CH 4 *Series*

- **Sequences** $\{a_n\}$

$$a_1, a_2, a_3, \cdots, a_n, \cdots,$$

- **Series** $\sum_{n=1}^{\infty} a_n$

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

- **Power Series** $\sum_{n=0}^{\infty} c_n x^n$

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots.$$

- **Taylor Series of** f

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Sequences

- A ***sequence*** of real numbers $\{a_n\}$:

$$a_1, a_2, a_3, \cdots, a_n, \cdots$$

(i) $a_n = n - 1$ → $0, 1, 2, \cdots, n-1, \cdots$

(ii) $a_n = \frac{1}{n}$ → $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$

(iii) $a_n = (-1)^{n+1}(\frac{1}{n})$ → $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \cdots$

Limits of *Sequences*

A number L is called the limit of a sequence $\{a_n\}$, if we can make the term a_n as close to L as we like by taking n sufficiently large, i.e., a_n tends to L as n becomes larger and larger.

Write $\boxed{\lim_{n\to\infty} a_n = L}$

or $\boxed{a_n \to L}$

Here L is a real number

Convergent* or *divergent

♣ The ***limit*** of $\{a_n\}$, if it ***exists***, is ***unique***.

☺ If $\lim_{n\to\infty} a_n = L$, we say that $\{a_n\}$ is ***convergent***,

& $\{a_n\}$ ***converges to*** L.

☻ If $\lim_{n\to\infty} a_n$ ***doesn't exist***, we say that $\{a_n\}$

is ***divergent***.

Here L is a real number

Ex

(i) $a_n = n - 1$

$0, 1, 2, \cdots, n-1, \cdots$

D

Although $a_n \to \infty$

but ∞ is NOT a real number

(ii) $a_n = \frac{1}{n}$

$1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$

C

$a_n \to 0$

(iii) $a_n = (-1)^{n+1}(\frac{1}{n})$

$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \cdots$

C

$a_n \to 0$

(iv) $a_n = \frac{n-1}{n}$

$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots$

C

$a_n \to 1$

(v) $\{a_n\} = \{1, 0, 1, 0, 1, \ldots\}$

D

The sequence
does not tend to
A FIXED VALUE

4.1 *Infinite Series*

4.1.1 *Definition*

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is called an <u>infinite series</u>.

The term a_n is the <u>nth</u> <u>term</u> of the series.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$

is an infinite series whose nth term is $\frac{1}{2^n}$.

$\bigcirc\!\!\!\!Q$ $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$

$= ?$

Which of the following is true?

(i) $(1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \ldots$

$= 0 + 0 + 0 + 0 + \ldots = 0.$

(ii) $1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots$

$= 1 + 0 + 0 + 0 + \ldots = 1.$

(iii) Because of (i) & (ii), the answer should be '½'.

[Grandi (1671-1742)]

What does $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ mean?

The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \end{aligned}$$

is called the sequence of partial sums of the series.

s_n is called the nth partial sum.

☺ If $s_n \to L$,

we say that

(i) the ***series*** $a_1 + a_2 + \cdots + a_n + \cdots$

is ***convergent*** &

(ii) its ***sum*** is L; & write

$$\sum_{n=1}^{\infty} a_n \ (= \sum a_n) = a_1 + a_2 + \cdots + a_n + \cdots = L.$$

☻ If $\{ s_n \}$ is ***divergent***, we say that the ***series***

$$a_1 + a_2 + \cdots + a_n + \cdots$$

is ***divergent***.

Answer to the ***Q***

What is

$$\mathbf{1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots ?}$$

The ***sequence*** of ***partial sums***:

$$1, 0, 1, 0, 1, 0, 1, 0, \ldots,$$

is ***divergent*** and so the ***series***

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

is ***divergent***.

4.1.2 *Geometric* Series

- ***Geometric*** series:

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

a $(\neq 0)$ the 1^{st} *term*, ***r*** the *common ratio*.

Q What are the ***values*** of **‘*r*’** for which the series is ***convergent*** ?

