MA 1505 Mathematics I
Tutorial 11 Solutions

1. We have

$$f(\mathbf{r}(u,v)) = f(2u+v, u-2v, u+3v) = 2u+v+u-2v+u+3v = 4u+2v,$$

$$\mathbf{F}(\mathbf{r}(u,v)) = \mathbf{F}(2u+v, u-2v, u+3v) = (2u+v)^2\mathbf{i} + (u-2v)^2\mathbf{j} + (u+3v)^2\mathbf{k},$$

$$\mathbf{r}_u \times \mathbf{r}_v = (2\mathbf{i}+\mathbf{j}+\mathbf{k}) \times (\mathbf{i}-2\mathbf{j}+3\mathbf{k}) = 5\mathbf{i}-5\mathbf{j}-5\mathbf{k},$$

$$||\mathbf{r}_u \times \mathbf{r}_v|| = 5\sqrt{3}.$$

Thus if R denotes the rectangular region

$$0 \le u \le 1, \quad 0 \le v \le 2,$$

$$\int\int_S f(x,y,z)\,dS = \int\int_R f(\mathbf{r}(u,v))\,||\mathbf{r}_u \times \mathbf{r}_v||\,dA$$

$$= \int_0^2 \int_0^1 (4u+2v)(5\sqrt{3})\,dudv = 40\sqrt{3},$$

and

$$\begin{aligned}
\int\int_S \mathbf{F}(x,y,z) \bullet d\mathbf{S} &= \int\int_R \mathbf{F}(\mathbf{r}(u,v)) \bullet (\mathbf{r}_u \times \mathbf{r}_v)\,dA \\
&= \int\int_R ((2u+v)^2\mathbf{i} + (u-2v)^2\mathbf{j} + (u+3v)^2\mathbf{k}) \bullet (5\mathbf{i}-5\mathbf{j}-5\mathbf{k})\,dA \\
&= \int_0^2 \int_0^1 (5(2u+v)^2 - 5(u-2v)^2 - 5(u+3v)^2)\,dudv \\
&= \int_0^2 \int_0^1 10(u^2+uv-6v^2)\,dudv = \int_0^2 -\frac{5}{3}(36v^2-3v-2)\,dv = -\frac{430}{3}
\end{aligned}$$

2. $z = 4 - x^2 - y^2$ parametrizes as $r(u,v) = u\mathbf{i} + v\mathbf{j} + (4-u^2-v^2)\mathbf{k}$.

$$\mathbf{r}_u \times \mathbf{r}_v = (\mathbf{i}-2u\mathbf{k}) \times (\mathbf{j}-2v\mathbf{k}) = 2u\mathbf{i}+2v\mathbf{j}+\mathbf{k} \Longrightarrow ||\mathbf{r}_u \times \mathbf{r}_v|| = \sqrt{4u^2+4v^2+1}$$

So $\displaystyle\int\int_S z\,dS = \int\int_S (4-u^2-v^2)dS. = \int\int_R (4-u^2-v^2)\sqrt{4u^2+4v^2+1}\,dA.$

R is the projection of S on the xy plane, which is the circular disk of radius 2.

Use polar coordinates:

$$R: \quad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi,$$

$$\int\int_R (4-x^2-y^2)\sqrt{4x^2+4y^2+1}\,dA = \int_0^{2\pi}\int_0^2 (4-r^2)\sqrt{4r^2+1}\,r\,drd\theta \quad (**)$$

$$= 2\pi\left[\frac{1}{120}(4r^2+1)^{3/2}(41-6r^2)\right]_0^2 = \frac{289}{60}\pi\sqrt{17} - \frac{41}{60}\pi.$$

$(**)$ To integrate

$$\int (4-r^2)\sqrt{4r^2+1}\;r\;dr,$$

you may use the substitution $u = 4r^2+1$.

3.
$$3x + 2y + z = 6 \Longrightarrow z = 6 - 3x - 2y,$$

So the surface S can be represented parametrically as

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + (6 - 3u - 2v)\mathbf{k}.$$

$$\mathbf{r}_u \times \mathbf{r}_v = (\mathbf{i} - 3\mathbf{k}) \times (\mathbf{j} - 2\mathbf{k}) = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

The projection of S on the xy plane is the region R bounded by the x-axis, y-axis and the line $3x + 2y = 6$ or $y = (6 - 3x)/2$:

$$R: \quad 0 \le v \le \frac{6-3u}{2}, \quad 0 \le u \le 2.$$

The required integral is

$$\int\int_S \mathbf{F} \bullet d\mathbf{S} = \int\int_R \mathbf{F}(\mathbf{r}(u,v)) \bullet (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

$$= \int\int_R (v\mathbf{i} + u^2\mathbf{j} + (6-3u-2v)^2\mathbf{k}) \bullet (3\mathbf{i} + 2\mathbf{j} + \mathbf{k})\, dA = \int_0^2 \int_0^{(6-3u)/2} (3v + 2u^2 + (6-3u-2v)^2)\, dv du$$

$$= \int_0^2 \frac{1}{8}(-60u^3 + 291u^2 - 540u + 396) du = 31$$

4. Let $\mathbf{F}(x,y,z) = \frac{1}{2}y^2\mathbf{i} + z\mathbf{j} + x\mathbf{k}$.

By Stoke's Theorem, the given integral is equal to $\int\int_S \text{curl } \mathbf{F} \bullet d\mathbf{S}$, where S is the surface on the plane $x + z = 0$ with C as boundary.

