PC2232 Physics for Electrical Engineers
Lecture 4: Input-Output Analysis and Interferometers (Supplemental Notes)

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Matrix Algebra

- Column vector: $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix}$
- Transpose of a matrix is defined by $(A^\top)_{nm} = A_{mn}$

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots \\ A_{21} & A_{22} & \dots \\ \vdots & & \end{pmatrix}, \qquad A^\top = \begin{pmatrix} A_{11} & A_{21} & \dots \\ A_{12} & A_{22} & \dots \\ \vdots & & \end{pmatrix}. \tag{1}$$

- Transpose of a column vector is a row vector:

$$\boldsymbol{v}^\top = \begin{pmatrix} v_1 & v_2 & \dots \end{pmatrix}. \tag{2}$$

- Transpose of matrix product is $(AB)^\top = B^\top A^\top$, because

$$(AB)_{nm} = \sum_l A_{nl} B_{lm}, \quad \left[(AB)^\top\right]_{nm} = \sum_l A_{ml} B_{ln} = \sum_l (B^\top)_{nl} (A^\top)_{lm} = \left(B^\top A^\top\right)_{nm}. \tag{3}$$

For example, if v is a column vector, A is a matrix, and u is another column vector,

$$u = Av, \qquad u^\top = v^\top A^\top. \tag{4}$$

Conjugate Transpose

- For matrices with complex elements, we often deal with an operation called conjugate transpose, denoted by †:

$$A^\dagger = (A^\top)^* = (A^*)^\top, \qquad \left(A^\dagger\right)_{nm} = A^*_{mn}. \tag{5}$$

  It is the transpose of a matrix, and then take the complex conjugate of each element.
- It inherits the product identity from the transpose:

$$(AB)^\dagger = B^\dagger A^\dagger. \tag{6}$$

50-50 Beam Splitter: One Input

- Consider the scattering matrix

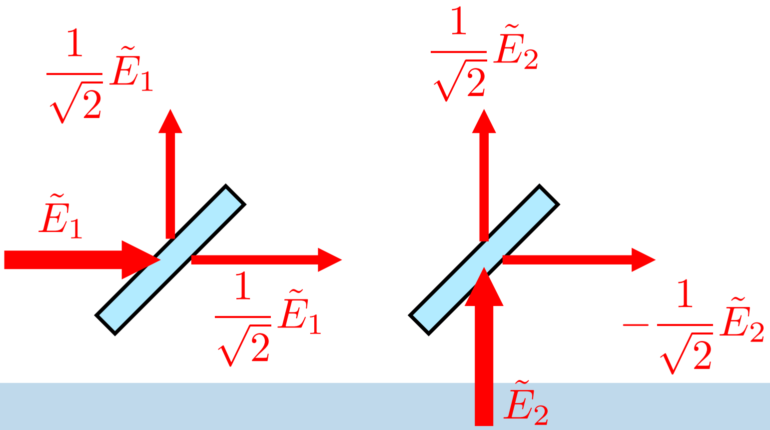
$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \tag{7}$$

- If there is just one input ($\tilde{E}_2 = 0$),

$$\tilde{E}_{\mathrm{out1}} = \frac{1}{\sqrt{2}}\tilde{E}_1, \quad \left|\tilde{E}_{\mathrm{out1}}\right|^2 = \frac{1}{2}\left|\tilde{E}_1\right|^2, \quad \tilde{E}_{\mathrm{out2}} = \frac{1}{\sqrt{2}}\tilde{E}_2, \quad \left|\tilde{E}_{\mathrm{out2}}\right|^2 = \frac{1}{2}\left|\tilde{E}_2\right|^2. \tag{8}$$

