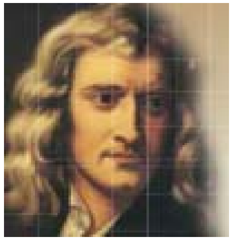
Chapter 3

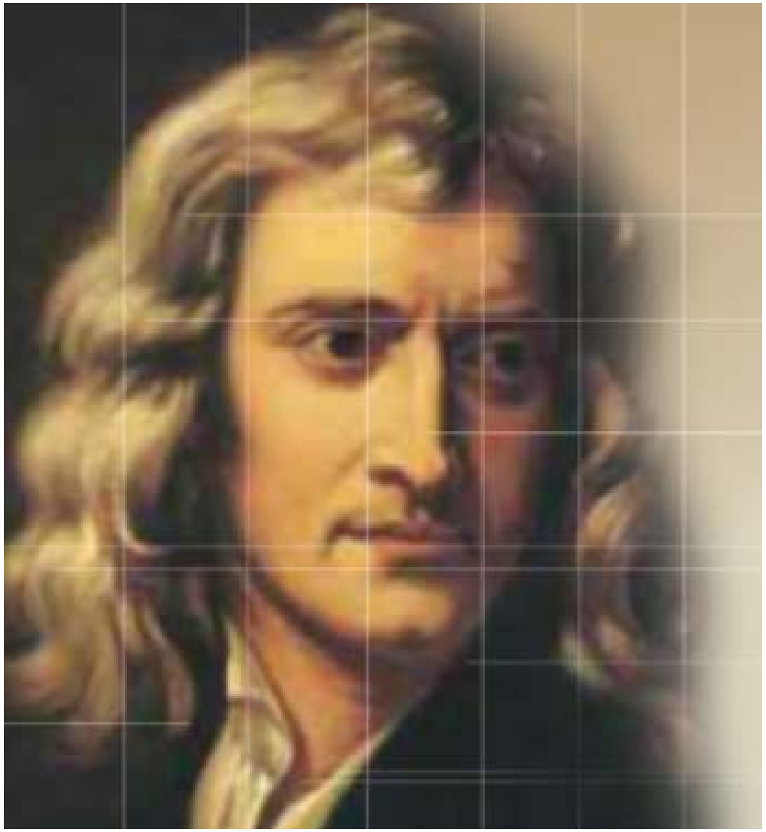
Integration

Key Results

- Indefinite integrals
- Fundamental Theorem of Calculus (FTC)
- Area of region under a graph

Great achievement of Newton and Leibniz to connect FTC to calculations of area and volume, among other applications.





Isaac Newton

(1643 – 1727)

Gottfried Wilhelm von Leibniz

(1646 – 1716)

Antiderivatives

A (differentiable) function $F(x)$ is an **antiderivative** of a function $f(x)$ if

$$\boxed{F'(x) = f(x)}$$

for all x in the domain of f.

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x. It is denoted by

$$\int \underbrace{\boxed{f(x)}}_{\text{integrand}} \overbrace{\boxed{dx}}^{\text{variable of integration } x}.$$

Fact: *Only the constant functions have zero derivative. Therefore, the antiderivatives of the zero function are all the constant functions.*

This leads to

If $\boxed{F'(x) = f(x) = G'(x)}$, then $\boxed{G(x) = F(x) + C,}$

$$\int f(x)dx = F(x) + C.$$

$F(x) = \sin x$

$F'(x) = \cos x = G'(x)$

e.g. $G(x) = \sin x + 1$

C here is called a **constant of integration**

Some Basic Integral Formulas

$$(1)\ \int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1,\ n \ \text{rational}$$

power rule

$$\int 1\,dx = \int dx = x + C \qquad (\text{Special case, } n = 0)$$

$$(2)\ \boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$

$$(3)\ \int \sin kx\,dx = -\frac{\cos kx}{k} + C, \quad k \neq 0$$

$$(4)\ \int \cos kx\,dx = \frac{\sin kx}{k} + C, \quad k \neq 0$$

Basic Formula List (cont'd)

$$(5)\ \int \sec^2 x\, dx = \tan x + C$$

$$(6)\ \int \csc^2 x\, dx = -\cot x + C$$

$$(7)\ \int \sec x \tan x\, dx = \sec x + C$$

$$(8)\ \int \csc x \cot x\, dx = -\csc x + C$$

Some Basic Rules

$$\int k\,f(x)\,dx = k\int f(x)\,dx,$$

$k = \text{constant (independent of } x)$

$$\int \left[f(x) \pm g(x)\right] dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Example

Find the curve in the xy-plane which passes through the $\boxed{\text{point } (9,4)}$ and whose $\boxed{\text{slope}}$ at each point (x, y) is $\boxed{3\sqrt{x}.}$

The curve is given by $y = y(x)$, satisfying

(i) $\boxed{\dfrac{dy}{dx} = 3\sqrt{x}}$ (ii) $\boxed{y(9) = 4.}$

$$y = \int 3\sqrt{x}\, dx = 3\,\frac{x^{3/2}}{3/2} + C = 2x^{3/2} + C.$$

Example (cont'd)

