2011/2012 SEMESTER 1 MID-TERM TEST

MA1505 MATHEMATICS I

27 September 2011

8:30pm to 9:30pm

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **Thirteen (13)** printed pages.
2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.
3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1/10).
4. Use **only 2B pencils** for FORM CC1/10.
5. On FORM CC1/10 (section B for matric numbers starting with A, section C for others), **write** your **matriculation number** and **shade** the corresponding numbered circles **completely**. Your FORM CC1/10 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.
6. **Write your full name** in the blank space for module code in section A of FORM CC1/10.
7. Only circles for answers 1 to 10 are to be shaded.
8. For each answer, the circle corresponding to your choice should be **properly** and **completely** shaded. If you change your answer later, you must make sure that the original answer is properly erased.
9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.
10. **Do not fold** FORM CC1/10.
11. Submit FORM CC1/10 before you leave the test hall.

Formulae List

1. The **Taylor series** of f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

2.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

3.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

4.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

5.

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

6.

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

1. Let $y = x^3 - x^2 + e^2 - \ln 3$. Then $\frac{dy}{dx} =$

(A) $3x^2 - 2x + 2e - \frac{1}{3}$

(B) $3x - 2$

(C) $\frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{3}e^3 - \frac{1}{3}$

(D) $3x^2 - 2x$

(E) None of the above

2. Let $x = \sin t$ and $y = \sin 2t$. Find $\dfrac{d^2y}{dx^2}$.

(A) $(-2\tan t)\left(2+\sec^2 t\right)$

(B) $(-2\sin t)\left(2+\sec^2 t\right)$

(C) $\dfrac{2\cos 2t\sin t - 4\sin 2t\cos t}{\cos^2 t}$

(D) $\dfrac{4\sin 2t\cos t - 2\cos 2t\sin t}{\cos^2 t}$

(E) None of the above

3. Let k be a nonzero constant.

Find the limit

$$\lim_{x\to 0} (\cos kx)^{\left(\frac{1}{x^2}\right)}$$

in terms of k if the limit exists.

(A) $\cos\left(k^2\right)$

(B) e^{-k}

(C) $e^{-k^2/2}$

(D) 1

(E) None of the above

4. Let a be a positive constant with $0 < a < 1$. Let M and m denote the absolute maximum value and absolute minimum value respectively of the function

$$f(x) = a^x - \ln(1+x),$$

in the domain $[0, a]$. Find $M - m$.

(Hint: You may want to use the formula $\frac{da^x}{dx} = (a^x)(\ln a)$.)

(A) $\ln(e + ae) - a^a$

(B) $\ln(1 + a) - a^a$

(C) $\ln(1 + a) + a^a$

(D) $\ln(e + ae) + a^a$

(E) None of the above

5. Suppose $0 < x < 1$. Then

$$\int x\sqrt{1-x}\,dx =$$

(A) $\frac{2}{15}(2+3x)\left(\sqrt{1-x}\right)^3 + C$

(B) $\frac{2}{15}(2+3x)\left(-\sqrt{1-x}\right)^3 + C$

(C) $\frac{2}{15}(2-3x)\left(\sqrt{1-x}\right)^3 + C$

(D) $\frac{2}{15}(2-3x)\left(-\sqrt{1-x}\right)^3 + C$

(E) None of the above

6. Let k be a real constant with $k > 3$. Find the value of the integral

$$\int_0^3 \left| x(x-2)(x-k) \right| dx.$$

(A) $\frac{5}{3}k - \frac{38}{9}$

(B) $\frac{8}{3}k - \frac{59}{12}$

(C) $\frac{4}{3}k - \frac{25}{6}$

(D) $\frac{9}{4}$

(E) None of the above

7. Let n denote a positive constant. The area of the finite region bounded by the curves $y = \frac{2}{x}$, $y = \frac{1}{x}$, and the vertical lines $x = \frac{1}{e}$ and $x = e^n$ is equal to 2011. What is the value of n?

(A) 2011

(B) 2008

(C) 2012

(D) 2010

(E) None of the above

8. A finite region R is bounded by the curve $x = \tan(\frac{\pi y}{4a})$, and the lines $x = 0$ and $y = a$, where a is a constant and $0 < a \leq 2$. Find the volume of the solid formed by revolving R one complete round about the y-axis.

