PC2232 Physics for Electrical Engineers
Lecture 4: Input-Output Analysis and Interferometers

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Intensity Calculation for Complex Fields

- Another way of writing the real part of a complex field:

$$\mathrm{Re}\,\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{E} + \boldsymbol{E}^*\right), \qquad \mathrm{Re}\,\boldsymbol{H} = \frac{1}{2}\left(\boldsymbol{H} + \boldsymbol{H}^*\right). \tag{1}$$

In terms of the complex fields,

$$\mathrm{Re}\,\boldsymbol{E} \times \mathrm{Re}\,\boldsymbol{H} = \frac{1}{4}\left(\boldsymbol{E} + \boldsymbol{E}^*\right) \times \left(\boldsymbol{H} + \boldsymbol{H}^*\right) \tag{2}$$

$$= \frac{1}{4}\left(\boldsymbol{E} \times \boldsymbol{H} + \boldsymbol{E}^* \times \boldsymbol{H}^* + \boldsymbol{E} \times \boldsymbol{H}^* + \boldsymbol{E}^* \times \boldsymbol{H}\right). \tag{3}$$

- Note that $\boldsymbol{E}$ and $\boldsymbol{H}$ are proportional to $\exp(-j\omega t)$, and $\boldsymbol{E}^*$ and $\boldsymbol{H}^* \propto \exp(j\omega t)$. When averaging over time,

$$\overline{\boldsymbol{E} \times \boldsymbol{H}} \propto \overline{e^{-2j\omega t}} = 0, \tag{4}$$

similarly for $\boldsymbol{E}^* \times \boldsymbol{H}^*$.

- On the other hand, $\boldsymbol{E} \times \boldsymbol{H}^* \propto e^{-j\omega t}e^{j\omega t}$ doesn't depend on time.
- The time-averaged Poynting vector is hence

$$\boxed{\bar{\boldsymbol{S}} = \overline{\mathrm{Re}\,\boldsymbol{E} \times \mathrm{Re}\,\boldsymbol{H}} = \frac{1}{4}\left(\boldsymbol{E} \times \boldsymbol{H}^* + \boldsymbol{E}^* \times \boldsymbol{H}\right) = \frac{1}{2}\mathrm{Re}\left(\boldsymbol{E} \times \boldsymbol{H}^*\right).} \tag{5}$$

Plane-Wave Intensity

- As a warm-up exercise, consider a plane wave:

$$\boldsymbol{E} = \hat{\boldsymbol{x}}\tilde{E}\exp(jkz - j\omega t), \quad \boldsymbol{H} = \hat{\boldsymbol{y}}\frac{\tilde{E}}{Z_0}\exp(jkz - j\omega t), \tag{6}$$

$$\boldsymbol{E} \times \boldsymbol{H}^* = \hat{\boldsymbol{z}}\frac{|\tilde{E}|^2}{Z_0}, \quad \mathrm{Re}\left(\boldsymbol{E} \times \boldsymbol{H}^*\right) = \hat{\boldsymbol{z}}\frac{|\tilde{E}|^2}{Z_0}, \tag{7}$$

$$\bar{\boldsymbol{S}} = \frac{1}{2}\mathrm{Re}\left(\boldsymbol{E} \times \boldsymbol{H}^*\right) = \hat{\boldsymbol{z}}\frac{|\tilde{E}|^2}{2Z_0}, \tag{8}$$

$$I = |\bar{\boldsymbol{S}}| = \frac{|\tilde{E}|^2}{2Z_0}. \tag{9}$$

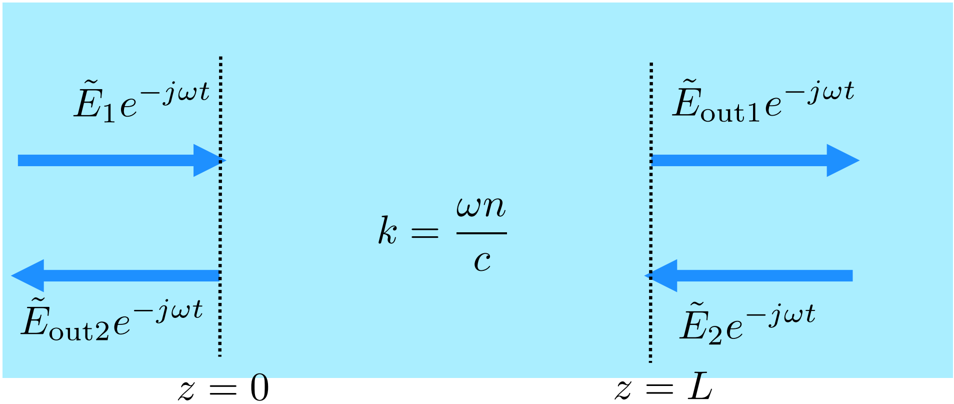
- The intensity is proportional to $|\tilde{E}|^2$, the **squared magnitude** of the complex amplitude $\tilde{E}$.
- For a plane wave in a dielectric with refractive index n,

$$\boldsymbol{E} = \hat{\boldsymbol{x}}\tilde{E}\exp(jkz - j\omega t), \quad \boldsymbol{H} = \hat{\boldsymbol{y}}\frac{\tilde{E}}{Z}\exp(jkz - j\omega t), \quad Z = \sqrt{\frac{\mu_0}{\epsilon}} = \frac{Z_0}{n}, \tag{10}$$

$$I = |\bar{\boldsymbol{S}}| = \frac{|\tilde{E}|^2}{2Z} = \frac{n|\tilde{E}|^2}{2Z_0}. \tag{11}$$

Intensity is now proportional to refractive index n.

Input and Output of Propagation



- Throughout this lecture, assume $\hat{\boldsymbol{x}}$-polarized light (electric field perpendicular to screen).
- For the right-propagating wave, we know the solution is $\tilde{E}\exp(jkz - j\omega t)$. If the input signal at $z = 0$ is $\tilde{E}_1 e^{-j\omega t}$, $\tilde{E} = \tilde{E}_1$, and the output signal at $z = L$ is $\tilde{E}\exp(jkL - j\omega t) = \tilde{E}_{\text{out}1} e^{-j\omega t}$. The output amplitude is then

$$\tilde{E}_{\text{out}1} = \tilde{E}_1 e^{jkL}. \tag{12}$$

  The output signal acquires a **phase** with respect to the input due to time delay.
- For the left-propagating wave, we know the solution is $\tilde{E}\exp(-jkz - j\omega t)$. At $z = L$, the input signal is $\tilde{E}_2 e^{-j\omega t}$, so we can substitute $\tilde{E}e^{-jkL} = \tilde{E}_2$. At $z = 0$, the signal is $\tilde{E}\exp(-j\omega t) = \tilde{E}_2 e^{jkL} e^{-j\omega t}$, so the output signal is

$$\tilde{E}_{\text{out}2} = \tilde{E}_2 e^{jkL}. \tag{13}$$

  The acquired phase due to time delay is the same.

