MA1506 TUTORIAL 4 SOLUTIONS

Question 1

(i) $\ddot{x} = \cosh(x)$. An equilibrium solution of an ODE is just a solution that is identically constant. That is not possible here because the cosh function never vanishes. So there is no equilibrium for this ODE.
(ii) $\ddot{x} = \cos(x)$. Equilibria are at $x = \pi/2$, $3\pi/2$, etc etc. Taylor expansion at $\pi/2$ is

$$\cos(x) = \cos'(\pi/2)[x - \pi/2] + ...$$

where we drop the higher order terms since we wish to consider a small perturbation. If we define $y = x - \pi/2$ then we have

$$\ddot{y} = -y$$

so we have simple harmonic motion [ie, stable equilibrium] with angular frequency [approximately] 1.

Taylor expansion at $3\pi/2$ is

$$\cos(x) = \cos'(3\pi/2)[x - 3\pi/2] + ...$$

where we drop the higher order terms since we wish to consider a small perturbation. If we define $y = x - 3\pi/2$ then we have

$$\ddot{y} \approx +y$$

so this equilibrium is unstable. The other equilibria are like these two; they alternate as we consider larger and smaller equilibrium values of x.
(iii) $\ddot{x} = \tan(\sin(x))$. Equilibria are at 0, π, 2π etc etc etc. Taylor expansion at 0 is

$$\tan(\sin(x)) = \cos(0)\sec^2(\sin(0))[x - 0] + ...$$

where we drop the higher order terms since we wish to consider a small perturbation. We have

$$\ddot{x} \approx +x$$

so we have an unstable equilibrium.

Taylor expansion at π is

$$\tan(\sin(x)) = \cos(\pi)\sec^2(\sin(\pi))[x - \pi] + ...$$

where we drop the higher order terms since we wish to consider a small perturbation. If we define $y = x - \pi$ then we have

$$\ddot{y} \approx -y$$

so we have simple harmonic motion [ie, stable equilibrium] with angular frequency approximately 1. The other equilibria are like these two; they alternate as we consider larger and smaller equilibrium values of x.

Question 2

If $\omega = 1$ then we have resonance [see the lecture notes, chapter 2]. So the relevant solution is

$$x = \frac{F_0 t}{2m\omega}\sin(\omega t).$$

In this case, F_0, the coefficient on the right side of the original ODE, is 1, and so are all of the other coefficients, so

$$x = \frac{t}{2}\sin(t).$$

The graph shows the uncontrolled growth characteristic of resonance in the absence of damping.

If ω is 0.9 then the relevant formula from the notes is

$$x = \frac{2F_0/m}{\alpha^2 - \omega^2}\sin\left[\left(\frac{\alpha-\omega}{2}\right)t\right]\sin\left[\left(\frac{\alpha+\omega}{2}\right)t\right].$$

Here we have

$$x = \frac{2}{(0.9)^2 - 1}\sin\left[\left(\frac{0.9-1}{2}\right)t\right]\sin\left[\left(\frac{0.9+1}{2}\right)t\right].$$

The graph is clearly the result of combining two sine waves, one with angular frequency 1/20 [and therefore with period 40π] and the other with angular frequency 0.95 [and therefore period a little more than 2π.] There are about 20 oscillations of one in each complete oscillation of the other. [Note: a complete oscillation of the "slow" wave encloses *two* of the "lumps" in the graph, so you have to be careful — don't measure the period by looking at the distance from one "lump" to the next. Confusingly, some physicists do that, getting a "beat frequency" twice the one we use. Please ignore the physicists.]

Question 3

The amplitude, as a function of the input frequency α, is given in Chapter 2 by

$$A(\alpha) = \frac{F_0/m}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}$$

Here F_0, m, and ω are to be regarded as fixed constants which determine the nature of the particular system. For sufficiently small values of the friction constant b, the shape of the graph of this function is as follows: it begins with a value of $F_0/m\omega$ at $\alpha = 0$, then it rises to a local maximum [this is the resonance situation] and then decreases monotonically towards zero. If b is too large, however, the function simply decreases monotonically from $F_0/m\omega$ — there is no resonance. In that case the maximum amplitude is just $F_0/m\omega$ at $\alpha = 0$.

Differentiating A with respect to α and setting the derivative equal to zero we get

$$4\alpha[\alpha^2 - \omega^2] + 2b^2\alpha/m^2 = 0.$$

Simplifying this we get

$$\alpha^2 = \omega^2 - \frac{b^2}{2m^2}.$$

Of course the left side cannot be negative, so if $b \geq \sqrt{2}m\omega$ then there is no resonance; this is the situation described above; in that case the maximum amplitude is at $\alpha = 0$ and is given by $F_0/m\omega$. Otherwise the maximal value of the amplitude is obtained by substituting this value of α into $A(\alpha)$. The result, after some simple algebra, is

$$A_{Resonance} = \frac{F_0/b\omega}{\sqrt{1 - (b^2/4m^2\omega^2)}}.$$

If $b^2/m^2\omega^2$ is negligible then this is approximately $F_0/b\omega$. That is, the resonance amplitude grows without limit as b becomes smaller.

Question 4

When the ship is at rest, the part of it which is under sea level has a volume of Ad [that is, the area of the base times the height]. Therefore, this is the volume of seawater that has been pushed aside by the ship. If the density of seawater is ρ, then the mass of seawater pushed aside is ρAd, and its weight is ρAdg. This upward force exactly balances the weight of the ship, so we have

$$\rho \, A \, d \, g \;=\; M \, g.$$

Thus

$$d \;=\; M/\rho \, A.$$

Now if the ship is moving and the distance from sea level to the bottom of the ship is d + x, where x is a function of time, we have to use Force = mass $\times$ acceleration. Taking the downwards direction to be positive, we find that the buoyancy force is now $-\rho A(d + x)g$, so we have

$$M\ddot{x} \;=\; Mg \;-\; \rho \, A \, (d + x)g,$$

which, using our formula for d, is just

$$\ddot{x} \;=\; -\frac{\rho \, A \, g}{M} \, x$$

This represents simple harmonic motion with angular frequency $\sqrt{\rho \, A \, g/M}$, as claimed. The ship will bob up and down at this frequency. Note the inverse dependence on M, which is to be expected, but also that the frequency increases if A is large, which is not so obvious.

Taking into account the force exerted by the waves, Force = mass $\times$ acceleration gives

$$M\ddot{x} \;=\; Mg \;-\; \rho \, A \, (d + x)g \;+\; F_0 \cos(\omega \, t)$$

or

$$M\ddot{x} \;+\; \rho \, A \, g \, x \;=\; F_0 \cos(\omega \, t).$$

This is exactly the equation studied in the notes when we studied resonance, except that k is replaced by ρAg. In the problem we are told that the input frequency [the frequency of the waves] is $\omega = \sqrt{\rho \, A \, g/M}$, the same as the natural frequency of the ship, so we do indeed have resonance here.

From the resonance notes we have [when x and its derivative are both zero at $t = 0$, which is also the case here]

$$x(t) = \frac{F_0 t}{2M\omega} \sin(\omega t).$$

Because of the factor of t, this will eventually be larger than any fixed number. As soon as it reaches the value H, this means that the ship has gone down by a greater distance than the height of the deck, which means that water washes over the deck and the ship sinks. So t_{sink} is the smallest positive solution of the equation

$$H = \frac{F_0 t_{\text{sink}}}{2M\omega} \sin(\omega t_{\text{sink}}).$$