CHAPTER 5

MATRICES AND THEIR USES

5.1 What is a Matrix?

A system of linear algebraic equations in two variables might look like this:

$$2x + 7y = 3$$

$$4x + 8y = 11$$

→ LINEAR because it just involves constant multiples of x and y, no x^2, no $\sin(y)$, etc.

→ ALGEBRAIC because no differentiation.

It's cool to write these systems using the following notation:

$$\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}.$$

Here $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 11 \end{bmatrix}$ are familiar - they are VEC-

TORS. But $\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix}$ is something new, called a MATRIX. We say that the PRODUCT of $\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix}$ with $\begin{bmatrix} x \\ y \end{bmatrix}$ gives you $\begin{bmatrix} 3 \\ 11 \end{bmatrix}$.

Every matrix has ROWS and COLUMNS. In this case, the rows are $[2 \quad 7]$ and $[4 \quad 8]$ and the columns are $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 7 \\ 8 \end{bmatrix}$. We call $[2 \quad 7]$ a ROW VECTOR and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ a column vector. We say that $\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix}$ is a 2 by 2 matrix since it has two rows and two columns. You can regard $[2 \quad 7]$ as having one row and 2 columns, etc. You can also have 3 by 3 matrices like $\begin{bmatrix} 1 & 7 & 9 \\ 7 & 8 & 2 \\ 4 & 10 & 12 \end{bmatrix}$ or even 2 by 3 matrices like $\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 9 \end{bmatrix}$ two rows, three columns.

A general 3 by 3 matrix can be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

so a_{ij} is the number in the i-th row and j-th column, Note $a_{ij} \neq a_{ji}$ usually!

Engineers and physicists like to talk about "the matrix a_{ij}". Strictly speaking, they mean "the matrix with entries a_{ij}" but we will talk in this sloppy way too! In the same way, any column vector can be written as $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$.

5.2 Matrix Arithmetic

[a] Addition and Subtraction.

Just add up or subtract the entries, as you would for a vector.

$$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 3 \\ 6 & 9 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 10 & 17 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} - \begin{bmatrix} 7 & 3 \\ 6 & 9 \end{bmatrix} = \begin{bmatrix} -6 & -1 \\ -2 & -1 \end{bmatrix}$$

In general, if a_{ij} and b_{ij} are matrices (both m by n, that is, both have m rows and n columns) then the sum is $a_{ij} + b_{ij}$ and the difference is $a_{ij} - b_{ij}$.

[b] Multiplying By a Number.

Just multiply every entry, as you would for a vector. $2 \cdot \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 16 \end{bmatrix}$. The product of the number c with the matrix a_{ij} is $c \cdot a_{ij}$.

[c] Transposition.

If you take a matrix and SWITCH THE FIRST ROW INTO THE FIRST COLUMN, second row into second column, and so on, the result is called the TRANSPOSE. We write $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 7 & 9 \\ 6 & 8 & 2 \\ 4 & 10 & 12 \end{bmatrix}^T = \begin{bmatrix} 1 & 6 & 4 \\ 7 & 8 & 10 \\ 9 & 2 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 6 \\ 2 & 8 \\ 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

and by looking at this example you can see

$a_{ij}^T = a_{ji} \rightarrow$ the order of the indices is reversed.

Notice that

$$\begin{aligned} \left((a_{ij})^T\right)^T &= (a_{ji})^T = a_{ij} \\ (a_{ij} + b_{ij})^T &= a_{ji} + b_{ji} = a_{ij}^T + b_{ij}^T \\ (c\, a_{ij})^T &= c\, a_{ji} = c\, (a_{ij})^T . \end{aligned}$$

[d] Multiplying Matrices.

We started by declaring that it was cool to write

$\begin{array}{c} 2x + 7y = 3 \\ 4x + 8y = 11 \end{array}$ as $\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$. Clearly this is a way of saying that the vector $\begin{bmatrix} 2x+7y \\ 4x+8y \end{bmatrix}$ equals $\begin{bmatrix} 3 \\ 11 \end{bmatrix}$, so $\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+7y \\ 4x+8y \end{bmatrix}$. Notice that ROWS of $\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix}$ multiply the COLUMN $\begin{bmatrix} x \\ y \end{bmatrix}$. We adopt this as our GENERAL RULE:

$$\boxed{\text{ROWS MULTIPLY COLUMNS!}}$$

$$\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 4+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+7 \\ 0+8 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 16 & -1 \\ 20 & 4 \end{bmatrix}$$

$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 11$, a 1 by 1 matrix! Also called a NUMBER!

