NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

MA1506 Laboratory 2 (MATLAB)

The aim of lab 2 is to demonstrate some tools available in MATLAB that help us to better understand solutions of differential equations. In Part A, we will graph direction fields and use them to visualize solutions to differential equations. In Part B, we shall learn how to use a numerical solver to approximate solutions to differential equations.

Part A: Direction Fields

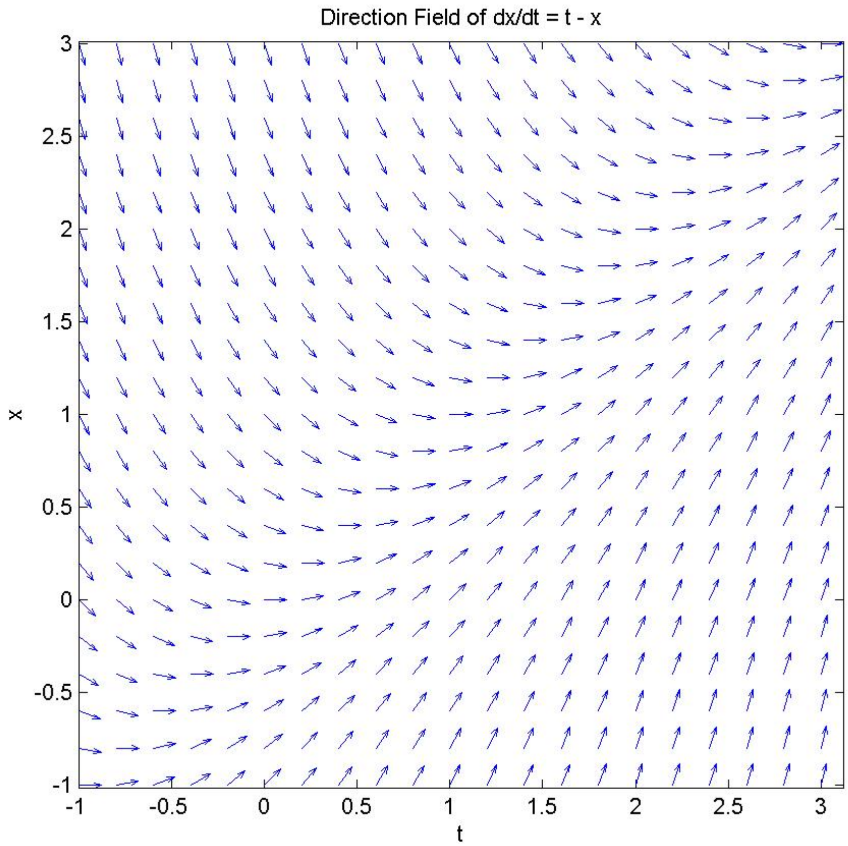
We should understand that it may not always be possible to solve any given first order differential equation. However, it is relatively easy to obtain the direction field of the differential equation and deduce qualitative information about the equation without solving it. The **direction field** (or slope field) of a first order differential equation of the form

$$\frac{dx}{dt} = f(t,x),$$

is obtained by drawing through each point in the (t,x) plane a short line segment with slope $x' = f(t,x)$.
For example, consider the equation $\frac{dx}{dt} = t - x$. To graph its direction field, we execute the following series of commands:

```
>> [T, X]=meshgrid(-1:0.2:3, -1:0.2:3);
>> S = T-X;            %the function f goes here.
>> L = sqrt(1+ S.^2);
>> quiver(T, X, 1./L, S./L, 0.5)
>> axis equal tight
>> xlabel('t')
>> ylabel('x')
>> title( 'Direction Field of dx/dt = t - x')
```

Before we study the resulting graph, let us explain the commands line by line. The meshgrid command creates a rectangular 21×21 grid of uniformly distributed points in the rectangle (actually a square) from -1 to 3 on both the horizontal and vertical axis. In this case, we created two arrays and called them T, representing the horizontal axis and X, representing the vertical axis. The array S contains the value of the derivative or slope $\frac{dx}{dt} = t - x$ and % is the comment character. MATLAB ignores everything that comes after %. The commands involving L merely normalizes the length of the arrows that appear and are not critical. The quiver command draws the resultant direction field. We only need to note that the 0.5 at the end denotes the length of the vectors. (Try the same quiver command with values 0.2 or 1.) Lastly, 'axis equal tight' ensures the same scale is used on both axis and removes the whitespace at the edges of the direction field.

