CH 3 - *Integration*

- ***Indefinite integral*** $\int f(x)\,dx$
- ***Definite integral*** $\int_a^b f(x)\,dx$
- **The *Fundamental Thm*** of ***Calculus***
- ***Techniques*** of ***Integration***
- **Applications**

3.1 *Indefinite Integrals*

3.1.1& 3.1.2 *Antiderivatives*

☺ $\boldsymbol{F(x)}$ —(differentiation)→ $\boldsymbol{F'(x) = f(x)}$

☻ Reverse procedure !!!

Let $\boldsymbol{F}$ & $\boldsymbol{f}$ be 2 functions defined on an interval I.

$\boldsymbol{F}$ is called an ***antiderivative*** of $\boldsymbol{f}$ on I if

$$\boldsymbol{F'(x) = f(x)} \qquad \text{for all } \boldsymbol{x} \text{ in } I.$$

The ***indefinite integral*** of $\boldsymbol{f}$ *wrt* $\boldsymbol{x}$

$= \int \boldsymbol{f(x)dx}$

= the ***set*** of all ***antiderivatives*** of $\boldsymbol{f}$

If $F'(x) = G'(x)$ for all x in I, then there exists constant C *s.t.* $G(x) = F(x) + C$ for all x in I.

If F is an ***antiderivative*** of f on I, then $F + C$ is also an ***antiderivative*** of f on I, & every ***antiderivative*** of f on I is of this form.

Thus,

$$\int f(x)dx = F(x) + C$$

Integral sign ***Integrand*** ***Constant of integration***

Geometrical Interpretation $\int f(x)dx = F(x) + C$

- The process on ***integration*** is to find all curves

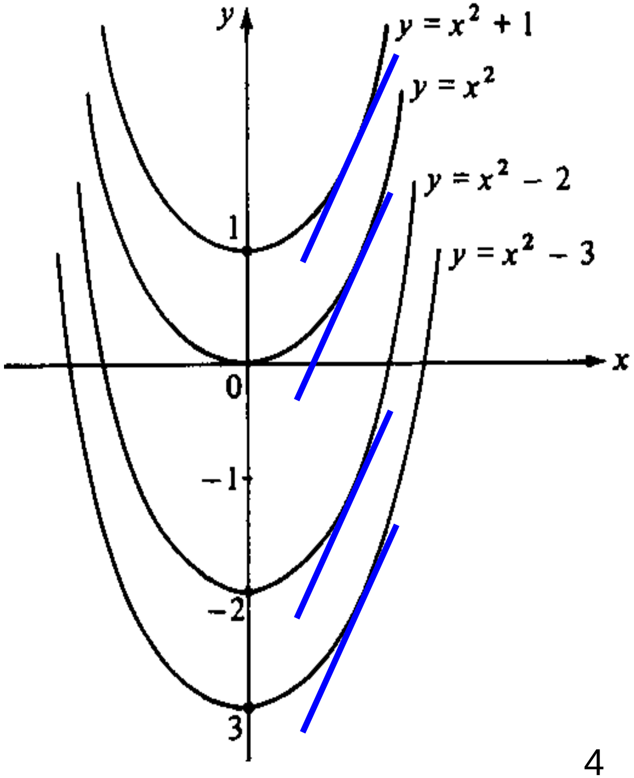
$$y = F(x) + C$$

s.t. their ***slopes*** at x are $f(x)$.

Example. $f(x) = 2x$

$$F(x) = x^2$$

$$F(x) + C = x^2 + C$$



3.1.3 *Integral Formulas*

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1,\ n \text{ rational}$

$\displaystyle\int 1\,dx = \int dx = x + C \quad \text{(Special case, } n = 0\text{)}$

2. $\displaystyle\int \sin kx\,dx = -\frac{\cos kx}{k} + C$

3. $\displaystyle\int \cos kx\,dx = \frac{\sin kx}{k} + C$

4. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

5. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$

6. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$

7. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \frac{1}{x}\,dx = \ln x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

3.1.4 *Basic Rules*

1. $$\int kf(x)\,dx = k\int f(x)\,dx,$$

   $k = \text{constant (independent of } x)$

2. $$\int -f(x)\,dx = -\int f(x)\,dx$$

   (Rule 1 with $k = -1$)

3. $$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Example

Find the curve in the xy-plane which passes through the point $(9, 4)$ and whose slope at each point (x, y) is $3\sqrt{x}$.

Solution. The curve is given by $y = y(x)$, satisfying

(i) $\dfrac{dy}{dx} = 3\sqrt{x}$ and (ii) $y(9) = 4$.

