2013/2014 SEMESTER 2 MID-TERM TEST

MA1506 MATHEMATICS II

March 2014

8:30pm - 9:30pm

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **Thirteen (13)** printed pages.

2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.

3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1/10).

4. Use **only 2B pencils** for FORM CC1/10.

5. On FORM CC1/10 (section B for matric numbers starting with A, section C for others), **write** your **matriculation number** and **shade** the corresponding numbered circles **completely**. Your FORM CC1/10 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.

6. **Write your full name** in the blank space for module code in section A of FORM CC1/10.

7. Only circles for answers 1 to 10 are to be shaded.

8. For each answer, the circle corresponding to your choice should be **properly** and **completely** shaded. If you change your answer later, you must make sure that the original answer is properly erased.

9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.

10. **Do not fold** FORM CC1/10.

11. Submit FORM CC1/10 before you leave the test hall.

Formulae Sheet

1. Integrating factor of $y' + Py = Q$ is given by

$$R = e^{\int P dx}.$$

2. The variation of parameter formulae for $y'' + py' + qy = r$:

$$u = \int \frac{-ry_2}{y_1 y_2' - y_2 y_1'} dx$$
$$v = \int \frac{ry_1}{y_1 y_2' - y_2 y_1'} dx.$$

1. Let $y = -3x^k$ be a solution of the differential equation

$$\frac{dy}{dx} = x^2y^2.$$

Then the value of the constant k must be

(A) -3

(B) 3

(**C**) 2

(**D**) -2

(**E**) None of the above

2. A perfectly spherical rain drop falls through very dry air and it evaporates in such a way that it always keeps its perfectly spherical shape and that the rate of reduction of its volume is directly proportional to its surface area. It is observed that the radius of the raindrop is halved at time $t = 30$ minutes and that the raindrop completely disappears at time $t = T$ minutes. What is the value of T ?

(A) $\sqrt{1800}$

(B) 45

(**C**) 60

(**D**) $\sqrt{7200}$

(**E**) None of the above

3. Suppose that y is a function of x that satisfies the differential equation

$$x\frac{dy}{dx} - \frac{1}{x}y = e^{-\frac{1}{x}}, \quad x > 0, \quad y(1) = 1.$$

Find the value of $y(2)$. Give your answer correct to three decimal places.

(A) 2.069

(B) 2.387

(C) 3.054

(D) 3.271

(E) None of the above

4. Romeo bought a cold beer which had a temperature of 4^o C and brought it to the lecture hall for Juliet. After half an hour, Juliet had not shown up and he noticed that the temperature of the beer was 10^o C. To his disappointment, Juliet never showed up for class that day and after the one hour lecture, he decided to bring the beer to his dorm room to drown his sorrow. Assume that the temperature of the air-conned lecture hall stayed constant at 20^o C, what was the temperature of the beer at the end of the lecture? Give your answer in degrees C correct to two decimal places.

(A) 13.25

(B) 13.75

(C) 14.25

(D) 14.75

(E) None of the above

5. Suppose that y is a function of x that satisfies the differential equation

$$x^2\frac{dy}{dx} + 2xy = y^3, \quad x > 0, \quad \text{and} \quad y(1) = 1.$$

Find the value of $y(2)$. Give your answer correct to three decimal places.

(A) 0.284

(B) 0.293

(C) 0.308

(D) 0.319

(E) None of the above

6. It is known that both $y = 3e^{-2x}$ and $y = 8e^{-4x}$ are solutions of the differential equation

$$ay'' + 2by' + cy = 0,$$

where a, b, c are three non-zero constants. Which one of the following equations must hold?

(A) $2b + c = 0$

(B) $3a - b = 0$

(C) $6a - c = 0$

(D) $8b + 3c = 0$

(E) None of the above

7. It is known that y is a function of x that satisfies the differential equation

$$y'' + y = \frac{1}{2}\cos y, \quad y(0) = y'(0) = 0.$$

What are the possible values of y' when $y = \frac{\pi}{4}$?

(Hint: you may want to use the formula $y'' = \frac{d}{dy}[\frac{1}{2}y'^2]$.)

