MA1506
Mathematics II
Chapter 2
Oscillations

2.1 Introduction

In this chapter , we study an important system called harmonic oscillator (HO) which is an application of 2nd order linear ODE.

The ODE for harmonic oscillator is given by $mx'' + bx' + kx = F(t)$

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2.1 Introduction

Notation

In this chapter

$\frac{dx}{dt}$ may be denoted by $\dot{x}$ or x'

$\frac{d^2x}{dt^2}$ may be denoted by $\ddot{x}$ or x''

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2.1 Introduction

We shall consider four types of HO:
(1)Simple harmonic oscillator

$$m\ddot{x} + kx = 0$$

where m (mass) and k (spring constant) are positive numbers.
Its motion (solution) is periodic, called simple harmonic motion (SHM).

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2.1 Introduction

(2)Damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0$$

m >0, *b* >0 , *k* > 0

Damping constant

spring constant

In real oscillator, friction (damping) slows down the motion of the system. The frictional force is given by $-b\dot{x}$

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2.1 Introduction

(3) Forced harmonic oscillator without damping

$$m\ddot{x} + kx = F(t)$$

The system is a simple harmonic oscillator driven by an EXTERNALLY applied force F(t).

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2.1 Introduction

(4) Forced harmonic oscillator with damping

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

The system is a damped harmonic oscillator driven by an EXTERNALLY applied force F(t).

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2.2 Simple harmonic oscillator

A simple harmonic oscillator is a system which obeys Hooke's law:

The restoring force F_r in a system is proportional to the displacement x from the **equilibrium point.**

$$F_r = -kx$$

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2.2 Simple harmonic oscillator

The equation of motion of such a system is given by

$$m\ddot{x} + kx = 0$$

Newton's 2nd law
$m\ddot{x} = -kx$

k>0, k called spring constant

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2.2 Simple harmonic oscillatoring

Simple harmonic motion SHM can serve as a mathematical model for a variety of motions, such as a mass on a spring and a pendulum swinging with small amplitudes.

For details, see
http://en.wikipedia.org/wiki/Simple_harmonic_motion

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2.2 Simple harmonic oscillator

In a SHM equation (i.e., $m\ddot{x} + kx = 0$) it is typical to define the quantity

$$\omega = \sqrt{\frac{k}{m}}$$

and rewrite the equation $m\ddot{x} + kx = 0$

as $$\ddot{x} + \omega^2 x = 0$$

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2.2 Simple harmonic oscillator

The general soln of the equation is
$$x(t) = C\cos(\omega t) + D\sin(\omega t)$$
which can be rewritten in the phase-amplitude form as

$$x(t) = A\cos(\omega t - \delta)$$

amplitude

phase or phase angle

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Note that

$$x = A\cos(\omega t - \delta) = A\cos(\omega t + 2\pi - \delta)$$

$$= A\cos(\omega(t + \frac{2\pi}{\omega}) - \delta)$$

Hence SHM is periodic with period

$$\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

and amplitude A

Some technical terms

- Period T= $\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

  It is the time taken for a single oscillation(cycle).

Some technical terms (ctd)

- Frequency f = $\frac{1}{T}$ = $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$

  It gives the number of cycles per unit time.

Some technical terms (ctd)

- Angular frequency = $2\pi f = \sqrt{\frac{k}{m}} = \omega$

  It is the number of cycles per 2π unit time.

- Amplitude is the maximal displacement from the equilibrium position.

Why $A\cos(\omega t - \delta)$?

