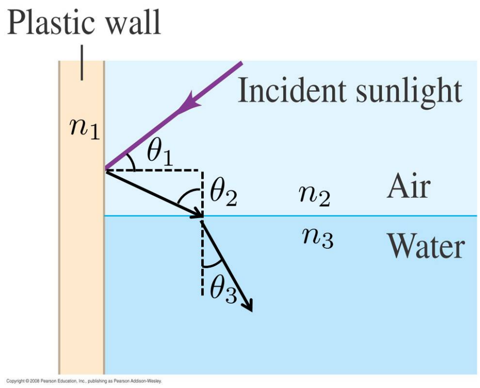
PC2232 – Tutorial 2 Solutions

1. Given/Known:

$n_1 = 1.61$ $\qquad n_2 = 1$ $\qquad n_3 = 1.33$



Question: $\theta_3 =?$

Approach:

(a) Brewster's angle: $\dfrac{n_1}{n_2} = \tan\theta_1$

(b) Geometry: $\theta_2 = 90 - \theta_1$

(c) Snell's law: $n_3 \sin\theta_3 = n_2 \sin\theta_2$

(d) Learning point: Refreshing your knowledge on Brewster's angle as well as Snell's law.

Solution:

(a)	$\dfrac{n_1}{n_2} = \tan\theta_1$	$\theta_1 = 58.2°$
(b)	$\theta_2 = 90 - \theta_1$	$\theta_2 = 31.8°$
(c)	$n_2 \sin\theta_2 = n_3 \sin\theta_3$	

Note the following:

- The equation for Brewster's angle is derived at the end of this document
- Understand that Brewster's angle is θ_1. It occurs at the air-plastic interface rather than the air-water interface because it's always the ***reflected*** light that is completely polarized, not the refracted light.

Answer:

$$\theta_3 = 23.4°$$

2. (a) Question: $\theta =?$

Approach:

i. Understand the question

ii. Use the Malus' equation

iii. Understand: $\cos^2$ graph is a symmetrical graph

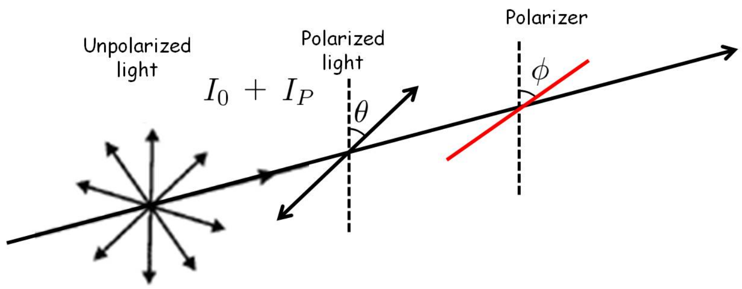
iv. Learning point:

Refreshing your knowledge on Malus' law.

Clarifying what the angle θ in Malus' law $I = I_0 \cos^2\theta$ actually represents

Solution:

i. Understand the question



Notice there are two sources of light.
The unpolarized light and the polarized light are separate and independent from each other.
Both light sources are going through the polarizer together

ii. Malus' law:

$$I_T = \frac{1}{2}I_0 + I_p \cos^2 |\theta - \phi|$$

Note:
The angle in Malus' law is always the angle between the polarized beam and the polarizer
From the diagram, one can note that this angle should be $|\phi - \theta|$

iii. Using data given to deduce θ

- For I to be maximum, $\cos^2(|\phi - \theta|) = 1 \quad \Rightarrow \quad \theta - \phi = 0$
  Therefore, when I_T is maximum, $\theta = \phi$
- Because it's a $\cos^2$ curve, it's a symmetrical curve
  Therefore $I_{\max}$ will occur right in between $30 < \phi < 40$: At $\phi = 35°$

Answer:

$$\theta = 35°$$

(b) Question:

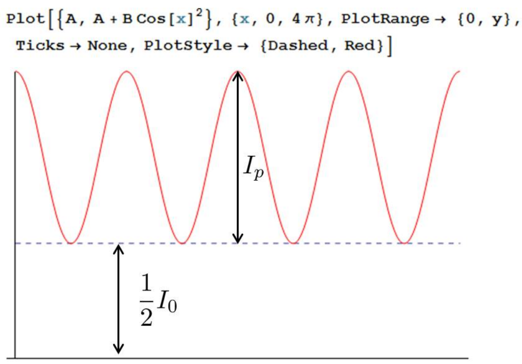
$$I_0 = ? \qquad\qquad I_p = ?$$

Approach:

i. Best way: Draw a graph.
ii. But simultaneous equations should solve the problem as well (not as accurate).

