ANSWERS TO MA1506 TUTORIAL 7

Question 1.

(a) We shall use the following s-Shifting property:

$L(f(t)) = F(s) \Rightarrow L(e^{ct}f(t)) = F(s-c)$

$\therefore \quad L(t^2) = \dfrac{2}{s^3} \Rightarrow$ use $L(t^n) = \dfrac{n!}{s^{n+1}}$

$\therefore \quad L(t^2e^{-3t}) = L(e^{-3t}t^2) = \dfrac{2}{(s+3)^3}$

(b) Here u denotes the Unit Step Function given by

$u(t-a)\begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$

We shall use the following t-Shifting property:

$L(f(t)) = F(s) \Rightarrow L\{f(t-a)u(t-a)\} = e^{-as}F(s)$

Let$f(t-2) = t$

$\therefore \quad f(t) = t + 2$

$\therefore \quad L(f(t)) = L(t+2) = L(t) + 2L(1)$

$= \dfrac{1}{s^2} + \dfrac{2}{s}$

$\therefore \quad L(tu(t-2)) = L\{f(t-2)u(t-2)\}$

$= e^{-2s}(\dfrac{1}{s^2} + \dfrac{2}{s})$

Question 2. (a)

$\dfrac{s}{s^2+10s+26} = \dfrac{s}{(s+5)^2+1} = \dfrac{(s+5)-5}{(s+5)^2+1}$

Let$F(s) = \dfrac{s-5}{s^2+1}$

$\therefore \quad L^{-1}(\dfrac{s}{s^2+10s+26}) = L^{-1}(F(s+5))$

$= L^{-1}(F(s-(-5))$

$= e^{-5t}L^{-1}(F(s)) \rightarrow$ use s-shifting

$= e^{-5t}L^{-1}(\dfrac{s}{s^2+1} - \dfrac{5}{s^2+1})$

$= e^{-5t}\{L^{-1}(\dfrac{s}{s^2+1}) - 5L^{-1}(\dfrac{1}{s^2+1})\}$

$= e^{-5t}(\text{cost} - 5\sin t)$

(b) Let $F(s) = \frac{1+2s}{s^3}$

$= \frac{1}{s^3} + \frac{2}{s^2}$

$\therefore \quad L^{-1}(F(s)) = \frac{t^2}{2} + 2t \quad \rightarrow \quad (\text{use } L(t^n) = \frac{n!}{s^{n+1}})$

Let $f(t) = \frac{t^2}{2} + 2t$

Using t-shifting,

$$L^{-1}(e^{-2s}\frac{1+2s}{s^3}) = L^{-1}(e^{-2s}F(s))$$
$$= f(t-2)u(t-2)$$
$$= \{\frac{(t-2)^2}{2} + 2(t-2)\}u(t-2)$$
$$= \frac{1}{2}(t^2-4)u(t-2)$$
$$= (\frac{1}{2}t^2 - 2)u(t-2)$$

Question 3. (a)

Let $L(y(t)) = Y(s)$

We shall use $L(y'(t)) = sY(s) - y(0)$.

We have

$$L(y') = L(tu(t-2))$$
$$\Rightarrow sY(s) - 4 = e^{-2s}(\frac{1}{s^2} + \frac{2}{s})$$
$$\Rightarrow Y(s) = e^{-2s}(\frac{1+2s}{s^3}) + \frac{4}{s}$$
$$\therefore \quad y(t) = L^{-1}(Y(s))$$
$$= L^{-1}\{e^{-2s}(\frac{1+2s}{s^3})\} + 4L^{-1}(\frac{1}{s})$$
$$= (\frac{1}{2}t^2 - 2)u(t-2) + 4$$

(by a previous question.)

(b) We shall use

$$L(y'') = s^2Y - sy(0) - y'(0)$$

We have

$$L(y'' - 2y') = L(4)$$
$$\Rightarrow s^2Y - sy(0) - y'(0) - 2\{sY - y(0)\} = \frac{4}{s}$$
$$\Rightarrow s^2Y - s - 2sY + 2 = \frac{4}{s}$$
$$\Rightarrow (s^2 - 2s)Y = \frac{4}{s} + s - 2 = \frac{4 + s^2 - 2s}{s}$$
$$\Rightarrow Y = \frac{s^2 - 2(s-2)}{s^2(s-2)}$$
$$= \frac{1}{s-2} - \frac{2}{s^2}$$
$$\therefore \quad y = L^{-1}(\frac{1}{s-2} - \frac{2}{s^2})$$
$$= e^{2t} - 2t$$

Question 4.

By definition,

$$F(s) = \int_0^\infty e^{-st} f(t)dt.$$

Differentiating both sides with respect to s [NOT t — think of t as a constant in this calculation, since s is doing the changing here] we get

$$F'(s) = -\int_0^\infty te^{-st} f(t)dt,$$

where the chain rule has been used. So

$$-F'(s) = \int_0^\infty e^{-st}[tf(t)]dt = L[tf(t)]$$

as required.

Now the Laplace transform of sin(t) is $1/(1 + \mathrm{s}^2)$. Hence the Laplace transform of t sin(t) is minus the derivative of this, ie

$$L[tsin(t)] = \frac{2s}{(1+s^2)^2}.$$

For the resonance equation

$$\ddot{y} + y = cos(t),$$

take the Laplace transform of both sides; with the given initial data, we get

$$s^2Y + Y = \frac{s}{(1+s^2)},$$

and so

$$Y(s) = \frac{s}{(1+s^2)^2}.$$

Thus

$$y(t) = \frac{1}{2} t sin(t),$$

which is indeed exactly the solution we got in Chapter 2 for resonance in this case. [The formula in the notes is $x = \frac{F_0 t}{2m\omega}\sin(\omega t)$; here $\mathrm{F}_0 = \omega = \mathrm{m} = 1$.] Notice how the Laplace method automatically takes care of the extra factor of t.