(ii) $\boxed{y(9) = 4.}$

$$y = 2x^{3/2} + C$$

$$4 \;=\; (2)9^{3/2} + C$$

$$C \;=\; 4 - 54 \;=\; -50$$

Thus, the curve has equation $\boxed{y = 2x^{3/2} - 50.}$

Area under a Curve

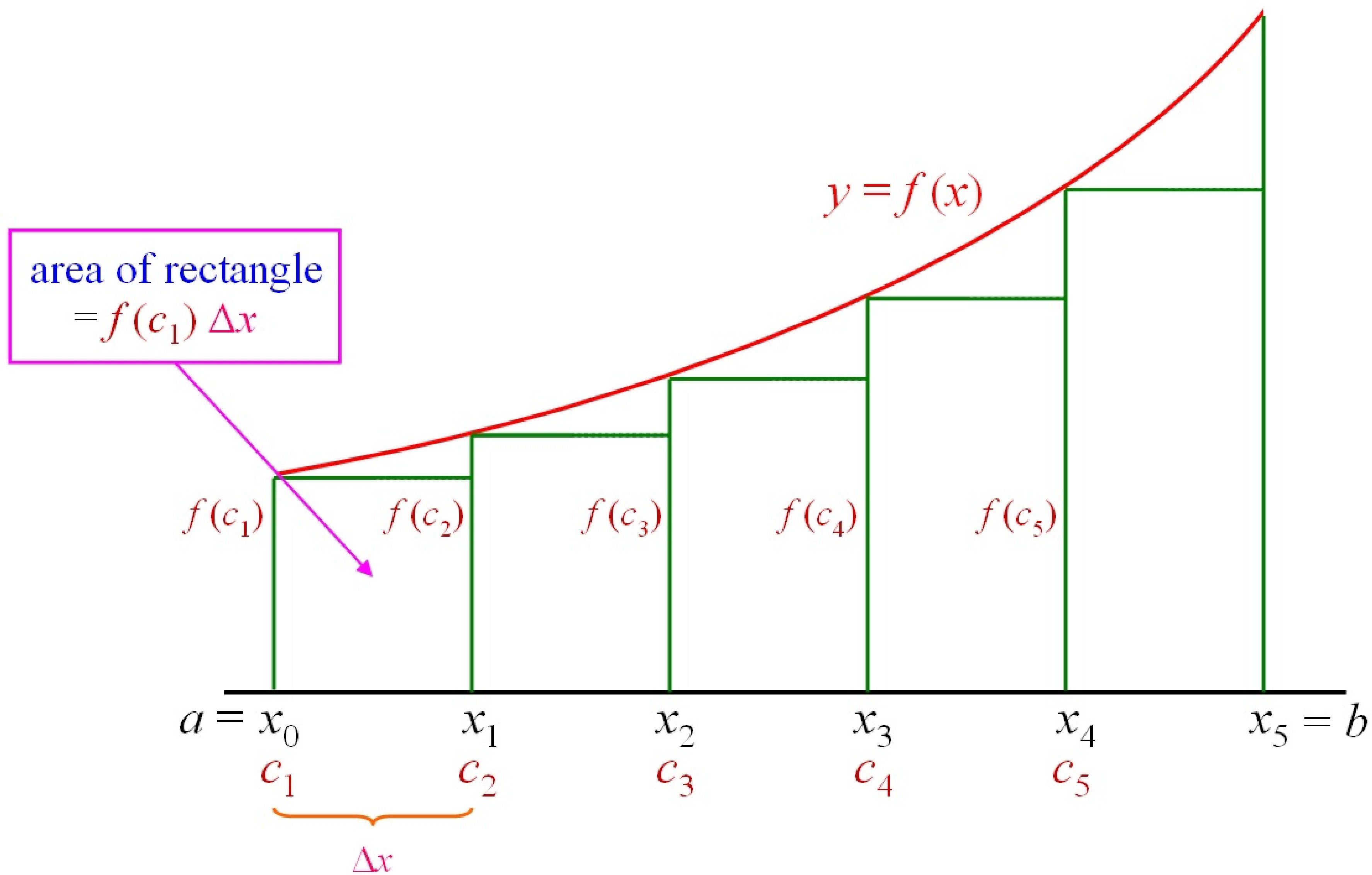
Special Case

Consider a non-negative continuous increasing function $f = f(x)$ over an interval $[a, b]$.

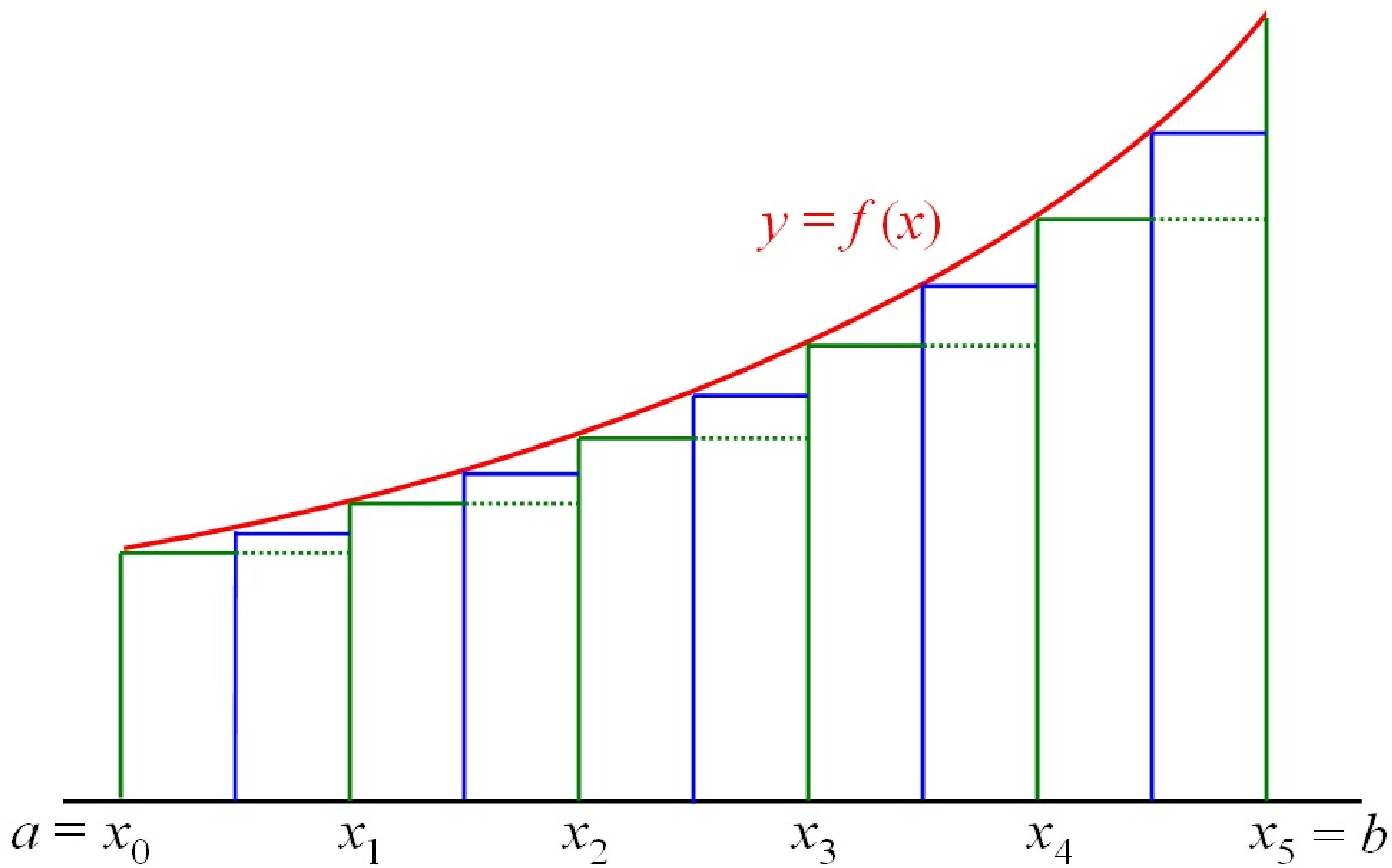
Partition $[a, b]$ into sub-intervals, say 5 sub-intervals, of equal length:

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], [x_3, x_4], [x_4, x_5]$$

Note that $x_0 = a, \quad x_5 = b$



$$\text{total area of rectangles } = \sum_{k=1}^{5} f(c_k)\Delta x$$



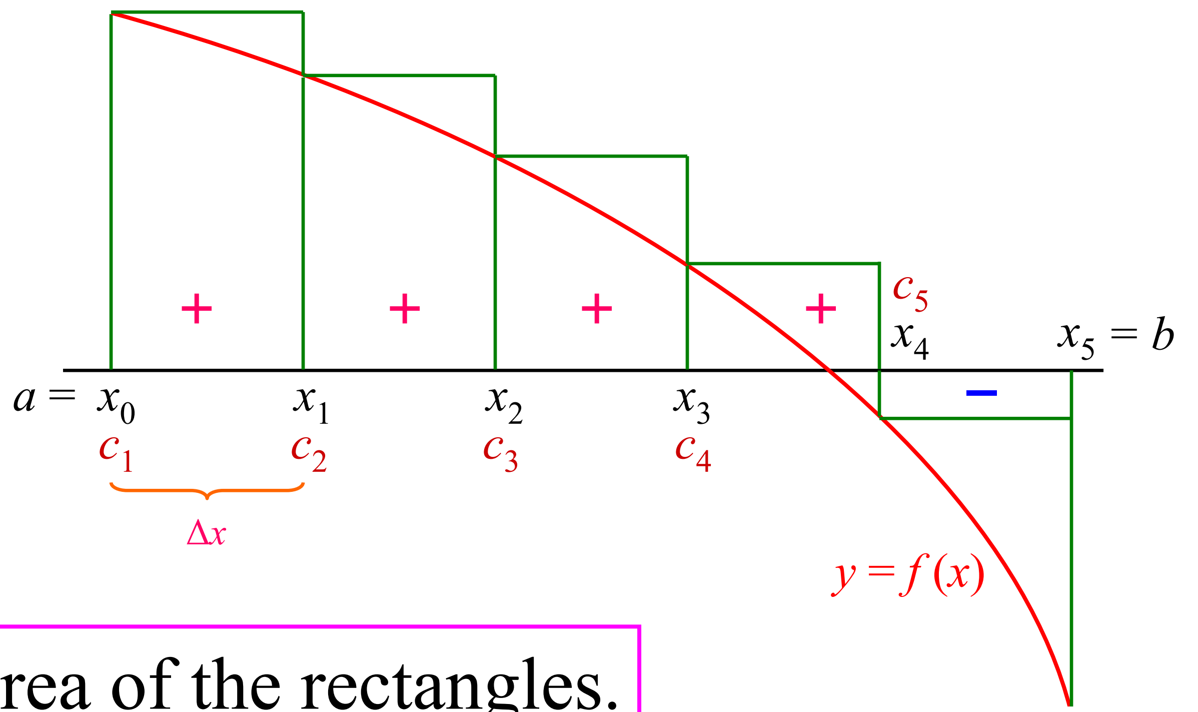
$$\sum_{k=1}^{5} f(c_k)\Delta x < \sum_{k=1}^{10} f(c_k)\Delta x$$

$$\sum_{k=1}^{5} f(c_k)\Delta x < \sum_{k=1}^{10} f(c_k)\Delta x < \sum_{k=1}^{20} f(c_k)\Delta x$$

$$< \cdots < \boxed{\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x = \text{exact area of region under the curve}}$$

More General Form

f continuous, but $f(x)$ may be negative for some values of x.



$$\sum_{k=1}^{n} f(c_k)\Delta x$$

gives total *signed* area of the rectangles.

(positive/negative)

Riemann Integral

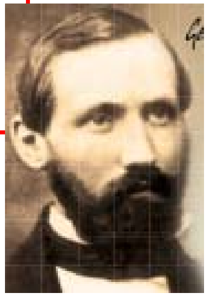
As more sub-intervals are used, the rectangles will approximate the region between the *x*-axis and *f* with increasing accuracy.

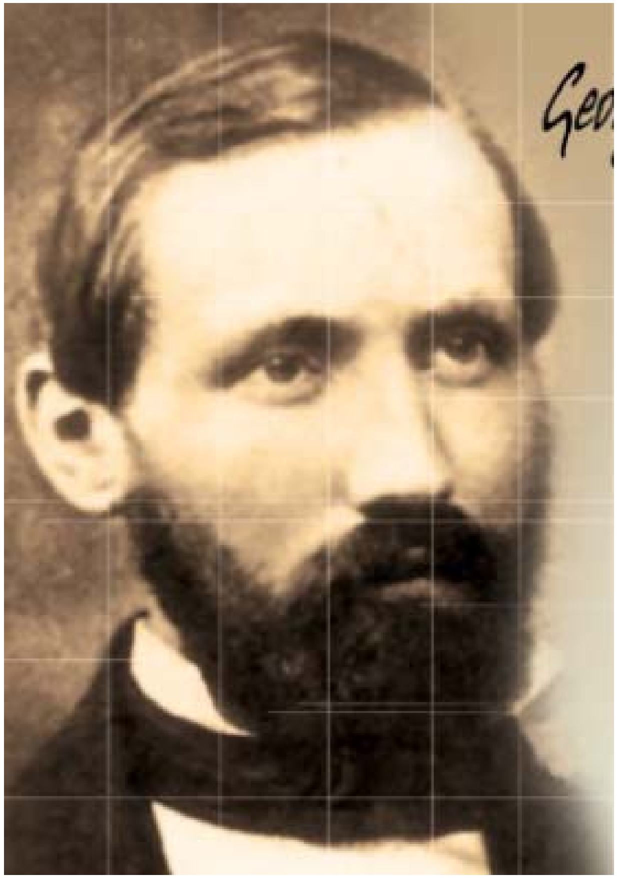
Obtain

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x = \int_a^b f(x)\,dx}$$

Riemann integral (or definite integral) of *f* over [*a*, *b*].

Riemann integral gives the signed area of the region under the graph of *f* over [*a*, *b*].





Georg Friedrich Bernhard Riemann

(1826 – 1866)

