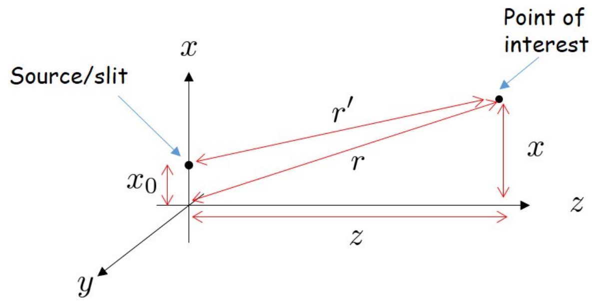
PC2232 – Tutorial 3 Solutions

Important Points/Equations for Lecture 3

I don't know if the derivation of these are important... I'm including them here because I think it helps you solve the tutorial problems if you first understand the concepts behind this[1]

1. Coordinate system



The E field equations below are all the E for/at the point of interest in the diagram above.

2. E field for a spherical wave with source at origin

$$E = \tilde{E}\frac{\mathrm{e}^{j\vec{k}\cdot\vec{r}}\mathrm{e}^{-j\omega t}}{r} = \tilde{E}\frac{\mathrm{e}^{jkr}\mathrm{e}^{-j\omega t}}{r} = \tilde{E}\frac{\mathrm{e}^{jk\sqrt{z^2+x^2}}\mathrm{e}^{-j\omega t}}{\sqrt{z^2+x^2}}$$

3. The above with paraxial approximation
   Paraxial approximation: $x \ll z$
   Therefore we can rewrite

$$E = \tilde{E}\frac{\mathrm{e}^{jk\sqrt{z^2+x^2}}\mathrm{e}^{-j\omega t}}{\sqrt{z^2+x^2}} = \tilde{E}\frac{\mathrm{e}^{jkz\sqrt{1+\frac{x^2}{z^2}}}\mathrm{e}^{-j\omega t}}{z\sqrt{1+\frac{x^2}{z^2}}}$$

   Taylor expansion:

$$z\sqrt{1+\frac{x^2}{z^2}} = z\left(1+\frac{x^2}{z^2}\right)^{\frac{1}{2}} \approx z\left(1+\frac{1}{2}\frac{x^2}{z^2}\dots\right)$$

   Keeping up to first order

$$E = \tilde{E}\frac{\mathrm{e}^{jkz}\mathrm{e}^{-j\omega t}}{z}$$

4. E field for a spherical wave where the source is x_0

$$E = \tilde{E}\frac{\mathrm{e}^{j\vec{k}\cdot\color{red}{\vec{r}'}}\mathrm{e}^{-j\omega t}}{\color{red}{r'}} = \tilde{E}\frac{\mathrm{e}^{jk\color{red}{\sqrt{z^2+(x-x_0)^2}}}\mathrm{e}^{-j\omega t}}{\color{red}{\sqrt{z^2+(x-x_0)^2}}}$$

5. The above, with paraxial approximation: $x \ll z$
   Therefore we can rewrite

$$E = \tilde{E}\frac{\mathrm{e}^{jk\sqrt{z^2+(x-x_0)^2}}\mathrm{e}^{-j\omega t}}{\sqrt{z^2+(x-x_0)^2}} = \tilde{E}\frac{\mathrm{e}^{jkz\sqrt{1+\frac{(x-x_0)^2}{z^2}}}\mathrm{e}^{-j\omega t}}{z\sqrt{1+\frac{(x-x_0)^2}{z^2}}}$$

   Taylor expansion:

$$z\sqrt{1+\frac{(x-x_0)^2}{z^2}} = z\left(1+\frac{(x-x_0)^2}{z^2}\right)^{\frac{1}{2}} \approx z\left(1+\frac{1}{2}\frac{(x-x_0)^2}{z^2}\dots\right)$$

   Keeping up to the second order in the numerator, and first order in the denominator

$$E = \tilde{E}\frac{\mathrm{e}^{jkz\left[1+\frac{(x-x_0)^2}{2z^2}\right]}\mathrm{e}^{-j\omega t}}{z}\tilde{E}\frac{\mathrm{e}^{jkz}\mathrm{e}^{jk\frac{(x-x_0)^2}{2z}}\mathrm{e}^{-j\omega t}}{z}$$

[1]The lecturer's ideas and thought processes are actually quite nice... But you have to stare at his lecture notes a long time to see the picture. I'm hoping this will help simplify the picture for you.