In fact it can also be rewritten in one of the following forms

$$A\sin(\omega t - \delta)$$

$$A\cos(\omega t + \delta)$$

$$A\sin(\omega t + \delta)$$

The two constants A and δ are determined by initial conditions

Suppose $x(0) = x_0 \quad \dot{x}(0) = v_0$

Then $x_0 = A\cos(-\delta)$

and $v_0 = (-A\omega)\sin(-\delta)$

So we can find A and δ , for example,

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$$

How to sketch the graph of

$$y = 3\cos(2x - \pi/4)$$

Amplitude

Angular frequency

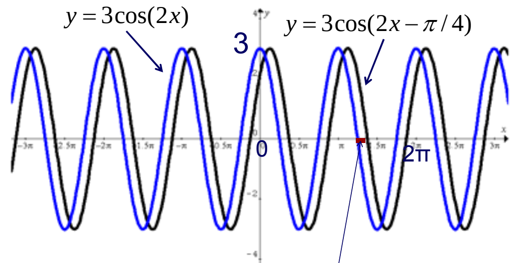
Phase angle

http://www.graphmatica.com

(cont)

$y = 3\cos(2x)$

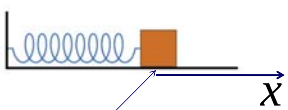
$y = 3\cos(2x - \pi/4)$



Amplitude=3
Angular frequency=2

Phase angle/angular freq=π/4/2=π/8

Example 1 Horizontal spring-mass system



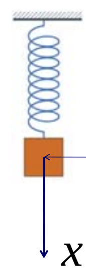
Equilibrium position

Newton's 2nd law

$$m\ddot{x} = -kx$$

$$m\ddot{x} + kx = 0$$

Example 2 Vertical spring-mass system



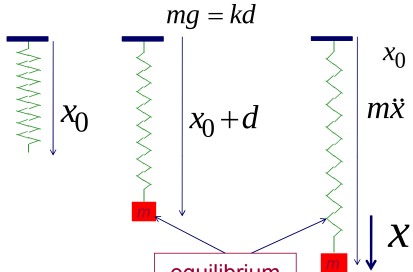
$$m\ddot{x} + kx = 0$$

equilibrium position

(cont)

$$mg + (-kd) = 0$$

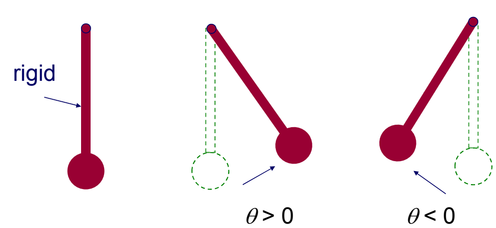
$$mg = kd$$



$$m\ddot{x} = -k(d + x) + mg = -kx$$

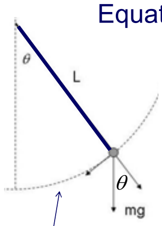
equilibrium position

2.2.1 Pendulum with small amplitude (small angle)



2.2.2 Pendulum with small amplitude

Equation of motion of pendulum is



$$m\frac{d^2(L\theta)}{dt^2} = -mg\sin\theta$$

Hence $mL\ddot{\theta} = -mg\sin\theta$

Non linear 2nd Order ODE.

arc length $L\theta$

where $\dot{\theta} = \frac{d\theta}{dt}$ $\ddot{\theta} = \frac{d^2\theta}{dt^2}$

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2.2.2 (cont)

$$mL\ddot{\theta} = -mg\sin\theta$$

We now approximate the above nonlinear ODE by linear ODE when θ is small

By Taylor series (at 0) of $\sin\theta$, we have

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \ldots$$

Thus, when θ is small, we have

$$\sin\theta \approx \theta$$

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2.2.2 (cont)

$$mL\ddot{\theta} \approx -mg\theta$$

$$\ddot{\theta} \approx -\frac{g}{L}\theta = -\omega^2\theta$$

where $\omega^2 = \frac{g}{L}$

minus is crucial

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2.2 Simple harmonic oscillator

So the given nonlinear ODE

$$mL\ddot{\theta} = -mg\sin\theta$$

can be approximated by $\ddot{\theta} = -\omega^2\theta$

General solution is

$$\theta = A\cos(\omega t - \delta)$$

amplitude

angular frequency

phase angle

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2.2 Simple harmonic oscillator