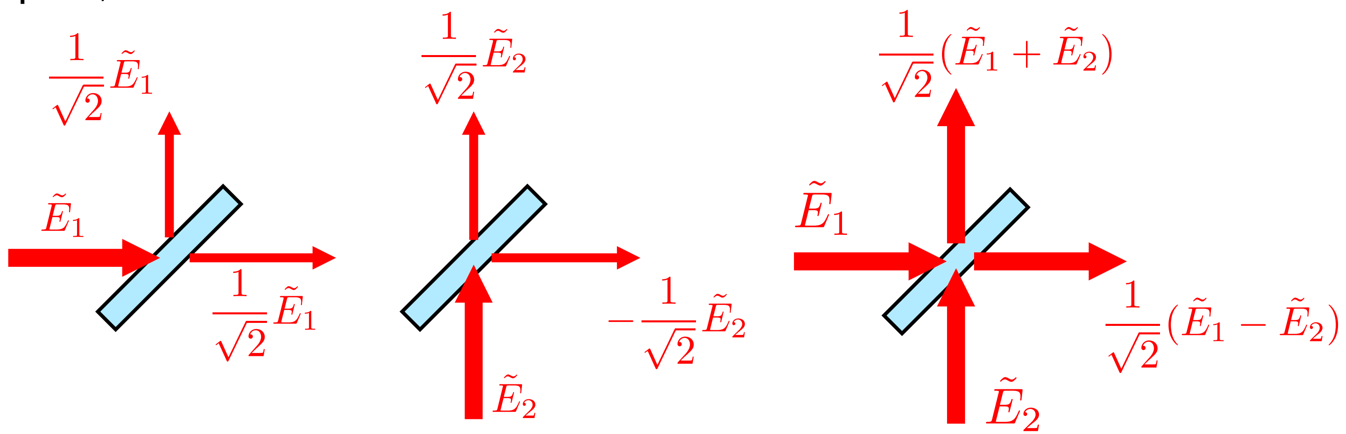
  Half of the power goes into 1 output port, and the other half goes into the other.
- Similarly, if $\tilde{E}_1 = 0$, there is just one input into the second port,

$$\tilde{E}_{\mathrm{out1}} = \frac{1}{\sqrt{2}}\tilde{E}_2, \quad \left|\tilde{E}_{\mathrm{out1}}\right|^2 = \frac{1}{2}\left|\tilde{E}_2\right|^2, \quad \tilde{E}_{\mathrm{out2}} = -\frac{1}{\sqrt{2}}\tilde{E}_2, \quad \left|\tilde{E}_{\mathrm{out2}}\right|^2 = \frac{1}{2}\left|\tilde{E}_2\right|^2. \tag{9}$$



50-50 Beam Splitter: Interference with Two Inputs

- With two inputs, we can add the two field solutions:



- For example, if $\tilde{E}_2 = \tilde{E}_1$, the inputs have the same power and phase. The total power $\propto 2|\tilde{E}_1|^2$, and the outputs are

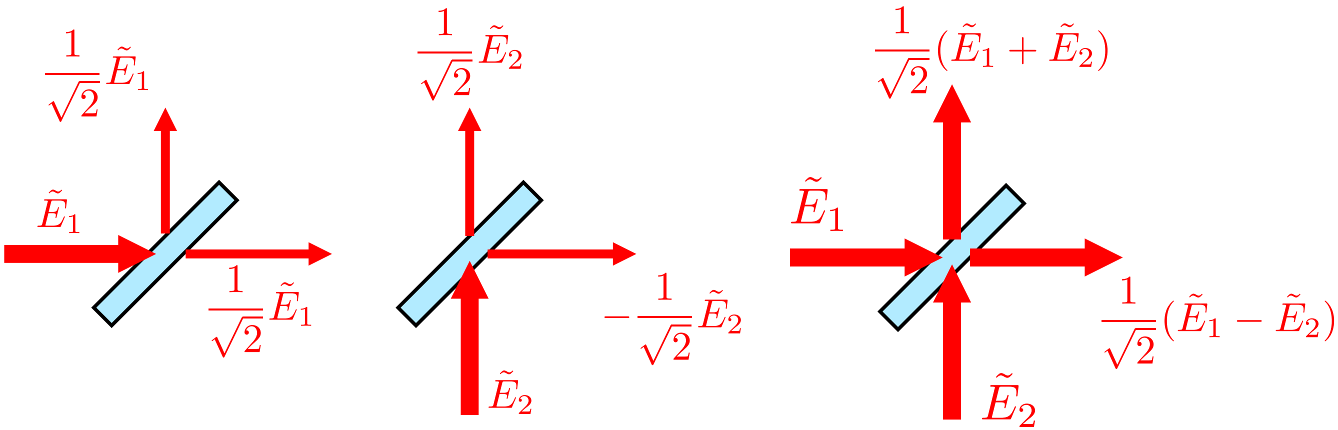
$$\tilde{E}_{\text{out}1} = \frac{1}{\sqrt{2}}\left(\tilde{E}_1 + \tilde{E}_1\right) = \sqrt{2}\tilde{E}_1, \qquad \tilde{E}_{\text{out}2} = \frac{1}{\sqrt{2}}\left(\tilde{E}_1 - \tilde{E}_1\right) = 0, \tag{10}$$

