MA1506 TUTORIAL 8 SOLUTIONS

Question 1

From the given hint, we see that we have to solve

$$\frac{d^4y}{dx^4} = -\frac{Mg}{EI}\delta(x - A),$$

subject to the given boundary conditions. [Note that $y(0) = y'(0) = 0$ since the pole is horizontal at the point where it joins the wall.] Taking the Laplace transform of both sides we get

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = -\frac{Mg}{EI}e^{-sA}$$

from which

$$Y(s) = -\frac{Mg}{EI}s^{-4}e^{-sA} - s^{-3}\frac{MgA}{EI} + s^{-4}\frac{Mg}{EI}.$$

Taking the inverse we have

$$y(x) = -\frac{Mg}{EI}\Big[\frac{1}{6}(x - A)^3u(x - A) + \frac{1}{2}x^2A - \frac{1}{6}x^3\Big],$$

where u(x - A) represents the unit step function as usual. So for $x \leq A$ we have

$$y(x) = -\frac{Mg}{EI}\Big[\frac{1}{2}x^2A - \frac{1}{6}x^3\Big],$$

but for $x \geq A$ we get

$$y(x) = -\frac{Mg}{EI}\Big[\frac{1}{6}(x - A)^3 + \frac{1}{2}x^2A - \frac{1}{6}x^3\Big] = -\frac{Mg}{EI}\Big[\frac{1}{2}xA^2 - \frac{1}{6}A^3\Big].$$

Question 2

We have to deal with the equation

$$V(t) = RI + L\dot{I} + \frac{1}{C}\int_0^t I\ dt.$$

The problem here is that we don't actually know V(t); all we know is that it is some multiple of the Dirac delta function $\delta(t)$ [since it was applied to the system, and turned off, almost instantaneously]. So we set $V(t) = A\delta(t)$ where A is some unknown constant. Thus we have

$$A\delta(t) = RI + L\dot{I} + \frac{1}{C}\int_0^t I\ dt.$$

Take the Laplace transform of both sides, and let Θ(s) denote the transform of I(t); then we have

$$A = R\Theta(s) + Ls\Theta(s) + \frac{1}{sC}\Theta(s),$$

recalling that the transform of the integral is given by (1/s) times the transform of the integrand. Solving for Θ(s) we get

$$\Theta(s) = \frac{As}{Ls^2 + Rs + (1/C)}.$$

But the Laplace transform of the given current is

$$\Theta(s) = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1} = \frac{s}{s^2+2s+2} = \frac{As}{As^2+2As+2A},$$

from which we see immediately that R = 2A; since TAL knows that R = 2, we see that A = 1. Clearly C must have been 1/2 and L must have been 1 in the appropriate units.

Question 3

B produces 2-dimensional vectors, but A can only eat 3-dimensional vectors, so AB is not defined.

Question 4

In lectures we saw that given any square matrix A, A + A^T is symmetric and A - A^T is antisymmetric. So if we write

$$A = \frac{1}{2}[A + A^T] + \frac{1}{2}[A - A^T],$$

we see that any square matrix can be expressed as a sum of a symmetric with an antisymmetric matrix. Furthermore, if B is any n $\times$ n matrix, we can write

$$B = \frac{Trace(B)}{n} \times I_n + \Big[B - \frac{Trace(B)}{n} \times I_n\Big],$$

where I_n is the identity matrix. Since the trace of the identity matrix is n, the second matrix in this equation is traceless. The given answer is obtained by applying the first equation to $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, and then applying the second equation to its symmetric part.

Question 5

As explained in the hint, the idea is to rotate the axis down to the x-axis, which is done by means of the matrix $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$. We then shear by 30 degrees, which since the shearing matrix is $\begin{pmatrix} 1 & tan\theta \\ 0 & 1 \end{pmatrix}$, corresponds to the matrix $\begin{pmatrix} 1 & 1/\sqrt{3} \\ 0 & 1 \end{pmatrix}$. Then we rotate back up to the original axis, by means of $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$. So the matrix we need is [note the order]

$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1/\sqrt{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

If you multiply it out, you get the result stated. The determinant is 1, as it should be because it is the very nature of a shear to preserve areas, and the determinant is the measure of how much areas change under the action of a linear transformation. Note also that each of the three matrices above has determinant equal to 1, so their product must also have determinant equal to 1.

Question 6

Clearly $A^2 = - I$, and $A^3 = - A$, and $A^4 = I$, and $A^5 = A$ and so on; all of the *even* powers are equal to $\pm I$, and all of the *odd* powers are $\pm$ A. So it makes sense to separate them as follows:

$$e^{\theta A} = I + \theta A + \frac{1}{2!}\theta^2 A^2 + \frac{1}{3!}\theta^3 A^3 + \frac{1}{4!}\theta^4 A^4 + ...$$

$$= I - \frac{1}{2!}\theta^2 I + \frac{1}{4!}\theta^4 I + ...$$

$$+\theta A - \frac{1}{3!}\theta^3 A + \frac{1}{5!}\theta^5 A - ....$$

Taking out the common factors of I and A, we get

$$e^{\theta A} = I[1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + ...]$$

$$+A[\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - ....$$

But we recognise these Taylor series: we have

$$e^{\theta A} = Icos\theta + Asin\theta$$

$$= \begin{pmatrix} cos\theta & 0 \\ 0 & cos\theta \end{pmatrix} + \begin{pmatrix} 0 & -sin\theta \\ sin\theta & 0 \end{pmatrix}$$

which is the rotation matrix.

The case of $\begin{pmatrix} 1 & tan\theta \\ 0 & 1 \end{pmatrix}$ is simpler: define $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and note that *all* powers of B vanish. So we have

$$e^{Btan\theta} = I + Btan\theta + 0 + 0 + ... = \begin{pmatrix} 1 & tan\theta \\ 0 & 1 \end{pmatrix}.$$