PC2232 Physics for Electrical Engineers
Lecture 6: Classical versus Quantum

Mankei Tsang
Department of Electrical and Computer Engineering
Department of Physics
National University of Singapore

mankei@nus.edu.sg
http://mankei.tsang.googlepages.com/

February 4, 2015

Classical Electrodynamics

- So far we have learned about the dynamics of electromagnetic fields *given* sources (ρ, $\boldsymbol{J}$) and medium properties (ϵ, n, etc.).
- The next question is how objects respond to electromagnetic fields. How do ρ, $\boldsymbol{J}$, ϵ, n, etc. respond to $\boldsymbol{E}$ and $\boldsymbol{H}$?
- For a charged particle at position $\boldsymbol{r}(t)$, it experiences the **Lorentz force** in free space:

$$\boldsymbol{F}(t) = q\left[\boldsymbol{E}(\boldsymbol{r}(t), t) + \boldsymbol{v}(t) \times \boldsymbol{B}(\boldsymbol{r}(t), t)\right], \tag{1}$$

  where q is the charge (in Coulomb) and $\boldsymbol{v}$ is the velocity:

$$\frac{d\boldsymbol{r}(t)}{dt} = \boldsymbol{v}(t). \tag{2}$$

- In classical physics, the particle would have a position $\boldsymbol{r}(t)$ and velocity $\boldsymbol{v}(t)$, and Newton's second law gives

$$\frac{d\boldsymbol{p}(t)}{dt} = \boldsymbol{F}(t), \tag{3}$$

  with the momentum $\boldsymbol{p}(t) \approx m\boldsymbol{v}(t)$ in Newtonian physics.
- This law is important in plasma physics, e.g., cathode-ray-tube (CRT) monitors.

Classical Interaction with Electromagnetic Waves

- Consider a monochromatic plane wave in free space and a charged particle. What is the order of magnitude of the magnetic-field component of the Lorentz force relative to the electric-field component?
- We know $\boldsymbol{H} \sim \boldsymbol{E}/Z_0$ for an EM wave, so

$$\boldsymbol{v} \times \boldsymbol{B} \sim v\mu_0 H \sim \frac{v\mu_0 E}{Z_0} \sim \frac{v}{c}E. \tag{4}$$

  The magnetic-field component of the force is smaller than the electric-field component by a factor of v/c.
- For particles with $v \ll c$, we can usually neglect the magnetic-field component.
- The magnetic-field component is also important if the particle is near a current source, where the magnetic field can be much stronger than the electric field.
- It turns out that this classical physics is not very accurate when dealing with the interaction of light with atoms, electrons, etc.

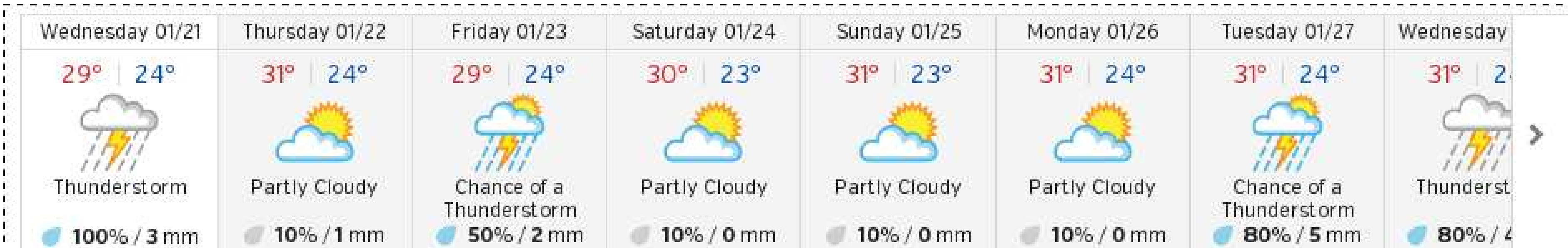
Classical Physics

- The general framework of classical physics suggests that the electromagnetic fields and the positions and velocities of charged particles are all fundamental physical quantities and have **definite values**.
- Other laws of classical physics: special relativity (e.g., $\boldsymbol{p}$ is modified for velocity near speed of light) and general relativity (dynamical generalization of Newton's law of gravitation)
- **Laplace's demon**:

> *We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.*
>
> —Pierre Simon Laplace, *A Philosophical Essay on Probabilities*

- **Determinism**: if we know perfectly the initial conditions of all the electromagnetic fields, gravitational fields, and the positions, velocities, charges, and masses of all particles, we can in principle solve the equations of motion and predict the future **perfectly**.
- `https://philosophicatz.wordpress.com/2008/05/01/laplaces-demon-makez-kitty-sad/`

- Laplace's demon is not possible in practice because no one knows everything (or we'd be able to predict stock market, lottery, weather, football scores, etc.)
- How do we perform predictions when we don't know everything?



