PC2232 Physics for Electrical Engineers
Lecture 5: Loss, Gain, Dispersion, Birefringence

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Permittivity

- We discussed a simple model of EM waves in media, where the permittivity is replaced by a scalar constant ϵ.
- A more realistic treatment is to define a displacement field $\boldsymbol{D}$:

$$\nabla \cdot \boldsymbol{D} = 0, \qquad \nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}, \tag{1}$$

  and write $\boldsymbol{D}$ as a function of $\boldsymbol{E}$.
- Suppose $\boldsymbol{E}$ oscillates in time as $\exp(-j\omega t)$:

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{\mathcal{E}}(\boldsymbol{r}) \exp(-j\omega t). \tag{2}$$

  In a more realistic model of linear optical medium, $\boldsymbol{D}$ can be written as

$$\boldsymbol{D}(\boldsymbol{r}, t) = \boldsymbol{\epsilon}(\omega)\boldsymbol{\mathcal{E}}(\boldsymbol{r}) \exp(-j\omega t), \tag{3}$$

  where $\boldsymbol{\epsilon}(\omega)$ may be **complex**, **frequency-dependent**, and a 3×3 **matrix**.

Complex Refractive Index

- First suppose that $\epsilon(\omega)$ is a complex scalar.
- Consider a plane wave:

$$\boldsymbol{E} = \tilde{\boldsymbol{E}} \exp(j\boldsymbol{k} \cdot \boldsymbol{r} - j\omega t), \qquad \boldsymbol{H} = \tilde{\boldsymbol{H}} \exp(j\boldsymbol{k} \cdot \boldsymbol{r} - j\omega t). \tag{4}$$

The Gauss laws lead to

$$\boldsymbol{k} \cdot \tilde{\boldsymbol{E}} = 0, \qquad \boldsymbol{k} \cdot \tilde{\boldsymbol{H}} = 0, \tag{5}$$

we still have transverse waves.

- The wave equation becomes

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} \to j\boldsymbol{k} \times \tilde{\boldsymbol{H}} = -j\omega\epsilon(\omega)\tilde{\boldsymbol{E}}, \qquad \nabla \times \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} \to j\boldsymbol{k} \times \tilde{\boldsymbol{E}} = j\omega\mu_0\tilde{\boldsymbol{H}}. \tag{6}$$

These lead to

$$\boldsymbol{k} \times (\boldsymbol{k} \times \tilde{\boldsymbol{E}}) = -(\boldsymbol{k} \cdot \boldsymbol{k})\tilde{\boldsymbol{E}} = -\omega^2\mu_0\epsilon(\omega)\tilde{\boldsymbol{E}}. \tag{7}$$

If $\epsilon(\omega)$ is complex, $\boldsymbol{k} \cdot \boldsymbol{k} = k^2$ is also complex,

$$k^2 = \omega^2\mu_0\epsilon(\omega), \qquad k = \omega\sqrt{\mu_0\epsilon(\omega)} = \frac{\omega n(\omega)}{c}, \qquad n(\omega) \equiv \sqrt{\frac{\epsilon(\omega)}{\epsilon_0}}. \tag{8}$$

$n(\omega)$ and therefore k are **complex** and **frequency-dependent**.

Optical Loss

- Let's write $n(\omega)$ and k in terms of their real and imaginary parts:

$$n(\omega) = n'(\omega) + jn''(\omega), \qquad k = k' + jk'' = \frac{\omega}{c}\left[n'(\omega) + jn''(\omega)\right]. \tag{9}$$

- Let's say $\boldsymbol{k} = k\hat{\boldsymbol{z}}$. Note that

$$\exp(jkz) = \exp(jk'z - k''z), \tag{10}$$

which decays exponentially if $n''(\omega)$ and therefore k'' are positive.

- Suppose $\tilde{\boldsymbol{E}} = \tilde{E}\hat{\boldsymbol{x}}$, $\tilde{\boldsymbol{H}} = \tilde{H}\hat{\boldsymbol{y}}$. The magnetic-field amplitude is $\tilde{H} = \frac{\tilde{E}}{Z} = n(\omega)\frac{\tilde{E}}{Z_0}$, and the intensity is

$$\overline{\boldsymbol{S}} = \frac{1}{2}\operatorname{Re}\left(\boldsymbol{E} \times \boldsymbol{H}^*\right) = \hat{\boldsymbol{z}}\frac{|\tilde{E}|^2}{2Z_0}\operatorname{Re}\left[n^*(\omega)e^{jkz-j\omega t}e^{-jk^*z+j\omega t}\right] = \hat{\boldsymbol{z}}\frac{|\tilde{E}|^2}{2Z_0}n'(\omega)e^{-2k''z}, \tag{11}$$

which decays exponentially with a decay constant of $2k'' = 2\omega n''(\omega)/c$.

- Power loss is often measured in dB. Decay coefficient α is usually measured in terms of power loss in dB per unit length.

$$\boxed{-\alpha z \text{ [dB]} = 10\log_{10}\frac{\bar{S}(z)}{\bar{S}(0)} = 10\log_{10}\exp(-2k''z) = -20k''z\log_{10}e.} \tag{12}$$

- For example, power loss in an typical optical fiber is around $\alpha = 0.2$ dB/km at $\lambda_0 = 1550$ nm (free-space wavelength, convert to frequency by $\omega = 2\pi\nu = 2\pi c/\lambda_0$).

