2009

Question 8 (a) [5 marks]

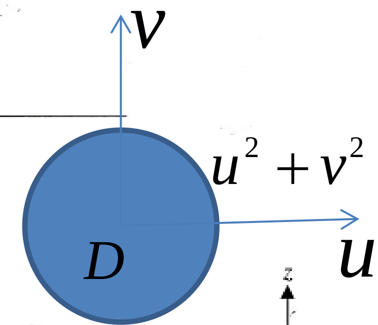
Find the **exact value** of the surface integral

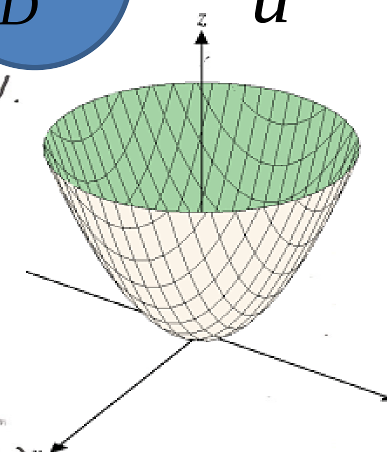
$$\int\int_S z\,dS,$$

where S is the surface $z = x^2 + y^2$ with $0 \le z \le 1$.

Answer 8(a)	$\left(\frac{5\sqrt{5}}{12} + \frac{1}{60}\right)\pi$

(Show your working below and on the next page.)





$S: \vec{r}(u,v) = u\vec{i} + v\vec{j} + (u^2+v^2)\vec{k}, \quad 0 \le u^2+v^2 \le 1.$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\vec{i} - 2v\vec{j} + \vec{k}$

$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{4u^2+4v^2+1}$

$\iint_S z\,dS = \iint_{u^2+v^2\le 1} (u^2+v^2)\sqrt{4u^2+4v^2+1}\,du\,dv$

$= \int_0^{2\pi}\int_0^1 r^2\sqrt{4r^2+1}\; r\,dr\,d\theta$

$= 2\pi \int_1^{\sqrt{5}} \frac{t^2-1}{4}\, t\, \frac{1}{4} t\,dt \qquad (\text{let } t = \sqrt{4r^2+1})$

$= \frac{\pi}{8}\int_1^{\sqrt{5}} (t^4 - t^2)\,dt$

$= \frac{\pi}{8}\left[\frac{1}{5}t^5 - \frac{1}{3}t^3\right]_1^{\sqrt{5}} = \frac{\pi}{8}\left\{5\sqrt{5} - \frac{5}{3}\sqrt{5} + \frac{2}{15}\right\}$

$= \frac{\pi}{8}\left(\frac{10\sqrt{5}}{3} + \frac{2}{15}\right) = \underline{\left(\frac{5\sqrt{5}}{12} + \frac{1}{60}\right)\pi}$

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA.$$

2011

Question 8 (a) [5 marks]

Find the **exact value** of the surface integral

$$\iint_S y^2 z \, dS,$$

where S is the portion of the cylinder $x^2 + y^2 = 4$ lying between the two planes $z = 3$ and $z = 0$.

Answer 8(a)	36π

(Show your working below and on the next page.)

$S: \vec{r}(u,v) = 2\cos u \, \vec{i} + 2\sin u \, \vec{j} + v \vec{k}, \quad 0 \le u \le 2\pi, \quad 0 \le v \le 3.$

$\vec{r}_u = -2\sin u \, \vec{i} + 2\cos u \, \vec{j} + 0 \vec{k}$

$\vec{r}_v = 0 \vec{i} + 0 \vec{j} + \vec{k}$

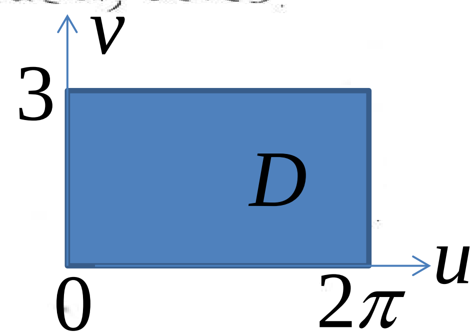
$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\cos u \, \vec{i} + 2\sin u \, \vec{j}$

$\|\vec{r}_u \times \vec{r}_v\| = 2$

$\iint_S y^2 z \, dS = \int_0^3 \int_0^{2\pi} (4\sin^2 u) v (2 \, du \, dv)$

$= \left[v^2 \right]_{v=0}^{3} \, 2\int_0^{2\pi} (1 - \cos 2u) \, du$

$= \underline{\underline{36\pi}}$



$$\iint_S f(x,y,z) \, dS = \iint_D f(\mathbf{r}(u,v)) \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA.$$

MA1505 Practice Examination

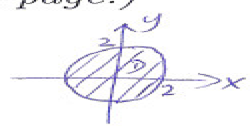
Question 5 (b) [5 marks]

Find the **exact value** of the surface area of that portion of the paraboloid $z = 1 + x^2 + y^2$ that lies below the plane $z = 5$.

