CHAPTER 6
LINEAR TRANSFORMATIONS

6.1 WHAT IS A LINEAR TRANSFORMATION?

You know what a function is - it's a RULE which turns NUMBERS INTO OTHER NUMBERS: $f(x) = x^2$ means "please turn 3 into 9, 12 into 144 and so on".

Similarly a TRANSFORMATION is a rule which turns VECTORS into other VECTORS. For example, "please rotate all 3-dimensional vectors through an angle of 90° clockwise around the z-axis". A LINEAR TRANSFORMATION T is one that ALSO satisfies these rules: if c is any scalar, and $\overrightarrow{u}$ and $\overrightarrow{v}$ are vectors, then

$$T(c\overrightarrow{u}) = cT(\overrightarrow{u}) \text{ and } T(\overrightarrow{u} + \overrightarrow{v}) = T(\overrightarrow{u}) + T(\overrightarrow{v}).$$

EXAMPLE: Let I be the rule $I\overrightarrow{u} = \overrightarrow{u}$ for all $\overrightarrow{u}$. You can check that I is linear! Called IDENTITY Linear Transformation.

EXAMPLE : Let D be the rule $D\overrightarrow{u} = 2\overrightarrow{u}$ for all $\overrightarrow{u}$.

$D(c\overrightarrow{u}) = 2(c\overrightarrow{u}) = c(2\overrightarrow{u}) = cD\overrightarrow{u}$

$D(\overrightarrow{u} + \overrightarrow{v}) = 2(\overrightarrow{u} + \overrightarrow{v}) = 2\overrightarrow{u} + 2\overrightarrow{v} = D\overrightarrow{u} + D\overrightarrow{v} \rightarrow$ LINEAR!

Note: Usually we write $D(\overrightarrow{u})$ as just $D\overrightarrow{u}$.

6.2. THE BASIC BOX, AND THE MATRIX OF A LINEAR TRANSFORMATION

The usual vectors $\hat{i}$ and $\hat{j}$ define a square:

Let's call this the BASIC BOX in two dimensions.

Similarly, $\hat{i}, \hat{j}$, and $\hat{k}$ define the BASIC BOX in 3 dimensions.

Now let T be any linear transformation. You know that any 2-dimensional vector can be written as $a\hat{i} + b\hat{j}$, for some numbers a and b. So for any vector, we have

$$T(a\hat{i} + b\hat{j}) = aT\hat{i} + bT\hat{j}.$$

This formula tells us something very important: IF I KNOW WHAT T DOES TO $\hat{i}$ and $\hat{j}$, THEN I KNOW EVERYTHING ABOUT T - because now I can tell you what T does to ANY vector.

EXAMPLE: Suppose I know that $T(\hat{i}) = \hat{i} + \frac{1}{4}\hat{j}$ and $T(\hat{j}) = \frac{1}{4}\hat{i} + \hat{j}$. Then what is $T(2\hat{i} + 3\hat{j})$?

Answer: $T(2\hat{i} + 3\hat{j}) = 2T\hat{i} + 3T\hat{j} = 2\left(\hat{i} + \frac{1}{4}\hat{j}\right) + 3\left(\frac{1}{4}\hat{i} + \hat{j}\right) = 2\hat{i} + \frac{1}{2}\hat{j} + \frac{3}{4}\hat{i} + 3\hat{j} = \frac{11}{4}\hat{i} + \frac{7}{2}\hat{j}$.

Since $T\hat{i}$ and $T\hat{j}$ tell me everything I need to know, this means that I can tell you everything about T by telling you WHAT IT DOES TO THE BASIC BOX.

EXAMPLE: Let T be the same transformation as above, $T(\hat{\imath}) = \hat{\imath} + \frac{1}{4}\hat{\jmath}$ and $T(\hat{\jmath}) = \frac{1}{4}\hat{\imath} + \hat{\jmath}$.

The basic box has been squashed a bit! Pictures of WHAT T DOES TO THE BASIC BOX tell us everything about T!

EXAMPLE: If D is the transformation $D\overrightarrow{u} = 2\overrightarrow{u}$, then the Basic Box just gets stretched:

So every LT can be pictured by

seeing what it does to the Basic Box.

There is another way!

Let $T\hat{\imath} = \begin{bmatrix} a \\ c \end{bmatrix}$ and $T\hat{\jmath} = \begin{bmatrix} b \\ d \end{bmatrix}$. Then we DEFINE

THE MATRIX OF T RELATIVE TO $\hat{\imath}, \hat{\jmath}$ as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, that is, the first COLUMN tells us what happened to $\hat{\imath}$, and the second column tells us what happened to $\hat{\jmath}$.

EXAMPLE: Let I be the identity transformation. Then $I\hat{\imath} = \hat{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $I\hat{\jmath} = \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so the matrix of the identity transformation relative to $\hat{\imath}\hat{\jmath}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

EXAMPLE: If $D\overrightarrow{u} = 2\overrightarrow{u}$, then $D\hat{\imath} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $D\hat{\jmath} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ so the matrix of D relative to $\hat{\imath}, \hat{\jmath}$ is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

EXAMPLE: If $T\hat{\imath} = \hat{\imath} + \frac{1}{4}\hat{\jmath}$ and $T\hat{\jmath} = \frac{1}{4}\hat{\imath} + j$, then the matrix is $\begin{bmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & 1 \end{bmatrix}$.

EXAMPLE: If $T\hat{\imath} = \hat{\jmath}$ and $T\hat{\jmath} = \hat{\imath}$, the matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Basic box is REFLECTED

EXAMPLE: Suppose in 3 dimensions $T\hat{i} = \hat{i} + 4\hat{j} + 7\hat{k}$, $T\hat{j} = 2\hat{i} + 5\hat{j} + 8\hat{k}$, $T\hat{k} = 3\hat{i} + 6\hat{j} + 9\hat{k}$, then the matrix is $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, relative to $\hat{i}\hat{j}\hat{k}$.

EXAMPLE: Suppose $T\hat{i} = \hat{i} + \hat{j} + 2\hat{k}$ and $T\hat{j} = \hat{i} - 3\hat{k}$. This is an example of a LT that eats 2-dimensional vectors but PRODUCES 3-dimensional vectors. But it still has a matrix, $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 2 & -3 \end{bmatrix}$. It's just that this matrix is not a SQUARE MATRIX, that is, it is not 2 by 2 or 3 by 3. Instead it is 3 by 2.

