MA1506 TUTORIAL 10

Question 1

Wherever possible, diagonalize the matrices in questions 4/5 of Tutorial 9. [That is, write them in the form PDP^{-1}, after finding P and D, where D is diagonal.] Using this, work out their 4th powers.

[Answers: The first one cannot be diagonalized; the fourth powers of the others are

$$\begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix}, \begin{pmatrix} 41 & 80 \\ 20 & 41 \end{pmatrix}, \begin{pmatrix} -7 & 24 \\ -24 & -7 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.]$$

Question 2

The great white shark likes to eat seals. Let Σ_k denote the number of sharks in a given area at year k, and let σ_k denote the number of seals in that area. In year k+1, the number of sharks depends on the number in year k and also on the number of seals [the more seals there are in year k, the more food for the sharks]. It is found that

$$\Sigma_{k+1} = \frac{\Sigma_k}{2} + \frac{\sigma_k}{100}.$$

Notice that this means that the sharks would die out if there were no seals for them to eat, which makes sense, since seals are their main food. Similarly the number of seals in year k+1 is determined by the number of sharks and by the number of seals in year k. It is found that

$$\sigma_{k+1} = \frac{5\sigma_k}{4} - \frac{50\Sigma_k}{4}.$$

Notice the minus sign: seals tend to disappear into the mouths of sharks. Notice too that the seal population would grow uncontrollably if the sharks were not kind enough to remove surplus seals. Suppose we start with 50 sharks and 1600 seals in year zero. How many sharks and seals will we have in the long run? [Answer: 14 sharks and 700 seals.]

Question 3

Let B be a diagonalizable matrix. Prove that $det(e^B) = e^{Tr(B)}$. Verify this for the 2×2 rotation matrix in question 6 of Tutorial 8.

Question 4

Suppose that

$$\frac{dx}{dt} = -4x + 3y$$
$$\frac{dy}{dt} = -2x + y$$

is a system of linear differential equations, with x(0) = y(0) = 1. [Note that this is a special case of a system solved in Chapter 7.] Find x(t) and y(t) using Laplace transforms.

Question 5

Classify the systems of ODEs with the following matrices. [That is, say whether they represent a nodal source, spiral sink, etc etc etc.]

$$\begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} -2 & -4 \\ 10 & 0 \end{pmatrix}, \begin{pmatrix} -5 & 4 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -10 & 0 \end{pmatrix}.$$

Go to http://www.aw-bc.com/ide/idefiles/media/JavaTools/romejuli.html to check your answers by looking at the phase diagrams. [You will have to divide all of these matrices by 10 to use that website.]

Question 6

Both Elves and Dwarves live in Rivendell, but there is a certain irrational resentment between the two groups [placed in their hearts, no doubt, by Sauron]. The amount of racial prejudice is not the same on both sides, Elves tending to be more intolerant than Dwarves, but both groups are inclined to move out of Rivendell at a rate controlled by the number of the other group. Apart from this, both groups reproduce as usual, but Elves can only be killed by violence –they never grow old– so their overall death rate per capita is lower than that of the Dwarves; also, the Elvish birth rate per capita is higher since Dwarf women are scarce [ever seen one?]. Both groups have more births than deaths. We can set up a model of this situation using a pair of simultaneous ODEs with a matrix [where the first row describes the rate of change of the Elf population] $\begin{pmatrix} 5 & -4 \\ -1 & 2 \end{pmatrix}$. Explain why this matrix does represent the situation we described. At a certain time, the number of Dwarves is slightly larger than the number of Elves. Predict what will happen to the population of Elves. This pattern has in fact often been observed in racially discriminatory societies.