Answer 5(b)	$\frac{\pi}{6}(17^{3/2} - 1)$

(Show your working below and on the next page.)

$z = 5 \Rightarrow x^2 + y^2 = 4 \Rightarrow$



$z_x = 2x$

$z_y = 2y$

surface area $= \iint_D \sqrt{1 + z_x^2 + z_y^2}\, dx\,dy$

$= \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dx\,dy$

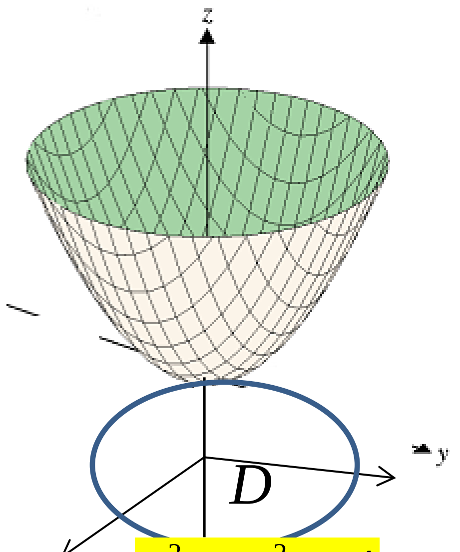
$= \int_0^{2\pi} \int_0^2 \sqrt{1+4r^2}\, r\,dr\,d\theta$

$= 2\pi \int_0^2 (1+4r^2)^{1/2}\, d(1+4r^2)/8$

$= \frac{\pi}{4} \cdot \frac{2}{3} (1+4r^2)^{3/2} \Big|_0^2$

$= \frac{\pi}{6}(17^{3/2} - 1)$

$$S = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA.$$



$x^2 + y^2 \leq 4$

You may use the following method with f=1,
but 1st method is better, since z is of the form z=f(x,y)

$$\iint_S f(x,y,z)\, dS = \iint_D f(\mathbf{r}(u,v)) \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA.$$

MA1505 Examination

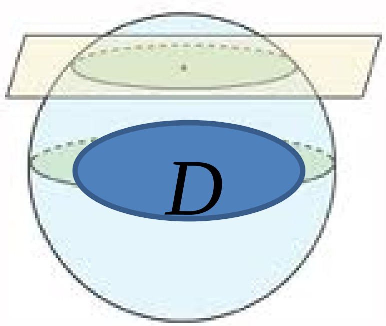
Question 5 (b) [5 marks]

Find the **exact value** of the surface area of that portion of the sphere $x^2 + y^2 + z^2 = 3$ that lies above the plane $z = 1$.

2012

Answer 5(b)	$(6-2\sqrt{3})\pi$

(Show your working below and on the next page.)



$z=1 \Rightarrow x^2+y^2+1=3 \Rightarrow x^2+y^2=2$

$z=\sqrt{3-x^2-y^2} \Rightarrow z_x = \dfrac{-x}{\sqrt{3-x^2-y^2}}, \quad z_y = \dfrac{-y}{\sqrt{3-x^2-y^2}}$

$\sqrt{1+z_x^2+z_y^2} = \sqrt{\dfrac{3}{3-x^2-y^2}}$

Surface area $= \displaystyle\iint_{0\le x^2+y^2\le 2} \sqrt{\frac{3}{3-x^2-y^2}}\, dx\,dy$

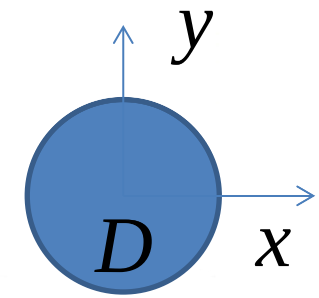
$= \displaystyle\int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{\frac{3}{3-r^2}}\, r\,dr\,d\theta$

$= 2\pi \displaystyle\int_0^{\sqrt{2}} \left(-\frac{\sqrt{3}}{2}\right)(3-r^2)^{-\frac{1}{2}}\, d(3-r^2)$

