MA1506
Mathematics II

Chapter 4
Laplace Transforms

In chapter one, we have learnt how to solve 2nd order ODE by

- Method of undetermined coefficients
- Variation of parameters

In this chapter, we shall study how to use the method of Laplace transform to solve some 2nd order ODE.

There are five parts in this chapter.
Part 1: Laplace transforms and inverse Laplace transforms
Part 2: Finding Laplace transforms and inverse Laplace transforms using known results
Part 3: Solving ODEs by Laplace transforms
Part 4: Laplace transforms of piecewise continuous functions and ODEs
Part 5: Laplace transforms of impulse functions and ODEs

4.1 Laplace transform

Laplace transform L(f) is a mapping L which maps a function f(t) to a function F(s), where F(s) is given by

$$L(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)dt$$

Example

Let f(t)=1 for t>0. Then

$$L(f) = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st} dt = \left[\frac{-e^{-st}}{s} \right]_{t=0}^{t=\infty} = \frac{1}{s}$$

where s>0.

Thus Laplace transform maps the constant function f(t)=1 to the function F(s)=1/s, where s>0.

In the above, if s<0, we will get

$$\int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st} dt = \left[\frac{-e^{-st}}{s} \right]_{t=0}^{t=\infty} = -\infty + \frac{1}{s}$$

if s=0, we will get

$$\int_0^\infty e^{-st} f(t)dt = \int_0^\infty f(t)dt = \int_0^\infty dt = \infty$$

As both integrals are infinite, we will only consider the case when s>0.

4.2 Inverse Laplace transform

If f(t)=1 for all t>0, then L(f)=1/s , s>0. Hence the inverse Laplace transform of 1/s is f(t)=1. We write

$$L^{-1}(1/s)=1$$

4.3 Basis Laplace transforms

$$L(e^{at})=\frac{1}{s-a},\quad s>a \qquad L^{-1}(1/(s-a))=e^{at}$$

$$L(t^n)=\frac{n!}{s^{n+1}},\quad n=0,1,2,\cdots \qquad L^{-1}\left(\frac{n!}{s^{n+1}}\right)=t^n$$

$$L(\sin\omega t)=\frac{\omega}{s^2+\omega^2} \qquad L^{-1}\left(\frac{\omega}{s^2+\omega^2}\right)=\sin\omega t$$

$$L(\cos\omega t)=\frac{s}{s^2+\omega^2} \qquad L^{-1}\left(\frac{s}{s^2+\omega^2}\right)=\cos\omega t$$

In the formula above, we assume t>0, s>0.

4.3 Basis Laplace transforms

$$L(e^{at})=\frac{1}{s-a}, s>a \qquad L^{-1}(1/(s-a))=e^{at}$$

$$L(t^n)=\frac{n!}{s^{n+1}}$$ n=0,1,2…

$$L^{-1}\left(\frac{n!}{s^{n+1}}\right)=t^n$$ n=0,1,2…

For t>0 and s>0, we have the following formula.

$$L(\sin\omega t)=\frac{\omega}{s^2+\omega^2} \qquad L^{-1}\left(\frac{\omega}{s^2+\omega^2}\right)=\sin\omega t$$

$$L(\cos\omega t)=\frac{s}{s^2+\omega^2} \qquad L^{-1}\left(\frac{s}{s^2+\omega^2}\right)=\cos\omega t$$

4.4 Linearity

We can apply L T term by term

$$L(af+bg)=aL(f)+bL(g)$$

We can also apply inverse L T term by term

$$L^{-1}(aF+bG)=aL^{-1}(F)+bL^{-1}(G)$$

where *a*, *b* are constants.

4.5 Operational properties

Let $L(f(t))=F(s)$

Then $L(tf(t))=-F'(s)$

$$L(t^n f(t))=(-1)^n F^{(n)}(s)$$

where n is a positive integer.

In Laplace transforms, we also have the following formula called the s-shifting formula, here we require s>a.

$$L\left(e^{at} f(t)\right) = F(s-a)$$

Part 2: Finding Laplace transforms and inverse Laplace transforms using known results

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4.6 Finding Laplace transforms

The following examples make use of the formula

$$L(t^n f(t)) = (-1)^n F^{(n)}(s)$$

Example 1

$$L(t\sin 2t) = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right] = \frac{4s}{(s^2+4)^2}$$

Example 2

$$L\left(t^2 \sin t\right) = \frac{d^2}{ds^2}\left[\frac{1}{s^2+1}\right] = \frac{2(-1+3s^2)}{(s^2+1)^3}$$

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Example 3

$$L(\cosh at) = L\left(\frac{1}{2}(e^{at}+e^{-at})\right) = \frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right)$$

$$L(\cosh at) = \frac{s}{s^2-a^2}, s > a \ge 0$$

Similarly,

$$L(\sinh at) = L\left(\frac{1}{2}(e^{at}-e^{-at})\right) = \frac{a}{s^2-a^2}$$

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4.6 Finding Laplace transforms

Example 4 Find $L(\sin^2 t)$

$$L(\sin^2 t) = L\left(\frac{1}{2}(1-\cos 2t)\right)$$

$$= \frac{1}{2}[L(1) - L(\cos 2t)] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= \frac{2}{s(s^2+4)}$$

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4.6 Finding Laplace transforms