there is a **constructive interference** at the first output and **destructive interference** at the second output. All power goes into the first output.
- For another example, say $\tilde{E}_2 = -\tilde{E}_1$,

$$\tilde{E}_{\text{out}1} = \frac{1}{\sqrt{2}}\left(\tilde{E}_1 - \tilde{E}_1\right) = 0, \qquad \tilde{E}_{\text{out}2} = \frac{1}{\sqrt{2}}\left(\tilde{E}_1 + \tilde{E}_1\right) = \sqrt{2}\tilde{E}_1, \tag{11}$$

that is, if the inputs are 180° out of phase, all power goes into the second output instead.

Phase Dependence of Interference



- Suppose that $\tilde{E}_2 = e^{j\theta}\tilde{E}_1$. The inputs have equal power, but there is a relative phase between them.

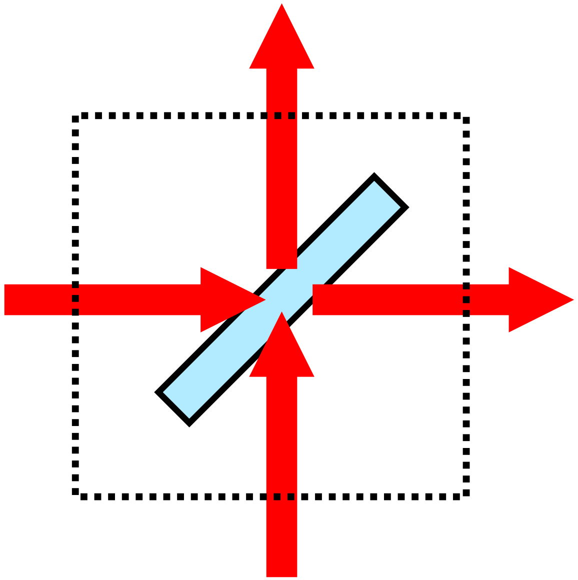
$$\tilde{E}_{\text{out}1} = \frac{1}{\sqrt{2}}\left(\tilde{E}_1 + e^{j\theta}\tilde{E}_1\right), \qquad \tilde{E}_{\text{out}2} = \frac{1}{\sqrt{2}}\left(\tilde{E}_1 - e^{j\theta}\tilde{E}_1\right). \tag{12}$$

In terms of power,

$$\left|\tilde{E}_{\text{out}1}\right|^2 = \frac{\left|\tilde{E}_1\right|^2}{2}\left|1 + e^{j\theta}\right|^2 = 2\cos^2\frac{\theta}{2}|\tilde{E}_1|^2, \quad \left|\tilde{E}_{\text{out}2}\right|^2 = \frac{\left|\tilde{E}_2\right|^2}{2}\left|1 - e^{j\theta}\right|^2 = 2\sin^2\frac{\theta}{2}|\tilde{E}_1|^2. \tag{13}$$

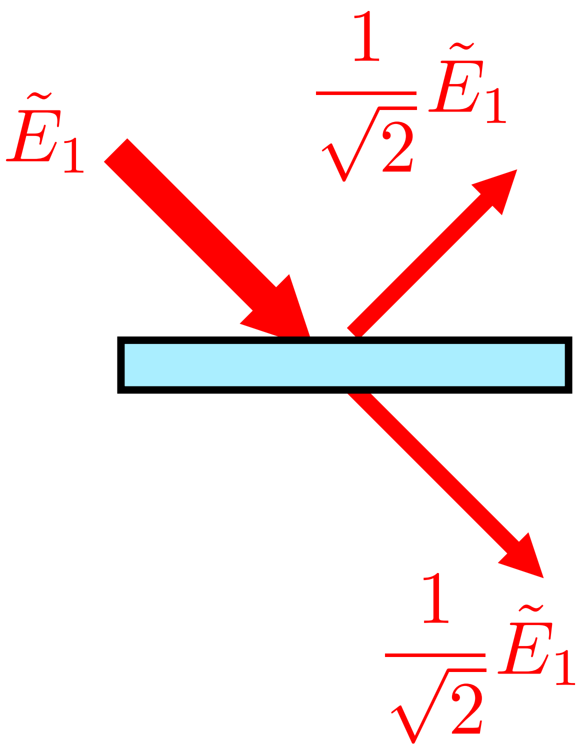
The output powers depend on the **relative phase** between the inputs.