Simple pendulum oscillation

http://en.wikipedia.org/wiki/File:Simple_Pendulum_Oscillator.gif

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2.3 Damped, Unforced Oscillators

$$m\ddot{x} + b\dot{x} + kx = 0$$

$m>0$, $k > 0$, $b > 0$

spring constant

damping constant

$$m\lambda^2 + b\lambda + k = 0$$

Auxiliary equation

Case 1: two real roots — Over damping

Case 2: double root — Critical damping

Case 3: complex roots — Under damping

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See damped oscillation at

http://www.aw-bc.com/ide/idefiles/media/JavaTools/vibedamp.html

2.3.1 Damped, Unforced Oscillators (2 negative real roots) Overdamping

Example $\ddot{x}+3\dot{x}+2x=0$

We have $\lambda=-1,-2$

General solution $x(t)=c_1e^{-t}+c_2e^{-2t}$

Overdamping

No oscillation

tend to zero rapidly

2.3.2 Damped, Unforced Oscillators (repeated roots) critical damping

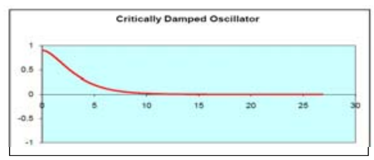
Example $\ddot{x}+6\dot{x}+9x=0$

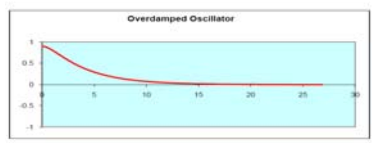
We have $\lambda=-3,-3$ (critical damping)

General solution $x(t)=c_1e^{-3t}+c_1te^{-3t}$

x(t) also goes to zero rapidly, and is called critical damping.

http://www.aw-bc.com/ide/idefiles/media/JavaTools/vibedamp.html





Damping is useful

"Don't let the door hit you on the way out."

A door-closer has two main parts: a spring to close the door, and a damper to prevent the door from slamming shut

Old western swing doors

Damping is useful

Suspension

Damper pedal

2.3 Damped, Unforced Oscillators

The graceful movement of everyday things
see

file:///F:/damping/The%20Graceful%20Movement%20of%20Everyday%20Things%20%20The%20Floating%20Bones%20Journal.htm

2.3.3 Damped, Unforced Oscillators (complex roots) underdamped

Example $\ddot{x} + 2\dot{x} + 26x = 0$

$$\lambda = -1 \pm 5i$$

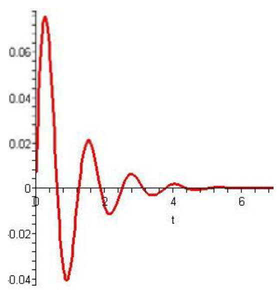
General solution

$$x(t) = e^{-t}\left[c_1\cos(5t) + c_2\sin(5t)\right]$$

which can be rewritten as

$$x(t) = Ae^{-t}\cos(5t - \delta)$$

Motion is under-damped.



Damped, Unforced Oscillators (complex roots) underdamped General case

$$\boxed{m\ddot{x} + b\dot{x} + kx = 0}$$

General soln is

$$x(t) = Ae^{\frac{-bt}{2m}}\cos(\beta t - \delta)$$

amplitude depends on time

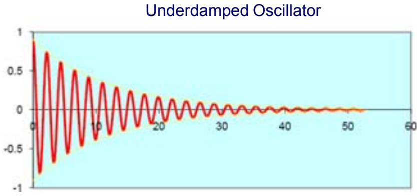
$\beta = \frac{1}{2m}\sqrt{4mk - b^2}$ = quasi-frequency

$\frac{2\pi}{\beta}$ is called the quasi-period.

