Chapter 6

Fourier Series

Key Results

- Derive formulas for Fourier coefficients
- Calculate the Fourier series of a periodic function
- Use Fourier series to approximate wave functions
- Approximate mathematical constants using series
- Half range expansions

Periodic Functions

A function $f(x)$ is called periodic if

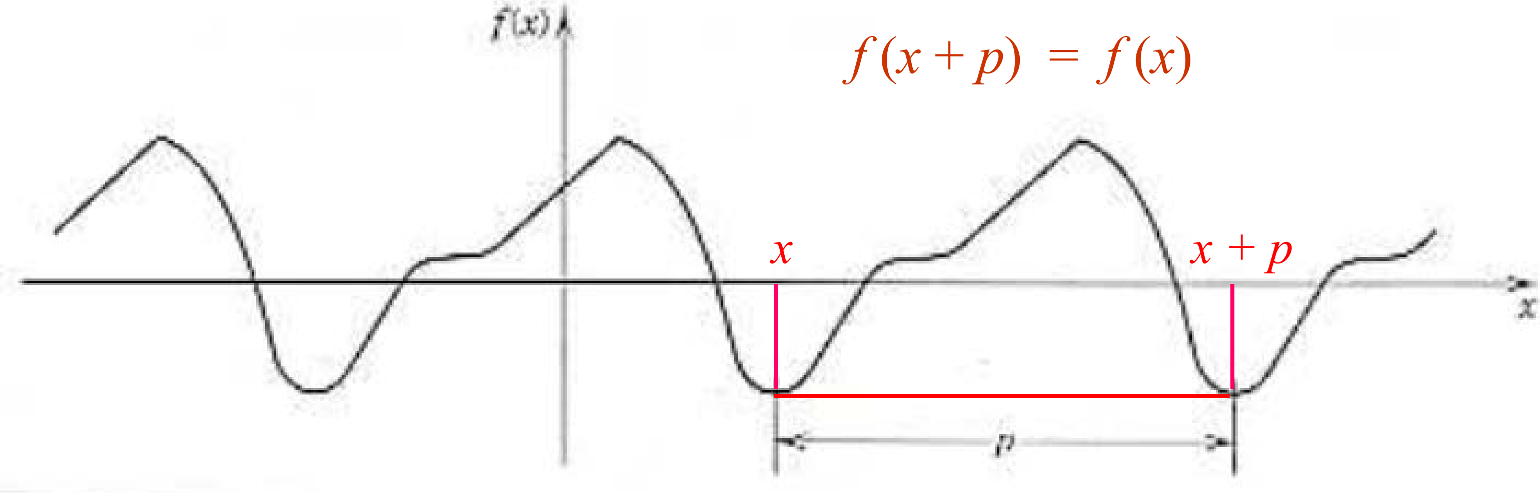
- f is defined for all real values x
- there is a positive number p such that

$$\boxed{f(x+p) = f(x) \quad \text{for all } x.}$$

The number p is called a period of f.

Graph of a Periodic Function

The graph of a periodic function can be obtained by periodic repetition of a portion of its graph over an interval of length p.



Examples

For a positive constant k, the functions

$$f(x) = \sin kx \quad \text{and} \quad g(x) = \cos kx$$

are periodic of $\boxed{\text{period } \dfrac{2\pi}{k}}$

But polynomials (e.g. x, x^2, x^3), exponential functions (e.g. a^x), logarithmic functions (e.g. $\log_a x$) are not periodic functions.

Some Properties

If *f* has period *p*, then

$$f(x+2p) = f(\underbrace{(x+p)}_{X} \underbrace{+\,p)}_{+\,p} = f(\underbrace{x+p}_{X}) = f(x)$$

Inductively, for any positive integer *n*,

$$\boxed{f(x+np) = f(x)}$$

Thus, $2p$, $3p$, … are also periods of *f*.

If f and g have period p, and a and b are constants, then the function

$$\boxed{h(x) = af(x) + bg(x)}$$

satisfies

$$\begin{aligned} h(x+p) &= af(x+p) + bg(x+p) \\ &= af(x) + bg(x) \\ &= h(x) \end{aligned}$$

for all x.

That is, h is also periodic with period p.

Even Functions

A function *f* is an even function if $\boxed{f(-x) = f(x).}$

e.g. $f(x) = x^2$.

Check: $f(-x) = (-x)^2 = x^2 = f(x)$

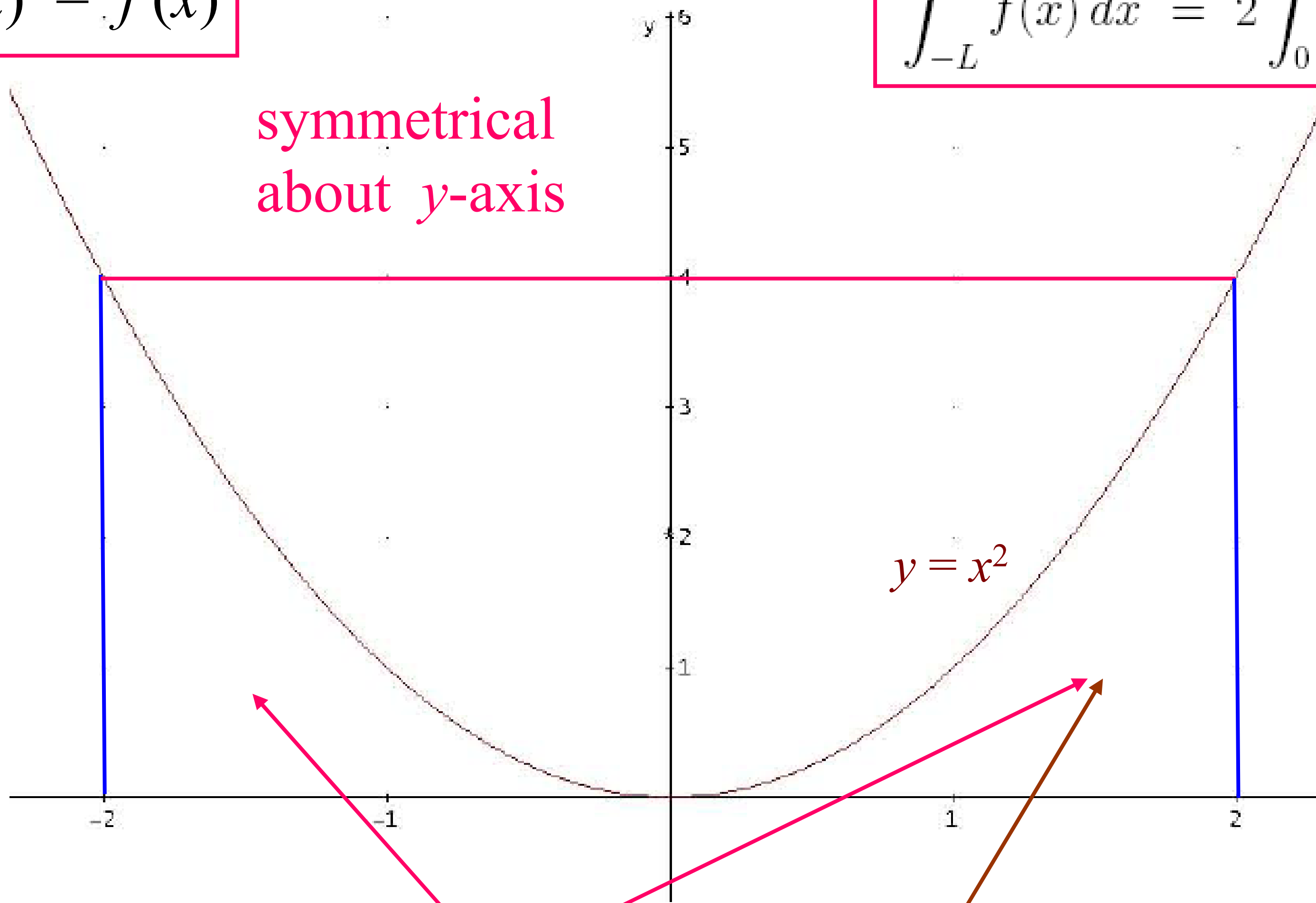
The graph of an even function is symmetrical about the *y*-axis.

Integration property:

$$\boxed{\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx}$$

$$f(-x) = f(x)$$

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$$



$$\int_{-2}^{2} x^2\,dx = 2\int_{0}^{2} x^2\,dx$$

Even Functions (cont'd)

