MA1506
Mathematics II

Chapter 3
Basic Mathematical Modelling

3.1 Introduction

Mathematical model uses mathematical language to describe a system.
In this module, we use ODE to describe some systems.
In the last two chapters, we have studied models for falling objects, cooling, and mass-spring systems.
In this chapter, we will introduce some more models.

3.1 Introduction

- Malthus model of population
- Logistic growth model — related
- Harvesting model — 1st order ODE
- Plug flow reactor model — 1st order ODE
- Cantilevered beam model — 4th order ODE

3.1 Introduction

- The process of developing a mathematical model is called mathematical modelling.
- In constructing math. models, there is almost always a tradeoff between accuracy and simplicity.
- We should use good judgment and common sense in constructing models and in making predictions.
- Begin with simple models, understand their weakness, and then improve them with more complicated and appropriate ones.

3.2 Malthus Model of Population

Thomas Malthus
1766 -1834

Total Population: *N(t)*

Per Capita Birth-Rate, *B*

babies born in $\delta t = BN\,\delta t$

Per Capita Death-Rate, *D*

deaths in $\delta t = DN\,\delta t$

δN = # births - # deaths = $(B-D)N\delta t$

$$\frac{\delta N}{\delta t} = (B-D)N$$

3.2 Malthus Model of Population

$$\frac{dN}{dt} = (B-D)N = kN$$

$$k = B-D$$

$$\int \frac{dN}{N} = \int k\,dt = kt + c$$

$$\ln N(t) = kt + c$$

$$N(t) = e^{kt+c} = e^{c}e^{kt} = N_0 e^{kt}$$

$$e^{c} = N_0 = N(0)$$

k > 0 : population explosion

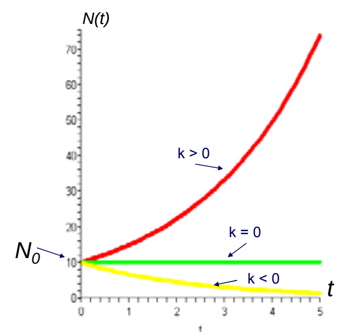
as $t \to \infty \quad e^{kt} \to \infty \quad \Rightarrow \quad N(t) \to \infty$

k = 0 : stable equilibrium population

because $N(t) = N(0)$ for all t.

k < 0 : population will extinct eventually

as $t \to \infty \quad e^{kt} \to 0 \quad \Rightarrow \quad N(t) \to 0$



In the logistic growth model, we impose one more condition that

death rate is a function of population

on Malthus model

We assume $D = sN$

which is called the logistic assumption.

The assumption says that, in a world with finite resources, large population will eventually cause starvation and disease.

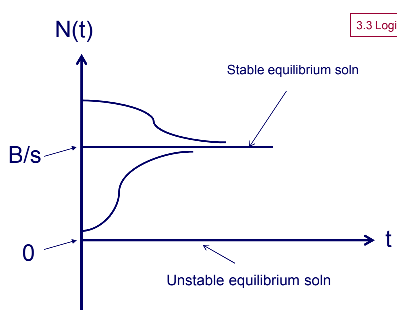
Since $D = sN$

The logistic equation is

$$\frac{dN}{dt} = BN - DN = BN - sN^2$$
$$= N(B - sN)$$

Notice that the ODE has two equilibrium solutions, namely, N(t)=0 and N(t)=B/s for all t.

Note also that N(t)=0 is unstable while N(t)=B/s is stable.



Phase line diagram

As our logistic equation is simple, we can solve it by separable method.

We may assume that $N(t) \neq 0$ and $N(t) \neq B/s$

Then

$$\frac{dN}{dt} = BN - sN^2 = N(B - sN)$$

$$dt = \frac{dN}{N(B - sN)}$$

By partial fraction we have

$$\frac{1}{N(B-sN)}=\frac{1}{BN}+\frac{s}{B(B-sN)}$$

$$t=\int\frac{dN}{N(B-sN)}=\frac{1}{B}\int\frac{dN}{N}+\frac{s}{B}\int\frac{dN}{B-sN}$$

$$=\frac{1}{B}\ln N-\frac{1}{B}\ln(|B-sN|)+c$$

$$=\frac{1}{B}\ln\left(\frac{N}{|B-sN|}\right)+c$$

Hence the solution is $\frac{N(t)}{|B-sN(t)|}=e^{-Bc+Bt}$

To find the value of c, we need to know N(0), i.e., the value of N(t) when t=0.

