Trigonometry Workbook Solutions

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NAMING ANGLES

■ 1. What do we call an angle of $\theta = 180^\circ$?

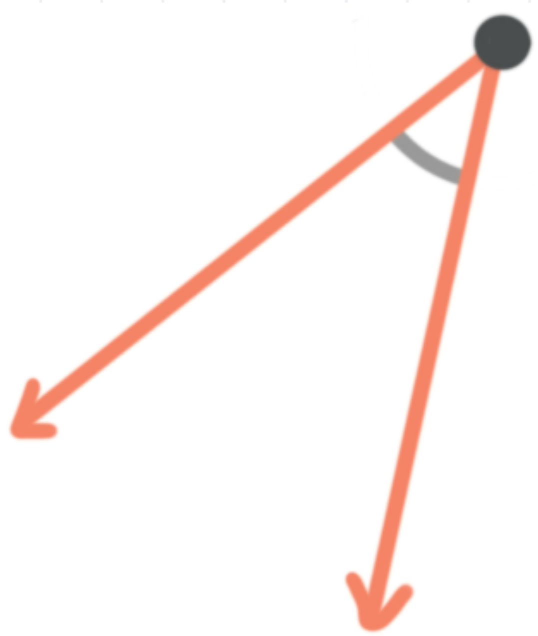
Solution:

Half-circle angles, or 180° angles, are called “straight” angles.

■ 2. Sketch an acute angle.

Solution:

Any angle that’s less than a quarter circle, or $0^\circ < \theta < 90^\circ$, is an acute angle.



3. What is the measure of a straight angle?

Solution:

A straight angle has a measure of 180°.

4. Name the angle $\theta = 6^\circ$.

Solution:

Any angle that's less than a quarter circle, or $0^\circ < \theta < 90^\circ$, is an acute angle. Therefore, $\theta = 6^\circ$ is acute angle.

5. Sketch a 360° angle.

Solution:

A 360° angle, or complete angle, is a full circle, which we sketch as

6. Give the full set of obtuse angles.

Solution:

Any angle that’s greater than a quarter circle but less than a half circle, or $90^\circ < \theta < 180^\circ$, is an obtuse angle.

COMPLEMENTARY AND SUPPLEMENTARY ANGLES

1. Find the supplement θ of $7\pi/8$.

Solution:

The angle θ and the angle $7\pi/8$ are supplementary, which means their sum is π.

$$\theta + \frac{7\pi}{8} = \pi$$

$$\theta = \pi - \frac{7\pi}{8}$$

$$\theta = \frac{8\pi}{8} - \frac{7\pi}{8}$$

$$\theta = \frac{\pi}{8}$$

2. Angles $\angle 1$ and $\angle 2$ are complementary. Find the supplement of $\angle 2$.

$$m\angle 1 = (2x + 5)^\circ$$

$$m\angle 2 = (x + 4)^\circ$$

Solution:

Because $\angle 1$ and $\angle 2$ are complementary, that means they sum to 90°.

$$m\angle 1 + m\angle 2 = 90^\circ$$

$$2x + 5 + x + 4 = 90^\circ$$

$$3x + 9 = 90^\circ$$

$$3x = 81$$

$$x = 27$$

Plugging this back into $\angle 2$ gives

$$m\angle 2 = (27 + 4)^\circ = 31^\circ$$

The supplement of $\angle 2$ is therefore

$$180^\circ - 31^\circ$$

$$149^\circ$$

■ 3. The complement of θ is $\pi/6$. Find the supplement of θ.

Solution:

Complementary angles sum to $\pi/2$. So

$$\theta + \frac{\pi}{6} = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{2} - \frac{\pi}{6}$$

$$\theta = \frac{3\pi}{6} - \frac{\pi}{6}$$

$$\theta = \frac{\pi}{3}$$

Since supplementary angles sum to π, the supplement of θ is

$$\pi - \frac{\pi}{3}$$

$$\frac{3\pi}{3} - \frac{\pi}{3}$$

$$\frac{2\pi}{3}$$

■ 4. Find the complement of 42°.

