MA1506
Mathematics II

Chapter 8
Systems of First Order ODEs

8.1 Solving linear system of ODEs

Consider

$$\frac{dx}{dt} = ax + by \qquad \frac{dy}{dt} = cx + dy$$

where *a,b,c,d* are constants.

i.e,
$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

We begin by guessing (as in ODE) that the solutions to

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

are of the form

$$x(t) = x_0 e^{\alpha t} \qquad y(t) = y_0 e^{\alpha t}$$

which can be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\alpha t}$$

Next, we have to find x_0, y_0, α

We first differentiate $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\alpha t}$

and get $\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \alpha e^{\alpha t}$

Thus the system $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

becomes $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \alpha e^{\alpha t} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} e^{\alpha t} = B\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} e^{\alpha t}$

where $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Let $u_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

Then we get

2x2 matrix

$$\alpha u_0 = B u_0$$

eigenvalue   eigenvector

To find the eigenvalues and eigenvectors of B, we have to solve

$$\det(B - \alpha I_2) = \det\begin{pmatrix} a-\alpha & b \\ c & d-\alpha \end{pmatrix} = 0$$

Hence $(a-\alpha)(d-\alpha) - bc = 0$

That is $\alpha^2 - (a+d)\alpha + (ad - bc) = 0$

Thus

Eigenvalue

$$\alpha = \frac{1}{2}\left[a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}\right]$$

$$= \frac{1}{2}\left[Tr[B] \pm \sqrt{(Tr[B])^2 - 4(\det[B])}\right]$$

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

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To find α we consider the following three cases

Case 1 $(Tr[B])^2 > 4Det[B]$

We have two distinct real roots (eigenvalues)
Suppose the two distinct real eigenvalues are α_1, α_2 and the corresponding eigenvectors are u_1, u_2

Then we have two solutions $u_1 e^{\alpha_1 t}$ and $u_2 e^{\alpha_2 t}$

The general solution is

$$c_1 u_1 e^{\alpha_1 t} + c_2 u_2 e^{\alpha_2 t} \quad .....(C1)$$

where c_1, c_2 are any real numbers.

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Case 2 $(Tr[B])^2 < 4Det[B]$

We have two complex roots (eigenvalues)

If one complex eigenvalue is $\lambda = \alpha + i\beta$
then the 2nd complex eigenvalue is $\bar{\lambda} = \alpha - i\beta$

Let the complex eigenvector corresponding to λ be w=u+iv. Then the general solution is

$$c_1 e^{\alpha t}\left[u\cos\beta t - v\sin\beta t\right] + c_2 e^{\alpha t}\left[u\sin\beta t + v\cos\beta t\right] \quad .....(C2)$$

where c_1, c_2 are real numbers.

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Case 3 $(Tr[B])^2 = 4Det[B]$

We will not consider Case 3 in this course.

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Examples

Solve $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = B\begin{pmatrix} x \\ y \end{pmatrix}$

(1) $B = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$

Eigenvalues are $\alpha_1 = -1$ $\alpha_2 = -2$

Corresponding eigenvectors are

$u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

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The general solution is

$$c_1 u_1 e^{-t} + c_2 u_2 e^{-2t} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$$

i.e.,

$$x(t) = c_1 e^{-t}$$

$$y(t) = c_1 e^{-t} + c_2 e^{-2t}$$

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(2) $B = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$

Eigenvalues are $\lambda = 1+3i \quad \bar{\lambda} = 1-3i$

We only have to consider $\lambda = 1+3i$

because the other eigenvalue will also give the same answer.