(http://www.wunderground.com)

Highlights | Sports | Lottery

Football | Motor Racing | < > refers to Opening Odds

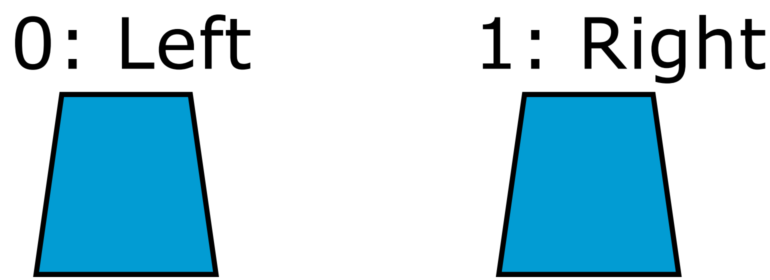
Date/Time	Match Number/Event	1	X	2	Bet Types	Stats
Mon 26/01/15 5:00PM	2329 ASIAN CUP KOREA REPUBLIC VS IRAQ	<1.60> 1.60	<3.35> 3.20	<4.85> 5.20	More Bets	
Tue 27/01/15 4:00AM	1340 ITALIAN LEAGUE NAPOLI VS GENOA	<1.35> 1.33	<4.30> 4.40	<6.50> 6.70	More Bets	
Tue 27/01/15 5:00PM	2330 ASIAN CUP AUSTRALIA VS UAE	<1.40> 1.40	<3.60> 3.60	<7.30> 7.30	More Bets	
Wed 28/01/15 3:45AM	1003 ENGLISH LEAGUE CUP CHELSEA VS LIVERPOOL	<1.50> 1.50	<3.60> 3.60	<5.50> 5.50	More Bets	
Tue 27/01/15 2:00AM	1339 ITALIAN LEAGUE EMPOLI VS UDINESE	<1.90> 1.92	<3.20> 3.15	<3.45> 3.45	More Bets	
Tue 27/01/15 2:00AM	3019 AFRICAN NATIONS CUP CAPE VERDE VS ZAMBIA	<2.60> 2.60	<2.90> 2.90	<2.50> 2.50	More Bets	
Tue 27/01/15 2:00AM	3020 AFRICAN NATIONS CUP CONGO DR VS TUNISIA	<3.20> 3.20	<3.05> 3.05	<2.05> 2.05	More Bets	
Tue 27/01/15 3:45AM	1540 SPANISH LEAGUE GETAFE VS CELTA DE VIGO	<2.77> 2.77	<3.00> 3.00	<2.30> 2.30	More Bets	

(http://www.singaporepools.com.sg/)

- `https://www.youtube.com/watch?v=4jRtK4_OgIE`, `https://www.youtube.com/watch?v=czCwt6fAalA`

Probabilities

- Suppose that the ball is in one of the cups initially ($t = t_0$). Let $x_0 \in \{0, 1\}$ denote the initial position of the ball.



- If I prepare the setup completely **randomly**, there would be 50% chance it's in one cup, and 50% chance for the other.

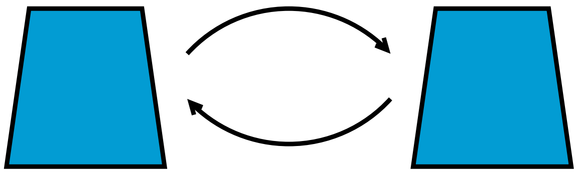
$$P(x_0 = 0, t_0) = \frac{1}{2}, \qquad P(x_0 = 1, t_0) = \frac{1}{2}. \tag{5}$$

- Now if you see the preparation above, you know for sure it's in the left cup:

$$P(x_0 = 0, t_0) = 1, \qquad P(x_0 = 1, t_0) = 0. \tag{6}$$

- $P(x_0 = 0, t_0) + P(x_0 = 1, t_0) = 1$; probabilities are normalized so that a certain event (ball has to be in one of the cups) has probability 1.

Deterministic Transition



- Suppose that the cups switch places once. Let t be the time after switching. Then the probability of x at time t, given that it was at x_0 at t_0, is

$$P(x=1,t|x_0=0,t_0)=1, \qquad P(x=0,t|x_0=1,t_0)=1, \tag{7}$$

and, by normalization of probabilities,

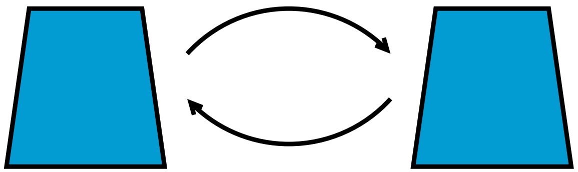
$$P(x=0,t|x_0=0,t_0)=1-P(x=1,t|x_0=0,t_0)=0, \tag{8}$$
$$P(x=1,t|x_0=1,t_0)=1-P(x=0,t|x_0=1,t_0)=1. \tag{9}$$

- Given the initial probabilities $P(x,t)$ and transition probabilities $P(x,t|x_0,t_0)$, what are the new probabilities after switching?

$$\boxed{P(x,t)=\sum_{x_0} P(x,t|x_0,t_0)P(x_0,t_0).} \tag{10}$$

- If $P(x_0=0,t_0)=1$, after switching, $P(x=1,t)=1$ and $P(x=0,t)=0$. Everything is deterministic; you know that the ball will be on the right.
- If $P(x_0=0,t_0)=P(x_0=1,t_0)=1/2$, after switching, $P(x=1,t)=1/2$ and $P(x=0,t)=1/2$. The position remains random.

