PC2232 – Tutorial 1 Solutions

1. (a) Given: $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

   Question: Show that $\vec{\nabla}(\vec{k}\cdot\vec{r}) = \vec{k}$

   Approach:
   - Write out $\vec{\nabla}$ and work through the terms
   - Note that k_x, k_y and k_z are constants
   - Understand the concept of partial differential, i.e. understand the following
     i. $\dfrac{\partial(k_y y)}{\partial x} = 0$
     ii. $\dfrac{\partial(k_x x)}{\partial x} = k_x$
   - Learning point: Familiarize yourself with vector algebra

   Solution:

$$
\begin{aligned}
\vec{\nabla}(\vec{k}\cdot\vec{r}) &= \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\left([k_x\hat{x} + k_y\hat{y} + k_z\hat{z}]\cdot[x\hat{x} + y\hat{y} + z\hat{z}]\right) \\
&= \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\left(k_x x + k_y y + k_z z\right) \\
&= \frac{\partial}{\partial x}\left(k_x x + k_y y + k_z z\right)\hat{x} + \frac{\partial}{\partial y}\left(k_x x + k_y y + k_z z\right)\hat{y} + \frac{\partial}{\partial z}\left(k_x x + k_y y + k_z z\right)\hat{z} \\
&= k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \\
&= \vec{k}
\end{aligned}
$$

   (b) Given: $\vec{E}_0$ is constant

   Question: Show that $\vec{\nabla}\cdot\left(\vec{E}_0 \mathrm{e}^{j\vec{k}\cdot\vec{r}}\right) = j\vec{k}\cdot\vec{E}_0\,\mathrm{e}^{j\vec{k}\cdot\vec{r}}$

   Approach:
   - The product rule for ∇ states: $\vec{\nabla}\cdot(a\vec{A}) = a\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}a$
   - Know how to differentiate exponential term[1]:
     $\dfrac{\mathrm{d}\mathrm{e}^{f(x)}}{\mathrm{d}x} = \dfrac{\mathrm{d}f(x)}{\mathrm{d}x}\mathrm{e}^{f(x)}$
   - Recall that $\vec{E}_0$ is a constant[2]
   - Learning point: Prove the equation in lecture notes as it will be useful later[3]

   Solution:

$$
\begin{aligned}
\vec{\nabla}\cdot\left(\vec{E}_0\mathrm{e}^{j\vec{k}\cdot\vec{r}}\right) &= \mathrm{e}^{j\vec{k}\cdot\vec{r}}\vec{\nabla}\cdot\vec{E}_0 + \vec{E}_0\cdot\vec{\nabla}\mathrm{e}^{j\vec{k}\cdot\vec{r}} \\
&= \vec{E}_0\cdot\vec{\nabla}\mathrm{e}^{j\vec{k}\cdot\vec{r}} \\
&= \vec{E}_0\cdot\vec{\nabla}(j\vec{k}\cdot\vec{r})\mathrm{e}^{j\vec{k}\cdot\vec{r}}
\end{aligned}
$$

   From part (a): $\vec{\nabla}(j\vec{k}\cdot\vec{r}) = j\vec{k}$
   Therefore:

$$
\vec{\nabla}\cdot\left(\vec{E}_0\mathrm{e}^{j\vec{k}\cdot\vec{r}}\right) = j\vec{k}\cdot\vec{E}_0\mathrm{e}^{j\vec{k}\cdot\vec{r}}
$$

2. Given: $(\vec{E}_1, \vec{H}_1)$ and $(\vec{E}_2, \vec{H}_2)$ are solutions to Maxwell equations

   (a) Show that $(a_1\vec{E}_1 + a_2\vec{E}_2, a_1\vec{H}_1 + a_2\vec{H}_2)$

   Approach:
   - Know the Maxwell equations

[1] If this confuses you, try differentiating e^{4x}. Notice that the equation I wrote is the general form of what you're doing there

[2] Knowing this makes my product rule above seem quite pointless.
Please understand that I show the product rule above in order to ensure that your $\cdot$ is in the right position.
Without the product rule, many would write the second term as $\vec{A}\vec{\nabla}\cdot a$ which does not make sense. The product rule is a known rule in vector calculus/algebra, so I believe we can use it without proof. Just wiki it to find out more info on it

[3] We use the cross version to obtain $\vec{H}$ from $\vec{E}$ as will be brought up in tutorial 2