6. E field for cylindrical waves where source is at x_0

$$E = \tilde{E}\frac{\Delta x \mathrm{e}^{j\vec{k}\cdot\vec{r}'}\mathrm{e}^{-j\omega t}}{\sqrt{r'}} = \tilde{E}\frac{\Delta x \mathrm{e}^{jk\sqrt{z^2+(x-x_0)^2}}\mathrm{e}^{-j\omega t}}{\left[z^2+(x-x_0)^2\right]^{\frac{1}{4}}}$$

where Δx is the slit width[2]

7. The above with paraxial approximation: $x \ll z$
Therefore we can rewrite

$$E = \tilde{E}\frac{\Delta x \mathrm{e}^{jk\sqrt{z^2+(x-x_0)^2}}\mathrm{e}^{-j\omega t}}{\left[z^2+(x-x_0)^2\right]^{\frac{1}{4}}} = \tilde{E}\frac{\Delta x \mathrm{e}^{jkz\sqrt{1+\frac{(x-x_0)^2}{z^2}}}\mathrm{e}^{-j\omega t}}{\sqrt{z}\left[1+\frac{(x-x_0)^2}{z^2}\right]^{\frac{1}{4}}}$$

Taylor expansion:

$$z\sqrt{1+\frac{(x-x_0)^2}{z^2}} = z\left(1+\frac{(x-x_0)^2}{z^2}\right)^{\frac{1}{2}} \approx z\left(1+\frac{1}{2}\frac{(x-x_0)^2}{z^2}\dots\right)$$

$$\sqrt{z}\left[1+\frac{(x-x_0)^2}{z^2}\right]^{\frac{1}{4}} \approx \sqrt{z}\left(1+\frac{1}{4}\frac{(x-x_0)^2}{z^2}\dots\right)$$

Keeping up to the second order in the numerator, and first order in the denominator

$$E = \tilde{E}\frac{\Delta x \mathrm{e}^{jkz\left[1+\frac{(x-x_0)^2}{2z^2}\right]}\mathrm{e}^{-j\omega t}}{\sqrt{z}} = \tilde{E}\frac{\Delta x \mathrm{e}^{jkz}\mathrm{e}^{jk\frac{(x-x_0)^2}{2z}}\mathrm{e}^{-j\omega t}}{\sqrt{z}}$$

8. The above with Franhouffer treatment[3]: $x_0 \ll x$
Please note that technically, this is not entirely accurate[4]
Therefore we can rewrite

$$E = \tilde{E}\frac{\Delta x \mathrm{e}^{jkz}\mathrm{e}^{jk\frac{(x-x_0)^2}{2z}}\mathrm{e}^{-j\omega t}}{\sqrt{z}} = \tilde{E}\frac{\Delta x \mathrm{e}^{jkz}\mathrm{e}^{\frac{jkx^2}{2z}\left(1-\frac{x_0}{x}\right)^2}\mathrm{e}^{-j\omega t}}{\sqrt{z}} = \tilde{E}\frac{\Delta x \mathrm{e}^{jkz}\mathrm{e}^{\frac{jkx^2}{2z}\left(1-\frac{2x_0}{x}+\frac{x_0^2}{x^2}\right)}\mathrm{e}^{-j\omega t}}{\sqrt{z}}$$

Because $x_0 \ll x$, therefore we can ignore the following term $\dfrac{x_0^2}{x^2}$

$$E \approx \tilde{E}\frac{\Delta x \mathrm{e}^{jkz}\mathrm{e}^{\frac{jkx^2}{2z}}\mathrm{e}^{-\frac{jkxx_0}{z}}\mathrm{e}^{-j\omega t}}{\sqrt{z}}$$

Note: I made an error in *all* my classes[5]. In class, I wrote the following expression on the board

$$E \approx \tilde{E}\frac{\Delta x \mathrm{e}^{jkz}\mathrm{e}^{\frac{jkx^2}{2z}}\mathrm{e}^{\frac{jkxx_0}{z}}\mathrm{e}^{-j\omega t}}{\sqrt{z}}$$

I left out the negative sign

9. From here on out,

$$A \equiv \tilde{E}\frac{\mathrm{e}^{jkz}\mathrm{e}^{\frac{jkx^2}{2z}}\mathrm{e}^{-j\omega t}}{\sqrt{z}}$$

10. Interference ideas:
Recall, this is the main idea of lecture 3

[2]For our purposes, we can assume that cylindrical waves are kind of caused by spherical waves going through a slit. It might help us understand the formulae better. Note that this may not strictly be true.

[3]As mentioned in class, the Franhouffer treatment is not the same as paraxial approximation. Generally, it applies only to diffraction

[4]I'm doing this because I think it makes it a little easier to talk about all the different set ups *after* you have the general formula for all the cylindrical waves... and I don't think the small technical difference will matter in a level 2 engineering physics course. So please note that this approximation is a simplified version to help you remember/understand.