(i) $\dfrac{dy}{dx} = 3\sqrt{x}$ and (ii) $y(9) = 4.$

Solving (i), we get

$$y = \int 3\sqrt{x}\,dx = 3\,\frac{x^{3/2}}{3/2} + C = 2x^{3/2} + C.$$

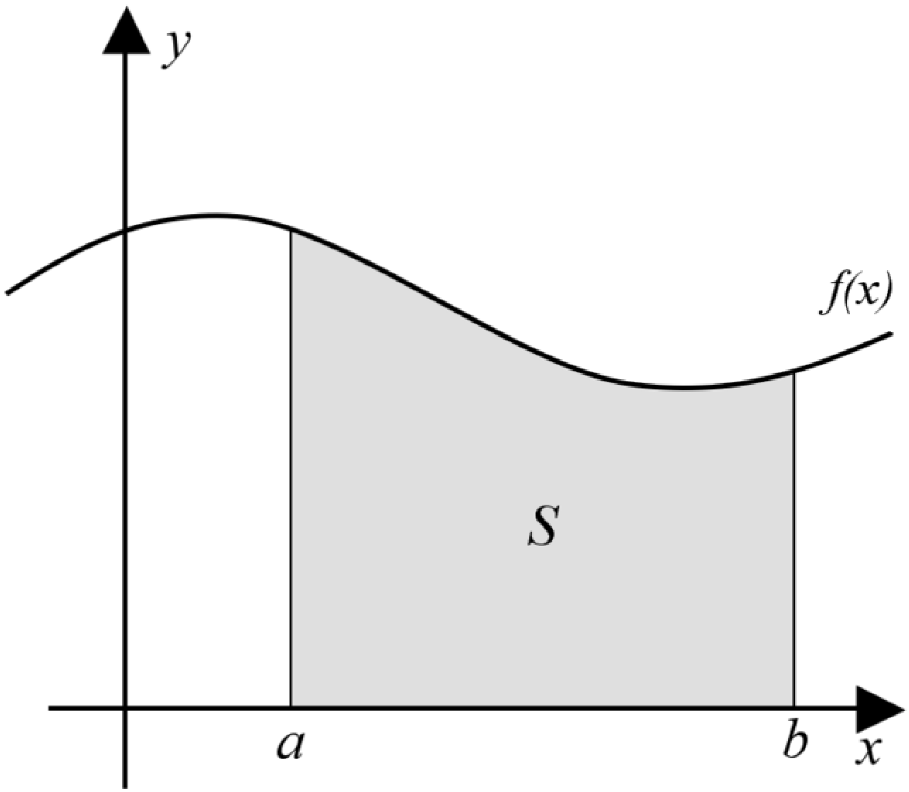
By (ii), $\quad 4 = (2)9^{3/2} + C = (2)27 + C,$

$C = 4 - 54 = -50.$

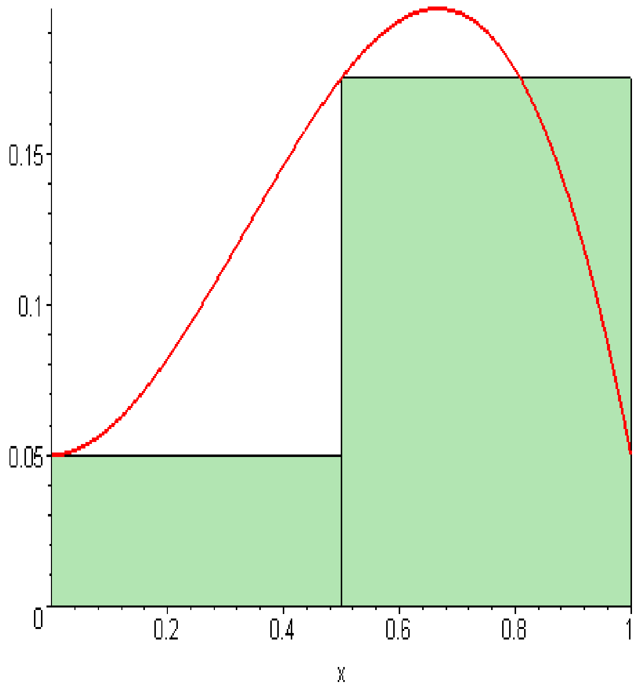
Hence $\quad y = 2x^{3/2} - 50.$

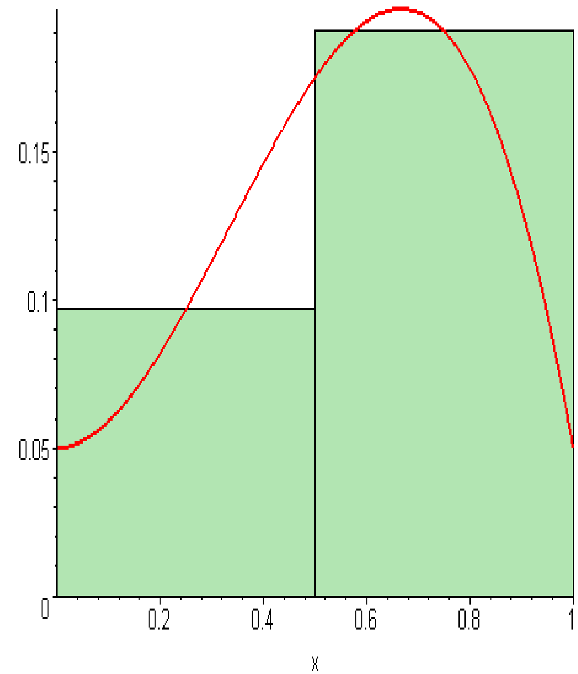
3.2 *Riemann Integrals*

- Find the ***area*** of the ***shaded*** region:

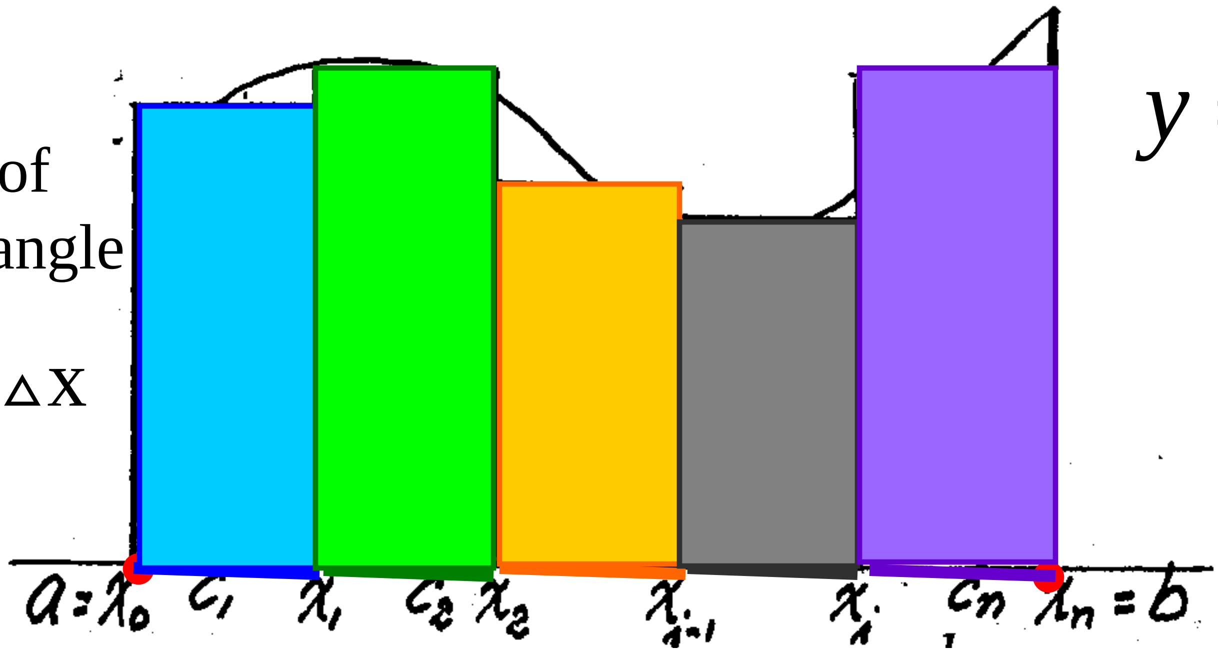


Riemann integral





Riemann Sum $= \sum_{k=1}^{n} f(c_k)\Delta x$ = total area of rectangles



$y = f(x)$

x-axis

Area of 1st rectangle

$f(c_1)\Delta x$

Length of each interval = $\Delta x = \dfrac{b-a}{n}$

Riemann Sum

- The ***area*** under the ***curve*** from *a* to *b*

$$\approx \quad \sum_{k=1}^{n} f(c_k)\Delta x$$

— a ***Riemann sum*** of f on [*a*, *b*].

The ***exact area*** is given by

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x.}$$

3.2.2 *Riemann* (*Definite*) *Integral*

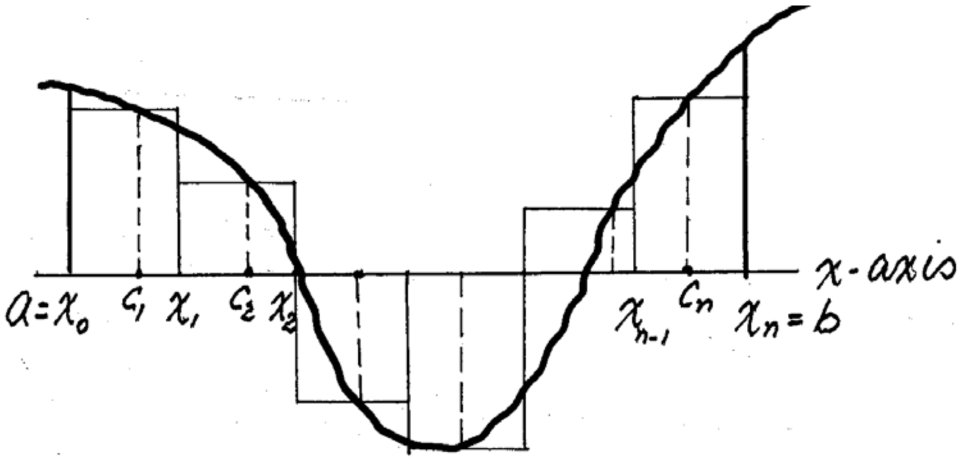
- We write

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x.}$$

& call it the ***Riemann*** (or ***Definite***) ***Integral*** of f over $[a, b]$.

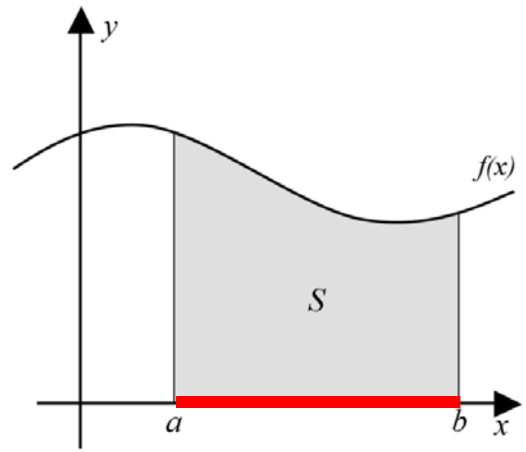
Note

($f(x)$ may be ***negative***)



3.2.3 *Terminology*

$$\int_a^b \boxed{f}(x)\,dx$$



$[a, b]$: the interval of integration

a : lower limit of integration

b : upper limit of integration

$f(x)$: the integrand

x : variable of integration (*dummy*)

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(t)\,dt, \quad \text{etc.}$$

3.2.4 *Basic Rules*

1. $\int_a^a f(x)\,dx = 0$ 2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx,$ (any constant k)

4. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

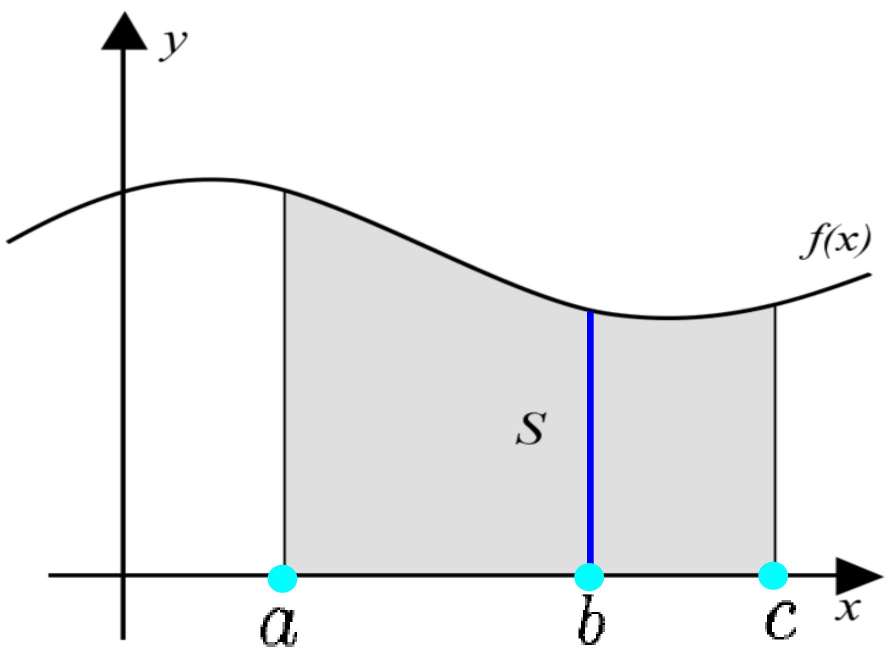
5. If $f(x) \geq g(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$

6. If f is continuous on the interval joining a, b and c, then

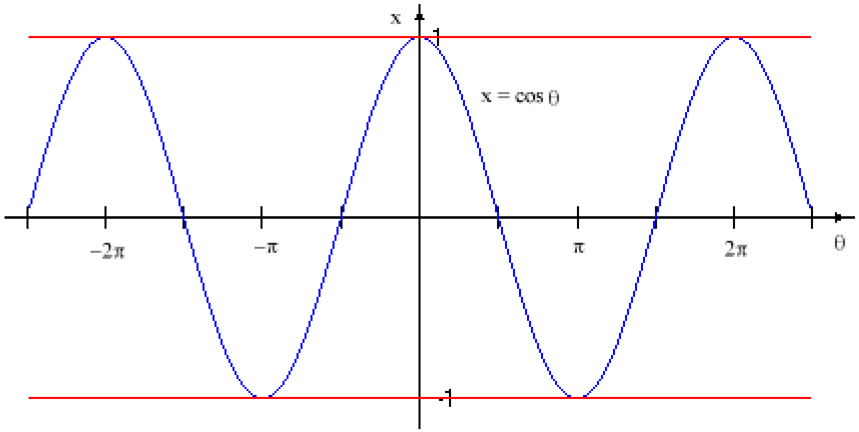
$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

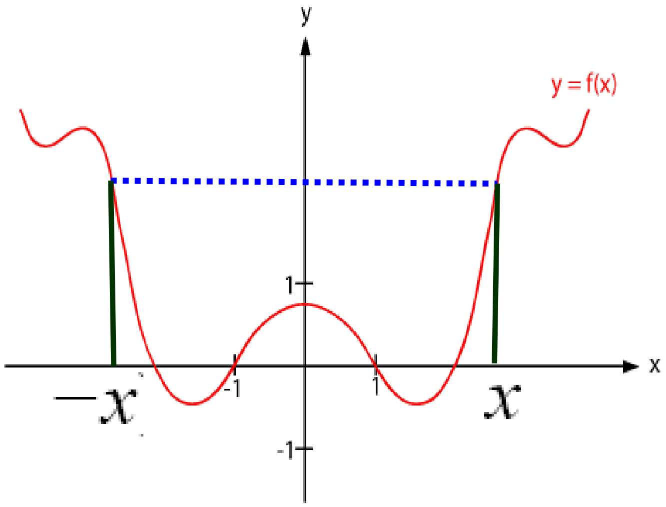


Even functions

- f is ***even*** if $f(-x) = f(x)$
  (x^2, **cos** x, $|x|$, etc)