Solution:

Complementary angles sum to 90°, so if we call θ the complement of 42°, then

$$\theta + 42^\circ = 90^\circ$$

$$\theta = 90^\circ - 42^\circ$$

$$\theta = 48^\circ$$

5. Find the angle that's supplementary to $2\pi/3$.

Solution:

Supplementary angles sum to π. So if we call the supplementary angle θ, then we can say

$$\theta + \frac{2\pi}{3} = \pi$$

$$\theta = \pi - \frac{2\pi}{3}$$

$$\theta = \frac{3\pi}{3} - \frac{2\pi}{3}$$

$$\theta = \frac{\pi}{3}$$

6. True or False? If x and y are complementary angles, then $2(x + y) = 180^\circ$.

Solution:

If x and y are complementary, then they sum to 90°.

$$x + y = 90^\circ$$

Then if we multiply both sides by 2, we get

$$2(x + y) = 180^\circ$$

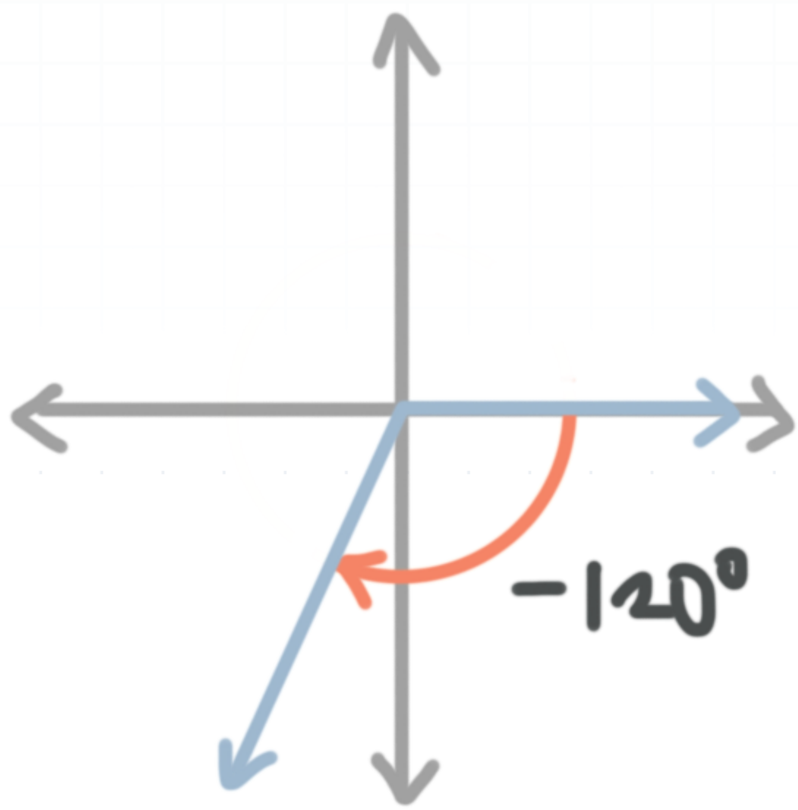
So that statement is true.