The proper version is in his lecture notes pg 14, 15 and 18. It looks very much the same. (His x' is our x_0). The technical difference is that it actually refers to the position of the multiple sources/slits within 1 single slit. Therefore there is a constraint on this position x', it must fall within the slit width... When I make the assumption before explaining the single slit, this constraint does not exist. If you're interested and confused, come consult me. If you're confused and not interested, stick with the statement I gave you, just know that it's not 100% accurate

[5]We never had problems with the later expressions because it does not really affect the final answer much when we do interference

(a) Write out all the E fields involved

(b) Sum up all the E fields at the point of interest

(c) Locate maxima and minima/determine conditions required for interference

(d) Obtain intensity

$$I = |\bar{S}| = \frac{1}{2} Re\left[EH^*\right] = \frac{1}{2Z_0} Re\left[EE^*\right] = \frac{1}{2} Re\left[|E|^2\right]$$

Where it is assumed that you know $|E|^2 = EE^*$

- The period of the intensity curve Λ is the x value for one cycle[6]
  Just the way the period T is the t value for one cycle
- For instance, interference of 2 plane waves has an intensity distribution of $I \propto \cos^2 k_x x$
  Therefore period $\Lambda = \dfrac{\pi}{k_x}$
  Just the way the period T of a $\cos \omega t$ graph would be $T = \dfrac{2\pi}{\omega}$
- He uses this again later[7]. For instance in pg 21: 'The intensity is periodic with a period of...'
  This turns out to be useful for us for question 4 of our tutorial

11. Therefore, example with double slit interference

(a) Write out all the E field involved
Let first slit be slit a with position: x_a and slit width Δx_a
Let second slit be slit b with position: x_b and slit width Δx_b

$$E_a = A\Delta x_a \mathrm{e}^{-\frac{jkxx_a}{z}} \qquad\qquad E_b = A\Delta x_b \mathrm{e}^{-\frac{jkxx_b}{z}}$$

(b) Summing the E at point of interest

$$E_T = E_a + E_b = A\left(\Delta x_a \mathrm{e}^{-\frac{jkxx_a}{z}} + \Delta x_b \mathrm{e}^{-\frac{jkxx_b}{z}}\right)$$

Assuming

- Slit width are the same size: $\Delta x_a = \Delta x_b = \Delta x$
- Position of the slit is symmetrical $x_a = -x_b$
  Therefore $|x_a| = |x_b| = \dfrac{d}{2}$

$$\begin{aligned} E_T &= A\Delta x \left(\mathrm{e}^{-\frac{jkxd}{2z}} + \mathrm{e}^{\frac{jkxd}{2z}}\right) \\ &= 2A\Delta x \cos\left(\frac{kxd}{2z}\right) \end{aligned}$$

(c) Obtaining location of maxima[8]

$$\begin{aligned} \cos\left(\frac{kxd}{2z}\right) &= \pm 1 \\ \frac{kxd}{2z} &= m\pi \qquad\qquad m \text{ is any integer} \\ \frac{\pi xd}{z\lambda} &= m\pi \\ d\frac{x}{z} &= m\lambda \\ d\sin\theta &= m\lambda \end{aligned}$$

Where due to the paraxial approx[9]: $\sin\theta \approx \tan\theta$

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[6] I think that some do not appreciate that the period is just the x value. So when we say the period $\frac{\lambda}{d}$, the LHS side of that equation is x. So: $x = \frac{\lambda}{d}$

[7] I think I mentioned in some classes that I did not see him bringing this up/using this. But I was mistaken. He doesn't use the function Λ again, but he says it in text

[8] Notice negative E can also give us an intensity max

[9] Throughout all out tutorial questions, I assumed this. Strictly speaking, this is actually the small angle approximation, and it should not hold for large θ. However, because our z is so huge, we reason out that our x still will not be that different even if θ is large. We revert to $\sin\theta$ because this is the formulae most of us are familiar with.

12. Single slit

    Idea: The single slit is divided into multiple slits of infinitesimal width

    (a) Therefore, multiple slit

$$E_T = \sum_i A\Delta x_i \mathrm{e}^{-\frac{jkxx_i}{z}}$$

    (b) If slit size is infinitesimally small, it ends up being a continuous variable, and we integrate

$$\begin{aligned} E_T &= \int_{-\frac{a}{2}}^{\frac{a}{2}} \mathrm{d}x_i\, A\mathrm{e}^{-\frac{jkxx_i}{z}} \\ &= A\, a \,\mathrm{sinc}\left(\frac{ax}{\lambda z}\right) \end{aligned}$$

    Please note that this integration should be easy enough for you to do. Try to do it on your own[10]

    (c) Locating minima: We are told that $\mathrm{sinc}\left(\frac{ax}{\lambda z}\right) = 0$ when $\frac{ax}{\lambda z} = \pm m$

    Where m is any integer

    Therefore, minima occurs at:

$$\begin{aligned} \frac{ax}{\lambda z} &= \pm m \\ a\frac{x}{z} &= \pm m\lambda \\ a\sin\theta &= m\lambda \end{aligned}$$

    (d) He draws out the intensity curve and points out that the width of the diffraction peak is $\frac{2\lambda z}{a}$ (pg 16 of lecture 3[11])

13. Double slit ***and*** single slit (pg 18 of lecture)

    - Notice that when we first did double slit, we were assuming that the slit was ideal – there was no diffraction through each slit[12]
    - This is simple.