- Confirm that the dot and cross product of $\vec{\nabla}$ is distributive[4]
- Apply the distributive principle to the 4 Maxwell equations
- Rearrange the terms such that it is in the same form as the original Maxwell equations
- Learning point:
  Familiarize yourself with vector algebra
  Understand and convince yourself about the linearity of Maxwell equations

Solution:

- Maxwell equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} \cdot \vec{H} = 0$$

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \qquad \vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

- Confirming that the dot and cross product of $\vec{\nabla}$ is distributive
  i. $\vec{\nabla}\cdot$ is distributive

$$\begin{aligned}
\vec{\nabla} \cdot (\vec{A} + \vec{B}) &= \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \right) \cdot \left([A_x + B_x]\hat{x} + [A_y + B_y]\hat{y} + [A_z + B_z]\hat{z} \right) \\
&= \frac{\partial}{\partial x}(A_x + B_x) + \frac{\partial}{\partial y}(A_y + B_y) + \frac{\partial}{\partial z}(A_z + B_z) \\
&= \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z + \frac{\partial}{\partial x}B_x + \frac{\partial}{\partial y}B_y + \frac{\partial}{\partial z}B_z \\
&= \frac{\partial}{\partial x}\hat{x} \cdot A_x\hat{x} + \frac{\partial}{\partial y}\hat{y} \cdot A_y\hat{y} + \frac{\partial}{\partial z}\hat{z} \cdot A_z\hat{z} + \frac{\partial}{\partial x}\hat{x} \cdot B_x\hat{x} + \frac{\partial}{\partial y}\hat{y} \cdot B_y\hat{y} + \frac{\partial}{\partial z}\hat{z} \cdot B_z\hat{z} \\
&= (\vec{\nabla} \cdot \vec{A}) + (\vec{\nabla} \cdot \vec{B})
\end{aligned}$$

  ii. $\vec{\nabla}\times$ is distributive
  First, know the cross rule for unit vectors

$$\begin{array}{lll}
\hat{x} \times \hat{x} = 0 & \hat{x} \times \hat{y} = \hat{z} & \hat{x} \times \hat{z} = -\hat{y} \\
\hat{y} \times \hat{x} = -\hat{z} & \hat{y} \times \hat{y} = 0 & \hat{y} \times \hat{z} = \hat{x} \\
\hat{z} \times \hat{x} = \hat{y} & \hat{z} \times \hat{y} = -\hat{x} & \hat{z} \times \hat{z} = 0
\end{array}$$

  Second, know how the write out cross products in component form.

$$\begin{aligned}
\vec{\nabla} \times \vec{C} &= \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \right) \times \left(C_x\hat{x} + C_y\hat{y} + C_z\hat{z} \right) \\
&= \left[\frac{\partial C_y}{\partial x}\hat{z} - \frac{\partial C_z}{\partial x}\hat{y} \right] + \left[-\frac{\partial C_x}{\partial y}\hat{z} + \frac{\partial C_z}{\partial y}\hat{x} \right] + \left[\frac{\partial C_x}{\partial z}\hat{y} - \frac{\partial C_y}{\partial z}\hat{x} \right] \\
&= \left[\frac{\partial C_z}{\partial y} - \frac{\partial C_y}{\partial z} \right]\hat{x} + \left[\frac{\partial C_x}{\partial z} - \frac{\partial C_z}{\partial x} \right]\hat{y} + \left[\frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y} \right]\hat{z}
\end{aligned}$$

  Finally, applying the above to the following:

$$\begin{aligned}
\vec{\nabla} \times (\vec{A} + \vec{B}) &= \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \right) \times \left([A_x + B_x]\hat{x} + [A_y + B_y]\hat{y} + [A_z + B_z]\hat{z} \right) \\
&= \left[\frac{\partial(A_z + B_z)}{\partial y} - \frac{\partial(A_y + B_y)}{\partial z} \right]\hat{x} + \left[\frac{\partial(A_x + B_x)}{\partial z} - \frac{\partial(A_z + B_z)}{\partial x} \right]\hat{y} + \left[\frac{\partial(A_y + B_y)}{\partial x} - \frac{\partial(A_x + B_x)}{\partial y} \right]\hat{z} \\
&= \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right]\hat{x} + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right]\hat{y} + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right]\hat{z} + \\
&\quad \left[\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right]\hat{x} + \left[\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right]\hat{y} + \left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right]\hat{z} \\
&= (\vec{\nabla} \times \vec{A}) + (\vec{\nabla} \times \vec{B})
\end{aligned}$$