The eigenvector corresponding to λ is

$$w = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

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The general solution is

$$c_1 e^t \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos 3t - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin 3t\right] + c_2 e^t \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 3t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos 3t\right]$$

as given in Case 2 solution (C2)

$$c_1 e^{\alpha t}\left[u\cos\beta t - v\sin\beta t\right] + c_2 e^{\alpha t}\left[u\sin\beta t + v\cos\beta t\right]$$

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Remark: 2nd order ODE is a special case of systems of ODE

The 2nd order ODE $\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$

can be rewritten as $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

where $\frac{dx}{dt} = y$

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Proof

Let $\frac{dx}{dt} = y$

Then $\frac{d^2x}{dt^2} = \frac{dy}{dt}$

Hence

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$$

$$\Rightarrow \frac{dy}{dt} = \frac{d^2x}{dt^2} = -a\frac{dx}{dt} - bx = -ay - bx$$

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Now rewrite the system of ODE

$$\frac{dx}{dt} = y \qquad \frac{dy}{dt} = -ay - bx$$

as

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

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8.2 Stability of equilibrium solution

Consider a system of ODE $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = B\begin{pmatrix} x \\ y \end{pmatrix}$

A solution of the ODE is said to be an equilibrium solution (equilibrium point) if it is a constant function.

For example, zero $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

is always a solution, so zero $\equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

is an equilibrium solution.

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Recall: eigenvalues are given by

$$\frac{1}{2}\left[Tr[B] \pm \sqrt{(Tr[B])^2 - 4(\det[B])}\right] \qquad Tr\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$$

Case 1 Two real eigenvalues $(TrB)^2 > 4\det B$

The general solution is $c_1 u_1 e^{\alpha_1 t} + c_2 u_2 e^{\alpha_2 t}$

Thus the zero equilibrium solution is stable if and only if the two eigenvalues are both less than or equal to zero.

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From $\alpha = \frac{1}{2}\left[Tr[B] \pm \sqrt{(Tr[B])^2 - 4(\det[B])}\right]$

we know that the two eigenvalues are less than or equal to zero if and only if

(1) $Tr(B) < 0$

(2) $Tr(B) + \sqrt{(Tr(B))^2 - 4\det(B)} \le 0$

$$\sqrt{(Tr(B))^2 - 4\det(B)} \le -Tr(B)$$

$$(Tr(B))^2 - 4\det(B) \le (Tr(B))^2$$

$$\det(B) \ge 0$$

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Hence for Case 1 $(TrB)^2 > 4\det B$

Zero equilibrium solution is stable iff

$Tr(B) < 0$ and $\det(B) \ge 0$

Note: In fact $\det(B) > 0$ since the inverse B exits ($\det(B) \ne 0$)

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Case 2: Two complex eigenvalues

$$(TrB)^2 < 4\det B$$

The general soln is

$$c_1 e^{\alpha t}\left[u\cos\beta t - v\sin\beta t\right]$$
$$+ c_2 e^{\alpha t}\left[u\sin\beta t + v\cos\beta t\right]$$

where $\alpha = \frac{1}{2}Tr(B)$

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Hence zero equilibrium solution is stable iff

$\alpha = \frac{1}{2}Tr(B) \le 0$ i.e., $Tr(B) \le 0$

Note that in this case $(TrB)^2 < 4\det B$

So $\det(B) > 0$

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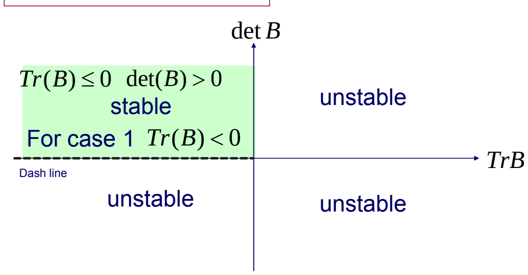
For Case1 and Case 2



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8.3 Phase plane

A solution of a 2-dim system of ODE is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

In this section, a phase plane is a x(t)-y(t) plane

The trajectories in the phase plane are the parametric curves described by x(t) and y(t).

Trajectories on a phase plane create a phase portrait.

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Phase planes are useful in visualizing the behavior (including stability of equilibrium soln) of physical system.

We shall only consider Case1 and Case 2 of 8.1.

