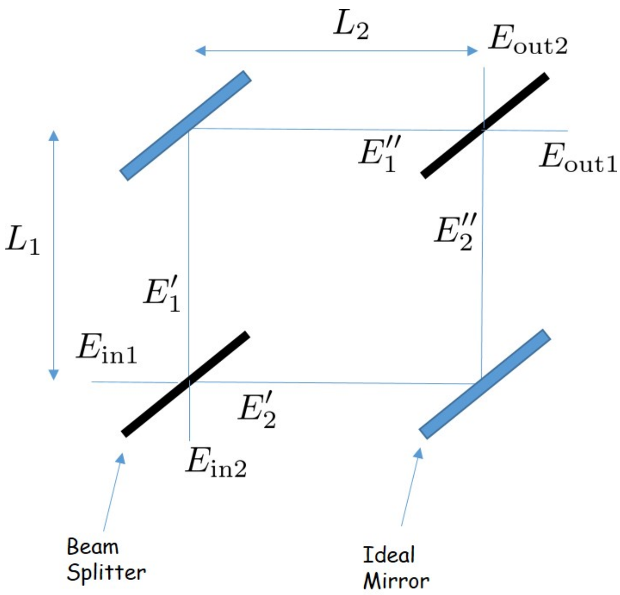
PC2232 – Tutorial 4 Solutions

1. Given:



(a) Express E_1' and E_2' in terms of E_{in1} and E_{in2}

Approach:
- Know the 50-50 beam splitter scattering matrix
- Do the math

Solution:
- 50-50 beam splitter: $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$
- Therefore, doing the math:

$$\begin{pmatrix} E_1' \\ E_2' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} E_{\text{in1}} \\ E_{\text{in2}} \end{pmatrix}$$

Answer:

$$E_1' = \frac{1}{\sqrt{2}}(E_{\text{in1}} + E_{\text{in2}}) \qquad E_2' = \frac{1}{\sqrt{2}}(E_{\text{in1}} - E_{\text{in2}})$$

(b) Express E_1'' and E_2'' in terms of E_{in1} and E_{in2}

Approach:
Know that as the light ray travels, it picks up a phase of e^{jkr} where r is the length traveled

Answer:

$$E_1'' = \mathrm{e}^{jk(L_1+L_2)}E_1' = \frac{1}{\sqrt{2}}\mathrm{e}^{jk(L_1+L_2)}(E_{\text{in1}} + E_{\text{in2}})$$
$$E_2'' = \mathrm{e}^{jk(L_1+L_2)}E_2' = \frac{1}{\sqrt{2}}\mathrm{e}^{jk(L_1+L_2)}(E_{\text{in1}} - E_{\text{in2}})$$

(c) Express E_{out1} and E_{out2} in terms of E_{in1} and E_{in2} in matrix form

Approach:
Pass the light through the beam splitter again
Reexpress it in terms of E_{in1} and E_{in2}

Solution:

$$\begin{pmatrix} E_{\text{out1}} \\ E_{\text{out2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} E_1'' \\ E_2'' \end{pmatrix}$$

Therefore

$$E_{\text{out1}} = \frac{1}{2}\mathrm{e}^{jk(L_1+L_2)}(E_{\text{in1}} + E_{\text{in2}}) + \frac{1}{2}\mathrm{e}^{jk(L_1+L_2)}(E_{\text{in1}} - E_{\text{in2}}) = \mathrm{e}^{jk(L_1+L_2)}E_{\text{in1}}$$
$$E_{\text{out2}} = \frac{1}{2}\mathrm{e}^{jk(L_1+L_2)}(E_{\text{in1}} + E_{\text{in2}}) - \frac{1}{2}\mathrm{e}^{jk(L_1+L_2)}(E_{\text{in1}} - E_{\text{in2}}) = \mathrm{e}^{jk(L_1+L_2)}E_{\text{in2}}$$