  iii. Scalar constant can be taken out of $\vec{\nabla}\cdot$ and $\vec{\nabla}\times$

$$\begin{aligned}
\vec{\nabla} \cdot (a\vec{B}) &= \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \right) \cdot \left(aB_x\hat{x} + aB_y\hat{y} + aB_z\hat{z} \right) \\
&= \frac{\partial}{\partial x}(aB_x) + \frac{\partial}{\partial y}(aB_y) + \frac{\partial}{\partial z}(aB_z)
\end{aligned}$$

[4] In class I called this the linearity of $\vec{\nabla}\cdot$ and $\vec{\nabla}\times$

$$
\begin{aligned}
&= a\frac{\partial}{\partial x}(B_x) + a\frac{\partial}{\partial y}(B_y) + a\frac{\partial}{\partial z}(B_z) \\
&= a\left[\frac{\partial}{\partial x}(B_x) + \frac{\partial}{\partial y}(B_y) + \frac{\partial}{\partial z}(B_z)\right] \\
&= a\left[\frac{\partial}{\partial x}\hat{x}\cdot(B_x\hat{x}) + \frac{\partial}{\partial y}\hat{y}\cdot(B_y)\hat{y} + \frac{\partial}{\partial z}\hat{z}\cdot(B_z\hat{z})\right] \\
&= a\vec{\nabla}\cdot\vec{B}
\end{aligned}
$$

Please do the proof for $\vec{\nabla}\times$ on your own

- Applying the distributive principle Maxwell equations[5]
  i. Proof for: $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

$$
\begin{aligned}
\vec{\nabla}\cdot(a_1\vec{E}_1 + a_2\vec{E}_2) &= \vec{\nabla}\cdot(a_1\vec{E}_1) + \vec{\nabla}\cdot(a_2\vec{E}_2) \\
&= a_1(\vec{\nabla}\cdot\vec{E}_1) + a_2(\vec{\nabla}\cdot\vec{E}_2) \\
&= a_1\frac{\rho_1}{\varepsilon_0} + a_2\frac{\rho_2}{\varepsilon_0} \\
&= \frac{a_1\rho_1 + a_2\rho_2}{\varepsilon_0} \\
&= \frac{\rho'}{\varepsilon_0}
\end{aligned}
$$

Where[6]: $\rho' = a_1\rho_1 + a_2\rho_2$

  ii. Proof for: $\vec{\nabla}\times\vec{H} = \vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}$

$$
\begin{aligned}
\vec{\nabla}\times(a_1\vec{H}_1 + a_2\vec{H}_2) &= \vec{\nabla}\times(a_1\vec{H}_1) + \vec{\nabla}\times(a_2\vec{H}_2) \\
&= a_1(\vec{\nabla}\times\vec{H}_1) + a_2(\vec{\nabla}\times\vec{H}_2) \\
&= a_1\left(\vec{J}_1 + \varepsilon_0\frac{\partial\vec{E}_1}{\partial t}\right) + a_2\left(\vec{J}_2 + \varepsilon_0\frac{\partial\vec{E}_2}{\partial t}\right) \\
&= (a_1\vec{J}_1 + a_2\vec{J}_2) + a_1\varepsilon_0\frac{\partial\vec{E}_1}{\partial t} + a_2\varepsilon_0\frac{\partial\vec{E}_2}{\partial t} \\
&= (a_1\vec{J}_1 + a_2\vec{J}_2) + \varepsilon_0\frac{\partial a_1\vec{E}_1}{\partial t} + \varepsilon_0\frac{\partial a_2\vec{E}_2}{\partial t} \\
&= (a_1\vec{J}_1 + a_2\vec{J}_2) + \varepsilon_0\frac{\partial(a_1\vec{E}_1 + a_2\vec{E}_2)}{\partial t} \\
&= \vec{J}' + \varepsilon_0\frac{\partial\vec{E}'}{\partial t}
\end{aligned}
$$

Note that the last few steps are necessary[7]. If you are confused, try working backwards

i.e. prove: $\varepsilon_0\frac{\partial(a_1\vec{E}_1 + a_2\vec{E}_2)}{\partial t} \quad\rightarrow\quad a_1\varepsilon_0\frac{\partial\vec{E}_1}{\partial t} + a_2\varepsilon_0\frac{\partial\vec{E}_2}{\partial t}$