2.3 Damped, Unforced Oscillators

http://www.aw-bc.com/ide/idefiles/media/JavaTools/vibedamp.html

Underdamped Oscillator

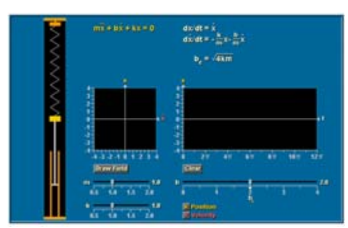


2.3.4 Damped, Unforced Oscillators Example 1

$$\boxed{m\ddot{x} + b\dot{x} + kx = 0}$$

spring constant damping constant

$m>0$, $k > 0$, $b > 0$

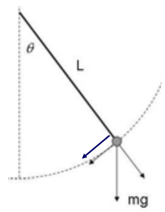


Damped Mass Spring Oscillator

Textbook p196

http://www.aw-bc.com/ide/idefiles/media/JavaTools/massprng.html

2.3.4 Damped pendulum Example 2



air resistance $\propto \dot{\theta}$

anti-motion

$$mL\ddot{\theta} = -mg\sin\theta - SL\dot{\theta} \approx -mg\theta - SL\dot{\theta}$$

$$\boxed{m\ddot{\theta} + S\dot{\theta} + \frac{mg}{L}\theta = 0}$$

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2.4 Forced Oscillators $m\ddot{x} + kx = F_0 \cos\alpha t$

$$m\ddot{x} + kx = F_0\, e^{i\alpha t}$$

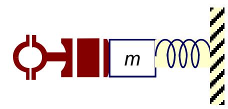
general solution $x = A\cos(\omega t - \delta) + \dfrac{F_0/m}{\omega^2 - \alpha^2}\cos(\alpha t)$

x_h $\quad$ x_p

(See appendix 3)

where $\omega = \sqrt{k/m}$

$F_0 \cos(\alpha t)$



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2.4 Forced Oscillators $m\ddot{x} + kx = F_0 \cos\alpha t$

$$x = A\cos(\omega t - \delta) + \frac{F_0/m}{\omega^2 - \alpha^2}\cos(\alpha t)$$

Assume initial condition

$$x(0) = \dot{x}(0) = 0$$

Then

$$x = \frac{F_0/m}{\omega^2 - \alpha^2}\big(\cos(\alpha t) - \cos(\omega t)\big)$$

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2.4 Forced Oscillators $m\ddot{x} + kx = F_0 \cos\alpha t$

$$x = \frac{F_0/m}{\omega^2 - \alpha^2}\big(\cos(\alpha t) - \cos(\omega t)\big)$$

Use $\cos A - \cos B = -2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)$

$$x = \underbrace{\frac{2F_0/m}{\alpha^2 - \omega^2}\sin\left[\left(\frac{\alpha - \omega}{2}\right)t\right]}_{A(t)} \sin\left[\left(\frac{\alpha + \omega}{2}\right)t\right]$$

Small angular frequency

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2.4 Forced Oscillators $m\ddot{x} + kx = F_0 \cos\alpha t$

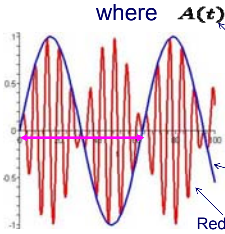
$$x = A(t)\sin\left[\left(\frac{\alpha + \omega}{2}\right)t\right]$$

Red curve

where $A(t) = \dfrac{2F_0/m}{\alpha^2 - \omega^2}\sin\left[\left(\dfrac{\alpha - \omega}{2}\right)t\right]$

Blue curve

$\dfrac{\alpha - \omega}{2}$ beat frequency



Blue curve

Red curve

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Beating

2.4 Forced Oscillators

If our ear is exposed to two sounds

1st sound $\quad$ 2nd sound

$$x = \frac{F_0/m}{\omega^2 - \alpha^2}\big(\cos(\alpha t) - \cos(\omega t)\big)$$

$$= A(t)\sin[\left(\frac{\alpha + \omega}{2}\right)t]$$

where

$$A(t) = \frac{2F_0/m}{\alpha^2 - \omega^2}\sin\left[\left(\frac{\alpha - \omega}{2}\right)t\right]$$

then we can only hear the term which contains the **difference** of the frequencies.

