# Chapter 10. Surface Integrals

## 10.1 Parametric Surfaces

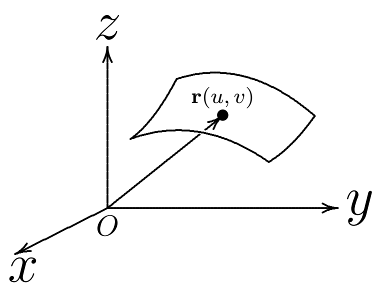
A **parametric representation** of a surface is given by the two-variable vector function

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad (1)$$

where $u$ and $v$ are two independent parameters.

The collection of points with position vectors (1) form a surface in the $xyz$-space.

The equations $x = x(u, v), y = y(u, v), z = z(u, v)$ are called the **parametric equations** of the surface.



### 10.1.1 Example (Planes)

For a general plane $ax+by+cz=d$, we can let two of the three components be $u$ and $v$ and obtain the remaining component in terms of $u$ and $v$ using the above equation.

E.g. $3x+2y-4z=6$: Let $x(u,v)=u,\ y(u,v)=v$. Then $z(u,v)=\frac{1}{4}(3x+2y-6)$. So the parametric representation of this plane is

$$\mathbf{r}(u,v)=u\mathbf{i}+v\mathbf{j}+\left(\frac{1}{4}(3u+2v-6)\right)\mathbf{k}.$$

If one variable is absent from the equation, we let the missing component be $u$ or $v$.

E.g. $2y+x=7$: Let $z(u,v)=u$. Then $y(u,v)=v$

and $x(u, v) = 7 - 2v$.

$$\mathbf{r}(u, v) = (7 - 2v)\mathbf{i} + v\mathbf{j} + u\mathbf{k}.$$

If two variables are absent from the equation, we let the two missing components be $u$ or $v$.

E.g. The $xy$-plane is given by

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + 0\mathbf{k}.$$

### 10.1.2 Example (Surfaces of the form $z = f(x, y)$)

A natural parametric representation of $S$ is

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}$$

E.g. The paraboloid $z = x^2 + y^2$.

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (u^2 + v^2)\mathbf{k}.$$

E.g. The upper cone $z = \sqrt{x^2 + y^2}$.

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{u^2 + v^2}\mathbf{k}.$$

### 10.1.3 Example (Spheres )

We have a standard parametric representation for a sphere $x^2 + y^2 + z^2 = a^2$ of radius $a$ centered at the origin:

$\mathbf{r}(u, v) = (a \sin u \cos v)\mathbf{i} + (a \sin u \sin v)\mathbf{j} + (a \cos u)\mathbf{k}.$

E.g. When $0 \le u \le \pi, \quad 0 \le v \le 2\pi$, the representation gives the full sphere.

When $0 \le u \le \pi/2, \quad 0 \le v \le 2\pi$, the representation gives the upper hemisphere.

### 10.1.4 Example (Circular cylinders)

We have a standard parametric representation for circular cylinder $x^2 + y^2 = a^2$ about the $z$-axis:

$$\mathbf{r}(u, v) = (a \cos u)\mathbf{i} + (a \sin u)\mathbf{j} + v\mathbf{k}.$$

Here $u$ measures the angle from the positive $x$-axis (about the $z$-axis) while $v$ measures the height from the $xy$-plane along the cylinder.

Similarly, for $x^2+z^2 = a^2$ and $y^2+z^2 = a^2$ (cylinders about $y$- and $x$-axes resp.), we have respectively

$$\mathbf{r}(u, v) = (a \cos u)\mathbf{i} + v\mathbf{j} + (a \sin u)\mathbf{k}$$

and

$$\mathbf{r}(u, v) = v\mathbf{i} + (a \cos u)\mathbf{j} + (a \sin u)\mathbf{k}.$$

### 10.1.5 Tangent planes and normal vectors

Let $S$ be a surface given by the parametric representation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \qquad (2)$$

We shall find the equation of the tangent plane to $S$ at a point $P_0$ with position vector $\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$.

Let us fix $v = v_0$ in (2) above.

