Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2013-2014

MA1506 MATHEMATICS II

April 2014 Time allowed: 2 hours

Examiners: Brett McInnes, Chew Tuan Seng, Leung Pui Fai, Quek Tong Seng

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INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** Do not write your name anywhere in this booklet. This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question. The marks for each question are indicated at the beginning of the question. The maximum possible total score for this examination paper is 80 marks.
4. This is a **closed book (with authorized material)** examination. Students are only allowed to bring into the examination hall **ONE** piece A4 size help-sheet which must be handwritten and can be written on both sides.
5. Candidates may use any non-programmable and non-graphing calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
(a)								
(b)								

Question 1 (a) (**Multiple Choice Question**) [5 marks]
You buy one gram of a newly discovered radio-active substance which decays with a half-life of ten years. How much of that substance do you still have after two years? Give your answer in gram correct to two decimal places.
(A) 0.78 (B) 0.83 (C) 0.87

Answer 1(a)	

(Show your working below and on the next page.)

(More working space for Question 1(a))

Question 1 (b) [5 marks]

Romeo is learning to solve Advanced Calculus problems. To measure his performance at any time t, the psychologist at his university uses the mathematical model

$$\frac{dP}{dt} = (\tan^{-1} t)(10 - P),$$

where his performance P is measured on a scale of 0 to 10 and time is measured in months. Assume that $P = 0$ at time $t = 0$, what is his performance at time $t = 3$ months? Give your answer correct to two decimal places.
(Suggestion: Treat the equation as a first order separable equation. You may also want to use the formula

$$\int \tan^{-1} t dt = t \tan^{-1} t - \frac{1}{2} \ln\left(t^2 + 1\right) + C. \quad)$$

Answer 1(b)	

(Show your working below and on the next page.)

(More working space for Question 1(b))

Question 2 (a) [5 marks]

An object moves along a straight line in such a way that the velocity of the object at a distance x from its starting position is equal to $\sqrt{x^3 + 2x^2 + 2x}$ for $x > 0$. Find the **exact value** of its acceleration when $x = 2$.

Answer 2(a)	

(Show your working below and on the next page.)

(More working space for Question 2(a))

Question 2 (b) [5 marks]

A buoy in the form of a 2 metre long right circular cylinder closed at both ends floats in equilibrium with its axis vertical and half submerged in the sea. The buoy is then further submerged so that it extends k metre above water level and then released from rest at time $t = 0$ second. It is known that there is no friction between the sides of the buoy and the sea water as the buoy bobs up and down in the sea with its axis keeping to a vertical position.

(i) Let x denote the length of the part of the buoy that is above water level at time t second, write down the second order differential equation that is satisfied by x. Give your answer in terms of g, the acceleration due to gravity, in the simplest form.

(ii) If $x = 0.61$ metre when $t = \frac{\pi}{3\sqrt{g}}$ second, find the **exact value** of the constant k.

Answer 2(b)(i)		**Answer 2(b)(ii)**	

(Show your working below and on the next page.)

(More working space for Question 2(b))

Question 3 (a) [5 marks]

A certain deer population on an isolated island where there is no natural predators follows a logistic growth model and reaches an equilibrium number of 5600. A group of wolves manage to land on the island and start to hunt the deer at a constant rate of 600 per year. The deer population eventually settles down to a new equilibrium number of 4000. What is the birth rate per capita of the deer population? Give your answer correct to three decimal places.

Answer 3(a)	

(Show your working below and on the next page.)

(More working space for Question 3(a))

Question 3 (b) [5 marks]

The Martian population on the planet Mars follows a modified logistic growth model with birth rate per capita equals to 1 per year and death rate per capita equals to $\frac{N}{\cosh t}$ per year, where t denotes time measured in years and N denotes the Martian population at that time. If at $t = 0$, the Martian population is one million, what is the Martian population at time $t = 4$? Give your answer in millions correct to one decimal place.
(Suggestion: You may want to use the formula

$$\int \frac{e^t}{\cosh t} dt = \ln(e^{2t} + 1) + C. \quad)$$

Answer 3(b)	

(Show your working below and on the next page.)

(More working space for Question 3(b))

Question 4 (a) [5 marks]

Let $y(t)$ be the solution of the initial value problem

$$y'' + y = u(t-1), \quad y(0) = 1, \quad y'(0) = 2,$$

where u denotes the unit step function.
(i) Find the value of $y(\frac{\pi}{4})$. Give your answer correct to two decimal places.
(ii) Find the value of $y(\frac{\pi}{3})$. Give your answer correct to two decimal places.

Answer 4(a)(i)		**Answer 4(a)(ii)**	

(Show your working below and on the next page.)

(More working space for Question 4(a))

Question 4 (b) [5 marks]

Let $y(t)$ be the solution of the initial value problem

$$y' = y - 2\delta(t-1) - 3\delta(t-2), \quad y(0) = 10,$$

where δ is the Dirac Delta function. Find the value of $y(3)$. Give your answer correct to two decimal places.

Answer 4(b)	

(Show your working below and on the next page.)