Solution:

i. If you plot out the graph, the following values will give you I_0 and I_p:



ii. Solving simultaneous equations

$$\begin{array}{rl} & 24.8 = \frac{1}{2}I_0 + I_p \cos^2(5) \\ - & 5.2 = \frac{1}{2}I_0 + I_p \cos^2(85) \\ \hline & 19.6 = I_p[\cos^2 5 - \cos^2 85] \\ & \therefore\ I_p = 19.9\text{Wm}^{-2} \end{array}$$

Substituting.

$$\begin{aligned} 24.8 &= \frac{1}{2}I_0 + 19.9\cos^2(5) \\ I_0 &= 10.1\text{Wm}^{-1} \end{aligned}$$

Answer:

$$I_0 = 10.1\text{Wm}^{-1} \qquad\qquad I_P = 19.9\text{Wm}^{-1}$$

3. Given:

$$\vec{E} = E_x\hat{x} + E_y\hat{y} \qquad E_x = E_0\sin(kz - \omega t) \qquad E_y = E_0\sin(kz - \omega t - \phi)$$

Question: Show that $E_y = E_x\cos\phi + \sqrt{E_0^2 - E_x^2}\sin\phi$

Approach:

(a) Since we need to prove the above equation, we start we the definition of E_y

(b) Use trigo formulae: $\sin(a+b) = \sin a\cos b + \cos a\sin b$

(c) Use trigo formulae: $\sin^2\theta + \cos^2\theta = 1$

(d) Learning point: No real learning point. It's a intermediate step to help us solve the later parts

Solution

$$\begin{aligned} E_y &= E_0\sin(\underbrace{kz - \omega t}_{a} + \underbrace{\phi}_{b}) \\ & E_0\Big[\sin(kz-\omega t)\cos(\phi) + \cos(kz-\omega t)\sin\phi\Big] \\ &= \underbrace{E_0\sin(kz-\omega t)}_{E_x}\cos(\phi) + \underbrace{E_0\cos(kz-\omega t)}_{\text{reexpressed as: }\sqrt{E_0^2+E_x^2}}\sin\phi \\ &= E_x\cos\phi + \sqrt{E_0^2\cos^2(kz-\omega t)}\sin\phi \\ &= E_x\cos\phi + \sqrt{E_0^2[1-\sin^2(kz-\omega t)]}\sin\phi \\ &= E_x\cos\phi + \sqrt{E_0^2 - E_x^2}\sin\phi \end{aligned}$$

(a) When $\phi = 0$ show that it is a linear polarization

Approach:
Substitute the value for ϕ in
Compare the E equation with the equations given for straight line: $y = x$

Solution:

$$E_y = E_x\cos(0) + \sqrt{E_0^2 - E_x^2}\sin(0) = E_x$$

(b) When $\phi = \frac{\pi}{2}$ show that it is a circular polarization

Approach:
Substitute the value for ϕ in
Compare the E equation with the equations given for a circle: $y^2 + x^2 = R^2$

Solution:

$$\begin{aligned} E_y &= E_x\cos\left(\frac{\pi}{2}\right) + \sqrt{E_0^2 - E_x^2}\sin\left(\frac{\pi}{2}\right) \\ E_y^2 &= E_0^2 - E_x^2 \\ E_x^2 + E_y^2 &= E_0^2 \end{aligned}$$

(c) When $\phi = \frac{\pi}{4}$ show that it is a elliptical polarization

Approach:

Substitute the value for ϕ in
Compare the E equation with the equations given for a ellipse[1]: $Ay^2 + Bxy + Cx^2 = 1$

Solution:

$$
\begin{aligned}
E_y &= E_x \cos\left(\frac{\phi}{4}\right) + \sqrt{E_0^2 - E_x^2}\sin\left(\frac{\phi}{4}\right) \\
E_y - \frac{E_x}{\sqrt{2}} &= \frac{\sqrt{E_0^2 - E_x^2}}{\sqrt{2}} \\
\sqrt{2}E_y - E_x &= \sqrt{E_0^2 - E_x^2} \\
2E_y^2 - 2\sqrt{2}E_yE_x + E_x^2 &= E_0^2 - E_x^2 \\
2E_y^2 - 2\sqrt{2}E_yE_x + 2E_x^2 &= E_0^2 \\
\frac{2}{E_0^2}E_y^2 - \frac{2\sqrt{2}}{E_0^2}E_yE_x + \frac{2}{E_0^2}E_x^2 &= 1
\end{aligned}
$$