(b) Generalize the results by showing that $\left(\sum_{i=1}^N a_i\vec{E}_i, \sum_{i=1}^N a_i\vec{H}_i\right)$ is also a solution to Maxwell's equations

Approach:

- Note that this is exactly the same as the above. Just pull both the a_1 and $\sum$ out of the $\vec{\nabla}\cdot$ and $\vec{\nabla}\times$ since we've proven the linearity of $\vec{\nabla}\cdot$ and $\vec{\nabla}\times$
- I'll show it for one of the equations
- Learning point: Fully comprehend[8] the linearity of $\vec{\nabla}\cdot$ and $\vec{\nabla}\times$

Solution:

[5] I showed for 2 of them. Do the other two on your own

[6] I find it a little difficult to bring across the idea this is not just mathematical manipulation. It makes sense that if I have a field that has the magnitude $a_1\vec{E}_1 + a_2\vec{E}_2$, we need a charge density of $a_1\rho_1 + a_2\rho_2$ to produce it. Just the way if I want to double the $\vec{E}$ that I have now, I have to double the charge density. So it makes physical sense.

[7] The $\vec{H}$ that goes into the left hand side of Maxwell's equation must be paired with the correct $\vec{E}$ that goes into the right hand side of Maxwell's equation. Remember, these are not just math equations, there is a physical meaning behind this equation – the $\vec{H}$ can only be created by the correct $\vec{E}$.

[8] Understand that pulling out the the $\sum$ is merely equivalent to the statement that we can do the addition ***after*** doing the dot product

$$
\begin{aligned}
\vec{\nabla} \cdot \left(\sum_{i=1}^{N} a_i \vec{E}_i \right) &= \sum_{i=1}^{N} a_i \left(\vec{\nabla} \cdot \vec{E}_i \right) \\
&= \sum_{i=1}^{N} a_i \frac{\rho_i}{\varepsilon_0} \\
&= \frac{\rho'}{\varepsilon_0}
\end{aligned}
$$

3. Verify that $f(x - vt)$ is a solution to the wave equation

   Approach:

   - Know the wave equation
   - Know the chain rule
   - Learning point: How to deal with generalized functions

   Solution:

   - Wave equation: $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{1}{c^2} \dfrac{\partial^2 f}{\partial t^2}$
   - Working out the LHS and RHS with the chain rule

     Let: $x - vt = u$

     Therefore:

$$
f(x - vt) = f(u) \qquad \frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial t} = -v
$$

   Looking at the LHS and RHS of the equation

$$
\begin{aligned}
\frac{\partial^2 f(u)}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f(u)}{\partial x} \right) \\
&= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \frac{\partial f(u)}{\partial u} \right) \\
&= \frac{\partial}{\partial x} \left(\frac{\partial f(u)}{\partial u} \right) \\
&= \frac{\partial u}{\partial x} \frac{\partial^2 f(u)}{\partial u^2} \\
&= \frac{\partial^2 f(u)}{\partial u^2}
\end{aligned}
\qquad
\begin{aligned}
\frac{\partial^2 f(u)}{\partial t^2} &= \frac{\partial}{\partial t} \left(\frac{\partial f(u)}{\partial t} \right) \\
&= \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \frac{\partial f(u)}{\partial u} \right) \\
&= \frac{\partial}{\partial t} \left(-v \frac{\partial f(u)}{\partial u} \right) \\
&= -v \left[\frac{\partial}{\partial t} \left(\frac{\partial f(u)}{\partial u} \right) \right] \\
&= -v \left[\frac{\partial u}{\partial t} \frac{\partial}{\partial u} \left(\frac{\partial f(u)}{\partial u} \right) \right] \\
&= v^2 \frac{\partial^2 f(u)}{\partial u^2}
\end{aligned}
$$

   - Putting them together:

     If v is c, we get

$$
\begin{aligned}
\frac{1}{c^2} \frac{\partial^2 f(x - vt)}{\partial t^2} &= \frac{1}{c^2} \left(v^2 \frac{\partial^2 f(u)}{\partial u^2} \right) \\
&= \frac{\partial^2 f(u)}{\partial u^2} && \because c = v \\
&= \frac{\partial^2 f(u)}{\partial x^2} && \text{As proven above} \\
&= \frac{\partial^2 f(x - vt)}{\partial x^2} && \text{Proven!}
\end{aligned}
$$

