MA 1505 Mathematics I

Tutorial 2 Solutions

1. (a) $$\lim_{x\to\pi/2}\frac{1-\sin x}{1+\cos 2x}=\lim_{x\to\pi/2}\frac{-\cos x}{-2\sin 2x}=\lim_{x\to\pi/2}\frac{\sin x}{-4\cos 2x}=\frac{1}{4}.$$

(b) $$\lim_{x\to 0}\frac{\ln(\cos ax)}{\ln(\cos bx)}=\lim_{x\to 0}\frac{\dfrac{-a\sin ax}{\cos ax}}{\dfrac{-b\sin bx}{\cos bx}}=\lim_{x\to 0}\frac{a\sin ax\ \cos bx}{b\sin bx\ \cos ax}=\frac{a^2}{b^2}.$$

(c) $$\lim_{x\to\infty}x\tan\frac{1}{x}=\lim_{x\to\infty}\frac{\tan(x^{-1})}{x^{-1}}=\lim_{x\to\infty}\frac{-x^{-2}\sec^2(x^{-1})}{-x^{-2}}=\lim_{x\to\infty}\cos^{-2}(x^{-1})=1.$$

(d) $$\lim_{x\to 0+}x^a\ln x=\lim_{x\to 0+}\frac{\ln x}{x^{-a}}=\lim_{x\to 0+}\frac{\dfrac{1}{x}}{-ax^{-a-1}}=\lim_{x\to 0+}\frac{x^a}{-a}=0.$$

(e) $\displaystyle\lim_{x\to 1}\ln x^{\frac{1}{1-x}}=\lim_{x\to 1}\frac{\ln x}{1-x}=\lim_{x\to 1}\frac{\frac{1}{x}}{-1}=-1.$ So $\lim_{x\to 1}x^{\frac{1}{1-x}}=e^{-1}$.

(f) Using (1d) we have

$$\lim_{x\to 0+}\ln x^{\sin x}=\lim_{x\to 0+}\sin x\ln x=\lim_{x\to 0+}\frac{\sin x}{x}\cdot x\ln x=\lim_{x\to 0+}\frac{\sin x}{x}\lim_{x\to 0+}x\ln x=0.$$

So $\displaystyle\lim_{x\to 0+}x^{\sin x}=e^0=1.$

(g) $$\lim_{x\to 0}\ln\left[\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}\right]=\lim_{x\to 0}\frac{\ln(\frac{\sin x}{x})}{x^2}=\lim_{x\to 0}\frac{(\frac{x}{\sin x})\cdot\frac{x\cos x-\sin x}{x^2}}{2x}$$
$$=\frac{1}{2}\lim_{x\to 0}\frac{x}{\sin x}\lim_{x\to 0}\frac{x\cos x-\sin x}{x^3}=\frac{1}{2}\lim_{x\to 0}\frac{\cos x-x\sin x-\cos x}{3x^2}$$
$$=-\frac{1}{6}\lim_{x\to 0}\frac{\sin x}{x}=-\frac{1}{6}.$$

So $\displaystyle\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}=e^{-1/6}.$

2. (a)

$$\int_1^{\sqrt{2}}\frac{s^2+\sqrt{s}}{s^2}\,ds=\int_1^{\sqrt{2}}(1+s^{-3/2})\,ds=(\sqrt{2}-1)-2s^{-1/2}\Big|_1^{\sqrt{2}}$$
$$=(\sqrt{2}-1)-\frac{2}{\sqrt[4]{2}}+2=1+\sqrt{2}-2^{3/4}.$$

(b)

$$\int_{-4}^{4}|x|\,dx=\int_0^4 x\,dx+\int_{-4}^{0}(-x)\,dx=\frac{1}{2}4^2+\frac{1}{2}4^2=16.$$