Further examples of even functions:

e.g. $f(x) = \cos kx$ where k is a constant

e.g. $f(x) = x^4 - x^2$

Odd Functions

A function f is an odd function if $\boxed{f(-x) = -f(x).}$

e.g. $f(x) = x^3.$

Check: $f(-x) = (-x)^3 = -x^3 = -f(x)$

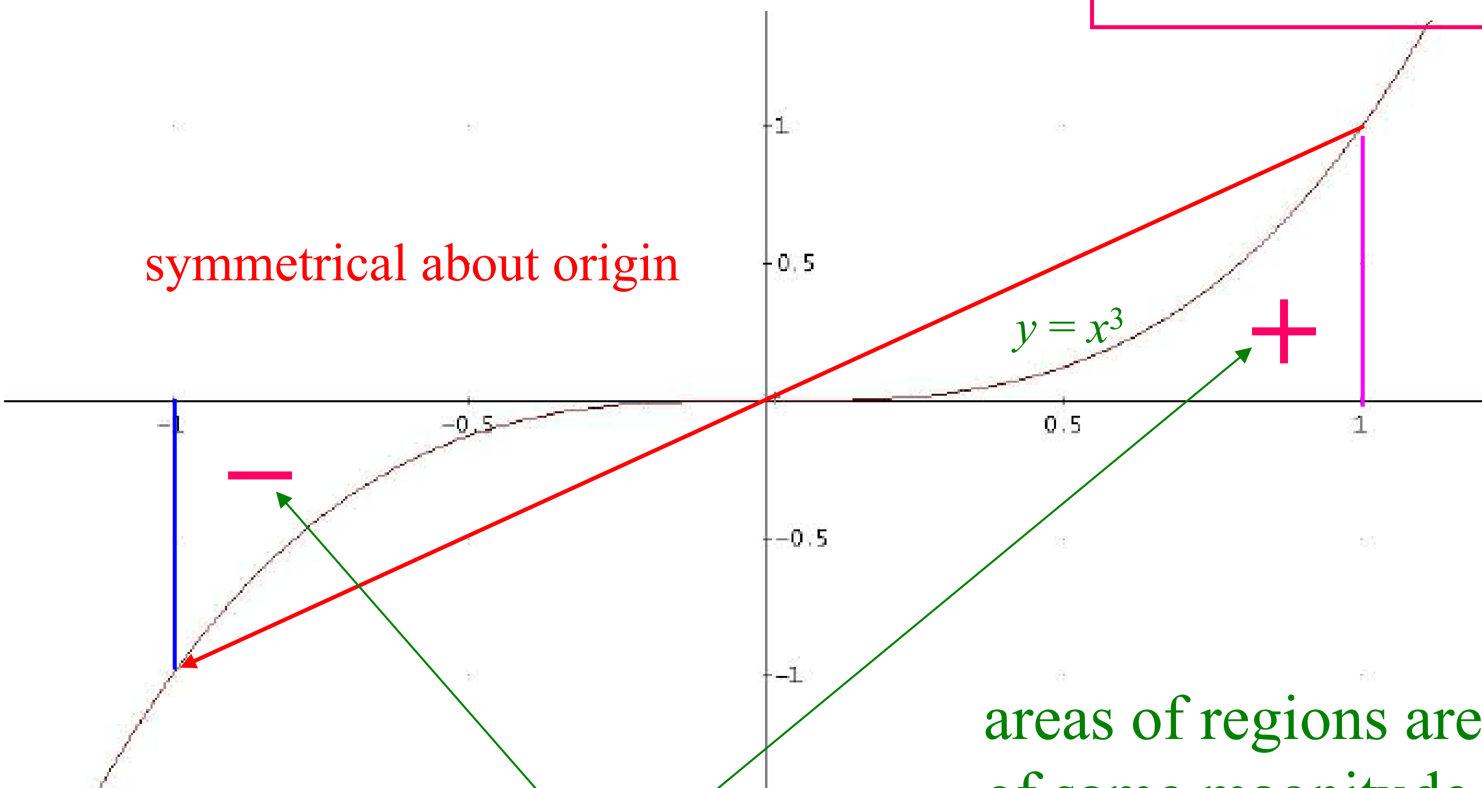
The graph of an odd function is symmetrical about the origin.

Integration property:

$$\boxed{\int_{-L}^{L} f(x)\, dx = 0}$$

$$f(-x) = -f(x)$$

$$\int_{-L}^{L} f(x)\, dx = 0$$



$$\int_{-1}^{1} x^3\, dx = 0$$

areas of regions are
of same magnitude

but of different signs

Odd Functions (cont'd)

Further examples of odd functions:

e.g. $f(x) = \sin kx$ where k is a constant

e.g. $f(x) = x^5 - x$

Product Properties

$f(x)$	$g(x)$	$f(x)\,g(x)$
odd	odd	even
odd	even	odd
even	odd	odd
even	even	even

e.g. $f(x) = \sin x \qquad g(x) = \sin 3x$

$$h(x) = f(x) \cdot g(x) = \boxed{\sin x \sin 3x}$$ is even

$$h(-x) = \sin(-x)\sin(-3x) = (-\sin x)(-\sin 3x)$$

$$= \boxed{\sin x \sin 3x} = h(x)$$

Trigonometric Series

Aim to represent periodic functions using simple periodic functions

$\boxed{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots}$

combined into a series

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

$$= \boxed{a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

where $a_0, a_1, a_2, \cdots, b_1, b_2, \cdots$ are real constants

called a trigonometric series.

Fourier Series

Since each of the terms

$$\boxed{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots}$$

is of period 2π, it follows that if the series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges, its sum will be a periodic function f of period 2π.

Write

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

Conversely, given a periodic function *f* of period 2π, find coefficients $a_0, a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ such that

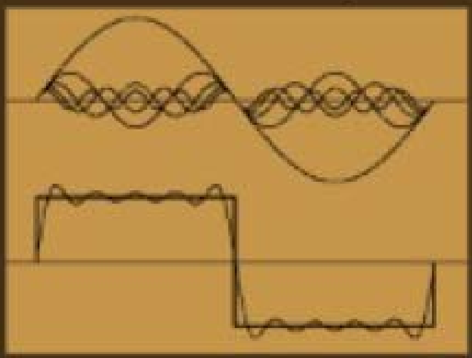
$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

The series on the right side of the equation is called the Fourier series of *f*.

Jean Baptiste Joseph Fourier

(1768 – 1830)

Jean Baptiste Joseph Fourier (1768 - 1830) was a French mathematician and physicist best known for his work on Fourier series and their applications to the problems of heat transfer. Joseph Fourier had a varied career. He was in turn a teacher, a secret policeman, a political prisoner, governor of Egypt, prefect of Isère and Rhône; and permanent secretary of the French Academy.



In 1822, he published his revolutionary treatise on the Theory of Heat, in which he showed how the conduction of heat in solid bodies could be analyzed in terms of trigonometric series (now called Fourier series).

Although the initial motivation of Fourier series was to solve the heat equation, it later became an important technique with many applications in electrical engineering, acoustics, optics, signal processing, image processing, quantum mechanics, econometrics, etc. He is also noted for the notion of dimensional analysis, and was the first to describe the Greenhouse Effect.

Source:
http://upload.wikimedia.org/wikipedia/commons/a/aa/Joseph_Fourier.jpg
http://home.nordnet.fr/~ajuhel/Fourier/Theo_Chaleur.JPG

Determine a_0

Recall $$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)}$$

Integrate both sides:

$$\int_{-\pi}^{\pi} f(x)\,dx = \boxed{\int_{-\pi}^{\pi}\left(a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right) dx}$$

Consider term-by-term integration on the right side.

$$\int_{-\pi}^{\pi} f(x)dx$$

$$= a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} (a_n \int_{-\pi}^{\pi} \cos nx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \, dx)$$

$$= 2\pi a_0 + \sum_{n=1}^{\infty} \left(a_n \cdot 0 \qquad + b_n \left[\frac{-\cos nx}{n} \right]_{-\pi}^{\pi} \right)$$

$$= 2\pi a_0 + \sum_{n=1}^{\infty}$$

$$\sin n\pi = 0 \qquad \sin(-n\pi) = 0$$

$$\sin \pi = 0 \qquad \sin(-\pi) = 0$$

$$\sin 2\pi = 0 \qquad \sin(-2\pi) = 0$$

$$\cdots \qquad \cdots$$

$$\boxed{\int_{-\pi}^{\pi} f(x)dx}$$

$$= a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos nx\, dx + b_n \int_{-\pi}^{\pi} \sin nx\, dx\right)$$

$$= 2\pi a_0 + \sum_{n=1}^{\infty} \left(a_n \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} + \boxed{b_n \left[\frac{-\cos nx}{n}\right]_{-\pi}^{\pi}} \right)$$

$$\cos(-\theta) = \cos\theta$$

$$= 2\pi a_0 + \sum_{n=1}^{\infty} b_n \cdot 0$$

$$\boxed{= 2\pi a_0}$$

$$\boxed{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx.}$$

$\sin n\pi = 0$	$\sin(-n\pi) = 0$
$\sin \pi = 0$	$\sin(-\pi) = 0$
$\sin 2\pi = 0$	$\sin(-2\pi) = 0$
$\cdots$	$\cdots$

Determine a_m, $m > 0$

Recall $$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Multiply both sides by cos *mx* and integrate:

$$\int_{-\pi}^{\pi} f(x) \cos\, mx\, dx$$
$$= a_0 \int_{-\pi}^{\pi} \cos\, mx\, dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos\, nx\, \cos\, mx\, dx \right.$$
$$\left. + b_n \int_{-\pi}^{\pi} \sin\, nx \cos\, mx\, dx \right)$$

Examine the three integrals on right side, one at a time.

First integral $\boxed{a_0 \int_{-\pi}^{\pi} \cos\ mx\, dx \ = \ 0}$

$$\int_{-\pi}^{\pi} \cos mx\, dx \ = \ \left[\frac{\sin mx}{m}\right]_{-\pi}^{\pi}$$

$$= \ \frac{\underbrace{\sin m\pi}_{0} - \underbrace{\sin(-m\pi)}_{0}}{m}$$

$$= \boxed{0}$$

Third integral easier than second integral, consider third integral now

$$b_n \int_{-\pi}^{\pi} \boxed{\sin\ nx}\boxed{\cos\ mx}\, dx \quad = \quad 0$$

sin *nx* is an odd function and cos *mx* is an even function.

Thus, the product sin *nx* cos *mx* is an odd function.