We shall say that a linear transformation is a 2-dimensional L.T. if it eats 2-dimensional vectors AND produces 2-dimensional vectors. A 2-dimensional L.T. has a square, 2 by 2 matrix relative to $\hat{i}, \hat{j}$. Similarly a 3-dimensional linear transformation has a 3 by 3 matrix. In Engineering applications, most linear transformations are 2-dimensional or 3-dimensional, so we are mainly interested in these two cases.

EXAMPLE: Suppose T is a linear transformation that eats 3-dimensional vectors and produces 2-dimensional vectors according to the rule $T\hat{\imath} = 2\hat{\imath}$, $T\hat{\jmath} = \hat{\imath} + \hat{\jmath}$, $T\hat{k} = \hat{\imath} - \hat{\jmath}$. What is its matrix?

Answer: $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$, a 2 by 3 matrix.

EXAMPLE: Suppose, in solid mechanics, you take a flat square of rubber and SHEAR it, as shown.

In other words, you don't

change its volume,

you just push it like a pack

of cards. The base stays fixed but the top moves a distance $\tan(\theta)$. (The height remains the same, 1 unit.) Clearly the shearing transformation S satisfies $S\hat{\imath} = \hat{\imath}$, $S\hat{\jmath} = \hat{\imath}\tan\theta + \hat{\jmath}$, so the matrix of S relative to $\hat{\imath}, \hat{\jmath}$ is $\begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}$.

EXAMPLE: Suppose $Ti = \hat{i} + \hat{j}$ and $T\hat{j} = \hat{i} + \hat{j}$. Matrix is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and basic box is SQUASHED FLAT!

EXAMPLE: Rotations in the plane. Suppose you ROTATE the whole plane through an angle θ (anti-clockwise). Then simple trigonometry shows you that

$R\hat{i} = \cos\theta\hat{i} + \sin\theta\hat{j}$

$R\hat{j} = -\sin\theta i + \cos\theta\hat{j}$

So the rotation matrix is

$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Application: Suppose an object is moving on a circle at constant angular speed ω. What is its acceleration?

Answer: Let its position vector at $t = 0$ be $\vec{r_0}$. Because the object is moving on a circle, its position at a later time t is given by rotating $\vec{r_0}$ by an angle $\theta(t)$. So

$$\vec{r}(t) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{r_0}$$

Differentiate

$\frac{d\vec{r}}{dt} = \dot{\theta} \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} \vec{r_0}$ by the chain rule. Here $\dot{\theta}$ is actually ω, so
$\frac{d\vec{r}}{dt} = \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} \omega\vec{r_0}$. Differentiate again,

$$\frac{d^2\vec{r}}{dt^2} = \begin{bmatrix} -\cos\theta & \sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix} \omega^2\vec{r_0}$$

.

$$= -\omega^2 \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{r_0}$$

Substitute the equation for $\overrightarrow{r}(t)$,

$$\frac{d^2\overrightarrow{r}}{dt^2} = -\omega^2\overrightarrow{r},$$

which is formula you know from physics.

6.3. COMPOSITE TRANSFORMATIONS AND MATRIX MULTIPLICATION.

You know what it means to take the COMPOSITE of two functions: if $f(u) = \sin(u)$, and $u(x) = x^2$, then $f \circ u$ means: "please do u FIRST, THEN f, so $f \circ u(x) = \sin(x^2)$. NOTE THE ORDER!!
$u \circ f(x) = \sin^2(x)$, NOT the same!

Similarly if A and B are linear transformations, then AB means "do B FIRST, then A".

NOTE: BE CAREFUL! According to our definition, A and B both eat vectors and both produce vectors. But then you have to take care that A can eat what

B produces!

EXAMPLE: Suppose A eats and produces 2-dimensional vectors, and B eats and produces 3-dimensional vectors. Then "AB" would not make sense!

EXAMPLE: Suppose B eats 2-d vectors and produces 3-d vectors (so its matrix relative to $\hat{i}\hat{j}\hat{k}$ looks like this: $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$, a 3 by 2 matrix) and suppose A eats 3-d vectors and produces 2-d vectors. Then AB DOES make sense, because A can eat what B produces. (In this case, BA also makes sense.).

IMPORTANT FACT: Suppose a_{ij} is the matrix of a linear transformation A relative to $\hat{i}\hat{j}\hat{k}$, and suppose b_{ij} is the matrix of the Linear Transformation B relative to $\hat{i}\hat{j}\hat{k}$. Suppose that AB makes sense. Then the matrix of AB relative to $\hat{i}\hat{j}$ or $\hat{i}\hat{j}\hat{k}$ is just the matrix product of a_{ij} and b_{ij}.

EXAMPLE: What happens to the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ if we shear 45° parallel to the x axis and then rotate 90° anticlockwise? What if we do the same in the reverse order?

Answer: Shear

$$\to \begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}$$

so in this case it is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. A rotation through θ has matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, so here it is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
Hence

SHEAR, THEN ROTATE

$$\to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

ROTATE, THEN SHEAR

$$\to \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

So shear, then rotate

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

Rotate, then shear

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Very different!

EXAMPLE: Suppose B is a LT with matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$ and A is a LT with matrix $\begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. What is the matrix of AB? Of BA?

Answer:

$$\underset{2 \text{ by } 3}{\begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}} \underset{3 \text{ by } 2}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}} = \underset{2 \text{ by } 2}{\begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}} = AB$$

$$\underset{3 \text{ by } 2}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}} \underset{2 \text{ by } 3}{\begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}} = \underset{3 \text{ by } 3}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}}$$

EXAMPLE: Suppose you take a piece of rubber in 2 dimensions and shear it parallel to the x axis by θ degrees, and then shear it again by ϕ degrees. What happens?

$$\begin{bmatrix} 1 & \tan\phi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \tan\theta + \tan\phi \\ 0 & 1 \end{bmatrix}$$

which is also a shear, but NOT through $\theta + \phi$!

The shear angles don't add up, since $\tan\theta + \tan\theta \neq \tan(\theta + \phi)$.

