# CHAPTER 7

# SYSTEMS OF FIRST-ORDER ODEs

## 7.1. ROMEO AND JULIET

We all know that many (most) relationships have their ups and downs. Let's try to model this fact.

Romeo loves Juliet, but Juliet believes in a more subtle approach and finds Romeo's excessive enthusiasm rather repulsive - the more he loves her, the less she likes him. On the other hand, when he loses interest, she fears losing him and begins to see his good side. Romeo is more straightforward: his love for Juliet increases when she is warm to him, and decreases when she is cold. Let $R(t)$ represent Romeo's feelings and $J(t)$ Juliet's. We can model the lovers' feelings by:

$$\begin{cases} \frac{dR}{dt} = aJ & R(0) = \alpha \\ \frac{dJ}{dt} = -bR & J(0) = \beta \end{cases} \tag{1}$$

where $a$, $b$ are positive constants and $\alpha$ and $\beta$ represent their feelings when they first meet. This is a SYSTEM OF SIMULTANEOUS FIRST-ORDER ODEs. In this case, the equations are linear, so it is easy to solve them. Put, as a TRIAL,

$$R = Ae^{\lambda t} \qquad J = Be^{\lambda t}$$

where $\lambda$ could turn out to be complex - if so, we will as usual interpret the exponential to mean that we are really dealing with sine and cosine functions. [The final solutions for R and J must be real — the feelings are real, not complex!]

So $A\lambda e^{\lambda t} = aBe^{\lambda t}$ and $B\lambda e^{\lambda t} = -bAe^{\lambda t}$. So we get $A\lambda = aB$, $B\lambda = -bA$ so ignoring special cases we get $\lambda^2 = -ab < 0$.

Since $e^{i\theta} = \cos\theta + i\sin\theta$, this is telling us that the solution is some combination of $\cos(\sqrt{ab}t)$ and $\sin(\sqrt{ab}t)$ *i.e.*:

$$R(t) = C\cos(\sqrt{ab}t) + D\sin(\sqrt{ab}t)$$

$$J(t) = E\cos(\sqrt{ab}t) + F\sin(\sqrt{ab}t)$$

So

$$R(0) = C \qquad \dot{R}(0) = \sqrt{ab}D$$

$$J(0) = E \qquad \dot{J}(0) = \sqrt{ab}F$$

But from equations (1), we see that

$$R(0) = \alpha \qquad \dot{R}(0) = aJ(0) = a\beta$$

$$J(0) = \beta \qquad \dot{J}(0) = -bR(0) = -b\alpha$$

so we have:

$$C = R(0) = \alpha \qquad D = \frac{\dot{R}(0)}{\sqrt{ab}} = \beta\sqrt{\frac{a}{b}}$$

$$E = J(0) = \beta \qquad F = \frac{\dot{J}(0)}{\sqrt{ab}} = -\alpha\sqrt{\frac{b}{a}}$$

Hence

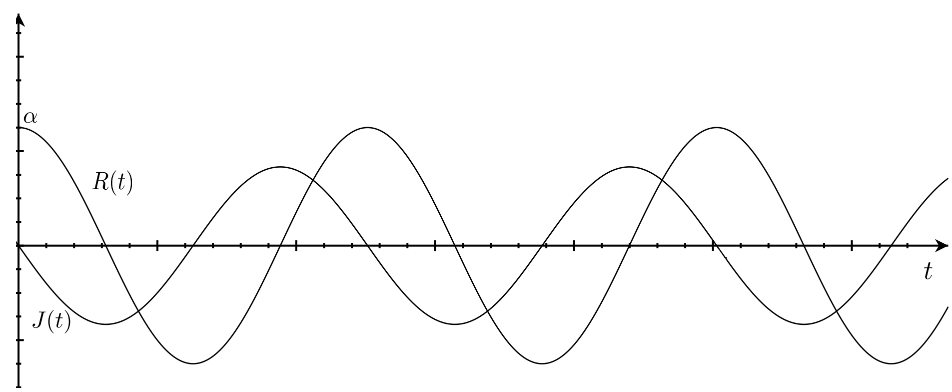
$$R(t) = \alpha\cos(\sqrt{ab}t) + \beta\sqrt{\frac{a}{b}}\sin(\sqrt{ab}t)$$

$$J(t) = \beta\cos(\sqrt{ab}t) - \alpha\sqrt{\frac{b}{a}}\sin(\sqrt{ab}t)$$

Suppose (for example) that $\beta = 0$ and $\alpha > 0$. Then

$$\begin{aligned} R(t) &= \alpha \cos(\sqrt{ab}t) \\ J(t) &= -\alpha\sqrt{\frac{b}{a}} \sin(\sqrt{ab}t) \end{aligned}$$

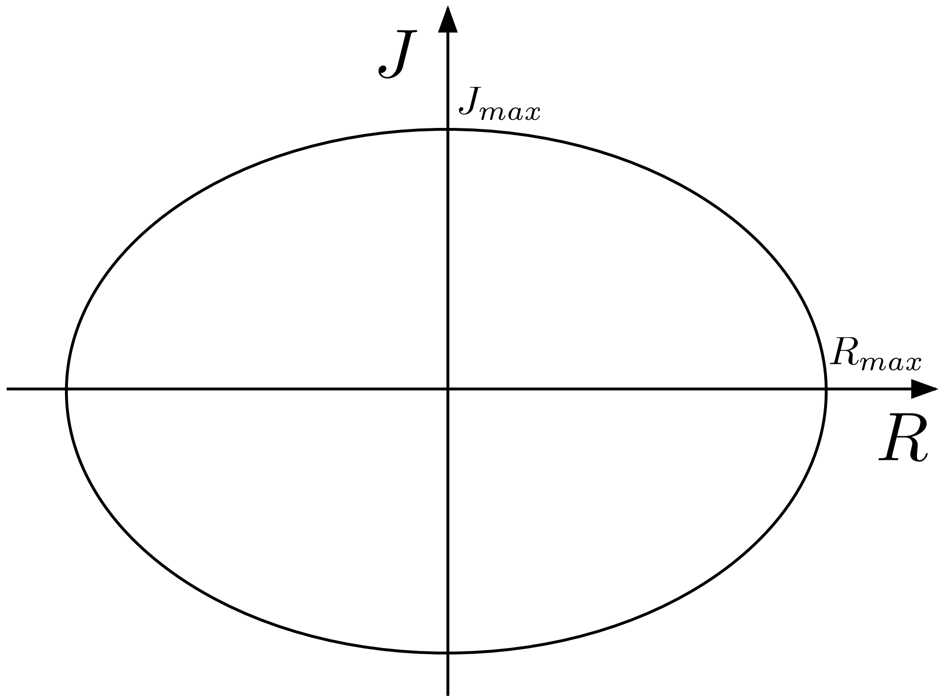
Then the graphs of $R(t)$ and $J(t)$ are:



