Chapter 10. Surface Integrals

10.1 Parametric Surfaces

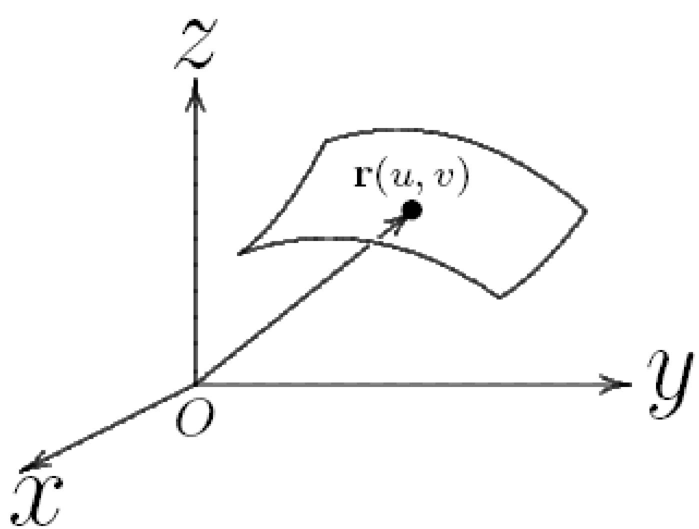
A **parametric representation** of a surface is given by the two-variable vector function

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad (1)$$

where u and v are two independent parameters.

The collection of points with position vectors (1) form a surface in the xyz-space.

The equations $x = x(u, v), y = y(u, v), z = z(u, v)$ are called the **parametric equations** of the surface.


z
r(u, v)
y
O
x

10.1.1 Example (Planes)

For a general plane $ax + by + cz = d$, we can let two of the three components be u and v and obtain the remaining component in terms of u and v using the above equation.

E.g. $3x+2y-4z=6$: Let $x(u,v)=u,\ y(u,v)=v$.

Then $z(u,v) = \frac{1}{4}(3x+2y-6)$. So the parametric representation of this plane is

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + \left(\frac{1}{4}(3u+2v-6)\right)\mathbf{k}.$$

If one variable is absent from the equation, we let the missing component be u or v.

E.g. $2y + x = 7$: Let $z(u, v) = u$. Then $y(u, v) = v$ and $x(u, v) = 7 - 2v$.

$$\mathbf{r}(u, v) = (7 - 2v)\mathbf{i} + v\mathbf{j} + u\mathbf{k}.$$

If two variables are absent from the equation, we let the two missing components be u or v.

E.g. The xy-plane is given by

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + 0\mathbf{k}.$$

10.1.2 Example (Surfaces of the form $z = f(x, y)$)

A natural parametric representation of S is

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}$$

E.g. The paraboloid $z = x^2 + y^2$.

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (u^2 + v^2)\mathbf{k}.$$

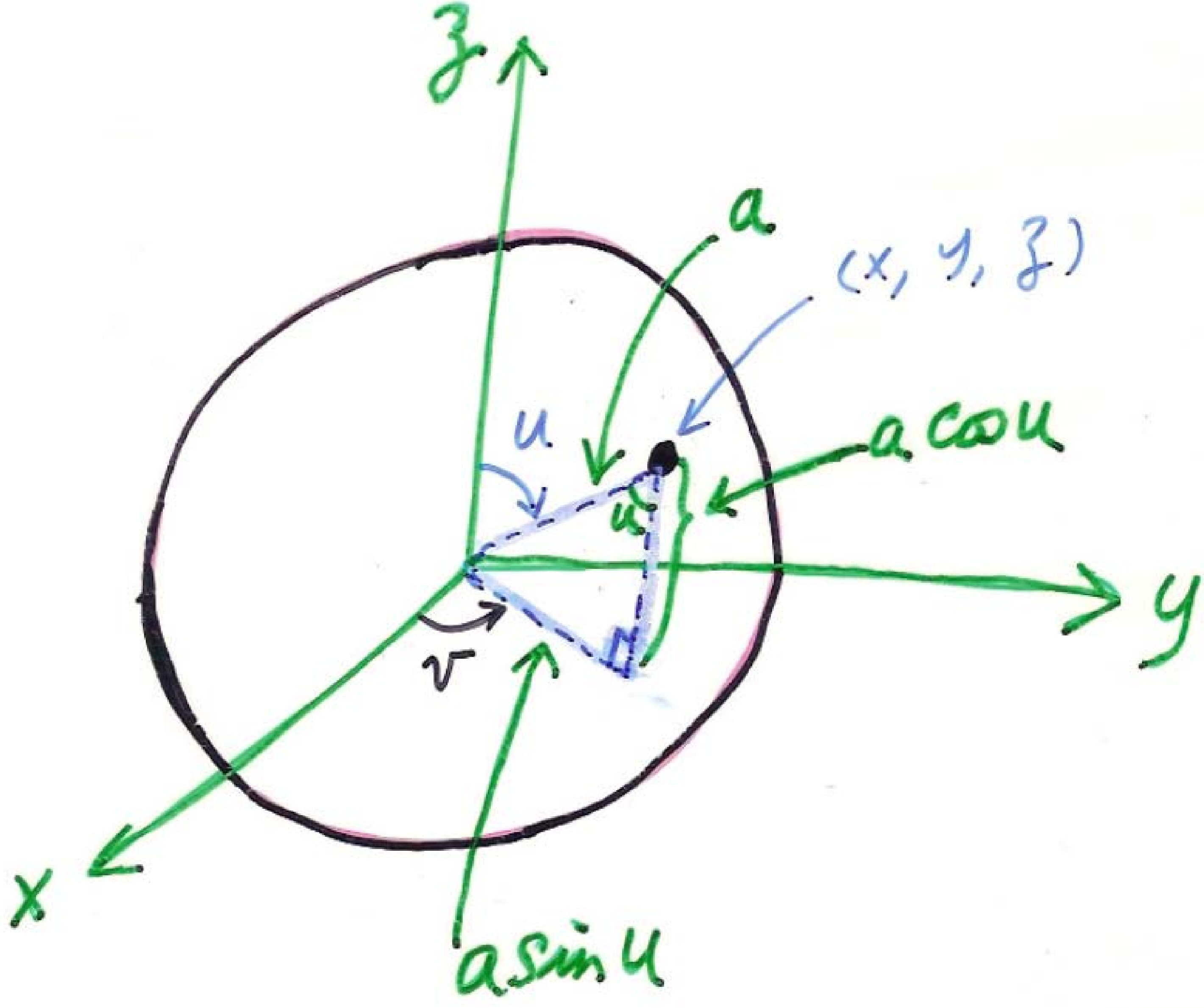
E.g. The upper cone $z = \sqrt{x^2 + y^2}$.

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{u^2 + v^2}\mathbf{k}.$$

10.1.3 Example (Spheres)

We have a standard parametric representation for a sphere $x^2 + y^2 + z^2 = a^2$ of radius a centered at the origin:

$$\mathbf{r}(u, v) = (a \sin u \cos v)\mathbf{i} + (a \sin u \sin v)\mathbf{j} + (a \cos u)\mathbf{k}.$$



z
a
(x, y, z)
u
u
a cos u
y
v
x
a sin u

$$\begin{cases} x = a\sin u\cos v \\ y = a\sin u\sin v \\ z = a\cos u \end{cases} \qquad \begin{array}{l} 0 \le u \le \pi \\ 0 \le v \le 2\pi \end{array}$$

$$\vec{r}(u,v) = a\sin u\cos v\,\vec{i} + a\sin u\sin v\,\vec{j} + a\cos u\,\vec{k}$$

E.g. When $0 \leq u \leq \pi, \quad 0 \leq v \leq 2\pi$, the representation gives the full sphere.

When $0 \leq u \leq \pi/2, \quad 0 \leq v \leq 2\pi$, the representation gives the upper hemisphere.

e.g. $\frac{\pi}{2} \le u \le \pi, \quad 0 \le v \le 2\pi$

gives the lower hemi-sphere.

e.g. $0 \le u \le \frac{\pi}{2}, \quad 0 \le v \le \frac{\pi}{2}$

gives the part of the sphere in the first octant.

10.1.4 Example (Circular cylinders)

We have a standard parametric representation for circular cylinder $x^2 + y^2 = a^2$ about the z-axis:

$$\mathbf{r}(u, v) = (a \cos u)\mathbf{i} + (a \sin u)\mathbf{j} + v\mathbf{k}.$$

Here u measures the angle from the positive x-axis (about the z-axis) while v measures the height from the xy-plane along the cylinder.

Similarly, for $x^2+z^2 = a^2$ and $y^2+z^2 = a^2$ (cylinders about y- and x-axes resp.), we have respectively

$$\mathbf{r}(u, v) = (a \cos u)\mathbf{i} + v\mathbf{j} + (a \sin u)\mathbf{k}$$

and

$$\mathbf{r}(u, v) = v\mathbf{i} + (a \cos u)\mathbf{j} + (a \sin u)\mathbf{k}.$$

10.1.5 Tangent planes and normal vectors

Let S be a surface given by the parametric representation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \qquad (2)$$

We shall find the equation of the tangent plane to S at a point P_0 with position vector $\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$.

Let us fix $v = v_0$ in (2) above.

Then the vector equation

$$\mathbf{r}(u, v_0) = x(u, v_0)\mathbf{i} + y(u, v_0)\mathbf{j} + z(u, v_0)\mathbf{k}$$

represents a space curve C_1 on S passing through the point P_0.

