MA1506 Tutorial 2 Solution

Question 1

$$\frac{dT}{dt} = -k(T - T_{env})$$

$$\frac{dT}{T - T_{env}} = -k\,dt$$

$$\int \quad \Rightarrow \quad \ln|T - T_{env}| = -kt + c$$

$$\Rightarrow \quad T - T_{env} = Ae^{-kt}$$

$$T(0) = 300 \Rightarrow 300 - 75 = A$$

$$\therefore T = 75 + 225e^{-kt}$$

$$T(\tfrac{1}{2}) = 200 \Rightarrow 200 = 75 + 225e^{-\frac{1}{2}k}$$

$$\Rightarrow e^{-\frac{1}{2}k} = \frac{125}{225} = \frac{5}{9}$$

$$\Rightarrow k = 2\ln\frac{9}{5}$$

$$\therefore T = 75 + 225e^{-(2\ln\frac{9}{5})t}$$

$$\therefore T(3) = 75 + 225e^{-6\ln\frac{9}{5}}$$

$$\approx 81.6^\circ F$$

Question 2

Let V = volume of the raindrop and A = surface area of the raindrop. Then

$$\text{Vol} \sim (\text{Surface area})^{3/2} \quad ---- \; (1)$$

$$\frac{dV}{dt} \sim A \quad ----- \; (2)$$

$$(1) \Rightarrow \quad V = aA^{3/2} \quad ---- \; (3) \qquad a > 0$$

$$(2) \Rightarrow \quad \frac{dV}{dt} = -bA \quad ---- \; (4) \qquad b > 0$$

$$\therefore (3) \Rightarrow \quad \frac{dV}{dt} = a\,\frac{3}{2}A^{\frac{1}{2}}\frac{dA}{dt}$$

combine with (4) $\Rightarrow \frac{3}{2}aA^{\frac{1}{2}}\frac{dA}{dt} = -bA$

$$\Rightarrow \frac{dA}{dt} = \frac{-2\sqrt{A}\,b}{3a}$$

$$\Rightarrow \frac{dA}{\sqrt{A}} = \frac{-2b}{3a}\,dt$$

$$\Rightarrow 2\sqrt{A} = \frac{-2b}{3a}t + C$$

Let $A(0) = A_0$

$$\therefore C = 2\sqrt{A_0}$$

$$\therefore \underline{\underline{\sqrt{A} = \sqrt{A_0} - \frac{b}{3a}t}} \quad ---- \; (5)$$

$$A = 0 \Rightarrow \quad \underline{\underline{t = \frac{3a}{b}\sqrt{A_0}}} \quad ---- \; (6)$$

Question 3

Starting with the differential equation $dP/dt = kP(200-P)$, we separate variables and integrate, noting that $P < 200$ because $P_0 = 100$:

$$\int \frac{dP}{P(200-P)} = \int k\,dt \quad \Rightarrow \quad \int \left(\frac{1}{P} + \frac{1}{200-P}\right) dP = \int 200k\,dt;$$

$$\ln \frac{P}{200-P} = 200kt + \ln C \quad \Rightarrow \quad \frac{P}{200-P} = Ce^{200kt}.$$

Now $P(0) = 100$ gives $C = 1$, and $P'(0) = 1$ implies that $1 = k \cdot 100(200-100)$, so we find that $k = 1/10000$. Substitution of these numerical values gives

$$\frac{P}{200-P} = e^{200t/10000} = e^{t/50},$$

and we solve readily for $P(t) = 200/\left(1+e^{-t/50}\right)$. Finally, $P(60) = 200/\left(1+e^{-6/5}\right) \approx 153.7$ million

Question 4

(a) $x' = 0.8x - 0.004x^2 = 0.004x(200-x)$, so the maximum amount that will dissolve is $M = 200$ g.

(b) With $M = 200$, $P_0 = 50$, and $k = 0.004$, Equation (4) in the text yields the solution

$$x(t) = \frac{10000}{50 + 150e^{-0.08t}}.$$

Substituting $x = 100$ on the left, we solve for $t = 1.25 \ln 3 \approx 1.37$ sec.

Note. (a) can also be solved by saying that x(t)=0 and x(t)=200 are equilibrium solutions. Using the phase line diagram, we know that x(t)=0 is unstable but x(t)=200 is stable. Thus the maximum amount of salt ever dissolves (i.e., eventually) is 200g.

Question 5

$$\frac{dR}{dt} - 1300KR = -KR^2$$

Let $z = R^{1-2} = R^{-1}$

$$\therefore \frac{dz}{dt} = -R^{-2}\frac{dR}{dt}$$

i.e. $$\frac{dR}{dt} = -R^2\frac{dz}{dt}$$

$$\therefore -R^2\frac{dz}{dt} - 1300KR = -KR^2$$

$$\frac{dz}{dt} + 1300KR^{-1} = K$$

$$\frac{dz}{dt} + 1300Kz = K$$

Integrating factor

$$= e^{\int 1300K\,dt} = e^{1300Kt}$$

$$z = e^{-1300Kt} \int e^{1300Kt} K\,dt$$

$$= e^{-1300Kt} \left\{ \frac{1}{1300} e^{1300Kt} + c \right\}$$

$$= \frac{1}{1300} + c e^{-1300Kt}$$

$$\therefore \frac{1}{R} = \frac{1}{1300} + c e^{-1300Kt}$$

$\because$ the rumour was started by one student

$\therefore R(0) = 1.$

i.e. $1 = \frac{1}{1300} + c$

$$c = 1 - \frac{1}{1300} = \frac{1299}{1300}$$

$$\therefore \frac{1}{R} = \frac{1}{1300} + \frac{1299}{1300} e^{-1300Kt}$$