MA1506
Mathematics II
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Chapter 1
Ordinary Differential Equations

1.1. Introduction

In this chapter, we study 1st order ordinary differential equations (ODE) and their applications.

We also study 2nd order ordinary differential equations.

Applications of 2nd ODE will be discussed in chapter two.

1.2 Separable equations

We first study ODE of the following form

$$\frac{dy}{dx} = \frac{M(x)}{N(y)}$$

We shall learn how to solve separable equations by examples.

Example 1

$$\frac{dy}{dx} = (1+y^2)e^x$$

We write $$e^x dx = \frac{1}{1+y^2} dy$$

Integrating both sides, we get

$$\int e^x dx = \int \frac{1}{1+y^2} dy$$

Example 1 (cont)

$$\int e^x dx = \int \frac{1}{1+y^2} dy$$

Hence

$$e^x = \tan^{-1} y + c$$

$$\tan^{-1} y = e^x - c$$

$$y = \tan(e^x - c)$$

Example 2

Radioactive Decay

A radioactive substance decays (i.e., the amount of substance is decreasing) at a rate proportional to the amount resent.

Find the amount at time t .

Example 2 (cont)

Let *x*(t) be the amount at time t.

Then $\frac{dx}{dt}$ represents the rate of change of the amount at time t.

$$\frac{dx}{dt} \propto -x \quad \Rightarrow \quad \frac{dx}{dt} = -kx$$

Example 2 (cont)

Note that we write $-x$ in

$$\frac{dx}{dt} \propto -x$$

because the amount *x* of the substance is decreasing.

Example 2 (cont)

$$\int \frac{dx}{x} = \int -kdt$$

$$\ln|x| = -kt + c$$

$$\ln x = -kt + c \quad (\because x \geq 0)$$

$$x(t) = e^{-kt+c} = e^{c}e^{-kt} = Ae^{-kt}$$

$$x(t) = Ae^{-kt}$$

Example 2 (cont)

To find A, we need *x*(0), which is the initial amount (at *t*=0).

Thus

$$x(0) = Ae^{-k0}$$

$$A = x(0)$$

$$x(t) = x(0)e^{-kt}$$

Example 2 (cont)

To find k, we need the half-life τ of the substance .

Half-life is the time taken for the substance to decay to half of its initial amount.

$$\frac{1}{2}x(0) = x(0)e^{-k\tau}$$

$$k = \frac{\ln 2}{\tau}$$

So

$$x(t) = x(0)e^{-\frac{\ln 2}{\tau}kt}$$

Example 3 Cooling Problem

A copper ball is heated to 100°C.

At t=0, it is placed in water maintained at 30°C.

At the end of 3 mins, temperature of the ball is reduced to 70°C.

Find the time at which the temperature of the ball is 31°C.

Example 3 (cont)

Physical information:

Newton's Law of Cooling

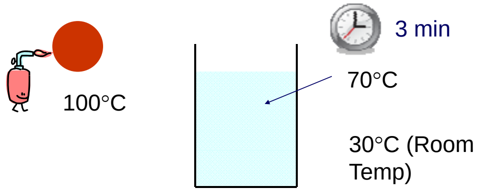
Rate of change *dT/dt* of the temperature *T*

of the ball is proportional to the difference between

T and the temp T_0 of the surrounding medium.

$$\frac{dT}{dt} = k(T - T_0)$$

Example 3 (cont)



$$\frac{dT}{dt} = k(T - T_0), \quad T_0 = 30$$

Question: T=100 when t=0, and T=70 when t=3.
Find t when T=31 .

$$\frac{dT}{dt} = k(T - T_0), \quad T_0 = 30$$

$$\int \frac{dT}{T - T_0} = \int k dt$$

$$\ln|T - T_0| = kt + c$$

$$\ln(T - T_0) = kt + c, \quad T > T_0$$

$$T(t) - T_0 = e^{kt + \ln 70} = 70e^{kt}$$

$$T(t) = 30 + 70e^{kt}$$

T_0=30

T=100 when t=0, hence

$$\ln 70 = 0 + c$$

Since T=70 when t=3, we have

$$70 - 30 = 70e^{3k}$$

$$\Rightarrow k = \frac{1}{3}\ln\frac{4}{7} = -0.1865$$

Solve

$$31 - 30 \approx 70e^{-0.1865 t_1}$$

$$\Rightarrow t_1 \approx 22.8$$

Example 4 (cont)

Physical assumptions and laws:
weight of the man + equipment = 712N
air resistance = bv^2,
where *b=30 kg/m=30kg/meter* and
v = velocity at time *t*.

Example 4 (cont)

Newton's
2nd Law

$$m\frac{dv}{dt} = \overset{712N}{mg} - bv^2$$

g=acceleration due to gravity =9.8m/s^2

$$\frac{dv}{dt} = -\frac{b}{m}(v^2 - k^2)$$

$$k^2 = \frac{mg}{b}$$

Example 4 (cont)

$$\frac{1}{v^2 - k^2}dv = -\frac{b}{m}dt$$

$$\frac{1}{2k}\left(\frac{1}{v-k} - \frac{1}{v+k}\right)dv = -\frac{b}{m}dt$$

$$\ln\left(\frac{v-k}{v+k}\right) = -\frac{2kb}{m}t + c$$

Example 4 (cont)

$$\frac{v-k}{v+k} = ce^{-pt},$$ where $$p = \frac{2kb}{m}$$

$$v = k\frac{1+ce^{-pt}}{1-ce^{-pt}}$$

How to find c?

$$\frac{v(0)-k}{v(0)+k} = ce^{-p0}$$

$$c = \frac{v(0)-k}{v(0)+k}$$

Example 4 (cont)

$$v = k\frac{1+ce^{-pt}}{1-ce^{-pt}}$$

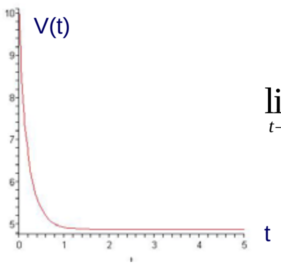
$$k^2 = \frac{mg}{b} = \frac{712}{30}$$

Now suppose $v(0) = 10$

Then $p = 4.02, \quad c = 0.345$

$$v = 4.87\frac{1+0.345e^{-4.02t}}{1-0.345e^{-4.02t}}$$

Example 4 (cont)



$$\lim_{t\to\infty} v(t) = 4.87$$

We can draw graph using
graphmatica

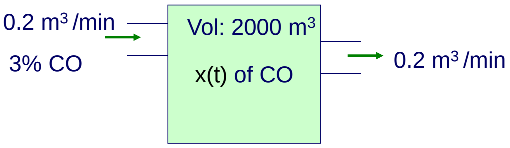
at http://www.graphmatica.com

Example 5 Mixture problem

A 2000m^3 room contains air with 0.002% CO at time t=0.
The ventilation system blows in air which contains 3% CO.
The system blows in and out air at a rate of 0.2m^3/min.
When is the air in the room containing 0.015% CO?