Example 5 *s*-shifting

Applying the formula $L\left(e^{at} f(t)\right) = F(s-a)$

to $L(t^n) = \frac{n!}{s^{n+1}}$ $\quad L(\sin\omega t) = \frac{\omega}{s^2+\omega^2}$ $\quad L(\cos\omega t) = \frac{s}{s^2+\omega^2}$

we have

$$L(e^{ct}t^n) = \frac{n!}{(s-c)^{n+1}}$$

$$L(e^{ct}\sin\omega t) = \frac{\omega}{(s-c)^2+\omega^2}$$

$$L(e^{ct}\cos\omega t) = \frac{s-c}{(s-c)^2+\omega^2}$$

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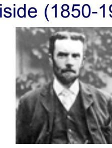
- The Laplace transform is named after the French mathematician Laplace, who studied this transform in 1782.

Pierre-Simon de Laplace (1749-1827)

- The techniques described in this chapter were developed primarily by Oliver Heaviside (1850-1925), an English electrical engineer.



Oliver Heaviside (1850-1925)

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Engineer Oliver Heaviside used Laplace transform in the nineteenth century to solve ODE.
However the mathematical community did not accept his work initially because he failed to justify his methods.
His reply: “ Should I refuse my dinner because I do not understand the process of digestion”.
Heaviside’s method (using Laplace transform to solve ODE) was eventually been justified in the twentieth century.

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4.7 Finding inverse Laplace transforms

Recall

$$L(e^{at}) = \frac{1}{s-a}, \quad s>a \qquad L(t^n) = \frac{n!}{s^{n+1}}, \quad n = 0,1,2,\cdots$$

$$L(\sin \omega t) = \frac{\omega}{s^2+\omega^2} \qquad L(\cos \omega t) = \frac{s}{s^2+\omega^2}$$

$$L(\cosh \omega t) = \frac{s}{s^2-\omega^2} \qquad L(\sinh \omega t) = \frac{\omega}{s^2-\omega^2}$$

where s>ω

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4.7 Finding inverse Laplace transforms

Recall

$$L(e^{ct}t^n) = \frac{n!}{(s-c)^{n+1}}$$

$$L(e^{ct}\sin \omega t) = \frac{\omega}{(s-c)^2+\omega^2}$$

$$L(e^{ct}\cos \omega t) = \frac{s-c}{(s-c)^2+\omega^2}$$

where s>c.

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4.7 Finding inverse Laplace transforms

We have

$$L(e^{at}) = \frac{1}{s-a} \qquad L(t^n) = \frac{n!}{s^{n+1}}$$

$$L(\sin \omega t) = \frac{\omega}{s^2+\omega^2} \qquad L(\cos \omega t) = \frac{s}{s^2+\omega^2}$$

$$L(\cosh \omega t) = \frac{s}{s^2-\omega^2} \qquad L(\sinh \omega t) = \frac{\omega}{s^2-\omega^2}$$

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$$L(e^{ct}t^n) = \frac{n!}{(s-c)^{n+1}}$$

$$L(e^{ct}\sin \omega t) = \frac{\omega}{(s-c)^2+\omega^2}$$

$$L(e^{ct}\cos \omega t) = \frac{s-c}{(s-c)^2+\omega^2}$$

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4.7 Finding inverse Laplace transforms

Example 6

From $L(t^n) = \frac{n!}{s^{n+1}}$ we get

$$L^{-1}\left\{\frac{3\ !}{s^4}\right\} = t^3$$

$$L^{-1}\left\{\frac{8}{s^3}\right\} = L^{-1}\left\{4\left(\frac{2\ !}{s^3}\right)\right\} = 4L^{-1}\left\{\frac{2\ !}{s^3}\right\} = 4t^2$$

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Example 7

From $L(\sin\omega t)=\dfrac{\omega}{s^2+\omega^2}$ $\quad L(\cos\omega t)=\dfrac{s}{s^2+\omega^2}$

$L(\cosh\omega t)=\dfrac{s}{s^2-\omega^2}$ $\quad L(\sinh\omega t)=\dfrac{\omega}{s^2-\omega^2}$

we get

$$L^{-1}\left\{\frac{4s+1}{s^2+9}\right\}=4L^{-1}\left\{\frac{s}{s^2+9}\right\}+\frac{1}{3}L^{-1}\left\{\frac{3}{s^2+9}\right\}=4\cos 3t+\frac{1}{3}\sin 3t$$

$$L^{-1}\left\{\frac{4s+1}{s^2-9}\right\}=4L^{-1}\left\{\frac{s}{s^2-9}\right\}+\frac{1}{3}L^{-1}\left\{\frac{3}{s^2-9}\right\}=4\cosh 3t+\frac{1}{3}\sinh 3t$$

Example 8

Let

$$Y(s)=-\frac{10}{(s+1)^3}=-\frac{10}{2!}\left[\frac{2!}{(s+1)^3}\right]=-5\left[\frac{2!}{(s+1)^3}\right]$$