Then the vector equation

$$\mathbf{r}(u,v_0) = x(u,v_0)\mathbf{i} + y(u,v_0)\mathbf{j} + z(u,v_0)\mathbf{k}$$

represents a space curve $C_1$ on $S$ passing through the point $P_0$.

The tangent vector to $C_1$ at $P_0$ is given by $\dfrac{d}{du}\mathbf{r}(u,v_0)\,|_{u=u_0}$, which is simply

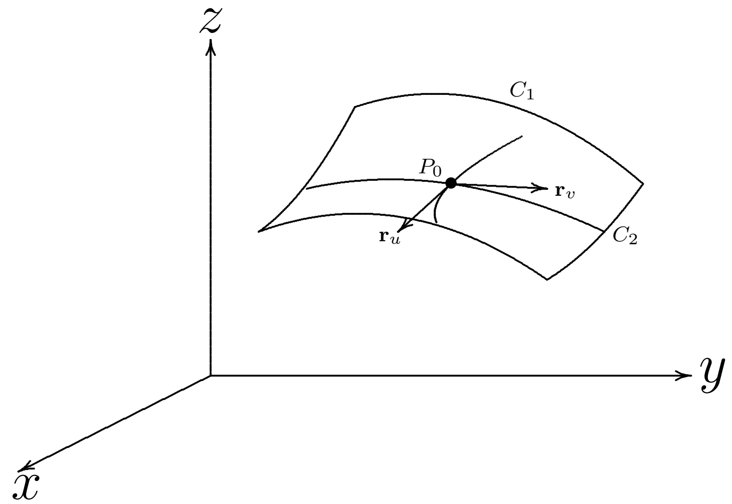
$$\mathbf{r}_u \equiv \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$

Similarly, if we fix $u = u_0$ in (2), we get another space curve $C_2$ with vector equation

$$\mathbf{r}(u_0, v) = x(u_0, v)\mathbf{i} + y(u_0, v)\mathbf{j} + z(u_0, v)\mathbf{k}.$$

The tangent vector to $C_2$ at $P_0$ is given by

$$\mathbf{r}_v \equiv \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$




Both vectors $\mathbf{r}_u$ and $\mathbf{r}_u$ lie in the tangent plane to $S$ at $P_0$. Therefore, the cross product $\mathbf{r}_u \times \mathbf{r}_v$, assuming it is nonzero, provides a normal vector to the tangent plane to $S$ at $P_0$. Therefore, $(\mathbf{r} - \mathbf{r}_0) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 0$ is the equation of the tangent plane.

### 10.1.6 Example

Find the equation of the tangent plane to the surface with parametric representation

$$\mathbf{r}(u, v) = u\mathbf{i} + v^2\mathbf{j} + (u^2 - v)\mathbf{k}$$

at the point $(1, 4, -1)$.

**Solution:** $\mathbf{r}_u = \mathbf{i} + 0\mathbf{j} + 2u\mathbf{k}$ and $\mathbf{r}_v = 0\mathbf{i} + 2v\mathbf{j} - \mathbf{k}$. Thus, a normal vector to the tangent plane is $\mathbf{r}_u \times \mathbf{r}_v = -4uv\mathbf{i} + \mathbf{j} + 2v\mathbf{k}$. The point $(1, 4, -1)$ corresponds to $\mathbf{r}(u, v) = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$. So, we have $(u, v) = (1, 2)$. Then, the normal vector at $(u, v) =$

$(1,2)$ is $-8\mathbf{i}+\mathbf{j}+4\mathbf{k}$. Therefore, the equation of the tangent plane to the surface at $(1,4,-1)$ is

$$[(x-1)\mathbf{i}+(y-4)\mathbf{j}+(z+1)\mathbf{k}]\cdot(-8\mathbf{i}+\mathbf{j}+4\mathbf{k})=0$$

or $-8x+y+4z+8=0$.