Partial Mirror

- For a glass coated with a very thin layer of metal (e.g., aluminum or silver), the reflection is not perfect and some is transmitted.
- Suppose there is just input from room 1, no input from room 2:

$$\text{from room 1: } E_{\text{in}1} = \tilde{E}_1 \exp(jkz - j\omega t). \tag{14}$$

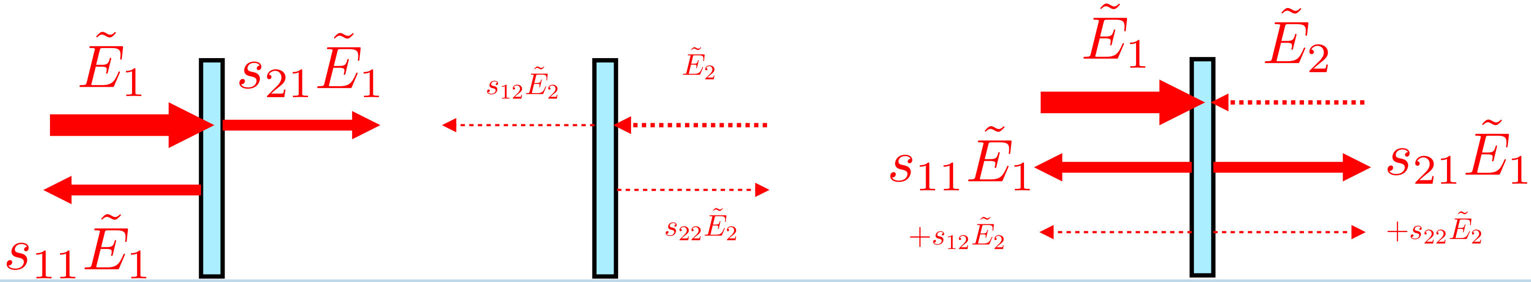
  In room 1: the reflected wave is $E_{\text{out}1} = s_{11}\tilde{E}_1 \exp(-jkz - j\omega t)$, where s_{11} is the reflection coefficient.
- In room 2, the transmitted wave is $E_{\text{out}2} = s_{21}\tilde{E}_1 \exp(jkz - j\omega t)$. Now if there is just input from room 2, no input from room 2:

$$\text{from room 2: } E_{\text{in}2} = \tilde{E}_2 \exp(-jkz - j\omega t), \tag{15}$$

$$\text{transmitted wave in room 2: } E_{\text{out}1} = s_{12}\tilde{E}_2 \exp(-jkz - j\omega t), \tag{16}$$

$$\text{reflected wave in room 2: } E_{\text{out}2} = s_{22}\tilde{E}_2 \exp(jkz - j\omega t). \tag{17}$$

- Focusing on the amplitudes of the input and output waves:



One-Way Mirror

- What if there are inputs from both rooms? Superposition:

$$E_{\text{out1}} = \tilde{E}_{\text{out1}} \exp(-jkz - j\omega t), \qquad \boxed{\tilde{E}_{\text{out1}} = s_{11}\tilde{E}_1 + s_{12}\tilde{E}_2,} \tag{18}$$

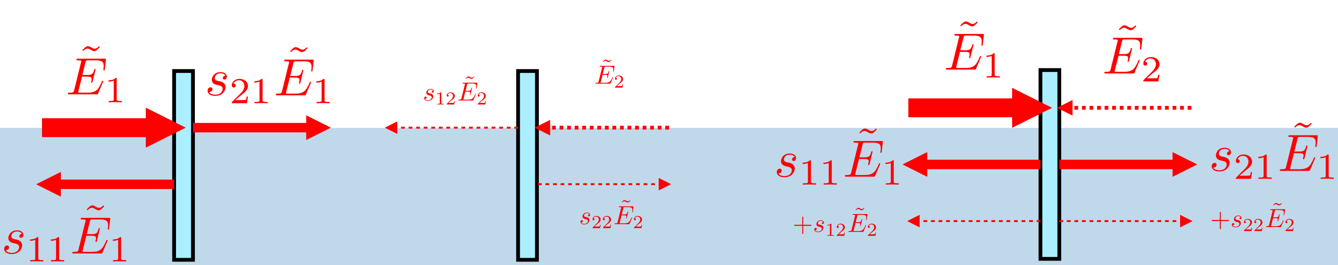
$$E_{\text{out2}} = \tilde{E}_{\text{out2}} \exp(jkz - j\omega t), \qquad \boxed{\tilde{E}_{\text{out2}} = s_{21}\tilde{E}_1 + s_{22}\tilde{E}_2.} \tag{19}$$

- Usually $|s_{12}| = |s_{21}|$, $|s_{11}| = |s_{22}|$.
- Let's say $s_{11} = 1/\sqrt{2}$ $s_{22} = -1/\sqrt{2}$, $s_{12} = s_{21} = 1/\sqrt{2}$. In terms of power, there is 50% reflection and 50% transmission.
- If room 1 is much brighter, $|\tilde{E}_1| \gg |\tilde{E}_2|$. Then

$$E_{\text{out1}} \approx s_{11}\tilde{E}_1 \exp(-jkz - j\omega t), \tag{20}$$

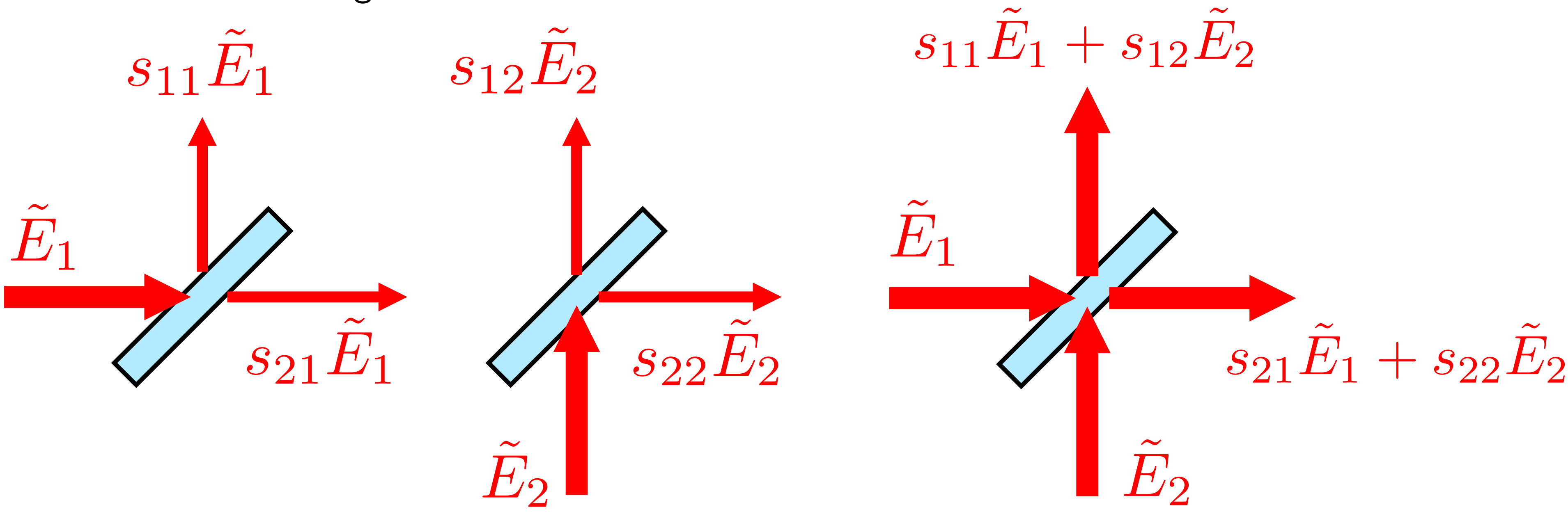
$$E_{\text{out2}} \approx s_{21}\tilde{E}_1 \exp(jkz - j\omega t). \tag{21}$$

In the bright room (1), you see mostly the reflection and not much transmission from the other room, but in the dark room, you see mostly the transmission from room 1 but not much from the reflection.



Beam Splitter

- Partial mirror at an angle:



$$\boxed{\begin{pmatrix} \tilde{E}_{\mathrm{out}1} \\ \tilde{E}_{\mathrm{out}2} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} \tilde{E}_1 \\ \tilde{E}_2 \end{pmatrix}.} \tag{22}$$

Power Conservation

- For an ideal beam splitter that does not absorb any energy, the output power should be equal to the input power. Since power $\propto$ intensity $\propto$ squared magnitude of wave amplitude,

$$\left|\tilde{E}_{\mathrm{out1}}\right|^2 + \left|\tilde{E}_{\mathrm{out2}}\right|^2 = \left|\tilde{E}_1\right|^2 + \left|\tilde{E}_2\right|^2 \tag{23}$$

for any $\tilde{E}_1$ and $\tilde{E}_2$.

