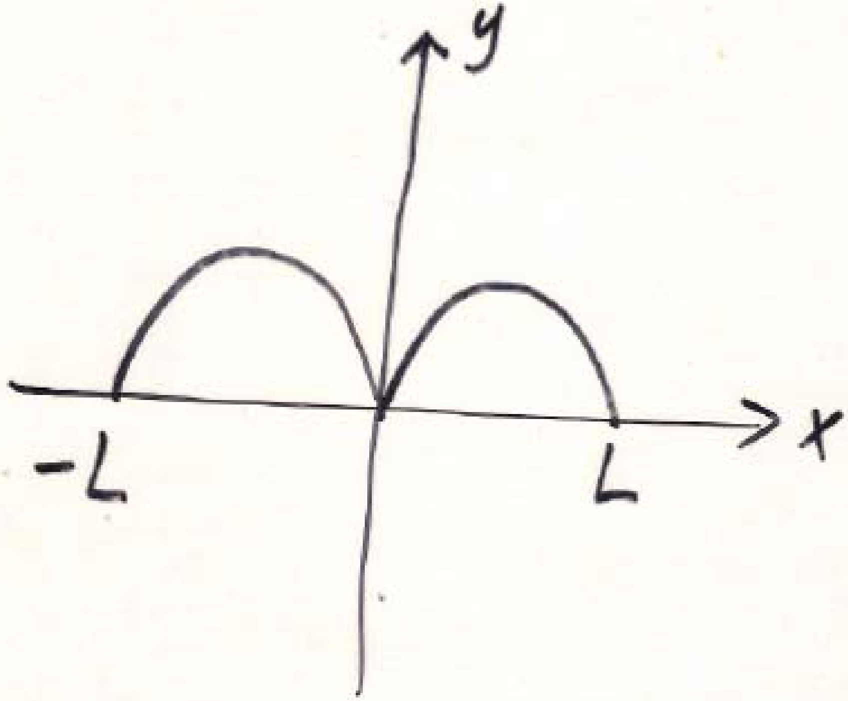
Chapter 6. Fourier Series

Even function

$$f(-x) = f(x)$$

Symmetry about the y-axis

e.g. $\cos x$, $|x|$, x^2



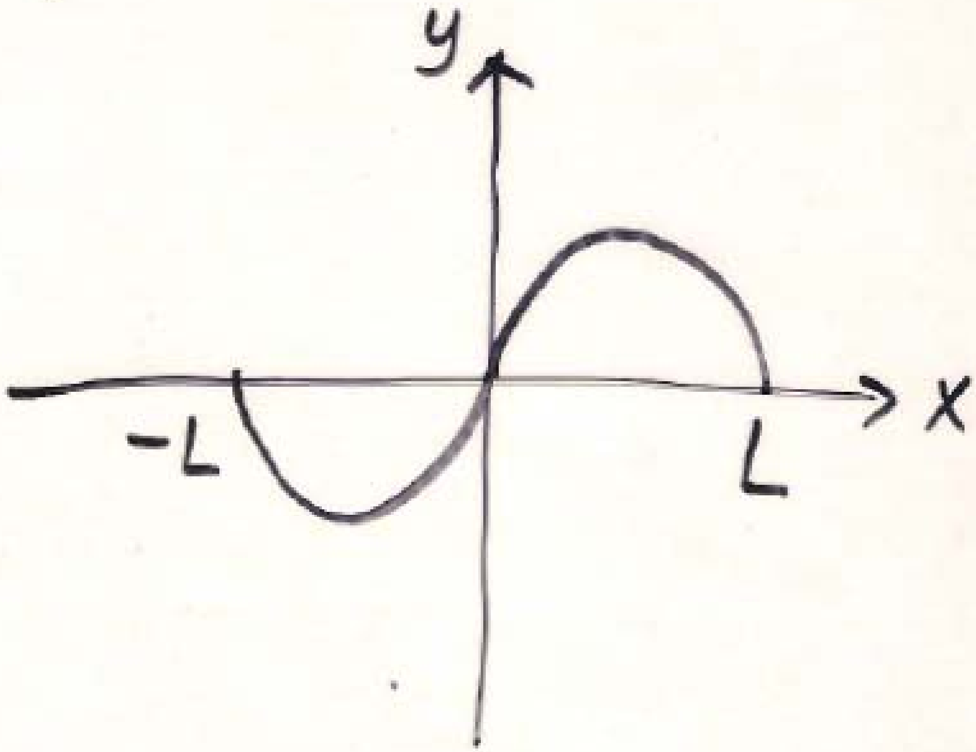
$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$$

Odd functions

$$f(-x) = -f(x)$$

symmetry about the origin

e.g. $\sin x$, x, x^3



$$\int_{-L}^{L} f(x)\,dx = 0$$

(even function)(even function) = even

(even ")(odd ") = odd

(odd ")(even ") = odd

(odd ")(odd ") = even

Any function can be written as an even part + an odd part like this:

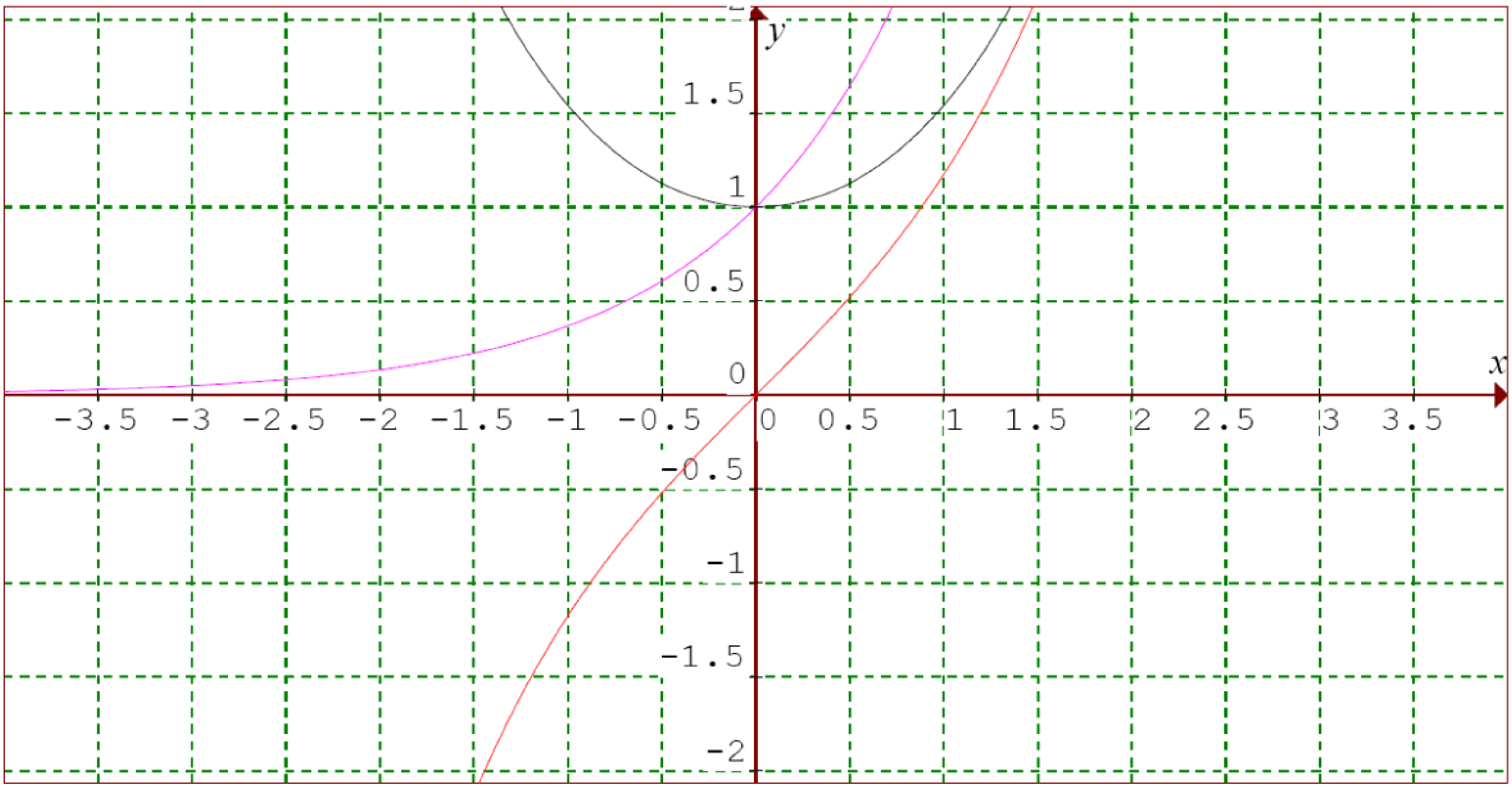
$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd}}$$

e.g. $f(x) = e^x$

$$e^x = f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$$

$$= \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$$

$$= \cosh x + \sinh x$$

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Equations on screen:
  1. y=sinh x
  2. y=cosh x
  3. y=e^x
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6.1 Periodic functions

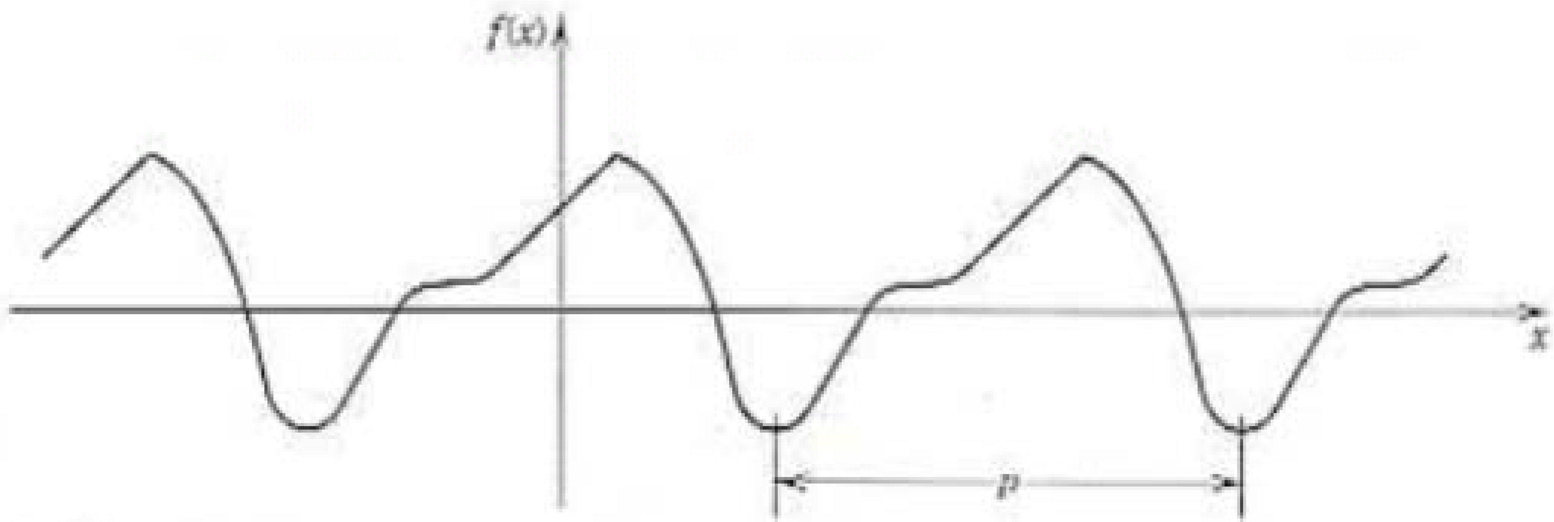
A function $f(x)$ is called *periodic* if it is defined for all real x and if there is some positive number p such that

$$f(x+p) = f(x) \quad \text{for all} \quad x. \tag{1}$$

The number p is called the *period* of $f(x)$.