Learning point:

- Parts (a) to (c) are there to help you visualize how to obtain circular polarized waves
- From here you should understand that combining two linearly polarized waves (one in the x and another in the y direction) can give you a variety of polarization depending on the phase difference of the two polarization
- We ***can*** get linear polarization along the diagonals by combining two different linear polarization if there's no phase difference
- But we can get circular and elliptical polarization as well[2].
- You should try to rewrite the equations for circular polarization using exponential functions. Check your answer with page Lecture 2 pg 4 and pg 5[3]
- Therefore, after doing this question, you should be able to immediately write out a circularly polarized $\vec{E}$ if you need to.

(d) Write out the Poynting vector of elliptically polarized light

Approach:

- Know $\vec{S} = \mathrm{Re}[\vec{E}] \times \mathrm{Re}[\vec{H}]$
- Know how to obtain $\vec{H}$ from $\vec{E}$
- Know trigo formula
- Learning point:
  Familiarize yourself with how to obtain $\vec{H}$ and $\vec{S}$ given $\vec{E}$
  See for yourself that the equations for $\vec{S}$ in the later lectures are the same as the given equation
  Refer to question 4 for more on $\vec{S}$ when we use the exponential form to express $\vec{E}$ and $\vec{H}$

Solution:

- Know $\vec{S} = \mathrm{Re}[\vec{E}] \times \mathrm{Re}[\vec{H}]$
  This is from lecture 1 (pg 20)
  Understand that S is rate of power flow per unit area (in the direction of power flow)[4]

- Determine $\vec{H}$ from $\vec{E}$
  - This is from lecture 1 (Page 17)

$$
\begin{aligned}
\vec{\nabla} \times \vec{E} &= -\mu_0 \frac{\partial \vec{H}}{\partial t} \\
j\vec{k} \times \vec{E} &= \mu_0 j\omega \vec{H}
\end{aligned}
$$

  LHS has been proven in tut 1 question 1 (we proved with $\vec{\nabla} \cdot \vec{E}$). Eq. shown in lecture 1 pg 15
  RHS is also in lecture 1 pg 15. We're just differentiating the t component of $\vec{H}$[5]

[1] Note that this is a generalized ellipse equation. The one given in your tutorial sheet was an ellipse along the x-axis or y-axis. I will rotate the equation given in your tutorial sheet at the end of this document to prove this to you

[2] This assumes you know the equations of a circle and ellipse. Therefore, solving this question helps you see that the particular value of ϕ reduces to a circle and an ellipse

[3] There it's the addition of y and z polarization propagating in the x direction. Ours is addition of x and y direction propagating in z.

[4] Notice that if we use the exponential form, we must cross the ***real*** components of E and H. This is because the imaginary component of E and H do not physically exist. We have the extra imaginary components because it's mathematically easier to work with exponential functions than trigo functions. In the trigo formulation (cos and sin) there is no imaginary part, so we need not worry

[5] We seem to be cheating here... it's like we already know $\vec{H}$ when we apply this formula. But recall that we always know the general form of $\vec{H}$ for plane waves. (Up till this tutorial we're only talking about plane waves). As shown in pg 14 of lecture 1, $\vec{H} = \tilde{H}\mathrm{e}^{j\vec{k}\cdot\vec{r}}\mathrm{e}^{-j\omega t}$. Right now we're trying to find the specific form of $\vec{H}$ given a particular $\vec{E}$

- Recall the following equations:

$$k = \frac{\omega}{c} \qquad\qquad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

  First is from the dispersion relation (lecture 1 pg 16)
    Notice also that k here is a scalar. Therefore, it's only the magnitude of k
  Second is something you should know by now... and it's in lecture 1 pg 7
- Therefore

$$\begin{aligned}\vec{k} \times \vec{E} &= \mu_0 \omega \vec{H} \\ &= \mu_0 \frac{|k|}{c} \mu_0 \vec{H} \\ \frac{\vec{k}}{|k| \vec{E}} &= \sqrt{\frac{\mu_0}{\varepsilon_0}} \vec{H} \\ \therefore \qquad \vec{H} &= \frac{\hat{k} \times \vec{E}}{Z_0}\end{aligned}$$