Example 5 (cont)

Let x(t) = vol of CO in the room at time t



0.2 m^3 /min

3% CO

Vol: 2000 m^3

x(t) of CO

0.2 m^3 /min

Example 5 (cont)

Let x(t) = vol of CO in the room at time t

$$\frac{dx}{dt} = \text{inflow - outflow}$$

CO per m^3

$$= 0.03 \times 0.2 - \frac{x}{2000} \times 0.2$$

$$= 0.006 - 0.0001x$$

Example 5 (cont)

$$\frac{dx}{dt} = 0.006 - 0.0001x$$

$$= 0.0001(60 - x)$$

$$\frac{dx}{60 - x} = 0.0001dt$$

$$-\ln(60 - x) = 0.0001t + c$$

$$\ln(60 - x) = -0.0001t - c$$

Example 5 (cont)

$$60 - x = e^{-0.0001t - c}$$

$$k = e^{-c}$$

$$= ke^{-0.0001t}$$

$$x(t) = 60 - ke^{-0.0001t}$$

Example 5 (cont)

Now we shall find k

As the 2000m^3 room contains air
with 0.002% CO at time t=0,
we have

x(0) = 2000x0.002/100 = 0.04

Example 5 (cont)

$$0.04 = x(0) = 60 - ke^{0}$$

$$k = 59.96$$

$$x = 60 - 59.96e^{-0.0001t}$$

Example 5 (cont)

To find the time t_1 when the air in the room contains 0.015% CO means

$x(t_1) = 0.00015 \text{ X } 2000 = 0.3$

Using $$x(t) = 60 - ke^{-0.0001t}$$

we have $$0.3 = 60 - 59.96e^{-0.0001t_1}$$

$$t_1 \approx 43.5\,\mathrm{min}$$

What happens when ODE is not separable?

Examples $$2xy\frac{dy}{dx} - y^2 + x^2 = 0$$

$$(2x - 4y + 5)\frac{dy}{dx} + x - 2y + 3 = 0$$

2 methods :

- reduction to separable
- linear change of variables

Reduction to separable form

Set $$\frac{y}{x} = v \Rightarrow y = vx$$

$$y' = v + xv'$$

$$y' = v + x\frac{dv}{dx}$$

$$\frac{dv}{y' - v} = \frac{dx}{x}$$

Example 6a : Reduction to separable form

$$2xy\frac{dy}{dx} - y^2 + x^2 = 0$$

➡ $$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = \frac{1}{2}\frac{y}{x} - \frac{1}{2}\frac{x}{y}$$

Let $$v = \frac{y}{x}$$

Then $$y = vx$$

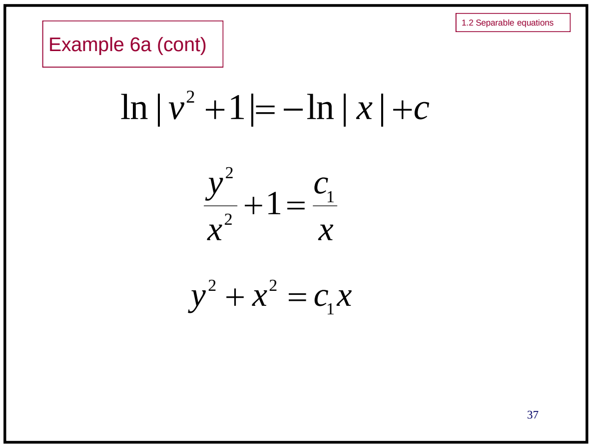
and $$y' = v + xv'$$

Example 6a (cont)

Hence $$v + xv' = \frac{v}{2} - \frac{1}{2v}$$

$$x\frac{dv}{dx} = -\frac{v}{2} - \frac{1}{2v}$$

$$\frac{2v}{v^2 + 1}dv = -\frac{1}{x}dx$$

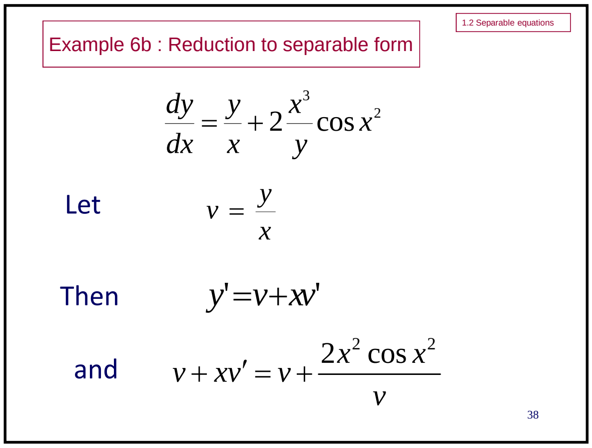


Example 6a (cont)

$$\ln|v^2+1| = -\ln|x| + c$$

$$\frac{y^2}{x^2} + 1 = \frac{c_1}{x}$$

$$y^2 + x^2 = c_1 x$$



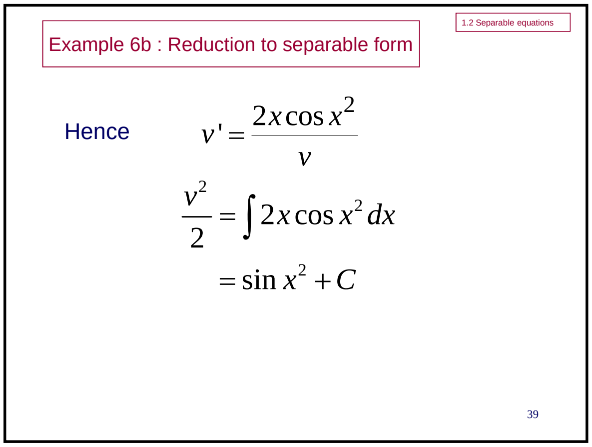
Example 6b : Reduction to separable form