POSITIVE AND NEGATIVE ANGLES

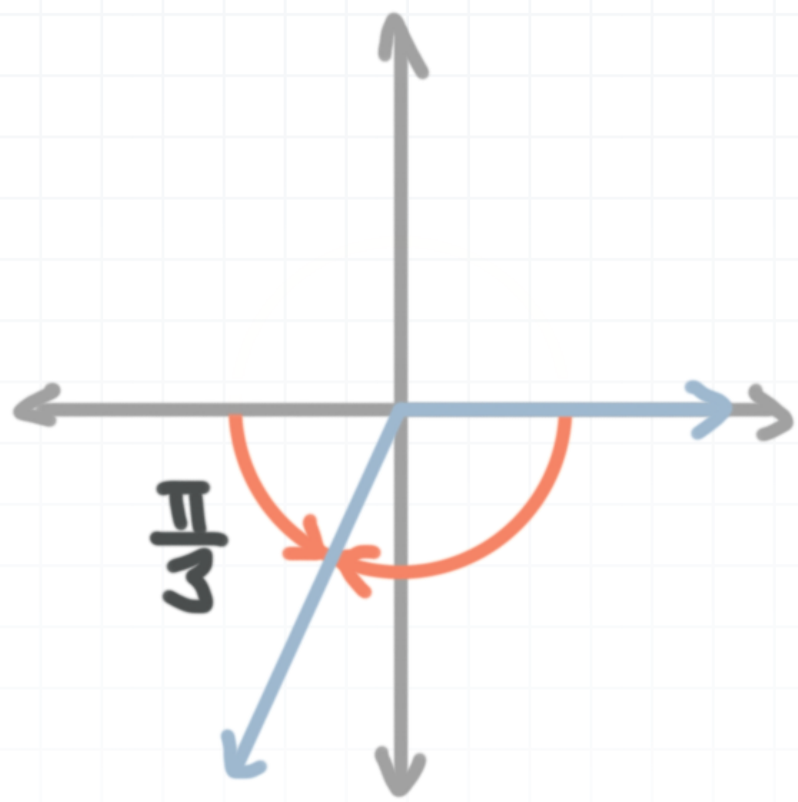
■ 1. Sketch -120° in standard position.

Solution:

The terminal side of the angle is found by rotating 120° in a negative (clockwise) direction. Since 120° is less than 180° but greater than 90°, the terminal side lies in quadrant III.



■ 2. Find the measure of the unknown negative angle in radians.



Solution:

The two angles shown are supplementary, which means they sum to π radians. So the measure of the unknown angle is

$$\pi - \frac{\pi}{3}$$

$$\frac{3\pi}{3} - \frac{\pi}{3}$$

$$\frac{2\pi}{3}$$

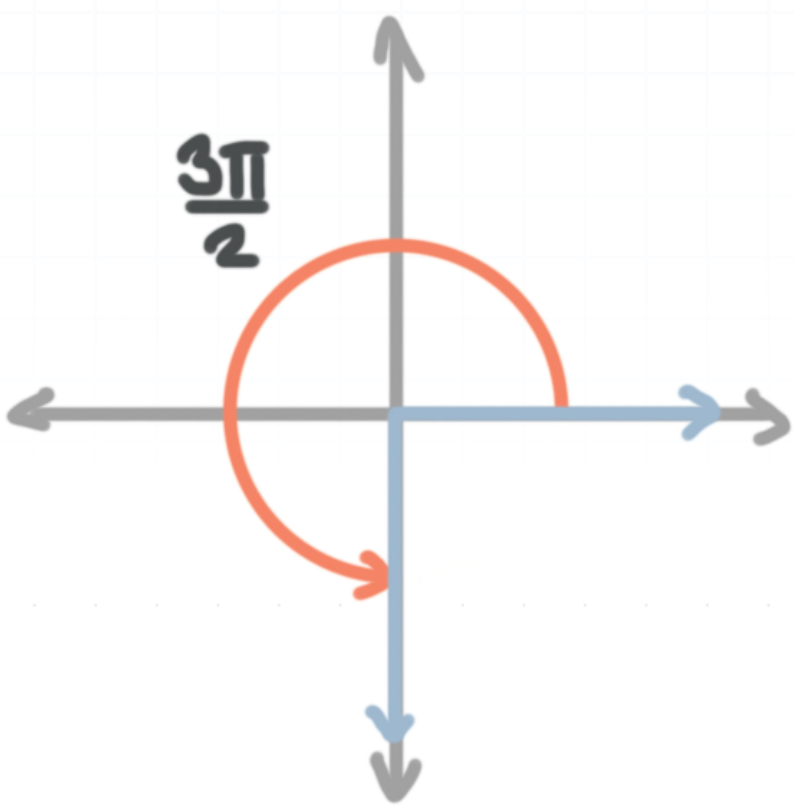
But because the angle shows clockwise rotation, that means it's the negative angle

$$-\frac{2\pi}{3}$$

■ 3. Sketch $3\pi/2$ in standard position.