Reexpressing it in terms of matrix[1]

$$\begin{pmatrix} E_{\text{out1}} \\ E_{\text{out2}} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{jk(L_1+L_2)} & 0 \\ 0 & \mathrm{e}^{jk(L_1+L_2)} \end{pmatrix} \begin{pmatrix} E_{\text{in1}} \\ E_{\text{in2}} \end{pmatrix}$$

Answer:

$$M = \begin{pmatrix} \mathrm{e}^{jk(L_1+L_2)} & 0 \\ 0 & \mathrm{e}^{jk(L_1+L_2)} \end{pmatrix}$$

(d) Check if M is unitary

Approach:
If M is unitary, $M^\dagger M = \mathbb{I}$

$$M^\dagger M = \begin{pmatrix} \mathrm{e}^{-jk(L_1+L_2)} & 0 \\ 0 & \mathrm{e}^{-jk(L_1+L_2)} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{jk(L_1+L_2)} & 0 \\ 0 & \mathrm{e}^{jk(L_1+L_2)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Learning point:

- Familiarize yourself with the scattering matrix
- Familiarize yourself with inteferometer
- Familiarize yourself with matrix manipulation

2. Given: Isotropic medium of permittivity ε
Find plane wave solution to Maxwell's equation and show that the wave vector satisfies

$$\frac{k^2}{\omega^2} = \mu_0 \varepsilon$$

What is the wavelength of wave in the medium?

Approach:

- Know that the general form of a plane wave solution is the same in vacuum and in isotropic medium
- Substitute the solution into the wave equation to obtain the dispersion relationship above
- Know how k is related to λ
- Learning point
  - Understand that an isotropic medium does not really change the wave equation and any other property of the wave except the speed of the wave
  - Thoroughly understand the idea that not all plane waves can satisfy Maxwell equations. Only the plane waves that obey the dispersion relationship can.
  - Notice how n is related to the various quantities

Solution:

- Deriving the wave equation – this is not necessary in this question. I'm doing this as a refresher
  - Start from Maxwell equation (Note this is source free):

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad\qquad \vec{\nabla} \cdot \vec{H} = 0$$
$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \qquad\qquad \vec{\nabla} \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

We will start with the $\vec{\nabla} \times \vec{E}$ equation.

[1] If you can't see this, convince yourself this is true by working out the matrix multiplication

- Using the identity: $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

$$\begin{aligned}
\vec{\nabla} \times \vec{\nabla} \times \vec{E} &= -\mu_0 \frac{\partial}{\partial t} \vec{\nabla} \times \vec{H} \\
\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} &= -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \\
\nabla^2 \vec{E} &= \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} && \because \vec{\nabla} \cdot \vec{E} = 0 \\
&= \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}
\end{aligned}$$

Note: $c = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}}$

- Understand that in an isotropic medium all the above still holds[2] The only change is to replace ε_0 with ε
  - Maxwell equations:

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \cdot \vec{H} = 0$$

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \qquad \vec{\nabla} \times \vec{H} = \boxed{\varepsilon} \frac{\partial \vec{E}}{\partial t}$$

  - Therefore, derive the wave equation

$$\begin{aligned}
\nabla^2 \vec{E} &= \mu_0 \varepsilon \frac{\partial \vec{E}}{\partial t} \\
&= \frac{1}{v^2} \frac{\partial \vec{E}}{\partial t}
\end{aligned}$$

Note: $v = \dfrac{1}{\sqrt{\varepsilon \mu_0}}$

Therefore, we define $n = \dfrac{c}{v} = \sqrt{\dfrac{\varepsilon}{\varepsilon_0}}$

