CH 6 – *Fourier Series*

Why Fourier series? First recall Power Series

***Power* series** about $x = a$

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

The **Taylor series** of f at a is

$$f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

♣ **Approximate** a function f by a ***polynomial*** in x

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

☺ **good** for points **near** 'a'

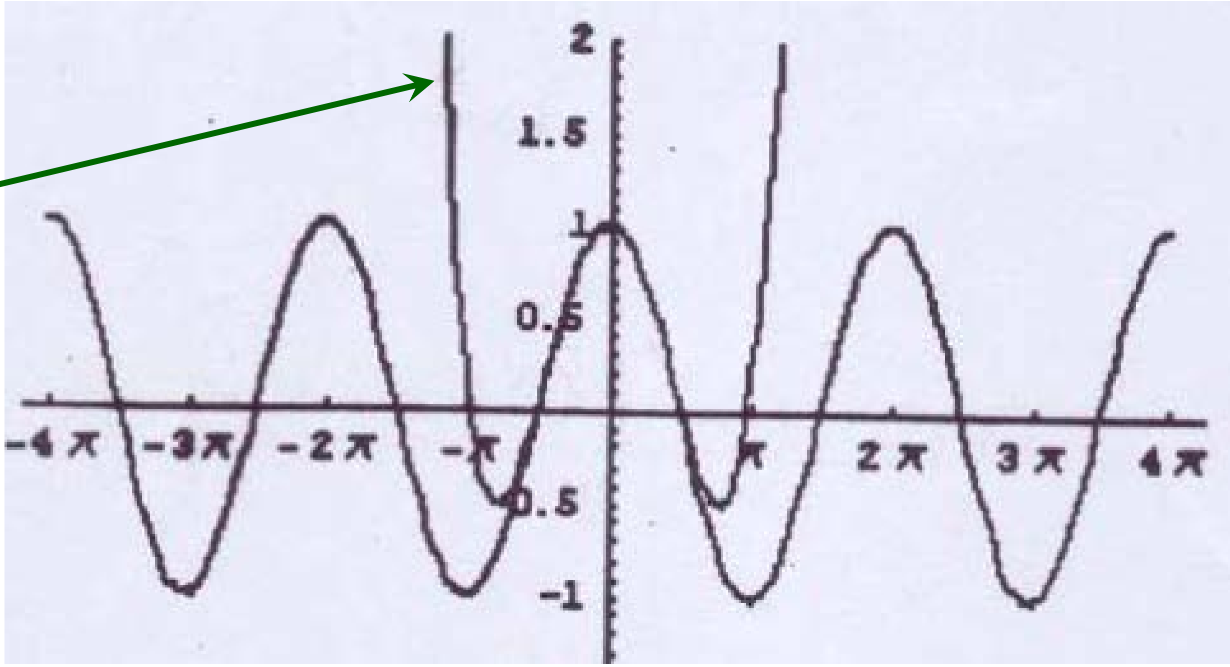
☻ **no good** for points **far away from** 'a'

♣ $y = \cos x$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad -\infty < x < \infty$$

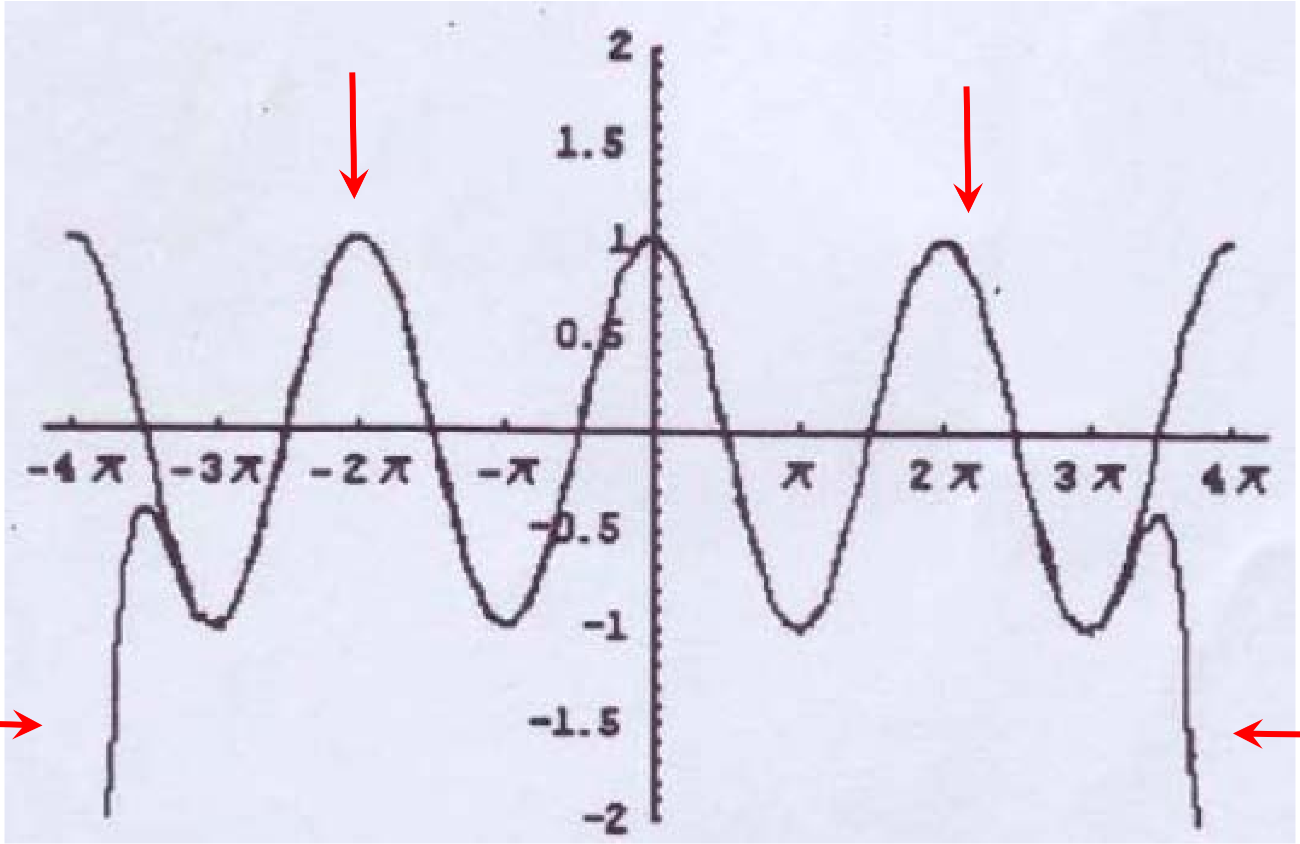
- ***degree = 4***

$$\boxed{1 - \frac{x^2}{2!} + \frac{x^4}{4!}}$$



$$y = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \qquad -\infty < x < \infty$$

- ***degree = 26***



Fourier Series

- Gives ***good*** approximations on ***wider*** intervals
- Often works for ***discontinuous*** functions (Taylor series fails to apply)

$$\boxed{\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k}$$

- Uses **sin *x*** & **cos *x*** (instead of powers of *x*)
- Good tool for solving problems such as ***heat transfer problems*** & many others in ***Engineering*** & ***Science***

Joseph Fourier (1768 –1830)

French mathematician

initiating the investigation of Fourier series and their application to problems of heat transfer.

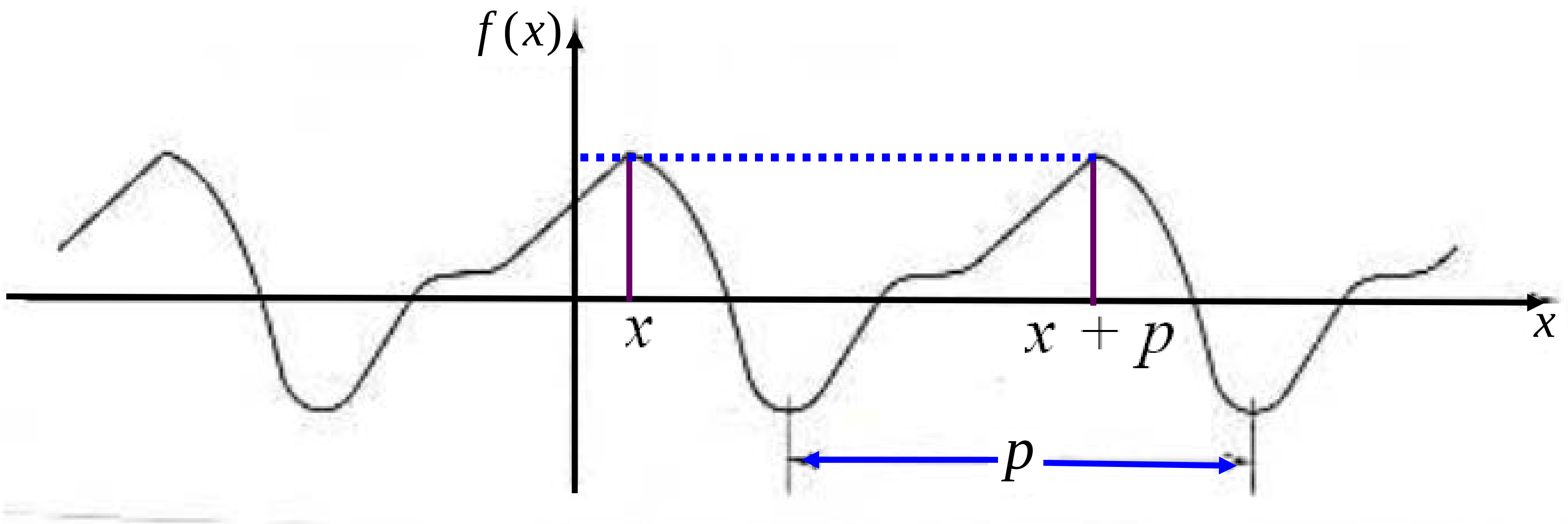
6.1. ***Periodic*** functions

6.1.1 Definition, Graph, Examples

- $\boldsymbol{f} : \boldsymbol{R} \to \boldsymbol{R}$ is ***periodic*** if

$$f(x + p) = f(x) \text{ for all } x \text{ in } \boldsymbol{R} \qquad \textbf{(1)}$$

p — the ***period*** of $\boldsymbol{f}$



Examples

- $\sin(x + 2\pi) = \sin x$
- $\cos(x + 2\pi) = \cos x$
- $f(x) = k$ (a constant, *any period*)
- $\{ 1, \sin x, \sin 2x, \cdots, \sin nx, \cdots,$