Mixing



- If a random number of shuffles have taken place (and you don't know the number),

$$P(x=0,t|x_0=0,t_0)=P(x=1,t|x_0=0,t_0)=\frac{1}{2}, \tag{11}$$

$$P(x=0,t|x_0=1,t_0)=P(x=1,t|x_0=1,t_0)=\frac{1}{2}. \tag{12}$$

- If $P(x_0=0,t_0)=1$, after the shuffle, $P(x=0,t)=P(x=1,t)=1/2$. You no longer know where the ball is.
- A more complicated scenario: If the ball is on the left initially, I don't switch:

$$P(x=0,t|x_0=0,t_0)=1, \qquad P(x=1,t|x_0=0,t_0)=0, \tag{13}$$

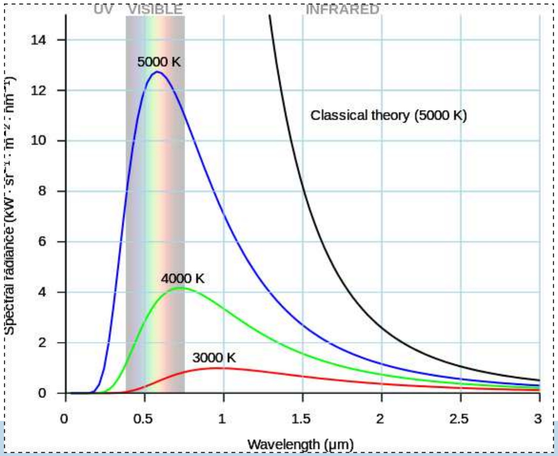
and if the ball is on the right initially, I make a random shuffle:

$$P(x=0,t|x_0=1,t_0)=\frac{1}{2}, \qquad P(x=1,t|x_0=0,t_0)=\frac{1}{2}. \tag{14}$$

The general rule is, again,

$$\boxed{P(x,t)=\sum_{x_0}P(x,t|x_0,t_0)P(x_0,t_0).} \tag{15}$$

Classical Thermodynamics/Statistical Mechanics



- Assigning probabilities to positions, velocities, etc. of particles, Boltzmann is able to combine classical physics with probability theory (statistical mechanics) to explain thermodynamical phenomena such as temperature, entropy, etc.
- Historically, experiments that did not agree with classical statistical mechanics (e.g., blackbody radiation) motivated the development of quantum theory by Planck, Einstein, Bohr *et al.*
  `http://en.wikipedia.org/wiki/Ultraviolet_catastrophe`

Quantum Mechanics

- Classical physics turns out to be inaccurate for very small particles, e.g., atoms, electrons, photons (at high optical frequencies).

> *Because atomic behavior is so unlike ordinary experience, it is very difficult to get used to, and it appears peculiar and mysterious to everyone–both to the novice and to the experienced physicist. Even the experts do not understand it the way they would like to, and it is perfectly reasonable that they should not, because all of direct, human experience and of human intuition applies to large objects. We know how large objects will act, but things on a small scale just do not act that way. So we have to learn about them in a sort of abstract or imaginative fashion and not by connection with our direct experience.*
>
> *In this chapter we shall tackle immediately the basic element of the mysterious behavior in its most strange form. We choose to examine a phenomenon which is impossible, absolutely impossible, to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the only mystery. We cannot make the mystery go away by explaining how it works. We will just tell you how it works. In telling you how it works we will have told you about the basic peculiarities of all quantum mechanics.*
>
> —Chap. 1, The Feynman Lectures on Physics, Vol. III

- For a quantum particle, it turns out that its position, momentum, etc. are not the most fundamental physical quantities.
- The most fundamental quantity is a complex **wavefunction** (a complex function of position, for example), analogous to the complex wave amplitudes we used to study EM waves.

- For the simplest example, consider just two possibilities for the position of the quantum particle, $x = 0$ (left), and $x = 1$ (right). x is called a (binary) **quantum observable**.
- There are many other types of observables that may have more possibilities, e.g., $x \in \{0, 1, 2, \ldots, N\}$, or $x \in \mathbb{R}$ (continuous). Also, the wavefunction can be a function of many observables. We shall focus on $x \in \{0, 1\}$ for now to illustrate the basic concepts.
- Instead of probabilities, quantum mechanics postulates a wavefunction $\psi(x)$, that is, two complex numbers (also called amplitudes):

$$\psi(x = 0) \qquad\qquad \psi(x = 1)$$

- As time evolves, the wavefunction evolves in time through complex transition amplitudes:

$$\boxed{\psi(x, t) = \sum_{x_0} \psi(x, t | x_0, t_0) \psi(x_0, t_0).} \tag{16}$$