Terminology

$$\int_a^b f(x)dx = \int_a^b f(u)\,du = \int_a^b f(t)\,dt, \quad \text{etc.}$$

$[a, b]$: the interval of integration

a : lower limit of integration

b : upper limit of integration

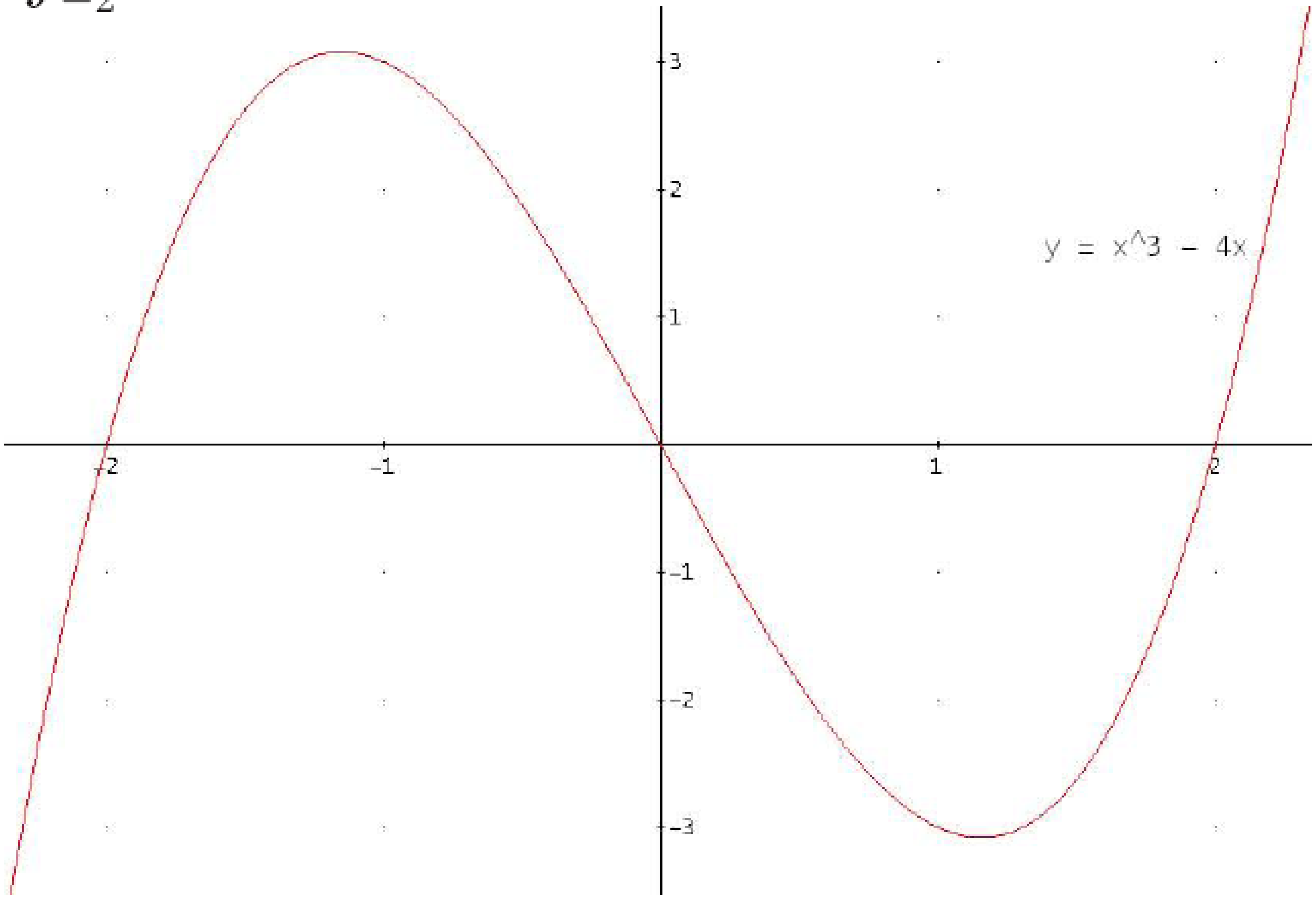
$f(x)$: the integrand

x : variable of integration

x is a *dummy* variable

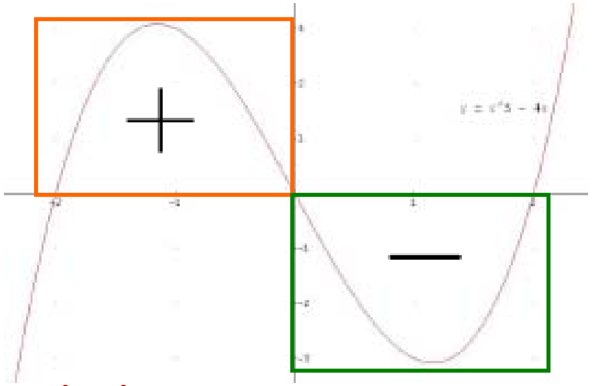
Example

Find $\int_{-2}^{2} (x^3 - 4x)\ dx$



Example (cont'd)

Find $\displaystyle\int_{-2}^{2} (x^3 - 4x)\ dx$



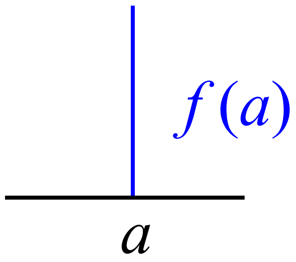
The graph is symmetrical about the origin.

The region under the graph over [–2, 0] is of the same (physical) area as the region under the graph over [0, 2]. But the ***areas*** are of '***different signs***'.

$$\int_{-2}^{2} (x^3 - 4x)\ dx = \boxed{0}$$

Definite Integral Rules

1. $\boxed{\int_a^a} f(x)\,dx \boxed{= 0}$



Interval $[a, a]$ is the point $x = a$.

Region under graph is a vertical line with ***no area***.

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$$\int_2^1 f(x)dx \overset{\text{define}}{=} -\int_1^2 f(x)dx$$

Does not make sense as the interval is [2, 1].

Other Routine Rules

3. $\displaystyle\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx, \quad$ (any constant k)

$$\left(\text{In particular, } \int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx\right)$$

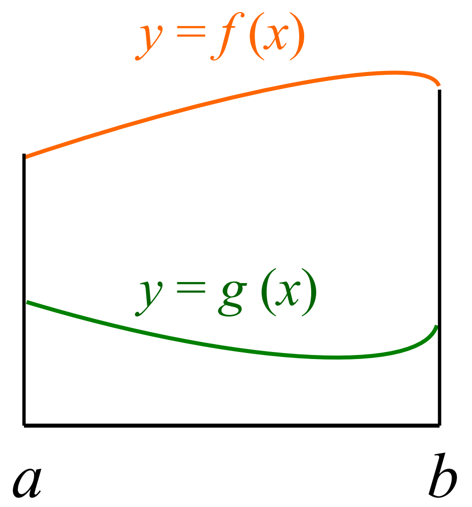
4. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. If $\boxed{f(x)}\boxed{\geq}\boxed{g(x)}$ on $[a,b]$, then

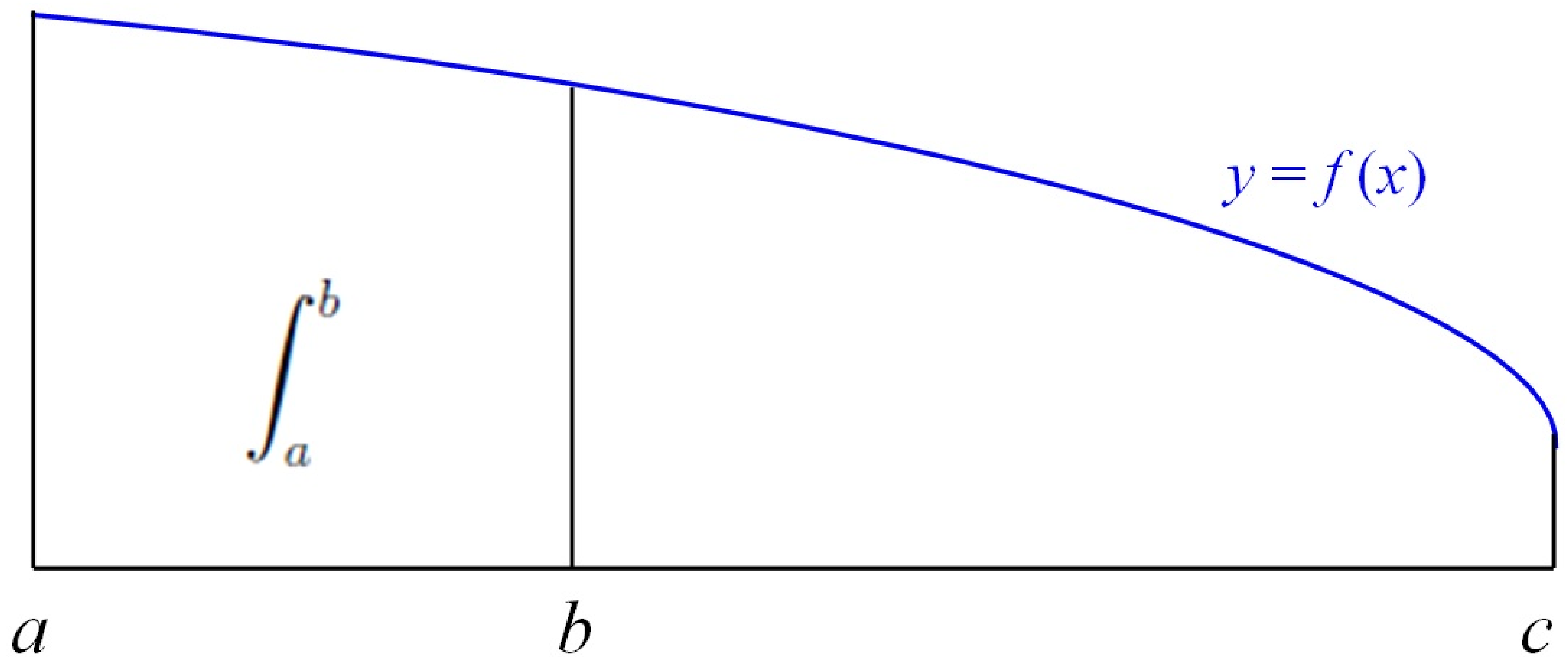
$$\boxed{\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.}$$

