<u>MA1506 Tutorial 1 Solutions</u>

(1a)
$$y' = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1} \to y = \ln\left|\frac{x}{x+1}\right| + c$$

(1b)
$$y' = \cos x \cos 5x = \frac{1}{2}\left[\cos 6x + \cos 4x\right] \to y = \frac{1}{2}\left[\frac{1}{6}\sin 6x + \frac{1}{4}\sin 4x\right] + c$$

(1c)
$$\frac{dy}{dx} = e^x e^{-3y} \Rightarrow e^{3y} dy = e^x dx \Rightarrow \frac{1}{3}e^{3y} = e^x + c$$

(1d)
$$\frac{1+y}{y^2} dy = (2x-1)dx \to \ln|y| - \frac{1}{y} = x^2 - x + c$$

In all these examples, the value of c is fixed by the given initial data. You can now use Graphmatica to graph these specific functions. If you feed the differential equations into Graphmatica with these data, you will get exactly the same graphs. [I checked!]

(2)
$\frac{dT}{dt} = -k(T - Tenv)$ where k is a positive constant. (If k were negative, then hot objects would get hotter when left to “cool”. That doesn’t happen – if it did, we would not be here to discuss it.) $T = Tenv$ is obviously a solution since $\frac{dTenv}{dt} = 0$ and this does make sense because objects do not spontaneously become hotter or colder. Having settled this case, we can assume $T \neq Tenv$ and so we can write
$\frac{dT}{T - Tenv} = -kdt$ so $\ln|T - Tenv| = -kt + c$.

In the case at hand, $T > Tenv$ so $|T - Tenv| = T - Tenv$ and so $T = Tenv + \alpha e^{-kt}$.

At $t = 0$, $T = 300$ so $300 = 75 + \alpha \Rightarrow \alpha = 225$. At $t = \frac{1}{2}$, $T = 200$, so
$$200 = 75 + 225e^{-k/2} \Rightarrow k = -2\ln\frac{125}{225} = 1.1756$$

Thus $T(3) = 75 + 225e^{-3k} \approx 81.6$.

(3)
The volume V is related to the area A by V = a $A^{(3/2)}$ where a is a positive constant with no units; this is reasonable because volume has units of cubic metres and area

has units of square metres. [Of course, ``reasonable" doesn't mean that it's always exactly true.] Then $dV/dt = (3a/2)(dA/dt)\ A^{(1/2)}$
The question tells us that $dV/dt = -bA$, where b is a positive constant with units of metres/sec. This is reasonable because evaporation takes place at the surface of the drop and so its rate can be expected to depend on the area. So we have $dA/\ A^{(1/2)}$ =

- $2bdt/3a$. Integrating from A_0, the initial area, up to zero, we find that the time taken for complete evaporation is $3aA_0^{(1/2)}/b$ which does indeed have units of time since the numerator has units of metres while the denominator has units of metres/sec.

If, instead of $dV/dt = -bA$, we propose that $dV/dt = -bA^2$, then we would have obtained $dA/\ A^{(3/2)} = -2bdt/3a$. When you try to integrate this from A_0 to zero, you will get a divergent integral, meaning that the evaporation would take infinite time and the rain would always reach the ground, contrary to the definition of Virga.

(4)
The moth flies in such a way that the angle ψ remains constant at all times, so we have a differential equation $\tan(\psi) = \text{constant} = rd\theta/dr$, hence $dr/r = d\theta / \tan(\psi)$, thus $r = R \exp(\theta / \tan(\psi))$, where we take it that $\theta = 0$ when the moth first sees the candle, and that her distance from the candle is R at that time. Remember that ψ is the angle between the radius vector of the moth [pointing outwards] and her velocity. From the point of view of the moth looking towards a candle in front of her, ψ would be an angle greater than 90 degrees. Draw a diagram if this is not obvious! Thus $\tan(\psi)$ will be negative and r will get steadily smaller as θ increases. Such a curve is called a spiral. So the unfortunate moth will spiral into the candle with tragic consequences. Of course if her first view of the candle is over her ``shoulder" then $\tan(\psi)$ will be positive and she will spiral outwards, something that would be a lot less noticeable. Finally if ψ is 90 degrees exactly, the moth will fly along a circle until it drops dead from exhaustion or starvation, whichever comes first.

(5)
These are examples of ODEs where a change of variable is needed.

(5a)
Let $v = 2x + y$ so $v' = 2 + y'$

$$\Rightarrow v' - 2 = \frac{1-2v}{1+v} \Rightarrow v' = \frac{3}{1+v}$$

$$\Rightarrow v + \frac{1}{2}v^2 = 3x + c$$

$$\Rightarrow (2x+y) + \frac{1}{2}(2x+y)^2 = 3x + c$$

(5b)

$$v = x + y \qquad y' = v' - 1 = \left(\frac{v+1}{v+3}\right)^2$$

$$\Rightarrow v' = 1 + \frac{v^2+2v+1}{(v+3)^2} = \frac{2v^2+8v+10}{(v+3)^2}$$

$$\Rightarrow \frac{(v+3)^2}{v^2+4v+5}dv = 2dx$$

$$\Rightarrow \frac{v^2+4v+5+2v+4}{v^2+4v+5}dv = 2dx$$

$$\Rightarrow \left(1+\frac{2v+4}{v^2+4v+5}\right)dv = 2dx$$

$$\Rightarrow v + \ln\left|\left(v^2+4v+5\right)\right| = 2x + c$$

$$\Rightarrow x + y + \ln\left|(x+y)^2 + 4x + 4y + 5\right| = 2x + c$$

(5c)

Here we need 2 changes of variables: first set $x = X + \alpha$, $y = Y + \beta$ so $x + y + 1 = X + Y + (\alpha + \beta + 1)$, $-x + y - 3 = -X + Y + (-\alpha + \beta - 3)$ so if we choose $\alpha + \beta + 1 = -\alpha + \beta - 3 = 0$ i.e. $\alpha = -2$, $\beta = 1$ then (i.e. $Y = y - 1$, $X = x + 2$)

$X + Y + (-X + Y)\frac{dY}{dX} = 0$ and this is homogeneous so set

$V = Y/X \Rightarrow Y' = V + XV'$

$$0 = 1 + V + (-1+V)(V + XV') \Rightarrow \frac{dX}{X} = \frac{1-V}{1+V^2}dV$$

$$\Rightarrow \ln|X| = \arctan V - \frac{1}{2}\ln\left|1+V^2\right| + c$$

$$\Rightarrow \ln|x+2| = \arctan\left(\frac{y-1}{x+2}\right) - \frac{1}{2}\ln\left(1+\left(\frac{y-1}{x+2}\right)^2\right) + c$$

since $V = \frac{Y}{X} = \frac{y-1}{x+2}$