(A) $(4 - \pi)\, a$

(B) $(4\pi - 10)\, a$

(C) $8 - a\pi$

(D) $8\pi - 10a$

(E) None of the above

9. Given that $5 - \frac{1}{3} + \ldots$ is a geometric series, what is its sum?

(A) $\dfrac{15}{16}$

(B) $\dfrac{15}{14}$

(C) $\dfrac{75}{16}$

(D) $\dfrac{75}{14}$

(E) None of the above

10. Using a Taylor series of $x \ln(1+x)$, find the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{2n}\, n(n+2)}\,.$$

(A) $\dfrac{3}{16} - \dfrac{3}{8} \ln \dfrac{3}{2}$

(B) $\dfrac{7}{4} - \dfrac{15}{2} \ln \dfrac{5}{4}$

(C) $\dfrac{7}{64} - \dfrac{15}{32} \ln \dfrac{5}{4}$

(D) $\dfrac{3}{8} - \dfrac{3}{4} \ln \dfrac{3}{2}$

(E) None of the above

END OF PAPER

Additional blank page for you to do your calculations

National University of Singapore
Department of Mathematics

2011-2012 Semester 1 MA1505 Mathematics I Mid-Term Test Answers

Question	1	2	3	4	5	6	7	8	9	10
Answer	D	A	C	A	B	B	D	A	C	B

1. Let $y = x^3 - x^2 + e^2 - \ln 3$. Then $\frac{dy}{dx} =$

(A) $3x^2 - 2x + 2e - \frac{1}{3}$

(B) $3x - 2$

(C) $\frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{3}e^3 - \frac{1}{3}$

(D) $3x^2 - 2x$

(E) None of the above

$$\frac{dy}{dx} = 3x^2 - 2x \quad //$$

2. Let $x = \sin t$ and $y = \sin 2t$. Find $\dfrac{d^2y}{dx^2}$.

(A) $(-2\tan t)\left(2+\sec^2 t\right)$

(B) $(-2\sin t)\left(2+\sec^2 t\right)$

(C) $\dfrac{2\cos 2t\sin t - 4\sin 2t\cos t}{\cos^2 t}$

(D) $\dfrac{4\sin 2t\cos t - 2\cos 2t\sin t}{\cos^2 t}$

(E) None of the above

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos 2t}{\cos t}$$

$$= \frac{4\cos^2 t - 2}{\cos t} = 4\cos t - 2\sec t$$

$$\frac{d^2y}{dx^2} = \left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right]\left(\frac{dt}{dx}\right)$$

$$= (-4\sin t - 2\sec t\tan t)/(\cos t)$$

$$= -4\tan t - 2\sec^2 t\tan t$$

$$= (-2\tan t)(2+\sec^2 t) \;//$$

3. Let k be a nonzero constant.

Find the limit

$$\lim_{x\to 0}(\cos kx)^{\left(\frac{1}{x^2}\right)}$$

in terms of k if the limit exists.

(A) $\cos\left(k^2\right)$

(B) e^{-k}

(C) $e^{-k^2/2}$

(D) 1

(E) None of the above

Let $y = (\cos kx)^{1/x^2}$

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(\cos kx)}{x^2}$$

$$= \lim_{x\to 0}\frac{\frac{1}{\cos kx}(-\sin kx)(k)}{2x}$$

$$= \left(\lim_{x\to 0}\frac{1}{\cos kx}\right)\left(\lim_{x\to 0}\frac{\sin kx}{kx}\right)\left(\frac{-k^2}{2}\right)$$

$$= -\frac{k^2}{2}$$

$$\therefore \lim_{x\to 0} y = e^{-k^2/2}$$

//

4. Let a be a positive constant with $0 < a < 1$. Let M and m denote the absolute maximum value and absolute minimum value respectively of the function

$$f(x) = a^x - \ln(1+x),$$

in the domain $[0, a]$. Find $M - m$.

(Hint: You may want to use the formula $\frac{da^x}{dx} = (a^x)(\ln a)$.)

