PC2232 Physics for Electrical Engineers: Tutorial 4

Question 1: Mach-Zender interferometer

A Mach-Zender interferometer built using 50–50 beam splitters and perfectly reflecting mirrors. The distance of each beam splitters from each mirrors is equal to are L_1 and L_2 respectively, as shown in Fig. 1. There are two input amplitudes initially having values $(E_{\text{in}1}, E_{\text{in}2})$.

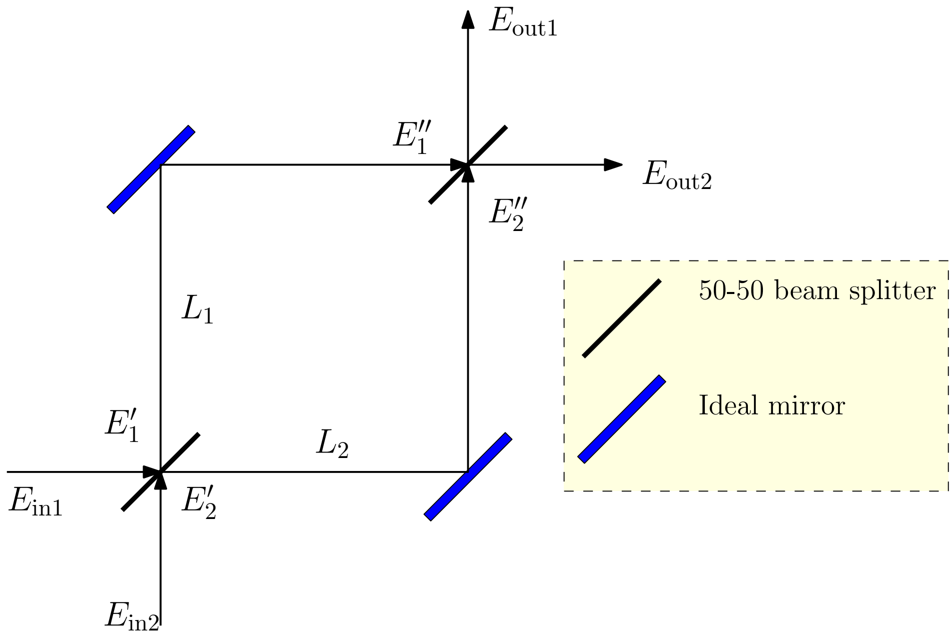

Figure 1: Mach-Zender interferometer

(a) Let light leaving the first beam splitter be expressed as (E_1', E_2'). Express E_1' and E_2' in terms of $E_{\text{in}1}$ and $E_{\text{in}2}$.

(b) Let the light just before entering the second beam splitter be (E_1'', E_2''). Express E_1'' and E_2'' in terms of $E_{\text{in}1}$ and $E_{\text{in}2}$.

(c) Calculate the output amplitudes $E_{\text{out}1}$ and $E_{\text{out}2}$ in terms of $E_{\text{in}1}$ and $E_{\text{in}2}$, and express in a matrix equation in the form

$$\begin{pmatrix} E_{\text{out}1} \\ E_{\text{out}2} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} E_{\text{in}1} \\ E_{\text{in}2} \end{pmatrix}. \tag{1}$$

(d) Is this matrix

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{2}$$

a unitary matrix?

Question 2: Isotropic medium

In an isotropic medium of permittivity ϵ, find the plane wave solution to Maxwell's equations. Show that the wave vector of the plane wave $\vec{k}$ satisfies

$$\frac{k^2}{\omega^2} = \mu_0 \epsilon, \quad \text{where} \quad k \equiv \sqrt{\vec{k} \cdot \vec{k}}. \tag{3}$$

What is the wavelength of the wave in the medium, and its ratio to its wavelength in vacuum?

Question 3: Optical pump

In an optical amplifier has $2k'' = -10\ \mathrm{cm}^{-1}$ [c.f. slide 6, Lecture 5, Eq. (13)], what is the length required to achieve 10 dB power gain?

Question 4: Anisotropic medium

The displacement field $\vec{D}$ is given as

$$\vec{D} = \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \equiv \overleftrightarrow{\epsilon} \cdot \vec{E}. \tag{4}$$

(a) Prove that, if $\vec{\nabla} \cdot \vec{D} = 0$, then $\vec{\nabla} \cdot \vec{E} \neq 0$ unless (i) $\epsilon_x = \epsilon_y = \epsilon_z$ (i.e., isotropic media.), or, (ii) $\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = \frac{\partial E_z}{\partial z} = 0$.

(b) Find the corresponding wave equation obeyed by $\vec{E}$.

(c) Show that

$$\vec{E} = \tilde{E}_x \exp\left(jk_x z - j\omega t\right)\ \hat{x} \tag{5}$$

is a solution to the wave equation.

Question 5: Expectation value

The expectation value of a probability density function $P(x)$ is defined as

$$\langle P \rangle = \int_{-\infty}^{\infty} \mathrm{d}x\, x P(x). \tag{6}$$

Given the probability density

$$P(x) = \begin{cases} A^2 \sin^2(kx), & 0 < x < L, \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

Find the expectation value given that $A = \sqrt{2/L}$ and $k = 2\pi/L$.

Question 6: Linearity of the Schrödinger equation (optional)

Suppose in a quantum system of potential U, there are two states which solve the time-independent Schrödinger equation, namely ψ_1 with energy E_1 and ψ_2 with energy E_2. Is the state defined by

$$\psi = A_1\psi_1 + A_2\psi_2 \tag{8}$$

also a solution to the time-independent Schrödinger's equation?