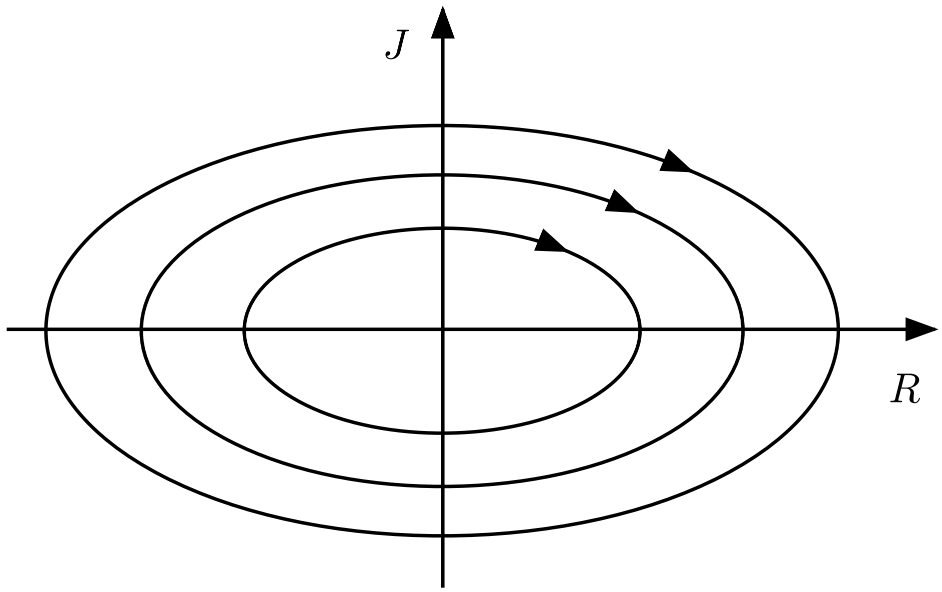

Actually it is more useful to eliminate $t$ and get a direct relation between $R$ and $J$. A bit of algebra will convince you that:

$$\frac{R^2}{R_{max}^2} + \frac{J^2}{J_{max}^2} = 1$$

which can be sketched in the $R - J$ plane: it is an ELLIPSE.




This is just for these particular initial conditions. Different initial conditions will result in a smaller or a larger ellipse. The full set of ALL possible love-affairs is represented by an infinite set of concentric ellipses: So this diagram tells

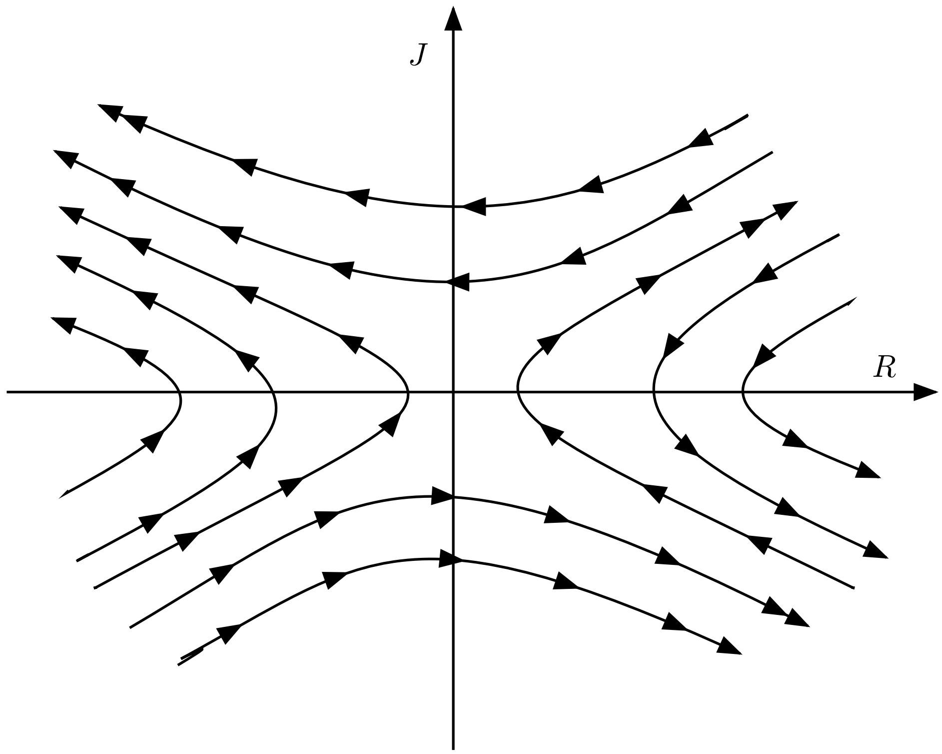



us everything there is to know about love.

