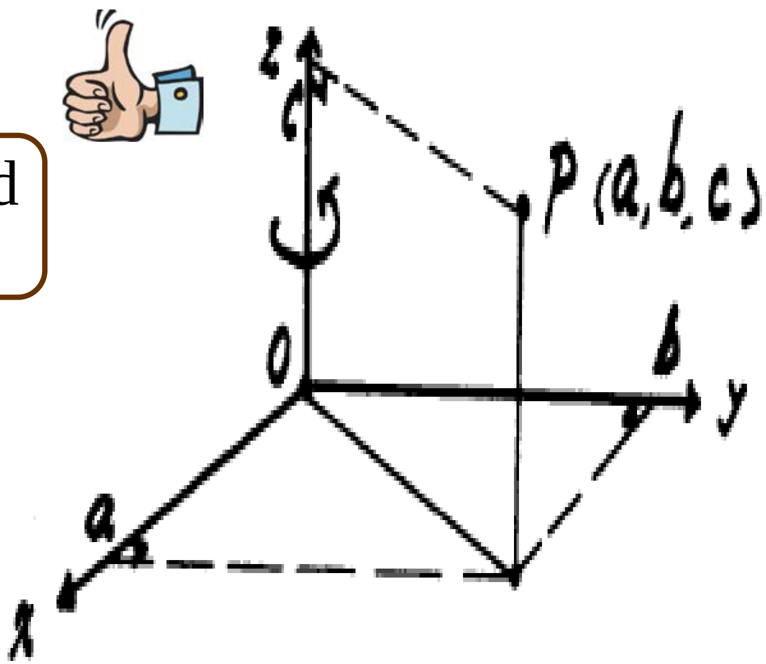
CH 5 - Three Dimensional Space

- ***Coordinate System***
- ***Vectors***
    - *- **dot** product*
    - *- **vector** product*
- ***Lines***
- ***Planes***
- ***Vector functions***
- ***Special curves***

5.1. The *Cartesian* ***Coordinate System***

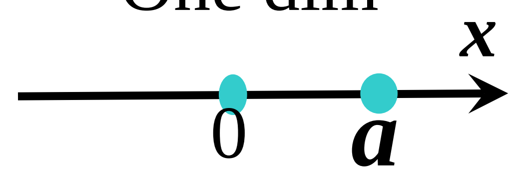
- **rectangular coordinates**

right-handed system

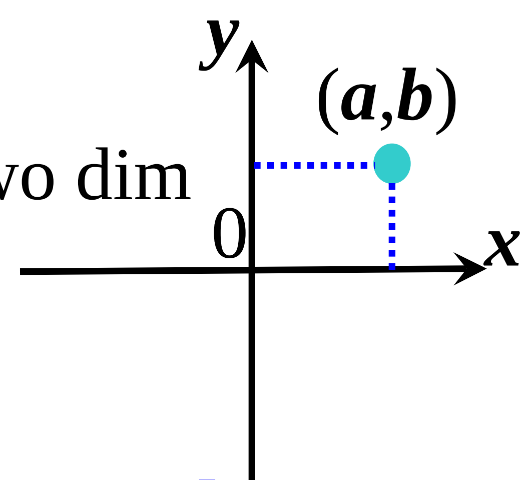


Three dim

One dim



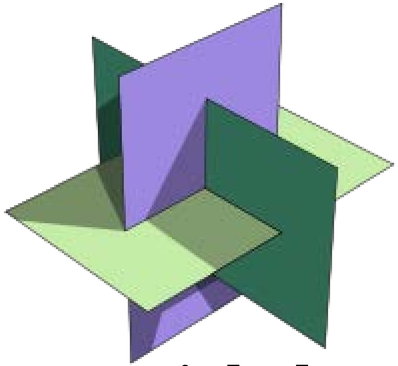
Two dim



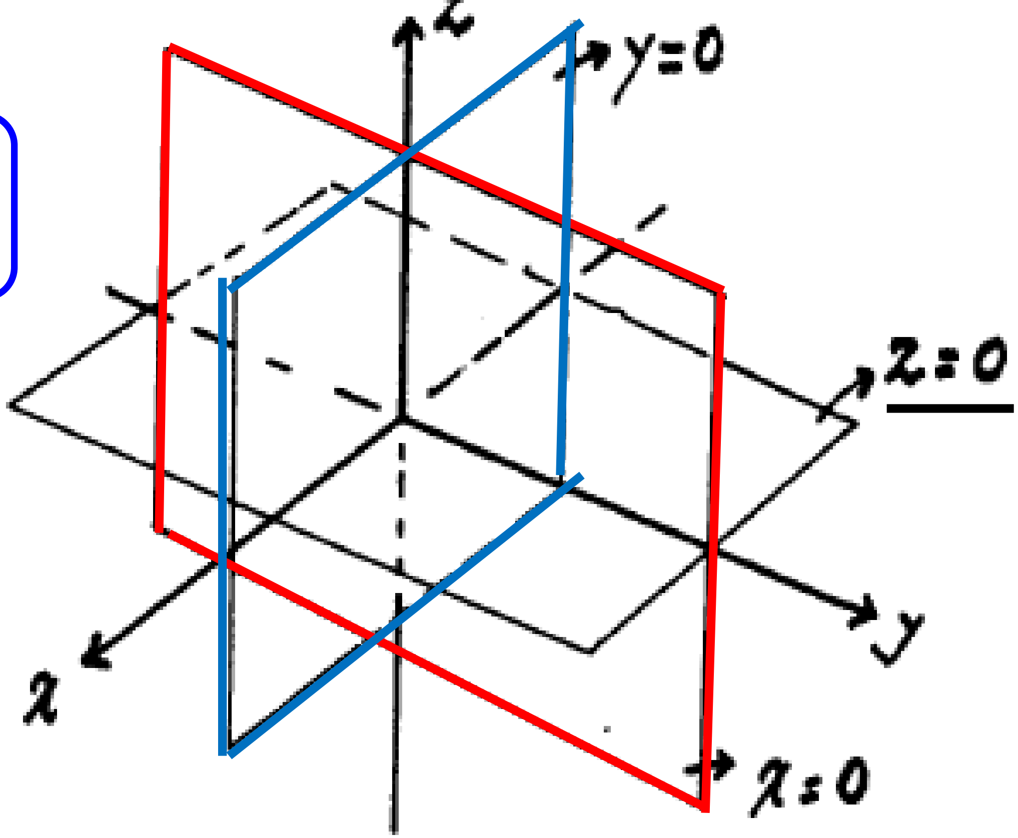
4 quadrants

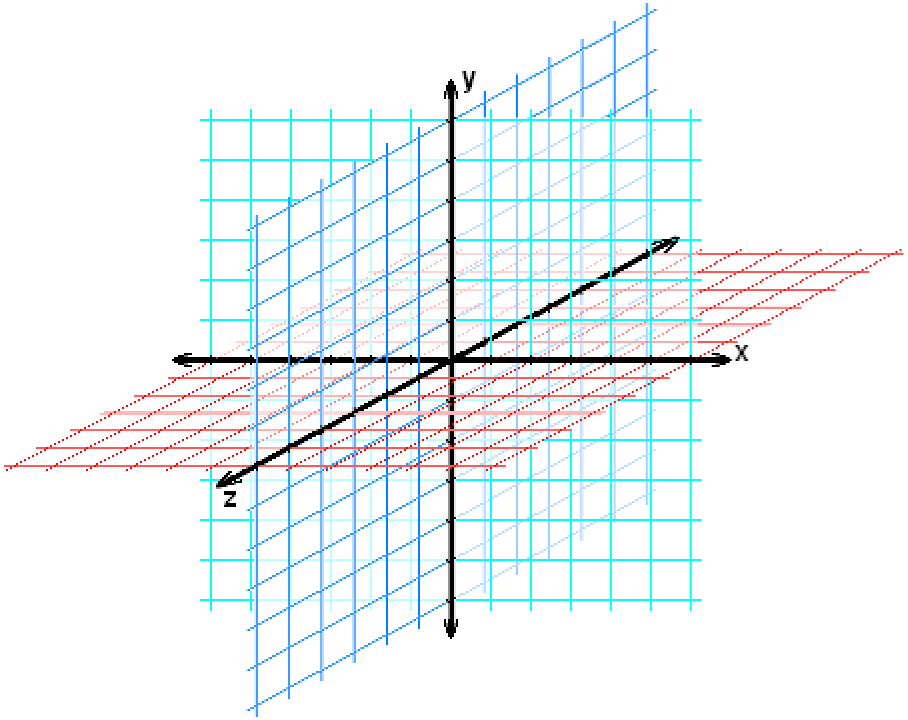
• planes: x = 0, y = 0, z = 0

eight *octants*



Three axial planes (x=0, y=0, z=0) divide space into eight equal octant



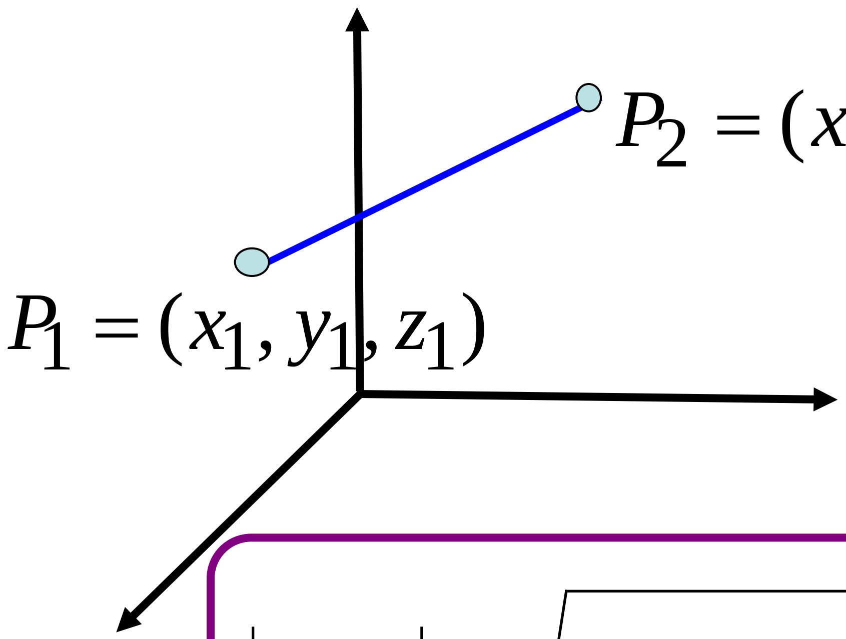


www.walter-fendt.de/m14e

Click

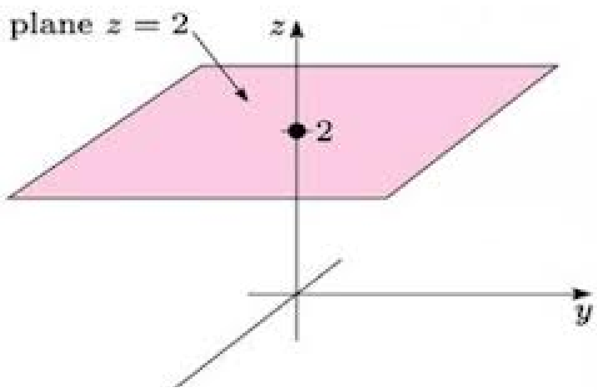
vector equation of a line in three- dim space

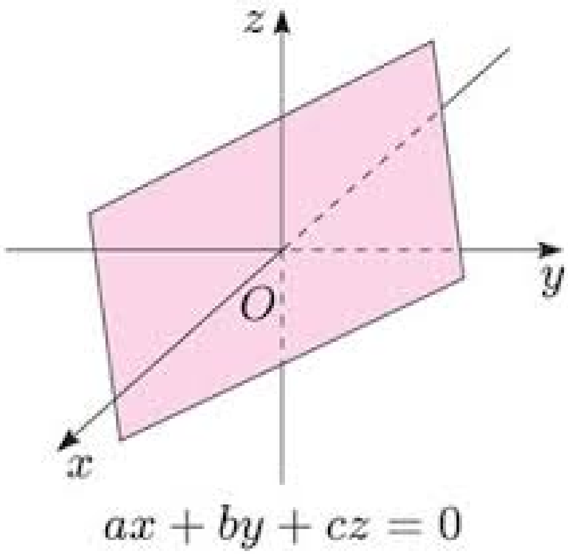
Distance between 2 points

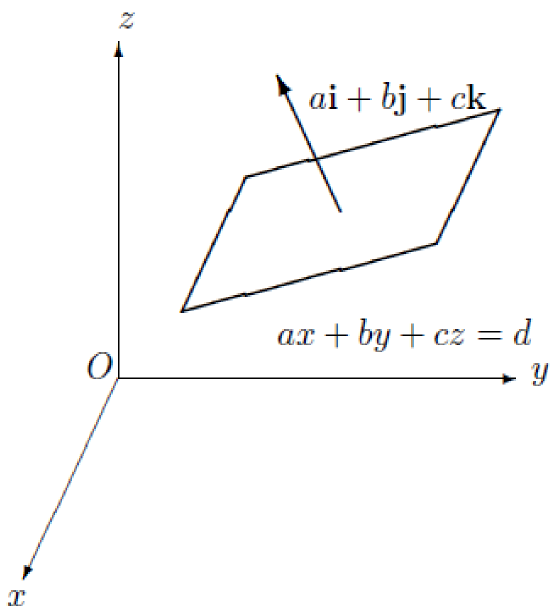


$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Motivation







: The plane $ax + by + cz = d$.

