Chapter 3. Integration

3.1 Indefinite Integral

Integration can be considered as the antithesis of differentiation, and they are subtly linked by the **Fundamental Theorem of Calculus**. We first introduce indefinite integration as an "inverse" of differentiation.

3.1.1 Antiderivatives

A (differentiable) function $F(x)$ is an *antiderivative* of a function $f(x)$ if

$$F'(x) = f(x)$$

for all x in the domain of f.

The set of all antiderivatives of f is the *indefinite integral* of f with respect to x, denoted by

$$\int f(x)\,dx.$$

Terminology:

f : *integrand* of the integral x : *variable* of integration

3.1.2 Constant of Integration

Any constant function has zero derivative. Hence the antiderivatives of the zero function are all the constant functions.

If $F'(x) = f(x) = G'(x)$, then $G(x) = F(x) + C$, where C is some constant. So

$$\int f(x)dx = F(x) + C.$$

C here is called the *constant of integration* or an *arbitrary constant.* Thus,

$$\int f(x)\,dx = F(x) + C$$

means the same as

$$\frac{d}{dx}F(x) = f(x).$$

In words,

indefinite integral and antiderivative (of a function) *differ by an arbitrary constant.*

3.1.3 Integral formulas

1. $$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1, \; n \text{ rational}$$

$$\int 1 \, dx = \int dx = x + C \qquad \text{(Special case, } n = 0\text{)}$$

2. $$\int \sin kx \, dx = -\frac{\cos kx}{k} + C$$

3. $$\int \cos kx \, dx = \frac{\sin kx}{k} + C$$

4. $\int \sec^2 x \, dx = \tan x + C$

5. $\int \csc^2 x \, dx = -\cot x + C$

6. $\int \sec x \tan x \, dx = \sec x + C$

7. $\int \csc x \cot x \, dx = -\csc x + C$

3.1.4 Rules for indefinite integration

1. $$\int k f(x)\,dx = k \int f(x)\,dx,$$

$k =$ constant (independent of x)

2. $$\int -f(x)\,dx = -\int f(x)\,dx$$

(Rule 1 with $k = -1$)

3. $\displaystyle\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

3.1.5 Example

Find the curve in the xy-plane which passes through the point $(9, 4)$ and whose slope at each point (x, y) is $3\sqrt{x}$.

Solution. The curve is given by $y = y(x)$, satisfying

(i) $\dfrac{dy}{dx} = 3\sqrt{x}$ and (ii) $y(9) = 4$.

Solving (i), we get

$$y = \int 3\sqrt{x}\, dx = 3\,\frac{x^{3/2}}{3/2} + C = 2x^{3/2} + C.$$

By (ii), $\qquad 4 = (2)9^{3/2} + C = (2)27 + C,$

$C = 4 - 54 = -50.$

$$\boxed{\text{Hence} \qquad y = 2x^{3/2} - 50.}$$

3.2 Riemann Integrals

3.2.1 Area under a curve

Let $f = f(x)$ be a non-negative continuous function $f = f(x)$ on an interval $[a, b]$.