Note that at a particular time $t$, $R(t)$ and $J(t)$ have definite values, giving a point $(R(t), J(t))$ in these pictures. The arrows indicate the direction of motion of such a point as time goes by. You can check this against the graphs on page 5 - notice that $J$ becomes negative immediately after $t = 0$ if the initial coordinates are

$(R, J) = (\alpha, 0)$. A picture like this, where we have two functions $R(t)$ and $J(t)$ but where we eliminate $t$ (and regard it as a PARAMETER) is called a PHASE PLANE DIAGRAM. It tells us many things that are not so obvious from the diagram on page 5. For example, suppose we ask: can Romeo and Juliet ever have a steady relationship with $R=$ constant and $J=$constant? The answer is clear from the picture on page 7: this is possible only if $R = J = 0$ (at the centre). We say that this is a point of EQUILIBRIUM, and clearly it is the only one. Furthermore, it is obvious from the diagram that this equilibrium is STA-

BLE. Contrast this with a phase diagram




like the one in the next picture: In this picture $(R, J) = (0, 0)$ is a point of equilibrium, but a slight push along the positive $J$ axis will cause the point to be swept off to infinity in the second quadrant, with $J \to +\infty$, $R \to -\infty$. Clearly

UNSTABLE equilibrium! So stability can be determined just by a glance at the phase diagram.

VERY OFTEN, WE DON'T CARE ABOUT THE DETAILS OF THE SOLUTIONS — THE PHASE PLANE DIAGRAM ALREADY TELLS US ALL WE NEED TO KNOW FOR MANY APPLIED PROBLEMS!

So in this part of the course, we are more interested in the shape of the phase plane diagram than in the explicit solution.

## 7.2. HOW TO SOLVE SYSTEMS OF SIMULTANEOUS ODEs

Let's consider the general system:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$ , with $a$, $b$, $c$ and $d$ constants

($i.e.$ $\frac{dx}{dt} = ax + by$ , $\frac{dy}{dt} = cx + dy$)

Trial solution:

$\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix} = e^{rt}\vec{u}_0$ , $\vec{u}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} =$ constant

So we get (defining $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$)

$$re^{rt}\vec{u}_0 = Be^{rt}\vec{u}_0$$

or

$$B\vec{u}_0 = r\vec{u}_0$$

so the possibilities for $r$ are given by the EIGENVALUES of $B$. We have

$$(B - rI_2)\vec{u}_0 = \vec{0}$$

so non-trivial solutions ($i.e.$ $\vec{u}_o \neq \vec{0}$) exist only if $\det(B - rI_2) = 0$ $i.e.$ $(a-r)(d-r) - bc = 0$

$$\Rightarrow r = \frac{1}{2}\left[a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}\right]$$

$$\Rightarrow r = \frac{1}{2}\left[Tr[B] \pm \sqrt{(Tr[B])^2 - 4(Det[B])}\right].$$

Except in the very special case where $(TrB)^2$ is exactly equal to $4\det B$, there are two solutions $r_+$, $r_-$ and so

$$\vec{u}(t) = c_+ e^{r_+ t}\vec{u}_+ + c_- e^{r_- t}\vec{u}_-$$

where $c_+$ and $c_-$ are constants and $\vec{u}_+$ and $\vec{u}_-$ are eigenvectors corresponding to $r_+$, $r_-$. Naturally $r_+$ and $r_-$ could be complex so we might have to interpret the exponentials as sin and cos.

**Example** Solve

$$\left\{ \begin{aligned} \frac{dx}{dt} &= -4x + 3y \\ \frac{dy}{dt} &= -2x + y \end{aligned} \right\}$$

Here $B = \begin{pmatrix} -4 & 3 \\ -2 & 1 \end{pmatrix} \to TrB = -3\ ,\ \det B = 2$
$r = \frac{1}{2}[-3 \pm \sqrt{9-8}] = -1 \text{ or } -2.$

First eigenvector satisfies:

$$\begin{pmatrix} -4 & 3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \to -4x_0 + 3y_0 = -x_0$$

So $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ will work. (Note: if $B\vec{u}_0 = r\vec{u}_0$, then $B(c\vec{u}_0) = r(c\vec{u}_0)$ for any number $c$, so clearly $\vec{u}_0$ CANNOT BE FOUND UNIQUELY. So if you prefer $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 13.59 \\ 13.59 \end{pmatrix}$, go ahead - it won't matter!)

Second eigenvector satisfies

$$\begin{pmatrix} -4 & 3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = -2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow -4x_0 + 3y_0 = -2x_0$$

*i.e.* $-2x_0 + 3y_0 = 0$ so take $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

General solution is now: $\begin{pmatrix} x \\ y \end{pmatrix} = c_+ e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} +$ $c_- e^{-2t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ where $c_+$ and $c_-$ are arbitrary constants. *i.e.*

$$\begin{cases} x = c_+ e^{-t} + 3c_- e^{-2t} \\ y = c_+ e^{-t} + 2c_- e^{-2t} \end{cases}$$

and we can work out $c_+$ and $c_-$ given $x(0)$ and $y(0)$.

**Example** Solve

$$\left\{ \begin{array}{l} \frac{dx}{dt} = 4x - 5y \\ \frac{dy}{dt} = 2x - 2y \end{array} \right\}$$

with the initial conditions x(0) = 0, y(0) = 2.

Here $B = \begin{pmatrix} 4 & -5 \\ 2 & -2 \end{pmatrix} \rightarrow TrB = 2$ , $\det B =$ $2$ $r = \frac{1}{2}[2 \pm \sqrt{4-8}] = 1 \pm i$

So we see that the eigenvalues are complex numbers here. This means that the eigenvectors too are complex, and this leads to a big mess. In this case, it is actually a lot easier to use Laplace transforms instead of the method used above, because it automatically takes care of finding the real part. We proceed as follows.