When t=0, we have

$$\frac{N(0)}{|B-sN(0)|}=e^{-Bc}$$

Thus our solution becomes

$$\frac{N(t)}{|B-sN(t)|}=\frac{N(0)}{|B-sN(0)|}e^{Bt}$$

Definition

Recall that B/s is the stable equilibrium solution of

$$\frac{dN}{dt}=BN-sN^2=N(B-sN)$$

B/s is also called the carrying capacity or sustainable population.

3.5 Harvesting

Assume the fish population N(t) follows the logistic growth model

$$\frac{dN}{dt}=BN-sN^2$$

Assume further that we catch E (constant) fish per year. Then our differential equation becomes

$$\boxed{\frac{dN}{dt}=BN-sN^2-E}$$

Basic Harvesting Model

Unlike the logistic model, we are unable to solve the harvesting model by separable method.

Instead, we shall study the solutions to harvesting model using equilibrium solutions and phase lines.

To find the equilibrium solutions of

$$\frac{dN}{dt}=BN-sN^2-E$$

we let $f(N)=-sN^2+BN-E$

and find the roots of $f(N)=0$

The discriminant for $f(N)=0$

is $B^2-4(-s)(-E)=B^2-4sE$

and we have to consider the following three cases

Case 1 $B^2-4sE>0$ two real roots

Case 2 $B^2-4sE=0$ one real root

Case 3 $B^2-4sE<0$ no real roots

Case 1 $B^2 - 4sE > 0$ i.e., $E < \frac{B^2}{4s}$

Let β_1, β_2 be the two roots of $f(N) = 0$ given by

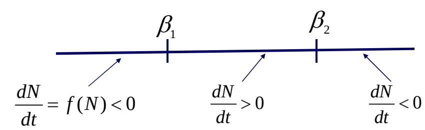
$$\beta_1 = \frac{-B + \sqrt{B^2 - 4(-s)(-E)}}{2(-s)}$$

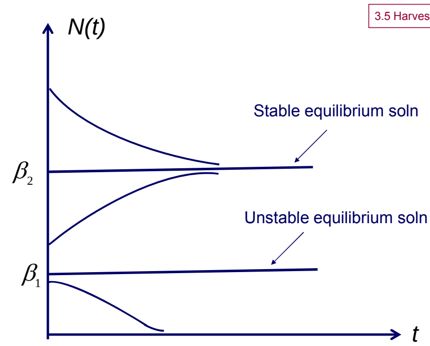
$$\beta_2 = \frac{-B - \sqrt{B^2 - 4(-s)(-E)}}{2(-s)}$$

Note that $\beta_1 < \beta_2$

$$f(N) = -sN^2 + BN - E$$

$$\frac{dN}{dt} = f(N) = (-s)(N - \beta_1)(N - \beta_2)$$





Phase line diagram for $E < \frac{B^2}{4s}$

Case 2 $E = \frac{B^2}{4s}$

$$f(N) = -sN^2 + BN - E = 0$$

has one repeated real root given by

$$\beta = \frac{-B \pm \sqrt{B^2 - 4(-s)(-E)}}{2(-s)} = \frac{B}{2s}$$

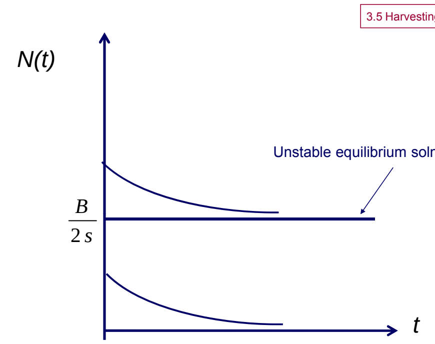
Thus $N(t) = \frac{B}{2s}$ is an equilibrium solution.

Since $E = \frac{B^2}{4s}$

we have $\frac{dN}{dt} = f(N) = (-s)(N - \beta)^2 < 0$

because s>0 which implies that –s is negative.

Thus the fish population N(t) is decreasing and all the fish will be wiped out.