$= 2\pi \left[-\sqrt{3}\,(3-r^2)^{\frac{1}{2}}\right]_{r=0}^{r=\sqrt{2}}$

$= 2\pi(-\sqrt{3}+3)$

$= \underline{\underline{(6-2\sqrt{3})\pi}}$



$x^2+y^2 \le 2$

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA.$$

You may use the following method with f=1,
but the 1st method is better, since z=f(x,y)

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA.$$

2010

Question 8 (a) [5 marks]

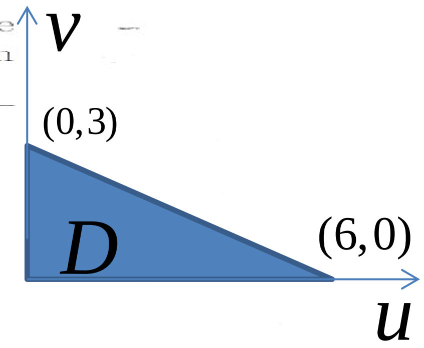
Find the **exact value** of the surface integral

$$\int\int_S \mathbf{F}\cdot d\mathbf{S}\ ,$$

where $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and S is the portion of the plane $x + 2y + 3z = 6$ in the first octant. The orientation of S is given by the downward normal vector.

Answer 8(a)	-33

(Show your working below and on the next page.)

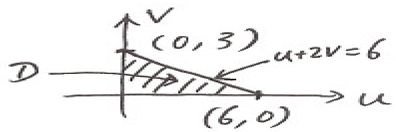


$x=u,\ y=v \Rightarrow u+2v+3z=6 \Rightarrow z=\frac{1}{3}(6-u-2v)$

$\vec{r}(u,v) = u\vec{i} + v\vec{j} + \frac{1}{3}(6-u-2v)\vec{k}$

$\vec{r}_u = \vec{i} + 0\vec{j} - \frac{1}{3}\vec{k}$

$\vec{r}_v = 0\vec{i} + \vec{j} - \frac{2}{3}\vec{k}$



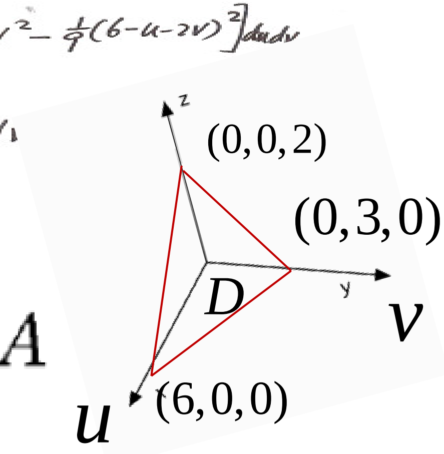
$\vec{r}_u \times \vec{r}_v = \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \vec{k}$ = upward normal

$\therefore$ orientation $= -\vec{r}_u \times \vec{r}_v = -\frac{1}{3}\vec{i} - \frac{2}{3}\vec{j} - \vec{k}$

$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(-\vec{r}_u\times\vec{r}_v)\,du\,dv = \iint_D \left[-\frac{1}{3}u^2 - \frac{2}{3}v^2 - \frac{1}{9}(6-u-2v)^2\right]du\,dv$

$= \int_0^3\int_0^{6-2v}\left(-\frac{4}{9}u^2 - \frac{10}{9}v^2 - 4 + \frac{4}{3}u + \frac{8}{3}v - \frac{4}{9}uv\right)du\,dv$

$= \underline{\underline{-33}}$



$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA$$

Question 7 (b) [5 marks]

Find the **exact value** of the surface integral

$$\iint_S (x\mathbf{i} + y\mathbf{j}) \cdot \, d\mathbf{S},$$

where S is the portion of the paraboloid $z = x^2 + y^2$ lying below the plane $z = 4$ and oriented with upward pointing normal vectors.

Answer 7(b)	-16π

(Show your working below and on the next page.)