Our equations can be written in matrix form:

$$\frac{d\overrightarrow{v}}{dt} = B\overrightarrow{v},$$

as we did earlier. Take the Laplace transform of both sides and let $\overrightarrow{V}(s)$ be the Laplace transform of $\overrightarrow{v(t)}$; then we have

$$s\overrightarrow{V(s)} - \overrightarrow{v(0)} = B\overrightarrow{V(s)}.$$

We can write this as

$$\overrightarrow{V(s)} = [sI - B]^{-1}\overrightarrow{v(0)},$$

or

$$\overrightarrow{V(s)} = \begin{pmatrix} s-4 & 5 \\ -2 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$= \frac{1}{s^2 - 2s + 2} \begin{pmatrix} s+2 & -5 \\ 2 & s-4 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

and so

$$X(s) = \frac{-10}{(s-1)^2+1}, \quad Y(s) = \frac{2(s-1)}{(s-1)^2+1} - \frac{6}{(s-1)^2+1}.$$

Taking the inverse Laplace transform in the usual way, we get x(t) $= -10sin(t)e^t$, y(t) $= 2e^t cos(t) - 6e^t sin(t)$.

WHAT TO NOTICE ABOUT THESE EXAMPLES:

First, the solutions are always combinations of $e^t$, $e^{-t}$, sin and cos. If it's $e^{-t}$ or sin or cos, the solution is BOUNDED as $t \to \infty$ and we associate that with STABLE BEHAVIOUR, *i.e.* if a system is in a stable state then [by definition] if you push it SLIGHTLY away from that state, it will not stray very far from the original state. On the other hand, $e^t$ soon becomes large even if it is small at first, so we associate $e^t$ (or any other positive exponent) with UNSTABLE behaviour.

Now in the FIRST example, we got $e^{-t}$ and $e^{-2t}$ because the eigenvalues of $B$ were $-1$ and $-2$, *i.e.* BOTH REAL AND BOTH NEGATIVE. That solution is stable because BOTH $e^{-t} \to 0$ and $e^{-2t} \to 0$ as $t \to \infty$. If the eigenvalues had been $+1$ and $+2$ then of course we would have an unstable situation, and similarly if they had been $+1$ and $-2$ or $-1$ and $+2$.

**SUMMARY**

**The stability of the linear system of ODEs with constant coefficient can be decided by examining the trace and determinant of the coefficient matrix**

$B$ [**because these numbers allow us to compute the eigenvalues.**

Now actually we can do much better than this. We saw that the phase diagram for Romeo and Juliet ALWAYS had the same general shape, a set of ellipses, no matter what the precise values of the coefficients might have been. It turns out that something similar happens for ANY coefficient matrix B: IF WE KNOW ITS EIGENVALUES [or its trace and determinant] THEN WE CAN PREDICT THE SHAPE OF THE PHASE DIAGRAM.

**7.3. PHASE PLANE: COMPLETE CLASSIFICATION** It turns out that all phase diagrams for $\frac{d\vec{u}}{dt} = B\vec{u}$, $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$, belong to one of the following families.

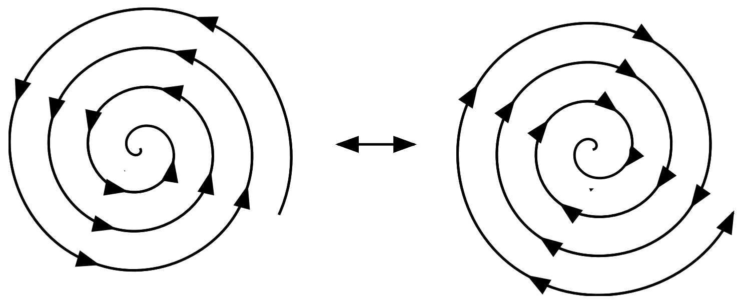


Figure 1: Spiral Sink

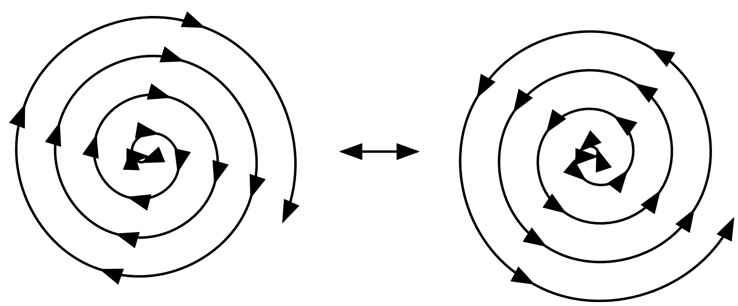


Figure 2: Spiral Source

(Note: both of the above can be either clockwise or anticlockwise)