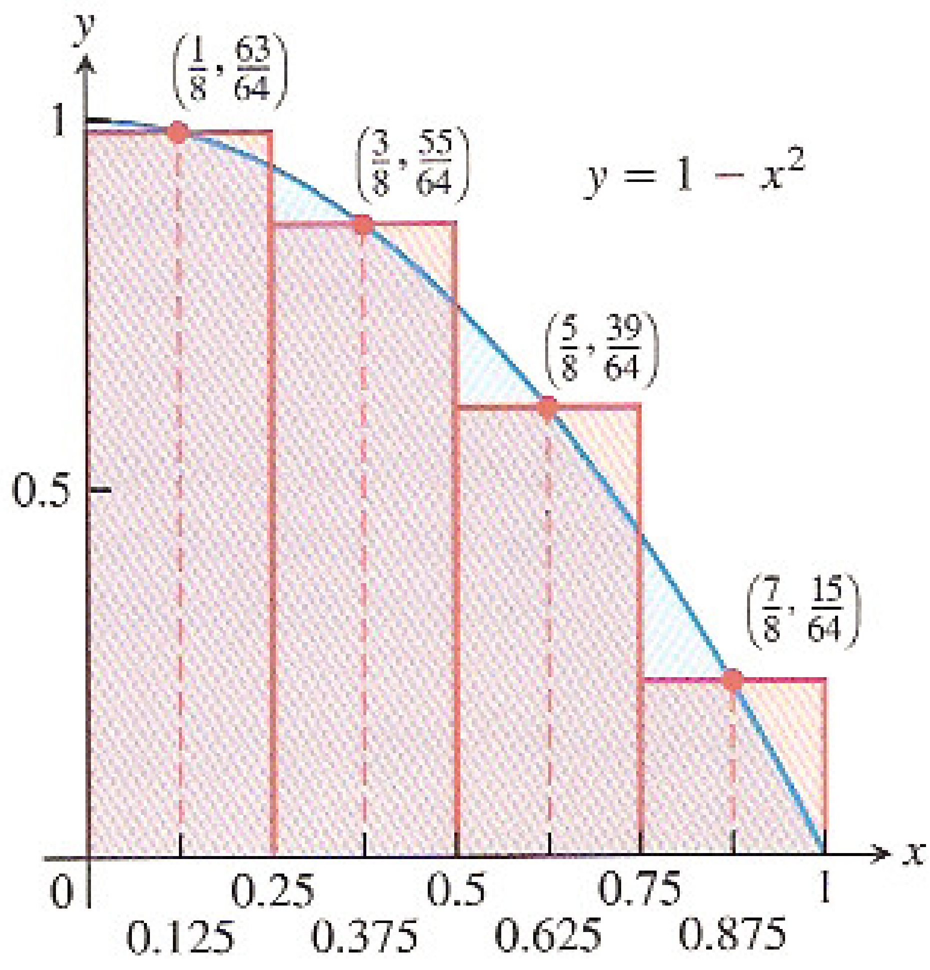
Partition $[a, b]$ into n consecutive sub-intervals $[x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$ each of length $\Delta x = \dfrac{b-a}{n}$, where we set $a = x_0$, $b = x_n$, and $x_1, x_2, \cdots, x_{n-1}$ to be successive points between a and b with $x_k - x_{k-1} = \Delta x$.

Let c_k be any intermediate point in the sub-interval $[x_{k-1}, x_k]$.

Then the sum

$$S = \sum_{k=1}^{n} f(c_k)\Delta x$$

gives an approximate area under the curve of $y = f(x)$ from $x = a$ to $x = b$.



y
x
$\left(\frac{1}{8}, \frac{63}{64}\right)$
$\left(\frac{3}{8}, \frac{55}{64}\right)$
$\left(\frac{5}{8}, \frac{39}{64}\right)$
$\left(\frac{7}{8}, \frac{15}{64}\right)$
$y = 1 - x^2$
1
0.5
0
0.25
0.5
0.75
1
0.125
0.375
0.625
0.875

The *exact* area A under the curve of $y = f(x)$ is achieved by letting the partition of the interval $[a, b]$ tends to infinity:

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \Delta x.$$

3.2.2 Riemann Integral

Let us continue with the notation as in the previous section and denote the limit by I.

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x = I.$$

We call I the **Riemann integral** (or **definite**

integral) of f over $[a, b]$ and we write

$$I = \int_a^b f(x)\,dx.$$

3.2.3 Terminology

$$\int_a^b f(x)dx$$

$[a, b]$: the interval of integration

a : lower limit of integration

b : upper limit of integration

x : variable of integration

$f(x)$: the integrand

x is a *dummy* variable, i.e.