(A) $\pm\sqrt{\frac{1}{\sqrt{2}} + \frac{\pi^2}{16}}$

(B) $\pm\sqrt{\frac{1}{\sqrt{2}} - \frac{\pi^2}{16}}$

(C) $\pm\sqrt{\frac{1}{2} + \frac{\pi^2}{16}}$

(D) $\pm\sqrt{\frac{1}{2} - \frac{\pi^2}{16}}$

(E) None of the above

8. Let y be a solution of the differential equation

$$y'' - 4y' - 5y = 6e^{-x}$$

such that

$$y(0) = 1, \text{ and } \quad y'(0) = 0.$$

Find the value of $y(\ln 2)$. Give your answer correct to two decimal places.

(A) 9.31

(B) 11.27

(C) 8.48

(D) 10.65

(E) None of the above

9. Let y be a solution of the differential equation

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Find the value of $y(\pi)$.

(A) $1 + \frac{\pi}{2}$

(B) $3 + \frac{\pi}{2}$

(C) $2 + \frac{\pi}{2}$

(D) $\frac{1}{2} + \frac{\pi}{2}$

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10. Let y be a solution of the differential equation

$$y'' - 2y' + y = e^x \cosh x,$$

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Find the value of $y(2)$. Give your answer correct to the nearest integer.

(Hint: You may use the fact that the general solution of the equation $y'' - 2y' + y = 0$ is $c_1 e^x + c_2 x e^x$ and you may want to use the formulae $\frac{d}{dx} \sinh x = \cosh x$ and $\frac{d}{dx} \cosh x = \sinh x$.)

(A) 30

(B) 40

(C) 50

(D) 60

(E) None of the above

END OF PAPER

Blank page for you to do your calculations

Answers to mid term test

1. A
2. C
3. A
4. B
5. D
6. B
7. B
8. D
9. A
10. C

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Then the value of the constant k must be

(A) -3

(B) 3

(C) 2

(D) -2

(E) None of the above

$y = -3x^k$

$\therefore \frac{dy}{dx} = -3kx^{k-1}$

$\therefore -3kx^{k-1} = x^2\{(-3x^k)^2\} = 9x^{2k+2}$

$\therefore \begin{cases} -3k = 9 \\ k-1 = 2k+2 \end{cases}$

$\therefore k = -3$

2. A perfectly spherical rain drop falls through very dry air and it evaporates in such a way that it always keeps its perfectly spherical shape and that the rate of reduction of its volume is directly proportional to its surface area. It is observed that the radius of the raindrop is halved at time $t = 30$ minutes and that the raindrop completely disappears at time $t = T$ minutes. What is the value of T ?

(A) $\sqrt{1800}$

(B) 45

(C) 60

(D) $\sqrt{7200}$

(E) None of the above

$V = \frac{4}{3}\pi r^3$, $A = 4\pi r^2$ and $\frac{dV}{dt} = kA$

$\Rightarrow 4\pi r^2 \frac{dr}{dt} = 4\pi k r^2 \Rightarrow \frac{dr}{dt} = k$

$\therefore r = kt + r_0$

$\frac{1}{2} r_0 = 30k + r_0 \Rightarrow k = -\frac{1}{60} r_0$

$\therefore r = -\frac{1}{60} r_0 t + r_0$

$r = 0 \Rightarrow t = 60$

3. Suppose that y is a function of x that satisfies the differential equation

$$x\frac{dy}{dx} - \frac{1}{x}y = e^{-\frac{1}{x}}, \quad x > 0, \quad y(1) = 1.$$

Find the value of $y(2)$. Give your answer correct to three decimal places.

(A) 2.069

(B) 2.387

(C) 3.054

(D) 3.271

(E) None of the above

$$\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x}e^{-\frac{1}{x}}$$

$$R = e^{\int -\frac{1}{x^2}dx} = e^{\frac{1}{x}}$$

$$y = e^{-\frac{1}{x}} \int e^{\frac{1}{x}} \frac{1}{x} e^{-\frac{1}{x}} dx$$

$$= e^{-\frac{1}{x}} (\ln x + C)$$

$$y(1) = 1 \Rightarrow 1 = e^{-1}(0 + C) \Rightarrow C = e$$

$$\therefore y = e^{-\frac{1}{x}} (\ln x + e)$$

$$y(2) = e^{-\frac{1}{2}} (\ln 2 + e) \simeq 2.069$$

4. Romeo bought a cold beer which had a temperature of 4^o C and brought it to the lecture hall for Juliet. After half an hour, Juliet had not shown up and he noticed that the temperature of the beer was 10^o C. To his disappointment, Juliet never showed up for class that day and after the one hour lecture, he decided to bring the beer to his dorm room to drown his sorrow. Assume that the temperature of the air-conned lecture hall stayed constant at 20^o C, what was the temperature of the beer at the end of the lecture? Give your answer in degrees C correct to two decimal places.