EXAMPLE: Rotate $90°$ around z-axis, then rotate $90°$ around x-axis in 3 dimensions. [Always anti-clockwise unless otherwise stated.] Is it the same if we reverse the order? Rotate about z axis $\rightarrow$ i becomes j, j becomes $-i, k$ stays the same, so $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Rotate about x axis, i stays the same,

j becomes k, k becomes $-j$, so $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\neq \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

so the answer is NO!

6.4 DETERMINANTS

You probably know that the AREA of the box defined by two vectors is

$|\vec{u} \times \vec{v}|$, magnitude

of the vector product.

If you don't know it, you can easily check it, since the area of any parallelogram is given by

$$
\begin{aligned}
AREA &= HEIGHT \times Base \\
&= \left(|\overrightarrow{v}|\sin\theta\right) \times |\overrightarrow{u}| \\
&= |\overrightarrow{u}|\ |\overrightarrow{v}|\sin\theta \\
&= |\overrightarrow{u} \times \overrightarrow{v}|.
\end{aligned}
$$

Similarly, the VOLUME of a "three-dimensional parallelogram" [called a PARALLELOPIPED!] is given by

$$\text{VOLUME} = (\text{AREA OF BASE}) \times \text{HEIGHT}.$$

If you take any 3 vectors in 3 dimensions, say $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$, then they define a 3-dimensional parallelogram. The area of the base is $|\overrightarrow{u} \times \overrightarrow{v}|$,

height is $|\overrightarrow{w}|\ |\sin\left(\frac{\pi}{2} - \theta\right)|$

where θ is the angle between

$\overrightarrow{u} \times \overrightarrow{v}$ and $\overrightarrow{w}$, so VOLUME

defined by $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ is just

$$
\begin{aligned}
&|\overrightarrow{u} \times \overrightarrow{v}| \;\; |\overrightarrow{w}| \;\; |\sin\left(\frac{\pi}{2} - \theta\right)| \\
=&|\overrightarrow{u} \times \overrightarrow{v}| \;\; |\overrightarrow{w}| \;\; |\cos\theta| \\
=&|\overrightarrow{u} \times \overrightarrow{v} \cdot \overrightarrow{w}|.
\end{aligned}
$$

[Check: Volume of Basic Box defined by $\hat{i}\hat{j}\hat{k}$ is $|\hat{i} \times \hat{j} \cdot \hat{k}| = |\hat{k} \cdot \hat{k}| = 1$, correct!

Now let T be any linear transformation in two dimensions. [This means that it acts on vectors in the xy plane and turns them into other vectors in the xy plane.]

We let T act on the Basic Box, as usual.

Now $T\hat{i}$ and $T\hat{j}$ still lie in the same plane as $\hat{i}$ and $\hat{j}$, so $(T\hat{i}) \times (T\hat{j})$ must be perpendicular to that plane. Hence it must be some multiple of $\hat{k}$. We define the DETERMINANT of T to be that multiple, that is, by definition, $\det(T)$ is the number given [STRICTLY IN 2 DIMENSIONS] by

$$(T\hat{i}) \times (T\hat{j}) = \det(T)\hat{k}.$$

EXAMPLE: If I = identity, then

$$I\hat{i} \times I\hat{j} = \hat{i} \times \hat{j} = \hat{k} = 1\hat{k}$$

so $\det(I) = 1$.

EXAMPLE: $D\overrightarrow{u} = 2\overrightarrow{u}$

$$D\hat{i} \times D\hat{j} = 4\hat{i} \times \hat{j} = 4\hat{k} \to \det(D) = 4$$

EXAMPLE: $T\hat{i} = \hat{i} + \frac{1}{4}\hat{j}$, $T\hat{j} = \frac{1}{4}\hat{i} + \hat{j}$,

$$T\hat{i} \times T\hat{j} = \left(\hat{i} + \frac{1}{4}\hat{j}\right) \times \left(\frac{1}{4}\hat{i} + \hat{j}\right) = \hat{i} \times \hat{j} + \frac{1}{16}\hat{j} \times \hat{i}$$
$$= \frac{15}{16}\hat{i} \times \hat{j} = \frac{15}{16}\hat{k} \to \det T = \frac{15}{16}.$$

EXAMPLE: $T\hat{i} = \hat{j}$, $T\hat{j} = \hat{i}$,

$$T\hat{i} \times T\hat{j} = \hat{j} \times \hat{i} = -\hat{k} \to \det T = -1$$

EXAMPLE: Shear, $S\hat{\imath} = \hat{\imath}$, $S\hat{\jmath} = \hat{\imath}\tan\theta + \hat{\jmath}$,

$$S\hat{\imath} \times S\hat{\jmath} = \hat{k} \rightarrow \det S = 1.$$

EXAMPLE: $T\hat{\imath} = \hat{\imath} + \hat{\jmath} = T\hat{\jmath}$,

$$T\hat{\imath} \times T\hat{\jmath} = \overrightarrow{0} \rightarrow \det T = 0.$$

EXAMPLE: Rotation

$$\begin{aligned} R\hat{\imath} \times R\hat{\jmath} &= (\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \times (-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}) \\ &= (\cos^2\theta - -\sin^2\theta)\hat{k} = \hat{k} \rightarrow \det(R) = 1. \end{aligned}$$

The area of the Basic Box is initially $|\hat{\imath} \times \hat{\jmath}| = 1$. After we let T act on it, the area becomes

$$|T\hat{\imath} \times T\hat{\jmath}| = |det\ T| \ \ |\hat{k}| = |\det\ T|.$$

So

$$\frac{\text{Final Area of Basic Box}}{\text{Initial Area of Basic Box}} = \frac{|\det T|}{1} = |\det T|$$

so $|\det T|$ TELLS YOU THE AMOUNT BY WHICH AREAS ARE CHANGED BY T. So $\det T = \pm 1$ means that the area is UNCHANGED (Shears, rotations, reflections) while $\det T = 0$ means that the Basic Box is squashed FLAT, zero area.

Take a general 2 by 2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We know that this means $M\hat{i} = a\hat{i} + c\hat{j}$, $M\hat{j} = b\hat{i} + d\hat{j}$. Hence $M\hat{i} \times M\hat{j} = \left(a\hat{i} + c\hat{j}\right) \times \left(b\hat{i} + d\hat{j}\right) = (ad - bc)\hat{k}$, so

$$\boxed{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.}$$

Check: $\det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4$, $\det \begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix} = 1$,

$$\det \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = 1,\ \det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0.$$

IN THREE dimensions there is a similar gadget.