- $\psi(x, t | x_0, t_0)$ is called the propagator. It can be written as a matrix. For $x \in \{0, 1\}$,

$$\begin{pmatrix} \psi(0, t) \\ \psi(1, t) \end{pmatrix} = \begin{pmatrix} \psi(0, t | 0, t_0) & \psi(0, t | 1, t_0) \\ \psi(1, t | 0, t_0) & \psi(1, t | 1, t_0) \end{pmatrix} \begin{pmatrix} \psi(0, t_0) \\ \psi(1, t_0) \end{pmatrix}. \tag{17}$$

Quantum Measurement

- If I perform a **measurement** of the position at time t, sometimes I will see it's on the left, sometimes I will see it's on the right. The probability of the outcome is equal to

$$\boxed{P(x,t) = |\psi(x,t)|^2.} \tag{18}$$

  This is called **Born's rule**.
- The wavefunction is normalized like the probability:

$$\sum_x |\psi(x,t)|^2 = \sum_x P(x,t) = 1. \tag{19}$$

- For the **conservation of probabilities**,

$$\sum_x |\psi(x,t)|^2 = 1 \text{ at any time } t. \tag{20}$$

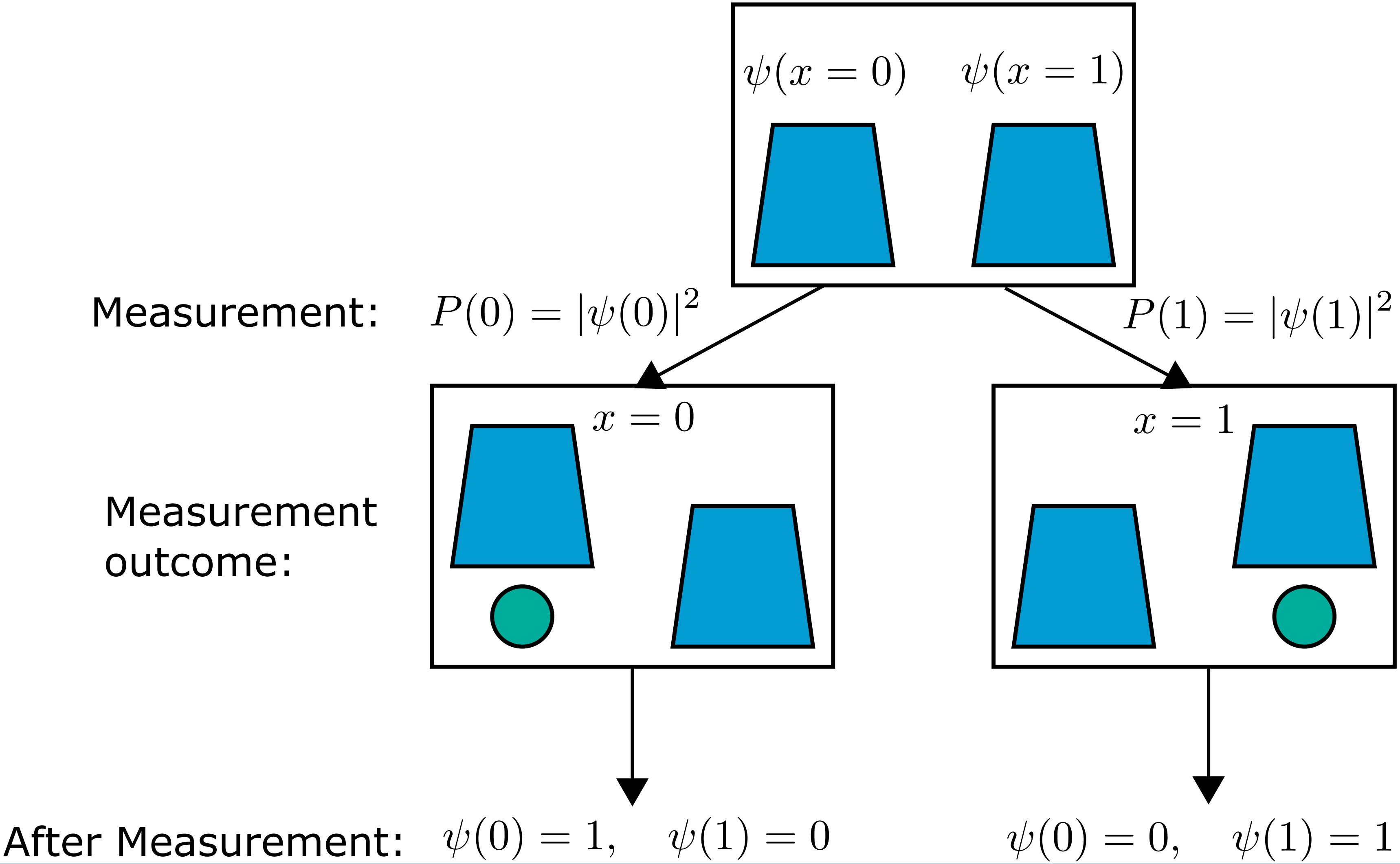
- This means that

$$|\psi(0,t)|^2 + |\psi(1,t)|^2 = |\psi(0,t_0)|^2 + |\psi(1,t_0)|^2, \tag{21}$$

  and $\psi(x,t|x_0,t_0)$ must be a **unitary matrix**.
- Measurement is done by a **macroscopic object**, e.g., eye, camera, ammeter, voltmeter, etc.