(A) 13.25

(B) 13.75

(C) 14.25

(D) 14.75

(E) None of the above

$\frac{dT}{dt} = k(T-20) \Rightarrow T-20 = Ae^{kt}$

$T(0)=4 \Rightarrow A=-16 \Rightarrow T=20-16e^{kt}$

$T(\frac{1}{2})=10 \Rightarrow 10=20-16e^{\frac{1}{2}k} \Rightarrow k=2(\ln 10-\ln 16)$

$\therefore T=20-16e^{2(\ln 10-\ln 16)t}$

$T(1)=20-16e^{2(\ln 10-\ln 16)}=13.75$

5. Suppose that y is a function of x that satisfies the differential equation

$$x^2\frac{dy}{dx} + 2xy = y^3, \quad x > 0, \quad \text{and} \quad y(1) = 1.$$

Find the value of $y(2)$. Give your answer correct to three decimal places.

(A) 0.284

(B) 0.293

(C) 0.308

(D) 0.319

(E) None of the above

$\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x^2}y^3 \Rightarrow z = y^{1-3} = y^{-2} \Rightarrow dz = -2y^{-3}dy$

$\therefore -\frac{1}{2}y^3\frac{dz}{dx} + \frac{2}{x}y = \frac{1}{x^2}y^3 \Rightarrow \frac{dz}{dx} - \frac{4}{x}z = -\frac{2}{x^2}$

$R = e^{\int -\frac{4}{x}dx} = e^{-4\ln x} = \frac{1}{x^4}$

$z = x^4\int \frac{1}{x^4}\left(-\frac{2}{x^2}\right)dx = x^4\int -\frac{2}{x^6}dx = x^4\left\{\frac{2}{5x^5} + C\right\}$

$\frac{1}{y^2} = \frac{2}{5x} + Cx^4$

$y = \frac{1}{\sqrt{\frac{2}{5x} + Cx^4}}$ $(\because y(1)=1, \therefore y \text{ is +ve})$

$y(1)=1 \Rightarrow C = \frac{3}{5} \Rightarrow y = \frac{1}{\sqrt{\frac{2}{5x} + \frac{3}{5}x^4}}$

$y(2) = \frac{1}{\sqrt{\frac{1}{5} + \frac{48}{5}}} = \sqrt{\frac{5}{49}} \approx \underline{0.319}$

6. It is known that both $y = 3e^{-2x}$ and $y = 8e^{-4x}$ are solutions of the differential equation

$$ay'' + 2by' + cy = 0,$$

where a, b, c are three non-zero constants. Which one of the following equations must hold?

(A) $2b + c = 0$

(B) $3a - b = 0$

(C) $6a - c = 0$

(D) $8b + 3c = 0$

(E) None of the above

$a(\lambda+2)(\lambda+4)$

$= a\lambda^2 + 6a\lambda + 8a$

$\therefore 6a = 2b, \quad 8a = c$

$\Rightarrow \underline{3a - b = 0}, \quad 8a - c = 0, \quad 8b - 3c = 0$

7. It is known that y is a function of x that satisfies the differential equation

$$y'' + y = \frac{1}{2}\cos y, \quad y(0) = y'(0) = 0.$$

What are the possible values of y' when $y = \frac{\pi}{4}$?

(Hint: you may want to use the formula $y'' = \frac{d}{dy}[\frac{1}{2}y'^2]$.)

(A) $\pm\sqrt{\frac{1}{\sqrt{2}} + \frac{\pi^2}{16}}$

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(C) $\pm\sqrt{\frac{1}{2} + \frac{\pi^2}{16}}$

(D) $\pm\sqrt{\frac{1}{2} - \frac{\pi^2}{16}}$

(E) None of the above

$$\frac{d}{dy}\left\{\tfrac{1}{2}y'^2\right\} = \tfrac{1}{2}\cos y - y$$

$$\tfrac{1}{2}y'^2 = \tfrac{1}{2}\sin y - \tfrac{1}{2}y^2 + C$$

$$y(0) = y'(0) = 0 \Rightarrow C = 0$$

$$\therefore y' = \pm\sqrt{\sin y - y^2}$$

$$y = \tfrac{\pi}{4} \Rightarrow y' = \pm\sqrt{\tfrac{1}{\sqrt{2}} - \tfrac{\pi^2}{16}}$$

8. Let y be a solution of the differential equation

$$y'' - 4y' - 5y = 6e^{-x}$$

such that

$$y(0) = 1, \text{ and } y'(0) = 0.$$

Find the value of $y(\ln 2)$. Give your answer correct to two decimal places.

