ANSWERS TO MA1506 TUTORIAL 5

Question 1

Following the standard equations for the Malthus Model [Chapter 3]:

$$
\begin{aligned}
N &= \hat{N}e^{kt}; N(0) = 10000 = \hat{N} \\
N(2.5) &= 10000e^{2.5k} = 11000 \\
\Rightarrow e^{2.5k} &= 1.1 \Rightarrow k = \frac{1}{2.5}\ell n(1.1) \\
&= 0.0381 \\
N(10) &= 10000e^{10k} = 10000e^{10(0.0381)} \approx 14600 \\
20000 &= 10000e^{kt} \to t = \frac{1}{k}\ell n(2) \\
&= 18.18 \text{ hours}
\end{aligned}
$$

Question 2

Let c be the number of emigrants per year. $\frac{dN}{dt} = kN - c$. You can solve this in the usual way, as a linear ODE, or use a trick: let $M = N - \frac{c}{k}$ so

$$
\begin{aligned}
\frac{dM}{dt} &= \frac{dN}{dt} = k(N - \frac{c}{k}) \\
&= kM
\end{aligned}
$$

So $M = Ae^{kt}$ i.e. $N - \frac{c}{k} = Ae^{kt}$. Let $N(0) = \hat{N}$, so

$$
A = \hat{N} - \frac{c}{k} \text{ i.e. } N = \frac{c}{k} + (\hat{N} - \frac{c}{k})e^{kt}
$$

Three cases: $\frac{c}{k} < \hat{N}$, so $\hat{N} - \frac{c}{k} > 0$
exponential growth $\to$ there is no point in sending out the emigrants, since the Earth's population continues to grow exponentially.
Next case: $\frac{c}{k} = \hat{N}$, $N = \frac{c}{k} = \hat{N}$. This is the desired situation, with a constant population on Earth.
Last case, $\frac{c}{k} > \hat{N}$, Earth's population decreases to zero! Presumably not what you really want.
Next, we assume an emigration rate proportional to t, so $\frac{dN}{dt} = kN - ct$. This is a linear ODE with an integrating factor e^{-kt}, [or you can use undetermined coefficients] so the solution is

$$
\begin{aligned}
&e^{kt}\int \frac{-ct}{e^{kt}} \\
&= -ce^{kt}\left[-\frac{1}{k}te^{-kt} - \left(\frac{1}{k^2}e^{-kt}\right) + constant\right]\begin{pmatrix}\text{integrate} \\ \text{by} \\ \text{parts}\end{pmatrix}. \\
&= \frac{c}{k}\left[t + \frac{1}{k}\right] + Ae^{kt}
\end{aligned}
$$

Now $\hat{N} \equiv N(0) = A + \frac{c}{k^2} \Rightarrow N(t) = \left(\hat{N} - \frac{c}{k^2}\right) e^{kt} + \frac{c}{k}\left[t + \frac{1}{k}\right]$

3 cases:

$\hat{N} - \frac{c}{k^2} > 0 \Rightarrow$ population explosion on Earth;

$\hat{N} = \frac{c}{k^2} \Rightarrow$ Earth's population grows linearly; not quite an "explosion" but still not so great;

$\hat{N} - \frac{c}{k^2} < 0$: population will grow for a while but eventually reach a maximum, followed by a decline to zero [because the exponential function will eventually defeat a linear one; note that we know that the population will grow initially because $\frac{dN}{dt}(t = 0) = \hat{N}k > 0$.] So none of the outcomes is really satisfactory in this case.

Question 3

$N = Ae^{(B-D)t}$ (constant B and D, so we have a Malthusian situation.)
Population doubles in 20 years, so

$$2 = e^{(B-D)20} \Rightarrow B - D = \frac{\ell n2}{20}$$

After the departure of the women, B is zero so $N = ce^{-Dt} \Rightarrow \frac{1}{2} = e^{-D\times 10} \Rightarrow D = \frac{\ell n2}{10} \Rightarrow B = \frac{\ell n2}{20} + \frac{\ell n2}{10} \Rightarrow B \approx 0.10397$ i.e. about 10.397% per year.
Assumptions:

1. men and old women have same death rate as young women, which is not true in reality because men smoke, get into fights etc while on the other hand old women are indestructible;
2. the death rate of the remaining population is not changed by the departure of the girls, a very questionable assumption since morale will be affected, etc.

Question 4

We have

$$\frac{dN}{dt} = BN - DN$$

where B and D are the birth and death rates per capita respectively. So

$$\frac{dN}{dt} = \Big[3 - 3tanh(3t)\Big]N - 0.2N^2.$$

This is a Bernoulli equation so we change the variable to Z = 1/N and get

$$\frac{dZ}{dt} + \Big[3 - 3tanh(3t)\Big]Z = 0.2.$$

This is first order linear. Integrating factor is $e^{3t}\,sech(3t)$ and so

$$\frac{d}{dt}\left[Z\,e^{3t}\,sech(3t)\right] = 0.2\,e^{3t}\,sech(3t).$$

Integrating and using N(0) = 1, we find

$$N(t) = \frac{e^{3t}\,sech(3t)}{\frac{0.2}{3}ln(e^{6t}+1)+1-\frac{0.2}{3}ln(2)}.$$

The graph of this rises after t = 0, reaches its maximum of about 1.65 around t = 0.47, and then asymptotically approaches zero. So there is a delay; the decline of the birth rate does not immediately stop the population from growing. It does however have a strong effect on the maximum population: if the birth rate per capita had remained fixed at 6, then the population would have risen to nearly 30 [in the usual logistic manner, B/s = 6/0.2 = 30].

