CH 2 - *Differentiation*

2.1 *Derivative*

2.1.1 *Definition of Derivative*

Let $f(x)$ be given

The derivative of f at the point a

is defined to be

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \text{ denoted by } f\,'(a),$$

provided the limit exists

An equivalent formulation of derivative is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Suppose we let $y = f(x)$

We may use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} = \frac{dy}{dx}(a) = f'(a)$$

2.1.2 *Differentiable functions*

- If $f'(a)$ exists,

we say : f is ***differentiable*** at the point a.

- If f is differentiable at ***every*** point in the domain,

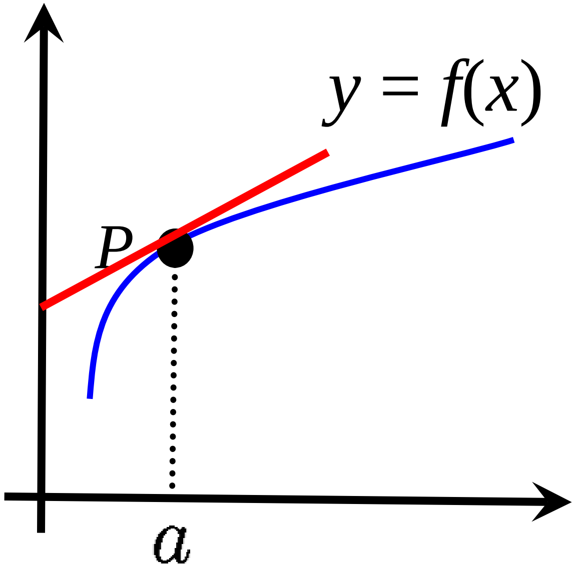
we say : f is ***differentiable*** in the domain

2.1.3 *Geometrical meaning*

Search tangent or derivative animation by internet, we can understand the derivative better

http://www.ima.umn.edu/~arnold/calculus/secants/secants2/secants-g.html

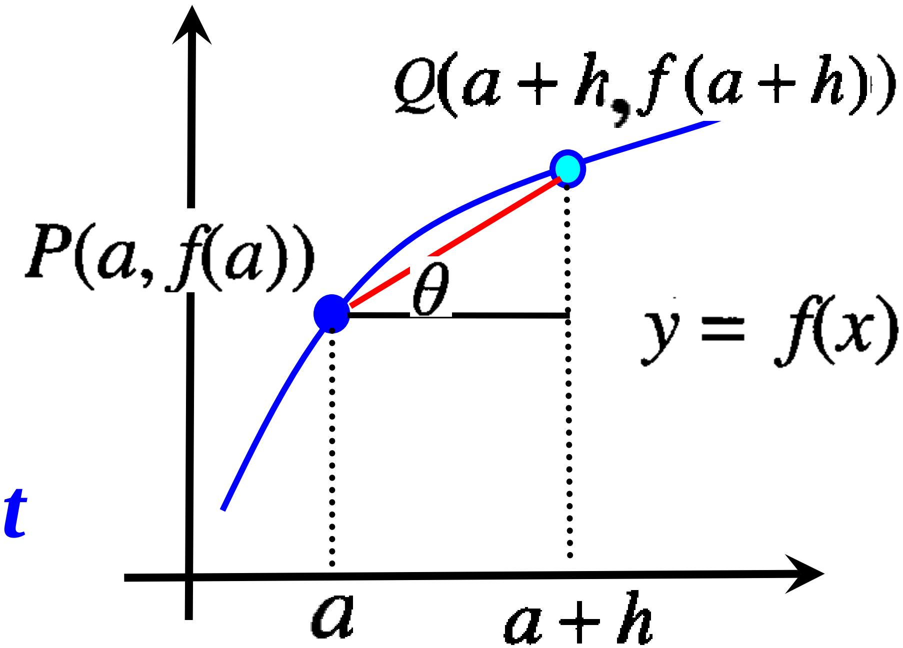
Problem Find the ***slope*** of the ***tangent*** to the curve $y = f(x)$ at $\boldsymbol{P}(a, f(a))$.



- The ***slope*** of ***PQ***

$= \tan\theta$

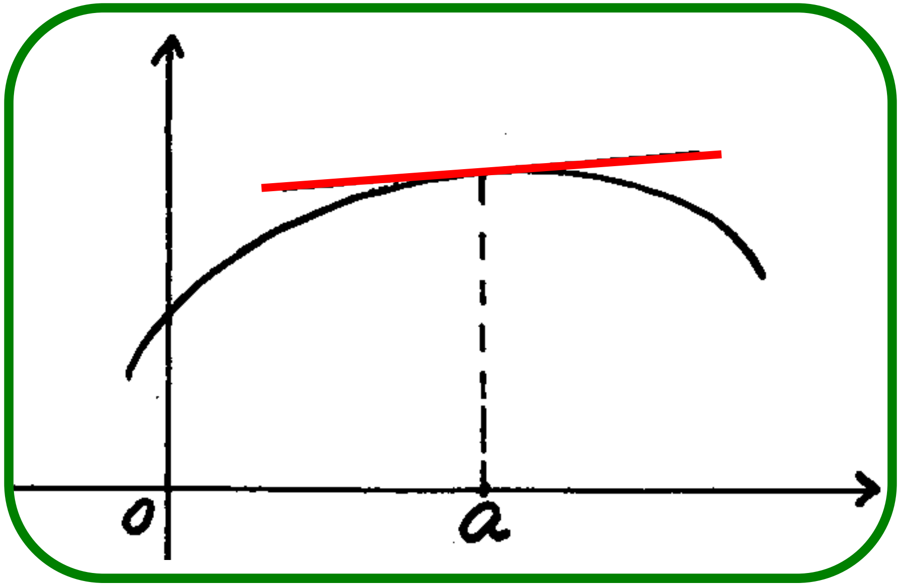
$$= \frac{f(a+h) - f(a)}{h}$$



The ***slope*** of the ***tangent*** to the curve $y = f(x)$ at ***P***$(a, f(a))$ is :

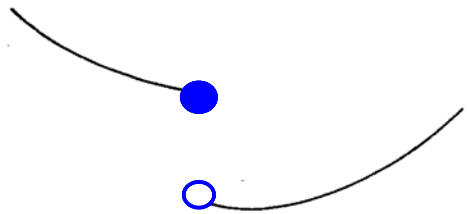
$$\boxed{\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}}$$

- If $f'(a)$ exists, then $f(x)$ is ***smooth*** at the point a.

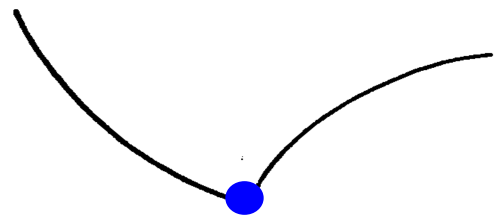


Some cases where $f'(a)$ fails to exist

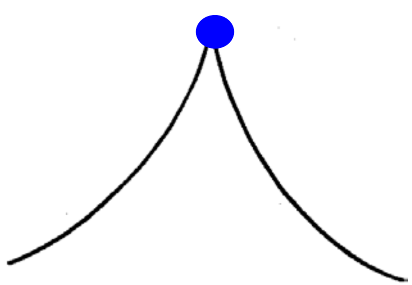
♣ ***Discontinuity***



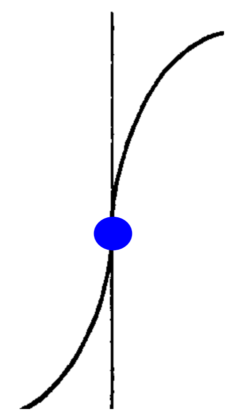
♥ ***Corner***



♠ ***Cusp***



♦ ***Vertical tangent***



Limit, Continuity & Differentiability

$f\,'(a)$ ***exists***

=> f is ***continuous*** at a

=> $\lim_{x \to a} f(x) = \mathrm{f(a)}$ ***exists***

The ***converses*** are in general not true.

