2012/2013 SEMESTER 2 MID-TERM TEST

MA1506 MATHEMATICS II

March 2013

8:30pm - 9:30pm

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **Thirteen (13)** printed pages.
2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.
3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1/10).
4. Use **only 2B pencils** for FORM CC1/10.
5. On FORM CC1/10 (section B for matric numbers starting with A, section C for others), **write** your **matriculation number** and **shade** the corresponding numbered circles **completely**. Your FORM CC1/10 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.
6. **Write your full name** in the blank space for module code in section A of FORM CC1/10.
7. Only circles for answers 1 to 10 are to be shaded.
8. For each answer, the circle corresponding to your choice should be **properly** and **completely** shaded. If you change your answer later, you must make sure that the original answer is properly erased.
9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.
10. **Do not fold** FORM CC1/10.
11. Submit FORM CC1/10 before you leave the test hall.

Formulae Sheet

1. Integrating factor of $y' + Py = Q$ is given by

$$R = e^{\int P dx}.$$

2. The variation of parameter formulae for $y'' + py' + qy = r$:

$$u = \int \frac{-ry_2}{y_1 y_2' - y_2 y_1'} dx$$
$$v = \int \frac{ry_1}{y_1 y_2' - y_2 y_1'} dx.$$

1. Let y be a solution of the differential equation

$$\frac{dy}{dx} = \frac{1+x^2}{e^y}, \quad x \geq 0,$$

such that

$$y(0) = 0.$$

Then the value of $y\,(3)$ is

(A) e^3

(B) $\ln(\frac{7}{3})$

(**C**) e

(**D**) $\ln(13)$

(**E**) None of the above

2. Let y be a solution of the differential equation

$$\frac{dy}{dt} + 2ty = t$$

such that

$$y(1) = 2.$$

Then $y(-1) =$

(A) e

(B) e^2

(**C**) 1

(**D**) 2

(**E**) None of the above

3. Let y be a solution of the differential equation

$$x\frac{dy}{dx} + y = xy^2 \ln x$$

such that

$$x > 0 \quad \text{and} \quad y(1) = 1.$$

Then, correct to two decimal places, $y\left(\sqrt{e}\right) =$

(A) 0.69

(B) 1.44

(C) 0.38

(D) 1.65

(E) None of the above

4. Juliet was standing directly below Romeo's balcony. The moment that Romeo stuck his head out of the balcony, Juliet threw a stone vertically upwards at him at a velocity of u m/s. To her delight, 0.46 seconds after she threw the stone, the stone hit Romeo on the face on its way up. Luckily for Romeo (and to the disappointment of Juliet), at the moment of impact the velocity of the stone was zero. Find the value of u correct to one decimal place, based on the following assumptions: the stone's mass is 0.3 kg, the gravitational constant g equals to 10 m/s^2 and the value of the air resistance at any time equals to $0.3v^2$ Newtons where v is the value of the velocity of the stone at that time measured in m/s.

(A) 23.6

(B) 16.3

(C) 27.1

(D) 19.8

(E) None of the above

5. A brilliant stock market player has a fortune which increases at a rate proportional to the square of the amount of money he has. One year ago he had one million dollars and currently he has two million. How many million will he have in eight months from now?

   (Suggestion: measure time in years and start the clock with time $t = 0$ one year ago.)

(A) 4

(B) 5

(C) 6

(D) 8

(E) None of the above

6. At time $t = 0$ a tank contains 25 grams of salt dissolved in 10 litres of water. Assume that water containing $\frac{1}{2}t$ grams of salt per litre, where time t is measured in minutes, is entering the tank at a rate of 2 litres per minute and the well stirred solution is leaving the tank at the same rate. Find the amount of salt in grams at time $t = 5$ minutes. Give your answer correct to two decimal places.