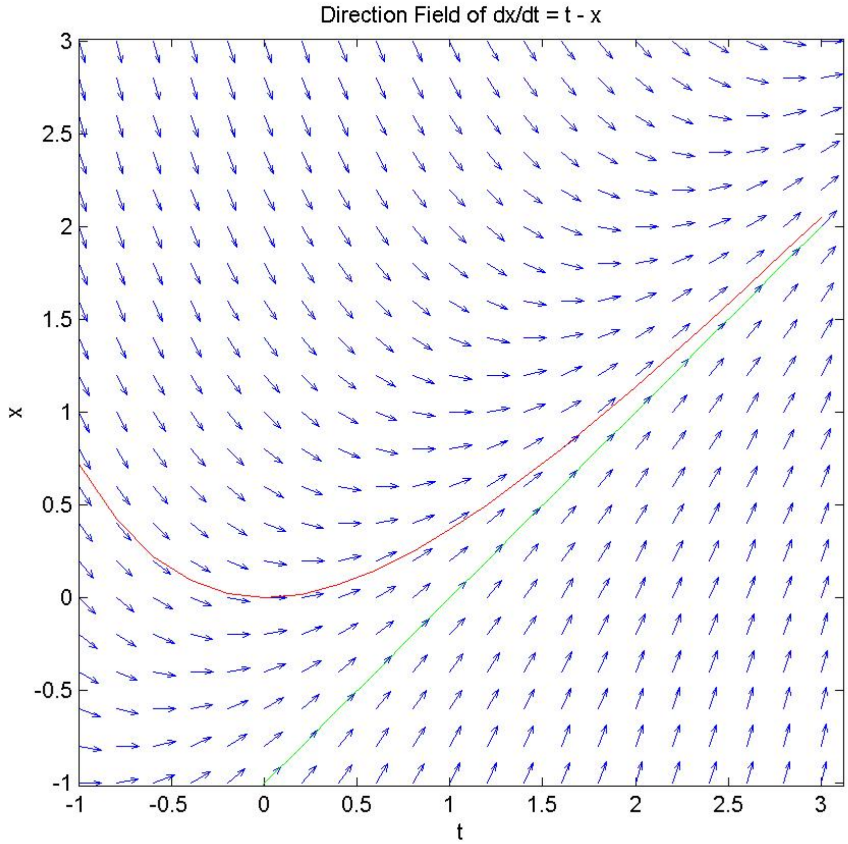


Let us now study the resulting graph. Note that $\frac{dx}{dt}$ seem to approach 1 as t gets large. This implies that solutions $x(t)$ seem to grow as t gets large. There does not appear to be any **equilibrium solutions**. Recall that equilibrium solutions $x(t)$ are those that do not change over time. Hence $x(t)$ should equal a constant. This implies $\frac{dx}{dt} = 0$. In a direction field, equilibrium solutions correspond to lines with slope 0, i.e. horizontal lines. As our differential equation is easy to solve, we know that the solutions are given by

$$x(t) = t - 1 + Ce^{-t}.$$

Let us plot two solutions corresponding to $C = 0$ and $C = 1$ with

```
>> t=[-1:0.2:3];
>> x1=t-1;
>> x2=t-1 + exp(-t);
>> hold on
>> plot(t,x1,'g')
>> plot(t,x2,'r')
```