Using $L(e^{ct}t^n)=\dfrac{n!}{(s-c)^{n+1}}$

we get

$$L^{-1}\{Y(s)\}=-5L^{-1}\left\{\frac{2!}{(s+1)^3}\right\}=-5t^2e^{-t}$$

Example 9

Find $L^{-1}\left[\dfrac{2s+13}{(s^2+5s+6)}\right]$

By partial fraction

$$\frac{2s+13}{(s+3)(s+2)}=\frac{A}{(s+3)}+\frac{B}{(s+2)}$$

$$2s+13=A(s+2)+B(s+3)$$

$$2s+13=(A+B)s+(2A+3B)$$

$$A+B=2,\ 2A+3B=13$$

$$A=-7,\ B=9$$

Hence

$$\frac{2s+13}{s^2+5s+6}=\frac{2s+13}{(s+3)(s+2)}$$

Recall $\dfrac{2s+13}{(s+3)(s+2)}=-\dfrac{7}{(s+3)}+\dfrac{9}{(s+2)}$

Applying $L(e^{at})=\dfrac{1}{s-a}$ we get

$$L^{-1}\left[\frac{2s+13}{(s+3)(s+2)}\right]=L^{-1}\left[-\frac{7}{(s+3)}+\frac{9}{(s+2)}\right]$$

$$=(-7)L^{-1}\left[\frac{1}{(s+3)}\right]+9L^{-1}\left[\frac{1}{(s+2)}\right]$$

$$=(-7)e^{-3t}+9e^{-2t}$$

Example 10

Using partial fraction

$$Y(s)=\frac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4}$$

we obtain A = 2, B = 5/3, C = 0, and D = -2/3.

By the formula $L(\cos\omega t)=\dfrac{s}{s^2+\omega^2}$ $\quad L(\sin\omega t)=\dfrac{\omega}{s^2+\omega^2}$

we obtain

$$Y(s)=\frac{2s}{s^2+1}+\frac{5/3}{s^2+1}-\frac{2/3}{s^2+4}$$

$$L^{-1}(Y(s))=2\cos t+\frac{5}{3}\sin t-\frac{1}{3}\sin 2t$$

Example 11

Find the inverse transform of $G(s)=\dfrac{s+1}{s^2+2s+5}$

We first complete the square:

$$G(s)=\frac{s+1}{s^2+2s+5}=\frac{s+1}{(s^2+2s+1)+4}=\frac{(s+1)}{(s+1)^2+4}$$

Applying the formula $L(e^{ct}\cos\omega t)=\dfrac{s-c}{(s-c)^2+\omega^2}$

we obtain $L^{-1}\{G(s)\}=e^{-t}\cos(2t)$

Example 12

Completing the square, we obtain

$$Y(s)=\frac{6}{s^2-8s+25}=\frac{6}{(s^2-8s+16)+9}$$

Thus $$Y(s)=2\left[\frac{3}{(s-4)^2+9}\right]$$

and we have $L^{-1}\{Y(s)\}=2e^{4t}\sin 3t$

by applying the formula

$$L(e^{ct}\sin\omega t)=\frac{\omega}{(s-c)^2+\omega^2}$$

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4.7 Finding inverse Laplace transforms

Example 13

$$Y(s)=\frac{4s-10}{s^2-6s+10}=\frac{4s-10}{(s^2-6s+9)+1}=\frac{4s-12+2}{(s-3)^2+1}$$
$$=\frac{4(s-3)+2}{(s-3)^2+1}=4\left[\frac{s-3}{(s-3)^2+1}\right]+2\left[\frac{1}{(s-3)^2+1}\right]$$

Thus

$$y(t)=4e^{3t}\cos t+2e^{3t}\sin t$$

Web site

http://wims.unice.fr/wims/wims.cgi

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Part 3 Solving ODE by Laplace transforms

4.8 Transform of derivatives

$$L(f')=sL(f)-f(0)$$
$$L(f'')=s^2L(f)-sf(0)-f'(0)$$
$$L(f^{(n)})=s^nL(f)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)}(0)$$

Note that derivatives disappear after taking Laplace transform. In this way, Laplace transform makes certain ODE or PDE become simpler.

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4.9 Solving ODE

Example 14 $y''+y=e^{2t}$ $y(0)=0, y'(0)=1$

Taking Laplace Transform

$$L(y''+y)=L(e^{2t})$$
$$L(y'')+L(y)=L(e^{2t})$$
$$s^2L(y)-sy(0)-y'(0)+L(y)=\frac{1}{s-2}$$
$$L(y)=\frac{1}{s^2+1}\left(1+\frac{1}{s-2}\right)=\frac{s-1}{(s-2)(s^2+1)}$$

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$$L(y)=\frac{1}{5}\frac{1}{s-2}-\frac{1}{5}\frac{s-3}{(s^2+1)}$$

To apply the formula

$$L(e^{at})=\frac{1}{s-a}\qquad L(\sin\omega t)=\frac{\omega}{s^2+\omega^2}\qquad L(\cos\omega t)=\frac{s}{s^2+\omega^2}$$

we rewrite $$L(y)=\frac{1}{5}\frac{1}{s-2}-\frac{1}{5}\frac{s}{(s^2+1)}+\frac{1}{5}\frac{3}{(s^2+1)}$$