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(t)\,dt, \quad \text{etc.}$$

3.2.4 Rules of algebra for definite integrals

1. $\displaystyle\int_a^a f(x)\,dx = 0$

2. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx, \quad \text{(any constant } k\text{)}$

$$\left(\text{In particular, } \int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx\right)$$

4. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. If $f(x) \geq g(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

In particular, if $f(x) \geq 0$ on $[a, b]$, then

$$\int_a^b f(x)\,dx \geq 0.$$

6. If f is continuous on the interval joining a, b and c, then

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

3.3 The Fundamental Theorem of Calculus

3.3.1 Part 1

If f is continuous on $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt \tag{1}$$

has a derivative at every point of $[a, b]$, and

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

3.3.2 Examples

$$\frac{d}{dx}\int_{-\pi}^{x} \cos t\, dt = \cos x$$

$$\frac{d}{dx}\int_{0}^{x} \frac{dt}{1+t^2} = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\int_{1}^{x^2} \cos t\, dt = \left[\frac{d}{d(x^2)}\int_1^{x^2} \cos t\, dt\right] \frac{d(x^2)}{dx} = (\cos x^2)\, 2x$$

$$\boxed{= 2x\cos(x^2)}$$

3.3.3 Part 2

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$,

then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Set $G(x) = \int_a^x f(t)\,dt.$

By the Fundamental Theorem of Calculus, Part 1,

above,

$$G'(x) = \frac{d}{dx}G(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

We also know that $F'(x) = f(x)$. Thus $G'(x) = F'(x)$ for $x \in [a, b]$.

Hence we have $F(x) = G(x) + c$ throughout $[a, b]$

for some constant c. Thus

$$\begin{aligned}
F(b) - F(a) &= G(b) + c - \big(G(a) + c\big) \\
&= G(b) - G(a) \\
&= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\
&= \int_a^b f(t)\,dt.
\end{aligned}$$

3.3.4 Examples

$$\int_0^{\pi} \cos x \, dx = \sin x \Big|_0^{\pi} = \sin \pi - \sin 0 = 0$$

$$\int_0^2 t^2 \, dt = \frac{1}{3} t^3 \Big|_0^2 = \frac{8}{3}$$

$$\int_{-2}^{2}(4-u^2)\,du = \left[4u-\frac{1}{3}u^3\right]\Big|_{-2}^{2}$$

$$= \left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)$$

$$= \frac{32}{3}$$

3.4 Integration by substitution

To evaluate $\displaystyle\int f\big(g(x)\big)\,g'(x)\,dx$ where f and g' are continuous:

1. Set $u = g(x)$. Then $g'(x) = \dfrac{du}{dx}$, the given integral becomes $\displaystyle\int f(u)\,du$.

2. Integrate with respect to u.

3. Replace u by $g(x)$ in the result of step 2.

3.4.1 Examples

$$I = \int (x^2+2x-3)^2 (x+1)\, dx$$

Let $u = x^2+2x-3$

$$du = (2x+2)\, dx$$

$$= 2(x+1)\, dx$$

$$I = \int u^2 \frac{1}{2}\, du = \frac{1}{2}\int u^2 \, du$$

$$= \frac{1}{6} u^3 + C$$

$$= \frac{1}{6}(x^2 + 2x - 3)^3 + C$$

$$I = \int \sin^4 x \cos x \, dx$$

Let $u = \sin x$

$$du = \cos x \, dx$$

$$I = \int u^4 \, du$$

$$= \frac{1}{5} u^5 + C$$

$$= \underline{\underline{\frac{1}{5} \sin^5 x + C}}$$

3.4.2 Substitution in definite integrals

The limits change accordingly:

$$\int_a^b f\left(g(x)\right) g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Note that in general we require $g' \geq 0$ or $g' \leq 0$ in

$\left[a, b\right].$

3.4.3 Example

$$I = \int_0^{\pi/4} \tan x \sec^2 x \, dx$$

Let $u = \tan x$

$$x = 0 \Rightarrow u = 0$$

$$x = \frac{\pi}{4} \Rightarrow u = 1$$

$$du = \sec^2 x \, dx$$

$$I = \int_0^1 u\,du$$

$$= \frac{1}{2}u^2 \Big|_0^1$$

$$= \underline{\underline{\frac{1}{2}}}$$

3.5 Integration by parts

Integration by parts is a technique for evaluating integrals of the form

$$\int f(x)g(x)\,dx$$

in which f can be differentiated repeatedly and g can be integrated without difficulty.

