Quizzes And Exercises On Partial Derivatives

EXERCISE 1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for each of the following functions.

(a) $z = x^2y^4$; (b) $z = (x^4 + x^2)y^3$, (c) $z = y^{\frac{1}{2}} \sin(x)$.

EXERCISE 2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for each of the following functions.

(a) $z = (x^2+3x)\sin(y)$, (b) $z = \frac{\cos(x)}{y^5}$, (c) $z = \ln(xy)$,

(d) $z = \sin(x)\cos(xy)$, (e) $z = e^{(x^2+y^2)}$, (f) $z = \sin(x^2 + y)$.

Quiz If $z = \cos(xy)$, which of the following statements is true?

(a) $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$, (b) $\frac{\partial z}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial y}$,

(c) $\frac{1}{y}\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$, (d) $\frac{1}{y}\frac{\partial z}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial y}$.

EXERCISE 3. For each of the following functions, verify that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

(a) $z = (x^2+3x)\sin(y)$, (b) $z = \frac{\cos(x)}{y^5}$, (c) $z = \ln(xy)$,

(d) $z = \sin(x)\cos(xy)$, (e) $z = e^{(x^2+y^2)}$, (f) $z = \sin(x^2 + y)$.

Notation For first and second order partial derivatives there is a compact notation. Thus $\frac{\partial f}{\partial x}$ can be written as f_x and $\frac{\partial f}{\partial y}$ as f_y. Similarly $\frac{\partial^2 f}{\partial x^2}$ is written f_{xx} while $\frac{\partial^2 f}{\partial x \partial y}$ is written f_{xy}.

Quiz If $z = e^{-y}\sin(x)$, which of the following is $z_{xx} + z_{yy}$?

(a) $e^{-y}\sin(x)$, (b) 0, (c) $-e^{-y}\sin(x)$, (d) $-e^{-y}\cos(x)$.

Solutions to Exercises

Exercise 1(a) To calculate the partial derivative $\frac{\partial z}{\partial x}$ of the function $z = x^2y^4$, the factor y^4 is treated as a constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^2y^4\right) = \frac{\partial}{\partial x}\left(x^2\right) \times y^4 = 2x^{(2-1)} \times y^4 = 2xy^4 .$$

Similarly, to find the partial derivative $\frac{\partial z}{\partial y}$, the factor x^2 is treated as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^2y^4\right) = x^2 \times \frac{\partial}{\partial y}\left(y^4\right) = x^2 \times 4y^{(4-1)} = 4x^2y^3 .$$

Exercise 1(b) To calculate $\frac{\partial z}{\partial x}$ for the function $z = (x^4 + x^2)y^3$, the factor y^3 is treated as a constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left((x^4 + x^2)y^3\right) = \frac{\partial}{\partial x}\left(x^4 + x^2\right) \times y^3 = (4x^3 + 2x)y^3 .$$

To find the partial derivative $\frac{\partial z}{\partial y}$ the factor $(x^4 + x^2)$ is treated as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left((x^4 + x^2)y^3\right) = (x^4 + x^2) \times \frac{\partial}{\partial y}y^3 = 3(x^4 + x^2)y^2 .$$

Exercise 1(c) If $z = y^{\frac{1}{2}}\sin(x)$ then to calculate $\frac{\partial z}{\partial x}$ the $y^{\frac{1}{2}}$ factor is kept constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(y^{\frac{1}{2}}\sin(x)\right) = y^{\frac{1}{2}} \times \frac{\partial}{\partial x}(\sin(x)) = y^{\frac{1}{2}}\cos(x) .$$

Similarly, to evaluate the partial derivative $\frac{\partial z}{\partial y}$ the factor $\sin(x)$ is treated as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(y^{\frac{1}{2}}\sin(x)\right) = \frac{\partial}{\partial y}y^{\frac{1}{2}} \times \sin(x) = \frac{1}{2}y^{-\frac{1}{2}}\sin(x) .$$

Exercise 2(a) The function $z = (x^2 + 3x)\sin(y)$ can be written as $z = uv$, where $u = (x^2 + 3x)$ and $v = \sin(y)$. The partial derivatives of u and v with respect to the variable x are

$$\frac{\partial u}{\partial x} = 2x + 3\,, \qquad \frac{\partial v}{\partial x} = 0\,,$$

while the partial derivatives with respect to y are

$$\frac{\partial u}{\partial y} = 0\,, \qquad \frac{\partial v}{\partial y} = \cos(y)\,.$$

Applying the *product rule*

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}v + u\frac{\partial v}{\partial x} = (2x+3)\sin(y)\,.$$

$$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y}v + u\frac{\partial v}{\partial y} = (x^2+3x)\cos(y)\,.$$

Exercise 2(b)

The function $z = \dfrac{\cos(x)}{y^5}$ can be written as $z = \cos(x)y^{-5}$.