Take inverse transform, we get

$$y(t)=\frac{1}{5}e^{2t}-\frac{1}{5}\cos t+\frac{3}{5}\sin t$$

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Exercise 1

$y''+2y'+5y=0$ $y(0)=2, y'(0)=-4$

Ans. $y(t)=2e^{-t}\cos 2t-e^{-t}\sin 2t$

Exercise 2

$y''-2y'+y=e^t+t$ $y(0)=1, y'(0)=0$

Ans. $y(t)=\frac{t^2}{2}e^t-e^t+t+2$

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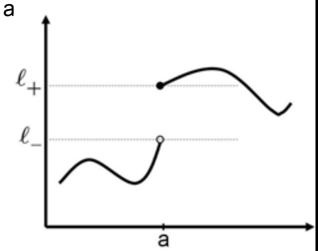
Part 4 Laplace transforms of piecewise continuous functions and ODEs

4.10 Piecewise Continuous Functions

Jump discontinuity at a

$$\lim_{t \to a^-} f(t) = \ell_-$$

$$\lim_{t \to a^+} f(t) = \ell_+$$



But *f(t)* is not continuous at a.

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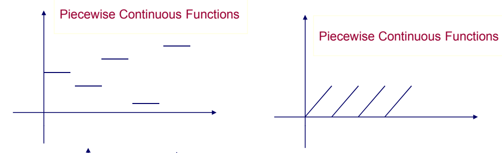
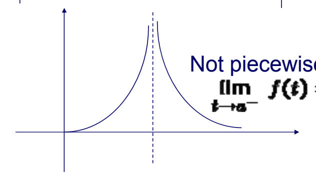
4.10 Piecewise Continuous Functions





Not piecewise cont

$$\lim_{t \to a^-} f(t) = \infty$$

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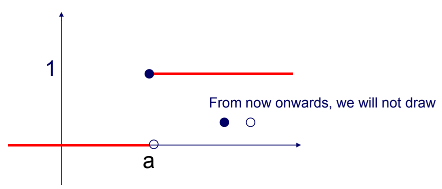
4.11 Unit Step (Heaviside) Function

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a. \end{cases}$$



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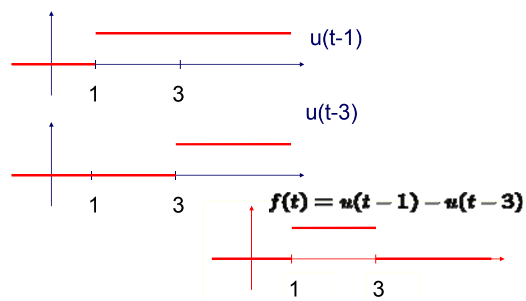
4.11 Unit Step (Heaviside) Function

Example 15 $f(t) = u(t-1) - u(t-3)$



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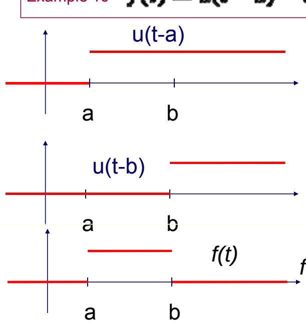
4.11 Unit Step (Heaviside) Function

Example 16 $f(t) = u(t-a) - u(t-b)$



$$f(t) = \begin{cases} 0, & t < a \\ 1, & a \leq t < b \\ 0, & t \geq b. \end{cases}$$

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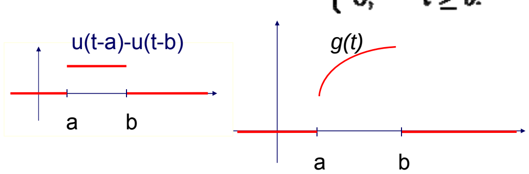
4.11 Unit Step (Heaviside) Function

Example 17

Let *g(t)* be a function of *t*. Then

$$g(t)(u(t-a) - u(t-b)) = \begin{cases} 0, & t < a \\ g(t), & a \leq t < b \\ 0, & t \geq b. \end{cases}$$



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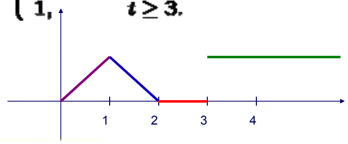
4.11 Unit Step (Heaviside) Function

Example 18 Piecewise cont. function in terms of *u(t)*

$$f(t)=\begin{cases} t, & 0\le t<1\\ 2-t, & 1\le t<2\\ 0, & 2\le t<3\\ 1, & t\ge 3.\end{cases}$$



$$t(u(t)-u(t-1))$$
$$+(2-t)(u(t-1)-u(t-2))$$
$$+u(t-3)$$

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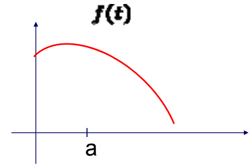
4.12 *t*-Shifting

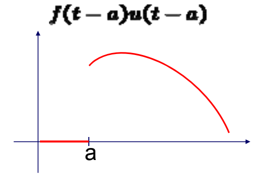
If *L(f(t)) = F(s)* then

$$L(f(t-a)u(t-a))=e^{-as}F(s)$$

$f(t)$



$f(t-a)u(t-a)$



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4.12 *t*-Shifting

If *L(f(t)) = F(s)* then

$$L(f(t-a)u(t-a))=e^{-as}F(s)$$

Hence

$$L(u(t-a))=\frac{e^{-as}}{s}$$

Recall $L(1)=\dfrac{1}{s}$

$u(t-a)$ is also denoted by $u_a(t)$

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$$L(f(t-a)u(t-a))=e^{-as}F(s)$$

