NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2007-2008

MA1506 Mathematics II

Nov 2007 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of a total of **SIX (6)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. Marks for each question are indicated at the beginning of the question.
3. Candidates may use non-graphing, non-programmable calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [20 marks]

(a) Find the inverse Laplace transform of

$$\frac{8}{(s-2)(s^2+4)}.$$

(b) Find the Laplace transform of the following function

$$f(t) = \begin{cases} t & 0 \le t < 3 \\ 6-t & 3 \le t < 6 \\ 0 & 6 \le t. \end{cases}$$

(c) Solve the following initial value problem

$$y' - 2y = 2\delta(t-3), \qquad y(0) = 0.$$

Question 2 [20 marks]

(a) Find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4(2x-1)\cos(2x).$$

(b) Find a particular solution of the following differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x \sec x.$$

Question 3 [15 marks]

(a) A library has three branches A, B and C. The library policy allows books to be borrowed from any branch to be returned to either branch A or B. Branch C will only accept returns of books that were borrowed from branch C. A statistical study revealed that the chances of books being returned to the same branch where they were borrowed are 70%, 50% and 70% for branch A, B and C respectively. There is also 10% probability of books borrowed from branch C being returned to branch A. What is the probability that a book originally at branch B will end up at branch A after exactly four loans?

(b) Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}.$$

By diagonalizing the matrix, calculate exactly A^{10}.

Question 4 [15 marks]

(a) Suppose that

$$x(t) = 2te^{-2t} + \sqrt{3}e^{-2t} + 2\cos(\sqrt{6}t - \delta)$$

is the solution of a mass-spring system

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \alpha t, \qquad x(0) = x_0, \dot{x}(0) = v_0.$$

Assume that the homogeneous solution is not identically zero.

(i) Which part of the solution is the transient solution?

(ii) Is the system underdamped, critically damped or overdamped?

(iii) If the mass is $1kg$, what is the damping constant b?

(iv) Find the angular frequency α of the forcing function.

(v) Find the forcing amplitude F_0.

(b) Find the total energy of this system:

$$3\ddot{x} + x = 0, \qquad x(0) = 1, \dot{x}(0) = -2.$$

Question 5 [15 marks]

(a) Suppose you obtained a sample of 1,000 cells. After one day, you discovered that the cell population grew to 1,400. You make the assumption that the birth and death rates should be constant and the population is growing at 40% daily. Now, you know that by adjusting the temperature, you can double the birthrate while keeping the death rate constant. After making this adjustment, you discovered that the cell population started to double every day from that point onwards. What is the **new** birthrate per day as a percentage of the population? (Leave your answer correct to 2 decimal places.)

(b) Suppose you have a sample of cells whose population is at a **logistic** equilibrium of 20,000 with birthrate of 20% per day. Suppose you extract 1,200 cells daily for experiment. Show that this will eventually wipe out your sample. Determine how many days of experiments can you perform.

Question 6 [15 marks]

(a) Classify the following phase plane diagrams of systems of linear differential equations having the following matrices:

$$A = \begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 5 \\ -2 & 1 \end{bmatrix}.$$

(b) Suppose that the humans are battling against aliens. Each human soldier can kill aliens at an average rate of 5 per unit time, while the aliens can only kill humans at the rate of 2/5 per unit time. However, each alien that is alive is able to reproduce an exact replica of itself per unit time. How many humans are needed to destroy 5000 aliens completely?

END OF PAPER

2007-2008 Semester 1 Examination

1). (a) To find

$$L^{-1}\left\{\frac{8}{(s-2)(s^2+4)}\right\}$$

Let $\dfrac{8}{(s-2)(s^2+4)} = \dfrac{A}{s-2} + \dfrac{Bs+C}{s^2+4}$

$\therefore 8 = A(s^2+4) + (Bs+C)(s-2)$

$s = 2 \Rightarrow 8 = 8A \Rightarrow A = 1$

Compare $s^2 \Rightarrow 0 = A + B \Rightarrow B = -1$

Compare constants $\Rightarrow 8 = 4A - 2C \Rightarrow C = -2$

$$L^{-1}\left\{\frac{8}{(s-2)(s^2+4)}\right\}$$

$$= L^{-1}\left(\frac{1}{s-2}\right) - L^{-1}\left(\frac{s+2}{s^2+4}\right)$$

$$= L^{-1}\left(\frac{1}{s-2}\right) - L^{-1}\left(\frac{s}{s^2+2^2}\right) - L^{-1}\left(\frac{2}{s^2+2^2}\right)$$