♥ **Symmetric** *wrt* y-axis

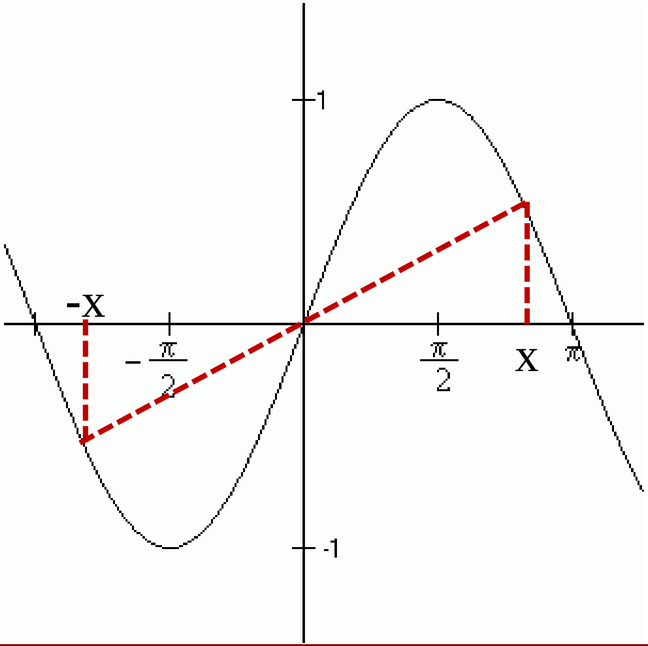


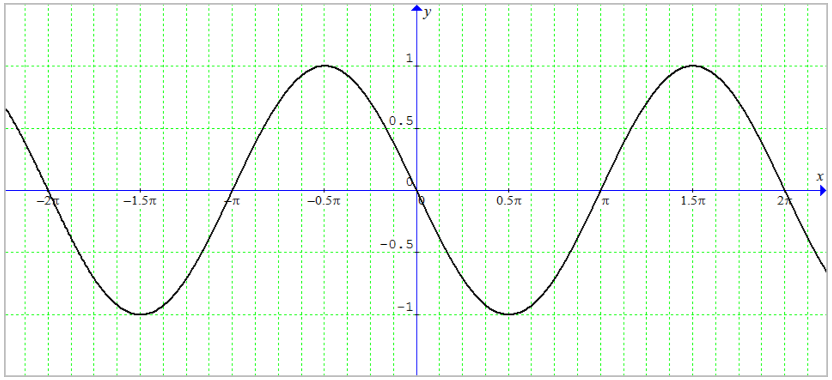


$$\boxed{\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx}$$

Odd functions

- ***f*** is ***odd*** if ***f*(−*x*) = −*f*(*x*)** (***x*, *x*³, sin *x*,** etc)
- **Symmetric** *wrt* **origin**





$$\int_{-a}^{a} f(x) = 0$$

- **How to *evaluate***

$$\int_a^b f(x)\,dx \quad ?$$

By

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x \quad ?$$

No, then how? See next slide

3. 3 *The Fundamental Thm* of *Calculus* (*FTC*)

3.3.1 & 3.33

- Let f be a ***continuous*** fn on $[a, b]$.

(I) Let

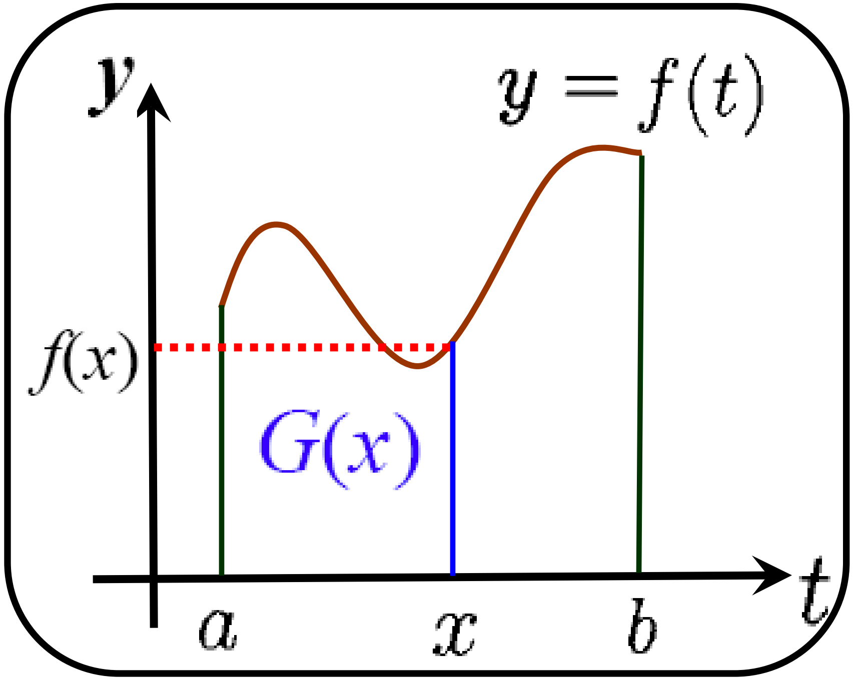
$$G(x) = \int_a^x f(t)\,dt$$

Then

$$\frac{d}{dx} G(x) = f(x)$$

i.e.,

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$



(II) If ***F*** is an ***antiderivative*** of ***f*** on [a, b], then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof Let $\boldsymbol{G(x)} = \int_a^x f(t)\,dt$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

By (I), $\boldsymbol{G'(x) = f(x).}$

As $\boldsymbol{G'(x) = f(x)} = \boldsymbol{F'(x)}$ for all $\boldsymbol{x}$ in [a, b],

there exists $\boldsymbol{c}$ *s.t.* $\boldsymbol{F(x)} = \boldsymbol{G(x) + c}$ on [a, b].

$$F(x) = G(x) + c$$

Thus

$$
\begin{aligned}
& F(b) - F(a) \\
&= G(b) + c - \big(G(a) + c\big) \\
&= G(b) - G(a) \\
&= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\
&= \int_a^b f(t)\,dt.
\end{aligned}
$$

$$\boxed{G(x) = \int_a^x f(t)\,dt}$$

Main Results

(I)

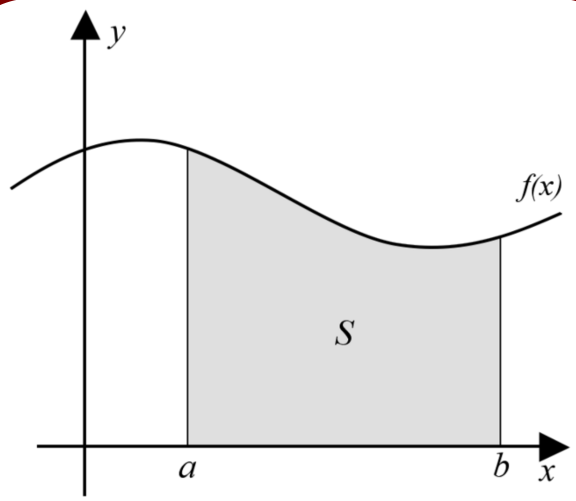
$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

(II) **(FTC)** If ***F*** is an ***antiderivative*** of ***f*** on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