Phase line diagram for $E = \frac{B^2}{4s}$

Case 3 Overfishing $B^2 - 4sE < 0$

Since $B^2 - 4sE < 0$

$$f(N) = -sN^2 + BN - E = 0$$

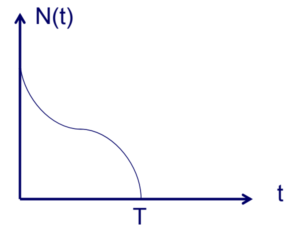
has no real roots.

Since s>0 which implies that –s is negative, we have $f(N) < 0$

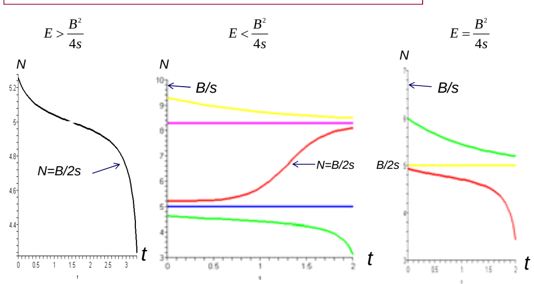
Thus $\frac{dN}{dt} = f(N) < 0$

and fish population N(t) is decreasing.

Fish population N(t) will be wiped out (i.e., N(t)=0) at a certain time T.



Summary: Solution curves for the three cases



Recall B/s=sustainable population in logistic model

No crossing principle

We notice that the above curves do not cross each other. This is because there is one and only one solution for each initial value problem for each 1st order ODE.

Suppose we have two curves crossing each other, then we can use the intersection point as an initial point and get two curves starting from this point. This means there are two solutions with the same initial point, a contradiction.

This observation is usually called the no crossing principle.

Remark

If there is no fishing for a long time, most likely the fish population is B/s.

Thus we may assume that the initial number of fish is B/s, i.e., N(0)= B/s.

If we start catching E fish per year, then the harvesting differentiation equation is

$$\frac{dN}{dt} = -sN^2 + BN - E$$

Our earlier discussions show that we will be overfishing if

$$E > \frac{B^2}{4s}$$

Otherwise, fish population will eventually tend to a fixed number β_2

3.6 Plug flow reactor (PFR) model

3.6.1 Chemical reactions

A chemical reaction consists of one or more starting material called reactants. They react to form new substances called products. For example,

$$2H_2 + O_2 \rightarrow 2H_2O$$

says 2 molecules of hydrogen and 1 molecule of oxygen react to form 2 molecules of water.

Now consider the reaction

$$A + B \rightarrow C$$

The concentration of reactant A is the number of moles per litre (1 mole = 6.022×10^{23} molecules).

3.6.2 Plug flow reactor (PFR) model

The **plug flow reactor** (**PFR**) model is used to describe chemical reactions in continuous, flowing systems.

PFR is like a long tube into which you push some mixture of chemicals which move through the tube while they react with each other.

33

PFR model

Assume

- Velocity u of flow is constant
- Cross-sectional area A is constant
- Temp constant
- Flow is plug flow: see next slide

34

PFR model

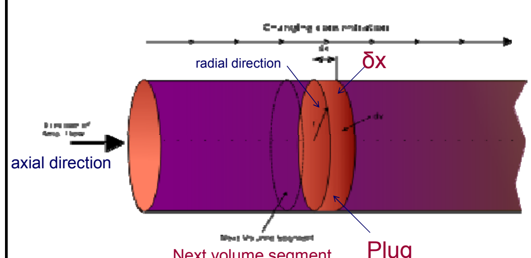
- Plug Flow:

  Fluid flowing through the reactor as a series of infinitely thin "plugs", each with a uniform composition, traveling in the axial direction of the reactor, with each plug having a different composition from the ones before and after it.