The Basic Box is defined by $\hat{i}\hat{j}\hat{k}$, and we can let any 3-dimensional L.T. act on it, to get a new box defined by $T\hat{i}$, $T\hat{j}$, $T\hat{k}$. We define

$$\det T = \left(T\hat{i}\right) \times \left(T\hat{j}\right) \cdot \left(T\hat{k}\right)$$

where the dot is the scalar product, as usual. Since $|T\hat{i} \times T\hat{j} \cdot T\hat{k}|$ is the volume of the 3-dimensional parallelogram defined by $T\hat{i}$, $T\hat{j}$, $T\hat{k}$, we see that

$$|\det T| = \frac{\text{Final Volume of Basic Box}}{\text{Initial Volume of Basic Box}},$$

that is, $|\det T|$ tells you how much T changes volumes. If T squashes the Basic Box flat, then $\det T = 0$.

Just as $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$, there is a formula for the determinant of a 3 by 3 matrix. The usual notation is this. We DEFINE

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Similarly $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is the determinant of $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and there is a formula for it, as follows:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

In other words, we can compute a three-dimensional determinant if we know how to work out 2-dimensional determinants.

COMMENTS:

[a] We worked along the top row. Actually, a THEOREM says that you can use ANY ROW OR ANY COLUMN!

[b] How did I know that a_{12} had to multiply the particular 2-dimensional determinant $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$? Easy:

I just struck out EVERYTHING IN THE SAME ROW AND COLUMN as a_{12} : $\begin{vmatrix} * & * & * \\ a_{21} & * & a_{23} \\ a_{31} & * & a_{33} \end{vmatrix}$ and just kept the survivors!

This is the pattern, for example if you expand along the second row you will get

$$-a_{21}\begin{vmatrix} * & a_{12} & a_{13} \\ * & * & * \\ * & a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & * & a_{13} \\ * & * & * \\ a_{31} & * & a_{33} \end{vmatrix}$$

$$-a_{23}\begin{vmatrix} a_{11} & a_{12} & * \\ * & * & * \\ a_{31} & a_{32} & * \end{vmatrix}$$

[c] What is the pattern of the + and − signs? It is an (Ang Moh) CHESSBOARD, starting with a + in the top left corner: $\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$

[d] You can do exactly the same thing in FOUR di-

mensions, following this pattern, using $\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$ because now you know how to work out 3-dimensional determinants. And so on!

Example:

$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 2 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} - -1\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}$$
$$= 0 + 2 + 0 = 2$$

(expanding along the top row) or, if you use the second row,

$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 2 & 0 & 0 \end{vmatrix}$$
$$= -1\begin{vmatrix} -1 & 0 \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - -1\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$$
$$= 0 + 0 + 2 = 2$$

or

$$
\begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 2 & 0 & 0 \end{vmatrix}
$$

$$
= 1 \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} - 1 \begin{vmatrix} -1 & 0 \\ 0 & 0 \end{vmatrix} + 2 \begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix}
$$

$$
= 0 + 0 + 2 = 2
$$

(expanding down the first column).

Important Properties of Determinants

[a] Let S and T be two linear transformations such that $\det S$ and $\det T$ are defined. Then

$$
\boxed{\det ST = \det TS = (\det S) \times (\det T).}
$$

Therefore, $\det[STU] = \det[UST] = \det[TUS]$ and so on: det doesn't care about the order. Remember however that this DOES NOT mean that $STU = UST$ etc etc.

[b] If M is a square matrix, then

$$\boxed{\det M^T = \det M.}$$

[c] If c is a number and M is an n by n matrix, then

$$\boxed{\det(cM) = c^n \det M.}$$

EXAMPLE: Remember from Section 2[g] of Chapter 5 that an ORTHOGONAL matrix satisfies $MM^T = I$. So $\det(MM^T) = \det I = 1$. But $\det(MM^T) = \det(M)\det(M^T) = \det(M) \times \det(M) = (\det M)^2$, thus

$$\det M = \pm 1$$

for any orthogonal matrix.

6.5. INVERSES.

If I give you a 3-dimensional vector $\overrightarrow{u}$ and a 3-dimensional linear transformation T, then T sends

$\overrightarrow{u}$ to a particular vector, it never sends $\overrightarrow{u}$ to two DIFFERENT VECTORS! So this picture is impossible:

$$T\overrightarrow{u}$$

$$\overrightarrow{u}$$

$$T\overrightarrow{u}$$

But what about this picture:

$$\overrightarrow{u}$$

$$T\overrightarrow{u} = T\overrightarrow{v}$$

$$\overrightarrow{v}$$

Can T send TWO DIFFERENT VECTORS TO ONE? Yes!

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So it can happen! Notice that this transformation destroys $\hat{j}$ (and also $\hat{k}$). In fact if $\overrightarrow{u} \neq \overrightarrow{v}$ and

$T\overrightarrow{u} = T\overrightarrow{v}$, then $T(\overrightarrow{u} - \overrightarrow{v}) = 0$, that is, $T\overrightarrow{w} = \overrightarrow{0}$ where $\overrightarrow{w}$ IS NOT THE ZERO VECTOR. So if this happens, T destroys everything in the $\overrightarrow{w}$ direction. That is, T SQUASHES 3-dimensional space down to two or even less dimensions. This means that T LOSES INFORMATION – it throws away all of the information stored in the $\overrightarrow{w}$ direction. Clearly T squashes the basic box down to zero volume, so

$$\boxed{\det T = 0}$$

and we say T is SINGULAR.

SUMMARY: A SINGULAR LINEAR TRANSFORMATION

[a] Maps two different vectors to one vector

[b] Destroys all of the vectors in at least one direction

[c] Loses all information associated with those directions

[d] Satisfies $\det T = 0$.

Conversely, a NON-SINGULAR transformation never maps 2 vectors to one,

$$\left.\begin{array}{ccc} \overrightarrow{u} & \longrightarrow & T\overrightarrow{u} \\ \overrightarrow{v} & \longrightarrow & T\overrightarrow{v} \end{array}\right\} \text{ Different}$$

Therefore if I give you $T\overrightarrow{u}$, THERE IS EXACTLY ONE $\overrightarrow{u}$. The transformation that takes you from $T\overrightarrow{u}$ back to $\overrightarrow{u}$ is called the INVERSE OF T. The idea is that since a NON-SINGULAR linear transformation does NOT destroy information, we can re-construct $\overrightarrow{u}$ if we are given $T\overrightarrow{u}$. Clearly T HAS AN INVERSE, CALLED T^{-1}, if and only if $\det T \neq 0$. All this works in other dimensions too.