- For example, a 50-50 beam splitter splits power equally. If there is just one nonzero input $\tilde{E}_1$,

$$|\tilde{E}_{\mathrm{out1}}|^2 = \frac{1}{2}\left|\tilde{E}_1\right|^2, \qquad |\tilde{E}_{\mathrm{out2}}|^2 = \frac{1}{2}\left|\tilde{E}_1\right|^2. \tag{24}$$

Similarly, if there is just input $\tilde{E}_2$:

$$|\tilde{E}_{\mathrm{out1}}|^2 = \frac{1}{2}\left|\tilde{E}_2\right|^2, \qquad |\tilde{E}_{\mathrm{out2}}|^2 = \frac{1}{2}\left|\tilde{E}_2\right|^2. \tag{25}$$

- This is an example of an ideal 50-50 beam splitter:

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}. \tag{26}$$

- Why does s_{22} have the minus sign?

Power-Conserving Input-Output Relations

- Write

$$\begin{pmatrix} \tilde{E}_{\text{out}1} \\ \tilde{E}_{\text{out}2} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} \tilde{E}_1 \\ \tilde{E}_2 \end{pmatrix}, \qquad \tilde{\boldsymbol{E}}_{\text{out}} = \boldsymbol{s}\tilde{\boldsymbol{E}}. \tag{27}$$

  $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{E}}_{\text{out}}$ are column vectors, $\boldsymbol{s}$ is a **scattering matrix**.
- In matrix form, the total output power can be written as the product of a column vector with **its complex transpose** (†):

$$\left|\tilde{E}_{\text{out}1}\right|^2 + \left|\tilde{E}_{\text{out}2}\right|^2 = \begin{pmatrix} \tilde{E}^*_{\text{out}1} & \tilde{E}^*_{\text{out}2} \end{pmatrix} \begin{pmatrix} \tilde{E}_{\text{out}1} \\ \tilde{E}_{\text{out}2} \end{pmatrix} = \tilde{\boldsymbol{E}}^\dagger_{\text{out}} \tilde{\boldsymbol{E}}_{\text{out}} = \tilde{\boldsymbol{E}}^\dagger \tilde{\boldsymbol{E}}. \tag{28}$$

- The complex transpose of $\tilde{\boldsymbol{E}}_{\text{out}}$ is

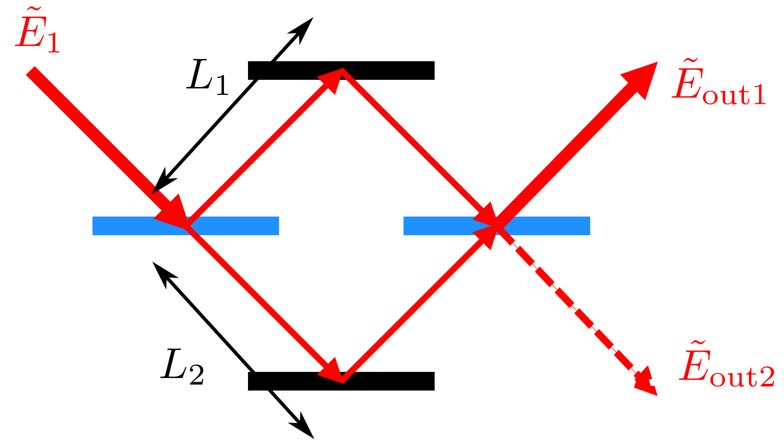
$$\tilde{\boldsymbol{E}}^\dagger_{\text{out}} = \tilde{\boldsymbol{E}}^\dagger \boldsymbol{s}^\dagger = \begin{pmatrix} \tilde{E}^*_1 & \tilde{E}^*_2 \end{pmatrix} \begin{pmatrix} s^*_{11} & s^*_{21} \\ s^*_{12} & s^*_{22} \end{pmatrix}, \tag{29}$$

$$\tilde{\boldsymbol{E}}^\dagger_{\text{out}} \tilde{\boldsymbol{E}}_{\text{out}} = \tilde{\boldsymbol{E}}^\dagger \boldsymbol{s}^\dagger \boldsymbol{s} \tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}^\dagger \tilde{\boldsymbol{E}}, \quad \tilde{\boldsymbol{E}}^\dagger \left(\boldsymbol{s}^\dagger \boldsymbol{s} - \boldsymbol{I}\right) \tilde{\boldsymbol{E}} = 0, \quad \boldsymbol{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{30}$$

  where $\boldsymbol{I}$ is the identity matrix.
- For this relation to hold for any $\tilde{\boldsymbol{E}}$, we require $\boxed{\boldsymbol{s}^\dagger \boldsymbol{s} = \boldsymbol{I}}$. This is the definition of a **unitary matrix** ($\boldsymbol{s}^{-1} = \boldsymbol{s}^\dagger$).

$$\begin{pmatrix} s^*_{11} & s^*_{21} \\ s^*_{12} & s^*_{22} \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{31}$$

Mach-Zehnder Interferometer



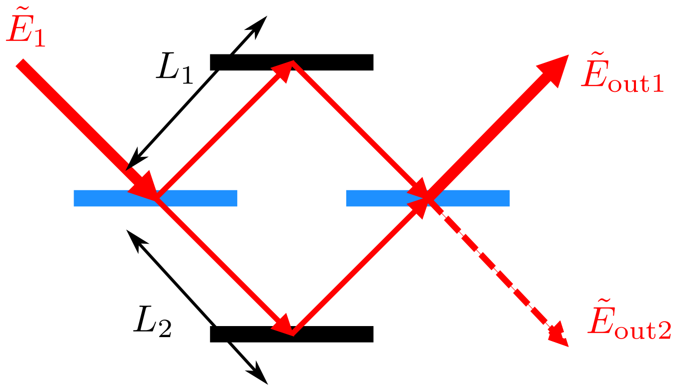
- Consider the setup with two identical beam splitters (blue) with reflection and transmission coefficients given by and two perfect mirrors (reflection coefficient $= 1$). Given an input $\tilde{E}_1 \exp(-j\omega t)$, what are the amplitudes at the final output ports?
- Suppose that the upper arm has a total length of $2L_1$. The input amplitude for the second beam splitter from the upper arm is $e^{2jkL_1} s_{11} \tilde{E}_1$.
- Suppose that the lower arm also has a total length of $2L_2$. The input amplitude for the second beam splitter from the lower arm is $e^{2jkL_2} s_{21} \tilde{E}_1$.
- The upper output amplitude for the second beam splitter is

$$\tilde{E}_{\text{out1}} = s_{11} e^{2jkL_1} s_{11} \tilde{E}_1 + s_{12} e^{2jkL_2} s_{21} \tilde{E}_1. \tag{32}$$

- Similarly,

$$\tilde{E}_{\text{out2}} = s_{21} e^{2jkL_1} s_{11} \tilde{E}_1 + s_{22} e^{2jkL_2} s_{21} \tilde{E}_1. \tag{33}$$

Destructive and Constructive Interference



- Suppose that the two arms have identical lengths $2L_1 = 2L_2 = 2L$, and the beam splitters are 50-50 beam splitters:

$$\left(\begin{array}{cc} s_{11} & s_{12} \\ s_{21} & s_{22} \end{array} \right) = \left(\begin{array}{cc} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{array} \right). \tag{34}$$

  The first output amplitude is

$$\tilde{E}_{\rm out1} = \left(\frac{1}{\sqrt{2}} e^{2jkL} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} e^{2jkL} \frac{1}{\sqrt{2}} \right) \tilde{E}_1 = e^{jkL} \tilde{E}_1. \tag{35}$$

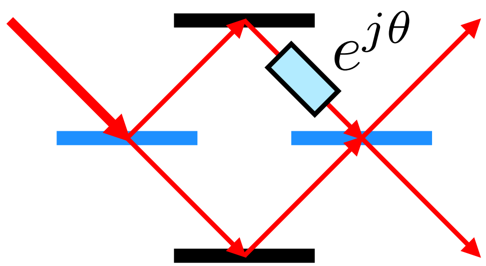
  All power appears in the first output! This is **constructive interference**.

- The second output amplitude is

$$\tilde{E}_{\rm out2} = \left(\frac{1}{\sqrt{2}} e^{2jkL} \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} e^{2jkL} \frac{1}{\sqrt{2}} \right) \tilde{E}_1 = 0. \tag{36}$$

The waves from upper and lower arms cancel each other. This is **destructive interference**.