### 10.1.7 Example

If $S$ has Cartesian equation $z=f(x,y)$. Then a parametric representation of $S$ is

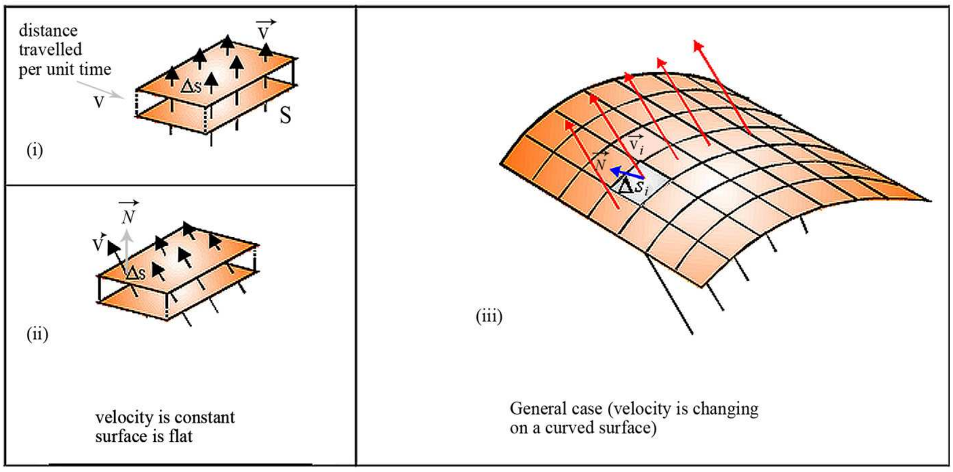
$$\mathbf{r}(u,v)=u\mathbf{i}+v\mathbf{j}+f(u,v)\mathbf{k}.$$

Thus, $\mathbf{r}_u=\mathbf{i}+0\mathbf{j}+f_u\mathbf{k}$ and $\mathbf{r}_v=0\mathbf{i}+\mathbf{j}+f_v\mathbf{k}$.

So the normal vector $\mathbf{r}_u\times\mathbf{r}_v=-f_u\mathbf{i}-f_v\mathbf{j}+1\mathbf{k}$.

## 10.2 Surface Integrals

Similar to line integrals, surface integrals involve integration over some (bounded) surfaces. Suppose $S$ is a surface and imagine a fluid with velocity $\mathbf{v}$ flows through $S$. We wish to calculate the total volume of fluid flowing out of $S$ per unit time.




Case (i): The fluid velocity is constant over flat surface $S$ and its direction is perpendicular to $S$. Then the volume flow rate is given by distance traveled per unit time multiplied with the area of $S$:

$$w = \|\mathbf{v}\|\Delta s.$$

Case (ii): The fluid velocity is constant over flat surface $S$ but its direction is not perpendicular to $S$. Then the volume flow rate is given by

$$w = \mathbf{v} \cdot \mathbf{N}\Delta s$$

where $\mathbf{N}$ is the unit normal vector to $S$.

Case (iii): The fluid velocity is changing over curved surface $S$. We can divide up the surface into small segments and then sum the volume flow rate of the individual segments to get the total flow rate. In a particular segment, we have

$$w_i \approx \mathbf{v}(x_i, y_i, z_i) \cdot \mathbf{N_i} \Delta s_i.$$

So the total flow rate is approximately

$$w \approx \sum_{1}^{n} \mathbf{v}(x_i, y_i, z_i) \cdot \mathbf{N_i} \Delta s_i \tag{3}$$

If we let $n$ goes to infinity, the RHS of (3) becomes an integral

$$\iint_S \mathbf{v}(x, y, z) \cdot \mathbf{N} ds$$

which represents the actual total volume flow rate. This integral is called a surface integral of the vector field $\mathbf{v}$.

There are two types of surface integrals, one for scalar functions and the other for vector fields.

### 10.2.1 Surface integrals of scalar functions

Let $f(x, y, z)$ be a function defined on a (bounded) surface $S$. Then for the parametric representation $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ of $S$, the corresponding set of ordered pairs $(u, v)$ come from a bounded domain $D$.

The **surface integral of a scalar function** $f$ over $S$ is

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v)) \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA.$$

The RHS of the above equation is a double integral over a domain $D$. Usually, $D$ can be described by giving the ranges of $u$ and $v$.

### 10.2.2 Example

Evaluate $\displaystyle\iint_S (xz + yz)\,dS$, where $S$ is part of the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.