EXAMPLE:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix},$$

two different vectors to one!

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 13.59 \\ -13.59 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It destroys everything

in that direction!

Finally det $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0$.

So it is SINGULAR and

has NO inverse. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

EXAMPLE: Take $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and suppose it acts on $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ and $\begin{bmatrix} a \\ b \end{bmatrix}$ and sends them to the same vector, so

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Then

$$\begin{bmatrix} -\beta \\ \alpha \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix} \rightarrow \begin{matrix} \alpha = a \\ \beta = b \end{matrix} \rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

so $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ and $\begin{bmatrix} a \\ b \end{bmatrix}$ are the same – this transformation never maps different vectors to the same vector. No vector is destroyed, no information is lost, nothing gets squashed! And $\det \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 1$, NON-SINGULAR.

How to FIND THE INVERSE.

By definition, T^{-1} sends $T\overrightarrow{u}$ to $\overrightarrow{u}$, i.e.

$$T^{-1}(T(\overrightarrow{u})) = \overrightarrow{u} = T(T^{-1}(\overrightarrow{u})).$$

But $\overrightarrow{u} = I\overrightarrow{u}$ (identity) so T^{-1} satisfies

$$T^{-1}T = TT^{-1} = I.$$

So to find the inverse of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ we just have to find a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\rightarrow b = 1$, $a = 0$, $d = 0$, $c = -1$ so answer is

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. In fact it's easy to show that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For example, when we needed to find the matrix S in Section 4 of Chapter 5, we needed to find a way of solving

$$S \begin{bmatrix} 0.7 & -0.4 \\ -0.5 & 0.7 \end{bmatrix} = I.$$

This just means that we need to inverse of $\begin{bmatrix} 0.7 & -0.4 \\ -0.5 & 0.7 \end{bmatrix}$, and the above formula does the job for us.

For bigger square matrices there are many tricks for finding inverses. A general [BUT NOT VERY PRACTICAL] method is as follows:

[a] Work out the matrix of COFACTORS. [A cofactor is what you get when you work out the smaller determinant obtained by striking out a row and a column, for example the cofactor of 6 in $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

is $\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = -6$. You can do this for each element in a given matrix, to obtain a new matrix of the same size. For example, the matrix of cofactors of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

[b] Keep or reverse the signs of every element according to $\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$ (you get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ above.)

[c] Take the TRANSPOSE, $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

[d] Divide by the determinant of the original matrix. THE RESULT IS THE DESIRED INVERSE. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ in this example. Check:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

INVERSE OF A PRODUCT:

$$(AB)^{-1} = B^{-1}A^{-1}$$

Note the order! Easily checked:

$$(AB)^{-1}AB = B^{-1}(A^{-1}A)B = B^{-1}IB = I.$$

APPLICATION: SOLVING LINEAR SYSTEMS.

Suppose you want to solve

$$\begin{aligned} x + 2y + 3z &= 1 \\ 4x + 5y + 6z &= 2 \\ 7x + 8y + 9z &= 4. \end{aligned}$$

One way is to write it as

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

Then all you have to do is find $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{-1}$ and multiply it on both sides, so you get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \text{answer}.$$

So this is a systematic way of solving such problems! Now actually $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 0$, and you can see why: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

So this transformation destroys everything in the direction of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. In fact it squashes 3-dimensional space down to a certain 2-dimensional space. [We say that the matrix has RANK 2. If it had squashed everything down to a 1-dimensional space, we would say that it had RANK 1.] Now actually $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$. DOES

NOT lie in that two-dimensional space. Since $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ squashes EVERYTHING into that two-dimensional space, it is IMPOSSIBLE for $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to be equal to $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$. Hence the system has NO solutions.

If we change $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, this vector DOES lie in the special 2-dimensional space, and the system $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ DOES have a solution – in fact it has infinitely many!

SUMMARY:

Any system of linear equations can be written as

$$M\overrightarrow{r} = \overrightarrow{a}$$

where M is a matrix, $\overrightarrow{r}$ = the vector of variables, and $\overrightarrow{a}$ is a given vector. Suppose M is square.

[a] If $\det M \neq 0$, there is exactly one solution,

$$\overrightarrow{r} = M^{-1}\overrightarrow{a}.$$

[b] If $\det M = 0$, there is probably no solution. But if there is one, then there will be many.

PRACTICAL ENGINEERING PERSPECTIVE:

In the REAL world, NOTHING IS EVER EXACTLY EQUAL TO ZERO! So if $\det M = 0$, either [a] you have made a mistake, OR [b] you are pretending that your data are more accurate than they really are!

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ REALLY means } \begin{bmatrix} 1.01 & 2.08 & 3.03 \\ 3.99 & 4.97 & 6.02 \\ 7.01 & 7.96 & 8.98 \end{bmatrix}$$

and of course the determinant of THIS is non-zero! Actually, $\det = 0.597835$!

6.6 EIGENVECTORS AND EIGENVALUES.

Remember we said that a linear transformation USUALLY changes the direction of a vector. But there may be some special vectors which DON'T have their direction changed!

EXAMPLE: $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ clearly DOES change the direction of $\hat{i}$ and $\hat{j}$, since $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not parallel to $\hat{i}$ and $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is not parallel to $\hat{j}$. BUT

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

which IS parallel to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

In general if a transformation T does not change the direction of a vector $\overrightharpoon{u}$, that is

$$T\overrightharpoon{u} = \lambda \overrightharpoon{u}$$

for some λ (SCALAR), then $\overrightarrow{u}$ is called an EIGENVECTOR of T. The scalar λ is called the EIGENVALUE of $\overrightarrow{u}$.