The tangent vector to C_1 at P_0 is given by $\dfrac{d}{du}\mathbf{r}(u, v_0)\,|_{u=u_0}$,

which is simply

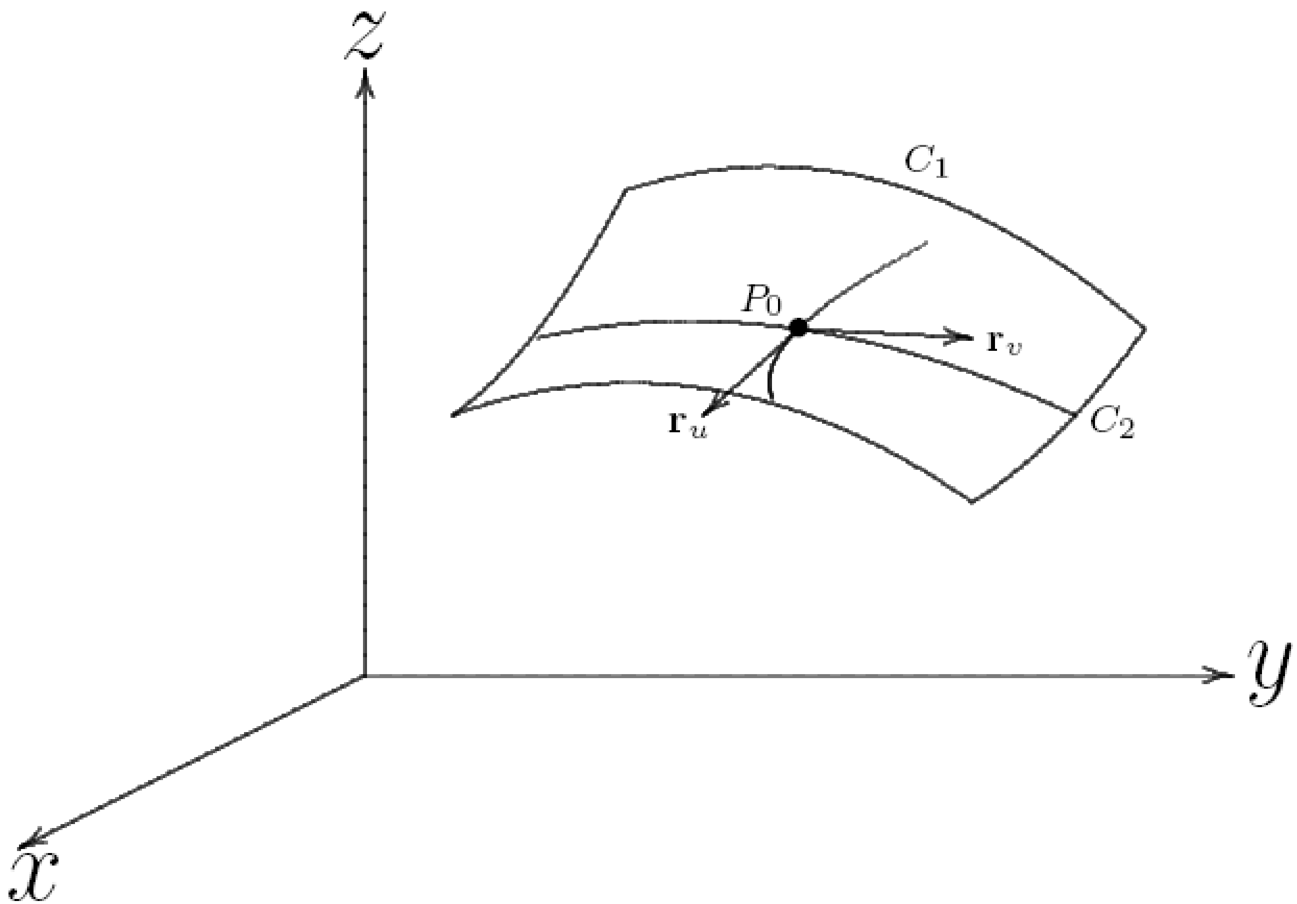
$$\mathbf{r}_u \equiv \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$

Similarly, if we fix $u = u_0$ in (2), we get another space curve C_2 with vector equation

$$\mathbf{r}(u_0, v) = x(u_0, v)\mathbf{i} + y(u_0, v)\mathbf{j} + z(u_0, v)\mathbf{k}.$$

The tangent vector to C_2 at P_0 is given by

$$\mathbf{r}_v \equiv \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$



z
C_1
P_0
$\mathbf{r}_v$
$\mathbf{r}_u$
C_2
y
x

Both vectors $\mathbf{r}_u$ and $\mathbf{r}_u$ lie in the tangent plane to S at P_0. Therefore, the cross product $\mathbf{r}_u \times \mathbf{r}_v$, assuming it is nonzero, provides a normal vector to the tangent plane to S at P_0. Therefore, $(\mathbf{r} - \mathbf{r}_0) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 0$ is the equation of the tangent plane.

10.1.6 Example

Find the equation of the tangent plane to the surface

with parametric representation

$$\mathbf{r}(u,v) = u\mathbf{i} + v^2\mathbf{j} + (u^2 - v)\mathbf{k}$$

at the point $(1, 4, -1)$.

Solution:

To find the tangent plane to

$$S: \quad \vec{r}(u,v) = u\vec{i} + v^2\vec{j} + (u^2 - v)\vec{k}$$

at the point $(1, 4, -1)$.

$$\vec{r}_u = \vec{i} + 0\vec{j} + 2u\vec{k}$$

$$\vec{r}_v = 0\vec{i} + 2v\vec{j} - \vec{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 2v & -1 \end{vmatrix} = -4uv\vec{i} + \vec{j} + 2v\vec{k}$$

at $(1, 4, -1)$, we have $u=1$, $v=2$

$$\therefore \vec{r}_u \times \vec{r}_v = -8\,\vec{i} + \vec{j} + 4\vec{k}$$

$\therefore$ Equation of tangent plane is

$$-8x + y + 4z = (1, 4, -1) \cdot (-8, 1, 4)$$
$$= -8 + 4 - 4 = -8$$

i.e. $$\underline{\underline{-8x + y + 4z + 8 = 0}}$$

10.1.7 Example

If S has Cartesian equation $z = f(x, y)$. Then a parametric representation of S is

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}.$$

Thus, $\mathbf{r}_u = \mathbf{i} + 0\mathbf{j} + f_u\mathbf{k}$ and $\mathbf{r}_v = 0\mathbf{i} + \mathbf{j} + f_v\mathbf{k}$.

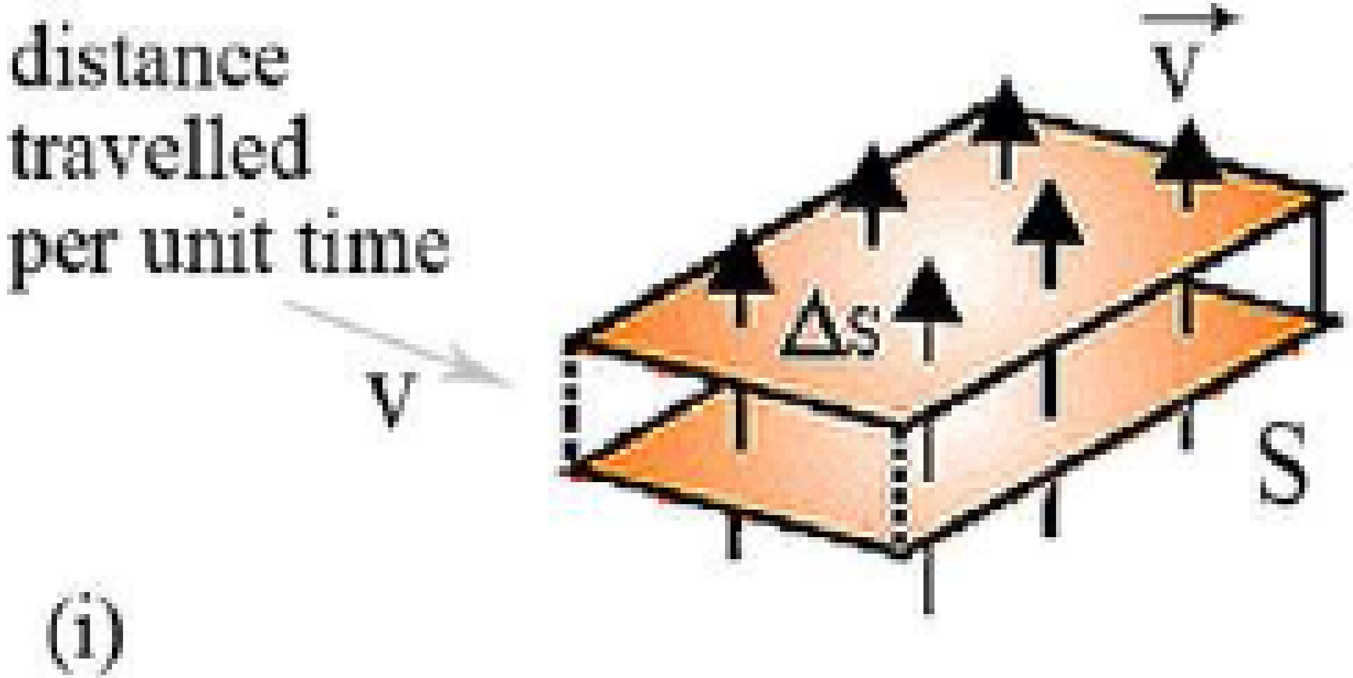
So the normal vector $\mathbf{r}_u \times \mathbf{r}_v = -f_u\mathbf{i} - f_v\mathbf{j} + 1\mathbf{k}$.

10.2 Surface Integrals

Similar to line integrals, surface integrals involve integration over some (bounded) surfaces. Suppose S is a surface and imagine a fluid with velocity $\mathbf{v}$ flows through S. We wish to calculate the total volume of fluid flowing out of S per unit time.