♥ ***FTC*** **(evaluating** $\int_a^b f(x)\,dx$ **)**

$$\boxed{\int_a^b f(x)dx} = F(b) - F(a).$$



$$F'(x) = f(x)$$

3.3.2 *Examples*

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\frac{d}{dx}\int_0^{\boxed{2}} t^2\,dt \quad = \mathbf{0}$$

$$\frac{d}{dx}\left(\int_0^x \sin\sqrt{t}\ dt\right) = \sin\sqrt{x}$$

♣ $$\frac{d}{dx}\left(\int_1^{x^4}\frac{t}{\sqrt{t^3+2}}dt\right)$$ ***The Chain Rule***

$$=\frac{d}{dx^4}\left(\int_1^{x^4}\frac{t}{\sqrt{t^3+2}}dt\right)\frac{d}{dx}x^4$$

$$=\frac{x^4}{\sqrt{(x^4)^3+2}}\,(4x^3)=\frac{4x^7}{\sqrt{x^{12}+2}}$$

Other Cases

$$\frac{d}{dx}\int_{x}^{a} f(t)dt \quad = -\frac{d}{dx}\int_{a}^{x} f(t)dt$$

$$\frac{d}{dx}\int_{x^2}^{x^4} f(t)dt$$

$$= \frac{d}{dx}\int_{x^2}^{a} f(t)dt + \frac{d}{dx}\int_{a}^{x^4} f(t)dt$$

8. $\frac{d}{d\boxed{x}}\int_0^{\boxed{x^2}} \sqrt{2-\sin^3 t}\,dt =$

(A) $2x\sqrt{2-\sin^3 x^2}$

(B) $2x\sqrt{2-\sin^3 x}$

(C) $\frac{2x}{\sqrt{2-\sin^3 x^2}}$

(D) $\frac{2x}{\sqrt{2-\sin^3 x}}$

(E) $\frac{-3\sin^2 x}{2\sqrt{2-\sin^3 x^2}}$

Find the value of

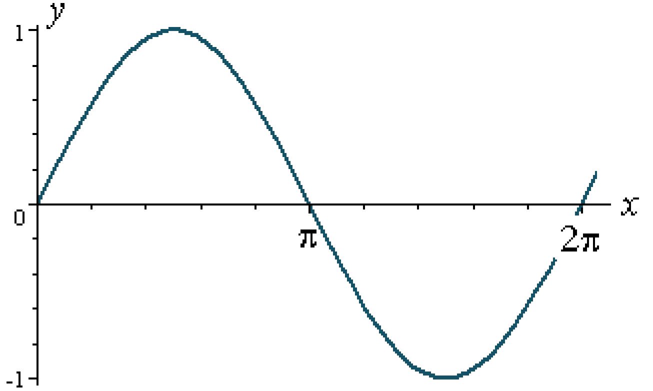
$$\lim_{x\to 3^+} \frac{x^2\int_3^x \sqrt{t^3+9}\,dt}{|3-x|}.\left(\frac{0}{0}\text{ form}\right)$$

(Ans. 54)

3.3.4 *Examples*

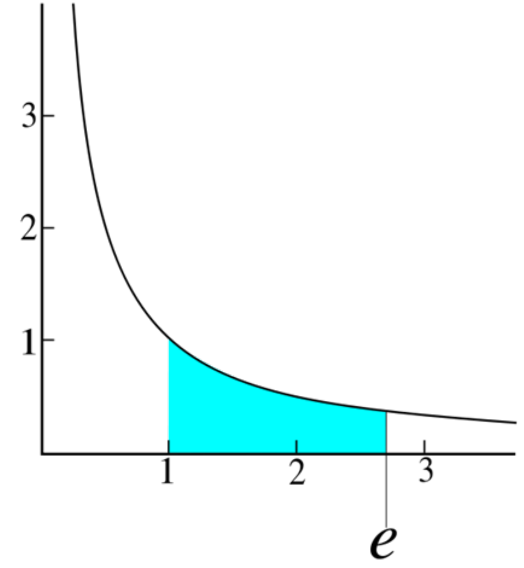
$$\int_0^{2\pi} \sin x dx \;= (-\cos x)]_0^{2\pi}$$

$$= \; -(\cos 2\pi - \cos 0) = 0$$



$$\int_1^e \frac{1}{x} dx \;= \ln x]_1^e$$

$$= \ln e - \ln 1 = 1.$$



Exercises

$$\int_0^{\pi} \cos x \, dx =$$

$$\int_0^{2} \mathrm{t}^2 \, dt =$$

$$\int_{-2}^{2} (4\text{-}\mathrm{u}^2) \, du =$$

3.4 *Integration* by ***Substitution***

Find **(♣)** $\int \frac{\ln^5 x}{x} dx$

Let **$u = \ln x$.**

Then $\boldsymbol{du/dx = 1/x}$, & so

$$(\clubsuit) = \int u^5 du$$

$$= \frac{u^6}{6} + C \quad = \frac{\ln^6 x}{6} + C$$

To evaluate $\boxed{\int f(g(x))g'(x)\,dx}$ where f and g' are continuous:

1. Set $u = g(x)$. Then $g'(x) = \dfrac{du}{dx}$, the given integral becomes $\int f(u)\,du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$ in the result of step 2.

$$\int e^{x+e^x}\,dx$$

$$= \int e^x e^{e^x}\,dx$$

$$\boxed{\begin{aligned} u &= e^x \\ \frac{du}{dx} &= e^x = u \end{aligned}}$$

$$= \int e^u\,du$$

$$= e^u + C \quad = e^{e^x} + C$$

3.4.1 *Examples*

$$\int (x^2+2x-3)^2(x+1)dx =$$

$$\int \sin^4 x \ \cos x \ dx =$$

3.4.2 &3.43

Evaluate $I = \int_0^{\pi/4} \tan x \cdot \sec^2 x \, dx$

$u = \tan x \qquad du = \sec^2 x dx$

$$\int \tan x \sec^2 x dx = \int u du = \frac{\tan^2 x}{2} + C$$

$$I = \left. \frac{\tan^2 x}{2} \right|_0^{\frac{\pi}{4}} = \frac{1}{2}$$

Let n be a positive odd integer.