$$\frac{dy}{dx} = \frac{y}{x} + 2\frac{x^3}{y}\cos x^2$$

Let $v = \frac{y}{x}$

Then $y' = v + xv'$

and $v + xv' = v + \frac{2x^2\cos x^2}{v}$

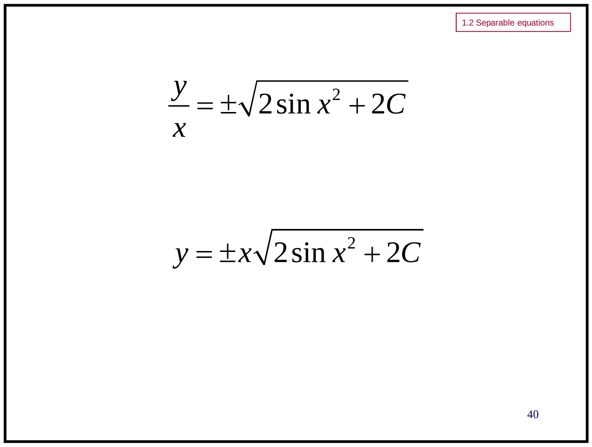


Example 6b : Reduction to separable form

Hence $v' = \frac{2x\cos x^2}{v}$

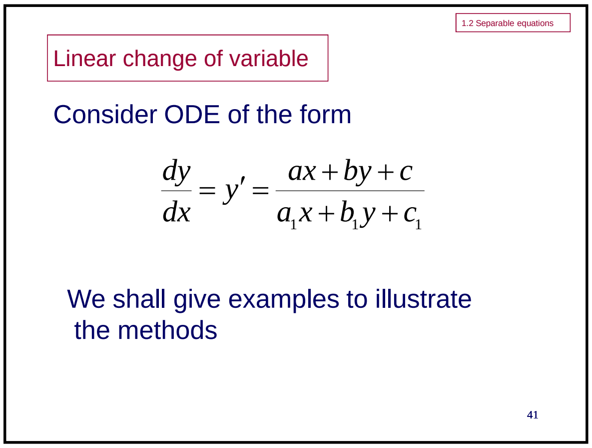
$$\frac{v^2}{2} = \int 2x\cos x^2\, dx$$

$$= \sin x^2 + C$$



$$\frac{y}{x} = \pm\sqrt{2\sin x^2 + 2C}$$

$$y = \pm x\sqrt{2\sin x^2 + 2C}$$

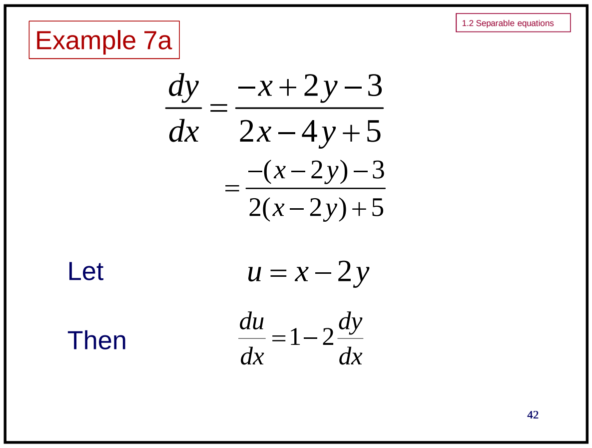


Linear change of variable

Consider ODE of the form

$$\frac{dy}{dx} = y' = \frac{ax + by + c}{a_1 x + b_1 y + c_1}$$

We shall give examples to illustrate the methods



Example 7a

$$\frac{dy}{dx} = \frac{-x + 2y - 3}{2x - 4y + 5}$$

$$= \frac{-(x-2y) - 3}{2(x-2y) + 5}$$

Let $u = x - 2y$

Then $\frac{du}{dx} = 1 - 2\frac{dy}{dx}$

Example 7a (cont)

Hence $$\frac{dy}{dx}=\frac{1}{2}(1-\frac{du}{dx})$$

$$\frac{1}{2}(1-\frac{du}{dx})=\frac{-u-3}{2u+5}$$

$$\frac{du}{dx}=\frac{4u+11}{2u+5}$$

$$\left(\frac{2u+5}{4u+11}\right)du=dx$$

Example 7a (cont)

Now $$\frac{2u+5}{4u+11}=\frac{1}{2}-\frac{1}{2}\frac{1}{4u+11}$$

$$\left(1-\frac{1}{4u+11}\right)du=2dx$$

$$u-\frac{1}{4}\ln(4u+11)=2x+c$$

$$(x-2y)-\frac{1}{4}\ln(4(x-2y)+11)=2x+c$$

Example 7b

$$\frac{dy}{dx}=\frac{y^2-x^2}{2xy}=\frac{1}{2}\frac{y}{x}-\frac{1}{2}\frac{x}{y}=\frac{v}{2}-\frac{1}{2v}$$

where $$\frac{y}{x}=v\ (\Rightarrow y=vx)$$

hence $$y'=v+xv'$$

$$v'x+v=\frac{v^2-1}{2v}$$

Example 7b (cont)

$$v'x=\frac{v^2-1}{2v}-v=-\frac{1+v^2}{2v}$$

$$\frac{2v}{v^2+1}dv=-\frac{1}{x}dx$$

$$\ln(v^2+1)=-\ln x+c$$

Example 7b (cont)

$$\ln((v^2+1)x)=c$$

$$(v^2+1)x=e^c=c_1$$

$$\left(\frac{y^2}{x^2}+1\right)x=c_1$$

1.3 Linear 1st Order ODE

$$\frac{dy}{dx}+p(x)y=Q(x)$$ Std form

Multiplingy both sides by the integrating factor

$$e^{\int p(x)dx}$$

$$\left(\frac{dy}{dx}+p(x)y\right)e^{\int p(x)dx}=Q(x)e^{\int p(x)dx}$$

We can check that

$$\frac{d}{dx}\left(ye^{\int p(x)dx}\right)=\frac{dy}{dx}e^{\int p(x)dx}+p(x)ye^{\int p(x)dx}$$