(c)

$$\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|)\,dx = \int_0^{\pi/2} \frac{1}{2}(\cos x + |\cos x|)\,dx + \int_{\pi/2}^{\pi} \frac{1}{2}(\cos x + |\cos x|)\,dx$$
$$= \int_0^{\pi/2} \cos x\,dx + 0 = \sin x\Big|_0^{\pi/2} = 1.$$

(d)

$$\int_0^{\pi} \sin^2\Big(1+\frac{\theta}{2}\Big)\,d\theta = \int_0^{\pi} \frac{1}{2}\big[1-\cos(2+\theta)\big]\,d\theta = \frac{1}{2}\pi - \frac{1}{2}\sin(2+\theta)\Big|_0^{\pi}$$
$$= \frac{1}{2}\pi - \frac{1}{2}\big[\sin(2+\pi) - \sin 2\big] = \frac{1}{2}\pi + \sin 2.$$

3. The Fundamental Theorem of Calculus (I) says that

$$\frac{d}{du}\int_a^u f(t)\,dt = f(u)$$

for a continuous function f. Here a is a fixed number. It is a sort of *chain rule* to find

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt\,.$$

To see this, let

$$F(u) = \int_a^u f(t)\,dt \qquad \text{and} \quad u = g(x)\,.$$

It follows that

$$\frac{dF}{du} = \frac{d}{du}\int_a^u f(t)\,dt = f(u)\,.$$

Furthermore,

$$F\circ g(x) = F(g(x)) = \int_a^{g(x)} f(t)\,dt\,.$$

By the chain rule, we have

$$\frac{dF(g(x))}{dx} = \frac{dF}{du}\frac{dg(x)}{dx} = f(u)\,g'(x) = f(g(x))\,g'(x)\,.$$

(a) $y = \displaystyle\int_0^{\sqrt{x}} \cos t\,dt;\quad \cos\sqrt{x}\cdot\frac{d}{dx}\sqrt{x} = \frac{\cos\sqrt{x}}{2\sqrt{x}}.$

(b) $y = \displaystyle\int_0^{x^2} \cos\sqrt{t}\,dt;\quad \cos\sqrt{x^2}\cdot 2x = 2x\cos|x| = 2x\cos x.$

(c) $y = \displaystyle\int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}},\quad |x| < \frac{\pi}{2}.\quad \frac{1}{\sqrt{1-\sin^2 x}}\cdot\frac{d}{dx}\sin x = \frac{1}{\cos x}\cos x = 1.$

4. (a) $\displaystyle\int x^{1/2}\sin(x^{3/2}+1)\,dx = \int \sin(x^{3/2}+1)\cdot\frac{2}{3}\,d(x^{3/2}+1) = -\frac{2}{3}\cos(x^{3/2}+1)+C.$

(b) $\displaystyle\int \csc^2 2t\,\cot 2t\,dt = \int \cot 2t\cdot\Big(-\frac{1}{2}\Big)d(\cot 2t) = -\frac{1}{4}\cot^2 2t + C.$

(c) $\displaystyle\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta = \int \sin\frac{1}{\theta}\cos\frac{1}{\theta}\cdot(-1)\,d\Big(\frac{1}{\theta}\Big) = -\int \sin\frac{1}{\theta}\,d\Big(\sin\frac{1}{\theta}\Big) = -\frac{1}{2}\sin^2\frac{1}{\theta}+C.$

(d) $\displaystyle\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)}\,dx = \int \frac{18\tan^2 x\,d(\tan x)}{(2+\tan^3 x)} = \int \frac{6\,d(\tan^3 x+2)}{(2+\tan^3 x)} = 6\,\ln|\tan^3 x+2|+C.$

(e) $\displaystyle\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\cos^3\sqrt{\theta}}\,d\theta = -2\int(\cos\sqrt{\theta})^{-3}\,d(\cos\sqrt{\theta}) = (\cos\sqrt{\theta})^{-2}+C = \sec^2\sqrt{\theta}+C.$