$S: \vec{r}(u,v) = u\vec{i} + v\vec{j} + (u^2+v^2)\vec{k}, \quad 0 \le u^2+v^2 \le 4$

$\vec{r}_u = \vec{i} + 0\vec{j} + 2u\vec{k}$

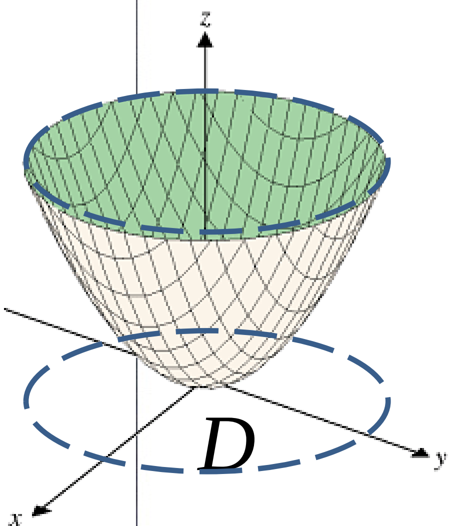
$\vec{r}_v = 0\vec{i} + \vec{j} + 2v\vec{k}$

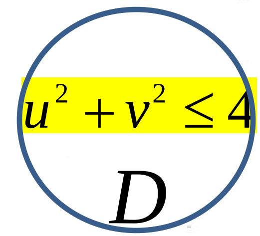
$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\vec{i} - 2v\vec{j} + \vec{k}$$

$\vec{r}_u \times \vec{r}_v \cdot \vec{k} = 1 > 0 \Rightarrow \vec{r}_u \times \vec{r}_v$ points upward.

$$\iint_S (x\vec{i} + y\vec{j}) \cdot d\vec{S} = \iint_{0 \le u^2+v^2 \le 4} (u\vec{i} + v\vec{j}) \cdot (-2u\vec{i} - 2v\vec{j} + \vec{k})\, du\, dv$$

$$= \iint_{0 \le u^2+v^2 \le 4} (-2u^2 - 2v^2)\, du\, dv$$





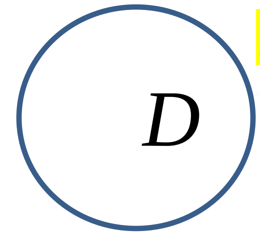
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

(More working space for Question 7(b))

$$= \int_0^{2\pi}\int_0^2 -2r^2\, r\,dr\,d\theta$$

$$= 2\pi\left[-\frac{1}{2}r^4\right]_0^2$$

$$= \underline{\underline{-16\pi}}$$



$u^2+v^2 \le 4$

2012

Question 7 (b) [5 marks]

Find the **exact value** of the surface integral

$$\iint_S \mathbf{F} \bullet d\mathbf{S},$$

where

$$\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$

and S is the portion of the plane

$$2x + y + z = 2$$

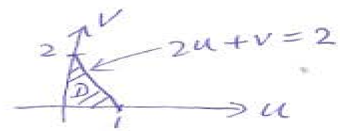
in the first octant. The orientation of S is given by the upward normal vector.

Answer 7(b)	$\frac{7}{3}$

(Show your working below and on the next page.)

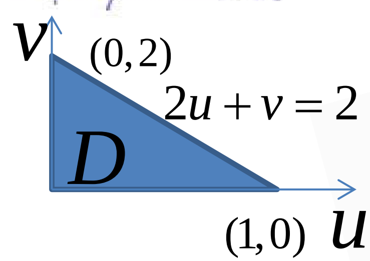
S : $\vec{r}(u,v) = u\vec{i} + v\vec{j} + (2-2u-v)\vec{k}$

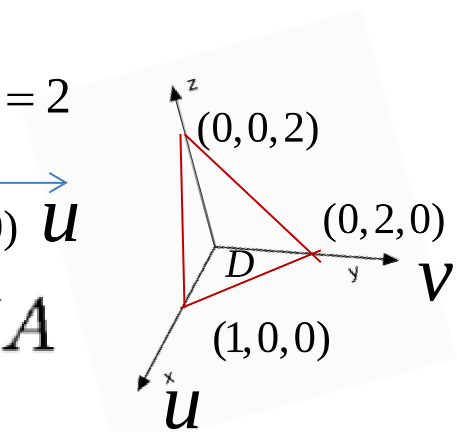
with $(u,v) \in D$.



$\vec{r}_u = \vec{i} + 0\vec{j} - 2\vec{k}$

$\vec{r}_v = 0\vec{i} + \vec{j} - \vec{k}$





$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

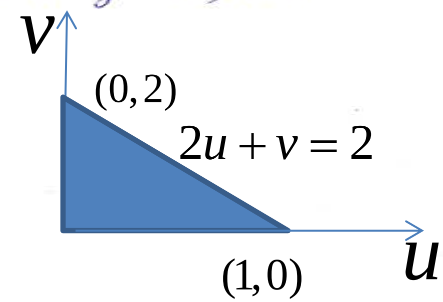
(More working space for Question 7(b))

2012

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix} = 2\vec{i} + \vec{j} + \vec{k}$$

