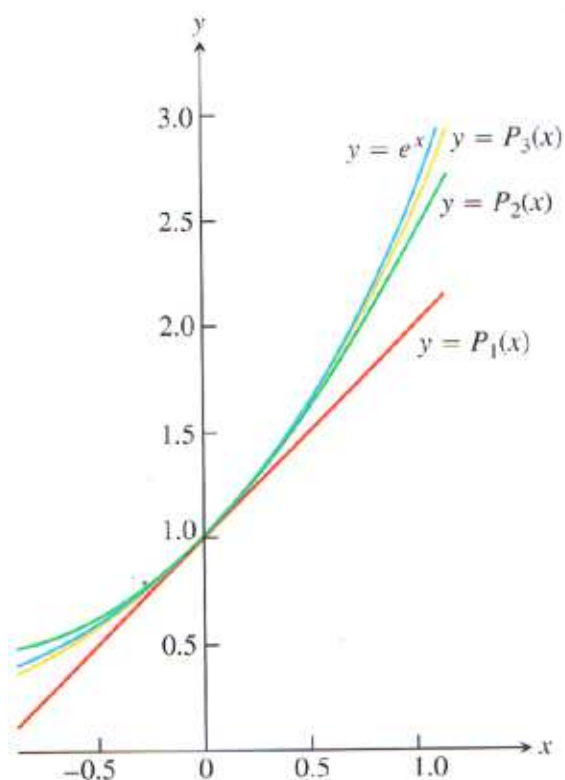
4.3.8 Example

The Taylor polynomials of e^x at $x = 0$ of order 1, 2 and 3:

$$P_1(x) = 1 + x$$

$$P_2(x) = 1 + x + \frac{x^2}{2!}$$
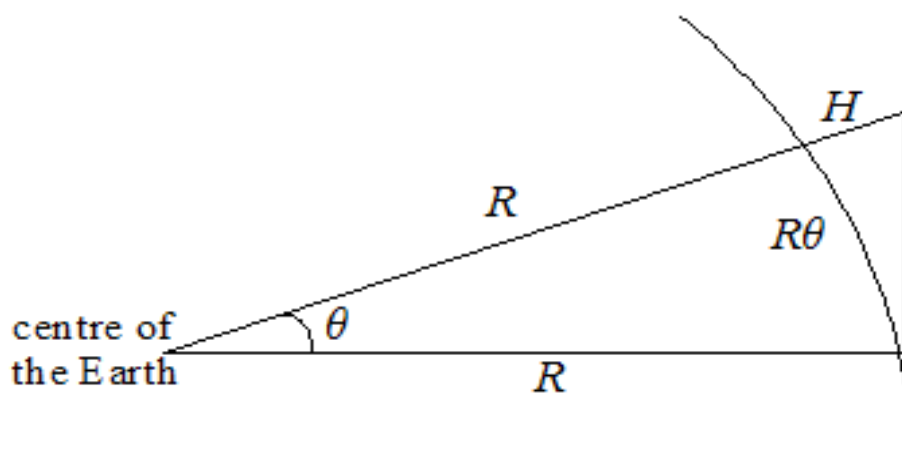
$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

In the diagram below, notice the very close agreement

between e^x and its Taylor polynomials near $x = 0$.

Note that the graph of $P_1(x)$ is in fact the tangent

line of e^x at $x = 0$.

4.3.9 An application of Taylor polynomials

Suppose you are at the top of a lighthouse, height H above sea level. How far out to sea can you see? The most distant spots are called the HORIZON.

In the diagram draw a tangent from the top of the lighthouse to the surface of the earth and complete a right-angled triangle with the centre of the earth as the third vertex. Simple trigonometry gives us

$$\frac{R}{R+H} = \cos\theta$$

and we can write this as

$$\left(1 + \frac{H}{R}\right)^{-1} = \cos\theta.$$

Now of course H/R and θ are extremely small numbers. The radius of the Earth is about 6370 km, while a very tall lighthouse might be 0.1 km tall.

For the left hand side, we approximate it with the order 1 Taylor polynomial:

$$\frac{1}{1 + \frac{H}{R}} \approx 1 - \frac{H}{R}$$

while for the right hand side, we approximate it with the order 2 Taylor polynomial (note that the order 1 Taylor polynomial of $\cos\theta$ is 1):

$$\cos\theta \approx 1 - \frac{\theta^2}{2}.$$

So we obtain

$$1 - \frac{H}{R} \approx 1 - \frac{\theta^2}{2}$$

approximately. Thus

$$R^2\theta^2 \approx 2RH$$

and so the distance to the horizon (measured along

the curved surface of the ocean) is

$$R\theta \approx (2R)^{1/2}H^{1/2} \approx 113H^{1/2}$$

where everything is measured in kilometers.

For a lighthouse 100m (=0.1km) in height, that comes

to about 35.7 km. Notice that if you double the

height of your lighthouse, you don't double the dis-

tance you can see (or from which you can be seen)!

4.3.10 Taylor's Theorem

Let $P_n(x)$ be the nth order Taylor polynomial of $f(x)$

at $x = a$. Then

$$f(x) = P_n(x) + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

The function $R_n(x)$ is called the **remainder of order n** or the **error term** for the approximation of

$f(x)$ by $P_n(x)$.

4.3.11 **Example**

The error term for the approximation of e^x by its Taylor polynomial $P_n(x)$ at $x = 0$ is:

$$R_n(x) = \frac{e^c}{(n+1)!} x^{n+1} \qquad (*)$$

for some c between 0 and x.

We can use $(*)$ to estimate the error for values of x near 0:

Say $x = 1$ and $n = 5$.

$$R_5(1) = \frac{e^c}{6!}(1)^6$$

with $0 < c < 1$.

Since e^x is an increasing function, $e^c < e^1 < 3$ and

$$R_5(1) < \frac{3}{6!} \approx 4.166 \times 10^{-3}.$$

In other words, the 5th order Taylor polynomial of e^x gives an approximate value for $e^1 = e$ with an error less than 0.005.