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(cont) Beating

2.4 Forced Oscillators

Thus a fast signal $\sin[(\frac{\alpha+\omega}{2})t]$

is modulated by a slower one

$$A(t)=\frac{2F_0/m}{\alpha^2-\omega^2}\sin\left[\left(\frac{\alpha-\omega}{2}\right)t\right]$$

This behaviour is called beating in physics.

http://www.school-for-champions.com/science/sound_beat.htm

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2.4 (cont) Forced Oscillators $m\ddot{x}+kx=F_0\cos\alpha t$

$$\lim_{\alpha\to\omega}A(t)=\lim_{\alpha\to\omega}\frac{2F_0/m}{\alpha+\omega}\times\frac{\sin\left[\frac{\alpha-\omega}{2}t\right]}{\alpha-\omega}$$

$$=\frac{F_0}{m\omega}\times\frac{t}{2}=\frac{F_0 t}{2m\omega}$$ L' Hospital Rule

When external frequency α is close to the natural frequency ω ,

A(t) tends to the straight line $\frac{F_0 t}{2m\omega}$

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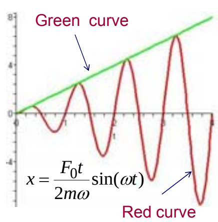
2.4 (cont) Forced Oscillators $m\ddot{x}+kx=F_0\cos\alpha t$

$$x=A(t)\sin\left[\left(\frac{\alpha+\omega}{2}\right)t\right]$$

Red curve Green curve

$$\lim_{\alpha\to\omega} x=\frac{F_0 t}{2m\omega}\sin(\omega t)$$

blue curve in slide 46 tends to green st. line when α tends to ω



Green curve

$x=\frac{F_0 t}{2m\omega}\sin(\omega t)$

Red curve

Oscillations go out of control when α is close to ω . It is said to be in resonance.

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Resonance

2.4 Forced Oscillators

If the driving (external) force has a frequency α close to the natural frequency ω of the system, the resulting amplitude can be very large even for small driving (external) amplitude.
The system is said to be in resonance.
It may cause violent swaying motions and even catastrophic failure in improperly constructed structures including bridges, buildings, and airplanes.

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(cont) Resonance

2.4 Forced Oscillators

Avoiding resonance disasters is a major concern in every building and bridge construction project.
As a countermeasure, a tuned mass damper can be installed to avoid disaster.
The Taipei 101 building relies on a 730-ton pendulum — a tuned mass damper — to avoid resonance.

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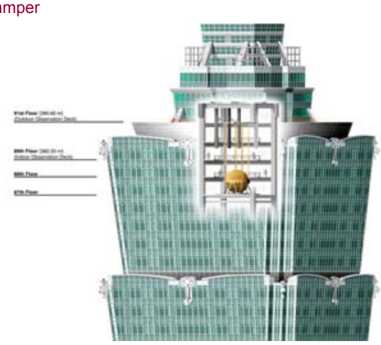
Tuned mass damper

Taipei 101 Tuned Mass Damper



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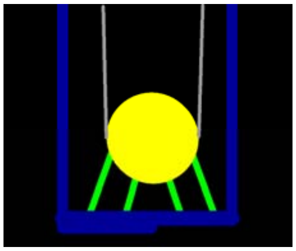
A **tuned mass damper is** a device mounted in structures to prevent damage caused by vibration



http://en.wikipedia.org/wiki/File:Tuned_mass_damper.gif

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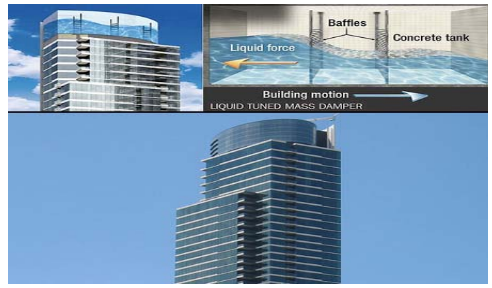
Liquid tuned mass damper