We shall not consider Case 3. You may refer to the book by Farlow, Chapter 6 for Case 3 (optional).

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Case 1: Two real eigenvalues

The general soln is

$$c_1 \underset{\text{eigenvector}}{u_1} e^{\overset{\text{eigenvalues}}{\alpha_1} t} + c_2 \underset{\text{eigenvector}}{u_2} e^{\alpha_2 t}$$

First note that if $c_2 = 0$ we have

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 u_1 e^{\alpha_1 t} = c_1 \begin{pmatrix} g \\ h \end{pmatrix} e^{\alpha_1 t}$$

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So $y(t) = \frac{h}{g} x(t)$ $\boxed{\text{Recall } u_1 = \begin{pmatrix} g \\ h \end{pmatrix}}$

Hence the straight line through 0 with gradient h/g is one of trajectories in the phase plane.

This straight line is in the same direction as the eigenvector.

Thus we should have two basic trajectories induced by two eigenvectors

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Examples

(1)Two negative eigenvalues

$$B = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

Eigenvalues are $\alpha_1 = -1, \alpha_2 = -3$

Corresponding Eigenvectors are

$$u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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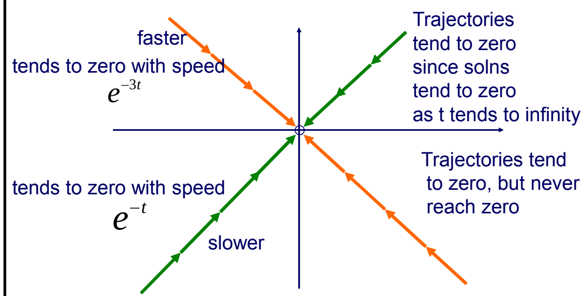
$\alpha_1 = -1, \alpha_2 = -3$ $u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



Trajectories induced by eigenvectors

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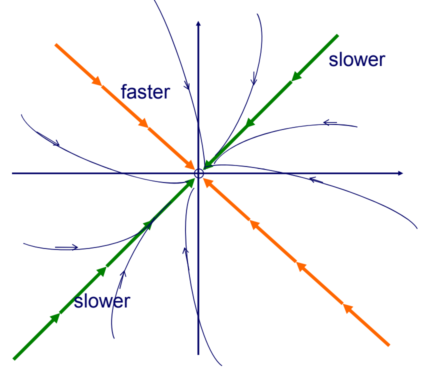
Trajectories tend to zero, but never reach zero
Why?
By
no crossing principle (See chapter 3)

We shall use the no crossing principle to sketch other trajectories.

Zero is a stable equilibrium soln called the nodal sink.



Web Application

- http://www.aw-bc.com/ide/idefiles/media/JavaTools/lnclmtrx.html

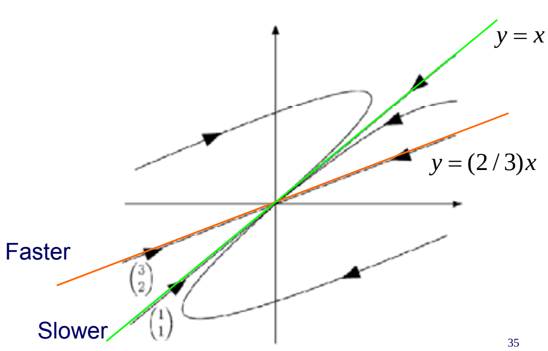
$$B = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

Another Example:

$$B = \begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

slower faster



Web Application

- http://www.aw-bc.com/ide/idefiles/media/JavaTools/lnclmtrx.html

$$B = \begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix}$$

(2) Two positive eigenvalues

$$\frac{dx}{dt} = 2x \qquad \frac{dy}{dt} = y \quad \Rightarrow \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Eigenvalues are 2, 1