(A) 9.31

(B) 11.27

(C) 8.48

(D) 10.65

(E) None of the above

$\lambda^2 - 4\lambda - 5 = 0 \Rightarrow \lambda = -1, 5$

Try $y = Axe^{-x} \Rightarrow \begin{cases} y' = Ae^{-x} - Axe^{-x} \\ y'' = -2Ae^{-x} + Axe^{-x} \end{cases}$

$\therefore -6Ae^{-x} = 6e^{-x}$

$\therefore A = -1$

$y = C_1 e^{-x} + C_2 e^{5x} - xe^{-x}$

$y' = -C_1 e^{-x} + 5C_2 e^{5x} - e^{-x} + xe^{-x}$

$y(0) = 1, \ y'(0) = 0 \Rightarrow C_1 = \frac{2}{3}, \ C_2 = \frac{1}{3}$

$\therefore y = \frac{2}{3}e^{-x} + \frac{1}{3}e^{5x} - xe^{-x}$

$y(\ln 2) \approx \underline{10.65}$

9. Let y be a solution of the differential equation

$$y'' + 4y = 2x$$

such that

$$y(0) = 1, \text{ and } y'(0) = 2.$$

Find the value of $y(\pi)$.

(A) $1 + \frac{\pi}{2}$

(B) $3 + \frac{\pi}{2}$

(C) $2 + \frac{\pi}{2}$

(D) $\frac{1}{2} + \frac{\pi}{2}$

(E) None of the above

$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$

Try $y = Ax + B \Rightarrow A = \frac{1}{2}, B = 0$

$\therefore y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{2}x$

$y' = -2C_1 \sin 2x + 2C_2 \cos 2x + \frac{1}{2}$

$y(0) = 1, y'(0) = 2 \Rightarrow C_1 = 1, C_2 = \frac{3}{4}$

$\therefore y = \cos 2x + \frac{3}{4}\sin 2x + \frac{1}{2}x$

$y(\pi) = 1 + \frac{\pi}{2}$

10. Let y be a solution of the differential equation

$$y'' - 2y' + y = e^x \cosh x,$$

such that

$$y(0) = 2, \text{ and } y'(0) = 3.$$

Find the value of $y(2)$. Give your answer correct to the nearest integer.

(Hint: You may use the fact that the general solution of the equation $y'' - 2y' + y = 0$ is $c_1 e^x + c_2 x e^x$ and you may want to use the formulae $\frac{d}{dx} \sinh x = \cosh x$ and $\frac{d}{dx} \cosh x = \sinh x$.)

(A) 30

(B) 40

(C) 50

(D) 60

(E) None of the above

END OF PAPER

Blank page for you to do your calculations

$$y_1 = e^x,\ y_2 = xe^x \Rightarrow \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$$

$$u = \int \frac{-xe^x(e^x \cosh x)}{e^{2x}}\,dx = \int -x\,d(\sinh x)$$

$$= -x\sinh x + \cosh x$$

$$v = \int \frac{e^x(e^x \cosh x)}{e^{2x}}\,dx = \int \cosh x\,dx = \sinh x$$

$$uy_1 + vy_2 = -xe^x \sinh x + e^x \cosh x + xe^x \sinh x$$

$$= e^x \cosh x$$

$$y = c_1 e^x + c_2 xe^x + e^x \cosh x$$

$$y' = c_1 e^x + c_2 e^x + c_2 xe^x + e^x \cosh x + e^x \sinh x$$

$$y(0) = 2,\ y'(0) = 3 \Rightarrow c_1 = c_2 = 1$$

$$\therefore y = e^x + xe^x + e^x \cosh x$$

$$y(2) = 3e^2 + e^2 \cosh 2 \approx \underline{\underline{49.966 \approx 50}}$$