Hence
$$\frac{d}{dx}\left(ye^{\int p(x)dx}\right)=Q(x)e^{\int p(x)dx}$$

S0
$$ye^{\int p(x)dx}=\int Q(x)e^{\int p(x)dx}dx$$

Example 8

$$\frac{dy}{dx}+p(x)y=Q(x)$$

$$xy'-3y=x^2$$

$$y'-3\frac{1}{x}y=x$$

Recall the formula

$$ye^{\int p(x)dx}=\int Q(x)e^{\int p(x)dx}dx$$

First compute integrating factor

$$e^{\int(-3\frac{1}{x})dx}=e^{-3\ln x}=e^{\ln x^{-3}}=x^{-3}$$

Then apply

$$ye^{\int p(x)dx}=\int Q(x)e^{\int p(x)dx}dx$$

to get

$$yx^{-3}=\int xx^{-3}dx=\int x^{-2}dx=-x^{-1}+c$$

Example 9 Retarded fall—air resistance

An object of mass m is dropped from rest, i,.e.,

$$v(0)=0,x(0)=0$$

Assume that the resistance to the object is proportional to the magnitude of the velocity of the object.

Find the position x(t) and velocity v(t) at time t.

Newton's 2nd law states that

$$m\frac{dv}{dt}=mg-kv$$

Hence

$$m\frac{dv}{dt}+kv=mg$$

$$\frac{dv}{dt}+\frac{k}{m}v=g$$

Integrating factor is
$$e^{\int\frac{k}{m}dt}=e^{\frac{kt}{m}}$$

By formula

$$ve^{\frac{kt}{m}}=\int ge^{\frac{kt}{m}}dt=g\frac{m}{k}e^{\frac{kt}{m}}+c$$

To find c we use v(0)=0 and get

$$c=-g\frac{m}{k}$$

cont.

$$ve^{\frac{kt}{m}} = g\frac{m}{k}e^{\frac{kt}{m}} - g\frac{m}{k}$$

$$= g\frac{m}{k}(e^{\frac{kt}{m}} - 1)$$

$$v = g\frac{m}{k}(1 - e^{-\frac{kt}{m}})$$

cont.

Since $v = \frac{dx}{dt}$

we have $\frac{dx}{dt} = g\frac{m}{k}(1 - e^{-\frac{kt}{m}})$

$$x(t) = g\frac{m}{k}\int(1 - e^{-\frac{kt}{m}})dt$$

$$x(t) == g\frac{m}{k}(t + \frac{m}{k}e^{-\frac{kt}{m}}) + d$$

cont.

We use x(0)=0 to find d. Thus

$$0 = x(0) == g\frac{m}{k}(0 + \frac{m}{k}e^{0}) + d$$

$$d = -g\frac{m^2}{k^2}$$

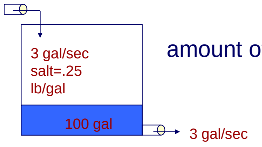
Hence

$$x(t) = g\frac{m}{k}(t + \frac{m}{k}e^{-\frac{kt}{m}}) - g\frac{m^2}{k^2}$$

Example 10 Mixture problem

At time t = 0 a tank contains 20 lbs of salt dissolved in 100 gal of water.
Assume that water containing 0.25 lb of salt per gallon is entering the tank at a rate of 3 gal/sec the solution is also leaving the tank at the same rate.
Find the amount of salt at time t.

Example 10 (cont)



amount of water = constant=100 gal

Let Q(t) be the amount of salt in the tank at time t.

Then Q(0)=20

$$\frac{dQ}{dt} = \text{inflow - outflow}$$

$$\frac{dQ}{dt} = 3\times 0.25 - 3\times\frac{Q}{100}$$

Example 10 (cont)

$$\frac{dQ}{dt} + \frac{3}{100}Q = 0.75$$

$$\boxed{e^{\int\frac{3}{100}dt} = e^{\frac{3t}{100}}}$$

$$Qe^{\frac{3t}{100}} = \int 0.75e^{\frac{3t}{100}}dt = 25e^{\frac{3t}{100}} + C$$

$$Q(t) = 25 - 5e^{-\frac{3t}{100}}$$

$$\lim_{t\to\infty} Q(t) = 25$$

Bernoulli Equations

$$y' + p(x)y = q(x)y^n$$

is called the Bernoulli equation.

If n=0 or n=1, Bernoulli equation is a 1st order linear ODE which has been discussed

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Bernoulli Equations

For n > 1, multiply

$$y' + p(x)y = q(x)y^n$$

by (1-n) y^{-n} and get

$$y'(1-n)y^{-n} + (1-n)y^{-n}p(x)y = (1-n)y^{-n}q(x)y^n$$

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cont.

Let $$z = y^{1-n}$$

Then $$z' = (1-n)y^{-n}y'$$

So, from

$$y'(1-n)y^{-n} + (1-n)y^{-n}p(x)y$$
$$= (1-n)y^{-n}q(x)y^n$$

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cont.

we get

$$z' + (1-n)p(x)z = (1-n)q(x)$$

which is a 1st order linear ODE in z.

Hence a Bernoulli equation becomes 1st order linear ODE by using the substitution

$$z = y^{1-n}$$

Example Bernoulli Equation

$$y' + y = x^2y^2$$

Set $$z = y^{1-2} = y^{-1}$$

With n=2 in the formula below

$$z' + (1-n)p(x)z = (1-n)q(x)$$

Example: Bernoulli Equation

we get $$z' + (1-2)z = (1-2)x^2$$

Solve this 1st order linear ODE to get

$$e^{-x}z = \int(-x^2)e^{-x}dx$$
$$= e^{-x}(x^2 + 2x + 2) + c$$

$$e^{-x}y^{-1} = e^{-x}(x^2 + 2x + 2) + c$$

Review: First Order ODE

- Separable $M(x)dx = N(y)dy$
- Linear $\frac{dy}{dx} + P(x)y = Q(x)$

Use integrating factor

What if neither applies?

Use the following substitutions

a) Reduction to separable, v = y/x

b) Linear change, u = ax+by +c

c) Bernoulli eq: $z = y^{1-n}$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

has many solutions. However if an initial condition $y(x_0)=y_{0,}$ (very often $x_0=0$) is given, then there is one and only one solution, i.e., the solution is unique.

Note that Q(x) may be a zero function, and we assume that P and Q are continuous.

1.4 Second-order linear ODE with constant coefficients

The general form is $\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = R(x)$

where A, B are constants.