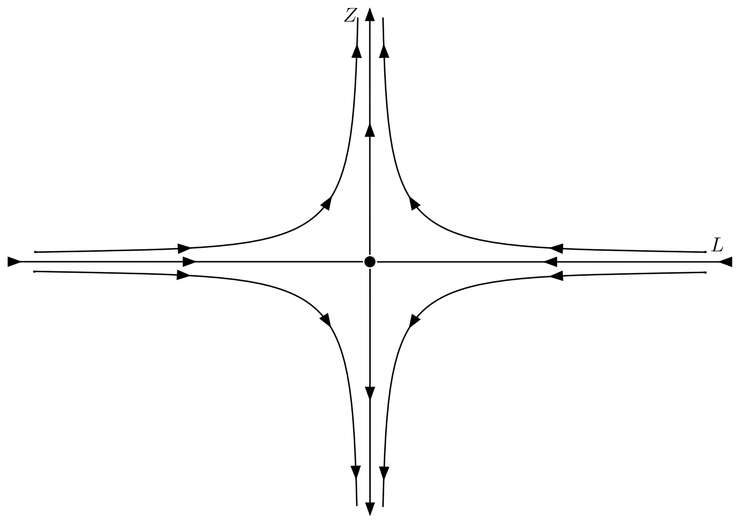



Figure 3: Saddle Point

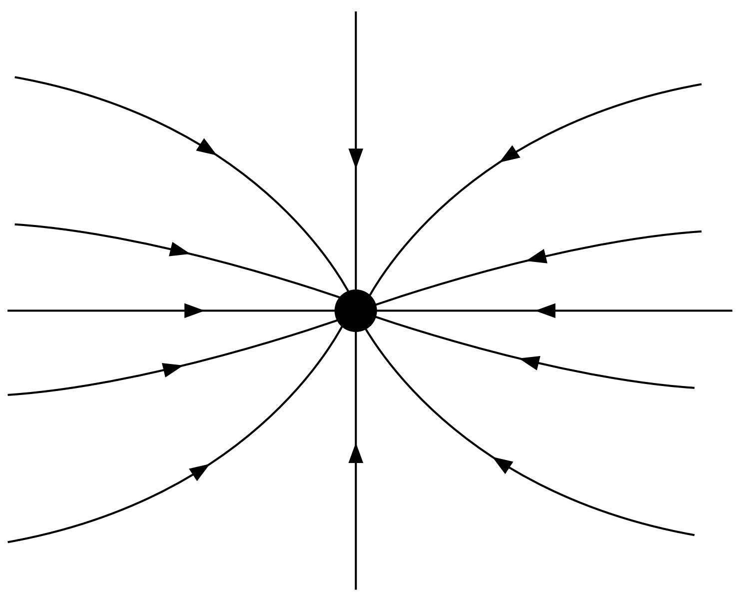


Figure 4: Nodal Sink

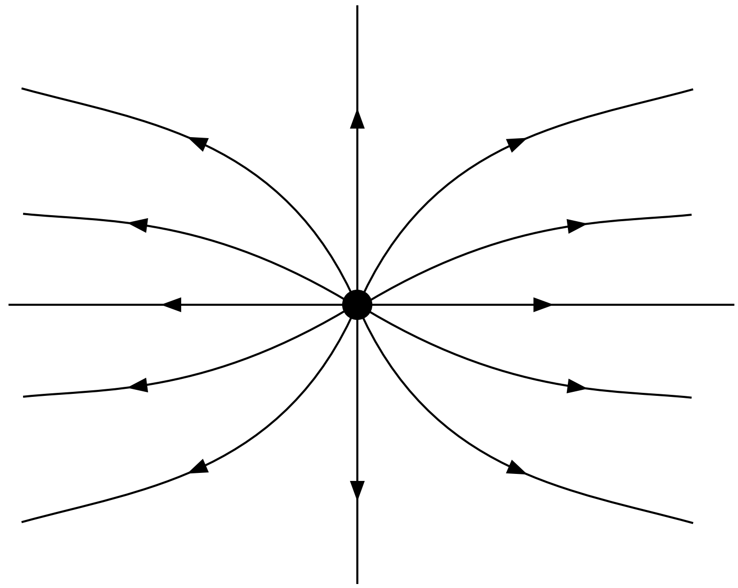


Figure 5: Nodal Source

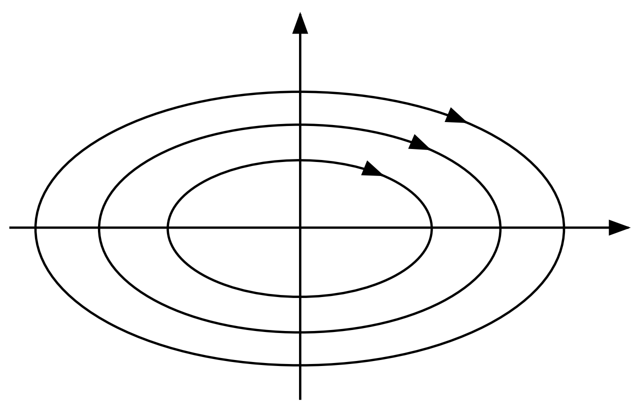


Figure 6: Centre

In words:

1. A spiral sink has all trajectories (*i.e.* solution curves) moving on a spiral TOWARDS

a certain equilibrium point.

2. A spiral source has all trajectories moving on a spiral AWAY FROM a certain equilibrium point.

3. A saddle has some trajectories moving toward, and some away from an equilibrium point

4. A nodal sink has all trajectories moving NOT on a spiral, TOWARDS a certain equilibrium point

5. A nodal source has all trajectories moving NOT on a spiral, AWAY FROM a certain equilibrium point

6. A centre has the trajectories orbiting around an equilibrium point [like Romeo and Juliet]

To understand what is happening in terms of eigenvalues, recall

$$r = \frac{1}{2}\left[TrB \pm \sqrt{(TrB)^2 - 4\det B}\right]$$

We know that if they are REAL, then we only get exponentials. Comparing this with the diagrams above, we see:

- Both positive $\longrightarrow$ Nodal source [everything moving away from the origin]
- Both negative $\longrightarrow$ Nodal sink [everything moving towards the origin]
- One positive, one negative $\longrightarrow$ saddle [things

move toward the origin in one direction, away from the origin in another direction]