Optical Switch



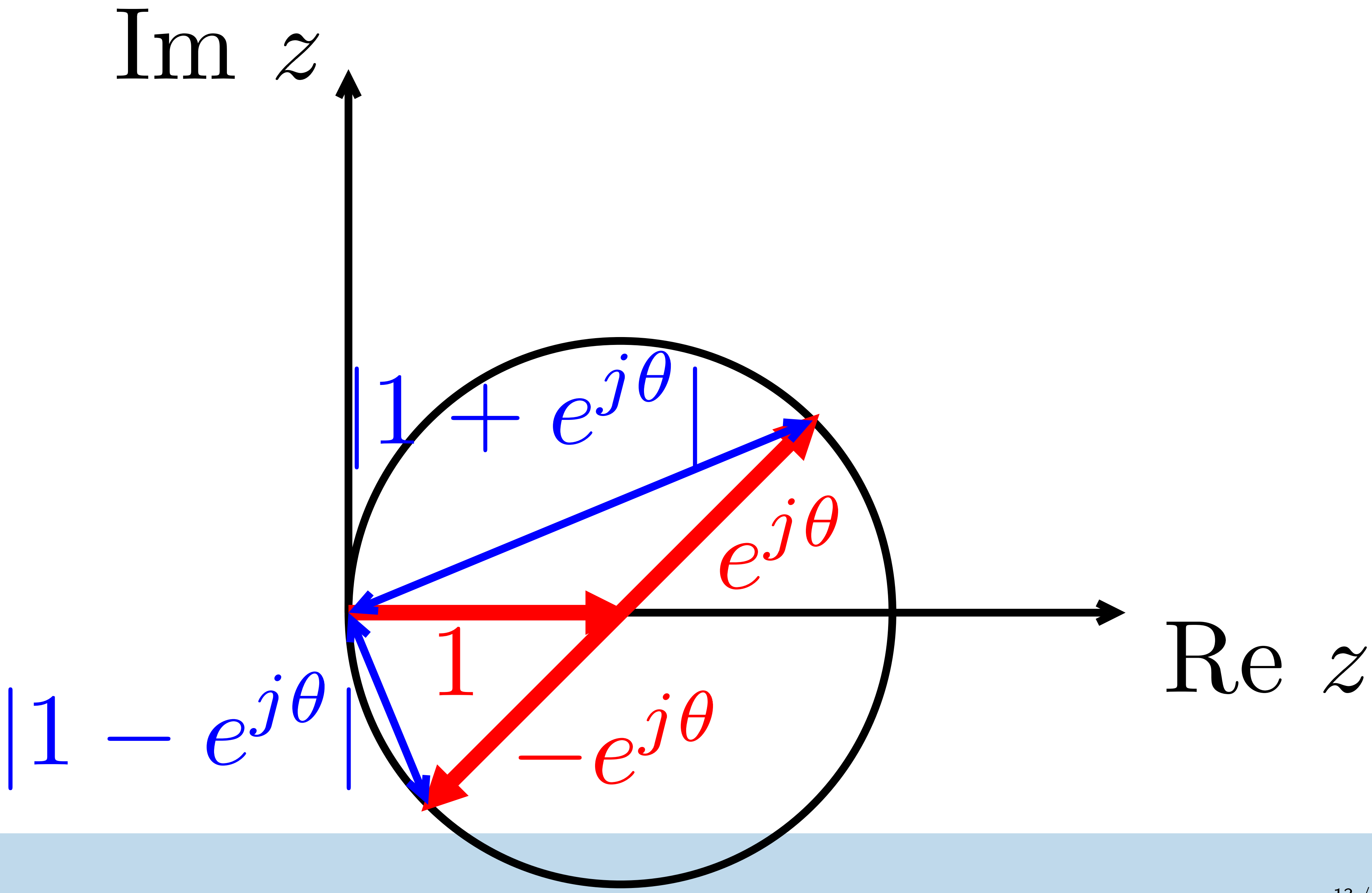
- Suppose there is an **additional phase shift** θ in the upper arm (e.g., change the length, change the refractive index, or add a phase modulator).

$$\tilde{E}_{\text{out1}} = \frac{1}{2}\left(e^{2jkL+j\theta} + e^{2jkL}\right)\tilde{E}_1, \qquad \tilde{E}_{\text{out2}} = \frac{1}{2}\left(e^{2jkL+j\theta} - e^{2jkL}\right)\tilde{E}_1. \tag{37}$$

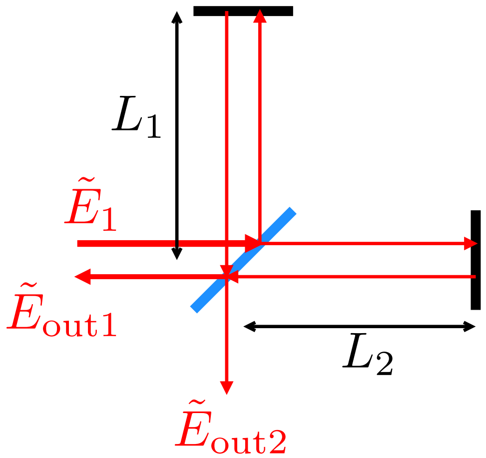
  Take the common factor e^{2jkL} out of the bracket, and consider the power $\propto |\tilde{E}|^2$,

$$\boxed{\left|\tilde{E}_{\text{out1}}\right|^2 = \frac{1}{4}\left|e^{j\theta}+1\right|^2\left|\tilde{E}_1\right|^2 = \cos^2\frac{\theta}{2}\left|\tilde{E}_1\right|^2,} \quad \boxed{\left|\tilde{E}_{\text{out2}}\right|^2 = \frac{1}{4}\left|e^{j\theta}-1\right|^2\left|\tilde{E}_1\right|^2 = \sin^2\frac{\theta}{2}\left|\tilde{E}_1\right|^2.} \tag{38}$$

  For $\theta = 0$, all power goes to the first port, but the outputs can be controlled by changing θ. When $\theta = \pi$, all power goes to the second port instead.
- Check that the power is conserved.
- Applications:
  - Optical switch: Route an optical signal to either output port by controlling θ.
  - Sensing: by measuring the intensity of either output port, an unknown phase shift θ can be inferred from the intensity.