  $\cos x, \cos 2x, \cdots, \cos nx, \cdots, \}$

  all have *period* 2π

☻ x^n $(n \geq 1)$, $\ln x$, e^x, etc, are *not periodic.*

6.1.2 ***Properties*** of ***periodic*** functions

♣ If $\boldsymbol{f}$ is of ***period*** p, then

$$\boldsymbol{f}(x + np) = \boldsymbol{f}(x), \text{ for all } x \text{ in } \boldsymbol{R},$$

i.e. $\boldsymbol{f}$ is also of ***period*** $2p$, $3p$, $\cdots$.

♣ If $\boldsymbol{f}$ & $\boldsymbol{g}$ are of ***period*** p, then for any constants a & b, the function

$$a\boldsymbol{f} + b\boldsymbol{g}$$

is also ***periodic*** of ***period*** p.

6.1.3 *Trigonometric Series*

Objective: Let f be a ***periodic*** function of ***period*** 2π. To express f in terms of

$$\begin{cases} \mathbf{1}, \cos x, \cos 2x, \cdots, \cos nx, \cdots, \\ \quad \sin x, \sin 2x, \cdots, \sin nx, \cdots, \end{cases} \tag{2}$$

that is,

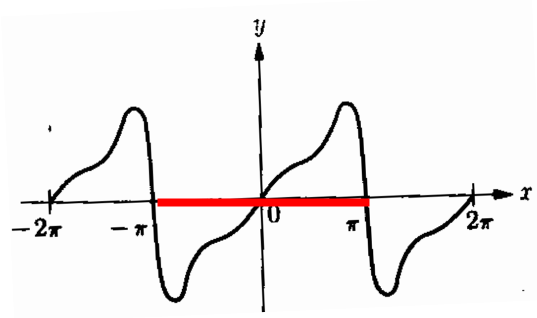
$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

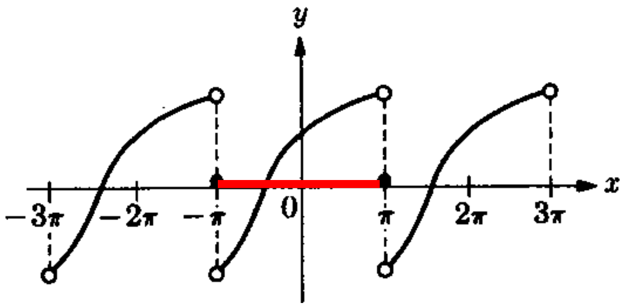
$$\boxed{= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)} \tag{3}$$

where $a_0, a_1, a_2, \cdots, b_1, b_2, \cdots$ are real constants.

6.2. *Fourier* Series

- Let ***f*** be a ***periodic*** function of ***period*** 2π (**from −π to π** as shown).



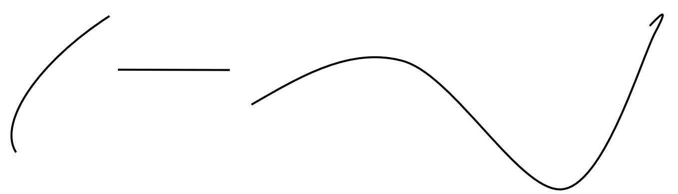


Suppose ***f***(*x*) = $a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

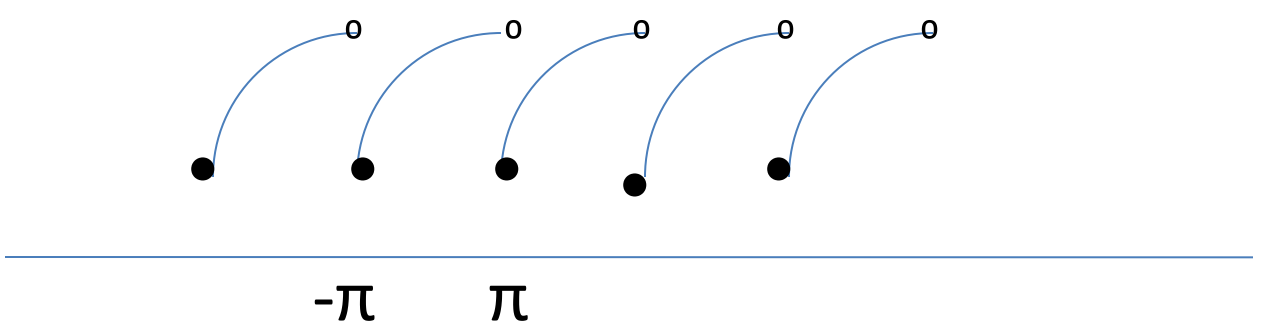
To **determine** a_n and b_n in terms of ***f***

SOME USEFUL FACTS

(1) Piecewise continuous

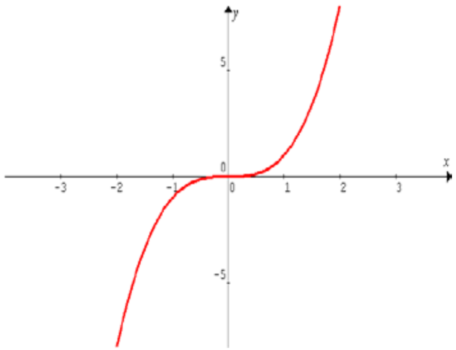


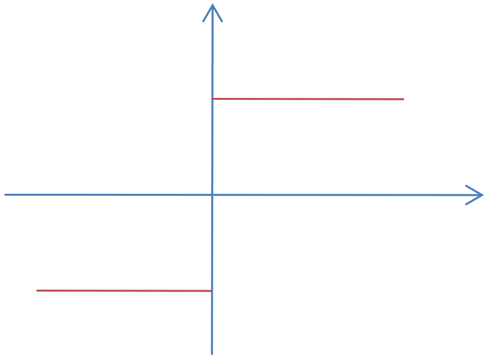
(2) 2π periodic function



(3) Odd function: symmetric w.r.t. origin

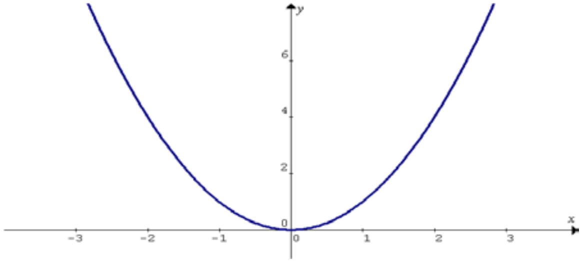
$$f(x) = -f(-x)$$

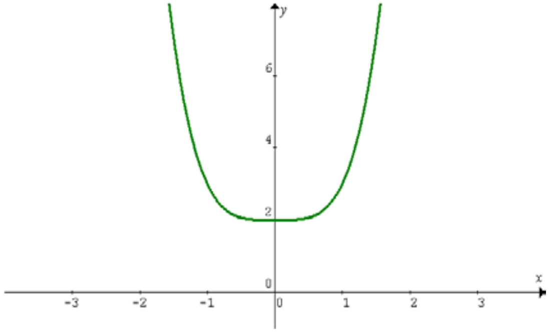




Even function: symmetric w.r.t. y-axis

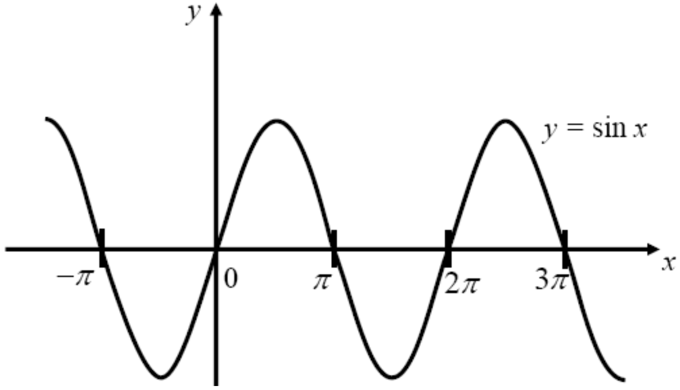
$$f(x) = f(-x)$$





♣ **sin *nx*** — ***odd*** *function*

$$\sin(x+2\pi)=\sin x$$



cos *nx* — ***even*** *function*

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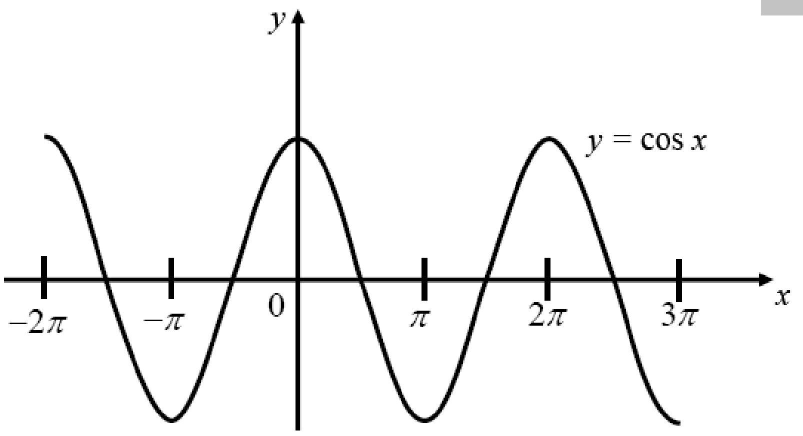