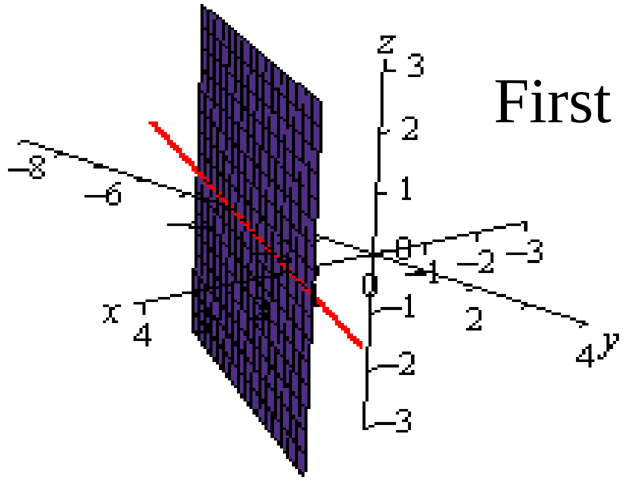
Equation of plane is given by

$$ax + by + cz = b$$

and vector $ai + bj + ck$

is perpendicular to the plane

The graph of y=2x-3 in three dim space



First , y=2x-3 is a line (red line) in the x-y plane

the graph is then a vertical plane that lies over the line

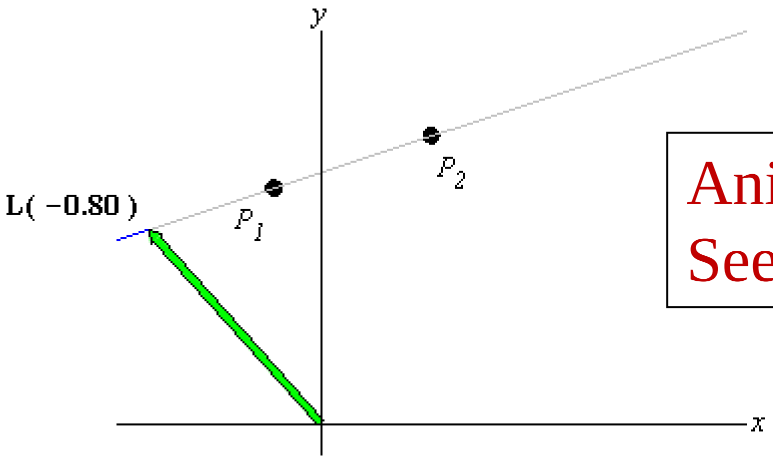
http://www.math.uri.edu/~bkaskosz/flashmo/tools/parsur/

With the above examples, we may ask what is an equation of a line in three dim space ? Expressing the equation in terms of x, y, z is not easy.

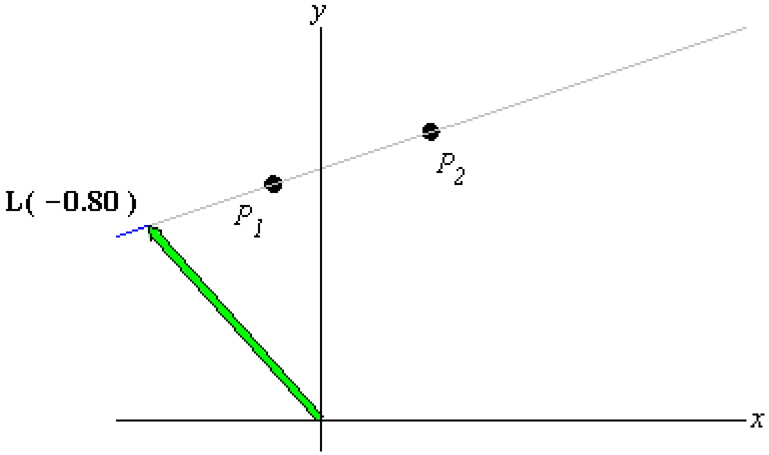
We shall use vector.

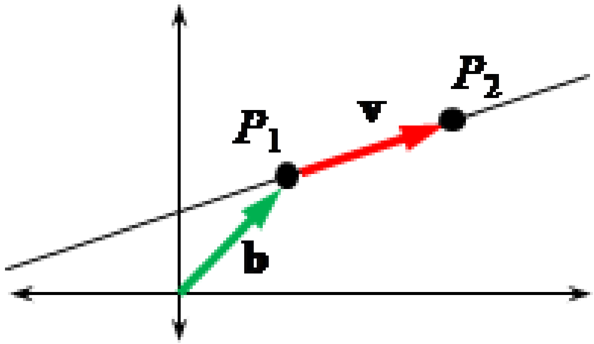
First look at the following example.

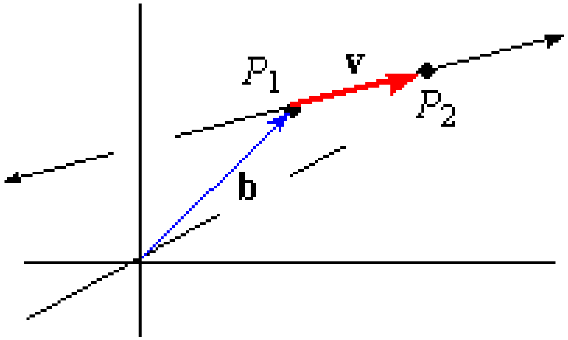
In this example, we use vectors to represent a line in two dim space

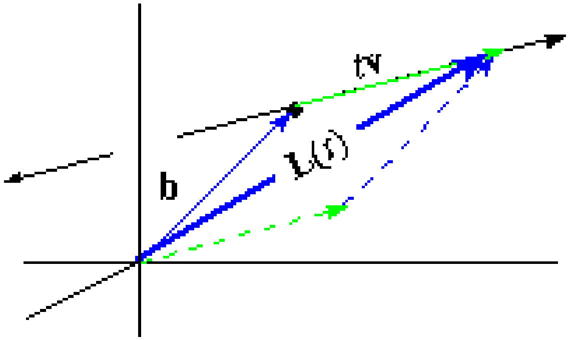


Animation
See power pt slide



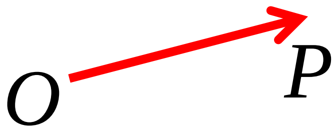




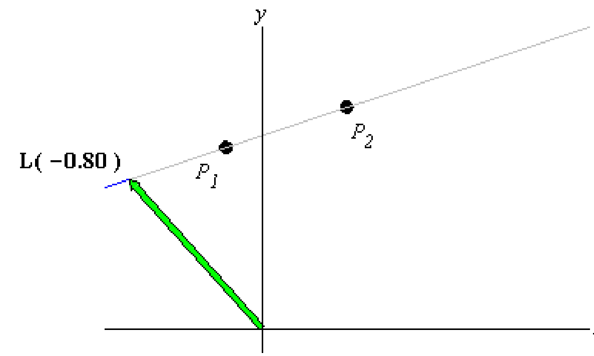


$$L(t) = b + tv$$

Each point $P(x,y)$ in R^2 corresponds to a position vector $OP = \begin{bmatrix} x \\ y \end{bmatrix}$



A line corresponds to the endpoints of a set of 2-dimensional position vectors.

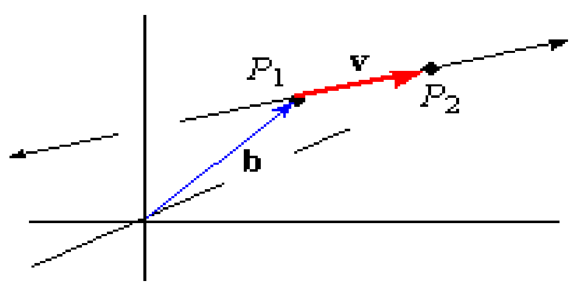


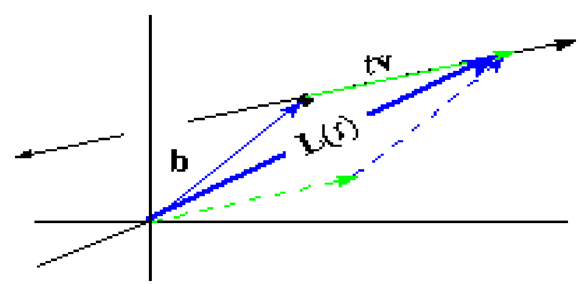
We use

$$L(t) = b + tv$$

to represent a line in two dim space

where b is a fixed position vector with end point on the line and v is a constant vector which is parallel to the line.

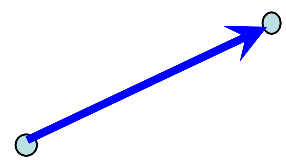




5.2 Vectors *(an important tool in three dim space)*

♣ ***Vector*** — a ***directed*** line segment PQ

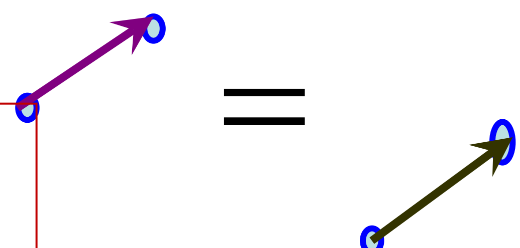
Q (***terminal*** point)



(***initial*** point) P

- ***direction of vector*** — **direction** of the arrow
- ***magnitude of vector***— **length** of the line segment PQ

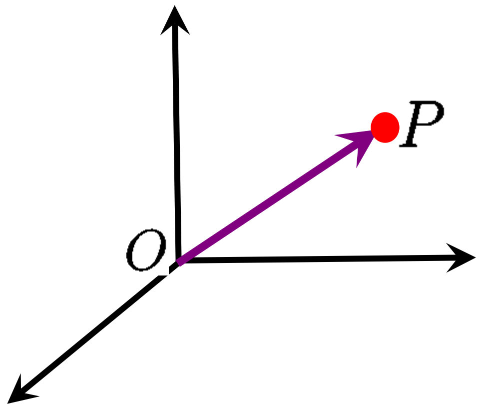
♣ Two vectors are ***equal*** if they have the same direction & magnitude.



They may have different initial and terminal points

5.2.1 *Terminologies and notations*

- The ***position*** vector of point $P(x, y, z)$



The initial point is always zero

$$\overrightarrow{op} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

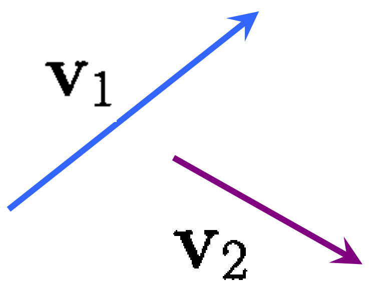
$$\left|\overrightarrow{op}\right| = \sqrt{x^2 + y^2 + z^2}$$ **(*magnitude*)**

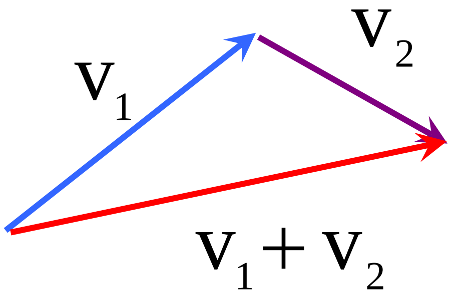
- The ***zero*** vector $\mathbf{O} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

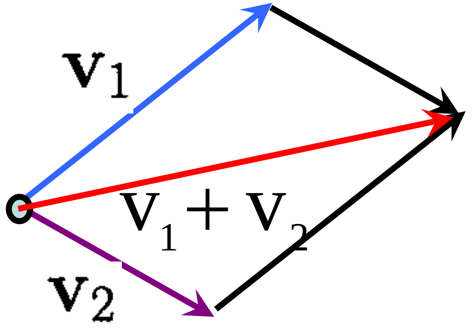
Addition

The sum of $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ is

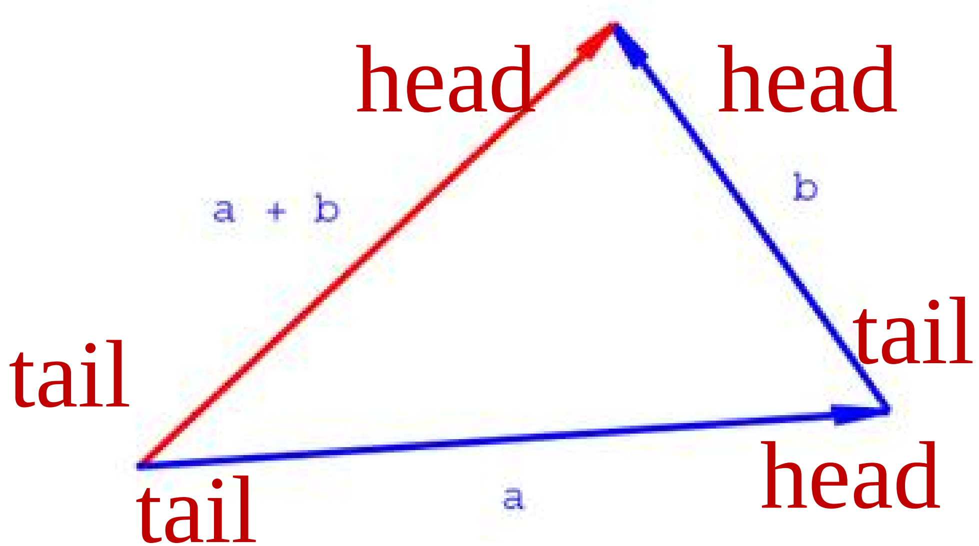
$$\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}.$$





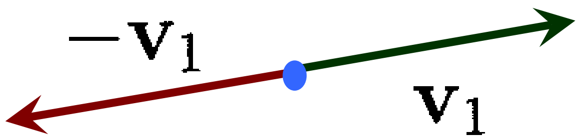


we *always* must connect vectors 'head to tail' and the *resultant* vector (which represents the vector sum) is drawn from the tail of the first vector to the head of the last vector

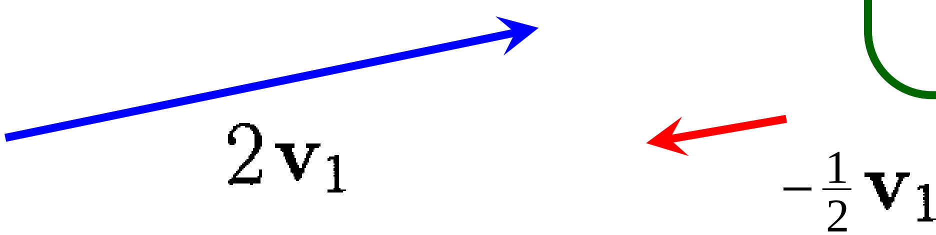


Negative & *Scalar Multiplication*

- The negative of $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ is $\boxed{-\mathbf{v}_1 = \begin{bmatrix} -x_1 \\ -y_1 \\ -z_1 \end{bmatrix}.}$