      Do the single slit for each slit separately

      Add the E due to individual slits to obtain the final E and I

$$E = 2Aa\,\mathrm{sinc}\left(\frac{ax}{\lambda z}\right)\cos\left(\frac{kxd}{2z}\right)$$

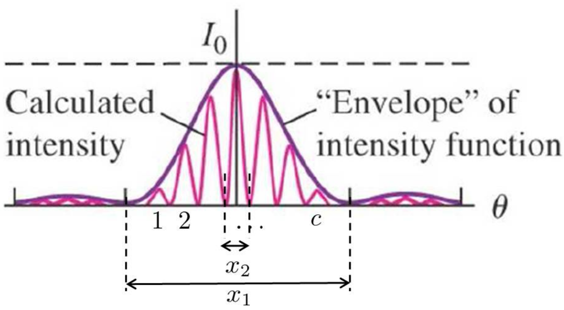
    - He comments on the period of the fringes inside the envelope as well as the width of the envelope[13]

      Notice that the width of the envelope: $x_1 = \frac{2\lambda z}{a}$

      And the period[14] of the fringes: $x_2 = \frac{\lambda z}{d}$

      Therefore if we want to know how many fringes there are inside one envelope

      Let c be the number of fringes inside one envelope



[10]Lecture 3 pg 15

[11]This is easy to see. Check if you understand this

[12]This is technically why he does not apply the Franhouffer treatment to his double slit in page 12. Remember than Franhouffer generally is applied to diffraction

[13]Stare at the equation until you can understand why his comments are true. If you don't get it, come see me

[14]This is lost on most people because they do not realize that the width and period are both x values. Note that x_1 is caused by the diffraction and x_2 is caused by the double slit interference

$$x_1 = cx_2 \qquad \Rightarrow \qquad c = \frac{x_1}{x_2} = \frac{2d}{a}$$

Therefore, the number of fringes inside one envelope is determined by the slit size and the separation between two slits[15] $\frac{d}{a}$

14. Diffraction grating:
    Idea: Multiple slit – Just use the given identity to obtain final result
    He mentions the period of the intensity curve here is $\frac{\lambda z}{d}$
    He also brings up the width of the principle maxima (pg 21 of lecture 3)

15. Lens

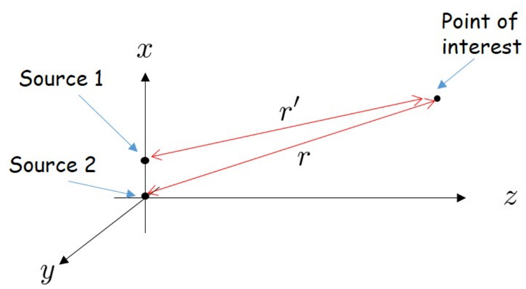
- For this, we do ***not*** apply the Franhouffer approximation
- He states that the phase change due to the lens is: $e^{-\frac{jkx'^2}{2f}}$
  His x' is our x_0 or x_a – basically position of the source/slit
  We have to sum over/integrate over x' because each x' introduces a slightly different phase
- At $z = f$, this quadratic phase factor is canceled out by the $e^{\frac{jkx'^2}{2z}}$ in our cylindrical wave[16]
- Simplifying and integrating, returns us the single slit diffraction equation
  Assuming the lens has a width if a
- Therefore, our lens gives us a diffraction pattern with the first minima obtained when $\frac{ax}{\lambda f} = 1$
  From here we can calculate the width of this spot[17]] Not many realize that this means our lens gives us a diffraction pattern
  Like all diffraction pattern, we can calculate width and et cetra

16. Phase difference due to media

- This is not strictly due in lecture 3, but we need this to solve question 2c)
- Recall: When light travels through a medium, $k' = \frac{\omega}{v} = \frac{\omega n}{c} = nk_0$ (lecture 2, pg 12)
- Note also that when we want to add a phase to our E fields, we do it by adding $e^{j\phi}$ to our E

$$E = \tilde{E}e^{jkr}e^{-j\omega t}e^{j\phi}$$

- Phase difference can be introduced in a variety of manners.
  - One way is to vary the path taken.



