ANSWERS TO MA1506 TUTORIAL 6

Question1.

First compare 80 with $\dfrac{B^2}{4s}$.

From Tutorial 5 we know $B = 1.5$ and and $N_\infty = 376$, so $N_\infty = B/s \Rightarrow s = \frac{1.5}{376} \Rightarrow \frac{B^2}{4s} = 141$. This is the maximum number we can kill without causing extinction.

Setting $E = 80$,

$$\begin{matrix}\beta_1 \\ \beta_2\end{matrix} = \frac{B \mp \sqrt{B^2 - 4Es}}{2s} = \begin{matrix}64 \\ 312\end{matrix}.$$

Since the initial number of bugs was 200, which is between these two values, we see that the limiting number is $\beta_2 = 312$, since this is the stable equilibrium.

Question 2.

We have $B_\infty = \frac{B}{s} = 194600$ so since $B = 0.09866$, $s = \frac{B}{N_\infty} = \frac{0.09866}{194600}$.

Maximum hunting rate is

$$\frac{B^2}{4s} = \frac{(0.09866)^2}{4 \times \frac{0.09866}{194600}} = 4800$$

Since $10000 > 4800$, birds are doomed.

Since $\frac{dN}{dt} = N(B - sN) - E$,

$$\int_0^T dt = \int_{194600}^{0} \frac{dN}{N\left(0.09866 - \frac{0.09866N}{194600}\right) - 10000}$$

We get 29.8 years if $E = 10000$.

Question 3.

For the fish to survive a 10% downward fluctuation, we must have (in the extreme case)

$\beta_1 = 90\%\ \ \beta_2$ i.e.

$$\frac{B - \sqrt{B^2 - 4Es}}{2s} = 0.9\left[\frac{B + \sqrt{B^2 - 4Es}}{2s}\right]$$

$$B - \sqrt{} = 0.9B + 0.9\sqrt{}$$

$$
\begin{aligned}
0.1B &= 1.9\sqrt{} \\
0.01B^2 &= 3.61(B^2 - 4Es) = 3.61B^2 - 14.44Es \\
14.44E &= 3.6B^2/s \\
E &= 0.2493074\frac{B^2}{s} \\
&= 0.997 \times \left(\frac{B^2}{4s}\right)
\end{aligned}
$$

So a less than 1% drop in the catch below E* will give a 10% margin of safety.

Question 4. The cubic in N on the right side clearly passes through the origin, and its slope is negative immediately to the right of the origin, so it [and therefore dN/dt] is negative for small values of N; therefore the population always decreases when N is sufficiently small, which describes depensation. So we have the right kind of equation. Roots of this cubic:

$$N = 0, N = [b \pm \sqrt{b^2 - 4ac}]/2a.$$

Since we are assuming $b^2 > 4ac$, both non-zero roots are real and, since $\sqrt{b^2 - 4ac} < b$, both are positive; the graph of the cubic goes down to a minimum [after passing through the origin], then up through the point N $= [b - \sqrt{b^2 - 4ac}]/2a$ on the N axis, reaching a maximum and then going down again to cut the N axis once more at N $= [b + \sqrt{b^2 - 4ac}]/2a$. So there are two non-zero equilibrium populations, one stable, one unstable, EVEN THOUGH THERE IS NO HARVESTING. The unstable equilibrium is at N $= [b - \sqrt{b^2 - 4ac}]/2a$; it is unstable because a population slightly above that will grow away from it, while a population slightly below it will drop towards zero. We see that the tigers will become extinct if the population ever falls below that value, EVEN IF WE DON'T HUNT THEM! So we have a good model of depensation.

Question 5

Follow the lecture notes [Chapter 2 section 6]. Again we have to use the fact that the difference between the rates at which hydrogen flows in and out of a plug [of thickness δx] is precisely the rate at which it is destroyed by the chemical reaction. Hydrogen flows IN at the point x at a rate $C_{H_2}(x)A(x)u$, where we note that A is now a function of x [whereas it was a constant in the notes]. H_2 molecules are flowing OUT at the point $x + \delta x$ at a rate $C_{H_2}(x + \delta x)A(x + \delta x)u$. H_2 molecules are being destroyed by the reaction inside the small piece of tube, so their number is changing at a rate

$-2rA(x)\delta x$. [Here it doesn't matter whether we write A(x) or A(x+δ x), because we are going to drop nonlinear terms in small quantities anyway.] Since the total matter content cannot increase or decrease,

$$C_{H_2}(x)A(x)u - C_{H_2}(x+\delta x)A(x+\delta x)u - 2rA(x)\delta x = 0.$$

Putting $\delta C_{H_2} \equiv C_{H_2}(x+\delta x) - C_{H_2}(x)$ and $\delta A \equiv A(x+\delta x) - A(x)$, and neglecting all terms quadratic in small quantities, we have

$$-\delta C_{H_2} Au - C_{H_2}\delta Au - 2rA\delta x = 0$$

so taking the limit as δx tends to zero we get [after dividing throughout by uAδx]

$$\frac{dC_{H_2}}{dx} = -\frac{dln(A)}{dx}C_{H_2} - \frac{2r}{u} = -\frac{dln(A)}{dx}C_{H_2} - \frac{2k}{u}C_{H_2},$$

where, as in the notes, we assume that the rate is a constant multiple of the concentration.

Now Ah Lian's reactor has a particular shape, $A(x) = A_0e^{-\gamma x}$, and substituting this into the above equation we find

$$\frac{dC_{H_2}}{dx} = \gamma C_{H_2} - \frac{2k}{u}C_{H_2} = -2[\frac{k - u\gamma/2}{u}]C_{H_2}$$

which is indeed exactly the equation we had in the lecture notes, except that k has been replaced by $k - (u\gamma/2)$. Thus while the new design may save money on construction, it slows down the reaction, and in fact the thing will not work at all if $u\gamma/2$ approaches k.

Question 6

If the weight per unit length is $2\alpha[1 - (x/L)]$, then the total weight is obtained by integrating this from 0 to L; the answer is αL, which is indeed exactly the same weight as Ah Huat's balcony [which has constant weight per length α.]

We have to solve Euler's equation

$$\frac{d^4y}{dx^4} = w(x)/EI$$

with exactly the same boundary conditions as in the lecture notes. The only difference is that instead of w(x) = constant = $-\alpha$ we now have w(x) = $-2\alpha[1 - (x/L)]$. You just have to integrate four times! I leave the details to you. You should find along the way that

$$y'' = -\frac{2\alpha}{EI}[\frac{x^2}{2} - \frac{x^3}{6L} - \frac{Lx}{2} + \frac{L^2}{6}],$$

and that

$$y = -\frac{\alpha x^2}{60EIL}[5Lx^2 - x^3 - 10L^2x + 10L^3].$$

Substituting x = L into the formula for y, you will find that the dip at the end of Ah Lian's balcony is just

$$y_{AL}(L) = -\frac{\alpha L^4}{15EI},$$

very much less [in absolute value] than Ah Huat's dip

$$y_{AH}(L) = -\frac{\alpha L^4}{8EI},$$

which is what we found in the lectures.