We will not prove the following formulae

Formulae

Functions	Derivatives	Functions	Derivatives
k	0	x^n	nx^{n-1}
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$	$\csc x$	$-\csc x \cot x$
a^x	$a^x \ln a$	e^x	e^x
$\log_a x$	$\frac{1}{x \ln a}$	$\ln x$	$\frac{1}{x}$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$		

Note

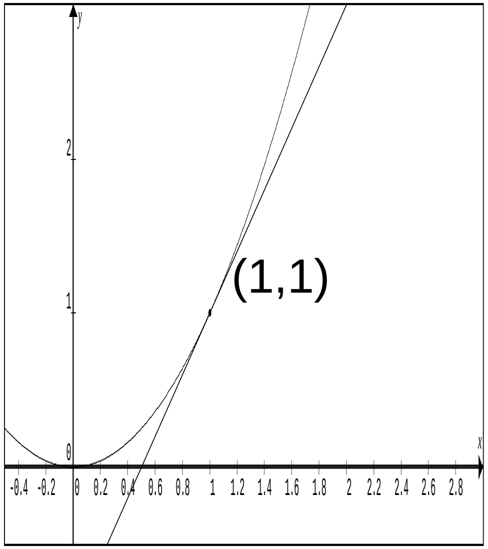
$$(\sin x)^{-1} = \frac{1}{\sin x}$$

$$\neq \sin^{-1} x$$

$$y = \sin^{-1} x \Leftrightarrow \sin y = x$$

2.1.4 *Example*

Find equations of the lines which are tangent and normal to the curve $y = x^2$ at $x = 1$ resp.



$$f'(x) = 2x,\ f'(1) = 2$$

The slope of the tangent is $f'(1) = 2$

The equation of tangent is

$$\frac{y-1}{x-1} = f'(1) = 2$$

The slope of the normal is $-\frac{1}{f'(1)} = -\frac{1}{2}$

The equation of normal is $\frac{y-1}{x-1} = -\frac{1}{2}$

2.1.5 ***Rules*** of **Differentiation**

- **Given**: ***f*** **&** ***g*** **are** ***differentiable*** **functions &**
  k **a** ***constant*****.**

♣ **Linearity**

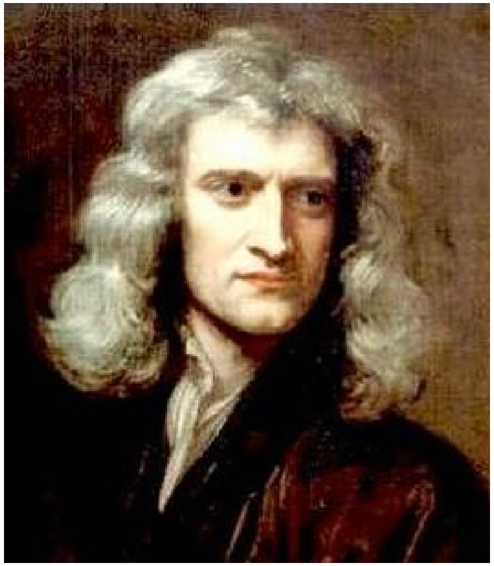
(i) $(k\,f)'(x) = k\,f\,'(x)$

(ii) $(f \pm g)'(x) = f\,'(x) \pm g'(x)$

Product

$$(u(x)\, v(x))' = u(x)'\, v(x)' \textbf{ ?} \quad \textit{No, No}$$

$$\frac{d}{dx} uv = u \frac{dv}{dx} + v \frac{du}{dx}$$



Newton

Leibniz

Rules of Differentiation

♥ **Product Rule**

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

♠ **Quotient Rule**

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

♣ Let $f(x)$ be a differentiable function which satisfies $f(5) = 3$ and $f'(5) = 2$. Find the value of the expression $\frac{d}{dx}\left[x^2 f(x)\right]$ at the point $x = 5$. **(Ans: 80; 2007/08 Sem 1 Mid-Term Test)**

♣ $$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$$

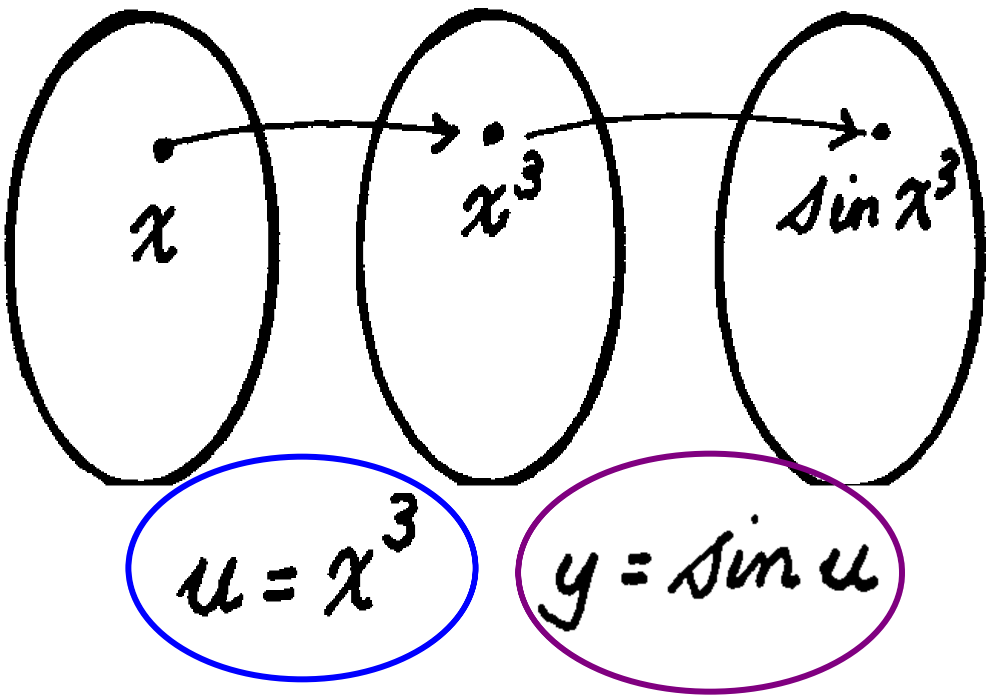
$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

$$= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

The Chain Rule

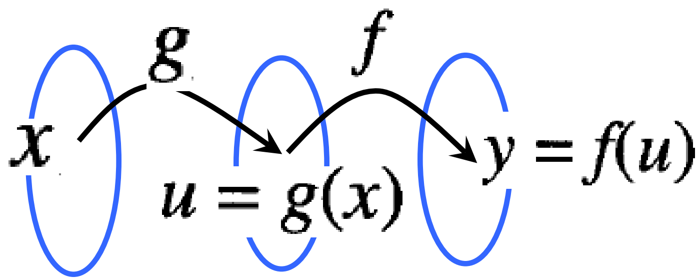
♣ $\frac{d}{dx}\sin(x^3) = \cos(x^3)\cdot\frac{d}{dx}x^3 = 3x^2\cos(x^3)$



The Chain Rule

$$\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}(f(g(x)))$$

$$= \frac{d}{dg(x)}(f(g(x)))\frac{d}{dx}\, g(x)$$



2.1.6 *Remark*

Let $y = f(u)$, $u = g(x)$. Then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

- **2006/07 Sem 1 Mid-Term Test**

Let $f(x) = \ln \frac{1+\sin x}{1-\sin x}$, where $0 < x < \frac{\pi}{2}$. Then $f'(x) =$

- **2008/09 Sem 1 Mid-Term Test**

Let $f(x)$ be a differentiable function which satisfies $f(1) = \sqrt{3}$ and $f'(1) = 10$. Find the value of the expression $\frac{d}{dx}\left[\sqrt{1+[f(x)]^2}\right]$ at the point $x = 1$. **(Ans. $5\sqrt{3}$)**

2.2 Other *Types* of Differentiation

2.2.1 Parametric Differentiation

If x & y are given by

$$\begin{cases} y = u(t) \\ x = v(t), \end{cases}$$

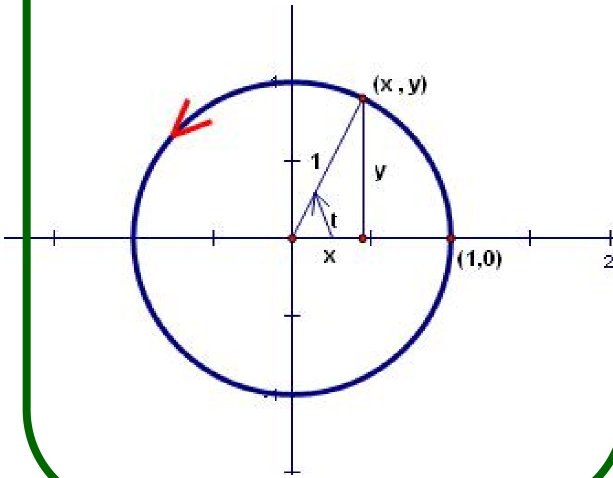
we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{u'(t)}{v'(t)}$$

The equations

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

describe a ***unit circle***.



2.2.2 *Example*

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

- Let $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t). \end{cases}$

$$\frac{dx}{dt} = \frac{d}{dt}[a(t - \sin t)]$$

$$\frac{dy}{dt} = \frac{d}{dt}[a(1 - \cos t)]$$

Then

$$\frac{dy}{dx} = \frac{a \sin t}{a(1 - \cos t)}$$

$$= \frac{2\sin(\frac{t}{2})\cos(\frac{t}{2})}{2\sin^2(\frac{t}{2})} = \cot\frac{t}{2}.$$

2.2.3 *Implicit Differentiation*

This method is used when the ***dependence*** of x & y is given ***implicitly*** by $\boldsymbol{F}(x, y) = 0$.

