PC2232 – Assignment 1 Solutions

1. (a) Given: Material with real permittivity $\varepsilon(\omega)$

   i. Express $n(\omega)$ in terms of $\varepsilon(\omega)$ and any other constants

   Approach:
   - Know the speed of light in vacuum
   - Know the how speed of light in a material changes due to the introduction of $\varepsilon(\omega)$
   - Know the definition of n

   Learning point:
   Familiarize yourself with all the equations and definitions that you should know
   This portion is just a guide to assist you in solving the last part of this question

   Solution:
   - Speed of light in vacuum: $c = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}}$
   - Speed of light in material due to : $v = \dfrac{1}{\sqrt{\varepsilon(\omega)\mu_0}}$
   - Definition of $n = \dfrac{c}{v}$

   Answer:

   $$n(\omega) = \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}$$

   ii. Explain why ω is constant when light passes through from vacuum into a material with refractive index $n(\omega)$

   Solution & Answer:
   Please note that this is just a suggested answer, and is not the only acceptable answer
   - At the boundary $\vec{E}$ must be continuous at all times
   - If $\vec{E}$ is continuous at all times, it is not possible for an 'extra' wave to suddenly appear at any point in time[1]
   - Therefore, the wavelength changes to match the speed in the new medium

   iii. Consider the plane wave solution to our Maxwell's equation in vacuum[2].:

   $$|\vec{E}_0| = \tilde{E} e^{j\vec{k}_0 \cdot \vec{r}} e^{-j\omega t} \tag{1}$$

   For a plane wave solution in a material of refractive index $n(\omega)$, express the following quantities in terms of $n(\omega)$[3]:

   A. $|\vec{E}|$
   B. $|\vec{H}|$
   C. $|\vec{\bar{S}}|$

   Approach:
   - Know how k changes in a material of refractive index $n(\omega)$
   - Understand that Z, and therefore $\tilde{H}$ is affected by the material change
     $Z = \sqrt{\dfrac{\mu_0}{\varepsilon(\omega)}} = \sqrt{\dfrac{\mu_0}{n(\omega)^2 \varepsilon_0}} = \dfrac{Z_0}{n(\omega)}$
   - Therefore, determine how $\bar{S}$ is affected by the medium change

   Learning point:
   - The whole point of all the parts of of question 1(a) is to guide you to obtain the average intensity of the EM wave in a medium[4]. Notice that there is a factor of n as compared to the equation of S in vacuum.

[1] If you don't really see this, consider this: At every single point in time, the wave that leaves one material enters the next. Therefore, when one cycle leaves the first material, only one cycle can enter the second material because the wave must be continuous at all times. It is just not possible for an 'extra' wave to appear.
[2] k_0 is the value of k in vacuum
[3] And any other quantity already given in the plane wave solution above
[4] Some students were confused about the magnitude sign. I added the magnitude sign in hopes that it would make it easier for you. In this way, you won't have to worry about the direction (unit vector) of E and H and S. I apologize for the confusion this may have caused.

- Notice also that the amplitude of the E field changes as it passes through from vacuum to another medium. Since we do not know the transmission coefficients here, we can just denote this new amplitude as $\tilde{E}'$

Solution & Answer:

$$|\vec{E}| = \tilde{E}'\mathrm{e}^{jn(\omega)\vec{k}_0\cdot\vec{r}}\mathrm{e}^{-j\omega t}$$

$$|\vec{H}| = \frac{\tilde{E}'}{Z}\mathrm{e}^{jn(\omega)\vec{k}_0\cdot\vec{r}}\mathrm{e}^{-j\omega t} = \frac{n(\omega)\tilde{E}'}{Z_0}\mathrm{e}^{jn(\omega)\vec{k}_0\cdot\vec{r}}\mathrm{e}^{-j\omega t}$$

$$|\vec{S}| = \frac{1}{2}\mathrm{Re}[E'H^*] = \frac{1}{2}\frac{n(\omega)\tilde{E}'^2}{Z_0}\left(\mathrm{e}^{jn(\omega)\vec{k}_0\cdot\vec{r}}\mathrm{e}^{-j\omega t}\times\mathrm{e}^{-jn(\omega)\vec{k}_0\cdot\vec{r}}\mathrm{e}^{j\omega t}\right) = \frac{1}{2}\frac{n(\omega)\tilde{E}'^2}{Z_0} \neq \frac{1}{2}n(\omega)|S_0|$$

(b) Given a material with complex permittivity. Therefore, k and n are also complex:

$$k = k' + jk'' \qquad\qquad n = n' + jn''$$

Show that this leads to power loss along the direction of propagation.
For simplicity, assume your wave propagates in the z direction.