6.7 FINDING EIGENVALUES AND EIGENVECTORS.

There is a systematic way of doing this. Take the equation

$$T\overrightarrow{u} = \lambda\overrightarrow{u}$$

and write $\overrightarrow{u} = I\overrightarrow{u}$, $I =$ identity. Then

$$(T - \lambda I)\overrightarrow{u} = \overrightarrow{0}$$

Let's suppose $\overrightarrow{u} \neq \overrightarrow{0}$ [of course, $\overrightarrow{0}$ is always an eigenvector, that is boring]. So the equation says that $T - \lambda I$ SQUASHES everything in the $\overrightarrow{u}$ direction. Hence

$$\det(T - \lambda I) = 0.$$

This is an equation which can be SOLVED to find λ.

EXAMPLE: Find the eigenvalues of $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$:

$$\det\left(\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\rightarrow \quad \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} = 0$$

$$\rightarrow \quad -(1-\lambda)(2+\lambda) - 4 = 0$$

$$\rightarrow \quad \lambda = 2 \ \text{ OR } \ -3$$

So there are TWO answers for a 2 by 2 matrix. Similarly, in general there are three answers for 3 by 3 matrices, etc.

What are the eigenvectors for $\lambda = 2$, $\lambda = -3$?

IMPORTANT POINT: Let $\overrightarrow{u}$ be an eigenvector of T. Then $2\overrightarrow{u}$ is also an eigenvector with the same eigenvalue!

$$T(2\overrightarrow{u}) = 2T\overrightarrow{u} = 2\lambda\overrightarrow{u} = \lambda \times (2\overrightarrow{u}).$$

Similarly $3\overrightarrow{u}$, $13.59\overrightarrow{u}$ etc are all eigenvectors! SO YOU MUST NOT EXPECT A UNIQUE ANSWER! OK, with that in mind, let's find an eigenvector for $\lambda = 2$. Let's call an eigenvector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. Then

$$
\begin{aligned}
& (T - \lambda I)\overrightarrow{u} = \overrightarrow{0} \\
\rightarrow\quad & \begin{bmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0 \\
\rightarrow\quad & \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0 \\
\rightarrow\quad & -\alpha + 2\beta = 0 \\
\rightarrow\quad & 2\alpha - 4\beta = 0
\end{aligned}
$$

But these equations are actually the SAME, so we really only have ONE equation for 2 unknowns. We aren't surprised, because we did not expect a unique answer anyway! We can just CHOOSE $\alpha = 1$ (or 13.59 or whatever) and then solve for β. Clearly $\beta = \frac{1}{2}$, so an eigenvector corresponding to $\lambda = 2$ is $\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$. But if you said $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 100 \\ 50 \end{bmatrix}$ that is also correct!

What about $\lambda = -3$?

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$$

$$\rightarrow \quad 4\alpha + 2\beta = 0$$

$$\rightarrow \quad 2\alpha + \beta = 0$$

Again we can set $\alpha = 1$, then $\beta = -2$, so an eigenvector corresponding to $\lambda = -3$ is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$ or $\begin{bmatrix} -10 \\ 20 \end{bmatrix}$ etc.

EXAMPLE: Find the eigenvalues, and corresponding eigenvectors, of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Answer: We have $\det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = 0 \rightarrow \lambda^2 + 1 = 0 \rightarrow \lambda = \pm i$, $i = \sqrt{-1}$.

Eigenvector for i: we set

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ \beta \end{bmatrix} = 0$$

$\rightarrow \ -i-\beta=0 \rightarrow \beta=-i$ so an eigenvector for i is $\begin{bmatrix} 1 \\ -i \end{bmatrix}$. For $\lambda=-i$ we have

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ \beta \end{bmatrix} = 0$$

$\rightarrow \ i-\beta=0 \rightarrow \beta=i$ so an eigenvector for $-i$ is $\begin{bmatrix} 1 \\ i \end{bmatrix}$. Note that a REAL matrix can have COMPLEX eigenvalues and eigenvectors! This is happening simply because $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a ROTATION through 90°, and of course such a transformation leaves NO [real] vector's direction unchanged (apart from the zero vector).

6.8. DIAGONAL FORM OF A LINEAR TRANSFORMATION.

Remember that we defined the matrix of a linear transformation T WITH RESPECT TO $\hat{i}, \hat{j}$ by letting T act on $\hat{i}$ and $\hat{j}$ and then putting the results in the columns. So to say that T has matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with respect to $\hat{i}, \hat{j}$ means that

$$T\hat{i} = a\hat{i} + c\hat{j}$$

$$T\hat{j} = b\hat{i} + d\hat{j}.$$

What's so special about the two vectors $\hat{i}$ and $\hat{j}$? Nothing, except that EVERY vector in two dimensions can be written as $\alpha\hat{i} + \beta\hat{j}$ for some α, β.

Now actually we only really use $\hat{i}$ and $\hat{j}$ for CONVENIENCE. In fact, we can do this with ANY pair of vectors $\overrightarrow{u}, \overrightarrow{v}$ in two dimensions,

PROVIDED that they are not parallel.

That is, any vector $\overrightarrow{w}$ can be

expressed as

$\overrightarrow{w} = \alpha\overrightarrow{u} + \beta\overrightarrow{v}$

for some scalars α, β. You can see this from the diagram – by stretching $\overrightarrow{u}$ to $\alpha\overrightarrow{u}$ and $\overrightarrow{v}$ to $\beta\overrightarrow{v}$, we can make their sum equal to $\overrightarrow{w}$.

We call $\overrightarrow{u}$, $\overrightarrow{v}$ a BASIS for 2-dimensional vectors. Let

$$\overrightarrow{u} = P_{11}\hat{i} + P_{21}\hat{j} = \begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix}$$

$$\overrightarrow{v} = P_{12}\hat{i} + P_{22}\hat{j} = \begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix}$$

Then the transformation that takes $\left(\hat{i}, \hat{j}\right)$ to $(\overrightarrow{u}, \overrightarrow{v})$ has matrix $\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = P$. In order for $\overrightarrow{u}$, $\overrightarrow{v}$ to be a basis, P must not squash the volume of the Basic Box down to zero, since otherwise $\overrightarrow{u}$ and $\overrightarrow{v}$ will be parallel. So we must have

$$\det P \neq 0.$$

The same idea works in 3 dimensions: ANY set of 3 vectors forms a basis PROVIDED that the matrix of components satisfies $\det P \neq 0$.

EXAMPLE: The pair of vectors $\overrightarrow{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \overrightarrow{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis, because $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$.

