Question 1(a)(i)

$$\frac{d}{dx}\ln(\ln x) = \frac{1}{\ln x}\left(\frac{d}{dx}\ln x\right) = \frac{1}{x\ln x}.$$

Question 1(a)(ii)

$$\frac{dy}{dx} + \frac{1}{x\ln x}y = \frac{x}{\ln x}.$$

Integrating factor $= e^{\int \frac{1}{x\ln x}} = e^{\ln\ln x} = \ln x$. Hence,

$$\begin{aligned} y\ln x &= \int \frac{x}{\ln x}\ln x = \frac{x^2}{2} + C \\ \implies\ y(e)\ln e = 0 &= \frac{e^2}{2} + C \implies C = -\frac{e^2}{2}. \end{aligned}$$

$$\begin{aligned} y\ln x &= \frac{x^2}{2} - \frac{e^2}{2} \\ \implies\ y(e^2)\ln e^2 &= \frac{e^4}{2} - \frac{e^2}{2} \\ \implies\ y(e^2) &= \frac{e^4 - e^2}{4}. \end{aligned}$$

Question 1(b)

$$\frac{dN}{N(N-100)} = 0.01dt \implies \left(\frac{1}{N-100} - \frac{1}{N}\right) dN = dt$$
$$\implies \ln|N-100| - \ln|N| = t + C$$
$$\implies \frac{N-100}{N} = Ae^t.$$

$N(0) = 101 \implies A = 1/101$. Therefore,

$$\frac{N-100}{N} = \frac{1}{101}e^t \implies N = \frac{10100}{101 - e^t}$$
$$\implies N(4.61) = \frac{10100}{101 - e^{4.61}} \approx 19579.$$

Question 2(a)

For homogeneous solution y_h, solve

$$\lambda^2 + 2\lambda + 1 = 0 \implies \lambda = -1.$$

Hence $y_h = (Ax + B)e^{-x}$. For y_p, try

$$\begin{aligned} y = ue^{-x} &\implies y' = u'e^{-x} - ue^{-x} \\ &\implies y'' = u''e^{-x} - 2u'e^{-x} + ue^{-x}. \end{aligned}$$

So

$$\begin{aligned} & u''e^{-x} - 2u'e^{-x} + ue^{-x} + 2\left(u'e^{-x} - ue^{-x}\right) + ue^{-x} = xe^{-x} \\ & \implies u'' = x \implies u = \frac{1}{6}x^3 + Cx + D. \end{aligned}$$

Thus $y = (Ax + B)e^{-x} + \frac{1}{6}x^3 e^{-x}$.

$y(0) = 1 \implies B = 1$.

Since

$$y' = Ae^{-x} - (Ax + 1)e^{-x} + \frac{1}{2}x^2 e^{-x} - \frac{1}{6}x^3 e^{-x}.$$

$y'(0) = 0 \implies A - 1 = 0$.

Finally,

$$y = (x + 1)e^{-x} + \frac{1}{6}x^3 e^{-x} = (1 + x + \frac{1}{6}x^3)e^{-x}.$$

Question 2(b)

For homogeneous solution y_h, solve

$$\lambda^2 + 1 = 0 \implies \lambda = \pm i.$$

Hence $y_h = A\cos x + B\sin x$.

Wronskian $W = \cos^2 x - (-\sin^2 x) = 1$, so

$$\begin{aligned} u &= -\int \sin x \sec x = -\int \frac{\sin x}{\cos x} = \ln(\cos x) \\ v &= \int \cos x \sec x = \int 1 = x. \end{aligned}$$

Hence $y = A\cos x + B\sin x + \ln(\cos x)\cos x + x\sin x$.

$y(0) = 1 \implies A = 1$.

$y' = -\sin x + B\cos x - \ln(\cos x)\sin x - \tan x\cos x + \sin x + x\cos x$.

$y'(0) = 1 \implies 1 = B$. Hence

$$y = \cos x + \sin x + \ln(\cos x)\cos x + x\sin x.$$

Question 3 (a)(i)

Let $f(x) = \cos(\pi \sin x)$.

Then $\ddot{x} = \cos(\pi \sin x) = f(x)$.

$$f(x) = 0 \quad \Rightarrow \quad \cos(\pi \sin x) = 0 \quad \Rightarrow \quad x = \frac{\pi}{6}$$

$$\begin{aligned} f'(\frac{\pi}{6}) &= (\pi \cos x)[-\sin(\pi \sin x)]\Big|_{x=\frac{\pi}{6}} \\ &= -\frac{\sqrt{3}\,\pi}{2} \\ &< 0 \end{aligned}$$

Hence $x = \dfrac{\pi}{6}$ is a stable equilibrium point.