Therefore

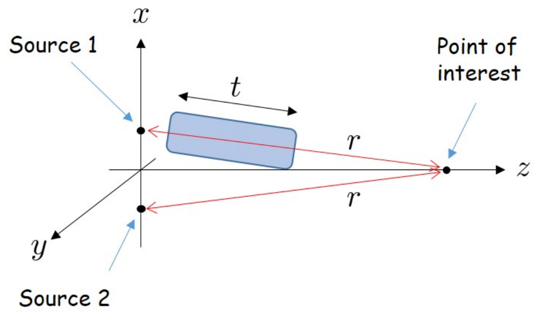
$$E_1 = \tilde{E}e^{jkr'}e^{-j\omega t}$$

$$\begin{aligned} E_2 &= \tilde{E}e^{jkr}e^{-j\omega t} \\ &= \tilde{E}e^{jkr'}e^{-j\omega t}e^{jk(r-r')} \\ &= \tilde{E}e^{jkr'}e^{-j\omega t}e^{j\phi} \end{aligned}$$

Notice that our phase difference here has been reexpressed[18] as: $\phi = k(r - r')$

[15] This assumes the double slit has the same slit size and symmetrical position
[16] For single slit, double slit and diffraction grating this term was ignored due to Franhouffer diffraction
[17] The spherical aperture gives you a mathematical function called the Airy function. I'm not sure how important that is. The spot size mentioned there is connected to the Rayleigh criterion that some of you may be familiar with
[18] To talk about phase difference, we must always express E_1 in terms of E_2 or vice versa. Only then can we pull out the phase difference

– One could also vary k by changing the n of the material



Where the blue slab in the picture is material of index of refraction n
Therefore

$$
\begin{aligned}
E_1 &= \tilde{E} \mathrm{e}^{j\left(k' t + k[r-t]\right)} \mathrm{e}^{-j\omega t} \\
&= \tilde{E} \mathrm{e}^{jkr} \mathrm{e}^{j[k't - kt]} \mathrm{e}^{-j\omega t} \\
&= \tilde{E} \mathrm{e}^{jkr} \mathrm{e}^{j\phi} \mathrm{e}^{-j\omega t}
\end{aligned}
\qquad
E_2 = \tilde{E} \mathrm{e}^{jkr} \mathrm{e}^{-j\omega t}
$$

where $\phi = k't - kr$

Tutorial 3 Solutions

1. Given:

$$T = 1\text{hour} \qquad v = 800\text{km}\,\text{h}^{-1}$$

   (a) Question: $\lambda =?$

   Approach:
   $$v = f\lambda = \frac{\lambda}{T}$$

   Answer:

   $$\lambda = 800\text{km}$$

   (b) Given:

   $$a_1 = 4500\text{km} \qquad a_2 = 3700\text{km}$$

   Question: Find the first minima for both these diffractions

   Approach:
   Understand that it's two different single slit diffraction question[19]
   Therefore refer to point 12, bullet (c) above for the derivation of the following equation

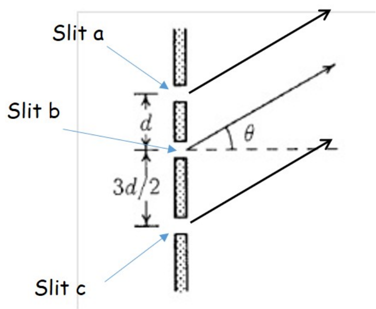
   Solution:

   $$a_1 \sin\theta_1 = \lambda \qquad a_2 \sin\theta_2 = \lambda$$

   Answer:

   $$\theta_1 = 10.2° \qquad \theta_2 = 12.5°$$

2. Given: $\lambda = \frac{2}{5}d$



   (a) Derive the condition θ must satisfy such that all 3 slits interfere constructively

   Approach:
   - Understand the ideas of interference (Point 10 above)
   - Write out the E due to the three different slits and sum them up
   - Deduce the correct condition for principle maxima
   - Learning point:
     Familiarize yourself with cylindrical wave
     Familiarize yourself with 4 steps of interference
     Applying the ideas in interference to 3 slits where the slit positions are not the same/symmetrical

   Solution:
   - Writing out and summing up E

$$\begin{aligned} E_T &= E_a + E_b + E_c \\ &= A\left(e^{-\frac{jkxx_a}{z}} + e^{-\frac{jkxx_b}{z}} + e^{-\frac{jkxx_c}{z}}\right) \\ &= A\Big(e^{-\frac{jkxd}{z}} + \underbrace{e^{-\frac{jkx(0)}{z}}}_{1} + e^{\frac{j3kxd}{2z}}\Big) \end{aligned}$$

[19] As opposed to a double slit + single slit problem

- Deriving condition for principle maxima

  Note: Because the middle term is 1, the first and third term must be 1 in order to get principle maxima[20]

$$e^{-\frac{jkxd}{z}} = 1 \qquad\qquad e^{\frac{j3kxd}{2z}} = 1$$

  Only taking real component of the exponential function

$$\begin{aligned} \cos\left(\frac{kxd}{z}\right) &= 1 & \cos\left(\frac{3kxd}{2z}\right) &= 1 \\ \frac{2\pi xd}{\lambda z} &= 2m_a\pi & \frac{2\pi(3xd)}{2\lambda z} &= 2m_c\pi \\ \frac{x}{z} &= \frac{m_a\lambda}{d} & \frac{x}{z} &= \frac{2m_c\lambda}{3d} \\ \sin\theta_a &= \frac{2m_a}{5} & \sin\theta_c &= \frac{4m_c}{15} \end{aligned}$$

  where, for now, m_a and m_c are any integer[21]

- Notice that these two conditions easily collapse into 1 condition[22].