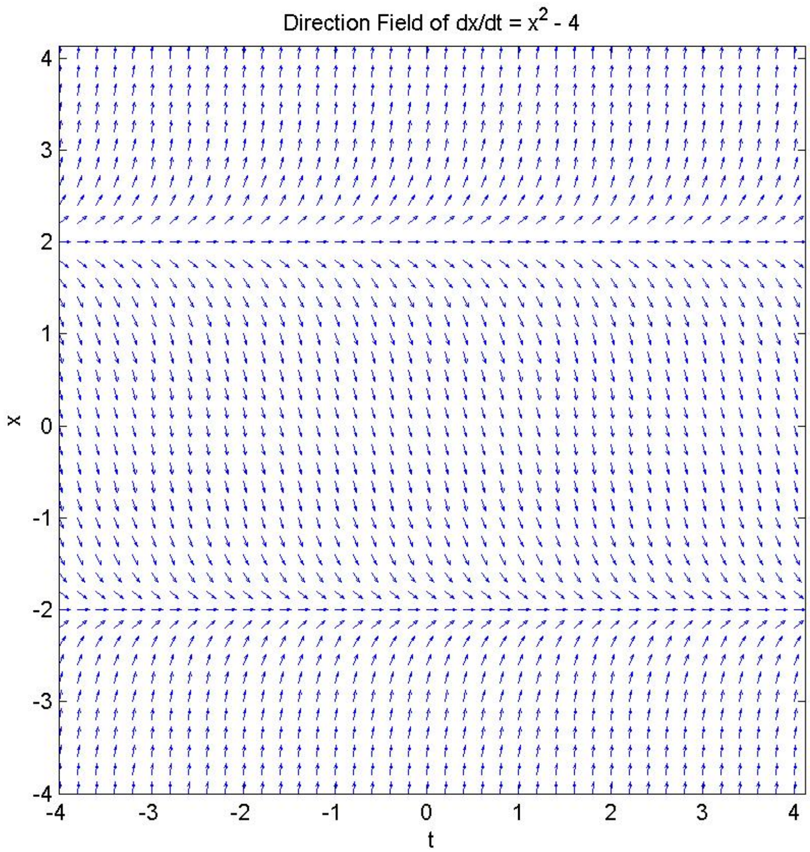
In conclusion, if we start with any initial point $x(t)$ on the graph and trace out the **trajectory** of the path, we will obtain the curve of a particular solution. Try this by starting with $x(0) = 1$ to compute the corresponding value of C, and plot the resulting particular solution into the direction field.
For our second example, let us consider the equation $\frac{dx}{dt} = x^2 - 4$. We graph the direction field with

```
>> [T, X]=meshgrid(-4:0.2:4, -4:0.2:4);
>> S = X.^2-4;
>> L = sqrt(1+ S.^2);
>> quiver(T, X, 1./L, S./L, 0.5)
>> axis equal tight
>> xlabel('t')
>> ylabel('x')
>> title( 'Direction Field of dx/dt = x^2 - 4')
```

Now, it is rather tedious to type the long string of command every time we want to graph a direction field. We can improve the process by making use of a MATLAB M-file. M-files are basically text files which end with a .m extension, and contain a list of MATLAB commands. (We will see later that we can also write new functions with M-files.) Use the drop-down menu → File → New → M-file. A new window will appear. Type the list of commands above line by line (without the "≫"). Now click → File → Save as, and name it "dfield.m". Remember to save the file in an appropriate directory, either your h: drive or into your own thumbdrive. Next change your MATLAB working directory to the location of the file and execute the file. For example

```
>> cd h:
>> dfield
```

If you wish to modify your M-file, you can use → File → Open, to edit your M-file.



By maximizing the figure window if necessary, observe that there are two horizontal lines at $x = -2$ and $x = 2$. These are precisely the equilibrium solutions. Note that solutions near $x = 2$ tend to move away whereas solutions near $x = -2$ tend to converge. Hence $x = -2$ is a **stable** equilibrium whereas $x = 2$ is unstable. Besides these two, what do the graphs of other solutions look like?

