MA1506
Mathematics II

Chapter 5
Matrices and their uses

This chapter consists of two parts.

Chapter 5
PART ONE

In part one, we shall study

Matrix operations

Some special matrices

Inverse matrix and unique solution of AX=B

Determinant and inverse matrix

Leontief Input-Output model

5.1 What is a Matrix?

The system of equations

$$2x+7y=3$$
$$4x+8y=11$$

can be rewritten as

$$\underbrace{\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix}}_{\text{2x2 Matrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

2x1 matrix

3x3 Matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

m x n Matrix: m rows, n columns

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

We may write $A=(a_{ij})$

Entry at i-th row j-th column

5.2 Matrix operations

- Matrix addition
- Scalar multiplication
- Matrix multiplication
- matrix transposition

Matrix Addition

$A = (a_{ij})$
$B = (b_{ij})$ } *m x n* matrices

Term by term addition

$$A + B = (a_{ij} + b_{ij})$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 3 \\ 6 & 9 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 10 & 17 \end{bmatrix}$$

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Scalar multiplication

$A = (a_{ij})$ *m x n* matrix

c real or complex number

Term by term multiplication

$$cA = (ca_{ij})$$

$$3\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 12 & 24 \end{bmatrix}$$

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Matrix multiplication

$A = (a_{ij})$ *m x n* matrix

$B = (b_{ij})$ *n x p* matrix

$AB = C$

$C = (c_{ij})$ *m x p* matrix

Multiplication is not term by term

but row to column as below

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

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Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times1+2\times2+3\times(-1) & 1\times1+2\times3+3\times(-2) \\ 4\times1+5\times2+6\times(-1) & 4\times1+5\times3+6\times(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix}$$

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In general $AB \neq BA$

$$AB = \begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 16 & -1 \\ 20 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 14 & 31 \\ 0 & 6 \end{bmatrix}$$

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Matrix transposition

Let

$$A = (a_{ij})$$

be a *m x n* matrix.

If we swap the rows with columns in *A*, we get

$$A^T = (a_{ji})$$

which is now a *n x m* matrix.

We call A^T the transpose of *A*.

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Example

$$\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 6 \\ 2 & 8 \\ 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 & 9 \\ 6 & 8 & 2 \\ 4 & 10 & 12 \end{bmatrix}^T = \begin{bmatrix} 1 & 6 & 4 \\ 7 & 8 & 10 \\ 9 & 2 & 12 \end{bmatrix}$$

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Properties of transpose

$$\left(A^T\right)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$(cA)^T = cA^T$$

$$(AB)^T = B^T A^T$$

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5.3 Special matrices

Symmetric matrix

A *n x n* matrix *A* is symmetric if

$$A^T = A$$

$$\begin{bmatrix} 1 & 7 & 9 \\ 7 & 8 & 2 \\ 9 & 2 & 12 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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Anti-Symmetric matrix

A *n x n* matrix *A* is anti-symmetric or skew symmetric if

$$A^T = -A$$

$$\begin{bmatrix} 0 & 7 & -9 \\ -7 & 0 & -2 \\ 9 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

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Identity matrix

$$I = I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

n x n identity matrix

In general, we have

$$AI = IA = A$$

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Orthogonal matrix

An *n x n* matrix, *B* is orthogonal if $BB^T = I$

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal because

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = I$$

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Vectors as special matrices

Matrices containing only one column are often called column vectors or vectors

Matrices containing only one row are often called row vectors or vectors

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

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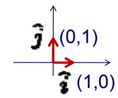
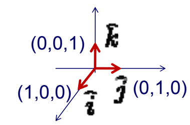
Vectors $\hat{i}$ $\hat{j}$ $\hat{k}$

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$





$$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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$$\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = a\hat{i} + b\hat{j}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a\hat{i} + b\hat{j} + c\hat{k}$$

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Shear parallel to x-axis

Let

$$S = \begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}$$

Then

$$S\hat{i} = \begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hat{i}$$

$$S\hat{j} = \begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \tan\theta \\ 1 \end{bmatrix} = \tan\theta\,\hat{i} + \hat{j}$$

Thus

$$S\,\hat{i} = \hat{i} \qquad S\,\hat{j} = \tan\theta\,\hat{i} + \hat{j}$$

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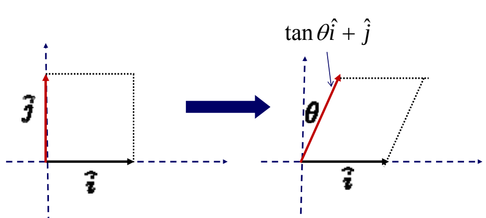
Thus $\quad S\,\hat{i} = \hat{i} \qquad S\,\hat{j} = \tan\theta\,\hat{i} + \hat{j}$

and we call S a shear matrix (with shear parallel to the x-axis).