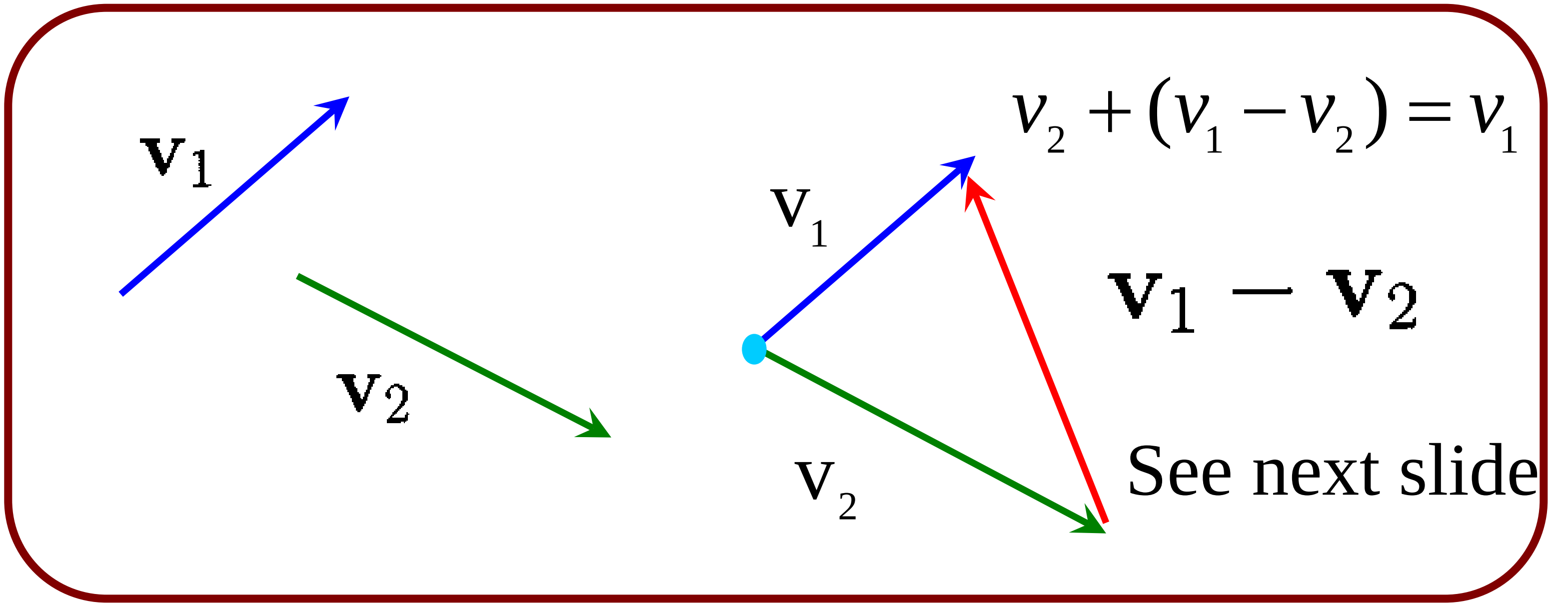
- The scalar multiplication of $\mathbf{v}_1$ is $\boxed{c\mathbf{v}_1 = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}.}$



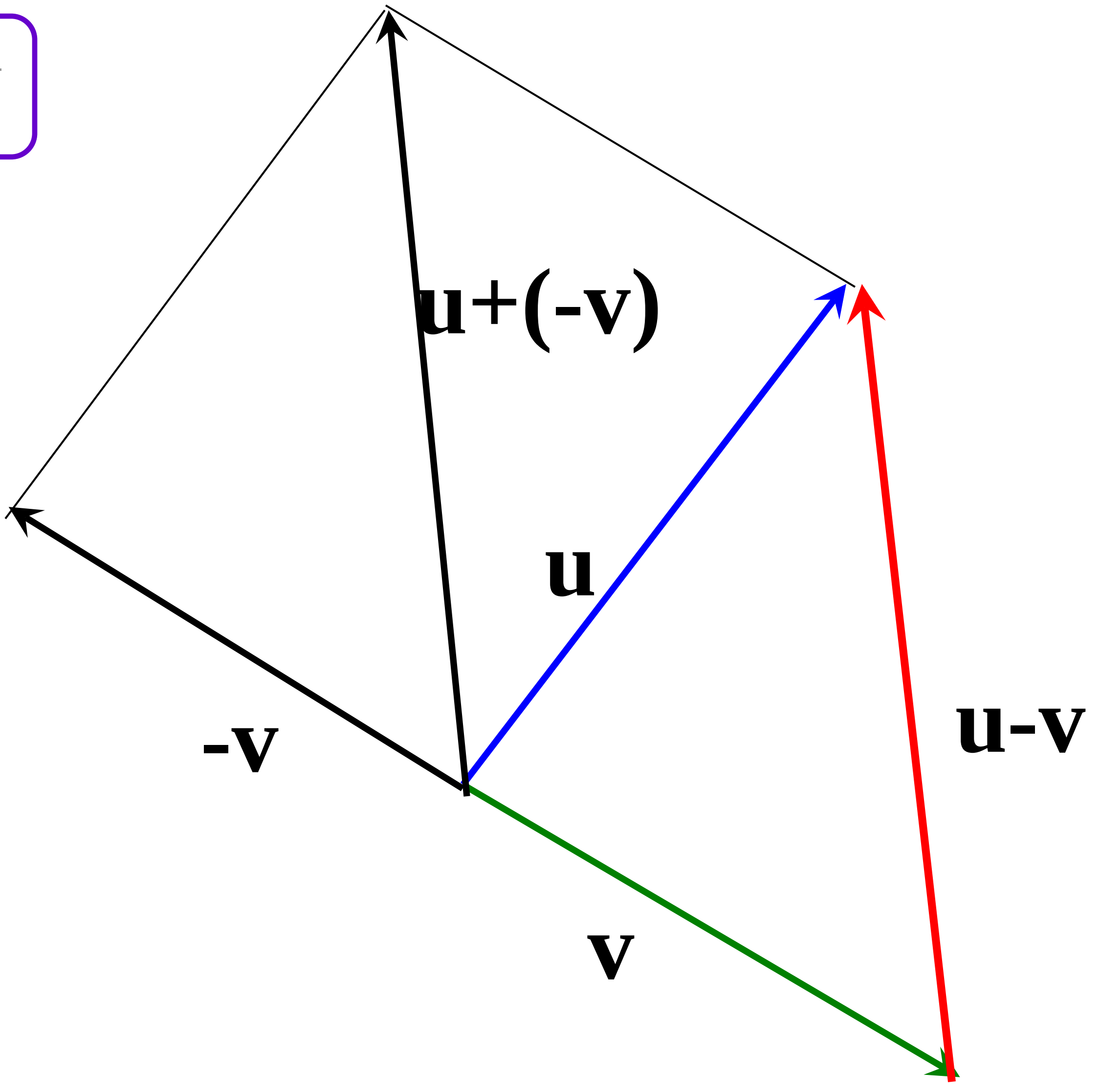
Difference

- The difference $\mathbf{v}_1 - \mathbf{v}_2$ is

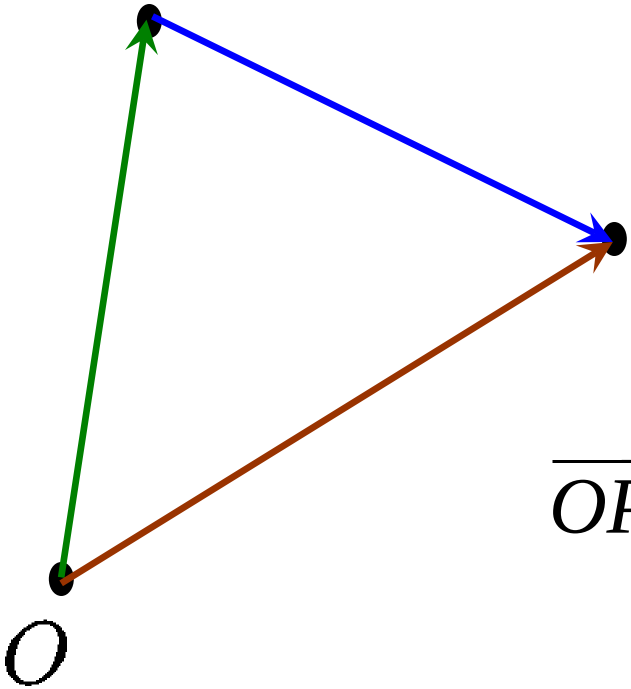
$$\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_1 + (-\mathbf{v}_2) = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ -y_2 \\ -z_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{bmatrix}$$



u-v



$$P = (x_1, y_1, z_1)$$



$$Q = (x_2, y_2, z_2)$$

$$\overrightarrow{OP} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \qquad \overrightarrow{OQ} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

$$\overrightarrow{OQ} - \overrightarrow{OP} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} = \overrightarrow{PQ}$$

$$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$$

5.2.2 Example

♣ Let P_1, P_2, Q_1 and Q_2 be the points $(3, 2, -1)$, $(0, 0, 0)$, $(5, 5, 4)$ and $(2, 3, 5)$ respectively.

$$\overrightarrow{P_1Q_1} = \begin{bmatrix} 5-3 \\ 5-2 \\ 4-(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$\overrightarrow{P_2Q_2} = \begin{bmatrix} 2-0 \\ 3-0 \\ 5-0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

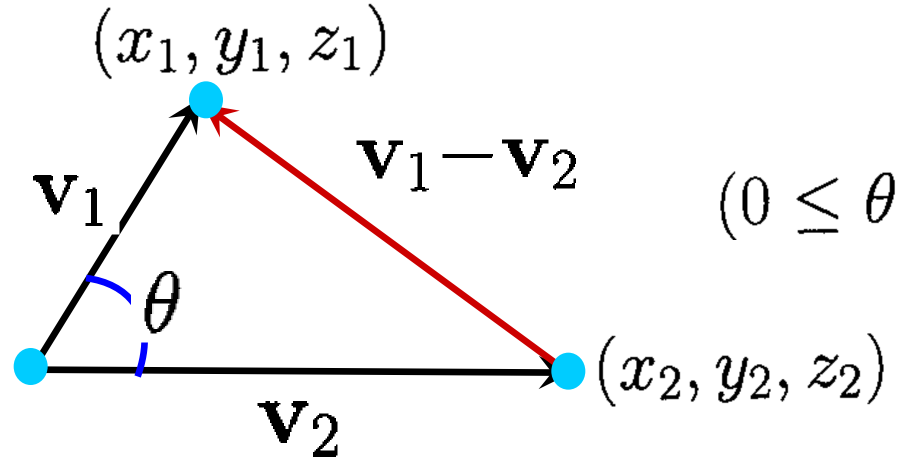
$$\Rightarrow \overrightarrow{P_1Q_1} = \overrightarrow{P_2Q_2}.$$

$$\boxed{|\overrightarrow{P_1Q_1}|}$$

The magnitude of $\overrightarrow{P_1Q_1}$ is

$$\boxed{||\overrightarrow{P_1Q_1}||} = \sqrt{(2)^2 + (3)^2 + (5)^2} = \sqrt{38}.$$

5.2.3 *Angle* between 2 vectors



$(0 \leq \theta \leq 180^0)$

$$||\mathbf{v}_1 - \mathbf{v}_2||^2 = ||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2 - 2||\mathbf{v}_1||\ ||\mathbf{v}_2|| \cos\theta$$

*p*7 (LN)

$$\boxed{\cos\theta = \frac{x_1\ x_2 + y_1\ y_2 + z_1\ z_2}{||\mathbf{v}_1||\ ||\mathbf{v}_2||}}$$

5.2.4 *Scalar* or *dot product*

- Given

$$\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

define (their ***dot product***)

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2.$$

$$\boxed{\cos\theta} = \frac{x_1\ x_2 + y_1\ y_2 + z_1\ z_2}{||\mathbf{v}_1||\ ||\mathbf{v}_2||} = \boxed{\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{||\mathbf{v}_1||\ ||\mathbf{v}_2||}.\quad (0 \le \theta \le 180^0)}$$

from previous slide

$\mathbf{v}_1$ and $\mathbf{v}_2$ are perpendicular
$\iff \mathbf{v}_1 \cdot \mathbf{v}_2 = 0.$

5.2.5 *Example*

♣ Find the ***angle*** between $\mathbf{v}_1$ and $\mathbf{v}_2$

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (2)(-1) + (4)(2) + (5)(3) = 21.$$

$$||\mathbf{v}_1|| = \sqrt{2^2 + 4^2 + 5^2} = \sqrt{45},$$

$$||\mathbf{v}_2|| = \sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{14}.$$

$$\boxed{\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{||\mathbf{v}_1||\ ||\mathbf{v}_2||}} = \frac{21}{\sqrt{45}\sqrt{14}} = \frac{\sqrt{7}}{\sqrt{10}}$$

Thus θ is approximately $33°13'$.

Perpendicular Vectors

- The vectors

$$\mathbf{w}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \text{ and } \mathbf{w}_2 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

are ***perpendicular*** as

$$\mathbf{w}_1 \cdot \mathbf{w}_2 = (2)(4) + (-5)(2) + (1)(2) = 0.$$

5.2.6 *Properties* of *dot product*

If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are vectors in xyz-space and c is a real number, then

(a) $\mathbf{v}_1 \cdot \mathbf{v}_1 = ||\mathbf{v}_1||^2 \geq 0$

(b) $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_1$

(c) $(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{v}_3 = \mathbf{v}_1 \cdot \mathbf{v}_3 + \mathbf{v}_2 \cdot \mathbf{v}_3$

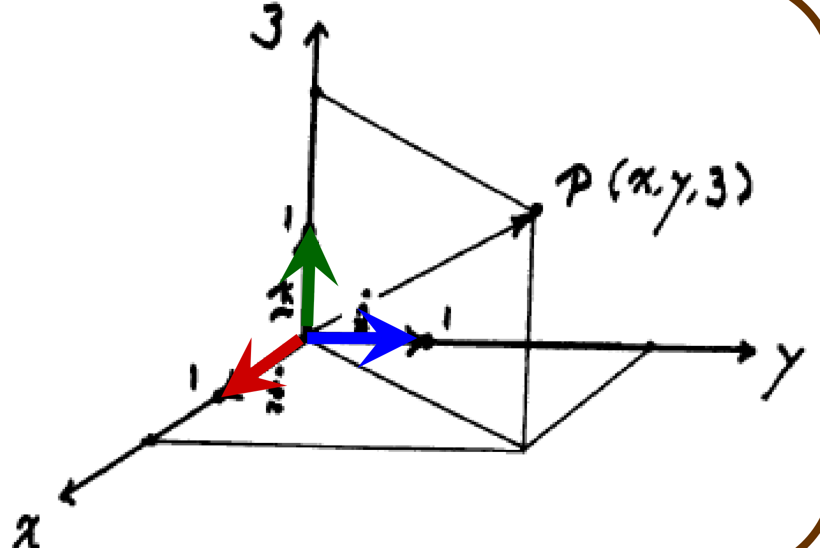
(d) $(c\mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot (c\mathbf{v}_2) = c(\mathbf{v}_1 \cdot \mathbf{v}_2)$.

Unit Vectors

Unit vector: a vector of *length* one.