  where $Z_0 \equiv \frac{\mu_0}{\varepsilon_0}$
- Note: The above is the proof of $\vec{H} = \frac{\hat{k} \times \vec{E}}{Z_0}$. If you don't have to prove this, you can immediately deduce that
    Magnitude of $\vec{H}$: $|\vec{H}| = \frac{|\vec{E}|}{Z_0}$
    Direction of $\vec{H}$: Perpendicular to $\vec{k}$ and $\vec{E} \quad \Rightarrow \quad \hat{k} \times \hat{E}$
    Therefore $\vec{H} = \frac{\hat{k} \times \hat{E} |\vec{E}|}{Z_0} = \frac{\hat{k} \times \vec{E}}{Z_0}$
- Once we know this, we can also claim that[6]: $\boxed{\vec{S} = \vec{E} \times \vec{H} = \frac{|E|^2}{Z_0} \hat{k}}$

- Using trigo formula:
  Recall: Our job – to obtain S for elliptical polarization
    Therefore $\vec{E}$ of elliptical polarization

$$\begin{aligned}E &= E_x \hat{x} + E_y \hat{y} \\ &= E_0 \sin(kz - \omega t) \hat{x} + E_0 \sin\Bigg(\underbrace{kz - \omega t}_{a} + \underbrace{\frac{\pi}{4}}_{b}\Bigg) \hat{y} \\ &= E_0 \sin(kz - \omega t) \hat{x} + E_0 \left[\frac{1}{\sqrt{2}} \sin(kz - \omega t) + \frac{1}{\sqrt{2}} \cos(kz - \omega t)\right] \hat{y}\end{aligned}$$

  where $\sin\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
  Notice: $\hat{k} = \hat{z}$
    Therefore, obtaining $\vec{H}$

$$\begin{aligned}\vec{H} &= \frac{\hat{k} \times \vec{E}}{Z_0} \\ &= \frac{E_0}{Z_0} \sin(kz - \omega t) \hat{y} - \left[\frac{E_0}{\sqrt{2} Z_0} \sin(kz - \omega t) + \frac{E_0}{\sqrt{2} Z_0} \cos(kz - \omega t)\right] \hat{x}\end{aligned}$$

  Therefore $\vec{S}$:

$$\begin{aligned}\vec{S} &= \vec{E} \times \vec{H} \\ &= \frac{E_0^2}{Z_0} \left[\sin^2(kz - \omega t) + \frac{1}{2} \sin^2(kz - \omega t) + \frac{1}{2} \cos^2(kz - \omega t) + \sin(kz - \omega t) \cos(kz - \omega t)\right] \hat{z} \\ &= \frac{E_0^2}{2 Z_0} \Big(3 \sin^2(kz - \omega t) + \cos^2(kz - \omega t) + \sin\big[2(kz - \omega t)\big]\Big) \hat{z}\end{aligned}$$

[6] Notice that:
  $|E|^2$ means $\vec{E} \cdot \vec{E}$
  It should be obvious that it's pointing the the direction of propagation
  In this question, I will work out $\vec{E} \times \vec{H}$. You should try the boxed up equation on your own to convince yourself that the equation really works

Try obtaining $\vec{S}$ via the boxed up equation to convince yourself it's the same.

Answer:

$$\vec{S} = \frac{E_0^2}{2Z_0}\Big(2\sin^2(kz-\omega t) + \sin\big[2(kz-\omega t)\big] + 1\Big)\hat{z}$$

4. Given:

$$\vec{E} = 2j\tilde{E}e^{-j\omega t}\sin(kx)\hat{y} \qquad \vec{H} = \frac{2\tilde{E}}{Z_0}e^{-j\omega t}\cos(kx)\hat{z}$$

Write out the Poynting vector for this standing wave

Approach:
Use the equation $\vec{S} = \mathrm{Re}[\vec{E}] \times \mathrm{Re}[\vec{H}]$
Learning point:
Familiarize yourself with Poynting vector $\vec{S}$

- In this question we start using the exponential terms to express our $\vec{E}$ and $\vec{H}$
- Therefore, we must be very careful to only cross the real values of $\vec{E}$ and $\vec{H}$
- Please note that the equation $\vec{S} = \frac{1}{2}\vec{E} \times \vec{H}^*$ can only be used to calculate average Poynting vector. $\vec{E} \times \vec{H}^*$ ***cannot*** be used to calculate general $\vec{S}$. Can you determine why[7]?