5. (a)

$$\begin{aligned}
\int x\sin\Big(\frac{x}{2}\Big)\,dx &= -2\int x\,d\Big[\cos\Big(\frac{x}{2}\Big)\Big] = -2\left[x\cos\Big(\frac{x}{2}\Big) - \int\cos\Big(\frac{x}{2}\Big)\,dx\right]+C\\
&= -2\left[x\cos\Big(\frac{x}{2}\Big) - 2\int\cos\Big(\frac{x}{2}\Big)\,d\Big(\frac{x}{2}\Big)\right]+C\\
&= -2\left[x\cos\Big(\frac{x}{2}\Big) - 2\sin\Big(\frac{x}{2}\Big)\right]+C\,.
\end{aligned}$$

(b)

$$\begin{aligned}
\int t^2e^{4t}\,dt &= \frac{1}{4}\int t^2\,d(e^{4t}) = \frac{1}{4}\left[t^2e^{4t} - 2\int t\,e^{4t}\,dt\right]+C = \frac{1}{4}\left[t^2e^{4t} - \frac{1}{2}\int t\,d(e^{4t})\right]+C\\
&= \frac{1}{4}\left[t^2e^{4t} - \frac{1}{2}\left(t\,e^{4t} - \int e^{4t}\,dt\right)\right]+C\\
&= \frac{1}{4}\left[t^2e^{4t} - \frac{1}{2}\left(t\,e^{4t} - \frac{e^{4t}}{4}\right)\right]+C\quad\text{(continue to simplify)}\,.
\end{aligned}$$

(c)

$$\begin{aligned}
\int e^{-y}\cos y\,dy &= \int e^{-y}\,d(\sin y) = e^{-y}\sin y + \int e^{-y}\sin y\,dy + C\\
&= e^{-y}\sin y - \int e^{-y}\,d(\cos y) + C = e^{-y}\sin y - e^{-y}\cos y - \int e^{-y}\cos y\,dy\\
\Rightarrow\quad \int e^{-y}\cos y\,dy &= \frac{e^{-y}}{2}(\sin y - \cos y) + C\,.
\end{aligned}$$

(There is no harm to rename $C/2$ as C.)

(d)

$$\begin{aligned}
\int\theta^2\sin(2\theta)\,d\theta &= -\frac{1}{2}\int\theta^2 d[\cos(2\theta)] = -\frac{1}{2}\left[\theta^2\cos(2\theta) - 2\int\theta\cos(2\theta)\,d\theta\right]+C\\
&= -\frac{1}{2}\left[\theta^2\cos(2\theta) - \int\theta\,d[\sin(2\theta)]\right]+C\\
&= -\frac{1}{2}\left[\theta^2\cos(2\theta) - \theta\sin(2\theta) + \int\sin(2\theta)\,d\theta\right]+C\\
&= -\frac{1}{2}\left[\theta^2\cos(2\theta) - \theta\sin(2\theta) - \frac{1}{2}\cos(2\theta)\right]+C\,.
\end{aligned}$$

(e)

$$\begin{aligned}\int z(\ln z)^2\,dz = \frac{1}{2}\int (\ln z)^2\,d(z^2) &= \frac{1}{2}\left[z^2(\ln z)^2 - 2\int z(\ln z)\,dz\right] + C\\
&= \frac{1}{2}\left[z^2(\ln z)^2 - \int (\ln z)\,d(z^2)\right] + C\\
&= \frac{1}{2}\left[z^2(\ln z)^2 - z^2(\ln z) + \int z\,dz\right] + C\\
&= \frac{1}{2}\left[z^2(\ln z)^2 - z^2(\ln z) + \frac{z^2}{2}\right] + C\,.\end{aligned}$$

(f)

$$\begin{aligned}\int \sin e^{-x}\,dz &= \int e^x\,d(\cos e^{-x})\\
&= e^x\cos e^{-x} - \int e^x \cos e^{-x}\,dx + C\,.\end{aligned}$$

6. (a) Observe that $\sec^2 x > 0$ and $-4\sin^2 x \le 0$ on $[-\pi/3,\ \pi/3]$.