$$= e^{2t} - \cos 2t - \sin 2t$$

1). (b) To find $L(f)$ where

$$f(t) = \begin{cases} t & 0 \le t < 3 \\ 6-t & 3 \le t < 6 \\ 0 & 6 \le t \end{cases}$$

Recall the formula: for $a < b$

$$g(t)\{u(t-a) - u(t-b)\} = \begin{cases} 0 & t < a \\ g(t) & a < t < b \\ 0 & b < t \end{cases}$$

$$\therefore f(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t < 3 \\ 0 & 3 < t \end{cases} + \begin{cases} 0 & t < 3 \\ 6-t & 3 < t < 6 \\ 0 & 6 < t \end{cases}$$

$$= t\{u(t-0) - u(t-3)\}$$
$$+ (6-t)\{u(t-3) - u(t-6)\}$$

$$= tu(t) - tu(t-3)$$
$$+ (6-t)u(t-3) - (6-t)u(t-6)$$

$$= tu(t) + (6-2t)u(t-3) + (t-6)u(t-6)$$

$$= tu(t) - 2(t-3)u(t-3) + (t-6)u(t-6)$$

$$\therefore L(f) = \frac{1}{s^2} - \frac{2}{s^2}e^{-3s} + \frac{1}{s^2}e^{-6s}$$

1). (c) To solve

$$\begin{cases} y' - 2y = 2\delta(t-3) \\ y(0) = 0 \end{cases}$$

$$L(y' - 2y) = L(2\delta(t-3))$$

$$\therefore \quad sY - y(0) - 2Y = 2e^{-3s}$$

$$Y = \frac{2}{s-2} e^{-3s}$$

Look up table:

$$L^{-1}\left(\frac{2}{s-2}\right) = 2e^{2t}$$

$$\therefore \quad y = L^{-1}\left(\frac{2}{s-2} e^{-3s}\right)$$

$$= 2e^{2(t-3)} u(t-3)$$

Question 2 [20 marks]

(a) Find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4(2x-1)\cos(2x).$$

Solution.

$$y'' - 4y' + 4y = 4(2x-1)\cos 2x \qquad (1)$$

$$y_h = Ae^{2x} + Bxe^{2x}$$

$$y_p = x(C\cos 2x + D\sin 2x) + (E\cos 2x + F\sin 2x)$$

$$y_p' = (C\cos 2x + D\sin 2x) + x(-2C\sin 2x + 2D\cos 2x) - 2E\sin 2x + 2F\cos 2x$$

$$\begin{aligned} y_p'' &= (4D\cos 2x - 4C\sin 2x) - 4x(C\cos 2x + D\sin 2x) - 4E\cos 2x - 4F\sin 2x \\ &= [(4D-4E)\cos 2x - (4C+4F)\sin 2x] - 4x(C\cos 2x + D\sin 2x) \end{aligned}$$

Substituting into equation 1 and equating coefficient to get

$$E = -\frac{1}{2}, \quad D = -1, \quad C = F = 0$$

Thus $\qquad y_p = -\frac{1}{2}\cos 2x - x\sin 2x$

Hence $\qquad y(x) = y_h(x) + y_p(x) = A\cos 2x + B\sin 2x - \frac{1}{2}\cos 2x - x\sin 2x$