Approach:

- Notice that substituting in complex k returns real exponential
- Understand therefore that when multiplying the conjugate of H, the real exponential does not cancel out like the complex exponential
- Understand that the real exponential decay causes power loss

Learning point:

Understanding how power loss comes about in materials

Solution:

$$\begin{aligned}
|E| &= \tilde{E}\mathrm{e}^{j(k'+k'')z}\mathrm{e}^{-j\omega t} = \tilde{E}'\mathrm{e}^{jk'z}\mathrm{e}^{-jk''z}\mathrm{e}^{-j\omega t}\\
|H| &= \frac{(n'+jn'')\tilde{E}}{Z_0}\mathrm{e}^{jk'z}\mathrm{e}^{-k''z}\mathrm{e}^{-j\omega t}\\
|\bar{S}| &= \frac{1}{2}\mathrm{Re}\big[|E||H|^*\big]\\
&= \frac{1}{2}\frac{\tilde{E}^2}{Z_0}\mathrm{Re}\left[\mathrm{e}^{jk'z}\mathrm{e}^{-k''z}\mathrm{e}^{-j\omega t}\times(n+jn'')\mathrm{e}^{-jk'z}\mathrm{e}^{-k''z}\mathrm{e}^{j\omega t}\right]\\
&= \frac{1}{2}\frac{\tilde{E}^2}{Z_0}n'\mathrm{e}^{-2k''z}
\end{aligned}$$

Answer:

$$|\bar{S}| = \frac{1}{2}\frac{\tilde{E}^2}{Z_0}n'\mathrm{e}^{-2k''z}$$

The real exponential power decay shows that power will be lost

(c) Given that the loss coefficient is defined as:

$$\alpha = -\frac{1}{z}\left(10\log_{10}\left[\frac{\bar{S}(z)}{\bar{S}(0)}\right]\right)$$

Show that for the complex k above, we will obtain the following equation

$$\alpha = 20k''\log_{10}\mathrm{e}$$

Approach:

Just substitute in results from previous part to obtain final result

Solution & Answer:

$$\alpha = -\frac{1}{z}10\log_{10}\left[\frac{\bar{S}(z)}{\bar{S}(0)}\right] = -\frac{1}{z}10\log_{10}\left[\mathrm{e}^{-2k''z}\right] = 20k''\log_{10}\mathrm{e}$$

(d) Given: $k'' = 0.04\mathrm{km}^{-1}$
Determine $\dfrac{S(z)}{S(0)}$ after $z = 5\mathrm{km}$

Approach:

Just calculate

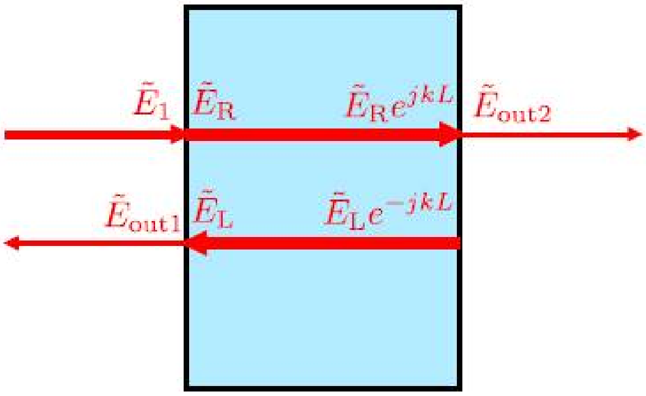
Solution:

$$20k'' \log_{10} \mathrm{e} = -\frac{10}{z} \log_{10} \left[\frac{\bar{S}(z)}{\bar{S}(0)}\right]$$

Answer:

$$\frac{\bar{S}(z)}{\bar{S}(0)} = 0.6703$$

2. A Fabry-Perot interferometer basically consists of a dielectric slab (refractive index of n) in free space. It works on the basis of multiple reflections[5].



(a) Work out the scattering matrix for the first interface if we have a TM input with normal incident. You are allowed to use equations from the lecture notes. State clearly where the equations you use come from.

Approach:

- Understand that the scattering matrix comes from the reflection and transmission coefficients in lecture 2[6]
- Work out using the TM equations for normal incidence: $\theta_i = \theta_t = 0°$

Learning point:

To understand that the scattering matrix is made up of reflection and transmission coefficients
To understand the notation in the scattering matrix[7]

Solution:

- From the lecture notes,

$$r_p = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t} \qquad t_p = \frac{2n_1 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t}$$

- Meaning of each term
  - s_{11} is r_p where $n_1 = 1$ and $n_2 = n$
  - s_{12} is t_p where $n_1 = n$ and $n_2 = 1$
  - s_{21} is t_p where $n_1 = 1$ and $n_2 = n$
  - s_{22} is r_p where $n_1 = n$ and $n_2 = 1$

