PC2232 Physics for Electrical Engineers
Lecture 1: Electromagnetic Waves

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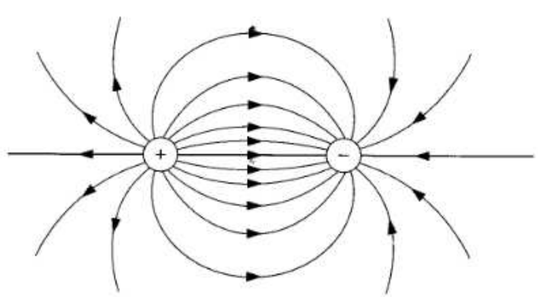
January 13, 2015

Maxwell's Equations

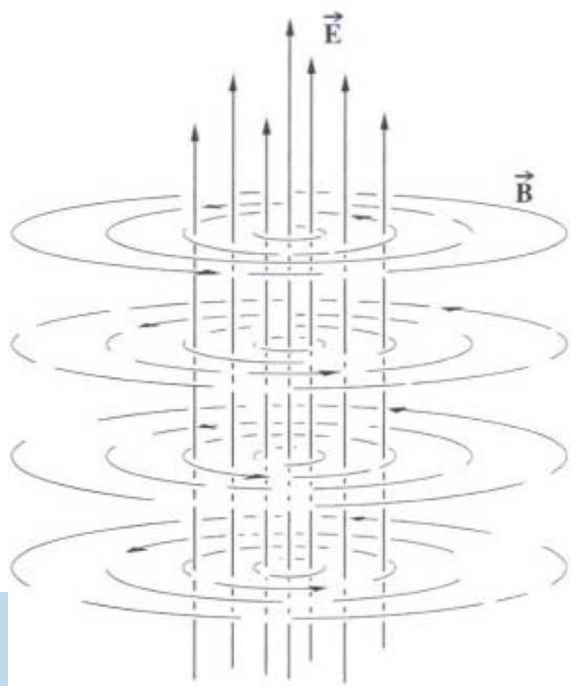
- Optics is the science of **light**. Light is **electromagnetic waves**, and electromagnetic waves obey the four **Maxwell equations**.
- In **vacuum (free space)**,

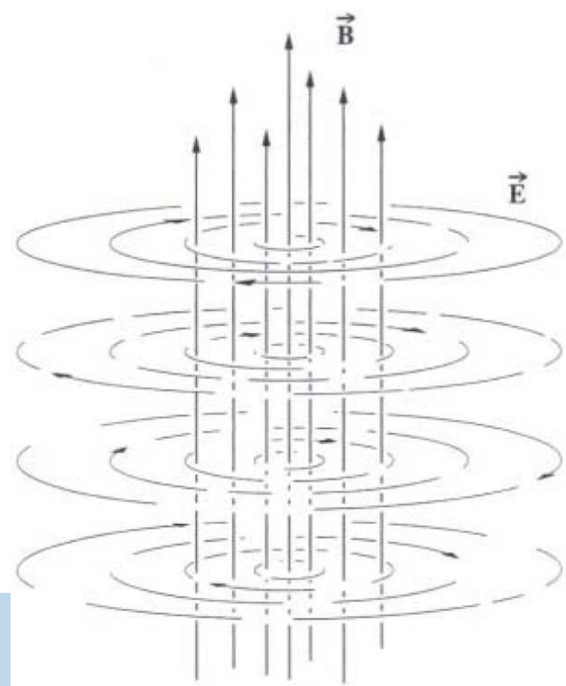
$$\boxed{\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \quad \text{(Gauss's law)}} \qquad \boxed{\nabla \cdot \boldsymbol{H} = 0 \quad \text{(Gauss's law for magnetism)}} \tag{1}$$

$$\boxed{\nabla \times \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} \quad \text{(Faraday's law)}} \qquad \boxed{\nabla \times \boldsymbol{H} = \boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \quad \text{(Modified Ampere's law)}} \tag{2}$$



Equal but opposite charges





Electric and Magnetic Fields

- The electromagnetic fields have the following units:

$$\boldsymbol{E}(\boldsymbol{r},t) = \text{Electric field (V/m)} \qquad \boldsymbol{H}(\boldsymbol{r},t) = \text{Magnetic field (A/m)} \tag{3}$$

- In physics, it is more often to define the magnetic field with a different unit:

$$\boldsymbol{B}(\boldsymbol{r},t) = \mu_0 \boldsymbol{H}(\boldsymbol{r},t) \quad \text{(T)} \tag{4}$$

- ρ and $\boldsymbol{J}$ are sources:

$$\rho(\boldsymbol{r},t) = \text{charge density (C/m}^3\text{)}, \qquad \boldsymbol{J}(\boldsymbol{r},t) = \text{current density (A/m}^2\text{)}. \tag{5}$$

- There are two constants:

$$\epsilon_0 = \text{free-space permittivity} \approx 8.854 \times 10^{-12} \text{ F/m} \tag{6}$$

$$\mu_0 = \text{free-space permeability} = 4\pi \times 10^{-17} \text{ Tm/A} \tag{7}$$

- C = Coulomb, V = Volt = 1 Joule/Coulomb, A = Ampere = Coulomb/s, T = Tesla = Volt $\cdot$ s/m^2, F = Farad = Coulomb/Volt (unit of capacitance)
- **Always** check the units of your solutions to make sure they are consistent!

Vectors

- $\boldsymbol{E}$, $\boldsymbol{H}$, ρ, $\boldsymbol{J}$ are functions of **three-dimensional space $\boldsymbol{r}$ and time** t.
- In Cartesian coordinates,

$$\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{8}$$

  $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ are unit vectors.
- Define $\boldsymbol{A} \equiv A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}}$, $\boldsymbol{B} \equiv B_x\hat{\boldsymbol{x}} + B_y\hat{\boldsymbol{y}} + B_z\hat{\boldsymbol{z}}$. Dot product:

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A} = A_x B_x + A_y B_y + A_z B_z, \tag{9}$$

$$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}} = \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{z}} = 1, \tag{10}$$

$$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}} = \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{x}} = 0 \quad \text{(perpendicular vectors have a zero dot product)}, \tag{11}$$

- Cross product:

$$\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}, \tag{12}$$

$$\hat{\boldsymbol{x}} \times \hat{\boldsymbol{x}} = \hat{\boldsymbol{y}} \times \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}} \times \hat{\boldsymbol{z}} = 0 \quad \text{(parallel vectors have a zero cross product)} \tag{13}$$

$$\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}, \quad \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}} = \hat{\boldsymbol{x}}, \quad \hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}. \tag{14}$$