Question 3 (a)(ii)

By Taylor's expansion, we have

$$\begin{aligned} f(x) &= f(\frac{\pi}{6}) + (x - \frac{\pi}{6}) f'(\frac{\pi}{6}) + \cdots \\ &\simeq 0 + (x - \frac{\pi}{6})(-\frac{\sqrt{3}\,\pi}{2}) \end{aligned}$$

Hence

$$\ddot{x} = \cos(\pi \sin x) = f(x) \simeq (-\frac{\sqrt{3}\,\pi}{2})(x - \frac{\pi}{6}).$$

Let $y = x - \dfrac{\pi}{6}$. Then

$$\ddot{x} = \cos(\pi \sin x) = f(x) \simeq (-\frac{\sqrt{3}\,\pi}{2})(x - \frac{\pi}{6})$$

$$\Rightarrow \quad \ddot{y} = -\frac{\sqrt{3}\,\pi}{2}\, y$$

Thus $\omega^2 \simeq \dfrac{\sqrt{3}\,\pi}{2}$

Question 3(b)

Before the pirates,

$$\begin{aligned} N &= \hat{N}e^{(B-D)t} \\ 2\hat{N} &= \hat{N}e^{(B-D)15} \implies B - D = \frac{1}{15}\ln 2. \end{aligned}$$

After the pirates,

$$\begin{aligned} N = \tilde{N}e^{(\tilde{B}-\tilde{D})t} & \\ \frac{1}{2}\tilde{N} = \tilde{N}e^{(\tilde{B}-\tilde{D})10} &\implies \tilde{B} - \tilde{D} = -\frac{1}{10}\ln 2 \\ &\implies 0.05B - 1.5D = -\frac{1}{10}\ln 2. \end{aligned}$$

Solving for B ,

$$1.5B - 0.05B = 2(\frac{1}{10}\ln 2) \implies B \approx 0.0956.$$

or $B \approx 9.56\%$.

Question 4(a)

$$B = 0.1, N_\infty = 200,000 \implies s = 0.1/200000 = 5 \times 10^{-7}.$$

Hence,

$$\begin{aligned}\frac{dN}{dt} &= BN - sN^2 - E \\ &= 0.1N - 5 \times 10^{-7}N^2 - 3000.\end{aligned}$$

Since $B^2/4s = BN_\infty/4 = 5000 > E$, we have two equilibriums given by the roots of $BN - sN^2 - E$. Solving, this quadratic gives us two roots, $\beta_1 \approx 36754$ (unstable) and $\beta_2 \approx 163246$ (stable).

Limiting population is then ≈ 163246.

Question 4(b)

$$\begin{aligned}\frac{dN}{dt} &= 0.001N(100-(N-100)^2) \\ &= -0.001N(N-90)(N-110).\end{aligned}$$

The curve $F(N) = \frac{dN}{dt}$ has 3 roots, giving us 3 equilibriums. $N = 90$ is an unstable equilibrium while $N = 110$ is stable.

(i) $A = 90$
(ii) If $\hat{N} > 110$ then $N \to 110$.

Question 5(a)

Apply Laplace transform to get,

$$\begin{aligned} L(y'') &= s^2Y - sy(0) - y'(0) = s^2Y - s \\ L(y) &= Y \\ L(0.5\delta(t-2\pi)) &= 0.5e^{-2\pi s}. \end{aligned}$$

Hence

$$(s^2+1)Y = s + 0.5e^{-2\pi s} \implies Y = \frac{s}{s^2+1} + \frac{0.5e^{-2\pi s}}{s^2+1}.$$

Since $L(f(t-a)u(t-a)) = e^{-as}F(s)$ and $L(sint) = \frac{1}{s^2+1}$,

$$y = L^{-1}(Y) = \cos t + \frac{1}{2}\sin(t-2\pi)u(t-2\pi).$$

Question 5(b)

Completing the square,

$$\frac{e^{-\pi s}}{s^2+2s+5} = \frac{e^{-\pi s}}{(s+1)^2+2^2}.$$

Using s-shifting,

$$L(e^{-t}\sin 2t) = \frac{2}{(s+1)^2+2^2}.$$

Using t-shifting, $L(f(t-a)u(t-a)) = e^{-as}F(s)$, we have

$$\begin{aligned} L^{-1}\left(\frac{e^{-\pi s}}{s^2+2s+5}\right) &= \frac{1}{2}e^{-(t-\pi)}\sin(2(t-\pi))u(t-\pi) \\ L^{-1}\left(\frac{e^{-\pi s}}{s^2+2s+5}\right)\Bigg|_{t=\frac{5\pi}{4}} &= \frac{1}{2}e^{-\frac{\pi}{4}}. \end{aligned}$$

Question 6(a)

