Precalculus Final Exam Solutions

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Precalculus Final Exam Answer Key

1. (5 pts)	A	B	C	D	■
2. (5 pts)	A	B	■	D	E
3. (5 pts)	A	■	C	D	E
4. (5 pts)	■	B	C	D	E
5. (5 pts)	A	B	■	D	E
6. (5 pts)	A	B	C	D	■
7. (5 pts)	A	B	C	■	E
8. (5 pts)	A	■	C	D	E

9. (15 pts) $z_1 = i$, $z_2 = -\frac{\sqrt{3}}{2} - \frac{i}{2}$, and $z_3 = \frac{\sqrt{3}}{2} - \frac{i}{2}$

10. (15 pts) $x = 2$ and $y = -2$

11. (15 pts) $f(x) = \frac{2x+1}{x^2+4} - \frac{2}{x+2}$

12. (15 pts) $e = 1$

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1. E. We'll rewrite the equation by squaring both sides.

$$r = 1 + \tan\theta$$

$$r^2 = (1 + \tan\theta)^2$$

We know that $\tan\theta = y/x$ and $r^2 = x^2 + y^2$, so we can substitute and get

$$x^2 + y^2 = \left(1 + \frac{y}{x}\right)^2$$

2. C. The equation of a cardioid is $r = c \pm c\cos\theta$ or $r = c \pm c\sin\theta$. So $r = 2\sin\theta + 2$ is a cardioid.

3. B. Since $a = 1 - 3i$ and $b = 1 + 3i$, then a/b is

$$\frac{a}{b} = \frac{1 - 3i}{1 + 3i}$$

$$\frac{a}{b} = \frac{(1 - 3i)(1 - 3i)}{(1 + 3i)(1 - 3i)}$$

$$\frac{a}{b} = \frac{1 - 6i + 9i^2}{1 - 3i + 3i - 9i^2}$$

$$\frac{a}{b} = \frac{1 - 6i + 9(-1)}{1 - 9(-1)}$$

$$\frac{a}{b} = \frac{-8 - 6i}{10}$$

$$\frac{a}{b} = -\frac{4}{5} - \frac{3}{5}i$$

4. A. The product of the matrices is

$$\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2-3 & 0-6 \\ 0+1 & 0+2 \end{bmatrix} = \begin{bmatrix} -1 & -6 \\ 1 & 2 \end{bmatrix}$$

5. C. Factor the denominator of the rational function.

$$f(x) = \frac{2}{x(x^2 - 1)} = \frac{2}{x(x+1)(x-1)}$$

The denominator is the product of three distinct linear factors, so the decomposition equation will be

$$\frac{2}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

To solve for A, we'll remove the factor of x from the left side, then evaluate what remains on the left at $x = 0$.

$$\frac{2}{(x+1)(x-1)} \to \frac{2}{(0+1)(0-1)} \to \frac{2}{(1)(-1)} \to -2$$

To solve for B, we'll remove the factor of $x+1$ from the left side, then evaluate what remains on the left at $x=-1$.

$$\frac{2}{x(x-1)} \to \frac{2}{-1(-1-1)} \to \frac{2}{2} \to 1$$

To solve for C, we'll remove the factor of $x-1$ from the left side, then evaluate what remains on the left at $x=1$.

$$\frac{2}{x(x+1)} \to \frac{2}{1(1+1)} \to \frac{2}{2} \to 1$$

Plugging $A=-2$, $B=1$, and $C=1$ back into the partial fractions decomposition gives,

$$f(x) = -\frac{2}{x} + \frac{1}{x+1} + \frac{1}{x-1}$$

$$f(x) = \frac{1}{x+1} + \frac{1}{x-1} - \frac{2}{x}$$

6. E. A hyperbola in the form

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

opens up and down. Comparing this standard form to the hyperbola

$$\frac{(y-1)^2}{2} - \frac{(x+1)^2}{3} = 1$$

lets us identify that $a = \sqrt{2}$, $b = \sqrt{3}$, and that the center is at $(h, k) = (-1,1)$. Its foci are $(h, k \pm c)$ where $c = \sqrt{a^2 + b^2} = \sqrt{5}$, so the foci are $(-1,1 \pm \sqrt{5})$.

7. D. If we solve $x = (t + 1)/2$ for t, we get $t = 2x - 1$, which we can then substitute into $y = t^2 + 1$.

$$y = t^2 + 1$$

$$y = (2x - 1)^2 + 1$$

$$y = 4x^2 - 4x + 2$$

8. B. We can match the equation of the ellipse,

$$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} = 1$$

to the Pythagorean identity $\sin^2 t + \cos^2 t = 1$, which gives

$$\sin^2 t = \frac{(y-2)^2}{9} \qquad \cos^2 t = \frac{(x-1)^2}{4}$$

$$\sin t = \frac{y-2}{3} \qquad \cos t = \frac{x-1}{2}$$

$$3 \sin t = y - 2 \qquad 2 \cos t = x - 1$$

$$y = 2 + 3 \sin t \qquad x = 1 + 2 \cos t$$

9. We'll rewrite the equation as

$$z^3 + i = 0$$

$$z^3 = -i$$

We know we can rewrite $-i$ as the complex number $0 - i$ in rectangular form. Then we can say

$$r = |z| = \sqrt{a^2 + b^2}$$

$$r = \sqrt{0^2 + (-1)^2}$$

$$r = 1$$

and that the angle θ is

$$\theta = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{-1}{0}\right) = \frac{3\pi}{2}$$