Solution:

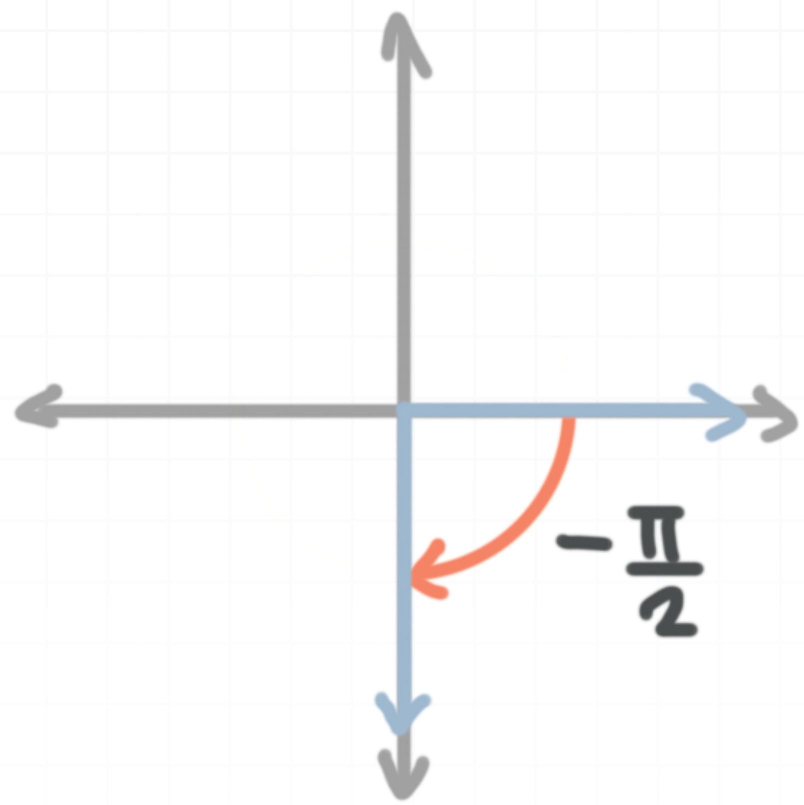
The angle $3\pi/2$ is positive, which means we rotate counterclockwise from the positive horizontal axis to the negative y-axis. Therefore, a sketch of the angle is



■ 4. Sketch $-\pi/2$ in standard position.

Solution:

The angle $-\pi/2$ is negative, which means we rotate clockwise from the positive horizontal axis to the negative y-axis. Therefore, a sketch of the angle is



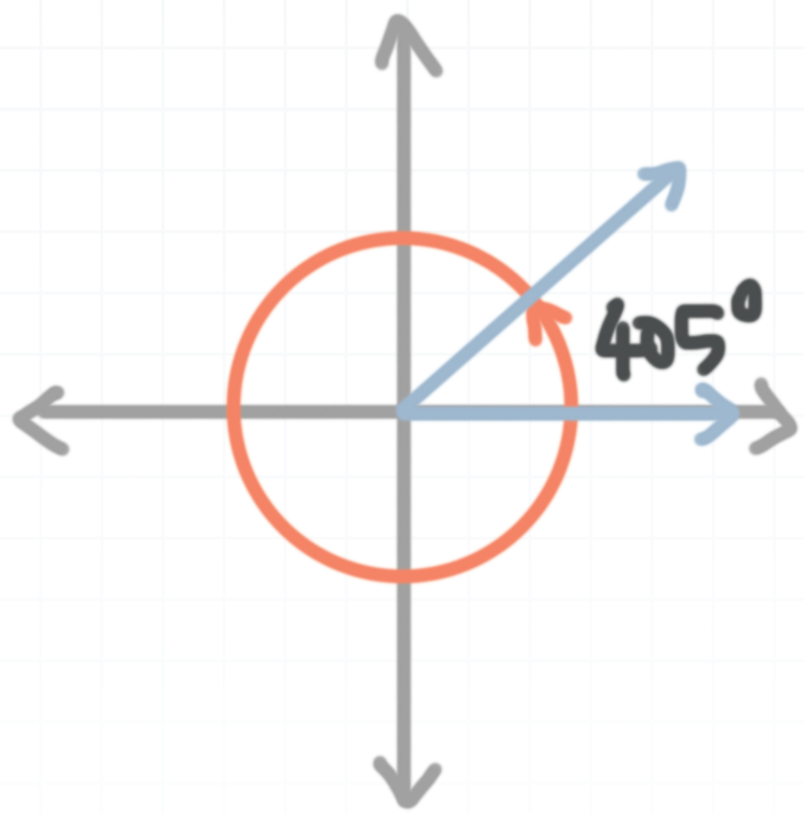
■ 5. Sketch 405° in standard position.