Power Conservation

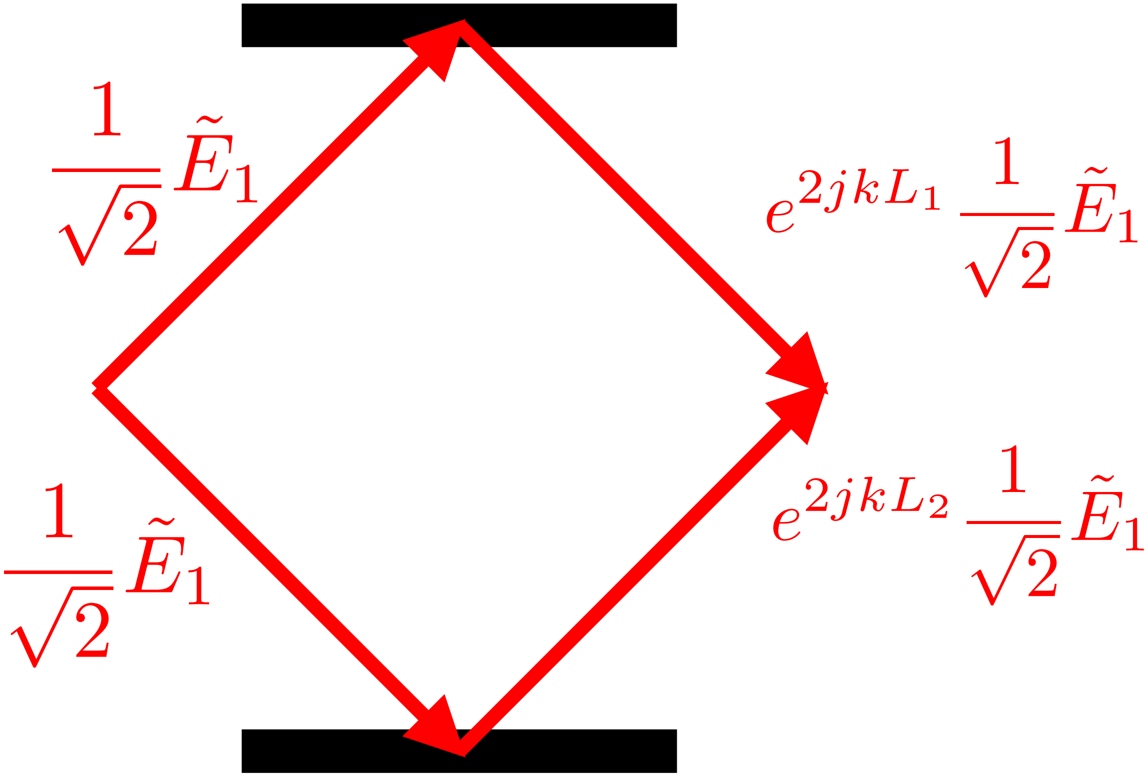


- Think of a box surrounding the beam splitter.
- If the beam splitter doesn't absorb (or provide) any power, input power = output power.
- Input power = power from the left + power from the bottom $\propto |\tilde{E}_1|^2 + |\tilde{E}_2|^2$.
- Output power = power out of the top + power out of the right $\propto |\tilde{E}_{\mathrm{out1}}|^2 + |\tilde{E}_{\mathrm{out2}}|^2$.