$\vec{r}_u \times \vec{r}_v \cdot \vec{k} = 1 > 0 \Rightarrow \vec{r}_u \times \vec{r}_v$ points upwards

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(v\vec{i} + (2-2u-v)\vec{j} + u\vec{k}\right) \cdot (2\vec{i} + \vec{j} + \vec{k})\, dA$$



$$= \iint_D (2v + 2 - 2u - v + u)\, dA$$

$$= \iint_D (v - u + 2)\, dA$$

$$= \int_0^1 \int_0^{2-2u} (v - u + 2)\, dv\, du$$

$$= \int_0^1 \left[\tfrac{1}{2}v^2 - uv + 2v\right]_{v=0}^{v=2-2u} du$$

$$= \int_0^1 (4u^2 - 10u + 6)\, du$$

$$= \left[\tfrac{4}{3}u^3 - 5u^2 + 6u\right]_0^1$$

$$= \frac{7}{3}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S}.$$

Question 8 (b) [5 marks]

Use Stokes' Theorem to find the **exact value** of the line integral

$$\oint_C (-yzdx + xzdy + xydz),$$

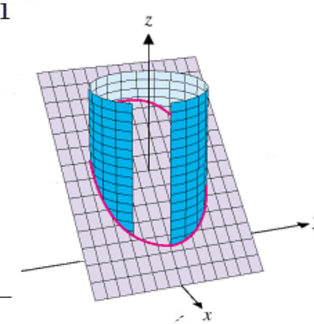
where C is the curve of intersection of the plane

$$x + y + z = 2$$

and the cylinder

$$x^2 + y^2 = 1,$$

oriented in the counterclockwise sense when viewed from above.



Answer 8(b)	4π

(Show your working below and on the next page.)

Let S = part of the plane $\{x+y+z=2\}$ bounded by C.

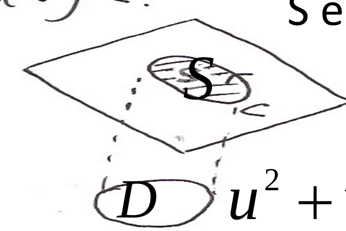
S: $\vec{r}(u,v) = u\vec{i} + v\vec{j} + (2-u-v)\vec{k}$, $0 \le u^2+v^2 \le 1$.

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k}$$

$\text{curl}\,(-yz\vec{i} + xz\vec{j} + xy\vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yz & xz & xy \end{vmatrix} = -2y\vec{j} + 2z\vec{k}$$

S ellipse



$u^2 + v^2 \le 1$

Bounded by circle

(More working space for Question 8(b))

Observe $\vec{r}_u \times \vec{r}_v$ points upwards

$\therefore$ orientations of S and C are consistent

$$\oint_C = \iint_S \text{curl} \cdot \vec{dS} = \iint_{u^2+v^2 \le 1} \{-2v + 2(2-u-v)\}\, du\, dv$$

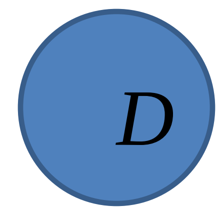
$$= \iint_{u^2+v^2 \le 1} (-4v + 4 - 2u)\, du\, dv$$

$$= \int_0^{2\pi} \int_0^1 (-4r\sin\theta + 4 - 2r\cos\theta)\, r\, dr\, d\theta$$

$$= 2\pi \int_0^1 4r\, dr$$

$$= 4\pi \left[r^2\right]_0^1 = \underline{\underline{4\pi}}$$

2009



$u^2 + v^2 \le 1$

END OF PAPER

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot d\mathbf{S}.$$

2010

Question 8 (b) [5 marks]

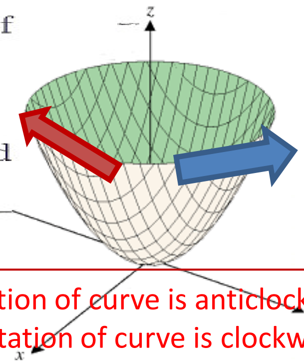
Use Stokes' Theorem to find the **exact value** of the surface integral

$$\iint_S (\text{curl } \mathbf{F})\cdot d\mathbf{S},$$

where $\mathbf{F} = -yz\mathbf{i} + x\mathbf{j} - e^x(\sin y)[\cos(z^2)]\mathbf{k}$, and S is the part of the elliptical paraboloid

$$z = x^2 + 4y^2$$

for which $z \leq 1$. The orientation of S is given by the outward normal vector.