2.2.4 *Example*

Given $x^2 + y^2 - a^2 = 0$ Find $\dfrac{dy}{dx}$

Solution $\dfrac{d}{dx}(x^2 + y^2 - a^2) = \dfrac{d}{dx}0$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 - \frac{d}{dx}a^2 = \frac{d}{dx}0$$

$$2x + 2y\frac{dy}{dx} - 0 = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

2.2.5 *Example*

Find $\frac{dy}{dx}$ if $2y = x^2 + \sin y$

Differentiate both sides with respect to x,

$$2\frac{dy}{dx} = 2x + \cos\ y\frac{dy}{dx}.$$

$$\frac{dy}{dx} = \frac{2x}{2 - \cos y}.$$

2.2.6 *Example*

Let $y = x^x,\ x > 0$. Find $\dfrac{dy}{dx}$

Solution $\ln y = \ln x^x = x \ln x$

$$\frac{d}{dx}\ln y = \frac{d}{dx}(x \ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = \ln x \frac{d}{dx}x + x\frac{d}{dx}\ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \ln x + x\frac{1}{x}$$

$$\frac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1)$$

Q: If $y^3 + xy - 1 = 0$, then $y' =$

Ans. $-y/(x + 3y^2)$

Question 1 (a) [5 marks]

Find the slope of the tangent to the curve $x = t - \sin t$, $y = 1 - \cos t$, at the point corresponding to $t = \frac{\pi}{3}$.

Question 1 (a) [5 marks]

Find the slope of the tangent to the curve $y^2 = x^3 + 2x^2 - 20$ at the point $(3, 5)$.

A leaky water tank is in the shape of an inverted right circular cone with depth 5 m and top radius 2 m. At a time when the water in the tank is 4 m deep, it is leaking out at a rate of $\frac{1}{10}$ $\mathrm{m}^3/\mathrm{min}$. How fast is the water level in the tank dropping at that time?

Ans. $5/(128\pi)$

Consider the curve $y = (\ln x)^{(\ln x)}$, which is defined on $x > 1$. Let L denote the tangent line to this curve at the point where $x = e^2$. Find the y-coordinate of the point of intersection of L with the y-axis.

Ans. -4ln2

2.2.7 *Higher Order* Derivatives

- Let ***y* = *f*(*x*).** Then

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x),$$

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x).$$

- The ***n*th derivative** is denoted by

$$\frac{d^ny}{dx^n} \quad \text{or} \quad f^{(n)}(x).$$

2.2.8 *Example*

Let $f(x) = \sqrt{x}$. Compute $f'''(x)$.

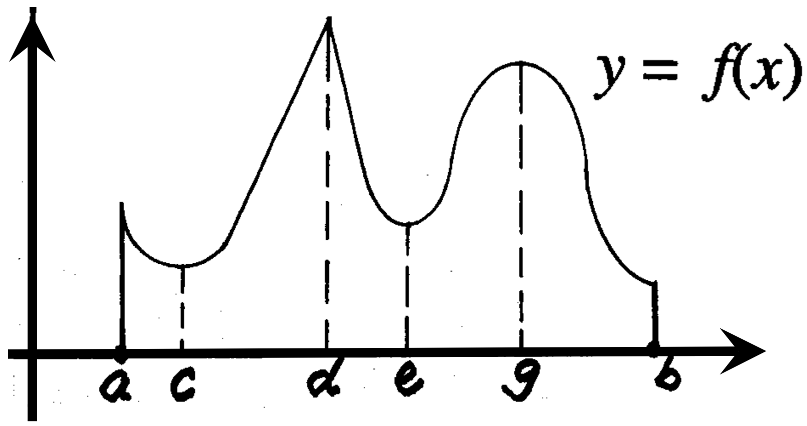
$$f'(x) = \frac{1}{2}x^{-1/2},$$

$$f''(x) = -\frac{1}{4}x^{-3/2},$$

$$f'''(x) = \frac{3}{8}x^{-5/2}.$$

2.3 *Maxima* & *Minima*

2.3.1 *Local & absolute extremes*



- *f* has ***local*** (***relative***) ***maximum*** values at
- *f* has ***local*** (***relative***) ***minimum*** values at
- *f* has the ***absolute maximum*** value at
- *f* has the ***absolute minimum*** value at

Note (1) ***a*** & ***b*** are ***end points*** of the domain,

(2) $f'(c) = f'(e) = f'(g) = 0$,

(3) $f'(d)$ ***doesn't exist***.

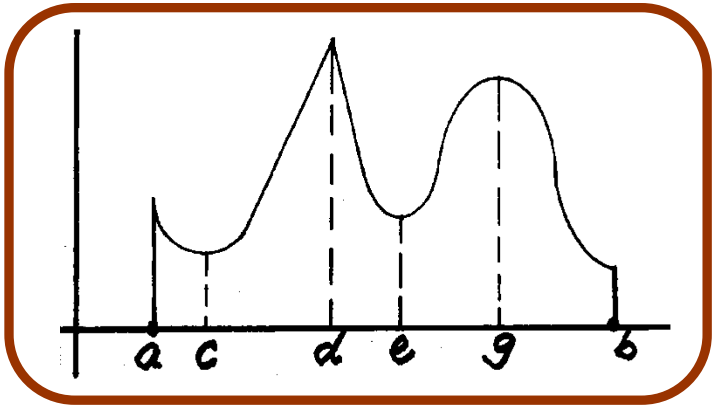
(2.3.2 & 2.3.3)

NOT end pt of D

Let ***f*** be a function with domain ***D***. An interior point ***c*** in ***D*** is called a ***critical point*** of ***f*** if

$f'(c) = 0$ or $f'(c)$ **doesn't exist.**

Finding extreme values of ***f***



check

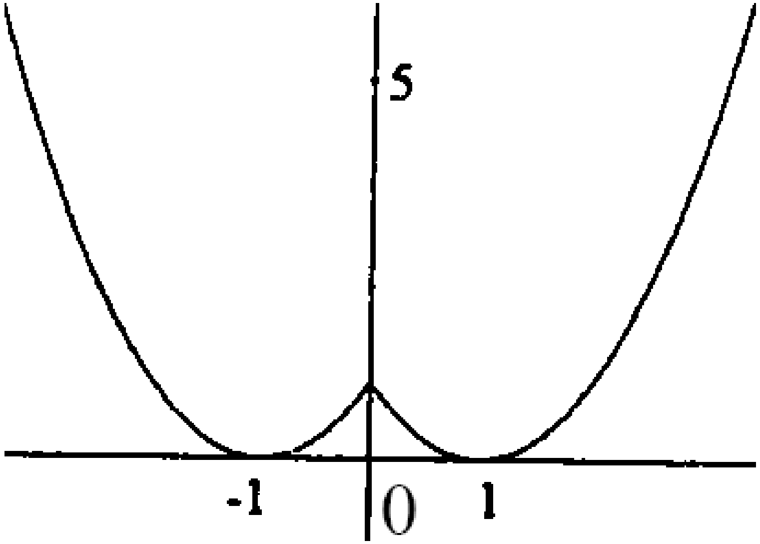
(1) ***critical points*** of ***f***

& (2) ***end points*** of ***D***.

2.3.4 *Example*

Let
$$f(x) = \begin{cases} (x-1)^2 & \text{if } x \geq 0, \\ (x+1)^2 & \text{if } x < 0. \end{cases}$$