Formula of the *n*th partial sum

$$\begin{aligned} s_n &= a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^{n-1}} \\ rs_n &= \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + \cancel{ar^{n-1}} + ar^n. \end{aligned}$$

Thus $s_n - rs_n = a - ar^n$,

and $$\boxed{s_n = a\frac{1-r^n}{1-r}, \qquad r \neq 1.}$$

Discussion (4 ***cases***)

(i) $\boldsymbol{r = 1}$

Then $s_n = a + ar + ar^2 + \cdots + ar^{n-1}$

$= \boldsymbol{na} \longrightarrow \infty$ (or $-\infty$)

i.e., the series is ***divergent***.

(ii) $\boldsymbol{r = -1}$

Then $\{ s_n \}$ is

$\boldsymbol{a}, 0, \boldsymbol{a}, 0, \cdots,$ $\qquad$ $(\boldsymbol{a - a + a - a + \cdots})$

& the series is ***divergent***.

Discussion

$$\boxed{a + ar + ar^2 + \cdots + ar^{n-1} + \cdots}$$

(iii) If $|r| > 1$, the series diverges.

Try $r = 2,\ r = -3$ *consider seq* r^n

$2^n \to \infty$ Divergent

$\{(-3)^n\} = \{(-1)^n 3^n\} = \{-3, 9, -27, 81, \ldots\}$

Divergent

$$\boxed{s_n = a\frac{1 - r^n}{1 - r}}$$

(iv) If $|r| < 1$, then $r^n \to 0$. Thus

$$s_n \to \frac{a}{1 - r},$$

and the sum of the series is $\dfrac{a}{1 - r}$.

4.1.3 ***Convergence*** of ***geometric*** **series**

$$\sum_{n=1}^{\infty} ar^{n-1} \begin{cases} = \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges if } |r| \geq 1. \end{cases}$$

4.1.4 *Example*

(i) $\dfrac{1}{9} + \dfrac{1}{27} + \dfrac{1}{81} + \cdots = \dfrac{1}{6}$ $\qquad (a = \dfrac{1}{9}, r = \dfrac{1}{3})$

(ii) $4 - 2 + 1 - \dfrac{1}{2} + \dfrac{1}{4} - \cdots$ $\qquad (a = 4, r = -\dfrac{1}{2})$

$= \dfrac{8}{3}$

4.1.5 *Rules* on *Series*

If $\sum a_n = A$, and $\sum b_n = B$, then

(1) Sum rule. $\sum(a_n + b_n) = A + B$.

(2) Difference rule. $\sum(a_n - b_n) = A - B$.

(3) Constant multiple rule. $\sum(ka_n) = kA$.

4.1.6 *Ratio Test*

Let $\sum a_n$ be a series, and let

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho.$$

Then

$$\text{the series} \begin{cases} \textbf{converges} & \text{if } \rho < 1 \\ \textbf{diverges} & \text{if } \rho > 1 \end{cases}$$

No conclusion can be drawn if $\rho = 1$

4.1.7 *Example*

♣ $$\sum \frac{(n!)^2}{(2n)!}$$

convergent

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!(n+1)!}{(2n+2)!}\frac{(2n)!}{n!n!}$$

$$= \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{1}{2}\frac{(1+1/n)}{(2+1/n)} \to \boxed{\frac{1}{4}.}$$

♣ $$\sum \frac{3^n}{2^n+5}$$

divergent

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{2^{n+1}+5}\frac{2^n+5}{3^n} = 3\cdot\frac{1+\frac{5}{2^n}}{2+\frac{5}{2^n}} \to \boxed{\frac{3}{2}.}$$

$$\boxed{\rho = 1}$$

$$\sum \frac{1}{n} \qquad \left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1} \longrightarrow 1.$$

Divergent
Proof omitted
see next slide

$$\sum \frac{1}{n^2} \qquad \left|\frac{a_{n+1}}{a_n}\right| = \frac{n^2}{(n+1)^2} \longrightarrow 1.$$

Convergent
Proof omitted
see next slide

We cannot draw conclusion from ratio test for the case $\rho = 1$

Another **Important** Series

- ***p-series***

$$\sum \frac{1}{n^p} \begin{cases} \textit{diverges} & 0 \le p \le 1 \\ \textit{converges} & p > 1 \end{cases}$$

4.2 *Power Series*

$$\sum a_n \Longrightarrow \sum f_n(x)$$

4.2.1 Power series about $x = 0$

is of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

where $c_0, c_1, \ldots, c_n, \ldots$ are constants

A ***power series*** is regarded as a ***function*** of ***x*** where it ***converges***

4.2.2 *Example*

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

Geometric Series

(a=1,r=x)

converges to $\dfrac{1}{1-x}$ when $|x| < 1$.