Wavefunction Collapse

- Although the measurement outcome can be random depending on $|\psi(x,t)|^2$, after the act of measurement and the outcome is known, the wavefunction **collapses** to match the outcome:



*Dirac's Bra-Ket Notation

- It is common to write the vector of complex amplitudes as **a ket**:

$$|\psi(t)\rangle = \begin{pmatrix} \psi(0,t) \\ \psi(1,t) \end{pmatrix}. \tag{22}$$

If I write the ket in the opposite direction, it is assumed to be the **conjugate transpose** of a ket:

$$\langle\psi(t)| = \begin{pmatrix} \psi^*(0,t) & \psi^*(1,t) \end{pmatrix}. \tag{23}$$

For example, if

$$|\psi\rangle = \begin{pmatrix} \psi(0) \\ \psi(1) \end{pmatrix}, \qquad |\phi\rangle = \begin{pmatrix} \phi(0) \\ \phi(1) \end{pmatrix}, \tag{24}$$

then the **inner product** between the two complex vectors is defined as

$$\langle\phi|\psi\rangle = \begin{pmatrix} \phi^*(0) & \phi^*(1) \end{pmatrix} \begin{pmatrix} \psi(0) \\ \psi(1) \end{pmatrix} = \phi^*(0)\psi(0) + \phi^*(1)\psi(1) = \sum_x \phi^*(x)\psi(x). \tag{25}$$

- Don't forget the conjugate!
- $|\psi(t)\rangle$ is often called a **quantum state**.

*Properties in Bra-Ket Notation

- We can define unit vectors (called **eigenkets**):

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \langle 0|0\rangle = \langle 1|1\rangle = 1, \qquad \langle 0|1\rangle = \langle 1|0\rangle = 0, \tag{26}$$

and the eigenkets are orthogonal to each other: $\langle x|x'\rangle = \langle x'|x\rangle = \delta_{xx'}$, where $\delta_{xx'}$ is the Kornecker delta; $\delta_{xx'} = 1$ if $x = x'$, $\delta_{xx'} = 0$ if $x \neq x'$.

- Any quantum state can be written as

$$|\psi\rangle = \psi(0)|0\rangle + \psi(1)|1\rangle = \sum_x \psi(x)|x\rangle. \tag{27}$$

(similar to how we write $\boldsymbol{E} = E_x\hat{\boldsymbol{x}} + E_y\hat{\boldsymbol{y}} + E_z\hat{\boldsymbol{z}}$).

- In terms of a quantum state, the wavefunction can be expressed as

$$\langle x|\psi\rangle = \langle x|\sum_{x'} \psi(x')|x'\rangle = \sum_{x'} \psi(x')\langle x|x'\rangle = \sum_{x'} \psi(x')\delta_{xx'} = \psi(x). \tag{28}$$

(whenever you see a bra followed by a ket, combine them to make it an inner product)

- Normalization:

$$\langle \psi(t)|\psi(t)\rangle = \sum_{x'} \psi^*(x')\langle x'|\sum_x \psi(x)|x\rangle = \sum_{x'}\sum_x \psi^*(x')\psi(x)\langle x'|x\rangle = \sum_{x'} \psi^*(x')\sum_x \psi(x)\delta_{x'x}$$
$$= \sum_{x'} \psi^*(x')\psi(x') = 1. \tag{29}$$

*Quantum Operators

- A matrix that transforms a ket to another ket is also called an operator. It is often denoted by a hat (e.g., $\hat{U}$).
- For example, the quantum evolution in time:

$$\psi(x,t) = \sum_{x_0} \psi(x,t|x_0,t_0)\psi(x_0,t_0). \tag{30}$$

can be written as

$$|\psi(t)\rangle = \hat{U}|\psi(t_0)\rangle, \tag{31}$$

where

$$\hat{U} = \sum_{x,x'} \psi(x,t|x',t_0)|x\rangle\langle x'|. \tag{32}$$

- To see why a matrix/operator can be written this way, recall

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{33}$$

so a ket-bra is

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad |0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{34}$$

$$|1\rangle\langle 0| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad |1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{35}$$

*Operator on a Ket

- So

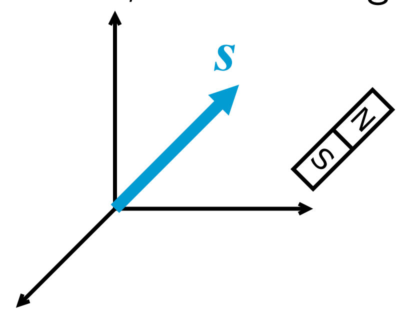
$$\begin{pmatrix} \psi(0,t|0,t_0) & \psi(0,t|1,t_0) \\ \psi(1,t|0,t_0) & \psi(1,t|1,t_0) \end{pmatrix} = \psi(0,t|0,t_0)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \psi(0,t|1,t_0)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$+ \psi(1,t|0,t_0)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \psi(1,t|1,t_0)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (36)$$
$$= \sum_{x,x'} \psi(x,t|x',t_0)|x\rangle\langle x'|. \quad (37)$$