Solution:

$$\begin{aligned}\vec{S} &= \mathrm{Re}[\vec{E}] \times \mathrm{Re}[\vec{H}] \\ &= \frac{4\tilde{E}^2}{Z_0}\Big(\sin(kx)\sin(\omega t)\hat{y} \times \cos(kx)\cos(\omega t)\hat{z}\Big) \\ &= \frac{4\tilde{E}^2}{Z_0}\sin(kx)\cos(kx)\sin(\omega t)\cos(\omega t)\hat{x} \\ &= \frac{\tilde{E}^2}{Z_0}\sin(2kx)\sin(2\omega t)\hat{x}\end{aligned}$$

Answer:

$$\vec{S} = \frac{\tilde{E}^2}{Z_0}\sin(2kx)\sin(2\omega t)\hat{x}$$

(a) When $x = 0$, $\vec{S} =$?

Approach:
Substitute value of x in
Learning point: Standing waves in a conducting cavity has no power flow outside the cavity

Solution

$$\vec{S} = \frac{\tilde{E}^2}{Z_0}\sin(2k[0])\sin(2\omega t)\hat{x}$$

Answer:

$$\vec{S} = 0$$

(b) Find the average $\vec{S}$

Approach:
This can be done by averaging over a cycle as in tutorial 1 question 4
Or by using the formula $S_{\mathrm{av}} = \frac{1}{2}\Big(\mathrm{Re}[\vec{E} \times \vec{H}^*]\Big)$
Learning point:
Familiarizing yourself with the average poynting vector[8]
Know that average poynting vector of a standing wave is 0

Solution:

[7] $\vec{S}$ depends on time. But $\vec{E} \times \vec{H}^*$ is a constant value. Therefore it can only be used for the average value of $\vec{S}$, which we know will be a constant

[8] Try the averaging way on your own to see that it returns us the same answer

$$S_{\text{av}} = \frac{1}{2}\left(\text{Re}\left[2j\tilde{E}\mathrm{e}^{-j\omega t}\sin(kx)\hat{y} \times \frac{2\tilde{E}}{Z_0}\mathrm{e}^{j\omega t}\cos(kx)\hat{z}\right]\right) = \text{Re}\left[\frac{\tilde{E}^2}{Z_0}j\sin[2kx]\hat{x}\right]$$

Note that the above expression has no real part **Answer:**

$$S_{\text{av}} = 0$$

5. Given: EM wave that propagates only in the x and y direction in the following cavity

$$L_x = 315\text{mm} \qquad L_y = 348\text{mm} \qquad L_z = 227\text{mm}$$

Question:
Write out the standing wave solutions
Write out the possible modes in the x and y direction

Approach:

- Understand that to find the solution to Maxwell equations for our particular question
  - We first know the general form of one of the possible solutions
    Our possible solution for now are plane wave solutions: $E = \tilde{E}\mathrm{e}^{j\vec{k}\cdot\vec{r}}\mathrm{e}^{-j\omega t}$
  - We then realize that the solution to our particular set up is the superposition of all possible scenarios
    In our case, we should have 4 possible cases
    * Wave moving in positive x direction
    * Wave moving in negative x direction after reflecting off the conducting wall
    * Wave moving in the positive y direction
    * Wave moving in the negative y direction after reflecting off the conducting wall
  - Finally, our solution must obey the boundary condition of our setup
    In this case, since we can assume our microwave walls are conductors, $\vec{E}$ must be 0 at

$$x = 0 \qquad x = L_x \qquad y = 0 \qquad y = L_y$$

    This generally gives us additional constraints to our solutions[9]
- Learning point:
  Understand these three steps that are required to solve Maxwell's equation for different set ups[10]

Solution:

- The 4 possible solutions[11]:

$$E_{+x} = \tilde{E}\mathrm{e}^{jk_x x}\mathrm{e}^{-j\omega t}\hat{z} \qquad E_{-x} = \tilde{E}\mathrm{e}^{-jk_x x}\mathrm{e}^{-j\omega t}\hat{z} \qquad E_y = \tilde{E}\mathrm{e}^{jk_y y}\mathrm{e}^{-j\omega t}\hat{z} \qquad E_{-y} = \tilde{E}\mathrm{e}^{-jk_y y}\mathrm{e}^{-j\omega t}\hat{z}$$