Solution:

Since $360^\circ < 405^\circ$, the angle 405° is more than one full rotation. We'll find out how much more by finding the difference between the angles.

$$405^\circ - 360^\circ = 45^\circ$$

So to sketch the angle, we'll put the initial side along the positive direction of the x-axis. Then we'll rotate counterclockwise, toward the first quadrant, and rotate one full rotation all the way around the circle, but then an additional 45°. Because 45° would normally land us in the first quadrant, we'll land in the first quadrant for the 405° angle as well.



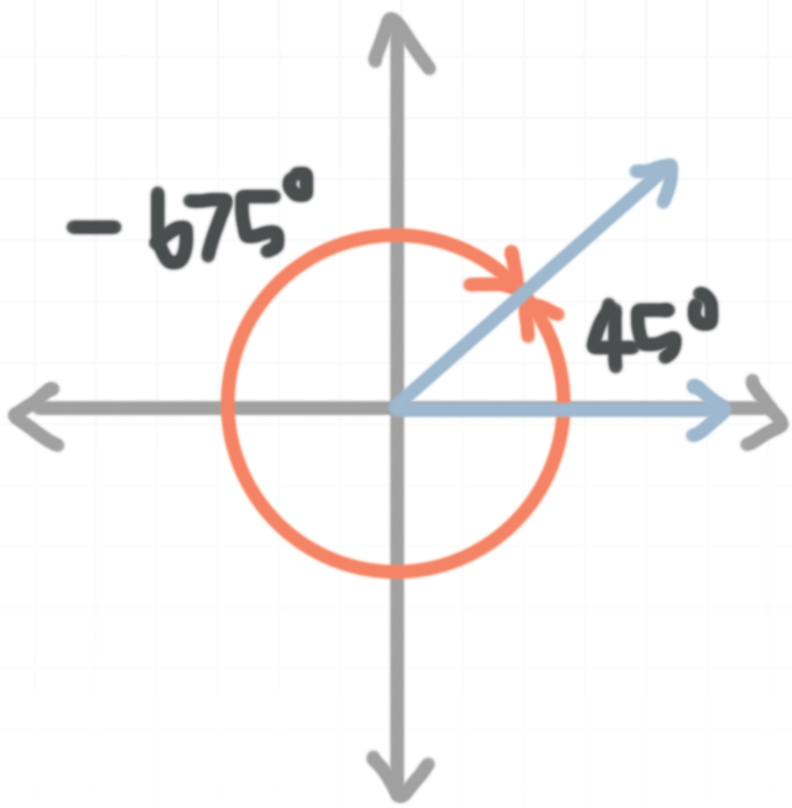
6. Find an angle between 0° and 360° that has the same terminal side as a -675° angle.

Solution:

First we need to sketch -675° in standard position. Since $360^\circ < 675^\circ$, the angle 675° is more than one full rotation. We'll find out how much more by finding the difference between the angles.

$$675^\circ - 360^\circ = 315^\circ$$

So to sketch the angle, we'll put the initial side along the positive direction of the x-axis. Then we'll rotate clockwise, toward the fourth quadrant, and rotate one full rotation all the way around the circle, but then an additional 315°. $315^\circ = 270^\circ + 45^\circ$, so we would land in the first quadrant for a -315° angle, and we'll land in the first quadrant for the -675° angle as well.