Im z
Re z
$|1 + e^{j\theta}|$
$e^{j\theta}$
1
$|1 - e^{j\theta}|$
$-e^{j\theta}$

Michelson Interferometer



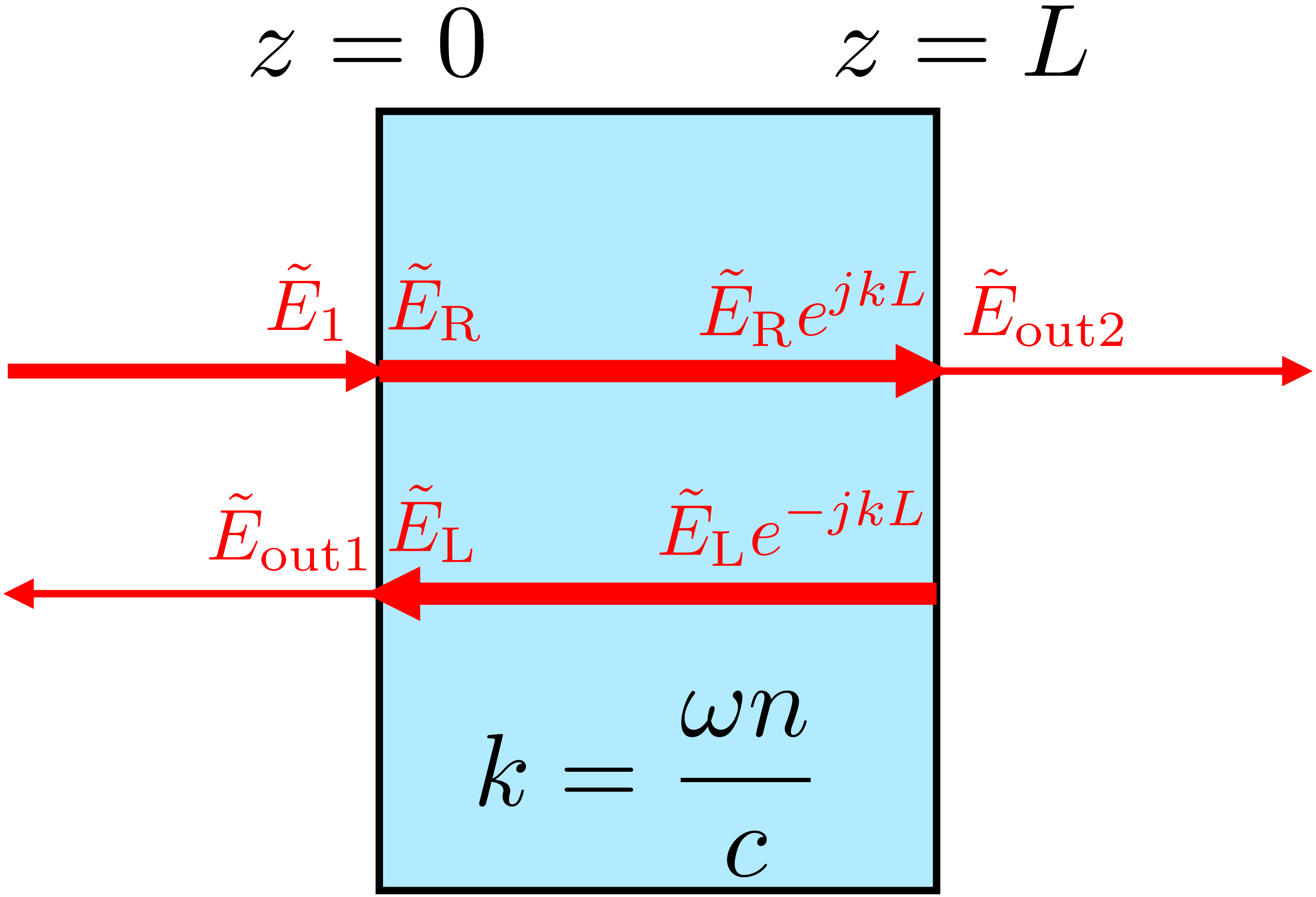
- This is simply a folded Mach-Zehnder interferometer. The input-output relations are the same:

$$\tilde{E}_{\mathrm{out1}} = \frac{1}{2}\left(e^{2jkL_1} + e^{2jkL_2}\right)\tilde{E}_1, \qquad \tilde{E}_{\mathrm{out2}} = \frac{1}{2}\left(e^{2jkL_1} - e^{2jkL_2}\right)\tilde{E}_1. \tag{39}$$

- The folded geometry makes it easier to change the relative lengths of the two arms.
- The phase difference between the two paths is $\theta = 2kL_1 - 2kL_2$. In terms of intensity,

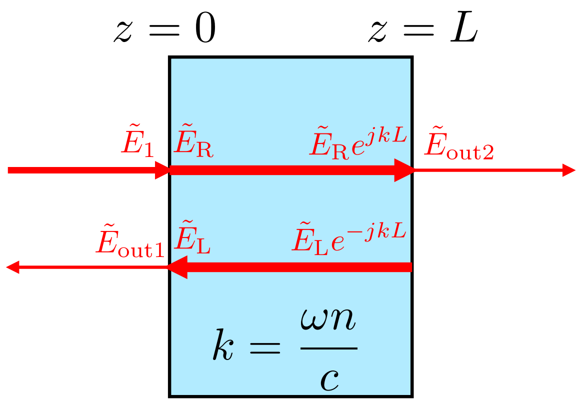
$$\left|\tilde{E}_{\mathrm{out1}}\right|^2 = \cos^2\left[k(L_1 - L_2)\right]\left|\tilde{E}_1\right|^2, \qquad \left|\tilde{E}_{\mathrm{out2}}\right|^2 = \sin^2\left[k(L_1 - L_2)\right]\left|\tilde{E}_1\right|^2. \tag{40}$$

- Historically this was used to measure the speed of light in different directions with respect to the motion of the earth. The measured constant speed of light motivated Einstein's discovery of special relativity.
- It is still being used in important scientific experiments, e.g., gravitational-wave detection.



- This is a different type of interferometer, where **multiple reflections** between the two interfaces occur.

Reflections and Transmissions at the Interfaces



- Consider a dielectric slab with refractive index n, free space on either side and a TE-polarization input with normal incidence. The scattering matrix for the first interface is

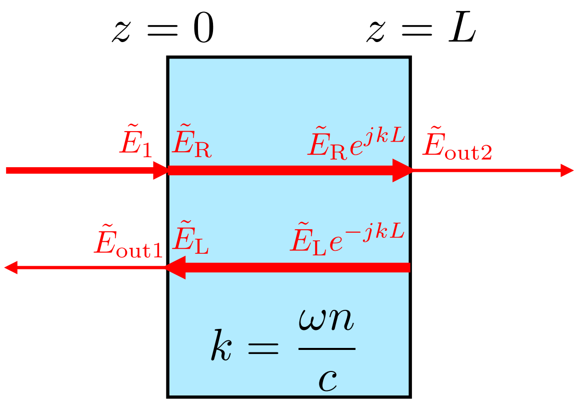
$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} \frac{1-n}{1+n} & \frac{2n}{1+n} \\ \frac{2}{1+n} & \frac{n-1}{1+n} \end{pmatrix}, \tag{41}$$

  which is obtained from reflection and refraction of two media.
- Note that this matrix is not unitary, because intensity in a dielectric is $|\tilde{E}|^2/(2Z) = n|\tilde{E}|^2/(2Z_0)$, which is proportional to n.
- The scattering matrix at the second interface is

$$\begin{pmatrix} s'_{11} & s'_{12} \\ s'_{21} & s'_{22} \end{pmatrix} = \begin{pmatrix} \frac{n-1}{1+n} & \frac{2}{1+n} \\ \frac{2n}{1+n} & \frac{1-n}{1+n} \end{pmatrix} = \begin{pmatrix} s_{22} & s_{21} \\ s_{12} & s_{11} \end{pmatrix}. \tag{42}$$

- What are the final output amplitudes on the left and the right?

Input-Output Relations



- Left free space: $\tilde{E}_1 e^{jk_0 z - j\omega t} + \tilde{E}_\mathrm{out1} e^{-jk_0 z - j\omega t}$, $k_0 = \omega/c$.
  Inside dielectric: $\tilde{E}_\mathrm{R} e^{jkz - j\omega t} + \tilde{E}_\mathrm{L} e^{-jkz - j\omega t}$, $k = \omega n/c$.
- Right free space: define the field as $\tilde{E}_\mathrm{out2} e^{jk_0(z-L) - j\omega t}$ (so that the field oscillates at $\tilde{E}_\mathrm{out2} e^{-j\omega t}$ at $z = L$, which makes the math simpler).
- Left interface at $z = 0$:

$$\tilde{E}_\mathrm{out1} = s_{11}\tilde{E}_1 + s_{12}\tilde{E}_\mathrm{L}, \tag{43}$$