Case (i): The fluid velocity is constant over flat surface S and its direction is perpendicular to S. Then the volume flow rate is given by distance traveled per unit time multiplied with the area of S:

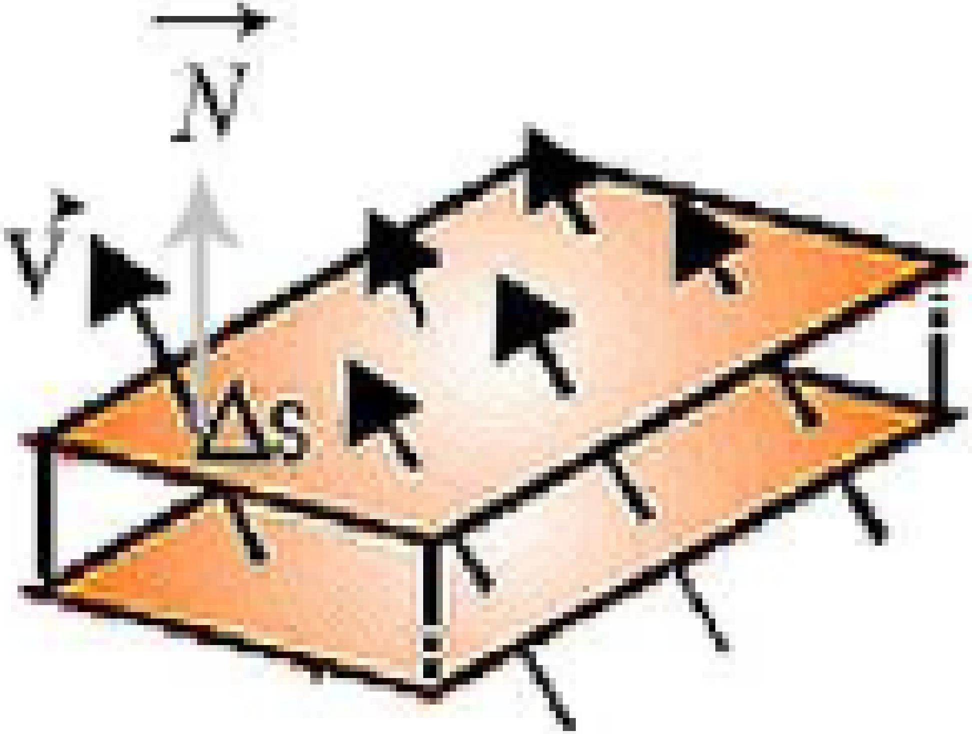
$$w = \|\mathbf{v}\|\Delta s.$$


distance
travelled
per unit time
v
$\vec{v}$
Δs
S
(i)

Case (ii): The fluid velocity is constant over flat surface S but its direction is not perpendicular to S. Then the volume flow rate is given by

$$w = \mathbf{v} \cdot \mathbf{N} \Delta s$$

where $\mathbf{N}$ is the unit normal vector to S.



(ii)

Case (iii): The fluid velocity is changing over curved surface S. We can divide up the surface into small segments and then sum the volume flow rate of the individual segments to get the total flow rate. In a particular segment, we have

$$w_i \approx \mathbf{v}(x_i, y_i, z_i) \cdot \mathbf{N_i} \Delta s_i.$$

So the total flow rate is approximately

$$w \approx \sum_{1}^{n} \mathbf{v}(x_i, y_i, z_i) \cdot \mathbf{N_i} \Delta s_i \tag{3}$$

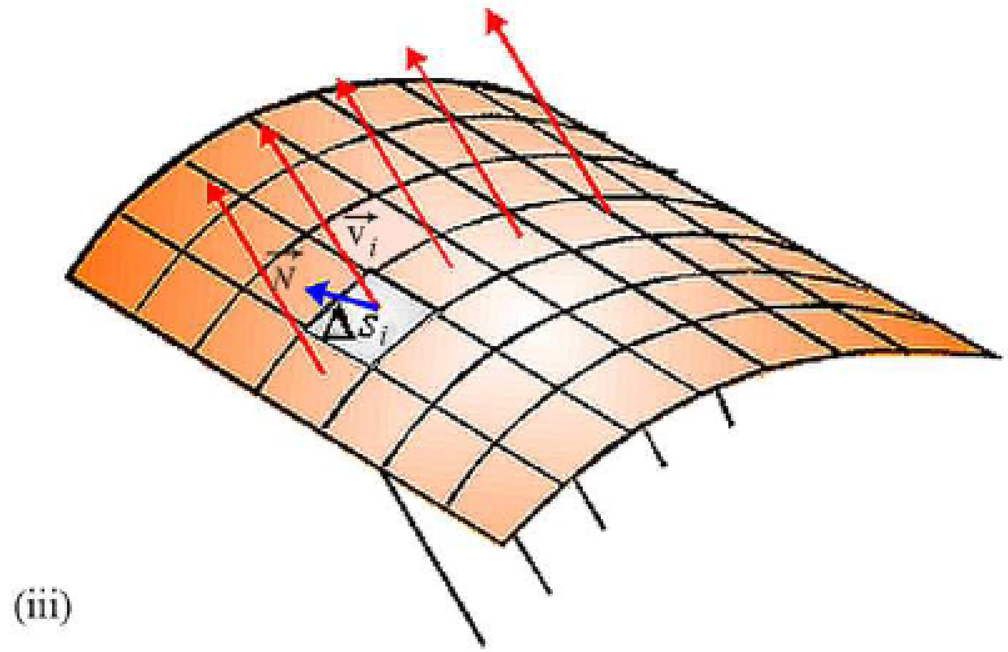
If we let n goes to infinity, the RHS of (3) becomes

an integral

$$\iint_S \mathbf{v}(x, y, z) \cdot \mathbf{N} ds$$

which represents the actual total volume flow rate.

This integral is called a surface integral of the vector field $\mathbf{v}$.



General case (velocity is changing on a curved surface)

There are two types of surface integrals, one for scalar functions and the other for vector fields.

10.2.1 Surface integrals of scalar functions

Let $f(x, y, z)$ be a function defined on a (bounded) surface S. Then for the parametric representation $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ of S, the corresponding set of ordered pairs (u, v) come from a bounded domain D.

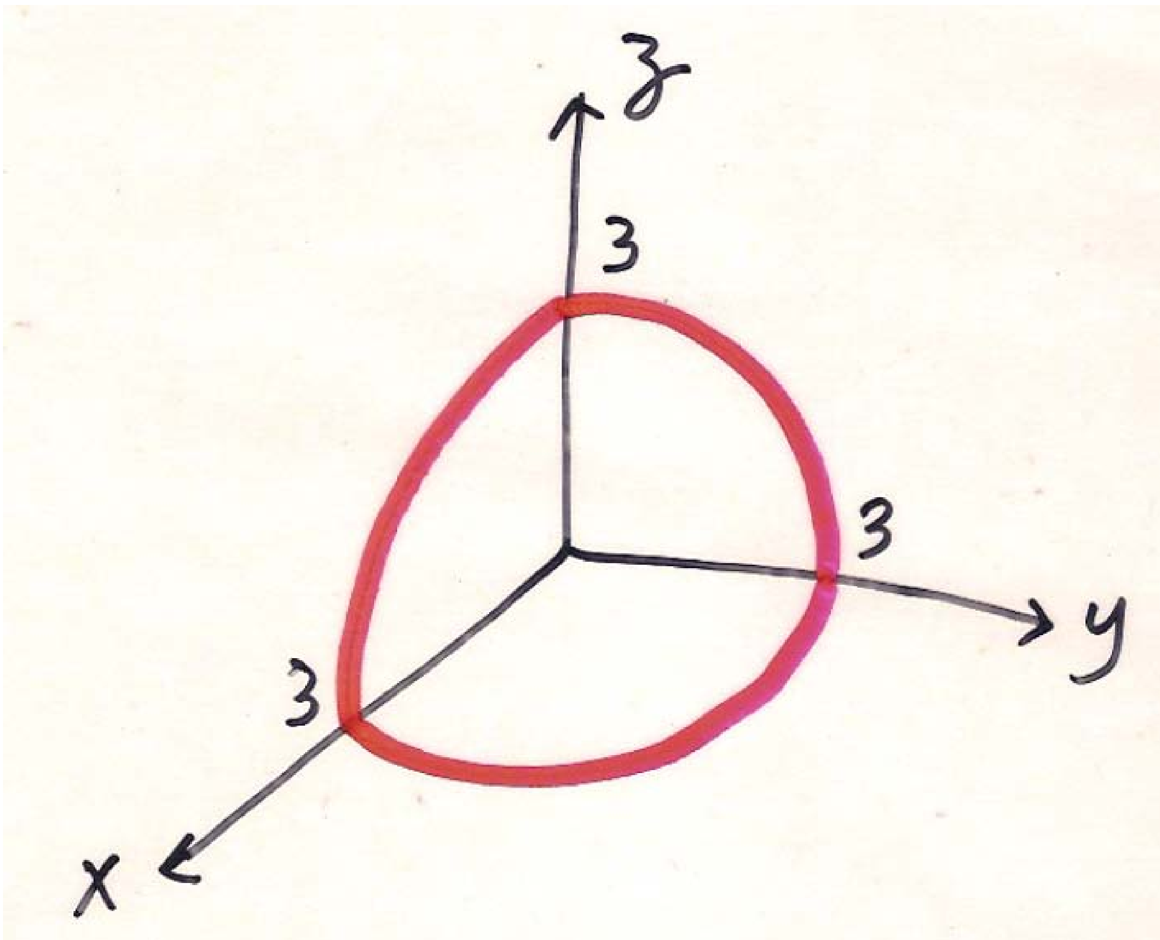
The **surface integral of a scalar function** f over S is

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA.$$

The RHS of the above equation is a double integral over a domain D. Usually, D can be described by giving the ranges of u and v.