Silica Optical Loss

- In reality, there are a lot of reasons behind optical loss of a material (absorption by material, absorption by impurities, scattering by small defects, etc.)
- The extremely low loss of high-purity silica enables modern optical fiber communications:

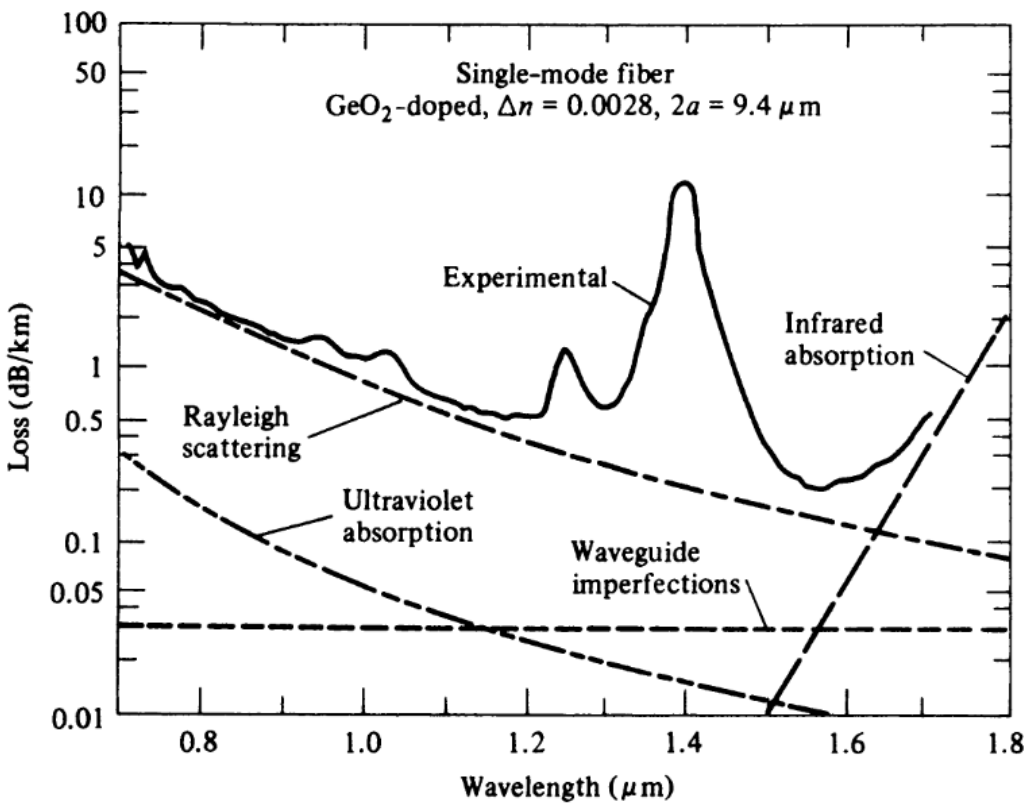


Figure 3.21 Observed loss spectrum of a germanosilicate single-mode fiber. Estimated loss spectra for various intrinsic material effects and waveguide imperfections are also shown. (From Reference [24].)

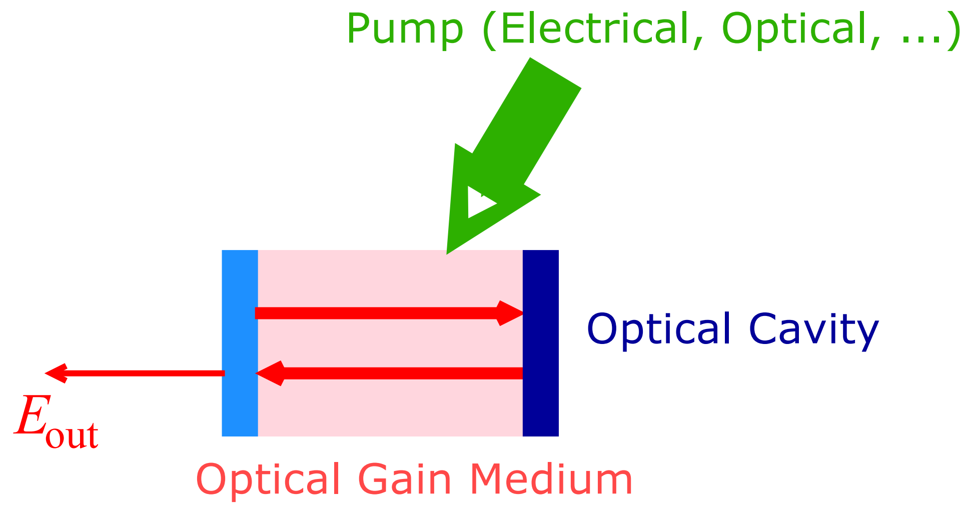
- Lowest loss happens around $\lambda_0 =$ 1530–1570 nm. Many optical devices and components are developed for optical communications around the 1550 nm wavelength.