**Solution:** A parametric representation of the sphere is given by (see Example 10.1.3)

$$\mathbf{r}(u, v) = 3\sin u\cos v\mathbf{i} + 3\sin u\sin v\mathbf{j} + 3\cos u\mathbf{k}.$$

To represent the first octant, the domain $D$ is given by $0 \le u \le \pi/2$ and $0 \le v \le \pi/2$.

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\cos u\cos v & 3\cos u\sin v & -3\sin u \\ -3\sin u\sin v & 3\sin u\cos v & 0 \end{vmatrix} \\ &= 9\sin^2 u\cos v\mathbf{i} + 9\sin^2 u\sin v\mathbf{j} + 9\sin u\cos u\mathbf{k}.\end{aligned}$$

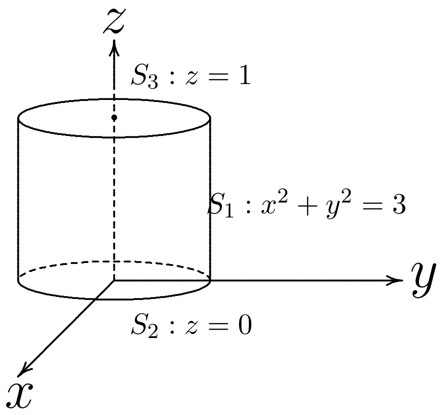
Therefore, $\|\mathbf{r}_u \times \mathbf{r}_v\| = 9\sin u$.

The surface integral is given by

$$
\begin{aligned}
&\iint_S (xz + yz)\, dS \\
&= \iint_D (9 \sin u \cos u \cos v + 9 \sin u \cos u \sin v) \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA \\
&= \int_0^{\pi/2} \int_0^{\pi/2} 81 \sin^2 u \cos u (\cos v \; + \sin v) du dv \\
&=81 \int_0^{\pi/2} \sin^2 u \cos u \; du \int_0^{\pi/2} (\cos v + \sin v) \; dv \\
&=81 \left[ \frac{1}{3} \sin^3 u \right]_0^{\pi/2} = 54.
\end{aligned}
$$

### 10.2.3 Example

Evaluate $\iint_S z\, dS$, where $S$ is the closed surface bounded laterally by $S_1$: the cylinder $x^2 + y^2 = 3$; bounded below by $S_2$: the $xy$-plane and bounded on top by $S_3$: the horizontal plane $z = 1$.




**Solution:** The surface integral is the sum of three surface integrals:

$$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS.$$

The surface $S_1$ is part of a circular cylinder. By Example 10.1.4, it has a parametric representation

$\mathbf{r}(u, v) = \sqrt{3}\cos u\mathbf{i} + \sqrt{3}\sin u\mathbf{j} + v\mathbf{k}.$

Thus, $\mathbf{r}_u \times \mathbf{r}_v = \sqrt{3}\cos u\mathbf{i} + \sqrt{3}\sin u\mathbf{j} + 0\mathbf{k}$

and $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{3}.$

Since $S_1$ is a full cylinder, the range of $u$ is given by

$0 \le u \le 2\pi.$

Moreover, $S_1$ is bounded above by the plane $z = 1$ and below by $z = 0$, so the range of $v$ is given by $0 \leq v \leq 1$.

Therefore,

$$\begin{aligned}\iint_{S_1} z\, dS &= \iint_D v \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA \\ &= \int_0^{2\pi} \int_0^1 \sqrt{3} v\, dv du = \int_0^{2\pi} \frac{\sqrt{3}}{2} du \\ &= \sqrt{3}\pi.\end{aligned}$$

$S_2$ is on the $xy$-plane, so we have $z = 0$. Thus the integrand of $\iint_{S_2} z\, dS$ is zero so that the integral has value zero. Therefore,

$$\iint_{S_2} z\, dS = 0.$$

The surface $S_3$ is on the horizontal plane $z = 1$. Thus

$$\iint_{S_3} z\,dS = \iint_{S_3} dS = \text{area of } S_3 = \pi(\sqrt{3})^2 = 3\pi.$$

Consequently,

$$\iint_{S} z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} zdS + \iint_{S_3} zdS = (3+\sqrt{3})\pi.$$

### 10.2.4 Surface integrals of vector fields

Let $\mathbf{F}$ be a continuous vector field defined on a surface $S$ with a unit normal vector $\mathbf{n}$. We have seen at the beginning of this section that the **surface integral** of $\mathbf{F}$ over $S$ is $\iint_S \mathbf{F} \cdot \mathbf{n}\,dS$. We usually simplify the notation as

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

This integral is also called the **flux** of $\mathbf{F}$ over $S$ as it is related to the volume flow rate of fluid.

