PC2232: Tutorial 8 solutions

Question 1: Stirling's approximation

(a) For $z = 10$,

$$\ln 10! = 15.105, \quad 10 \ln 10 - 10 = 13.026. \tag{1}$$

Therefore, the error is $0.1376 \sim 13.8\%$.

(b) By trial, we find that for $z = 23$, the error is 4.8%.

Question 2: Two-dimensional density of states

For two dimensions, the energy is given by

$$E = \frac{\hbar^2}{2m} k^2, \quad k^2 = k_x^2 + k_y^2. \tag{2}$$

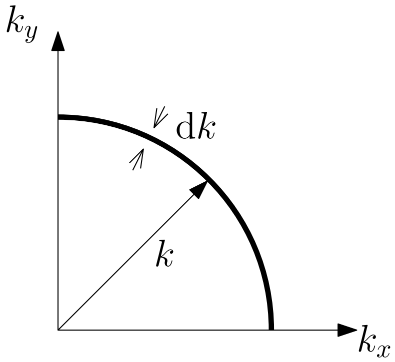
Recall that

$$k_x = \frac{n_x \pi}{L_x}, \quad k_y = \frac{n_y \pi}{L_y}. \tag{3}$$

Therefore the number of states per k-space *area* is

$$C(k_x, k_y) = 2 \frac{L_x L_y}{\pi^2}. \tag{4}$$

Consider a very thin ring in the positive quadrant:



The circular arc has length $\frac{1}{4} \times 2\pi k$, and the thickness is $\mathrm{d}k$. Therefore the infinitesimal area is

$$\text{area} = \frac{1}{4} 2\pi k \, \mathrm{d}k = \frac{\pi}{2} k \mathrm{d}k. \tag{5}$$

The number of states within that area is

$$\begin{aligned}\text{No. of states} &= (\text{States per area}) \times (\text{area}) \\ &= C(k_x, k_y) \times \frac{\pi}{2} k\mathrm{d}k = \frac{2L_xL_y}{\pi^2} \times \frac{\pi}{2} k\mathrm{d}k = \frac{L_xL_y}{\pi} k\mathrm{d}k. \end{aligned} \tag{6}$$

To find the density of states $\rho(E)$, we need to express the variables in terms of E instead of k. Using

$$\begin{aligned} E &= \frac{\hbar^2}{2m} k^2 \\ \mathrm{d}E &= \frac{\hbar^2}{m} k\mathrm{d}k \\ k\mathrm{d}k &= \frac{2m\mathrm{d}E}{\hbar^2}. \end{aligned} \tag{7}$$

Substitute Eq. (7) into (6),

$$\text{No. of states} = \frac{L_xL_y\,2m}{\pi\hbar^2}\mathrm{d}E = D(E)\mathrm{d}E, \tag{8}$$

where $D(E) = \frac{L_xL_y\,2m}{\pi\hbar^2}$. The density of states therefore $D(E)$ divided by real-space area,

$$\rho(E) = \frac{D(E)}{L_xL_y} = \frac{2m}{\pi\hbar^2}. \tag{9}$$

Question 3: Bose-Einstein distrubtion

Given W, the logarithm is

$$\begin{aligned} \ln W(N_1, \ldots, N_N) &= \ln \prod_{n=1}^{\infty} \frac{(N_n + d_n - 1)!}{N_n!(d_n - 1)!} = \sum_{n=1}^{\infty} \ln \left[\frac{(N_n + d_n - 1)!}{N_n!(d_n - 1)!} \right] \\ &= \sum_{n=1}^{\infty} \left[\ln(N_n + d_n - 1)! - \ln N_n! - \ln(d_n - 1)! \right] \end{aligned} \tag{10}$$

Using Stirling's approximation, the expression simplifies to

$$\ln W \simeq \sum_n \left[(N_n + d_n - 1)\ln(N_n + d_n - 1) - N_n \ln N_n - (d_n - 1)\ln(d_n - 1) \right] \tag{11}$$

Similar to the procedure for Fermi-Dirac distribution, we wish to maximize $\ln W$ with constraints:

$$N = \sum_n N_n, \quad E = \sum_n N_n E_n, \tag{12}$$

using Lagrange multipliers α and β. We define

$$\begin{aligned} G &= \ln W + \alpha \left(N - \sum_n N_n \right) + \beta \left(E - \sum_n N_n E_n \right) \\ &= \sum_n \left[(N_n + d_n - 1) \ln(N_n + d_n - 1) - N_n \ln N_n - (d_n - 1) \ln(d_n - 1) \right] \\ &\quad + \alpha \left(N - \sum_n N_n \right) + \beta \left(E - \sum_n N_n E_n \right) \end{aligned} \tag{13}$$

The maximization under constraint is obtained by the condition

$$\begin{aligned} \frac{\partial G}{\partial N_n} &= 0 = \ln(N_n + d_n - 1) - \ln N_n - \alpha - \beta E_n \\ N_n &= \frac{d_n - 1}{\mathrm{e}^{\alpha + \beta E_n} - 1}. \end{aligned} \tag{14}$$