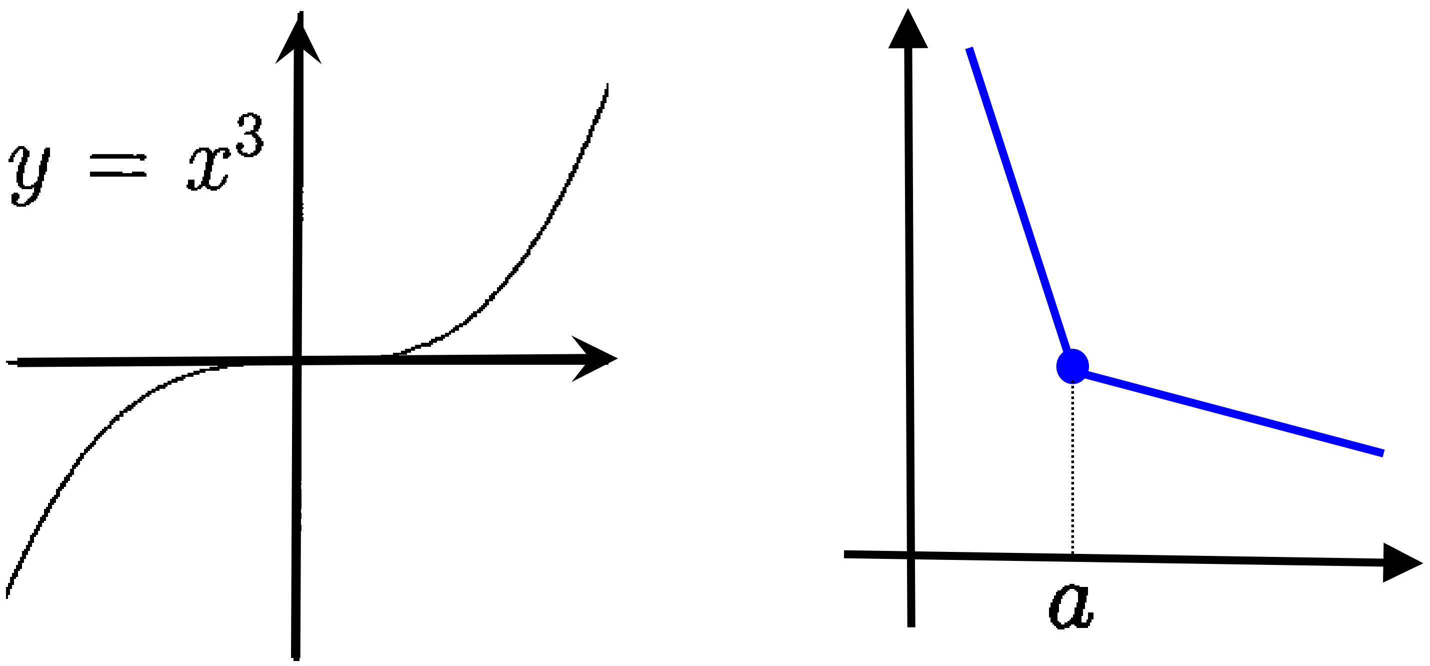
- $f'(x) = \begin{cases} 2(x+1), \ x < 0 \\ 2(x-1), \ x > 0 \end{cases}$
- $f'(0)$ doesn't exist



Critical points: $x = -1, 0, 1.$

Note

- A function may ***not*** have a local extreme at a ***critical*** point.



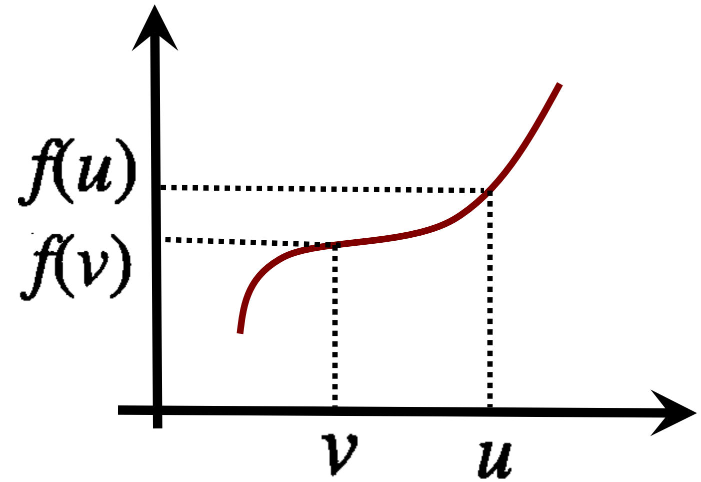
2.4 *Increasing and Decreasing* Functions

Let $f : I \text{ (interval)} \longrightarrow \mathbb{R}$.

- f is ***increasing*** *on I* if

  $v < u \Rightarrow f(v) < f(u)$

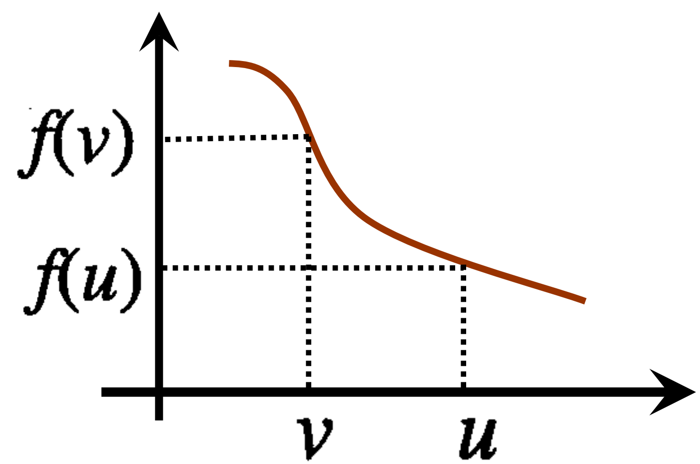
  (v, u in I)



- f is ***decreasing*** *on I* if

  $v < u \Rightarrow f(v) > f(u)$

  (v, u in I)



Test for ***Monotonic (increasing/decreasing)*** *Functions*

- $f'(x) > 0$ for any x in I => f is ***increasing*** on I
- $f'(x) < 0$ for any x in I => f is ***decreasing*** on I

2.4.3 *Example*

See Lecture Notes

Example

Let $f(x) = x^3(x-1)^2.$

Then $f'(x) = x^2(x-1)(5x-3)$

& $f'(x) = 0 \quad \Leftrightarrow \quad x = 0,\ 1$ or $3/5.$

$f'(x) = x^2(x-1)(5)(x-3/5)$

$f'(x)$	(+)(-)(-)	(+)(-)(-)	(+)(-) (+)	(+)(+)(+)
$f(x)$	increasing	increasing	decreasing	increasing

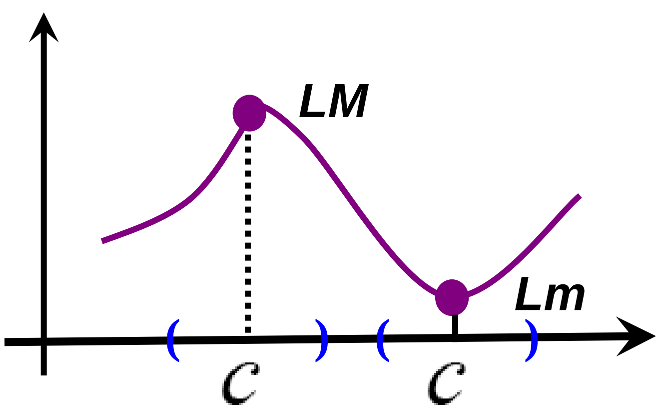
0 3/5 1

2.4.4 *First Derivative Test*

- Assume ***c*** in (a, b), a ***critical point*** of ***f***. If

(i) $\boldsymbol{f'(x) > 0}$ for x in (a, c), & $\boldsymbol{f'(x) < 0}$ for x in (c, b), then ***f*(*c*)** is a local ***maximum***;

(ii) $\boldsymbol{f'(x) < 0}$ for x in (a, c), & $\boldsymbol{f'(x) > 0}$ for x in (c, b), then ***f*(*c*)** is a local ***minimum***.

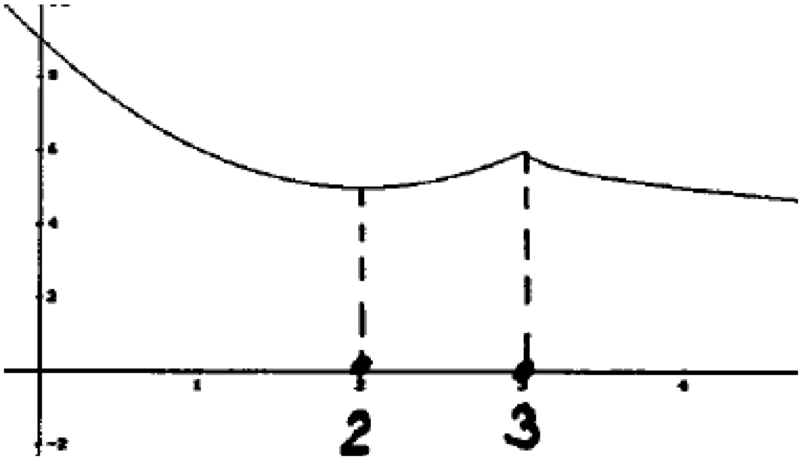


♣ Let $f(x)=\begin{cases} x^2-4x+9, & x\le 3 \\ 6-\sqrt{x-3}, & x>3 \end{cases}$

Then

$$f'(x)=\begin{cases} 2(x-2) & x<3 \\ -\dfrac{1}{2\sqrt{x-3}} & x>3 \end{cases}$$

- $f'(2)=0$, $f'(3)$ not exist
- ***critical points***: $x = 2, 3$



$f'(x)$ (-) (+) (-)

Conclusion:

Local min at '2',

local max at '3'

2.5 *Concavity*

2.5.1 & 2.5.3 *Definition and example*

- Let $y = f(x) = x^3$, x in $\mathbb{R}$.

