Remarks on Tutorial 2

Q2 Find function y(x) such that

Derivative at pt x

Derivative at pt t

$$\frac{dy}{dx} = \frac{\mu}{T}\int_0^x \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\,dt$$

i.e., solving the above **integral** equation

Need to change to solving **differential** equation

Let u(x)=dy/dx at pt x, then

$$u(x) = \frac{\mu}{T}\int_0^x \sqrt{u(t)^2 + 1}\,dt$$

Diff the eq wrt x, get

Note that u(0)=0 which is an initial condition for the following ODE

$$\frac{d}{dx}u(x) = \frac{\mu}{T}\sqrt{u^2(x)+1}$$

So now solving ODE

$$\frac{1}{\sqrt{u^2+1}}du = \frac{\mu}{T}dx$$

get $u = \frac{dy}{dx}$ Then integrate u, get y

Formulae: $$\int \frac{1}{\sqrt{u^2+a^2}}du = \sinh^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sinh(ax) = \frac{1}{a}\cosh(ax) + c$$

$$\cosh z = \frac{e^z + e^{-z}}{2} \qquad \sinh z = \frac{e^z - e^{-z}}{2} \qquad \tanh z = \frac{\sinh z}{\cosh z}$$

Q3 (i) $$\frac{dP}{dt} = C[M-P]$$

What does this constant C measure?

