NATIONAL UNIVERSITY OF SINGAPORE

MA1506 - MATHEMATICS II

(Semester 1 : AY2014/2015)

Name of examiner : Prof Brett McInnes

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Please write your matriculation/student number only. Do not write your name.
2. This examination paper contains **EIGHT** questions and comprises 4 printed pages.
3. Answer **ALL** questions.
4. PLEASE START EACH QUESTION ON A NEW PAGE.
5. This is a CLOSED BOOK (with helpsheet) examination.
6. Students are allowed to use one A4 size helpsheet, handwritten on both sides.
7. Candidates may use NON-PROGRAMMABLE calculators.

Question 1

[a] If y(x) satisfies

$$y'' \;+\; y \;=\; \cos(x), \;\; y(0) = 0, \; y'(0) = 0,$$

where the dash denotes a derivative with respect to x, find y(2).

[b] Your bank offers an investment product which works like this: whenever you have S dollars in your account, the amount grows at a rate given by $k\sqrt{S}$, where k is a constant equal to (2/year). If you start with \$10000, how much money do you have after two years?

Question 2

[a] A certain system is governed by a differential equation of the form

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0.$$

Is this system overdamped or underdamped?

[b] Consider the second-order ordinary differential equation

$$\frac{d^2x}{dt^2} + \frac{1}{2}\cos(x) + \frac{1}{2}\sin(x) = 0, \quad x(0) = \frac{dx}{dt}(0) = 1.$$

Find $\left(\frac{dx}{dt}\right)^2$ when $x = 0$.

Question 3

[a] On a certain island there is a population of bears governed by the equation

$$\frac{dN}{dt} \;=\; -KN(N-1)(N-2)(N-3),$$

where N is measured in thousands and K is a constant equal to 10/year. It is known that the bears have a stable equilibrium population of one thousand; however, the wildlife managers fear that this is too small and hope to import a number of bears to establish a new, higher, but still stable equilibrium. How many new bears should they import?

[b] There are 10000 cows on a certain farm, the population being governed by a standard logistic-with-harvesting model. The population is stable with a birth rate per capita of 10% per year, and the farmer sells 100 cows per year. One year, 2000 wild cows wander onto the farm, which cannot support such a large population. What is the death rate per capita in the cow population immediately after the arrival of the wild cows?

Question 4

[a] Find y($\pi/4$) if y(t) satisfies

$$\frac{d^2y}{dt^2} + y = 4\delta(t - 2\pi), \quad y(0) = -1, \quad \frac{dy}{dt}(0) = 0,$$

where $\delta(t)$ is the Dirac delta function.

[b] Let $f(t)$ denote the inverse Laplace transform of

$$\frac{1}{(s-1)(s^2-1)}.$$

Evaluate $f(1)$.

Question 5

[a] Compute the 2×2 matrix

$$\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 8 & 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & -2 \end{pmatrix}.$$

[b] The economy of the Central Asian Republic is based on the manufacture of cars and guns. The technology matrix for this economy, expressed in terms of the Central Asian Yen, is given by

$$T = \begin{pmatrix} 0.2 & 0.3 \\ 0.3 & 0.2 \end{pmatrix},$$

the ordering being cars first, guns second. The government wants to export 20 million yen of cars and 30 million yen of guns each year. How many yen of cars and guns should it arrange to produce each year?

Question 6

[a] Express the vector $2\,\mathbf{i} + \mathbf{j}$ in terms of the basis vectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

[b] Find the eigenvalues of the linear transformation with matrix $\begin{pmatrix} 2 & 4 \\ 4 & -4 \end{pmatrix}$.

Question 7

The element Kryptonite decays into Marvellium at a rate equal (in certain units to be used throughout this problem) at a rate equal to -2 times the amount of Kryptonite present. Marvellium itself decays into lead at a rate equal to -5 times the amount of Marvellium present. The equations for the amounts of Kryptonite and Marvellium are a system of first order linear ordinary differential equations.

[a] Classify this system (that is, say whether it is a nodal source, or a saddle etc.)

[b] Initially, you have exactly one kg of Kryptonite and 2/3 of a kg of Marvellium. Later you find that you have exactly 1/2 of a kg of Kryptonite. How much Marvellium do you then have? Hint: draw a phase diagram, including the eigenvectors of the system matrix.

Question 8

[a] Let $y\left(t, x\right)$ be the solution of the wave equation

$$y_{tt} = 9y_{xx}$$

with $y\left(t, 0\right) = y\left(t, \pi\right) = 0$, $y\left(0, x\right) = x\left(\pi - x\right)$, $y_t\left(0, x\right) = 0$.
Using d'Alembert's equation, find the value of $y\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

[b] The Laplace equation,

$$u_{xx} + u_{yy} = 0,$$

for a function of x and y defined on the square domain $0 \leq \mathrm{x} \leq \pi$, $0 \leq \mathrm{y} \leq \pi$, with boundary conditions u(x,0) = u(0,y) = u(π,y) = 0, and u(x,π) = sin(9x), has a solution of the form

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin(nx) \sinh(ny).$$

Find $u(\pi/2, \pi/2)$, expressing your answer in terms of $\cosh(9\pi/2)$. (Hint: sinh(2A) = 2sinh(A)cosh(A).)