2009/2010 SEMESTER 2 MID-TERM TEST

MA1506 MATHEMATICS II

March 1, 2010

8:30pm - 9:30pm

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **Fifteen (15)** printed pages.
2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.
3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1).
4. Use **only 2B pencils** for FORM CC1.
5. On FORM CC1 (section B), **write** your **matriculation number** and **shade** the corresponding numbered circles carefully. Your FORM CC1 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.
6. Write your full name in section A of FORM CC1.
7. Only circles for answers 1 to 10 are to be shaded.
8. For each answer, the circle corresponding to your choice should be properly and completely shaded. If you change your answer later, you must make sure that the original answer is properly erased.
9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.
10. **Do not fold** FORM CC1.
11. Submit FORM CC1 before you leave the test hall.

Formulae Sheet

1. Integrating factor for $y'+Py=Q$ is given by

$$R=\exp\left(\int P dx\right).$$

2. The variation of parameters formulae for $y''+py'+qy=r$:

$$u=\int\frac{-ry_2}{y_1y_2'-y_2y_1'}dx$$

$$v=\int\frac{ry_1}{y_1y_2'-y_2y_1'}dx.$$

1. Let y be a solution of the differential equation

$$\frac{dy}{dx} = \frac{y-1}{x+1}$$

such that

$$y(0) = 0.$$

Then $y\,(1) =$

(A) -1

(B) $-\frac{1}{2}$

(**C**) $\frac{1}{2}$

(**D**) 1

(**E**) None of the above

2. Let $y > 0$ be a solution of the differential equation

$$\frac{dy}{dx} = \frac{4x + 2y - 1}{2x + y + 1}$$

such that

$$y(1) = 1.$$

If $y\,(2) = a$, then a satisfies the equation

(A) $a + 3\ln|17 + 4a| = 3 + 3\ln 13$

(B) $a + 3\ln|17 + 4a| = 3 + 4\ln 13$

(**C**) $4a + 3\ln|17 + 4a| = 12 + 3\ln 13$

(**D**) $4a + 4\ln|17 + 4a| = 12 + 3\ln 13$

(**E**) None of the above

3. Let y be a solution of the differential equation

$$\frac{dy}{dx} - xy = xy^3$$

such that

$$y(0) = 1, \text{ and } 0 \leq x < \sqrt{\ln 2}$$

Then $y\left(\sqrt{\ln \frac{3}{2}}\right) =$

(A) $\frac{1}{\sqrt{2}}$

(B) $\sqrt{3}$

(C) $\frac{1}{\sqrt{3}}$

(D) $\sqrt{2}$

(E) None of the above

4. A body was found at the Engineering Canteen. You are a member of the CSI team and you arrived at the crime scene at 8am. Immediately upon arrival, you took the temperature of the victim and found that it was 80^oF. At 9am you took the temperature of the victim again and found that it was 75^oF. You estimated that the victim's temperature was 98.6^oF just before death and that the temperature at the Engineering Canteen stayed approximately constant at 70^oF. What is your estimate on the time of death?

(A) 5:45am

(B) 6:00am

(C) 6:15am

(D) 6:30am

(E) 6:45am

5. A certain fungus grows on fruit in circular patches. The rate at which the area of a patch grows is proportional to its circumference, with constant of proportionality 2 cm/hour. If a given patch has an area of π cm^2 at time $t = 0$, then its area at time $t = 2$ hours is

(A) 4π cm^2

(B) 16π cm^2

(C) 21π cm^2

(D) 25π cm^2

(E) None of the above

6. At time $t = 0$ a tank contains 30 lbs of salt dissolved in 200 gal of water. Assume that water containing 0.3 lb of salt per gallon is entering the tank at a rate of 2 gallons per minute and the well stirred solution is leaving the tank at the same rate. Find the amount of salt in the tank at time $t = 2$ hours. Give your answer in lbs and correct to 3 decimal places.

(A) 29.393

(B) 33.392

(C) 50.964

(D) 30.594

(E) 43.535

7. Let y be a solution of the differential equation

$$y'' + 4y' + 5y = 0$$

such that

$$y(0) = 0,\ y'(0) = 1.$$

Then $y(\frac{\pi}{2}) =$

(A) $e^{-\pi}$

(B) $e^{-\frac{\pi}{2}}$

(C) e^{π}

(D) $e^{\frac{\pi}{2}}$

(E) None of the above

8. Let y be a solution of the differential equation

$$y'' - 3y' + 2y = 4$$

such that

$$y(0) = 3,\ y'(0) = 1.$$

Then $y(\ln 2) =$

(A) 2

(B) $2\ln 2$

(C) 3

(D) $2\ln 3$

(E) None of the above

9. Let y be a solution of the differential equation

$$y'' - 6y' + 9y = 2e^{3x}$$

such that

$$y(0) = 1, \text{ and } y'(0) = 4.$$

Then $y(1) =$

(A) e^3

(B) e^{-3}

(C) $\cosh(3)$

(D) $\sinh(3)$

(E) $3e^3$

10. Let y be a solution of the differential equation

$$y'' + y = \sec x \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2},$$

such that

$$y(0) = y'(0) = 0.$$

Then $y\left(\frac{\pi}{3}\right) =$

(Hint: $\int(-\tan^2 x)dx = x - \tan x$)