In particular, if $f(x) \geq 0$ on $[a,b]$, then

$$\int_a^b f(x)\,dx \geq 0.$$

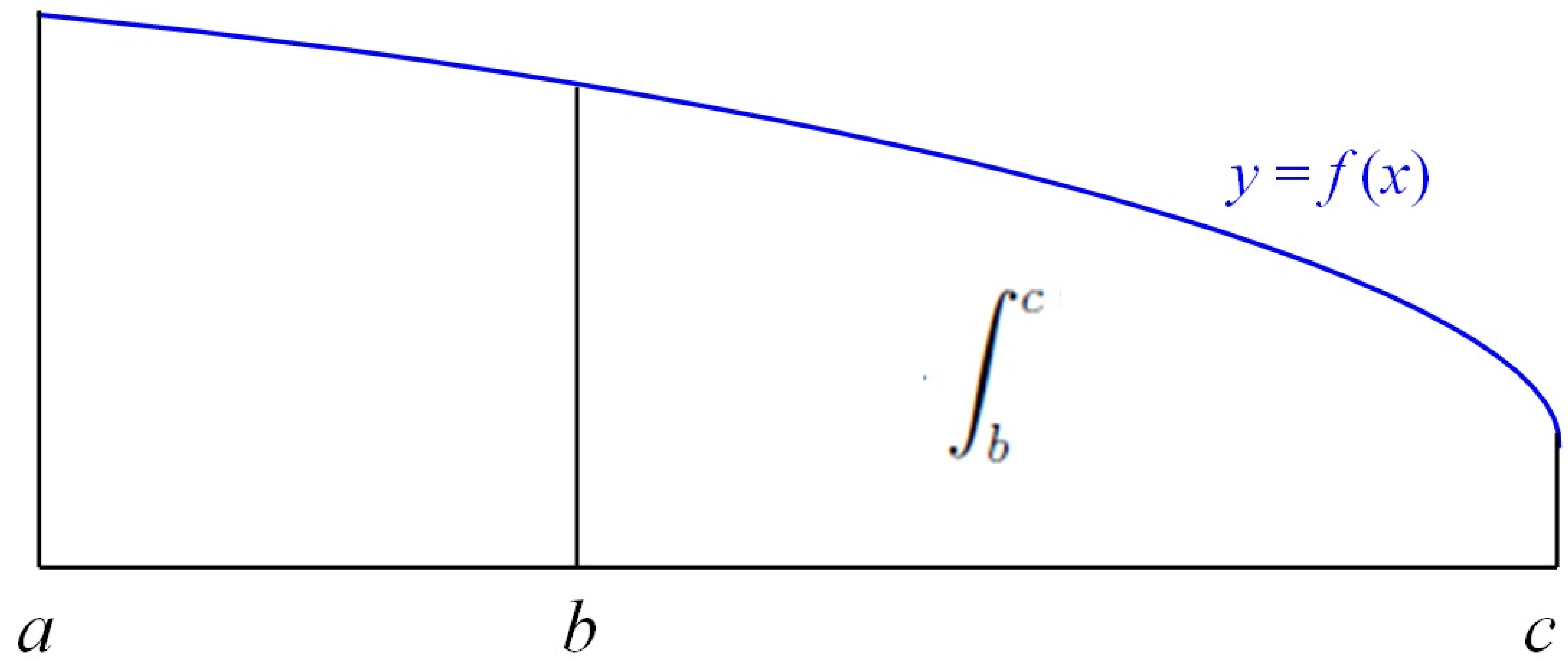


6. If f is continuous on the interval joining a, b and c, then



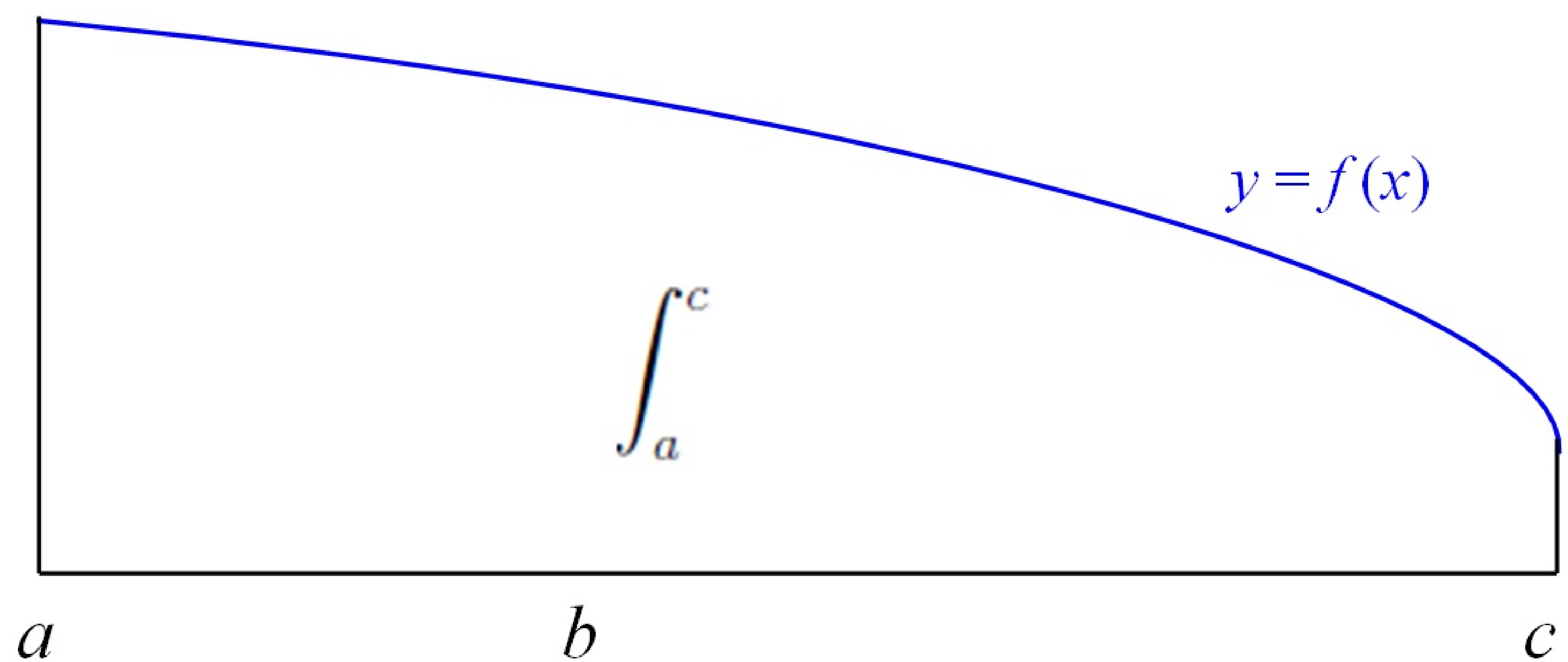
$$\boxed{\int_a^b f(x)\,dx} + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

6. If f is continuous on the interval joining a, b and c, then



$$\int_a^b f(x)\,dx + \boxed{\int_b^c f(x)\,dx} = \int_a^c f(x)\,dx$$

6. If f is continuous on the interval joining a, b and c, then



$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \boxed{\int_a^c f(x)\,dx}$$

Fundamental Theorem of Calculus (FTC)