Recall the product rule

$$\frac{d}{dx}(uv) = u\,\frac{dv}{dx} + v\frac{du}{dx}$$

In differential form it becomes

$$d(uv) = u\,dv + v\,du$$

or, equivalently,

$$u\,dv = d(uv) - v\,du.$$

Thus we have the **Integration-by-parts Formula**:

$$\int u\,dv = uv - \int v\,du$$

or,

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

3.5.1 Example

Evaluate $I = \displaystyle\int x \cos x \, dx$.

Solution.

$$\begin{aligned} I &= \int x \cos x \, dx = \int x \, d(\sin x) \\ &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + C \end{aligned}$$

3.5.2 Exercise

(a) $\displaystyle\int \ln x \, dx$

$$\int \underbrace{\ln x}_{u} \, \underbrace{dx}_{v}$$

$$= x \ln x - \int x \frac{1}{x} \, dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C$$

(b) $\int x^2 e^x \, dx$

$$\int x^2 e^x dx$$

$$= \int x^2 d(e^x)$$

$$= x^2 e^x - 2\int x e^x dx$$

$$= x^2 e^x - 2\int x \, d(e^x)$$

$$= x^2 e^x - 2x e^x + 2\int e^x dx$$

$$= x^2 e^x - 2x e^x + 2e^x + C$$

(d) $\int e^x \cos x \, dx$ (*Hint:* Consider also $\int e^x \sin x \, dx$.)

To find: $A = \int e^x \cos x \, dx$

Let $B = \int e^x \sin x \, dx$

$$A = \int e^x \cos x \, dx$$

$$= \int e^x \, d(\sin x)$$

$$= e^x \sin x - \int e^x \sin x \, dx$$

$$A + B = e^x \sin x \quad \cdots\cdots \quad (1)$$

$$B = \int e^x \sin x \, dx$$

$$= -\int e^x \, d(\cos x)$$

$$= -e^x \cos x + \int e^x \cos x \, dx$$

$$A - B = e^x \cos x \quad \text{- - - -} \quad (2)$$

$$A = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right)$$

$$B = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right)$$

3.6 Area between two curves

If f_1 and f_2 are continuous functions with $f_1(x) \leq f_2(x)$ in the interval $a \leq x \leq b$, then the area of the region between the curves $y = f_1(x)$ and $y = f_2(x)$ from a to b is the integral of $f_2 - f_1$ from a to b,

i.e.

$$\text{Area} = \int_a^b \left[f_2(x) - f_1(x)\right] dx. \tag{1}$$

This is the basic formula.

If the curves only cross at one or both end points of $[a,b]$, we apply (1) once to find the area. If the curves cross within the interval $[a,b]$, we need to apply (1) more than once. Thus, to find the area of the region between two curves

(i) Sketch the curves and determine the crossing points.

(ii) Evaluate the area(s) using (1). **Or**, integrate $|f_2 - f_1|$ over $[a,b]$.