When R(x) is a zero function, we have

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

The ODE is called homogeneous.

When R(x) is not a zero function,

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = R(x)$$

is called nonhomogenous.

We shall consider the homogeneous case first.

1.4.1 Second-order homogeneous linear ODE

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

It is clear that the zero function is a solution to the differential equation and this zero solution is called the trivial solution.

Now we shall look for nontrivial solutions. Recall that the general solution of first-order linear homogeneous ODE

$$\frac{dy}{dx} + p(x)y = 0$$

is $y = Ce^{-\int P(x)dx}$

Consider a special case: when p(x) is a constant , say B. Then the general solution is

$$y = Ce^{-Bx}$$

From this solution , we may guess that a nontrivial solution of

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

is of the form $y = e^{\lambda x}$

Then we have

$$\frac{dy}{dx} = \lambda e^{\lambda x} \qquad \frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x}$$

Substituting into $\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$ to get

$$\lambda^2 e^{\lambda x} + A\lambda e^{\lambda x} + Be^{\lambda x} = 0$$

$$\lambda^2 + A\lambda + B = 0$$

We call $\lambda^2 + A\lambda + B = 0$ the characteristic equation or auxiliary equation of the ODE.

When solving

$$\lambda^2 + A\lambda + B = 0$$

There are three cases:

- Two distinct real roots
- Only one real root
- Two distinct complex roots

Two distinct real roots

Suppose that the two distinct real roots are λ_1 and λ_2

Then we have two distinct (linearly independent, see Appendix 1) solutions

$$y = e^{\lambda_1 x} \qquad y = e^{\lambda_2 x}$$

General soln is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

The above property is called superposition principle (see Appendix 2)

In fact , we can prove that every solution is of the form

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

Here C_1 and C_2 are any constants.

Example: Solve $y'' - y' - 6y = 0$

Solution: Let $y = e^{\lambda x}$

Substitute this y into the given ODE to get

$$\lambda^2 - \lambda - 6 = 0$$

We have two distinct real roots,

$$\lambda_1 = 2, \lambda_2 = 3$$

Thus the general solution of the differential equation is

$$y = c_1 e^{2x} + c_2 e^{3x}$$

(b) Only one real root

Suppose that the only one real root is λ_1

Then we have a solution $y = e^{\lambda_1 x}$

It can be proved that there are two distinct (linearly independent) solutions in a 2nd ordered ODE.

What is the 2nd solution?

The 2nd solution is $y = xe^{\lambda_1 x}$

We can verify that

$$y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$$

is also a solution (superposition principle)

In fact , we can prove that every solution is of the form

$$y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$$

Example:

Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

Solution

The auxiliary equation is

$$\lambda^2 - 4\lambda + 4 = 0$$

We have only one solution $\lambda_1 = 2.$

Hence the general solution is

$$y = c_1 e^{2x} + c_2 x e^{2x}$$

Two distinct complex roots

Suppose that we have two distinct complex roots, namely λ_1 and λ_2

Then we have two distinct (linearly independent) complex-valued solutions:

$$y = e^{\lambda_1 x} \qquad y = e^{\lambda_2 x}$$

Suppose that $\lambda_1 = a + ib$

Then $\lambda_2 = a - ib$

Note that these two solutions are complex valued . Since we want real valued solutions, we have to look at the real part and

imaginary part of the solution

$$y = e^{\lambda_1 x} = e^{ax} e^{ibx} \qquad \lambda_1 = a + ib$$

$$= e^{ax}(\cos bx + i \sin bx)$$

$$= e^{ax} \cos bx + i \, e^{ax} \sin bx$$

Now the real part $y = e^{ax}\cos bx$

and the imaginary part $y = e^{ax}\sin bx$

can be shown to be two real valued solutions.

We can also prove that every solution is of the form

$$y = c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx$$
$$= e^{ax}(c_1\cos bx + c_2\sin bx)$$

Example:

Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

Solution

The complex roots of the auxiliary equation $\lambda^2 - 2\lambda + 2 = 0$

are $\lambda_1 = 1 + i$

and $\lambda_2 = 1 - i$

Hence the general solution is

$$y = e^{x}(c_1\cos x + c_2\sin x)$$

that is

$$y = c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx$$
$$= e^{ax}(c_1\cos bx + c_2\sin bx)$$

Remark:

As in the case for 1st order ODE, the 2nd order ODE

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

has many solutions.

If initial conditions are given, then there is ONLY one solution, as in the next example.

Example Initial value problem (IVP)

$$y'' - y = 0, \qquad \overbrace{y(0) = 5,\ y'(0) = 3}^{\text{Initial value conditions}}$$

e^{x}, e^{-x} are two linearly indep solutions

$y = c_1 e^{x} + c_2 e^{-x}$ is the general solution

$$y' = c_1 e^{x} - c_2 e^{-x}$$

$$\begin{matrix} 5 = c_1 + c_2 \\ 3 = c_1 - c_2 \end{matrix} \quad \Rightarrow \quad y = 4e^{x} + e^{-x}$$

1.4.2 Second-order nonhomogeneous linear ODE

The general form is

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = R(x)$$

The following three steps will be used to solve the ODE.

1.4.2 Second-order nonhomogeneous linear ODE

1.Find the general solution to the homogeneous equation

$$\frac{d^2y}{dx^2}+A\frac{dy}{dx}+By=0$$

Let the solution be y_h

2. Find a particular solution y_p to the nonhomogeneous equation

$$\frac{d^2y}{dx^2}+A\frac{dy}{dx}+By=R(x)$$

3. Add the solutions from step 1 and step 2 to get y_h+y_p which is the general solution to

$$\frac{d^2y}{dx^2}+A\frac{dy}{dx}+By=R(x)$$

(see Appendix 3)

We have learnt step 1. There are two methods for step 2.

Method 1.

The method of undetermined coefficients.

Method 2.

The method of variation of parameters.

Method 1. Method of undetermined coefficients

Example 1

Solve $y''-y'-2y=4x^2$

Can we guess a solution?

$$y_p=Ax^2$$

or $y_p=A+Bx+Cx^2$

Method 1 (cont)

It can be seen by direct substitution that $y_p=Ax^2$ is NOT a solution

To verify that $y_p=A+Bx+Cx^2$ is a solution, we have to find A, B, C.