(More working space for Question 4(b))

Question 5 (a) [5 marks]

Let a, b, c denote three constants and M a 2×2 matrix given by

$$M = \begin{pmatrix} -\frac{\sqrt{2}}{2} & a \\ b & c \end{pmatrix}.$$

It is known that M represents an anti-clockwise rotation about the origin.

(i) Find the **exact value** of c.

(ii) Find M^{20}. Give **exact values** for all entries in M^{20} in your answer in their **simplest** form.

Answer 5(a)(i) $c =$		**Answer 5(a)(ii)** $M^{20} =$	

(Show your working below and on the next page.)

(More working space for Question 5(a))

Question 5 (b) [5 marks]

Let

$$M = \begin{pmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{pmatrix}.$$

We know that M is diagonalizable and 1 is an eigenvalue of M.

(i) What is the **exact value** of the other eigenvalue of M?

(ii) Suppose that $\begin{pmatrix} a \\ 1 \end{pmatrix}$ and $\begin{pmatrix} b \\ 1 \end{pmatrix}$ are two linearly independent eigenvectors of M, find the **exact value** of $a + b$.

Answer 5(b)(i)		**Answer 5(b)(ii)**	

(Show your working below and on the next page.)

(More working space for Question 5(b))

Question 6 (a) [5 marks]

Let $M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{pmatrix}$. It is known that $\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ are eigenvectors of M corresponding to the eigenvalues 3, 2, 1 respectively and it is also known that $\begin{pmatrix} 0 & -1 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{pmatrix}^{-1}$. Let $M^7 = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & k \end{pmatrix}$, where the nine letters a to k denote constants.

(i) Find the **exact value** of c.

(ii) Find the **exact value** of k.

Answer 6(a)(i) $c =$		**Answer 6(a)(ii)** $k =$	

(Show your working below and on the next page.)

(More working space for Question 6(a))

Question 6 (b) (**Multiple Choice Question**) [5 marks]

Let $\frac{d\mathbf{r}}{dt} = M\mathbf{r}$ denote a system of first order linear differential equations.

(i) If $M = \begin{pmatrix} -a & -b \\ -c & 0 \end{pmatrix}$, where a, b, c are positive constants, classify this system.

(A) saddle (B) nodal source (C) nodal sink

(ii) If $M = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}$, where a is a constant that satisfies $-1 < a < 0$, classify this system.

(A) nodal source (B) spiral source (C) spiral sink

Answer 6(b)(i)		**Answer 6(b)(ii)**	

(Show your working below and on the next page.)

(More working space for Question 6(b))

Question 7 (a) [5 marks]

Use Laplace Transform to solve the initial value problem:

$$\begin{cases} \frac{dx}{dt} = -x + 5y \\ \frac{dy}{dt} = x - 5y \end{cases}$$

with $x(0) = y(0) = 1$.
[NOTE: Zero marks if you do not use Laplace Transform.]

Answer 7(a)	

(Show your working below and on the next page.)

(More working space for Question 7(a))

Question 7 (b) [5 marks]

A certain nuclear plant consists of two 500-gallon tanks. Tank A is initially filled with water in which a large amount of uranium hexafluoride is dissolved, while tank B is initially filled with pure water. The well mixed solution from tank A is constantly being pumped to tank B at a rate of 15 gal/min, and the well mixed solution in tank B is constantly being pumped back to tank A at the same rate of 15 gal/min.

(i) Suppose the amount of uranium hexafluoride in tank A and tank B at time t minutes are x and y lbs respectively, write down the system of first order ordinary differential equations which x and y must satisfy.

(ii) If the amount of uranium hexafluoride in tank A and tank B at time $t = 50$ minutes are α and β lbs respectively, find the value of $\frac{\alpha}{\beta}$. Give your answer correct to one decimal place.

(Hint: You may want to use the fact that the eigenvalues are -0.06 and 0 with corresponding eigenvectors $\begin{pmatrix} 0.7 \\ -0.7 \end{pmatrix}$ and $\begin{pmatrix} -0.7 \\ -0.7 \end{pmatrix}$ respectively for the matrix $\begin{pmatrix} -0.03 & 0.03 \\ 0.03 & -0.03 \end{pmatrix}$.)

Answer 7(b)(i)		**Answer 7(b)(ii)**	

(Show your working below and on the next page.)

(More working space for Question 7(b))

Question 8 (a) [5 marks]

Use the method of separation of variables to find $u(x, y)$ that satisfies the partial differential equation

$$xu_x - (y+1)u_y = 0 \ , \quad x > 0, \quad y > 0,$$

given that $u(x, 0) = x^2$ for all $x > 0$.

Answer 8(a)	

(Show your working below and on the next page.)

(More working space for Question 8(a))

Question 8 (b) [5 marks]

Let $y(t,x)$ be the solution of the wave equation

$$y_{tt} = y_{xx} \quad ,$$

with $y(t,0) = y(t,\pi) = 0$, $y(0,x) = \sin^5 x$, $y_t(0,x) = 0$.
Find the value of $y\left(\frac{\pi}{8}, \frac{\pi}{4}\right)$. Give your answer correct to two decimal places.
(Suggestion: You may want to use d'Alembert's solution to the wave equation.)

Answer 8(b)	

(Show your working below and on the next page.)

(More working space for Question 8(b))

END OF PAPER