Vector Calculus

- **Del operator**: $\nabla \equiv \frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial}{\partial z}\hat{\boldsymbol{z}}$.

$$\nabla U = \frac{\partial U}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial U}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial U}{\partial z}\hat{\boldsymbol{z}} \text{ (gradient)}, \tag{15}$$

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \text{ (divergence)}, \qquad \nabla \times \boldsymbol{A} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \text{ (curl)}, \tag{16}$$

$$\nabla^2 \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \text{ (Laplacian)}. \tag{17}$$

- Some vector calculus identities:
  - Linearity:

$$\nabla \cdot (\boldsymbol{A} + \boldsymbol{B}) = \nabla \cdot \boldsymbol{A} + \nabla \cdot \boldsymbol{B}, \qquad \nabla \times (\boldsymbol{A} + \boldsymbol{B}) = \nabla \times \boldsymbol{A} + \nabla \times \boldsymbol{B}, \text{ etc.} \tag{18}$$

  - Order of partial derivatives doesn't matter:

$$\nabla \cdot \frac{\partial \boldsymbol{A}}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}), \qquad \nabla \times \frac{\partial \boldsymbol{A}}{\partial t} = \frac{\partial}{\partial t}(\nabla \times \boldsymbol{A}), \text{ etc.} \tag{19}$$

  - Chaining of operators:

$$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0, \qquad \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}. \tag{20}$$

- Maxwell equations admit **wave** solutions. To see this, let us assume no external source ($\rho = 0$, $\boldsymbol{J} = 0$):
- Let's take the curl of the Faraday law $\nabla \times \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}$:

$$\nabla \times \nabla \times \boldsymbol{E} = -\mu_0 \nabla \times \frac{\partial \boldsymbol{H}}{\partial t}. \tag{21}$$

- Using a vector calculus identity, the left-hand side becomes

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E}. \tag{22}$$

Gauss law is $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$, and since we already assumed $\rho = 0$, the term $\nabla(\nabla \cdot \boldsymbol{E})$ is zero, and

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\nabla^2 \boldsymbol{E}. \tag{23}$$

- Taking the curl of the right-hand side of Faraday law, and interchanging the order of the curl and the time derivative,

$$-\nabla \times \mu_0 \frac{\partial \boldsymbol{H}}{\partial t} = -\mu_0 \frac{\partial}{\partial t} (\nabla \times \boldsymbol{H}). \tag{24}$$

Wave Equation

- Now if we use the modified Ampere law with $\boldsymbol{J} = 0$ ($\nabla \times \boldsymbol{H} = \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$),

$$-\mu_0 \frac{\partial}{\partial t} (\nabla \times \boldsymbol{H}) = -\mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}. \tag{25}$$

- Equating Eq. (23) and Eq. (25), we get

$$\boxed{\nabla^2 \boldsymbol{E} = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2}.} \tag{26}$$

  As we shall see later, this admits wave solutions, i.e., electromagnetic waves, and is called a wave equation.
- Similarly, convince yourself that the magnetic field also obeys a wave equation: $\nabla^2 \boldsymbol{H} = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{H}}{\partial t^2}$.
- c is the speed of the wave, i.e., it is the **speed of light** in free space. It is defined **exactly** as

$$\boxed{c \equiv \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 299,792,458 \text{ m/s} \approx 3 \times 10^8 \text{ m/s}} \tag{27}$$

  The second is defined using atomic oscillations, and the meter is defined in terms of c and the second.
- Convince yourself that $1/\sqrt{\mu_0 \epsilon_0}$ indeed has the unit of velocity.

(1+1)D Wave Equation

- Remember $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. For simplicity, suppose first that $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{H}(\boldsymbol{r}, t)$ are constant along y and z, such that

$$\frac{\partial \boldsymbol{E}}{\partial y} = \frac{\partial \boldsymbol{E}}{\partial z} = 0, \qquad \frac{\partial \boldsymbol{B}}{\partial y} = \frac{\partial \boldsymbol{B}}{\partial z} = 0. \tag{28}$$

The wave equation becomes

$$\frac{\partial^2 \boldsymbol{E}}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2}. \tag{29}$$

This is actually three equations:

$$\frac{\partial^2 E_x}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2}, \qquad \frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2}, \qquad \frac{\partial^2 E_z}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 E_z}{\partial t^2}. \tag{30}$$

- Let us focus on the E_y equation. One possible solution is

$$\boxed{E_y(x, t) = f(x - vt),} \tag{31}$$

where f is any *single-variable* function. Convince yourself that $v^2 = c^2$. We call a wave solution that propagates in one direction (x here) and is independent of the other two directions (y and z here) a **plane wave**.

- Assume that $E_y(x,t) = f(x-vt)$, and v is positive. At $t=0$,

$$E_y(x,0) = f(x). \tag{32}$$

- Then after some time T, the field distribution is **shifted to the right** :

$$E_y(x,T) = f(x-vT). \tag{33}$$

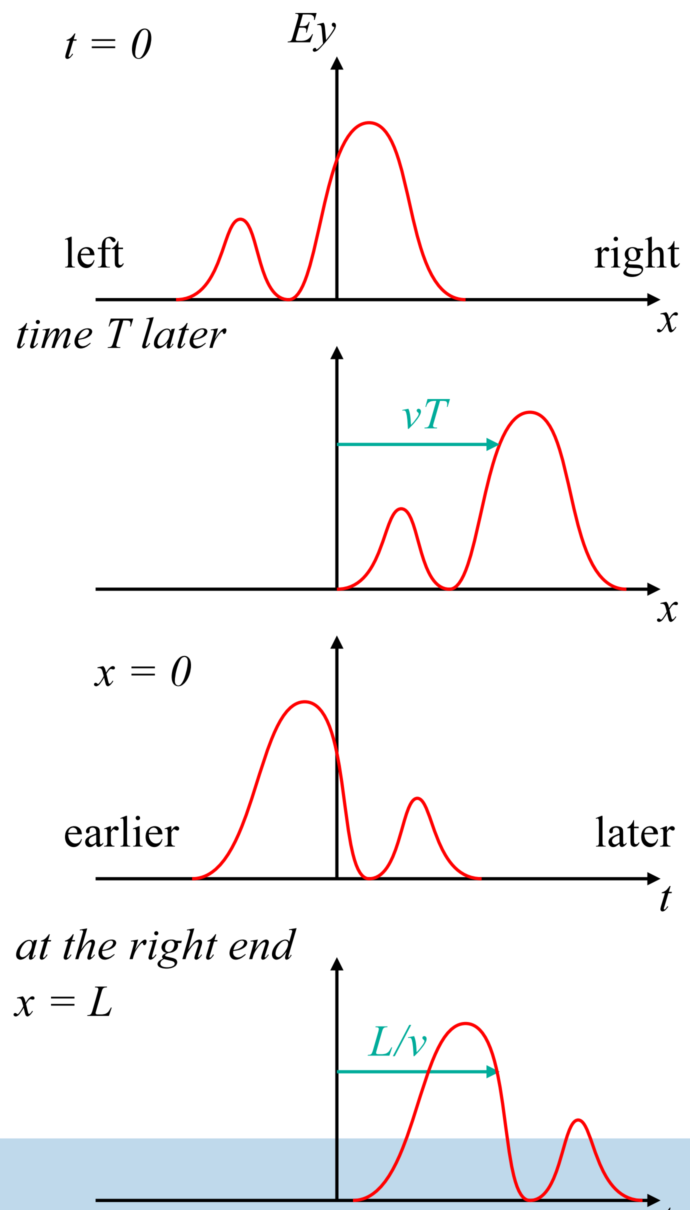
  The velocity is v.
- Now let's fix $x=0$ and observe the field as it changes in time:

$$E_y(0,t) = f(-vt). \tag{34}$$

  Note that the temporal shape is flipped with respect to the spatial shape; for a right-propagating wave, the *right* part arrives *earlier*.
- At some distance L to the right, the field changes in time as

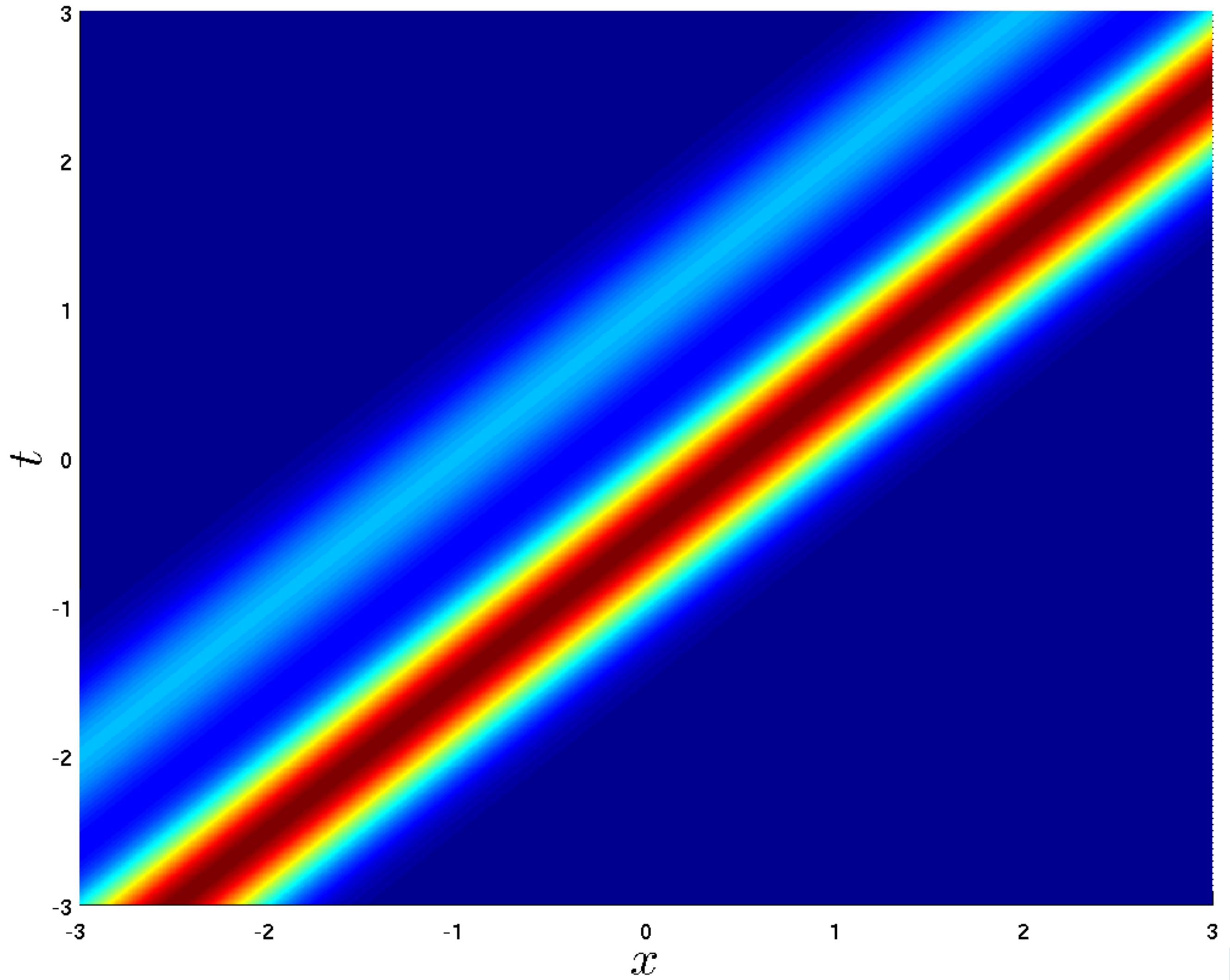
$$E_y(L,t) = f(L-vt) = f\left(-v\left(t-\frac{L}{v}\right)\right). \tag{35}$$

  The **time delay** is L/v. The faster the speed v, the smaller the delay.
- PhET/sims/radio-waves/radio-waves_all.jar



2D Plot

Here's what a wave looks like in 2D plot in (x, t):



Sinusoidal EM Wave

- A very important wave solution is the sinusoidal wave:

$$E_y(x,t) = E_{\mathsf{max}} \cos(kx - \omega t). \tag{36}$$

- The spatial period

$$\boxed{\lambda = \frac{2\pi}{k}} \tag{37}$$

is called the **wavelength** (m). k is called the **wavenumber**.