Substitute $z = -x$ into $x^2 + 2y^2 + z^2 = 1$ to get

$$x^2 + 2y^2 + (-x)^2 = 1 \iff x^2 + y^2 = \frac{1}{2}.$$

The projection of C onto the xy plane is a circle centred at origin with radius $1/\sqrt{2}$.

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{2}y^2 & z & x \end{vmatrix} = -\mathbf{i} - \mathbf{j} - y\mathbf{k}.$$

S can be described as

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + (-u)\mathbf{k}$$

with

$$\mathbf{r}_u \times \mathbf{r}_v = (\mathbf{i} - \mathbf{k}) \times \mathbf{j} = \mathbf{i} + \mathbf{k}.$$

and R is the plane region enclosed by C:

$$R: \quad 0 \le r \le \frac{1}{\sqrt{2}}, \quad 0 \le \theta \le 2\pi.$$

Hence the required answer is

$$\begin{aligned}
\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S} &= \iint_R (-\mathbf{i} - \mathbf{j} - v\mathbf{k}) \bullet (\mathbf{i} + \mathbf{k})\ dA \\
&= -\iint_R (1+v)\ dA = -\int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} (1 + r\sin\theta)\ r\ drd\theta \\
&= -\int_0^{2\pi} \left[\frac{1}{4} + \frac{1}{6\sqrt{2}}\sin\theta\right]\ d\theta \\
&= -\left[\frac{1}{4}\theta - \frac{1}{6\sqrt{2}}\cos\theta\right]_0^{2\pi} = -\frac{\pi}{2}.
\end{aligned}$$

5. Using Stoke's Theorem:
Let C be given by

$$\mathbf{r}(\theta) = \frac{1}{2}\cos\theta\mathbf{i} + \frac{1}{2}\sin\theta\mathbf{j} + \frac{1}{2}\mathbf{k}, \quad (0 \le \theta \le 2\pi).$$

Then

$$\mathbf{r}'(\theta) = -\frac{1}{2}\sin\theta\mathbf{i} + \frac{1}{2}\cos\theta\mathbf{j}.$$

Note that the orientation of C given by this vector equation is anticlockwise. In order to match the orientation of S given by the outer normal vector, we need to take the negative orientation. Therefore, we will integrate from 2π to 0 in the line integral.

$$\mathbf{F}(\mathbf{r}) = \frac{1}{2}\sin\theta\mathbf{i} - \frac{1}{2}\cos\theta\mathbf{j} + \frac{1}{4}\sin\theta\mathbf{k},$$

so that

$$\int\int_S \text{curl } \mathbf{F}\bullet d\mathbf{S} = \oint_C \mathbf{F}(\mathbf{r})\bullet d\mathbf{r}$$

$$= \int_{2\pi}^0 -\frac{1}{4}(\sin^2\theta + \cos^2\theta)\ d\theta = \int_{2\pi}^0 -\frac{1}{4}\ d\theta = \frac{\pi}{2}.$$

6. By the divergence theorem,

$$\iint_S \mathbf{F} \bullet d\mathbf{S} = \iiint_D \text{div } \mathbf{F} \; dV$$

where D is the rectangular region given by

$$D: \quad 0 \le x \le 1, \; 0 \le y \le 2, \; -3 \le z \le 0$$

and

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(x^3 y^3) = 3x.$$

So

$$\begin{aligned}
\iint_S \mathbf{F} \bullet d\mathbf{S} &= \int_0^1 \int_0^2 \int_{-3}^0 3x \; dz dy dx \\
&= 3 \int_0^1 x \; dx \int_0^2 \; dy \int_{-3}^0 \; dz \\
&= 3(\frac{1}{2})(2)(3) = 9
\end{aligned}$$

7. Let $\mathbf{F} = \frac{1}{2}(bz - cy)\,\mathbf{i} + \frac{1}{2}(cx - az)\,\mathbf{j} + \frac{1}{2}(ay - bx)\,\mathbf{k}$.

Then curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{2}(bz - cy) & \frac{1}{2}(cx - az) & \frac{1}{2}(ay - bx) \end{vmatrix} = \mathbf{n}$.

By Stoke's Theorem, we have

$\frac{1}{2} \int_C (bz - cy)\, dx + (cx - az)\, dy + (ay - bx)\, dz$

$= \int_C \mathbf{F} \cdot d\mathbf{r}$

$= \int \int_D \text{curl } \mathbf{F} \cdot d\mathbf{S}$

$= \int \int_D \text{curl } \mathbf{F} \cdot \mathbf{n} dS$

$= \int \int_D \mathbf{n} \cdot \mathbf{n} dS$

$=$ Area (D)