Treating the factor y^{-5} as a constant and differentiating with respect to x:

$$\frac{\partial z}{\partial x} = -\sin(x)y^{-5} = -\frac{\sin(x)}{y^5}\,.$$

Treating $\cos(x)$ as a constant and differentiating with respect to y:

$$\frac{\partial v}{\partial y} = \cos(x)(-5y^{-6}) = -5\frac{\cos(x)}{y^6}\,.$$

Exercise 2(c) The function $z = \ln(xy)$ can be rewritten as

$$z = \ln(xy) = \ln(x) + \ln(y)\,.$$

Thus the partial derivative of z with respect to x is

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\ln(x) + \ln(y)) = \frac{\partial}{\partial x}\ln(x) = \frac{1}{x}\,.$$

Similarly the partial derivative of z with respect to y is

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\ln(x) + \ln(y)) = \frac{\partial}{\partial y}\ln(y) = \frac{1}{y}\,.$$

Exercise 2(d) To calculate the partial derivatives of the function $z = \sin(x)\cos(xy)$ the *product rule* has to be applied

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \cos(xy)\frac{\partial}{\partial x}\sin(x) + \sin(x)\frac{\partial}{\partial x}\cos(xy)\,,\\
\frac{\partial z}{\partial y} &= \cos(xy)\frac{\partial}{\partial y}\sin(x) + \sin(x)\frac{\partial}{\partial y}\cos(xy)\,.
\end{aligned}$$

Using the *chain rule* with $u = xy$ for the partial derivatives of $\cos(xy)$

$$\begin{aligned}
\frac{\partial}{\partial x}\cos(xy) &= \frac{\partial\cos(u)}{\partial u}\frac{\partial u}{\partial x} = -\sin(u)y = -y\sin(xy)\,,\\
\frac{\partial}{\partial y}\cos(xy) &= \frac{\partial\cos(u)}{\partial u}\frac{\partial u}{\partial y} = -\sin(u)x = -x\sin(xy)\,.
\end{aligned}$$

Thus the partial derivatives of $z = \sin(x)\cos(xy)$ are

$$\frac{\partial z}{\partial x} = \cos(xy)\cos(x) - y\sin(x)\sin(xy)\,, \qquad \frac{\partial z}{\partial y} = -x\sin(x)\sin(xy)\,.$$

Exercise 2(e) To calculate the partial derivatives of $z = \mathrm{e}^{(x^2+y^2)}$ the *chain rule* has to be applied with $u = (x^2+y^2)$:

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial}{\partial u}(\mathrm{e}^u)\,\frac{\partial u}{\partial x} = \mathrm{e}^u\,\frac{\partial u}{\partial x}\,,\\
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial u}(\mathrm{e}^u)\,\frac{\partial u}{\partial y} = \mathrm{e}^u\frac{\partial u}{\partial y}\,.
\end{aligned}$$

The partial derivatives of $u = (x^2+y^2)$ are

$$\frac{\partial u}{\partial x} = \frac{\partial(x^2)}{\partial x} = 2x\,, \qquad \frac{\partial u}{\partial y} = \frac{\partial(y^2)}{\partial y} = 2y\,.$$

Therefore the partial derivatives of the function $z = \mathrm{e}^{(x^2+y^2)}$ are

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \mathrm{e}^u\frac{\partial u}{\partial x} = 2x\,\mathrm{e}^{(x^2+y^2)}\,,\\
\frac{\partial z}{\partial x} &= \mathrm{e}^u\frac{\partial u}{\partial x} = 2y\,\mathrm{e}^{(x^2+y^2)}\,.
\end{aligned}$$

Exercise 2(f) Applying the *chain rule* with $u = x^2 + y$ the partial derivatives of the function $z = \sin(x^2+y)$ can be written as

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial}{\partial u}(\sin(u))\,\frac{\partial u}{\partial x} = \cos(u)\,\frac{\partial u}{\partial x}\,,\\
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial u}(\sin(u))\,\frac{\partial u}{\partial y} = \cos(u)\frac{\partial u}{\partial y}\,.
\end{aligned}$$

The partial derivatives of $u = x^2 + y$ are

$$\frac{\partial u}{\partial x} = \frac{\partial x^2}{\partial x} = 2x\,, \qquad \frac{\partial u}{\partial y} = \frac{\partial y}{\partial y} = 1\,.$$

Thus the partial derivatives of the function $z = \sin(x^2+y)$ are

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \cos(u)\frac{\partial u}{\partial x} = 2x\,\cos(x^2+y)\,,\\
\frac{\partial z}{\partial y} &= \cos(u)\frac{\partial u}{\partial y} = \cos(x^2+y)\,.
\end{aligned}$$

Solution to Quiz:

To determine which of the options is correct, the partial derivatives of $z = \cos(xy)$ must be calculated. From the calculations of exercise 2(d) the partial derivatives of $z = \cos(xy)$ are

$$\begin{aligned}
\frac{\partial}{\partial x}\cos(xy) &= \frac{\partial \cos(u)}{\partial u}\frac{\partial u}{\partial x} = -\sin(u)y = -y\sin(xy)\,, \\
\frac{\partial}{\partial y}\cos(xy) &= \frac{\partial \cos(u)}{\partial u}\frac{\partial u}{\partial y} = -\sin(u)x = -x\sin(xy)\,.
\end{aligned}$$