The standard unit vectors:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



$$\mathrm{i} \cdot \mathrm{j} = \mathrm{j} \cdot \mathrm{k} = \mathrm{k} \cdot \mathrm{i} = 0$$

Notice that every vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.}$$

For example,

$$\mathbf{w} = \begin{bmatrix} 4 \\ -5 \\ 22 \end{bmatrix} = 4\mathbf{i} - 5\mathbf{j} + 22\mathbf{k}.$$

The unit vector with the same direction as $\mathbf{w}$ is

$$\left\| \frac{\mathrm{w}}{\|\mathrm{w}\|} \right\| = \frac{1}{\|\mathrm{w}\|}\|\mathrm{w}\| = 1$$

$$\begin{aligned} \frac{1}{||\mathbf{w}||}\mathbf{w} &= \frac{1}{\sqrt{4^2 + 5^2 + 22^2}}(4\mathbf{i} - 5\mathbf{j} + 22\mathbf{k}) \\ &= \frac{4}{\sqrt{525}}\mathbf{i} - \frac{5}{\sqrt{525}}\mathbf{j} + \frac{22}{\sqrt{525}}\mathbf{k}. \end{aligned}$$

$$\left\| \frac{\mathrm{w}}{\|\mathrm{w}\|} \right\| = \frac{1}{\|\mathrm{w}\|} \|\mathrm{w}\| = 1$$

$$\mathrm{w} = \|\mathrm{w}\| \frac{\mathrm{w}}{\|\mathrm{w}\|}$$

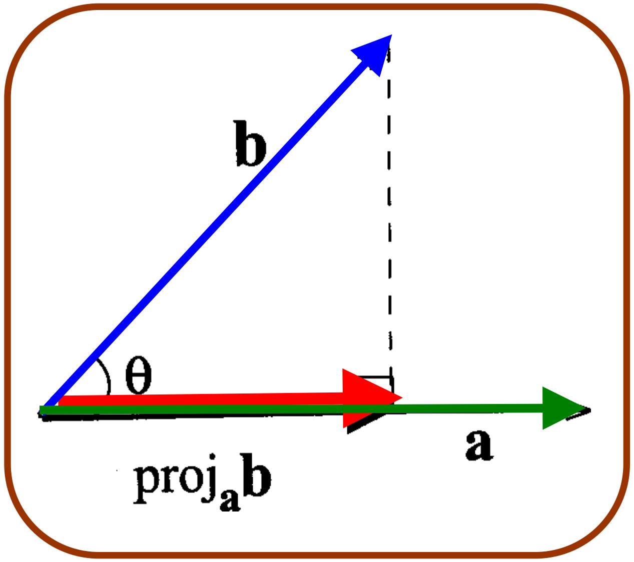
$$= \|\mathrm{w}\| \ (\textit{unit vector in } \ \mathrm{w} \ \text{direction})$$

$$= \|\mathrm{w}\| \ (\textit{unit vector in } \ \text{vector v direction})$$

if v and w are in the same direction

5.2.8 *Projection*

Let **a** & **b** be vectors. The ***projection*** $\mathrm{proj}_{\mathbf{a}}\mathbf{b}$ of **b** onto **a** is illustrated below:



Note that $||\mathrm{proj}_{\mathbf{a}}\mathbf{b}|| = ||\mathbf{b}|| \cos\theta$

$$= ||\mathbf{b}|| \frac{(\mathbf{a}\cdot\mathbf{b})}{||\mathbf{a}||\,||\mathbf{b}||} = \frac{(\mathbf{a}\cdot\mathbf{b})}{||\mathbf{a}||}.$$

Note that

See last slide

$$\boxed{\mathrm{proj}_{\mathbf{a}}\mathbf{b}} = (||\mathrm{proj}_{\mathbf{a}}\mathbf{b}||)\cdot(\text{unit vector along } \mathbf{a})$$

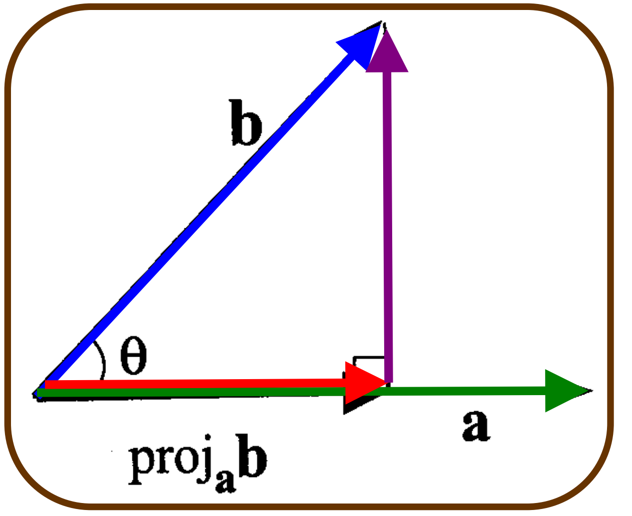
Hence

$$= \frac{(\mathbf{a}\cdot\mathbf{b})}{||\mathbf{a}||}\left(\frac{\mathbf{a}}{||\mathbf{a}||}\right) = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||^2}\,\mathbf{a} = \boxed{\frac{\mathbf{b}\cdot\mathbf{a}}{||\mathbf{a}||^2}\,\mathbf{a}}$$

Express **b** *as the* ***sum*** *of* ***vectors parallel*** *&* ***perpendicular*** *to* **a**

♣

$$\boxed{\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{b}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}}$$



$$\mathbf{b} = \text{proj}_{\mathbf{a}}\mathbf{b} + \boxed{\mathbf{b} - \text{proj}_{\mathbf{a}}\mathbf{b}}$$

Parallel to **a** ***perpendicular*** to **a**

5.2.9 *Example*

Find the ***projection*** of $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$ onto $\mathbf{b} = \mathbf{i} + \mathbf{j}$.

Solution.

$$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\,\mathbf{b}$$

$$= \frac{7}{2}(\mathbf{i} + \mathbf{j})$$

5.3 *Vector* or *Cross* **Product**

Given $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$,

their **vector product** or **cross product** is the vector

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$= (y_1 z_2 - y_2 z_1)\mathbf{i} - (x_1 z_2 - x_2 z_1)\mathbf{j} + (x_1 y_2 - x_2 y_1)\mathbf{k}$$

Example

- Given

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix},$$

their vector product is the vector

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 3 & -1 & -3 \end{vmatrix} = -\mathbf{i} + 12\mathbf{j} - 5\mathbf{k}.$$

5.3.1 *Properties* of *Cross* Product

Let $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ be vectors in xyz-space, and let c be a real number. Then

$\boxed{\text{(a)}}$ $\mathbf{v}_1 \times \mathbf{v}_2 = -\mathbf{v}_2 \times \mathbf{v}_1$.

(b) $\mathbf{v}_1 \times (\mathbf{v}_2 + \mathbf{v}_3) = \mathbf{v}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \mathbf{v}_3$.

(c) $(\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_3 + \mathbf{v}_2 \times \mathbf{v}_3$.

(d) $c(\mathbf{v}_1 \times \mathbf{v}_2) = (c\mathbf{v}_1) \times \mathbf{v}_2 = \mathbf{v}_1 \times (c\mathbf{v}_2)$.

$\boxed{\text{(e)}}$ $\mathbf{v}_1 \times \mathbf{v}_1 = \mathbf{O}$.

(f) $\mathbf{O} \times \mathbf{v}_1 = \mathbf{v}_1 \times \mathbf{O} = \mathbf{O}$.

Geometrical Interpretation

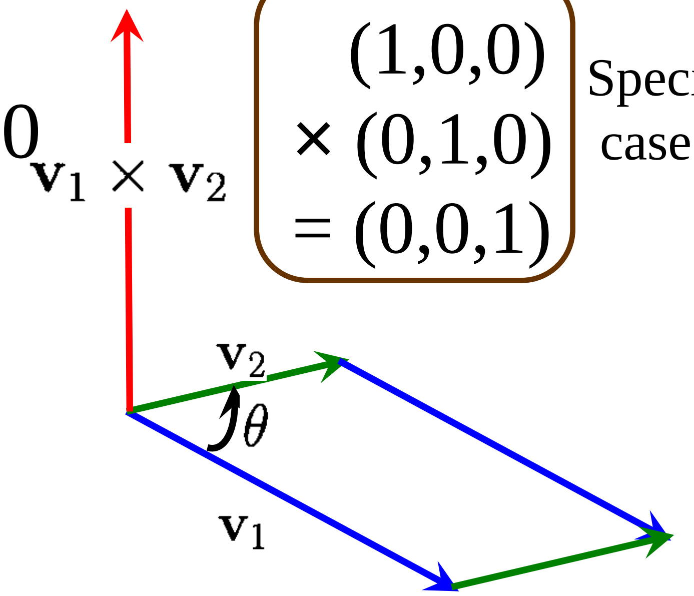
5.3.2 *Direction* $\mathbf{v}_1 \times \mathbf{v}_2$

$$\boxed{(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_1 = 0 = (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_2}$$

$$\mathbf{v}_1 \times \mathbf{v}_2 = (\underline{y_1 z_2} - y_2 z_1)\mathbf{i} - (\underline{x_1 z_2} - x_2 z_1)\mathbf{j} + (x_1 y_2 - x_2 y_1)\mathbf{k}$$

$$\mathbf{v}_1 = \underline{x_1}\ \mathbf{i} + \underline{y_1}\ \mathbf{j} + z_1\ \mathbf{k}$$

$$(v_1 \times v_2) \bullet v_1 = (y_1 z_2 - y_2 z_1) x_1 + \ldots = 0$$



$$\boxed{\begin{array}{r} (1,0,0) \\ \times\ (0,1,0) \\ = (0,0,1) \end{array}}$$

Special case

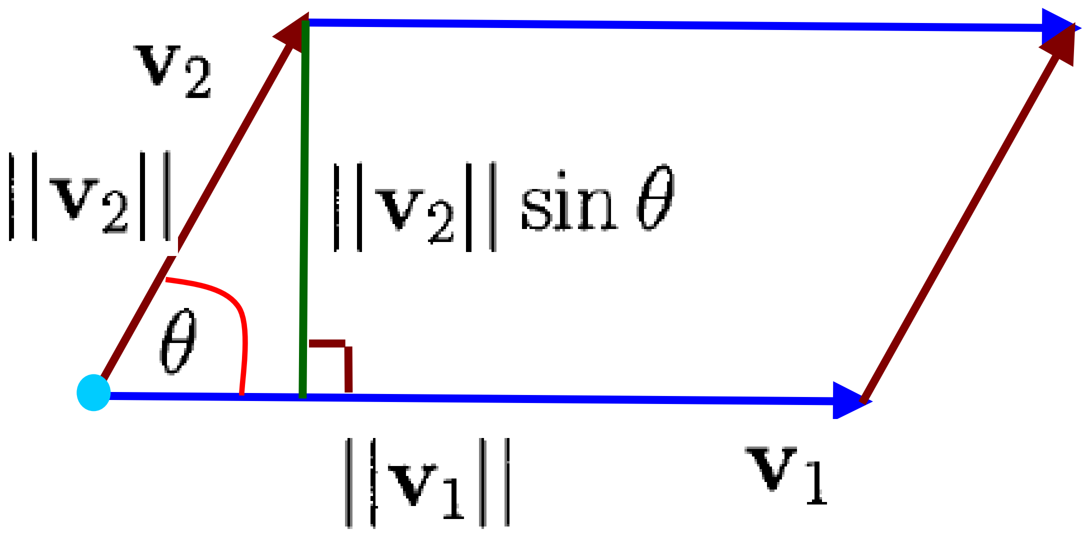
5.3.2 *Magnitude* $\mathbf{v}_1 \times \mathbf{v}_2$

$$\boxed{||\mathbf{v}_1 \times \mathbf{v}_2|| = ||\mathbf{v}_1||\,||\mathbf{v}_2||\sin\theta}$$

Proof omitted

♣ $$\boxed{||\mathbf{v}_1 \times \mathbf{v}_2|| = ||\mathbf{v}_1||\,||\mathbf{v}_2|| \sin\theta}$$

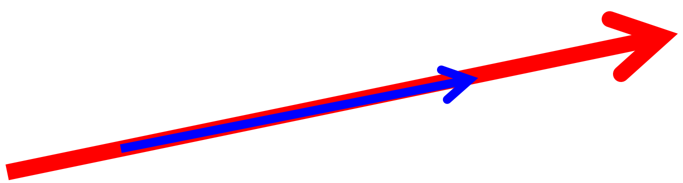
= the ***area*** of the following ***parallelogram***