  The key assumption is that the fluid is perfectly mixed in the radial direction but not in the axial direction (forwards or backwards). Each plug of small volume is considered as an independent, separate entity,

35

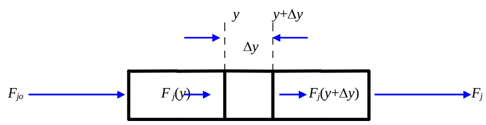
PFR model



36

PFR model

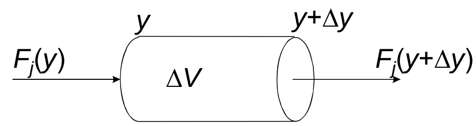
Plug flow reactor



PFR model

- Material balance $F_{j0} - F_j + \int_V r_j dV = \frac{dN_j}{dt}$

- Let us inspect a volume ΔV, where the reaction rate can be assumed constant.



38

PFR model

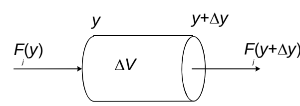
Reaction rate is constant in ΔV $\qquad \int_V r_j dV = r_j \Delta V$

Steady state assumption, i.e. no accumulation $\qquad \frac{dN_j}{dt} = 0$

39

PFR model

For the volume ΔV we can write the material balance for component *j*.



$$F_j(\Delta y) - F_j(y + \Delta y) = -r_j \Delta V$$

40

PFR model

A = cross-sectional area

$$\Delta V = A \Delta y$$

$$F_j(y) - F_j(y + \Delta y) = -r_j A \Delta y$$

$$F_j(y) - F_j(y + \Delta y) = -r_j \Delta V$$

$$\Delta V \to 0$$

$$-\frac{dF_j}{dV} = -r_j \Leftrightarrow \frac{dF_j}{dV} = r_j$$

41

PFR model

Separation of variables

$$dV = F_{j0} \frac{dX}{-r_j}$$

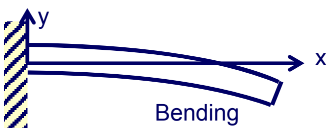
$$\int_0^V dV = F_{j0} \int_0^X \frac{dX}{-r_j}$$

$$V = F_{j0} \int_0^X \frac{dX}{-r_j}$$

42

3.7 Cantilevered Beams

A cantilevered beam is a beam supported at only one end and whose length is much larger than its width and thickness.



In this section, we will discuss the bending of a cantilevered beam due to its weight and other forces acting on the beam called the load.
We denote the load by

W(x)=force **per until length at point x**

Load=weight of springboard

Load=weight of bamboo stick+clothes

Load=weight of springboard + swimmers

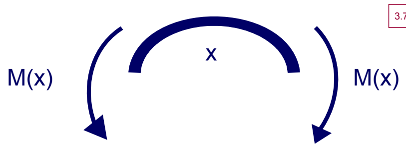
Recall that 2nd derivative $\frac{d^2y}{dx^2}$ gives us information about the way the graph bends.

In fact, a graph is concave up if $\frac{d^2y}{dx^2}>0$ and concave down if $\frac{d^2y}{dx^2}<0$.

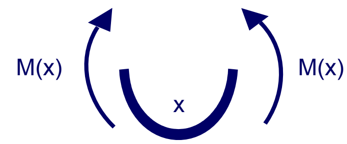
$\frac{d^2y}{dx^2}>0$

$\frac{d^2y}{dx^2}<0$

When the beam is supported at one end, internal forces, called the shear forces, and bending moments M(x) are developed.
First we consider the bending moment M(x) at x, here x is fixed.



The beam at position x is concave down.
Thus the 2nd derivative is negative and the sign of M(x) is taken to be negative.



The beam at x is concave up.
Thus 2nd derivative is positive and the sign of M(x) is taken to be positive.

The value (always positive) of $\dfrac{M(x)}{\dfrac{d^2y}{dx^2}}$ depends on stiffness.
From physics we have

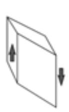
$$\frac{M(x)}{\dfrac{d^2y}{dx^2}} = EI$$

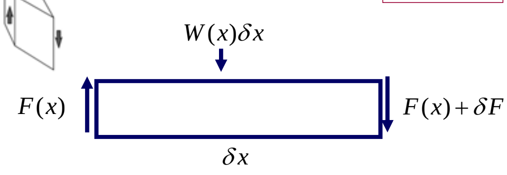
Where E is a constant called the Young's modulus and I is a constant called the moment of inertia.