$$\begin{aligned}\text{Area } &= \int_{-\pi/3}^{\pi/3}\left[\frac{1}{2}\sec^2 x - (-4\sin^2 x)\right]dx\\
&= \left[\frac{1}{2}\tan x + \int (2 - 2\cos 2x)\,dx\right]_{-\pi/3}^{\pi/3}\\
&= \tan\frac{\pi}{3} + (2x - \sin 2x)\Big|_{-\pi/3}^{\pi/3}\\
&= \sqrt{3} + \frac{4}{3}\pi - 2\sin\frac{\pi}{3} = \frac{4}{3}\pi.\end{aligned}$$

(b) The points of intersection: $x = x^2/4$ implies $x = 0$ or $x = 4$. Hence the points of intersection are $(0,0)$ and $(4,4)$.

Note that $y = x^2/4 \Leftrightarrow x = 2\sqrt{y}$.

$$\text{The required area } = \int_0^1 \left[2\sqrt{y} - (y)\right]dy = \left[\frac{4}{3}y^{3/2} - \frac{1}{2}y^2\right]_0^1 = \frac{4}{3} - \frac{1}{2} = \frac{5}{6}.$$

(c) We have that $(2-x)-(4-x^2) = x^2 - x - 2 = (x+1)(x-2)$
is negative if and only if $x \in (-1, 2)$.

Hence

$$\begin{aligned}
\text{Area} &= \int_{-2}^{3} \left|(2-x)-(4-x^2)\right| dx \\
&= \Big[\int_{-2}^{-1} + \int_{2}^{3}\Big](x^2-x-2)\,dx + \int_{-1}^{2} -(x^2-x-2)\,dx \\
&= \Big[\int_{-2}^{3} - 2\int_{-1}^{2}\Big](x^2-x-2)\,dx \\
&= \Big[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\Big]_{-2}^{3} - 2\Big[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\Big]_{-1}^{2} \\
&= \frac{1}{3}\big[(27+8) - 2(8+1)\big] - \frac{1}{2}\big[(9-4) - 2(4-1)\big] - 2\big[5 - 2(3)\big] \\
&= \frac{1}{3}17 + \frac{1}{2} + 2 = \frac{49}{6}.
\end{aligned}$$

7. Since $x > 0$, therefore we can let $y = \ln x$. That is $x = e^y$.

We have

$\int \frac{1}{x^7+x} dx = \int \frac{e^y dy}{e^{7y}+e^y}$
$= \int \frac{dy}{e^{6y}+1}$
$= \int \frac{e^{-6y} dy}{1+e^{-6y}}$
$= -\frac{1}{6} \int \frac{d\left(1+e^{-6y}\right)}{1+e^{-6y}}$
$= -\frac{1}{6} \ln\left(1+e^{-6y}\right) + C$
$= -\frac{1}{6} \ln\left(1+x^{-6}\right) + C$
$= -\frac{1}{6} \ln\left(1+x^{6}\right) + \ln x + C$

In the following, we outline two other ways to solve this problem (you can fill in the details yourself):

Since $x > 0$, therefore we can let $y = \frac{1}{x}$. Then
$\int \frac{1}{x^7+x} dx = -\int \frac{y^5}{1+y^6} dy$
$= -\frac{1}{6} \ln\left(1+y^6\right) + C$
$= -\frac{1}{6} \ln\left(\frac{x^6+1}{x^6}\right) + C$
$= -\frac{1}{6} \ln\left(1+x^6\right) + \ln x + C$

We can also solve the problem by observing that $\frac{1}{x^7+x} = \frac{1}{x(x^6+1)} = \frac{1}{x} - \frac{x^5}{x^6+1}$. Then
$\int \frac{1}{x^7+x} dx = \int \left(\frac{1}{x} - \frac{x^5}{x^6+1}\right) dx$
$= -\frac{1}{6} \ln\left(1+x^6\right) + \ln x + C$