Second integral $a_n \int_{-\pi}^{\pi} \cos\ nx\ \cos\ mx\, dx$

$$\int_{-\pi}^{\pi} \cos\ nx\ \cos\ mx\, dx$$

$\cos 2mx$ $\cos 0 = 1$

$$= \frac{1}{2}\int_{-\pi}^{\pi} (\cos(m+n)x + \cos(m-n)x)\, dx$$

$$= \begin{cases} \dfrac{1}{2}\left[\dfrac{\boxed{\sin(m+n)x}}{m+n} + \dfrac{\boxed{\sin(m-n)x}}{\boxed{m-n}}\right]_{-\pi}^{\pi} = \boxed{0} \text{ if } \boxed{m \neq n} \\ \boxed{\dfrac{1}{2}}\left[\dfrac{\boxed{\sin 2mx}}{2m} + x\right]_{-\pi}^{\pi} = \boxed{\pi} \text{ if } \boxed{m = n} \end{cases}$$

$$\frac{1}{2}\int_{-\pi}^{\pi} (\cos 2mx + 1)\, dx$$

$\sin k\pi = 0$

$k = 0, \pm 1, \pm 2, \pm 3, \ldots$

Equation (5) now simplifies

$$\int_{-\pi}^{\pi} f(x) \cos\ mx\, dx$$

$$= \quad 0 \quad + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos\ nx\ \cos\ mx\, dx \right.$$

$$\left. + \quad 0 \quad \right)$$

$$= \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\ \cos mx\, dx$$

$$\int_{-\pi}^{\pi} f(x)\cos\ mx\, dx \;=\; \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\ \cos mx\, dx$$

a_m : want a_1, set $m = 1$

$$a_1 \int_{-\pi}^{\pi} \cos x \quad \cos\ x \quad dx \;=\; a_1 \cdot \pi$$

$$a_2 \int_{-\pi}^{\pi} \cos 2x\ \cos\ x \quad dx \;=\; a_2 \cdot 0$$

$$a_3 \int_{-\pi}^{\pi} \cos 3x\ \cos\ x \quad dx$$

$$a_4 \int_{-\pi}^{\pi} \cos 4x\ \cos\ x \quad dx$$

$$\ldots$$

$$\int_{-\pi}^{\pi} \cos nx\ \cos mx\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\boxed{\int_{-\pi}^{\pi} f(x)\cos\ mx\,dx} = \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\ \cos mx\,dx$$

$m = 1$ $\qquad$ a_m : want a_1, set $m = 1$

$$a_1 \int_{-\pi}^{\pi} \cos x \quad \cos\ x \quad dx = a_1 \cdot \pi \boxed{= a_1\,\pi}$$

$$a_2 \int_{-\pi}^{\pi} \cos 2x\ \cos\ x \quad dx = a_2 \cdot 0 = 0$$

$$a_3 \int_{-\pi}^{\pi} \cos 3x\ \cos\ x \quad dx = a_3 \cdot 0 = 0$$

$$a_4 \int_{-\pi}^{\pi} \cos 4x\ \cos\ x \quad dx = a_4 \cdot 0 = 0$$

$\cdots$

$$\boxed{\int_{-\pi}^{\pi} f(x)\cos\, mx\, dx} = \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\, \cos mx\, dx$$

$m = 2$

a_m : want a_2, set $m = 2$

$$a_1 \int_{-\pi}^{\pi} \cos x \quad \cos\, 2x\,\, dx = a_1 \cdot 0 = 0$$

$$a_2 \int_{-\pi}^{\pi} \cos 2x\, \cos\, 2x\,\, dx = a_2 \cdot \pi \boxed{= a_2\,\pi}$$

$$a_3 \int_{-\pi}^{\pi} \cos 3x\, \cos\, 2x\,\, dx = a_3 \cdot 0 = 0$$

$$a_4 \int_{-\pi}^{\pi} \cos 4x\, \cos\, 2x\,\, dx = a_4 \cdot 0 = 0$$

$\cdots$

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\ \cos mx\,dx$$

$$= a_m \pi$$

$$\boxed{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx, \qquad m = 1, 2, \cdots}$$

Determine b_m, $m > 0$

Recall $$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Multiply both sides by sin mx and integrate:

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx$$
$$= a_0 \int_{-\pi}^{\pi} \sin mx \, dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos nx \sin mx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx\right)$$

Examine the three integrals on right side, one at a time.

First integral $\boxed{a_0 \int_{-\pi}^{\pi} \sin\ mx\, dx \ = \ 0}$

sin *mx* is an odd function

Second integral $\boxed{a_n \int_{-\pi}^{\pi} \boxed{\cos\ nx}\boxed{\sin\ mx} dx \ = \ 0}$

cos *nx* is an even function and sin *mx* is an odd function.

Thus, the product cos *nx* sin *mx* is an odd function.

Third integral $\quad b_n \int_{-\pi}^{\pi} \sin\, nx \sin\, mx\, dx$

$$\int_{-\pi}^{\pi} \sin nx\ \sin mx\, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}(\underbrace{\cos(n-m)x}_{\cos 0\,=\,1} - \underbrace{\cos(n+m)x}_{\cos 2mx})dx$$

$$= \begin{cases} \frac{1}{2}\left[\frac{\boxed{\sin(n-m)x}}{\boxed{n-m}} - \frac{\boxed{\sin(n+m)x}}{n+m}\right]_{-\pi}^{\pi} = \boxed{0} \text{ if } \boxed{m \neq n} \\ \boxed{\frac{1}{2}}\left[x - \frac{\boxed{\sin 2mx}}{2m}\right]_{-\pi}^{\pi} = \boxed{\pi} \text{ if } \boxed{m = n} \end{cases}$$

$$\frac{1}{2}\int_{-\pi}^{\pi}(1-\cos 2mx)\, dx$$

$\sin k\pi = 0$

$k = 0, \pm 1, \pm 2, \pm 3, \ldots$

Equation (6) now simplifies

$$\int_{-\pi}^{\pi} f(x)\sin\, mx\, dx$$

$$= \quad 0 \quad + \sum_{n=1}^{\infty}\left(\quad 0 \quad + b_n \int_{-\pi}^{\pi} \sin\, nx \sin\, mx\, dx \right)$$

$$= \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx\, \sin mx\, dx$$

$$= \quad b_m \pi$$

$$\int_{-\pi}^{\pi} \sin nx\, \sin mx\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

Equation (6) now simplifies

$$\int_{-\pi}^{\pi} f(x)\sin\, mx\, dx$$

$$= \quad 0 \quad + \sum_{n=1}^{\infty}\Big(\quad 0$$

$$+ b_n \int_{-\pi}^{\pi} \sin\, nx \sin\, mx\, dx \Big)$$

$$= \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx\, \sin mx\, dx$$

$$= \quad b_m \pi$$

$$\Rightarrow \boxed{b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\, dx \qquad m = 1, 2, \cdots}$$

Euler Formulas

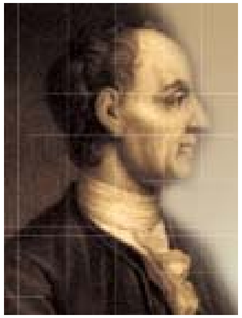
A periodic function *f*(*x*) of period 2π with Fourier series

$$\boxed{a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

has Fourier coefficients given by

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \quad n = 1, 2, \cdots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx, \quad n = 1, 2, \cdots \end{aligned}}$$

These formulas are known as Euler formulas.



$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\pi} + 1 = 0$$

Leonhard Euler

(1707 – 1783)

Leonhard Euler (1707 – 1783) invented functional notation, the natural logarithm base e (which he proved irrational), $i = \sqrt{-1}$, and wrote the first book containing the symbol π, all of which he related by:

$$e^{i\pi} + 1 = 0$$

In about 800 papers and 20 books (nearly half written after becoming blind at age 59), he also worked in science, architecture and music. His three-body problem work helped calculation of longitude at sea. Euler's Introductio in analysin infinitorum has been called the greatest modern textbook in mathematics. His polyhedral formula $V - E + F = 2$ has been generalized to the Euler characteristic in contemporary topology. His solution of the Königsberg bridge problem is considered to have launched graph theory.

In number theory, he discovered the law of quadratic reciprocity, and introduced the prime-producing polynomial $n^2 + n + 41$.

That no greater number than 41 also works, was proved only in 1967. Euler also founded analytic number theory, showing that the sum of the reciprocals of primes diverges, and using generating functions to prove that

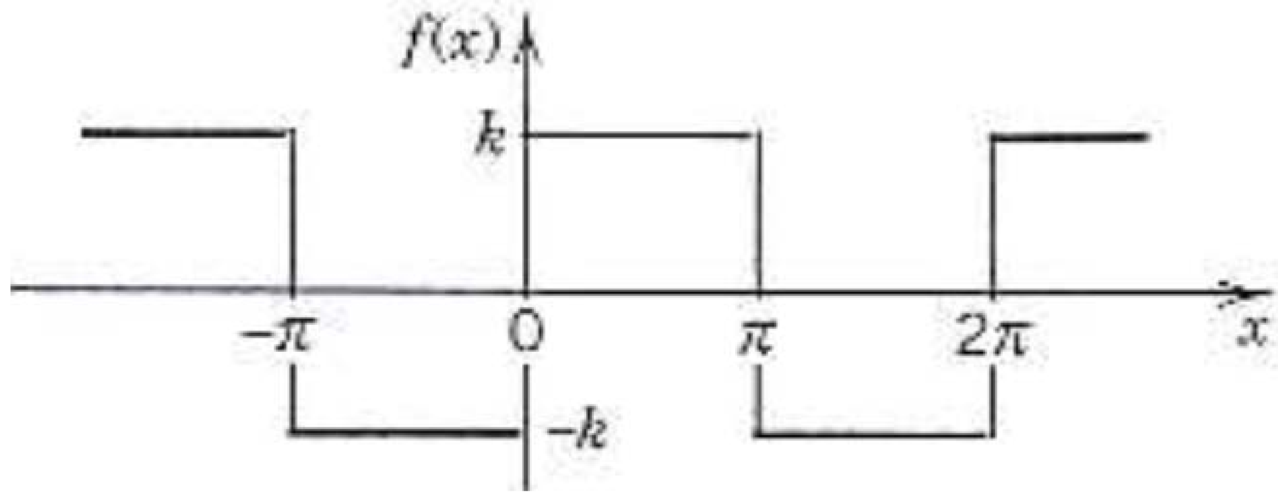
$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots..$$