Optical Gain

- Under special conditions (more on this later), a medium can also provide gain to the EM wave. Then $n''(\omega)$ is negative, and the power grows exponentially along z:

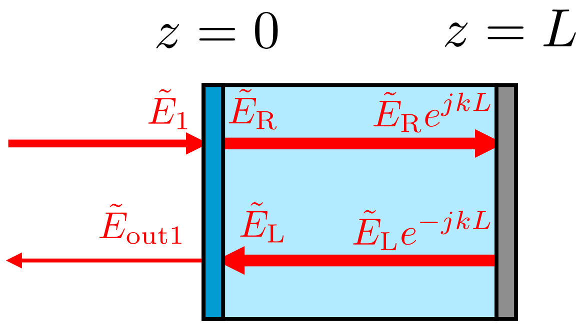
$$\bar{S} = \hat{\boldsymbol{z}}\frac{|\tilde{E}|^2}{2Z_0}n'(\omega)e^{-2k''z} = \hat{\boldsymbol{z}}\frac{|\tilde{E}|^2}{2Z_0}n'(\omega)e^{2|k''|z} \quad (k'' < 0) \tag{13}$$

- The increase in optical energy must be supplied by a **pump**.
- An **optical amplifier** uses gain to increase the power of optical signals. For example, erbium-doped fiber amplifiers (EDFA) provide optical amplification near 1550 nm.
- A laser consists of an optical amplifier inside a **Fabry-Perot cavity**.

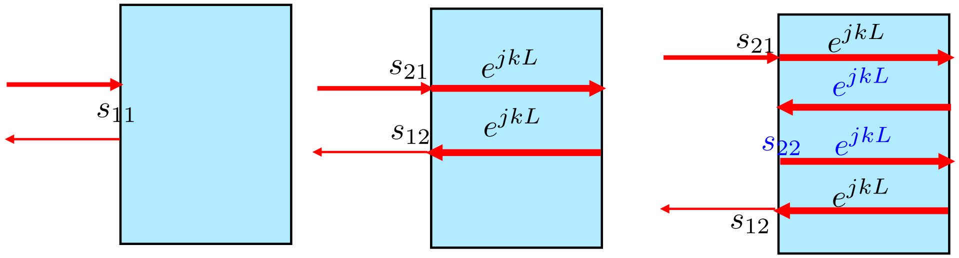


- LASER = Light Amplification by Stimulated Emission of Radiation.
- We will learn about the quantum phenomenon of stimulated emission later.

- Consider a partial mirror on the left, a perfect mirror on the right (reflection coefficient = 1 for simplicity), and an optical gain medium in-between.



- Sum over partial waves to obtain final reflection coefficient:



$$\tilde{E}_{\mathrm{out1}} = \left[s_{11} + s_{12} s_{21} e^{2jkL} \left(1 + G + G^2 + \dots \right) \right] \tilde{E}_1, \tag{14}$$

where

$$G = s_{22} e^{2jkL} = s_{22} e^{2jk'L} e^{-2k''L} \tag{15}$$

is the factor corresponding to one round trip.

Threshold Condition

- When $k'' \propto n''(\omega) < 0$ (gain), and the gain overcomes the loss at the partial mirror ($|s_{22}| < 1$), the round-trip factor can have a magnitude $|G| \geq 1$. The infinite series

$$\sum_{m=0}^{\infty} G^m \tag{16}$$

  does not converge!
- There is something wrong with our assumptions: we assumed solutions $\exp(-j\omega t)$ at all times, but that does not work if the system is **unstable**.
- When $|G| > 1$, the system is **unstable**, and the solution should grow **exponentially in time** starting from an initial condition.
- $|G| > 1$ is known as **the threshold condition for a laser**. It leads to huge increase in energy transfer from the pump to the optical beam.
- The exponential increase cannot last forever. Eventually the gain will saturate ($n''(\omega)$ becomes less negative in response to high optical energy inside cavity), and the laser settles into a steady state.

Initial Condition?

- Riddle?
  - Q: A student decides to start working hard. To ramp up his work load, he decides that each day he/she will spend twice as much time to study compared with the day before. On the Nth day, how long is he going to study?
  - A: If he doesn't study at all to begin with, he will never study!
- Exponential increase in time $f(x) = f(0)\exp(\gamma t)$ is zero if $f(0)$ is zero.
- For an unstable system like a laser above the threshold condition, it still needs a nonzero initial condition to start according to our (classical) analysis.
- The full laser operation will require a quantum theory of light-matter interaction to explain.