- The operator applied on a ket then becomes

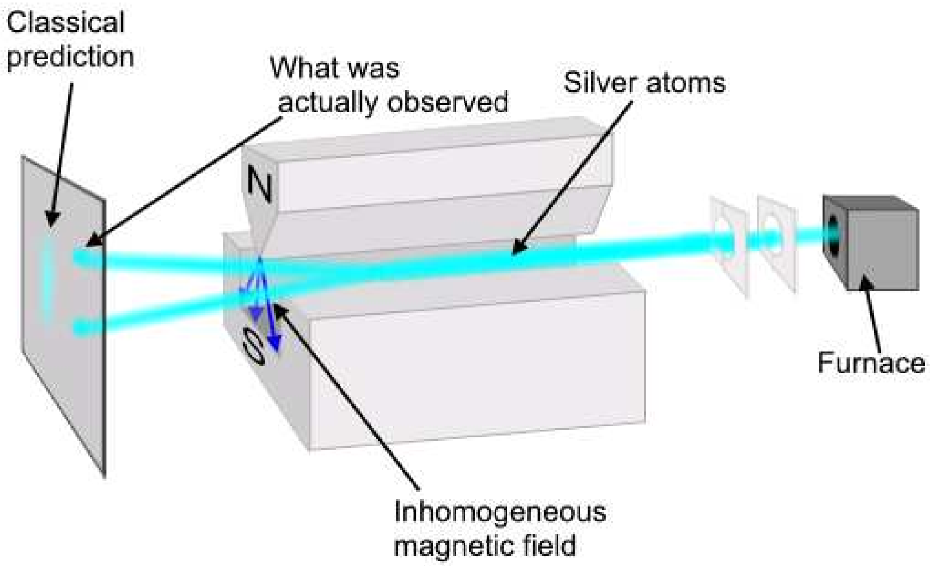
$$\hat{U}|\psi(t_0)\rangle = \sum_{x,x'} \psi(x,t|x',t_0)|x\rangle\langle x'| \sum_{x_0} \psi(x_0,t_0)|x_0\rangle \quad (38)$$
$$= \sum_{x,x'} \psi(x,t|x',t_0)|x\rangle \sum_{x_0} \psi(x_0,t_0)\langle x'|x_0\rangle \quad (39)$$
$$= \sum_{x,x'} \psi(x,t|x',t_0)|x\rangle \sum_{x_0} \psi(x_0,t_0)\delta_{x'x_0} \quad (40)$$
$$= \sum_{x,x'} \psi(x,t|x',t_0)\psi(x',t_0)|x\rangle \quad (41)$$

Spin

- Each silver atom has a spin (magnetic moment), like a magnet.
- Classically, a magnetic moment is a vector, it has a magnitude and a direction.



- Consider the Stern-Gerlach experiment, where the atoms are deflected depending on their magnetic moments along the external inhomogeneous magnetic field (imagine throwing tiny magnets along the path) and can be used to measure a spin component.



(http://en.wikipedia.org/wiki/File:Stern-Gerlach_experiment.PNG)

Quantized Spin

- For a beam of silver atoms, the magnetic moments are expected to be in random directions. The measured distribution of a spin component is expected to be continuous.
- Instead, the experiment would find that each spin component has only two values; I will call them $+1/2$ and $-1/2$ (The actual magnetic moment in mks units is $\pm 1/2$ times a constant). This is true for measurements along any direction.
- In terms of a spin component, say, s_z, there are two possible values, and there are two complex amplitudes: $\psi_z(s_z = +1/2)$ and $\psi_z(s_z = -1/2)$.
- Given $\psi_z(s_z)$, the probability distribution of measuring s_z is

$$P_z\left(s_z = +\frac{1}{2}\right) = \left|\psi_z\left(s_z = +\frac{1}{2}\right)\right|^2, \qquad P_z\left(s_z = -\frac{1}{2}\right) = \left|\psi_z\left(s_z = -\frac{1}{2}\right)\right|^2. \tag{42}$$

Incompatible Observables

- If instead the x-spin component s_x is measured, it turns out that the wavefunction of s_x depends on the wavefunction of s_z in the following way:

$$\begin{pmatrix} \psi_x\left(s_x = +\frac{1}{2}\right) \\ \psi_x\left(s_x = -\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \psi_z\left(s_z = +\frac{1}{2}\right) \\ \psi_z\left(s_z = -\frac{1}{2}\right) \end{pmatrix}, \tag{43}$$

such that probability distribution of measured s_x is

$$P_x(s_x) = |\psi_x(s_x)|^2 \tag{44}$$

- Similarly, the wavefunction of s_y depends on the wavefunction of s_z as follows:

$$\begin{pmatrix} \psi_y\left(s_y = +\frac{1}{2}\right) \\ \psi_y\left(s_y = -\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix} \begin{pmatrix} \psi_z\left(s_z = +\frac{1}{2}\right) \\ \psi_z\left(s_z = -\frac{1}{2}\right) \end{pmatrix} \tag{45}$$

such that probability distribution of measured s_y is

$$P_y(s_y) = |\psi_y(s_y)|^2 . \tag{46}$$

- Here is the second important concept: some observables, such as s_x, s_y, s_z, have wavefunctions that are related to one another by unitary matrices.