4.12 *t*-Shifting

Example 19

$$L(t^2u(t-1))=L((t-1+1)^2u(t-1))$$
$$=L\big(((t-1)^2+2(t-1)+1)u(t-1)\big)$$
$$=L\big((t-1)^2u(t-1)\big)+2L\big((t-1)u(t-1)\big)$$
$$+L\big(u(t-1)\big)$$
$$=e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right).$$

$$\boxed{f(t)=t^2\Rightarrow F(s)=\frac{2}{s^3}}\qquad \boxed{f(t)=t\Rightarrow F(s)=\frac{1}{s^2}}$$

$$\boxed{L(u(t-a))=\frac{e^{-as}}{s}}$$

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4.12 *t*-Shifting

$$L(f(t-a)u(t-a))=e^{-as}F(s)$$

Example 20

$$L((e^t+1)u(t-2))$$
$$=L((e^{t-2}e^2+1)u(t-2))$$
$$=e^2L(e^{t-2}u(t-2))+L(u(t-2))$$
$$=e^{-2s}\left(\frac{e^2}{s-1}+\frac{1}{s}\right)$$

$$\boxed{f(t)=e^t\Rightarrow F(s)=\frac{1}{s-1}}\qquad \boxed{L(u(t-a))=\frac{e^{-as}}{s}}$$

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4.13 *t*-Shifting and ODE

Example 21

$$y''+3y'+2y=g(t)\qquad y(0)=0,\quad y'(0)=1$$

where $g(t)=\begin{cases}1, & 0\le t<1\\ 0, & t\ge 1.\end{cases}$

$$g(t)=u(t)-u(t-1)$$

$$L(g(t))=\frac{1}{s}-\frac{e^{-s}}{s}$$

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Example 22

$$y'' + 3y' + 2y = g(t) \qquad y(0) = 0, \quad y'(0) = 1$$

where $g(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1. \end{cases}$

LHS:

$$s^2 L(y) - sy(0) - y'(0) + 3(sL(y) - y(0)) + 2L(y)$$
$$= (s^2 + 3s + 2)L(y) - 1$$

$$(s^2 + 3s + 2)L(y) - 1 = L(g(t)) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$L(y) = \underbrace{\frac{s+1}{s(s^2+3s+2)}} - e^{-s}\left(\frac{1}{s(s^2+3s+2)}\right)$$

$$\frac{s+1}{s(s+1)(s+2)} = \frac{1}{s(s+2)} = \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right)$$

$$\Rightarrow \quad L^{-1}\left(\frac{1}{s(s+2)}\right) = \frac{1}{2}(1 - e^{-2t})$$

To find $L^{-1}\left(\dfrac{e^{-s}}{s(s+1)(s+2)}\right)$

Recall

$$L(f(t-a)u(t-a)) = e^{-as}F(s)$$

So $F(s) = \dfrac{1}{s(s+1)(s+2)}$

So need to find $L^{-1}\left(\dfrac{1}{s(s+1)(s+2)}\right)$

$$L(f(t-a)u(t-a)) = e^{-as}F(s)$$

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2}\left(\frac{1}{s} + \frac{1}{s+2}\right) - \frac{1}{s+1}$$

$$\Rightarrow \quad L^{-1}\left(\frac{1}{s(s+1)(s+2)}\right) = \frac{1}{2}(1 + e^{-2t}) - e^{-t}$$

$$L^{-1}\left(\frac{e^{-s}}{s(s+1)(s+2)}\right)$$
$$= \left(\frac{1}{2}(1 + e^{-2(t-1)}) - e^{-(t-1)}\right)u(t-1)$$

Example 23

$$y'' + 3y' + 2y = g(t) \qquad y(0) = 0, \quad y'(0) = 1$$

$$L(y) = \frac{s+1}{s(s^2+3s+2)} - e^{-s}\left(\frac{1}{s(s^2+3s+2)}\right)$$

$$y(t) = \frac{1}{2}(1 - e^{-2t}) - \left(\frac{1}{2}(1 + e^{-2(t-1)}) - e^{-(t-1)}\right)u(t-1)$$

http://www.aw-bc.com/ide/idefiles/navigation/toolindexes/14.htm#14

Example 24

- Find the solution to the initial value problem

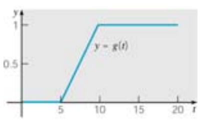
$$y'' + 4y = g(t), \quad y(0) = 0, \; y'(0) = 0$$

where

$$g(t) = u_5(t)\frac{t-5}{5} - u_{10}(t)\frac{t-10}{5} = \begin{cases} 0, & 0 \le t < 5 \\ (t-5)/5 & 5 \le t < 10 \\ 1, & t \ge 10 \end{cases}$$

$$\frac{t-5}{5}\{U_5(t) - U_{10}(t)\} + U_{10}(t)\}$$

- The graph of forcing function $g(t)$ is given on right, and is known as ramp loading.



$$y'' + 4y = u_5(t)\frac{t-5}{5} - u_{10}(t)\frac{t-10}{5}, \quad y(0) = 0, \; y'(0) = 0$$

Then

$$L(y'') + 4L(y) = \left[L\{u_5(t)(t-5)\}\right]/5 - \left[L\{u_{10}(t)(t-10)\}\right]/5$$

$$\left[s^2 L(y) - sy(0) - y'(0)\right] + 4L(y) = \frac{e^{-5s} - e^{-10s}}{5s^2}$$