(A) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}(1 - \ln 2)$

(B) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}(1 - \ln 2)$

(C) $\frac{\pi}{6} + \frac{\sqrt{3}}{2}(1 - \ln 2)$

(D) $\frac{\pi}{3} + \frac{\sqrt{3}}{2}\ln 2$

(E) None of the above

END OF PAPER

Answers to mid term test

1. A
2. C
3. B
4. D
5. D
6. C
7. A
8. E
9. E
10. A

1). A

$$\frac{dy}{y-1} = \frac{dx}{x+1} \Rightarrow \ln|y-1| = \ln|x+1| + C$$

$$\Rightarrow \ln\left|\frac{y-1}{x+1}\right| = C \Rightarrow y-1 = A(x+1)$$

$$y(0)=0 \Rightarrow 0-1 = A \Rightarrow A=-1$$

$$\therefore y = 1-(x+1) = -x$$

$$\therefore y(1) = \underline{\underline{-1}}$$

2). C

$$\text{Let } 2x+y=u \Rightarrow 2+y'=u'$$

$$\therefore u'-2 = \frac{2u-1}{u+1} \Rightarrow u' = \frac{2u-1}{u+1} + 2 = \frac{4u+1}{u+1}$$

$$\frac{u+1}{4u+1}du = dx \Rightarrow \frac{4u+4}{4u+1}du = 4dx$$

$$\left(1+\frac{3}{4u+1}\right)du = 4dx$$

$$u + \frac{3}{4}\ln|4u+1| = 4x + C$$

$$2x+y+\frac{3}{4}\ln|8x+4y+1| = 4x+C$$

$$y(1)=1 \Rightarrow 3+\frac{3}{4}\ln|13| = 4+C$$

$$\Rightarrow C = \frac{3}{4}\ln 13 - 1$$

$$\therefore y + \frac{3}{4}\ln|8x+4y+1| = 2x + \frac{3}{4}\ln 13 - 1$$

$$y(2)=a \Rightarrow a + \frac{3}{4}\ln|17+4a| = 3 + \frac{3}{4}\ln 13$$

$$\Rightarrow \underline{\underline{4a + 3\ln|17+4a| = 12 + 3\ln 13}}$$

3). B

Let $z = y^{1-3} = y^{-2} \Rightarrow z' = -2y^{-3}y' \Rightarrow y' = -\frac{y^3}{2}z'$

$\therefore -\frac{y^3}{2}z' - xy = xy^3 \Rightarrow z' + 2xy^{-2} = -2x$

$\therefore z' + 2xz = -2x$

$R = e^{\int 2x\,dx} = e^{x^2}$

$z = e^{-x^2}\int e^{x^2}(-2x)\,dx = e^{-x^2}\{-e^{x^2} + c\}$

$\frac{1}{y^2} = -1 + ce^{-x^2}$

$y(0) = 1 \Rightarrow 1 = -1 + c \Rightarrow C = 2$

$\therefore y^2 = \dfrac{1}{2e^{-x^2} - 1} \Rightarrow y = \dfrac{1}{\sqrt{2e^{-x^2} - 1}}$ $(\because y(0) = 1 = +ve)$

$\therefore y(\sqrt{\ln\frac{3}{2}}) = \dfrac{1}{\sqrt{2e^{-\ln\frac{3}{2}} - 1}}$

$= \dfrac{1}{\sqrt{2(\frac{2}{3}) - 1}} = \underline{\underline{\sqrt{3}}}$

4). D

Set time $t=0$ at 8 am, t measured in hours.