- Suppose

$$\begin{pmatrix} \psi_z\left(s_z = +\frac{1}{2}\right) \\ \psi_z\left(s_z = -\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{47}$$

I know $s_z = +1/2$. What if I now measure s_x of this state?

$$\begin{pmatrix} \psi_x\left(s_x = +\frac{1}{2}\right) \\ \psi_x\left(s_x = -\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}. \tag{48}$$

- Now I will get a 50% chance of obtaining $s_x = +1/2$, and a 50% chance of obtaining $s_x = -1/2$.
- For a given state, the probability distributions $P_z(s_z)$ and $P_x(s_x)$ are related. If s_z is known, then s_x becomes random. s_x, s_y, s_z are known as **incompatible observables**. Their values cannot be measured simultaneously.

- Instead of a binary observable, consider now a continuous observable, such as the one-dimensional position $x \in \mathbb{R}$.
- The **probability density** of measuring x is

$$P_x(x,t) = |\psi(x,t)|^2, \tag{49}$$

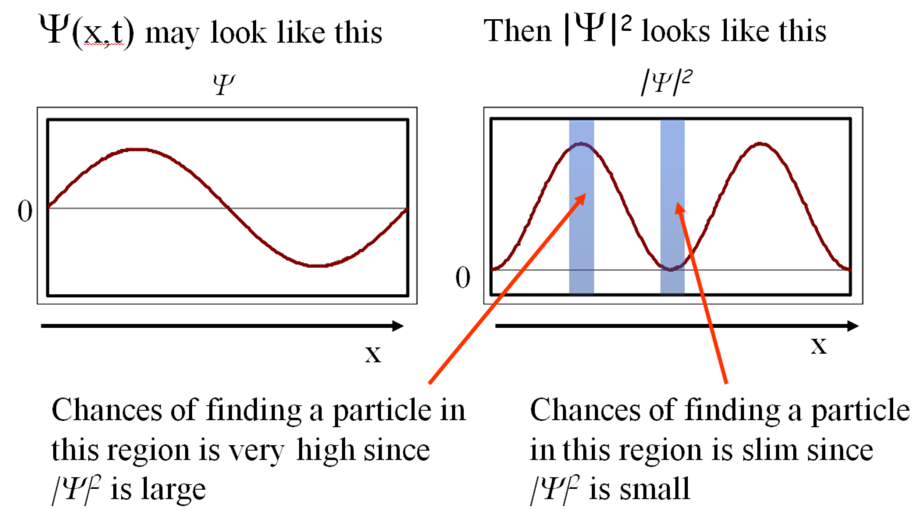
but note that the normalization condition is

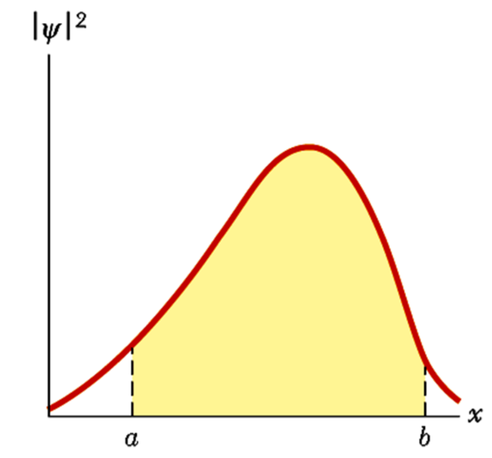
$$\boxed{\int_{-\infty}^{\infty} dx P_x(x,t) = \int_{-\infty}^{\infty} dx |\psi(x,t)|^2 = 1.} \tag{50}$$

- $P_x(x,t)$ is a probability density, meaning that

$$P_x(x,t)dx$$
$$= \text{probability of finding } x \text{ in } [x, x+dx), \tag{51}$$

$$\int_a^b P_x(x,t)dx$$
$$= \text{probability of finding } x \text{ in } [a,b). \tag{52}$$





Position and Momentum

- It turns out that the momentum wavefunction is the **Fourier transform** of the position wavefunction:

$$\boxed{\phi(k,t) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp(-ikx)\psi(x,t).} \tag{53}$$

- This kind of transform is a generalization of unitary matrix (think of this as the continuous limit of $\sum_m \frac{\Delta x}{\sqrt{2\pi}} \exp(ik_n x_m)\psi(x_m)$).
- In quantum mechanics, the unitary matrix and the integration with a propagator are both examples of so-called **operators**: They operate on a vector (or a function) to give another vector (or another function).
- The probability density of measured momentum k is then

$$P_k(k,t) = |\phi(k,t)|^2, \qquad \int_{-\infty}^{\infty} dk P_k(k,t) = \int_{-\infty}^{\infty} dk\, |\phi(k,t)|^2 = 1. \tag{54}$$