Letting $Y(s) = L(y)$, and substituting in initial conditions,

$$\left(s^2 + 4\right)Y(s) = \left(e^{-5s} - e^{-10s}\right)/5s^2$$

Thus

$$Y(s) = \frac{\left(e^{-5s} - e^{-10s}\right)}{5s^2\left(s^2 + 4\right)}$$

$f(t) = t$

$$\boxed{f(t) = t \Rightarrow F(s) = \frac{1}{s^2}}$$

$$\boldsymbol{L(f(t-a)u(t-a)) = e^{-as}F(s)}$$

We have

$$Y(s) = \frac{\left(e^{-5s} - e^{-10s}\right)}{5s^2\left(s^2 + 4\right)} = \frac{e^{-5s} - e^{-10s}}{5} H(s)$$

where

$$H(s) = \frac{1}{s^2\left(s^2 + 4\right)}$$

If we let $h(t) = L^{-1}(H(s))$, then

$$y = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right]$$

Recall

$$\boldsymbol{L(f(t-a)u(t-a)) = e^{-as}F(s)}$$

Thus we examine $H(s)$, as follows.

$$H(s) = \frac{1}{s^2\left(s^2 + 4\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 4}$$

This partial fraction expansion yields the equations

$$(A + C)s^3 + (B + D)s^2 + 4As + 4B = 1$$

$$\Rightarrow A = 0, \; B = 1/4, \; C = 0, \; D = -1/4$$

Thus

$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2 + 4}$$

Thus

$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2 + 4} = \frac{1}{4}\left[\frac{1}{s^2}\right] - \frac{1}{8}\left[\frac{2}{s^2 + 4}\right]$$

and hence

$$h(t) = L^{-1}\{H(s)\} = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

For $h(t)$ as given above, and recalling our previous results, the solution to the initial value problem is then

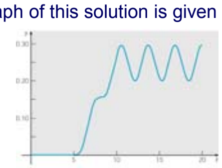
$$y = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right]$$

Thus the solution to the initial value problem is

$$y = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right], \quad \text{where}$$

$$h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

The graph of this solution is given below.



Part 5: Laplace transforms of impulse functions and ODEs

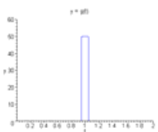
- In some applications, it is necessary to deal with an **impulsive** force.
- For example, a mechanical system subject to a sudden large force $g(t)$ over a short time interval about t_0.
- The differential equation will then have the form

$$ay'' + by' + cy = g(t),$$

where

$$g(t) = \begin{cases} \text{big}, & t_0 - \tau < t < t_0 + \tau \\ 0, & \text{otherwise} \end{cases}$$

and $\tau > 0$ is small.



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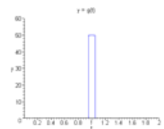
4.14 Measuring Impulse

In a mechanical system, where $g(t)$ is a force, the total **impulse** of this force is measured by the integral

$$I(\tau) = \int_{-\infty}^{\infty} g(t)dt = \int_{t_0-\tau}^{t_0+\tau} g(t)dt$$



Note that if $g(t)$ has the form

$$g(t) = \begin{cases} c, & t_0 - \tau < t < t_0 + \tau \\ 0, & \text{otherwise} \end{cases}$$

then

$$I(\tau) = \int_{-\infty}^{\infty} g(t)dt = \int_{t_0-\tau}^{t_0+\tau} g(t)dt = 2\tau c, \quad \tau > 0$$

In particular, if $c = 1/(2\tau)$, then $I(\tau) = 1$ (independent of τ).

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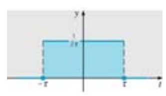
4.14 Measuring Impulse

Unit Impulse Function

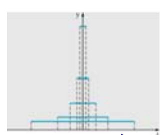
Suppose the forcing function $d_\tau(t)$ has the form

$$d_\tau(t) = \begin{cases} 1/2\tau, & -\tau < t < \tau \\ 0, & \text{otherwise} \end{cases}$$



Then as we have seen, $I(\tau) = 1$.

We are interested $d_\tau(t)$ acting over shorter and shorter time intervals (i.e., $\tau \to 0$). See graph on right.



Note that $d_\tau(t)$ gets taller and narrower as $\tau \to 0$. Thus for $t \neq 0$, we have

$$\lim_{\tau\to 0} d_\tau(t) = 0, \text{ and } \lim_{\tau\to 0} I(\tau) = 1$$

t≠0 why? t

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4.14 Measuring Impulse

Dirac Delta Function

Thus for $t \neq 0$, we have $\lim_{\tau\to 0} d_\tau(t) = 0, \text{ and } \lim_{\tau\to 0} I(\tau) = 1$

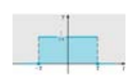
The **unit impulse function** δ is defined to have the properties

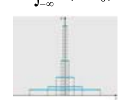
$$\delta(t) = 0 \text{ for } t \neq 0, \text{ and } \int_{-\infty}^{\infty} \delta(t)dt = 1$$

The unit impulse function is an example of a generalized function and is usually called the **Dirac delta function**.