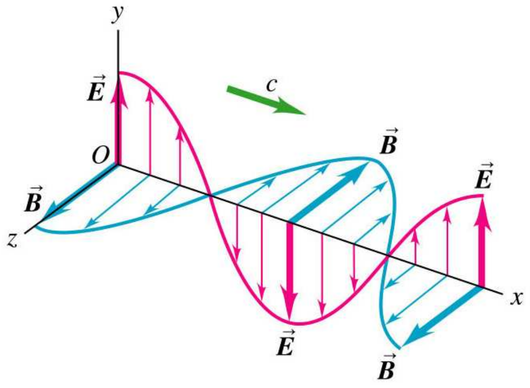
- The temporal frequency in Hertz (Hz) is

$$\boxed{\nu = \frac{\omega}{2\pi}.} \tag{38}$$

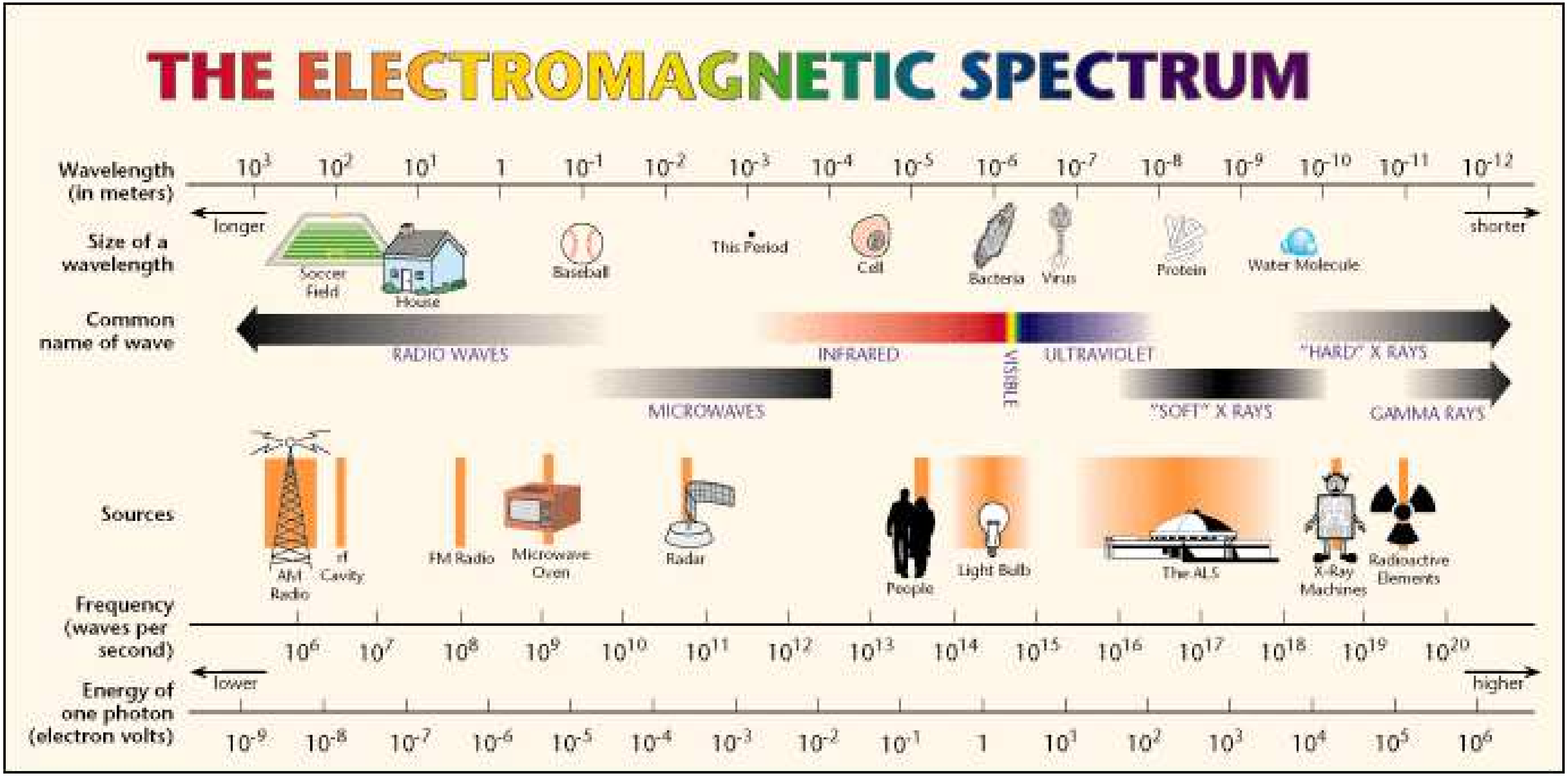
ω is the angular frequency (rad/s).

- Speed of light is related to frequency and wavelength by

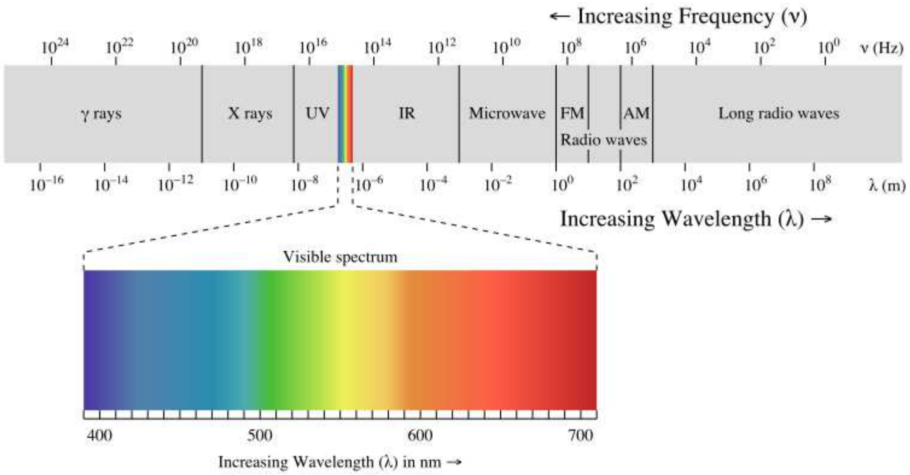
$$\boxed{c = \frac{\omega}{k} = \nu\lambda.} \tag{39}$$



- http://en.wikipedia.org/wiki/Electromagnetic_radiation


THE ELECTROMAGNETIC SPECTRUM
Wavelength (in meters)
10^3 10^2 10^1 1 10^{-1} 10^{-2} 10^{-3} 10^{-4} 10^{-5} 10^{-6} 10^{-7} 10^{-8} 10^{-9} 10^{-10} 10^{-11} 10^{-12}
longer
shorter
Size of a wavelength
Soccer Field
House
Baseball
This Period
Cell
Bacteria
Virus
Protein
Water Molecule
Common name of wave
RADIO WAVES
INFRARED
VISIBLE
ULTRAVIOLET
"HARD" X RAYS
MICROWAVES
"SOFT" X RAYS
GAMMA RAYS
Sources
AM Radio
rf Cavity
FM Radio
Microwave Oven
Radar
People
Light Bulb
The ALS
X-Ray Machines
Radioactive Elements
Frequency (waves per second)
10^6 10^7 10^8 10^9 10^{10} 10^{11} 10^{12} 10^{13} 10^{14} 10^{15} 10^{16} 10^{17} 10^{18} 10^{19} 10^{20}
lower
higher
Energy of one photon (electron volts)
10^{-9} 10^{-8} 10^{-7} 10^{-6} 10^{-5} 10^{-4} 10^{-3} 10^{-2} 10^{-1} 1 10^1 10^2 10^3 10^4 10^5 10^6

