MA1506 Tutorial 3 Solutions

(1a)

$y''+6y'+9y=0$ Set $y = e^{\lambda t}$

$\lambda^2+6\lambda+9=0 \rightarrow \lambda=-3$

$\rightarrow y=(A+Bx)e^{-3x} \rightarrow y'=Be^{-3x}-3(A+Bx)e^{-3x}$

$y(0)=1 \Rightarrow A=1$

$y'(0)=-1 \Rightarrow B-3A=-1 \Rightarrow B=2 \rightarrow y=(1+2x)e^{-3x}$

(1b)

$\lambda^2-2\lambda+(1+4\pi^2)=0 \rightarrow \lambda=1\pm 2\pi i$

$\rightarrow y=e^x[A\cos 2\pi x+B\sin 2\pi x]$

$y'=y+e^x[-2\pi A\sin 2\pi x+2\pi B\cos 2\pi x]$

$y(0)=-2 \Rightarrow A=-2$

$y'(0)=2(3\pi-1) \Rightarrow 2(3\pi-1)=-2+2\pi B$

$\Rightarrow B=3 \Rightarrow y=e^x[-2\cos 2\pi x+3\sin 2\pi x]$

(2a)

Try $y=Ax^2+Bx+C$

$y''+2y'+10y$

$=2A+2(2Ax+B)+10(Ax^2+Bx+C)$

$=25x^2+3$

$\rightarrow 10A=25, \quad 4A+10B=0, \quad 2A+2B+10C=3$

$\rightarrow A=5/2, \quad \mathrm{B}=-1, \quad \mathrm{C}=0$

$\rightarrow \mathrm{y}=\frac{5}{2}x^2-x$

(2b)

Try $y=(Ax^2+Bx+C)e^{3x}$

$y'=(2Ax+B)e^{3x}+3(Ax^2+Bx+C)e^{3x}$

$y''=2Ae^{3x}+3(2Ax+B)e^{3x}+3(2Ax+B)e^{3x}+9(Ax^2+Bx+C)e^{3x}$

$9A-18A+8A=1 \rightarrow A=-1$

$6A+6A+9B-12A-18B+8B=0 \rightarrow B=0$

$2A+3B+3B+9C-6B-18C+8C=0 \rightarrow C=-2$

$y=(-x^2-2)e^{3x}$

(2c)

$$y''-y=2x\,\mathrm{Im}\,e^{ix} \qquad (\mathrm{Im} = \text{imaginary part})$$

if we can solve the complex equation $z''-z=2xe^{ix}$ then Imz satisfies the above.

Try $z=(Ax+B)e^{ix}$

$$z'=Ae^{ix}+i(Ax+B)e^{ix}$$

$$z''=Aie^{ix}+iAe^{ix}-(Ax+B)e^{ix}$$

$$z''-z=(2Ai-Ax-B-Ax-B)e^{ix}=2xe^{ix}$$

$$\to A=-1$$

$$-2i-2B=0\to B=-i$$

$$\to z=(-x-i)e^{ix}=-x\cos x+\sin x+i[-\cos x-x\sin x]$$

$$\mathrm{Im}\,z=-\cos x-x\sin x\to y=-\cos x-x\sin x$$

(2d)

$$y''+4y=\frac{1}{2}(1-\cos 2x)=\frac{1}{2}-\frac{1}{2}\mathrm{Re}(e^{2ix})$$

Solve $z''+4z=\frac{1}{2}-\frac{1}{2}e^{2ix}$ $\Leftrightarrow$ Try $z=A+Bxe^{2ix}$

$$z''=-4Bxe^{2ix}+4iBe^{2ix}\to z''+4z=-4Bxe^{2ix}+4A+4Bxe^{2ix}+4iBe^{2ix}$$

$$\to 4A=\frac{1}{2}\to A=\frac{1}{8}$$

$$\to -\frac{1}{2}=4iB\to B=\frac{1}{8}i$$

$$z=\frac{1}{8}+\frac{1}{8}ixe^{2ix}=\frac{1}{8}(1+x(i\cos 2x-\sin 2x))$$

$$y=\mathrm{Re}\,z=\frac{1}{8}-\frac{1}{8}x\sin 2x$$

(3a)

Variation of parameters: first solve $y''+4y=0\to y=A\cos 2x+B\sin 2x$

Promote A and B to functions A(x), B(x).