- The position and momentum obey an **uncertainty relation**: you cannot measure them both simultaneously.
- If the position is known precisely, the momentum has a large uncertainty, and vice versa.
- k here has the unit of m^{-1}. In mks units, the momentum is

$$\boxed{p = \hbar k.} \tag{55}$$

*Uncertainty Relations

- Suppose $\psi(x) \propto \delta(x - x_0)$, where δ is the Dirac delta function centered at x_0.
- Never mind that the probability density is unnormalizable, let's focus on qualitative picture here.
- The position is known quite precisely to be near x_0.
- The momentum wavefunction is

$$\phi(k) \propto \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp(-ikx)\delta(x - x_0) = \frac{1}{\sqrt{2\pi}} e^{-ikx_0}, \qquad |\phi(k)|^2 \propto 1, \tag{56}$$

  The momentum distribution is constant (i.e., its momentum is random).
- Suppose instead that $\psi(x) \propto e^{ik_0 x}$ (a plane wave) $|\psi(x)|^2 \propto 1$, The position is random, but

$$\phi(k) \propto \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp(-ikx) e^{ik_0 x} = \sqrt{2\pi}\delta(k - k_0), \tag{57}$$

  and the momentum has a definite value at k_0.
- More generally, define the standard deviation of $|\psi(x)|^2$ and the standard deviation of $|\phi(k)|^2$ as

$$\Delta x \equiv \left[\int_{-\infty}^{\infty} dx (x - \langle x \rangle)^2 |\psi(x)|^2\right]^{\frac{1}{2}}, \qquad \Delta k \equiv \left[\int_{-\infty}^{\infty} dk (k - \langle k \rangle)^2 |\phi(k)|^2\right]^{\frac{1}{2}}, \tag{58}$$

$$\langle x \rangle \equiv \int_{-\infty}^{\infty} dx\, x |\psi(x)|^2, \qquad \langle k \rangle \equiv \int_{-\infty}^{\infty} dk\, k |\phi(k)|^2. \tag{59}$$

  It can be **proven** that, for any $\psi(x)$,

$$\boxed{\Delta x \Delta k \geq \frac{1}{2}.} \tag{60}$$

Time Evolution

- For the wavefunction of a continuous observable, time evolution of the wavefunction is given by

$$\boxed{\psi(x,t) = \int_{-\infty}^{\infty} dx_0 \psi(x,t|x_0,t_0)\psi(x_0,t_0).} \tag{61}$$

- The propagator $\psi(x,t|x_0,t_0)$ is like a matrix, except that here x and x_0 are continuous variables instead of two discrete indices.
- *Normalization condition $\int_{-\infty}^{\infty} dx|\psi(x,t)|^2 = \int_{-\infty}^{\infty} dx_0|\psi(x_0,t_0)|^2$ means that the propagator must satisfy the condition

$$\int_{-\infty}^{\infty} dx \psi^*(x,t|x_0,t_0)\psi(x,t|x_0',t_0) = \delta(x_0 - x_0'). \tag{62}$$

This is a generalization of the unitary matrix property.

Suggested Problems

- Electron Mach-Zehnder interferometer [Ji *et al.*, Nature **422**, 415 (2003)]: Suppose that the position of an electron is binary $x \in \{0, 1\}$. Initial wavefunction is $\psi(0, t_0) = 1$, $\psi(1, t_0) = 0$. A "beam splitter" for the electron splits it into two arms according to the unitary matrix

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}. \tag{63}$$

  Path 0 introduces a phase shift $e^{i\theta_0}$, and path 1 introduces a phase shift $e^{i\theta_1}$. Then the two paths go into another beam splitter identical to the first. Calculate the probabilities of observing the electron in the final two outputs.
- Spin: Suppose $\psi_z(s_z = 1/2) = \cos(\theta/2)$, $\psi_z(s_z = -1/2) = e^{i\phi}\sin(\theta/2)$, where θ and ϕ are arbitrary angle parameters. Verify normalization condition. Find $\psi_x(s_x)$ and $\psi_y(s_y)$.
- What are the θ and ϕ that lead to $\psi_x(s_x = 1/2) = 1$? How about $\psi_y(s_y = 1/2) = 1$?
- Now suppose $\psi_x(s_x = 1/2) = \cos(\theta'/2)$, $\psi_x(s_x = -1/2) = e^{i\phi'}\sin(\theta'/2)$. What are $\psi_z(s_z)$ and $\psi_y(s_y)$?
- Uncertainty relations: Consider the following wavefunction for a continuous position:

$$\psi(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(-\frac{x^2}{4\sigma^2}\right). \tag{64}$$

  Verify normalization condition. Find the standard deviation of the distribution $|\psi(x)|^2$. Compute $\phi(k)$ by performing Fourier transform on $\psi(x)$. Find the standard deviation of the distribution $|\phi(k)|^2$. What is the product of the two deviations? Look up the normal distribution, standard deviation, and Fourier transform of a Gaussian function on wikipedia if you don't know them.