First $(y_p)'=B+2Cx$

$(y_p)''=2C$

Substituting into $y''-y'-2y=4x^2$

to get $2C-B-2Cx-2A-2Bx-2Cx^2=4x^2$

So $C=-2, B=2, A=-3$

Hence $y_p = -3+2x-2x^2$

is a particular solution of

$$y''-y'-2y=4x^2$$

On the other hand

$$y_h = C_1e^{2x}+C_2e^{-x}$$

is the general solution of

$$y''-y'-2y=0$$

Therefore

$$y_h+y_p = C_1e^{2x}+C_2e^{-x}-3+2x-2x^2$$

is the general solution of the nonhomogeneous ODE

$$y''-y'-2y=4x^2$$

Here C_1 and C_2 can be any constant

Example 2

Solve $y''-3y'-4y=2\sin x$

We guess $y_p = A\cos x + B\sin x$

to be a particular solution.

As in Example 1, the values of A and B are

$$A=\frac{3}{17} \qquad B=\frac{-5}{17}$$

Hence a particular solution is

$$y_p = \frac{3}{17}\cos x + \frac{-5}{17}\sin x$$

On the other hand , the general solution of

$$y''-3y'-4y=0$$

is

$$y_h = C_1e^{4x}+C_2e^{-x}$$

So the general solution of

$$y''-3y'-4y=2\sin x$$

is

$$y_h + y_p$$

Example 3

Consider $y''+py'+qy=e^{ax}$

We guess a particular solution to be

$$y_p = Ae^{ax}$$

Subst $y_p = Ae^{ax}$ into the ODE to get

$$A(a^2+pa+q)e^{ax}=e^{ax}$$

Hence $A = \dfrac{1}{a^2+pa+q}$

We have to assume that

Case1 $\quad a^2+pa+q \neq 0$

i.e., e^{ax} is NOT a solution of the corresponding homogeneous equation

$$y''+py'+qy=0$$

Note that $a^2+pa+q=0$ iff

e^{ax} is a solution of $y''+py'+qy=0$

Case 2

Suppose that e^{ax} is a solution of

$$y''+py'+q=0 \qquad (a^2+pa+q=0)$$

Then we guess a particular solution to be

$$y_p = xAe^{ax}$$

Subst $\quad y_p = xAe^{ax}$

into $\quad y''+py'+qy=e^{ax}$

$$A(a^2+pa+q)xe^{ax}+A(2a+p)e^{ax}=e^{ax}$$

Since $\quad a^2+pa+q=0$

we have $\quad A(2a+p)=1$

So $\quad A=\frac{1}{2a+p}$

We assume that $\quad 2a+p\neq 0$

i.e., a is NOT a double root of

$$\lambda^2+p\lambda+q=0$$

Double root?

If a is a double root of

$$\lambda^2+p\lambda+q=0$$

then

$$a=\frac{-p\pm\sqrt{p^2-4q}}{2}=\frac{-p}{2} \qquad 2a+p=0$$

i.e., a is a double root if and only if a is the only root.

Case 3

Suppose that a is a double root

i.e., $\quad 2a+p=0$

Then we guess a particular solution is

$$y_p = x^2Ae^{ax}$$

Subst this solution into $\quad y''+py'+qy=e^{ax}$

to get $\quad A=\frac{1}{2}.$

Hence $\quad y_p=\frac{1}{2}x^2e^{ax}$

Summary

(A) The general solution of $y''-3y'-4y=0$

is $\quad C_1e^{4x}+C_2e^{-x}$

(1) A particular solution of $y''-3y'-4y=e^{2x}$

is of the form $\quad Ae^{2x}$

(2) A particular solution of $y''-3y'-4y=e^{4x}$

is of the form $\quad xAe^{4x}$

Summary (cont)

(B) The general solution of $\quad y''+2y'+1=0$

is $\quad C_1e^{-x}+C_2xe^{-x}$

So a particular soln of $\quad y''+2y'+1=e^{-x}$

is of the form $\quad x^2Ae^{-x}$

Example 4 Find a particular soln of

$$y''+y=\sin x$$

First the general soln of $y''+y=0$
is $C_1\sin x+C_2\cos x$

As in the summary, a particular soln is of the form

$$y_p=x(A\sin x+B\cos x)$$

(cont)

We can check that $A=0, B=\frac{-1}{2}$

Hence a particular soln is

$$y_p=x(\frac{-1}{2}\cos x)$$

The general soln of $y''+y=\sin x$
is

$$C_1\sin x+C_2\cos x-\frac{1}{2}x\cos x$$

Example 5

Consider $y''-4y'+2y=2x^3e^{2x}$

We can guess that a particular solution is

$$(Ax^3+Bx^2+Cx+D)e^{2x}$$

By the method used in previous examples, we can find A, B, C, D.However the computation is very involved. We will use the following method to simplify the computation

---

(cont)

Let $u(x)=Ax^3+Bx^2+Cx+D$

A particular soln is $y=ue^{2x}$

We have $y'=u'e^{2x}+2ue^{2x}$

$$y''=u''e^{2x}+4u'e^{2x}+4ue^{2x}$$

Subst the above into the given ODE and get

$$u''-2u=2x^3$$

Subst $u(x)=Ax^3+Bx^2+Cx+D$

into $u''-2u=2x^3$

We can find A,B, C, D and get

$$u(x)=-x^3-3x$$

Thus a particular soln is

$$y_p=(-x^3-3x)e^{2x}$$

Example 6

Consider $y''-4y'+4y=20x^3e^{2x}$

First note that $\lambda^2-4\lambda+4=0$
has only one root 2 (double root)

the general soln of $y''-4y'+4y=0$

is $C_1e^{2x}+C_2xe^{2x}$

Example 6 (cont)

So a particular soln of

$$y''-4y'+4y=20x^3e^{2x}$$

is of the form $x^2(Ax^3+Bx^2+Cx+D)e^{2x}$

Note that we have extra term x^2 above

By method used in Example 5 , we can get A=1,B=C=D=0

Simple examples

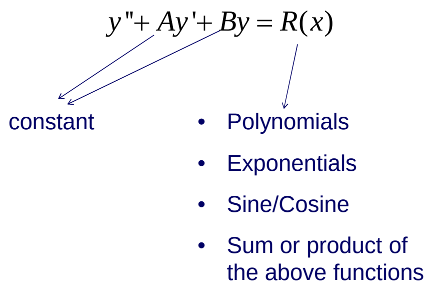
$$y''+y=10$$

$$y_p=10$$

$$3y''+2y=10$$

$$y_p=5$$

Remark: Method of undetermined coeff only works for the following case



$$y''+Ay'+By=R(x)$$

constant

- Polynomials
- Exponentials
- Sine/Cosine
- Sum or product of the above functions

Method 2: Variation of parameters

Let $y_h=c_1y_1(x)+c_2y_2(x)$

be the general solution of homogeneous ODE $y''+Ay'+By=0$

Then a particular solution of the corresponding nonhomogeneous ODE $y''+Ay'+By=r(x)$

is

$$y_p=u(x)y_1(x)+v(x)y_2(x)$$

(cont)

How to find u(x) and v(x)?