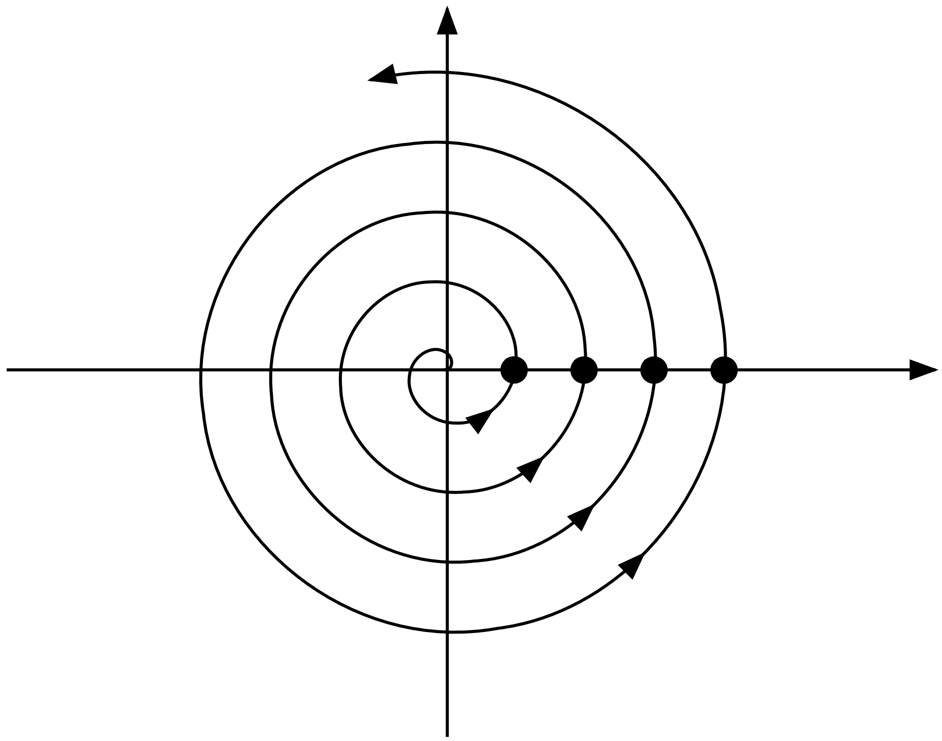
What if the eigenvalues are complex? Then

$$\underset{\text{exponential}\uparrow}{\tfrac{1}{2}\left[TrB\right.}\ \ \underset{\text{sin, cos}\uparrow}{\left.\pm\sqrt{(TrB)^2-4\det B}\right]}$$

Clearly $TrB$ controls the exponential part and $\pm\sqrt{...}$ controls sin, cos part.

Now a SPIRAL, by definition, is a curve which is (a) steadily moving away from (or towards) the origin and (b) moves AROUND the origin infinitely many times.

So WE GET SPIRALS WHEN THE EIGENVALUES ARE COMPLEX AND WHEN $TrB \neq 0$ (Source if $TrB > 0$, sink if $TrB < 0$). What



if $TrB = 0$? then if $r$ is complex we still have sin and cos, but we have no exponential, e.g.

$$\left.\begin{array}{l} x = cos(t) \\ y = sin(t) \end{array}\right] \text{CIRCLE.}$$

In general, WE GET A CENTRE WHEN THE EIGENVALUES ARE PURE IMAGINARY *i.e.* $TrB = 0$.

## SUMMARY

Given $\begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{cases}$ *i.e.* $\frac{d\vec{u}}{dt} = B\vec{u}$, if you want to sketch the phase plane, you can do it as follows:

Find eigenvalues of $B$, given by

$$r = \frac{1}{2}\left[TrB \pm \sqrt{(TrB)^2 - 4\det B},\right]$$

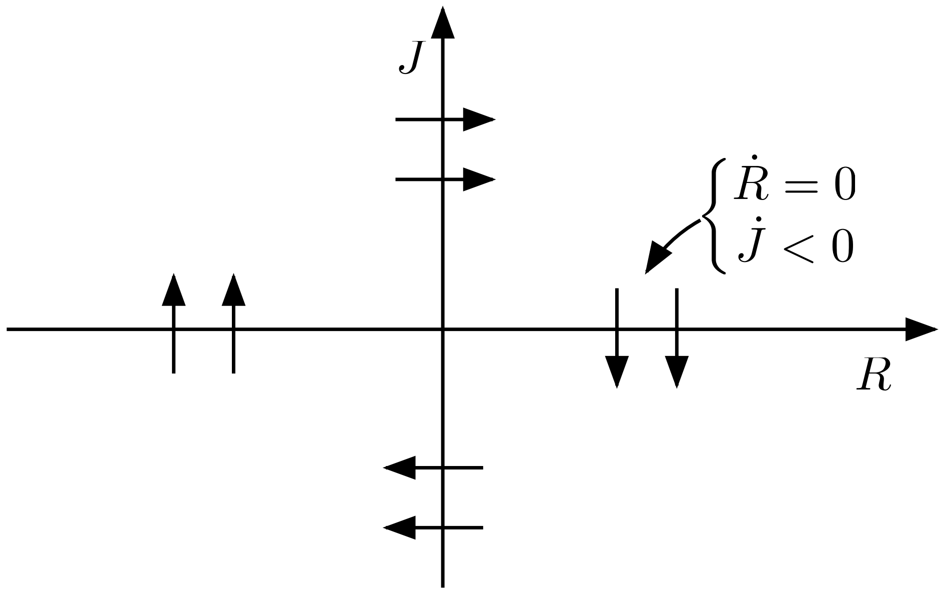
and then classify as follows:

**EIGENVALUES**

- both real
  - BOTH $> 0 \rightarrow$ Nodal Source
  - BOTH $< 0 \rightarrow$ Nodal Sink
  - Opposite Signs $\rightarrow$ Saddle
- both complex
  - Real Part $> 0 \rightarrow$ Spiral Source
  - Real Part $< 0 \rightarrow$ Spiral Sink
  - Real Part $= 0 \rightarrow$ Centre

EXAMPLE Romeo + Juliet

$$B = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}, \quad a, b > 0, \quad TrB = 0,$$