Direction of curve is anticlockwise, orientation of curve is clockwise

Answer 8(b)	$-\pi$

(Show your working below and on the next page.)

Boundary of $S = C$: $\vec{r}(t) = \cos t\,\vec{i} + \frac{1}{2}\sin t\,\vec{j} + \vec{k}$, $0 \leq t \leq 2\pi$.

$\therefore \vec{r}'(t) = -\sin t\,\vec{i} + \frac{1}{2}\cos t\,\vec{j}$

$\because$ orientation of S = outward normal

$\therefore$ orientation of S is compatible to orientation of $(-C)$.

$\therefore \iint_S (\text{curl } \vec{F})\cdot d\vec{S} = \int_{-C} \vec{F}\cdot d\vec{r}$

$= -\int_0^{2\pi} \left(\frac{1}{2}\sin^2 t + \frac{1}{2}\cos^2 t\right) dt$

$= \underline{\underline{-\pi}}$

Direction of curve is anticlockwise, orientation of curve is clockwise

$$\mathrm{F}\cdot \mathrm{dr} = \left[\mathrm{F}\cdot \mathrm{r}'(t)\right]dt \qquad \mathrm{r}'(t) = -\sin t\,\mathrm{i} + (1/2)\cos t\,\mathrm{j} + 0\mathrm{k}$$

MA1505 Examination

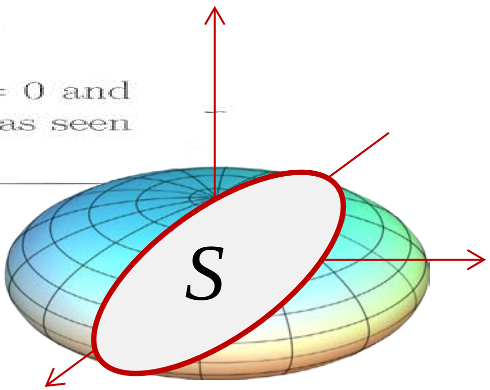
2011

Question 8 (b) [5 marks]

Use Stokes' Theorem to find the **exact value** of the line integral

$$\oint_C (y\,dx + z^2\,dy + x\,dz)\,,$$

where C is the curve of intersection of the plane $2x + z = 0$ and the ellipsoid $x^2 + 5y^2 + z^2 = 1$, oriented counterclockwise as seen from above.



Answer 8(b)	$-\frac{\pi}{5}$

(Show your working below and on the next page.)

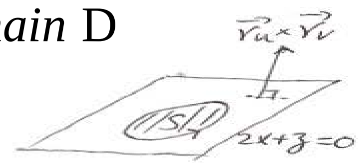
$$C: \begin{cases} 2x + z = 0 \\ x^2 + 5y^2 + z^2 = 1 \end{cases} \Rightarrow x^2 + 5y^2 + 4x^2 = 1 \Rightarrow 5x^2 + 5y^2 = 1$$

Let S = region on $2x+z=0$ bounded by C.

$$\therefore S: \vec{r}(u,v) = u\vec{i} + v\vec{j} - 2u\vec{k}, \quad u^2 + v^2 \le \left(\tfrac{1}{\sqrt{5}}\right)^2$$

Domain D

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{vmatrix} = 2\vec{i} + \vec{k}$$



$\vec{r}_u \times \vec{r}_v$ points up $\Rightarrow$ correct orientation

$$\vec{F} = y\vec{i} + z^2\vec{j} + x\vec{k}$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z^2 & x \end{vmatrix} = -2z\vec{i} - \vec{j} - \vec{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl }\mathbf{F}) \cdot d\mathbf{S}.$$

The projection of S is a region D bounded by a circle $u^2 + v^2 = \dfrac{1}{5}$

(More working space for Question 8(b))

2011

, Stoke's theorem

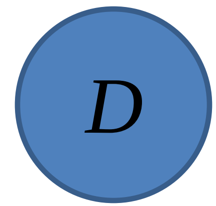
$$\Rightarrow \oint_C y\,dx + z^2\,dy + x\,dz = \iint_{u^2+v^2 \le \left(\frac{1}{\sqrt{5}}\right)^2} (8u - 1)\,du\,dv$$

$$= \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{5}}} (8r\cos\theta - 1)\,r\,dr\,d\theta$$

$$= \int_0^{\frac{1}{\sqrt{5}}}\int_0^{2\pi} (8r^2\cos\theta - r)\,d\theta\,dr$$

$$= 2\pi\left[-\tfrac{1}{2}r^2\right]_0^{\frac{1}{\sqrt{5}}}$$

$$= -\frac{\pi}{5}$$



$$u^2 + v^2 \le \frac{1}{5}$$

END OF PAPER

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S}.$$

Question 8 (a) [5 marks]