6.1.1 Graphs of periodic functions

The graph of such a function can be obtained by periodic repetition of its graph in any interval of length p.



f(x)
x
p

For example, sine and cosine functions are periodic

2π.

$f(x) = c$, c constant, is a periodic function of period

p for every positive number p.

$x,\ x^2,\ x^3, \cdots, e^x,\ \ln x$ are not periodic.

6.1.2 Some algebraic properties of periodic functions

From (1),

$$f(x+2p) = f((x+p)+p) = f(x+p) = f(x).$$

Thus (by induction) for any positive integer n,

$f(x+np) = f(x)$, for all x.

Hence $2p,\, 3p, \cdots$ are also periods of f.

Further, if f and g have period p, then the function $h(x) = a\, f(x) + bg(x)$ with $a,\, b$ constants also has period p.

6.1.3 Trigonometric series

Our aim is to represent various periodic functions of

period 2π in terms of simple functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots, \cos nx, \sin nx, \cdots \quad (2)$$

which have period 2π.

The series that arises in this connection will be of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

$$= a_0 + \textstyle\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad (3)$$

where $a_0, a_1, a_2, \cdots, b_1, b_2, \cdots$ are real constants.

Series (3) is called a *trigonometric series, and* a_n *and* b_n are called *coefficients* of the series.

The set of functions (2) is often called a *trigonometric system*.

We note that each term of the series (3) has period 2π. Hence if the series converges, its sum will be a periodic function of period 2π.

6.2 Fourier Series

Assume that $f(x)$ is a periodic function of period 2π and that it can be represented by a trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \qquad (4)$$

That is, we assume that the series on the right converges and has $f(x)$ as its sum.

We say the right hand side of (4) is the Fourier series of $f(x)$.

Given $f(x)$, our task now is to determine the coefficients a_n and b_n.

6.2.1 Determine a_0

We integrate both sides of (4) from $-\pi$ to π :

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} (a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx))\, dx.$$

Assuming that term by term integration is allowed,

we obtain

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x)dx \\
&= a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \, dx\right) \\
&= 2\pi a_0 + \sum_{n=1}^{\infty} \left(\left[a_n \frac{\sin\ nx}{n} \right]_{-\pi}^{\pi} + \left[b_n \frac{\cos\ nx}{-n} \right]_{-\pi}^{\pi} \right) \\
&= 2\pi a_0
\end{aligned}$$

So $a_0 = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x)dx.$

6.2.2 Determine $a_m,\ m > 0$

We multiply both sides of (4) by $\cos mx$ and integrate term by term from $-\pi$ to π :

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x)\cos mx\,dx\\
=&a_0\int_{-\pi}^{\pi}\cos mx\,dx+\sum_{n=1}^{\infty}\Bigg(a_n\int_{-\pi}^{\pi}\cos nx\ \cos mx\,dx\\
&\qquad\qquad\qquad+b_n\int_{-\pi}^{\pi}\sin nx\cos mx\,dx\Bigg) \qquad (5)
\end{aligned}$$

Computing the three integrations on the right hand

sides of (5):

(i) $\displaystyle\int_{-\pi}^{\pi} \cos mx \, dx = \left[\frac{\sin mx}{m}\right]_{-\pi}^{\pi} = 0.$

(ii) $\displaystyle\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0,$

since $\sin nx$ is odd and $\cos mx$ is even

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \qquad\qquad + $$

$$\cos A \cos B = \frac{1}{2}\left\{\cos(A+B) + \cos(A-B)\right\}$$

$$\sin A \sin B = \frac{1}{2}\left\{\cos(A-B) - \cos(A+B)\right\}$$

(iii) $$\int_{-\pi}^{\pi} \cos nx \, \cos mx \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} (\cos(m+n)x + \cos(m-n)x)\, dx$$

$$= \begin{cases} \dfrac{1}{2}\left[\dfrac{\sin(m+n)x}{m+n} + \dfrac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} & m \neq n \\ \frac{1}{2m}[mx + \sin\, mx \, \cos\, mx]_{-\pi}^{\pi} & m = n \end{cases}$$

$$= \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

Calculation for $n=m$:

$$\frac{1}{2}\int_{-\pi}^{\pi}(\cos 2mx + 1)\,dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2mx}{2m}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\frac{2mx + 2\sin mx \cos mx}{2m}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2m}\left[mx + \sin mx \cos mx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2m}\left[2m\pi\right] = \underline{\underline{\pi}}$$

Substituting the above results back in (5), we get

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx, \;\; m = 1, 2, \cdots$$

6.2.3 Determine $b_m,\ m > 0$

We multiply (4) by $\sin mx$ and integrate from $-\pi$

to π :

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x) \sin mx \, dx \\
=&a_0 \int_{-\pi}^{\pi} \sin mx \, dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \sin mx \, dx \right. \\
&\qquad\qquad \left. + \, b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \right) \qquad (6)
\end{aligned}$$

$$= \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx$$

as the first two integrands on the right hand side of (6) are odd functions.

Now $\displaystyle\int_{-\pi}^{\pi} \sin nx \ \sin mx \, dx$

$$
\begin{aligned}
&= \frac{1}{2}\int_{-\pi}^{\pi} (\cos(n-m)x - \cos(n+m)x) dx \\
&= \begin{cases} \frac{1}{2}\left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m}\right]_{-\pi}^{\pi} & m \neq n \\ \frac{1}{2m}\left[mx - \sin mx \ \cos mx\right]_{-\pi}^{\pi} & m = n \end{cases} \\
&= \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}
\end{aligned}
$$

Thus $\displaystyle b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx \, dx, \ m = 1, 2, \cdots .$

6.2.4 Euler formulas

Given a periodic function $f(x)$ of period 2π with Fourier series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Its coefficients are known as *Fourier coefficients* and are given by

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ \ n = 1, 2, \cdots \quad (7) \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\ \ n = 1, 2, \cdots
\end{aligned}
$$

(7) are known as Euler formulas.