[Note: it is of course not realistic for the birth rate per capita to decline to *zero*. But it is easy to change the problem so that it declines to some small but positive value. However, in that case it is very difficult to solve the differential equation exactly, though it can still be done by a computer. The general outcome is much the same.]

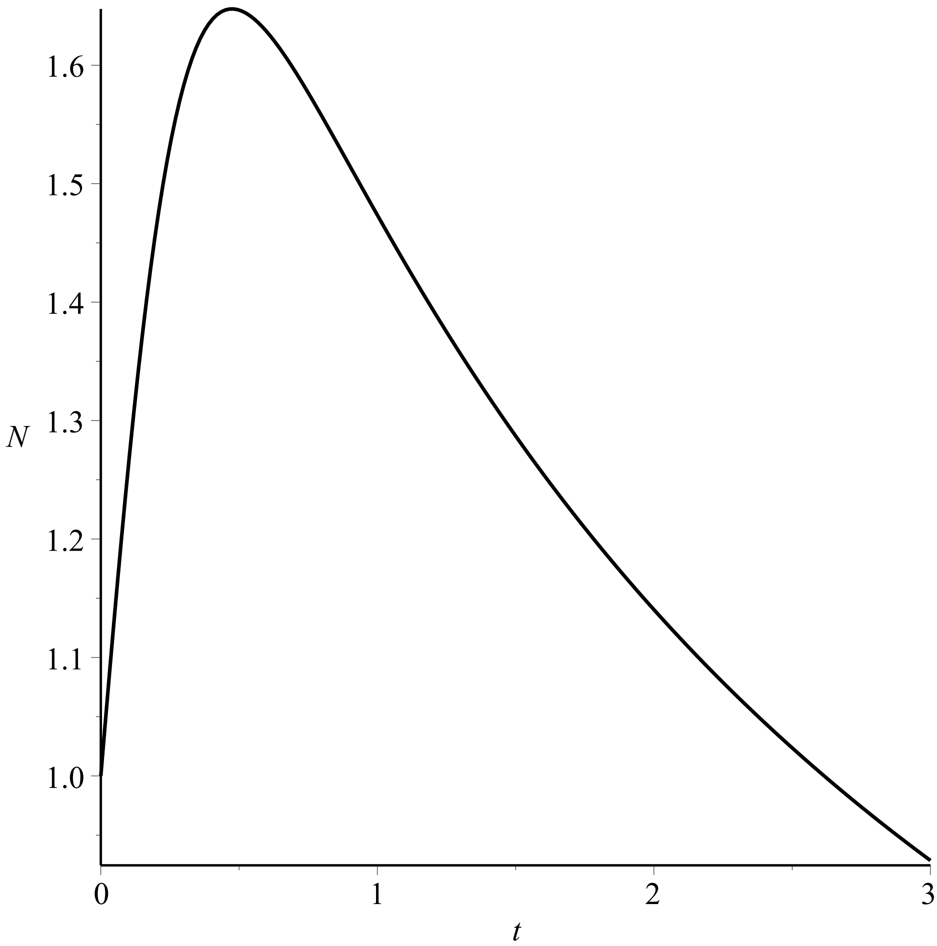

Figure 1: Population Momentum.

Question 5.
The logistic equation has 3 kinds of solution, one increasing, one constant, and one decreasing. Since the number of bugs in this problem clearly increases, the relevant solution of the logistic equation is

$$N = \frac{B}{s + \left(\frac{B}{\hat{N}} - s\right) e^{-Bt}} = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right) e^{-Bt}}$$

Here $\hat{N} = 200$, $B = 1.5$, so at $t = 2$ we have

$$\begin{aligned}
360 &= \frac{N_\infty}{1 + \left(\frac{N_\infty}{200} - 1\right) e^{-1.5\times 2}} \\
&\Rightarrow 360 + \frac{360}{200} e^{-3} N_\infty - 360 e^{-3} = N_\infty \\
N_\infty &= \frac{360(1 - e^{-3})}{1 - \frac{360}{200} e^{-3}} \approx 376 \\
N(3) &= \frac{N_\infty}{1 + \left(\frac{N_\infty}{200} - 1\right) e^{-4.5}} \approx 372
\end{aligned}$$

Question 6
Using the suggested form of D, we have

$$\begin{aligned}
\frac{dN}{dt} &= BN - DN = BN - (D_0 + SN)N \\
&= (B - D_0)N - sN^2 = B^* N - sN^2
\end{aligned}$$

$(B^* \equiv B - D_0)$ so $\frac{dN}{dt} = B^* N - sN^2$ which is the logistic differential equation. So nothing new in that sense.

However, note that in the logistic model, $B < 0$ is obviously impossible, but here it IS possible to have $B^* < 0$. So we have a new case. Clearly if $B^* < 0$ then

$$\frac{dN}{dt} = B^* N - sN^2 < 0$$

so N always DECREASES. However, it cannot reach zero. To see this, note that $N \equiv 0$ is a solution of this equation. According to the "no-crossing principle", this means that no other solution can reach zero. So the solutions of this equation just look like the solutions of the standard logistic equation, but now there are new solutions which just decrease monotonically, approaching zero asymptotically.

Note: the students may ask: how is it possible that D can be non-zero even when N is zero? The answer is the usual one: this is just a model, which works in some circumstances but not in all. For extremely low populations, the whole concept of a death rate per capita loses its usefulness anyway.