Therefore

(d) $$\frac{1}{y}\frac{\partial}{\partial x}\cos(xy) = -\sin(xy) = \frac{1}{x}\frac{\partial}{\partial y}\cos(xy)\,.$$

Exercise 3(a)
From exercise 2(a), the first order partial derivatives of $z = (x^2+3x)\sin(y)$ are

$$\frac{\partial z}{\partial x} = (2x+3)\sin(y)\,, \qquad \frac{\partial z}{\partial y} = (x^2+3x)\cos(y)\,.$$

The *mixed* second order derivatives are

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left((x^2+3x)\cos(y)\right) = (2x+3)\cos(y)\,, \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left((2x+3)\sin(y)\right) = (2x+3)\cos(y)\,.
\end{aligned}$$

Exercise 3(b)

From exercise 2(b), the first order partial derivatives of $z = \dfrac{\cos(x)}{y^5}$ are

$$\frac{\partial z}{\partial x} = -\frac{\sin(x)}{y^5}\,, \qquad \frac{\partial z}{\partial y} = -5\frac{\cos(x)}{y^6}\,,$$

so the *mixed* second order derivatives are

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(-5\frac{\cos(x)}{y^6}\right) = 5\frac{\sin(x)}{y^6}\,, \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\sin(x)}{y^5}\right) = 5\frac{\sin(x)}{y^6}\,.
\end{aligned}$$

Exercise 3(c)
From exercise 2(c), the first order partial derivatives of $z = \ln(xy)$ are

$$\frac{\partial z}{\partial x} = \frac{1}{x}\,, \qquad \frac{\partial z}{\partial y} = \frac{1}{y}\,.$$

The *mixed* second order derivatives are

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{y}\right) = 0\,, \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x}\right) = 0\,.
\end{aligned}$$

Exercise 3(d) From exercise 2(d), the first order partial derivatives of $z = \sin(x)\cos(xy)$ are

$$\frac{\partial z}{\partial x} = \cos(x)\cos(xy) - y\sin(x)\sin(xy)\,, \qquad \frac{\partial z}{\partial y} = -x\sin(x)\sin(xy)\,.$$

The *mixed* second order derivatives are

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(-x\sin(x)\sin(xy)) \\
&= -\sin(x)\sin(xy) - x\cos(x)\sin(xy) - xy\sin(x)\cos(xy)\,, \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(\cos(x)\cos(xy) - y\sin(x)\sin(xy)) \\
&= -x\cos(x)\sin(xy) - \sin(x)\sin(xy) - xy\sin(x)\cos(xy)\,.
\end{aligned}$$

Exercise 3(e) From exercise 2(e), the first order partial derivatives of $z = \mathrm{e}^{(x^2+y^2)}$ are

$$\frac{\partial z}{\partial x} = 2x\mathrm{e}^{(x^2+y^2)}\,, \qquad \frac{\partial z}{\partial y} = 2y\mathrm{e}^{(x^2+y^2)}\,.$$

The *mixed* second order derivatives are thus

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(2y\mathrm{e}^{(x^2+y^2)}\right) = 4xy\mathrm{e}^{(x^2+y^2)}\,, \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(2x\mathrm{e}^{(x^2+y^2)}\right) = 4yx\mathrm{e}^{(x^2+y^2)}\,.
\end{aligned}$$

Exercise 3(f) From exercise 2(f), the first order partial derivatives of $z = \sin(x^2+y)$ are

$$\frac{\partial z}{\partial x} = 2x\cos(x^2+y)\,, \qquad \frac{\partial z}{\partial y} = \cos(x^2+y)\,.$$

The *mixed* second order derivatives are thus

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\cos(x^2+y)\right) = -2x\sin(x^2+y)\,, \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(2x\cos(x^2+y)\right) = -2x\sin(x^2+y)\,.
\end{aligned}$$

Solution to Quiz:
The first order derivatives of $z = \mathrm{e}^{-y}\sin(x)$ are

$$z_x = \mathrm{e}^{-y}\cos(x)\,, \qquad z_y = -\mathrm{e}^{-y}\sin(x)\,,$$

where e^{-y} is kept constant for the first differentiation and $\sin(x)$ for the second. Continuing in this way, the second order derivatives z_{xx} and z_{yy} are given by the expressions

$$\begin{aligned}
z_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\mathrm{e}^{-y}\cos(x)\right) = -\mathrm{e}^{-y}\sin(x)\,, \\
z_{yy} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\mathrm{e}^{-y}\sin(x)\right) = \mathrm{e}^{-y}\sin(x)\,.
\end{aligned}$$

Adding these two equations together gives

(b)

$$z_{xx} + z_{yy} = 0\,.$$