- ♣ ***Product*** of 2 ***even*** functions is ***even***
- ♣ ***Product*** of 2 ***odd*** functions is $\boxed{\textbf{\textit{even}}}$
- ♣ ***Product*** of an ***even*** fn & an ***odd*** fn is $\boxed{\textbf{\textit{odd}}}$

They are different from numbers

- ♣ ***Product*** of 2 ***even* numbers** is ***even***
- ♣ ***Product*** of 2 ***odd*** numbers is $\boxed{\textbf{\textit{odd}}}$
- ♣ ***Product*** of an ***even*** fn & an ***odd*** fn is $\boxed{\textbf{\textit{even}}}$

(4) How to write even numbers

2n   n=1,2,3,…………

2,4,6,…………..

(5) How to write odd numbers

2n-1   n=1,2,3,……..

1,3,5,……..

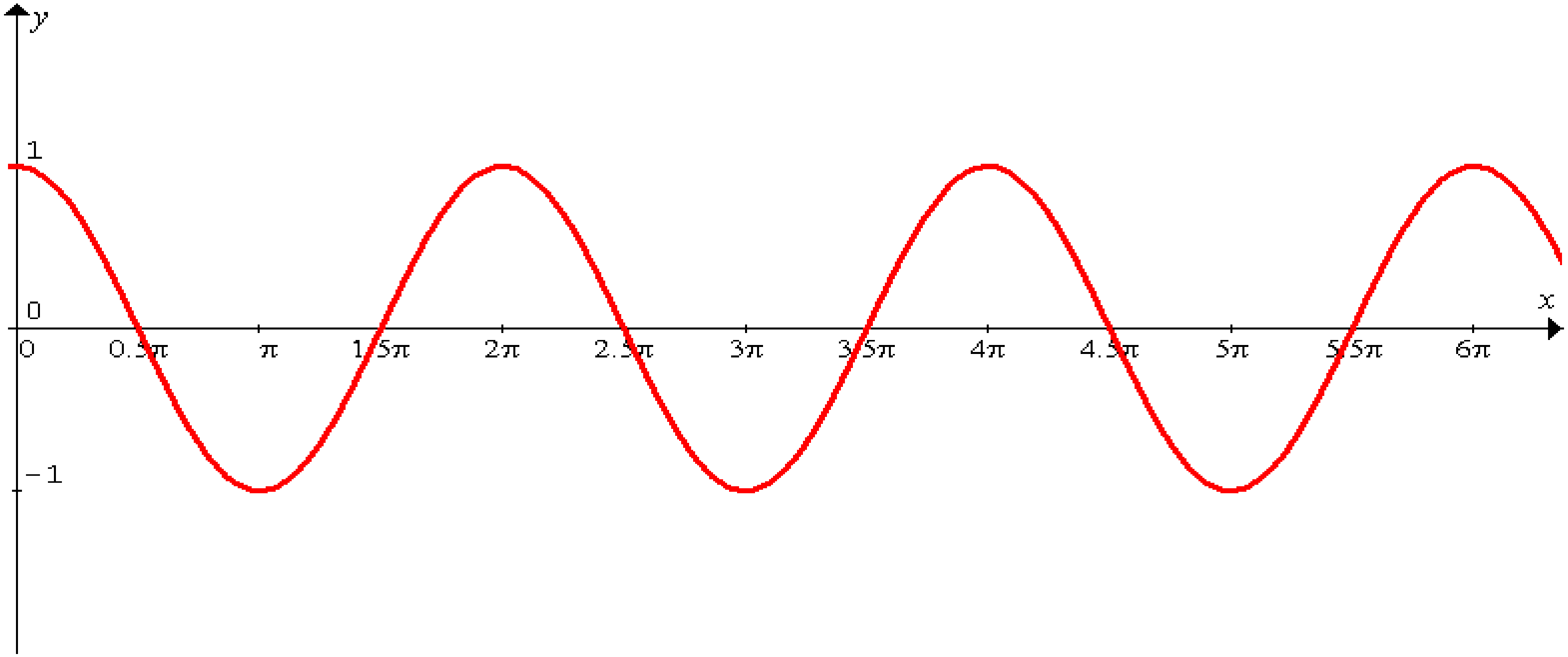
(6) How to write even terms

a_{2n} n=1,2,3,…………

(7) How to write odd terms

a_{2n-1} n=1,2,3,…………

(8)

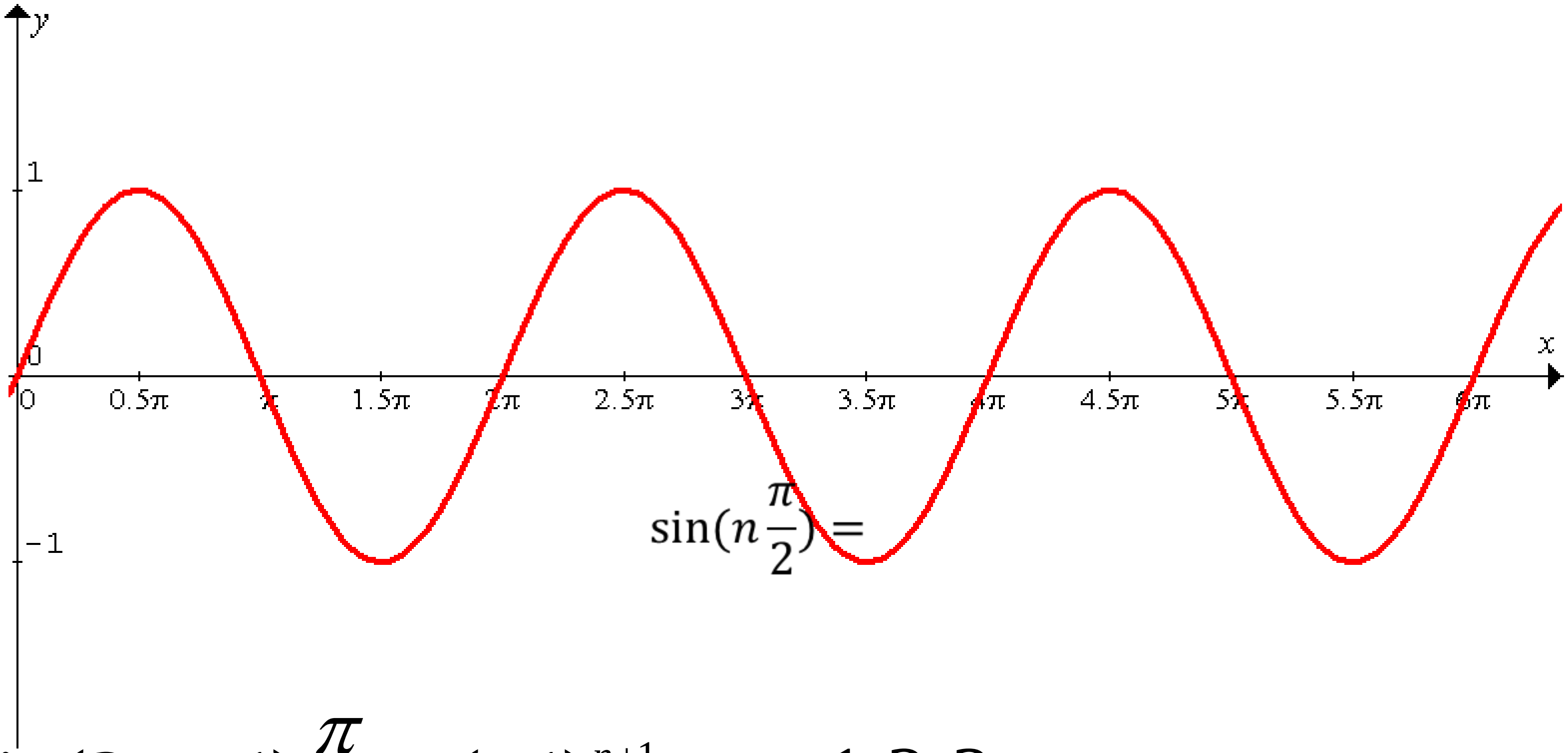


$$\cos n\pi = (-1)^n \qquad n = 1, 2, 3, \ldots$$

$$\cos(2n-1)\frac{\pi}{2} = 0 \qquad n = 1, 2, 3, \ldots$$

$$\cos(n\frac{\pi}{2}) =$$

(9)



$$\sin(n\frac{\pi}{2}) =$$

$$\sin(2n-1)\frac{\pi}{2} = (-1)^{n+1} \quad n = 1, 2, 3, \ldots$$

$$\sin n\pi = 0 \qquad n = 1, 2, 3, \ldots$$

$$\sin(n\frac{\pi}{2}) =$$

(10) Some special integrals

♣ $$\frac{1}{L}\int_{-L}^{L}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx=\begin{cases}0 & \text{if } n\neq m\\ 1 & \text{if } n=m\end{cases}$$