Then we get

$$z^3 = \sin\left(\frac{3\pi}{2}\right)i$$

$r = 1$, so $\sqrt[3]{r} = 1$

Once the complex number is in polar form, then its nth roots are given in radians by

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

$$\sqrt[3]{z^3} = \sqrt[3]{r}\left[\cos\left(\frac{\frac{3\pi}{2} + 2\pi k}{3}\right) + i\sin\left(\frac{\frac{3\pi}{2} + 2\pi k}{3}\right)\right]$$

$$z = \cos\left(\frac{\frac{3\pi}{2} + 2\pi k}{3}\right) + i\sin\left(\frac{\frac{3\pi}{2} + 2\pi k}{3}\right)$$

The first root for $k = 0$ is

$$z = \sin\left(\frac{\frac{3\pi}{2}}{3}\right)i + \cos\left(\frac{\frac{3\pi}{2}}{3}\right)$$

$$z = \sin\left(\frac{\pi}{2}\right)i + \cos\left(\frac{\pi}{2}\right)$$

$$z_1 = i$$

The second root for $k = 1$ is

$$z = \sin\left(\frac{\pi}{2} + \frac{2\pi}{3}\right)i + \cos\left(\frac{\pi}{2} + \frac{2\pi}{3}\right)$$

$$z = \cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)$$

$$z_2 = -\frac{\sqrt{3}}{2} - \frac{i}{2}$$

The third root for $k = 2$ is

$$z = \sin\left(\frac{\pi}{2} + \frac{4\pi}{3}\right)i + \cos\left(\frac{\pi}{2} + \frac{4\pi}{3}\right)$$

$$z = \cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)$$

$$z_3 = \frac{\sqrt{3}}{2} - \frac{i}{2}$$

10. We can solve the system

$$x - 3y = 8$$

$$2x + y = 2$$

by setting up an augmented matrix and then applying Gauss-Jordan elimination.

$$\begin{bmatrix} 1 & -3 & | & 8 \\ 2 & 1 & | & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & | & 8 \\ 0 & 7 & | & -14 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & | & 8 \\ 0 & 1 & | & -2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & -2 \end{bmatrix}$$

From this resulting matrix, we get the solution set

$1x + 0y = 2$, or $x = 2$

$0x + 1y = -2$, or $y = -2$

That’s all it took to find that the solution to the system is $x = 2$ and $y = -2$.

11. The denominator of the rational function,

$$f(x) = \frac{5x - 6}{(x^2 + 4)(x + 2)}$$

is the product of one distinct quadratic factor and one distinct linear factor, so its partial fractions decomposition equation can be written as

$$\frac{5x - 6}{(x^2 + 4)(x + 2)} = \frac{Ax + B}{x^2 + 4} + \frac{C}{x + 2}$$

To solve for C, remove the factor of $x + 2$ from the left side, then evaluate what remains on the left at $x = -2$.

$$\frac{5x - 6}{x^2 + 4} \to \frac{5(-2) - 6}{(-2)^2 + 4} \to \frac{-16}{8} \to -2$$

We’ll combine the fractions on the right side of the equation by finding a common denominator.

$$\frac{5x - 6}{(x^2 + 4)(x + 2)} = \frac{Ax + B}{x^2 + 4} - \frac{2}{x + 2} = \frac{Ax^2 + Bx + 2Ax + 2B - 2x^2 - 8}{(x^2 + 4)(x + 2)}$$

$$\frac{5x - 6}{(x^2 + 4)(x + 2)} = \frac{Ax + B}{x^2 + 4} - \frac{2}{x + 2} = \frac{(A - 2)x^2 + (B + 2A)x + 2B - 8}{(x^2 + 4)(x + 2)}$$

Now that the denominators of the left and right sides are equivalent, we can set the numerators equal to each other to get

$$5x - 6 = (A - 2)x^2 + (B + 2A)x + 2B - 8$$

Equating coefficients on the left and right sides gives the system of equations

$$A - 2 = 0$$

$$B + 2A = 5$$

$$2B - 8 = -6$$

Therefore, $A = 2$ and $B = 1$. Plugging these constant values back into the partial fractions decomposition gives

$$f(x) = \frac{2x + 1}{x^2 + 4} - \frac{2}{x + 2}$$

12. By matching the equation of the conic

$$2x^2 - 4xy + 2y^2 - x + 5 = 0$$

to the standard form of a conic, we can identify $A = 2$, $B = -4$, and $C = 2$, so the value of the discriminant is

$$B^2 - 4AC = (-4)^2 - 4(2)(2) = 0$$

Because the discriminant is 0, we know the conic section is a parabola, and therefore that the eccentricity is $e = 1$.