$$\frac{dT}{dt} = k(T-70)$$

$$\frac{dT}{T-70} = k\,dt$$

$$\ln|T-70| = kt + C$$

$$T-70 = Ae^{kt}$$

$$T(0)=80 \Rightarrow A = 80-70 = 10$$

$$\therefore T-70 = 10e^{kt}$$

$$T(1) = 75 \Rightarrow 5 = 10e^{k} \Rightarrow k = -\ln 2$$

$$\therefore T-70 = 10e^{(-\ln 2)t}$$

$$T(\tau) = 98.6 \Rightarrow 28.6 = 10e^{(-\ln 2)\tau}$$

$$\Rightarrow 2.86 = e^{(-\ln 2)\tau}$$

$$\Rightarrow \tau = -\frac{\ln 2.86}{\ln 2}$$

$$\approx -1.5 \text{ hours}$$

$\therefore$ Time of death $\approx$ 6:30 am

5). D

$$A=\pi r^2 \text{ and } \frac{dA}{dt}=2(2\pi r)$$

$$\Rightarrow 2\pi r\frac{dr}{dt}=4\pi r$$

$$\Rightarrow \frac{dr}{dt}=2$$

$$\therefore r=2t+C$$

at $t=0$, $A=\pi \Rightarrow 1=C$

$$\therefore r=2t+1$$

at $t=2 \Rightarrow r=5$

$$\Rightarrow \underline{\underline{A=25\pi}}$$

6). C

$$\frac{dQ}{dt}=0.6-\frac{2Q}{200}=0.6-0.01Q=-0.01(Q-60)$$

$$\therefore \frac{dQ}{Q-60}=-0.01\,dt$$

$$\therefore Q-60=Ae^{-0.01t}$$

$$Q(0)=30 \Rightarrow A=-30$$

$$\therefore Q=60-30e^{-0.01t}$$

$t=2$ hours $=120$ min.

$$\Rightarrow Q=60-30e^{-1.2}$$

$$\approx \underline{\underline{50.964}}$$

7). A

$$\lambda^2+4\lambda+5=0$$

$$\lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$$

$$y = c_1 e^{-2x}\cos x + c_2 e^{-2x}\sin x$$

$$y(0)=0 \Rightarrow c_1 = 0$$

$$\therefore y = c_2 e^{-2x}\sin x$$

$$y' = -2c_2 e^{-2x}\sin x + c_2 e^{-2x}\cos x$$

$$y'(0)=1 \Rightarrow 1 = c_2$$

$$\therefore y = e^{-2x}\sin x$$

$$y\left(\tfrac{\pi}{2}\right) = \underline{\underline{e^{-\pi}}}$$

8). E

$$\lambda^2-3\lambda+2=0 \Rightarrow \lambda = 1, 2$$

Observe $y=2$ is a particular solution

$$\therefore y = c_1 e^{x} + c_2 e^{2x} + 2$$

$$y' = c_1 e^{x} + 2c_2 e^{2x}$$

$$y(0)=3 \Rightarrow c_1 + c_2 = 1$$

$$y'(0)=1 \Rightarrow c_1 + 2c_2 = 1$$

$$\therefore c_1 = 1,\ c_2 = 0$$

$$\therefore y = e^{x}+2$$

$$\therefore y(\ln 2) = e^{\ln 2} + 2 = \underline{\underline{4}}$$

9). E

$$\lambda^2 - 6\lambda + 9 = 0$$

$$\Rightarrow \lambda = 3 \text{ double root.}$$

Try $y = (Ax^2 + Bx + C)e^{3x}$

$$y' = (2Ax + B)e^{3x} + 3(Ax^2 + Bx + C)e^{3x}$$

$$y'' = 2Ae^{3x} + 6(2Ax + B)e^{3x} + 9(Ax^2 + Bx + C)e^{3x}$$

$$y'' - 6y' + 9y = 2Ae^{3x}$$

$$\therefore 2Ae^{3x} = 2e^{3x} \Rightarrow A = 1$$

$$y = c_1 e^{3x} + c_2 x e^{3x} + x^2 e^{3x}$$

$$y' = 3c_1 e^{3x} + c_2 e^{3x} + 3c_2 x e^{3x} + 2xe^{3x} + 3x^2 e^{3x}$$

$$y(0) = 1 \Rightarrow c_1 = 1$$

$$y'(0) = 4 \Rightarrow 3c_1 + c_2 = 4 \Rightarrow c_2 = 1$$

$$\therefore y = e^{3x} + xe^{3x} + x^2 e^{3x}$$

$$y(1) = e^3 + e^3 + e^3 = \underline{\underline{3e^3}}$$

10). A

$$\lambda^2+1=0 \Rightarrow \lambda = \pm i$$

Let $y_1 = \cos x$, $y_2 = \sin x \Rightarrow \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 1$

$$u = \int \frac{-\sec x \tan x \sin x}{1} dx$$

$$= \int -\tan^2 x \, dx = x - \tan x$$

$$V = \int \sec x \tan x \cos x = -\ln|\cos x| = \ln \sec x$$

$$\therefore y_p = u y_1 + V y_2 = x\cos x - \sin x + \sin x \ln \sec x$$

$$\therefore y = C_1 \cos x + C_2 \sin x + x\cos x - \sin x + \sin x \ln \sec x$$

$$y' = -C_1 \sin x + C_2 \cos x + \cos x - x \sin x - \cos x$$
$$+ \cos x \ln \sec x + \sin x \cos x \sec x \tan x$$

$$y(0) = 0 \Rightarrow C_1 = 0$$

$$y'(0) = 0 \Rightarrow 0 = C_2 + 1 - 1 \Rightarrow C_2 = 0$$

$$\therefore y = x\cos x - \sin x + \sin x \ln \sec x$$

$$y\left(\frac{\pi}{3}\right) = \frac{\pi}{3}\frac{1}{2} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\ln 2$$

$$= \underline{\underline{\frac{\pi}{6} - \frac{\sqrt{3}}{2}(1 - \ln 2)}}$$