6.2.5 Representation by a Fourier series

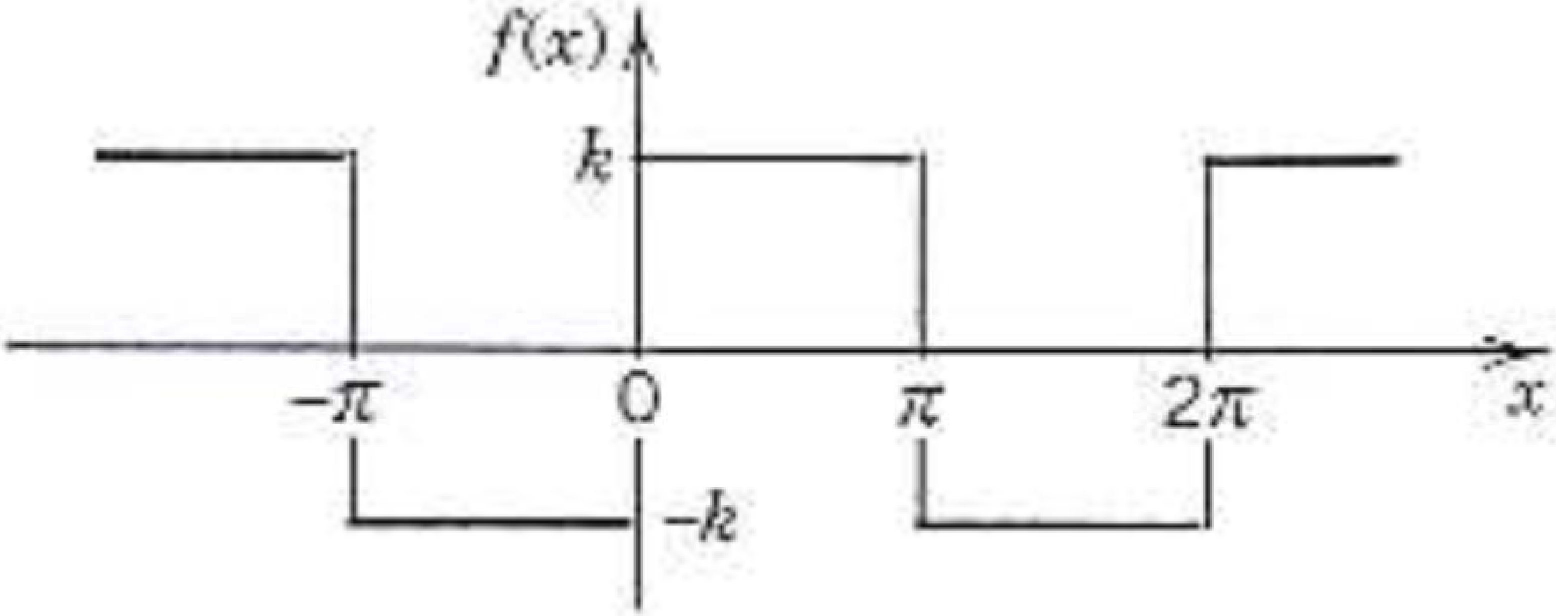
If a periodic function $f(x)$ with period 2π is piecewise continuous in the interval $-\pi \leq x \leq \pi$ and has a left hand derivative and right hand derivative at each point of the interval, then the Fourier series with coefficients (7) is convergent. Its sum is $f(x)$ except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left hand and right hand limits of f at x_0.

6.2.6 Example

Find the Fourier series of $f(x)$ given by

$$f(x) = \begin{cases} -k, & \text{if } -\pi < x < 0 \\ k, & \text{if } 0 < x < \pi \end{cases}$$

and $f(x) = f(x + 2\pi)$.



f(x)
k
-k
x
−π
0
π
2π

Solution. We observe that over the interval $(-\pi, \pi)$, f is an odd function. Thus $f(x)\cos nx$ is also an odd function. Thus by (7), $a_n = 0$ for $n = 0, 1, 2, \cdots$, and

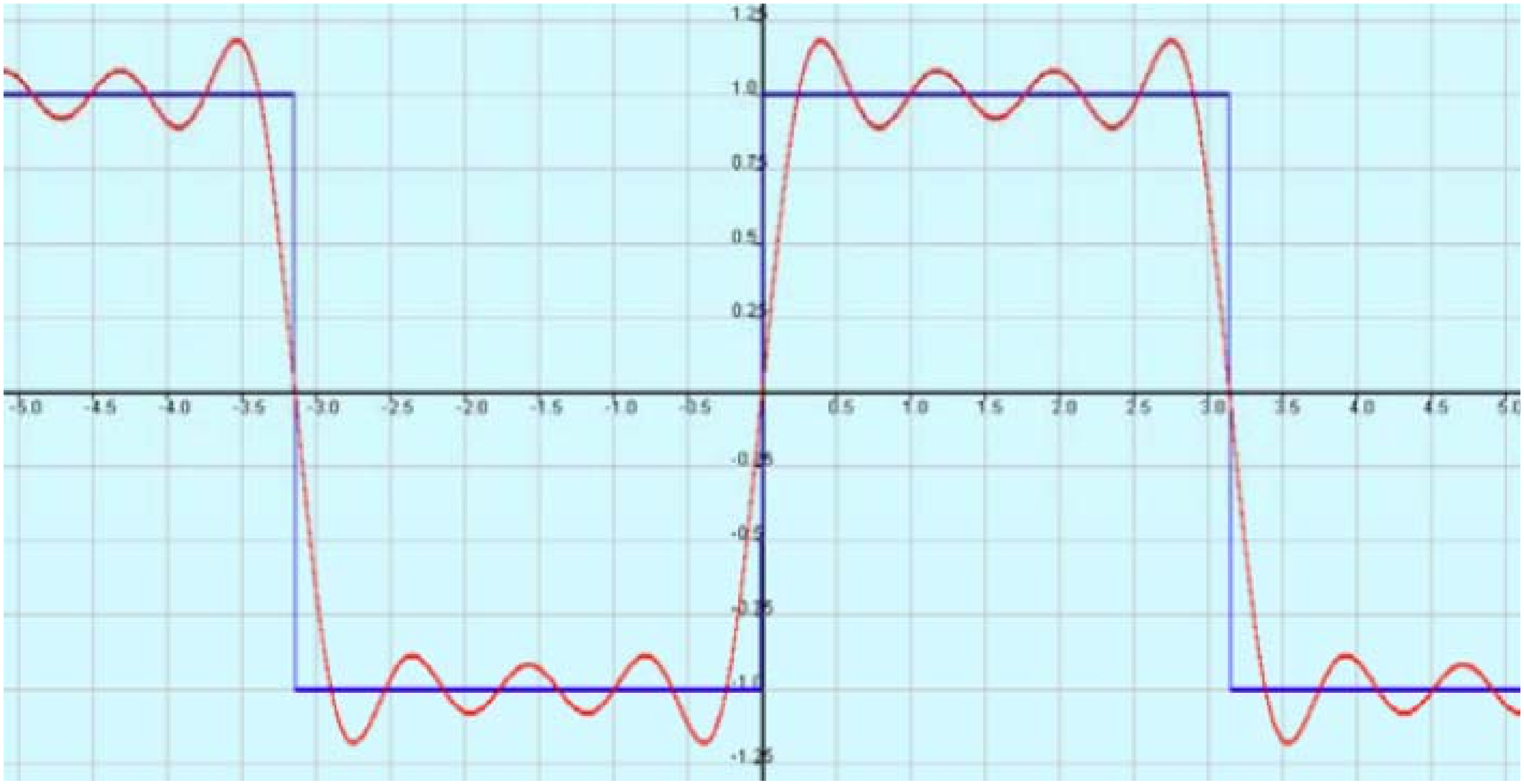
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$
\begin{aligned}
&= \frac{2}{\pi}\int_0^{\pi} k\ \sin nx\, dx = \frac{2k}{n\pi}(1-\cos n\pi) \\
&= \frac{2k}{n\pi}(1-(-1)^n). \\
b_1 &= \frac{4k}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{4k}{3\pi}, \\
b_4 &= 0, \quad b_5 = \frac{4k}{5\pi}, \cdots .
\end{aligned}
$$