We state

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1.$$

4.2.3 ***Power* series** about ***x* = *a***

is of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

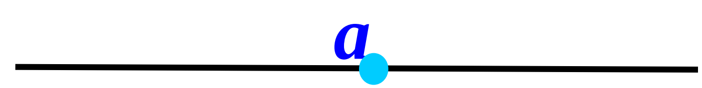
$$+ c_n(x-a)^n + \cdots .$$

'***a***' is called the ***centre*** of the ***power series.***

4.2.4 Convergence of $\sum c_n(x-a)^n$

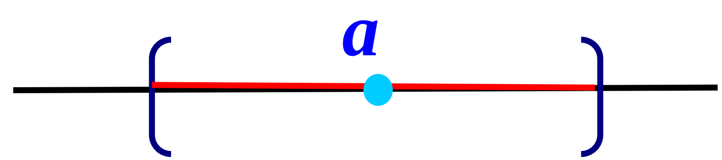
4.2.5 *Radius* of *convergence* (*R*)

(1) ***Converges* only** at ‘***a***’



***R* = 0**

(2) ***Converges*** in (***a-h***, ***a+h***) but ***diverges*** outside [***a-h***, ***a+h***] (the series ***may*** converge at ‘***a-h***’ or ‘***a+h***’)



R* = *h

(3) ***Converges*** at ***every x***

R $= \infty$

Find the radius of convergence of the power series by ***Ratio Test***

4.2.6 *Example*

(i) Find the radius of convergence of the power series

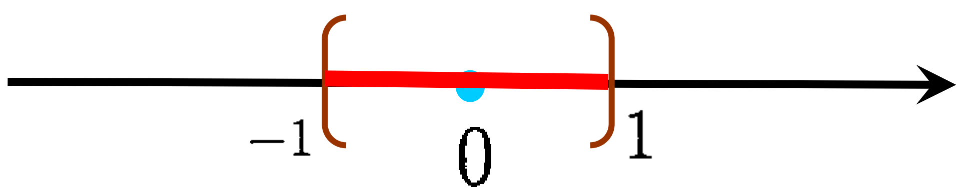
$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots .$$

We apply ratio test to the series with $\boxed{u_n = (-1)^{n-1}\frac{x^n}{n}.}$

$$\left|\frac{u_{n+1}}{u_n}\right| = \boxed{\frac{n}{n+1}|x|} \to |x| \text{ as } n \to \infty.$$

The series ***converges*** if $|x| < 1$; ***diverges*** if $|x| > 1$.

Thus, $\boldsymbol{R} = 1$.



(ii) $$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\,\frac{n!}{x^n}\right| = \frac{|x|}{n+1}$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty.$$

for any x, the limit always 0, which is less than 1

Therefore, the series ***converges*** for ***any*** *x*.

Thus, ***R*** $= \infty$.

(iii) $\displaystyle\sum_{n=0}^{\infty} n!x^n = 1+x+2!x^2+3!x^3+\cdots.$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x|$$

$$\to \infty \quad \text{as } n \to \infty$$

(***unless*** ***x*** = 0)

The series ***diverges*** for any ***x*** ***except*** ***x*** = 0.

Thus, ***R*** = 0.

4.2.7 ***Differentiation and Integration*** of Power Series

Let $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n, \quad a-h<x<a+h$

where ***h*** is the ***radius*** of convergence.