We can find u and v by using the following two equations

$$u'y_1+v'y_2=0$$

$$u'y_1'+v'y_2'=r(x)$$

Solving these two equations, get

$$u'=-\frac{y_2r}{y_1y_2'-y_1'y_2} \qquad v'=\frac{y_1r}{y_1y_2'-y_1'y_2}$$

(cont)

$$u=-\int\frac{y_2r}{y_1y_2'-y_1'y_2}dx$$

$$v=\int\frac{y_1r}{y_1y_2'-y_1'y_2}dx$$

Example 7 Solve $y'' + y = \tan x$

First note that $y_h = c_1 \cos x + c_2 \sin x$

is the general soln of $y'' + y = 0$

Hence a particular soln of $y'' + y = \tan x$

is

$$y_p = u(x)\cos x + v(x)\sin x$$

By

$$u = -\int \frac{y_2 r}{y_1 y_2' - y_1' y_2} dx$$

$$v = \int \frac{y_1 r}{y_1 y_2' - y_1' y_2} dx$$

Thus

$$u = -\int \sin x \tan x dx$$

$$= -\int \frac{\sin^2 x}{\cos x} dx = \int \frac{\cos^2 x - 1}{\cos x} dx$$

$$= \int (\cos x - \sec x) dx = \sin x - \ln|\sec x + \tan x|$$

$$v = \int \cos x \tan x dx = \int \sin x dx = -\cos x$$

General soln of $y'' + y = \tan x$ is

$y_h + y_p$

$$= c_1 \cos x + c_2 \sin x + u \cos x + v \sin x$$

$$= c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|$$

Example 8 $y'' - 3y' + 2y = \dfrac{1}{1 + e^{-x}}$

$$y_h = c_1 e^x + c_2 e^{2x}$$

$$y_p = u e^x + v e^{2x}$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}$$

$$\begin{aligned} u &= -\int \frac{e^{2x}}{1 + e^{-x}} \left(\frac{1}{e^{3x}}\right) dx \\ &= \ln(1 + e^{-x}) \end{aligned}$$

$$\begin{aligned} v &= \int \frac{e^{x}}{1 + e^{-x}} \left(\frac{1}{e^{3x}}\right) dx \\ &= \int \frac{e^{-2x} + e^{-x}}{1 + e^{-x}} - \frac{e^{-x}}{1 + e^{-x}} dx \\ &= -e^{-x} + \ln(1 + e^{-x}) \end{aligned}$$

$$\begin{array}{r l} & e^{-x} \\ e^{-x}+1 \,\big) & e^{-2x} \\ & \underline{e^{-2x}+e^{-x}} \\ & -e^{-x} \end{array}$$

$$\frac{e^{-2x}}{e^{-x}+1}=e^{-x}-\frac{e^{-x}}{e^{-x}+1}$$

Example 9 $y''+\frac{1}{x}y'-\frac{1}{x^2}y=\frac{1}{x^2+x^3}$

$$y_h=c_1x+c_2\frac{1}{x}$$

$$y_p=ux+v\frac{1}{x}$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}=-\frac{2}{x}$$

Recall
$$\begin{aligned} u &= -\int\frac{y_2r}{y_1y_2'-y_1'y_2}dx, \\ v &= \int\frac{y_1r}{y_1y_2'-y_1'y_2}dx. \end{aligned}$$

$$\begin{aligned} u &= -\int\frac{\frac{1}{x}}{-\frac{2}{x}}\left(\frac{1}{x^2+x^3}\right) \\ &= -\int\frac{1}{2}\left(-\frac{1}{x}+\frac{1}{x^2}+\frac{1}{1+x}\right) \\ &= \frac{1}{2}\left(\ln(1+x)-\ln x-\frac{1}{x}\right) \end{aligned}$$

$$\begin{aligned} v &= \int\frac{x}{-\frac{2}{x}}\left(\frac{1}{x^2+x^3}\right) \\ &= -\frac{1}{2}\ln(1+x) \end{aligned}$$

$$y = c_1x+c_2\frac{1}{x}-\frac{1}{2}-\frac{x}{2}\ln x+\left(\frac{x}{2}-\frac{1}{2x}\right)\ln(1+x).$$

Summary: 2nd Order Linear D.E.

$$y''+p(x)y'+q(x)y=F(x)$$

$$\begin{aligned} y &= y_h+y_p \\ &= \underbrace{c_1y_1+c_2y_2}+y_p \end{aligned}$$

If *p* and *q* are constants, use charac. equation

Method of undetermined coeff

Variation of parameters

1.5 Equilibrium solutions

In applications, most differential equations are rather complicated and are expressed in terms of the rate of changes.

It is common that we may be unable to solve such differential equations. In such case, we normally study some of its solutions, called the equilibrium solutions.

In this section, we shall consider the equilibrium solutions of some first and second order ODE.

(1) First order ODE

We begin with a concrete example.
Consider the differential equation:

$$\frac{dx}{dt} = x - x^2$$

To facilitate our explanations, we let

$$f(x) = x - x^2$$

Then our differential equation becomes

$$\frac{dx}{dt} = f(x)$$

To find the equilibrium solutions, we solve

$$f(x) = 0$$

That is $x - x^2 = 0$

Hence $x = 0,1$

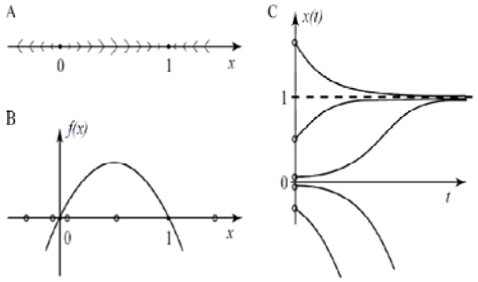
Stability of equilibrium solutions

Once we have found the equilibrium solutions, we will ask how solutions that start close to the equilibrium solutions behave.