  To find the θ that obeys both equations, we have

$$\begin{aligned} \frac{2}{5}m_a &= \frac{4}{15}m_c \\ m_c &= \frac{3}{2}m_a \end{aligned}$$

  Therefore,

$$\sin\theta_c = \frac{4m_c}{15} = \frac{4}{15}\left(\frac{3}{2}m_a\right) = \frac{2}{5}m_a$$

  We have reduced it into 1 condition!

  However, not all m_a are allowed now. Recall that both m_a and m_c must be integers.

  Therefore, to ensure $m_c = \frac{3}{2}m_a$, only even valued m_a are allowed: $m_a = 2m$

  Therefore, condition for principle maxima from all three slits are:

$$\sin\theta = \frac{4}{5}m \qquad \text{where } m \text{ is any integer}$$

- Therefore, only allowed values for θ are when $m = 0$ and ± 1

Answer:

$$\begin{aligned} \sin\theta &= \frac{4}{5}m \\ \theta &= 0, \pm 53.13° \end{aligned}$$

(b) The third ray is given an additional phase of π.

Question: How many principle maximas are observed and what are the θ?

Approach:

Mostly a repeat of the part (a)

Know how to add in a phase to one particular ray

Learning point: To be able to add phase difference to one of the rays involved in double/triple slit

Solution:

- Sum up all E from the 3 slits

  Add a phase only to ray 3

$$\begin{aligned} E_T &= E_a + E_b + E_c \\ &= A\left(e^{-\frac{jkxx_a}{z}} + e^{-\frac{jkxx_b}{z}} + e^{-\frac{jkxx_c}{z}}e^{j\pi}\right) \\ &= A\Big(e^{-\frac{jkxd}{z}} + \underbrace{e^{-\frac{jkx(0)}{z}}}_{1} + e^{j\left(\frac{3kxd}{2z}+\pi\right)}\Big) \end{aligned}$$

[20]As compared to the conditions derived in the page before where the condition for E can be ± 1

[21]Recall, we consider $\sin\theta \approx \tan\theta$

[22]Notice that this question specifically asked for the condition, whereas in part (b) they only asked for the value of θ and how many principle maximas. This is because the two conditions here easily collapse into 1 condition

- Determining position for maxima

  Due to slit a, the condition from part (a) holds: $\sin\theta_a = \frac{2}{5}m_a$. ($m_a$ can be any integer)
  Due to slit c,

$$\begin{aligned}
\cos\left(\frac{3kxd}{z}+\pi\right) &= 1\\
\frac{2\pi(3xd)}{2\lambda z}+\pi &= 2m_c\pi\\
\frac{\pi(3xd)}{\lambda z} &= (2m_c-1)\pi\\
\frac{x}{z} &= \frac{(2m_c-1)\lambda}{3d}\\
\sin\theta_c &= \frac{2(2m_c-1)}{15}
\end{aligned}$$

  Therefore the allowed θ

$$\theta_a = 0^\circ,\ \pm 23.58^\circ,\ \pm 53.13^\circ \qquad\qquad \theta_c = \pm 7.66^\circ,\ \pm 23.58^\circ\ \pm 41.81^\circ,\ \pm 68.96^\circ$$

Answer:

$$\theta = \pm 23.58^\circ$$

Therefore there's only one principle maxima[23]

(c) If part (b) is achieved with a filter of $n = 1.33$, what is the minimum thickness t of the filter required?

Approach:

- Before attempting the question:
  - Refer to point 16 above to aid you in expressing phase difference
  - Note that the two rays referred to when we speak of 'phase difference' are
    Ray from slit c without the filter: E_{c_1}
    Ray from slit c with the filter: E_{c_2}
  - Know that $k' = nk$
- Write out both E fields
- Rearrange the expression of E to enable you to pull out the phase difference factor
- Calculate t from the phase difference
- Learning point: Understand and apply point 16 as derived in page 5

Solution:

- Writing out both E fields

$$E_{c_1} = A\mathrm{e}^{jkr}\mathrm{e}^{-j\omega t} \qquad\qquad E_{c_2} = A\mathrm{e}^{j\left(k't+k[r-t]\right)}\mathrm{e}^{-j\omega t}$$

- Reexpressing E_{c_2} such that we can pull out the phase difference factor

$$E_{c_2} = A\mathrm{e}^{jkr}\mathrm{e}^{j(k't-kt)}\mathrm{e}^{-j\omega t} = E_{c_1}\mathrm{e}^{j\phi}$$

  Therefore: $\phi = k't - kt$
- Calculating t from the phase difference

$$\begin{aligned}
\phi &= k't - kt\\
\pi &= nkt - kt\\
&= (n-1)\frac{2\pi}{\lambda}t\\
t &= \frac{\lambda}{2(n-1)}\\
&= \frac{d}{5(n-1)}
\end{aligned}$$

