PC2232: Basics of Quantum Mechanics

(Lecture 6)

15/03/2015

Review of Probability Laws

If $S = A \cup B \cup C$, where A, B, C are disjoint

$$P(D) = P(A \cap D) + P(B \cap D) + P(C \cap D) \quad \text{---(1)}$$

Also,
$$\left.\begin{aligned} P(A \cap D) &= P(D|A)\,P(A) \\ P(B \cap D) &= P(D|B)\,P(B) \\ P(C \cap D) &= P(D|C)\,P(C) \end{aligned}\right\} \quad \text{---(2)}$$

Using (1) & (2):

$$\left.\begin{aligned} P(D) &= P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C) \\ \Rightarrow P(D) &= \sum_{\text{all } X} P(D|X)\,P(X) \quad \text{where } X = A, B, C \end{aligned}\right\} \quad \text{---(3)}$$

Quantities in Quantum Mechanics:

In quantum mechanics, measurable quantities such as position (x), momentum (k) etc. are reprented by their wavenumbers $\psi(x,t)$, $\psi(k,t)$ etc.

These are functions of complex quantities known as complex amplitudes and time.

Case I: x is a discrete random variable

Born's rule:

Probability of finding x with wave function $\psi(x,t)$:

$$P(x,t) = |\psi(x,t)|^2 \quad \text{---(4)}$$

Normalization:

$$\sum_{x} P(x,t) = \sum_{x} |\psi(x,t)|^2 = 1 \quad \text{---(5)}$$

all possible values of x.

Eqn (5) is valid for any time 't', such that:

$$\sum_{x} |\psi(x,t_1)|^2 = \sum_{x} |\psi(x,t_0)|^2 = 1 \quad \text{--- (5a)}$$

t_0 = initial time
t_1 = any other time

(b) <u>Time evolution:</u>

Analogus to (3), we can write:

$$\psi(x,t) = \sum_{x_0} \psi(x,t \mid x_0,t_0)\, \psi(x_0,t_0) \quad \text{--- (6)}$$

Here $\psi(x,t)$, $\psi(x_0,t_0)$ are column vectors and $\psi(x,t \mid x_0,t_0)$ known as the propagator is a unitary matrix. For $x \in \{0,1\}$, Equation (6) can be expanded to:

Complex amplitudes

$$\underbrace{\begin{pmatrix} \psi(0,t) \\ \psi(1,t) \end{pmatrix}}_{\psi(x,t)} = \underbrace{\begin{bmatrix} \psi(0,t \mid 0,t_0) & \psi(0,t \mid 1,t_0) \\ \psi(1,t \mid 0,t_0) & \psi(1,t \mid 1,t_0) \end{bmatrix}}_{\psi(x,t, \mid x_0,t_0)} \underbrace{\begin{pmatrix} \psi(0,t_0) \\ \psi(1,t_0) \end{pmatrix}}_{\psi(x_0,t_0)} \quad \text{--- (6a)}$$

<u>Review of Concepts from Linear Algebra</u>

For a matrix A with real elements:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ its transpose } A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

For a matrix C with Complex elements

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \text{ its (Complex) transpose } C^H = \begin{pmatrix} c_{11}^* & c_{21}^* \\ c_{12}^* & c_{22}^* \end{pmatrix}$$

Inner product of real vectors: $A = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}$
(dot product).

$$A^T B = \begin{pmatrix} a_{11} & a_{21} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = a_{11} b_{11} + a_{21} b_{21}$$

Inner product of complex vectors: $C = \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix}$, $D = \begin{pmatrix} d_{11} \\ d_{21} \end{pmatrix}$
(dot product)

$$C^H D = \begin{pmatrix} c_{11}^* & c_{21}^* \end{pmatrix} \begin{pmatrix} d_{11} \\ d_{21} \end{pmatrix} = c_{11}^* d_{11} + c_{21}^* d_{21}$$

Orthogonal matrix Q is one with real elements with the property: $Q^T Q = I$. (identity matrix)

Unitary matrix U is one with complex elements with the property: $U^H U = I$. (identity matrix)

<u>Generalizing:</u>

If $A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix}$ where $a_1, a_2, a_3, \ldots, a_n$ are complex

Then $|A|^2 = A^H A = \begin{pmatrix} a_1^* & a_2^* & a_3^* \cdots a_n^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

$$= a_1^* a_1 + a_2^* a_2 + \cdots + a_n^* a_n$$

$$= \sum_{i=1}^{n} a_i^* a_i$$

Wavefunction and complex amplitudes

Let $\psi(x,t)$ be a wavefunction of x. If x can assume values $x_1, x_2, \ldots, x_n$ such that $x \in \{x_1, x_2, \ldots, x_n\}$

Then
$$\psi(x,t) = \begin{bmatrix} \psi(x_1,t) \\ \psi(x_2,t) \\ \vdots \\ \psi_n(x_n,t) \end{bmatrix}$$

$\psi(x_1,t)$, $\psi(x_2,t)$, ..., $\psi(x_n,t)$ are complex numbers and known as complex amplitudes of the wave function $\psi(x,t)$.