  Then

  $$f'(x) = 3x^2 > 0 \text{ for all } x.$$

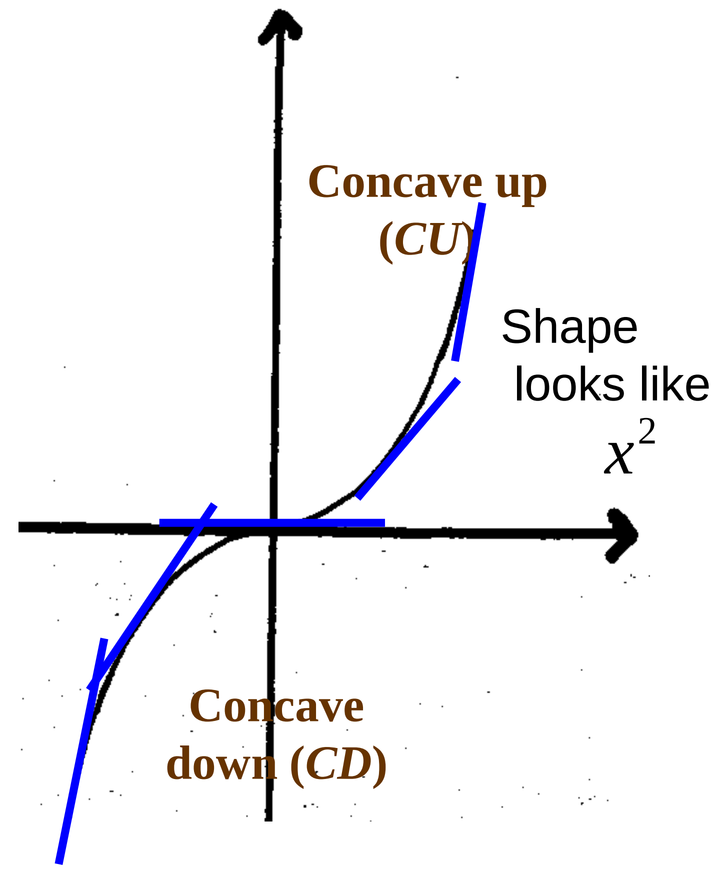
  slope of the tangent =f'(x)>0

Note that

$$f''(x) = 6x \begin{cases} < 0, & x < 0 \\ > 0, & x > 0 \end{cases}$$

Slope tanθ decreasing

Slope tanθ increasing

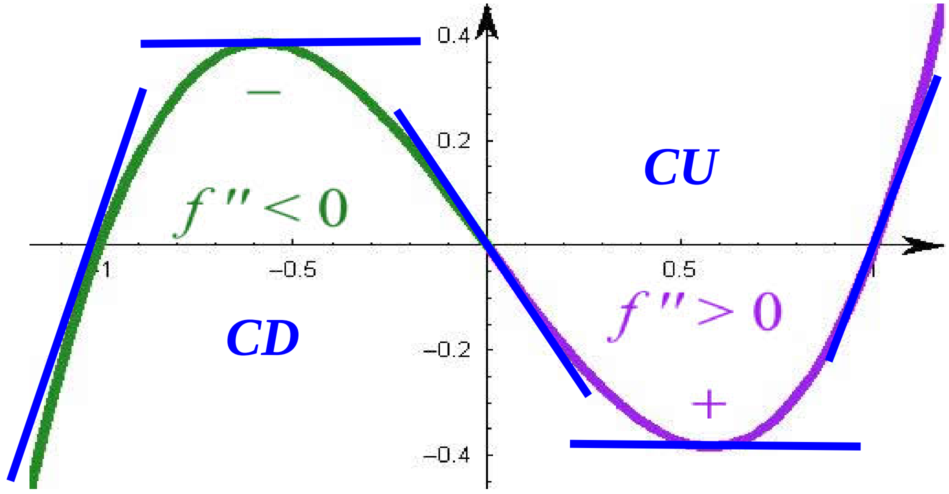


Shape looks like $-x^2$

2.5.2 *Concavity Test*

Let ***I*** be an open interval. The curve ***y* = *f*(*x*)** defined on ***I*** is

- ***CU*** on *I* if ***f*′ is *increasing* on *I*** ($f''(x) > 0$ on I)
- ***CD*** on *I* if ***f*′ is *decreasing* on *I*** ($f''(x) < 0$ on I)

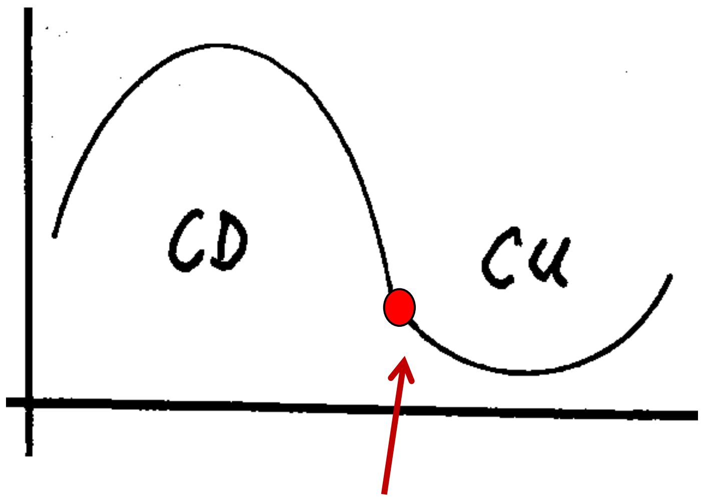


Slope tanθ decreasing Slope tanθ increasing

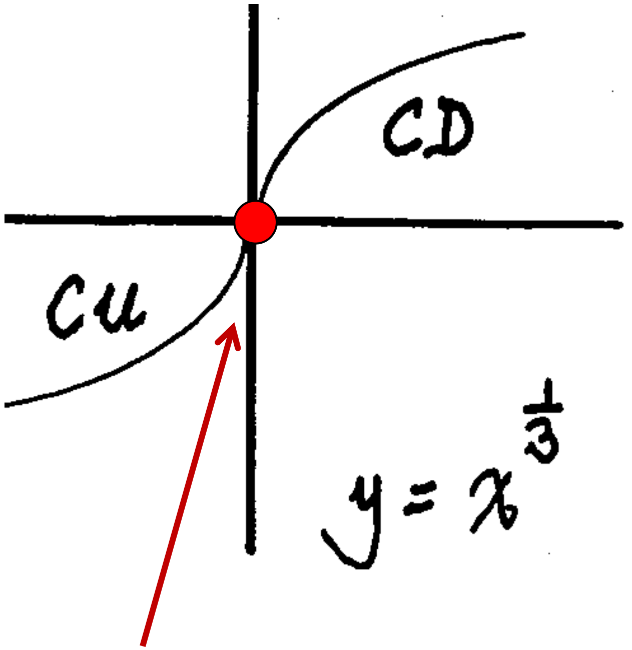
2.5.4 &2.5.5 *Points of Inflection & Examples*

- Let $f: I \to \mathbb{R}$ & c in I. We call c a ***point* of *inflection*** of f if f is *continuous* at c & the ***concavity*** of f *changes* at c.