An angle measuring 45° has the same terminal side as an angle of -675°.

QUADRANT OF THE ANGLE

1. In which quadrant is the angle $\pi/5$ located?

Solution:

The angle $\pi/5$ is less than the angle $\pi/2$, which means $\pi/5$ has to lie in the first quadrant.

2. In which quadrant is the angle -820° located?

Solution:

One full rotation is -360°, so we know -820° is more than one full rotation. To figure out how many rotations are made by -820°, divide -820° by -360°.

$$\frac{-820^\circ}{-360^\circ}$$

$$2.28$$

So -820° is almost 2 and 0.3 rotations in the negative direction. If we start along the positive x-axis in standard position and make 2 full rotations in the negative direction, we'll end up right back in the same place, on the positive x-axis.

Two full rotations is

$$2(-360^\circ)$$

$$-720^\circ$$

and once we've made a -720° rotation, to get to -820°, we'll need an additional -100° of rotation. From standard position, we know -90° puts us along the negative vertical axis, and then -180° puts us along the negative horizontal axis. So a rotation of -100° will put us in the third quadrant, just past the negative vertical axis.

3. In which quadrant is the angle $-13\pi/4$ located?

Solution:

Remember that, in radians, one full rotation is 2π. So to determine how many full rotations are included in $-13\pi/4$, divide $-13\pi/4$ by 2π.

$$\frac{-\frac{13\pi}{4}}{2\pi} = -\frac{13\pi}{4} \cdot \frac{1}{2\pi} = -\frac{13\pi}{8\pi} = -1.625$$

This tells us that $-13\pi/4$ includes 1 full rotation in the negative direction, plus an additional 0.625 rotation in the negative direction. We just need to figure out how much is 0.625 of 2π.