(A) 20.68

(B) 18.39

(C) 19.56

(D) 17.43

(E) None of the above

7. The general solution of the differential equation

$$y'' + 4y' + 5y = 0$$

is

(A) $y = c_1 e^{-2x} + c_2 x e^{-2x}$

(B) $y = c_1 e^{-x} + c_2 e^{-5x}$

(C) $y = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x$

(D) $y = c_1 e^{-x} \cos 2x + c_2 e^{-5x} \sin 2x$

(E) None of the above

8. It is known that $y = 1506xe^{-3x}$ is a solution of the differential equation

$$y'' + ay' + by = 0,$$

where a and b are two real constants. Find the value of the product ab.

(A) 36

(B) 54

(C) 1506

(D) 4518

(E) None of the above

9. Let y be a solution of the differential equation

$$y'' - 4y = xe^{2x}$$

such that

$$y(0) = 0, \text{ and } \quad y'(0) = \frac{63}{16}.$$

Find the value of $y(1)$. Give your answer correct to one decimal place.

(A) 8.8

(B) 7.7

(C) 5.5

(D) 6.6

(E) None of the above

10. Let y be a solution of the differential equation

$$y'' - 2y' + y = -\frac{e^x}{x^2}, \quad x > 0$$

such that

$$y(1) = 2e, \text{ and } \quad y'(1) = 4e.$$

Find the value of $y(2)$. Give your answer correct to the nearest integer.

(A) 27

(B) 16

(C) 48

(D) 35

(E) None of the above

END OF PAPER

Blank page for you to do your calculations

Answers to mid term test

1. D
2. D
3. A
4. C
5. C
6. B
7. C
8. B
9. B
10. A

1). D

$$e^y dy = (1+x^2)\,dx$$

$$e^y = x + \tfrac{1}{3}x^3 + C$$

$$y(0)=0 \Rightarrow 1 = C$$

$$\therefore\ e^y = x + \tfrac{1}{3}x^3 + 1$$

$$y = \ln\left(x + \tfrac{1}{3}x^3 + 1\right)$$

$$y(3) = \ln(3+9+1)$$

$$= \underline{\underline{\ln 13}}$$

2). D

$$R = e^{\int 2t\,dt} = e^{t^2}$$

$$y = \frac{1}{R}\int R t\,dt$$

$$= e^{-t^2}\int t e^{t^2}\,dt$$

$$= e^{-t^2}\int \frac{1}{2} e^{t^2}\,d(t^2)$$

$$= \frac{1}{2} e^{-t^2}\left\{e^{t^2} + c\right\}$$

$$= \frac{1}{2} + C e^{-t^2}$$

$$y(1) = 2 \Rightarrow 2 = \frac{1}{2} + \frac{C}{e} \Rightarrow C = \frac{3e}{2}$$

$$\therefore\ y = \frac{1}{2} + \frac{3e}{2} e^{-t^2}$$

$$y(-1) = \frac{1}{2} + \frac{3e}{2} e^{-1} = \underline{\underline{2}}$$

Another way to get the answer 2:

$$\because\ y = \frac{1}{2} + Ce^{-t^2}$$

$\therefore$ y is an even function

$$\therefore\ y(-1) = y(1) = \underline{\underline{2}}$$

3) A

$$\frac{dy}{dx} + \frac{1}{x} y = (\ln x) y^2 , \quad x > 0$$

$$\text{Let } z = y^{1-2} = \frac{1}{y}$$

$$\therefore dz = -\frac{1}{y^2} dy \Rightarrow dy = -y^2 dz$$

$$\therefore \frac{-y^2 dz}{dx} + \frac{1}{x} y = (\ln x) y^2$$

$$\frac{dz}{dx} - \frac{1}{x} z = -\ln x$$

$$R = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

$$z = x \int \frac{1}{x} (-\ln x)\, dx$$

$$= x \int (-\ln x)\, d(\ln x)$$

$$= x \left\{ -\frac{1}{2} (\ln x)^2 + C \right\}$$

$$y = \frac{1}{z} = \frac{1}{x \left\{ -\frac{1}{2} (\ln x)^2 + C \right\}}$$

$$y(1) = 1 \Rightarrow C = 1$$

$$y(\sqrt{e}) = \frac{1}{\sqrt{e} \left\{ -\frac{1}{8} + 1 \right\}} = \frac{8}{7\sqrt{e}}$$