Then $\int_0^{\frac{\pi}{2}} \cos x \sqrt{\sin^n x}\, \mathrm{d}x =$

(E) $\frac{2}{n+2}$

7. Let n be a positive integer which is bigger than 1505. Then

$$\int_1^2 \frac{1}{x\left(1+x^n\right)} dx =$$

(B) $\ln 2 - \frac{1}{n} \ln\left(1+2^n\right) + \frac{1}{n} \ln 2$

'partial fraction'

$$\frac{1}{x(1+x^n)} = \frac{1}{x} - \frac{x^{n-1}}{(1+x^n)}$$

3.5 ***Integration*** by ***Parts***

Recall the product rule

$$\frac{d}{dx}(uv) = u\,\frac{dv}{dx} + v\frac{du}{dx}$$

$$d(uv) = u\,dv + v\,du$$

$$u\,dv = d(uv) - v\,du.$$

Integration-by-**parts** ***formula***

$$\int u\,dv = uv - \int v\,du$$

2nd version

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

3.5.1 Examples

$$\int u\,dv = uv - \int v\,du$$

$$\int \ln x dx$$

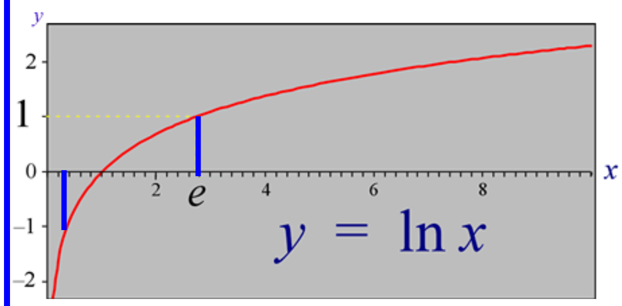
$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}dx \qquad v = x$$

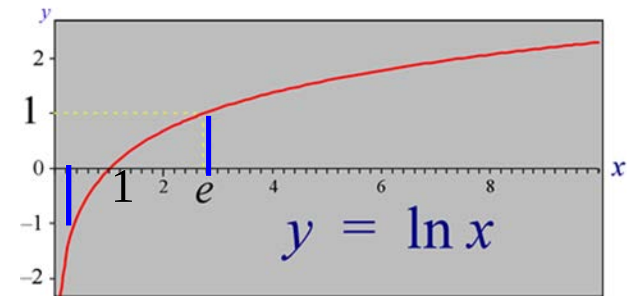
$$= x\ln x - \int x\frac{1}{x}dx \qquad = x\ln x - x + C$$

Evaluate $\displaystyle\int_{\frac{1}{e}}^{e} |\ln x|\, dx$

$$2\left(1 - e^{-1}\right)$$



Evaluate $\displaystyle\int_{\frac{1}{e}}^{e} |\ln x|\, dx$



$$\int_{1/e}^{e} |\ln x| dx = \int_{1/e}^{1} |\ln x| + \int_{1}^{e} |\ln x| dxdx$$

$$= \int_{1/e}^{1} -\ln x \ dx + \int_{1}^{e} \ln x \ dx$$

Ans: $2\left(1 - e^{-1}\right)$

$$=$$

♣ **Show** that

$$\int u\,dv = uv - \int v\,du$$

$$\int e^x \sin x\, dx = \frac{1}{2} e^x(\sin x - \cos x) + C$$

Let $u = e^x$ & $dv = \sin x\, dx.$

Then $du = e^x\, dx$ & $v = -\cos x.$

Thus $\int e^x \sin x\, dx$

$$= -e^x \cos x - \int -\cos x\, e^x\, dx$$

$$\boxed{\begin{aligned}&\int e^x \sin x\, dx \\ &= -e^x \cos x + \int \cos x\, e^x\, dx\end{aligned}}$$

Likewise,

$$\boxed{\int \cos x\, e^x\, dx = e^x \sin x - \int e^x \sin x\, dx.}$$

Hence

$$2\int e^x \sin x\, dx = e^x \sin x - e^x \cos x$$

& the identity follows.

☺ The ***method*** is suitable for other ***integrands***:

$$\underset{u}{\boxed{x^n}}\, e^x, \quad x^n \underset{u}{\boxed{\ln x}}, \quad \underset{u}{\boxed{x^n}} \cos x, \quad \underset{u}{\boxed{x^n}} \sin x, \quad \text{etc.}$$