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Example

S: shear 45 degrees parallel to x axis

Recall

$$S(\theta) = \begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}$$

Then

$$S(45^\circ) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

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Rotation

Let $R(\theta)=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

Then
$$R\hat{i}=\cos\theta\hat{i}+\sin\theta\hat{j}$$
$$R\hat{j}=-\sin\theta\hat{i}+\cos\theta\hat{j}$$

We call $R(\theta)$ a rotation matrix (through an anti-clockwise angle θ)

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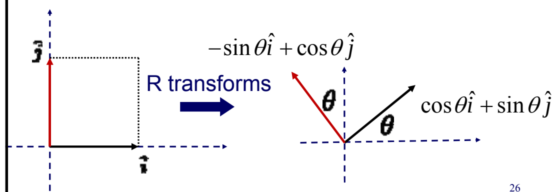
$$R\hat{i}=\cos\theta\hat{i}+\sin\theta\hat{j}$$
$$R\hat{j}=-\sin\theta\hat{i}+\cos\theta\hat{j}$$



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Example

R: rotate 90 degrees anticlockwise

Recall $R(\theta)=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

Then
$$R(90^\circ)=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

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Summary

symmetric matrix	$A^T=A$
anti symmetric matrix	$A^T=-A$
identity matrix	I
orthogonal matrix	$BB^T=I$
vector	$\begin{bmatrix}1\\2\end{bmatrix}$ $[1\ \ 2]$
shear matrix	$S(\theta)=\begin{bmatrix}1 & \tan\theta\\ 0 & 1\end{bmatrix}$
rotation matrix	$R(\theta)=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

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5.4 Inverse matrix and unique solution

Let $A=\begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{nn}\end{pmatrix}$ be a $n\times n$ matrix

If C is a matrix such that $AC=CA=I_n$

then C is called the inverse matrix of A .

The inverse matrix of A is also denoted by A^{-1}

Thus
$$AA^{-1}=A^{-1}A=I_n$$

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Example

Let
$$A=\begin{pmatrix}1 & 4 & 2\\ -2 & -8 & 3\\ 0 & 1 & 1\end{pmatrix}$$

Suppose that we know that A has an inverse.

Then we can use row operations (next slide) to find A^{-1}

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First write $\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ -2 & -8 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$

Try to get zero as many as possible for the lower triangular part.

$$\xrightarrow{2R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

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$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 7 & 2 & 1 & 0 \end{array}\right)$$

Now get 1 on diagonal

$$\xrightarrow{\frac{1}{7}R_3} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2/7 & 1/7 & 0 \end{array}\right)$$

$$\xrightarrow{-R_3+R_2} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2/7 & -1/7 & 1 \\ 0 & 0 & 1 & 2/7 & 1/7 & 0 \end{array}\right)$$

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Next we try to get as many zeroes as possible for the upper triangular part.

$$\xrightarrow{-2R_3+R_1} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & 3/7 & -2/7 & 0 \\ 0 & 1 & 0 & -2/7 & -1/7 & 1 \\ 0 & 0 & 1 & 2/7 & 1/7 & 0 \end{array}\right)$$

$$\xrightarrow{-4R_2+R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 11/7 & 2/7 & -4 \\ 0 & 1 & 0 & -2/7 & -1/7 & 1 \\ 0 & 0 & 1 & 2/7 & 1/7 & 0 \end{array}\right)$$

$$\underbrace{\begin{pmatrix} 11/7 & 2/7 & -4 \\ -2/7 & -1/7 & 1 \\ 2/7 & 1/7 & 0 \end{pmatrix}}_{A^{-1}}$$

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Hence we have

$$A^{-1} = \begin{pmatrix} 11/7 & 2/7 & -4 \\ -2/7 & -1/7 & 1 \\ 2/7 & 1/7 & 0 \end{pmatrix}$$

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Consider linear system AX=B where

$$A = \begin{pmatrix} 1 & 4 & 2 \\ -2 & -8 & 3 \\ 0 & 1 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad B = \begin{pmatrix} -2 \\ 32 \\ 1 \end{pmatrix}$$

Since the inverse of A exists, we have

$$AX = B \implies A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

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Therefore

$$(11/7)(-2)+(2/7)(32)+(-4)(1) = 2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11/7 & 2/7 & -4 \\ -2/7 & -1/7 & 1 \\ 2/7 & 1/7 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 32 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

$$(2/7)(-2)+(1/7)(32)+(0)(1) = 4 \qquad (-2/7)(-2)+(-1/7)(32)+(1)(1) = -3$$

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5.5 Determinants and inverse

In this section, we introduce determinants, and study some of its properties.