(A) $\ln(e + ae) - a^a$

(B) $\ln(1 + a) - a^a$

(C) $\ln(1 + a) + a^a$

(D) $\ln(e + ae) + a^a$

(E) None of the above

$f'(x) = a^x \ln a - \frac{1}{1+x} = \text{negative}$ $\left(\because 0<a<1 \; \therefore \ln a = -ve \right)$

$\therefore f$ is decreasing.

$\therefore M = f(0) = 1 = \ln e$

$m = f(a) = a^a - \ln(1+a)$

$\therefore M - m = \ln e - a^a + \ln(1+a)$

$= \ln(e + ea) - a^a$ //

5. Suppose $0 < x < 1$. Then

$$\int x\sqrt{1-x}\,dx =$$

(A) $\frac{2}{15}(2+3x)\left(\sqrt{1-x}\right)^3 + C$

(B) $\frac{2}{15}(2+3x)\left(-\sqrt{1-x}\right)^3 + C$

(C) $\frac{2}{15}(2-3x)\left(\sqrt{1-x}\right)^3 + C$

(D) $\frac{2}{15}(2-3x)\left(-\sqrt{1-x}\right)^3 + C$

(E) None of the above

Let $u = 1-x$

$\therefore\ x = 1-u$ and $dx = -du$

$$\int x\sqrt{1-x}\,dx = \int (1-u)\,u^{1/2}\,(-du)$$
$$= \int (u^{3/2} - u^{1/2})\,du$$
$$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$$
$$= \frac{2}{15}(-u^{1/2})^3(5-3u) + C$$
$$= \frac{2}{15}(2+3x)(-\sqrt{1-x})^3 + C \quad //$$

6. Let k be a real constant with $k > 3$. Find the value of the integral

$$\int_0^3 \left| x(x-2)(x-k) \right| dx.$$

(A) $\frac{5}{3}k - \frac{38}{9}$

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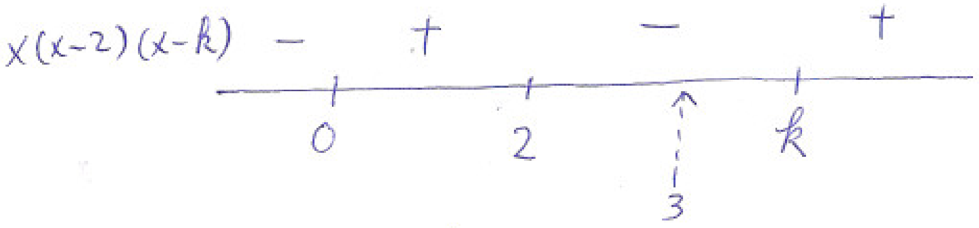


Let $F(x) = \int x(x-2)(x-k)\,dx = \int (x^3 - (2+k)x^2 + 2kx)\,dx$

$= \frac{1}{4}x^4 - \frac{1}{3}(2+k)x^3 + kx^2$

$\int_0^3 |x(x-2)(x-k)|\,dx = \int_0^2 x(x-2)(x-k)\,dx - \int_2^3 x(x-2)(x-k)\,dx$

$= \{F(2) - F(0)\} - \{F(3) - F(2)\}$

$= 2F(2) - F(3)$

$= \{8 - \frac{16}{3}(2+k) + 8k\} - \{\frac{81}{4} - (2+k)9 + 9k\}$

$= \frac{8}{3}k - \frac{59}{12}$ //

7. Let n denote a positive constant. The area of the finite region bounded by the curves $y = \frac{2}{x}$, $y = \frac{1}{x}$, and the vertical lines $x = \frac{1}{e}$ and $x = e^n$ is equal to 2011. What is the value of n?