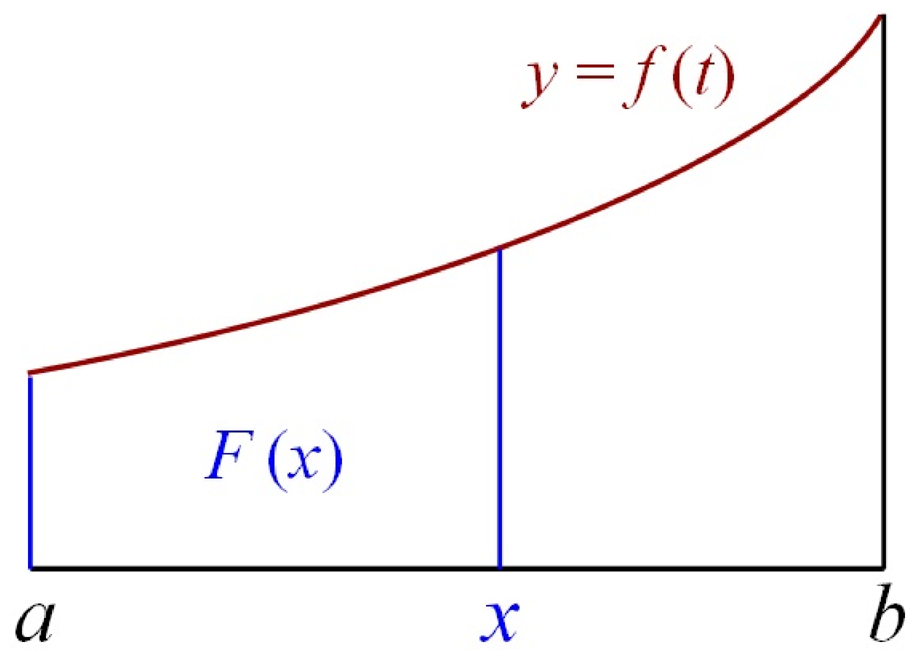
Part 1

If f is continuous on $[a, b]$, then the function

$$\boxed{F(x) = \int_a^x f(t)\,dt}$$

Area function of x

has a derivative at every point of $[a, b]$, and

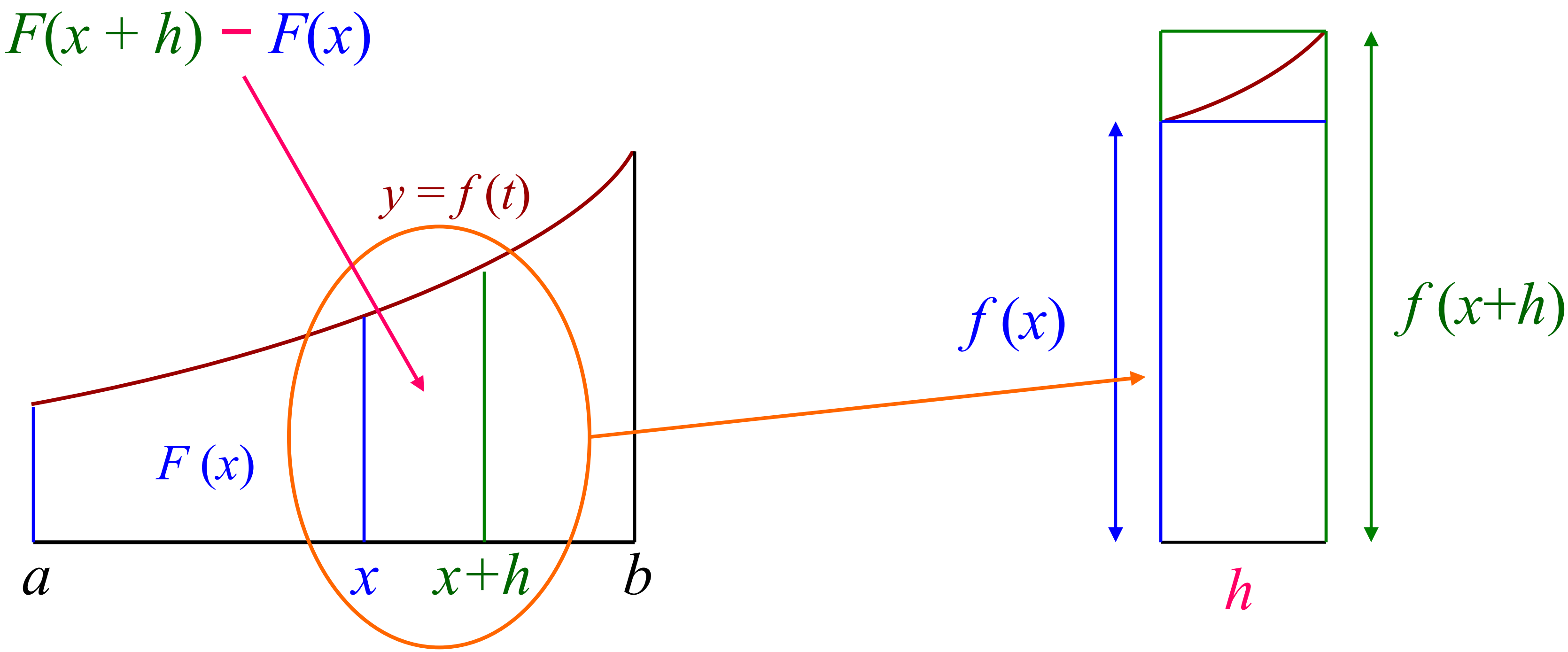


$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x).}$$

Main idea in proof: consider areas of rectangles

First consider a change in value of x:

$$\boxed{f(x)\,h \;\le\; F(x+h) - F(x) \;\le\; f(x+h)\,h}$$



$$f(x)\,h \;\le\; F(x+h)-F(x) \;\le\; f(x+h)\,h$$

Divide by *h*

$$f(x) \;\le\; \frac{F(x+h)-F(x)}{h} \;\le\; f(x+h)$$

Now take limits as $h \longrightarrow 0$.

Noting that $\lim\limits_{h\longrightarrow 0} f(x+h) \;=\; f(x)$, obtain

f is continuous

$$f(x) \le \lim_{h\longrightarrow 0} \frac{F(x+h)-F(x)}{h} \le f(x)$$

Therefore,

$$\boxed{f(x)} = \lim_{h\longrightarrow 0} \frac{F(x+h)-F(x)}{h} = \boxed{\frac{d}{dx}F(x)}$$

$$f'(a) = \lim_{h\longrightarrow 0} \frac{f(a+h)-f(a)}{h}.$$

Examples

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \qquad a = -\pi \qquad f(t) = \cos t$$

(1) $$\frac{d}{dx}\int_{-\pi}^x \cos t\,dt \quad = \boxed{\cos x}$$

Examples

$$\frac{d}{dx}\int_a^x f(t)\,dt = \boxed{f(x)} \qquad a = 0 \qquad \boxed{f(t) = \frac{1}{1+t^2}}$$