Let us solve for $x(t)$, first assuming $x \neq 2$ or -2:

$$\begin{aligned} \frac{dx}{dt} = x^2 - 4 \quad &\Longrightarrow \quad \int \frac{dx}{x^2-4} = \int 1dt \\ &\Longrightarrow \quad \log|x-2| - \log|x+2| = 4t + C. \end{aligned}$$

Recalling the *no-crossing principle*, i.e. the solution curves should not cross, we have three cases:

$$\frac{x-2}{x+2} = \begin{cases} Ae^{4t} & \text{if } x > 2, \\ -Ae^{4t} & \text{if } -2 < x < 2, \\ Ae^{4t} & \text{if } x < -2. \end{cases}$$

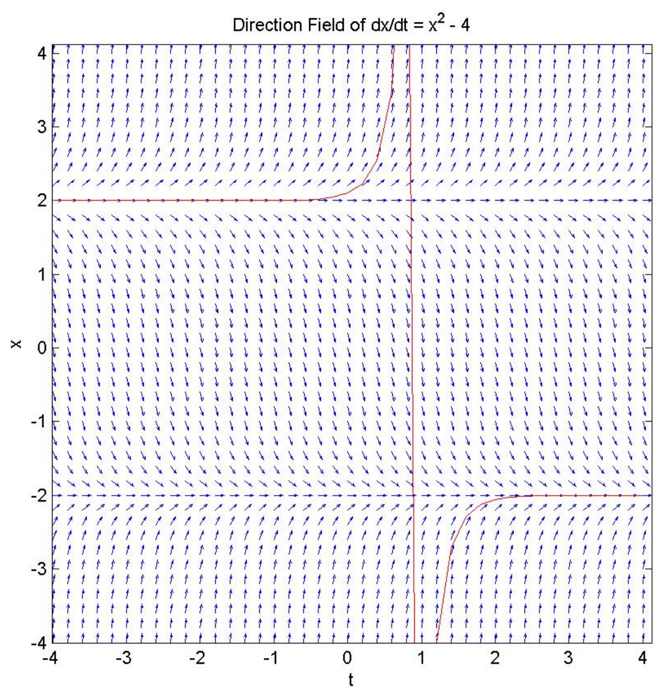
Here, A is a positive constant.
In the first case $x > 2$, some rearrangement gives us

$$x = \frac{2 + 2Ae^{4t}}{1 - Ae^{4t}}.$$

Beginning with the condition, $x(0) = 2.1$, we calculate $A = 1/41 \approx 0.0244$. Suppose we try to plot this graph with

```
>> hold on
>> t1=-4:0.2:4;
>> x1 = 2*(1+0.0244*exp(4*t1))./(1-0.0244*exp(4*t1));
>> plot(t1,x1,'r')
```

We will obtain:



You should be suspicious when you see that vertical red line and the portion of the curve that lies below $x = -2$.

Exercise 2A

1. This solution curve clearly violates the *no-crossing principle*. What went wrong?
2. Use appropriate initial values of $x(0)$ to plot at least 6 solution curves on the direction field. (Hint: For each solution curve, restrict your range of t so that only the correct portion of the curve appears.)
3. Plot the direction field for
$$\frac{dy}{dx} = 3\sin y + y - 2$$
on a suitably large rectangle. Discuss the behaviour as $x \to \infty$ and find the equilibriums.

Part B: Numerical Solvers

Consider the following example of a differential equation which cannot be easily solved by known techniques.
$$\frac{dx}{dt} = t + x^2.$$
While the direction field helps us visualize the family of solutions, we sometimes need numerical values. In such cases, it is often very useful to use numerical methods to approximate solutions. We shall learn how to do this using MATLAB's numerical solver called **ode45** which solves initial value problems (IVP) of the following form:
$$\frac{dx}{dt} = f(t, x), \qquad a \leq x \leq b,\ x(a) = x_0.$$
Note that the initial condition should always be supplied because we are approximating a unique particular solution and not the whole family of solutions.
Let us start with the following simple IVP first.
$$\frac{dx}{dt} = \frac{t}{x}, \qquad 0 \leq t \leq 2,\ x(0) = 1.$$

```
>>f = inline ('t/x', 't', 'x')
>>ode45(f, [0 2], 1)
```