Next we shall prove that

$$\frac{d^2M}{dx^2} = -W(x)$$

To prove this formula, we need to consider shear force F(x), which is parallel to the cross section and causes the internal structure of a material to slide against itself



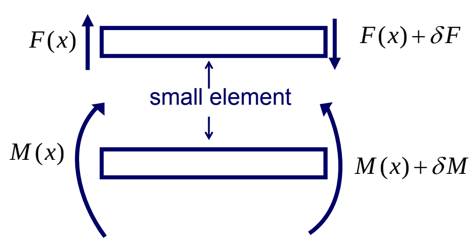


W(x) = Load = force per unit length

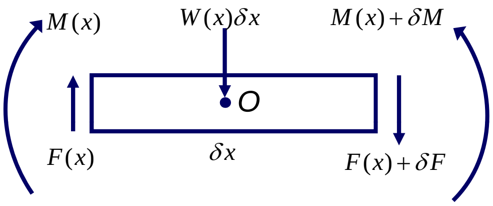
$$F(x) = F(x) + \delta F + W(x)\delta x$$

$$\frac{dF}{dx} = -W(x)$$

Standard convention



Moment= force x distance



Take moment about centre O, we get

$$M(x)+F(x)\frac{\delta x}{2}+(F(x)+\delta F)\frac{\delta x}{2}=M(x)+\delta M$$

$$M(x)+F(x)\frac{\delta x}{2}+(F(x)+\delta F)\frac{\delta x}{2}=M(x)+\delta M$$

$$F(x)\delta x+(\delta F)(\delta x)/2=\delta M$$

Too small, ignore

$$F(x)\delta x=\delta M$$

$$\frac{dM}{dx}=F(x)$$

Since

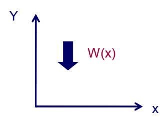
$$\frac{M(x)}{\frac{d^2y}{dx^2}}=EI \qquad \frac{dM}{dx}=F(x) \qquad \frac{dF}{dx}=-W(x)$$