Note $f'(c)$ may not exist

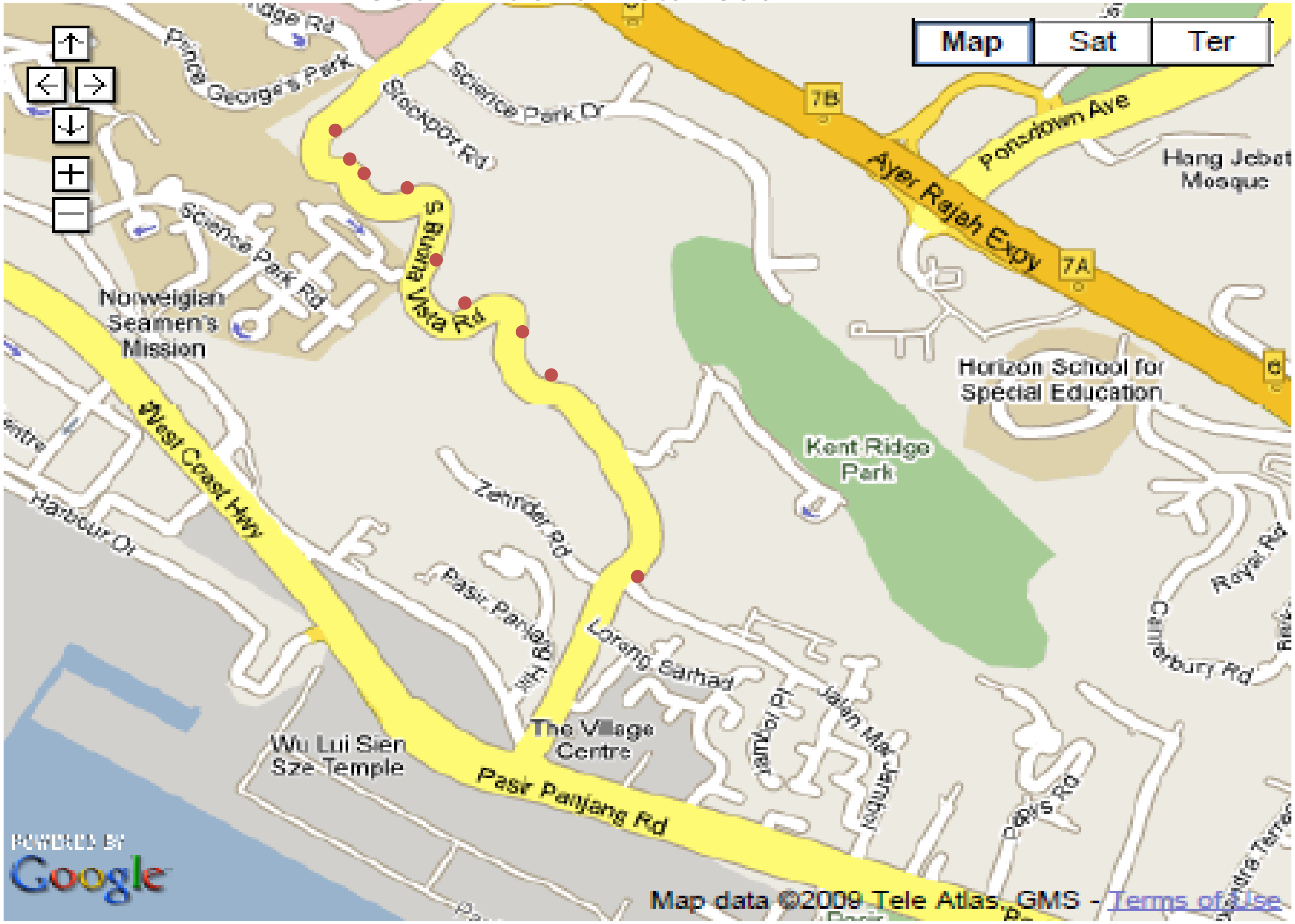


concavity of f *changes at*



concavity of f *changes* at

South Buona Vista road



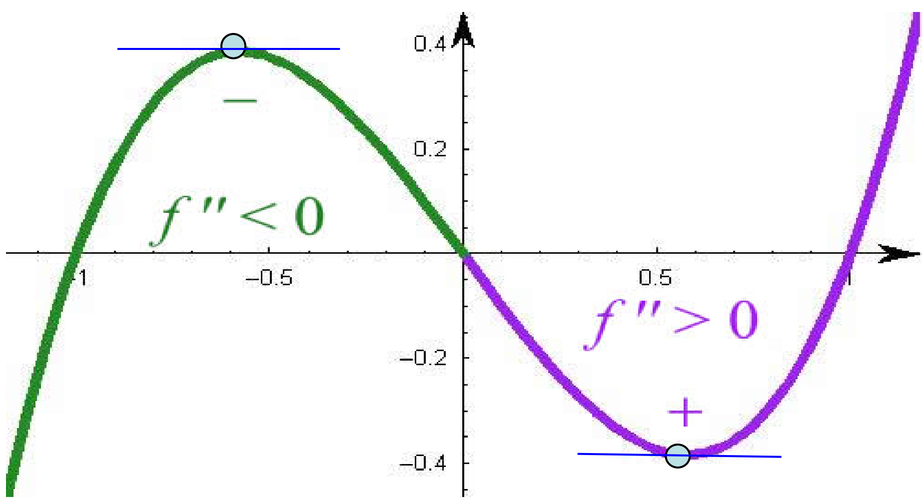
famous (or infamous) hilly winding road
buona vista means "good sight" in Italian

How many inflection pts there? 9

2.5.7 *Second Derivative Test*

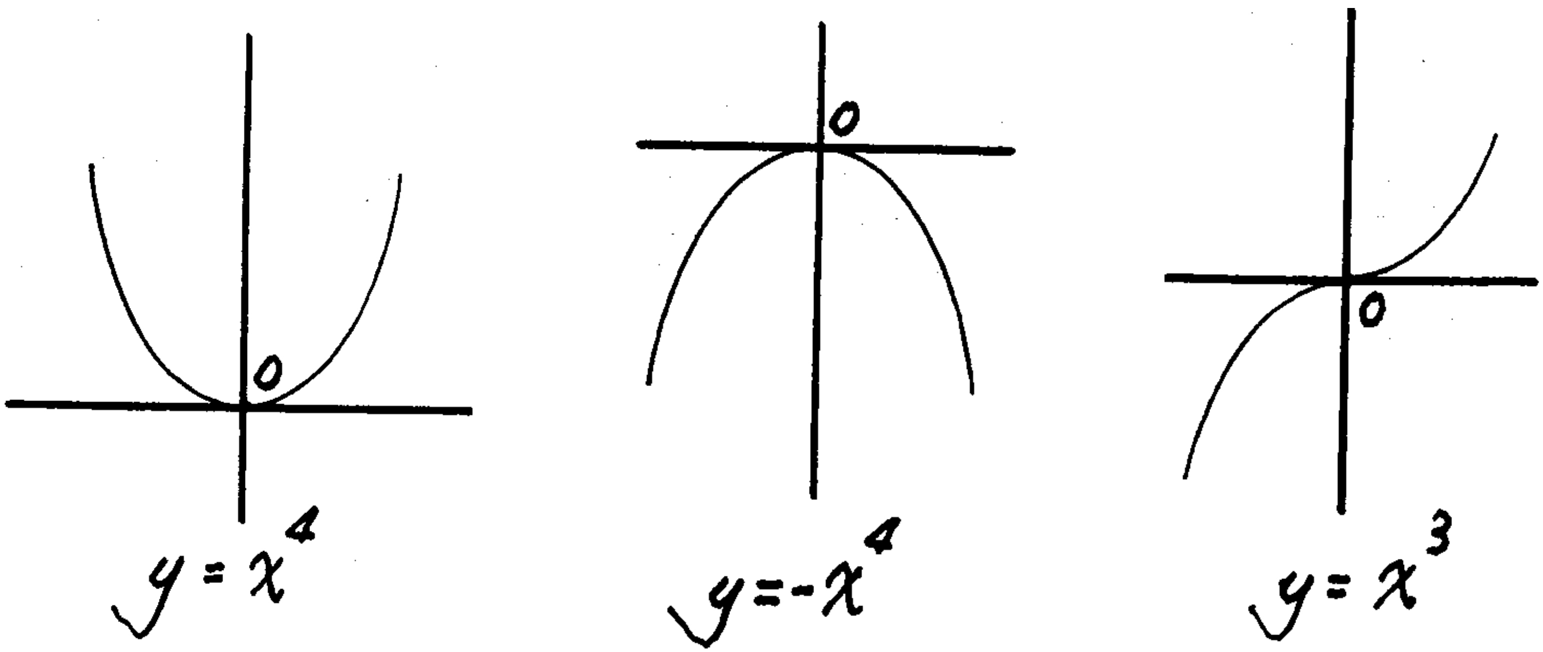
$f'(c) = 0$ → $f''(c) < 0$ => f has local ***max*** at 'c'

$f'(c) = 0$ → $f''(c) > 0$ => f has local ***min*** at 'c'



Note

- **If $f'(c) = 0$ & $f''(c) = 0$**, then the test fails



- In each case, $f'(c) = f''(c) = 0$.

2.5.7 *Example*

Let $y = x^3 - 3x + 2$ defined on $(-\infty, \infty)$

Domain has no endpoints and
f is differentiable everywhere

Therefore local extrema can occur
only where $y' = 3x^2 - 3 = 0$

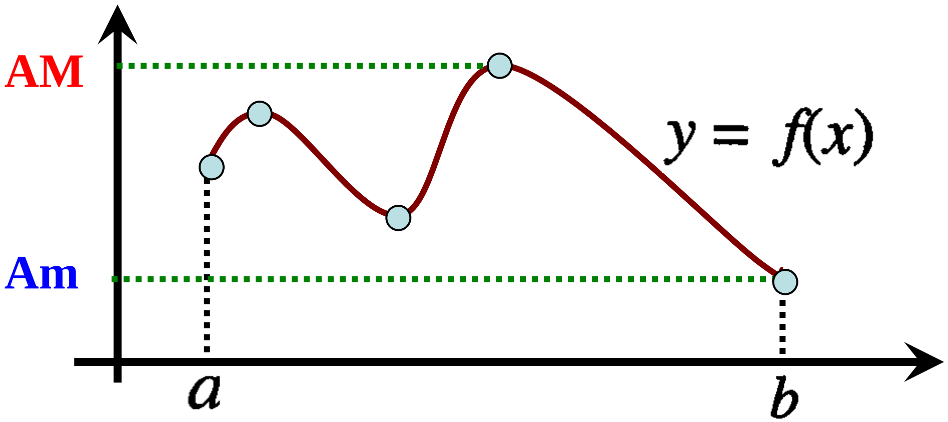
which means $x = 1$ and $x = -1$

$y'' = 6x$ $\qquad y''(1) = 6 > 0,\ y''(-1) = -6 < 0$

$\mathrm{y}(1) = 0$ local mini $y(-1) = 4$ local maxi

2.6 *Optimization Problems*

2.6.1 Finding Absolute Extreme Values



STEPS

(1)Find all ***critical points*** in the interior.