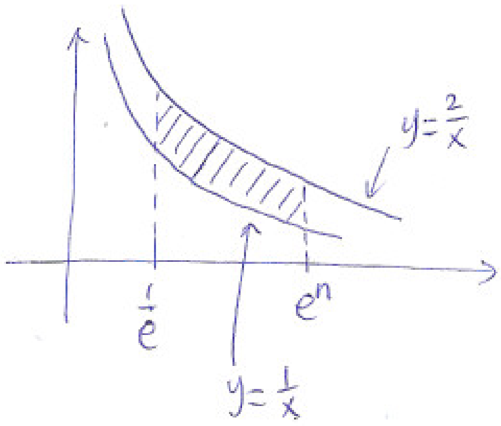
(A) 2011

(B) 2008

(C) 2012

(D) 2010

(E) None of the above



$$\int_{\frac{1}{e}}^{e^n} \left(\frac{2}{x} - \frac{1}{x}\right) dx = 2011$$

$$\Rightarrow \left[2\ln x - \ln x\right]_{1/e}^{e^n} = 2011$$

$$\Rightarrow n + 1 = 2011$$

$$\Rightarrow n = 2010$$

8. A finite region R is bounded by the curve $x = \tan(\frac{\pi y}{4a})$, and the lines $x = 0$ and $y = a$, where a is a constant and $0 < a \leq 2$. Find the volume of the solid formed by revolving R one complete round about the y-axis.

(A) $(4 - \pi)\, a$

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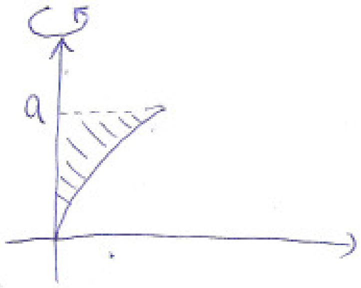

$$V = \int_0^a \pi x^2 dy = \pi \int_0^a \tan^2\left(\frac{\pi y}{4a}\right) dy$$

$$= \pi \int_0^a \left[\sec^2\left(\frac{\pi y}{4a}\right) - 1\right] dy$$

$$= \pi \left[\frac{4a}{\pi} \tan \frac{\pi y}{4a} - y\right]_0^a$$

$$= \pi \left[\frac{4a}{\pi} - a\right] = (4 - \pi) a \,//$$

9. Given that $5 - \frac{1}{3} + \ldots$ is a geometric series, what is its sum?

(A) $\frac{15}{16}$

(B) $\frac{15}{14}$

(C) $\frac{75}{16}$

(D) $\frac{75}{14}$

(E) None of the above

$a = 5$

$r = \frac{-1/3}{5} = -\frac{1}{15}$

$S = \frac{5}{1-(-\frac{1}{15})} = \frac{75}{16}$ //

10. Using a Taylor series of $x\ln(1+x)$, find the exact value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{2n}\, n(n+2)}.$$

(A) $\frac{3}{16} - \frac{3}{8}\ln\frac{3}{2}$

(B) $\frac{7}{4} - \frac{15}{2}\ln\frac{5}{4}$

(C) $\frac{7}{64} - \frac{15}{32}\ln\frac{5}{4}$

(D) $\frac{3}{8} - \frac{3}{4}\ln\frac{3}{2}$

(E) None of the above

END OF PAPER

$$\int_0^{\frac{1}{2^2}} x\ln(1+x)\,dx = \int_0^{\frac{1}{2^2}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n+1}}{n}\,dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+2}}{n(n+2)} \Big|_0^{\frac{1}{2^2}} = \frac{1}{16} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{2n}\, n(n+2)}$$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{2n}\, n(n+2)} = 16\int_0^{\frac{1}{4}} x\ln(x+1)\,dx = 16\int_0^{\frac{1}{4}} \ln(x+1)\,d\left(\frac{x^2}{2}\right)$$

$$= 8x^2\ln(x+1)\Big|_0^{\frac{1}{4}} - 8\int_0^{\frac{1}{4}} \frac{x^2}{x+1}\,dx$$

$$= \frac{1}{2}\ln\frac{5}{4} - 8\int_0^{\frac{1}{4}} \left(x - 1 + \frac{1}{x+1}\right)dx$$

$$= \frac{1}{2}\ln\frac{5}{4} - 8\left[\frac{1}{2}x^2 - x + \ln(x+1)\right]_0^{\frac{1}{4}}$$

$$= \frac{1}{2}\ln\frac{5}{4} - 8\left(\frac{1}{32} - \frac{1}{4} + \ln\frac{5}{4}\right] = \frac{7}{4} - \frac{15}{2}\ln\frac{5}{4} \quad //$$