Question 5.

The original ODE describing this situation in the absence of friction was

$$M\ddot{x} \;=\; Mg \;-\; \rho A\,(d+x)g,$$

where the downward direction is positive. We need an extra term to account for the sudden force exerted by the rogue wave. Since the force is exerted suddenly, this suggests that we need a Dirac delta function, so the force will be proportional to $\delta(t-T)$. Now F = ma, Newton's law, can be written as F = [time derivative of mv], where v is the velocity, so the change in the momentum is equal to the time integral of the force. Recall that $\int_0^\infty \delta(t-T)dt = 1$, so clearly in our case $F = -P\delta(t-T)$, since P is the given change in the momentum. [Integrate both sides to verify this, and remember that x(t) here is the DOWNWARD displacement so the upward force of the wave [as stated in the problem] is negative, like the buoyancy force. Note that the units here are correct since the delta function has units of 1/time; this is because the time integral of the delta function is a pure number.] So we have

$$M\ddot{x} \;=\; Mg \;-\; \rho A\,(d+x)g - P\delta(t-T),$$

which, as in Tutorial 4, simplifies to

$$\ddot{x} \;=\; -\frac{\rho\, A\, g}{M}\, x - \frac{P}{M}\delta(t-T).$$

Taking the Laplace transform of both sides, remembering that the ship is initially at rest, we have

$$s^2 X = -\frac{\rho A g}{M} X - \frac{P}{M} e^{-Ts},$$

or

$$X(s) = -\frac{P}{M}\frac{e^{-Ts}}{s^2+\omega^2} = -\frac{P}{\omega M}\frac{\omega e^{-Ts}}{s^2+\omega^2},$$

where ω is the natural frequency of oscillation of the ship, $\sqrt{\rho A g/M}$. Using the t-shifting theorem, we can find the inverse Laplace transform:

$$x(t) = -\frac{P}{\omega M} sin[\omega(t-T)]u(t-T).$$

The graph is flat until t = T, then you get the usual Simple Harmonic Motion with angular frequency ω. That makes sense — the sudden impulse given by the wave should trigger off SHM. The amplitude is P/ωM, so this is the maximum distance the ship goes down if it doesn't sink.

Question 6

The force suddenly switches on at t = T and then switches off at t = T + τ. It should therefore be proportional to u(t - T) - u(t - (T + τ)), since this function behaves in just that way. Note that its integral from 0 to ∞ is τ. So we should set

$$F = -\frac{P}{\tau}[u(t-T) - u(t-(T+\tau))]$$

— again you can check this by integrating both sides and remembering that the change in momentum should be the integral of the force, and that up is negative in this problem. So now the differential equation in question 5 becomes

$$\ddot{x} = -\frac{\rho A g}{M}x - \frac{P}{M\tau}[u(t-T) - u(t-(T+\tau))].$$

Take the Laplace transform of both sides to get

$$s^2 X(s) = -\frac{\rho A g}{M}X(s) - \frac{P}{Ms\tau}[e^{-Ts} - e^{-(T+\tau)s}],$$

and so

$$X(s) = \frac{-\frac{P}{Ms\tau}[e^{-Ts} - e^{-(T+\tau)s}]}{s^2+\omega^2},$$

where ω is defined as before. You can obtain the solution using the t-shifting theorem as before.

Using L'Hopital's rule, one sees that

$$\frac{1}{\tau}[e^{-Ts} - e^{-(T+\tau)s}] \to se^{-Ts}$$

as $\tau \to 0$. Hence our expression for X(s) does indeed tend to the same expression for X(s) as in Question 5 if you let τ tend to zero. That's as expected, because the impulse should be like a delta function if τ is very short. So you can always think of a delta function as a shorthand for a difference of two step functions in this manner.

Question 7

Fourier coefficients for any function f(t) with period T are given by

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)cos(\frac{2\pi nt}{T})dt,$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)sin(\frac{2\pi nt}{T})dt,$$

and the Fourier series is then

$$\frac{a_0}{2} + \sum_{1}^{\infty}\Big[a_n cos(\frac{2\pi nt}{T}) + b_n sin(\frac{2\pi nt}{T})\Big].$$

Now in the definition of Dirac's comb, only one of the terms in the infinite sum is non-zero in the interval $-T/2$ to $T/2$, namely the k $= 0$ term, so we only have to include that term when we work out the integrals. We have

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \delta(t)cos(\frac{2\pi nt}{T})dt = \frac{2}{T}cos(0) = \frac{2}{T},$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} \delta(t)sin(\frac{2\pi nt}{T})dt = \frac{2}{T}sin(0) = 0,$$

using the property of the delta function explained in the notes. So the Fourier series of the comb is just

$$\frac{1}{T} + \frac{2}{T}\sum_{1}^{\infty} cos(\frac{2\pi nt}{T}).$$

This may look ridiculous mathematically, but if you use graphmatica/matlab/whatever to graph, for example,

$$y = 1/2 + cos(x) + cos(2x) + cos(3x) + cos(4x) + cos(5x) + cos(6x) + cos(7x) + cos(8x) + cos(9x),$$

you can convince yourself that if you take enough terms you will indeed end up with something that looks like Dirac's comb. You are summing waves which interfere destructively everywhere except at regularly spaced points, where they suddenly interfere constructively.