10.2.2 Example

Evaluate $\iint_S (xz + yz)\, dS$, where S is part of the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.


z
3
3
y
3
x

Solution: A parametric representation of the sphere is given by (see Example 10.1.3)

$$\mathbf{r}(u, v) = 3 \sin u \cos v \mathbf{i} + 3 \sin u \sin v \mathbf{j} + 3 \cos u \mathbf{k}.$$

To represent the first octant, the domain D is given by $0 \leq u \leq \pi/2$ and $0 \leq v \leq \pi/2$.

$$
\begin{aligned}
\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\cos u \cos v & 3\cos u \sin v & -3\sin u \\ -3\sin u \sin v & 3\sin u \cos v & 0 \end{vmatrix} \\
&= 9\sin^2 u \cos v\mathbf{i} + 9\sin^2 u \sin v\mathbf{j} + 9\sin u \cos u\mathbf{k}.
\end{aligned}
$$

Therefore, $\|\mathbf{r}_u \times \mathbf{r}_v\| = 9\sin u$.

The surface integral is given by

$$\iint_S (xz + yz)\, dS$$

$$= \iint_D (9 \sin u \cos u \cos v + 9 \sin u \cos u \sin v) \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} 81 \sin^2 u \cos u (\cos v \ + \sin v) du dv$$

$$=81 \int_0^{\pi/2} \sin^2 u \cos u \ du \int_0^{\pi/2} (\cos v + \sin v) \ dv$$

$$=81 (\left[\frac{1}{3} \sin^3 u \right]_0^{\pi/2})(2) = 54.$$

Some additional details:

$$\int_0^{\frac{\pi}{2}} \sin^2 u \cos u \, du = \int_0^{\frac{\pi}{2}} \sin^2 u \, d(\sin u)$$

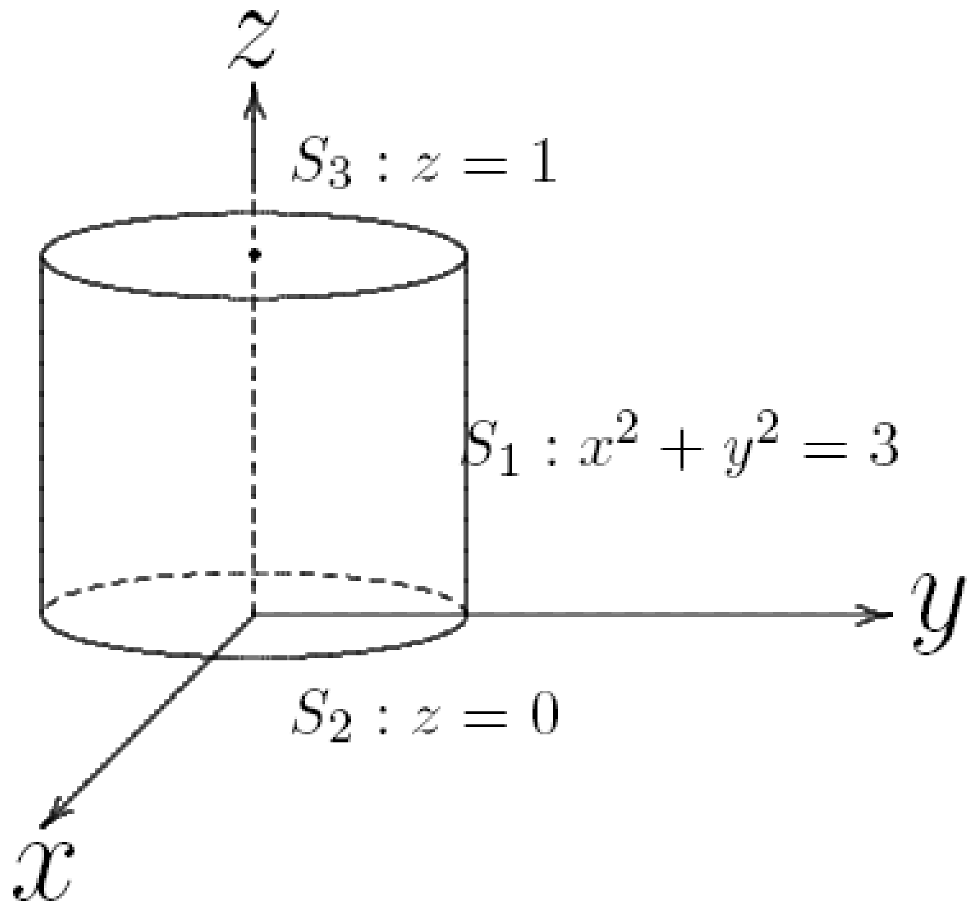
$$= \frac{1}{3}\left[\sin^3 u\right]_0^{\frac{\pi}{2}} = \frac{1}{3}$$

$$\int_0^{\frac{\pi}{2}} (\cos v + \sin v)\, dv$$

$$= \left[\sin v - \cos v\right]_0^{\frac{\pi}{2}} = 1 - (-1) = 2$$

10.2.3 Example

Evaluate $\iint_S z\, dS$, where S is the closed surface bounded laterally by S_1: the cylinder $x^2 + y^2 = 3$; bounded below by S_2: the xy-plane and bounded on top by S_3: the horizontal plane $z = 1$.



z
$S_3 : z = 1$
$S_1 : x^2 + y^2 = 3$
y
$S_2 : z = 0$
x

Solution: The surface integral is the sum of three surface integrals:

$$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS.$$

The surface S_1 is part of a circular cylinder. By Example 10.1.4, it has a parametric representation

$\mathbf{r}(u, v) = \sqrt{3}\cos u\mathbf{i} + \sqrt{3}\sin u\mathbf{j} + v\mathbf{k}.$

Thus, $\mathbf{r}_u \times \mathbf{r}_v = \sqrt{3}\cos u\mathbf{i} + \sqrt{3}\sin u\mathbf{j} + 0\mathbf{k}$

and $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{3}$.

Since S_1 is a full cylinder, the range of u is given by

$0 \leq u \leq 2\pi$.

Moreover, S_1 is bounded above by the plane $z = 1$

and below by $z = 0$, so the range of v is given by

$0 \leq v \leq 1$.

Therefore,

$$\begin{aligned}
\iint_{S_1} z\,dS &= \iint_D v\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA \\
&= \int_0^{2\pi}\int_0^1 \sqrt{3}v\,dv\,du = \int_0^{2\pi} \frac{\sqrt{3}}{2}du \\
&= \sqrt{3}\pi.
\end{aligned}$$

S_2 is on the xy-plane, so we have $z = 0$. Thus the integrand of $\iint_{S_2} z\,dS$ is zero so that the integral has value zero. Therefore,

$$\iint_{S_2} z\,dS = 0.$$

The surface S_3 is on the horizontal plane $z = 1$. Thus

$$\iint_{S_3} z\, dS = \iint_{S_3} dS = \text{area of } S_3 = \pi(\sqrt{3})^2 = 3\pi.$$

Consequently,

$$\iint_{S} z\, dS = \iint_{S_1} z\, dS + \iint_{S_2} z dS + \iint_{S_3} z dS = (3+\sqrt{3})\pi.$$

10.2.4 Surface integrals of vector fields

Let $\mathbf{F}$ be a continuous vector field defined on a surface S with a unit normal vector $\mathbf{n}$. We have seen at the beginning of this section that the **surface integral** of $\mathbf{F}$ over S is $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$. We usually simplify the notation as

$$\iint_S \mathbf{F}\cdot d\mathbf{S}$$

This integral is also called the **flux** of $\mathbf{F}$ over S as it is related to the volume flow rate of fluid.

If S is given by the parametric representation $\mathbf{r} = \mathbf{r}(u, v)$ with domain D,

then

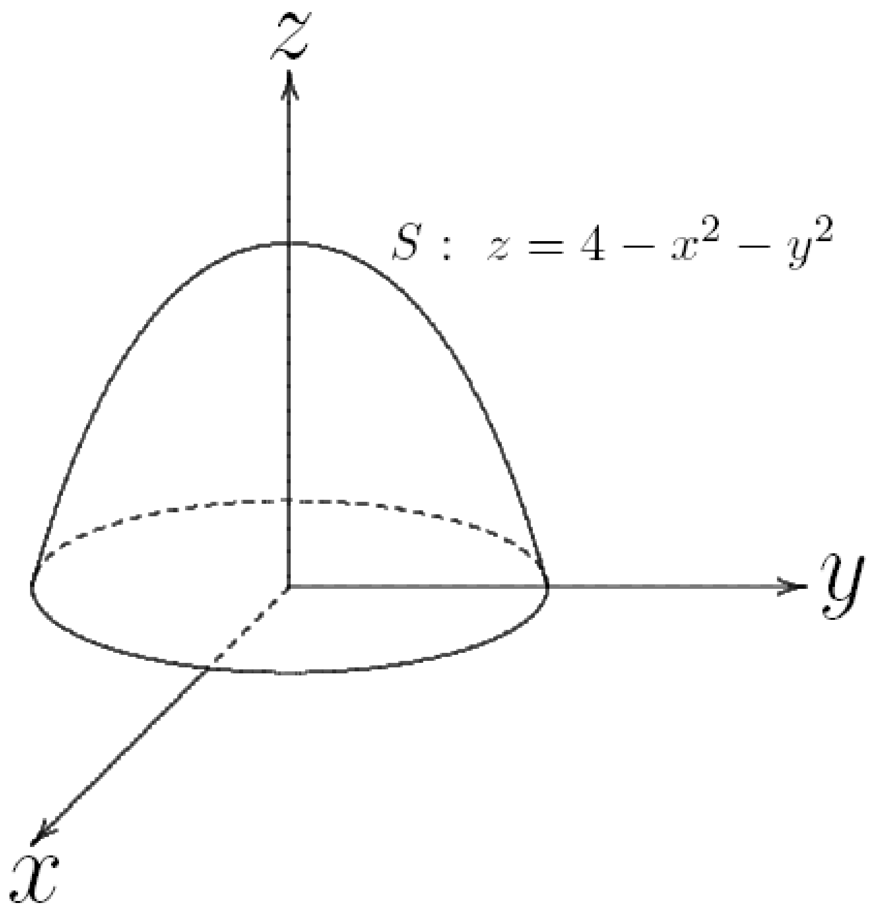
$$
\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \, dS \\
&= \iint_D \left[\mathbf{F}(\mathbf{r}(u,v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \right] \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA \\
&= \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA.
\end{aligned}
$$

Therefore,

$$
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA
$$

10.2.5 Example

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + xy\mathbf{k}$, and S is the part of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane.



z
$S: \; z = 4 - x^2 - y^2$
y
x

Solution: Since S has Cartesian equation $z = 4 - x^2 - y^2$, the parametric representation is

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (4 - u^2 - v^2)\mathbf{k}.$$

The region D is then the projection onto the xy-plane, which is the disk of radius 2.