In general, for a unit impulse at an arbitrary point t_0,

$$\delta(t - t_0) = 0 \text{ for } t \neq t_0, \text{ and } \int_{-\infty}^{\infty} \delta(t - t_0)dt = 1$$





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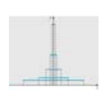
4.15 Laplace Transform of δ

The Laplace Transform of δ is defined by

$$L\{\delta(t - t_0)\} = \lim_{\tau\to 0} L\{d_\tau(t - t_0)\}, \; t_0 > 0$$



and thus

$$L\{\delta(t - t_0)\} = \lim_{\tau\to 0}\int_0^{\infty} e^{-st} d_\tau(t - t_0)dt = \lim_{\tau\to 0}\frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} e^{-st}dt$$

$$= \lim_{\tau\to 0}\frac{-e^{-st}}{2s\tau}\bigg|_{t_0-\tau}^{t_0+\tau} = \lim_{\tau\to 0}\frac{1}{2s\tau}\left[-e^{-s(t_0+\tau)} + e^{-s(t_0-\tau)}\right]$$

$$= \lim_{\tau\to 0}\frac{e^{-st_0}}{s\tau}\left[\frac{e^{s\tau} - e^{-s\tau}}{2}\right] = e^{-st_0}\left[\lim_{\tau\to 0}\frac{\sinh(s\tau)}{s\tau}\right]$$

$$= e^{-st_0}\left[\lim_{\tau\to 0}\frac{s\cosh(s\tau)}{s}\right] = e^{-st_0}$$

o/o form

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4.15 Laplace Transform of δ

Thus the Laplace Transform of δ is

$$L\{\delta(t - t_0)\} = e^{-st_0}, \; t_0 > 0$$

For example, when $t_0 = 10$, we have $L\{\delta(t-10)\} = e^{-10s}$.

For Laplace Transform of δ at $t_0 = 0$, take limit as follows:

$$L\{\delta(t)\} = \lim_{t_0\to 0} L\{d_\tau(t - t_0)\} = \lim_{t_0\to 0} e^{-st_0} = 1$$

Why take limit?

Ans: see last slide.

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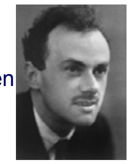
Paul A. M. Dirac (1902-19840), an English physicist, received the Nobel prize at age 31 for his work in quantum theory
When Dirac introduced his delta functions in early 1930s, again mathematical community was somewhat suspect
Finally his delta functions have been Justified. Today, delta functions have a solid place.
Mathematicians should get ideas from Engineers and physicists



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4.16 impulse functions and ODE

Example 25

Consider the solution to the initial value problem

$$2y'' + y' + 2y = \delta(t-7), \quad y(0) = 0,\ y'(0) = 0$$

Then

$$2L\{y''\} + L\{y'\} + 2L\{y\} = L\{\delta(t-7)\}$$

Letting $Y(s) = L\{y\}$,

$$\left[2s^2Y(s) - 2sy(0) - 2y'(0)\right] + \left[sY(s) - y(0)\right] + 2Y(s) = e^{-7s}$$

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4.16 impulse functions and ODE

$$\left[2s^2Y(s) - 2sy(0) - 2y'(0)\right] + \left[sY(s) - y(0)\right] + 2Y(s) = e^{-7s}$$

Substituting in the initial conditions, we obtain

$$\left(2s^2 + s + 2\right)Y(s) = e^{-7s}$$

$$Y(s) = \frac{e^{-7s}}{2s^2 + s + 2}$$

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4.16 impulse functions and ODE

Recall $L(f(t-a)u(t-a)) = e^{-as}F(s)$

We have

$$Y(s) = \frac{e^{-7s}}{2s^2 + s + 2}$$

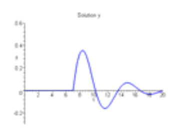
$$L(e^{ct}\sin\omega t) = \frac{\omega}{(s-c)^2 + \omega^2}$$

The partial fraction expansion of $Y(s)$ yields

$$Y(s) = \frac{e^{-7s}}{2\sqrt{15}}\left[\frac{\sqrt{15}/4}{(s+1/4)^2 + 15/16}\right] \qquad f(t) = \frac{1}{2\sqrt{15}}e^{-\frac{1}{4}t}\sin\frac{\sqrt{15}}{4}t$$

and hence

$$y(t) = \frac{1}{2\sqrt{15}}u_7(t)e^{-(t-7)/4}\sin\frac{\sqrt{15}}{4}(t-7)$$



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4.16 impulse functions and ODE

Example 26

A mass m=1 is attached to a spring with constant k=4 and released from rest 3 units below its equilibrium position. Hence we have

$$x'' + 4x = 0, x(0) = 3, x'(0) = 0$$

Now at the instant t=2π the mass is struck with a Hammer in the downwards direction, exerting an impulse of 8 units on the mass. So we have

$$x'' + 4x = 8\delta_{2\pi}(t) = 8\delta(t-2\pi), x(0) = 3, x'(0) = 0$$

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4.16 impulse functions and ODE