Suppose two vectors u and v are parallel

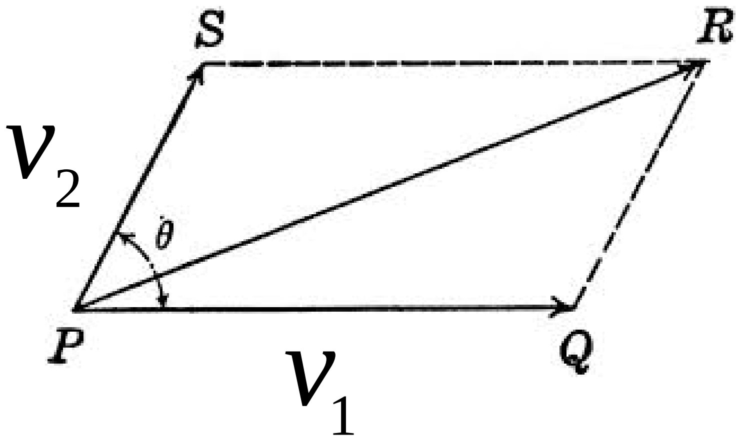
Then there exists a real number t such that

$$u = tv$$



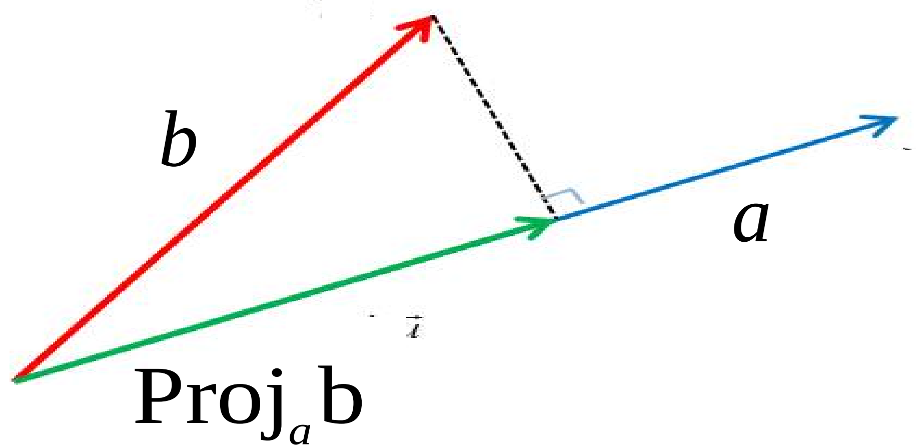
Summary

$$\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\|\ \|\mathbf{v}_2\|}$$



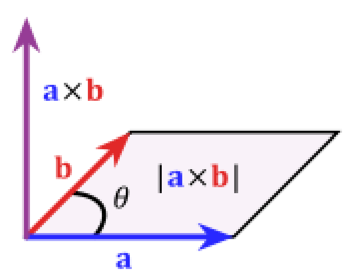
$$\text{proj}_a b = \frac{b \bullet a}{\|a\|^2} a = \frac{b \bullet a}{\|a\|} \frac{a}{\|a\|}$$

$$\|\text{Proj}_a b\| = \frac{\|a \bullet b\|}{\|a\|}$$



$$\|a \times b\| = area\ of\ \text{parallelogram}$$

induced by a and b

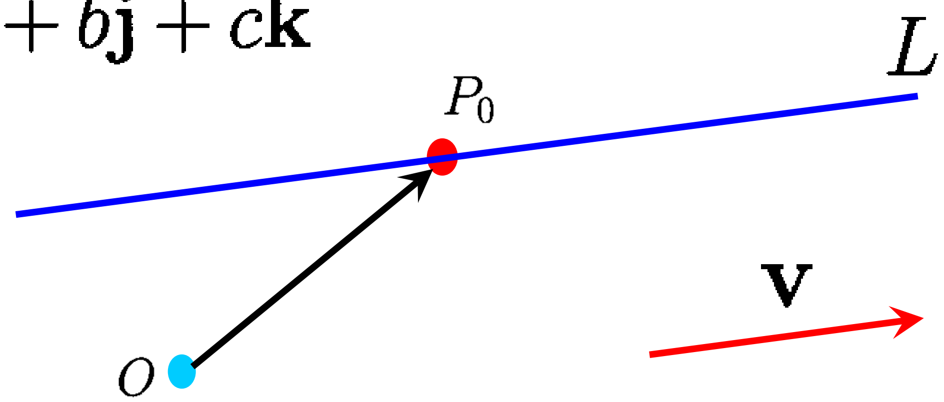


5.4 *Lines* in Space

Problem Given point P_0 with position vector

$$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$$

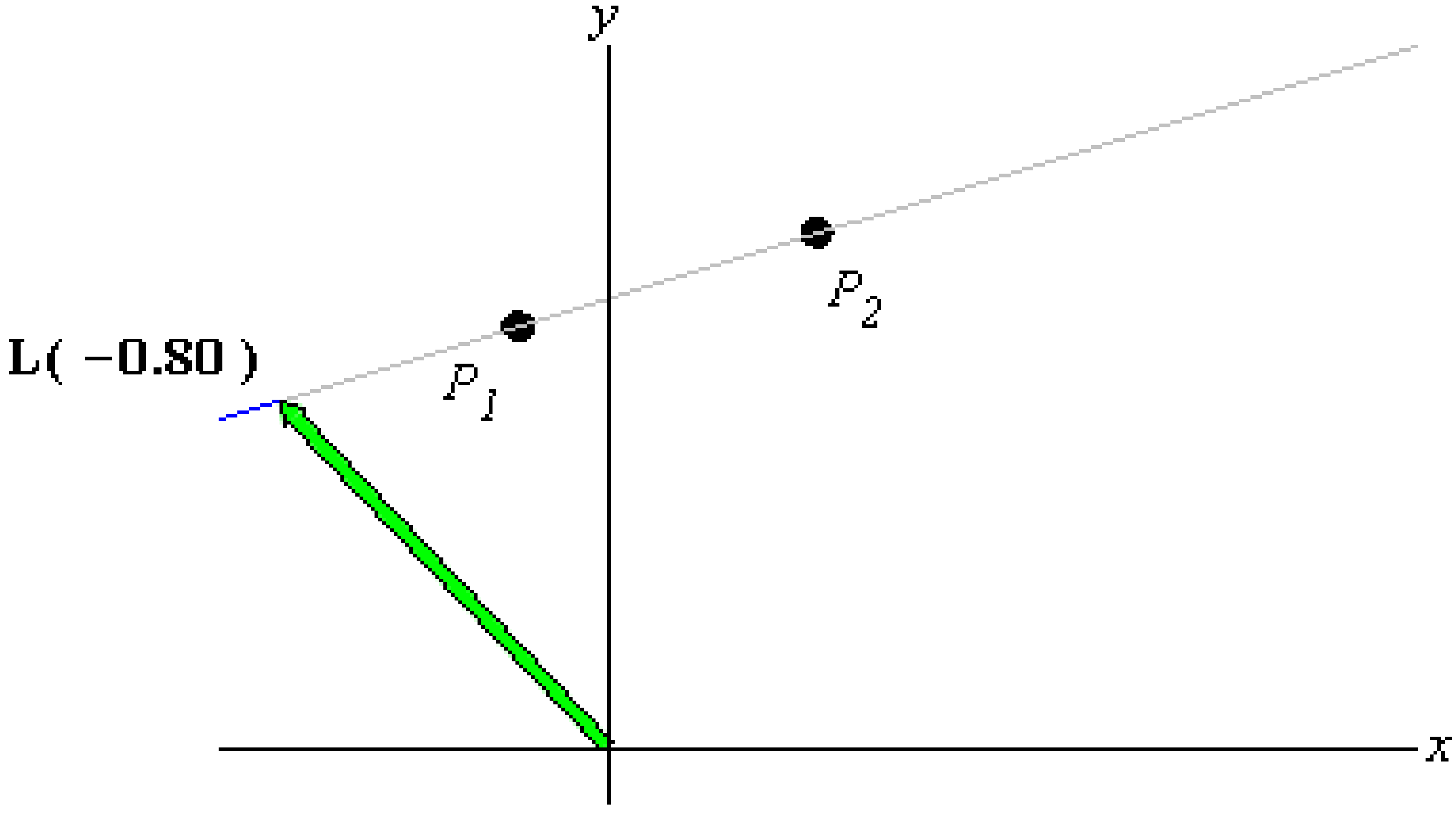
& vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$



find the ***equation*** of the ***line*** ***L*** passing through P_0

& parallel to $\mathbf{v}$

Recall line equation in 2-dim space

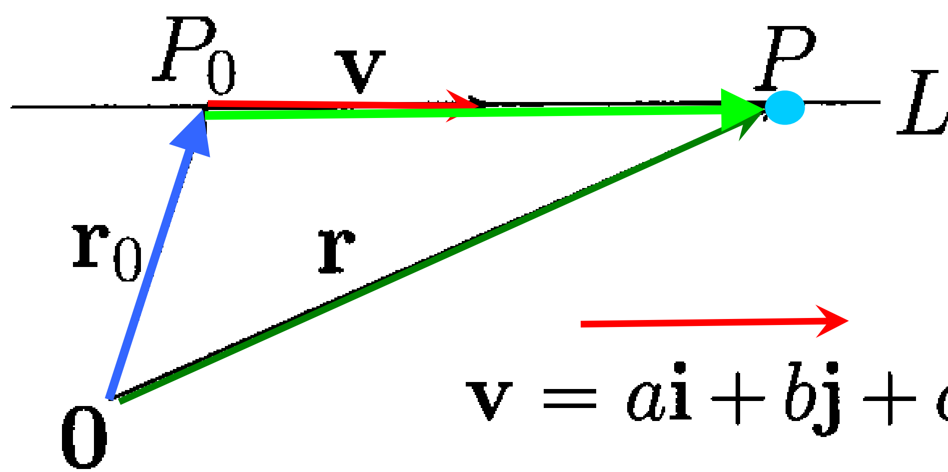


y
L(−0.80)
P_1
P_2
x

5.4.1 *Vector Equation* of a ***Line L***

- Let ***P*** be a point on L with ***position*** vector

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.



$\overrightarrow{P_0P}$ and v are parallel

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$\overrightarrow{OP} = r = r_0 + \overrightarrow{P_0P} = r_0 + tv \text{ for some } t \in \mathbf{R}.$$

$$= (x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}) + t(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \quad (3)$$

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \quad (t \in \mathbf{R}) \qquad \textbf{vector equation of } L$$

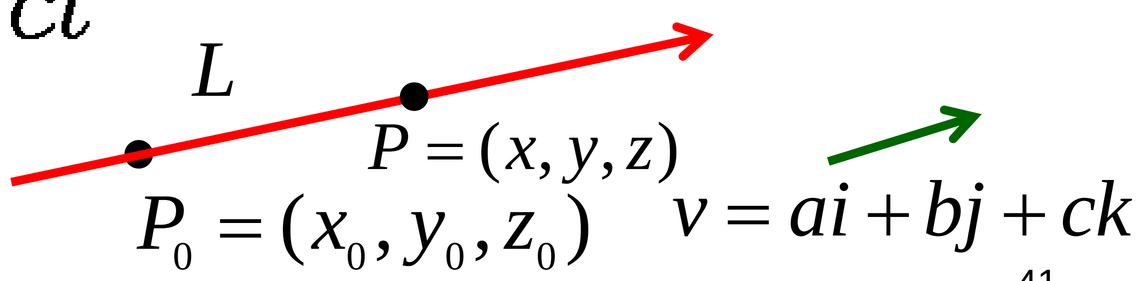
5.4.2 *Parametric equation* (another version) of *L*

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}) + t(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}).$$

$$\begin{cases} x = x_0 + at, \\ y = y_0 + bt, \\ z = z_0 + ct \\ t \in \mathbf{R} \end{cases}$$

— ***Parametric equation*** of ***L***

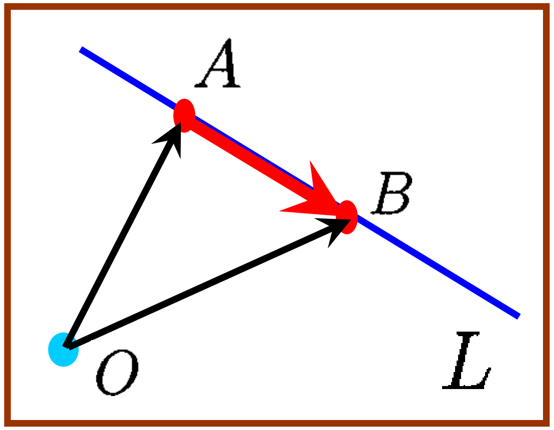


5.4.3 *Example*

A and B have position vectors

$$-3\mathbf{i}+2\mathbf{j}-3\mathbf{k} \quad \text{and} \quad \mathbf{i}-\mathbf{j}+4\mathbf{k}$$

Write down the parametric equations of L passing through A and B.



$$\overrightarrow{AB}=(\mathbf{i}-\mathbf{j}+4\mathbf{k})-(-3\mathbf{i}+2\mathbf{j}-3\mathbf{k})$$
$$=4\mathbf{i}-3\mathbf{j}+7\mathbf{k}.$$

$\overrightarrow{AB}$ and L are parallel

Parametric equation **of** ***L***

$$\begin{cases} x=-3+4t, \\ y=2-3t, \\ z=-3+7t \end{cases} \qquad t\in\mathbf{R}$$

$$\boxed{\begin{aligned} x&=x_0+at, \\ y&=y_0+bt, \\ z&=z_0+ct \end{aligned}}$$

5.4.4 *Example*

Given the following lines

$$L_1 : \ \mathbf{r} = \mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}),$$

$$L_2 : \ \mathbf{r} = (2\mathbf{i} + \mathbf{j}) + t_2\left(3\mathbf{i} + \tfrac{9}{2}\mathbf{j} + \tfrac{9}{2}\mathbf{k}\right) \text{ and}$$

$$L_3 : \ \mathbf{r} = (2\mathbf{i} + \mathbf{j}) + t_3(3\mathbf{i} + \mathbf{j})$$

(a) Find the position vector of the point of intersection of L_1 and L_2.

(b) Show that L_1 and L_3 are skew, i.e. do not intersect and they are not parallel

(a)

$$L_1: \ \mathbf{r} = \mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}),$$
$$L_2: \ \mathbf{r} = (2\mathbf{i} + \mathbf{j}) + t_2\left(3\mathbf{i} + \tfrac{9}{2}\mathbf{j} + \tfrac{9}{2}\mathbf{k}\right)$$

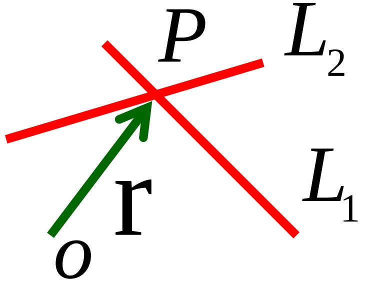
Eliminating r (finding the parameter for point of intersection P) :

$$\mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = (2\mathbf{i} + \mathbf{j}) + t_2\left(3\mathbf{i} + \frac{9}{2}\mathbf{j} + \frac{9}{2}\mathbf{k}\right)$$

Comparing the components :

$$t_1 = 1 + 3t_2, \quad 2t_1 = 1 + \frac{9}{2}t_2, \quad 3t_1 = \frac{9}{2}t_2$$

Solving : $t_1 = -1, \quad t_2 = -2/3.$



The required position vector :

$$\mathbf{r} = \mathbf{i} + (-1)(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = \boxed{-2\mathbf{j} - 3\mathbf{k}.}$$

(b) $$L_1: \ \mathbf{r} = \mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}),$$

$$L_3: \ \mathbf{r} = (2\mathbf{i} + \mathbf{j}) + t_3(3\mathbf{i} + \mathbf{j})$$

Eliminating r :

$$\mathbf{i} + t_1(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = (2\mathbf{i} + \mathbf{j}) + t_3(3\mathbf{i} + \mathbf{j})$$

It follows that

$$t_1 = 1 + 3t_3, \quad 2t_1 = 1 + t_3, \quad 3t_1 = 0$$

which are ***inconsistent***.