Source:
http://www-gap.dcs.st-and.ac.uk/~history/Biographies/Euler.html

Example

Consider $f(x) = \begin{cases} -k, & \text{if } -\pi < x < 0 \\ k, & \text{if } 0 < x < \pi \end{cases}$

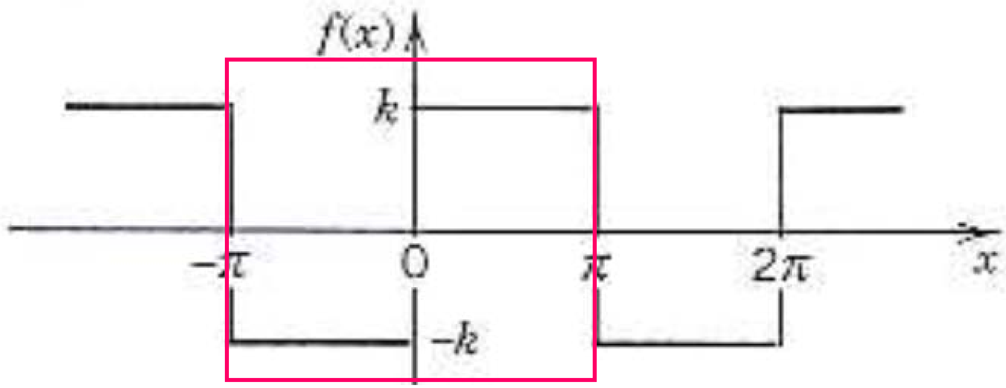
$f(x) = f(x + 2\pi)$



$f(x)$ is piecewise continuous

Omitting a single point (x value) does not affect integrals

Leave f undefined at $x = 0$, $x = \pm\pi$



Over the interval (–π, π), graph is symmetrical about the origin, *f* is an odd function.

$$\boxed{a_0 =} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \quad = \boxed{0}$$

Now, *f*(*x*) cos *nx* is also an odd function. Thus,

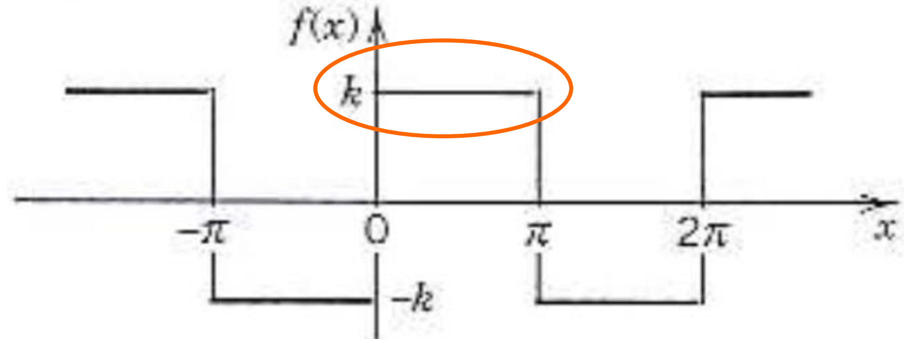
$$\boxed{a_n =} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \quad = \boxed{0} \qquad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \boxed{\frac{2}{\pi}\int_0^{\pi}} k\ \sin nx\,dx$$

$$= \frac{2k}{n\pi}(1 - \boxed{\cos n\pi})$$

$$= \frac{2k}{n\pi}(1 - (-1)^n)$$



$f(x)$, $\sin nx$ both odd functions

$f(x)\sin nx$ is an even function

$$\cos n\pi = \boxed{(-1)^n}$$

$$\cos \pi = -1 \qquad \cos 2\pi = 1$$

$$\cos 3\pi = -1 \qquad \cos 4\pi = 1$$

$$b_1 = \frac{4k}{\pi} \qquad b_3 = \frac{4k}{3\pi} \qquad b_5 = \frac{4k}{5\pi}$$

$$b_2 = 0 \qquad b_4 = 0 \qquad \ldots$$

$$\boxed{a_0 = 0 \qquad a_n = 0 \quad n = 1, 2, 3, \ldots}$$

$$\boxed{b_1 = \frac{4k}{\pi},} \quad \boxed{b_2 = 0,} \quad \boxed{b_3 = \frac{4k}{3\pi},} \quad \boxed{b_4 = 0,} \quad \boxed{b_5 = \frac{4k}{5\pi},} \cdots$$

(page 10) $\boxed{a_0} + \sum_{n=1}^{\infty} (\boxed{a_n} \cos nx + \boxed{b_n \sin nx})$