We have $\mathbf{r}_u = \mathbf{i} + 0\mathbf{j} - 2u\mathbf{k}$, $\mathbf{r}_v = 0\mathbf{i} + \mathbf{j} - 2v\mathbf{k}$ and

$\mathbf{r}_u \times \mathbf{r}_v = 2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k}$.

Therefore,

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA \\
&= \iint_D (u\mathbf{i} + v\mathbf{j} + uv\mathbf{k}) \cdot (2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k})\, dA \\
&= \iint_D (2u^2 + 2v^2 + uv)\, dA
\end{aligned}$$

Change to Polar Coordinates

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^2 \left(2r^2 + \frac{r^2 \sin 2\theta}{2}\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2} r^4 + \frac{1}{8} r^4 \sin 2\theta\right]_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} (8 + 2\sin 2\theta)\, d\theta$$

$$= \left[8\theta - \cos 2\theta \right]_0^{2\pi}$$

$$= \underline{\underline{16\pi}}$$

10.2.6 Example

Let $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$. Evaluate $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$,

where S is the sphere $x^2 + y^2 + z^2 = 1$.

Solution: A parametric representation of the unit sphere is given by

$$\mathbf{r}(u,v) = \sin u \cos v\mathbf{i} + \sin u \sin v\mathbf{j} + \cos u\mathbf{k},$$

with D given by $0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$.

We have

$$\mathbf{r}_u \times \mathbf{r}_v = \sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \sin u \cos u\mathbf{k}$$

and

$$\mathbf{F}(\mathbf{r}(u, v)) = \sin u \sin v\mathbf{i} + \sin \ \ u \cos v\mathbf{j} + \cos u\mathbf{k}.$$

Thus,

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 2 \sin^3 u \sin v \cos v + \sin u \cos^2 u.$$

Therefore,

$$
\begin{aligned}
&\iint_S \mathbf{F} \cdot d\mathbf{S} \\
&= \int_0^{2\pi} \int_0^{\pi} (2 \sin^3 u \sin v \cos v + \sin u \cos^2 u) \, du dv \\
&= \int_0^{\pi} \sin^3 u \, du \int_0^{2\pi} \sin 2v \, dv + \int_0^{\pi} \sin u \cos^2 u \, du \int_0^{2\pi} \, dv \\
&= 4\pi/3.
\end{aligned}
$$

Some additional details:

$$\int_0^{2\pi} \sin 2v \, dv = -\frac{1}{2}\Big[\cos 2v\Big]_0^{2\pi} = 0$$

$$\int_0^{\pi} \sin u \cos^2 u \, du = -\int_0^{\pi} \cos^2 u \, d(\cos u)$$

$$= -\frac{1}{3}\Big[\cos^3 u\Big]_0^{\pi} = \frac{2}{3}$$

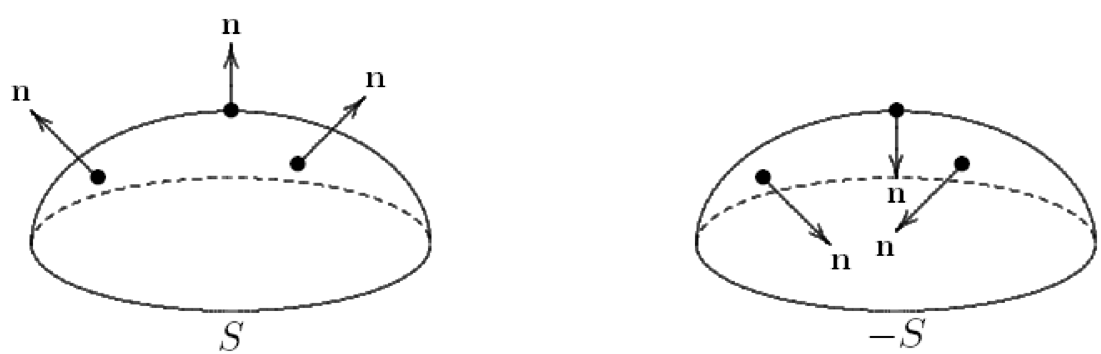
10.2.7 Orientation of surfaces

Note that, in the above example, if we switch the order of u and v, then

$$\mathbf{r}_v \times \mathbf{r}_u = -\mathbf{r}_u \times \mathbf{r}_v$$

and the surface integral will be evaluated to $-4\pi/3$.

Therefore, for surface integral of a vector field, the value depends on the choice of the normal vector, which is known as the **orientation** of the surface.



If S is a surface given in parametric form by $\mathbf{r} = \mathbf{r}(u, v)$, then the normal vector $\mathbf{r}_u \times \mathbf{r}_v$ automatically supply an orientation to S.

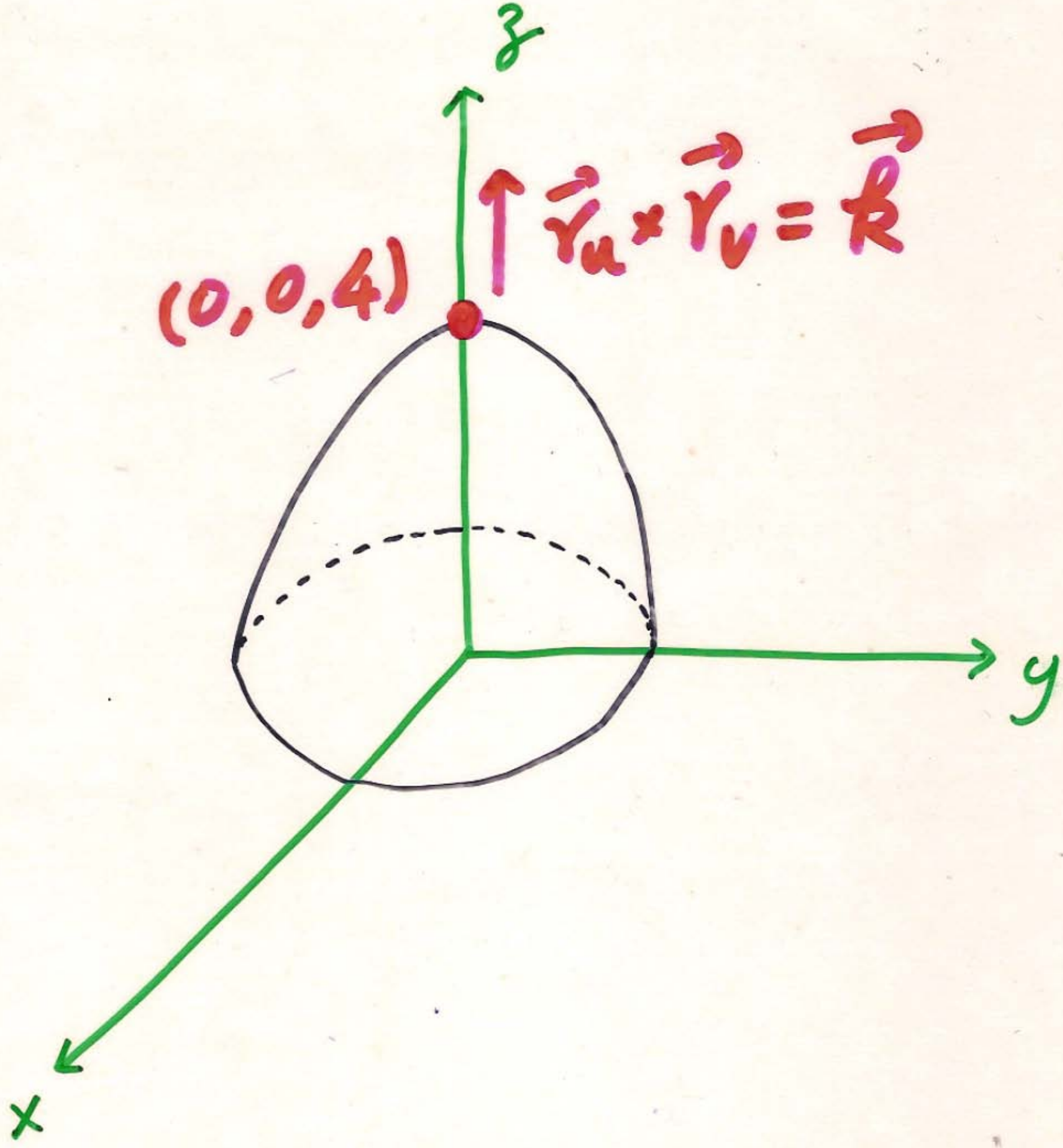
The opposite orientation is given by $\mathbf{r}_v \times \mathbf{r}_u = -\mathbf{r}_u \times \mathbf{r}_v$ and the corresponding oriented surface is denoted by $-S$. Then

$$\iint_{-S} \mathbf{F} \cdot d\mathbf{S} = -\iint_{S} \mathbf{F} \cdot d\mathbf{S}.$$

10.2.8 Example

In example 10.2.5, the normal vector we used is $\mathbf{r}_u \times \mathbf{r}_v = 2u\mathbf{i}+2v\mathbf{j}+\mathbf{k}$. Consider the point $(0, 0, 4)$ on the paraboloid. This point corresponds to $u = 0, v = 0$.