Thus $\boxed{L_1 \text{ and } L_3 \text{ do not intersect.}}$

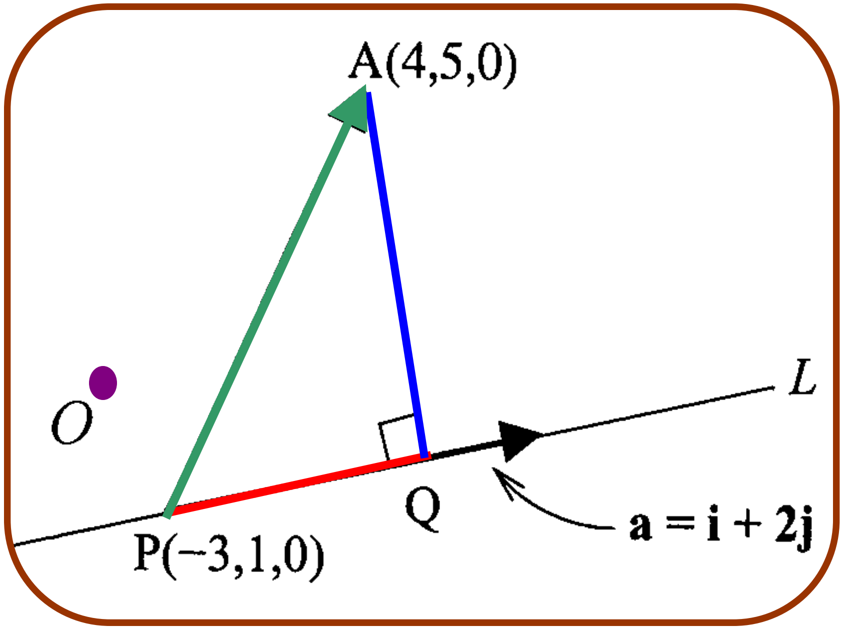
$i + 2j + 3k$ and $3i + j$ are not parallel,

so L_1 and L_3 not parallel

5.4.5 *Example*

Find the (***shortest***) ***distance*** from *A* (4**i** + 5**j**) to the line *L* : **r** = (−3**i** + **j**) + *t*(**i** + 2**j**).

$$\overrightarrow{PA} = \overrightarrow{OA} - \overrightarrow{OP} = (4\mathbf{i} + 5\mathbf{j}) - (-3\mathbf{i} + \mathbf{j})$$
$$= 7\mathbf{i} + 4\mathbf{j}.$$



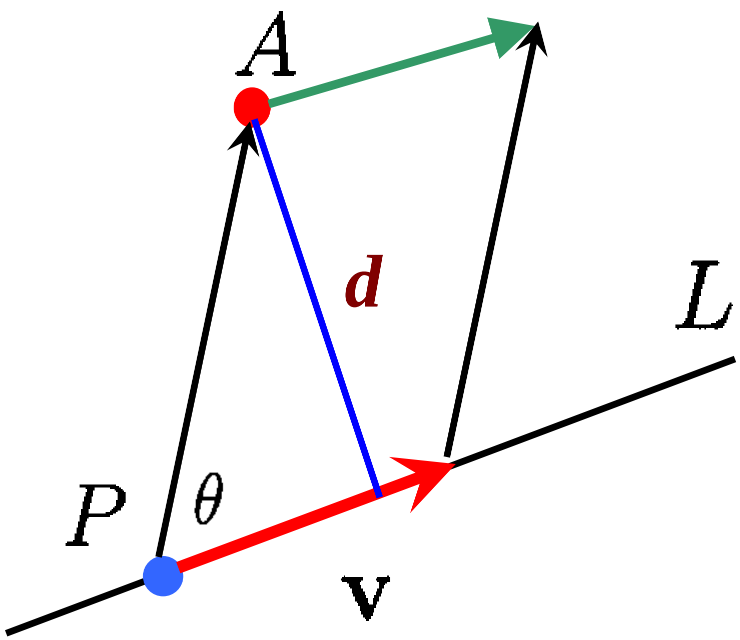
$$|PQ| = \frac{\mathbf{b} \cdot \mathbf{a}}{||\mathbf{a}||}$$

$$= \frac{(\mathbf{i} + 2\mathbf{j}) \cdot (7\mathbf{i} + 4\mathbf{j})}{\sqrt{1^2 + 2^2}} = \frac{15}{\sqrt{5}}.$$

$$|AQ| = \sqrt{||\mathbf{b}||^2 - \left(\frac{15}{\sqrt{5}}\right)^2} = 2\sqrt{5}.$$

Distance from ***point A*** to ***line L***

$$= \boldsymbol{d} = \boxed{\frac{\|\overrightarrow{PA} \times \mathbf{v}\|}{\|\mathbf{v}\|}}$$



Proof:

$$\|\overrightarrow{PA} \times \mathbf{v}\|$$
$$= \|\overrightarrow{PA}\| \, \|\mathbf{v}\| \sin\theta$$
$$= \boldsymbol{d}\,\|\mathbf{v}\|$$

$$\boxed{\left\|\overrightarrow{PA} \times \mathrm{v}\right\| = \textit{area} \text{ of parallelogram}}$$

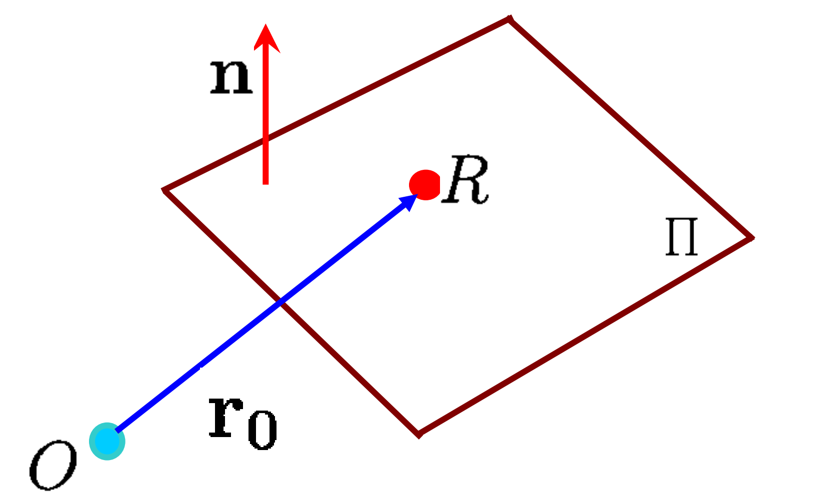
5.5 ***Planes*** in **Space**

♣ A ***plane*** **Π** in space is determined by

(i) a ***point*** on the plane &

(ii) its ***orientation*** (indicated by a ***normal*** to **Π**)

Problem Given point ***R*** in **Π** with position vector $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ & normal $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ to **Π** :



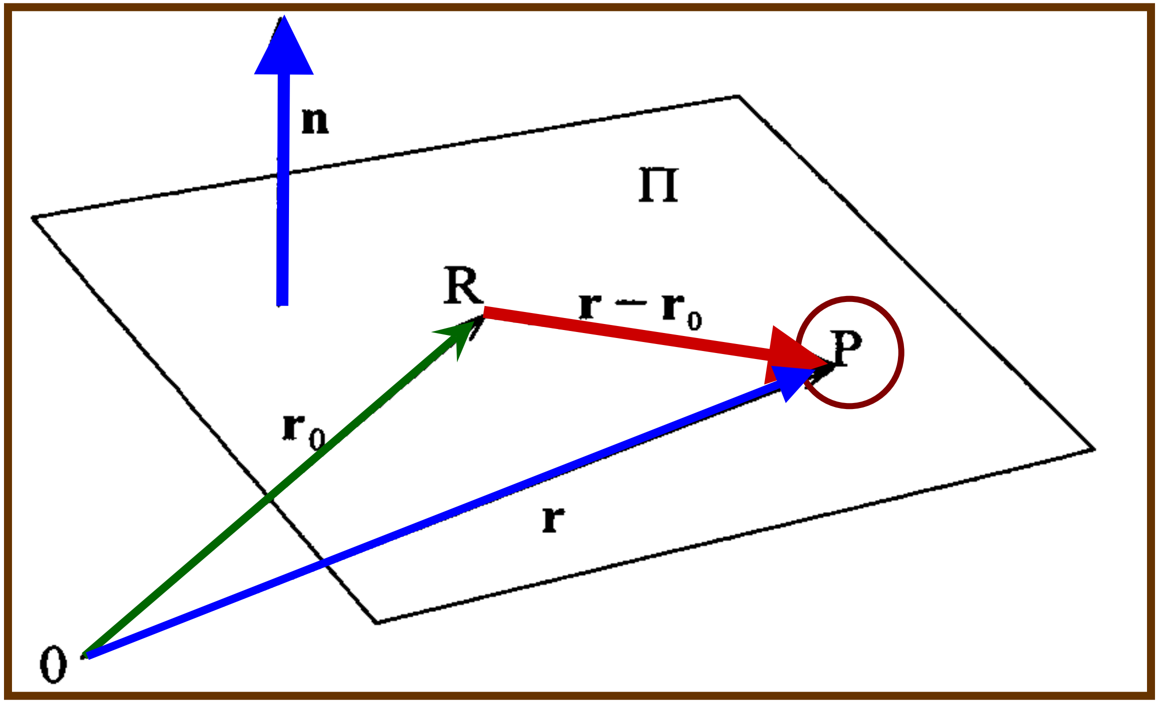
find an ***equation*** for **Π**

Vector equation for **Π**

- Let ***P*** be a point in **Π** with position vector

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Then



$$\boxed{(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0.}$$

5.5.1 *Cartesian* equation for $\mathbf{\Pi}$

$$\boxed{(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{n}=0}$$

$$\mathbf{r}-\mathbf{r_0}=(x-x_0)\mathbf{i}+(y-y_0)\mathbf{j}+(z-z_0)\mathbf{k}$$

$$\mathbf{n}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$$

$$(\mathbf{r}-\mathbf{r_0})\cdot\mathbf{n}=a(x-x_0)+b(y-y_0)+c(z-z_0)$$

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

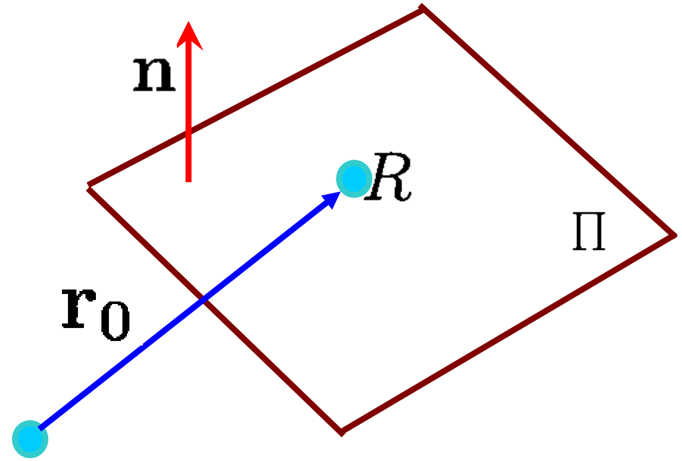
or

$$ax+by+cz=d,$$

where $d=ax_0+by_0+cz_0$

SUMMARY

Equations for Π



$$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$$

$$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

♣ **Vector equation :**

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$$

♣ **Cartesian equation :**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

♣ **Cartesian equation simplified :**

$$ax + by + cz = d,$$

where $d = ax_0 + by_0 + cz_0$

5.5.2 & 5.5.3 *Examples*

Find the equation of the plane passing through the point $(0, 2, -1)$ normal to the vector $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

$$\mathbf{r_0} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$$
$$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$ax + by + cz = d,$$
where $d = ax_0 + by_0 + cz_0$

Solution The required equation is

$$3x + 2y - z$$
$$= 3(0) + 2(2) - (-1)$$

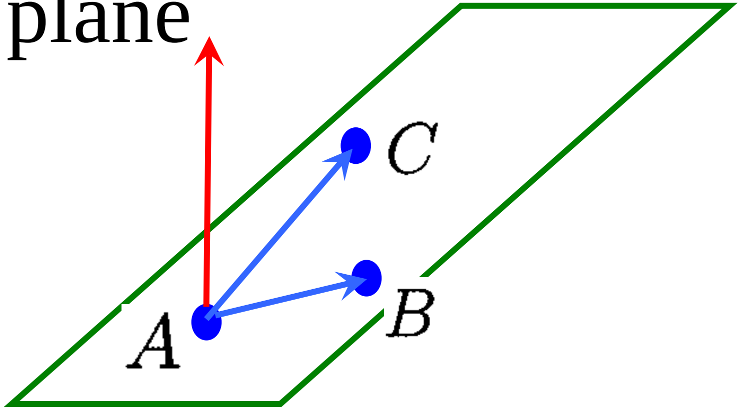
or $$3x + 2y - z = 5.$$

Find the vector equation of the plane passing through the points $A(0,0,1)$, $B(2,0,0)$ and $C(0,3,0)$.

We need a point and a normal, a point is there, so need to find a normal

Solution. A ***normal*** to the plane is

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix}$$



$$= 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}.$$

- The plane contains ***A***(0, 0, 1) with ***normal*** $3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

The ***equation*** of the plane is given by

$$\mathbf{r_0} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$$

$$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$ax + by + cz = d,$$

where $d = ax_0 + by_0 + cz_0$

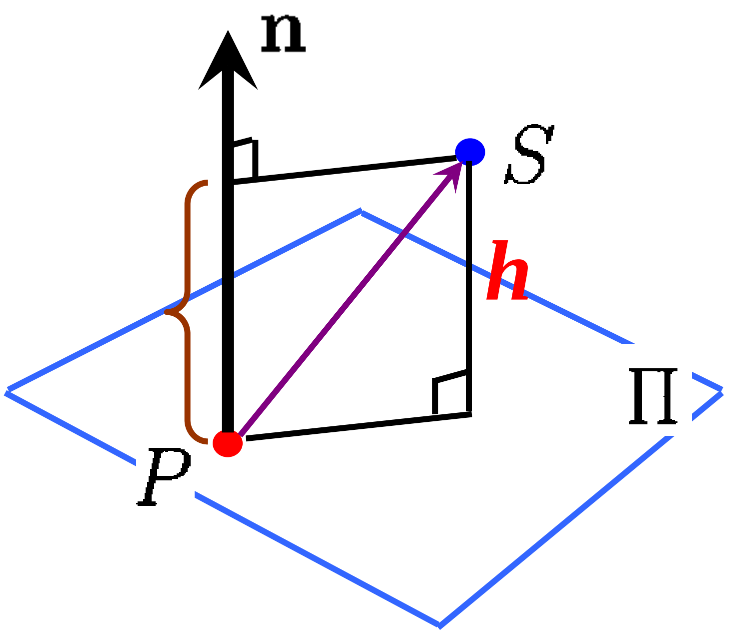
$$3x + 2y + 6z = 3(0) + 2(0) + 6(1),$$

or $$3x + 2y + 6z = 6.$$

5.5.4 *Distance h* from *point* **S**$(x_0,\ y_0,\ z_0)$ to *plane* **Π** $(ax + by + cz = d)$

Let $P(x_1,\ y_1,\ z_1)$ be a point in **Π**. Then ***h*** is the ***length*** of the ***projection*** of $\overrightarrow{PS}$ onto **n** :

$$\frac{|\overrightarrow{PS}\cdot\mathbf{n}|}{||\mathbf{n}||}$$



$$\boxed{h = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}}}$$

(p31)

$$\overrightarrow{PS} = \begin{bmatrix} x_0 - x_1 \\ y_0 - y_1 \\ z_0 - z_1 \end{bmatrix} = (x_0 - x_1)i + (y_0 - y_1)j + (z_0 - z_1)k$$

Plane equation is $ax + by + cz = d$

So normal **n** to the plane is $\mathrm{n} = ai + bj + ck$

$$\overrightarrow{PS} \bullet \mathrm{n} = (x_0 - x_1)a + (y_0 - y_1)b + (z_0 - z_1)c$$

$$= x_0 a + y_0 b + z_0 c - (x_1 a + y_1 b + z_1 c)$$

$$= x_0 a + y_0 b + z_0 c - d$$

5.5.5 *Example*

♣ Find the ***distance*** from the ***point*** (2, −3, 4) to the ***plane***: $x + 2y + 3z = 13$.