Eigenvectors are $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

General soln is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}$

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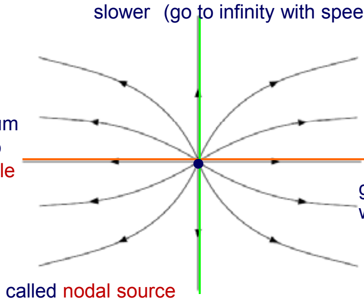
All trajectories tend to infinity as t tends to infinity

slower (go to infinity with speed e^t)



Equilibrium soln zero is unstable

Faster go to infinity with speed e^{2t}

Zero is called nodal source

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Web Application

- http://www.aw-bc.com/ide/idefiles/media/JavaTools/lnclmtrx.html

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

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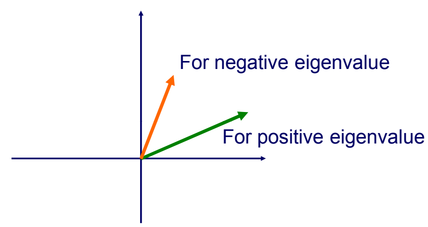
(3) Two real eigenvalues with opposite signs

Suppose we have eigenvectors as follows



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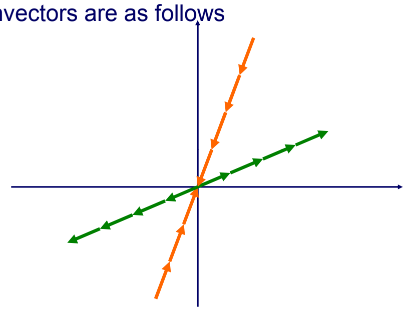
The two basic trajectories induced by the eigenvectors are as follows



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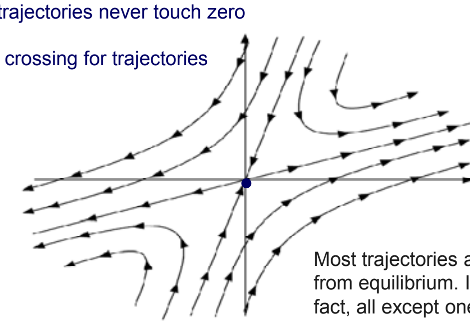
All trajectories never touch zero

No crossing for trajectories



Most trajectories away from equilibrium. In fact, all except one

Zero is called a saddle point and is an unstable equilibrium solution.

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Web Application

- http://www.aw-bc.com/ide/idefiles/media/JavaTools/lnclmtrx.html

$$B = \begin{bmatrix} -1 & -2 \\ -3 & -1 \end{bmatrix}$$

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Summary of Case 1

Two Real Eigenvalues

- Opp signs : Saddle
- Both > 0 : Nodal source
- Both < 0 : Nodal sink

Next we shall use $Tr(B), \det(B)$
to classify the above three cases,
which is easier

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Eigenvalues are given by

$$\alpha = \frac{1}{2}\left[Tr[B] \pm \sqrt{(Tr[B])^2 - 4(\det[B])}\right]$$

Real roots $(TrB)^2 > 4\det B$
Then from above , we have

Nodal Source	Both > 0	$TrB > 0$	$\det B > 0$
Nodal Sink	Both < 0	$TrB < 0$	$\det B > 0$
Saddle	Opp Signs		$\det B < 0$

Recall: eigenvector must be non-zero.

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Case 2: Complex Eigenvalues $(Tr(B))^2 < 4\det(B)$

$$r = \frac{1}{2}\left[Tr(B) \pm i\sqrt{4\det(B) - (Tr(B))^2}\right]$$

$$= \alpha \pm i\beta$$

The general solution is

$$e^{\alpha t}\begin{bmatrix} c_1\left(u\cos\beta t - v\sin\beta t\right) \\ +c_2\left(u\sin\beta t + v\cos\beta t\right) \end{bmatrix}$$