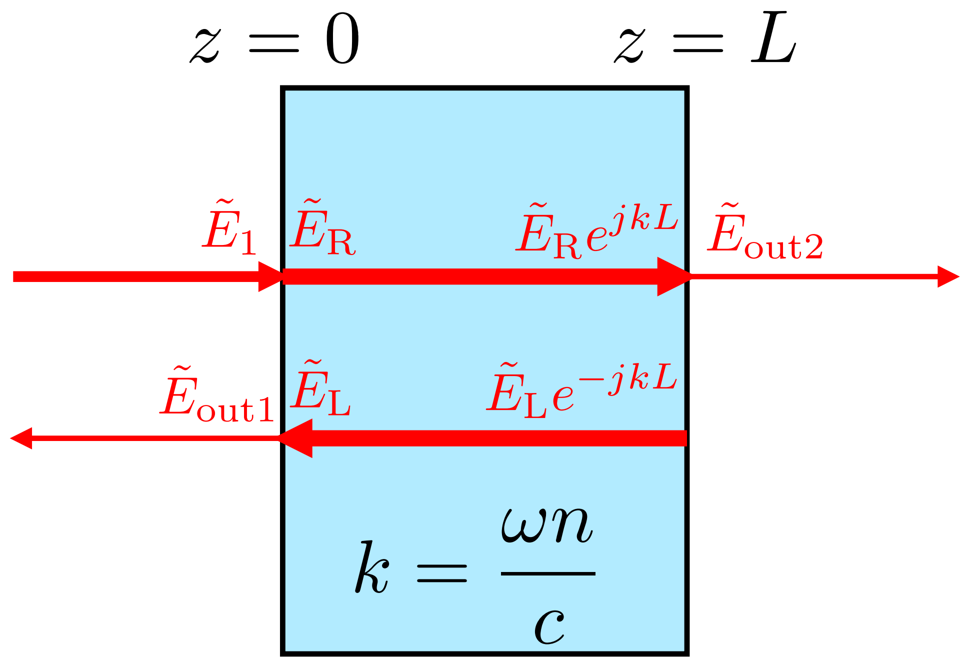
$$\tilde{E}_\mathrm{R} = s_{21}\tilde{E}_1 + s_{22}\tilde{E}_\mathrm{L}. \tag{44}$$

  Notice that here $\tilde{E}_\mathrm{L}$ is an input and $\tilde{E}_\mathrm{R}$ is an output.
- Right interface: At $z = L$, the fields on the left are $\tilde{E}_\mathrm{R} e^{jkL} e^{-j\omega t} + \tilde{E}_\mathrm{L} e^{-jkL} e^{-j\omega t}$ and the fields on the right are $\tilde{E}_\mathrm{out2} e^{jk_0 L} e^{-j\omega t}$. There are additional phase factors when we write the input-output relations:

$$\tilde{E}_\mathrm{L} e^{-jkL} = s'_{11}\tilde{E}_\mathrm{R} e^{jkL} = s_{22}\tilde{E}_\mathrm{R} e^{jkL}, \qquad \tilde{E}_\mathrm{L} = s_{22} e^{2jkL}\tilde{E}_\mathrm{R}, \tag{45}$$

$$\tilde{E}_\mathrm{out2} = s'_{21}\tilde{E}_\mathrm{R} e^{jkL} = s_{12}\tilde{E}_\mathrm{R} e^{jkL}, \qquad \tilde{E}_\mathrm{out2} = s_{12} e^{jkL}\tilde{E}_\mathrm{R}. \tag{46}$$

Notice that here $\tilde{E}_\mathrm{R}$ is an input and $\tilde{E}_\mathrm{L}$ is an output!



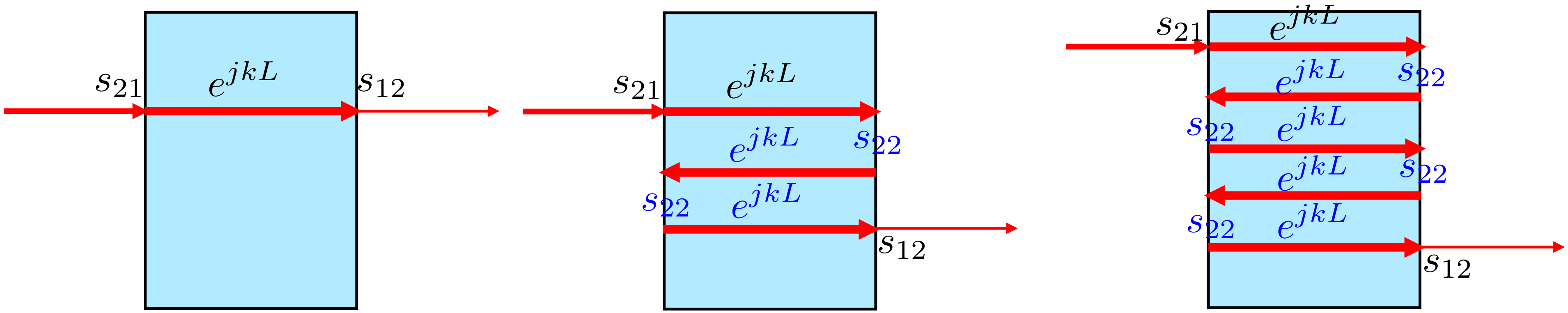
- Let's focus on $\tilde{E}_{\mathrm{out2}}$:

$$\begin{aligned}
\tilde{E}_{\mathrm{out2}} &= s_{12} e^{jkL} \tilde{E}_{\mathrm{R}} && (47)\\
&= s_{12} e^{jkL} \left(s_{21} \tilde{E}_1 + s_{22} \tilde{E}_{\mathrm{L}} \right) && (48)\\
&= s_{12} e^{jkL} \left[s_{21} \tilde{E}_1 + s_{22} \left(s_{22} e^{2jkL} \tilde{E}_{\mathrm{R}} \right) \right] && (49)\\
&= s_{12} e^{jkL} \left\{ s_{21} \tilde{E}_1 + s_{22} \left[s_{22} e^{2jkL} \left(s_{21} \tilde{E}_1 + s_{22} \tilde{E}_{\mathrm{L}} \right) \right] \right\} && (50)\\
&= \ldots && (51)
\end{aligned}$$

- It is **an infinite series**!

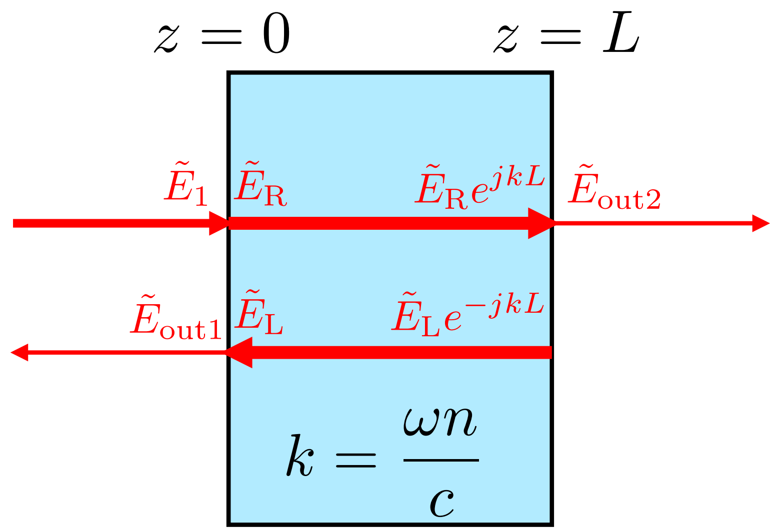
- We can represent each term in the series using a figure:

$$\tilde{E}_{\text{out2}} = \left[s_{12} e^{jkL} s_{21} + s_{12} \left(s_{22}^2 e^{2jkL} \right) e^{jkL} s_{21} + s_{12} \left(s_{22}^2 e^{2jkL} \right)^2 e^{jkL} s_{21} + \dots \right] \tilde{E}_1. \quad (52)$$



- The final output is the sum of amplitudes from **all possible paths** from input to output.
- Each partial wave undergoes a number of **round trips**. Each round trip picks up a factor of $s_{22}^2 e^{2jkL}$.