The Fourier series for the square wave is, therefore,

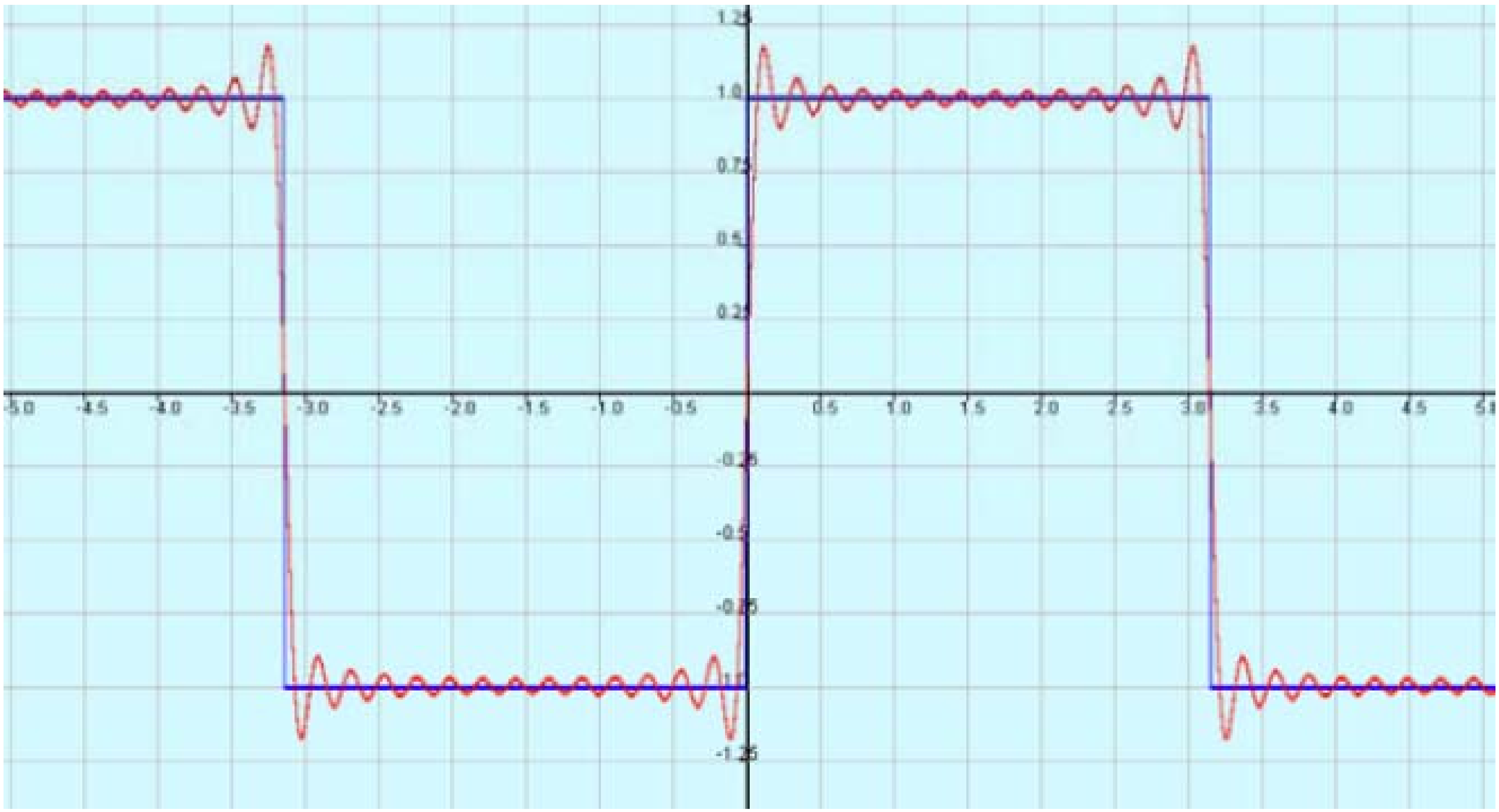
$$\frac{4k}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots).$$

Fourier series approximation cut off at n=8



Fourier series approximation cut off at n=18

Fourier series approximation cut off at n=28



6.2.7 An approximation for π

From the previous section, the series converges to $f(x)$ in $(0, \pi)$.