(1) $$\frac{d}{dx}\int_{-\pi}^x \cos t\,dt = \boxed{\cos x}$$

(2) $$\frac{d}{dx}\int_0^x \frac{dt}{1+t^2} = \boxed{\frac{1}{1+x^2}}$$

Example

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \qquad a = 1 \qquad f(t) = \cos t$$

$$(3)\ \frac{d}{dx}\int_1^{x^2} \cos t\,dt$$

$$= \frac{d}{dx}\int_1^{u} \cos t\,dt \qquad \text{Let } u = x^2.$$

Example

$$\frac{d}{du}\int_a^u f(t)\,dt = f(u) \qquad a = 1 \qquad f(t) = \cos t$$

$$(3)\ \frac{d}{dx}\int_1^{x^2} \cos t\,dt$$

$$= \frac{d}{dx}\boxed{\int_1^u \cos t\,dt} = y \qquad \text{Let } u = x^2.$$

$$= \frac{d}{du}\int_1^u \cos t\,dt \cdot \boxed{\frac{du}{dx}} \qquad \text{(chain rule)}$$

$$\frac{d\boxed{y}}{dx} = \frac{dy}{du}\cdot\boxed{\frac{du}{dx}}$$

Example

$$\frac{d}{du}\int_a^u f(t)\,dt \;=\; f(u) \qquad a = 1 \qquad f(t) = \cos t$$

$$(3)\ \frac{d}{dx}\int_1^{x^2} \cos t\,dt$$

$$= \frac{d}{dx}\int_1^{u} \cos t\,dt \qquad \text{Let } u = x^2. \ \text{Then } \frac{du}{dx} = 2x.$$

$$= \frac{d}{du}\int_1^{u} \cos t\,dt \cdot \frac{du}{dx}$$

(chain rule)

$$= \cos u \cdot 2x$$

$$= \boxed{2x \cos x^2}$$

FTC Part 2

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\boxed{\int_a^b f(x)dx = \Big[F(x)\Big]_a^b} = F(b) - F(a).$$

Main idea of proof is to compare two antiderivatives, one given as $F(x)$ and the other from FTC Part 1:

$$G(x) = \int_a^x f(t)\,dt$$

From page 2, noting that

$$F'(x) = f(x) = G'(x)$$

obtain

$$\boxed{F(x) = G(x) + c}$$

Thus,

$$\begin{aligned} \boxed{F(b) - F(a)} &= G(b) + c - \big(G(a) + c\big) \\ &= \boxed{G(b) - \boxed{G(a)}} \\ &= \int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{=\,\mathbf{0}} \\ &= \boxed{\int_a^b f(t)\,dt} \end{aligned}$$

$$G(x) = \int_a^x f(t)\,dt$$

Examples

$$(1)\quad \int_0^{\pi} \cos x\, dx = \Big[\sin x\Big]_0^{\pi} = \sin\pi - \sin 0 = \boxed{0}$$

$$(2)\quad \int_0^{2} t^2\, dt = \left[\frac{1}{3}t^3\right]_0^{2} = \frac{1}{3}(2^3) - \frac{1}{3}(0^3) = \boxed{\frac{8}{3}}$$

$$(3)\quad \int_{-2}^{2} (4-u^2)\, du = \left[4u - \frac{1}{3}u^3\right]_{-2}^{2}$$

$$= \left[4(2) - \frac{1}{3}(2^3)\right] - \left[4(-2) - \frac{1}{3}(-2)^3\right] = \boxed{\frac{32}{3}}$$

Integration by Substitution

Consider $$\boxed{\int f\big(g(x)\big)g'(x)\,dx}$$

Set $u = g(x)$. Then $g'(x) = \dfrac{du}{dx}$

Given integral becomes $$\boxed{\int f(u)\,du}$$

which may be simpler to calculate

Example

Let $\boxed{u = \sin x}$

$$\frac{du}{dx} = \cos x$$

$$\boxed{du = \cos x\, dx}$$

$$\int \boxed{\sin^4 x}\boxed{\cos x\, dx}$$

$$= \int u^4\, du = \frac{1}{5}u^5 + c = \boxed{\frac{1}{5}\sin^5 x + c}$$

Example

Let $\boxed{u = x^2 + 2x - 3}$

$$\frac{du}{dx} = 2x + 2 = 2(x+1)$$

$$\boxed{\frac{1}{2}du = (x+1)dx}$$

$$\int \boxed{(x^2 + 2x - 3)^2}\boxed{(x+1)\,dx}$$

$$= \int u^2 \cdot \frac{1}{2}\,du = \frac{1}{2}\cdot\frac{1}{3}u^3 + c = \boxed{\frac{1}{6}(x^2 + 2x - 3)^3 + c}$$

Changing Limits of Integration

$$\boxed{\int_a^b f\left(g(x)\right) g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du}$$

Substitution formula requires g' to be continuous on $[a, b]$ and f to be continuous on the range of g.

Set $u = g(x)$ $\qquad g'(x) = \dfrac{du}{dx}$

Example

Let $\boxed{u = \tan x}$

$$\frac{du}{dx} = \sec^2 x$$

$$\boxed{du = \sec^2 x \, dx}$$

$$\boxed{u = 0 \quad \text{when} \quad x = 0}$$

$$\boxed{u = 1 \quad \text{when} \quad x = \frac{\pi}{4}}$$

$$\int_0^{\pi/4} \boxed{\tan x} \cdot \boxed{\sec^2 x \, dx} = \int_0^1 u \, du = \left[\frac{1}{2}u^2\right]_0^1 = \boxed{\frac{1}{2}}$$

Integration by Parts

Consider

$$\int \boxed{u(x)}\,\boxed{w(x)}\,dx = \int u \boxed{\frac{dv}{dx}}\,dx$$

$u(x)$ can be differentiated (repeatedly)

$w(x)$ can be integrated easily, with antiderivative $v(x)$,

i.e.

$$\boxed{w(x) = \frac{dv}{dx}}$$

Begin with product rule

$$\frac{d}{dx}(uv) = \boxed{u\frac{dv}{dx}} + v\frac{du}{dx}$$

$$\boxed{u\frac{dv}{dx}} = \boxed{\frac{d}{dx}(uv) - v\frac{du}{dx}}$$

Integrate both sides of the above equation

$$\int u(x)\,w(x)\,dx = \boxed{\int \boxed{u\frac{dv}{dx}}dx = \boxed{uv - \int v\frac{du}{dx}\,dx}}$$

More compact form:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example

Many choices for u and dv. But only a few choices may work. For instance,

$$\boxed{u = x \quad \Rightarrow \quad du = dx}$$

$$\boxed{dv = \cos x \, dx \quad \Rightarrow \quad v = \int \cos x \, dx = \sin x}$$

$$\int x \cos x \, dx$$

$$= x \sin x - \int \sin x \, dx$$

$$= \boxed{x \sin x + \cos x + C}$$

$$\int u \, dv = uv - \boxed{\int v \, du}$$

Example

Consider

$$\boxed{u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}\,dx}$$

$$\boxed{dv = dx \quad \Rightarrow \quad v = x}$$

$$\int_1^e \ln x \, dx$$

$$= \Big[(\ln x)\,x\Big]_1^e - \int_1^e x\,\frac{1}{x}\,dx$$

$$= \big[e - 0\big] - \big[x\big]_1^e$$

$$= \boxed{1}$$

$$\int u\,dv = uv - \boxed{\int v\,du}$$

Area between Curves

If f_1 and f_2 are continuous functions with

$$\boxed{f_1(x) \quad \leq \quad f_2(x)}$$

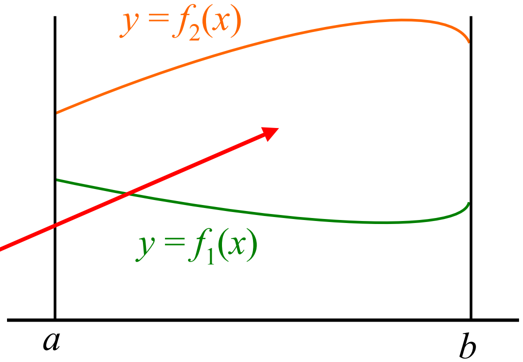
graph of $y = f_2(x)$ is higher than graph of $y = f_1(x)$