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\implies \quad A^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\implies \quad A^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$A^4 = A^5 = \cdots = A^n = 0$$

and so

$$e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \ldots + \frac{A^n}{n!} + \ldots$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & \frac{3}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Question 6(b)

$$M = \begin{pmatrix} R \to R & F \to R & S \to R \\ R \to F & F \to F & S \to F \\ R \to S & F \to S & S \to S \end{pmatrix} = \begin{pmatrix} 0.3 & 0.7 & 0.5 \\ 0.4 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0.3 \end{pmatrix}.$$

$$\begin{aligned} M^2 &= \begin{pmatrix} 0.3 & 0.7 & 0.5 \\ 0.4 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0.3 \end{pmatrix} \begin{pmatrix} 0.3 & 0.7 & 0.5 \\ 0.4 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.52 & 0.4 & 0.44 \\ 0.26 & 0.34 & 0.3 \\ 0.22 & 0.26 & 0.26 \end{pmatrix} \\ M^3 &= \begin{pmatrix} 0.52 & 0.4 & 0.44 \\ 0.26 & 0.34 & 0.3 \\ 0.22 & 0.26 & 0.26 \end{pmatrix} \begin{pmatrix} 0.3 & 0.7 & 0.5 \\ 0.4 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0.3 \end{pmatrix} \\ &= \begin{pmatrix} 0.448 & 0.448 & 0.472 \\ 0.304 & 0.28 & 0.288 \\ 0.248 & 0.232 & 0.24 \end{pmatrix} \end{aligned}$$

Answer $= 0.28$.

Question 7(a)

Rotate -30°, shear 45° parallel to x-axis, then rotate 30°.

$$\begin{aligned}
T &= \begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix} \begin{pmatrix} 1 & \tan 45 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 30 & \sin 30 \\ -\sin 30 & \cos 30 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} - \frac{1}{2} & \frac{\sqrt{3}}{2} + \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 - \frac{\sqrt{3}}{4} & \frac{3}{4} \\ -\frac{1}{4} & 1 + \frac{\sqrt{3}}{4} \end{pmatrix}.
\end{aligned}$$

$$T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{\sqrt{3}}{4} & \frac{3}{4} \\ -\frac{1}{4} & 1 + \frac{\sqrt{3}}{4} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ 1 + \frac{\sqrt{3}}{4} \end{pmatrix}.$$

Answer $= (\frac{3}{4}, 1 + \frac{\sqrt{3}}{4})$.

Question 7(b)

$$\det(A-\lambda I)=\begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}=0$$

$$\Longrightarrow \quad \lambda^2+\lambda-6=0$$

$$\Longrightarrow \quad \lambda=-3,2.$$

$$\lambda=-3,\ 2 \quad \Rightarrow \text{ Eigenvectors } \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

So

$$P=\begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{2} \end{pmatrix} \Longrightarrow P^{-1}=\frac{2}{5}\begin{pmatrix} \frac{1}{2} & -1 \\ 2 & 1 \end{pmatrix}.$$

Hence

$$\begin{aligned} A^4 &= \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} (-3)^4 & 0 \\ 0 & 2^4 \end{pmatrix}\frac{2}{5}\begin{pmatrix} \frac{1}{2} & -1 \\ 2 & 1 \end{pmatrix} \\ &= \frac{2}{5}\begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{81}{2} & -81 \\ 32 & 16 \end{pmatrix} \\ &= \begin{pmatrix} 29 & -26 \\ -26 & 68 \end{pmatrix}. \end{aligned}$$

Question 8(a)

$$\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$$

Trace $= -1$, Det $= -4 \implies$ Saddle.

$$\begin{pmatrix} 2 & -2 \\ 8 & 0 \end{pmatrix}$$

Trace $= 2$, Det $= 16 > 0$
Tr^2 - 4Det $= -60 < 0 \implies$ Spiral Source.

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Trace $= 4$, Det $= 3 > 0$
Tr^2 - 4Det $= 4 > 0 \implies$ Nodal Source.

$$\begin{pmatrix} 0 & -2 \\ 8 & 0 \end{pmatrix}$$

Trace $= 0$, Det $= 16 > 0$
Tr^2 - 4Det $= -64 < 0 \implies$ Centre.

Question 8(b)

Let P, S be the number of Persians and Spartans respectively.

$$\frac{dP}{dt} = -P - \frac{15}{2}S,\ \frac{dS}{dt} = -\frac{1}{2}P \Rightarrow \begin{bmatrix} \frac{dP}{dt} \\ \frac{dS}{dt} \end{bmatrix} = \begin{bmatrix} -1 & -\frac{15}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} P \\ S \end{bmatrix}$$

Since det < 0,we have a saddle.

$$\begin{vmatrix} -1-\lambda & -\frac{15}{2} \\ -\frac{1}{2} & -\lambda \end{vmatrix} = 0 \implies \lambda^2 + \lambda - \frac{15}{4} = 0$$

$$\implies \lambda = -\frac{5}{2}, \frac{3}{2}.$$

The corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ \frac{1}{5} \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{1}{3} \end{pmatrix}.$$

Using the first eigenvector, we see that as long as $S > 10000$, the solution curve will intersect the vertical axis where $P = 0$.