Then for $a-h<x<a+h$,

$$
\begin{aligned}
f'(x) &= \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}, \\
f''(x) &= \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}, \ldots .
\end{aligned}
$$

Examples , see Lecture Notes

Integration of Power Series

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n,\ a-h<x<a+h$$

For $\boldsymbol{a-h<x<a+h}$,

$$\int f(x)dx = \sum_{0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + c.$$

♣ $$\frac{1}{1+x} = 1-x+x^2-x^3+\cdots, \qquad -1<x<1$$

$$\boxed{\ln(1+x)} = \int \frac{dx}{1+x}$$

$$\boxed{= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots \qquad -1<x<1}$$

4.3 ***Taylor*** & ***Maclaurin*** Series

From previous slides

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1.$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots,$$

♣ Can a function ***f*** be expressed as $\sum_{n=0}^{\infty} c_n x^n$ **?**

♣ If '**yes**', what is the ***relation*** between ***f*(*x*)** & c_n **?**

4.3.1 *Definition* of *Taylor* **Series**

- Let f be a function *s.t.* the ***derivatives*** of ***all*** orders exist for all x in an open interval containing 'a'.

The **Taylor series** of f at a is

$$f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \qquad (1)$$

4.3.2 *Example*

♣ Let $f(x) = e^x$. Then $f'(x) = f''(x) = \cdots = e^x$

& $f(0) = f'(0) = f''(0) = \cdots = 1.$

Thus, the ***Taylor*** series of e^x at $x = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\boxed{\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k}$$

The radius of convergence of this series is ∞.

Note The ***Taylor*** series of *f* at '0' is called the ***Maclaurin*** series of *f*.

Taylor **(1685 – 1731)**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Maclaurin **(1698 – 1746)**

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

4.3.3-4.3.6 *Example*

♣ $f(x) = \sin x.$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\begin{array}{ll}
f(x) = \sin x & f(0) = 0 \\
f'(x) = \cos x & f'(0) = 1 \\
f''(x) = -\sin x & f''(0) = 0 \\
f^{(3)}(x) = -\cos x & f^{(3)}(0) = -1 \\
f^{(4)}(x) = \sin x & f^{(4)}(0) = 0 \\
\vdots & \vdots
\end{array}$$

$$f^{(2k)}(0) = 0 \quad \& \quad f^{(2k+1)}(0) = (-1)^k$$

The ***Maclaurin*** series of **sin *x*** is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}}$$

Maclaurin series

- $$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1.$$

- $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \qquad -1 < x < 1$$

- $$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots \qquad -1 < x < 1$$

- $$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots, \quad -1 < x < 1$$

- $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \qquad -\infty < x < \infty$$

- $$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \qquad -\infty < x < \infty$$

♣ An application – *Evaluate* $\int_0^1 \sin(x^2)\, dx$

(This integral arises in the study of light diffraction.)

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$$

$$\begin{aligned}\int_0^1 \sin(x^2)\, dx &= \int_0^1 (x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots)\, dx \\ &= (\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \dots)\Big|_0^1 \\ &= \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \dots \\ &\approx \mathbf{0.31026.}\end{aligned}$$

Taylor series of $\frac{1}{2x+1}$ at $x = -2$

- $$\sum_{k=0}^{\infty} \boxed{\frac{f^{(k)}(a)}{k!}} (x - \boxed{a})^k$$

- $$\frac{1}{1-\boxed{x}} = 1 + \boxed{x} + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1.$$

$$\frac{1}{2x+1} = \boxed{\frac{q}{1-p(x+2)}} \quad ?$$

Taylor series of $\dfrac{1}{2x+1}$ at $x = -2$

$$\frac{1}{2x+1} = \frac{1}{2(x+2)-3}$$

$$\frac{q}{1-p(x+2)}$$

$$= \left(-\frac{1}{3}\right) \cdot \frac{1}{1-\boxed{\frac{2}{3}(x+2)}}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots ,$$
$$-1 < x < 1$$

$$= \left(-\frac{1}{3}\right) \sum_{n=0}^{\infty} \left(\frac{2}{3}(x+2)\right)^n \quad = \sum_{n=0}^{\infty} \left(-\frac{2^n}{3^{n+1}}\right)(x+2)^n$$

$$\boxed{\left|\frac{2}{3}(x+2)\right| < 1} \Leftrightarrow |x+2| < \frac{3}{2}$$

$$R = \frac{3}{2}$$

4.3.7 ***Taylor*** Polynomials

The *n*th order ***Taylor polynomial*** of *f* at '*a*':

$$P_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

It gives a good ***polynomial approximation*** of order *n*.