$$\boxed{\int u\, dv} = uv - \int v\, du$$

tough ***easier***

3.5.2 *Exercise*

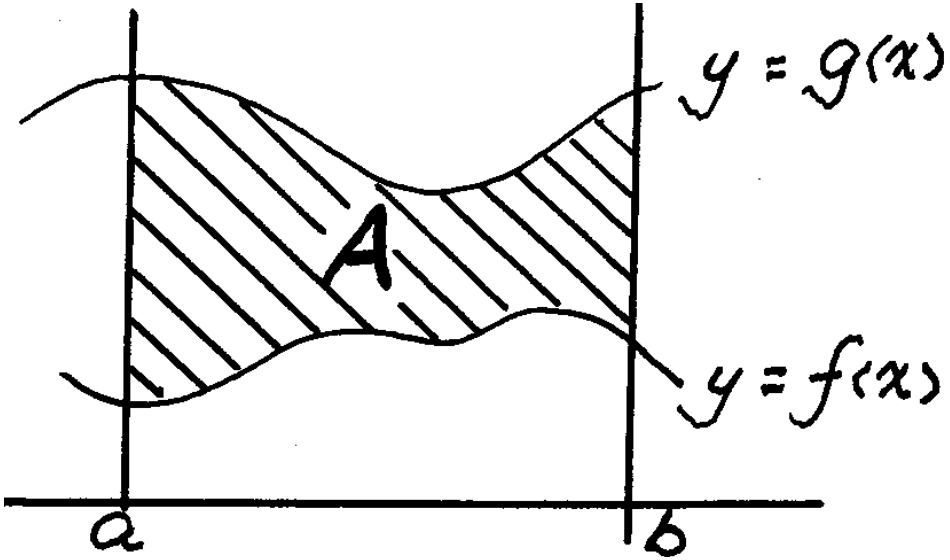
$$\int x^2 e^x dx =$$

$$\int_0^1 xe^x dx =$$

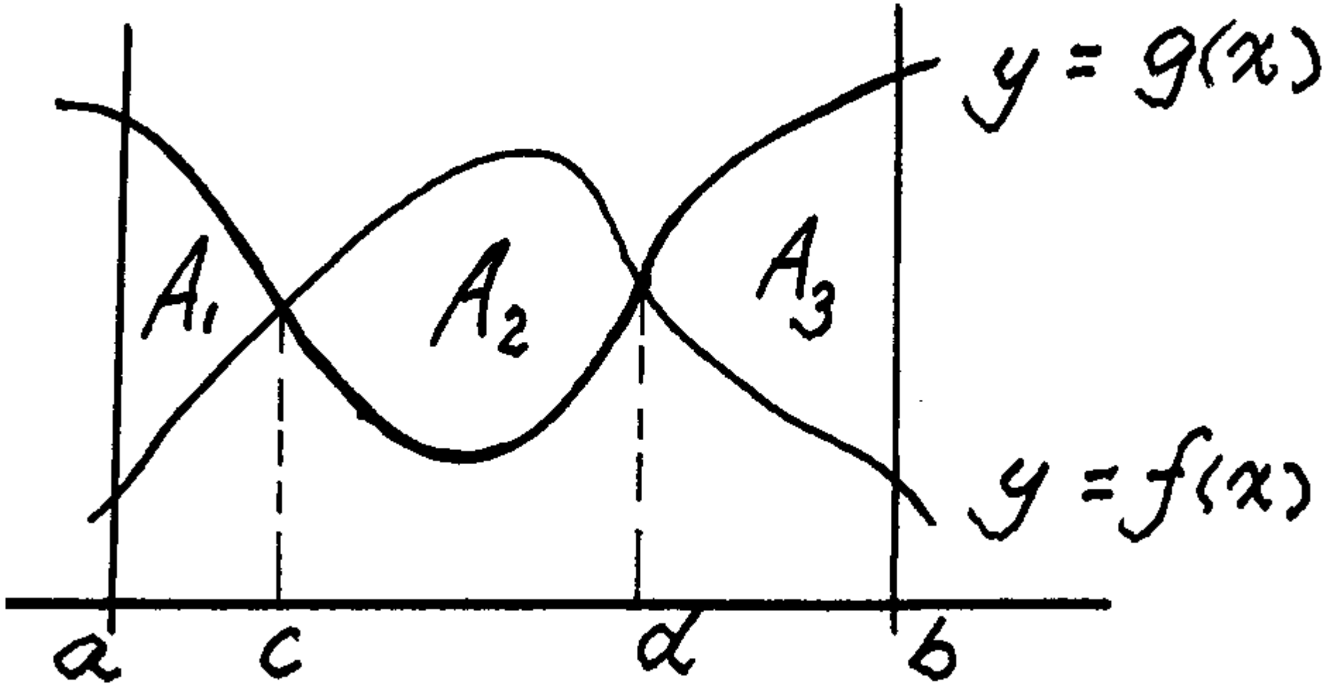
- **Show that**

$$\int \sin(\ln x)\,dx = \tfrac{1}{2}\,x(\sin(\ln x) - \cos(\ln x)) + C$$

3.6 *Area* between 2 *Curves*



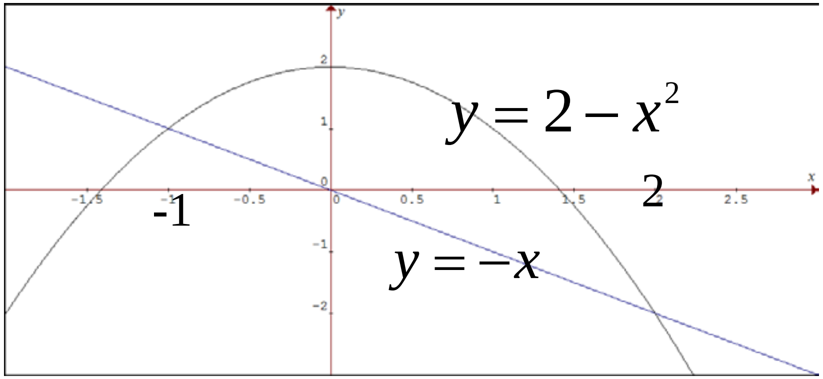
$$A = \int_a^b (g(x) - f(x))\,dx$$



$$\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = \int_a^c (g(x) - f(x))\,dx + \int_c^d (f(x) - g(x))\,dx + \int_d^b (g(x) - f(x))\,dx$$

3.6.1 *Examples*

♣ Find area enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.



- Sometimes we may like to view the curve as $x = g(y)$

  (instead of $y = f(x)$) when evaluating area.

  The area will be $A = \int_c^d [g_2(y) - g_1(y)]\, dy.$

Find the area of the finite region bounded by the straight line

$y = x - 2$ and the curve $y^2 = x$.

(Ans. 4.5)

pts of intersection: (1,-1), (4,2)