Optical Frequencies and Wavelengths



- Optics usually refer to wavelengths and frequencies near the visible ($\lambda = 400 - 700$ nm).

$$\lambda \sim 1\ \mu\text{m}\ (10^{-6}\ \text{m}), \qquad \nu \sim 300\ \text{THz}\ (3 \times 10^{14}\ \text{Hz}) \tag{40}$$

  This frequency is **extremely high**. Compare it, for example, to your microwave oven, CPU clock speed (GHz), radio frequency (kHz-MHz), audio frequency (20 Hz-20 kHz), AC power source (60 Hz), heartbeat (Hz), etc.
- An EM wave at optical frequencies is often labeled by its wavelength λ *in free space*.
- Compare with width of human hair ($\sim 50\ \mu$m)

(3+1)D Sinusoidal Waves

- Although the electromagnetic fields must be real, it is often extremely convenient to express them in terms of complex signals. For example, for a sinusoidal wave,

$$E_y(x,t) = E_{\mathsf{max}} \cos(kx - \omega t + \theta) = \mathrm{Re}\left[\tilde{E} \exp\left(jkx - j\omega t\right)\right], \qquad \tilde{E} = E_{\mathrm{max}} e^{j\theta}. \tag{41}$$

  The complex $\tilde{E} \exp(jkx - j\omega t)$ is also a solution of the wave equation. $\tilde{E}$ is called the **wave amplitude**. Very often we will simply focus on complex solutions, and it is understood implicitly that the actual fields are the real part.
- Let us now turn to 3-dimensional space. Consider the solution

$$\boldsymbol{E}(\boldsymbol{r},t) = \tilde{\boldsymbol{E}} \exp\left(j\boldsymbol{k}\cdot\boldsymbol{r} - j\omega t\right), \qquad \boldsymbol{H}(\boldsymbol{r},t) = \tilde{\boldsymbol{H}} \exp\left(j\boldsymbol{k}\cdot\boldsymbol{r} - j\omega t\right). \tag{42}$$

  $\tilde{\boldsymbol{E}}$ is a **complex** vector that doesn't depend on $(\boldsymbol{r}, t)$.
- $\boldsymbol{k} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}} + k_z\hat{\boldsymbol{z}}$ is a real vector called the **wavevector**.
- Suppose that $\hat{\boldsymbol{k}}$ is the unit vector pointing in the same direction as $\boldsymbol{k}$ and $k = |\boldsymbol{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2}$ is the magnitude. Then I can write

$$\boldsymbol{k} = k\hat{\boldsymbol{k}}, \qquad \boldsymbol{k}\cdot\boldsymbol{r} = k(\hat{\boldsymbol{k}}\cdot\boldsymbol{r}), \qquad \exp(j\boldsymbol{k}\cdot\boldsymbol{r} - j\omega t) = \exp[jk(\hat{\boldsymbol{k}}\cdot\boldsymbol{r}) - j\omega t]. \tag{43}$$

  $\hat{\boldsymbol{k}}\cdot\boldsymbol{r}$ is the spatial coordinate along $\hat{\boldsymbol{k}}$. For example, if $\hat{\boldsymbol{k}} = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{k}}\cdot\boldsymbol{r} = x$, and the wave propagates in the x direction.
- The spatial period of the wave along that direction of $\boldsymbol{k}$, or the wavelength, is

$$\lambda = \frac{2\pi}{k}. \tag{44}$$

Sinusoidal Waves and Maxwell's Equations

- To check that $\tilde{\boldsymbol{E}}\exp(j\boldsymbol{k}\cdot\boldsymbol{r} - j\omega t)$ is a solution of the Maxwell's equations, it is helpful to note that

$$\nabla\cdot\left(\tilde{\boldsymbol{E}}e^{j\boldsymbol{k}\cdot\boldsymbol{r}}\right) = j\boldsymbol{k}\cdot\tilde{\boldsymbol{E}}e^{j\boldsymbol{k}\cdot\boldsymbol{r}}, \qquad \nabla\times\left(\tilde{\boldsymbol{E}}e^{j\boldsymbol{k}\cdot\boldsymbol{r}}\right) = j\boldsymbol{k}\times\tilde{\boldsymbol{E}}e^{j\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{45}$$

$$\nabla^2\left(\tilde{\boldsymbol{E}}e^{j\boldsymbol{k}\cdot\boldsymbol{r}}\right) = -k^2\tilde{\boldsymbol{E}}e^{j\boldsymbol{k}\cdot\boldsymbol{r}}, \qquad \frac{\partial}{\partial t}e^{-j\omega t} = -j\omega e^{-j\omega t}. \tag{46}$$

  Differential operations become algebraic operations (rule: replace ∇ with $j\boldsymbol{k}$, replace $\frac{\partial}{\partial t}$ with $-j\omega$).
- The wave equation then becomes

$$\nabla^2\boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2} \to -k^2\tilde{\boldsymbol{E}} = -\frac{\omega^2}{c^2}\tilde{\boldsymbol{E}}. \tag{47}$$

- There are two solutions. Either $\tilde{\boldsymbol{E}} = 0$ (which is not very interesting) or