Rotating involving angle βt

Stretch / Shrink

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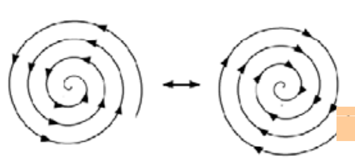
(1) $Tr(B) < 0$



Spiral Sink: (Clockwise or anticlockwise)

Trajectories spiraling towards equilibrium soln zero

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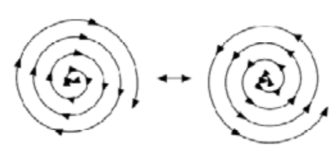
(2) $Tr(B) > 0$



2) Spiral Source: (Clockwise or anticlockwise)

Trajectories spiraling away from equilibrium soln zero

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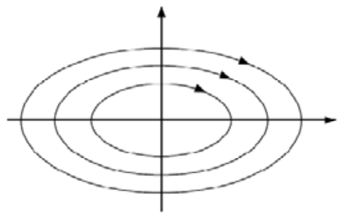
(3) $Tr(B)=0$

Centre



Trajectories orbiting around equilibrium soln zero

SUMMARY of Case 2

Complex eigenvalues

$$(Tr(B))^2 < 4\det(B)$$

Spiral Source	$TrB>0$
Spiral Sink	$TrB<0$
Centre	$TrB=0$

How to determine clockwise or anticlockwise

We shall use one example to illustrate it

Example:

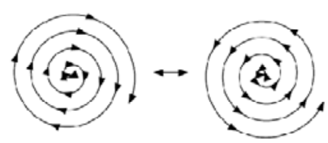
$$B=\begin{bmatrix} 4 & -5 \\ 2 & -2 \end{bmatrix}$$

$$(Tr(B))^2-4\det(B)=4-4(-8+10)=-4$$

Complex Eigenvalues

$Tr(B)=2$

$Tr(B)=2$ ➡ Spiral Source



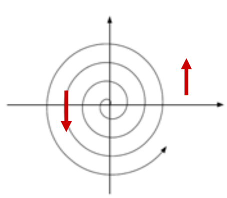
In the above, which one we shall choose

How to determine clockwise or anticlockwise

$$\frac{dy}{dt}=2x-2y \qquad \left.\frac{dy}{dt}\right|_{y=0}=2x$$

It means, near the positive x-axis, the value of y increases; near the negative x-axis, the value of y decreases

Hence anticlockwise



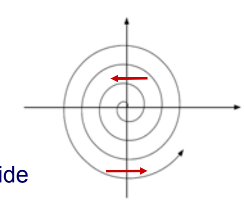
We may also look at

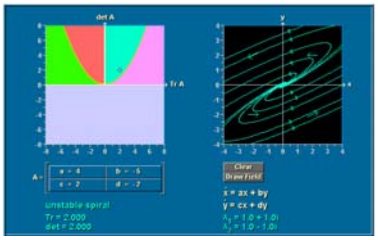
$$\frac{dx}{dt}=4x-5y \qquad \left.\frac{dx}{dt}\right|_{x=0}=-5y$$

It means, near the positive y-axis, the value of x decreases; near the negative y-axis, the value of x increases

Hence anticlockwise

The graph just indicates zero is a spiral source anticlockwise, for more precise graph, see next slide





Summary

Method 1: Find Eigenvalues

EIGENVALUES

both real:
- BOTH $> 0 \to$ Nodal Source
- BOTH $< 0 \to$ Nodal Sink
- Opposite Signs $\to$ Saddle

both complex:
- Real Part $> 0 \to$ Spiral Source
- Real Part $< 0 \to$ Spiral Sink
- Real Part $= 0 \to$ Centre

Method 2: $r = \frac{1}{2}\left[Tr[B] \pm \sqrt{(Tr[B])^2 - 4(\det[B])}\right]$

$(TrB)^2 - 4\det B$

$+$:
- $TrB > 0$, $\det B > 0 \to$ Nodal Source
- $TrB < 0$, $\det B > 0 \to$ Nodal Sink
- $\det B < 0 \to$ Saddle

$-$:
- $TrB > 0 \to$ Spiral Source
- $TrB < 0 \to$ Spiral Sink
- $TrB = 0 \to$ Centre

8.5 Romeo and Juliet

My bounty is as boundless as the sea,
My love as deep; the more I give to thee,
The more I have, the more I give to thee

—Again pledging her love to Romeo, Juliet says that the more she gives love to him, the more she has.