Consider

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Then

$$a_{11}x_1 + a_{12}x_2 = b_1 \qquad (1)$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \qquad (2)$$

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Perform $a_{22}(1) - a_{12}(2)$

and get

$$(a_{11}a_{22} - a_{12}a_{21})x_1 = b_1a_{22} - a_{12}b_2$$

Similarly

$$(a_{21}a_{12} - a_{22}a_{11})x_2 = b_2a_{12} - a_{22}b_1$$

Therefore the linear system of equations has unique solution if and only if

$$a_{11}a_{22} - a_{12}a_{21} \neq 0$$

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We define the determinant of $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

to be $a_{11}a_{22} - a_{12}a_{21}$

We write

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

or

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

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Inverse of a 2x2 matrix

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Inverse of A

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note that

$$AA^{-1} = A^{-1}A = I$$

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Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then we define the determinant of A, (denoted by det(A) or $|A|$) as

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= (a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31}) - (a_{13}a_{22}a_{31} + a_{23}a_{32}a_{11} + a_{12}a_{21}a_{33})$$

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Important Properties of Determinants

$$\det(ST) = (\det S)(\det T) = \det(TS)$$

$$\det M^T = \det M$$

$$\det(cM) = c^n \det M$$

where M is a nxn matrix.

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Theorem 1 (proof omitted)

A nxn matrix A has an inverse if and only if det(A)≠0

$$A=\begin{pmatrix}1&1&1\\0&2&3\\0&0&3\end{pmatrix} \qquad \det A = 1\times 2\times 3 = 6$$

A has an inverse

$$B=\begin{pmatrix}1&1&1\\0&0&3\\0&0&2\end{pmatrix} \qquad \det B = 1\times 0\times 2 = 0$$

B has no inverse

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Theorem 2 (proof omitted)

Let A be an nxn matrix. Then AX=B has a unique solution if and only if det(A)≠0

Thus
$$\begin{pmatrix}1&1&1\\0&2&3\\0&0&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}4\\5\\6\end{pmatrix}$$

has a unique solution

because
$$\det\begin{pmatrix}1&1&1\\0&2&3\\0&0&3\end{pmatrix}=1\times 2\times 3=6\neq 0$$

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5.6 Leontief Input-Output Model

A way to analyze **economics of interdependent sectors**

Example ---Oil and Transportation industries

(1) Transportation industry requires
- (i) gasoline from the oil industry
- (ii) transportation of equipment from the transportation industry

(2) Oil industry requires
- (i) transportation of gasoline from the transportation industry
- (ii) oil-based fuels for processing from the oil industry

We will look at a single oil company and a single transportation company as a closed system

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Information

Oil Industry

Cost of producing $1 worth of gas:
- $0.32 in oil costs
- $0.12 in transportation costs

Transportation industry

Cost of producing $1 worth of transportation:
- $0.50 in gas costs
- $0.20 in transportation costs

Suppose that the demand from the outside sector of the economy (all consumers outside of oil and transportation) is:
- $15 billion for oil
- $1.2 billion for transportation

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Leontief model allows us to calculate how much each company should produce to meet a given demand

Let x = the total output from oil company

Let y = the total output from transportation company

	Internal		External
	From oil company	From transportation company	
Oil demand	.32x	.50y	d1 = $15 b
Transportation demand	.12x	.20y	d2 = $1.2 b

The *internal demand* for each is the combined demand from the oil industry and from the transportation industry

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Setting up the demand equations

- The total output of each company will equal the sum of the internal and external demands:

$$\begin{cases} x = .32x + .50y + d_1 \\ y = .12x + .20y + d_2 \end{cases}$$

- Expressed as a matrix equation:

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}.32&.50\\.12&.20\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}d_1\\d_2\end{bmatrix}$$

$$X = MX + D$$

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Solving the demand equations

- Solve for X: $X = MX + D$
  $X = (I - M)^{-1} D$

$X = MX + D$
$IX - MX = D$
$(I - M)X = D$

- In our example: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .32 & .50 \\ .12 & .20 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

$M = \begin{bmatrix} .32 & .50 \\ .12 & .20 \end{bmatrix}$ $\quad I - M = \begin{bmatrix} .68 & -.50 \\ -.12 & .80 \end{bmatrix}$

Technology matrix

$(I - M)^{-1} = \begin{bmatrix} 1.65 & 1.03 \\ 0.25 & 1.40 \end{bmatrix}$

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5.6 Leontief Input-Output Model