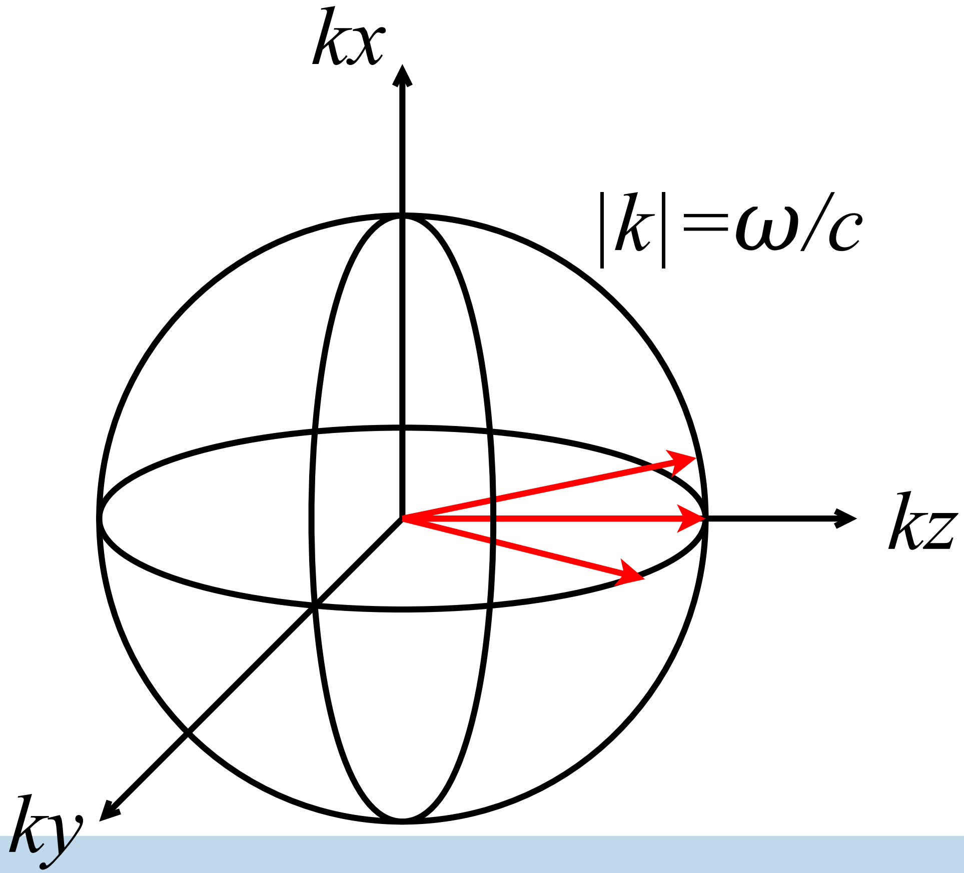
$$k^2 = |\boldsymbol{k}|^2 = k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}. \tag{48}$$

  This is called a dispersion relation, relating the frequency ω to the wavevector $\boldsymbol{k}$.

- For a given ω, the magnitude of $\boldsymbol{k}$ is fixed:

$$k^2 = |\boldsymbol{k}|^2 = k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}. \tag{49}$$

- This can be expressed as a **sphere** in $\boldsymbol{k}$-space:



Transverse Waves

- If we plug the plane-wave solution in the Gauss laws, we see that $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{H}}$ are perpendicular to $\boldsymbol{k}$ (transverse waves):

$$\nabla \cdot \boldsymbol{E} = 0, \qquad \nabla \cdot \boldsymbol{H} = 0, \qquad \rightarrow \qquad \boldsymbol{k} \cdot \tilde{\boldsymbol{E}} = 0, \qquad \boldsymbol{k} \cdot \tilde{\boldsymbol{H}} = 0. \tag{50}$$

- $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{H}}$ are related and have the same phase:

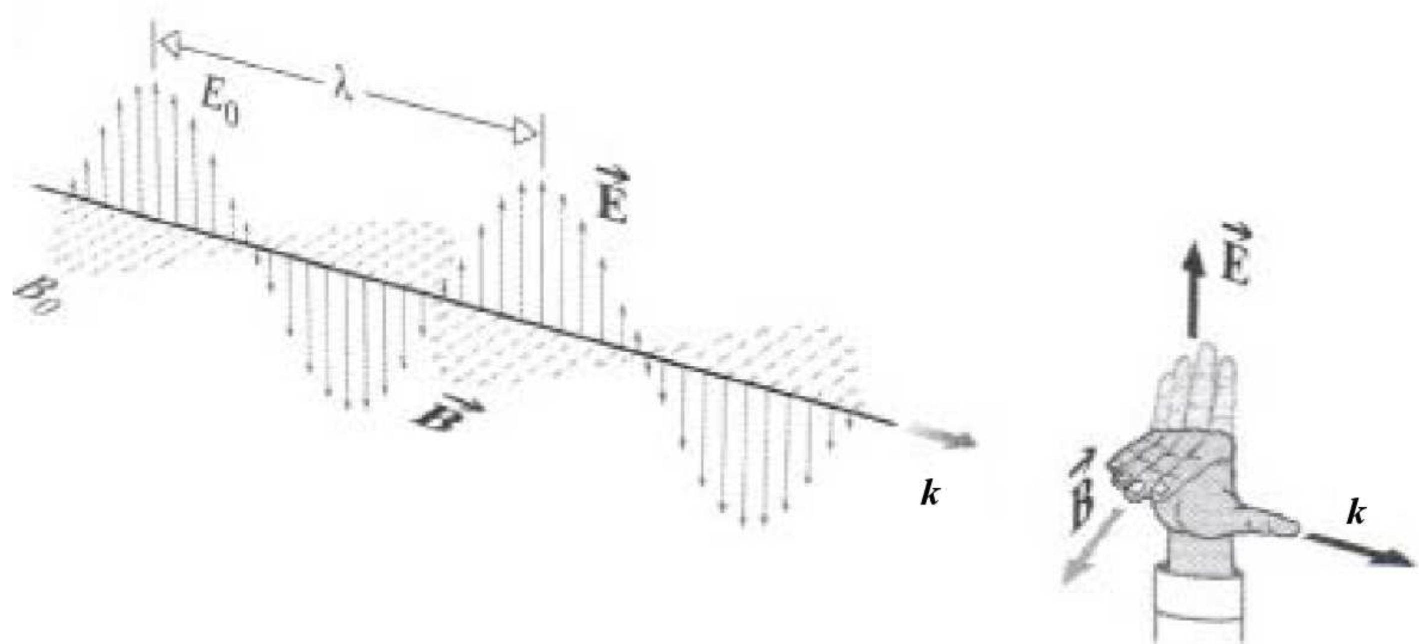
$$\nabla \times \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}, \quad \nabla \times \boldsymbol{H} = \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}, \quad \rightarrow \quad \boldsymbol{k} \times \tilde{\boldsymbol{E}} = \mu_0 \omega \tilde{\boldsymbol{H}}, \quad \boldsymbol{k} \times \tilde{\boldsymbol{H}} = -\epsilon_0 \omega \tilde{\boldsymbol{E}}. \tag{51}$$

- The ratio between the electric-field and magnetic-field amplitudes is called the free-space impedance:

$$\boxed{\frac{|\tilde{\boldsymbol{E}}|}{|\tilde{\boldsymbol{H}}|} \equiv Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 119.9169832\pi \text{ Ohm} \approx 120\pi \text{ Ohm } (\Omega)} \tag{52}$$