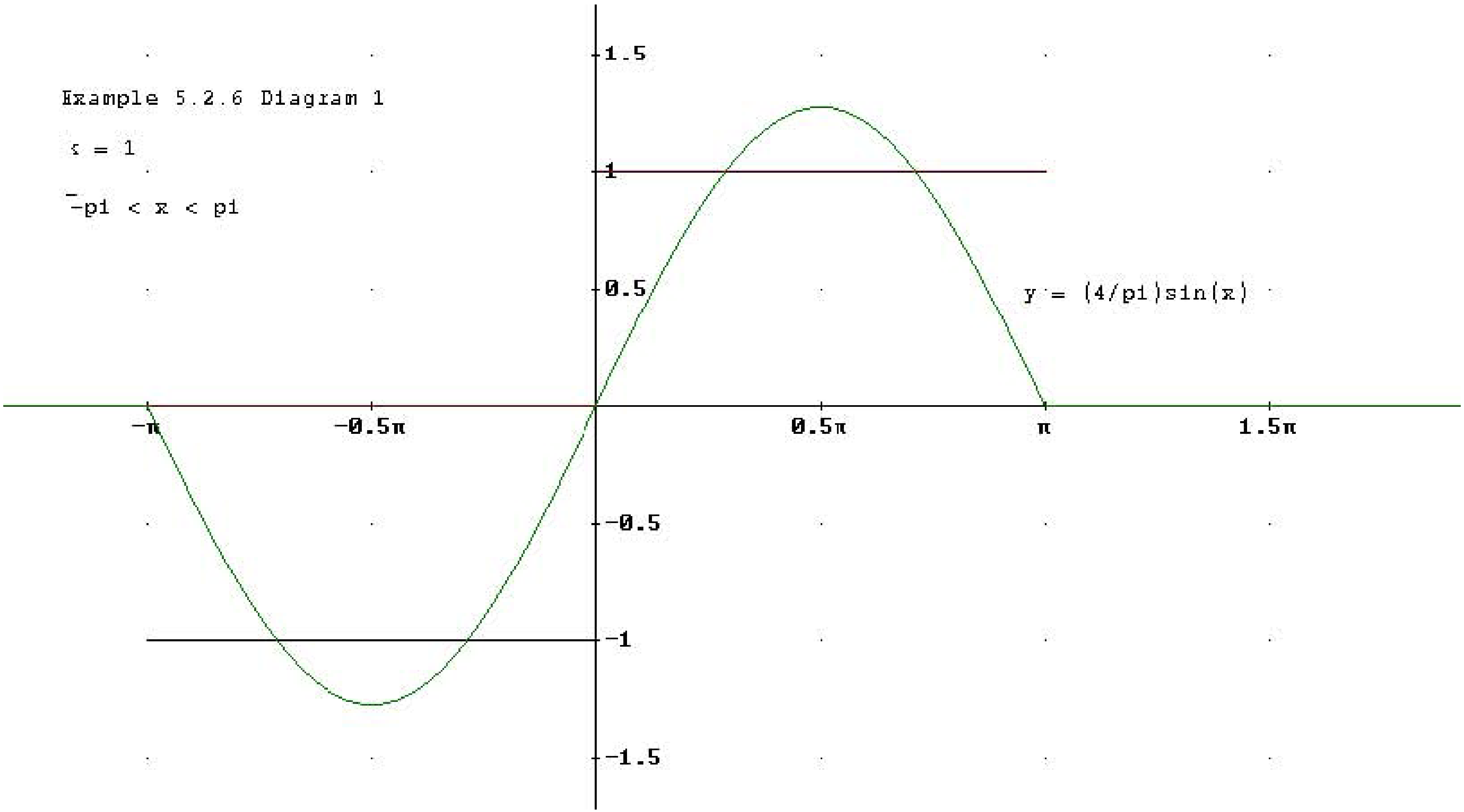
only these terms left

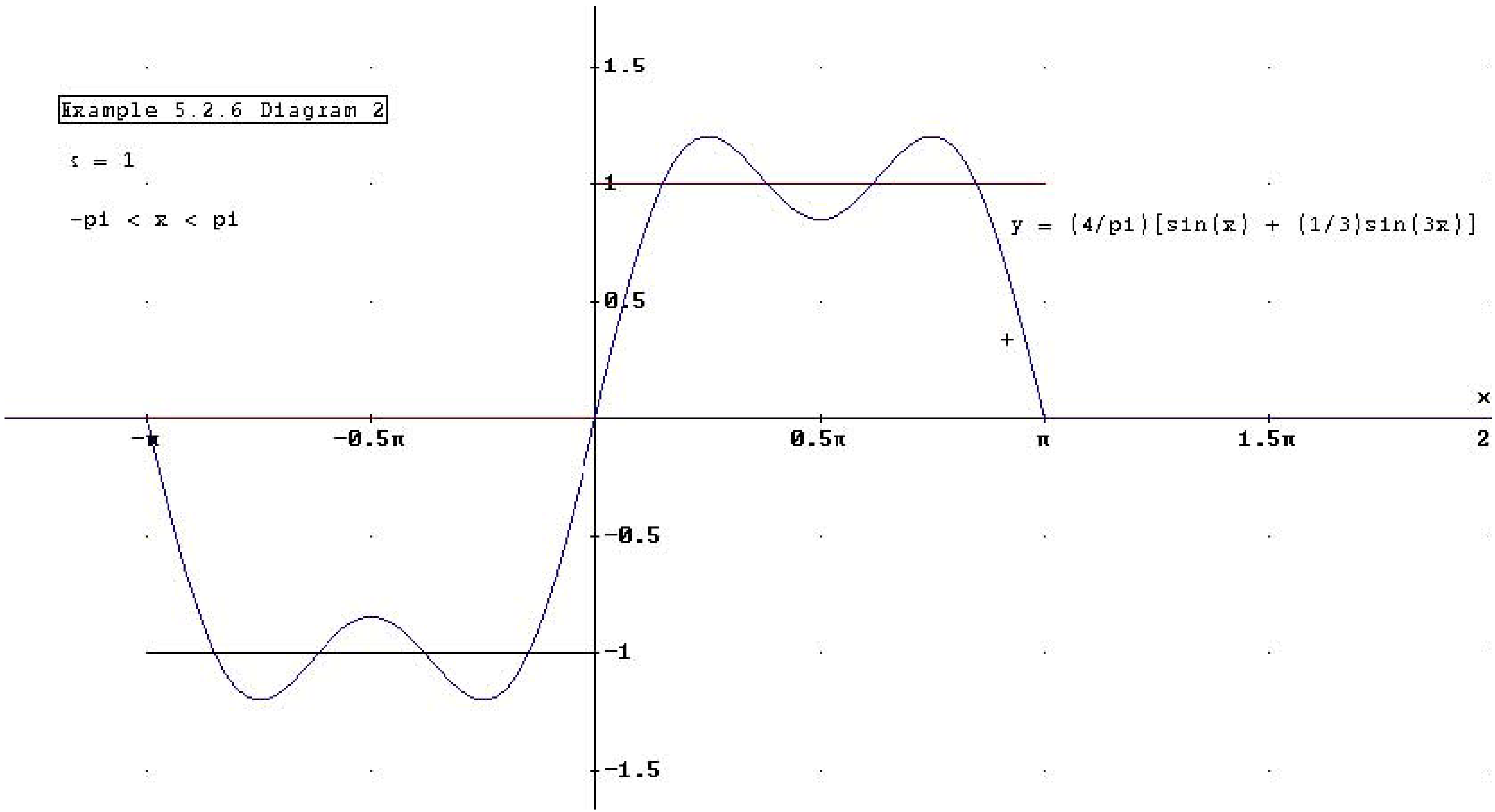
Fourier series for the square wave

$$\underbrace{\frac{4k}{\pi} \sin x}_{n=1} + \underbrace{0 \cdot \sin 2x}_{n=2} + \underbrace{\frac{4k}{3\pi} \sin 3x}_{n=3} + \underbrace{0 \cdot \sin 4x}_{n=4}$$

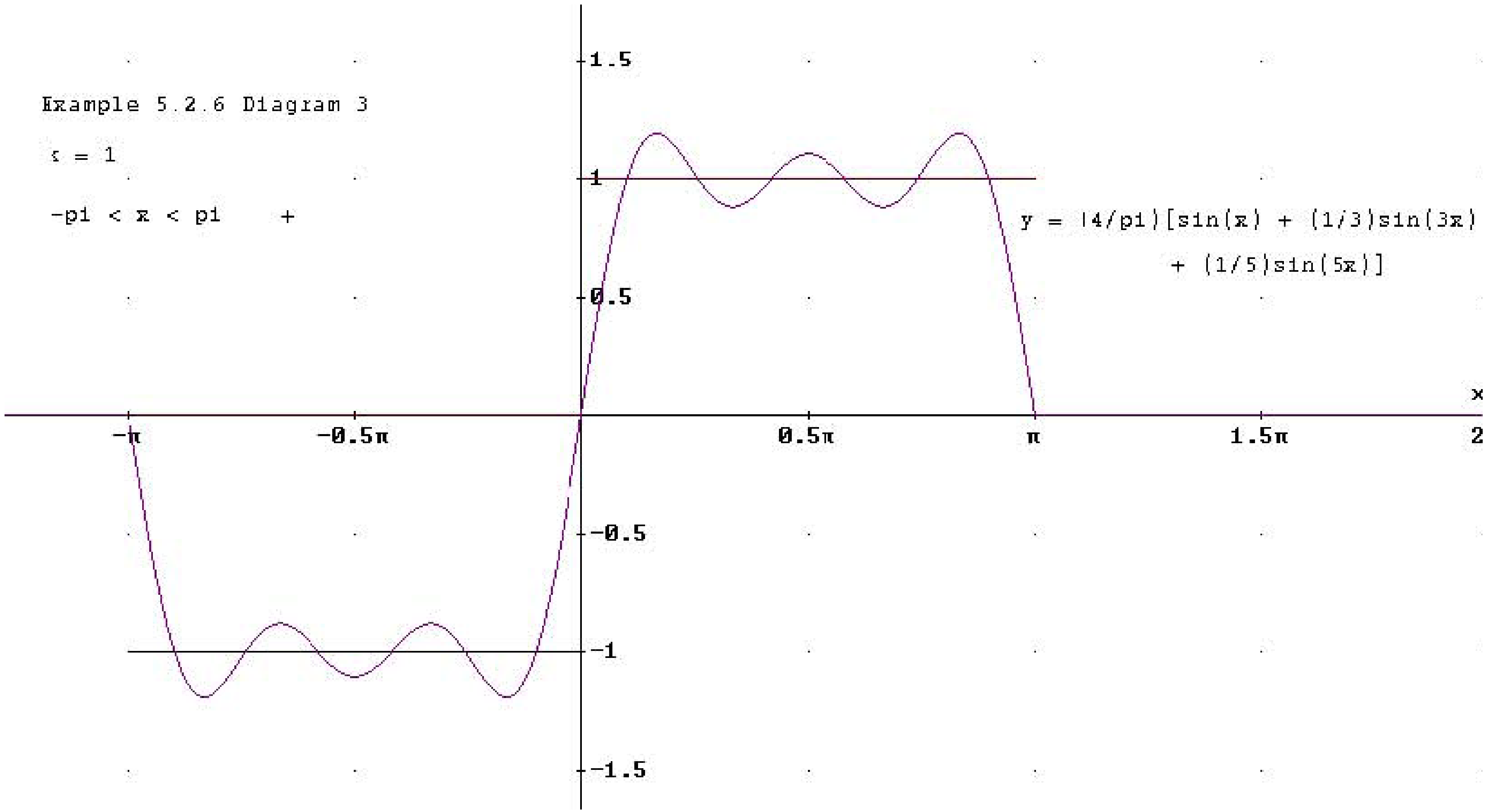
$$+ \underbrace{\frac{4k}{5\pi} \sin 5x}_{n=5} + \cdots$$

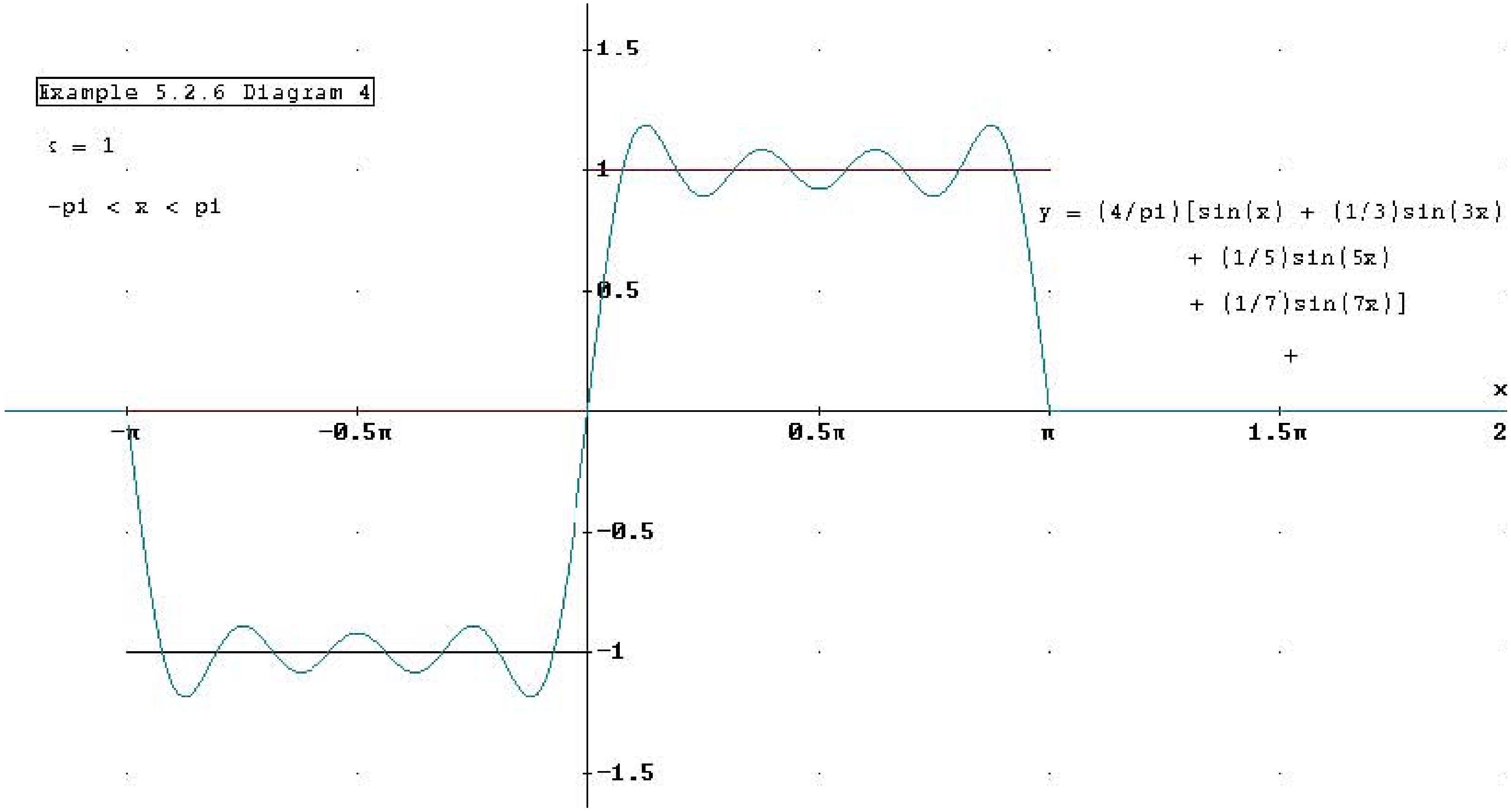
$$= \boxed{\frac{4k}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right)}$$