Setting $x = \frac{\pi}{2}$, we get

$$k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$$

i.e. $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$ (Leibniz).

Note that at all the points of discontinuity $(0, \pi, etc)$ of f, the sum of the series is equal to 0, which is the average of the left hand and right hand limits of f (e.g. they are $-k$ and k respectively at $x = 0$).

6.2.8 Periodic functions of period $p = 2L$

Let $f(x)$ be a periodic function of period $p = 2L$.

We set $v = \frac{\pi x}{L}$. Then $x = \frac{vL}{\pi}$ and at $x = \pm L, v =$

$\pm\pi$.

We now view f as a function of v and put $f(x) = g(v)$. Then g becomes a periodic function of period 2π.

Proof $g(v) = f(x) = f\left(\frac{vL}{\pi}\right)$

$$\therefore g(v+2\pi) = f\left[\frac{(v+2\pi)L}{\pi}\right]$$

$$= f\left(\frac{vL}{\pi} + 2L\right)$$

$$= f\left(\frac{vL}{\pi}\right) \quad (\because f \text{ is } 2L\text{-periodic})$$

$$= g(v)$$

If $f(x)$ has a fourier series, then so has $g(v)$. We have

$$g(v) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nv + b_n \sin nv)$$

with

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(v)dv = \frac{1}{2\pi}\int_{-L}^{L} g(v)\frac{\pi}{L}\,dx \\ &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \end{aligned}$$

and for $n = 1, 2, 3, \cdots$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\cos nv\, dv \\
&= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx.
\end{aligned}
$$

Since $g(v) = f(x)$, we get

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

with a_0, a_n and b_n as given above.

The interval of integration in the above formula can be replaced by any interval of length $p = 2L$, for example, by $0 \leq x \leq 2L$ or $L \leq x \leq 3L$.

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx.$$

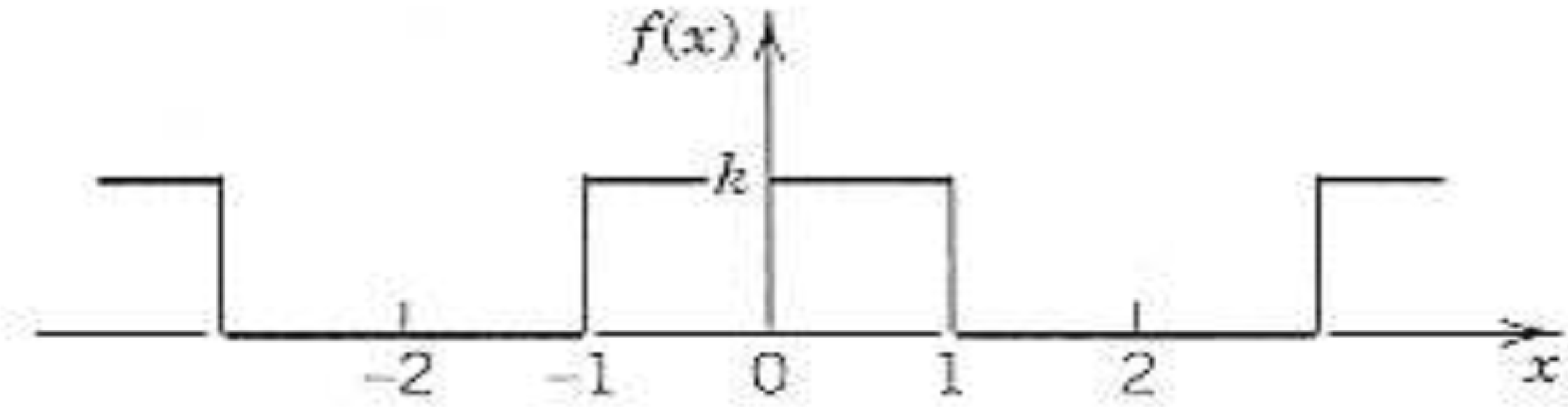
for $n = 1, 2, 3, \cdots$

6.2.9 Example

Let f be a periodic square wave of period $p = 2L = 4$

defined as follows :

$$f(x) = \begin{cases} 0, & \text{if } -2 < x < -1 \\ k, & \text{if } -1 < x < 1 \\ 0, & \text{if } 1 < x < 2 \end{cases}$$



To find the Fourier series of f, we compute

$$
\begin{aligned}
a_0 &= \frac{1}{4}\int_{-2}^{2} f(x)dx \\
a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx
\end{aligned}
$$

and since f is even,

$$b_n = 0 \quad \text{for} \quad n = 1, 2, \cdots .$$

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)dx = \frac{1}{4}\int_{-1}^{1} k dx = \frac{k}{2}$$

$$
\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}dx \\
&= \frac{2k}{n\pi}\sin\frac{n\pi}{2}
\end{aligned}
$$

Hence $a_n = 0$ if n is even and

$$a_n = \begin{cases} \dfrac{2k}{n\pi} & \text{if } n = 1, 5, 9, \cdots \\ -\dfrac{2k}{n\pi} & \text{if } n = 3, 7, 11, \cdots \end{cases}$$

Hence

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - \cdots\right).$$

6.2.10 Fourier cosine and sine series

Using

$$\int_{-L}^{L} f(x)dx = \begin{cases} 0 & \text{if } f \text{ is odd} \\ 2\int_0^L f(x)dx & \text{if } f \text{ is even.} \end{cases}$$

we obtain the following two series.

The Fourier series of an even function $f(x)$ of period $2L$ is the *Fourier cosine series*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

with

$$\begin{aligned} a_0 &= \frac{1}{L} \int_0^L f(x) dx, \\ a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \ n = 1, 2, \cdots . \end{aligned}$$

The Fourier series of an odd function $f(x)$ of period $2L$ is a *Fourier sine series*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

with $b_n = \dfrac{2}{L} \displaystyle\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx.$

6.2.11 Sum and Scalar multiplication

The Fourier coefficients of $f_1 + f_2$ are the sums of corresponding Fourier coefficients of f_1 and f_2.