Solution

Apply L T to the above ODE, get

$$L(x'') + 4L(x) = 8L(\delta_{2\pi}(t)), x(0) = 3, x'(0) = 0$$

$$s^2L(x) - 3s + 4L(x) = 8e^{-2\pi s} \qquad L\{\delta(t-t_0)\} = e^{-st_0},\ t_0 > 0$$

$$L(x) = \frac{3s}{s^2+4} + \frac{8e^{-2\pi s}}{s^2+4} \qquad L(f(t-a)u(t-a)) = e^{-as}F(s)$$

$$x(t) = 3\cos 2t + 4u(t-2\pi)\sin 2(t-2\pi)$$

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Example 27

A mass m=1 is attached to a spring with constant k=1 and released from rest at its equilibrium position. Hence we have

$$x''+x=0, x(0)=0, x'(0)=0$$

Now at the instant t=0,π, 2π, 3π,…nπ,… the mass is struck with a Hammer in the downwards direction, exerting one unit impulse on the mass. So we have

$$x''+x=\sum_{n=0}^{\infty}\delta_{n\pi}(t), x(0)=0, x'(0)=0$$

Solution

First $L(\delta_{n\pi}(t))=e^{-n\pi s}$

Apply L T to ODE, get

$$s^2L(x)+L(x)=\sum_{n=0}^{\infty}L(\delta_{n\pi}(t))=\sum_{n=0}^{\infty}e^{-n\pi s}$$

$$L(x)=\sum_{n=0}^{\infty}\frac{e^{-n\pi s}}{s^2+1}$$

$$x(t)=L^{-1}\left(\sum_{n=0}^{\infty}\frac{e^{-n\pi s}}{s^2+1}\right)=\sum_{n=0}^{\infty}L^{-1}\left(\frac{e^{-n\pi s}}{s^2+1}\right)$$

$$x(t)=\sum_{n=0}^{\infty}u(t-n\pi)\sin(t-n\pi)$$

Note that If t<nπ, $u(t-n\pi)=0$

If t>nπ, $u(t-n\pi)=1$

So if $n\pi<t<(n+1)\pi$, then we have

$$u(t-0\pi)=u(t-\pi)=...=u(t-n\pi)=1$$

$$u(t-(n+1)\pi)=u(t-(n+2)\pi)=.....=0$$

Cont

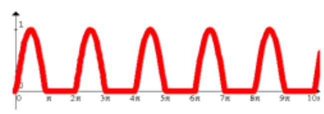
if $n\pi<t<(n+1)\pi$ then

$$x(t)=\sum_{k=0}^{n}u(t-k\pi)\sin(t-k\pi)=\sum_{k=0}^{n}\sin(t-k\pi)$$

$$=\sum_{k=0}^{n}(-1)^k\sin t=\sin t\sum_{k=0}^{n}(-1)^k$$

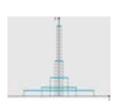
Hence $x(t)=\sin t$ when n is even

$x(t)=0$ when n is odd



4.17 Product of Continuous Functions and δ

The product of the delta function and a continuous function *f* can be integrated, using the mean value theorem for integrals:



$$\int_{-\infty}^{\infty}\delta(t-t_0)f(t)dt=\lim_{\tau\to 0}\int_{-\infty}^{\infty}d_\tau(t-t_0)f(t)dt$$
$$=\lim_{\tau\to 0}\frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau}f(t)dt$$
$$=\lim_{\tau\to 0}\frac{1}{2\tau}[2\tau f(t^*)]\quad(\text{where } t_0-\tau<t^*<t_0+\tau)$$
$$=\lim_{\tau\to 0}f(t^*)$$
$$=f(t_0)$$

Cont.

Thus $\int_{-\infty}^{\infty}\delta(t-t_0)f(t)dt=f(t_0)$

$$\int_{-\infty}^{\infty}\delta(t-t_0)\sin t dt=\sin(t_0)$$

$$\int_{-\infty}^{\infty}\delta(t-t_0)\cos t dt=\cos(t_0)$$

Example 28 Injections

t=0 , 100 mg of morphine into a patient,
almost instantly (impulsive injection)

t=1 , 100 mg……………………………………………………

half-life of morphine = 18 hours = 0.75 days

Y(t)= amount of morphine y(0.75)=y(0)/2

Without injections $\frac{dy}{dt} = -ky$

$$y(t) = y(0)e^{-kt}$$

$$0.5 = e^{-0.75k} \Rightarrow k = \frac{\ln 2}{0.75} = 0.924$$

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t=0 , 100 mg of morphine into a patient, almost instantly
t=1 , 100 mg
half-life of morphine = 18 hours = 0.75 days

$$\frac{dy}{dt} = -ky + 100\delta(t) + 100\delta(t-1)$$

$$sL(y) - y(0) = -0.924L(y) + 100 \times 1 + 100e^{-s}$$

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$$L(e^{ct}t^n) = \frac{n!}{(s-c)^{n+1}} \qquad \boxed{L(f(t-a)u(t-a)) = e^{-as}F(s)}$$

$$sL(y) - \overset{0}{\overset{\shortparallel}{y(0)}} = -0.924L(y) + 100 \times 1 + 100e^{-s}$$

$$(s + 0.924)L(y) = 100(1 + e^{-s})$$

$$L(y) = \frac{100}{s + 0.924} + \frac{100e^{-s}}{s + 0.924}$$

$$y = 100e^{-0.924t} + 100e^{-0.924(t-1)}u(t-1)$$

$$= \begin{cases} 100e^{-0.924t}, & 0 < t < 1 \\ 100(1 + e^{0.924})e^{-0.924t}, & t > 1. \end{cases}$$

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