- Doing a superposition for all the 4 solutions:

$$\begin{aligned}\vec{E} &= E_{+x} + E_{-x} + E_{+y} + E_{-y} \\ &= \tilde{E}\mathrm{e}^{-j\omega t}\big(\mathrm{e}^{jk_x x} + \mathrm{e}^{-jk_x x} + \mathrm{e}^{jk_y y} + \mathrm{e}^{-jk_y y}\big)\hat{z} \\ &= \tilde{E}\mathrm{e}^{-j\omega t}\big(2j\sin[k_x x] + 2j\sin[k_y y]\big)\hat{z}\end{aligned}$$

- Checking boundary conditions:
  - Note that our boundary conditions means that E at $x = 0$ and any y must be 0

$$\begin{aligned}\vec{E}(0, y, z, t) &= \tilde{E}\mathrm{e}^{-j\omega t}\big(2j\sin[k_x 0] + 2j\sin[k_y y]\big)\hat{z} \\ &= \tilde{E}\mathrm{e}^{-j\omega t}\big(2j\sin[k_y y]\big)\hat{z} \\ &\neq 0\end{aligned}$$

[9] We will find that our k values can only take certain values in order to obey boundary conditions

[10] I think it's often not so clear that this ***is*** how we solve Maxwell's equations. We don't solve the 4 Maxwell differential equations for each situation. That would be crazy. We identify the form of one general solution. Then we identify what possible form that general solution takes for the physical case/setup we're given. Then we check the boundary conditions to see if there are any other additional constraints to it

Technically this question is not a great question, because the solution to this is not a simple superposition of the possible solutions. You have not been taught of any other options (beyond simple superposition). However, I think this question forces you to take a step back to appreciate how exactly we tackle the problem of solving Maxwell equations. If the superposition does not work, is there something else that can work? How can we check that it will work? How did we know that superposition of $\vec{E}$ could work anyway? (Answered in the next page)

[11] I assume that $\vec{E}$ is polarized in the z direction for this. You can have other answers to this question if you prefer. It's easier if they are all polarized in the z direction however

- Therefore, we notice that a simple superposition of the two waves does not work.
  It does not obey boundary condition
- If we want the boundary condition to work, we can guess that the following form might work:
  $\vec{E} \propto \tilde{E}\mathrm{e}^{-j\omega t}\big(\sin[k_x x]\sin[k_y y]\big)\hat{z}$
  Obviously if it is multiplied, the boundary condition for $x = 0$ will work for any y
- But it's not immediately clear to us that this sort of solution satisfy Maxwell's equations[12].
  Check that this solution is allowed on your own.
- Therefore, we conclude: $\vec{E} = \tilde{E}'\mathrm{e}^{-j\omega t}\Big(\sin[k_x x]\sin[k_y y]\Big)\hat{z}$ is a valid solution
  It immediately obeys the following two boundary conditions:

$$\vec{E}(0,y,z,t) = \tilde{E}'\mathrm{e}^{-j\omega t}\Big(\sin[k_x 0]\sin[k_y y]\Big)\hat{z} = 0$$
$$\vec{E}(x,0,z,t) = \tilde{E}'\mathrm{e}^{-j\omega t}\Big(\sin[k_x x]\sin[k_y 0]\Big)\hat{z} = 0$$

- Checking the other two boundary conditions:

$$\begin{aligned}\vec{E}(L_x,y,z,t) &= \tilde{E}'\mathrm{e}^{-j\omega t}\Big(\sin[k_x L_x]\sin[k_y y]\Big)\hat{z}\\ 0 &= \tilde{E}'\mathrm{e}^{-j\omega t}\Big(\sin[k_x L_x]\sin[k_y y]\Big)\hat{z}\end{aligned}$$

  For this to be true, $\sin[k_x L_x]$ must be 0

$$\begin{aligned}\sin[k_x L_x] &= 0\\ k_x L_x &= q_x \pi\\ k_x &= \frac{q_x \pi}{L_x} \qquad \text{where } q_x \text{ must be integers}\end{aligned}$$

  Similarly, demanding that $\vec{E}(x,L_y,z,t) = 0$, we find that $k_y = \dfrac{q_y \pi}{L_y}$

Answer:

- Standing wave solution: $\vec{E}(x,y,z,t) = \tilde{E}'\mathrm{e}^{-j\omega t}\left(\sin\left[\frac{q_x\pi}{L_x}x\right]\sin\left[\frac{q_y\pi}{L_y}y\right]\right)\hat{z}$
- Possible modes: q_x and q_y may be any positive integer