If $S$ is given by the parametric representation $\mathbf{r} = \mathbf{r}(u, v)$ with domain $D$,

then

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}\, dS \\
&= \iint_D \left[ \mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \right] \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA \\
&= \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.
\end{aligned}$$

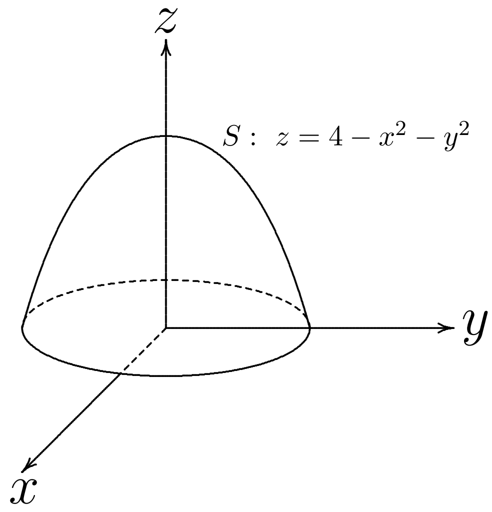
Therefore,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

### 10.2.5 Example

Evaluate $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + xy\mathbf{k}$, and $S$ is the part of the paraboloid $z = 4 -$

$x^2 - y^2$ above the $xy$-plane.




**Solution:** Since $S$ has Cartesian equation $z = 4 - x^2 - y^2$, the parametric representation is

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (4 - u^2 - v^2)\mathbf{k}.$$

The region $D$ is then the projection onto the $xy$-plane, which is the disk of radius 2.

We have $\mathbf{r}_u = \mathbf{i} + 0\mathbf{j} - 2u\mathbf{k}$, $\mathbf{r}_v = 0\mathbf{i} + \mathbf{j} - 2v\mathbf{k}$ and $\mathbf{r}_u \times \mathbf{r}_v = 2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k}$.

Therefore,

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA \\
&= \iint_D (u\mathbf{i} + v\mathbf{j} + uv\mathbf{k}) \cdot (2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k})\, dA \\
&= \iint_D (2u^2 + 2v^2 + uv)\, dA \\
&= \int_0^{2\pi} \int_0^2 (2r^2 + r^2 \cos\theta \sin\theta)\, r dr d\theta = 16\pi.
\end{aligned}$$

Note that as the region $D$ is a circular disk, we compute the double integral in polar coordinates.

### 10.2.6 Example

Let $\mathbf{F}(x,y,z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$. Evaluate $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $S$ is the sphere $x^2 + y^2 + z^2 = 1$.

**Solution:** A parametric representation of the unit sphere is given by

$$\mathbf{r}(u,v) = \sin u \cos v \mathbf{i} + \sin u \sin v \mathbf{j} + \cos u \mathbf{k},$$

with $D$ given by $0 \le u \le \pi$ and $0 \le v \le 2\pi$.

We have

$$\mathbf{r}_u \times \mathbf{r}_v = \sin^2 u \cos v \mathbf{i} + \sin^2 u \sin v \mathbf{j} + \sin u \cos u \mathbf{k}$$

and

$$\mathbf{F}(\mathbf{r}(u,v)) = \sin u \sin v \mathbf{i} + \sin\ \ u \cos v \mathbf{j} + \cos u \mathbf{k}.$$

Thus,

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 2 \sin^3 u \sin v \cos v + \sin u \cos^2 u.$$