Dispersion

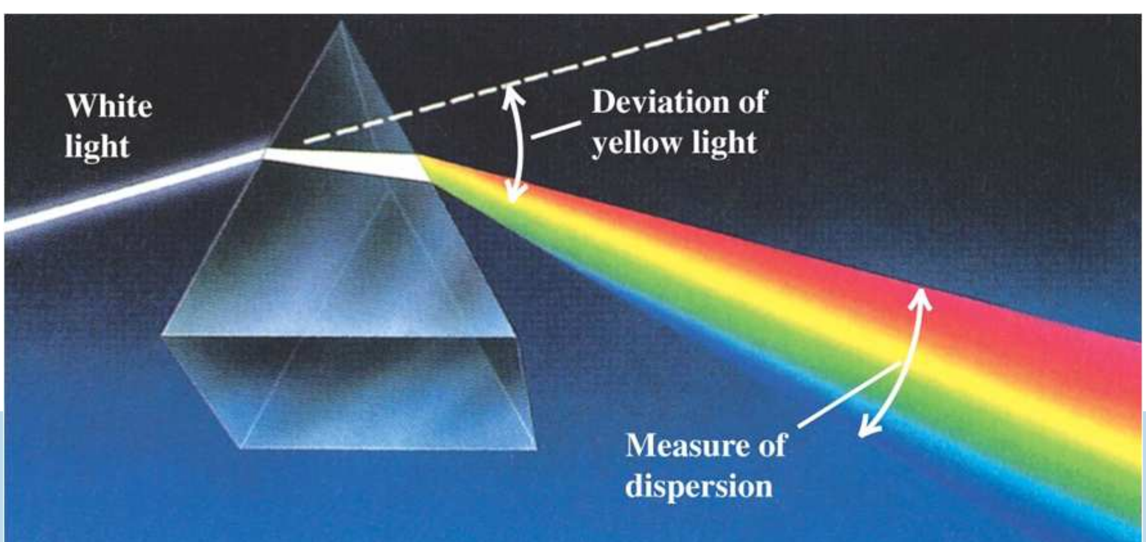
- Let us now focus on the real part of $n(\omega)$, assume that the imaginary part is negligible (i.e. $n(\omega)$ is real), and consider the effect of frequency-dependent refractive index.
- With

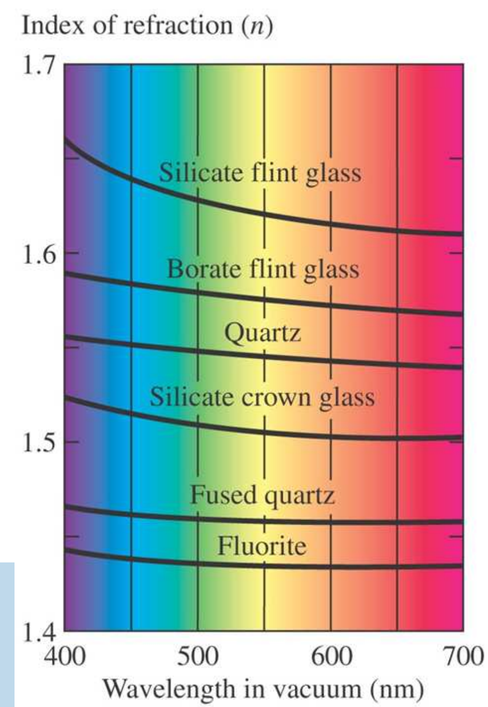
$$\boldsymbol{E} = \tilde{\boldsymbol{E}} \exp\left[j\frac{\omega n(\omega) z}{c} - j\omega t\right], \tag{17}$$

different frequencies propagate at different speeds (phase velocity):

$$v(\omega) = \frac{c}{n(\omega)}. \tag{18}$$

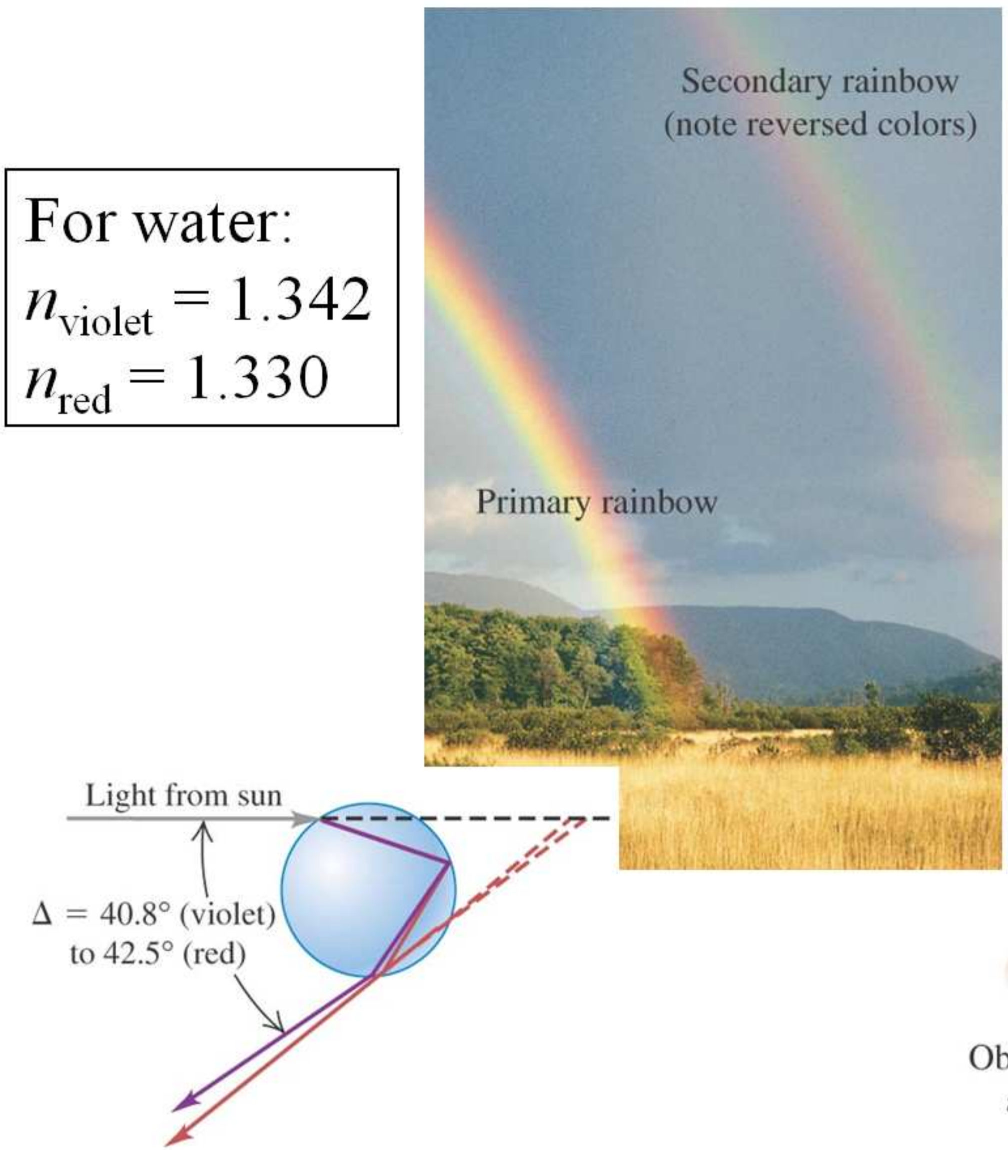
- The refracted angle will depend on frequency:

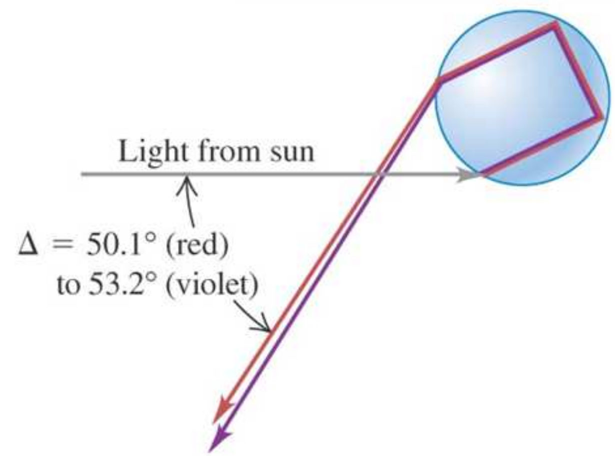


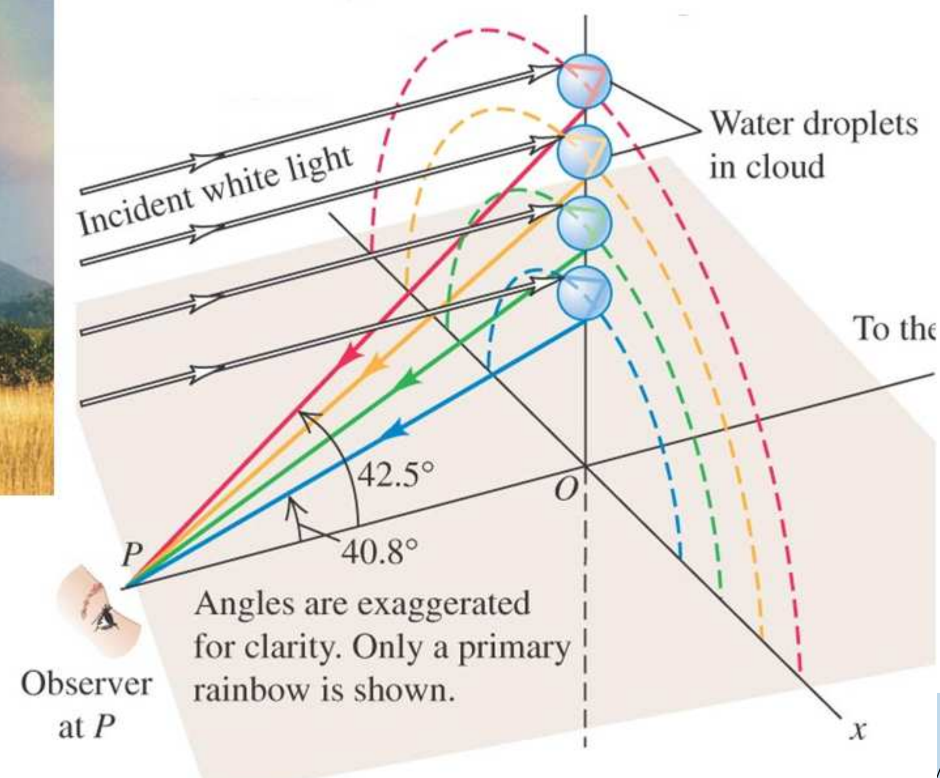


Rainbows

For water:
$n_{\text{violet}} = 1.342$
$n_{\text{red}} = 1.330$







*Pulse Propagation

- Remarkably, in some materials the phase velocity can exceed c.
- One cannot send information using just $\exp(-j\omega t)$ with one frequency however. There must be some additional variations in time (e.g., switch the signal on and off, or $a(t)\exp(-j\omega t)$) to send information.
- At $z = 0$, suppose that the input signal is $E(0,t)$. I can write it as an **inverse Fourier transform**:

$$E(0,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{E}(\omega)\exp(-j\omega t), \qquad \tilde{E}(\omega) = \int_{-\infty}^{\infty} dt E(0,t)\exp(j\omega t). \tag{19}$$