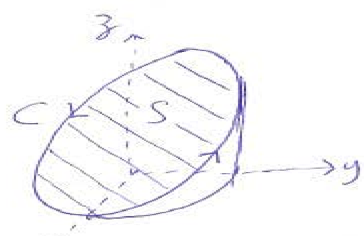
Use Stokes' Theorem to find the **exact value** of the line integral

$$\oint_C (-y\, dx + x^2\, dy + z^3\, dz),$$

where C is the curve of intersection of the plane $x + z = 3$ and the cylinder $x^2 + y^2 = 4$, oriented counterclockwise as seen from above.

Answer 8(a)	4π

(Show your working below and on the next page.)



Let $S = \{x+z=3\} \cap \{x^2+y^2=4\}$

$\therefore S: \vec{r}(u,v) = u\vec{i} + v\vec{j} + (3-u)\vec{k}$

$0 \le u^2+v^2 \le 4$

$\vec{r}_u = \vec{i} + 0\vec{j} - \vec{k}$

$\vec{r}_v = 0\vec{i} + \vec{j} + 0\vec{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \vec{i} + \vec{k}$$

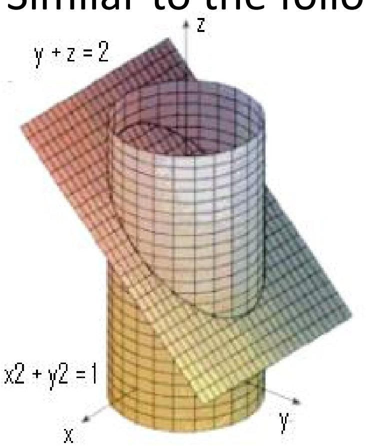
$\vec{r}_u \times \vec{r}_v \cdot \vec{k} = 1 > 0 \Rightarrow \vec{r}_u \times \vec{r}_v$ points upwards

$\therefore$ using $\vec{r}_u \times \vec{r}_v$ as the orientation of S is compatible to the orientation of C in Stoke's Theorem.

D

$u^2 + v^2 \le 4$

Similar to the following



(More working space for Question 8(a))

$$\operatorname{curl}\left(-y\vec{i} + x^2\vec{j} + z^3\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x^2 & z^3 \end{vmatrix} = (2x+1)\vec{k}$$

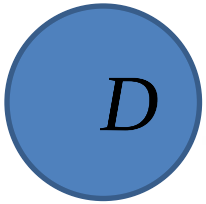
Stoke's Theorem $\Rightarrow \oint_C -y\,dx + x^2\,dy + z^3\,dz$

$$= \iint_{0 \le u^2+v^2 \le 4} (2u+1)\,du\,dv$$

$$= \int_0^2 \int_0^{2\pi} (2r\cos\theta + 1)\,d\theta\; r\,dr$$

$$= 2\pi \left[\tfrac{1}{2} r^2\right]_0^2$$

$$= \underline{\underline{4\pi}}$$



$u^2 + v^2 \le 4$

2012

Question 8 (a) [5 marks]

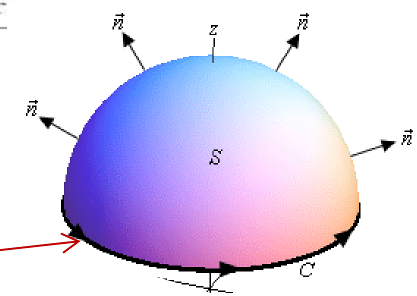
Let S be the upper hemisphere with equation

$$S \quad : \quad z = \sqrt{1 - x^2 - y^2}.$$

If

$$\mathbf{F}(x, y, z) = z^2\mathbf{i} - 2x\mathbf{j} + y^3\mathbf{k},$$

find the **exact value** of the surface integral $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S}$, where the orientation of S is given by the outer normal vector



Answer 8(a)	-2π

(Show your working below and on the next page.)

$C = \partial S$: $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + 0\,\vec{k}$
$0 \le t \le 2\pi$

orientation of C is compatible to the orientation of S in Stoke's Theorem.