Now of course the COMPONENTS of a vector will change if you choose a different basis. For example,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\hat{\imath} + 2\hat{\jmath} \quad \text{BUT}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq 1\overrightarrow{u} + 2\overrightarrow{v}.$$

Instead, $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\overrightarrow{u} + 2\overrightarrow{v}$, so the components of this vector relative to $\overrightarrow{u}, \overrightarrow{v}$ are $\begin{bmatrix} -1 \\ 2 \end{bmatrix}_{(\overrightarrow{u}, \overrightarrow{v})}$. Where did I get these numbers?

As usual, set $\overrightarrow{u} = P\hat{\imath}$, $\overrightarrow{v} = P\hat{\jmath}$ where $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

We want to find α, β such that $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \alpha\overrightarrow{u} + \beta\overrightarrow{v}$.

Substituting $\overrightarrow{u} = P\hat{\imath}$, $\overrightarrow{v} = P\hat{\jmath}$ into this equation we get

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \alpha P\hat{\imath} + \beta P\hat{\jmath} = P[\alpha\hat{\imath} + \beta\hat{\jmath}] = P\begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

We know P is not singular, so we can take P over to the left side by multiplying both sides of this equation by the inverse of P. So we get

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = P^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and this is our answer: this is how we find α and β! So to get α and β we just have to work out

$$P^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

that is, the components of this vector relative to $\overrightarrow{u}, \overrightarrow{v}$ are found as

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}_{(\overrightarrow{u}, \overrightarrow{v})} = P^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(\hat{\imath}, \hat{\jmath})}$$

THE COMPONENTS RELATIVE TO $\overrightarrow{u}, \overrightarrow{v}$ ARE OBTAINED BY MULTIPLYING P^{-1} INTO THE COMPONENTS RELATIVE TO $\hat{\imath}, \hat{\jmath}$. Similarly for linear transformations – if a certain linear transformation T has matrix $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}_{\hat{i}\hat{j}}$ relative to $\hat{i}, \hat{j}$ it will have a DIFFERENT matrix relative to $\vec{u}, \vec{v}$. We have $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}_{(\hat{i},\hat{j})} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(\hat{i},\hat{j})} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}_{(\hat{i},\hat{j})}$

That is, the matrix of T relative to $\hat{i}, \hat{j}$ sends $\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(\hat{i}\hat{j})}$ to $\begin{bmatrix} 5 \\ -2 \end{bmatrix}_{(\hat{i}\hat{j})}$. In the same way, the matrix of T relative to $(\vec{u}, \vec{v})$, which we don't know and want to find, sends $\begin{bmatrix} -1 \\ 2 \end{bmatrix}_{(\vec{u},\vec{v})}$ to $\begin{bmatrix} 7 \\ -2 \end{bmatrix}_{(\vec{u},\vec{v})}$, because these are the components of these two vectors relative to $\vec{u}, \vec{v}$, as you can show by multiplying P^{-1} into $\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(\hat{i}\hat{j})}$ and $\begin{bmatrix} 5 \\ -2 \end{bmatrix}_{(\hat{i}\hat{j})}$ respectively.

So the unknown matrix we want satisfies

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}_{(\vec{u},\vec{v})} \begin{bmatrix} -1 \\ 2 \end{bmatrix}_{(\vec{u},\vec{v})} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}_{(\vec{u},\vec{v})}$$

But we know

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}_{(\vec{u},\vec{v})} = P^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(\hat{\imath},\hat{\jmath})} \quad \text{and}$$

$$\begin{bmatrix} 7 \\ -2 \end{bmatrix}_{(\vec{u},\vec{v})} = P^{-1} \begin{bmatrix} 5 \\ -2 \end{bmatrix}_{(\hat{\imath},\hat{\jmath})} \quad \text{so}$$

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}_{(\vec{u},\vec{v})} P^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(\hat{\imath}\hat{\jmath})} = P^{-1} \begin{bmatrix} 5 \\ -2 \end{bmatrix}_{(\hat{\imath}\hat{\jmath})}.$$

Multiply both sides by P and get

$$P \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}_{(\vec{u},\vec{v})} P^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(\hat{\imath}\hat{\jmath})} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}_{(\hat{\imath}\hat{\jmath})}$$

Compare this with

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}_{(\hat{\imath},\hat{\jmath})} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{(\hat{\imath},\hat{\jmath})} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}_{(\hat{\imath},\hat{\jmath})}$$

$$\rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}_{(\hat{\imath},\hat{\jmath})} = P \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}_{(\vec{u},\vec{v})} P^{-1}$$

$$\rightarrow \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}_{(\vec{u},\vec{v})} = P^{-1} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}_{(\hat{\imath},\hat{\jmath})} P.$$

[In the last step, we multiplied both sides on the LEFT by P^{-1}, and on the RIGHT by P.]

We conclude that THE MATRIX OF T RELATIVE TO $\vec{u}, \vec{v}$, IS OBTAINED BY MULTIPLYING P^{-1} ON THE LEFT AND P ON THE RIGHT INTO THE MATRIX OF T RELATIVE TO $\hat{\imath}, \hat{\jmath}$. In this example,

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}_{(\vec{u},\vec{v})} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}.$$

So now we know how to work out the matrix of any linear transformation relative to ANY basis.

Now let T be a linear transformation in 2 dimensions, with eigenvectors $\vec{e_1}, \vec{e_2}$, eigenvalues λ_1, λ_2. Now $\vec{e_1}$ and $\vec{e_2}$ may or may not give a basis for 2-dimensional space. But suppose they do.

QUESTION: What is the matrix of T relative to $\overrightarrow{e_1}$, $\overrightarrow{e_2}$?

ANSWER: As always, we see what T does to $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$, and put the results into the columns!

By definition of eigenvectors and eigenvalues,

$$T\overrightarrow{e_1} = \lambda_1\overrightarrow{e_1} = \lambda_1\overrightarrow{e_1} + 0\overrightarrow{e_2}$$

$$T\overrightarrow{e_2} = \lambda_2\overrightarrow{e_2} = 0\overrightarrow{e_1} + \lambda_2\overrightarrow{e_2}$$

So the matrix is $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}_{(\overrightarrow{e_1},\overrightarrow{e_2})}$.

We say that a matrix of the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ or $\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$ is DIAGONAL. So we see that THE MATRIX OF A TRANSFORMATION RELATIVE TO ITS OWN EIGENVECTORS (assuming that these form a basis) is DIAGONAL.