- Plane wave solution: $\vec{E} = \tilde{\vec{E}} \mathrm{e}^{-j\vec{k}\cdot\vec{r}} \mathrm{e}^{-j\omega t} = \tilde{\vec{E}} \mathrm{e}^{j(k_x x + k_y y + k_z z)} \mathrm{e}^{-j\omega t}$
- Substituting this into the new wave equation to obtain new dispersion relationship

$$\nabla^2 \vec{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left(\tilde{\vec{E}} \mathrm{e}^{j(k_x x + k_y y + k_z z)} \mathrm{e}^{-j\omega t} \right) = -(k_x^2 + k_y^2 + k_z^2) \vec{E}$$

$$\mu_0 \varepsilon \frac{\partial \vec{E}}{\partial t} = \mu_0 \varepsilon \frac{\partial}{\partial t} \tilde{\vec{E}} \mathrm{e}^{j(k_x x + k_y y + k_z z)} \mathrm{e}^{-j\omega t} = -\omega^2 \mu_0 \varepsilon \vec{E}$$

Therefore, a plane wave solution that satisfies the relationship is a solution.

$$\vec{k} \cdot \vec{k} = \omega^2 \mu_0 \varepsilon$$

- Determine how λ is related to λ_0

Let the one with subscript 0 be in vacuum

$$k = \frac{\omega}{v} = \frac{\omega n}{c} = n k_0$$

We know that $k = \dfrac{2\pi}{\lambda}$

$$\begin{aligned}
k &= n k_0 \\
\frac{2\pi}{\lambda} &= \frac{2\pi n}{\lambda_0} \\
\lambda &= \frac{\lambda_0}{n}
\end{aligned}$$

Answer:

Plane wave solution: $\vec{E} = \tilde{\vec{E}} \mathrm{e}^{-j\vec{k}\cdot\vec{r}} \mathrm{e}^{j\omega t} = \tilde{\vec{E}} \mathrm{e}^{-j(k_x x + k_y y + k_z z)} \mathrm{e}^{j\omega t}$

Relationship for λ: $\lambda = \dfrac{\lambda_0}{n}$

[2]ref: Lecture 2 pg 12.

3. Given: $2k'' = -10\text{cm}^{-1}$
   Question: What is the length required to achieve 10dB power gain?

   Approach:

   - Know $\vec{E}$ for plane wave
   - Know how to find $\bar{S}$
   - Know how to define power loss
   - Learning point:
     - Understand what happens when ε can be imaginary – how it affects $\vec{E}$ and other things
     - Know how to obtain $\bar{S}$
     - Know the power loss equation

   Solution:

   - Electric field: $\vec{E} = \vec{\tilde{E}} \mathrm{e}^{jk'x} \mathrm{e}^{5x} \mathrm{e}^{-j\omega t}$
   - Obtaining $\bar{S}$
     Recall: $Z = \dfrac{\mu_0}{\varepsilon}$

$$\bar{S}(x) = \frac{1}{2}\mathrm{Re}\left[\frac{EE^*}{Z}\right] = \frac{1}{2}\mathrm{Re}\left[\frac{EE^*}{Z_0}n\right] = \mathrm{Re}\left[\frac{1}{2Z_0}|E|^2 n\right] = \mathrm{Re}\left[\frac{\tilde{E}^2}{2Z_0}n\right]\mathrm{e}^{5x}$$

   - Power loss/gain: αx dB

$$\alpha x = 10\log_{10}\frac{\bar{S}(x)}{\bar{S}(0)} = 10\log_{10}\left(\mathrm{e}^{5x}\right) = 50x\log_{10}\mathrm{e}$$
$$\therefore \quad \alpha = 50\log_{10}\mathrm{e} = 43.43$$

   Therefore, to find x to achieve 10dB,

$$10 = \alpha x$$

   Answer:

$$x = 0.2303\text{cm}$$

4. Given:

$$\vec{D} = \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

   (a) Prove that: If $\vec{\nabla}\cdot\vec{D} = 0$, then $\vec{\nabla}\cdot\vec{E} \neq 0$ unless
   - Medium is isotropic
   - $\dfrac{\partial E_x}{\partial x} = \dfrac{\partial E_y}{\partial y} = \dfrac{\partial E_z}{\partial z}$

   Approach:
   Mathematically write out the equation and conclude from the equation
   Learning point
   Learn about the Maxwell equation in media: $\vec{\nabla}\cdot\vec{D} = 0$ and $\vec{\nabla}\cdot\vec{E} \neq 0$

