

MA 1505 Mathematics I
Tutorial 1 Solutions

1. Note that $(g \circ f)(x) = \sqrt{|3 - \frac{6}{x}|}$ and $(f \circ g)(x) = \frac{6}{\sqrt{|3-x|}}$.

2. (a) $y = \frac{ax+b}{cx+d}, \quad y' = \frac{a(cx+d) - c(ax+b)}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}$ (use quotient rule)
- (b) $y = \sin^n x \cos mx, \quad y' = n \sin^{n-1} x \cos x \cos mx - m \sin^n x \sin mx$
(use product rule and chain rule)
- (c) $y = e^{x^2+x^3} \quad y' = e^{x^2+x^3} (2x+3x^2)$ (use chain rule)
- (d) $y = x^3 - 4(x^2 + e^2 + \ln 2), \quad y' = 3x^2 - 8x$ (note that e^2 and $\ln 2$ are constants)

Similarly, we find the derivatives in (e) - (h).

- (e) $-2 \sin \theta (\cos \theta - 1)^{-2}$ (use quotient and chain rule)
- (f) $\sqrt{t} \sec^2(2\sqrt{t}) + \tan(2\sqrt{t})$ (use product and chain rule)
- (g) $\frac{2\sqrt{\theta+1}+1}{2\sqrt{\theta+1}} \cos(\theta + \sqrt{\theta+1})$ (use chain rule)
- (h) $4 \tan x \sec x - \csc^2 x$ (use quotient rule)

3. Let $V_c(t)$ be the volume of coffee in the cone at time t and $V_p(t)$ be the volume of coffee in the pot at time t .

Note that the rate of volume change in the cone $\frac{dV_c}{dt}$ is equal the rate of volume change in the pot $\frac{dV_p}{dt}$.

Let $h_c(t)$ be the level of coffee in the cone at time t and $h_p(t)$ be the level of coffee in the pot at time t .

- (a) $V_p = \text{base area} \times h_p = 9\pi h_p$.

$$\begin{aligned} \frac{dV_p}{dt} &= 9\pi \frac{dh_p}{dt} \\ \Rightarrow 10 &= 9\pi \frac{dh_p}{dt} \\ \Rightarrow \frac{dh_p}{dt} &= \frac{10}{9\pi} \end{aligned}$$

$$(b) V_c = \frac{1}{3} \text{base area} \times h_c = \frac{1}{3} \pi r^2 h_c = \frac{1}{3} \pi \left(\frac{h_c}{2}\right)^2 h_c = \frac{\pi h_c^3}{12}.$$

Note that the base radius r of the cone is half that of the height h_c .

$$\begin{aligned} \frac{dV_c}{dt} &= \frac{\pi h_c^2}{4} \frac{dh_c}{dt} \\ \Rightarrow 10 &= \frac{\pi 5^2}{4} \frac{dh_c}{dt} \\ \Rightarrow \frac{dh_c}{dt} &= \frac{8}{5\pi} \end{aligned}$$

4. (a) $x^{2/3} + y^{2/3} = a^{2/3}$. Differentiating the equality we get

$$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \frac{dy}{dx} = 0.$$

Since $0 < x < a$ and $0 < y$, we have

$$\begin{aligned} \frac{dy}{dx} &= -\frac{y^{1/3}}{x^{1/3}} = -\frac{\sqrt{a^{2/3} - x^{2/3}}}{x^{1/3}} = -\sqrt{\left(\frac{a}{x}\right)^{2/3} - 1}; \\ \frac{d^2y}{dx^2} &= -\frac{1}{2} \frac{1}{\sqrt{\left(\frac{a}{x}\right)^{2/3} - 1}} \left(-\frac{2}{3}\right) a^{2/3} x^{-5/3} = \frac{a^{2/3}}{3x^{5/3} \sqrt{\left(\frac{a}{x}\right)^{2/3} - 1}} = \frac{a^{2/3}}{3x^{4/3} \sqrt{a^{2/3} - x^{2/3}}}. \end{aligned}$$

- (b) $y = (\sin x)^{\sin x}$, $0 < x < \frac{\pi}{2}$, so $\sin x > 0$.

$$\begin{aligned} \ln y &= \sin x \ln \sin x, \quad \frac{y'}{y} = \cos x \ln \sin x + \cos x, \quad y' = y(1 + \ln \sin x) \cos x, \\ y'' &= y'(1 + \ln \sin x) \cos x + y \left[(1 + \ln \sin x)(-\sin x) + \frac{\cos^2 x}{\sin x} \right] \\ &= y(1 + \ln \sin x)^2 \cos^2 x + y \left[\frac{\cos^2 x}{\sin x} - (1 + \ln \sin x) \sin x \right]. \end{aligned}$$

Hence

$$\begin{aligned} y' &= (\sin x)^{\sin x} (1 + \ln \sin x) \cos x, \\ y'' &= (\sin x)^{\sin x} \left[(1 + \ln \sin x)^2 \cos^2 x + \frac{\cos^2 x}{\sin x} - (1 + \ln \sin x) \sin x \right]. \end{aligned}$$

- (c) $x = a \cos t$, $y = a \sin t$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a \cos t}{-a \sin t} = -\cot t, \\ \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt} (-\cot t)}{\frac{dx}{dt}} = \frac{\frac{1}{\sin^2 t}}{-a \sin t} = -\frac{1}{a \sin^3 t}. \end{aligned}$$