2). (b) To solve

$$y'' - 2y' + 2y = e^x \sec x$$

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Let $y_1 = e^x \cos x$

$y_2 = e^x \sin x$

Then

$$u = \int \frac{-r y_2}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \int \frac{-e^{2x} \tan x}{\begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix}}$$

$$= \int \frac{-e^{2x} \tan x}{e^{2x}} = \int -\tan x = \ln|\cos x|$$

$$v = \int \frac{r y_1}{e^{2x}} = \int \frac{e^{2x}}{e^{2x}} = x$$

$$\therefore \; y = c_1 e^x \cos x + c_2 e^x \sin x + (\ln|\cos x|) e^x \cos x + x e^x \sin x$$

3). (a) Prob. $(A \to A) = 0.7$ $(B \to A) = ?$ $(C \to A) = 0.1$

Prob. $(A \to B) = ?$ $(B \to B) = 0.5$ $(C \to B) = ?$

Prob. $(A \to C) = 0$ $(B \to C) = 0$ $(C \to C) = 0.7$

$$M = \begin{pmatrix} 0.7 & 0.5 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0 & 0.7 \end{pmatrix} \quad (\because \text{sum of each column} = 1)$$

$$M^4 = \begin{pmatrix} 0.6256 & 0.624 & 0.4332 \\ 0.3744 & 0.376 & 0.3267 \\ 0 & 0 & 0.2401 \end{pmatrix}$$

$$\therefore \text{Prob.}\,(B \xrightarrow{4} A) = \underline{\underline{0.624}}$$

3). (b) $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) = 0$$

$$\Rightarrow \lambda = 1, 2.$$

$\lambda = 1 \Rightarrow \quad 0x + 2y = 0 \Rightarrow x = t, \; y = 0 \Rightarrow t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda = 2 \Rightarrow \quad -x + 2y = 0 \Rightarrow x = 2t, \; y = t \Rightarrow t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

$\therefore \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

and $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}^{10} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^{10} \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2046 \\ 0 & 1024 \end{pmatrix}$$

4). (a)

$$x = 2te^{-2t} + \sqrt{3}e^{-2t} + 2\cos(\sqrt{6}t - \delta)$$

solves

$$\begin{cases} m\ddot{x} + b\dot{x} + kx = F_0 \cos \alpha t \\ x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases}.$$

(i) Transient solution $= \underline{\underline{2te^{-2t} + \sqrt{3}e^{-2t}}}$

(ii) Observe that from the transient solution, $m\lambda^2 + b\lambda + k = 0$ has double root $\lambda = -2$.

$\therefore$ $\underline{\underline{\text{critically damped}}}$.

(iii) $m = 1 \Rightarrow$ sum of roots $= -b$

$\Rightarrow -2-2 = -b$

$\Rightarrow \underline{\underline{b = 4}}$

(iv) angular frequency $\alpha = \underline{\underline{\sqrt{6}}}$

(V) Recall the formula for the steady state solution is

$$\frac{\frac{1}{m} F_0 \cos(\alpha t - \delta)}{\sqrt{\left(\frac{k}{m} - \alpha^2\right)^2 + \frac{b^2 \alpha^2}{m^2}}}$$

$$\therefore \quad 2 = \frac{\frac{1}{m} F_0}{\sqrt{\left(\frac{k}{m} - \alpha^2\right)^2 + \frac{b^2 \alpha^2}{m^2}}}$$

$$= \frac{F_0}{\sqrt{(4-6)^2 + 16 \times 6}}$$

$\left(\because k = \text{product of roots} = (-2)(-2) = 4 \right)$

$$\therefore \ F_0 = 20$$

4). (b) $3\ddot{x} + x = 0$

$$\because \ddot{x} = \frac{d}{dt}(\dot{x}) = \left\{\frac{d}{dx}(\dot{x})\right\}\left(\frac{dx}{dt}\right) = \dot{x}\frac{d\dot{x}}{dx}$$

$$\therefore 3\dot{x}\frac{d\dot{x}}{dx} + x = 0$$

$$\therefore \frac{3}{2}\dot{x}^2 + \frac{1}{2}x^2 = \text{constant} = \text{total energy}$$

at $t = 0 \Rightarrow x(0) = 1,\ \dot{x}(0) = -2$

$$\therefore \frac{3}{2}(-2)^2 + \frac{1}{2}(1)^2 = \frac{13}{2}$$

$\therefore$ total energy $= \frac{13}{2}$ units

5). (a)