over the interval $[a, b]$, then the area of the region

between the graphs of $y = f_1(x)$ and $y = f_2(x)$

from $x = a$ to $x = b$ is

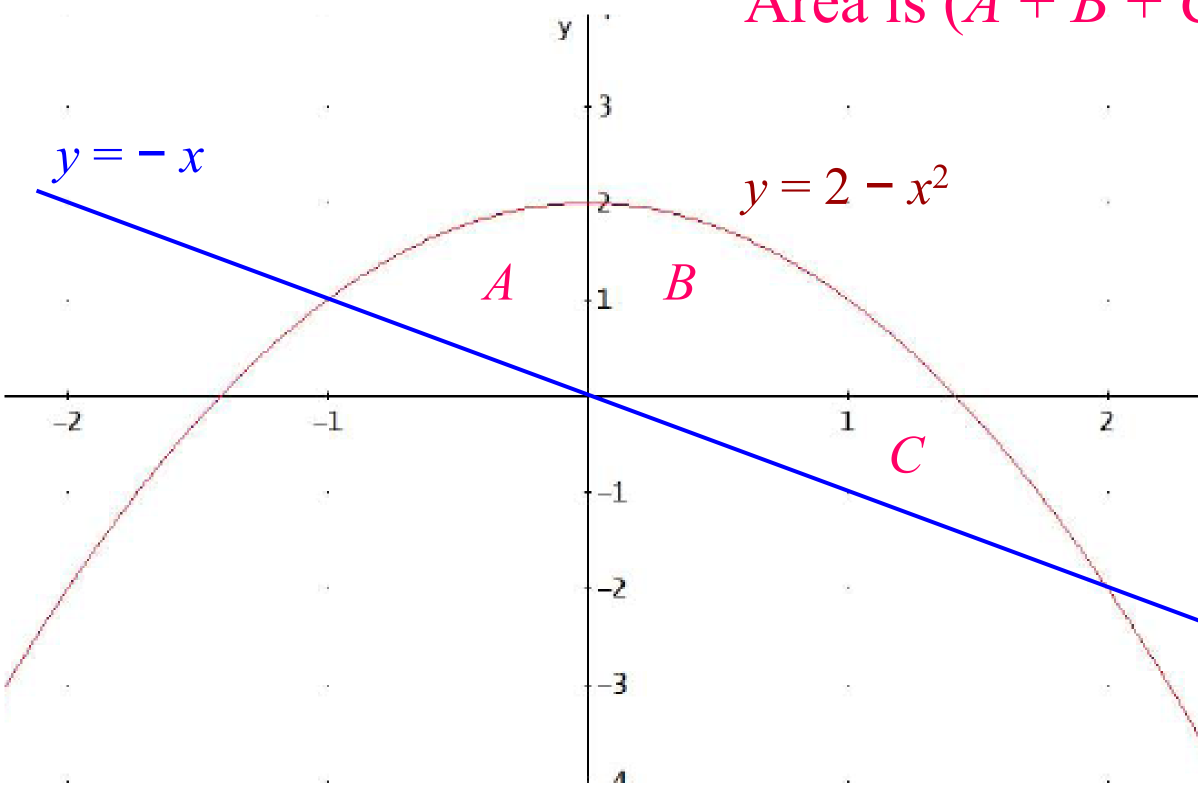
$$\boxed{\text{Area} \ = \int_a^b [f_2(x) - f_1(x)]\, dx}$$



Example

Find the area of the (finite) region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

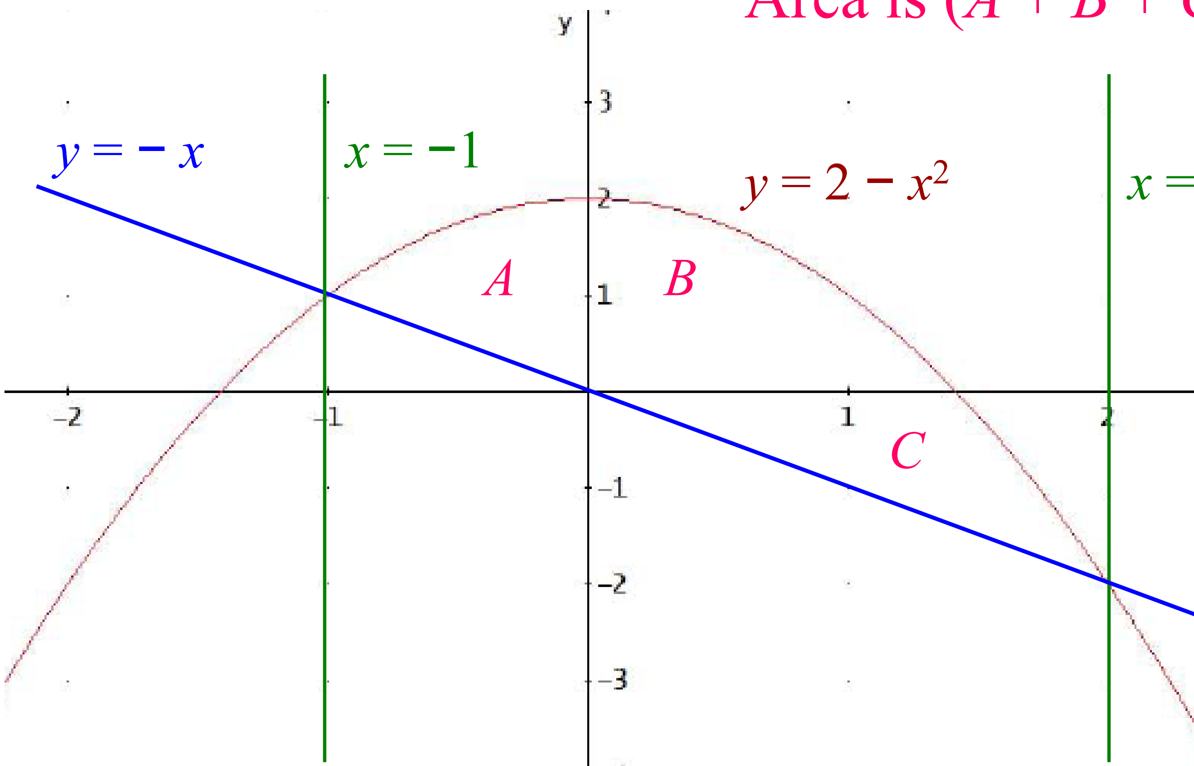
Area is $(A + B + C)$



Find the points of intersection by setting

$$\begin{aligned} 2 - x^2 &= -x \\ 0 &= x^2 - x - 2 = (x+1)(x-2) \\ x &= -1 \quad \text{or} \quad x = 2 \end{aligned}$$

Area is $(A + B + C)$



Parabola is higher than the straight line over the interval [–1, 2].

Parabola is higher than the straight line over the interval [–1, 2].

$$\begin{aligned}
\text{Area} &= \int_{-1}^{2} \left[(2 - x^2) - (-x)\right]\, dx \\
&= \left[2x - \frac{1}{3}x^3 + \frac{1}{2}x^2\right]_{-1}^{2} \\
&= \left(4 - \frac{8}{3} + 2\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right) \\
&= \boxed{\frac{9}{2}}
\end{aligned}$$

End of Chapter 3