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Tuned mass damper

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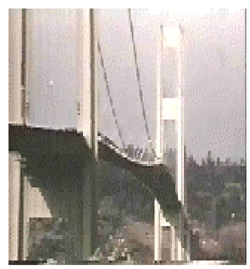
Collapse of the *Tacoma Narrow Bridge*.
http://www.youtube.com/watch?v=3mclp9QmCGs



Bridge
tuned mass damper

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2.4 Forced Oscillators

Forced , NO damped, Oscillator

$$m\ddot{x} + kx = F_0 \cos \alpha t$$

has two important phenomena

beating and resonance

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2.5 Forced Damped Oscillators

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\alpha t)$$

A particular solution is

$$x(t) = \frac{F_0(k - m\alpha^2)\cos(\alpha t) + F_0 b\alpha \sin(\alpha t)}{(k - m\alpha^2)^2 + b^2\alpha^2}$$

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2.5 Forced Damped Oscillators

General soln is

$$x(t) = \frac{F_0(k - m\alpha^2)\cos(\alpha t) + F_0 b\alpha \sin(\alpha t)}{(k - m\alpha^2)^2 + b^2\alpha^2}$$

+ Gen Sol of $m\ddot{x} + b\dot{x} + kx = 0$

The 2nd part(damped oscillation) tends to zero rapidly.

Hence 2nd part is called the transient solution.

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2.5 Forced Damped Oscillators

So when t is big enough, the general soln becomes

$$x(t) = \frac{F_0(k - m\alpha^2)\cos(\alpha t) + F_0 b\alpha \sin(\alpha t)}{(k - m\alpha^2)^2 + b^2\alpha^2}$$

$$x(t) = \frac{\frac{1}{m}F_0 \cos(\alpha t - \gamma)}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}} \qquad \omega = \sqrt{k/m}$$

Oscillation at angular frequency α

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2.5 Forced Damped Oscillators (ctd)

$x(t)$ is called the steady-state solution (or response) and is equal to the particular solution.

http://www.aw-bc.com/ide/idefiles/media/JavaTools/vibefdmp.html

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2.5 (cont) Forced Damped Oscillators

Although the steady-state oscillation has the same frequency as the external force it is NOT in phase with the external force.

The amplitudes of the steady-state solution and the external force are also different.

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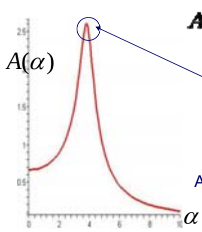
2.5 (cont) Forced Damped Oscillators

$$x(t) = \frac{\frac{1}{m}F_0 \cos(\alpha t - \gamma)}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}$$ steady-state soln (response)

$$A(\alpha) = \frac{F_0/m}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}$$



Amplitude is max when????

Amplitude of steady-state response curve

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2.5.1 Example 1 Forced Damped Harmonic motion

air resistance $\propto \dot{\theta}$

External force

$$mL\ddot{\theta} = -mg\sin\theta - SL\dot{\theta} + F(t)$$

$$\boxed{m\ddot{\theta} + S\dot{\theta} + \frac{mg}{L}\theta = \frac{1}{L}F(t)}$$

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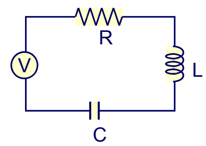
2.5.1 Example 2 LCR Circuit

The electrical analog of the forced damped mass-spring oscillator is the LCR circuit.
An **LCR circuit** (also known as a resonant circuit, tuned circuit, or RCL circuit) is an electrical circuit consisting of a resistor (R), an inductor (L), and a capacitor (C).

(cont) ODE for LCR circuit

Define $Q = \int I\, dt$



$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t)$$

ODE for LCR circuit

(cont) LCR circuit

LCR circuits have many applications particularly for oscillating circuits and in radio and communication engineering.

For example, AM/FM radios typically use an LCR circuit to tune a radio frequency.

