Review of Chapter 4 (2nd part) Power series

$\sum_{n=0}^{\infty} c_n(x-a)^n$ called Power series

(A) Convergence of $\sum c_n(x-a)^n$

There are only three cases

(1)Converges only at one point, then this point should be point a.

Radius R of convergence=0

(2) Converges for all x.

Radius R of convergence= ∞

(3) Converges in (a-h,a+h) but diverges outside [a-h,a+h].

The series may or may not converge at end points a-h,a+h.

Radius of convergence= h

Radius of convergence can be computed by ratio test in this chapter

(B) We can differentiate power series one term by one term.

$$\frac{1}{1-x}=1+x+x^2+\ldots+x^n+\ldots \text{ for } -1<x<1$$

$$\frac{d}{dx}\left[\frac{1}{1-x}\right]=\frac{d}{dx}\left[1+x+x^2+\ldots+x^n+\ldots\right] \quad \text{for } -1<x<1$$

$$=\frac{d}{dx}1+\frac{d}{dx}x+\frac{d}{dx}x^2+\ldots+\frac{d}{dx}x^n+\ldots \quad \text{for } -1<x<1$$

$$=0+1+2x+\ldots+nx^{n-1}+\ldots \text{ for } -1<x<1$$

$$\frac{1}{(1-x)^2}=1+2x+\ldots+nx^{n-1}+\ldots \text{ for } -1<x<1$$

Should learn how to write the infinite sum in terms of $\sum$

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n \quad \text{for } -1<x<1$$

$$\frac{1}{(1-x)^2} = 1 + 2x + \ldots + nx^{n-1} + \ldots = \sum_{n=1}^{\infty} nx^{n-1} \quad \text{for } -1<x<1$$

$$\frac{d}{dx}\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{d}{dx} x^n \quad \text{for } -1<x<1$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} \quad \text{for } -1<x<1$$

(C) We can integrate power series one term by one term

From previous slide

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots \quad \text{for } -1<x<1$$

In the above, replace x by $-x^2$, we get

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + \ldots + (-1)^n x^{2n} + \ldots \quad \text{for } -1<x<1$$

Integrate the above equality, get

$$\int_0^t \frac{1}{1+x^2} dx = \int_0^t 1 - x^2 + x^4 + \ldots + (-1)^n x^{2n} + \ldots \mathrm{d}\, x \qquad \text{for } -1<t<1$$

$$\tan^{-1} t = \int_0^t 1 dx - \int_0^t x^2 dx + \int_0^t x^4 dx + \ldots + \int_0^t (-1)^n x^{2n} dx + \ldots \qquad \text{for } -1<t<1$$

$$= t - \frac{t^3}{3} + \frac{t^5}{5} - \ldots + (-1)^n \frac{t^{2n+1}}{2n+1} + \ldots \qquad \text{for } -1<t<1$$

Should learn how to write the infinite sum in terms of $\sum$

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n \quad \text{for } -1<x<1$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{for } -1<x<1$$

$$\int_0^t \frac{1}{1+x^2}\,dx = \sum_{n=0}^{\infty} (-1)^n \int_0^t x^{2n}\,dx \quad \text{for } -1<x<1$$

(D) Taylor series and Maclaurin series

(1) Taylor series of f at point a

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

(2) Taylor series of f at point 0, called Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

Maclaurin Series

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n \qquad \text{for } -1<x<1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-x)^n + \ldots = \sum_{n=0}^{\infty} (-1)^n x^n \qquad \text{for } -1<x<1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{for all x}$$

What is the coefficient of x^{71}

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{for all x}$$

Maclaurin Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all x}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + .. = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + .. = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}. \qquad -1 \le x \le 1$$

What is the coefficient of x^{35}

$$(1+\mathrm{x})^{\mathrm{p}} = 1 + \binom{p}{1} x + \binom{p}{2} x^2 + \binom{p}{3} x^3 + \ldots \qquad -1 < x < 1$$