For stony limits cannot hold love out,
And what love can do that dares love attempt.

—When Juliet asks how he managed to get over the high wall to Capulet's garden, Romeo replies that he flew on the wings of love, which is stronger than "stony limits."

The story of Romeo and Juliet in this module is different from Shakespeare's

R(t), Romeo's feelings *J(t)*, Juliet's feelings

$$\frac{dR}{dt} = aJ, \quad R(0) = \alpha$$
$$\frac{dJ}{dt} = -bR, \quad J(0) = \beta$$

a, b >0

The above system of equations says that
Romeo's love grows in proportion to Juliet's love for him
Juliet's love decreases in proportion to Romeo's love for her
How does their relationship evolve?

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$$B = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix} \qquad Tr(B) = 0 \qquad \det(B) = ab$$

$$(Tr(B))^2 - 4\det(B) = -4ab < 0$$

Complex eigenvalues

$$r = \frac{1}{2}\left[Tr(B) \pm i\sqrt{4\det(B) - (Tr(B))^2}\right] = \pm i\sqrt{ab}$$

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For eiganvalue $i\sqrt{ab}$

The corresponding eigenvector is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ \sqrt{\frac{b}{a}} \end{pmatrix}$$

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The general soln is

$$c_1\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos\sqrt{abt} - \begin{pmatrix} 0 \\ \sqrt{\frac{b}{a}} \end{pmatrix}\sin\sqrt{abt}\right)$$

$$+c_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin\sqrt{abt} + \begin{pmatrix} 0 \\ \sqrt{\frac{b}{a}} \end{pmatrix}\cos\sqrt{abt}\right)$$

See 8.1 solution (C2)

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$$R(t) = c_1\cos\sqrt{abt} + c_2\sin\sqrt{abt}$$
$$J(t) = -c_1\sqrt{\frac{b}{a}}\sin\sqrt{abt} + c_2\sqrt{\frac{b}{a}}\cos\sqrt{abt}$$

Now $R(0) = \alpha \quad J(0) = \beta$

so $c_1 = \alpha \qquad c_2 = \sqrt{\frac{a}{b}}\beta$

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Hence

$$R(t) = \alpha\cos(\sqrt{ab}t) + \beta\sqrt{\frac{a}{b}}\sin(\sqrt{ab}t)$$
$$J(t) = \beta\cos(\sqrt{ab}t) - \alpha\sqrt{\frac{a}{b}}\sin(\sqrt{ab}t)$$

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$$\frac{dR}{dt} = aJ, \qquad R(0) = \alpha$$

$$\frac{dJ}{dt} = -bR, \quad J(0) = \beta$$

$$R(0) = \alpha, \qquad \dot{R}(0) = aJ(0) = a\beta$$

$$J(0) = \beta, \qquad \dot{J}(0) = -bR(0) = -b\alpha$$

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Consider $\alpha > 0, \beta = 0$

$$R(t) = \alpha\cos(\sqrt{ab}t)$$

$$J(t) = -\alpha\sqrt{\frac{a}{b}}\sin(\sqrt{ab}t)$$

Hence $$\frac{R^2}{\alpha^2} + \frac{J^2}{\beta^2} = 1$$

where $\beta = \alpha\sqrt{\frac{b}{a}}$

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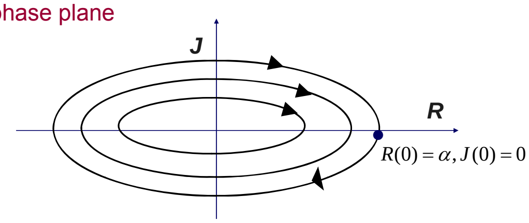
phase plane