we have

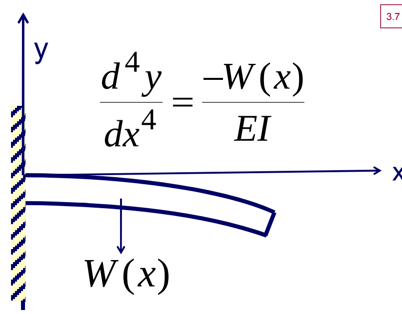
$$M(x)=\frac{d^2y}{dx^2}EI$$

$$\frac{d^2M}{dx^2}=\frac{dF}{dx}=-W(x)$$

$$\frac{d^4y}{dx^4}=\frac{1}{EI}\frac{d^2M}{dx^2}=-\frac{W(x)}{EI}$$



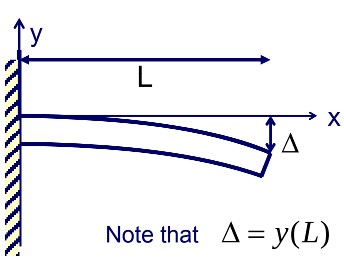
$$\frac{d^4y}{dx^4}=\frac{-W(x)}{EI}$$



Maximum deflection Δ

$$\frac{d^4y}{dx^4}=\frac{-W(x)}{EI}$$

Assume: uniform mass $W(x)=\alpha$



Note that $\Delta=y(L)$

To find Δ i.e., to find $y(L)$

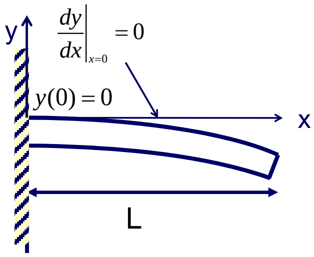
we need to solve

$$\frac{d^4y}{dx^4}=\frac{-W(x)}{EI}=\frac{-\alpha}{EI}$$

This is a 4th order ODE and we need FOUR initial conditions which are

$$y(0)=0 \qquad \left.\frac{dy}{dx}\right|_{x=0}=0$$

$$M(L)=0 \qquad F(L)=0$$



$M(L) = 0$ $\qquad$ $F(L) = 0$

No moment at L $\qquad$ No shear force at L

Recall $\quad \frac{d^2y}{dx^2} = \frac{M(x)}{EI}$

So $\quad \frac{d^3y}{dx^3} = \frac{1}{EI}\frac{dM}{dx} = \frac{F(x)}{EI}$

Hence $\quad \left.\frac{d^2y}{dx^2}\right|_{x=L} = \frac{M(L)}{EI} = 0$

$$\left.\frac{d^3y}{dx^3}\right|_{x=L} = \frac{F(L)}{EI} = 0$$

Thus we can rewrite the initial conditions as

$y(0) = 0 \qquad \left.\frac{dy}{dx}\right|_{x=0} = 0$

$\left.\frac{d^2y}{dx^2}\right|_{x=L} = 0 \qquad \left.\frac{d^3y}{dx^3}\right|_{x=L} = 0$

$\frac{d^4y}{dx^4} = \frac{-\alpha}{EI}$ implies $\frac{d^3y}{dx^3} = \frac{-\alpha x}{EI} + A$

Since

$$\left.\frac{d^3y}{dx^3}\right|_{x=L} = 0$$

we get

$$A = \frac{\alpha L}{EI}$$

$\frac{d^3y}{dx^3} = \frac{-\alpha x}{EI} + \frac{\alpha L}{EI}$ implies

$$\frac{d^2y}{dx^2} = \frac{-\alpha x^2}{2EI} + \frac{\alpha L x}{EI} + B$$

Since

$$\left.\frac{d^2y}{dx^2}\right|_{x=L} = 0$$

we get

$$B = \frac{\alpha L^2}{2EI} - \frac{\alpha L^2}{EI} = -\frac{\alpha L^2}{2EI}$$

$$\frac{d^2y}{dx^2} = \frac{-\alpha x^2}{2EI} + \frac{\alpha L x}{EI} - \frac{\alpha L^2}{2EI}$$

implies $\quad \frac{dy}{dx} = \frac{-\alpha x^3}{6EI} + \frac{\alpha L x^2}{2EI} - \frac{\alpha L^2 x}{2EI} + C$

Since

$$\left.\frac{dy}{dx}\right|_{x=0} = 0$$

we get $\quad C = 0$

3.7 Cantilevered Beams

$$\frac{dy}{dx}=-\frac{\alpha x^3}{6EI}+\frac{\alpha L x^2}{2EI}-\frac{\alpha L^2 x}{2EI}$$

implies

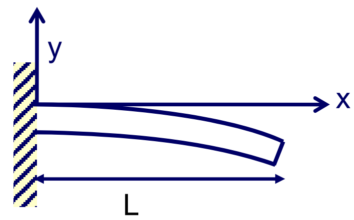
$$y=-\frac{\alpha x^4}{24EI}+\frac{\alpha L x^3}{6EI}-\frac{\alpha L^2 x^2}{4EI}+D$$

Since $y(0)=0$

we get $D=0$

67

3.7 Cantilevered Beams



Beam equation $(W(x)=\alpha)$

$$y(x)=\frac{\alpha L^4}{2EI}\left(-\frac{1}{12}\left(\frac{x}{L}\right)^4+\frac{1}{3}\left(\frac{x}{L}\right)^3-\frac{1}{2}\left(\frac{x}{L}\right)^2\right)$$

68

3.7 Cantilevered Beams

For $W(x)=\alpha$

$$y(x)=\frac{\alpha L^4}{2EI}\left(-\frac{1}{12}\left(\frac{x}{L}\right)^4+\frac{1}{3}\left(\frac{x}{L}\right)^3-\frac{1}{2}\left(\frac{x}{L}\right)^2\right)$$

$$\Delta=y(L)=\frac{\alpha L^4}{2EI}\left(-\frac{1}{12}+\frac{1}{3}-\frac{1}{2}\right)=-\frac{\alpha L^4}{8EI}$$

Cantilever beam deflection formula

69

3.7 Cantilevered Beams

Remarks on cantilever beam of length L

Differential equation

$$\frac{d^4y}{dx^4}=\frac{-W(x)}{EI} \qquad \frac{d^2y}{dx^2}=\frac{M(x)}{EI}$$

Initial conditions

$$y(0)=0 \qquad \left.\frac{dy}{dx}\right|_{x=0}=0$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=L}=0 \qquad \left.\frac{d^3y}{dx^3}\right|_{x=L}=0$$

70