$$\approx \underline{\underline{0.693\ldots}}$$

4). C

$$v \uparrow \bigcirc \downarrow 0.3v^2$$

$$\downarrow mg = (0.3)(10) = 3$$

$$0.3\frac{dv}{dt} = -3 - 0.3v^2$$

$$\frac{dv}{dt} = -10 - v^2 \Rightarrow \frac{dv}{v^2+10} = -dt$$

$$v(0) = u, \quad v(0.46) = 0$$

$$\int_u^0 \frac{dv}{v^2+10} = \int_0^{0.46} -dt = -0.46$$

$$\therefore 0.46 = \int_0^u \frac{dv}{v^2+10} = \frac{1}{\sqrt{10}}\tan^{-1}\left(\frac{u}{\sqrt{10}}\right)$$

$$u = \sqrt{10}\tan\left(0.46\sqrt{10}\right)$$

$$\approx \underline{\underline{27.10}}\cdots$$

5). C

Let x = amount of money in million at time t years, where $t=0$ one year ago.

$$\frac{dx}{dt} = kx^2$$

$$\frac{dx}{x^2} = k\,dt$$

$$-\frac{1}{x} = kt + C$$

$$x(0) = 1 \Rightarrow C = -1$$

$$\therefore\ x = \frac{1}{1-kt}$$

$$x(1) = 2 \Rightarrow 2 = \frac{1}{1-k} \Rightarrow k = \frac{1}{2}$$

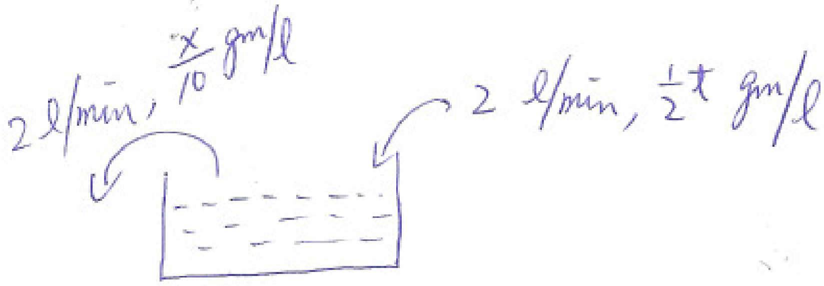
$$\therefore\ x = \frac{2}{2-t}$$

$$x\left(1+\frac{8}{12}\right) = x\left(\frac{5}{3}\right)$$

$$= \frac{2}{2-\frac{5}{3}}$$

$$= \underline{\underline{6}}$$

6). B.



Let x = amount of salt in gm at time t min.

$$\Delta x = 2\left(\tfrac{1}{2}t\right)\Delta t - 2\left(\tfrac{x}{10}\right)\Delta t$$

$$\frac{dx}{dt} = t - \frac{1}{5}x$$

$$\frac{dx}{dt} + \frac{1}{5}x = t$$

$$R = e^{\int \frac{1}{5}dt} = e^{\frac{1}{5}t}$$

$$x = e^{-\frac{1}{5}t}\int t e^{\frac{1}{5}t}\,dt$$

$$= e^{-\frac{1}{5}t}\int t\, d\left(5e^{\frac{1}{5}t}\right)$$

$$= e^{-\frac{1}{5}t}\left\{ t\left(5e^{\frac{1}{5}t}\right) - \int 5e^{\frac{1}{5}t}dt\right\}$$