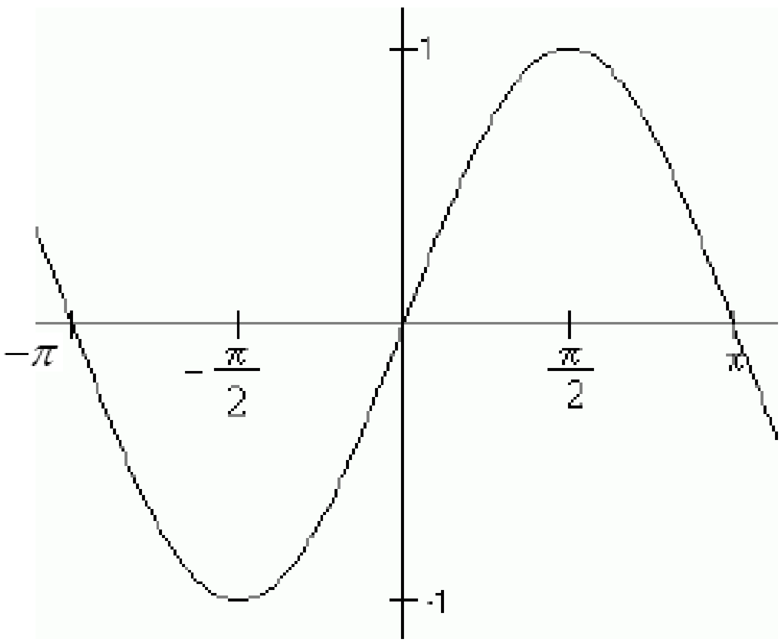
where m, n are positive integers

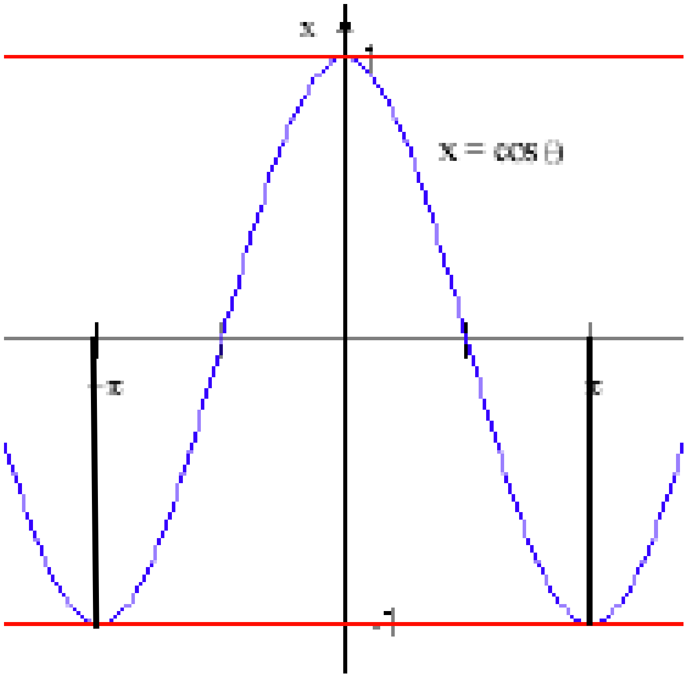
♣ $$\frac{1}{L}\int_{-L}^{L}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx=\begin{cases}0 & \text{if } n\neq m\\ 1 & \text{if } n=m\end{cases}$$

where m, n are positive integers

♣ $$\frac{1}{L}\int_{-L}^{L}\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx=0$$ for any positive integers m, n

♣ $$\int_{-\pi}^{\pi} \sin nx \, dx = \int_{-\pi}^{\pi} \cos nx \, dx = 0$$





$$\int_{-L}^{L} f(x)dx = \begin{cases} 0 & \text{if } f \text{ is odd} \\ 2\int_0^L f(x)dx & \text{if } f \text{ is even.} \end{cases}$$

$$\int_{-L}^{L} f(x)\sin(nx)dx = 0 \text{ if f is even}$$

$$\int_{-L}^{L} f(x)\cos(nx)dx = 0 \text{ if f is odd}$$

♣ $\int x \sin kx dx$

$$u = x,\ dv = \sin kx dx$$
$$du = dx,\ v = -\frac{1}{k}\cos kx$$

$$= -\frac{1}{k}x\cos kx + \frac{1}{k}\int \cos kx dx$$

$$= -\frac{1}{k}x\cos kx + \frac{1}{k^2}\sin kx$$

♣ $\int x\cos kx\,dx$

$$u = x,\quad dv = \cos kx\,dx$$
$$du = dx,\quad v = \frac{1}{k}\sin kx$$

$$= \frac{1}{k}x\sin kx - \frac{1}{k}\int \sin kx\,dx$$

very often
$$k = \frac{n\pi}{L}$$

$$= \frac{1}{k}x\sin kx + \frac{1}{k^2}\cos kx$$

♣ $$\int x\sin(\lambda x)dx = \frac{\sin(\lambda x)}{\lambda^2} - \frac{x\cos(\lambda x)}{\lambda}$$

♣ $$\int x^2\sin(\lambda x)dx = \frac{2x\sin(\lambda x)}{\lambda^2} + \frac{(2-\lambda^2x^2)\cos(\lambda x)}{\lambda^3}$$

♣ $$\int x\cos(\lambda x)dx = \frac{\cos(\lambda x)}{\lambda^2} + \frac{x\sin(\lambda x)}{\lambda}$$

♣ $$\int x^2\cos(\lambda x)dx = \frac{2x\cos(\lambda \mathrm{x})}{\lambda^2} + \frac{(\lambda^2x^2-2)\sin(\lambda x)}{\lambda^3}$$

(11)

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\sin x \cos x = \frac{1}{2}\sin 2x$$

$$\sin x \sin y = \frac{1}{2}[\cos(x-y)-\cos(x+y)]$$

$$\cos x \cos y = \frac{1}{2}[\cos(x-y)+\cos(x+y)]$$

$$\sin x \cos y = \frac{1}{2}[\sin(x-y)+\sin(x+y)]$$

Recall

The **Taylor series** of f at a is

$$f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

coefficient

$$= \sum_{k=0}^{\infty} \boxed{\frac{f^{(k)}(a)}{k!}} (x-a)^k \qquad (1)$$

Suppose $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

To **determine** a_n and b_n in terms of f

We shall use ***Integrals*** to determine

6.2.1 *Determine* a_0

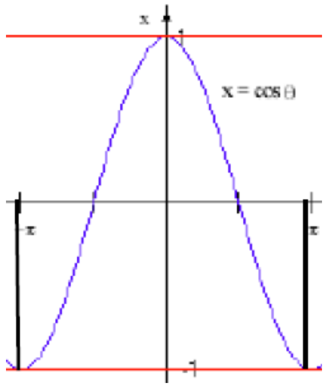
$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

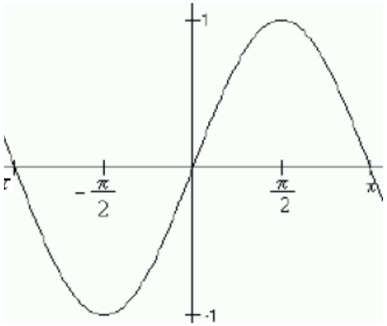
$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}\left(a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right) dx$$

$$= a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos nx\, dx + b_n \int_{-\pi}^{\pi} \sin nx\, dx\right)$$

$$= 2\pi a_0$$

$$\boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx}$$





6.2.2 *Determine* $a_m, m \geq 1$

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)}$$

$$\int_{-\pi}^{\pi} f(x) \cos\, mx\, dx$$

$$= a_0 \int_{-\pi}^{\pi} \cos\, mx\, dx + \sum_{n=1}^{\infty}\left(a_n \boxed{\int_{-\pi}^{\pi} \cos\, nx\, \cos\, mx\, dx} \right.$$

$$\left. + b_n \int_{-\pi}^{\pi} \underbrace{\sin\, nx \cos\, mx}_{\textit{odd}}\, dx \right) \qquad (5)$$

$$\int_{-\pi}^{\pi} \cos\, nx\, \cos\, mx\, dx \qquad \sum_{n=1}^{\infty}\left(a_n \boxed{\int_{-\pi}^{\pi} \cos\, nx\, \cos\, mx\, dx}\right.$$

2cos*A* cos*B* = cos(*A*+*B*) + cos(*A*−*B*)

$$= \frac{1}{2}\int_{-\pi}^{\pi} (\cos(m+n)x + \cos(m-n)x)\, dx$$

$$= \begin{cases} \dfrac{1}{2}\left[\dfrac{\sin(m+n)x}{m+n} + \dfrac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} (m \neq n) \quad = 0 \\ \dfrac{1}{2}\int_{-\pi}^{\pi} (1 + \cos 2mx)\, dx \end{cases}$$

- $$\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

$$= a_0 \int_{-\pi}^{\pi} \cos mx\,dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos nx \cos mx\,dx + b_n \int_{-\pi}^{\pi} \underbrace{\sin nx \cos mx}\,dx\right)$$

$$= a_m \pi$$

$$\boxed{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx,\ m = 1, 2, \cdots}$$

