MA1506

Tutorial 1 Solution

1. (a)

$$\begin{aligned}
\frac{y}{\sqrt{1-y^2}}\,dy &= \frac{-x}{\sqrt{1-x^2}}\,dx \\
\int \frac{y}{\sqrt{1-y^2}}\,dy &= \int \frac{-x}{\sqrt{1-x^2}}\,dx
\end{aligned}$$

Let $u = 1 - y^2$. Then $du = -2ydy$

$$\begin{aligned}
\left(-\frac{1}{2}\right)\frac{(1-y^2)^{\frac{1}{2}}}{\frac{1}{2}} &= \left(\frac{1}{2}\right)\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + C \\
(1-y^2)^{\frac{1}{2}} + (1-x^2)^{\frac{1}{2}} &= C
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{y}{1+y^2}\,dy &= \frac{1}{x+x^3}\,dx = \left(\frac{1}{x} - \frac{x}{1+x^2}\right)\,dx \\
\int \frac{y}{1+y^2}\,dy &= \int \frac{1}{x}\,dx - \int \frac{x}{1+x^2}\,dx
\end{aligned}$$

Let $u = 1 + y^2$. Then $du = 2ydy$

$$\begin{aligned}
\frac{1}{2}\ln(1+y^2) &= \ln|x| - \frac{1}{2}\ln(1+x^2) + C \\
\ln(1+y^2) &= 2\ln x - \ln(1+x^2) + 2C \\
\ln\frac{(1+y^2)(1+x^2)}{x^2} &= 2C \\
\frac{(1+y^2)(1+x^2)}{x^2} &= e^{2C} \\
(1+y^2)(1+x^2) &= C_1 x^2
\end{aligned}$$

(c) The integrating factor is $e^{\int (-\tan x)\, dx} = e^{\ln \cos x} = \cos x$.
Hence,

$$\cos x \frac{dy}{dx} - (\tan x) \cos x \; y = \cos^2 x$$

$$\begin{aligned} y \cos x &= \int \cos^2 x \; dx \\ &= \int \frac{1}{2}(1 + \cos 2x) \; dx \\ &= \frac{1}{2}x + \frac{1}{4} \sin 2x + C. \end{aligned}$$

$$y = \frac{1}{\cos x} \left[\frac{1}{2}x + \frac{1}{4} \sin 2x + C \right]$$

(d) The integrating factor is

$$e^{\int 2t \; dt} = e^{t^2}$$

$$e^{t^2} \frac{dy}{dt} + 2te^{t^2} y = te^{t^2}$$

$$e^{t^2} y = \int te^{t^2} \; dt = \frac{1}{2} \int e^{t^2} \; d(t^2) = \frac{1}{2} e^{t^2} + C$$

Now when $t = 1$, $y = 2$, so

$$2e = \frac{1}{2}e + C$$

Hence $C = \frac{3}{2}e$. Thus $y = \frac{1}{2} + \frac{3}{2}e^{1-t^2}$.

(e)

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x}e^x$$

The integrating factor is

$$e^{\int \frac{1}{x} \; dx} = e^{\ln x} = x.$$

$$x \frac{dy}{dx} + y = e^x$$

$$xy = \int e^x \, dx = e^x + C.$$

Now when $x = 1$, $y = e$

$$e = e + C$$

Thus $C = 0$. So

$$xy = e^x$$

Question 2

(a)

$$y' = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1} \to y = \ln\left|\frac{x}{x+1}\right| + c$$

(b)

$$y' = \cos x \cos 5x = \frac{1}{2}[\cos 6x + \cos 4x] \to y = \frac{1}{2}\left[\frac{1}{6}\sin 6x + \frac{1}{4}\sin 4x\right] + c$$

(c)

$$\frac{dy}{dx} = e^x e^{-3y} \Rightarrow e^{3y}dy = e^x dx \Rightarrow \frac{1}{3}e^{3y} = e^x + c$$

(d)

$$\frac{1+y}{y^2}dy = (2x-1)dx \to \ln|y| - \frac{1}{y} = x^2 - x + c$$

In all these examples, the value of c is fixed by the given initial data. You can now use Graphmatica to graph these specific functions. If you feed the differential equations into Graphmatica with these data, you will get exactly the same graphs.

Question 3

(a)

$$y' + (1+\frac{1}{x})y = \frac{1}{x}e^{-x}$$

Integrating factor is

$$e^{\int(1+1/x)dx} = e^{(x+\ln x)} = e^x x$$

$$\Rightarrow yxe^x = x + c \Rightarrow y = e^{-x} + cx^{-1}e^{-x}$$

(b) Integrating factor is $\frac{1}{x^3}e^{-x}$

$$\frac{y}{x^3}e^{-x} = \int \frac{e^{-x}}{x^2} + 2\int \frac{e^{-x}}{x^3} + c$$

$$= \int \frac{e^{-x}}{x^2} + \frac{-e^{-x}}{x^2} - \int \frac{e^{-x}}{x^2} + c$$

$$= \frac{-e^{-x}}{x^2} + c$$

$$y = -x + cx^3e^x$$

$$\text{since } y(1) = e - 1 = -1 + ce \Rightarrow c = 1$$

$$y = -x + x^3e^x$$

(c)

$z = y^2 \qquad z' = 2yy' \qquad y' = z'/2y$

$$\frac{z'}{2y} + y + \frac{x}{y} = 0 \Rightarrow \frac{1}{2}z' + z + x = 0 \Rightarrow z' + 2z = -2x$$

$$\Rightarrow ze^{2x} = (-x + \frac{1}{2})e^{2x} + c \Rightarrow y^2 = \frac{1}{2} - x + ce^{-2x}$$

(d)

Since $2yy' = (y^2)'$ we define $Y = y^2$, $Y' + (1 - \frac{1}{x})Y = xe^x$, $\exp\int(1 - \frac{1}{x}) = \frac{1}{x}e^x$

We have $$Y = \frac{1}{2}xe^x + cxe^{-x}$$

i.e., $$y^2 = \frac{1}{2}xe^x + cxe^{-x}$$

Question 4

(a)

Let $v = 2x + y$ so $v' = 2 + y'$

$$\Rightarrow v' - 2 = \frac{1 - 2v}{1 + v} \Rightarrow v' = \frac{3}{1 + v}$$

$$\Rightarrow v + \frac{1}{2}v^2 = 3x + c$$

$$\Rightarrow (2x + y) + \frac{1}{2}(2x + y)^2 = 3x + c$$

(b)

$$v = x + y \qquad\qquad y' = v' - 1 = \left(\frac{v+1}{v+3}\right)^2$$

$$\Rightarrow v' = 1 + \frac{v^2 + 2v + 1}{(v+3)^2} = \frac{2v^2 + 8v + 10}{(v+3)^2}$$

$$\Rightarrow \frac{(v+3)^2}{v^2 + 4v + 5} dv = 2dx$$

$$\Rightarrow \frac{v^2 + 4v + 5 + 2v + 4}{v^2 + 4v + 5} dv = 2dx$$

$$\Rightarrow \left(1 + \frac{2v + 4}{v^2 + 4v + 5}\right) dv = 2dx$$

$$\Rightarrow v + \ln\left|\left(v^2 + 4v + 5\right)\right| = 2x + c$$

$$\Rightarrow x + y + \ln\left|(x+y)^2 + 4x + 4y + 5\right| = 2x + c$$