The last one is optional

where $\binom{p}{k} = \frac{p(p-1)\ldots(p-k+1)}{k!}$

(E) Find Taylor or Maclaurin series from known results

(1) Find Maclaurin series of $\ln(1+x+x^2)$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$$

$$\ln(1+x+x^2)=\ln\left(\frac{1-x^3}{1-x}\right)$$

$$=\ln(1-x^3)-\ln(1-x)$$

$$=\ln(1+(-x^3))-\ln(1-(-x))$$

$$=(-x^3)-\frac{(-x^3)^2}{2}+\frac{(-x^3)^3}{3}-\frac{(-x^3)^4}{4}+\ldots$$

$$-\left((-x)-\frac{(-x)^2}{2}+\frac{(-x)^3}{3}-\frac{(-x)^4}{4}+\ldots\right)$$

$$=-(x^3+\frac{x^6}{2}+\frac{x^9}{3}+\frac{x^{12}}{4}+\ldots)+(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}\ldots)$$

(2) Find the Taylor series of $\dfrac{1}{x}$ at $x=2$

$$\frac{1}{1+x}=1-x+x^2-\ldots+(-x)^n+\ldots \qquad \text{for -1<x<1}$$

$$\frac{1}{x}=\frac{1}{2+(x-2)}=\frac{1}{2}\frac{1}{\left[1+\dfrac{(x-2)}{2}\right]}$$

$$=\frac{1}{2}\left[1-\left(\frac{x-2}{2}\right)+\left(\frac{x-2}{2}\right)^2-\ldots+(-1)^n\left(\frac{(x-2)}{2}\right)^n+\ldots\right]$$

$$=\frac{1}{2}-\frac{1}{2^2}(x-2)+\frac{1}{2^3}(x-2)^2+\ldots+(-1)^n\frac{(x-2)^n}{2^{n+1}}+\ldots$$

for $\left|\dfrac{x-2}{2}\right|<1$ i.e., $|x-2|<2$

Should learn how to write the infinite sum in terms of $\sum$

$$\frac{1}{x}=\frac{1}{2+(x-2)}=\frac{1}{2}\frac{1}{\left[1+\frac{(x-2)}{2}\right]}$$

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n \qquad \text{for } -1<x<1$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(x-2)^n}{2^n}$$

$$==\sum_{n=0}^{\infty}(-1)^n\frac{(x-2)^n}{2^{n+1}}$$

(F) Applications of Taylor series

(1) Evaluate $\int_0^{0.5} \frac{1}{(1+x^{10})} dx$

$$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-x)^n + \ldots \qquad \text{for -1<x<1}$$

$$\frac{1}{1+x^{10}} = 1 - x^{10} + x^{20} - \ldots + (-1)^n (x)^{10n} + \ldots \qquad \text{for -1<x<1}$$

$$\int_0^{0.5} \frac{1}{1+x^{10}} dx = \int_0^{0.5} 1 dx - \int_0^{0.5} x^{10} dx + \int_0^{0.5} x^{20} dx - \ldots + \int_0^{0.5} (-1)^n x^{10n} dx + \ldots \qquad \text{for -1<x<1}$$

How many terms on the right side should be used depending on the degree of accuracy

(2)

$$\lim_{x\to 0}\frac{\sin x - x + x^3}{x^3\cos x} = \lim_{x\to 0}\frac{-x^3/3! + x^5/5! + \ldots + x^3}{x^3(1 - x^2/2! + x^4/4! + \ldots)}$$

$$= \lim_{x\to 0}\frac{-1/3! + x^2/5! + \ldots + 1}{(1 - x^2/2! + \ldots)} = \frac{-1}{3!} + 1$$

(G) Approximation and error

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(\mathrm{x}-\mathrm{a})^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

$$where\ R_n(x) = \frac{f^{(n+1)}(\mathrm{c})}{(n+1)!}(x-a)^{n+1}$$

We know c is between a and x, but we don't know the exact value

Example, see Lecture Notes

TEST 2011

Use a Taylor series of $x\ln(1+x)$

find the exact value of $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n(n+2)}\left(\frac{1}{2^2}\right)^{n+2}$

Solution

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$

$$x\ln(1+x)=x\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{n+1}}{n}$$

$$\int_0^{\frac{1}{2^2}}x\ln(1+x)dx=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\frac{1}{2^2}}x^{n+1}dx$$