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$= \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ so we have

$c_{11} = a_{11}b_{11} + a_{12}b_{21} = \sum_j a_{1j}b_{j1}$

$c_{12} = a_{11}b_{12} + a_{12}b_{22} = \sum_j a_{1j}b_{j2}$

$c_{21} = a_{21}b_{11} + a_{22}b_{21} = \sum_j a_{2j}b_{j1}$

$c_{22} = a_{21}b_{12} + a_{22}b_{22} = \sum_j a_{2j}b_{j2}$

Can you see the pattern?

$$c_{mn} = \sum_j a_{mj}b_{jn}.$$

This is true for all matrices, not just 2 by 2 matrices.

NOTE that

$\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 16 & -1 \\ 20 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 14 & 31 \\ 0 & 6 \end{bmatrix}$ completely different!

So the ORDER OF MATRIX MULTIPLICATION

is IMPORTANT. If A and B are matrices, USUALLY $AB \neq BA$.

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[e] Transposition and Matrix Multiplication.

According to our rules,

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}.$$

But

$$\begin{bmatrix} 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 & 1 \end{bmatrix}^T,$$

$$\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}^T \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix}^T.$$

In general, if A and B are matrices of any kind, the rule is

$$(AB)^T = B^T A^T$$

DON'T FORGET TO REVERSE THE ORDER!

A matrix is said to be SYMMETRIC if

$$A^T = A.$$

$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 16 \\ 16 & 10^9 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are all symmetric. Any matrix of the form $B + B^T$, where B is ANY matrix, is symmetric. [Proof: $(B + B^T)^T = B^T + (B^T)^T = B^T + B$.] If A is symmetric, so is BAB^T for any B [Proof: $(BAB^T)^T = (B^T)^T A^T B^T = BA^T B^T = BAB^T$.] A matrix is said to be ANTISYMMETRIC if

$$A^T = -A.$$

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 16 \\ -16 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are all antisymmetric. Any matrix of the form $B - B^T$ is antisymmetric, and BAB^T is antisymmetric if A is antisymmetric.

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[f] SCALAR AND VECTOR PRODUCTS IN TERMS OF MATRICES.

You are familiar with the scalar or dot product,

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 .$$

This is actually a MATRIX PRODUCT, because you can write it as

$$\begin{aligned} \vec{u}^{\,T} \vec{v} &= \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ &= u_1 v_1 + u_2 v_2 + u_3 v_3 = \vec{u} \cdot \vec{v} . \end{aligned}$$

Thus, in particular, the length of a vector can be expressed as

$$\left| \vec{u} \right| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{\vec{u}^{\,T} \vec{u}} .$$

You are also familiar with the VECTOR or CROSS product of two vectors, $\vec{u} \times \vec{v}$. This is also a matrix product! To see this, notice that any three-dimensional vector can be used to define an anti-symmetric three by three matrix as follows:

$$\overrightarrow{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ defines } \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}.$$

Let's call this matrix A. Then you can check that, for any vector $\overrightarrow{v}$,

$$\overrightarrow{u} \times \overrightarrow{v} = A\overrightarrow{v}.$$

So the vector product is really just a special kind of matrix multiplication. Notice that A is always antisymmetric.

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[g] ORTHOGONAL MATRICES.

A matrix B is said to be ORTHOGONAL if it satisfies

$$B^T B = I,$$

where I is the IDENTITY MATRIX, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in two dimensions, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ in three, etc. Note that $IA = A = AI$ for any matrix A. In two dimensions, $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal for any θ. Since

$$
\begin{aligned}
&\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Another example is $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$.

5.3 Application: Markov Chains.

Let's construct a simple MODEL of weather forecasting. We assume that each day is either RAINY or SUNNY.

Rainy today $\rightarrow$ probably rainy tomorrow (probability 60%).

Sunny today $\rightarrow$ probably sunny tomorrow (probability 70%).

Since probabilities have to add up to 100%, you can easily see that Rainy $\rightarrow$ Sunny has probability 40% and Sunny $\rightarrow$ Rainy has probability 30%. We can organise these data into a matrix

$$\begin{aligned} M &= \begin{bmatrix} \text{Rainy} \rightarrow \text{Rainy} & \text{Sunny} \rightarrow \text{Rainy} \\ \text{Rainy} \rightarrow \text{Sunny} & \text{Sunny} \rightarrow \text{Sunny} \end{bmatrix} \\ &= \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}. \end{aligned}$$

Question: Suppose today is sunny. What is the probability that it will be rainy 4 days from now? To see how to proceed, we make a "tree" like this: [R = rain, S = sun] for the first two days:

We have, similarly, that the probability of Rainy $\rightarrow$ Sunny over 2 days is

$$RS_2 = 0.6 \times 0.4 + 0.4 \times 0.7$$

By constructing a tree starting with S, you will find that the probability of rain 2 days after a sunny day is

$$SR_2 = 0.6 \times 0.3 + 0.3 \times 0.7$$

and similarly

$$SS_2 = 0.4 \times 0.3 + 0.7 \times 0.7$$

So now the matrix of probabilities is

$$\begin{bmatrix} RR_2 & SR_2 \\ RS_2 & SS_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0.6 \times 0.6 + 0.3 \times 0.4 & 0.6 \times 0.3 + 0.3 \times 0.7 \\ 0.6 \times 0.4 + 0.4 \times 0.7 & 0.4 \times 0.3 + 0.7 \times 0.7 \end{bmatrix}.$$

But this is exactly $\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} = M^2.$

So matrix multiplication actually allows you to compute all of the probabilities in this “Markov Chain”. To predict the weather 4 days from now, we need

$$\begin{bmatrix} RR_4 & SR_4 \\ RS_4 & SS_4 \end{bmatrix} = M^4 = M^2M^2 = \begin{bmatrix} 0.43 & 0.43 \\ 0.57 & 0.57 \end{bmatrix}.$$

So if it is rainy today, the probability of rain in 4 days is 0.43=43%. If you want 20 days, just compute M^{20}. A very complicated problem without matrix multiplication!

5.4 Application: Leontief Model of Manufacturing

The Leontief model describes the economics of INTERDEPENDENT companies. For example, the electric company MUST sell electricity to the factory that makes generators, which in turn MUST sell generators to the electric company. Let x be the number of dollars' worth of electricity generated, and let y be the number of dollars' worth of generators made by the factory. Assume

[a] The electric company has to sell \$150 of electricity to the city, and the generator factory wants to sell \$100 to outsiders.

[b] Each dollar of electricity costs 30 cents to make [fuel].

[c] Each dollar's worth of generator needs 40 cents of electricity.

[d] Each dollar's worth of generator costs 30 cents [parts].

[e] Each dollar's worth of electricity needs 50 cents' worth of generator.

Then x splits into 3 parts:

0.3 x for fuel

0.4 y goes to the generator factory

150 goes for sale

So we have

$x = 0.3x + 0.4y + 150$

and similarly

$y = 0.5x + 0.3y + 100$

or

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.3 & 0.4 \\ 0.5 & 0.3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 150 \\ 100 \end{bmatrix}$$

$$\overrightharpoon{u} = T\overrightharpoon{u} + \overrightharpoon{c}$$

where $\overrightharpoon{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, $T = \begin{bmatrix} 0.3 & 0.4 \\ 0.5 & 0.3 \end{bmatrix}$, $\overrightharpoon{c} = \begin{bmatrix} 150 \\ 100 \end{bmatrix}$.

The matrix T contains all of the INTERNAL INFORMATION about the two companies and their contractual relationship. It is called the TECHNOLOGY MATRIX. An easy way to construct the matrix is as follows: think of the COLUMNS as the costs of making a unit of something. For example, in this case the first column refers to electricity, so the costs of making one dollar of electricity go down that column [30 cents to MAKE one dollar of electricity, and 50 cents to pay for the generator needed to MAKE one dollar of electricity.] Similarly the costs of making one dollar's worth of generator go down the second column. In a more complicated problem the matrix may be much larger, but this idea of using the columns to keep track of costs will always work.

Write $\overrightarrow{u} = I\overrightarrow{u}$ where I is the identity matrix. Then

$$(I - T)\overrightarrow{u} = \overrightarrow{c}.$$

Now if these were NUMBERS, I could easily solve this for $\overrightarrow{u}$ $\rightarrow$ just divide both sides by $I - T$, or multiply both sides by $(I-T)^{-1}$. But can we do that for matrices? In our case, $I - T = \begin{bmatrix} 0.7 & -0.4 \\ -0.5 & 0.7 \end{bmatrix}$. CAN WE FIND A MATRIX S such that $S(I-T) = I$?? Actually you can $\rightarrow$ $S = \frac{1}{29}\begin{bmatrix} 70 & 40 \\ 50 & 70 \end{bmatrix}$ Check it!

So multiplying both sides of our equation by S, we get

$$S(I-T)\overrightarrow{u} = S\overrightarrow{c} \rightarrow \overrightarrow{u} = \frac{1}{29}\begin{bmatrix} 70 & 40 \\ 50 & 70 \end{bmatrix}\begin{bmatrix} 150 \\ 100 \end{bmatrix}$$

$\rightarrow \overrightarrow{u} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 500 \\ 500 \end{bmatrix} \rightarrow x = y = \$500$, both companies should produce \$500 worth of their products

\$500 electricity = \$150 fuel + \$200 to factory + \$150 sold.

\$500 generators = \$150 parts + \$250 to electric + \$100 sold.

Given a matrix M, how do you find another matrix S such that $SM = I$? We need a systematic way of doing that! Come back to this later.