Then $A(x)\cos(2x)+B(x)\sin 2x$ is *a* solution of $y''+4y=\frac{1}{2}(1-\cos 2x)$

if A(x) and B(x) are chosen to satisfy

$$A'=\frac{-[\frac{1}{2}\overbrace{(1-\cos 2x)}^{(RHS)}]\sin 2x}{W}, \qquad B'=\frac{+[\frac{1}{2}\overbrace{(1-\cos 2x)}^{(RHS)}]\cos 2x}{W}$$

where W = (cos2x)x(sin2x)'-(cos2x)' sin2x = 2

so

$$A' = -\frac{1}{4}\sin 2x + \frac{1}{4}\cos 2x \sin 2x = -\frac{1}{4}\sin 2x + \frac{1}{8}\sin 4x$$

$$B' = \frac{1}{4}\cos 2x - \frac{1}{4}\cos^2(2x) = \frac{1}{4}\cos 2x - \frac{1}{8}(\cos 4x + 1)$$

$$\to A = \frac{1}{8}\cos 2x - \frac{1}{32}\cos 4x$$

$\to B = \frac{1}{8}\sin 2x - \frac{1}{32}\sin 4x - \frac{x}{8}$ so the solution is

$$\text{Acos 2x} + \text{Bsin2x} = \frac{1}{8}[(\cos 2x - \frac{1}{4}\cos 4x)\cos 2x + (\sin 2x - \frac{1}{4}\sin 4x - x)\sin 2x]$$

which is the same as in (2d) since

$$\frac{1}{8}(\cos^2 2x - \frac{1}{4}\cos 4x \cos 2x + \sin^2 2x - \frac{1}{4}\sin 4x \sin 2x) - \frac{x}{8}\sin 2x$$

$= \frac{1}{8}(1 - \frac{1}{4}\cos 2x) - \frac{x}{8}\sin 2x$ and the extra $-\frac{1}{32}\cos 2x$ can be absorbed into the general solution

(arbitrary constant) x cos2x + (arbitrary constant) x sin2x

(3b)

$A(x)\cos x + B(x)\sin(x)$ where

$$\text{A}' = \frac{-[\sec(x)]\,\text{sinx}}{w} \qquad \text{B}' = \frac{+[\sec(x)]\,\text{cosx}}{w}$$

$$\text{w} = (\text{cosx})(\text{sinx})' - (\text{cosx})'(\text{sinx}) = +1$$

$$\text{A} = -\int \frac{\text{sinx}}{\text{cosx}}\,dx = \ln|\cos x|,\ \text{B} = \text{x}$$

$$\to \text{y} = \text{cosx}\ln|\text{cosx}| + \text{xsinx}$$

(4)

$\frac{\mathbf{d}^2 y}{dx^2} = \frac{dy}{dx}\frac{d}{dy}y' = \frac{d}{dy}[(y'^2)/2]$ so the given equation can be written as

$\frac{d}{dy}[(y'^2)/2]$ = F(y). Just integrate both sides with respect to y, and you get a separable first-order ODE.

Using this trick in the present case we have

$\frac{d}{dr}[(\dot{r}^2)/2]$ = - GM/ r^2 [Sorry there should be a dot over the r on the left side, but it’s not very clear.] From the problem statement it is clear that the Earth’s initial speed is zero when r = R, where we use R to denote the original radius of the Earth’s orbit

[150 billion metres]. Using that to fix the arbitrary constant, we get upon integrating both sides with respect to r:

$$(\dot{r}^2)/2 = \frac{GM}{r} - \frac{GM}{R}.$$

Since the Earth will of course fall inward, the speed must always be negative after the first instant, so when we take the square root we must be careful to take the negative one. Hence

$\dfrac{-dr}{\sqrt{\dfrac{2GM}{r} - \dfrac{2GM}{R}}} = dt$, where we have to integrate from r = R [Earth orbit] to r =2R/3 [Venus' orbit radius, 100 billion metres]. It's convenient to change the variable from r to x = r/R. You then get

$$t = \frac{R^{3/2}}{\sqrt{2GM}} \int_{2/3}^{1} \frac{dx}{\sqrt{\dfrac{1}{x} - 1}}.$$

This is an improper integral, which is why the website recommended will sometimes refuse to do it [though usually it does work!] If it is in a bad mood, just integrate from 2/3 up to 0.999 or something like that. With the given data you should get about 44.74 days. So that's how long it takes to reach the orbit of Venus. Remember that a day is 24 times 3600 seconds.