Answer:

$$\begin{pmatrix} \frac{n-1}{n+1} & \frac{2n}{n+1} \\ \frac{2}{n+1} & \frac{1-n}{1+n} \end{pmatrix}$$

[5]Refer to pg 15 of lecture 4

[6]Some students pointed out that at normal incidence, the plane of incidence is not defined, therefore TE and TM waves are physically identical. This is true. We used TM waves here just so you'd have to go back to lecture 2 to obtain the equation rather than copy from lecture notes in lecture 4. Although the TE and TM waves are physically identical, the scattering matrix looks slightly different. This arises from the definition of the unit vector for our reflected TE and TM wave. This definition is probably not in the scope of our course, however. This footnote is just extra information for those who are interested

[7]That is, to understand what the subscripts mean

(b) Given:

$$E_{\text{out}_2} = s_{12}s_{21}\mathrm{e}^{jkL}\sum_{m}^{\infty}\left(s_{22}^2\mathrm{e}^{2jkL}\right)^m \tilde{E}_{\text{in}}$$

Using the identity for the infinite geometric series (given in lecture notes pg 20, lecture 4), obtain the final expression for E_{out_2}

Approach:
Just use the identity and substitute it in

Learning point:
Basically to work through the lecture notes to reach the final form of the E_{out_2}

Solution & Answer:
From $\sum_{m=0}^{\infty} a^m = \dfrac{1}{1-a}$
Therefore:

$$E_{\text{out}_2} = s_{12}s_{21}\mathrm{e}^{jkL}\sum_{m}^{\infty}\left(s_{22}^2\mathrm{e}^{2jkL}\right)^m \tilde{E}_{\text{in}} = \frac{s_{12}s_{21}\mathrm{e}^{jkL}}{1-s_{22}^2\mathrm{e}^{2jkL}}\tilde{E}_{\text{in}}$$

(c) For $n = 4$, obtain the conditions for L for minimum transmission. Give your answer in terms of wavelength of light in free space (λ_0)

Approach:

- Intensity is proportional to $\text{Re}[EE^*]$
- Work out the $|E_{\text{out}_2}|^2 = E_{\text{out}_2}E^*_{\text{out}_2}$
- Understand that for minimum transmission, denominator should be maximum and numerator minimum
- Work out the condition needed

Learning Point:
To notice that the method to obtain maxima here is the same as the ones covered in Lect. 3
To work out and obtain the condition of L

Solution:

$$\begin{aligned}
|E_{\text{out}_2}|^2 &= E_{\text{out}_2}E^*_{\text{out}_2} \\
&= \left(\frac{s_{12}s_{21}\mathrm{e}^{jkL}}{1-s_{22}^2\mathrm{e}^{2jkL}}\tilde{E}_{\text{in}}\right)\left(\frac{s_{12}s_{21}\mathrm{e}^{-jkL}}{1-s_{22}^2\mathrm{e}^{-2jkL}}\tilde{E}_{\text{in}}\right) \\
&= \frac{s_{12}^2 s_{21}^2}{(1-s_{22}^2\mathrm{e}^{2jkL})(1-s_{22}^2\mathrm{e}^{-2jkL})}\tilde{E}_{\text{in}}^2 \\
&= \frac{s_{12}^2 s_{21}^2}{1+s_{22}^4-s_{22}^2\mathrm{e}^{2jkL}-s_{22}^2\mathrm{e}^{-2jkL}}\tilde{E}_{\text{in}}^2
\end{aligned}$$

Therefore,

$$\text{Re}[E_{\text{out}_2}E^*_{\text{out}_2}] = \frac{s_{12}^2 s_{21}^2}{1+s_{22}^2-2s_{22}^2\cos(2kL)}\tilde{E}_{\text{in}}^2$$

And hence, minimum transmission is obtained when $\cos(2kL) = -1$

$$\cos(2kL) = -1 \qquad 2kL = (2q+1)\pi \qquad \text{where } q \text{ goes from } 0,\ 1,\ 2...$$

Therefore

$$L = \frac{(2q+1)\pi}{2k} = \frac{2q+1}{4}\lambda = \frac{2q+1}{4n}\lambda_0$$

Answer:

$$L = \frac{(2q+1)\lambda_0}{16}$$

(d) Given that the refractive index of the medium inside a Fabry-Perot interferometer is complex such that $k = k' + jk''$. Show that it is now possible to end up with a non-converging sum for E_{out_2}.

Approach:

Know that the geometric series only converges if $|a \leq 1|$

Solution & Answer:
Therefore, the above result will only converge if the following holds true: $s_{22}^2 \mathrm{e}^{-k''L} \leq 1$

Hence, if $\mathrm{e}^{-k''L} > \dfrac{1}{s_{22}^2}$, we will end up with a non-converging sum