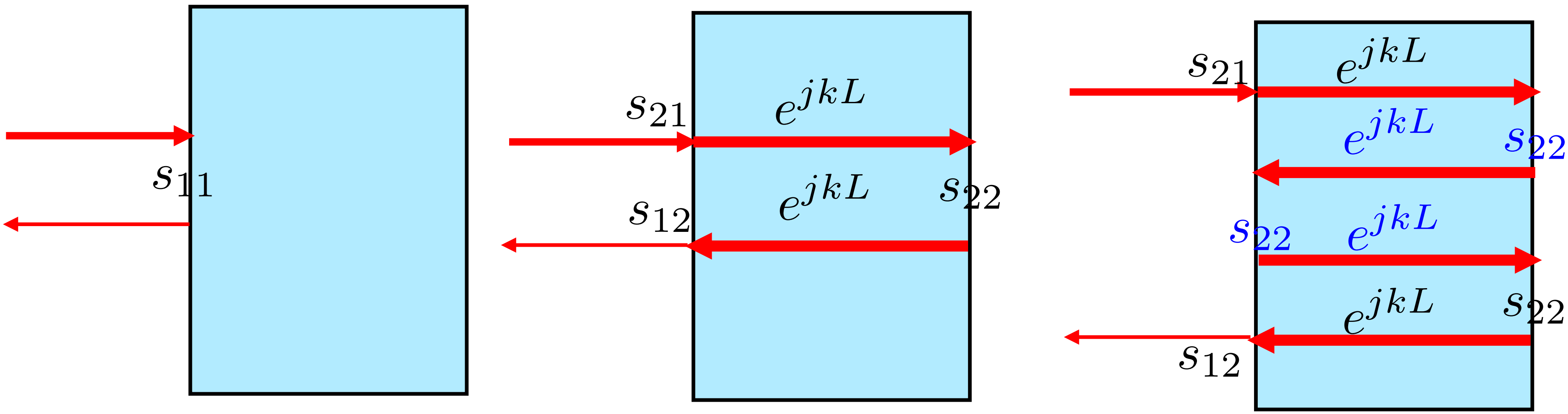
Final Solution



- Identity for infinite geometric series:

$$\sum_{m=0}^{\infty} a^m = \frac{1}{1-a} \text{ for } |a| < 1. \qquad \tilde{E}_{\mathrm{out2}} = \frac{s_{12} s_{21} e^{jkL}}{1 - s_{22}^2 e^{2jkL}} \tilde{E}_1. \tag{53}$$

Note that this depends on frequency $k = \omega n / c$.



- Similarly,

$$\tilde{E}_{\text{out}1} = \left\{ s_{11} + s_{12}s_{22}s_{21}e^{2jkL} \left[1 + s_{22}^2 e^{2jkL} + \left(s_{22}^2 e^{2jkL} \right)^2 + \dots \right] \right\} \tilde{E}_1 \tag{54}$$

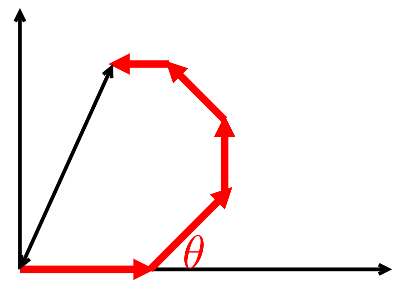
$$= \left(s_{11} + \frac{s_{12}s_{22}s_{21}e^{2jkL}}{1 - s_{22}^2 e^{2jkL}} \right) \tilde{E}_1. \tag{55}$$

- If the incident angle is oblique, the calculation is pretty much the same except that the scattering matrices at the interfaces should be modified for oblique incidence and k should be replaced by $k \cos\theta_t$.

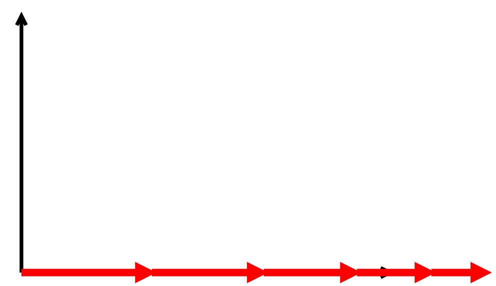
Transmission of Fabry-Perot Interferometer

- When is power transmission $\propto |\tilde{E}_{\text{out2}}|^2$ maximum if we change ω? Two ways of thinking about it:
  1. Think of the infinite series:

$$\tilde{E}_{\text{out2}} = s_{12} e^{jkL} s_{21} \left[1 + s_{22}^2 e^{2jkL} + \left(s_{22}^2 e^{2jkL} \right)^2 + \ldots \right] \tilde{E}_1. \tag{56}$$



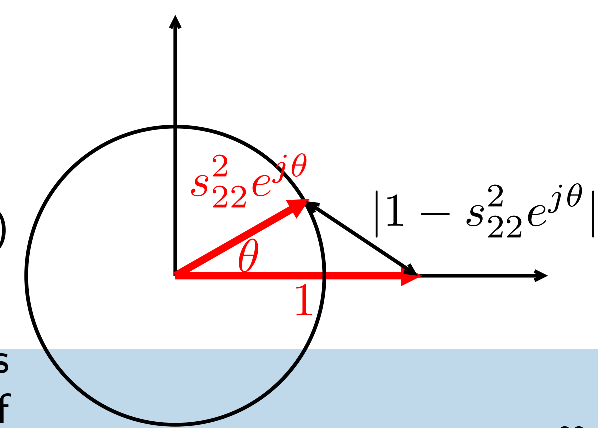
Each term is multiplied by $s_{22}^2 e^{j\theta}$, $\theta = 2kL = 2\omega nL/c$. To make the output maximum, the partial waves should have constructive interference, i.e., they should all have the same phase. For a dielectric slab, s_{22}^2 is real and positive and < 1, so constructive interference occurs when



$$\angle \left(s_{22}^2 e^{2jkL} \right) = 2kL = 2\pi q, \quad q \text{ integer.} \tag{57}$$

  2. Since

$$\left| \tilde{E}_{\text{out2}} \right|^2 = \frac{|s_{12} s_{21}|^2}{\left| 1 - s_{22}^2 e^{2jkL} \right|^2} |\tilde{E}_1|^2, \tag{58}$$



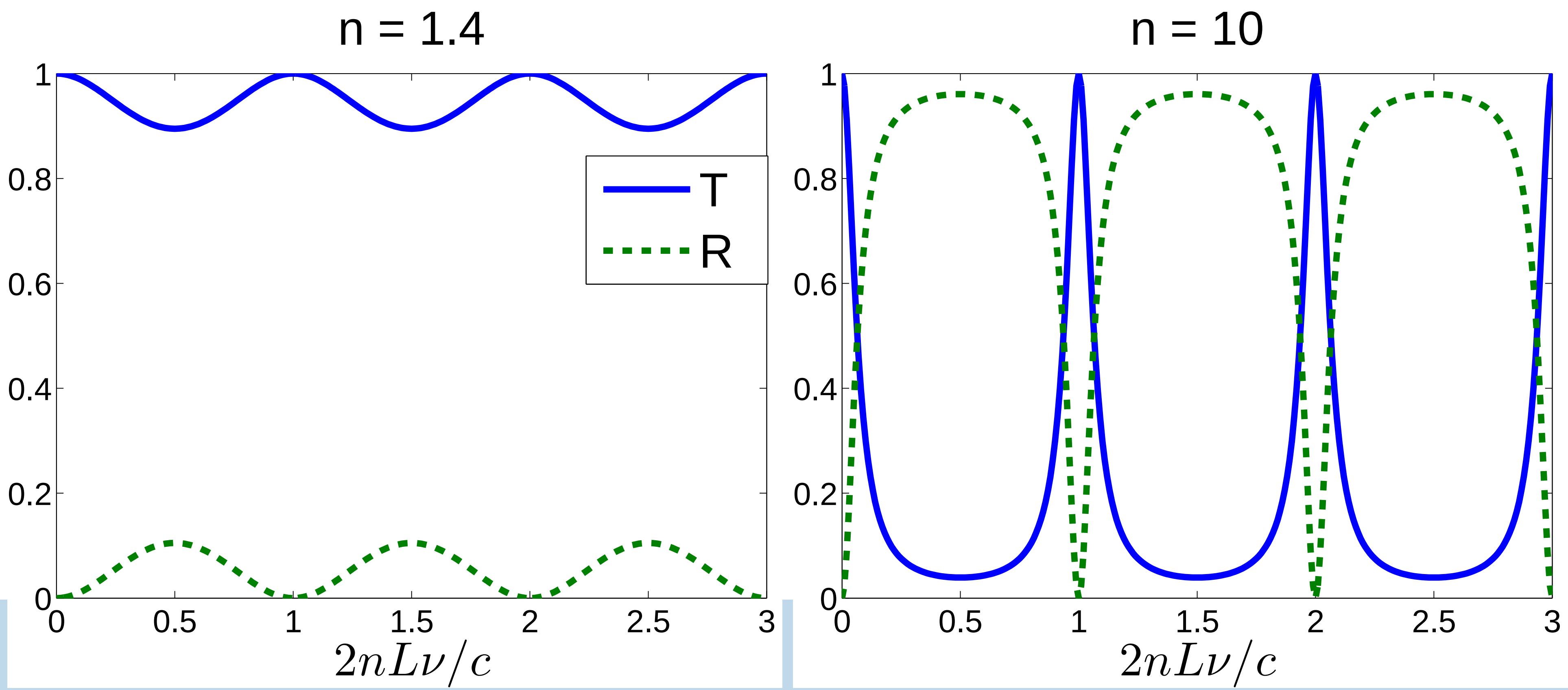
The transmitted power is maximum when the denominator is the smallest. This happens when $\angle s_{22}^2 e^{2jkL}$ is multiples of 2π.