The solution

- Putting all together: $D = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 1.2 \end{bmatrix}$

$$X = (I - M)^{-1} D = \begin{bmatrix} 1.65 & 1.03 \\ 0.25 & 1.40 \end{bmatrix} \begin{bmatrix} 15 \\ 1.2 \end{bmatrix} = \begin{bmatrix} 26.0 \\ 5.4 \end{bmatrix}$$

- In order to meet the demand the companies need to produce
  - $26,0 billion of oil
  - $5.4 billion of transportation

End of part I

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Chapter 5
PART TWO

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In part II, we shall study

Eigenvalues and eigenvectors
Diagonalization of matrix

Weather forecasting model
Discrete linear population model

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5.7 Eigenvalues and eigenvectors

Consider

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Vectors are in different directions

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Vectors are in different directions

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

parallel

Vectors are in same direction

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We rewrite what we have discussed by letting

$$T = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

Then

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \qquad T \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We are interested in the last case, i.e.,

$$T \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

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Eigenvector

Let T be a nxn matrix. Suppose there is a non zero vector $\vec{u}$ and a real number λ such that

$$T\vec{u} = \lambda\vec{u}$$

eigenvalue eigenvector

Then we call λ an eigenvalue for T and

$\vec{u}$ the corresponding eigenvector.

Note that the zero vector $\vec{0}$

is not an eigenvector though $T\vec{0} = \lambda\vec{0}$

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How to find Eigenvalues and Eigenvectors

First note that $T\vec{u} = \lambda\vec{u} = \lambda I\vec{u}$

$$(T - \lambda I)\vec{u} = \vec{0}$$

we want nonzero vector

To find a non-zero vector $\vec{u}$ we have to solve

$$\det(T - \lambda I) = 0$$

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Example

Find the eigenvalues of $T = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\Rightarrow \quad \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \quad -(1-\lambda)(2+\lambda) - 4 = 0$$

$$\lambda = 2, -3$$

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Finding Eigenvectors

Let $\vec{u} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

be an eigenvector corresponding to $\lambda = 2$.

Then $(T - 2I)\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1-2 & 2 \\ 2 & -2-2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \quad \left.\begin{aligned} -\alpha + 2\beta &= 0 \\ 2\alpha - 4\beta &= 0 \end{aligned}\right\} \text{Two identical equations}$$

Thus there are infinitely many solutions, i.e., there are infinitely many eigenvectors.

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In fact, the eigenvectors associated to $\lambda = 2$

are of the form

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \frac{\alpha}{2} \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \qquad \alpha \neq 0$$

Thus all the eigenvectors are parallel.

Now we may choose $\alpha = 1$

and get the eigenvector $\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$

Note that we need only one eigenvector.

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Next we let $\vec{u} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ be an eigenvector

corresponding to $\lambda = -3$

$$(T - (-3)I)\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1+3 & 2 \\ 2 & -2+3 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 4\alpha + 2\beta &= 0 \\ 2\alpha + \beta &= 0 \end{aligned}\right\} \text{two identical equations}$$

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Thus the eigenvectors associated to $\lambda = -3$ are of the form

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\alpha\\ -2\alpha\end{bmatrix} = \alpha\begin{bmatrix}1\\ -2\end{bmatrix} \qquad \alpha \neq 0$$

Now choose $\alpha = 1$

and get the eigenvector $\begin{bmatrix}1\\ -2\end{bmatrix}$

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Example

Let $T = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

Then $\det(T - \lambda I) = 0$

$$\begin{vmatrix}-\lambda & -1\\ 1 & -\lambda\end{vmatrix} = 0$$
$$\Rightarrow \lambda^2 + 1 = 0$$
$$\Rightarrow \lambda = \pm i$$

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Consider $\lambda = i$

$$(T - \lambda I)\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = 0$$

$$\begin{bmatrix}-i & -1\\ 1 & -i\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = 0$$

$$\left.\begin{array}{r}-i\alpha - \beta = 0\\ \alpha - i\beta = 0\end{array}\right\} \text{two identical equations}$$

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Since $\alpha - i\beta = 0$

eigenvector $\begin{bmatrix}i\beta\\ \beta\end{bmatrix} = \beta\begin{bmatrix}i\\ 1\end{bmatrix}$

choose $\beta = 1$

and get the complex eigenvector $\begin{bmatrix}i\\ 1\end{bmatrix}$

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Next we consider second eigenvalue $\lambda = -i$

and get

$$\begin{bmatrix}i & -1\\ 1 & i\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = 0 \quad \Rightarrow \quad \begin{array}{l} i\alpha - \beta = 0\\ \alpha + i\beta = 0\end{array}$$

Use $\alpha + i\beta = 0$

and get the eigenvector $\begin{bmatrix}-i\beta\\ \beta\end{bmatrix} = \beta\begin{bmatrix}-i\\ 1\end{bmatrix}$

Choose $\beta = i$

and get complex eigenvector $\begin{bmatrix}1\\ i\end{bmatrix}$

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5.7 Eigenvalues and eigenvectors

Recall $R(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

So the matrix $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ represents a rotation through 90 degrees.