Example 5.2.6 Diagram 1
ε = 1
-pi < x < pi
y = (4/pi)sin(x)
1.5
1
0.5
-0.5
-1
-1.5
-π
-0.5π
0.5π
π
1.5π



Example 5.2.6 Diagram 2
k = 1
-pi < x < pi
y = (4/pi)[sin(x) + (1/3)sin(3x)]
1.5
1
0.5
-0.5
-1
-1.5
x
-π
-0.5π
0.5π
π
1.5π
2


Example 5.2.6 Diagram 3
s = 1
-pi < x < pi +
y = (4/pi)[sin(x) + (1/3)sin(3x)
+ (1/5)sin(5x)]
1.5
1
0.5
-0.5
-1
-1.5
x
-π
-0.5π
0.5π
π
1.5π
2


Example 5.2.6 Diagram 4
ε = 1
-pi < x < pi
y = (4/pi)[sin(x) + (1/3)sin(3x)
+ (1/5)sin(5x)
+ (1/7)sin(7x)]
1.5
1
0.5
−0.5
−1
−1.5
x
−π
−0.5π
0.5π
π
1.5π
2

An Approximation for π

From $$f(x) = \begin{cases} -k, & \text{if } -\pi < x < 0 \\ \boxed{k, \quad \text{if } 0 < x < \pi} \end{cases}$$

$$= \boxed{\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)}$$

let $\boxed{x = \frac{\pi}{2}}$

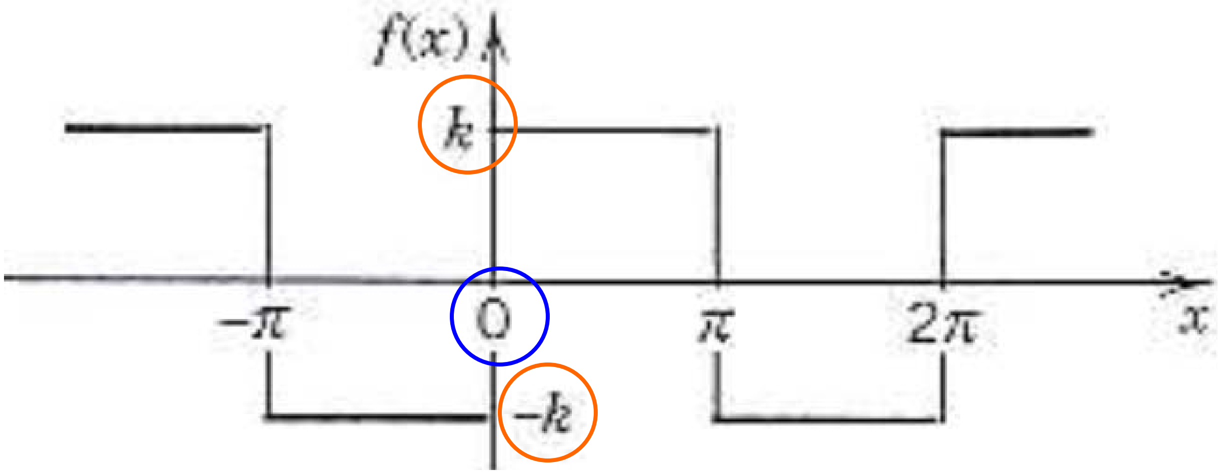
$$k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right)$$

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - + \cdots}$$

$$\boxed{f\left(\frac{\pi}{2}\right) = k} \quad \sin\frac{\pi}{2} = 1 \quad \sin\frac{3\pi}{2} = -1 \quad \sin\frac{5\pi}{2} = 1 \quad \cdots$$

At Points of Discontinuity

At points of discontinuity of *f*, the Fourier series has sum equal to the average of the left limit and right limit.



For example, at $x = 0$,

$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right) = \boxed{0}$$

Check indeed that the average of $-k$ and k is 0.

General Period $p = 2L$

Let $f(x)$ be a periodic function of period $p = 2L$.

To find the Fourier series of $f(x)$, ***use a substitution*** (change of variable) in the Euler formulas on page 10.

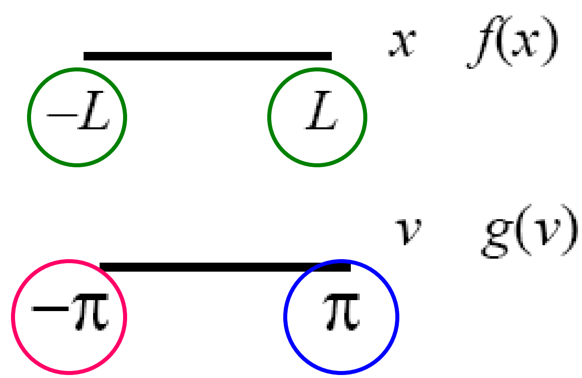
Let $\boxed{x = \frac{L}{\pi} v}$

$f(x)$ becomes a periodic function $\boxed{g(v) \text{ with period } 2\pi}$

Note $\boxed{v = \frac{\pi}{L} x}$ gives

$\boxed{v = -\pi}$ when $x = -L$

$\boxed{v = \pi}$ when $x = L$



General Fourier Formulas (derivation)

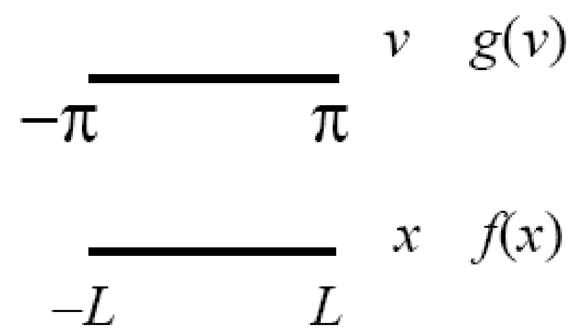
Fourier series of $g(v)$ (from page 10, v for x, g for f):

$$g(v) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv) \qquad \text{period } 2\pi$$

$$\boxed{a_0 =} \frac{1}{2\pi} \int_{-\pi}^{\pi} g(v)\, dv \qquad v = \frac{\pi}{L} x \qquad \frac{dv}{dx} = \frac{\pi}{L}$$

$$= \frac{1}{2\pi} \int_{-L}^{L} g(v) \frac{\pi}{L}\, dx \qquad dv = \frac{\pi}{L} dx$$

$$= \boxed{\frac{1}{2L} \int_{-L}^{L} f(x)\, dx}$$



General Fourier Formulas (derivation)

Fourier series of $g(v)$ (from page 10, v for x, g for f):

$$g(v) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv)$$

period 2π

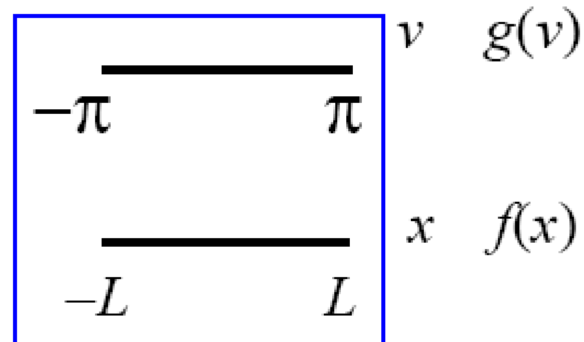
for $n = 1, 2, 3, \cdots$

$$\boxed{v = \frac{\pi}{L} x} \qquad \frac{dv}{dx} = \frac{\pi}{L} \qquad \boxed{dv = \frac{\pi}{L} dx}$$

$$\boxed{a_n =} \frac{1}{\pi} \boxed{\int_{-\pi}^{\pi}} g(v) \cos nv \, dv$$

$$= \frac{1}{\pi} \int_{-L}^{L} g(v) \cos \left(n \frac{\pi}{L} x \right) \frac{\pi}{L} dx$$

$$= \boxed{\frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx}$$



General Fourier Formulas

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

Period $p = 2L$

$$v = \frac{\pi}{L} x$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx \qquad \text{for } n = 1, 2, 3, \cdots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx \qquad \text{for } n = 1, 2, 3, \cdots$$

General Fourier Formulas

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Period $p = 2L$ Let $L = \pi$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \qquad \text{for } n = 1, 2, 3, \cdots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \qquad \text{for } n = 1, 2, 3, \cdots$$