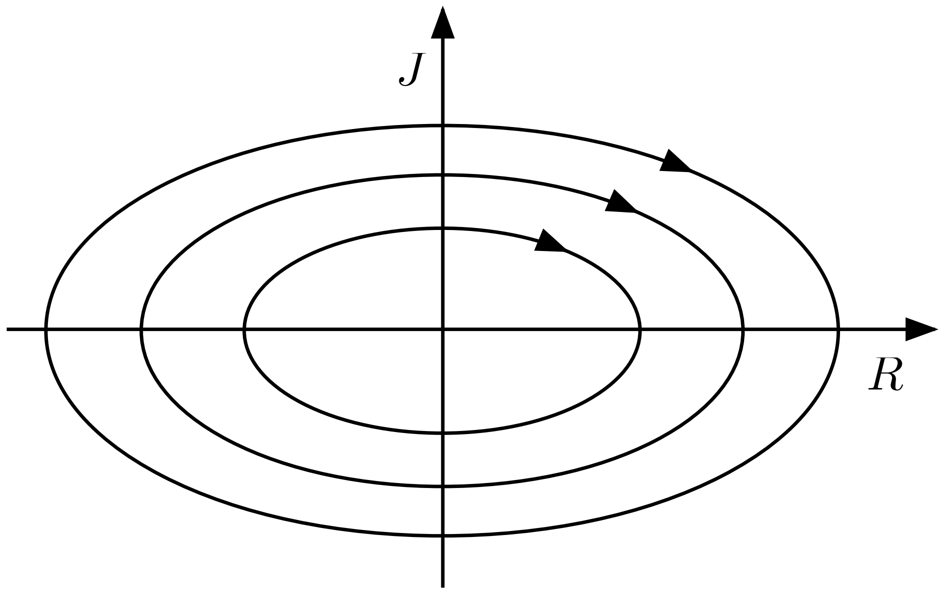

$\det B = ab > 0$

$$(TrB)^2 - 4 \det B < 0 \rightarrow \text{ complex}$$

$$\sqrt{0 - 4ab} = \sqrt{ab}i, \text{ Real part } = 0$$

$\Rightarrow$ CENTRE

Clearly, from the equations themselves, J is decreasing anywhere along the positive R axis, so the arrows have to be pointing DOWN there [because down is the direction of decreasing J],




and this tells us that the direction of the arrows must be clockwise, as in the diagram.

But now suppose we modify the story a little. [To be concrete, let's suppose that a = 5/hour and b = 3/hour.] The original equations mean that, for example, if Juliet feels nothing for Romeo, then his feelings for her will remain static. But in reality his patience is not infinite: if she persists in feeling nothing for him then his affection will decay away. So a better equation might be $dR/dt = -0.8\ R + 5\ J$. This means that, if Juliet feels nothing for him, then his feeling for her will decay exponentially, as you know from the Malthus model. OK, he's not very patient.

Juliet, on the other hand, being somewhat eccentric and perhaps a little confused, might become more and more attached to Romeo even if he is unaware of her existence, eg in the way some girls fall in love with members of boy bands. So perhaps we really have $dJ/dt = -3\ R + 0.7\ J$. This time we see that, even if $R = 0$, Juliet's affections will grow exponentially.

The matrix is now $\begin{pmatrix} -0.8 & 5 \\ -3 & 0.7 \end{pmatrix}$. You can verify in this case that despite Juliet's crazed love for Romeo, this system is a spiral sink [complex eigenvalues, negative trace], so the lovers are doomed to a spiral downwards, leading finally to utter indifference on both sides. The spiral is clockwise, as you can verify; you can also see this by regarding this example as a small perturbation of the previous one, where the motion in the phase plane was clockwise. [In other words, spirals are what you get instead of a centre when the trace is small but not exactly zero.]

EXAMPLE (page 13)

$$B = \begin{pmatrix} -4 & 3 \\ -2 & 1 \end{pmatrix} \rightarrow TrB = -3\ ,\ \det B = 2$$

$$(TrB)^2 - 4\det B = 1 > 0 \rightarrow \text{ real}$$

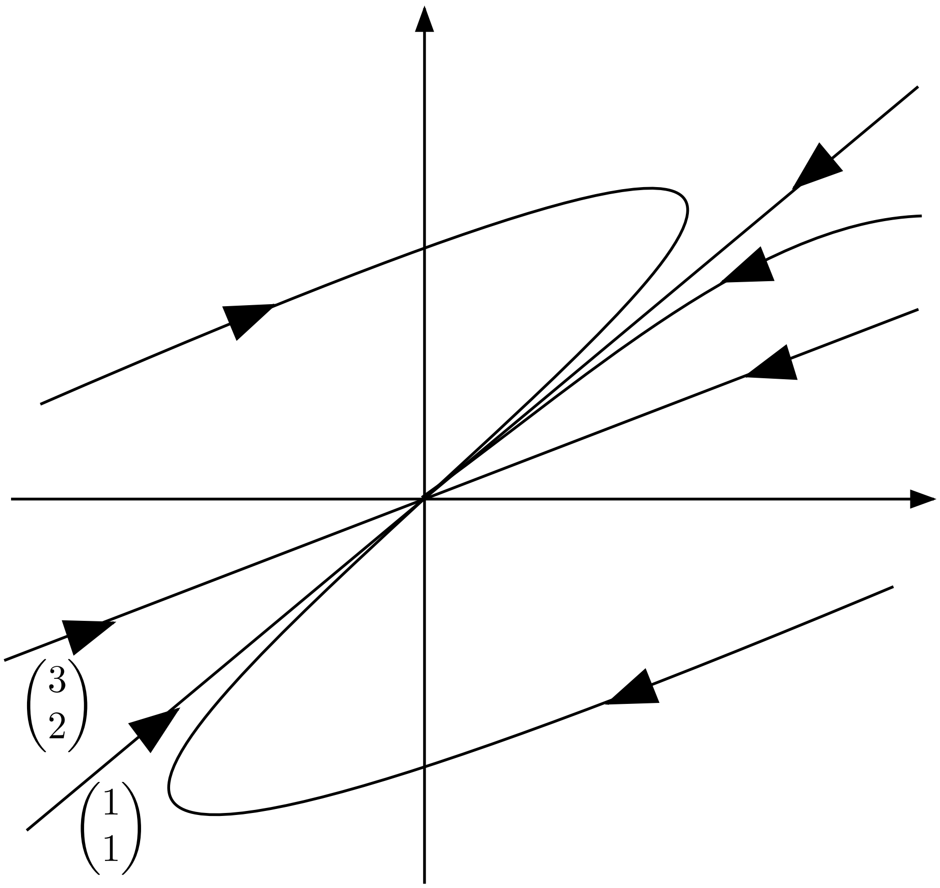
$TrB < 0$ , both eigenvalues negative $\rightarrow$ nodal sink