2.6 Conservation of Energy

First note that

$$\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = \frac{d}{dx}\left(\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right)$$
$$= \frac{d}{dt}\left(\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right)\frac{dt}{dx} \quad \text{Chain rule}$$
$$= \frac{1}{2}2\frac{dx}{dt}\frac{d^2x}{dt^2}\frac{dt}{dx}$$
$$= \frac{d^2x}{dt^2} = \ddot{x}$$

2.6 (cont) Conservation of Energy

SHM: $m\ddot{x} = m\frac{d}{dx}(\frac{1}{2}\dot{x}^2) = -kx$ $\quad \boxed{d(\frac{1}{2}m\dot{x}^2) = -kx\,dx}$

Integrate $\frac{1}{2}m\dot{x}^2 = -\frac{1}{2}kx^2 + E$

$$\Rightarrow E = \underbrace{\frac{1}{2}m\dot{x}^2}_{\text{kinetic}} + \underbrace{\frac{1}{2}kx^2}_{\text{potential}}$$

Conservation of energy : The total mechanical energy E remains constant for SHM

2.6 Conservation of Energy

SHM: $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$

Let $y = \dot{x}$ get $\frac{1}{2}kx^2 + \frac{1}{2}my^2 = E$ Ellipse

Unstable motion: $\ddot{x} = +\omega^2 x$ Hyperbolic curve

As in SHM case $\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 = E$ = Constant

2.6 Conservation of Energy (friction)

Damped HM: $m\ddot{x} = -kx - b\dot{x}$

$$\boxed{\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = \ddot{x}}$$

$$m\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) + kx = -b\dot{x}$$

Integrate $E = \int -b\dot{x}dx$

$$\boxed{E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2}$$

$$\Rightarrow \frac{dE}{dx} = -b\dot{x}$$

$$\frac{dE}{dt} = \frac{dx}{dt}\frac{dE}{dx} = \dot{x}\frac{dE}{dx} = -b\dot{x}^2 \le 0$$

For damped HM, E is not constant, E is decreasing

http://www.aw-bc.com/ide/idefiles/media/JavaTools/vibedenr2.html

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Appendix 1

Use the formula

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

to get

$$A\cos(\omega t - \delta) = A\cos(\omega t)\cos(-\delta) + A\sin(\omega t)\sin(-\delta)$$

Hence $C\cos(\omega t) + D\sin(\omega t)$

can be written as $A\cos(\omega t - \delta)$

where $C = A\cos(-\delta)$

$D = A\sin(-\delta)$

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Appendix 2

Formulae

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$(\cosh x)^2 - (\sinh x)^2 = 1$$

$$\frac{d\sinh x}{dx} = \cosh x \qquad \frac{d\cosh x}{dx} = \sinh x$$

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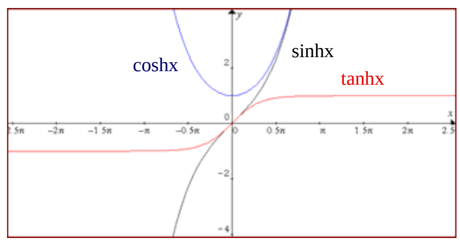
Appendix 2 (cont) Graphs of sinhx, coshx, tanhx

http://www.graphmatica.com



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Appendix 3 $m\ddot{x} + kx = F_0\cos\alpha t$

$$x = A\cos(\omega t - \delta) + \frac{F_0/m}{\omega^2 - \alpha^2}\cos(\alpha t)$$

$$\dot{x} = -A\omega\sin(\omega t - \delta) - \frac{\alpha F_0/m}{\omega^2 - \alpha^2}\sin(\alpha t)$$

Assume initial condition $x(0) = \dot{x}(0) = 0$

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Appendix 3 (cont) $m\ddot{x} + kx = F_0\cos\alpha t$

By the initial conditions in previous slide, get

$$0 = A\cos(\delta) + \frac{F_0/m}{\omega^2 - \alpha^2} \Rightarrow A = -\frac{F_0/m}{\omega^2 - \alpha^2}$$

$$0 = A\omega\sin(\delta) \Rightarrow \delta = 0$$

$$x = \frac{F_0/m}{\omega^2 - \alpha^2}\left(\cos(\alpha t) - \cos(\omega t)\right)$$

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