If all the solutions that start sufficiently close to an equilibrium solution eventually approach the equilibrium solution, then the equilibrium solution is called a stable equilibrium solution.

If there are solutions starting arbitrarily close to the equilibrium solution and leaving the area near the equilibrium solution, then the equilibrium solution is called unstable equilibrium solution.

We normally use the phase line approach to study the nature (stable or unstable) equilibrium solutions.
In our example, we have



It is clear from graph C that $x(t) = 1$ is a stable equilibrium solution and $x(t) = 0$ is an unstable equilibrium solution.

(2) Second order ODE

Consider the second ODE $\frac{d^2x}{dt^2} = f(x)$

To find the equilibrium solution of the differential equation, we have to solve f(x)=0.

Let $x = a$ be a solution of f(x)=0.

Then $x(t) = a$ is an equilibrium solution of the ODE.

To consider the nature of the equilibrium solution, we consider the sign of $f'(a)$.

If $f'(a) < 0$ then $x(t) = a$ is a stable equilibrium solution of the ODE.

If $f'(a) > 0$ then $x(t) = a$ is an unstable equilibrium solution of the ODE.

Examples

(1) Consider the ODE $\frac{d^2x}{dt^2} + \sin x = 0$

Let $f(x) = -\sin x$

Then $f(0) = 0$ $f(\pi) = 0$

Hence $x(t) = 0$ $x(t) = \pi$

are equilibrium solutions.

Now $f'(x) = -\cos x$

(i) $f'(0) = -\cos 0 = -1 < 0$

implies $x(t) = 0$

is a stable equilibrium solution.

(ii) $f'(\pi) = -\cos\pi = 1 > 0$

implies $x(t) = \pi$

is an unstable equilibrium solution.

Examples

(2) Consider the ODE $\frac{d^2x(t)}{dt^2} = -x(t)$

Let $f(x) = -x$

Then $f(0) = 0$

Hence $x(t) = 0$

is an equilibrium solution.

Now $f'(x) = -1$

Since $f'(0) = -1 < 0$

$x(t) = 0$ is a stable equilibrium solution.

(3) Consider the ODE $\frac{d^2x(t)}{dt^2} = x(t)$

Let $f(x) = x$

Then $f(0) = 0$

Hence $x(t) = 0$

is an equilibrium solution.

Now $f'(x) = 1$

Since $f'(0) = 1 > 0$

$x(t) = 0$ is an unstable equilibrium solution.

Appendix 1 (Optional) Linearly independent soln

- Two solutions u(x) and v(x) are said to be linearly dependent if we can find a constant c such that u(x)=cv(x), for all x, otherwise they are linearly independent
- For examples, sinx and cosx are linearly indep; sinx and 2sinx are linearly dep.

Appendix 2 (Optional) Superposition principle

$$y'' + Ay' + By = 0$$

If y_1 and y_2 are solutions then so is $c\,y_1 + d\,y_2$

$$(cy_1+dy_2)'' + A(cy_1+dy_2)' + B(cy_1+dy_2)$$
$$= cy_1'' + dy_2'' + Acy_1' + Ady_2' + Bcy_1 + Bdy_2$$
$$= cy_1'' + Acy_1' + Bcy_1 + dy_2'' + Ady_2' + Bdy_2$$
$$= c(y_1'' + Ay_1' + By_1) + d(y_2'' + Ay_2' + By_2)$$
$$= c\cdot 0 + d\cdot 0 = 0$$

Appendix 2 (cont) Caution

Superposition may not be true for non-homogeneous ODE

$$y'' + y = 1$$

$1+\cos x$ and 1 are both solutions, but

$$1+1+\cos x$$

is NOT a solution

Appendix 3 (Optional) General soln of non-homogeneous ODE

$$y'' + Ay' + By = r(x)$$

General solution is $y = y_h + y_p$

where $y_h'' + Ay_h' + By_h = 0$

$$y_p'' + Ay_p' + By_p = r(x)$$

We can check that

$$(y_h+y_p)'' + A(y_h+y_p)' + B(y_h+y_p) = r(x)$$

Appendix 4

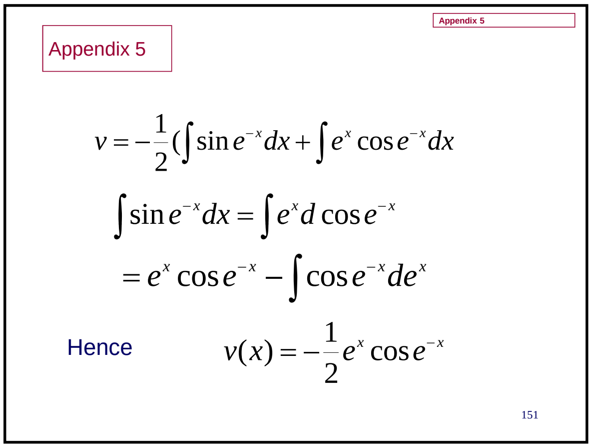
$$u = \frac{1}{2}\int e^{-2x}\sin e^{-x}dx + \frac{1}{2}\int e^{-x}\cos e^{-x}dx$$

$$\int e^{-2x}\sin e^{-x}dx = \int e^{-x}d\cos e^{-x}$$
$$= e^{-x}\cos e^{-x} - \int \cos e^{-x}de^{-x}$$
$$= e^{-x}\cos e^{-x} - \sin e^{-x}$$

Appendix 4 (cont)

$$\int e^{-x}\cos e^{-x}dx = -\int \cos e^{-x}de^{-x}$$
$$= -\sin e^{-x}$$

$$u(x) = \frac{1}{2}(e^{-x}\cos e^{-x} - 2\sin e^{-x})$$



Appendix 5

$$v = -\frac{1}{2}(\int \sin e^{-x} dx + \int e^{x} \cos e^{-x} dx$$

$$\int \sin e^{-x} dx = \int e^{x} d \cos e^{-x}$$

$$= e^{x} \cos e^{-x} - \int \cos e^{-x} de^{x}$$

Hence $$v(x) = -\frac{1}{2} e^{x} \cos e^{-x}$$