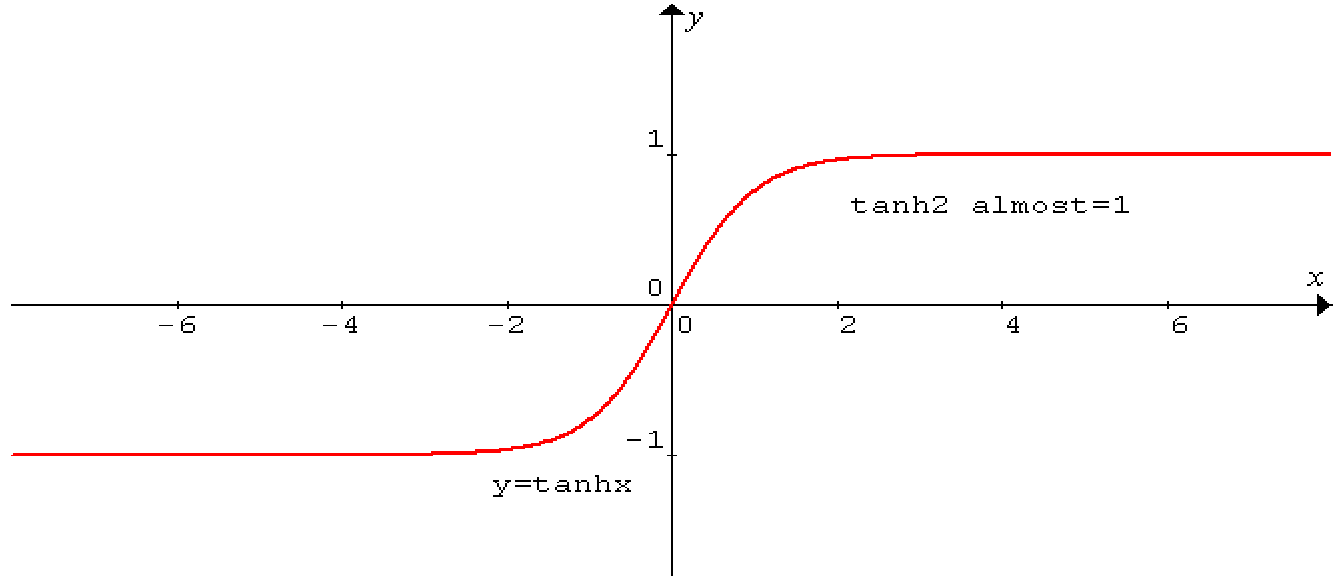
$$C = \frac{\frac{dP}{dt}}{M-P}$$

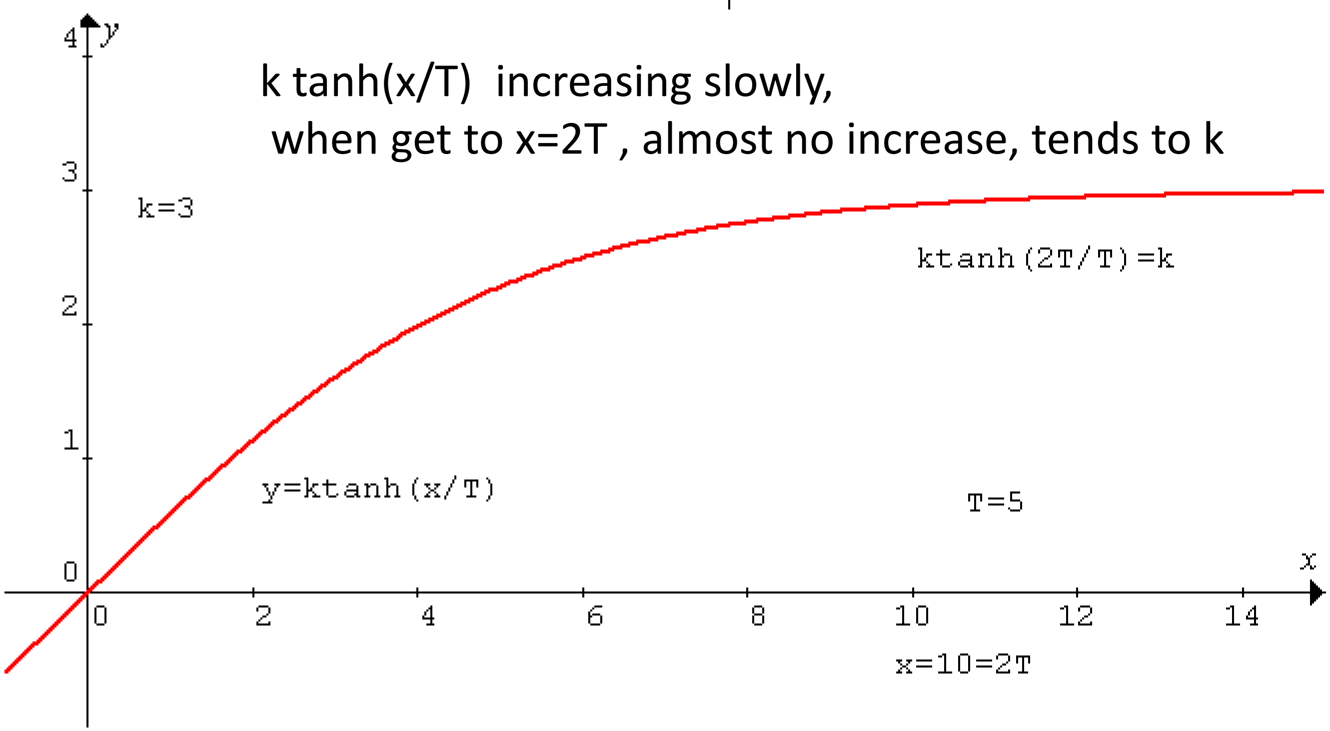
Write down $\dfrac{\frac{dP}{dt}}{M-P}$ in English

(ii) $$\frac{dP}{dt} = C(t)[M-P]$$ where $$C(t) = K \tanh\left(\frac{t}{T}\right)$$

Is it reasonable? What are the meanings of K and T?

To answer these questions,
it is good to look at the graph of tanh


y
1
tanh2 almost=1
0
x
−6
−4
−2
0
2
4
6
−1
y=tanhx


4
y
k tanh(x/T) increasing slowly,
when get to x=2T , almost no increase, tends to k
3
k=3
ktanh(2T/T)=k
2
1
y=ktanh(x/T)
T=5
0
x
0
2
4
6
8
10
12
14
x=10=2T

$$\tanh x \approx 1 \text{ when } x = 2$$

$$\int \tanh x dx = \int \frac{\sinh x}{\cosh x} dx = \int \frac{1}{\cosh x} d(\cosh x)$$

Q4 R(t)=# of students who have heard the rumour

Hence R(t) is a nonnegative integer,
so we can't differentiate the function R(t)

However we can construct a smooth curve
passing through those integer pts R(t)

This smooth curve is also denoted by R(t),
so in this Q, when we solve ODE, R(t) is a smooth curve.

$$\frac{dR}{dt} = KR(1500 - R)$$

dR/dt is small when R or (1500-R) is small