At this point, $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{k}$, which is pointing "upwards". Hence the orientation of the paraboloid we used in this example is given by the **upward normal vector**.


z
(0,0,4)
$\vec{r}_u \times \vec{r}_v = \vec{k}$
y
x

10.2.9 Example

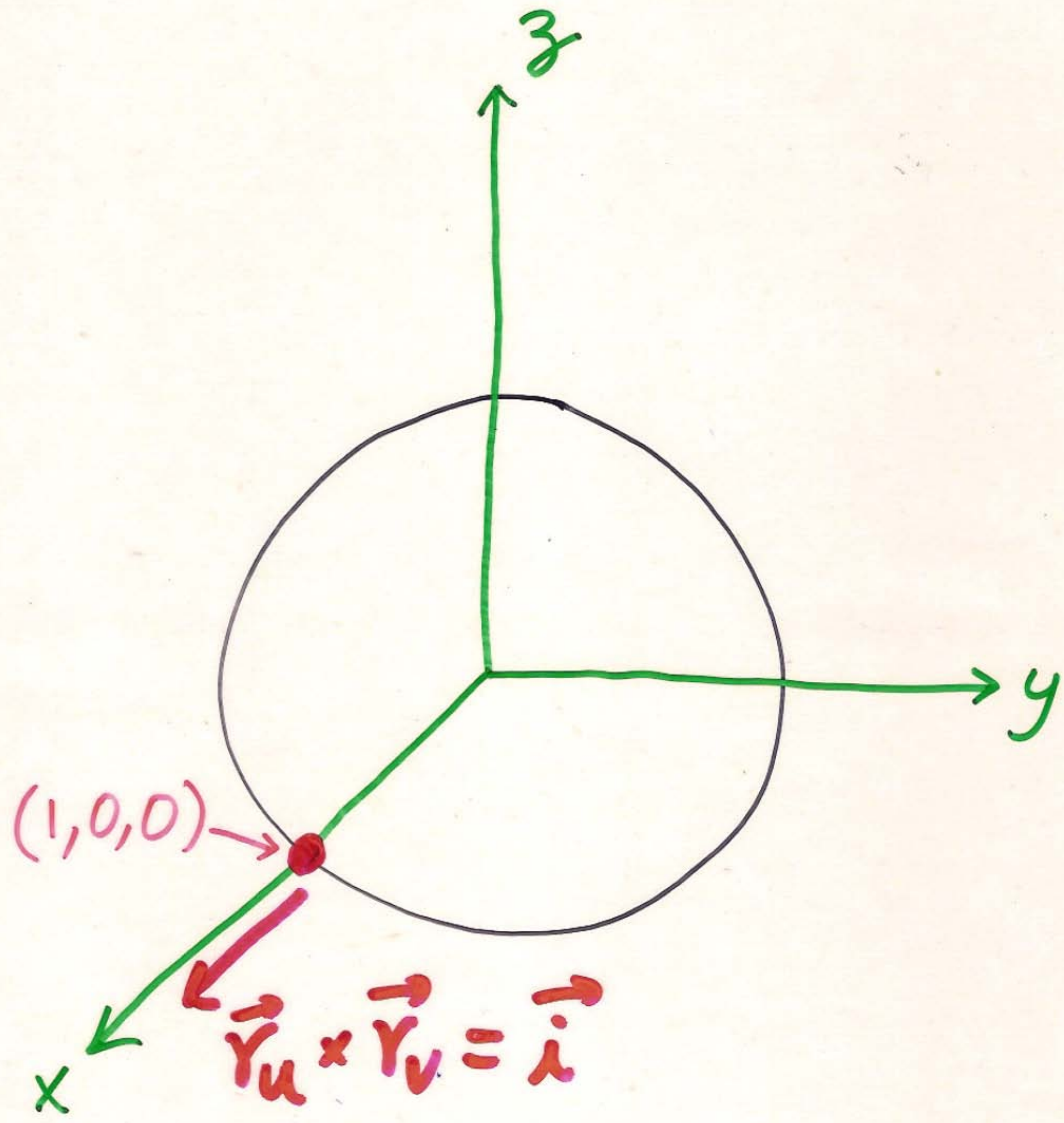
In Example 10.2.6, the normal vector we used is

$$\mathbf{r}_u \times \mathbf{r}_v = \sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \sin u \cos u\mathbf{k}.$$

Consider the point $(1, 0, 0)$ on the sphere.

This point corresponds to $u = \pi/2, v = 0$.

At this point, $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{i}$, which is pointing "outwards" away from the sphere. Hence the orientation of the sphere we used in this example is given by the **outward normal vector**.


z
y
x
(1,0,0)
$\vec{r}_u \times \vec{r}_v = \vec{i}$

10.3 Curl and Divergence

In this section, we introduce two operators on vector fields which will be used in the subsequent sections.

10.3.1 Curl

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field in the xyz-space. The **curl** of $\mathbf{F}$ is defined by

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

is a vector field.

10.3.2 Divergence

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field in the xyz-space. The **divergence** of $\mathbf{F}$ defined by

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

is a scalar function.

10.3.3 Del operator

The curl and divergence operators can be expressed in terms of the **del operator**:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Then

(i) taking the cross product of ∇ with a vector field

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k},$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

So

$$\nabla \times \mathbf{F} = \text{curl } \mathbf{F}$$

(ii) taking the dot product of ∇ with a vector field

$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$,

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \\
&= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
\end{aligned}$$

So $\nabla \cdot \mathbf{F} = \text{div } \mathbf{F}$.

10.3.4 Example

Let $\mathbf{F}(x, y, z) = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}$.

$$\begin{aligned}
\text{curl } \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} \\
&= (xz^2 - xy^2)\mathbf{i} + (x^2y - yz^2)\mathbf{j} + (y^2z - x^2z)\mathbf{k}.
\end{aligned}$$

$$\begin{aligned}
\text{div } \mathbf{F} &= \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2) \\
&= 6xyz.
\end{aligned}$$

For a scalar function f

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

is the gradient of f.

10.3.5 Example

Show that curl $(\nabla f) = \mathbf{0}$.

Solution:

$$\text{curl}\ (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k}$$

$$= \mathbf{0}$$

since $f_{xy} = f_{yx}$ etc.

10.3.6 Curl and conservative fields

Let $\mathbf{F}$ be a vector field in the xyz-space.

$$\text{If curl } \mathbf{F} = \mathbf{0}, \text{ then } \mathbf{F} \text{ is a conservative field.}$$

The converse is also true.

Note: Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\nabla \times \vec{F} = \vec{0}$$

$$\Longleftrightarrow \begin{cases} \dfrac{\partial R}{\partial y} = \dfrac{\partial Q}{\partial z} \\[2ex] \dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x} \\[2ex] \dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y} \end{cases}$$

$\Longleftrightarrow$ $\vec{F}$ is conservative.

10.3.7 Example

Find the curl of the velocity vector fields defined by

(a) $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$, (b)$\mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$, (c)$\mathbf{F}_3 = \cos y\mathbf{i} + \sin x\mathbf{j}$.

Solution:

(a) curl $\mathbf{F}_1 = \mathbf{0}$, (b) curl $\mathbf{F}_2 = 2\mathbf{k}$, (c) curl $\mathbf{F}_3 = (\cos x + \sin y)\mathbf{k}$.

$$\text{Curl } \vec{F}_3 = \nabla \times \vec{F}_3$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos y & \sin x & 0 \end{vmatrix}$$

$$= \underline{\underline{(\cos x + \sin y)\,\vec{k}}}$$

10.3.8 Example

Find the divergence of the velocity vector fields defined by (a) $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$, (b) $\mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$, (c) $\mathbf{F}_3 = -x^2\mathbf{i} + y^2\mathbf{j}$.

Solution: (a) div $\mathbf{F}_1 = 2$, (b) div $\mathbf{F}_2 = 0$, (c) div $\mathbf{F}_3 = 2(y - x)$.