$\begin{bmatrix}i\\ 1\end{bmatrix}$ $\begin{bmatrix}1\\ i\end{bmatrix}$ are the eigenvectors which are **complex.**

In fact, when rotating through 90 degrees, every **real** vector should change direction, so NO **real** eigenvector exists.

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5.8 Diagonalization

We say a matrix A is diagonalizable if

$$\boxed{A = PDP^{-1}}$$

Matrix of eigenvectors

Diagonal matrix of eigenvalues

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$

Then $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

eigenvalues

$P = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & -2 \end{bmatrix}$ $\quad P^{-1} = -\frac{2}{5}\begin{bmatrix} -2 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix}$

eigenvectors

$$\begin{aligned} P^{-1}\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}P &= -\frac{2}{5}\begin{bmatrix} -2 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ \frac{1}{2} & -2 \end{bmatrix} \\ &= -\frac{2}{5}\begin{bmatrix} -2 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix} \\ &= -\frac{2}{5}\begin{bmatrix} -5 & 0 \\ 0 & -\frac{15}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \end{aligned}$$

Example

$\begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}$ has only one eigenvalue since

$$\det\left(\begin{bmatrix} 1-\lambda & \tan\theta \\ 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 = 0$$

Since we have only one eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

it is not possible to diagonalize the matrix.

Finding M^n

$$\begin{aligned} M &= PDP^{-1} \\ M^2 &= PDP^{-1}PDP^{-1} = PD^2P^{-1} \\ M^3 &= MM^2 = PDP^{-1}PD^2P^{-1} = PD^3P^{-1} \end{aligned}$$

$$\Rightarrow \quad M^n = PD^nP^{-1} = P\begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}P^{-1}$$

5.9 Model weather forecasting

Today	Tomorrow	Probability
Rainy	Rainy	60%
	Sunny	40%
Sunny	Rainy	30%
	Sunny	70%

$$M = \begin{bmatrix} R \to R & S \to R \\ R \to S & S \to S \end{bmatrix} = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}.$$

Current : R S

$$M = \begin{bmatrix} R\to R & S\to R \\ R\to S & S\to S \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \end{array}. \begin{array}{l} \text{Next:} \\ R \\ S \end{array}$$

Transition matrix

columns add to 1

The problem we consider here is an example of Markov Chains

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Transition matrix

Current : R S

$$\begin{bmatrix} 0.3 & 0.8 \\ 0.7 & 0.2 \end{bmatrix} \begin{array}{l} \text{Next:} \\ R \\ S \end{array}$$

columns add to 1

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$$M = \begin{bmatrix} R\to R & S\to R \\ R\to S & S\to S \end{bmatrix} = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}.$$

Today is Sunny, will it be rainy 2 days later?

S —0.7→ S —0.7→ S 0.49
S —0.7→ S —0.3→ R 0.21
S —0.3→ R —0.4→ S 0.12
S —0.3→ R —0.6→ R 0.18

0.61 

0.39 

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S —0.7→ S —0.7→ S 0.49
S —0.7→ S —0.3→ R 0.21
S —0.3→ R —0.4→ S 0.12
S —0.3→ R —0.6→ R 0.18

0.61 

0.39 

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} =$$

$$\begin{bmatrix} 0.6\times 0.6 + 0.3\times 0.4 & \boxed{0.3\times 0.7 + 0.6\times 0.3} \\ 0.4\times 0.6 + 0.7\times 0.4 & \boxed{0.7\times 0.7 + 0.4\times 0.3} \end{bmatrix}$$

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$$M = \begin{bmatrix} R\to R & S\to R \\ R\to S & S\to S \end{bmatrix} = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}.$$

Today is Rainy, will it be rainy 4 days later?

$$M^4 = \begin{bmatrix} R\to R_4 & S\to R_4 \\ R\to S_4 & S\to S_4 \end{bmatrix}$$

$$M^4 = \begin{bmatrix} 0.4332 & 0.4251 \\ 0.5668 & 0.5749 \end{bmatrix}$$

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Today is Rainy, will it be rainy 30 days later?