(2)Evaluate ***f*(*c*)**, where ***c*** is a ***critical*** or ***end*** point

(3)The ***largest*** & ***smallest*** of these values will be the ***absolute max*** & ***min*** values respectively

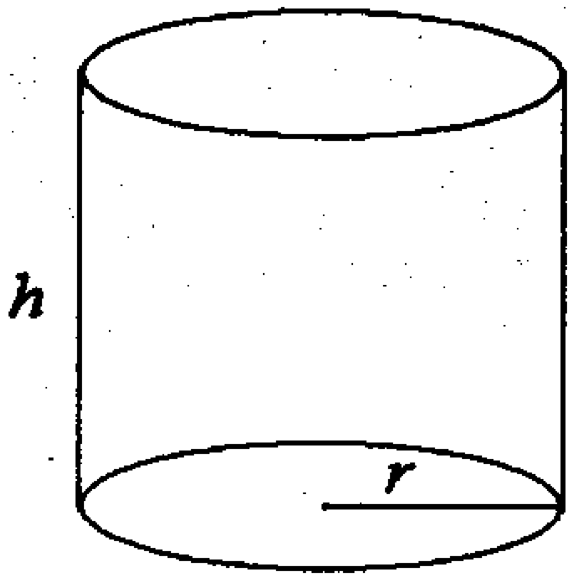
2.6.2 *Example*

We are asked to design a 1000cm^3 can shaped like a right circular cylinder. What dimensions will use the least material?

The surface area is minimum

Solution

Let r be the radius of the circular base and h the height of the can.



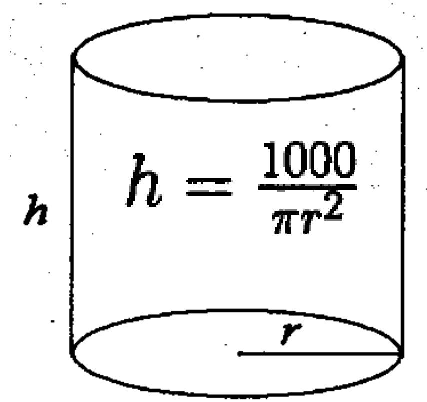
We have volume

$$V = \pi r^2 h = 1000$$

and so $h = \frac{1000}{\pi r^2}$.

The surface area

$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{2000}{r}, \qquad r > 0.$$



$$A' = 4\pi r - \frac{2000}{r^2} = 0 \quad \Rightarrow \quad r = \left(\frac{500}{\pi}\right)^{1/3}$$

$$A'' = 4\pi + \frac{4000}{r^3} > 0, \qquad \text{for } r > 0.$$

Thus $r = \left(\frac{500}{\pi}\right)^{\frac{1}{3}}$ leads to minimum of A. This value of r gives $\boxed{h = 2r.}$

Why? see next slide

$$h = \boxed{\frac{1000}{\pi r^2}}$$

$$\boxed{r = \left(\frac{500}{\pi}\right)^{1/3}}$$

$$r^3 = \frac{500}{\pi}$$

$$h = \frac{1000}{\pi r^2} = \frac{1000r}{\pi r^3} = \frac{1000r}{500} = 2r$$

A wire 10 m long is cut into two pieces. One piece is used to form a square. The other piece is used to form a rectangle with length twice as long as its width. If the total area enclosed by the two figures is minimum, then this minimum area in square metres equals

Ans. 50/17

A lighthouse L is located on a small island 5 km north of a point A on a straight east-west shoreline. A cable is to be laid from L to a point B on the shoreline 10 km east of A. The cable will be laid through the water in a straight line from L to a point C on the shoreline between A and B, and then from C to B along the shoreline. The part of the cable lying in the water costs \$5000 per km, and the part along the shoreline costs \$4000 per km. Find the minimum total cost of the cable.

Ans. 55000

2.7 Indeterminate Forms

☺ $$\lim_{x\to 1}\frac{x-1}{9x}=$$

☺ $$\lim_{x\to 1^+}\frac{3x}{x-1}= \qquad \lim_{x\to 1^-}\frac{3x}{x-1}=$$

♥ $$\lim_{x\to 1}\frac{x^2-1}{x-1}=$$

☻ $$\lim_{x\to 1}\frac{\ln x}{x-1}$$

Forms:

$\frac{0}{0}$	$\frac{\infty}{\infty}$	$0\cdot\infty$	$\infty-\infty$	1^{∞}	∞^{0}	0^{0}

2.7.1 *L´Hôpital’*s **Rule**

Suppose **(1)** $\boldsymbol{f}$ & $\boldsymbol{g}$ are ***differentiable*** on an open interval I containing pt a,

(2) $\boldsymbol{f}(a) = \boldsymbol{g}(a) = 0$, &

(3) $\boldsymbol{g}'(x) \neq 0$ for all x in $I \setminus \{a\}$.

Then

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

(1661- 1704)

2.7.2 Example

$$(i)\ \lim_{x\to 0}\frac{3x-\sin x}{x}=\left.\frac{3-\cos x}{1}\right|_{x=0}=2$$

$$(ii)\ \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\left.\frac{(1/2)(1+x)^{-1/2}}{1}\right|_{x=0}=\frac{1}{2}$$

$$(iii)\ \lim_{x\to 0}\frac{x-\sin x}{x^3}=\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\lim_{x\to 0}\frac{\sin x}{6x}=\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6}$$

$$(iv)\lim_{x\to 0}\frac{1-\cos x}{x+x^2}=\lim_{x\to 0}\frac{\sin x}{1+2x}=0$$

Examples

- ♣ $$\lim_{x\to 1}\frac{\ln x}{x-1}$$

- ♣ $$\lim_{x\to 0}\frac{\sin x}{x}$$

$$\lim_{x\to 1}\frac{x^2-1}{x^3+x-2} = \boxed{\lim_{x\to 1}\frac{2x}{3x^2+1}}$$

STOP HERE

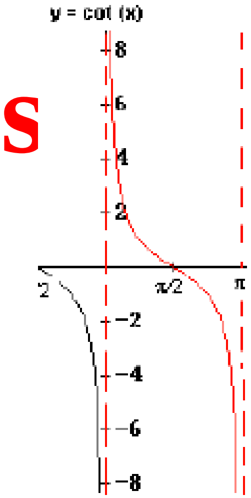
$$\frac{1}{2}$$

$$= \lim_{x\to 1}\frac{2}{6x} = \lim_{x\to 1}\frac{0}{6} = 0$$

WRONG

The rule doesn't apply when the '*numerator*' or '*denominator*' has a *finite nonzero* limit.

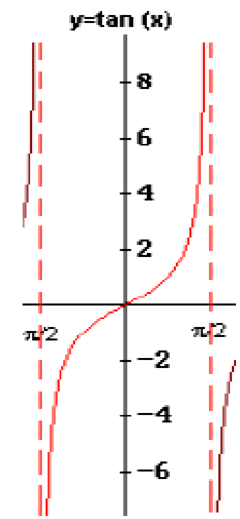
2.7.3, 2.7.4, 2.4.5 Other Forms



- ∞ / ∞

♣ $$\lim_{x\to\infty} \frac{x-2x^2}{3x^2+5} = \lim_{x\to\infty} \frac{1-4x}{6x} = \lim_{x\to\infty} \frac{-4}{6} = -\frac{2}{3}.$$

♣ $$\lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{1+\tan x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\sec^2 x}{\sec^2 x} = 1.$$



- $0 \cdot \infty$ (change to 0/0 or ∞/∞)

$$\lim_{x\to 0^+} x\cot x = \lim_{x\to 0^+} \frac{x}{\tan x} = \lim_{x\to 0^+} \frac{1}{\sec^2 x} = 1.$$

- $\infty - \infty$

(change to 0/0 or ∞/∞) $\lim_{x\to 0^+} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$