Since typically d_n is a very large number, $d_n - 1 \simeq d_n$. Therefore

$$N_n = \frac{d_n}{\mathrm{e}^{\alpha + \beta E_n} - 1} \tag{15}$$

Question 4: Blackbody radiation

Given that

$$C = \frac{2V}{\pi^3}, \tag{16}$$

the degeneracy, i.e., number of states having value k, is

$$\begin{aligned} d_k &= (\text{No. of states per volume}) \times (\text{volume of 1 thin, octant shell of radius } k) \\ &= C(k_x, k_y, k_z) \times \frac{1}{8} 4\pi k^2 \mathrm{d}k = \frac{V k^2 \, \mathrm{d}k}{\pi^2} \end{aligned} \tag{17}$$

The Bose-Einstein distribution is

$$N_k = \frac{d_k}{\mathrm{e}^{\alpha + \beta E_k} - 1} \tag{18}$$

For photons, for temperature T, we have $\beta = \frac{1}{k_\mathrm{B} T}$, the number of photons is not conserved, $\alpha = 0$. Substitute $\alpha = 0$ and (17) into (18),

$$N_k = \frac{V k^2 \, \mathrm{d}k}{\pi^2 \left(\mathrm{e}^{E / k_\mathrm{B} T} - 1 \right)} \tag{19}$$

Using Planck's formula,

$$k = \frac{\omega}{c}, \quad \mathrm{d}k = \frac{\mathrm{d}k}{c}, \quad E = \hbar\omega, \tag{20}$$

we may express in terms of ω, we have

$$N_\omega = \frac{V\omega^2 \mathrm{d}\omega}{\pi^2 c^3 \left(\mathrm{e}^{\hbar\omega/k_\mathrm{B}T} - 1\right)} \tag{21}$$

The energy density in the interval ω to $\omega + \mathrm{d}\omega$ for a given value of ω, is defined by total energy per volume. Therefore

$$\begin{aligned}
\rho(\omega)\mathrm{d}\omega &= (\text{Number of states}) \times (\text{energy per state}) \times \frac{1}{V} \\
&= \frac{N_\omega}{V} \times \hbar\omega \\
\rho(\omega)\mathrm{d}\omega &= \frac{\hbar\omega^3 \, \mathrm{d}\omega}{\pi^2 c^3 \left(\mathrm{e}^{\hbar\omega/k_\mathrm{B}T} - 1\right)}.
\end{aligned} \tag{22}$$

Question 5: Wien's Law

(a) To convert $\rho(\omega)\mathrm{d}\omega \to \bar{\rho}(\lambda)\mathrm{d}\lambda$, we use the relation

$$\omega = kc = \frac{2\pi c}{\lambda}, \quad \mathrm{d}\omega = -\frac{2\pi c}{\lambda^2}\mathrm{d}\lambda, \tag{23}$$

Substitute into (22),

$$\begin{aligned}
\frac{\hbar\omega^3 \, \mathrm{d}\omega}{\pi^2 c^3 \left(\mathrm{e}^{\hbar\omega/k_\mathrm{B}T} - 1\right)} &= \frac{\hbar \left(\frac{2\pi c}{\lambda}\right)^3 \left(-\frac{2\pi c}{\lambda^2}\right) \mathrm{d}\lambda}{\pi^2 c^3 \left(\mathrm{e}^{2\pi\hbar c/k_\mathrm{B}T} - 1\right)} \\
&= -\frac{16\pi^2 \hbar c \, \mathrm{d}\lambda}{\lambda^5 \left(\mathrm{e}^{\hbar 2\pi c/k_\mathrm{B}T} - 1\right)}
\end{aligned} \tag{24}$$

Therefore

$$\bar{\rho}(\omega)\mathrm{d}\omega = -\frac{16\pi^2 \hbar c \, \mathrm{d}\lambda}{\lambda^5 \left(\mathrm{e}^{\hbar 2\pi c/\lambda k_\mathrm{B}T} - 1\right)} \tag{25}$$

(b) Ignoring the unimportant constants,

$$\rho(\bar{\lambda}) \propto \frac{1}{\lambda^5 \left(\mathrm{e}^{2\pi\hbar c/\lambda k_\mathrm{B}T} - 1\right)} \tag{26}$$

To simplify, let $x = \frac{2\pi\hbar c}{\lambda k_\mathrm{B}T}$, then

$$\rho(\bar{\lambda}) \propto \frac{\left(\frac{k_\mathrm{B}T}{2\pi\hbar c}\right)^5 x^5}{\mathrm{e}^x - 1} \tag{27}$$

For maximum intensity,

$$\begin{aligned}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}x} &= 0 = \frac{5x^4}{\mathrm{e}^x - 1} - \frac{x^5\mathrm{e}^x}{\left(\mathrm{e}^x - 1\right)^2} \\ 5\left(\mathrm{e}^x - 1\right) &= x\mathrm{e}^x \\ (5 - x)\mathrm{e}^x &= 5 \end{aligned} \tag{28}$$

The roots to this equation is $x = 4.965$

$$x = 4.965 = \frac{2\pi\hbar c}{\lambda k_{\mathrm{B}} T}. \tag{29}$$

Substituting in the numerical values for h, c and k_{B},

$$\lambda T = 2.90 \times 10^{-3}\ \mathrm{mK} \tag{30}$$