Therefore,

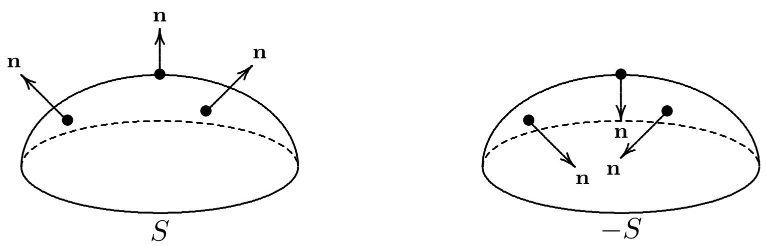
$$\begin{aligned}
&\iint_S \mathbf{F} \cdot d\mathbf{S} \\
&= \int_0^{2\pi} \int_0^{\pi} (2 \sin^3 u \sin v \cos v + \sin u \cos^2 u)\, du dv \\
&= \int_0^{\pi} \sin^3 u\, du \int_0^{2\pi} \sin 2v\, dv + \int_0^{\pi} \sin u \cos^2 u\, du \int_0^{2\pi} dv \\
&= 4\pi/3.
\end{aligned}$$

### 10.2.7 Orientation of surfaces

Note that, in the above example, if we switch the order of $u$ and $v$, then

$$\mathbf{r}_v \times \mathbf{r}_u = -\mathbf{r}_u \times \mathbf{r}_v$$

and the surface integral will be evaluated to $-4\pi/3$. Therefore, for surface integral of a vector field, the value depends on the choice of the normal vector, which is known as the **orientation** of the surface.




If $S$ is a surface given in parametric form by $\mathbf{r} = \mathbf{r}(u, v)$, then the normal vector $\mathbf{r}_u \times \mathbf{r}_v$ automatically supply an orientation to $S$.

The opposite orientation is given by $\mathbf{r}_v \times \mathbf{r}_u = -\mathbf{r}_u \times \mathbf{r}_v$ and the corresponding oriented surface is denoted by $-S$. Then

$$\iint_{-S} \mathbf{F} \cdot d\mathbf{S} = -\iint_{S} \mathbf{F} \cdot d\mathbf{S}.$$

### 10.2.8 Example

In example 10.2.5, the normal vector we used is $\mathbf{r}_u \times \mathbf{r}_v = 2u\mathbf{i}+2v\mathbf{j}+\mathbf{k}$. Consider the point $(0,0,4)$ on the paraboloid. This point corresponds to $u = 0, v = 0$. At this point, $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{k}$, which is pointing "upwards". Hence the orientation of the paraboloid we used in this example is given by the **upward normal vector**.

### 10.2.9 Example

In Example 10.2.6, the normal vector we used is

$$\mathbf{r}_u \times \mathbf{r}_v = \sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \sin u \cos u\mathbf{k}.$$

Consider the point $(1, 0, 0)$ on the sphere.

This point corresponds to $u = \pi/2, v = 0$.

At this point, $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{i}$, which is pointing "outwards" away from the sphere. Hence the orientation of the sphere we used in this example is given by the **outward normal vector**.

## 10.3 Curl and Divergence

In this section, we introduce two operators on vector fields which will be used in the subsequent sections.

### 10.3.1 Curl

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field in the $xyz$-space. The **curl** of $\mathbf{F}$ is defined by

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

is a vector field.

### 10.3.2 Divergence

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field in the $xyz$-space. The **divergence** of $\mathbf{F}$ defined by

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

is a scalar function.

### 10.3.3 Del operator

The curl and divergence operators can be expressed in terms of the **del operator**:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Then

(i) taking the cross product of $\nabla$ with a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

So

$$\nabla \times \mathbf{F} = \text{curl } \mathbf{F}$$

(ii) taking the dot product of $\nabla$ with a vector field

$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$,

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\end{aligned}$$

So $\nabla \cdot \mathbf{F} = \text{div } \mathbf{F}$.

#### 10.3.4 Example

Let $\mathbf{F}(x, y, z) = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}$.

$$\begin{aligned}\text{curl } \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} \\ &= (xz^2 - xy^2)\mathbf{i} + (x^2y - yz^2)\mathbf{j} + (y^2z - x^2z)\mathbf{k}.\end{aligned}$$

$$\begin{aligned}\text{div } \mathbf{F} &= \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2) \\ &= 6xyz.\end{aligned}$$

### 10.3.5 Example

Show that curl $(\nabla f) = \mathbf{0}$.

