Parameterizing the Intersection of a Sphere and a Plane

Problem: Parameterize the curve of intersection of the sphere $\mathcal{S}$ and the plane $\mathcal{P}$ given by

$$\begin{array}{ll} (\mathcal{S}) & x^2 + y^2 + z^2 = 9 \\ (\mathcal{P}) & x + y = 2 \end{array}$$

Solution: There is no foolproof method, but here is one method that works in this case and many others where we are intersecting a cylinder or sphere (or other "quadric" surface, a concept we'll talk about Friday) with a plane.

Step 1: *Find an equation satisfied by the points of intersection in terms of two of the coordinates.*

We'll eliminate the variable y. Note that the equation $(\mathcal{P})$ implies $y = 2 - x$, and substituting this into equation $(\mathcal{S})$ gives:

$$\begin{aligned} x^2 + (2-x)^2 + z^2 &= 9 \\ x^2 + 4 - 4x + x^2 + z^2 &= 9 \\ 2x^2 - 4x + z^2 &= 5 \end{aligned}$$

Step 2: *Parameterize the equation from Step 1, writing the two coordinates (here, x and z) in terms of a parameter t.*

There are many ways to parameterize

$$2x^2 - 4x + z^2 = 5 \tag{1}$$

but one way is to note that an equation of the form

$$u^2 + v^2 = R^2$$

for a constant R and variables u and v is the equation of a circle of radius R in (u, v)-coordinates which can be parameterized using trigonometric functions as

$$\begin{cases} u = R\cos t \\ v = R\sin t, \end{cases} \qquad 0 \le t \le 2\pi \tag{2}$$

For equation (1), we can complete the square:

$$\begin{aligned} 2x^2 - 4x + z^2 &= 5 \\ 2(x^2 - 2x) + z^2 &= 5 \\ 2[(x-1)^2 - 1] + z^2 &= 5 \\ 2(x-1)^2 + z^2 &= 7 \end{aligned}$$

which is of the correct form for $u = \sqrt{2}(x-1)$, $v = z$, and $R = \sqrt{7}$. The parameterization (2) becomes:

$$\begin{cases} \sqrt{2}(x-1) = \sqrt{7}\cos t \\ z = \sqrt{7}\sin t, \end{cases} \qquad 0 \le t \le 2\pi$$

and solving for x and z we get a parameterization for two of the three coordinates:

$$\begin{cases} x = \sqrt{\tfrac{7}{2}}\cos t + 1 \\ z = \sqrt{7}\sin t, \end{cases} \qquad 0 \le t \le 2\pi$$

Step 3: *The final step (which is barely even a step) is to add a parameterization for the final coordinate.*

From the plane equation ($\mathcal{P}$), we know $y = 2 - x$, so we can substitute in the parameterization for x to get:

$$y = 2 - x = 2 - (\sqrt{\tfrac{7}{2}}\cos t + 1) = 1 - \sqrt{\tfrac{7}{2}}\cos t$$

The final parameterization for all three coordinates is:

$$\begin{cases} x = \sqrt{\tfrac{7}{2}}\cos t + 1 \\ y = 1 - \sqrt{\tfrac{7}{2}}\cos t \\ z = \sqrt{7}\sin t, \end{cases} \qquad 0 \le t \le 2\pi$$

and so

$$\mathbf{r}(t) = \langle \sqrt{\tfrac{7}{2}}\cos t + 1, 1 - \sqrt{\tfrac{7}{2}}\cos t, \sqrt{7}\sin t \rangle, \qquad 0 \le t \le 2\pi$$