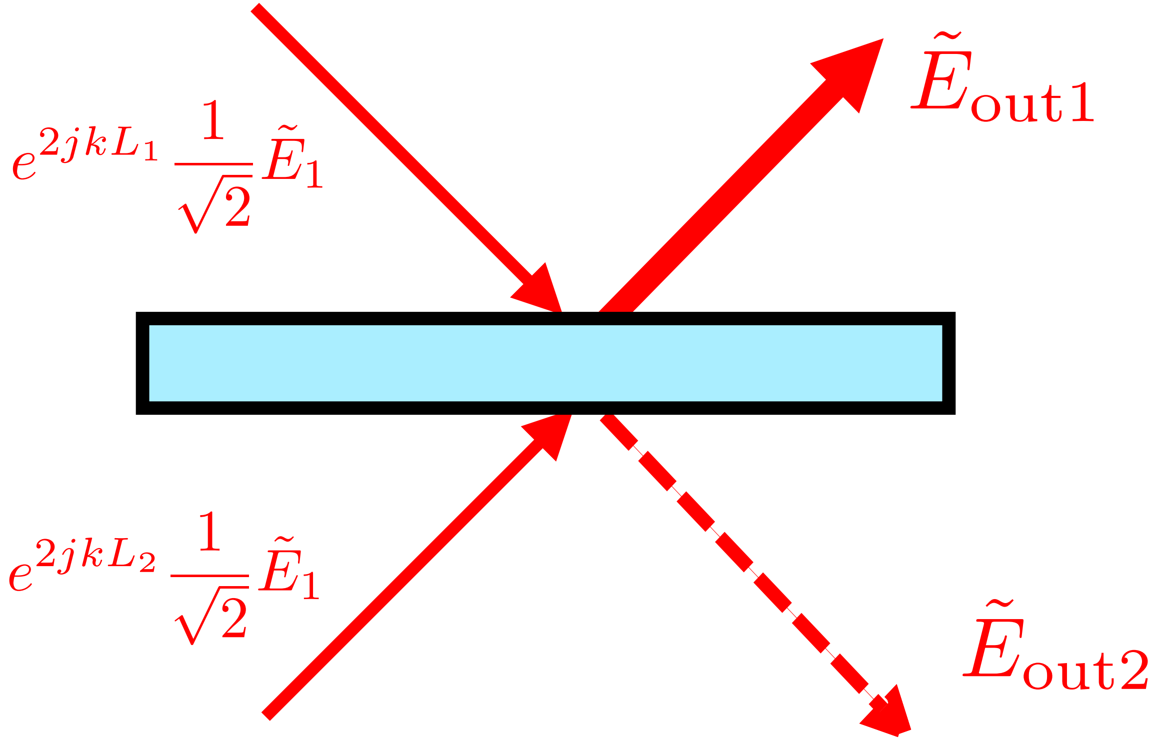
Mach-Zehnder Interferometer: First Beam Splitter



- Take the outputs of the first beam splitter as the inputs for propagation in the two arms:

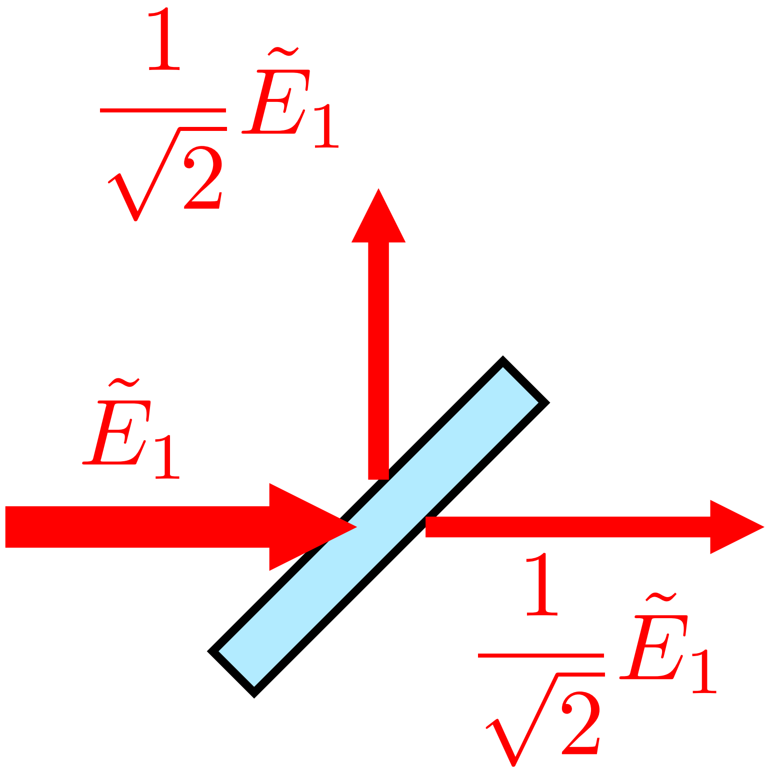


- Take the outputs of propagation as inputs to the second beam splitter:

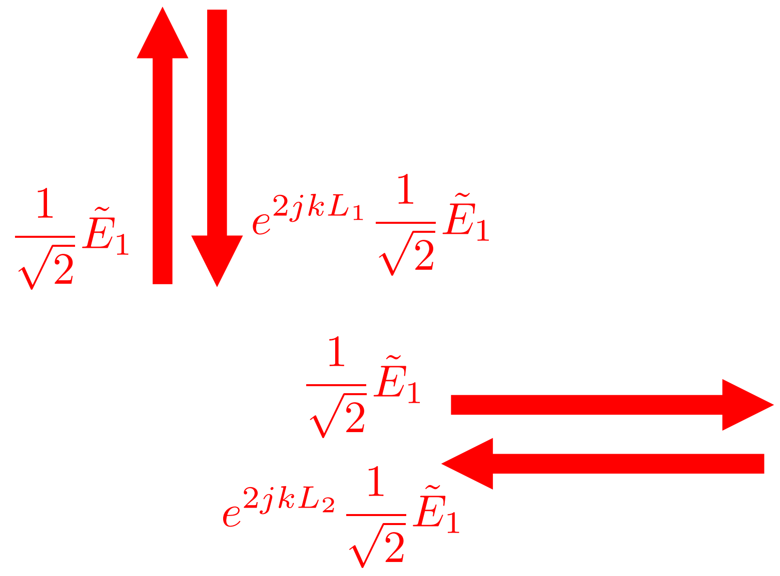


$$\tilde{E}_{\rm out1} = \frac{1}{\sqrt{2}}\left(e^{2jkL_1}\frac{1}{\sqrt{2}}\tilde{E}_1\right) + \frac{1}{\sqrt{2}}\left(e^{2jkL_2}\frac{1}{\sqrt{2}}\tilde{E}_1\right), \tag{14}$$

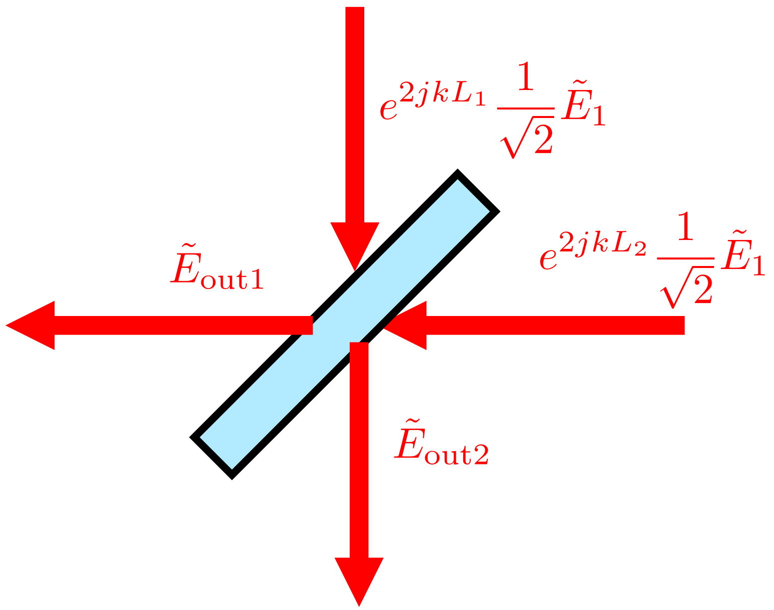
$$\tilde{E}_{\rm out2} = \frac{1}{\sqrt{2}}\left(e^{2jkL_1}\frac{1}{\sqrt{2}}\tilde{E}_1\right) - \frac{1}{\sqrt{2}}\left(e^{2jkL_2}\frac{1}{\sqrt{2}}\tilde{E}_1\right). \tag{15}$$


$\frac{1}{\sqrt{2}}\tilde{E}_1$
$\tilde{E}_1$
$\frac{1}{\sqrt{2}}\tilde{E}_1$

Michelson Interferometer: Propagation in Two Arms



Michelson Interferometer: “Second” Beam Splitter



$$\tilde{E}_{\mathrm{out1}} = \frac{1}{\sqrt{2}}\left(e^{2jkL_1}\frac{1}{\sqrt{2}}\tilde{E}_1\right) + \frac{1}{\sqrt{2}}\left(e^{2jkL_2}\frac{1}{\sqrt{2}}\tilde{E}_1\right), \tag{16}$$

$$\tilde{E}_{\mathrm{out2}} = \frac{1}{\sqrt{2}}\left(e^{2jkL_1}\frac{1}{\sqrt{2}}\tilde{E}_1\right) - \frac{1}{\sqrt{2}}\left(e^{2jkL_2}\frac{1}{\sqrt{2}}\tilde{E}_1\right). \tag{17}$$