There is a probability associated with each complex amplitude:

$$P(x=x_1) = |\psi(x_1,t)|^2$$
$$P(x=x_2) = |\psi(x_2,t)|^2$$
$$\vdots$$
$$P(x=x_n) = |\psi(x_n,t)|^2$$

Such that: $P(x=x_1) + P(x=x_2) + \cdots + P(x=x_n)$

$$= |\psi(x_1,t)|^2 + |\psi(x_2,t)|^2 + \cdots + |\psi(x_n,t)|^2 = 1$$

or, $$\sum_x P(x,t) = \sum_x |\psi(x,t)|^2 = 1.$$

<u>Spin</u>: Like position (x), momentum (k), spin (s) is another quantum quantity and is discrete in nature

$s \in \{+\frac{1}{2}, -\frac{1}{2}\}$ are the complex amplitudes for s.

The z-component of the corresponding wavefunction:

$$S_z = \psi_z(s,t) = \begin{pmatrix} \psi_z(\frac{1}{2},t) \\ \psi_z(-\frac{1}{2},t) \end{pmatrix} \quad --- (7)$$

The x component is related to the z-component:

$$S_x = \psi_x(s,t) = \begin{pmatrix} \psi_x(\frac{1}{2},t) \\ \psi_x(-\frac{1}{2},t) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \psi_z(\frac{1}{2},t) \\ \psi_z(-\frac{1}{2},t) \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \psi_z(s,t) \quad --- (8)$$

The y component is related to the z-component

$$S_y = \psi_y(s,t) = \begin{pmatrix} \psi_y(1/2,t) \\ \psi_y(-1/2,t) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -j/\sqrt{2} \\ 1/\sqrt{2} & j/\sqrt{2} \end{pmatrix} \begin{pmatrix} \psi_z(1/2,t) \\ \psi_z(-1/2,t) \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & -j/\sqrt{2} \\ 1/\sqrt{2} & j/\sqrt{2} \end{pmatrix} \psi_z(s,t) \quad --- (9)$$

<u>Uncertainties of Incompatible Observables</u>

Let $S_z = \begin{pmatrix} \psi_z(1/2,t) \\ \psi_z(-1/2,t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at some t.

This means at time 't' $\begin{cases} P(S_z = 1/2) = |1|^2 = 1 \\ P(S_z = -1/2) = |0|^2 = 0 \end{cases}$

Then using (8), at time t,

$$S_x = \begin{pmatrix} \psi_x(1/2, t) \\ \psi_x(-1/2, t) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

So, at time 't' $\begin{cases} P(S_x = 1/2) = (|1/\sqrt{2}|)^2 = 1/2 \\ P(S_x = -1/2) = (|1/\sqrt{2}|)^2 = 1/2 \end{cases}$

Also using (9), at time 't'

$$S_y = \begin{pmatrix} \psi_y(1/2, t) \\ \psi_y(-1/2, t) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -j/\sqrt{2} \\ 1/\sqrt{2} & j/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

So, at time 't' $\begin{cases} P(S_y = 1/2) = (1/\sqrt{2})^2 = 1/2 \\ P(S_y = -1/2) = (1/\sqrt{2})^2 = 1/2 \end{cases}$

So, even if S_z is deterministic; S_x, S_y are not.

Case II : x is a continuous random variable

Born's Rule

Probability density of finding x with wavefunction $\psi(x,t)$

$$f(x,t) = |\psi(x,t)|^2 \quad \text{--- (10)}$$

so that $$P(a \leq x < b) = \int_a^b f(x,t)\,dx \quad \text{--- (11)}$$

Normalization

$$\int_{-\infty}^{\infty} f(x,t)\,dx = \int_{-\infty}^{\infty} |\psi(x,t)|^2\,dx = 1 \quad \text{--- (12)}$$

Time evolution:

$$\psi(x,t) = \int_{-\infty}^{\infty} \psi(x,t \mid x_0,t_0)\,\psi(x_0,t_0)\,dx_0 \quad \text{---} (12)$$

Position and Momentum wave functions:

Momentum wavefunction is the Fourier Transform of the position wave function

$$\phi(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(-jkx)}\,\psi(x,t)\,dx \quad \text{----} (13)$$

Momentum in m.k.s. units

$$p = \hat{h}k \quad \text{----} (14)$$

$\hat{h} = \frac{h}{2\pi}$, h = Plank's const. $= 6.626 \times 10^{-34}$ Js