Before:

$$\frac{dN}{dt} = (B-D)N$$

$$\Rightarrow N = N_0 e^{(B-D)t} = 1000 e^{(B-D)t}$$

$$N(1) = 1400 \Rightarrow 1.4 = e^{(B-D)}$$

$$\Rightarrow B - D = \ln 1.4 \quad \text{----} \; ①$$

After:

$$\frac{d\tilde{N}}{dt} = (2B-D)\tilde{N}$$

$$\Rightarrow \tilde{N} = \tilde{N}_0 e^{(2B-D)t}$$

$$\tilde{N}(1) = 2\tilde{N}_0 \Rightarrow 2 = e^{(2B-D)}$$

$$\Rightarrow 2B - D = \ln 2 \quad \text{----} \; ②$$

$$② - ① \Rightarrow B = \ln 2 - \ln 1.4 \simeq 0.35667$$

$$\therefore \text{New rate} = 2B \simeq 0.71334$$

$$= 71.33\,\%$$

5). (b) $N_\infty = 20000$

$B = 0.2$

$\because N_\infty = \dfrac{B}{S}$

$\therefore S = \dfrac{B}{N_\infty} = \dfrac{1}{100000}$

$$\frac{B^2}{4S} = \frac{0.2 \times 0.2}{4} \times 100000 = 1000$$

$\because E = 1200 > 1000$

$\therefore$ Sample will be wiped out.

$$\frac{dN}{dt} = 0.2N - \frac{1}{100000}N^2 - 1200$$

$$= -\frac{1}{100000}\left\{N^2 - 20000N + 120000000\right\}$$

$$= -\frac{1}{100000}\left\{N^2 - 20000N + 100000000 + 20000000\right\}$$

$$= -\frac{1}{100000}\left\{(N-10000)^2 + 20000000\right\}$$

$$\frac{dN}{(N-10000)^2+20000000} = -\frac{1}{100000}dt$$

$$\int_{20000}^{0} \frac{dN}{(N-10000)^2+20000000} = \int_0^T -\frac{1}{100000}dt$$

$$\therefore\ T = \int_0^{20000} \frac{100000\ dN}{(N-10000)^2+20000000}$$

$$= \frac{1}{200}\int_0^{20000} \frac{dN}{\left(\frac{N-10000}{\sqrt{20}\times 1000}\right)^2+1}$$

$$= \frac{\sqrt{20}\times 1000}{200}\tan^{-1}\left(\frac{N-10000}{\sqrt{20}\times 1000}\right)\Bigg|_0^{20000}$$

$$= \frac{100}{\sqrt{20}}\left\{\tan^{-1}\frac{10}{\sqrt{20}} + \tan^{-1}\frac{10}{\sqrt{20}}\right\}$$

$$= 20\sqrt{5}\tan^{-1}\sqrt{5}$$

$$\approx \underline{\underline{51.441}}$$

6). (a)