Solution. As

$$\boldsymbol{h} = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

$(a, b, c, d) = (1, 2, 3, 13)$ and

$$(x_0,\ y_0,\ z_0) = (2, -3, 4)\,,$$

we have

$$\boldsymbol{h} = \frac{|1(2) + 2(-3) + 3(4) - 13|}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{5}{\sqrt{14}}$$

2007

Question 3 (b) [5 marks]

Find the shortest distance from the point $(-1, 1, 2)$ to the plane

$$2x + 3y - z - 10 = 0.$$

Question 4 (a) [5 marks]

Let S be the plane which passes through the points $(1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 3)$. Find the distance from the point $(-1, -2, -3)$ to S.

Solutions

Q3(b)

$$d = \frac{|2(-1) + 3(1) - (2) - 10|}{\sqrt{4+9+1}} = \frac{11}{\sqrt{14}}$$

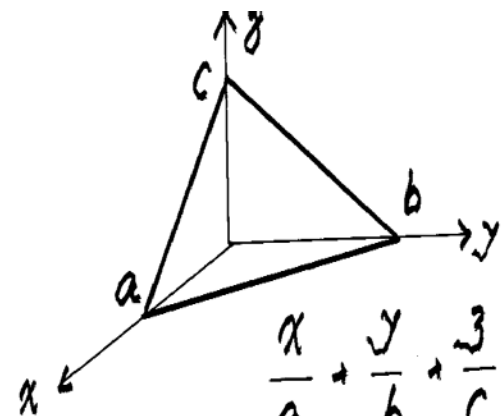
Q4(a) By inspection, or by a straight-forward calculation

$$S: \quad \frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$$

$$\text{i.e. } 6x + 3y + 2z = 6$$

$$\therefore \text{distance} = \frac{|6(-1) + 3(-2) + 2(-3) - 6|}{\sqrt{6^2 + 3^2 + 2^2}}$$

$$= \frac{24}{7}$$



$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

2008

Question 4 (a) [5 marks]

Let L_1 be a straight line which passes through the point $(-1, 0, 1)$ and suppose that L_1 is perpendicular to the plane $2x - y + 7z = 12$. Let L_2 be the line $\mathbf{r}(t) = (3 + t)\,\mathbf{i} + (-2 + 2t)\,\mathbf{j} + (15 - 3t)\,\mathbf{k}$. Find the coordinates of the point of intersection of L_1 and L_2.

Solution

Q4(a)

- $L_1: (x, y, z) = (-1, 0, 1) + s(2, -1, 7)$

$$= (-1+2s, -s, 1+7s)$$

$$\begin{cases} 3+t = -1+2s & \text{---- } ① \\ -2+2t = -s & \text{---- } ② \\ 15-3t = 1+7s & \text{---- } ③ \end{cases}$$

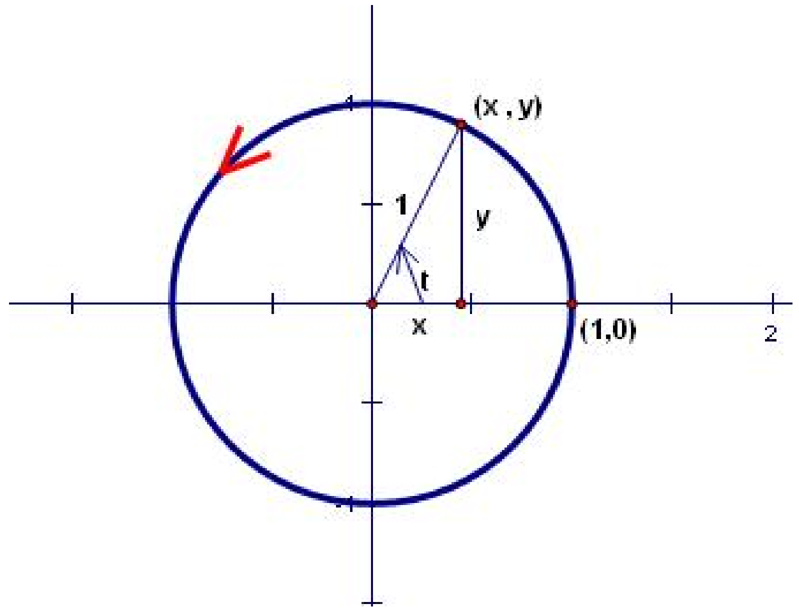
$① + 2② \Rightarrow -1+5t = -1 \Rightarrow t = 0$

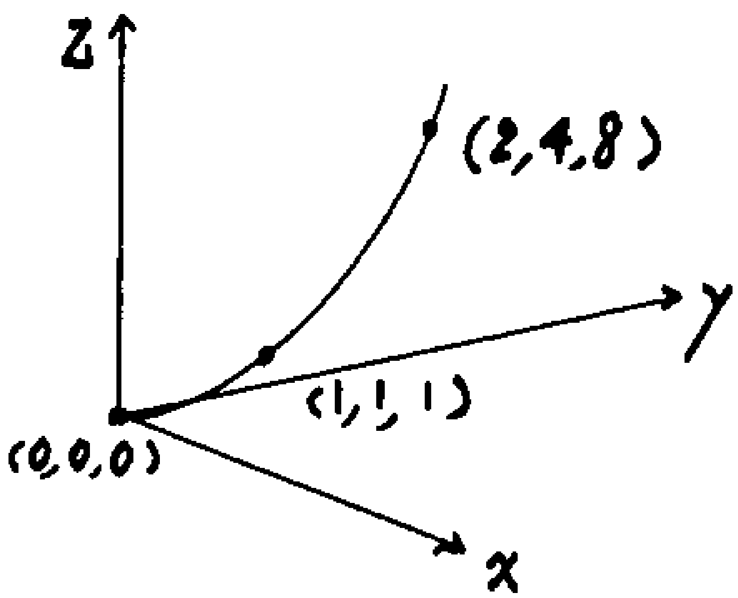
$\therefore$ point of intersection $= (3, -2, 15)$

5.6 *Vector* functions of *one* variable

- $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$

  $0 \le t$



- 

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k},\ \ t \ge 0$$

- A ***vector*** function

$$\mathbf{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

is a function *s.t.* the outputs are ***vectors***. The real-valued functions $f(t)$, $g(t)$ & $h(t)$ are called the ***component functions*** of $\mathbf{r}(t)$.

5.6.1 Example $\mathbf{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j} + (2 - 7t)\mathbf{k}.$

Then $\mathbf{r}(2) = 2\mathbf{i} + 5\mathbf{j} - 12\mathbf{k}.$

5.6.2 *Derivative*

If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$,

where *f*, *g* & *h* are ***differentiable***, then

$$(\mathbf{r})'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

5.6.3 *Example*

$$\mathbf{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j} + (2 - 7t)\mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + (2t)\mathbf{j} - 7\mathbf{k}$$

5.6.4 *Integral*

- The ***definite integral*** of a ***continuous*** vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

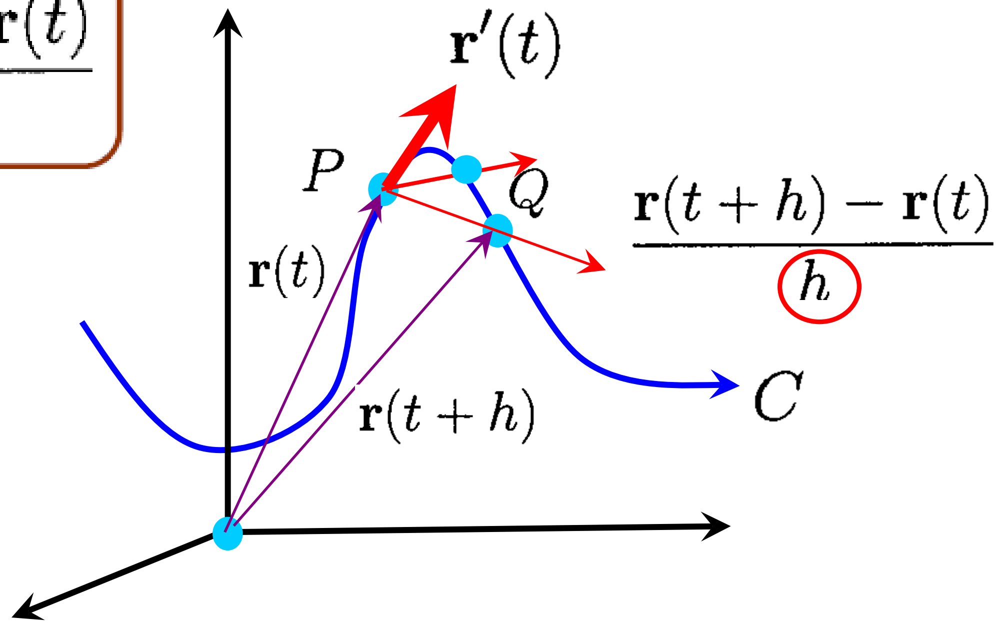
on the interval [*a*, *b*] is $\int_a^b \mathbf{r}(t)\,dt$

$$= \int_a^b f(t)\,dt\,\mathbf{i} + \int_a^b g(t)\,dt\,\mathbf{j} + \int_a^b h(t)\,dt\,\mathbf{k}.$$

$$\boxed{\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})dt = \left[t^2\right]_{t=0}^{t=2}\mathbf{i} + \left[t^3\right]_{t=0}^{t=2}\mathbf{j} = 4\mathbf{i} + 8\mathbf{j}.}$$

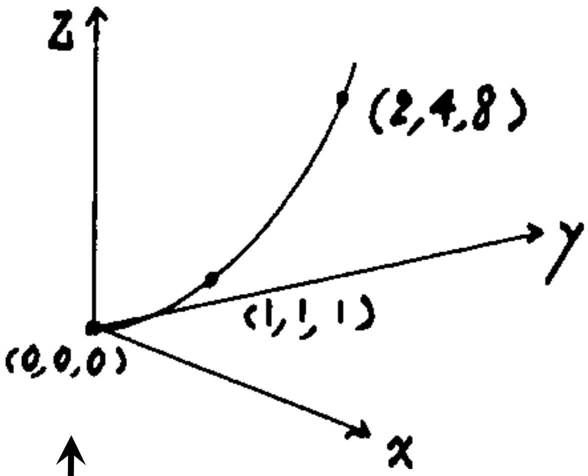
Geometrical interpretation of $\mathbf{r}'(t)$

$$\lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$$

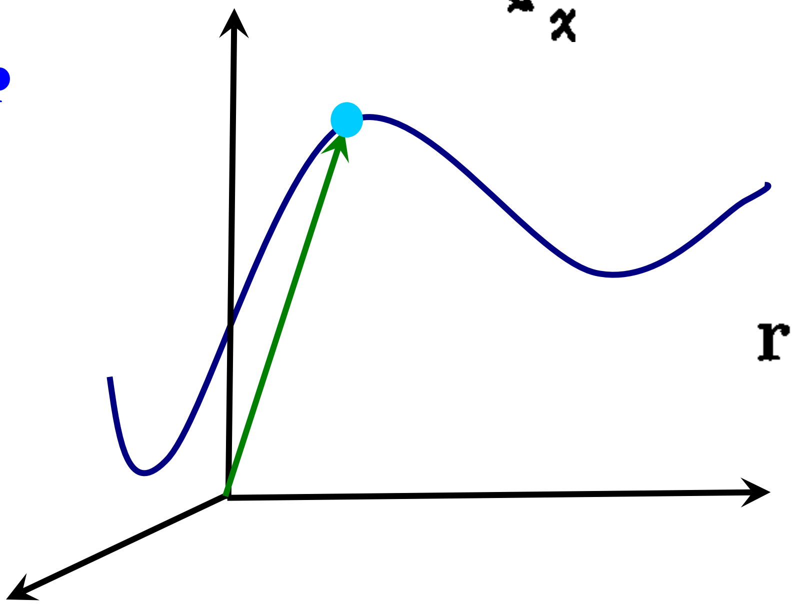


As $h \to 0$, $Q \to P$ along $\boldsymbol{C}$ & $\overrightarrow{PQ}/h$ becomes the ***tangent vector*** $\mathbf{r}'(t)$

5.7 *Curves in space*

♣ 

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k},\ t \geq 0$$

♣ 

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

- A ***curve*** ***C*** in space can be represented by

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

s.t. a point *P* lies on the curve if $\overrightarrow{OP}$ is the ***image*** of **r**(*t*), i.e., $\overrightarrow{OP} = \mathbf{r}(t_0)$ for some $t_0 \in \mathbf{R}$.

$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$	$\begin{cases} x = f(t), \\ y = g(t), \\ z = h(t) \end{cases}$
Vector eqn. of ***C***	***Parametric eqn.*** of ***C***

http://www.math.uri.edu/~bkaskosz/flashmo/tools/parcur/

5.7.1 *Examples*

♣ The ***vector*** eqn.