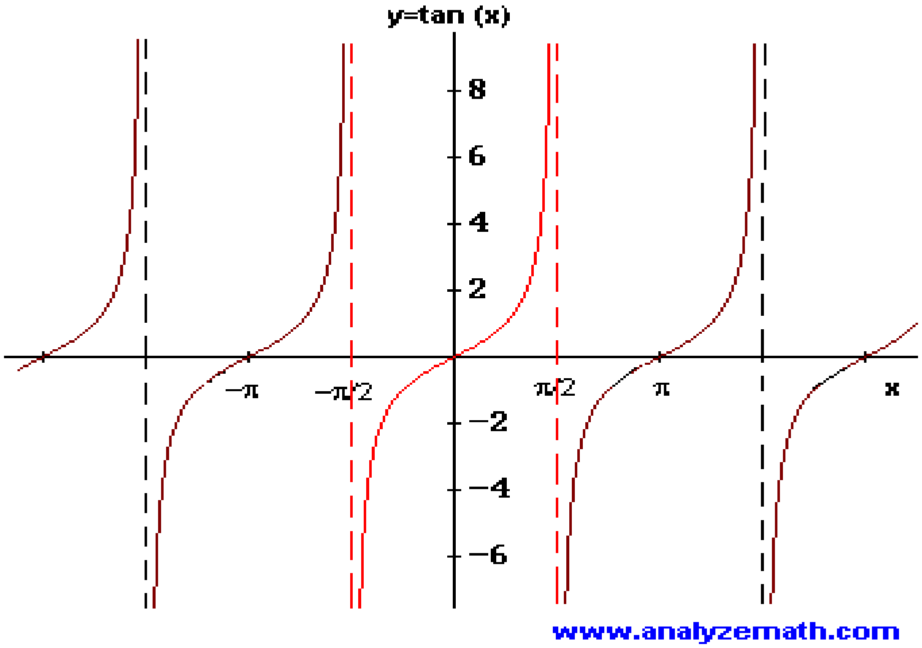
$$= \lim_{x\to 0^+} \frac{x - \sin x}{x \sin x} = \lim_{x\to 0^+} \frac{1 - \cos x}{x \cos x + \sin x}$$

$$= \lim_{x\to 0^+} \frac{\sin x}{-x \sin x + \cos x + \cos x}$$

$$= 0$$

$$\boxed{1^{\infty}, \infty^{0}, 0^{0}}$$ (change to 0/0 or ∞/∞)

♣ $$\lim_{x \to \frac{\pi}{2}^-} (\sin x)^{\tan x}$$

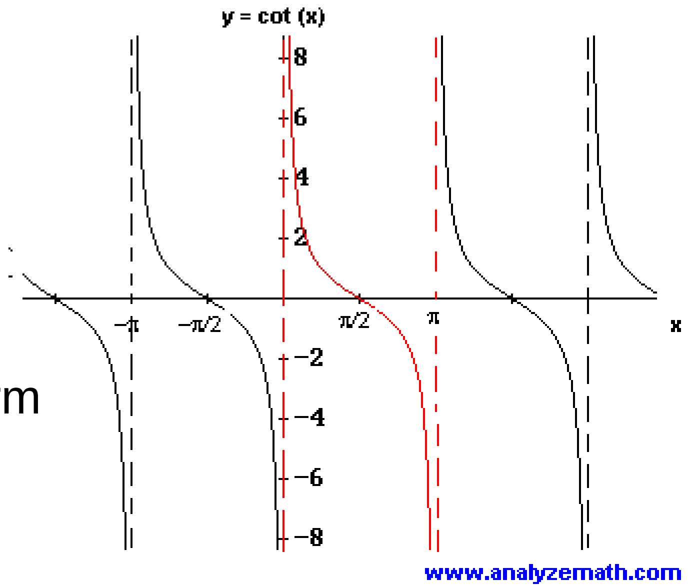


$$y = (\sin x)^{\tan x}$$

$$\ln y = \tan x \ln(\sin x) \quad = \frac{\ln(\sin x)}{\cot x}$$

$$\lim_{x \to \frac{\pi}{2}^-} \frac{\ln(\sin x)}{\cot x}$$

0/0 form



$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x} \qquad -\left(\frac{\cos x}{\sin x}\right) \sin^2 x$$

$$= \lim_{x \to \frac{\pi}{2}^-} -\cos x \sin x \quad = 0$$

$$\lim_{x \to \frac{\pi}{2}^-} (\ln y) = 0 \quad \Rightarrow \quad \ln\left[\lim_{x \to \frac{\pi}{2}^-} y\right] = 0$$

$$\Rightarrow \boxed{\lim_{x \to \frac{\pi}{2}^-} y = e^0 = 1}$$

Notes

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

(1) The rule applies to the forms $0/0$ & ∞/∞ ONLY; & ‘$x\to a$’ may be replaced by ‘$x\to\infty$’.

(2) Continue to ***differentiate*** f & g as long as we get the form $0/0$ (or ∞/∞).

(3) The rule doesn’t apply when the ‘*numerator*’ or ‘*denominator*’ has a ***finite nonzero*** limit.

(4) To apply the rule to f/g, we do $f'(x)/g'(x)$ not $(f(x)/g(x))'$

(5) Convert the forms $0\cdot\infty$ & $\infty-\infty$ to $0/0$ or ∞/∞ by algebraic manipulations before applying the rule.

(6) Convert the forms $1^{\infty}, \infty^{0}, 0^{0}$ to $0/0$ or ∞/∞ by first taking ‘**ln**’.

END

Appendix

Show that

$$\lim_{x\to\infty}(1+\frac{1}{x})^{x}=e$$

$$\lim_{x\to\infty} x^{\frac{1}{x}}=1$$

$$\lim_{x\to\infty}(x^{\frac{1}{x}}-1)^{\frac{1}{x}}=1.$$

Evaluate (*) $\lim_{x\to 1}\dfrac{\sin(\ln\sqrt{x})^3}{(x-1)^3}$

$$(*) = \lim_{x\to 1}\frac{\sin(\ln\sqrt{x})^3}{(\ln\sqrt{x})^3}\,\frac{(\ln\sqrt{x})^3}{(x-1)^3}$$

Let $\theta = (\ln\sqrt{x})^3$

$$= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{x\to 1}\left(\frac{\ln\sqrt{x}}{x-1}\right)^3 = 1\cdot\lim_{x\to 1}\left(\frac{\frac{1}{2}\ln x}{x-1}\right)^3$$

$$= \frac{1}{8}\lim_{x\to 1}\left(\frac{\ln x}{x-1}\right)^3 = \frac{1}{8}\left(\lim_{x\to 1}\frac{\ln x}{x-1}\right)^3$$

$$= \frac{1}{8}\left(\lim_{x\to 1}\frac{\frac{1}{x}}{1}\right)^3 = \frac{1}{8}\,.$$

Find the limit $\lim_{x\to+\infty}(x+e^x+e^{2x})^{\frac{1}{x}}$ if it exists

Let $y=(x+e^x+e^{2x})^{1/x}$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln(x+e^x+e^{2x})}{x}$$

$$=\lim_{x\to\infty}\frac{\left(\frac{1}{x+e^x+e^{2x}}\right)(1+e^x+2e^{2x})}{1}$$

$$=\lim_{x\to\infty}\frac{1+e^x+2e^{2x}}{x+e^x+e^{2x}} \qquad \frac{\infty}{\infty}$$

L H rule does not work here

$(\div e^{2x})$

$$=\lim_{x\to\infty}\frac{e^{-2x}+e^{-x}+2}{xe^{-2x}+e^{-x}+1}=2.$$

$$\lim_{x\to\infty}y=e^2.$$

Past Exam Question

Find the value of

$$\lim_{x \to 0} \frac{\cos^2 8x - \cos^2 5x}{x^2}.$$

$$= \left(\lim_{x \to 0} \frac{\cos 8x - \cos 5x}{x^2} \right) \left(\lim_{x \to 0} (\cos 8x + \cos 5x) \right)$$

$$= 2 \lim_{x \to 0} \frac{-8 \sin 8x + 5 \sin 5x}{2x}$$

$$= \lim_{x \to 0} \left(-64 \cos 8x + 25 \cos 5x \right)$$

$$= \underline{\underline{-39}}$$

Show that $\ln(1+x) < x$ for all $x > 0$

Let $f(x) = x - \ln(1 + x)$, where $x \geq 0$.

Observe

$$f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0 \ \forall\ x > 0$$

$\Rightarrow$ f is increasing on $[0, \infty)$

$$\Rightarrow \quad f(x) > f(0) = 0 \qquad \forall\ x > 0$$

$$\Rightarrow \quad x - \ln(1 + x) > f(0) = 0 \qquad \forall\ x > 0$$

$$\Rightarrow \quad x > \ln(1 + x) \qquad \forall\ x > 0$$