6. There is some misunderstanding with the terms here
   So let me rewrite this question according to the terms in our lecture notes.
   Our job: Prove that

$$\tilde{E}_r = \frac{n_1 - n_2}{n_1 + n_2}\tilde{E}_i$$

- Basically we are proving equation (44) in lecture 2 pg 21 for when $\theta_i = \theta_t = 0$
- We will chose $\vec{E}$ to be polarized in the x direction, and $\vec{B}$ to be polarized in the y direction[13]
- Our boundary between material 1 and material 2 is at $z = 0$
- We were told to consider the continuity conditions – for us, it's the boundary condition[14]

$$\begin{aligned}\vec{E}_1 &= \vec{E}_2 & \qquad \vec{B}_1 &= \vec{B}_2\\ \vec{E}_i + \vec{E}_r &= \vec{E}_t & \qquad \vec{B}_i + \vec{B}_r &= \vec{B}_t\end{aligned}$$

Basically the E in medium 1 must be equivalent to E in medium 2.
Therefore the addition of incident and reflected E must be equivalent to transmitted E

[12]Recall: How did we decide that a superposition (simple addition) of E fields ***is*** also a solution to Maxwell's equation? We did this for a general case in question 2 of tutorial 1. There we proved that if $\vec{E}_1$ and $\vec{E}_2$ are solutions to Maxwell's equations, the addition of both solutions are also solutions. That proof allows us to do the second step here. Remember our first step is a general form of $\vec{E}$ that solves Maxwell equations. In the second step, we add up the different possibilities that works for our scenario. We know this works because of question 2 in tutorial 1. To check that this new form of $\vec{E}$ in this question also solves Maxwell's equation, you should plug this solution into the LHS of the 4 different Maxwell equations and check if you can get the RHS. Alternatively, we can (and should since it's a lot easier) just plug it into the wave equation (since the Maxwell equation reduces to the wave equation when there are no external sources). I leave this to you. Refer to lecture 1 pg 15 if you're stuck. You should get the dispersion relationship after you plug this new $\vec{E}$ into the wave equation

[13]You can change this to $\vec{H}$ on your own. I'm doing this in $\vec{B}$ to make it slightly easier for me

[14]Lecture 2 pg 14. First two equations of eq (28)

Approach:

- We were told to write out equations for E_i, E_r, E_t, B_i, B_r and B_t
- Then we were told to consider the boundary conditions above
- Trigo manipulation
- Learning point: See for ourselves the derivation of eq (44) of lecture 2 for a simple case ($\theta_i = 0°$)

Solution

- Writing down the expressions. Note these are general equations. I chose sin, but you can use cos

$$\vec{E}_i = \tilde{E}_i \sin(k_1 z - \omega t)\hat{x} \qquad \vec{E}_t = \tilde{E}_t \sin(k_2 z - \omega t)\hat{x} \qquad \vec{E}_r = \tilde{E}_r \sin(k_1 z + \omega t)\hat{x}$$
$$\vec{B}_i = \tilde{B}_i \sin(k_1 z - \omega t)\hat{y} \qquad \vec{B}_t = \tilde{B}_t \sin(k_2 z - \omega t)\hat{y} \qquad \vec{B}_r = -\tilde{B}_r \sin(k_1 z + \omega t)\hat{y}$$

  Notice that there is a negative sign on my reflected B.
  This is to ensure that the Poynting vector of $\vec{E}_r \times \vec{B}_r$ points in the correct direction[15]
  Notice also when my k has a different subscript and when the ωt has a negative sign

- Boundary conditions

$$\vec{E}_i + \vec{E}_r = \vec{E}_t \qquad \vec{B}_i + \vec{B}_r = \vec{B}_t$$

  Dividing both equations[16]:

$$\begin{aligned}\frac{\vec{E}_i + \vec{E}_r}{\vec{B}_i + \vec{B}_r} &= \frac{\vec{E}_t}{\vec{B}_t}\\ \frac{\tilde{E}_i \sin(k_1 z - \omega t) + \tilde{E}_r \sin(k_1 z + \omega t)}{\tilde{B}_i \sin(k_1 z - \omega t) - \tilde{B}_r \sin(k_1 z + \omega t} &= \frac{\tilde{E}_t \sin(k_2 z - \omega t)}{\tilde{B}_t \sin(k_2 z - \omega t)}\end{aligned}$$