The first line defines a two variable inline function $f(t, x)$. Unlike previously where we had to input specific values into an array, the inline command allows us to define a function. The first argument in single quotation gives an expression for f, and the subsequent arguments list the independent variables. Note that there is no need to use "./" for dividing in inline functions. The second command plots an approximate solution to f on the interval $[0, 2]$

with the last argument specifying the initial value, i.e. the value of x at the left end point of the interval. We remark that ode45 is one of several numerical solvers available and is sufficient for our purposes. More advanced users can try to study the difference between ode45 and ode23 or ode113 etc.
In the graph, the circles indicate points at which ode45 computed the approximate solution and the centers of these circles are connected by line segments.
Since this d.e. is separable, we can solve the IVP to get $x(t) = \sqrt{t^2+1}$. In order to compare the two graphs, we get the numerical values from ode45 by specifying

```
>>[t, xa]= ode45(f, [0 2], 1);
>> X = sqrt( t.^2+1);
>> plot(t, xa, 'g')
>> hold on
>> plot(t, X, 'r')
```

The array **t** stores the values where approximations were computed with the corresponding approximated values stored in **xa**. Note that you actually cannot see any difference graphically. To see how good are the approximations, we use the following commands and note how many decimal places the two sets of values agree to.

```
>>format long
>>[X, xa]
```

We can also use ode45 to approximate solutions to IVP involving second order differential equations.

$$x'' = f(t, x, x'), \qquad , a \leq x \leq b, x(a) = \alpha_0, x'(a) = \alpha_1.$$

Consider the following past year midterm question:

$$x'' - 3x' + 2x = te^t, \qquad x(0) = 0, x'(0) = -2.$$

Since the solution is $x = e^t - e^{2t} + (-\frac{1}{2}t^2 - t)e^t$, we have $x(1) = -e^2 - \frac{1}{2}e \approx -8.7482$.
To solve this numerically, instead of creating an inline function, we write our new function as a M-file. Create the following and save it as myfunction.m .

```
function xdot = myfunction(t,x)
xdot = zeros (2,1);
xdot(1) = x(2);
xdot(2) = 3*x(2) - 2*x(1)+t.*exp(t); %only need to change this line
```

Now input the following. (Remember to change your current directory.)

```
>> [ta , xa] = ode45('myfunction', [0 1] , [0 -2] );
>> xa(end,1)
```

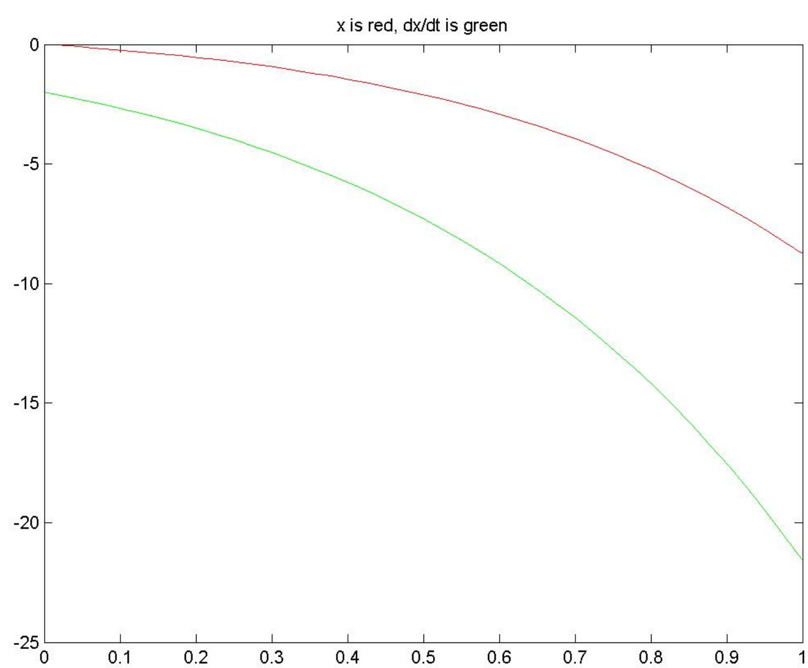
From your workspace, check that you now have an array called **ta** which contains increasing values from 0 to 1, and a double array **xa** which contains two columns of values. (Double click **xa**.) The last value in the first column of **xa** is exactly the solution that we want! We obtain its value with the command **xa(end,1)**. The first argument indicates the row, with **end** signifying the last row, the second argument indicates the column.

