NATIONAL UNIVERSITY OF SINGAPORE

MA1506 - MATHEMATICS II

(Semester 1 : AY2013/2014)

Name of setter : Assoc Prof Fellow Quek Tong Seng

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Please write your matriculation/student number only. Do not write your name.
2. This examination paper contains **FOUR** questions and comprises **FIVE** printed pages.
3. Answer **ALL** questions. The maximum score for this examination is **80 Marks**.
4. Please start each question on a new page.
5. This is a CLOSED BOOK (with helpsheet) examination.
6. Students are allowed to use two handwritten A4 size helpsheets.
7. Candidates may use scientific calculators. However, they should lay out systematically the various steps in the calculations.

Answer all the questions.

Marks for each question are indicated at the beginning of the question.

Question 1 [20 marks]

(a) Consider the differential equation

$$2y\frac{dy}{dx} = Ax^2 - 3, \qquad x \geq 0.$$

Let $y(x)$ be the solution to the differential equation such that $y(0) = 3$ and $y(1) = \sqrt{7}$.

(i) Find the value of A.

(ii) With the value of A found above, find the value of $y(3)$.

(b) (i) By using the substitution $y(x) = u(x)e^{2x}$, or otherwise, find a particular solution to the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 2y = 2x^3e^{2x}.$$

(ii) Hence, or otherwise, find the general solution to the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 2y = 2x^3e^{2x}.$$

(c) Show that $u(x, y) = 1 + x + xy - ye^y$ is a solution to the partial differential equation

$$\frac{\partial u}{\partial y} + e^y\frac{\partial u}{\partial x} = x, \qquad u(x, 0) = 1 + x.$$

Question 2 [20 marks]

(a) The motion of an object of unit mass along the x-axis under the influence of a variable force $F(x)$ satisfies the differential equation

$$\frac{d^2x}{dt^2} = F(x),$$

where $x(t)$ is the distance of the object from the origin at time t. If the velocity of the object at a distance x from the origin is $\sqrt{x^3 - 2x + 2}$, find $F(x)$ for $x \geq 1$.

(b) A certain salt dissolves in methanol. The number $x(t)$ of grams of the salt in the solution at time t seconds satisfies the differential equation

$$\frac{dx}{dt} = x - 0.005x^2.$$

(i) Find the equilibrium solutions of the differential equation and use the phase line diagram to determine their stability.

(ii) If $x(0) = 50$, how long will it take for an additional 50 grams of the salt to dissolve?

(iii) Find the amount of salt in the methanol after a long time. Justify your answer.

(c) In an RLC circuit, the charge $q(t)$ on the capacitor and the current $i(t)$ satisfies the differential equation

$$\frac{di}{dt} + 90i + 2000q = 1100.$$

Suppose the initial current and charge in the circuit are both zero.

(i) Use the formula $i = \frac{dq}{dt}$ to express $Q(s)$ in terms of $I(s)$ and s, where $Q(s)$ and $I(s)$ are the Laplace transforms of $q(t)$ and $i(t)$, respectively.

(ii) Find $I(s)$ in terms of s.

(iii) Find $\lim_{t\to\infty} i(t)$. Justify your answer.

Question 3 [20 marks]

(a) Find the solution to the initial value problem

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} = \delta(t-1), \qquad y(0) = 0, \quad y(2) = e - 1,$$

where δ is the Dirac delta function.

(b) Using the quiver command, or otherwise, write a MATLAB programme to plot the direction field of

$$\frac{dx}{dt} = t - x$$

for $-1 \leq x, t \leq 3$.

(c) Find the Laplace transform of f given by

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ t & \text{if } 1 \leq t < 2 \\ 0 & \text{if } 2 \leq t < \infty. \end{cases}$$

(d) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\mathbf{i} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}, \qquad T\mathbf{j} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}, \qquad T\mathbf{k} = \mathbf{i} - \mathbf{j},$$

where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are the unit vectors along the x,y and z axes respectively.

(i) Find the matrix representation of T.

(ii) Find the rank of T. Justify your answer.

(iii) Solve the equation

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Question 4 [20 marks]

(a) Let A and B be two non-singular matrices. Prove, or disprove by giving a suitable example, that

$$(A+B)^{-1} = A^{-1} + B^{-1}.$$

(b) Consider the system of differential equations

$$\begin{bmatrix} \frac{dx}{dt} \\ \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ \\ y \end{bmatrix}.$$

(i) Find the equilibrium point (or points) and classify it (or them) as one of the six types discussed in class.

(ii) Sketch the phase plane diagram for the system.

(c) Consider two tanks, Tank A and Tank B. Each tank contains 100 litres of pure water initially. A salt solution with salt concentration 1 gram per litre is pumped into Tank A at a rate of 3 litres per minute. The solution in Tank A is well stirred and pumped into Tank B at a rate of 4 litres per minute. The well mixed solution in Tank B is pumped back into Tank A at a rate of 1 litre per minute and also to the outside at a rate of 3 litre per minute.

(i) How much salt is in each tank at any time?

(ii) How much salt is in each tank after a long time? Justify your answer.

END OF PAPER