Linearly Polarized Wave

- e.g., If $\tilde{\boldsymbol{E}} = E_y \hat{\boldsymbol{y}}$ is along the y-axis and $\boldsymbol{k} = k\hat{\boldsymbol{x}}$ is along the x-axis, the EM wave would look like this along a line:



Electromagnetic Energy

- The EM **energy density** in free space is

$$\boxed{u(\boldsymbol{r},t) = \frac{1}{2}\epsilon_0 \boldsymbol{E}(\boldsymbol{r},t)\cdot\boldsymbol{E}(\boldsymbol{r},t) + \frac{1}{2}\mu_0 \boldsymbol{H}(\boldsymbol{r},t)\cdot\boldsymbol{H}(\boldsymbol{r},t) \quad (\text{Joule/m}^3).} \tag{53}$$

  It is a **quadratic** function of the electromagnetic fields. We should be careful when calculating it for complex EM fields; **always take the real part first** before taking the products.
- To obtain the EM energy in a volume, one can compute the volume integral $\int_V d^3r u(\boldsymbol{r},t)$.
- Consider the real sinusoidal wave solution $\boldsymbol{E} = E_{\max}\hat{\boldsymbol{y}}\cos(kx-\omega t)$ and $\boldsymbol{H} = \frac{E_{\max}}{Z_0}\hat{\boldsymbol{z}}\cos(kx-\omega t)$.
- Noting that $Z_0 = \sqrt{\mu_0/\epsilon_0}$ and $\cos^2\theta = [1+\cos(2\theta)]/2$, the energy density is

$$u(\boldsymbol{r},t) = \epsilon_0 E_{\max}^2 \cos^2(kx-\omega t) = \epsilon_0 E_{\max}^2 \frac{1}{2}\left\{1+\cos\left[2(kx-\omega t)\right]\right\}. \tag{54}$$

  At a given x, it oscillates between 0 and $\epsilon_0 E_{\max}^2$.
- Averaged over a long time relative to the temporal period $2\pi/\omega$, the **average** energy density is

$$\bar{u}(\boldsymbol{r}) \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T dt u(\boldsymbol{r},t) = \frac{1}{2}\epsilon_0 E_{\max}^2, \tag{55}$$

  which is constant **everywhere**.
- Question: What is the total energy of this sinusoidal wave solution?

Electromagnetic Power Flow

- What is the power flow across a surface in $y - z$ plane with surface area A for the sinusoidal wave?
- For a short (infinitesimal) time dt, A volume of EM energy flows past the surface. The volume is $Acdt$, and since the length along x is very small, so we can assume that the EM energy U is constant along x.

$$dU = uAcdt, \tag{56}$$

$$\text{Power} = \frac{dU}{dt} = uAc = A\epsilon_0 c E_{\max}^2 \cos^2(kx - \omega t). \tag{57}$$

Hence, the direction of the power flow is $\hat{\boldsymbol{x}}$, and the magnitude is

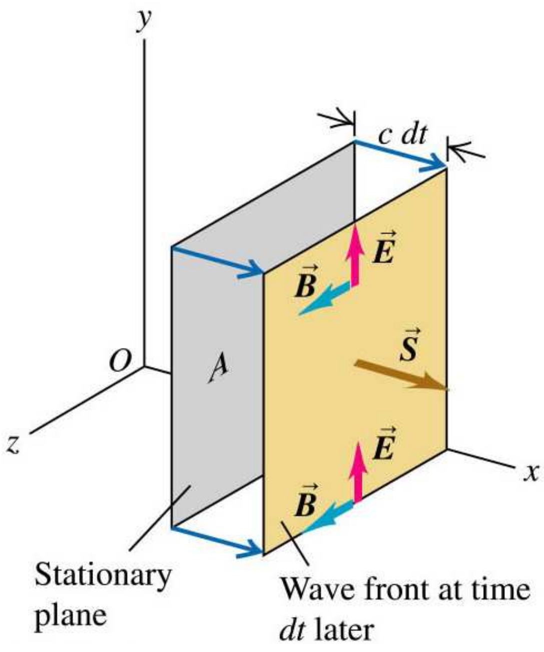
$$\text{Power flow/unit area} = \epsilon_0 c E_{\max}^2 \cos^2(kx - \omega t). \tag{58}$$

The power flow per unit area averaged over time (intensity) is $\bar{S} = \epsilon_0 c E_{\max}^2/2$.

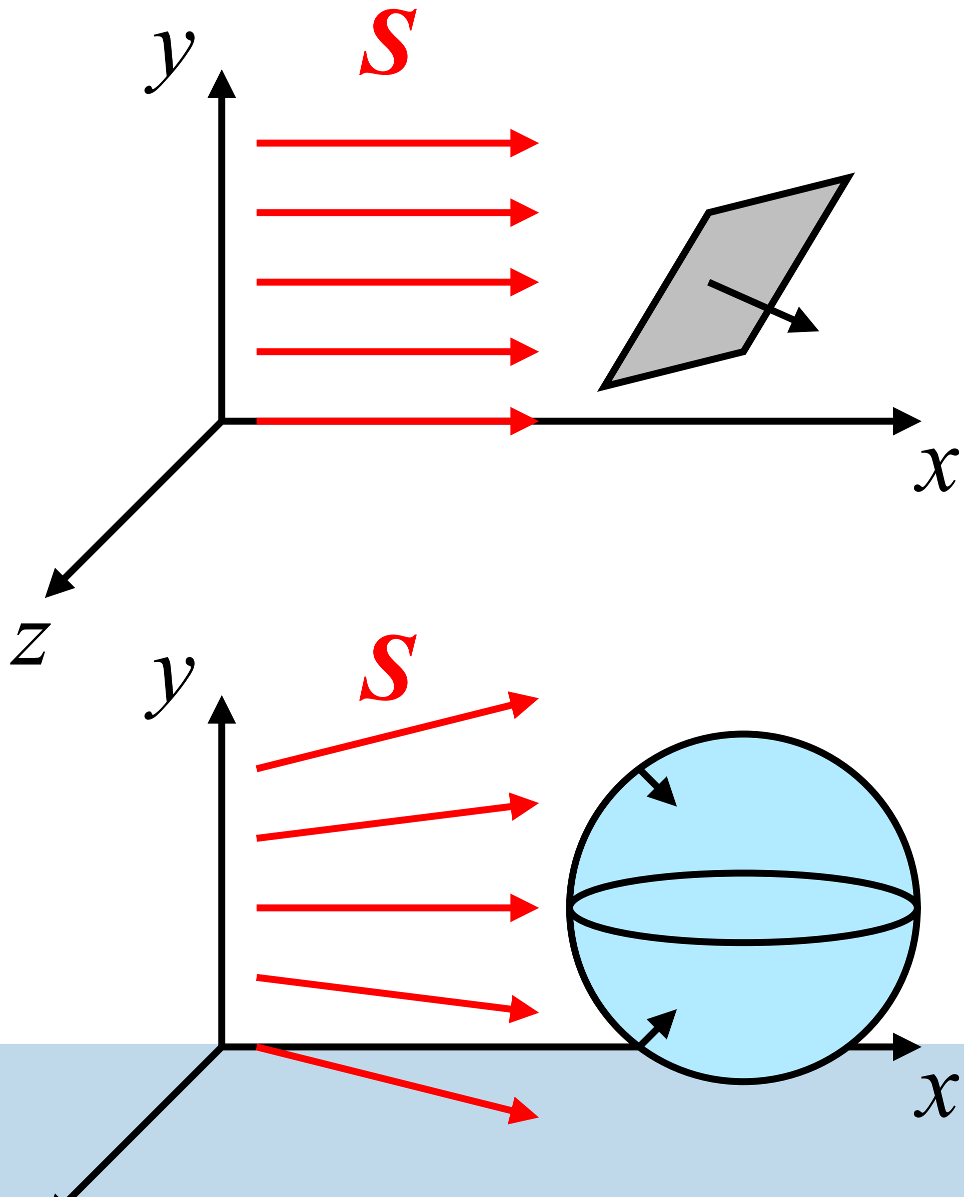
- In general, the vector for the power flow per unit area is