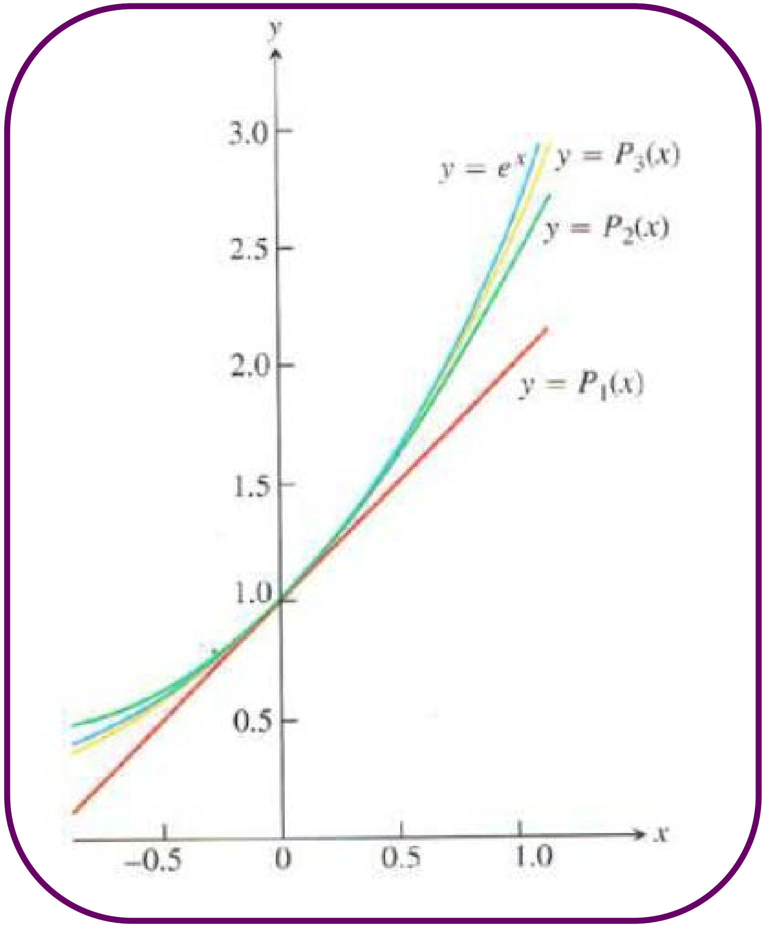
4.3.8 *Example*

The Taylor polynomials of $\boxed{e^x}$ at x = 0 of order 1, 2 and 3 :

$$P_1(x) = 1 + x$$

$$P_2(x) = 1 + x + \frac{x^2}{2!}$$

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$



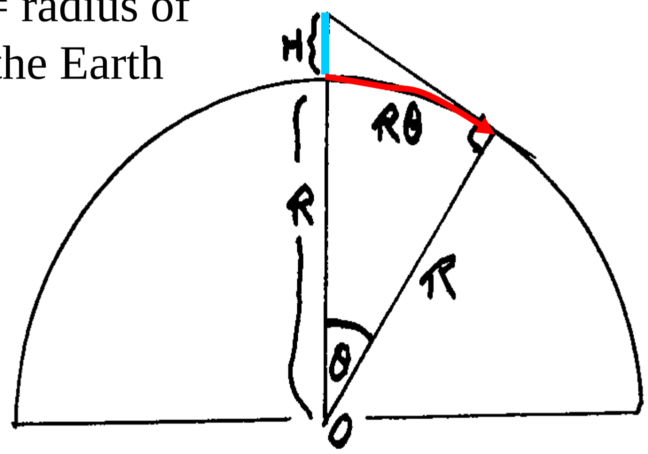
4.3.9 *Application*

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

You are at the top of a lighthouse, height *H* above sea level. How far out to sea can you see ?