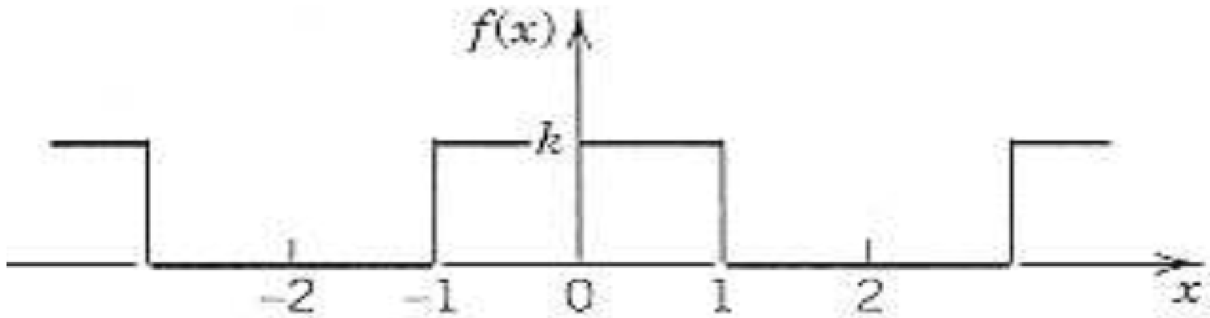
These are the Euler formulas on page 10

Example

Consider $\quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$

$p = 2L = 4$

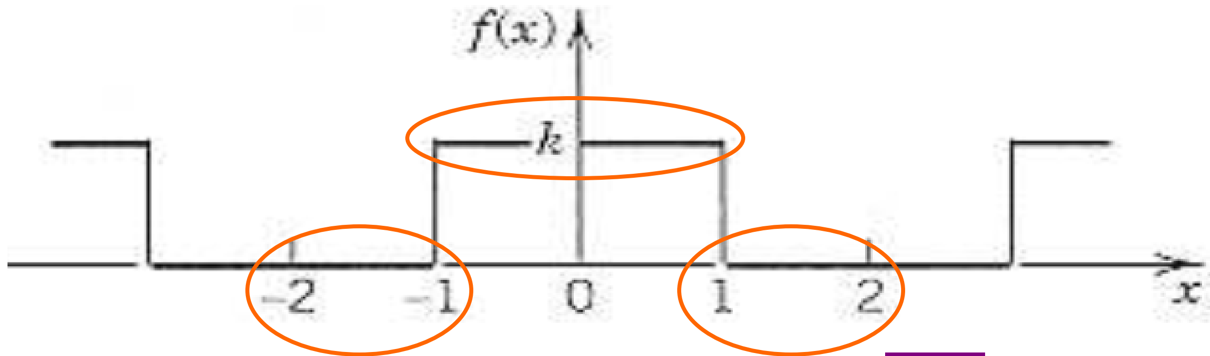
i.e. $L = 2$



Graph is symmetrical about *y*-axis, i.e. *f*(*x*) is an even function

$f(x)\sin\frac{n\pi x}{2}$ is an odd function

$$\boxed{b_n =} \frac{1}{2}\int_{-2}^{2}\boxed{f(x)\sin\frac{n\pi x}{2}}dx = \boxed{0} \qquad n = 1, 2, 3, \ldots$$



$L = 2$

$$\boxed{a_0 =} \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\int_{-1}^{1} k\,dx = \boxed{\frac{k}{2}}$$

$$f(x) = 0 \quad \text{for} \quad -2 < x < -1 \quad 1 < x < 2$$

$$n = 1, 2, 3, \ldots$$

$$\boxed{a_n =} \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}dx$$

$$= \boxed{\frac{2k}{n\pi}\sin\frac{n\pi}{2}}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

$$a_0 = \frac{k}{2} \qquad a_n = \frac{2k}{n\pi}\sin\frac{n\pi}{2} \qquad b_n = 0 \qquad L = 2$$

n	1	2	3	4	5	$\cdots$
$\sin\frac{n\pi}{2}$	1	0	-1	0	1	$\cdots$
a_n	$\frac{2k}{\pi}$	0	$-\frac{2k}{3\pi}$	0	$\frac{2k}{5\pi}$	$\cdots$

Fourier series

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\cos\frac{\pi}{2}x - \frac{2k}{3\pi}\cos\frac{3\pi}{2}x + \frac{2k}{5\pi}\cos\frac{5\pi}{2}x - + \cdots$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x\right)$$

Fourier Cosine and Sine Series

The Fourier series of an odd function or an even function simplifies to only involve sine terms or cosine terms.

Simplification comes from the following key integration properties

$$\int_{-L}^{L} g(x)\,dx = \begin{cases} 0 & \text{if } g \text{ is odd} \\ 2\displaystyle\int_{0}^{L} g(x)\,dx & \text{if } g \text{ is even} \end{cases}$$

Fourier Cosine Series

The Fourier series of an even function *f*(*x*) of period 2*L* is

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x}$$

Fourier cosine series

because

$$b_n = \frac{1}{L}\int_{-L}^{L} \boxed{f(x)\sin\frac{n\pi x}{L}}\,dx$$

f(*x*) is an even function

$\sin\frac{n\pi x}{L}$ is an odd function

$$= \boxed{0}$$

$f(x)\sin\frac{n\pi x}{L}$ is an odd function

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

Fourier Cosine Series (cont'd)

Also,

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$n = 1, 2, 3, \ldots$$

$f(x)$ is an even function

$\cos\dfrac{n\pi x}{L}$ is an even function

$f(x)\cos\dfrac{n\pi x}{L}$ is an even function

$$a_n = \frac{1}{L}\int_{-L}^{L} \boxed{\boxed{f(x)}\cos\frac{n\pi x}{L}}\,dx \qquad a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

Fourier Sine Series

Similar to the development of the Fourier cosine series, there is a Fourier sine series for an odd function $f(x)$ of period $2L$.

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x}$$

$$\boxed{b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \qquad n = 1, 2, 3, \ldots}$$

Exercise: Using ideas involving integrals of even functions and odd functions, show that $a_n = 0$ for $n = 0, 1, 2, \ldots$

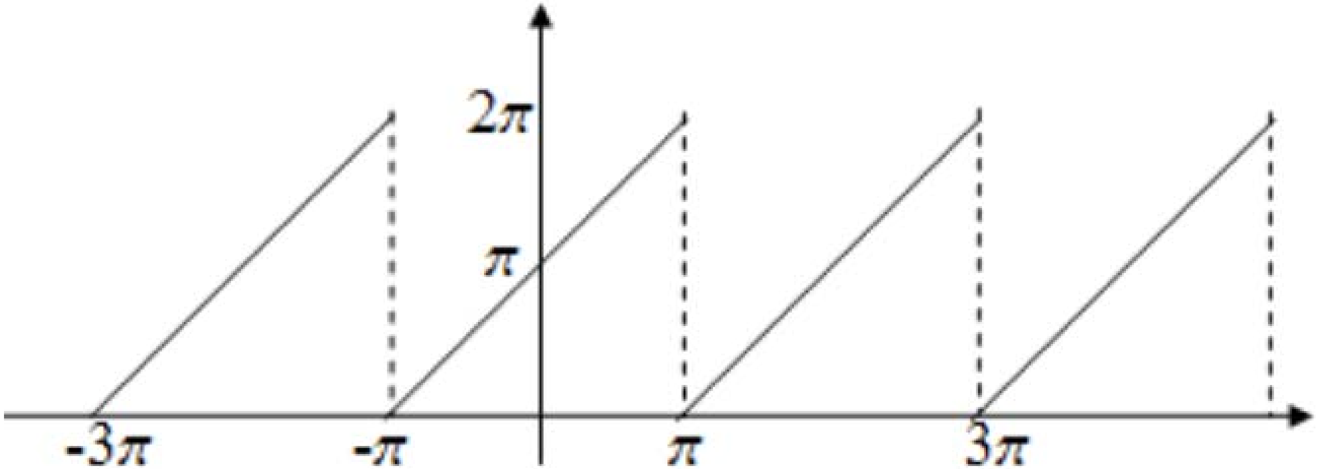
Sum and Scalar Multiplication

The Fourier coefficients of $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2.

For any constant c, the Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.

Saw Tooth Function

$$f(x) = x + \pi, \quad -\pi < x < \pi, \qquad f(x) = f(x + 2\pi)$$



$$f = f_1 + f_2, \text{ where } \; f_1 = x, \; \boxed{f_2 = \pi}$$

Fourier coefficients for f_2 are $a_0 = \pi$ $\boxed{a_n = 0 = b_n,}$ $n \geq 1$

$$\pi = a_0 + \sum_{n=1}^{\infty} \boxed{(a_n \cos nx + b_n \sin nx)}$$

Fourier series of f_1

First note that the function $f_1(x) = x$ is odd.

Thus, $a_n = 0$ for all $n = 0, 1, 2, 3, \ldots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

x and sin *nx* are odd functions

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin nx \, dx$$

x sin *nx* is an even function

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Fourier series of f_1

First note that the function $f_1(x) = x$ is odd.