  $\tilde{E}(\omega)$ is a frequency-domain amplitude.
- At another z,

$$E(z,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{E}(\omega)\exp\left[jk(\omega)z - j\omega t\right], \qquad k(\omega) = \frac{\omega n(\omega)}{c}. \tag{20}$$

  This is a **continuous superposition of plane waves** and matches the boundary condition.
- Suppose that

$$E(0,t) = a(t)\exp(-j\omega_0 t), \qquad \tilde{E}(\omega) = \int_{-\infty}^{\infty} dt a(t)\exp[j(\omega - \omega_0)t] = \tilde{a}(\omega - \omega_0), \tag{21}$$

  where $a(t)$ is a pulse envelope and $\tilde{a}(\omega)$ is its Fourier transform. Let's expand $k(\omega)$ in the first order near ω_0:

$$k(\omega) \approx k(\omega_0) + k^{(1)}(\omega_0)(\omega - \omega_0), \qquad k^{(1)}(\omega) = \left.\frac{dk(\omega)}{d\omega}\right|_{\omega=\omega_0}. \tag{22}$$

*Group Velocity

- $$E(z,t) \approx \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{E}(\omega) \exp\left\{j\left[k(\omega_0) + k^{(1)}(\omega_0)(\omega-\omega_0)\right] z - j\omega t\right\} \tag{23}$$
$$= e^{jk(\omega_0)z - j\omega_0 t} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{a}(\omega-\omega_0) e^{jk^{(1)}(\omega_0)(\omega-\omega_0)z - j(\omega-\omega_0)t} \tag{24}$$
$$= a(t - k^{(1)}(\omega_0)z) e^{jk(\omega_0)z - j\omega_0 t}. \tag{25}$$

  The envelope propagates with **group velocity** $1/k^{(1)}(\omega_0)$, while the phase propagates with **phase velocity** $\omega_0/k(\omega_0)$.
- Remarkably, the group velocity can also exceed c in some materials.
- In reality, $n(\omega)$ has higher orders and is complex. It can be shown that, information cannot travel faster than c when we remove the above approximations. See L. Brillouin, *Wave Propagation and Group Velocity* (Academic Press, New York, 1960).
- The higher-order terms in $n(\omega)$ are important in fiber-optic communications, as they can distort the signals.

Anistropic Medium

- Consider now the possibility that $\boldsymbol{\epsilon}$ is a matrix. The medium is then called anisotropic. Suppose we align our coordinate system such that $\boldsymbol{\epsilon}$ is diagonal:

$$\boldsymbol{\mathcal{D}} = \begin{pmatrix} \mathcal{D}_x \\ \mathcal{D}_y \\ \mathcal{D}_z \end{pmatrix} = \begin{pmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{pmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \\ \mathcal{E}_z \end{pmatrix}. \tag{26}$$

  Consider $\epsilon_{x,y,z}$ to be real for simplicity. The different values for $\epsilon_{x,y,z}$ mean that the wave speed is different depending on the polarization.
- Suppose that a plane wave propagates in the z direction. An x-polarized wave will propagate with a speed of

$$v_x = \frac{1}{\sqrt{\mu_0 \epsilon_x}} = \frac{c}{n_x}, \qquad n_x \equiv \sqrt{\frac{\epsilon_x}{\epsilon_0}}, \tag{27}$$

  and a y-polarized wave will propagate with a speed of

$$v_y = \frac{1}{\sqrt{\mu_0 \epsilon_y}} = \frac{c}{n_y}, \qquad n_y \equiv \sqrt{\frac{\epsilon_y}{\epsilon_0}}. \tag{28}$$

Typical Values

- "Isotropic" refers to $\epsilon_x = \epsilon_y = \epsilon_z$, i.e. the matrix is equivalent to a scalar. "Uniaxial" refers to two of the three components being equal (e.g., $\epsilon_x = \epsilon_y \neq \epsilon_z$).