4. Verify that $\left\langle \cos^2(kx - \omega t) \right\rangle = \dfrac{1}{2}$

   Approach:

   - Know the formula for time average

- Know that to integrate $\cos^2$ function we must use the following: $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$
- Please note that I used a simpler explanation in class because I did not have enough time to show the integration[9]
- Learning point: Understand why S_{av} and u_{av} have a factor of $\frac{1}{2}$

Solution:

$$\begin{aligned}
\langle\cos^2(kx-\omega t)\rangle &= \frac{1}{T}\int_0^T \cos^2(kx-\omega t)\mathrm{d}t \\
&= \frac{1}{T}\int_0^T \left(\frac{1}{2} + \frac{1}{2}\cos\big(2[kx-\omega t]\big)\right)\mathrm{d}t \\
&= \frac{1}{T}\left[\left(\frac{1}{2}t - \frac{1}{2}\frac{\sin\big(2[kx-\omega t]\big)}{2\omega}\right)\right]_0^T \\
&= \frac{1}{T}\left(\frac{1}{2}T\right) \qquad\qquad \text{this step is explained below} \\
&= \frac{1}{2}
\end{aligned}$$

Note that:

$$[\sin(2kx-2\omega t)]_0^T = \left[\sin\left(2kx - 2\frac{2\pi}{T}t\right)\right]_0^T = \sin 2kx - \sin(2kx-4\pi) = 0$$

Because $\sin(\theta \pm n\pi) = \sin\theta$

5. Given: $\vec{S} = 10kx\,\hat{z}\,\mathrm{Wm}^{-2}$
   Question: $P =$?

   Approach:

   - Know the equation $P = \int \vec{S}\cdot\mathrm{d}\vec{A}$
   - Know that the $\vec{A}$ direction points perpendicular to the surface
   - Learning point: Familiarize yourself with surface integration

   Solution:

$$\begin{aligned}
P &= \int \vec{S}\cdot\mathrm{d}\vec{A} \\
&= \int_0^b\int_0^a 10kx\,\hat{z}\cdot\mathrm{d}x\,\mathrm{d}y\,\hat{z} \\
&= \int_0^b 5ka^2\,\mathrm{d}y \\
&= 5ka^2b\,\mathrm{W}
\end{aligned}$$

   Answer:

$$P = 5ka^2b\,\mathrm{W}$$

6. Given:

$$P = 316\times10^3\,\mathrm{W} \qquad\qquad r = 5\times10^3\mathrm{m}$$

   (a) Question: P_{rad}=?

   Approach:

   i. Understand (derive[10] on your own: pg 19-23 of lecture 1) the equation: $P_{\text{rad}} = \frac{1}{2}\varepsilon_0 E_0^2 = \frac{S_{\text{av}}}{c}$

[9] In class I did the integral over 1 period, and therefore the $\cos(2[kx-\omega t])$ term immediately drops out. Here I did it over a general value of T

[10] The equation between P_{rad} and S_{av} is not in the lecture notes, but you should be able to derive it since you know how S is related to E

ii. Know that average Poynting Vector (S_{av}) is intensity (I)
iii. Know how intensity relates to power (P)
iv. Potential errors:
   Forget that it's a hemisphere: $A = 2\pi r^2$
   Forget that totally reflecting wave has a factor of 2

Solution:

$$P_{\text{rad}} = \frac{2S_{\text{av}}}{c} = \frac{2I}{c} = \frac{2P}{Ac}$$
$$\therefore\ P_{\text{rad}} = \frac{2P}{2\pi r^2 c}$$

Answer:

$$P_{\text{rad}} = 1.34 \times 10^{-11}\text{Pa}$$

(b) Question: $E_0 = ?$
$B_0 = ?$

Approach:

i. Understand how to obtain the equation $S = uc$ from definition: $\dfrac{\text{Rate of flow of energy}}{\text{Area}}$
ii. Know the equation for u (energy density)
iii. Know that that average Poynting Vector (S_{av}) is intensity (I) and how it relates to power (P)
iv. Know the relationship between E_0 and H_0 as well as between H_0 and B_0
v. Potential error:
   Do not realize that $\vec{S}$ is not constant ($\vec{E}$ is not a constant)
   Therefore, forget S_{av} has a factor of half (E_0 is not a constant)

