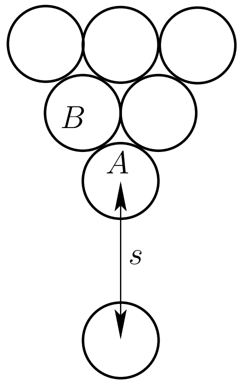
PC2232 Physics for Electrical Engineers: Tutorial 7

Question 1: Scanning Tunneling Microscope (STM)

In the STM, the tunneling current depends exponentially $I = I_0 \mathrm{e}^{-2ks}$ on the distance, s, between the tip atom and the surface atom (center-to-center distance). Let I_A be the tunneling current due to tunneling between the atom (A) and the surface atom. Let I_B be the tunneling current due to tunneling between the atom (B) (one level higher than tip atom) and the surface atom. Calculate the ratio I_B/I_A. (Assume $k = 5 \times 10^9$ m^{-1}, and diameter of the atoms is 0.5nm and $s = 2$ nm.)



Question 2: Hydrogen Atom

Consider an electron in the ground state of a hydrogen atom,

$$\psi_{100}(r, \theta, \phi) = \frac{1}{\sqrt{\pi a_0^3}} \mathrm{e}^{-r/a_0}. \tag{1}$$

(a) Sketch the plots of E and $U(r)$ on the same axes.

(b) Show that, classically, an electron with this energy should not be able to get further than $2a_0$ from the proton.

(c) What is the probability of the electron being found in the classically forbidden region? *Hint*: The formula for an infinitesimal volume element in spherical coordinates is given by (Lecture 9, page 10):

$$\mathrm{d}V = r^2 \sin\theta \; \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi. \tag{2}$$

Also, you might find the following indefinite integral formula useful:

$$\int \mathrm{d}x \; x^2 \mathrm{e}^{ax} = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right) \mathrm{e}^{ax}. \tag{3}$$

Question 3: Average angular momentum

A simplified approach to the question of how l is related to angular momentum can be stated as follows: If L_z can take on only those values $m_l \hbar$, where $m_l = 0, \pm 1, \ldots, \pm l$, then its square is allowed only values $m_l^2 \hbar^2$, and the average of L_z^2 should be the sum of its allowed values divided by the number of values $2l + 1$. Because there is no preferred direction in space, the averages of L_x^2 and L_y^2 should be the same, and the sum of all three should give the average of L^2. Given the sum

$$\sum_{n=1}^{N} n^2 = \frac{1}{6} N \left(N+1\right) \left(2N+1\right), \tag{4}$$

show that by these arguments, the average of L^2 should be $l(l+1)\hbar^2$.

Question 4: Particle in a 3D box

Consider a cubic 3D infinite well of side length L. There are 15 identical particles of mass m in the well. For some reason, no more than two particles may occupy the same wave function.

(a) What is the lowest possible *total* energy?

(b) In this minimum total-energy state, at what point(s) would the highest energy particle most likely be found?

(c) What is the occupation number N_{n_x,n_y,n_z} for each value of energy involved?

Question 5: Degeneracy counting

Consider a particle in a 2-dimensional square box of sides $L_x = L_y = L$. The Schrödinger equation was solved in the previous tutorial. Recall that the energy states and quantum numbers are given by

n_x	n_y	E_n	d_n
1	1	$E_1 = 2K$	1
1	2	$E_2 = 5K$	2
2	1		
$\vdots$	$\vdots$	$E_3 =$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

where we have introduced the abbreviation

$$K = \frac{\pi^2 \hbar^2}{2mL^2}. \tag{5}$$

Suppose that this box is occupied by three ($N = 3$) *fermions* of spin 1/2. The total energy of the system is observed to be $E = 20K$. What are the possible configurations $(N_1, N_2, N_3, N_4, N_5)$ that carries this energy? For each configuration, calculate each of their respective multiplicities $W(N_1, N_2, N_3, N_4, N_5)$.

Question 6: (Optional) Hybrid states

The $\psi_{2,1,0}$ state (the $2p$ state in which $m_l = 0$) has most of its probability density along the z-axis, and so is often referred to as the $2p_z$ state. To allow its probability density to stick out in other ways, and thus facilitate various kinds of molecular bonding with other atoms, an atomic electron may assume a wave function that is an algebraic combination of multiple wave functions open to it. One such "hybrid state" is the sum $\psi_{2,1,+1} + \psi_{2,1,-1}$. (*Note*: because the Schrödinger equation is a linear differential equation, a sum of solutions with the same energy is also a solution with that energy. Also, normalization constants may be ignored in the following questions.)

(a) Write this wave function and its probability density in terms of r, θ and ϕ. (Use the Euler formula to simplify your results.)

(b) In which of the following ways does this state differ from its parts (i.e., $\psi_{2,1,+1}$ and $\psi_{2,1,-1}$) and from the $2p_z$ state: Energy? Radial dependence of its probability density? Angular dependence of its probability density?

(c) The state is often referred to as the $2p_y$. Why?

(d) How might we produce a $2p_x$ state?

Question 7: (Optional) Orbital angular momentum

Consider a hydrogen-like atom where the electron orbital has angular momentum given by

$$\vec{L} = L_x\,\hat{x} + L_y\,\hat{y} + L_z\,\hat{z} \tag{6}$$

(a) If the value of L_z is known, we cannot know either L_x or L_y precisely. But we can know the value of the quantity $\sqrt{L_x^2 + L_y^2}$. Write an expression for this quantity in terms of l, m_l and $\hbar$.

(b) What is the meaning of $\sqrt{L_x^2 + L_y^2}$?

(c) For a state of non-zero orbital angular momentum, find the maximum and minimum values of $\sqrt{L_x^2 + L_y^2}$. Explain your results.