MA1506 Tutorial 3 Solution

Question 1

(a) The auxiliary equation is

$$\lambda^2 + 2\lambda - 3 = 0$$

$$(\lambda - 1)(\lambda + 3) = 0$$

Hence $\lambda = 1$ or $\lambda = -3$.

Therefore the general solution is $y = c_1 e^x + c_2 e^{-3x}$.

(b) The auxiliary equation is

$$\lambda^2 - 2\lambda + 1 = 0.$$

Hence $(\lambda - 1)^2 = 0$.

Thus $\lambda = 1$.

The general solution is $c_1 e^x + c_2 x e^x$.

(c) The auxiliary equation is

$$\lambda^2 + 4\lambda + 5 = 0$$

$$\begin{aligned}\lambda =& \frac{-4 \pm \sqrt{4^2 - 4 \times 5}}{2} \\ =& \frac{-4 \pm \sqrt{-4}}{2} \\ =& -2 \pm i\end{aligned}$$

Hence the general solution is

$$e^{-2x}(c_1 \cos x + c_2 \sin x).$$

(d)

The characteristic equation is

$$0 = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3).$$

The general solution is

$$y = c_1 e^{2x} + c_2 e^{-3x}.$$

Hence $y' = 2c_1 e^{2x} - 3c_2 e^{-3x}$.

By $y(0) = 0$, $y'(0) = 5$, we get

$$c_1 + c_2 = 0$$

$$2c_1 - 3c_2 = 5.$$

Hence $c_1 = 1$, $c_2 = -1$.

Therefore the solution is

$$y = e^{2x} - e^{-3x}.$$

2. (a) $y = (1+2x)e^{-3x}$ $(\lambda = -3, -3)$

(b) $y = -2e^{x}\cos 2\pi x + 3e^{x}\sin 2\pi x$

$(\lambda = 1 \pm 2\pi i)$

3 (a) $y_p = \frac{5}{2}x^2 - x$

(b) $y_p = (-x^2 - 2)e^{3x}$

Additional Question → (c) $y'' - y' = 2x\sin x$

Let $y_p = (Ax+B)\sin x + (Cx+D)\cos x$

$$y_p' = (A-D)\sin x + (B+C)\cos x + Ax\cos x - Cx\sin x$$

$$y_p'' = (-B-2C)\sin x + (2A-D)\cos x - Ax\sin x - Cx\cos x$$

$$\underline{2x\sin x} = y_p'' - y_p' = (\overbrace{2A - D - B - C}^{=0})\cos x$$
$$+ (\overbrace{-B - 2C - A + D}^{=0})\sin x$$
$$+ \underline{(\overbrace{C - A}^{=2})x\sin x} + (\overbrace{-C - A}^{=0})x\cos x$$

$\therefore A = -1,\ B = -2,\ C = 1,\ D = -1$

Question 4

The characteristic equation is

$$\lambda^2 + 4 = 0$$

Hence

$$\lambda = \pm 2i.$$

Therefore the general solution of $y'' + 4y = 0$ is $y_h = c_1 \cos 2t + c_2 \sin 2t$.

Let a particular solution of $y'' + 4y = \sin 2t$ be $y_p = V_1(t)\cos 2t + V_2(t)\sin 2t$.

Consequently

$$V_1(t) = -\frac{1}{4}t + \frac{1}{16}\sin 4t$$
$$V_2(t) = -\frac{1}{16}\cos 4t$$

Therefore

$$\begin{aligned} y_p &= V_1(t)\cos 2t + V_2(t)\sin 2t \\ &= \left(-\frac{1}{4}t + \frac{1}{16}\sin 4t\right)\cos 2t + \left(-\frac{1}{16}\cos 4t\right)\sin 2t \end{aligned}$$

Hence the general solution is

$$y = y_h + y_p = c_1 \cos 2t + c_2 \sin 2t + \left(-\frac{1}{4}t + \frac{1}{16}\sin 4t\right)\cos 2t + \left(-\frac{1}{16}\cos 4t\right)\sin 2t.$$