TABLE 1.3 Refractive Indices[a] of Some Typical Solid Crystals

Isotropic		Fluorite		1.392	
		Sodium chloride, NaCl		1.544	
		Diamond, C		2.417	
		CdTe		2.69	
		GaAs		3.40	
		Ge		3.40	
		InP		3.61	
		GaP		3.73	
Uniaxial			n_o		n_e
	Positive	MgF_2	1.378		1.390
		Quartz, SiO_2	1.544		1.553
		Beryllium oxide, BeO	1.717		1.732
		$La_3Ga_5SiO_{14}$	1.90		1.91
		ZnO	1.94		1.96
		SnO_2	2.01		2.10
		YVO_4	1.96		2.16
		$LiTaO_3$	2.183		2.188
		ZnS	2.354		2.358
		Rutile, TiO_2	2.616		2.903
	Negative	KDP, KH_2PO_4	1.507		1.467
		ADP, $(NH_4)\ H_2PO_4$	1.522		1.478
		Beryl, $Be_3Al_2(SiO_3)_6$	1.598		1.590
		Sodium nitrate, $NaNO_3$	1.587		1.366
		Calcite, $CaCO_3$	1.658		1.486
		β-BaB_2O_4 (BBO)	1.67		1.55
		Sapphire, Al_2O_3	1.768		1.760
		Lithium niobate, $LiNbO_3$	2.300		2.208
		$PbMoO_3$	2.40		2.27
		Proustite, Ag_3AsS_3	3.019		2.739
Biaxial			n_x	n_y	n_z
		Gypsum	1.520	1.523	1.530
		Feldspar	1.522	1.526	1.530
		Mica	1.552	1.582	1.588
		Topaz	1.619	1.620	1.627
		Sodium nitrite, $NaNO_2$	1.344	1.411	1.651
		$YAlO_3$	1.923	1.938	1.947
		SbSI	2.7	1.7	3.8

[a] The refractive indices of most materials depend on the wavelength (dispersion). The listed numbers are typical values.

From Yariv and Yeh, *Photonics: Optical Electronics in Modern Communications*

Wave Plate

- Suppose that, at $z = 0$, the input is

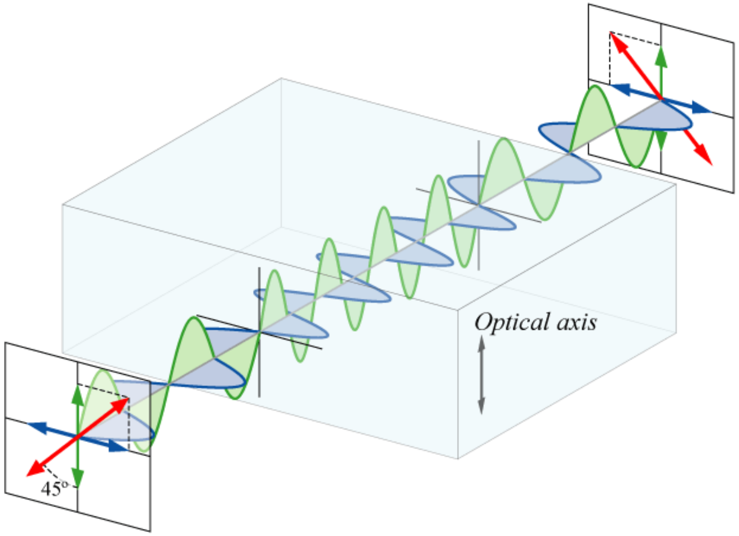
$$\boldsymbol{E}(0,t) = \left(\hat{\boldsymbol{x}}\tilde{E}_x + \hat{\boldsymbol{y}}\tilde{E}_y\right) e^{-j\omega t}. \tag{29}$$

What is the output at $z = L$?

$$\boldsymbol{E}(z,t) = \hat{\boldsymbol{x}}\tilde{E}_x \exp\left(j\frac{\omega n_x}{c}z - j\omega t\right) + \hat{\boldsymbol{y}}\tilde{E}_y \exp\left(j\frac{\omega n_y}{c}z - j\omega t\right), \tag{30}$$

$$\boldsymbol{E}(L,t) = \left(\hat{\boldsymbol{x}}\tilde{E}_x + \hat{\boldsymbol{y}}\tilde{E}_y e^{j\theta}\right) e^{j\omega n_x L/c - j\omega t}, \quad \theta = \frac{\omega L}{c}(n_y - n_x). \tag{31}$$

There is a relative phase delay between the two polarizations.



http://en.wikipedia.org/wiki/File:Waveplate.png

Half-Wave Plate

- Suppose that the thickness L is made such that the relative phase becomes $2\pi q + \pi$, q an integer,

$$\theta = 2\pi q + \pi, \qquad \frac{\omega L}{c}(n_y - n_x) = \frac{2\pi L}{\lambda}(n_y - n_x) = 2\pi q + \pi. \tag{32}$$

  This is called a half-wave plate.
- Suppose that the input is linearly polarized 45° with respect to $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$:

$$\boldsymbol{E}(0,t) = \tilde{E}\frac{\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}}{\sqrt{2}} e^{-j\omega t}. \tag{33}$$

- At the output,

$$\boldsymbol{E}(L,t) = \tilde{E}\frac{\hat{\boldsymbol{x}} + e^{j\theta}\hat{\boldsymbol{y}}}{\sqrt{2}} e^{j\omega L n_x z/c - j\omega t} = \tilde{E}\frac{\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}}{\sqrt{2}} e^{j\omega L n_x z/c - j\omega t}, \tag{34}$$

  the sign of the y component is flipped with respect to the x component, and the polarization is rotated by 90°.
- n_x and n_y of the liquid crystal in a LCD are controlled by the applied voltage. Depending on the voltage, the polarization of the light can be rotated by 90°, leading to voltage-dependent transmission of light through the cross-polarizers.