Clockwise since $\dot{J}(0) = -b\alpha$

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Note that we do not have to solve ODE.
We can sketch the phase plane using the following two conditions

$$(Tr(B))^2 - 4\det(B) = -4ab < 0$$

$$Tr(B) = 0$$

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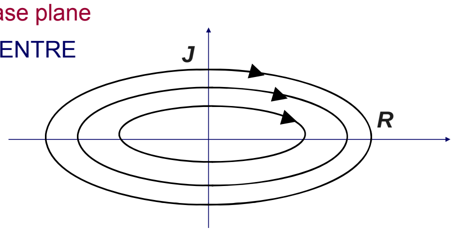
Phase plane

CENTRE



The more Romeo loves her, the more she wants to run away and hide. But when he dislike her, she begins to find him strangely attractive.
He, on the other hand, tends to warm up when she loves him and cools down when she hates him.

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Web Application

- http://www.aw-bc.com/ide/idefiles/media/JavaTools/lnclmtrx.html

$$B = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

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8.6.Ancient Warfare

G(t)= # of Gondorians M(t)= # of Mordorians

$$\frac{dG}{dt} = -G - 0.75M$$

G(0)=15000
M(0)= 11000

$$\frac{dM}{dt} = -G$$

$$\frac{dM}{dt} = -G \qquad \frac{\frac{dM}{dt}}{G} = -1$$

The above equation says Mordorians die (killed by Gondorians) at a rate proportional to the number of Gondorians

Suppose unit of t is day. Then every Gondorian kills one Mordorian per day.

$$\frac{dG}{dt} = -G - 0.75M$$

The above equation says one Mordorian kills 0.75 Gondorians per day

The death rate per capita of Gondorians due to disease is 1

We shall not solve this ODE, but only look sketch its phase plane.

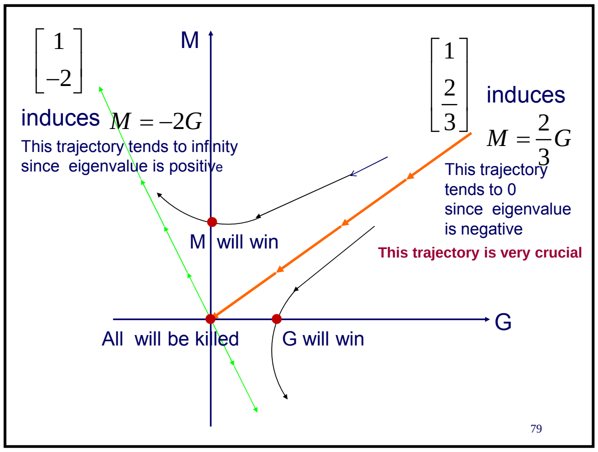
$$\frac{dG}{dt} = -G - 0.75M \qquad \frac{dM}{dt} = -G$$

$$B = \begin{bmatrix} -1 & -0.75 \\ -1 & 0 \end{bmatrix} \Rightarrow TrB = -1, \ \det B = -0.75$$

➡ Saddle

Eigenvalues 1/2 -3/2

Eigenvectors $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}$ induce two basic trajectories