Thus, $a_n = 0$ for all $n = 0, 1, 2, 3, \dots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

x and $\sin nx$ are odd functions

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin nx \, dx$$

$x \sin nx$ is an even function

integrate by parts

$$= \frac{2}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi} \frac{-\cos nx}{n} dx\right\}$$

$$\boxed{\cos n\pi = (-1)^n}$$

$$= \frac{2}{\pi}\left\{\frac{-(-1)^n \pi}{n} - \left[\frac{-\sin nx}{n^2}\right]_0^{\pi}\right\}.$$

$$b_n = \frac{2}{\pi}\left\{\frac{-(-1)^n\pi}{n} - \underbrace{\cancel{\left[\frac{-\sin nx}{n^2}\right]_0^\pi}}_{0}\right\}$$

$$= (-1)^{n+1}\frac{2}{n}$$

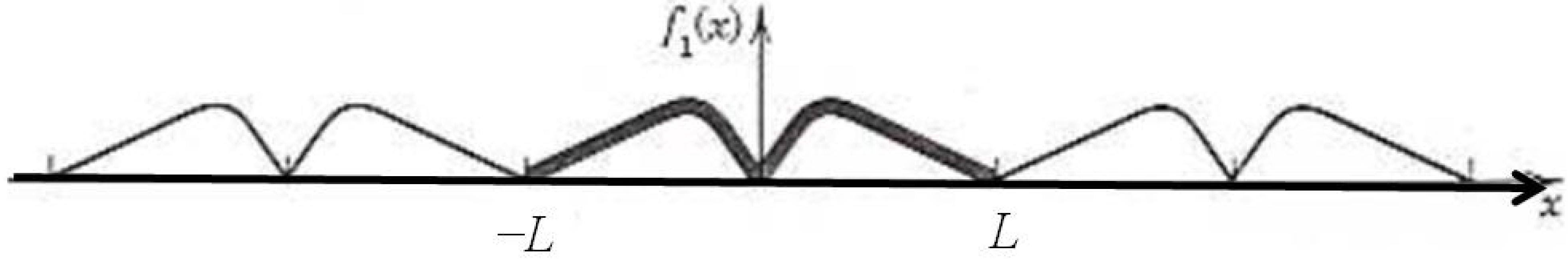
Fourier series of $f(x) = f_1(x) + f_2(x)$

$$f(x) = f_1(x) + f_2(x)$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx \quad + \quad \pi$$

$$= \boxed{\pi + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx}$$

Cosine Half Range Expansion



Consider a function $f(x)$ that is only defined on the interval $[0, L]$.

Suppose we wish to express $f(x)$ as a *simple* Fourier series,

e.g. a series that only involves cosine terms.

Such a series enjoys the following properties:

(1) it is an even function, i.e. graph is symmetrical about y-axis;

(2) it is periodic, e.g. of period $2L$.

This series is called the cosine half range expansion of $f(x)$.

The cosine half range expansion is

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x}$$

where

$$\boxed{\begin{aligned} a_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \cdots \end{aligned}}$$

Example

Find the cosine half range expansion of

$$f(x) = \begin{cases} \boxed{0, \quad 0 < x < \frac{\pi}{2}} \\ \boxed{1, \quad \frac{\pi}{2} < x < \pi} \end{cases}$$

$L = \pi$

$$a_0 = \frac{1}{\pi}\left\{\int_0^{\pi/2} 0dx + \int_{\pi/2}^{\pi} 1dx\right\} = \frac{1}{2}$$

0

$$a_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} 0\cos nx dx + \int_{\pi/2}^{\pi} 1\cos nx dx\right\}$$

0

$$= \frac{2}{\pi}\cdot\frac{1}{n}\left[\sin n\pi \; - \; \sin\frac{1}{2}n\pi\right] = -\frac{2}{n\pi}\sin\frac{1}{2}n\pi$$

0

$$a_n = \frac{2}{L}\int_0^L \boxed{f(x)\cos\frac{n\pi x}{L}} dx \qquad a_0 = \frac{1}{L}\boxed{\int_0^L f(x)\,dx}$$

$$a_0 = \frac{1}{2} \qquad a_n = -\frac{2}{n\pi}\sin\frac{1}{2}n\pi \qquad n \geq 1$$

If n is even and $n \geq 2$, then $a_n = 0$.

If n is odd, then consider

$$\boxed{a_1 = -\frac{2}{\pi} \qquad a_3 = \frac{2}{3\pi} \qquad a_5 = -\frac{2}{5\pi} \qquad \cdots}$$

$m = 1$ $\qquad$ $m = 2$ $\qquad$ $m = 3$

odd integers are expressed as $\boxed{2m - 1}$

alternating sign ± 1 are expressed as $(-1)^m$

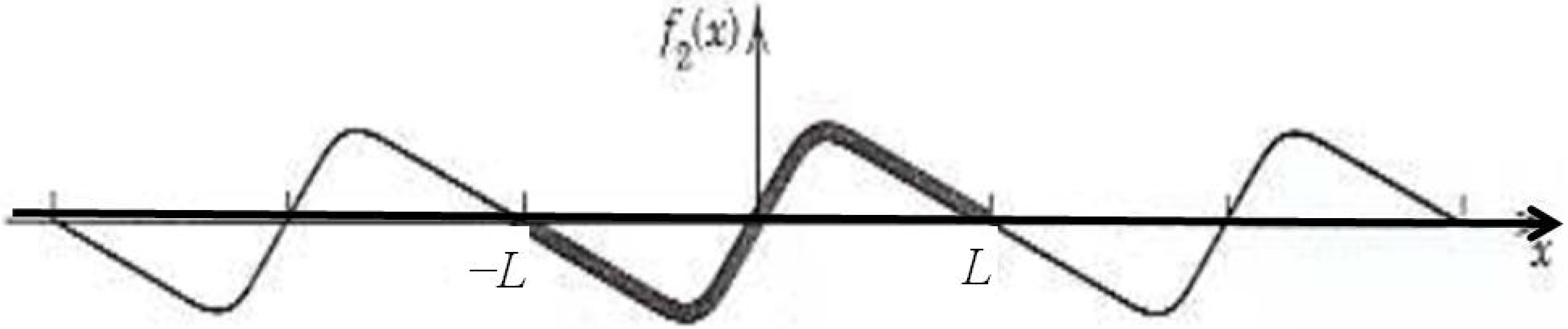
Thus, the cosine half range expansion is

$$\boxed{f(x) = \frac{1}{2} + 2\sum_{m=1}^{\infty} \frac{(-1)^m}{(2m-1)\pi}\cos(2m-1)x}$$

$L = \pi$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

Sine Half Range Expansion



Consider a function $f(x)$ that is only defined on the interval $[0, L]$.

Suppose we wish to express $f(x)$ as a *simple* Fourier series,

e.g. a series that only involves sine terms.

Such a series enjoys the following properties:

(1) it is an odd function, i.e. graph is symmetrical about origin;

(2) it is periodic, e.g. of period $2L$.

This series is called the sine half range expansion of $f(x)$.

The sine half range expansion is

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x}$$

where

$$\boxed{b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx \qquad n = 1, 2, \cdots}$$

Example

Find the sine half range expansion of

$$f(x) = \begin{cases} 0, & 0 < x < \frac{\pi}{2} \\ \\ 1, & \frac{\pi}{2} < x < \pi \end{cases}$$

$L = \pi$

$$b_n = \frac{2}{\pi} \left\{ \int_0^{\pi/2} 0 \sin nx dx + \int_{\pi/2}^{\pi} 1 \sin nx dx \right\}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Example

Find the sine half range expansion of

$$f(x) = \begin{cases} 0, & 0 < x < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < x < \pi \end{cases} \qquad L = \pi$$

$$b_n = \frac{2}{\pi}\left\{ \cancelto{0}{\int_0^{\pi/2} 0 \sin nx\,dx} + \int_{\pi/2}^{\pi} 1 \sin nx\,dx \right\}$$

$$= \frac{2}{\pi}\left[\frac{-\cos nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{n\pi}\left[-\cos n\pi + \cos \frac{1}{2}n\pi\right]$$

$$= \frac{2}{n\pi}\boxed{\left[(-1)^{n+1} + \cos \frac{1}{2}n\pi\right]}$$

For $n = 1, 2, 3, \ldots,$
what is the pattern of resulting values?

values of n		b_n	$(-1)^{n+1} + \cos\frac{1}{2}n\pi$
1 5 9 …	$4m-3$	$\frac{2}{(4m-3)\pi}\cdot 1$	$1 + 0 = \boxed{1}$
2 6 10 …	$4m-2$	$\frac{2}{(4m-2)\pi}\cdot(-2)$	$-1 + (-1) = \boxed{-2}$
3 7 11 …	$4m-1$	$\frac{2}{(4m-1)\pi}\cdot 1$	$1 + 0 = \boxed{1}$
4 8 12 …	$4m$	$\frac{2}{4m\pi}\cdot 0$	$-1 + 1 = \boxed{0}$

1 2 3

values of m

$$b_n = \frac{2}{n\pi}\boxed{\left[(-1)^{n+1} + \cos\frac{1}{2}n\pi\right]}$$

values of n		b_n	$(-1)^{n+1} + \cos\frac{1}{2}n\pi$
1 5 9 ...	$4m-3$	$\frac{2}{(4m-3)\pi} \cdot 1$	$1 + 0 = 1$
2 6 10 ...	$4m-2$	$\frac{2}{(4m-2)\pi} \cdot (-2)$	$-1 + (-1) = -2$
3 7 11 ...	$4m-1$	$\frac{2}{(4m-1)\pi} \cdot 1$	$1 + 0 = 1$
4 8 12 ...	$4m$	$\frac{2}{4m\pi} \cdot 0$	$-1 + 1 = 0$

1 2 3

values of m **(block numbers)**

The sine half range expansion is $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$ $\quad L = \pi$

$$\boxed{f(x) = \sum_{m=1}^{\infty}\left[\frac{2}{(4m-3)\pi}\sin(4m-3)x - \frac{4}{(4m-2)\pi}\sin(4m-2)x + \frac{2}{(4m-1)\pi}\sin(4m-1)x\right]}$$

End of Chapter 6