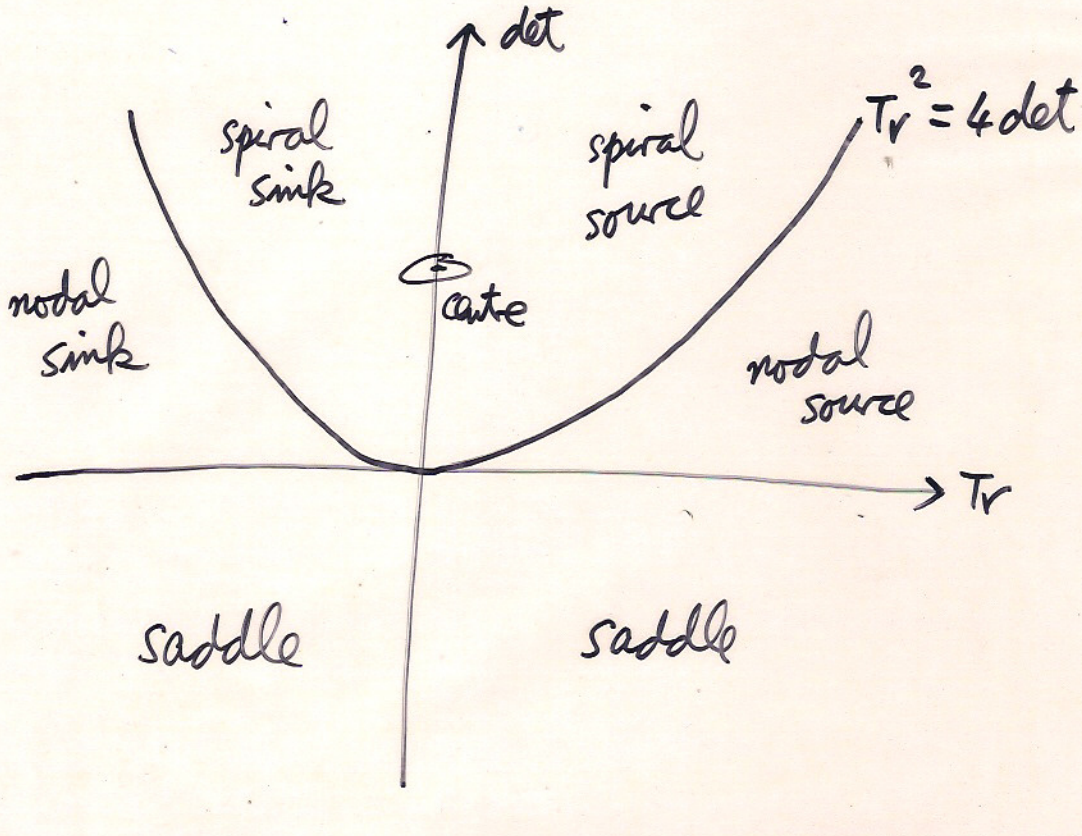
$$\begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix} = 1, \quad Tr = 0 \Rightarrow \text{centre}$$

$$\begin{vmatrix} -2 & 4 \\ 0 & 1 \end{vmatrix} = -2 \Rightarrow \text{saddle}$$

$$\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 4, \quad Tr = 5, \quad Tr^2 - 4\det = +ve \Rightarrow \text{nodal source}$$

$$\begin{vmatrix} -3 & 5 \\ -2 & 1 \end{vmatrix} = 7, \quad Tr = -2, \quad Tr^2 - 4\det = -ve \Rightarrow \text{spiral sink}$$

Recall:



6). (b) A = aliens

H = humans

$$\begin{cases} \dfrac{dA}{dt} = A - 5H \\ \\ \dfrac{dH}{dt} = \dfrac{-2}{5} A \end{cases}$$

$A(0) = 5000$

$$\begin{vmatrix} 1 & -5 \\ -\frac{2}{5} & 0 \end{vmatrix} = -2 \Rightarrow \text{saddle}.$$

$$\begin{vmatrix} 1-\lambda & -5 \\ -\frac{2}{5} & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda + \lambda^2 - 2 = 0$$

$$\Rightarrow \lambda^2 - \lambda - 2 = 0$$

$$\Rightarrow (\lambda+1)(\lambda-2) = 0$$

$$\Rightarrow \lambda = -1, \quad \lambda = 2.$$

When $\lambda = -1$.