The solution is actually

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_+ e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_- e^{-2t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

and the phase diagram is shown in the next diagram.

Don't worry too much about the details of this picture: the main thing is for you to understand that a nodal sink has this kind of shape, with everything being sucked into the origin along a sort of S-shaped pattern.



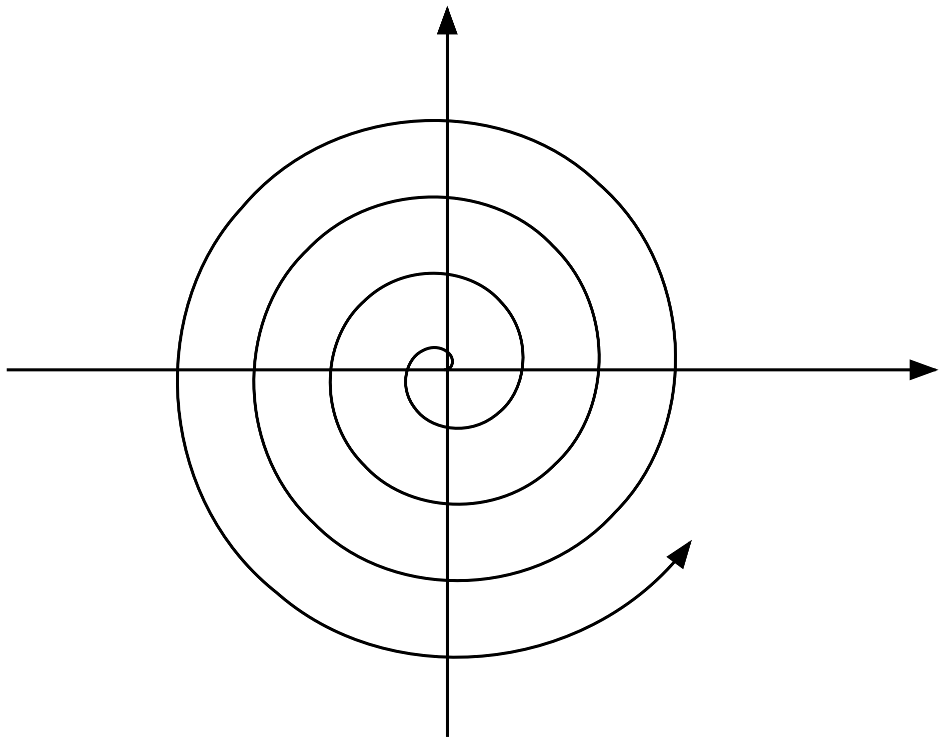


Example (page 14)

$$B = \begin{pmatrix} 4 & -5 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} Tr B = 2 \\ \det B = 2 \end{pmatrix} \to \text{spiral source}$$

The phase diagram is shown in the next diagram.

How did I know that the spirals move in the



anticlockwise direction? Easy: just check what happens along the positive x axis. On that axis, y is zero so dy/dt = 2x - 2y = 2x which is of course positive, so y must be INCREASING there, which means that the direction is upward, that is, anticlockwise!

## 7.4. WARFARE

A long and bitter battle is being fought on the slopes of Mount Doom between 15,000 Men of Gondor and 11,000 Orcs of Mordor. The Men die at a rate proportional to the number of Orcs, and also from a dread disease spread among them by the servants of Sauron, while the Orcs only die at a rate proportional to the number of Men – Orcs never get sick.

Let G(t) denote the number of Gondorians and M(t) denote the number of Mordor citizens in the battle. Then the above information tells us that we have a pair of differential equations which might have this form:

$$\frac{d}{dt}\begin{pmatrix} G \\ M \end{pmatrix} = \begin{pmatrix} -1 & -0.75 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} G \\ M \end{pmatrix},$$

where the $-1$ in the top left corner is the death rate per capita of the Gondorians due to disease, the $-0.75$ describes the rate at which Mordorians kill Gondorians, and the $-1$ in the second row describes the rate at which Gondorians kill Mordorians. [So the Gondorians are better soldiers.]

ODEs have actually been used to study real battles in this way! Of course, AS USUAL, you would want to make the model a lot more complicated than this model if you are really serious.

You can easily show that this is a saddle. The two eigenvectors are $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$, but we are only interested in the first quadrant, since G and M cannot be negative! So we focus on the second vector. If we take the G axis to be horizontal and the M axis to be vertical, you can show that this vector points along the line M = (2/3)G. The points in the phase plane move towards the origin along this line, because this vector has a negative eigenvalue.

If you draw the phase plane diagram for this saddle, you will soon see that any initial point lying BELOW this line will lead to victory for Gondor. Any initial value ABOVE it results in victory for Mordor. [Any initial point ON this line results in a situation where the last Gondorian falls dead while killing the last Mordorian!] Alas, you should be able to see that MORDOR WINS this particular battle! That wasn't obvious, since the Gondorians heavily outnumbered the Mordorians, and furthermore they are better fighters. Sad.