3.6.1 Example

Find area enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

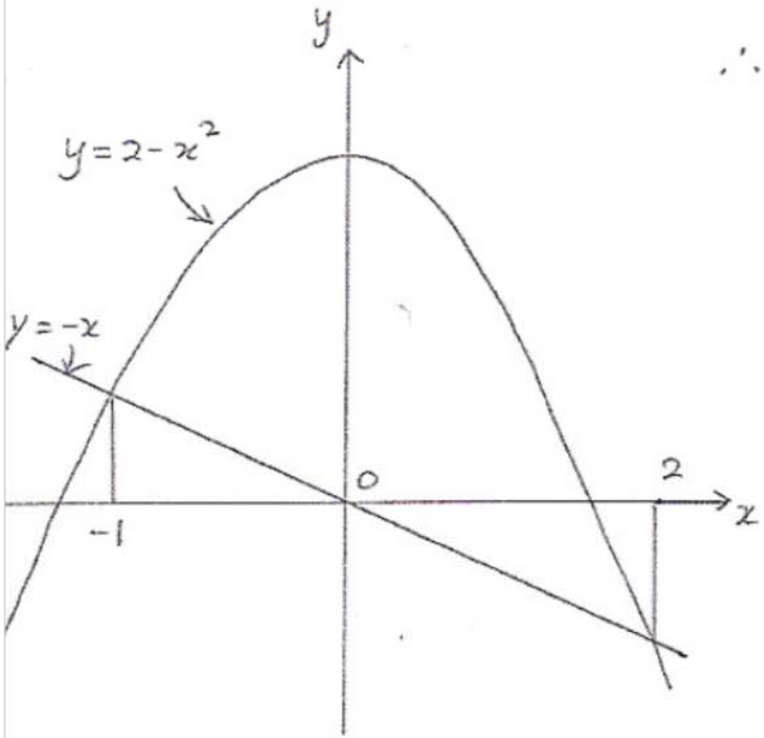
$y = 2 - x^2, \quad y = -x$

Points of intersection: Set $2 - x^2 = -x$

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0$$

$$\therefore x = -1, \quad x = 2.$$



$$\text{Area} = \int_{-1}^{2} \{(2-x^2) - (-x)\}\,dx$$

$$= \int_{-1}^{2} (2 - x^2 + x)\,dx$$

$$= \left[2x - \frac{1}{3}x^3 + \frac{1}{2}x^2\right]_{-1}^{2}$$

$$= \left(4 - \frac{8}{3} + 2\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{9}{2} \;//$$

Some examples from past test papers

2007 Question 9

9. Let a be a positive constant. Find the area of the bounded plane region between the parabola $y^2 = x + 12a^2$ and the straight line $y = \frac{1}{a}x$.

(A) $\frac{397}{6}a^3$

(B) $\frac{665}{6}a^3$

(C) $\frac{343}{6}a^3$

(D) $\frac{301}{6}a^3$

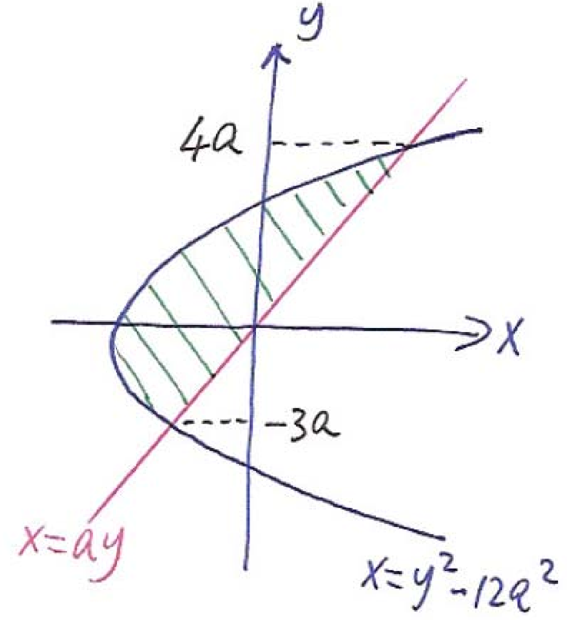
(E) $\frac{73}{6}a^3$

9). $x = y^2 - 12a^2$ and $x = ay$

$\Rightarrow y^2 - ay - 12a^2 = 0$

$\Rightarrow (y - 4a)(y + 3a) = 0$

$\Rightarrow y = 4a\,,\ y = -3a$



$$\text{Area} = \int_{-3a}^{4a} \{ay - (y^2 - 12a^2)\}\,dy$$

$$= \underline{\underline{\frac{343}{6}a^3}}$$

C

2008 Question 5

5. Evaluate

$$\int_{\frac{1}{e}}^{e} |\ln x| \, dx$$

(A) $2\,(1+e)$

(B) $2\,(e-1)$

(C) $2\left(1+e^{-1}\right)$

(D) $2\left(e-e^{-1}\right)$

(E) $2\left(1-e^{-1}\right)$

5). $\displaystyle\int_{1/e}^{e} |\ln x|\, dx = \int_{1/e}^{1} -\ln x\, dx + \int_{1}^{e} \ln x\, dx$

$$= -x\ln x\Big|_{1/e}^{1} + \int_{1/e}^{1} dx + x\ln x\Big|_{1}^{e} - \int_{1}^{e} dx$$

$$= -\frac{1}{e} + 1 - \frac{1}{e} + e - e + 1$$

$$= 2 - \frac{2}{e} = \underline{\underline{2(1-e^{-1})}}$$

E