Quarter-Wave Plate

- Now suppose that the relative phase becomes $2\pi q + \pi/2$:

$$\theta = 2\pi q + \frac{\pi}{2}, \qquad \frac{\omega L}{c}(n_y - n_x) = \frac{2\pi L}{\lambda}(n_y - n_x) = 2\pi q + \frac{\pi}{2}. \tag{35}$$

  This is called a quarter-wave plate.
- Suppose that the input is linearly polarized 45° with respect to $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$:

$$\boldsymbol{E}(0,t) = \tilde{E}\frac{\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}}{\sqrt{2}} e^{-j\omega t}. \tag{36}$$

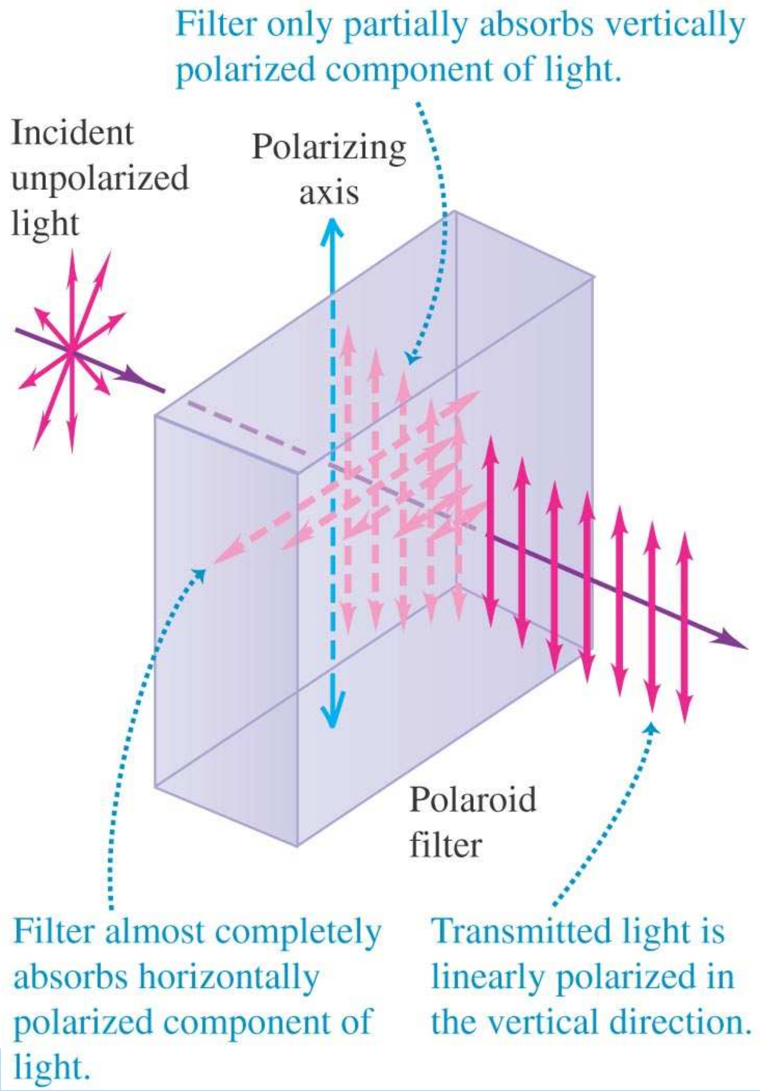
- At the output,

$$\boldsymbol{E}(L,t) = \tilde{E}\frac{\hat{\boldsymbol{x}} + e^{j\theta}\hat{\boldsymbol{y}}}{\sqrt{2}} e^{j\omega L n_x z/c - j\omega t} = \tilde{E}\frac{\hat{\boldsymbol{x}} + j\hat{\boldsymbol{y}}}{\sqrt{2}} e^{j\omega L n_x z/c - j\omega t}, \tag{37}$$

  The output polarization becomes circularly polarized.

Polarizer

- Very high loss for one polarization (say, $n''_x \gg 0$), and low loss for the other polarization (say, $n''_y \ll n''_x$).



Suggested Problems

- Loss coefficients: if $\alpha = 0.2$ dB/km, what is k'' and n'' at $\lambda_0 = 1550$ nm? What is the fraction of power that remains after 50 km?
- Gain: If an amplifier has $2k'' = -10$ cm^{-1}, what is the length required to achieve 10dB power gain?
- Bandwidth: For λ_0 =1530–1570 nm, what is the bandwidth in Hertz? Compare this to the 2.4–2.5 GHz band in Wi-Fi communication.
- Wave plates: Assume the refractive indices for quartz given on slide 15. What is the minimum thickness needed to make a half-wave plate if $\lambda_0 = 1550$ nm? How about quarter-wave plate?
- Wave plates: what happens if I put circularly-polarized light through a half-wave plate? quarter-wave plate?
- Consider a plane wave in a medium with anisotropic permittivity matrix and free-space permeability. Gauss's law is $\nabla \cdot \boldsymbol{D} = 0$ (assume no charge). Is $\tilde{\boldsymbol{E}}$ perpendicular to $\boldsymbol{k}$? Is $\tilde{\boldsymbol{H}}$ perpendicular to $\boldsymbol{k}$? Is the Poynting vector $\overline{\boldsymbol{E} \times \boldsymbol{H}}$ parallel to $\boldsymbol{k}$?