Answer:

$$t = 0.61d$$

[23] 2 maximas, if you consider the one at $\theta = -23.58^\circ$ as another maxima.
Note that 0 is *not* a maxima

3. Given:

$$\lambda = 500\text{nm} \qquad z = 90\text{cm} \qquad x_2 = 1\text{cm} \qquad x_1 = 6\text{cm}$$

Question:
$a = ?$
$d = ?$

Approach:

(a) Refer to point 13 above
Note that I have already expressed the information from the question in terms of the correct x_1 and x_2

(b) Once you have understood point 13, just apply the equations

(c) Learning point:
- Understand the idea of having double slit and single slit together and how it differs from double slit alone/single slit alone in points 11 and 12[24].
- Notice the usefulness of his comments and statements in the lecture notes. Be able to picture what the width of the envelope and the period of the fringes actually mean.
- Learn to interpret the information given in the question correctly. For instance, understanding why $x_1 = 6\text{cm}$

Solution:

$$x_1 = \frac{2\lambda z}{a} \qquad x_2 = \frac{\lambda z}{d}$$
$$a = \frac{2\lambda z}{x_1} \qquad d = \frac{\lambda z}{x_2}$$

Alternatively, we can look at the equation for E and determine position for maxima and minima[25]

$$E = 2Aa\,\text{sinc}\left(\frac{ax}{\lambda z}\right)\cos\left(\frac{kxd}{2z}\right)$$

Understand that when $x = 3\text{cm}$, $\text{sinc}\left(\frac{ax}{\lambda z}\right) = 0$. When $x = 1\text{cm}$, $\cos\left(\frac{kxd}{2z}\right) = 1$

Answer:

$$a = 1.5 \times 10^{-5}\text{m} \qquad d = 4.5 \times 10^{-5}\text{m}$$

4. Given:

$$400\text{nm} < \lambda < 700\text{nm} \qquad d = \frac{10^{-3}}{350}\text{m}$$

Question: What is the angular width of the first and third order

Approach

Understand that the period of the intensity curve: $x = \frac{\lambda z}{d}$
As mentioned in point 14 in page 5.

Approach:

- For first order:

$$\left(\frac{x}{z}\right)_{\text{min}} = \frac{\lambda_{\text{min}}}{d} \qquad \left(\frac{x}{z}\right)_{\text{max}} = \frac{\lambda_{\text{max}}}{d}$$
$$\sin(\theta_{\text{min}}) = \frac{\lambda_{\text{min}}}{d} \qquad \sin(\theta_{\text{max}}) = \frac{\lambda_{\text{max}}}{d}$$

- For third order:
We are looking for the the third period.

$$\left(\frac{x}{z}\right)_{\text{min}} = \frac{3\lambda_{\text{min}}}{d} \qquad \left(\frac{x}{z}\right)_{\text{max}} = \frac{3\lambda_{\text{max}}}{d}$$
$$\sin(\theta_{\text{min}}) = \frac{3\lambda_{\text{min}}}{d} \qquad \sin(\theta_{\text{max}}) = \frac{3\lambda_{\text{max}}}{d}$$

[24] Of course, double slit alone does not exist in real life... it's an idealized situation
[25] Point 10 above, step 3

Answer:

$$\begin{aligned} \text{First order :} \qquad & \Delta\theta = \theta_{\max} - \theta_{\min} = 6.1^\circ \\ \text{Third order :} \qquad & \Delta\theta = \theta_{\max} - \theta_{\min} = 22.5^\circ \end{aligned}$$

5. Given: Two slits of different width such that E of the second slit is double the E of the first[26]

(a) Show that: $I = I_0 \left(\frac{5}{9} + \frac{4}{9}\cos\phi \right)$
where ϕ is the phase difference between the two waves

Approach:

- Know how to write out cylindrical waves (point 8 above)
- Understand the idea of interference, and how to solve problems with it (point 10 above)
  For this question, we need to complete up to the last step – obtain intensity
- Able to reexpress E from one of the slits such that we can factor out the phase difference

Solution:

- Writing out the E from both slits

$$E_a = A\Delta x \mathrm{e}^{\frac{jkxx_a}{z}} \qquad \begin{aligned} E_b &= 2A\Delta x \mathrm{e}^{\frac{jkxx_b}{z}} \\ &= 2A\Delta x \mathrm{e}^{\frac{jkxx_a}{z}} \mathrm{e}^{\frac{jkx(x_b - x_a)}{z}} \\ &= 2A\Delta x \mathrm{e}^{\frac{jkxx_a}{z}} \mathrm{e}^{j\phi} \end{aligned}$$

- Summing up both E fields

$$E_T = E_a + E_b = A\Delta \mathrm{e}^{\frac{jkxx_a}{z}} \left(1 + 2\mathrm{e}^{j\phi}\right) = A' \left(1 + 2\mathrm{e}^{j\phi}\right)$$