$\begin{bmatrix}1\\-2\end{bmatrix}$ induces $M=-2G$
This trajectory tends to infinity since eigenvalue is positive
$\begin{bmatrix}1\\ \frac{2}{3}\end{bmatrix}$ induces $M=\frac{2}{3}G$
This trajectory tends to 0 since eigenvalue is negative
This trajectory is very crucial
M
G
M will win
All will be killed
G will win
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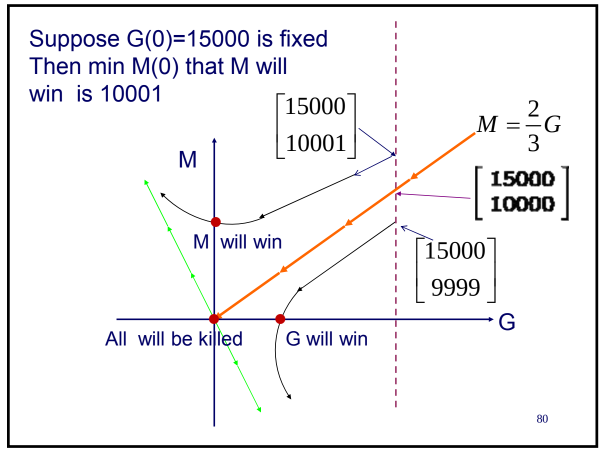
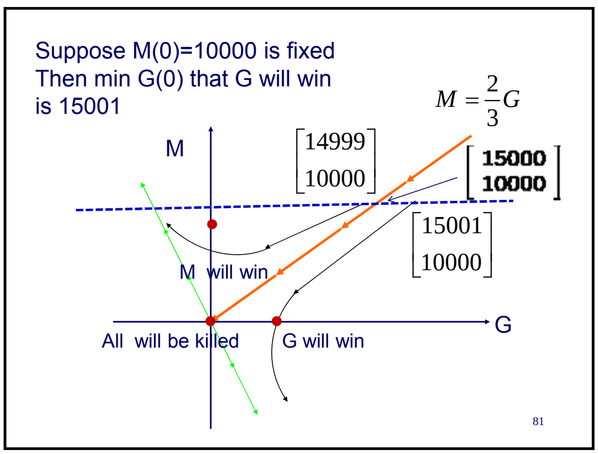
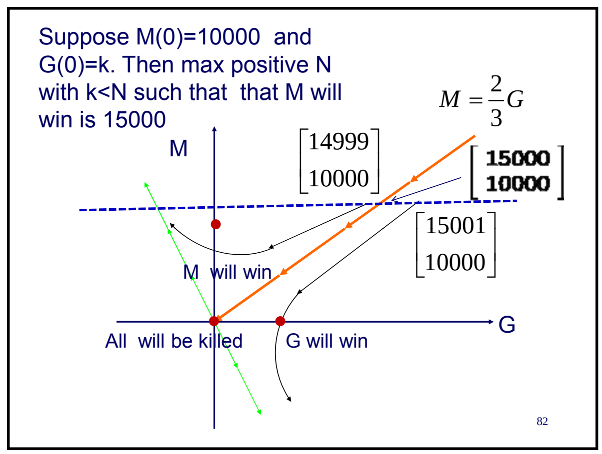

Suppose G(0)=15000 is fixed
Then min M(0) that M will win is 10001
$\begin{bmatrix}15000\\10001\end{bmatrix}$
$M=\frac{2}{3}G$
$\begin{bmatrix}15000\\10000\end{bmatrix}$
$\begin{bmatrix}15000\\9999\end{bmatrix}$
M
G
M will win
All will be killed
G will win
80


Suppose M(0)=10000 is fixed
Then min G(0) that G will win is 15001
$M=\frac{2}{3}G$
$\begin{bmatrix}14999\\10000\end{bmatrix}$
$\begin{bmatrix}15000\\10000\end{bmatrix}$
$\begin{bmatrix}15001\\10000\end{bmatrix}$
M
G
M will win
All will be killed
G will win
81


Suppose M(0)=10000 and G(0)=k. Then max positive N with k<N such that that M will win is 15000
$M=\frac{2}{3}G$
$\begin{bmatrix}14999\\10000\end{bmatrix}$
$\begin{bmatrix}15000\\10000\end{bmatrix}$
$\begin{bmatrix}15001\\10000\end{bmatrix}$
M
G
M will win
All will be killed
G will win
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