PC2232 Physics for Electrical Engineers: Tutorial 6

Question 1: Harmonic oscillator ground state

For the quantum harmonic oscillator, the potential of the system is given by

$$U(x) = \frac{1}{2}m\omega^2 x^2. \tag{1}$$

Suppose that $\psi(x)$ is a solution to the quantum harmonic oscillator, i.e., the Schrödinger equation is satisfied:

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{1}{2}m\omega^2 x^2\psi = E\psi. \tag{2}$$

(a) Show that a new wavefunction $\phi(x)$ defined by

$$\phi = \left(\hbar\frac{\mathrm{d}}{\mathrm{d}x} + m\omega x\right)\psi \tag{3}$$

Is also a solution to the Schrödinger equation of energy $E' = E - \hbar\omega$. That is, ϕ is a state of energy lower than ψ by a value of $\hbar\omega$.

(b) The ground state ψ_0 is defined as a state of the lowest possible energy. That means, if we use the above method to find a lower energy state ϕ by calculating

$$\phi = \left(\hbar\frac{\mathrm{d}}{\mathrm{d}x} + m\omega x\right)\psi_0, \tag{4}$$

does not exist. Therefore, set $\phi = 0$ and solve Eq. (4) to derive the ground state wavefunction of the harmonic oscillator.

Question 2: Step potential

Particles of energy E are incident from the left, where $U(x) = 0$, and at the origin encounter an abrupt drop in potential energy, whose depth is $-3E$.

(a) Classically what would the particles do, and what would happen to their kinetic energy?

(b) Applying quantum mechanics: Assume an incident wave of the form $\psi_{\text{inc}}(x) = \mathrm{e}^{\mathrm{i}kx}$, where the normalization constant has been given the simple value of 1. Determine completely the wave function everywhere, including numeric values for the multiplicative constants.

(c) What is the probability that incident particles will be reflected?

Question 3: Transmission resonance

In the $E > U_0$ potential barrier, there should be no reflected wave when the incident wave is at one of the transmission resonances. Prove this by assuming that a beam of particles is incident at the first transmission resonance, $E = U_0 + \frac{\pi^2\hbar^2}{2mL^2}$ prove that this is true by utilizing the continuity conditions. *Hint*: Your aim is to show that $B = 0$. (Note: k' is particularly simple in this special case.)

Question 4: Two-dimensional well

Consider an infinite, two-dimensional square well given by the potential

$$U(x, y) = \begin{cases} 0, & \text{if } 0 < x < L, \quad 0 < y < L \\ \infty, & \text{otherwise.} \end{cases} \tag{5}$$

(a) Solve the Schrödinger equation and find the possible wave functions.

(b) What are the allowed energies E?

(c) List the five lowest energy levels. For each level, write down the number of different states that correspond to that energy.

Question 5: Tunneling

An electron with initial kinetic energy 5.5 eV encounters a square potential barrier with height 10.0 eV. What is the width of the barrier if the electron has a 0.10% probability of tunneling through the barrier? What is the energy of the electron after it tunnels through the barrier?