Find M^{30}

Should use

$$M^{30} = PD^{30}P^{-1}$$

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Eigenvalues of $M = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$

are $\lambda_1 = 0.3$ $\lambda_2 = 1$

Corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ \frac{4}{3} \end{bmatrix}$$

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$$P = \begin{bmatrix} 1 & 1 \\ -1 & \frac{4}{3} \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} \frac{4}{7} & -\frac{3}{7} \\ \frac{3}{7} & \frac{3}{7} \end{bmatrix}$$

$$D^{30} = \begin{bmatrix} 0.3^{30} & 0 \\ 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 2 \times 10^{-16} & 0 \\ 0 & 1 \end{bmatrix}$$

$$M^{30} = \begin{bmatrix} 1 & 1 \\ -1 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 2 \times 10^{-16} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{7} & -\frac{3}{7} \\ \frac{3}{7} & \frac{3}{7} \end{bmatrix} \approx \begin{bmatrix} \frac{3}{7} & \frac{3}{7} \\ \frac{4}{7} & \frac{4}{7} \end{bmatrix}.$$

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Transition matrix

$$\begin{array}{cc} & \text{current} \quad S \quad R \quad \text{Next} \\ M = & \begin{bmatrix} & \\ & \end{bmatrix} \begin{matrix} S \\ R \end{matrix} \end{array} \qquad M = \begin{bmatrix} S \to S & R \to S \\ S \to R & R \to R \end{bmatrix}$$

$$M^6 = \begin{bmatrix} S \to S_6 & R \to S_6 \\ S \to R_6 & R \to R_6 \end{bmatrix}$$

$S \to S_6$ means today sunny, 6 days later sunny

The entry here represents the probability that $S \to S_6$ happens

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5.10Trace of a Matrix

Let *M* be a square matrix.

The trace of *M* , denoted *Tr (M),* is the sum of the diagonal entries

$$Tr\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2, \quad Tr\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 15,$$

$$Tr\begin{bmatrix} 1 & 5 & 16 \\ 7 & 2 & 15 \\ 11 & 9 & 8 \end{bmatrix} = 11$$

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$$Tr(MN) = Tr(NM)$$

$$Tr(M) = Tr(PDP^{-1}) = Tr(D)$$

Given matrix

Representation of M wrt new basis

For a diagonalizable matrix *M*,

Tr(M) = ***Tr(D)*** =sum of its eigenvalues.

Use this to check your calculations of eigenvalues and to find the remaining eigenvalue

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$$M = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$$

eigenvalue $\lambda_1 = 0.3$

Tr(M)=0.6+0.7=1.3

Hence the 2nd eigenvalue is

$\lambda_2 = 1$

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5.11 Discrete Linear Population Models

Discrete data of pigeon population

Year	Juveniles	Adults
1	J_1	A_1
2	J_2	A_2
k	J_k	A_k

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J_k # of Juvenile at year k A_k # of adult at year k

Reproduction Rules

A) Juvenile becomes Adult after 1 year

1) Reproduction rate = 2 $J_{k+1} = 2A_k$
2) Half of adults die each year
3) A quarter of juveniles survive

$$A_{k+1} = \frac{A_k}{2} + \frac{J_k}{4}$$

If a building has 100 adults and 40 juveniles in year 0, how does the population change?

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$$\left.\begin{aligned} J_{k+1} &= 2A_k \\ A_{k+1} &= \frac{A_k}{2} + \frac{J_k}{4} \end{aligned}\right\} \quad \vec{V}_{k+1} = B\vec{V}_k$$

where

$$\vec{V}_k = \begin{bmatrix} A_k \\ J_k \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{bmatrix}$$

Then $\vec{V}_{k+2} = B\vec{V}_{k+1} = BB\vec{V}_k = B^2\vec{V}_k$

$$\Downarrow$$

$$\vec{V}_{k+m} = B^m\vec{V}_k \quad \Longrightarrow \quad \vec{V}_m = B^m\vec{V}_0$$

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$$\vec{V}_k = \begin{bmatrix} A_k \\ J_k \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{bmatrix} \quad \vec{V}_k = B^k\vec{V}_0$$

Eigenvalues of B: 1, $-\frac{1}{2}$

Eigenvectors of B: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$$B^k = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-\frac{1}{2})^k \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 4 + 2(-\frac{1}{2})^k & 1 - (-\frac{1}{2})^k \\ 8 - 8(-\frac{1}{2})^k & 2 + 4(-\frac{1}{2})^k \end{bmatrix}$$

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$$\vec{V}_k = \begin{bmatrix} A_k \\ J_k \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{bmatrix} \quad \vec{V}_k = B^k\vec{V}_0$$

$$\vec{V}_0 = \begin{bmatrix} 100 \\ 40 \end{bmatrix}$$

$$B^k\vec{V}_0 \Longrightarrow \quad A_k = \frac{220}{3} + \frac{80}{3}(-\frac{1}{2})^k, \quad J_k = \frac{440}{3} - \frac{320}{3}(-\frac{1}{2})^k$$

In the long run $(-\frac{1}{2})^k \to 0$

Adults = 73.33 , Juveniles = 146.66

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$$B^k = P\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} P^{-1}$$

Summary

1) In this case (eigenvalue=1 and eigenvalue=-1/2), population will oscillate but eventually converge to some fixed value.