EXAMPLE: We know that the eigenvectors of $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ are $\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. So here $P = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & -2 \end{bmatrix}$,

$$P^{-1} = -\tfrac{2}{5}\begin{bmatrix} -2 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix},$$

$$\begin{aligned} P^{-1}\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} P &= -\frac{2}{5}\begin{bmatrix} -2 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ \frac{1}{2} & -2 \end{bmatrix} \\ &= -\frac{2}{5}\begin{bmatrix} -2 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix} \\ &= -\frac{2}{5}\begin{bmatrix} -5 & 0 \\ 0 & \frac{15}{2} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \text{as expected since the} \\ &\quad \text{eigenvalues are 2 and -3.} \end{aligned}$$

EXAMPLE: The shear matrix $\begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}$.

Eigenvalues: $\det\begin{bmatrix} 1-\lambda & \tan\theta \\ 0 & 1-\lambda \end{bmatrix} = 0 \rightarrow (1-\lambda)^2 = 0 \rightarrow \lambda = 1$. Only one eigenvector, namely $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so the eigenvectors DO NOT give us a basis in this case $\rightarrow$ NOT possible to diagonalize this matrix!

EXAMPLE: Suppose you want to do a reflection ρ

of the entire 2-dimensional plane around a straight line that passes through the origin and makes an angle of θ with the x-axis. Then the vector $\begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix}$ lies along this line, so it is left unchanged by the reflection; in other words, it is an eigenvector of ρ with eigenvalue 1. On the other hand, the vector $\begin{bmatrix} -\ sin\theta \\ cos\theta \end{bmatrix}$ is perpendicular to the first vector [check that their scalar product is zero] so it is reflected into its own negative by ρ. That is, it is an eigenvector with eigenvalue $-\ 1$.

So ρ has a matrix with these eigenvectors and eigenvalues. The P matrix in this case is

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

and clearly $P^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, and since the eigenvalues are ± 1, we have

$$\rho = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Doing the matrix multiplication and using the trigonometric identities for cos2θ and sin2θ, you will find that

$$\rho = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

Notice that the determinant is -1, as is typical for a reflection. Check that this gives the right answer for reflections around the 45 degree diagonal and around the x-axis.

6.9 APPLICATION – MARKOV CHAINS.

We saw back in Section 3 of Chapter 5 that to predict the weather 4 days from now, we needed the 4th power of the matrix

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}.$$

But suppose I want the weather 30 days from now – I need M^{30}! There is an easy way to work this out using eigenvalues.

Suppose I can diagonalize M, that is, I can write $P^{-1}MP = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ for some matrix P. Then

$$\begin{aligned} M &= PDP^{-1} \\ M^2 &= (PDP^{-1})(PDP^{-1}) \\ &\quad = PDP^{-1}PDP^{-1} = PD^2P^{-1} \\ M^3 &= MM^2 = PDP^{-1}PD^2P^{-1} = PD^3P^{-1} \text{ etc} \\ M^{30} &= PD^{30}P^{-1}. \end{aligned}$$

But D^{30} is very easy to work out – it is just $\begin{bmatrix} \lambda_1^{30} & 0 \\ 0 & \lambda_2^{30} \end{bmatrix}$.

Let's see how this works!

Eigenvectors and eigenvalues of $\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$ are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (eigenvalue 0.3) and $\begin{bmatrix} 1 \\ \frac{4}{3} \end{bmatrix}$ (eigenvalue 1) so

$$P = \begin{bmatrix} 1 & 1 \\ -1 & \frac{4}{3} \end{bmatrix}, D = \begin{bmatrix} 0.3 & 0 \\ 0 & 1 \end{bmatrix}, P^{-1} = \begin{bmatrix} \frac{4}{7} & -\frac{3}{7} \\ \frac{3}{7} & \frac{3}{7} \end{bmatrix}$$

$$D^{30} = \begin{bmatrix} (0.3)^{30} & 0 \\ 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 2 \times 10^{-16} & 0 \\ 0 & 1 \end{bmatrix}$$

so

$$M^{30} = \begin{bmatrix} 1 & 1 \\ -1 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 2 \times 10^{-16} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{7} & -\frac{3}{7} \\ \frac{3}{7} & \frac{3}{7} \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 3 + 8 \times 10^{-16} & 3 - 6 \times 10^{16} \\ 4 - 8 \times 10^{-16} & 4 + 6 \times 10^{-16} \end{bmatrix} \approx \begin{bmatrix} \frac{3}{7} & \frac{3}{7} \\ \frac{4}{7} & \frac{4}{7} \end{bmatrix}$$

So if it is rainy today, the probability of rain tomorrow is 60%, but the probability of rain 30 days from now is only $\frac{3}{7} \approx 43\%$. As we go forward in time, the fact that it rained today becomes less and less important! The probability of rain in 31 days is almost the same as the probability of rain in 30 days!

6.10 APPLICATION: THE TRACE OF A DIAGONALIZABLE MATRIX IS THE SUM OF ITS EIGENVALUES.

Let M be any square matrix. Then the TRACE of M, denoted TrM, is defined as the sum of the diagonal entries: $Tr\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2$, $Tr\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 15$, $Tr\begin{bmatrix} 1 & 5 & 16 \\ 7 & 2 & 15 \\ 11 & 9 & 8 \end{bmatrix} = 11$, etc.

In general it is NOT true that $Tr(MN) = TrM\ TrN$ BUT it is true that $TrMN = TrNM$.

Proof: $TrM = \sum_i M_{ii}$ so

$$TrMN = \sum_i \sum_j M_{ij} N_{ji} = \sum_j \sum_i N_{ji} M_{ij} = TrNM.$$

Hence $Tr(P^{-1}AP) = Tr(APP^{-1}) = TrA$ so THE TRACE OF A MATRIX IS ALWAYS THE SAME NO MATTER WHICH BASIS YOU USE! This is why the trace is interesting: it doesn't care which basis you use. In particular, if A is diagonalizable, $TrA = Tr\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \lambda_1 + \lambda_2$.

So the trace is equal to the sum of the eigenvalues. This gives a quick check that you have not made a mistake in working out the eigenvalues: they have to add up to the same number as the trace of the original matrix. Check this for the examples in this chapter.