$$= 5t - e^{-\frac{1}{5}t}\left\{25e^{\frac{1}{5}t} + C_1\right\}$$

$$= 5t - 25 + Ce^{-\frac{1}{5}t}$$

$x(0) = 25 \Rightarrow C = 50$

$$x(5) = 25 - 25 + 50e^{-1} = \frac{50}{e}$$

$$\approx \underline{\underline{18.39}}3\ldots$$

7). C

$$\lambda^2+4\lambda+5=0$$

$$\lambda=\frac{-4\pm\sqrt{16-20}}{2}=-2\pm i$$

$$\underline{\underline{y=C_1e^{-2x}\cos x+C_2e^{-2x}\sin x}}$$

8) B

$y = 1506\,x e^{-3x}$ is a solution

$\Rightarrow -3$ is a double root of $\lambda^2 + a\lambda + b = 0$

$\therefore a = -(\text{sum of roots}) = -(-3+(-3)) = 6$

$b = \text{product of roots} = (-3)(-3) = 9$

$\therefore ab = \underline{\underline{54}}$

9). B

$$\lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2$$

Try $y = (Ax^2 + Bx)e^{2x}$

$$y' = (2Ax + B)e^{2x} + 2(Ax^2 + Bx)e^{2x}$$

$$y'' = 2Ae^{2x} + 4(2Ax + B)e^{2x} + 4(Ax^2 + Bx)e^{2x}$$

$$y'' - 4y = xe^{2x} \Rightarrow 2A + 4(2Ax + B) = x$$

$$\Rightarrow A = \frac{1}{8}, \ B = -\frac{A}{2} = -\frac{1}{16}$$

$$\therefore y = C_1 e^{2x} + C_2 e^{-2x} + \left(\frac{1}{8}x^2 - \frac{1}{16}x\right)e^{2x}$$

$$y' = 2C_1 e^{2x} - 2C_2 e^{-2x} + \left(\frac{1}{4}x - \frac{1}{16}\right)e^{2x} + 2\left(\frac{1}{8}x^2 - \frac{1}{16}x\right)e^{2x}$$

$$\left.\begin{array}{l} y(0) = 0 \Rightarrow C_1 + C_2 = 0 \\ y'(0) = \frac{63}{16} \Rightarrow 2C_1 - 2C_2 - \frac{1}{16} = \frac{63}{16} \end{array}\right\} \Rightarrow \begin{array}{l} C_1 = 1 \\ C_2 = -1 \end{array}$$

$$\therefore y = e^{2x} - e^{-2x} + \left(\frac{1}{8}x^2 - \frac{1}{16}x\right)e^{2x}$$

$$\therefore y(1) = e^2 - e^{-2} + \frac{1}{16}e^2 = \frac{17}{16}e^2 - \frac{1}{e^2}$$

$$\approx \underline{\underline{7.71}}\cdots$$

10). A

$$y'' - 2y' + y = -\frac{1}{x^2}e^x, \quad x > 0$$

$$\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda = 1 \text{ double root}$$

Let $y_1 = e^x$, $y_2 = xe^x$

Then $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$

$$u = \int \frac{-(-\frac{1}{x^2})e^x (xe^x)}{e^{2x}} dx$$

$$= \int \frac{1}{x} dx = \ln x$$

$$v = \int \frac{-\frac{1}{x^2}e^x e^x}{e^{2x}} dx$$

$$= \int -\frac{1}{x^2} dx = \frac{1}{x}$$

$$\therefore uy_1 + vy_2 = e^x \ln x + e^x$$

$$y = \tilde{c}_1 e^x + c_2 xe^x + e^x \ln x + e^x$$

$$\therefore y = c_1 e^x + c_2 xe^x + e^x \ln x$$

$$y' = c_1 e^x + c_2 e^x + c_2 xe^x + e^x \ln x + \frac{1}{x} e^x$$

$$\left.\begin{aligned} y(1) = 2e &\Rightarrow c_1 e + c_2 e = 2e \\ y'(1) = 4e &\Rightarrow c_1 e + c_2 e + c_2 e + e = 4e \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 1 \\ c_2 &= 1 \end{aligned}$$

$$\therefore y = e^x + xe^x + e^x \ln x$$

$$y(2) = e^2 + 2e^2 + e^2 \ln 2 = e^2(3 + \ln 2)$$

$$\approx \underline{\underline{27.2}}\cdots$$