5. (a) $y = \frac{x+1}{x^2+1}, x \in [-3, 3].$

$$y' = \frac{2-(x+1)^2}{(x^2+1)^2} \text{ and } y' = 0 \text{ if } x = -1 \pm \sqrt{2}.$$

So critical points are $x = -1 \pm \sqrt{2}$ and endpoints are $x = \pm 3$.

$$y' \begin{cases} < 0 & \text{if } -3 \leq x < -1 - \sqrt{2}, \\ = 0 & \text{if } x = -1 - \sqrt{2}, \\ > 0 & \text{if } -1 - \sqrt{2} < x < -1 + \sqrt{2}, \\ = 0 & \text{if } x = -1 + \sqrt{2}, \\ < 0 & \text{if } -1 + \sqrt{2} < x \leq 3. \end{cases}$$

Hence y is decreasing in $[-3, -1 - \sqrt{2})$, increasing in $(-1 - \sqrt{2}, -1 + \sqrt{2})$, and decreasing in $(-1 + \sqrt{2}, 3]$.

So local min is:

$$y(-1 - \sqrt{2}) = -\frac{1}{2(\sqrt{2}+1)}, \quad y(3) = \frac{2}{5}$$

and local max is:

$$y(-1 + \sqrt{2}) = \frac{1}{2(\sqrt{2}-1)}, \quad y(-3) = -\frac{1}{5}.$$

Since

$$-\frac{1}{2(\sqrt{2}+1)} < -\frac{1}{5} < \frac{2}{5} < \frac{1}{2(\sqrt{2}-1)},$$

so absolute min. is $\min_{x \in [-3, 3]} y = -\frac{1}{2(\sqrt{2}+1)}$ at $x = -1 - \sqrt{2}$

and absolute max. is $\max_{x \in [-3, 3]} y = \frac{1}{2(\sqrt{2}-1)}$ at $x = -1 + \sqrt{2}$.

(b) $y = (x-1)\sqrt[3]{x^2}, x \in (-\infty, \infty).$

$$y' = x^{2/3} + \frac{2}{3}(x-1)x^{-1/3} = \frac{5x-2}{3x^{1/3}}$$

and $y' = 0$ if $x = \frac{2}{5}$.

Note that y' does not exist at $x = 0$.

So the critical points are $x = 0$ and $x = \frac{2}{5}$.

$$y' \begin{cases} > 0 & \text{if } x < 0, \\ \text{does not exist} & \text{if } x = 0, \\ < 0 & \text{if } 0 < x < \frac{2}{5}, \\ = 0 & \text{if } x = \frac{2}{5}, \\ > 0 & \text{if } x > \frac{2}{5}. \end{cases}$$

Hence y is increasing in $(-\infty, 0)$, decreasing in $(0, \frac{2}{5})$, and increasing in $(\frac{2}{5}, \infty)$.

So local max. is $y(0) = 0$

and local min. is $y(\frac{2}{5}) = -\frac{3}{5}(\frac{2}{5})^{2/3}$.

Since $\lim_{x \rightarrow -\infty} y = -\infty$, $\lim_{x \rightarrow \infty} y = \infty$, so there is no absolute extremes.

6. Let x be the distance between B and C. Suppose the energy that it takes to fly over land is 1 unit per km, then it will take 1.4 unit per km to fly over water.

The total energy is given by the function

$$f(x) = 1.4\sqrt{5^2 + x^2} + (13 - x).$$

Then

$$f'(x) = \frac{1.4x - \sqrt{5^2 + x^2}}{\sqrt{5^2 + x^2}}.$$

Solving $f'(x) = 0$, we have $x = 5.103$ and the First Derivative Test shows that this point is an absolute minimum.

7. We have

$$\begin{aligned}\sin x &= 2 \cos \frac{x}{2} \sin \frac{x}{2} \\ &= 2^2 \cos \frac{x}{2} \cos \frac{x}{2^2} \sin \frac{x}{2^2} \\ &= 2^n \cos \frac{x}{2} \cos \frac{x}{2^2} \dots \cos \frac{x}{2^n} \sin \frac{x}{2^n}.\end{aligned}$$

Therefore, we have

$$\cos \frac{x}{2} \cos \frac{x}{2^2} \dots \cos \frac{x}{2^n} = \frac{\sin x}{2^n \sin \frac{x}{2^n}},$$

and taking \ln on both sides, we obtain

$$\ln \cos \frac{x}{2} + \ln \cos \frac{x}{2^2} + \dots + \ln \cos \frac{x}{2^n} = \ln \sin x - \ln 2^n - \ln \sin \frac{x}{2^n}.$$

Differentiate now respect to x , we find that

$$\frac{-\frac{1}{2} \sin \frac{x}{2}}{\cos \frac{x}{2}} + \frac{-\frac{1}{2^2} \sin \frac{x}{2^2}}{\cos \frac{x}{2^2}} + \dots + \frac{-\frac{1}{2^n} \sin \frac{x}{2^n}}{\cos \frac{x}{2^n}} = \frac{\cos x}{\sin x} - \frac{\frac{1}{2^n} \cos \frac{x}{2^n}}{\sin \frac{x}{2^n}}$$

which simplifies to

$$\frac{1}{2} \tan \frac{x}{2} + \frac{1}{2^2} \tan \frac{x}{2^2} + \dots + \frac{1}{2^n} \tan \frac{x}{2^n} = \frac{1}{2^n} \cot \frac{x}{2^n} - \cot x.$$