**Solution:**

$$\text{curl } (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k}$$

$$= \mathbf{0}$$

since $f_{xy} = f_{yx}$ etc.

### 10.3.6 Curl and conservative fields

Let $\mathbf{F}$ be a vector field in the $xyz$-space.

$$\text{If curl } \mathbf{F} = \mathbf{0}, \text{ then } \mathbf{F} \text{ is a conservative field.}$$

The converse is also true.

### 10.3.7 Example

Find the curl of the velocity vector fields defined by (a) $\mathbf{F}_1 = x\mathbf{i}+y\mathbf{j}$, (b)$\mathbf{F}_2 = -y\mathbf{i}+x\mathbf{j}$, (c)$\mathbf{F}_3 = \cos y\mathbf{i}+\sin x\mathbf{j}$.

**Solution:**

(a) curl $\mathbf{F}_1 = \mathbf{0}$, (b) curl $\mathbf{F}_2 = 2\mathbf{k}$, (c) curl $\mathbf{F}_3 = (\cos x + \sin y)\mathbf{k}$.

### 10.3.8 Example

Find the divergence of the velocity vector fields defined by (a) $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$, (b) $\mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$, (c) $\mathbf{F}_3 = -x^2\mathbf{i} + y^2\mathbf{j}$.

**Solution:** (a) div $\mathbf{F}_1 = 2$, (b) div $\mathbf{F}_2 = 0$, (c) div $\mathbf{F}_3 = 2(y - x)$.

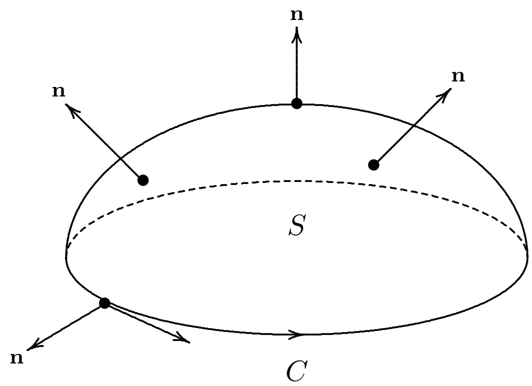
## 10.4 Stokes' Theorem

Let $S$ be an oriented piecewise-smooth surface that is bounded by a closed, piecewise-smooth boundary curve $C$.

Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on $S$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S}.$$

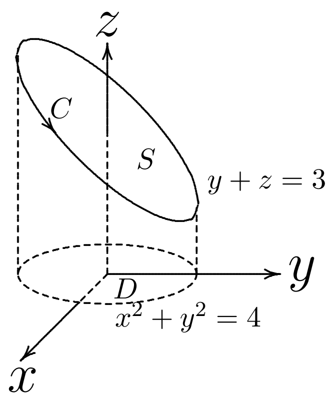
**Note**: In the above equation, the orientation of $C$ must be consistent with that of $S$: when you walk in the direction (orientation) around $C$ with your head pointing in the direction of the normal vector of $S$, the corresponding orientation of $S$ should be on your left.




### 10.4.1 Example

Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + xy\mathbf{k}$ and $C$ is the curve of intersection of the plane $y + z = 3$ and the cylinder $x^2 + y^2 = 4$. ($C$ is oriented in the counterclockwise sense when viewed from above.)

**Solution:** Let $S$ be the (bounded) surface enclosed by $C$ on the plane $y + z = 3$. So $S$ has parametric representation $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (3 - v)\mathbf{k}$ and the region $D$ is the disk of radius 2.




We have $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{j} + \mathbf{k}$, which is the upward normal vector of $S$. This gives the orientation of $S$ which agrees with that of $C$.

Also curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & xy \end{vmatrix} = 2x\mathbf{i} - 2z\mathbf{k}$.