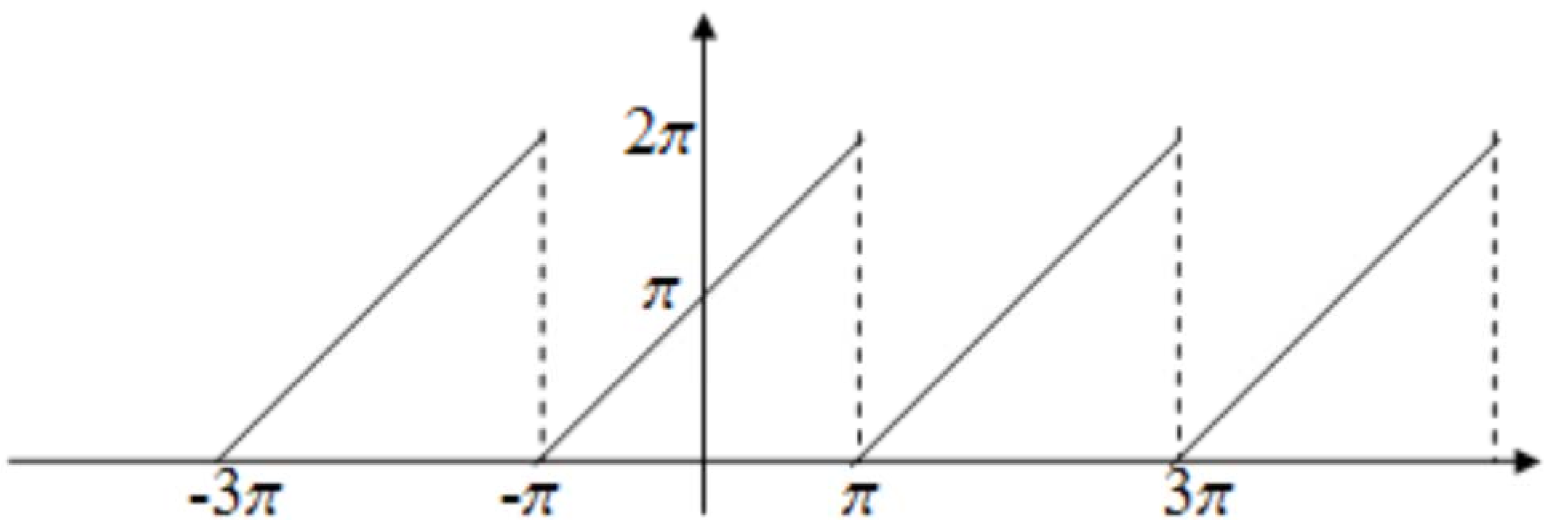
The Fourier coefficients of cf (c a constant) are c times the corresponding Fourier coefficients of f.

6.2.12 Example

Saw tooth function

$$f(x) = x + \pi, \quad -\pi < x < \pi,$$

$$f(x) = f(x + 2\pi).$$



We note that $f = f_1 + f_2$, where $f_1 = x,\ f_2 = \pi$.

The Fourier coefficients for f_2 are $a_0 = \pi$ and

$a_n = 0 = b_n$, $n \geq 1$.

The function $f_1 = x$ is odd.

Thus $a_n = 0$ for all n, and

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi x\sin nx\,dx \\
&= \frac{2}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^\pi - \int_0^\pi \frac{-\cos nx}{n}dx\right\} \\
&= \frac{2}{\pi}\left\{\frac{-(-1)^n\pi}{n} - \left[\frac{-\sin nx}{n^2}\right]_0^\pi\right\} \\
&= \frac{(-1)^{n+1}2}{n}
\end{aligned}$$

So

$$
\begin{aligned}
f(x) &= f_1(x) + f_2(x) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2}{n} \sin \frac{n\pi x}{\pi} + \pi \\
&= \pi + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx
\end{aligned}
$$

6.3 Half-range Expansions

In various applications there is a practical need to use Fourier series in connection with functions f that are given on some interval only, say, $0 \le x \le L$.

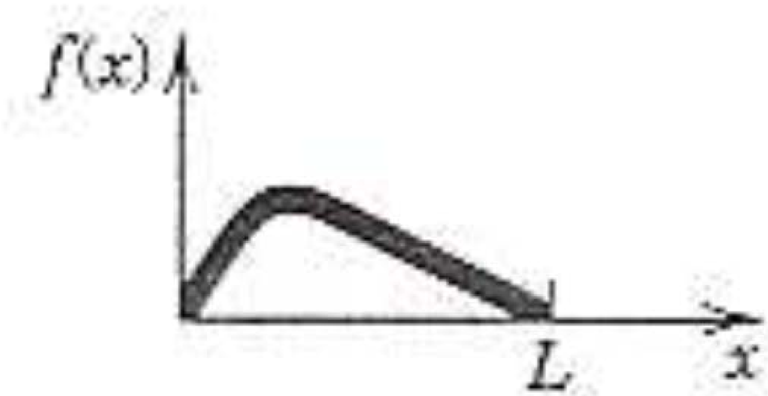


Figure (a)

6.3.1 Extension of $f(x)$

We could extend $f(x)$ as a periodic function with period L and then represent the extended function by a Fourier series, which in general would involve both sine and cosine terms. We can do better and always get a cosine series by first extending $f(x)$ from $0 \leq x \leq L$ as an even function on the interval $-L \leq x \leq L$ as in figure (b) and then extend this new function as a periodic function of period $2L$, and since it is even, we can represent it by a Fourier cosine series.