(c) $\vec{F_3} = -x^2 \vec{i} + y^2 \vec{j}$

$\operatorname{div} \vec{F_3}$

$= \nabla \cdot \vec{F_3}$

$= \frac{\partial}{\partial x}(-x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(0)$

$= \underline{\underline{-2x + 2y}}$

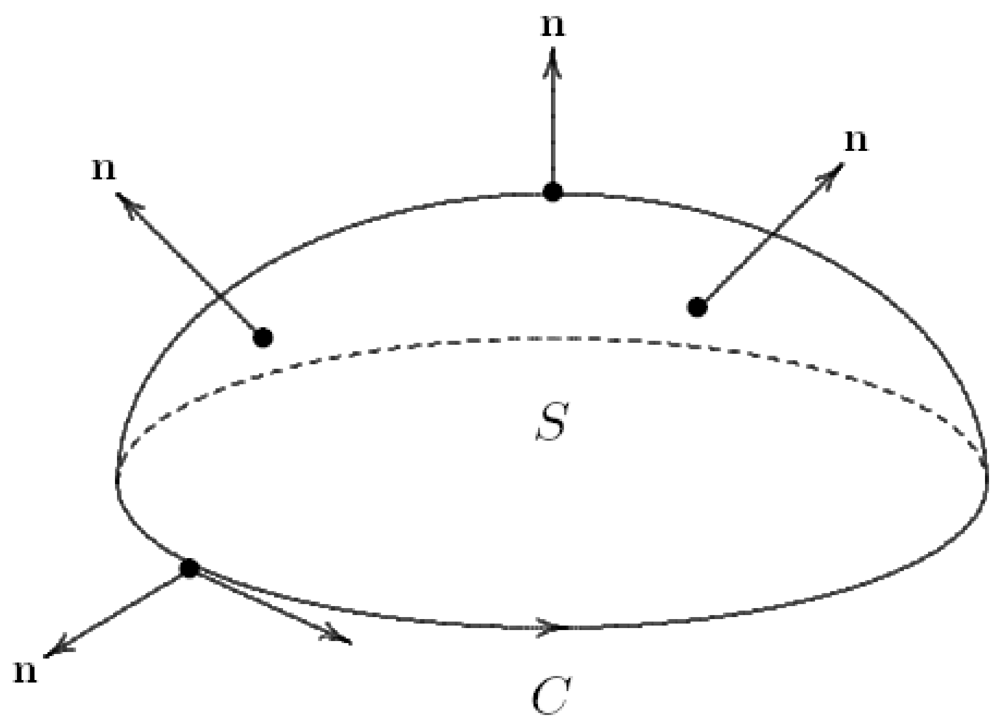
10.4 Stokes' Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a closed, piecewise-smooth boundary curve C.

Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on S. Then

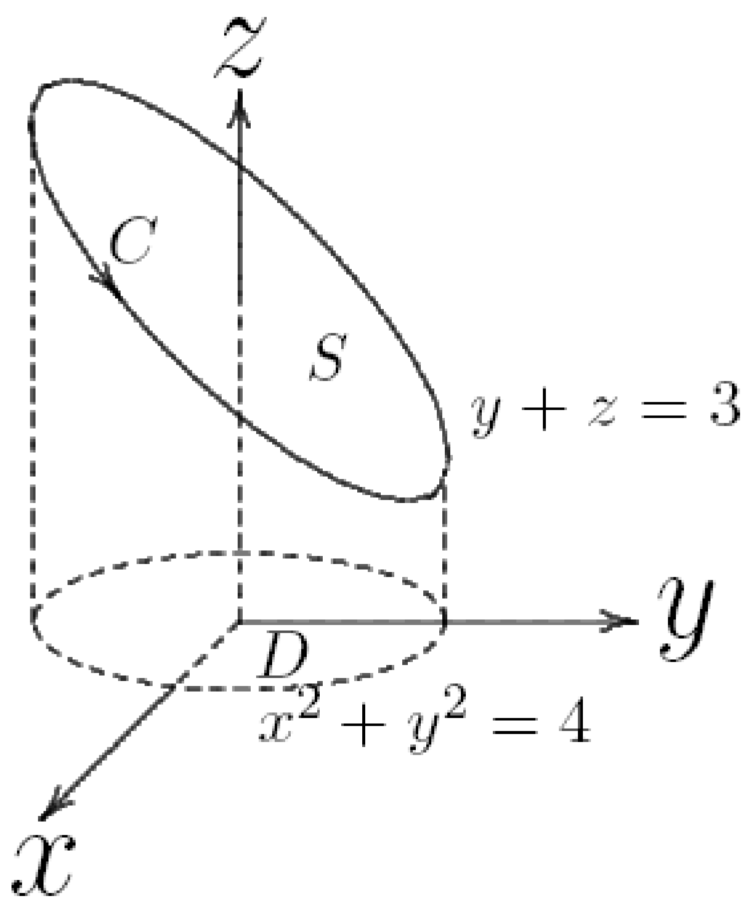
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\mathrm{curl}\ \mathbf{F}) \cdot d\mathbf{S}.$$

Note: In the above equation, the orientation of C must be consistent with that of S: when you walk in the direction (orientation) around C with your head pointing in the direction of the normal vector of S, the corresponding orientation of S should be on your left.


n
n
n
S
n
C

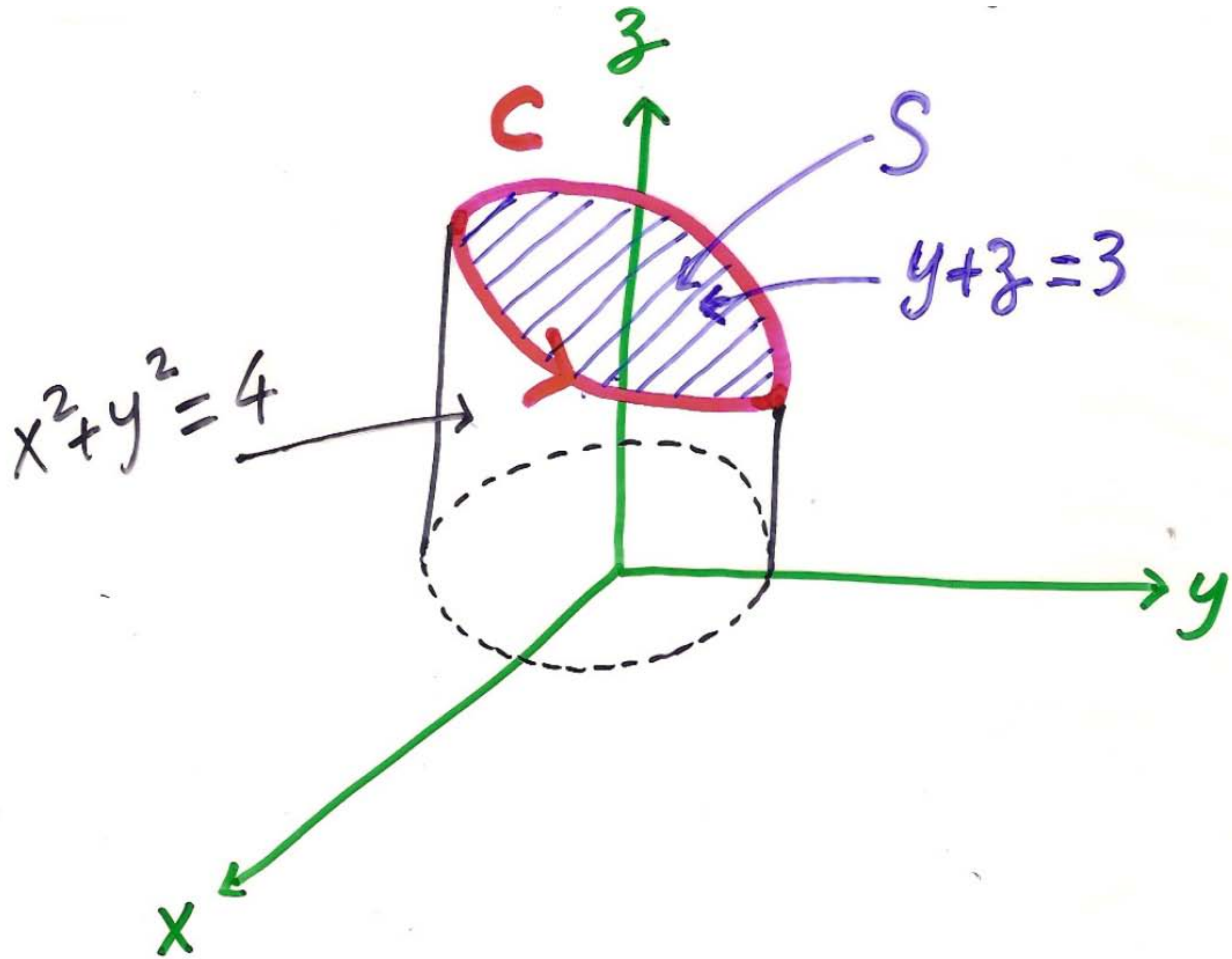
10.4.1 Example

Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + xy\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 3$ and the cylinder $x^2 + y^2 = 4$. (C is oriented in the counterclockwise sense when viewed from above.)



z
C
S
$y + z = 3$
y
D
$x^2 + y^2 = 4$
x

Solution: Let S be the (bounded) surface enclosed by C on the plane $y + z = 3$. So S has parametric representation $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (3 - v)\mathbf{k}$ and the region D is the disk of radius 2.



z
C
S
y+z=3
x²+y²=4
y
x

We have $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{j} + \mathbf{k}$, which is the upward normal vector of S. This gives the orientation of S which agrees with that of C.

Also $\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & xy \end{vmatrix} = 2x\mathbf{i} - 2z\mathbf{k}$.