Transmission and Reflection Spectra

- Both ways lead to

$$2kL = \frac{2\omega nL}{c} = 2\pi q, \qquad \omega = \frac{2\pi qc}{2nL}, \qquad \boxed{\nu = \frac{\omega}{2\pi} = \frac{qc}{2nL},} \qquad q = 0, 1, 2, \ldots. \tag{59}$$

- The spacing is $c/(2nL)$. Note that this spacing is the same as the spacing of resonant frequencies in an optical cavity.
- $T = |\tilde{E}_{\text{out2}}|^2/|\tilde{E}_1|^2$, $R = |\tilde{E}_{\text{out1}}|^2/|\tilde{E}_1|^2$:



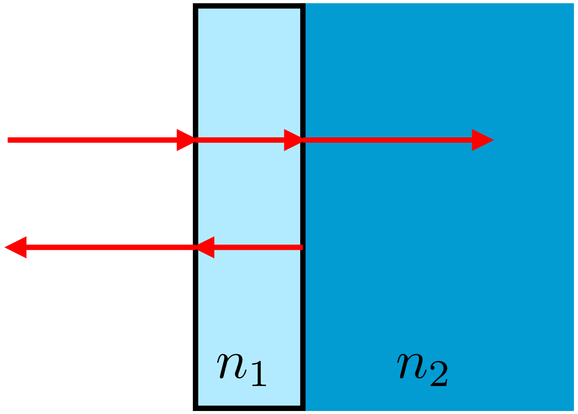
Thin Film

- For thin films such as oil on water or soap bubbles, the effect of multiple reflections at the interfaces is to transmit some frequencies and reflect others.
- The transmission and reflection depend on frequency and incident angle, so this is why they appear colorful.

- Fabry-Perot interferometers (dielectric thin film) are often used as **color filters** in optical experiments.

Anti-Reflective Coating

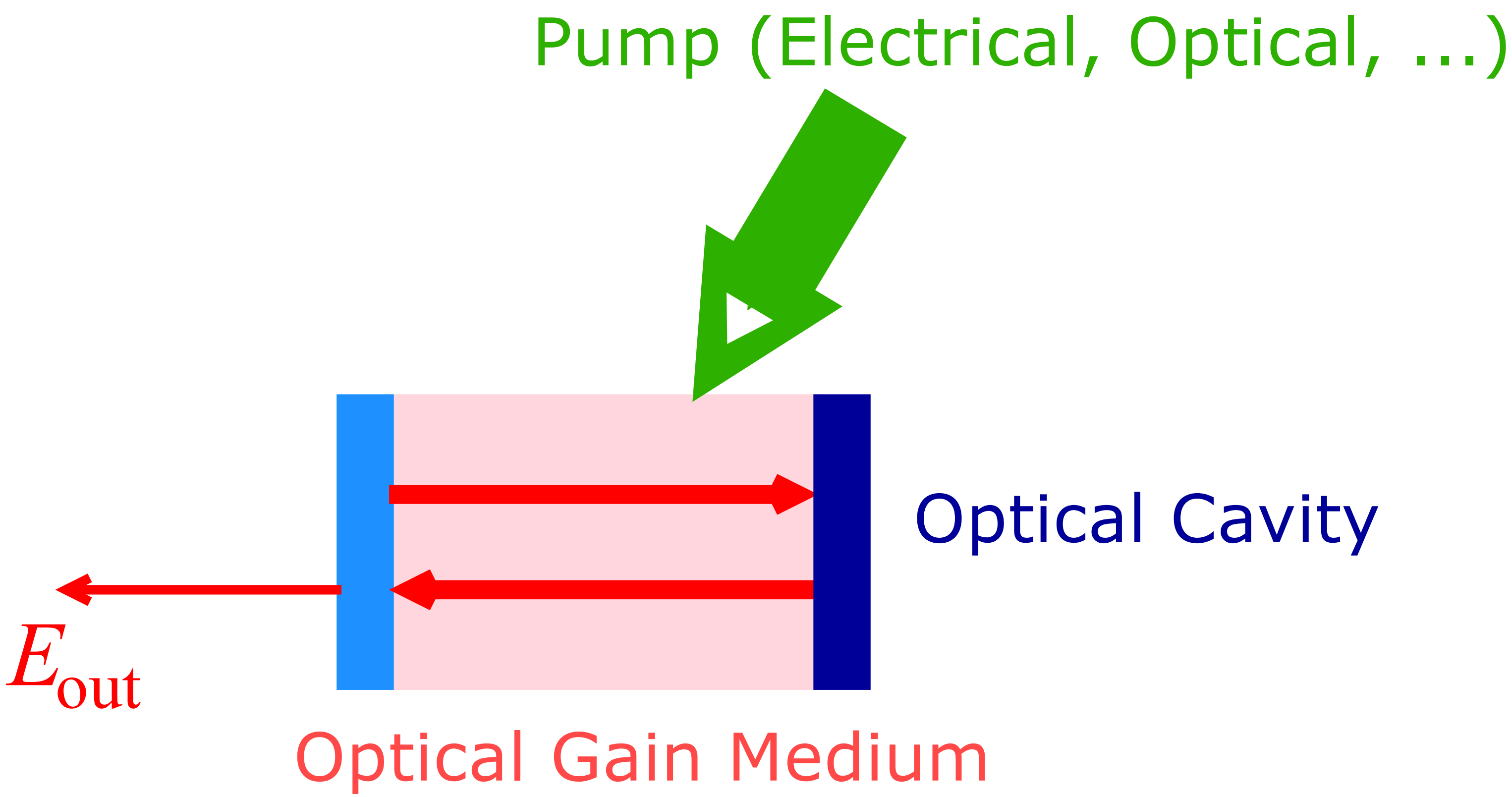
- You can find anti-reflective coatings on glasses, camera lens, etc.
- The purpose is minimize reflection by choosing the refractive index and thickness of the coating appropriately.



- Question: If you look at your glasses or a camera lens at an angle, why do they have a red/blue tint?

Laser Cavity

- A Fabry-Perot optical cavity is an important component of a laser. A typical laser cavity consists of one partially transmitting mirror and a perfectly reflective mirror.
- There is also an optical gain medium inside to provide power. More on this later.
- Multiple reflections inside the cavity enhance the gain when on resonance.



Suggested Problems

- Propagation: for a given ω, what is the phase shift for a sinusoidal plane wave after propagating in a dielectric medium with refractive index n? The phase shift is proportional to index.
- Beam splitter: confirm input-output power conservation if

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{60}$$

  for any ϕ.
- Mach-Zehnder: If there are two nonzero input amplitudes, what are the output amplitudes? Write in matrix form. Confirm that the matrix is unitary.
- Fabry-Perot: In terms of input $\tilde{E}_1$, what is $|\tilde{E}_{\mathrm{out1}}|^2 + |\tilde{E}_{\mathrm{out2}}|^2$ for a lossless dielectric slab? What if there is another input from the right with amplitude $\tilde{E}_2$? What can you conclude about the scattering matrix relating $\tilde{E}_{\mathrm{out1}}$ and $\tilde{E}_{\mathrm{out2}}$ to the inputs? No calculation allowed!