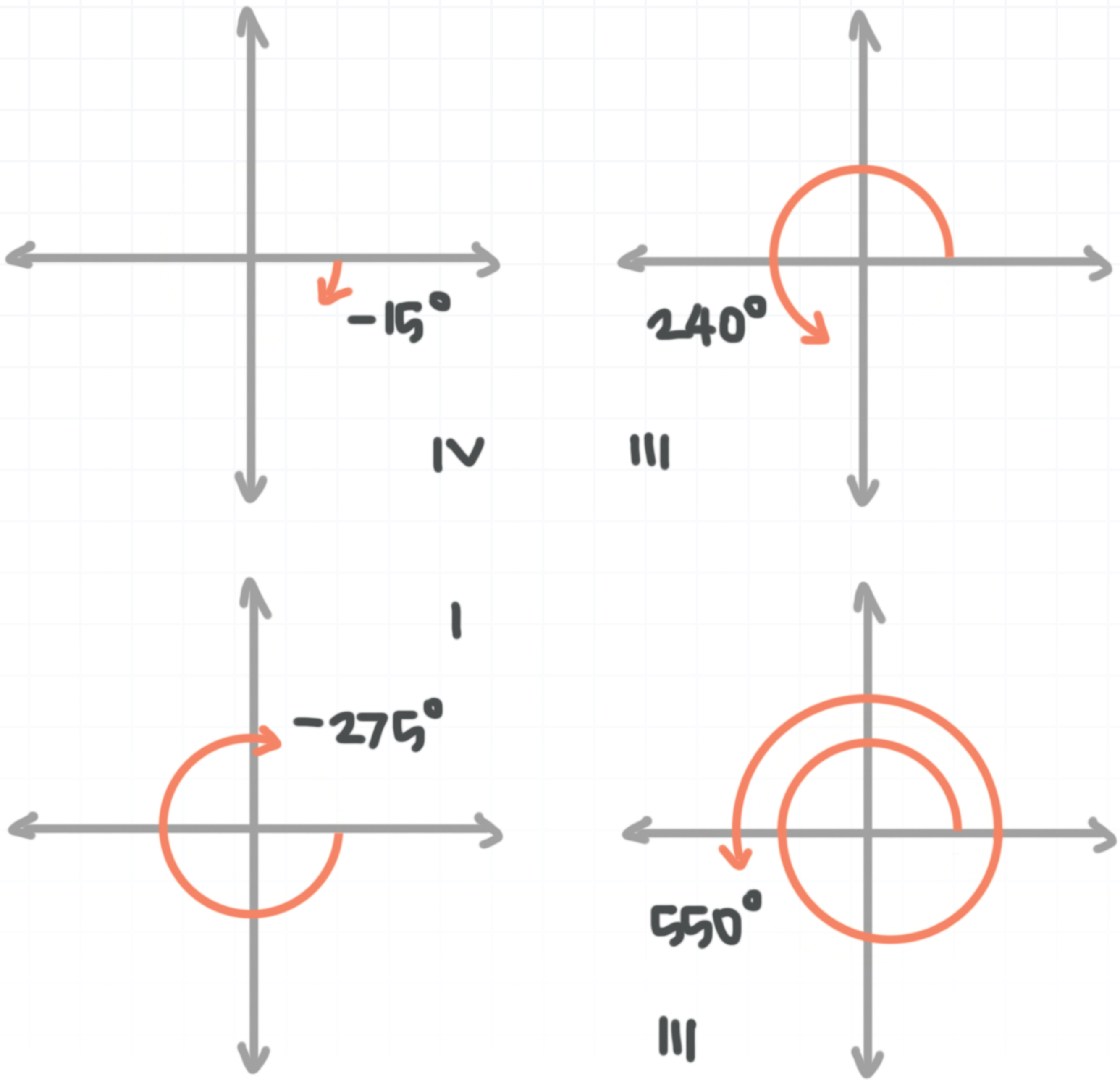
$$0.625(2\pi) = 1.25\pi$$

So, from the starting point of the positive direction of the x-axis, we complete 1 full rotation in the negative direction, which gets us back to the same starting point, and then we rotate an additional 1.25π in the negative direction, which is further than $-1.0\pi = -\pi$, but not as far as $-1.5\pi = -3\pi/2$. Which means the terminal side of the angle will land in the second quadrant.

■ 4. Of the angles -15°, 240°, -275°, and 550°, assuming all four are sketched in standard position, which one has its terminal side in the fourth quadrant?

Solution:

If we sketch each angle, we see that -15° lies in the fourth quadrant, $240^\circ = 180^\circ + 60^\circ$ lies in the third quadrant, $-275^\circ = -180^\circ - 90^\circ - 5^\circ$ lies in the first quadrant, and $550^\circ = 360^\circ + 180^\circ + 10^\circ$ lies in the third quadrant.



■ 5. In which quadrant is the angle $1,200^\circ$ located?

Solution:

Subtract multiples of 360° from $1,200^\circ$ until we find an angle in the interval $[0^\circ,360^\circ)$.

$$1,200^\circ - 360^\circ = 840^\circ$$

$$840^\circ - 360^\circ = 480^\circ$$

$$480^\circ - 360^\circ = 120^\circ$$

This tells us that $1{,}200^\circ$ includes 3 full rotations in the positive direction, plus an additional 120° rotation in the positive direction.

The angle 120° falls between angles of 90° and 180°, which means 120° lies in the second quadrant. Therefore, $1{,}200^\circ$ also lies in the second quadrant.

■ 6. On which axis does the angle -7π lie?

Solution:

An angle of -7π is more than one full negative rotation, so we'll add multiples of 2π to -7π until we find an angle in the interval $[0,2\pi)$.

$$-7\pi + 2\pi = -5\pi$$

$$-5\pi + 2\pi = -3\pi$$

$$-3\pi + 2\pi = -\pi$$

This tells us that -7π includes 3 full rotations in the negative direction, plus an additional π rotation in the negative direction.

The angle $-\pi$ falls on the negative horizontal axis, which means -7π also lies on the negative horizontal axis.

DEGREES, RADIANS, AND DMS

1. Convert $65^\circ 13' 12''$ to a decimal angle.

Solution:

To convert from DMS to degrees, we only need to convert the minutes and seconds parts, since the degree part is already given in degrees.