6.2.3 Determine $b_m,\ m \geq 1$

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx$$

$$= a_0 \int_{-\pi}^{\pi} \sin mx \, dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \sin mx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \right) \quad (6)$$

$$= \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx$$

$$\text{Now} \int_{-\pi}^{\pi} \sin nx \ \sin mx \, dx$$

2sin*A* sin*B* = cos(*A*−*B*) − cos(*A*+*B*)

$$= \frac{1}{2}\int_{-\pi}^{\pi}(\cos(n-m)x - \cos(n+m)x)dx$$

$$= \begin{cases} \frac{1}{2}\left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m}\right]_{-\pi}^{\pi} & m \neq n \\ \frac{1}{2m}[mx - \sin mx \ \cos mx]_{-\pi}^{\pi} & m = n \end{cases}$$

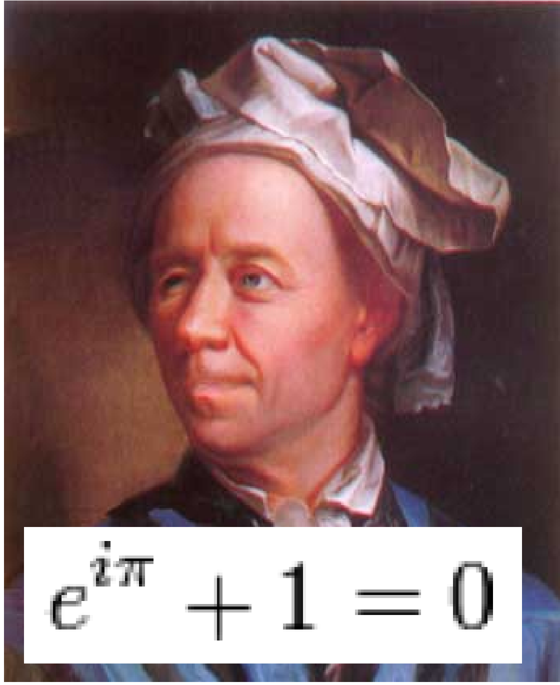
$$= \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

$$\boxed{b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx \, dx, \ m = 1, 2, \cdots .}$$

6.2.4 *Euler* Formulas

- Let ***f*** be a ***periodic*** function of ***period*** 2π (from −π to π) with ***Fourier*** series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$



(1707 – 1783)

Then

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \;\; n = 1, 2, \cdots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx, \;\; n = 1, 2, \cdots \end{aligned}\right\} \quad (7)$$

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx,\ \ n = 1, 2, \cdots \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx,\ \ n = 1, 2, \cdots
\end{aligned}
$$

$a_n = 0$ for all n, if f is odd

$b_n = 0$ for all n ,if f is even

6.2.5 ***Representation*** by ***Fourier*** series

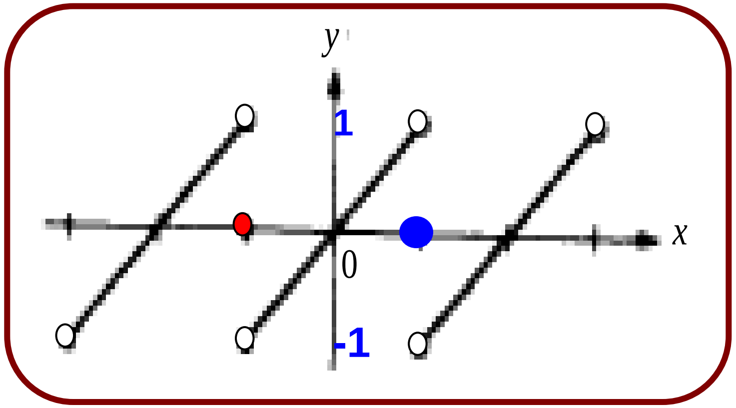
- A ***piecewise continuous*** fn on [a, b] is a fn which is ***continuous*** **except** at a ***finite*** number of points where it has ***jumps*** (one-side limits exist from each side).

Let f be a fn *s.t.* f & f' are ***piecewise continuous*** on [-π, π]. Then $\boxed{f(x) \mathbf{?}\ a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)}$

(1) at any point x where f is ***continuous***, $f(x)$ equals to its ***Fourier*** series;

(2) at c where f is ***discontinuous***, the ***Fourier*** series ***converges*** to $\frac{1}{2}(f(c^+) + f(c^-))$

where $f(c^+)$ & $f(c^-)$ are respectively the ***RH*** & ***LH*** limits of f at c



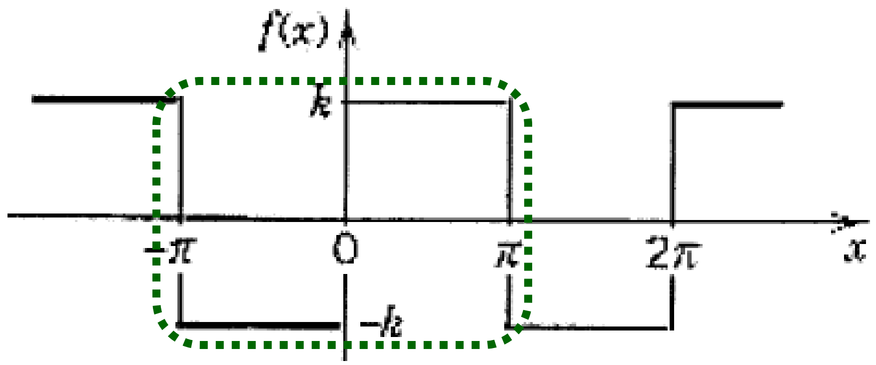
6.2.6 *Example*

♣

A ***square wave*** is a fn f defined by

$$f(x) = \begin{cases} -k, & \text{if } -\pi < x < 0 \\ k, & \text{if } 0 < x < \pi \end{cases}$$

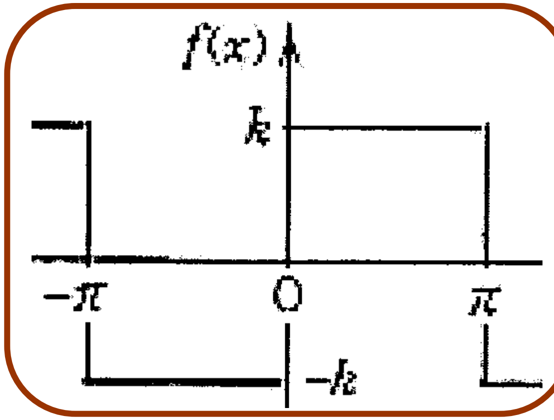
and $f(x) = f(x + 2\pi)$.



f is ***odd*** !

Find the Fourier series of $f(x)$

- By ***Euler*** formulas,



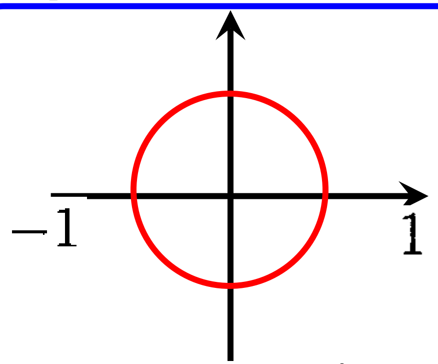
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ n = 1, 2, \cdots = 0$$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx} = \frac{2}{\pi}\int_{0}^{\pi} k\ \sin nx\,dx$$

$$= \frac{2k}{n\pi}(1 - \cos n\pi)$$

$$= \frac{2k}{n\pi}(1 - (-1)^n).$$



$$\cos n\pi = (-1)^n$$

$$\boxed{b_n = \frac{2k}{n\pi}(1-(-1)^n).}$$

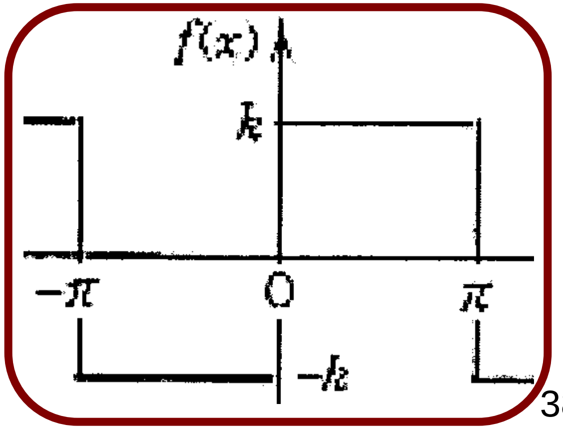
$$b_1 = \frac{4k}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{4k}{3\pi}, \quad b_4 = 0; \quad b_5 = \frac{4k}{5\pi}, \cdots .$$

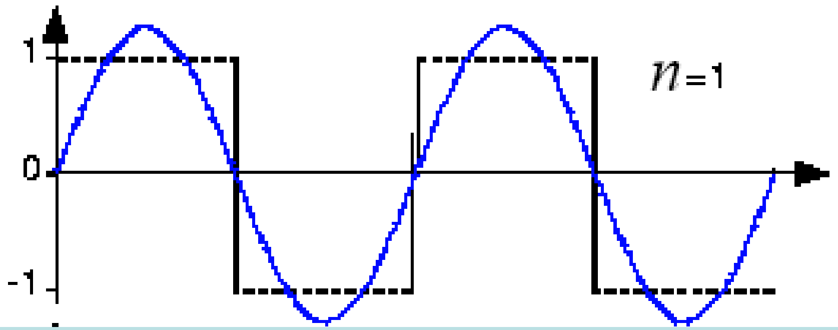
$$\boxed{f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).}$$