- Obtaining intensity

$$\begin{aligned} I &= \frac{1}{2Z_0} EE^* \\ &= \frac{A'^2}{2Z_0} \left(1 + 2\mathrm{e}^{j\phi}\right)\left(1 + 2\mathrm{e}^{-j\phi}\right) \\ &= \frac{A'^2}{2Z_0} \left(1 + 2\mathrm{e}^{j\phi} + 2\mathrm{e}^{-j\phi} + 4\right) \\ &= \frac{A'^2}{2Z_0} (5 + 4\cos\phi) \end{aligned}$$

- Reexpressing in terms of I_0
  This occurs when $\phi = 0$

$$\begin{aligned} I_0 &= 9\frac{A'^2}{2Z_0} \\ \frac{A'^2}{2Z_0} &= \frac{1}{9} I_0 \end{aligned}$$

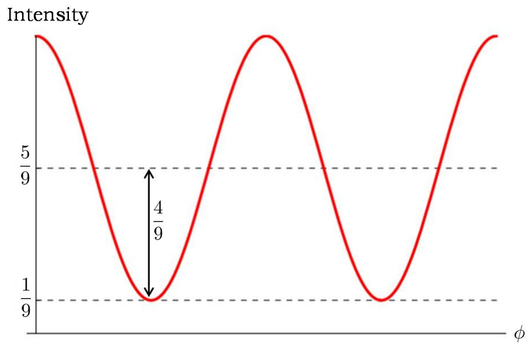
Therefore

$$I = \frac{1}{9} I_0 (5 + 4\cos\phi)$$

(b) Sketch graph of I versus ϕ and determine the value of $I_{\min}$ and the ϕ where they occur

Solution & Answer:

[26]You should realize that this means $\Delta x_2 = 2\Delta x_1$



$$I_{\min} = \frac{1}{9} I_0$$
$$\phi_{\min} = \pi, 3\pi \ldots (2m+1)\pi$$

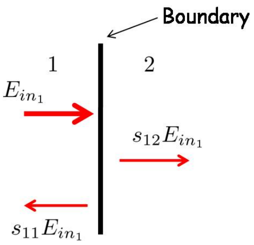
6. Show that beam splitter conserves power if $\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Approach:

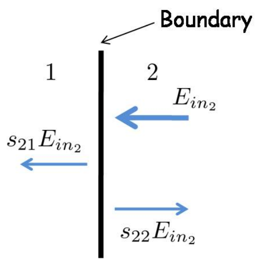
- Know what the terms in the scattering matrix represents
- Know that $I \propto EE^*$. Since intensity is related to power, if it conserves power, it must conserve EE^*
- Know how all these translate to matrix representation

Solution:

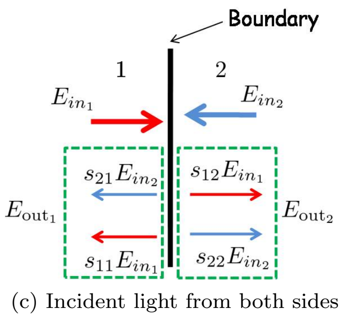
- Know the terms in the scattering matrix



(a) Incident light from 1



(b) Incident light from 2



(c) Incident light from both sides

Therefore we can express E_{out}

$$E_{out_1} = s_{11}E_{in_1} + s_{21}E_{in_2}$$
$$E_{out_2} = s_{12}E_{in_1} + s_{22}E_{in_2}$$

Therefore, this can be rewritten in matrix form as

$$\begin{pmatrix} E_{out_1} \\ E_{out_2} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} E_{in_1} \\ E_{in_2} \end{pmatrix}$$
$$\mathbf{E_{out}} = \mathbf{s}\,\mathbf{E_{in}}$$

where the last equation is in matrix

- Therefore, intensity and power will now be proportional to $\mathbf{E}^\dagger\mathbf{E}$
  Understand that when we convert it to matrix form[27], the conjugate E in $I \propto EE^*$ becomes a conjugate transpose
- Therefore, if this scattering matrix is power conserving[28],

$$\begin{aligned} \mathbf{E_{in}^\dagger E_{in}} &= \mathbf{E_{out}^\dagger E_{out}} \\ &= \mathbf{E_{in}}\,\mathbf{s}^\dagger\mathbf{s}\,\mathbf{E_{in}} \end{aligned}$$

  Conclusion: $\mathbf{s}^\dagger\mathbf{s} = \mathbb{I}$ if $\mathbf{s}$ is power conserving
- Showing that this is true for the given matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^\dagger \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

[27] We do this because it's easier to do the math this way – just the way we do our cos and sin in complex exponential form... because it's easier to handle. If you're not familiar with all these, you should familiarize yourself with it/consult me for help. We use this form quite extensively in quantum mechanics

[28] Here we assume you know that $(\mathbf{sE})^\dagger = \mathbf{E}^\dagger\mathbf{s}^\dagger$