2) In general

If abs. value |eigenvalue| of one of eigenvalues > 1, population explosion

If both |eigenvalues| < 1, population goes to zero

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Appendix

Taylor's Theorem

1 variable:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \ldots$$

2 variable:

$$f(x,y) \;=\; f(0,y) + x\frac{\partial f}{\partial x}(0,y) + \ldots$$

Keep y fixed, expand about x

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Taylor's Theorem

$$\begin{aligned} f(x,y) &= \boxed{f(0,y)} + \boxed{x\frac{\partial f}{\partial x}(0,y)} + \ldots \\ &= \left(f(0,0) + y\frac{\partial f}{\partial y}(0,0) + \ldots\right) \\ &\quad + x\left(\frac{\partial f}{\partial x}(0,0) + y\frac{\partial^2 f}{\partial y \partial x}(0,0) + \ldots\right) \\ &= f(0,0) + yf_y + xf_x + \\ &\quad \frac{1}{2}(y^2 f_{yy} + 2xyf_{xy} + x^2 f_{xx}) + \ldots \end{aligned}$$

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Taylor's Theorem

$$\begin{aligned} f(x,y,z) &= f(0,0,0) + xf_x + yf_y + zf_z \ldots \\ g(x,y,z) &= g(0,0,0) + xg_x + yg_y + zg_z \ldots \\ h(x,y,z) &= h(0,0,0) + xh_x + yh_y + zh_z \ldots \end{aligned}$$

$$\vec{u} = \begin{bmatrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z) \end{bmatrix}$$

Vector function

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Taylor's Theorem

$$\begin{aligned} f(x,y,z) &= f(0,0,0) + xf_x + yf_y + zf_z \ldots \\ g(x,y,z) &= g(0,0,0) + xg_x + yg_y + zg_z \ldots \\ h(x,y,z) &= h(0,0,0) + xh_x + yh_y + zh_z \ldots \end{aligned}$$

$$\begin{bmatrix} f \\ g \\ h \end{bmatrix} = \begin{bmatrix} f(\vec{0}) \\ g(\vec{0}) \\ h(\vec{0}) \end{bmatrix} + \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ldots$$

+ negligible terms

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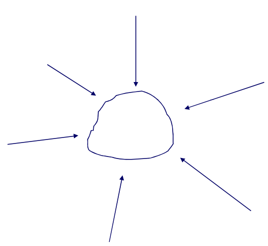
Strain Tensor

Underground rock, distorted by stress



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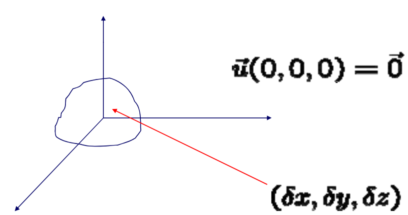
Strain Tensor

$$\vec{u} = \begin{bmatrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z) \end{bmatrix}$$

$\vec{u}(x,y,z)$ displacement vector

i.e. how much each point moves.



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Strain Tensor

$$\begin{bmatrix} f \\ g \\ h \end{bmatrix} = \begin{bmatrix} f(\vec{0}) \\ g(\vec{0}) \\ h(\vec{0}) \end{bmatrix} + \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdots$$

$$\vec{u} = \begin{bmatrix} f \\ g \\ h \end{bmatrix} \approx \underbrace{\begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix}}_{S=(S_{ij})} \underbrace{\begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}}_{\vec{r}}$$

Strain Tensor: $S = (S_{ij})$ $\quad \vec{r}$

$\delta x, \delta y, \delta z$ small, 2nd order terms vanishes

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Strain Tensor

$$\vec{u} = \begin{bmatrix} f \\ g \\ h \end{bmatrix} \approx \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = S\vec{r}$$

$$\boxed{u_i = \sum_{j=1}^{3} S_{ij} r_j} \qquad S = (S_{ij})$$

$$\sum_{i=1}^{3} S_{ii} = f_x + g_y + h_z = \text{div}\, \vec{u} = \vec{\nabla} \cdot \vec{u}$$