Note that at all the points of discontinuity $(0, \pi, etc)$

$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right). = 0$$

$$\frac{1}{2}(f(c^+) + f(c^-)) = 0$$




1
0
-1
t
n=1

6.2.7 *An approximation for π*

$$f(x) = \frac{4k}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots).$$

Let $x = \dfrac{\pi}{2}$ we get

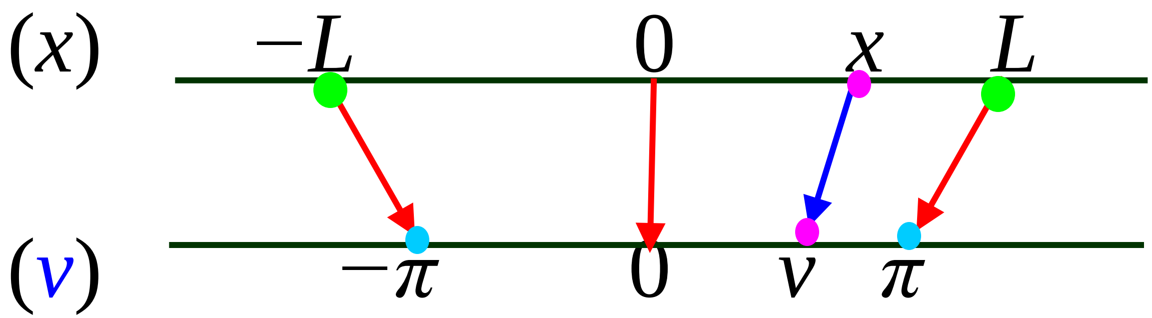
$$f(\pi/2) = \frac{4k}{\pi}(\sin(\pi/2) + \frac{1}{3}\sin(3(\pi/2) + \ldots)$$

Hence $k = \dfrac{4k}{\pi}(1 - \dfrac{1}{3} + \dfrac{1}{5} - \ldots)$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$$

6.2.8 *Periodic* functions of *period* 2*L*

f **(*x*):** a ***periodic*** fn of period $2L$ (from $-L$ to L).



Set $\boxed{\dfrac{x}{L} = \dfrac{v}{\pi}}$ or $\boxed{v = \dfrac{\pi x}{L}}$ & $\boxed{g(v) = f(x).}$

Then ***g*** is a ***periodic*** fn of period $\mathbf{2\pi}$.

$g(v)=f(x)$ & g: *periodic* from $-\pi$ to π

$$g(v) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nv + b_n \sin nv)$$

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(v)dv \\ &= \frac{1}{2\pi}\int_{-L}^{L} g(v)\frac{\pi}{L}dx \\ &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \end{aligned}$$

$$v = \frac{\pi x}{L}$$

For $n \geq 1$,
$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\cos nv\, dv \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx. \end{aligned}$$

$$v = \frac{\pi x}{L}$$

$$g(v) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nv + b_n \sin nv)$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

with a_0, a_n and b_n as given above.

f* (*x*)**: a ***periodic fn of period $2L$ (from $-L$ to L).

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

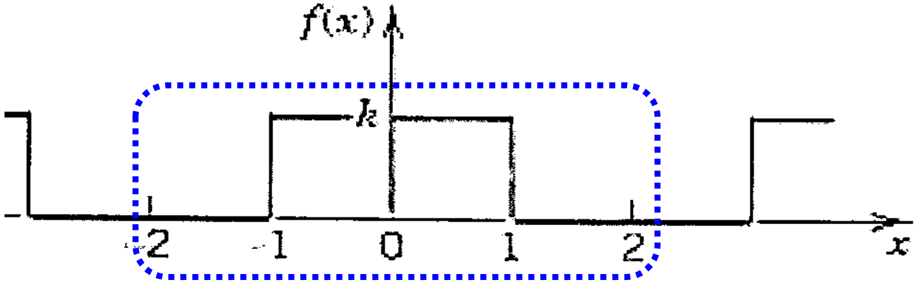
$$n = 1, 2, \cdots .$$

$$a_n = 0 \text{ for all n, if f is odd}$$

$$b_n = 0 \text{ for all n ,if f is even}$$

6.2.9 *Example*

♣ Let $$f(x) = \begin{cases} 0, & \text{if } -2 < x < -1 \\ k, & \text{if } -1 < x < 1 \\ 0, & \text{if } 1 < x < 2 \end{cases}$$



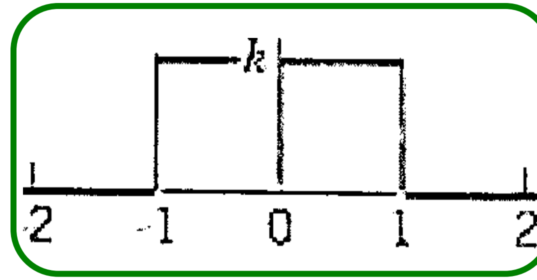
period $p = 2L = 4$

f is even,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

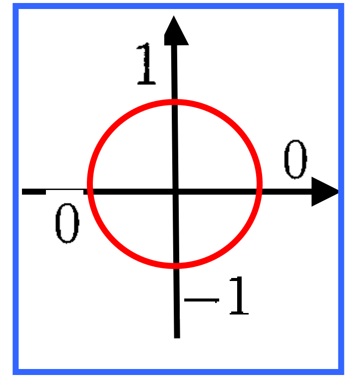
$$= 0 \quad \text{for} \quad n = 1, 2, \cdots .$$

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)dx = \frac{1}{4}\int_{-1}^{1} k dx = \frac{k}{2}$$



$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx$$

$$= \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}dx = \frac{2k}{n\pi}\boxed{\sin\frac{n\pi}{2}}$$



$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}}$$

$$\sin\frac{n\pi}{2} = \begin{cases} 0 & n \text{ even} \\ 1 & n = 1,5,9,\cdots \\ -1 & n = 3,7,11,\cdots \end{cases}$$

$$= \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - \cdots\right).$$

Euler Formulas

(1707 – 1783)

- Let ***f*** be a ***periodic*** function of ***period*** 2π (from −π to π) with ***Fourier*** series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Then

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx,\ \ n = 1, 2, \cdots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx,\ \ n = 1, 2, \cdots \end{aligned}\right\} \quad (7)$$

f*(*x*)**: a ***periodic fn of period $\boxed{2L}$ (from $-L$ to L).

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos \boxed{\frac{n\pi}{L}} x + b_n \sin \boxed{\frac{n\pi}{L}} x\right)$$

$$\boxed{a_0} = \frac{1}{2\boxed{L}}\int_{\boxed{-L}}^{\boxed{L}} f(x)\,dx$$

$$\boxed{\pi \longrightarrow L}$$

$$\boxed{a_n} = \frac{1}{L}\int_{-L}^{L} f(x)\cos \boxed{\frac{n\pi x}{L}}\,dx$$

$$\boxed{b_n} = \frac{1}{L}\int_{-L}^{L} f(x)\sin \boxed{\frac{n\pi x}{L}}\,dx$$

$$n = 1, 2, \cdots.$$

6.2.10 Fourier cosine & sine series

Fourier cosine series

f : *periodic*, period **2*L*** (from ***–L*** to ***L***)

☺ ***f even*** then $b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\frac{n\pi x}{L}}_{odd} dx. = 0$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L},$$

with

$$a_0 = \frac{1}{L}\int_0^L f(x)dx,$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \; n = 1, 2, \cdots .$$

If ***f*(*x*)** = ***c*** (a ***constant*** fn), the ***Fourier series*** of '***f***' is '***c***'.

Fourier sine series

☺ ***f odd*** then $a_n = \frac{1}{L}\int_{-L}^{L} \underline{f(x)}\, \underline{\cos\frac{n\pi x}{L}}\, dx = 0$

odd

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$$

with $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$

6.2.11 *Sum* & *Scalar Multiplication*

The Fourier coefficients of $f_1 + f_2$ are the sums of corresponding Fourier coefficients of f_1 and f_2.

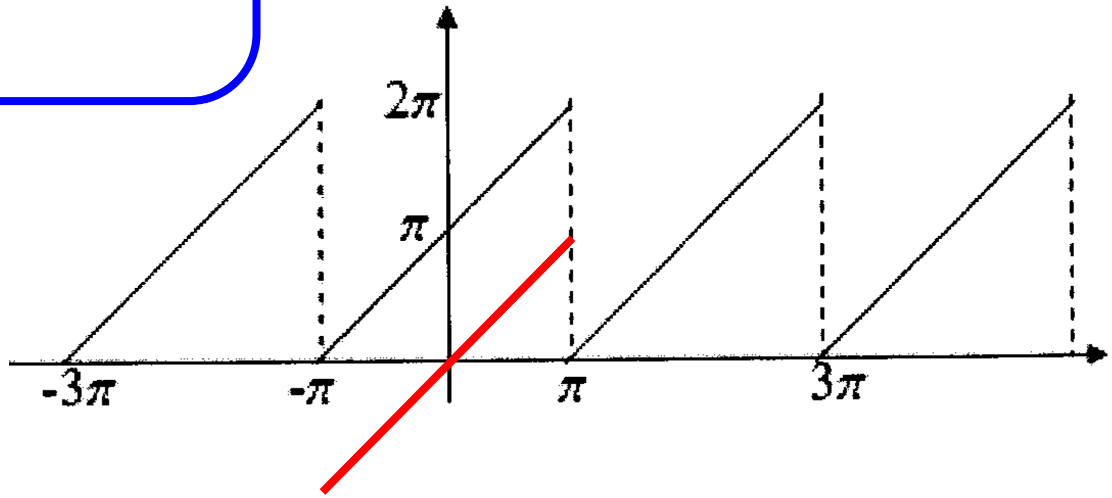
The Fourier coefficients of cf (c a constant) are c times the corresponding Fourier coefficients of f.