  Our end result does not have $\vec{B}$, therefore we need to use an equation to convert B to E: $\vec{E} = v\vec{B}$[17]

$$\begin{aligned}v_1 \frac{\tilde{E}_i \sin(k_1 z - \omega t) + \tilde{E}_r \sin(k_1 z + \omega t)}{\tilde{E}_i \sin(k_1 z - \omega t) - \tilde{E}_r \sin(k_1 z + \omega t)} &= v_2 \frac{\tilde{E}_t \sin(k_2 z - \omega t)}{\tilde{E}_t \sin(k_2 z - \omega t)}\\ \tilde{E}_r \sin(k_1 z + \omega t) &= \frac{v_2 - v_1}{v_1 + v_2} \tilde{E}_i \sin(k_1 z - \omega t)\\ \vec{E}_r &= \frac{v_2 - v_1}{v_1 + v_2} \vec{E}_i\\ &= \frac{n_1 - n_2}{n_1 + n_2} \vec{E}_i\end{aligned}$$

  where the last step was achieved by using the equation $n = \frac{c}{v}$

[15]Things like this you must notice. Alternatively, you can always be careful by writing out your $\vec{H}$ the long way as was done in question 3

[16]I don't think you will be tested on something like this. I understand it's not immediately obvious why you should divide the two...

[17]You should know how B relates to H, and how H relates to E

1. Obtaining the equation for Brewster's angle from the reflection and transmission coefficient in the lecture notes. Refer to pg 21 of lecture 2

   - Verifying this (as opposed to deriving it)
     - Note that for Brewster's angle, the reflected wave will be fully TE polarized
     - This means r_p should go to 0
     - From lecture notes:

$$\begin{aligned} r_p &= \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_2 \cos\theta_t} \\ 0 &= \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_2 \cos\theta_t} \\ n_2 \cos\theta_i &= n_1 \cos\theta_t \end{aligned}$$

     - Recall, that I pointed out in tutorial that Brester's angle occurs when the angle between the reflected and transmitted ray is 90°
       Therefore: $\theta_t = 90 - \theta_i$

$$\begin{aligned} n_2 \cos\theta_B &= n_1 \cos(90 - \theta_B) \\ &= n_1 \sin\theta_B \\ \tan\theta_B &= \frac{n_2}{n_1} \end{aligned}$$

       This may look different from the equation used in question 1.
       That's because n_1 is the incident material; n_2 is the transmitted material in the lecture notes
       In question 1, n_1 should be the transmitted material.

   - Deriving it
     Notice that in the above, I already assumed that $\theta_t = 90 - \theta_i$ for Brewster's angle to occur.
     Therefore if you want a more legitimate proof, come see me.
     I don't think you'll be required to prove this, so I won't bother to type this is. It's quite algebra heavy

2. Obtaining the generalized equation for an ellipse

   - Please note that this too is probably not testable material.
   - First, you should be able to write out a rotation matrix.[18]

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

   - Equation of the ellipse that was given to us: $\dfrac{E_x^2}{a^2} + \dfrac{E_y^2}{b^2} = 1$
   - Therefore E_x and E_y are coordinates of an ellipse along the x or y axis
   - Let E'_x and E'_y be the coordinates of a diagonal ellipse. We apply the rotation matrix to E'_x and E'_y to rotate that ellipse such that it lies along the x plane.
   - Because it now lies along the x plane, we know it should the E'_x is rotated to E_x and E'_y is rotated to E_y

$$\begin{aligned} \begin{pmatrix} E_x \\ E_y \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} E'_x \\ E'_y \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} E'_x \\ E'_y \end{pmatrix} \end{aligned}$$

     where I have assumed $\theta = 45°$ just to give me a constant value of $\dfrac{1}{\sqrt{2}}$. Let $\dfrac{1}{\sqrt{2}} \equiv c$

$$\begin{aligned} E_x &= cE'_x - cE'_y & E_y &= cE'_x + cE'_y \\ E_x^2 &= c^2 E'^2_x - 2cE'_x E'_y + c^2 E_y^2 & E_y^2 &= c^2 E'^2_x + 2cE'_x E'_y + c^2 E'^2_y \end{aligned}$$

     Therefore, placing them back into the ellipse equation,

$$\frac{c^2 E_x^2 - 2cE_x E_y + c^2 E_y^2}{a^2} + \frac{c^2 E_x^2 + 2cE_x E_y + c^2 E_y^2}{b^2} = 1$$

     The equation above can be rearranged to take the form of $AE_x^2 + BE_x E_y + CE_y^2 = 1$

[18] Wiki it if you can't