By Stokes' Theorem,

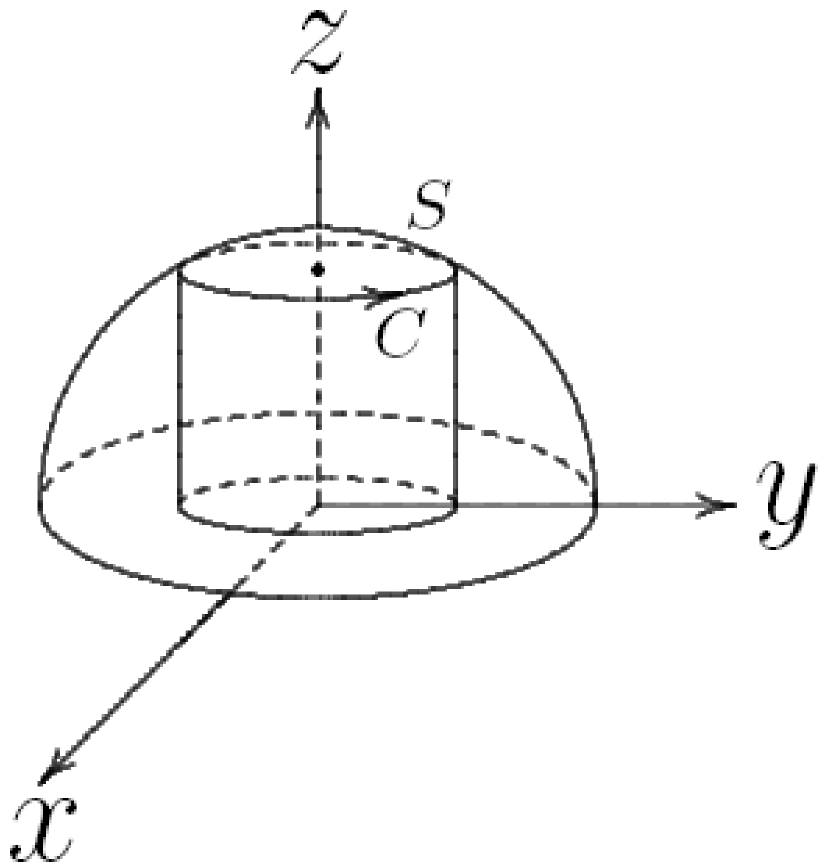
$$
\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \\
&= \iint_D (2u\mathbf{i} - 2(3 - v)\mathbf{k}) \cdot (\mathbf{j} + \mathbf{k}) \, dA \\
&= \iint_D (-6 + 2v) \, dA
\end{aligned}
$$

Since D is the disk of radius 2, we may use polar coordinates:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \int_0^2 (-6 + 2r \sin\theta) r\,dr\,d\theta \\ &= \int_0^{2\pi} \left(-12 + \frac{16}{3} \sin\theta\right) d\theta = -24\pi.\end{aligned}$$

10.4.2 Example

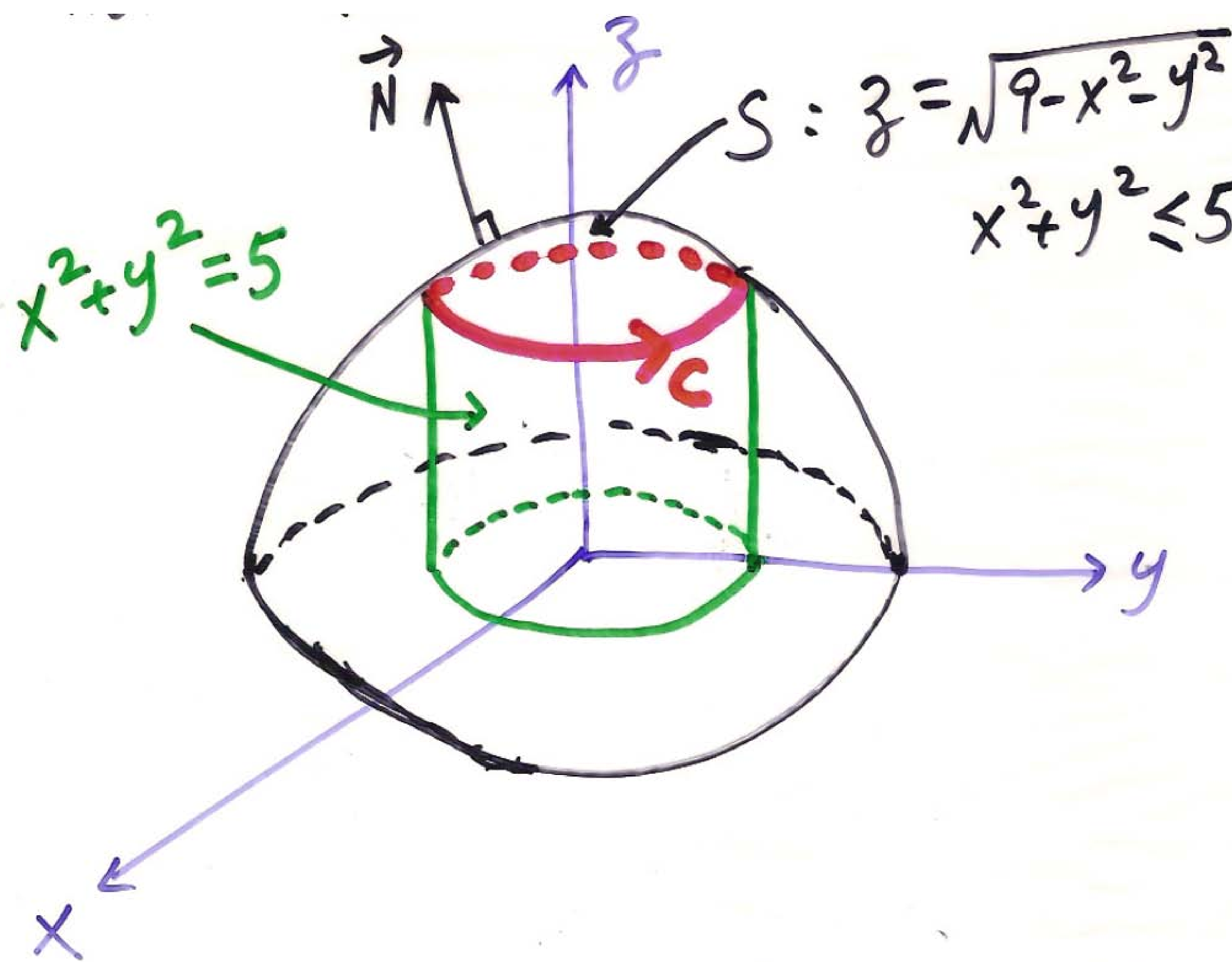
Use Stokes' Theorem to compute $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y^2 z\mathbf{i} + x\mathbf{j} + (x + y)\mathbf{k}$ and S is the part of the upper hemisphere $z = \sqrt{9 - x^2 - y^2}$ that lies within the cylinder $x^2 + y^2 = 5$ and the orientation of S is given by the upward normal vector.



z
S
C
y
x

Solution: The boundary C of S is given by the intersection of the cylinder $x^2 + y^2 = 5$ and the upper hemisphere $z = \sqrt{9 - x^2 - y^2}$. Solving the two equations, we have $z = 2$. So the curve C has a vector equation given by

$$\mathbf{r}(t) = \sqrt{5}\cos t\mathbf{i} + \sqrt{5}\sin t\mathbf{j} + 2\mathbf{k}.$$


$\vec{N}$
z
$S: z=\sqrt{9-x^2-y^2}$
$x^2+y^2\leq 5$
$x^2+y^2=5$
C
y
x

With this vector equation, the curve traverses in anticlockwise direction when viewed from top. This agrees with the given orientation of S.

Now $\mathbf{r}'(t) = -\sqrt{5}\sin t\mathbf{i} + \sqrt{5}\cos t\mathbf{j} + 0\mathbf{k}$ and

$\mathbf{F}(\mathbf{r}(t)) = 10\sin^2 t\mathbf{i} + \sqrt{5}\cos t\mathbf{j} + \sqrt{5}(\cos t + \sin t)\mathbf{k}$.

By Stokes' Theorem,

$$\begin{aligned}
\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\
&= \int_0^{2\pi} (-10\sqrt{5}\sin^3 t + 5\cos^2 t)\, dt
\end{aligned}$$

$$= \int_{-\pi}^{\pi} \left(\overbrace{-10\sqrt{5}\sin^3 t}^{\text{odd}} + \overbrace{5\cos^2 t}^{\text{even}} \right) dt$$

$\Big($ $\because$ the integrand is 2π periodic

$\therefore \int_0^{2\pi} = \int_{-\pi}^{\pi}$ $\Big)$

$$= 10 \int_0^{\pi} \cos^2 t \, dt$$

$$= 10 \int_0^{\pi} \frac{1 + \cos 2t}{2} \, dt$$

$$= 5 \left[t + \frac{1}{2} \sin 2t \right]_0^{\pi}$$

$$= \underline{\underline{5\pi}}$$

10.5 Divergence Theorem (or Gauss' Theorem)

Let E be a solid region and let S be the boundary of E, given with the **outward orientation***. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives in E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV.$$

* The outward orientation of the boundary surface of a solid region E is the one for which the normal vector point outward from E.

10.5.1 Example

Let $\mathbf{F}(x, y, z) = (x+y)\mathbf{i}+(y+z)\mathbf{j}+(z+x)\mathbf{k}$. Evaluate $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the sphere $x^2 + y^2 + z^2 = 1$ with orientation given by the outward normal vector.

Solution: By the Divergence Theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E 3 \, dV$$

$$= 3 \times \text{volume of the unit ball} = 4\pi.$$

10.5.2 Example

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F} = x^2\mathbf{i} + (xy + x\cos z)\mathbf{j} + e^{xy}\mathbf{k}$$

and S is the surface of the cubic region E bounded by the three coordinate planes $x = 0, y = 0, z = 0$ and the three planes $x = 1,\ y = 1,\ z = 1$. The orientation of S is given by the outward normal vector.

Solution: The cubic region E can be described as

$$E: \quad 0 \leq x \leq 1, \;\; 0 \leq y \leq 1, \;\; 0 \leq z \leq 1$$

By the Divergence Theorem, we have

$$
\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F}\, dV \\
&= \iiint_E 3x\, dV = 3 \int_0^1 \int_0^1 \int_0^1 x\, dx dy dz \\
&= \frac{3}{2}.
\end{aligned}
$$