Example

- ***Saw tooth function*** :

$f(x) = x + \pi\,, \quad -\pi < x < \pi$

$f(x) = f(x + 2\pi)$



$$f = f_1 + f_2, \text{ where } f_1 = x, \;\; f_2 = \pi$$

♣ $\int x \sin kx\,dx$

$$\boxed{\begin{aligned} &u = x,\ dv = \sin kx\,dx \\ &du = dx,\ v = -\frac{1}{k}\cos kx \end{aligned}}$$

$$= -\frac{1}{k}x\cos kx + \frac{1}{k}\int \cos kx\,dx$$

$$= -\frac{1}{k}x\cos kx + \frac{1}{k^2}\sin kx$$

$$f_1(x) = x \quad (\textbf{\textit{odd}})$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi} \left\{ \left[x \left(\frac{-\cos nx}{n} \right) \right]_0^{\pi} - \int_0^{\pi} \frac{-\cos nx}{n} dx \right\}$$

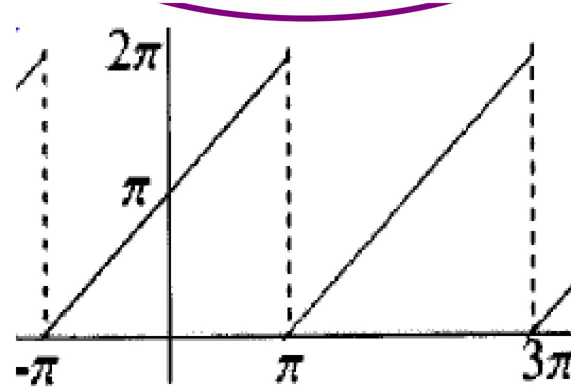
$$= \frac{(-1)^{n+1} 2}{n}$$

$$f(x) = f_1(x) + f_2(x)$$

$$= \pi + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

Check $x = \pi$ **!**

$$\frac{1}{2}\left(f(c^+) + f(c^-) \right)$$



♣ $$b_n = \frac{2}{\pi}\int_0^\pi x \sin nx \, dx$$
$$= \frac{2}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^\pi - \int_0^\pi \frac{-\cos nx}{n} dx\right\}$$

- $\int_0^\pi \frac{\cos nx}{n} dx = \frac{1}{n}\frac{\sin nx}{n}\Big|_0^\pi = 0.$

$\cos n\pi = (-1)^n$

- $\left[x\frac{-\cos nx}{n}\right]_0^\pi = \frac{-\pi}{n}\cos n\pi = \frac{-\pi}{n}(-1)^n = \frac{\pi}{n}(-1)^{n+1}$

- Thus, $b_n = \frac{2}{\pi}\cdot\frac{\pi}{n}(-1)^{n+1} = \frac{2}{n}(-1)^{n+1}.$

6.3. *Half-range Expansions*

6.3.1 & 6.3.2 *Extension of f(x)*

Given ***f*** as shown below:

To ***expand*** it in
a ***Fourier series***.

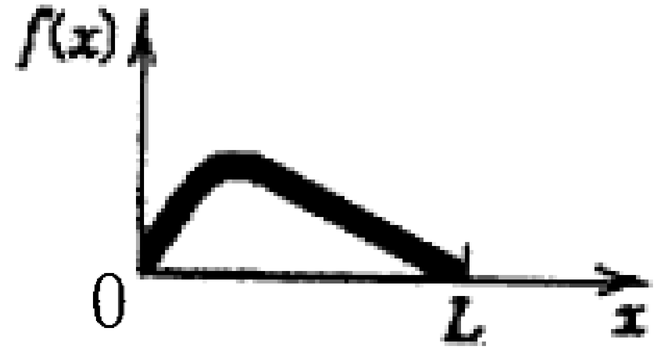


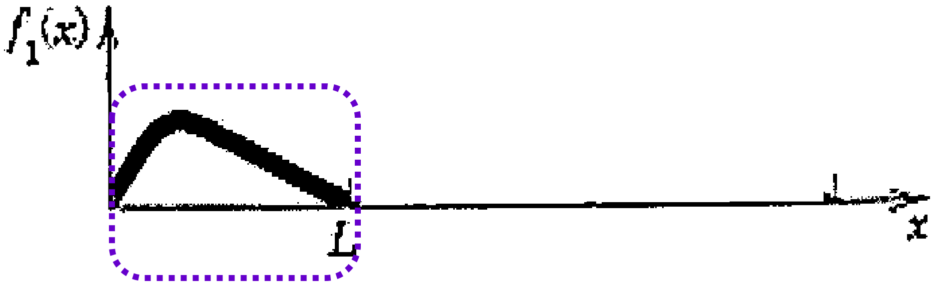
Figure (a)

Extend ***f*** to $[-L, L]$ *s.t.*

(1) ***f*** is ***even*** on $[-L, L]$ or

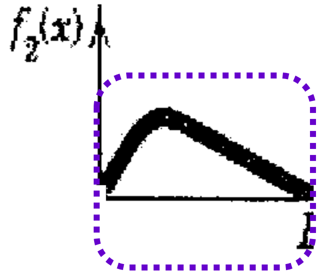
(2) ***f*** is ***odd*** on $[-L, L]$

(1) ***f*** is ***even*** on $[-L, L]$



Represent it by ***Fourier cosine series***.

(2) ***f*** is ***odd*** on $[-L, L]$



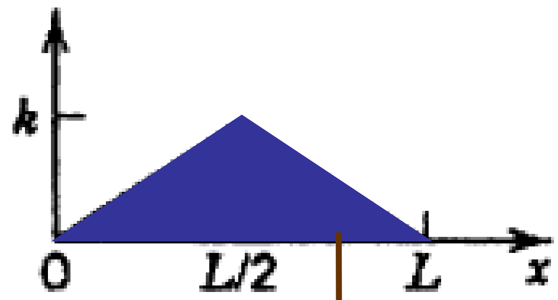
Represent it by ***Fourier sine series***.

6.3.3 *Example (self study). Give another example*

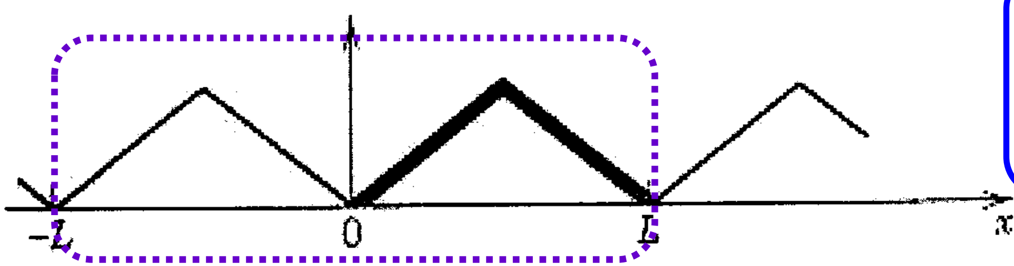
♣ ‘***Triangle***’ function

Find the two half range expansions for

$$f(x) = \begin{cases} \frac{2}{L} kx, & 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L - x), & \frac{L}{2} < x < L. \end{cases}$$



(1) *Cosine half-range expansion*



$$a_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$a_0 = \frac{1}{L} \left\{ \int_0^{L/2} \frac{2k}{L} x dx + \int_{L/2}^{L} \frac{2k}{L}(L - x) dx \right\}$$

♣ $\int x \cos kx dx$

$$u = x, \quad dv = \cos kx dx$$
$$du = dx, \quad v = \frac{1}{k}\sin kx$$

$$= \frac{1}{k}x\sin kx - \frac{1}{k}\int \sin kx dx$$

$$= \frac{1}{k}x\sin kx + \frac{1}{k^2}\cos kx$$

$$a_n = \frac{2}{L}\left\{\int_0^{L/2} \frac{2k}{L} x \cos\frac{n\pi x}{L}dx + \int_{L/2}^{L} \frac{2k}{L}(L-x)\cos\frac{n\pi x}{L}dx\right\}$$

$$= \frac{4k}{L^2}\left\{\boxed{\int_0^{L/2} x\cos\frac{n\pi x}{L}dx} + \boxed{\int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}dx}\right\}$$

$$\boxed{\frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)}$$

by parts

$$\boxed{-\frac{L^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)}$$

Thus $a_n = \dfrac{4k}{n^2\pi^2}\left(2\cos\dfrac{n\pi}{2} - \cos n\pi - 1\right)$

Check: $a_2 = \dfrac{-16k}{2^2\pi^2}, \quad a_6 = \dfrac{-16k}{6^2\pi^2}, \quad a_{10} = \dfrac{-16k}{10^2\pi^2}, \quad , \cdots$

and $a_n = 0$ if $n \geq 1$ and $n \neq 2, 6, 10, \cdots$.

