PC2232 Physics for Electrical Engineers: Tutorial 5

Question 1: Spin probabilities

Consider a particle in a quantum state given by

$$\Psi_z = \begin{pmatrix} \psi_z(+\frac{1}{2}) \\ \psi_z(-\frac{1}{2}) \end{pmatrix}. \tag{1}$$

The probability that the particle having spin $s_z = +1/2$ is $|\psi_z(+1/2)|^2$, and the probability it is in spin $s_z = -\frac{1}{2}$ is $|\psi_z(-1/2)|^2$.

(a) Show that

$$\left|\psi_z\left(+\frac{1}{2}\right)\right|^2 = |(1\ \ 0)\Psi_z|^2, \quad \text{and,} \quad \left|\psi_z\left(-\frac{1}{2}\right)\right|^2 = |(0\ \ 1)\Psi_z|^2. \tag{2}$$

(b) For a particular state

$$\Psi_z = \begin{pmatrix} \sqrt{3/5} \\ c \end{pmatrix}, \tag{3}$$

Where c is an unknown complex number. What is the value of c? What is the probability of the particle to be in the state $s_z = -1/2$?

Next we wish to measure the y-component of the spin, we obtain the wavefunction of s_y as* [c.f. Lecture 6, Eq. (43)]

$$\Psi_y = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{\mathrm{i}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} \end{pmatrix} \Psi_z. \tag{4}$$

(c) Write down the expression of Ψ_y. What is the probability of the particle being in $s_y = +1/2$? What is the probability of the particle being in $s_y = -1/2$?

Question 2: Seperation of variables

The time-dependent Schrödinger equation is given as

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = \left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + U(\vec{r})\right]\Psi(\vec{r},t). \tag{5}$$

Show that $\Psi(\vec{r},t) = \mathrm{e}^{-\mathrm{i}Et/\hbar}\psi(\vec{r})$ is a solution to (5) if $\psi(\vec{r})$ satisfies

$$\left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + U(\vec{r})\right]\psi(\vec{r}) = E\psi(\vec{r}), \tag{6}$$

*Here, $\mathrm{i} = j = \sqrt{-1}$.

where $\psi(\vec{r})$ is a function that only depends on position $\vec{r}$. [Equation (6) is sometimes called the time-independent Schrödinger equation].

Suppose that ψ actually only depends on the x-direction; i.e., $\psi(\vec{r}) = \psi(x)$ and $U(\vec{r}) = U_0$ is a constant. Show that

$$\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} = -\frac{2m}{\hbar}(E - U_0)\,\psi(x). \tag{7}$$

Question 3: Infinite well

Consider a particle of mass m in a potential defined by (see Fig. 1)

$$U(x) = \begin{cases} 0, & 0 < x < L, \\ \infty, & \text{otherwise} \end{cases} \tag{8}$$

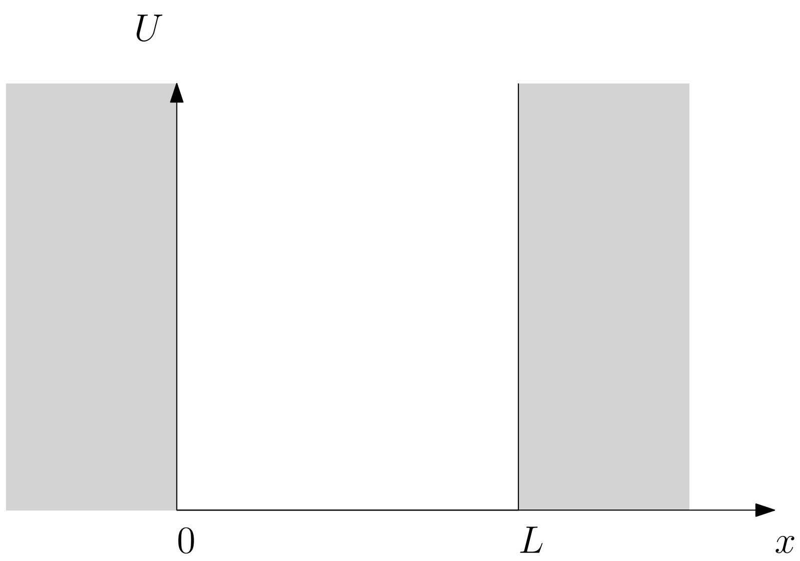

Figure 1: Infinite well

Find the wave function that represents the particle $\psi(x)$, by solving the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + U(x)\psi(x) = E\psi(x), \tag{9}$$

under the boundary condition and normalization condition

$$\psi(0) = 0, \quad \psi(L) = 0, \quad \int_{-\infty}^{\infty} |\psi(x)|^2\,\mathrm{d}x = 1. \tag{10}$$

Normalize the wavefunction such that $\int_{-\infty}^{\infty} |\psi(x)|^2\,\mathrm{d}x = 1$. What are the possible values of E for the particle?

Question 4: Finite well

Consider a particle of mass m in a potential defined by (see Fig. 2)

$$U(x) = \begin{cases} U_0, & x < 0, \\ 0, & 0 < x < L, \\ U_0 & x > L \end{cases} \tag{11}$$

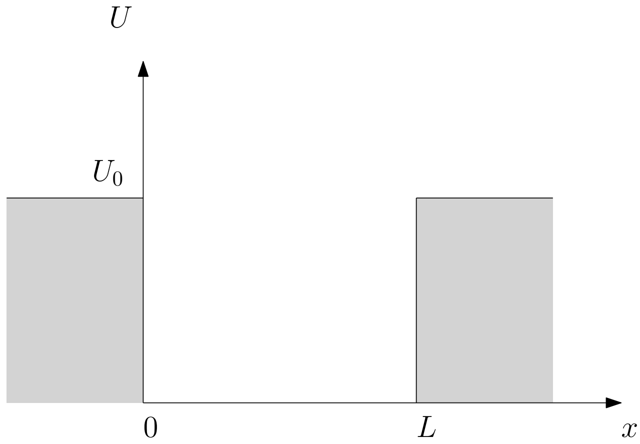

Figure 2: Finite well

Find a general solution to Schrödinger's equation for a wavefunction $\psi(x)$ if

(a) $E < U_0$,

(b) $E > U_0$.

[You do not need to determine the normalization constants as well as the two arbitrary constants (k and α).] What is the qualitative nature of the resulting probability distributions for the two cases?

Question 5: Schrödinger equation $\leftrightarrow$ Conservation of energy

Consider a wavefunction of the form

$$\psi(x) = A \sin kx, \tag{12}$$

which is a solution to Schrödinger's equation for some potential U, with A and k as constants. Show that, if $p = \hbar k$,

$$\frac{p^2}{2m} + U = E, \quad \text{(Kinetic energy+Potential energy=total energy)} \tag{13}$$

which is the typical equation of Newtonian mechanics.

Question 6: (Optional)

A quantum particle of mass m_1 is in a square well with infinitely high walls and length 3 nm. Rank the situation (a) through (f) according to the energy from highest to lowest, noting any cases of equality.

(a) The particle of mass m_1 is in the ground state ($n = 1$) of the well.

(b) The same particle is in the $n = 2$ excited state of the same well.

(c) A particle of mass $2m_1$ is in the ground state of a well of length 1.5 nm.

(d) A particle of mass m_1 is in the ground state of a well of length 3 nm but with finite potential wells.

(e) A particle of mass m_1 is in the $n = 2$ state of a well of length 3 nm but with finite potential wells.

(f) A particle of mass m_1 is in the ground state of a well of length 3 nm, and the uncertainty principle has become inoperative, that is, Planck's constant has been reduced to zero.