Let us now examine the commands we used. In our M-file, we are creating a new function called "myfunction" with two variables **t** and **x**. The variable **x** actually contains two columns, **x(1)** which represent x in our d.e. and **x(2)** which represents x'. So the last line of our M-file, essentially reads $x'' = f(t, x, x')$ where t, x, x' are replaced by **t, x(1), x(2)** respectively.

The ode45 command approximates the solution to 'myfunction' (the single quotes are essential) on the interval $0 \leq t \leq 1$, with the two initial conditions. The numerical solutions are stored in the double array **xa** where the first column contains x and the second column contains x'. To plot these solutions we use

```
>> plot(ta, xa(:,1), 'r')
>> hold on
>> plot(ta, xa(:,2), 'g')
>> title('x is red, dx/dt is green')
```

Note that **xa(:,1)** means all the values in the first column. Actually, you should be able to tell which solution curve is x or x' by looking at the initial values at $t = 0$.



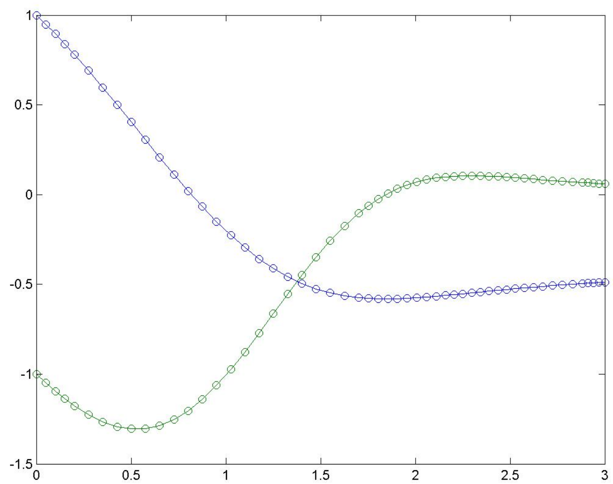
Let us solve another example, this time involving a differential equation with non constant coefficients.

$$x'' + t^2x' + x = 0, \qquad x(0) = 1, x'(0) = -1.$$

Use MATLAB to open your myfunction.m file and change the last line to:

```
xdot(2) = -t^2*x(2) -x(1) ; %only need to change this line
```

```
>> ode45('myfunction', [0 3] , [1 -1] )
```



Use the initial values to decide which curve is $x(t)$ and which is $x'(t)$.

$\boxed{\textit{Exercise 2B}}$

1. Let $f(x)$ be a solution of the differential equation $f'(x) = \frac{x+1}{\sqrt{x}}$, $x > 0$, such that $f(1) = 2$. Find $f(4)$. (This was a past year midterm question. Use ode45 to get an approximate answer.)

2. Use ode45 to find solutions to

$$\frac{dx}{dt} = t + x^2, \qquad x(0) = 1.$$

(Hint: You might get an error. Try plotting your graph on the rectangle, $0 \leq t \leq 1, 0 \leq x \leq 100$.)

3. $$x'' + 2x' + x = 2 + e^{2t}, \qquad x(0) = 37/9, x'(0) = -7/9.$$

Find $x(1)$.

—The End—