The ***cosine half-range expansion*** of ***f*** is

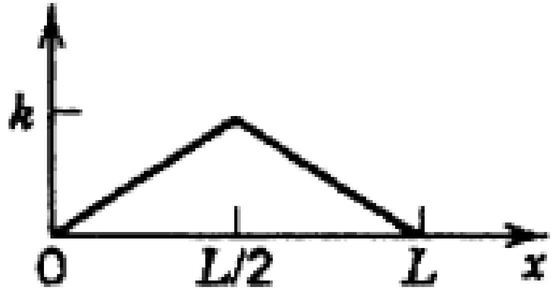
$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}}$$

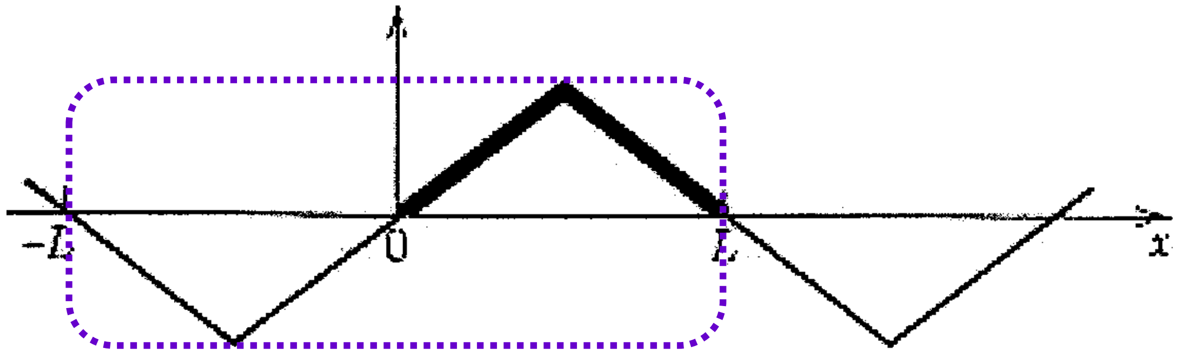
$$\frac{k}{2} - \frac{16k}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(4m-2)^2}\cos\frac{(4m-2)\pi x}{L}$$

$$= \frac{k}{2} - \frac{4k}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\frac{2(2m-1)\pi x}{L}$$

(2) *Sine half-range expansion*

$$f(x)=\begin{cases}\frac{2}{L}kx, & 0<x<\frac{L}{2}\\[2ex] \frac{2k}{L}(L-x), & \frac{L}{2}<x<L\end{cases}$$





$$b_n=\frac{2}{L}\left\{\int_0^{L/2}\underline{\frac{2k}{L}}x\sin\frac{n\pi x}{L}dx+\int_{L/2}^{L}\underline{\frac{2k}{L}}(L-x)\sin\frac{n\pi x}{L}dx\right\}$$

$$b_n = \frac{4k}{L^2}\left\{\boxed{\int_0^{L/2} x \sin\frac{n\pi x}{L}dx} + \boxed{\int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}dx}\right\}$$

$$\boxed{-\cancel{\frac{L^2}{2n\pi}\cos\frac{n\pi}{2}} + \frac{L^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)}$$

$$\boxed{\cancel{\frac{L^2}{2n\pi}\cos\frac{n\pi}{2}} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}}$$

$$\boxed{b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}}$$

$$\boxed{\sin\frac{n\pi}{2} = \begin{cases} 0 & n \text{ even} \\ 1 & n = 1,5,9,\cdots \\ -1 & n = 3,7,11,\cdots \end{cases}}$$

The ***sine half-range expansion*** is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\boxed{\sin\frac{n\pi}{2}}\sin\frac{n\pi x}{L}$$

$$= \frac{8k}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^2}\sin\frac{(2m-1)\pi x}{L}$$

Write $n = 2m - 1$, the above is $(-1)^{m-1}$

Appendix
Sum & Scalar Multiplication

The Fourier coefficients of $f_1 + f_2$ are the sums of corresponding Fourier coefficients of f_1 and f_2.

The Fourier coefficients of cf (c a constant) are c times the corresponding Fourier coefficients of f.

$$f_1(x) = x \quad (\textbf{\textit{odd}})$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi} \frac{-\cos nx}{n} dx\right\}$$

$$= \frac{(-1)^{n+1}2}{n}$$

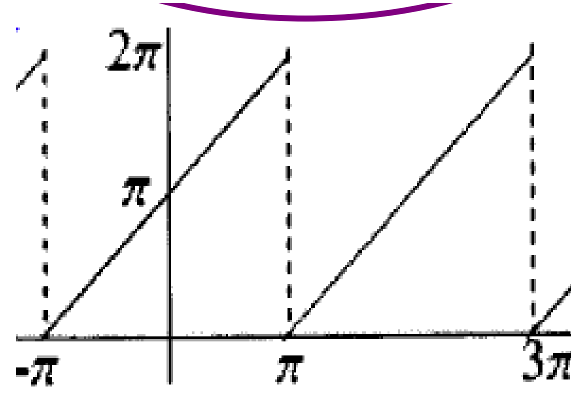
$$f_2(x) = \pi$$

$$f(x) = f_1(x) + f_2(x)$$

$$= \pi + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

Check $x = \pi$!

$$\frac{1}{2}\left(f(c^+) + f(c^-)\right)$$



Let $f(x) = 2x + 1$ for all $x \in (-\pi, \pi)$ and $f(x) = f(x + 2\pi)$. Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$ **(2005)**

be the Fourier Series which represents $f(x)$. Find the value of $a_0 + a_5 + b_5$.

Note that $x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$

Hence $f(x) = 2x + 1$

$$= 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{n} \sin nx$$

and so $a_0 + a_5 + b_5 = 1 + 0 + (-1)^6 \frac{4}{5} = \frac{9}{5}$.

Let

$$f(x) = x^2\sqrt{\pi^2 - x^2}, \quad -\pi \le x \le \pi,$$

and $f(x + 2\pi) = f(x)$ for all x. Let

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

(2008)

be the Fourier Series which represents $f(x)$. Find the **exact value** of $b_2 + b_3 + \sum_{n=1}^{\infty} a_n$.

Since f is even, $b_n = 0$ for each $n = 1, 2, 3, \cdots$.

By assumption,

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = f(x) = x^2\sqrt{\pi^2 - x^2}$$

Putting $x = 0$, we have $a_0 + \sum_{n=1}^{\infty} a_n = f(0) = 0$.

That is, $\sum_{n=1}^{\infty} a_n = -a_0$.

Now,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \quad = \frac{1}{\pi}\int_0^{\pi} x^2\sqrt{\pi^2 - x^2}\,dx$$

(let $x = \pi\sin\theta$)

$$= \frac{1}{\pi}\int_0^{\pi/2} (\pi^2\sin^2\theta)(\pi\cos\theta)(\pi\cos\theta d\theta)$$

$$= \frac{\pi^3}{4}\int_0^{\pi/2} \sin^2 2\theta d\theta \quad = \frac{\pi^3}{8}\int_0^{\pi/2}(1-\cos 4\theta)d\theta \quad = \frac{\pi^4}{16} .$$

Thus, $b_2 + b_3 + \sum_{n=1}^{\infty} a_n = -\frac{\pi^4}{16}$

Fourier sine series

☺ ***f*** ***odd*** then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

$$\text{with} \quad b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx.$$

♣ **(2007)**

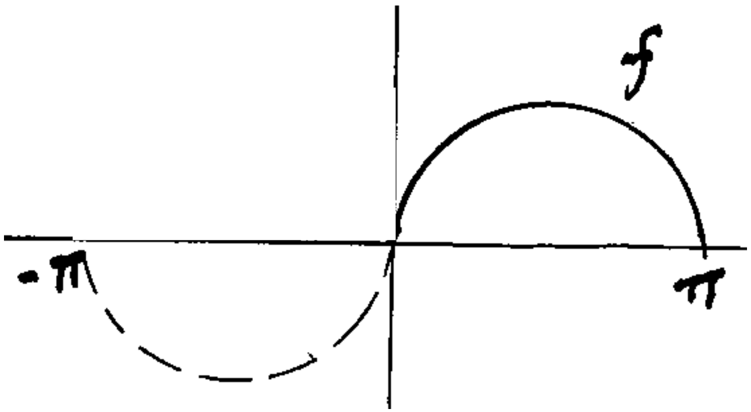
Let $f(x) = |\sin x|$ for all $x \in (-\pi, \pi)$, and $f(x+2\pi) = f(x)$ for all x. Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier Series which represents $f(x)$. Let m denote a fixed positive integer. Find $a_0 + a_2 + a_{2m+1} + b_m$.

As f is even,

$$b_n = 0 \quad \forall\, n = 1, 2, \ldots$$



Observe that

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{2}{\pi},$$

- $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx$$
  $(n \geq 2)$

$$\boxed{\begin{aligned} &2\cos A \sin B \\ &= \sin(A+B) - \sin(A-B) \end{aligned}}$$

$$= \frac{1}{\pi}\int_0^{\pi} \left\{ \sin(n+1)x - \sin(n-1)x \right\} dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right]_0^{\pi}$$

$$= \frac{1}{\pi}\left\{ \frac{(-1)^{n+2}}{n+1} + \frac{1}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n-1} \right\}$$

thus $\quad a_{2m+1} = 0 \quad \forall\, m = 1, 2, \ldots$

$$a_2 = \frac{1}{\pi}\left(\frac{2}{3} - 2 \right) = -\frac{4}{3\pi}$$

Hence $\quad a_0 + a_2 + a_{2m+1} + b_m = \frac{2}{\pi} - \frac{4}{3\pi} = \frac{2}{3\pi}$

♣ (2005)

Let $f(x) = 2x + 1$ for all $x \in (-\pi, \pi)$ and $f(x) = f(x + 2\pi)$. Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier Series which represents $f(x)$. Find the value of $a_0 + a_5 + b_5$.

Let $g(x) = x$. **(odd)** Thus $g(x) = \sum_{n=1}^{\infty} c_n \sin nx$,

where $c_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx = (-1)^{n+1} \frac{2}{n}$.

Hence $f(x) = 2x + 1$

$$= 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{n} \sin nx$$

and so $a_0 + a_5 + b_5 = 1 + 0 + (-1)^6 \frac{4}{5} = \boxed{\frac{9}{5}}$.