$$\boxed{\boldsymbol{S}(\boldsymbol{r}, t) = \boldsymbol{E} \times \boldsymbol{H} \ (\text{W/m}^2).} \tag{59}$$

This is true for any EM solution, not just the sinusoidal solution. It is called the **Poynting vector**.



- Power flow across any surface can be obtained by computing the surface integral $P = \int_{\mathcal{A}} d\boldsymbol{A} \cdot \boldsymbol{S}$. The power is maximized if the surface is **perpendicular** to the Poynting vector.



32.18 These rooftop solar panels are tilted to be face-on to the sun—that is, face-on to the Poynting vector of electromagnetic waves from the sun, so that the panels can absorb the maximum amount of wave energy.

Momentum of EM Waves

- **Energy conservation** and **momentum conservation** are among the most important principles in physics.
- When EM waves interact with matter (e.g., absorption, reflection, refraction, emission), the total energy and the total momentum of the whole system must be conserved.
- What is momentum?
    - For Newtonian massive objects, $\boldsymbol{p} \approx m\boldsymbol{v}$.
    - For relativistic massive objects, $\boldsymbol{p} = m\boldsymbol{v}/\sqrt{1 - v^2/c^2}$ ($m =$rest mass).
    - For sinusoidal wave, it is intuitive that the direction of momentum should be along $\boldsymbol{S}$, but we don't really know the magnitude, as EM waves have zero mass and speed of light $v = c$.
- We have to rely on the relativistic energy-momentum relation: $U^2 = m^2c^4 + p^2c^2$. With $m = 0$, we know that the magnitude of $\boldsymbol{p}$ must be

$$p = \frac{U}{c}. \tag{60}$$

  EM momentum is proportional to energy (compare with the relation with mass or Newtonian $U = p^2/2m$). The momentum density is thus $u(\boldsymbol{r}, t)/c$. Including the direction, the EM momentum density turns out to be equal to

$$\boldsymbol{g}(\boldsymbol{r}, t) = \frac{\boldsymbol{S}(\boldsymbol{r}, t)}{c^2}. \tag{61}$$

- See `http://www.feynmanlectures.caltech.edu/II_27.html#Ch27-S6` for a more detailed explanation.

Radiation Pressure

- If an object completely **absorbs** the sinusoidal wave propagating in x direction, the wave momentum must be transferred to the object.
- Suppose that the object has a flat surface in the $y-z$ plane with area A. In time dt, a volume of EM waves is absorbed; the volume is $Acdt$. The object must acquire a momentum given by $dp = \frac{u}{c}Acdt = uAdt$ along the x direction. Over time, the momentum of the object must keep increasing. This is equivalent to a force:

$$F = \frac{dp}{dt} = Au = A\epsilon_0 E_{\max}^2 \cos^2(kx - \omega t) \text{ (N)}. \tag{62}$$

- The **radiation pressure** (force/unit area) is then

$$P = \frac{F}{A} = \epsilon_0 E_{\max}^2 \cos^2(kx - \omega t), \qquad \bar{P} = \frac{1}{2}\epsilon_0 E_{\max}^2. \tag{63}$$

- If the wave is completely **reflected** by a perpendicular mirror, the wave reverses its direction, that is, the wave momentum change is -$2p\hat{\boldsymbol{x}}$. The radiation pressure is then

$$\bar{P}_{\text{reflect}} = \epsilon_0 E_{\max}^2, \tag{64}$$

  which is twice the value for absorption.
- Radiation pressure is actually miniscule and mostly negligible in everyday life, but has some specialized applications (optical tweezing, solar sail, Casimir force, etc.).

Suggested Problems

- Dimensional analysis: e.g., check that $1/\sqrt{\epsilon_0\mu_0}$ has the unit of velocity, check $\sqrt{\mu_0/\epsilon_0}$ has unit of resistance (Ohm).
- Basic vector calculus to refresh memory: e.g., work out $\nabla \times \boldsymbol{A}$ in Cartesian coordinates
- Linearity: if $(\boldsymbol{E}_1, \boldsymbol{H}_1)$ and $(\boldsymbol{E}_2, \boldsymbol{H}_2)$ are solutions of Maxwell's equations, $(a_1\boldsymbol{E}_1 + a_2\boldsymbol{E}_2, a_1\boldsymbol{H}_1 + a_2\boldsymbol{H}_2)$ is also a solution for any complex constants a_1 and a_2. Generalize to superposition of multiple solutions. Generalize to integrals of solutions (in $\boldsymbol{k}$ and ω for example).
- Wave equation: verify that $f(x - vt)$ is a solution of the 2nd-order wave equation. Matlab: plot $f(x - vt)$ for different v, negative v, etc.
- Frequency and wavelength: compute frequencies of microwave, infrared, red, green, blue, ultraviolet, x-ray, etc. given wavelengths, vice versa
- Time average: verify time average of $\cos^2(kx - \omega t) = 1/2$ for any x.
- Poynting vector: do some surface integrals to compute power flow. Compute power flow for a solar panel at different angles.
- Radiation pressure: compute force on a solar panel. Note how small the force is (compared with gravity for example).