Solution:

$$S = \frac{1}{A}\frac{\mathrm{d}U}{\mathrm{d}t} \qquad \text{where } \mathrm{d}U = u\,A\,c\mathrm{d}t$$
$$= uc \qquad \text{where } u = \frac{1}{2}\varepsilon_0\vec{E}\cdot\vec{E} + \frac{1}{2}\mu_0\vec{H}\cdot\vec{H} = \varepsilon_0\vec{E}\cdot\vec{E}$$
$$= \varepsilon_0 c\vec{E}\cdot\vec{E}$$

Therefore:

$$S_{\text{av}} = \frac{1}{2}\varepsilon_0 c E_0^2$$
$$\frac{P}{A} = \frac{1}{2}\varepsilon_0 c E_0^2$$
$$E_0 = \sqrt{\frac{2P}{A\varepsilon_0 c}}$$

And:

$$H_0 = \frac{E_0}{Z_0} \qquad B_0 = \mu_0 H_0$$

Answer:

$$E_0 = 1.23\text{Vm}^{-1} \qquad B_0 = 4.10 \times 10^{-9}\text{T}$$

(c) Question: $u_{\text{av}} = ?$

Approach:

i. Know equation for u
ii. Potential error:
   Do not realize that u is not constant
   Therefore, forget that u_{av} has a factor of half

Solution:

$$u_{\text{av}} = \frac{1}{2}\epsilon_0 E_0^2$$

Answer:

$$u_{\text{av}} = 6.69 \times 10^{-12}\text{Jm}^{-3}$$

(d) What percentage of the energy is due to electric field and magnetic field?

Approach:

i. Know it's always half:half
ii. If you don't know i., know the equation for u_{av}
iii. Divide the term due to E by total u_{av}
iv. Repeat iii. for H

Solution:

$$u_{av} = \frac{1}{2}\left(\frac{1}{2}\varepsilon_0 E_0^2 + \frac{1}{2}\mu_0 H_0^2\right)$$
$$\%_E = \frac{\frac{1}{4}\varepsilon_0 E_0^2}{u_{\text{av}}} = \frac{\frac{1}{4}\varepsilon_0 E_0^2}{6.69 \times 10^{-12}}$$

Answer:

It's always 50:50

7. Given:

$$\vec{E}(t) = (-6\hat{x} + 3\sqrt{5}\hat{y})(10^4)\mathrm{e}^{\sqrt{5}x+2y(\frac{\pi}{3})(10^7)-9.42\times10^{15}t}$$

Question:

(a) Direction of E field oscillation

Approach:
Unit vector of $\vec{E}(t)$

Solution:

$$\hat{E} = \frac{\vec{E}}{|E|} = \frac{(-6\hat{x} + 3\sqrt{5}\hat{y})(10^4)}{\sqrt{6^2 + 9(5)}(10^4)}$$

Answer:

$$\hat{E} = -\frac{2}{3}\hat{x} + \frac{\sqrt{5}}{3}\hat{y}$$

(b) Scalar value of amplitude

Approach:
Already calculated in part (a)

Solution:

$$E_0 = 9 \times 10^4 \mathrm{Vm}^{-1}$$

(c) Direction of propagation

Approach:
Direction of $\vec{k}$
Able to identify $\vec{k} \cdot \vec{r}$ in the exponential term, extract $\vec{k}$ and find it's unit vector

Solution:

$$\hat{k} = \frac{\vec{k}}{|k|} = \frac{(\sqrt{5}\hat{x} + 2\hat{y})\left(\frac{\pi}{3} \times 10^7\right)}{\sqrt{5+4})\left(\frac{\pi}{3} \times 10^7\right)}$$

Answer:

$$\hat{k} = \frac{\sqrt{5}}{3}\hat{x} + \frac{2}{3}\hat{y}$$

(d) Propagation number and wavelength

Approach:
Propagation number is $|k|$

Answer:

$$|k| = \pi \times 10^7 \qquad \lambda = 2 \times 10^{-7}\mathrm{m}$$

(e) Frequency and angular frequency

Answer:

$$\omega = 9.42 \times 10^{15} \qquad\qquad f = 1.5 \times 10^{15}\text{Hz}$$

(f) Velocity

Approach:
$v = f\lambda$

Answer:

$$v \approx 3 \times 10^{8}\text{ms}^{-1}$$