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Strain Tensor

Recall: Div, Grad, Curl

$$S = \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix}$$

$$\sum_{i=1}^{3} S_{ii} = f_x + g_y + h_z = \text{div}\, \vec{u} = \vec{\nabla} \cdot \vec{u}$$

$$\vec{\nabla} = \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \\ \frac{d}{dz} \end{bmatrix} \Rightarrow \vec{\nabla}(\vec{u}) = \begin{bmatrix} \frac{df}{dx} \\ \frac{dg}{dy} \\ \frac{dh}{dz} \end{bmatrix}$$

$$\vec{\nabla} \times \vec{u} = \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \\ \frac{d}{dz} \end{bmatrix} \times \begin{bmatrix} f \\ g \\ h \end{bmatrix}$$

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Strain Tensor

$$S = \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix}$$

$$S - S^T = \begin{bmatrix} 0 & -(g_x - f_y) & (f_z - h_x) \\ (g_x - f_y) & 0 & -(h_y - g_z) \\ -(f_z - h_x) & (h_y - g_z) & 0 \end{bmatrix}$$

- Anti-symmetric
- Associated to a vector

$$\begin{bmatrix} h_y - g_z \\ f_z - h_x \\ g_x - f_y \end{bmatrix} = \vec{\nabla} \times \vec{u} = \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \\ \frac{d}{dz} \end{bmatrix} \times \begin{bmatrix} f \\ g \\ h \end{bmatrix}$$

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Strain Tensor

$$S = \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix}$$

$S - S^T$ Measures the twisting deformation

No twisting means $S = S^T$

➡ Strain tensor is symmetric

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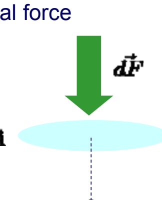
Strain Tensor

$$S = \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix}$$

Strain depends on amount of external force

Stress is force per unit area



$$d\vec{F} = T d\vec{A}$$

$$dF_i = \sum_j T_{ij} dA_j$$

$$\begin{bmatrix} dF_1 \\ dF_2 \\ dF_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} dA_1 \\ dA_2 \\ dA_3 \end{bmatrix}$$

Stress tensor

$d\vec{F}$ $\quad dA$ $\quad d\vec{A}$

directional vector

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Strain Tensor

Relationship between stress and strain

$$T_{ij} = \sum_k \sum_j Y_{ijkl} S_{kl}$$

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Orthogonal matrix

An n x n matrix, B is orthogonal if

$$\boxed{BB^T = I}$$

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = I$$

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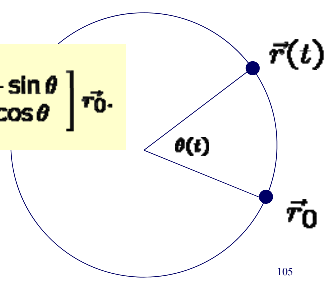
Application of rotation matrix

Suppose an object is moving in a circle at constant angular speed ω . What is its acceleration?

$$\vec{r}(t) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\vec{r_0}.$$

θ depends on t



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Application (cont.)

$$\vec{r}(t) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\vec{r_0}$$

$$\frac{d\cos\theta}{dt} = -\sin\theta\frac{d\theta}{dt} = (-\sin\theta)\dot{\theta}$$

$$\frac{d\vec{r}}{dt} = \dot{\theta}\begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}\vec{r_0}$$

$$\boxed{\dot{\theta} = \omega}$$

ω is constant

$$\frac{d^2\vec{r}}{dt^2} = \begin{bmatrix} -\cos\theta & \sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}\omega^2\vec{r_0} = -\omega^2\vec{r}(t)$$

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Composing two shears

S: shear θ degrees parallel to x axis

S: shear ϕ degrees parallel to x axis

$$S(\phi)S(\theta) = \begin{bmatrix} 1 & \tan\phi \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \tan\phi + \tan\theta \\ 0 & 1 \end{bmatrix}.$$

Still a shear but note that $\tan(\phi+\theta) \neq \tan\phi + \tan\theta$

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Rotation in 3D

Rotate 90 degrees (anticlockwise) about z-axis $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rotate 90 degrees (anticlockwise) about x-axis $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

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Determinant of Orthogonal Matrix $MM^T = I$

$$\begin{aligned}\det(MM^T) &= \det(M)\det(M^T)\\ &= \det(M) \times \det(M)\\ &= (\det M)^2\end{aligned}$$

$$\boxed{\det M = \pm 1}$$

$$\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} \quad \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix} \quad \begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix}$$

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Moment of Inertia Tensor

$$J\left(\frac{d\vec{\Omega}}{dt}\right) + \vec{\Omega} \times J(\vec{\Omega}) = \vec{0}$$

Freely rotating objects spin steadily only around an axis defined by an eigenvector of the moment of inertia tensor

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