   Solution:

$$\begin{aligned} \vec{\nabla}\cdot\vec{D} &= \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\cdot(D_x\hat{x} + D_y\hat{y} + D_z\hat{z}) \\ 0 &= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \\ &= \frac{\partial \varepsilon_x E_x}{\partial x} + \frac{\partial \varepsilon_y E_y}{\partial y} + \frac{\partial \varepsilon_z E_z}{\partial z} \\ &= \varepsilon_x\frac{\partial E_x}{\partial x} + \varepsilon_y\frac{\partial E_y}{\partial y} + \varepsilon_z\frac{\partial E_z}{\partial z} \end{aligned}$$

   Notice that given the constraint $\vec{\nabla}\cdot\vec{D} = 0$, it is not necessary for $\vec{\nabla}\cdot\vec{E}$ to be 0.
   $\vec{\nabla}\cdot\vec{E}$ will be 0 under the following two conditions:

- If $\varepsilon_x = \varepsilon_y = \varepsilon_z = \varepsilon$,

$$\varepsilon \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) = 0$$

Because $\varepsilon \neq 0$, therefore $\vec{\nabla} \cdot \vec{E} = 0$

- If $\dfrac{\partial E_x}{\partial x} = \dfrac{\partial E_x}{\partial x} = \dfrac{\partial E_x}{\partial z} = 0$

(b) Find the corresponding wave equation

Approach:
- Start from Maxwell equation
- Using the identity: $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

Solution
- Start from Maxwell equation (Note this is source free):

$$\vec{\nabla} \cdot \vec{E} = C \qquad \vec{\nabla} \cdot \vec{H} = 0$$

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \qquad \vec{\nabla} \times \vec{H} = \boldsymbol{\varepsilon} \frac{\partial \vec{E}}{\partial t}$$

Note that $\vec{\nabla} \cdot E \neq 0$, but some scalar C
Note also that $\boldsymbol{\varepsilon}$ is a matrix
We will start with the $\vec{\nabla} \times \vec{E}$ equation.

- Using the identity:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial^2}{\partial t^2} \vec{\nabla} \times \vec{H}$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu_0 \boldsymbol{\varepsilon} \frac{\partial^2 \vec{E}}{\partial t^2}$$

Note: Left hand side has two terms now because $\vec{\nabla} \cdot \vec{E} \neq 0$

Answer:
Wave equation:

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu_0 \boldsymbol{\varepsilon} \frac{\partial^2 \vec{E}}{\partial t^2}$$

(c) Given: $\vec{E} = \tilde{E}_x \mathrm{e}^{jk_{x_{\mathrm{pol}}} z - j\omega t}$
Show that the above is a solution to the wave equation

Approach:
- First note that $k_{x_{\mathrm{pol}}}$ is pointing in the z direction
- Substituting into the wave equation:

$$\vec{\nabla}(\vec{\nabla}) \cdot \vec{E} = 0$$

Because E is in the $\hat{x}$ direction and therefore $\vec{\nabla} \cdot E = \dfrac{\partial E}{\partial x} = 0$ since E does not depend on x

$$\nabla^2 \vec{E} = \frac{\partial^2}{\partial z^2} \tilde{E}_x \mathrm{e}^{jk_{x_{\mathrm{pol}}} z - j\omega t} = -k_z^2 \tilde{E}_x \mathrm{e}^{jk_{x_{\mathrm{pol}}} z - j\omega t}$$

$$\mu_0 \boldsymbol{\varepsilon} \frac{\partial^2 \vec{E}}{\partial t^2} = \omega^2 \mu_0 \varepsilon_0 \tilde{E}_x \mathrm{e}^{jk_{x_{\mathrm{pol}}} z - j\omega t}$$

- Therefore, this is a solution to the wave equation if it satisfies the following condition

$$\mu_0 \boldsymbol{\varepsilon} \omega^2 = k_{x_{\mathrm{pot}}}^2$$