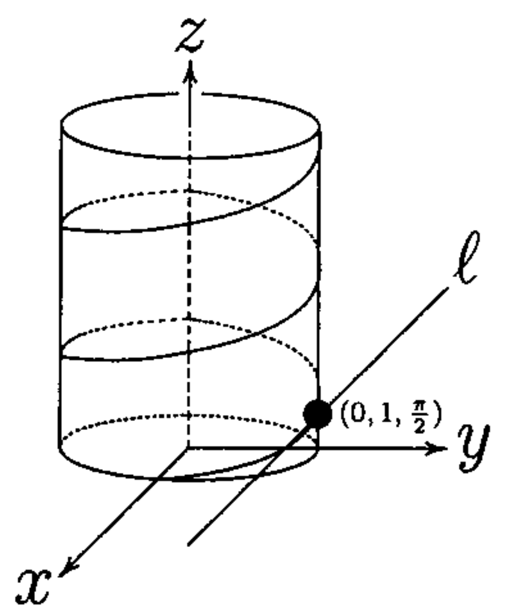
$$\mathbf{r}(t) = (1+t)\mathbf{i} + (2+t)\mathbf{j} + (3+t)\mathbf{k}$$

$$= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

represents a line passing through (1, 2, 3) & parallel to $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

♣ The ***circular helix***

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$



5.7.2 *Smooth curves*

- Let C be a curve with vector fn $\mathbf{r}(t)$ on an interval I (t in I). We say that C is ***smooth*** if

  (i) $\mathbf{r}'(t)$ is continuous &

  (ii) $\mathbf{r}'(t) \neq \mathbf{0}$

  (that is, **$f'(t)$, $g'(t)$ & $h'(t)$ are all continuous & are not 0 simultaneously**)

Note The condition that $\mathbf{r}'(t) \neq \mathbf{0}$ is to make sure that the curve has a continuously turning tangent at every point (& thus has no sharp corners or cusps).

Example

- $\mathbf{r}(t) = (1+t^3)\mathbf{i} + t^2\mathbf{j}$

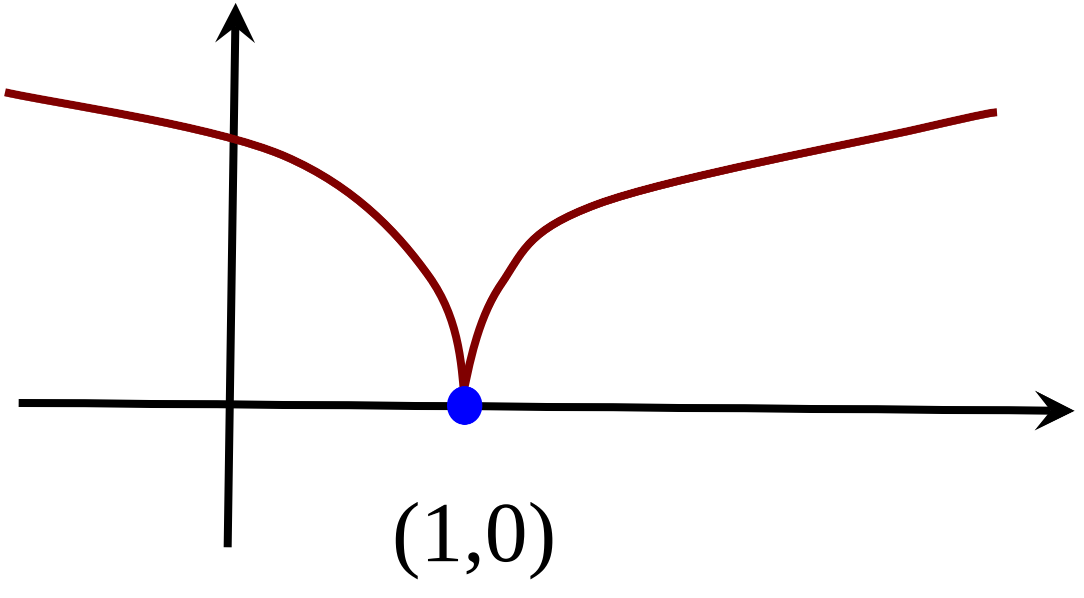
  $\mathbf{r}'(t) = 3t^2\mathbf{i} + 2t\,\mathbf{j}$

  $\mathbf{r}'(0) = \mathbf{0}$

$t = 0$

$\Rightarrow 1 + t^3 = 1$

$t^2 = 0$



5.7.3 *Example*

♣ The vector eqn.

$$\mathbf{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j} + (2 - 7t)\mathbf{k}$$

represents a ***smooth*** curve since

$$\mathbf{r}'(t) = \mathbf{i} + (2t)\mathbf{j} - 7\mathbf{k} \neq \mathbf{0}$$

♣ The vector eqn.

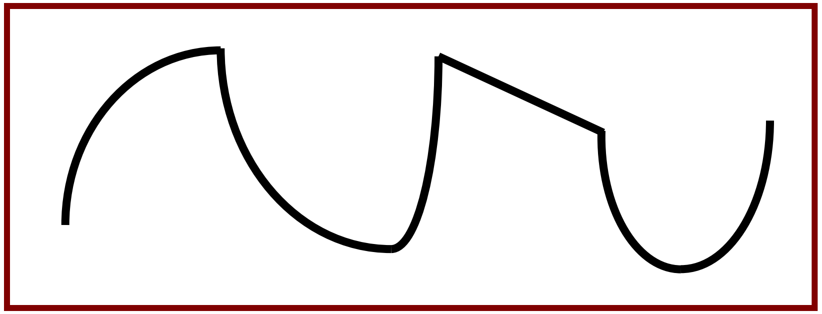
$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

also represents a ***smooth*** curve as

$$\mathbf{r}'(t) \neq \mathbf{0}$$

Piecewise smooth curves

- A ***curve*** in space is said to be ***piecewise smooth*** if it is made up of a ***finite*** number of ***smooth*** pieces.

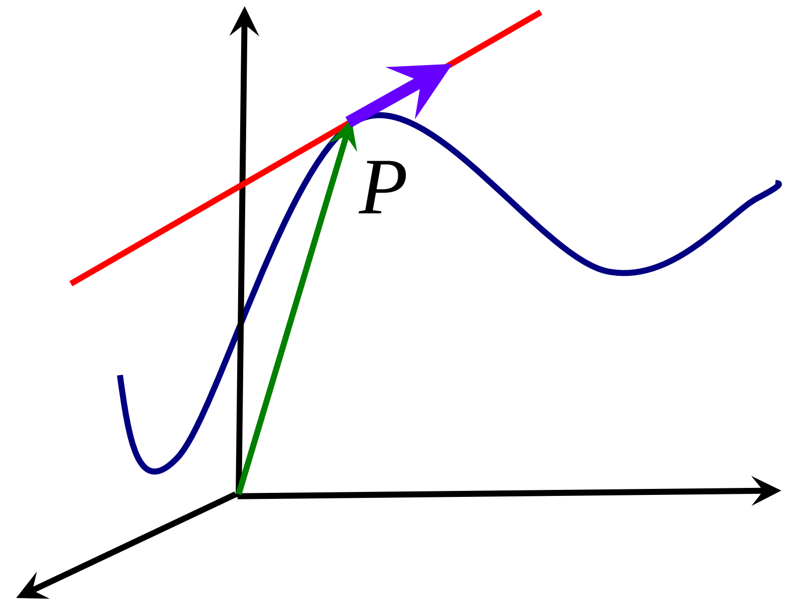


5.7.4 Example The vector function:

$$\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} & \text{if } 0 \le t \le 1 \\ (2t-1)\mathbf{i} + t^2\mathbf{j} + (t^2+t-1)\mathbf{k} & \text{if } 1 < t \le 2 \end{cases}$$

represents a **piecewise smooth** curve

5.7.5 *Tangent vector* & *tangent line* **to a curve**

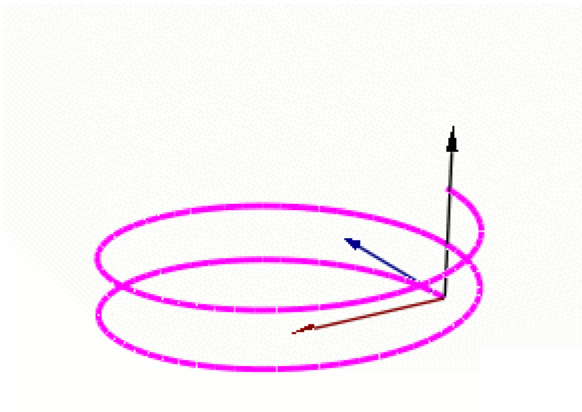


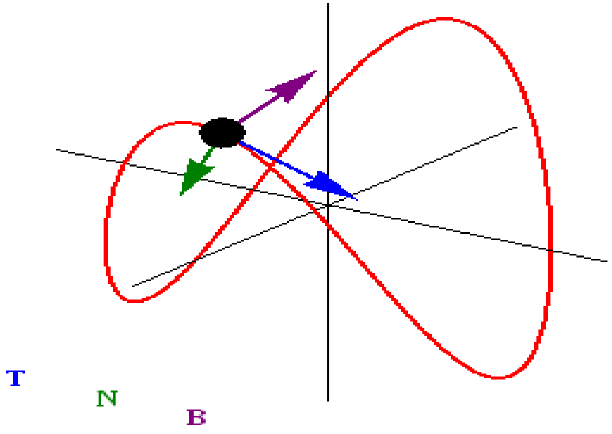
- ***Curve*** : $\mathbf{r}(t)$
- ***Tangent vector*** :

  $\mathbf{r}'(t)\ (\neq \mathbf{0})$
- ***Unit tangent vector*** to the curve at $t = t_0$:

  $$\frac{\mathbf{r}'(t_0)}{||\mathbf{r}'(t_0)||}$$
- ***Tangent line*** to $\mathbf{r}(t)$ at a point P **(**$= \mathbf{r}(t_0)$**)** :

  line through P & parallel to $\mathbf{r}'(t_0)$

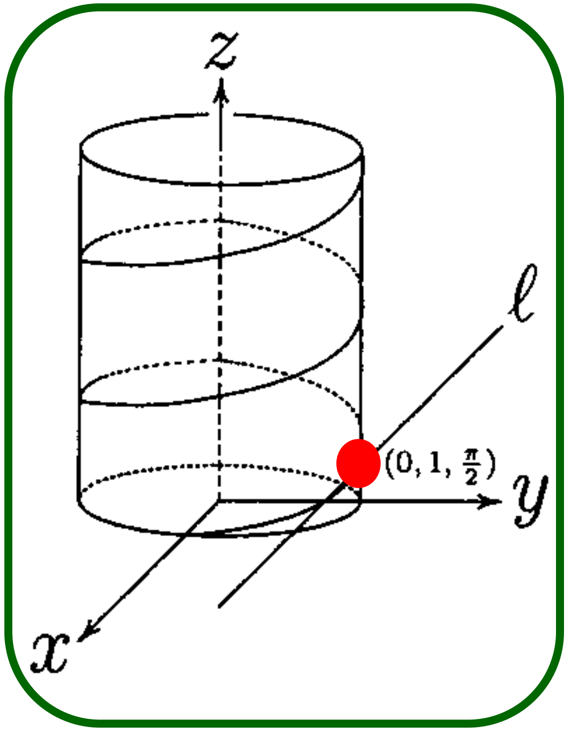



T
N
B

5.7.6 Example

♣ The ***circular helix***

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$



- $\mathbf{r}(\frac{\pi}{2}) = ?$

$$\mathbf{r}(\frac{\pi}{2}) = (\cos\frac{\pi}{2})\mathbf{i}+(\sin\frac{\pi}{2})\mathbf{j}+\frac{\pi}{2}\mathbf{k}$$

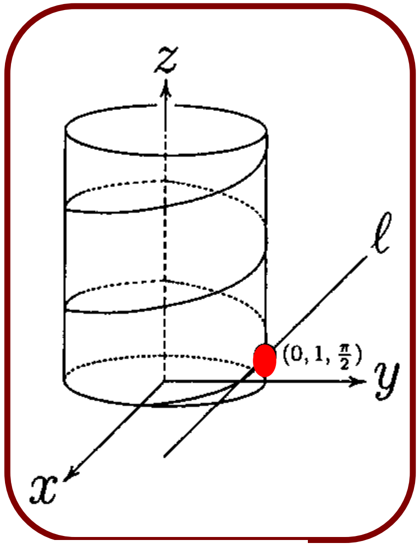
$$= 0\mathbf{i}+1\mathbf{j}+\frac{\pi}{2}\mathbf{k} = \mathbf{j}+\frac{\pi}{2}\mathbf{k}$$

The point $(0, 1, \frac{\pi}{2})$ lies on the curve.

- ***Unit tangent vector*** at $(0, 1, \frac{\pi}{2})$:

$$\mathbf{r}'(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \neq \mathbf{0} \quad \text{for all } t \in \mathbf{R}$$

$$\mathbf{r}'(\frac{\pi}{2}) = (-1)\mathbf{i} + (0)\mathbf{j} + (1)\mathbf{k} = -\mathbf{i} + \mathbf{k}$$



The required unit tangent vector is

$$\frac{-i+k}{\sqrt{(-1)^2+1^2}} = \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{k})$$

- ***Tangent line*** at $(0, 1, \frac{\pi}{2})$:

line through $(0, 1, \frac{\pi}{2})$ & parallel to $\frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{k})$

$$\mathbf{r}(t) = \mathbf{j} + \frac{\pi}{2}\mathbf{k} + t(-\mathbf{i} + \mathbf{k})$$

or $$\left\{ x = -t, \quad y = 1, \quad z = \frac{\pi}{2} + t \right\}$$

5.7.7 *Arc length* of ***a curve***

- ***Curve*** ***C*** **:** $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

  where $f'(t)$, $g'(t)$ and $h'(t)$ are continuous functions

  & $a \le t \le b$.

The ***arc length*** of ***C*** is

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\; dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\; dt$$

$$= \int_a^b ||\mathbf{r}'(t)||\; dt$$

Proof omitted

5.7.8 Example

♣ Given the ***helix*** : $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$

find its ***arc length*** from $t = 0$ to $t = 2\pi$.

$$\boxed{L = \int_a^b ||\mathbf{r}'(t)||\ dt}$$

$$\mathbf{r}'(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}$$

$$||\mathbf{r}'(t)|| = \sqrt{(-\sin\ t)^2 + (\cos\ t)^2 + 1^2} = \sqrt{2}$$

$$L = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi.$$

Arc length of *a curve* in two dim

- ***Curve*** ***C* :** $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, $a \le t \le b$.

Then $h(t) \equiv 0$

The ***arc length*** of ***C*** is

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2} \quad dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad dt$$

$$= \int_a^b ||\mathbf{r}'(t)|| \; dt$$

END