To find: $R\theta$

R= radius of the Earth



$$\cos\theta = \frac{R}{R+H} = \frac{1}{1+\frac{H}{R}}$$

$$1 - \frac{\theta^2}{2} \approx 1 - \frac{H}{R}$$

$$R^2\theta^2 \approx 2RH$$

$$R\theta \approx \sqrt{2RH}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

4.3.10 *Taylor's Theorem*

Let $P_n(x)$ be the nth order Taylor poly of

$f(x)$ at $x = a$

Then $\quad f(x) = P_n(x) + R_n(x)$

where
$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x

$R_n(x)$ called remainder of order n or error term

4.3.11 *Example*

Let $f(x) = e^x$

Error term for the approximation of f(x) by $P_n(x)$ at x=0 is

$$R_n(x) = \frac{e^c}{(n+1)!}x^{n+1} \qquad (*)$$

for some c between 0 and x

We can use (*) to estimate the error

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + R_5(x)$$

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + R_5(1)$$

$$R_5(1) = \frac{e^c}{6!}(1)^6 \text{ where } 0 < c < 1$$

$$0 < R_5(1) = \frac{e^c}{6!}(1)^6 < \frac{3}{6!} \approx 4.166 \times 10^{-3} < 0.005$$

If we use $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$ to esimate e

then the error is less than 0.005

END

Appendix

Another way

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots, \qquad -1 < t < 1$$

$$\boxed{\begin{aligned} \ln(1+x) &= \int_0^x \frac{dt}{1+t} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots, \\ & -1 < x < 1. \end{aligned}}$$

The ***Maclaurin*** series of arc tan x

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots \qquad -1 < x < 1$$

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}$$

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\,dt \;\; = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^{2n} \right) dt$$

$$\boxed{\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad -1 < x < 1}$$

♣ Let $f(x) = \ln(1 + x + x^2)$ & $\sum_{n=0}^{\infty} c_n x^n$

be the Taylor series of f at $x = 0$.

Find $\boxed{c_{2010} + c_{2011}.}$

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

$$f(x) = \ln(1 + x + x^2) = \ln\left(\frac{1 - x^3}{1 - x}\right)$$

$$= \ln(1 - x^3) - \ln(1 - x)$$

$$= \ln(1 + (-x^3)) - \ln(1 + (-x))$$

$$= (-x^3) - \frac{(-x^3)^2}{2} + \frac{(-x^3)^3}{3} - \cdots\cdots$$

$$- \left((-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \cdots\cdots\right)$$

$$\boxed{\begin{gathered}\ln(1 + x) = \\ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots \\ -1 < x < 1\end{gathered}}$$

- $$= -\left(x^3 + \frac{x^6}{2} + \frac{x^9}{3} + \cdots\cdots\right)$$

$$+ \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\cdots\right).$$

Note that $\boxed{2010 = 3 \cdot 670.}$

Thus, $c_{2010} + c_{2011}$

$$= -\frac{1}{670} + \frac{1}{2010} + \frac{1}{2011}.$$

Final Exam (08/09, Sem 1)

Question 2 (b) [5 marks]

A car is moving with speed 20 m/s and acceleration k m/s^2 at a given instant. The car is observed to have moved a distance of 29 m in the next second. Using a second degree Taylor polynomial, estimate the value of k.

We may assume that the car is at the origin with $t=0$ when $v=20$ m/s and acceleration $=k$ m/s^2.

Let $x=$ distance from origin at time t.

$$\therefore \quad \frac{dx}{dt}(0)=20, \quad \frac{d^2x}{dt^2}(0)=k$$

$$\therefore \quad x \simeq 0+20t+\frac{k}{2!}t^2 = 20t+\frac{k}{2}t^2$$

$x=29$ when $t=1 \Rightarrow 29 = 20+\frac{k}{2}$

$$\Rightarrow k = \underline{\underline{18}}$$

More Examples

9. Evaluate the sum

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+2)}.$$

(Hint: Integrate the Taylor series of xe^{-x}.)

10. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \left(\frac{1}{3^n + (-2)^n}\right) \frac{x^n}{(n+1)}.$$

$$\sum \frac{1}{n}$$

Harmonic series

$$\sum \frac{1}{n} = 1 + (\frac{1}{2}) + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + \ldots$$

$$> 1 + (\frac{1}{2}) + (\frac{1}{2}) + (\frac{1}{2}) + \ldots$$

Divergent !