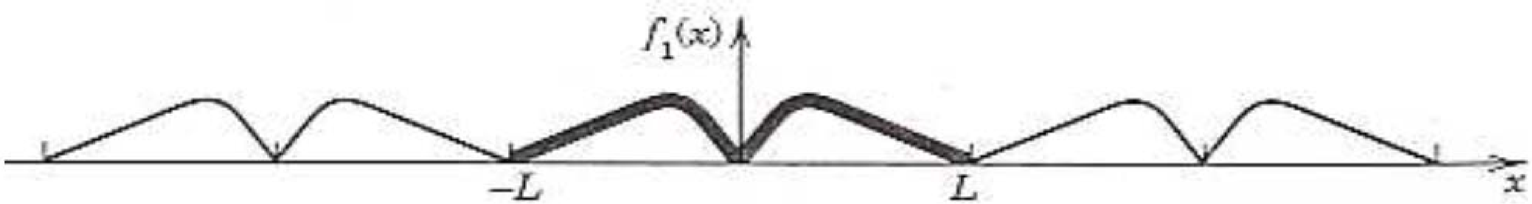


Figure (b)

Also, we can extend $f(x)$ from $0 \leq x \leq L$ as an odd function on $-L \leq x \leq L$ as in figure (c) and then extend this new function as a periodic function of period $2L$, and since it is odd, it is represented by a Fourier sine series.

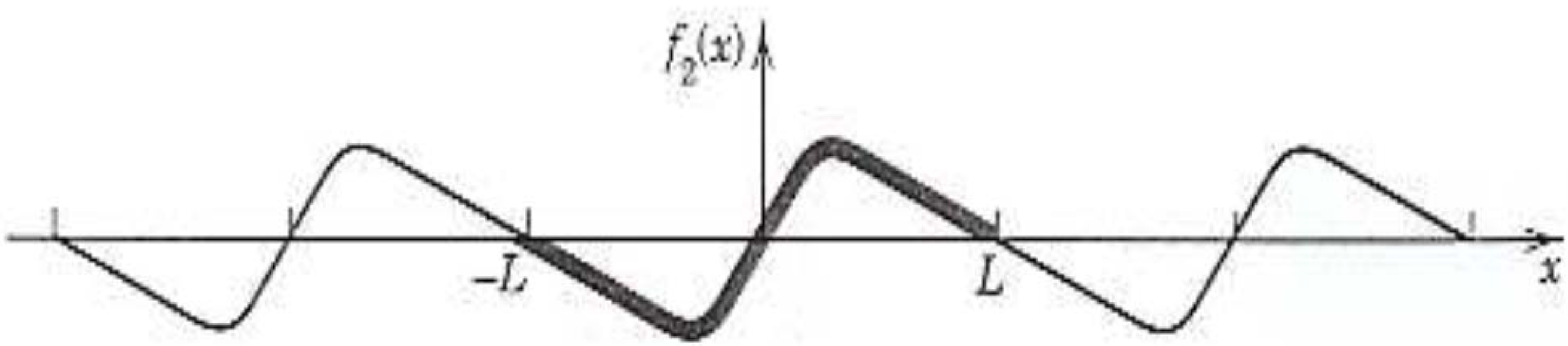


Figure (c)

6.3.2 Half range expansion

Such two series are called the two 'half range expansions' of the function f which is given only on 'half the range'.

The cosine half range expansion is

$$f(x) = a_0 + \sum_{1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

with

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L f(x)dx, \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \; n = 1, 2, \cdots \end{aligned}$$

The sine half range expansion is

$$f(x) = \sum_{1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

with

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \; n = 1, 2, \cdots .$$

6.3.3 Example

Find the two half range expansions for

$$f(x) = \begin{cases} 0, & 0 < x < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < x < \pi. \end{cases}$$

For the cosine half range expansion, we have

$$a_0 = \frac{1}{\pi}\left\{\int_0^{\pi/2} 0dx + \int_{\pi/2}^{\pi} 1dx\right\} = \frac{1}{2}$$

and

$$
\begin{aligned}
& a_n \\
= \; & \frac{2}{\pi}\left\{\int_0^{\pi/2} 0 \cos nx dx + \int_{\pi/2}^{\pi} 1 \cos nx dx\right\} \\
= \; & \frac{2}{\pi}[\frac{\sin \pi n - \sin \frac{1}{2}\pi n}{n}]
\end{aligned}
$$

Thus a_n simplifies to

$$a_n = \frac{2}{n\pi}\left(-\sin\frac{n\pi}{2}\right)$$

Indeed,

$$a_1 = \frac{-2}{\pi}, \quad a_3 = \frac{2}{3\pi}, \quad a_5 = \frac{-2}{5\pi}, \quad , \cdots$$

and $a_n = 0$ if $n \geq 1$ and n is even.

The cosine half range expansion is

$$f(x) = \frac{1}{2} + 2\sum_{m=1}^{\infty} \frac{(-1)^m}{(2m-1)\pi} \cos(2m-1)x$$

For the sine half range expansion, we have

$$
\begin{aligned}
& b_n \\
= \;& \frac{2}{\pi}\left\{\int_0^{\pi/2} 0 \sin nx dx + \int_{\pi/2}^{\pi} 1 \sin nx dx\right\} \\
= \;& \frac{2}{\pi}[\frac{-\cos \pi n + \cos \frac{1}{2}\pi n}{n}]
\end{aligned}
$$

Thus b_n simplifies to

$$b_n = \frac{2}{n\pi}\left[(-1)^{n+1} + \cos\frac{n}{2}\pi\right]$$

Indeed,

$$b_1 = \tfrac{2}{\pi}, \quad b_2 = \tfrac{-4}{2\pi}, \quad b_3 = \tfrac{2}{3\pi}, \quad b_4 = 0,$$

$$b_5 = \tfrac{2}{5\pi}, \quad b_6 = \tfrac{-4}{6\pi}, \quad b_7 = \tfrac{2}{7\pi}, \quad b_8 = 0, \cdots .$$

The sine half range expansion is

$f(x) =$

$\frac{2}{\pi}\sum_{m=1}^{\infty}\left\{\frac{\sin(4m-3)x}{4m-3} - \frac{2\sin(4m-2)x}{4m-2} + \frac{\sin(4m-1)x}{4m-1}\right\}$