By Stokes' Theorem,

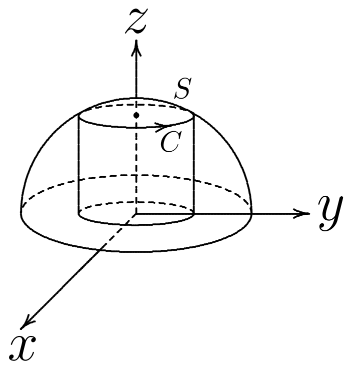
$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \\ &= \iint_D (2u\mathbf{i} - 2(3-v)\mathbf{k}) \cdot (\mathbf{j} + \mathbf{k})\, dA \\ &= \iint_D (-6 + 2v)\, dA \end{aligned}$$

Since $D$ is the disk of radius 2, we may use polar coordinates:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi}\int_0^2 (-6+2r\sin\theta) r\,dr\,d\theta \\ &= \int_0^{2\pi} (-12+\frac{16}{3}\sin\theta)\sin\theta)\,d\theta = -24\pi.\end{aligned}$$

### 10.4.2 Example

Use Stokes' Theorem to compute $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = y^2z\mathbf{i} + x\mathbf{j} + (x+y)\mathbf{k}$ and $S$ is the part of the upper hemisphere $z=\sqrt{9-x^2-y^2}$ that lies within the cylinder $x^2+y^2=5$ and the orientation of $S$ is given by the upward normal vector.



**Solution:** The boundary $C$ of $S$ is given by the intersection of the cylinder $x^2 + y^2 = 5$ and the upper hemisphere $z = \sqrt{9 - x^2 - y^2}$. Solving the two equations, we have $z = 2$. So the curve $C$ has a vector equation given by

$$\mathbf{r}(t) = \sqrt{5}\cos t\mathbf{i} + \sqrt{5}\sin t\mathbf{j} + 2\mathbf{k}.$$

With this vector equation, the curve traverses in anticlockwise direction when viewed from top. This agrees with the given orientation of $S$.

Now $\mathbf{r}'(t) = -\sqrt{5}\sin t\mathbf{i} + \sqrt{5}\cos t\mathbf{j} + 0\mathbf{k}$ and

$\mathbf{F}(\mathbf{r}(t)) = 10\sin^2 t\mathbf{i} + \sqrt{5}\cos t\mathbf{j} + \sqrt{5}(\cos t + \sin t)\mathbf{k}.$

By Stokes' Theorem,

$$\begin{aligned}\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\ &= \int_0^{2\pi} (-10\sqrt{5}\sin^3 t + 5\cos^2 t)\, dt = 5\pi.\end{aligned}$$

## 10.5 Divergence Theorem (or Gauss' Theorem)

Let $E$ be a solid region and let $S$ be the boundary of $E$, given with the **outward orientation***. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives in $E$. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F}\, dV.$$

* The outward orientation of the boundary surface of a solid region $E$ is the one for which the normal

vector point outward from $E$.

### 10.5.1 Example

Let $\mathbf{F}(x, y, z) = (x+y)\mathbf{i}+(y+z)\mathbf{j}+(z+x)\mathbf{k}$. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $S$ is the sphere $x^2 + y^2 + z^2 = 1$ with orientation given by the outward normal vector.

**Solution:** By the Divergence Theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} \, dV = \iiint_E 3 \, dV$$
$$= 3 \times \text{volume of the unit ball} = 4\pi.$$

### 10.5.2 Example

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F} = x^2\mathbf{i} + (xy + x\cos z)\mathbf{j} + e^{xy}\mathbf{k}$$

and $S$ is the surface of the cubic region $E$ bounded by the three coordinate planes $x = 0, y = 0, z = 0$ and the three planes $x = 1,\ y = 1,\ z = 1$. The orientation of $S$ is given by the outward normal vector.

**Solution:** The cubic region $E$ can be described as

$$E: \quad 0 \le x \le 1, \;\; 0 \le y \le 1, \;\; 0 \le z \le 1$$

By the Divergence Theorem, we have

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F}\, dV \\ &= \iiint_E 3x\, dV = 3\int_0^1 \int_0^1 \int_0^1 x\, dxdydz \\ &= \frac{3}{2}.\end{aligned}$$