$$2x - 5y = 0$$

$$x = 5 \Rightarrow y = 2$$

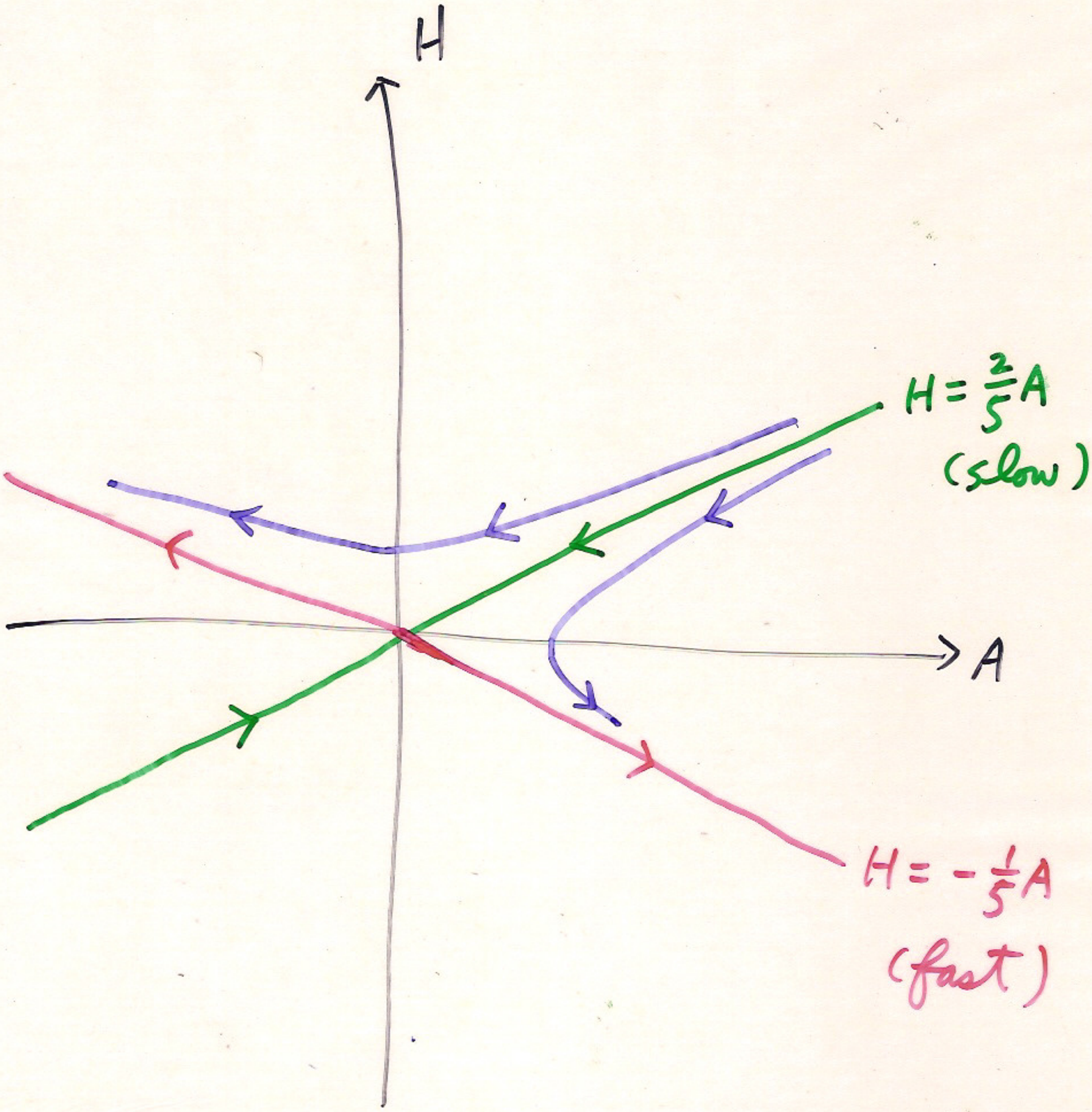
$$\therefore \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \vec{u}_1 .$$ (slow direction)

When $\lambda = 2$.

$$-x - 5y = 0$$

$$x = 5 \Rightarrow y = -1$$

$$\therefore \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \vec{u}_2$$ (fast direction)



In order to destroy A, i.e. for A to reach zero, we must start above the green line $H = \frac{2}{5}A$. i.e. $H > \frac{2}{5}A$

$\because A(0) = 5000$

$\therefore H(0) > \frac{2}{5} \times 5000 = \underline{\underline{2000}}$