Stoke's Theorem

$$\Rightarrow \iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} (0^2\,\vec{i} - 2\cos t\,\vec{j} + \sin^3 t\,\vec{k}) \cdot (-\sin t\,\vec{i} + \cos t\,\vec{j} + 0\,\vec{k})\,dt$$

$$= \int_0^{2\pi} -2\cos^2 t\,dt$$

$$= \int_0^{2\pi} (-1 - \cos 2t)\,dt = \underline{\underline{-2\pi}}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S}.$$

Question 8 (b) [5 marks]

Use the divergence theorem to evaluate $\iint_S \mathbf{F} \bullet d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = -y^2 z\mathbf{i} + y^2\mathbf{j} + x^3y^3\mathbf{k}$$

and S is the surface of the rectangular region bounded by the three coordinate planes and the planes $x = -1$, $y = 2$, $z = -3$. The orientation of S is given by the outer normal vector.

Answer 8(b)	12

(Show your working below and on the next page.)

$\operatorname{div} \vec{F} = 2y$

orientation of S is compatible in the Divergence Theorem.

Divergence Theorem $\Rightarrow \iint_S \vec{F} \cdot d\vec{S}$

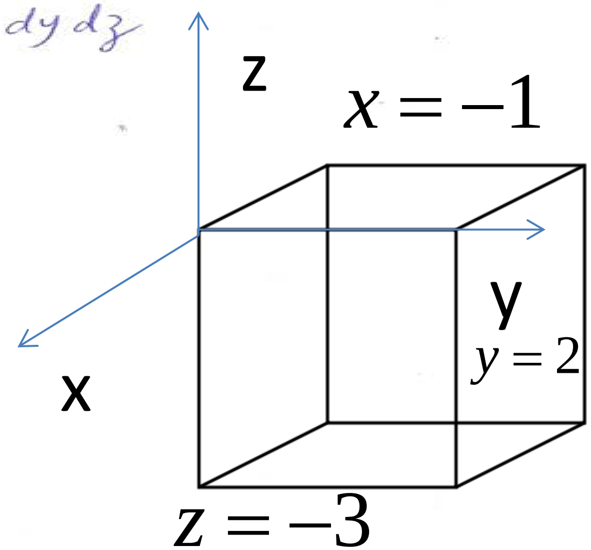
$$= \int_{-3}^{0} \int_{0}^{2} \int_{-1}^{0} 2y \, dx \, dy \, dz$$

$$= \int_{-3}^{0} \int_{0}^{2} 2y \, dy \, dz$$

$$= \int_{-3}^{0} \left[y^2 \right]_{y=0}^{y=2} dz$$

$$= \int_{-3}^{0} 4 \, dz = \underline{\underline{12}}$$

rectangular box
enclosed by six planes



$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV.}$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV.$$

2012

MA1505

Question 8 (b) [5 marks]

Let W be the cube bounded by the three coordinate planes $x = 0$, $y = 0$, $z = 0$ and the three planes $x = 2$, $y = 2$, $z = 2$. Let S be the surface consisting of five sides of W, excluding the side where $z = 0$. Orient S with outward pointing normal vector. Let $\mathbf{F}$ be the vector field given by

$$\mathbf{F}(x, y, z) = (10x - 3xy + \cos y^2)\,\mathbf{i} + (z^2 e^x + \cos x^2)\,\mathbf{j} + (3zy - 1)\,\mathbf{k}.$$

Find the **exact value** of the surface integral $\iint_S \mathbf{F} \bullet d\mathbf{S}$.

Answer 8(b)	76

(Show your working below and on the next page.)

Let S_1 be the side of W with $z = 0$.

S_1: $\vec{r}(u,v) = u\vec{i} + v\vec{j} + 0\vec{k}$, $0 \le u \le 2$, $0 \le v \le 2$

$\vec{r}_u \times \vec{r}_v = \vec{i} \times \vec{j} = \vec{k}$

Orient S_1 with $\vec{k}$.

Note that $\vec{k}$ points inward to W on S_1.

$\therefore$ ∂W with positive orientation $= S - S_1$

$\operatorname{div} \vec{F} = 10x - 3y + 3y = 10$

Divergence Theorem $\Rightarrow \iint_{S-S_1} \vec{F} \cdot d\vec{S} = \iiint_W (\operatorname{div} \vec{F})\, dV$

$= 10 (\text{vol. of } W) = 80$

$\therefore \iint_S \vec{F} \cdot d\vec{S} = 80 + \iint_{S_1} \vec{F} \cdot d\vec{S}$

$= 80 + \iint_{\substack{0 \le u \le 2 \\ 0 \le v \le 2}} (-1)\, du\, dv = 80 - 4 = \underline{\underline{76}}$