First we convert seconds to minutes, then minutes to degrees. We'll convert the seconds part first. We need to convert $12''$ from seconds to minutes. We know that $1' = 60''$, so we'll multiply $12''$ by $1'/60''$ in order to cancel the seconds and be left with just minutes.

$$12''\left(\frac{1'}{60''}\right)$$

$$\left(\frac{12}{60}\right)'$$

$$0.2'$$

Then the total minutes in $65^\circ 13' 12''$ is

$$(13 + 0.2)'$$

$$13.2'$$

To convert this value for minutes into degrees, we'll multiply by $1^\circ/60'$ in order to cancel the minutes and be left with an approximate value for degrees.

$$13.2'\left(\frac{1^\circ}{60'}\right)$$

$$\left(\frac{13.2}{60}\right)^\circ$$

$$0.22^\circ$$

Putting this together with the 65° from the original angle, we get approximately

$$(65 + 0.22)^\circ$$

$$65.22^\circ$$

■ 2. Find the sum $20.25^\circ + 20^\circ 2' 5''$ in DMS. Note: The sum of two DMS angles is found by adding the two degree parts, adding the two minutes parts, adding the two seconds parts, and then combining the separate sums into one angle.

Solution:

First convert 20.25° to DMS.

$$20^\circ + 0.25^\circ$$

$$20^\circ + \left(\frac{25}{100}\right)^\circ \left(\frac{60'}{1^\circ}\right)$$

$$20^\circ + 15'$$

$$20^\circ 15'$$

Then find the sum.

$$20.25^\circ + 20^\circ 2' 5''$$

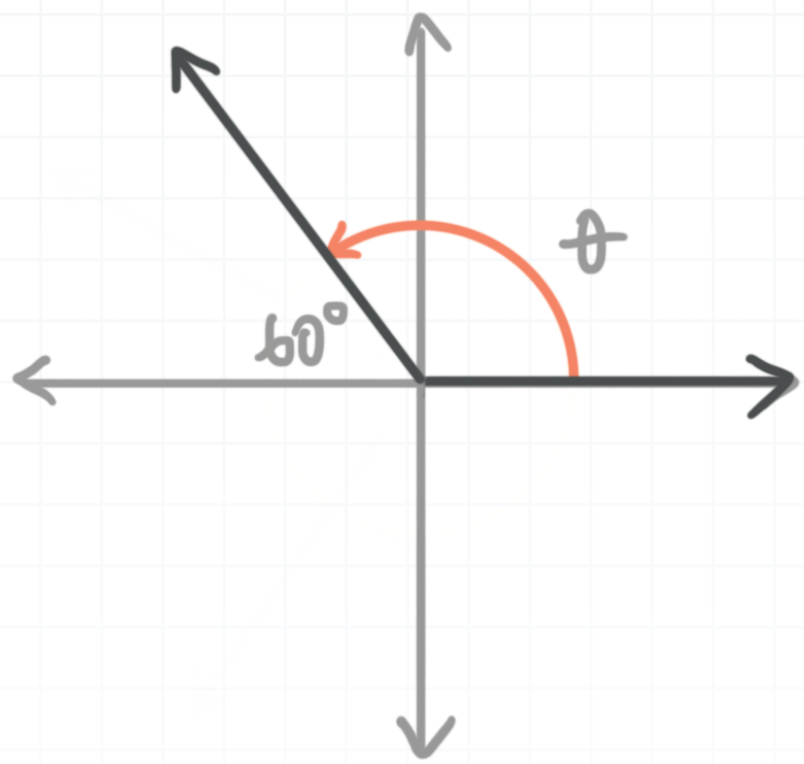
$$20^\circ 15' + 20^\circ 2' 5''$$

$$20^\circ + 20^\circ + 15' + 2' + 5''$$

$$40^\circ + 17' + 5''$$

$$40^\circ 17' 5''$$

■ 3. Find the measure of θ in radians.



Solution:

Because θ is supplementary with 60°, its measure in degrees is

$$180^\circ - 60^\circ$$

$$120^\circ$$

Convert this to radians.

$$120^\circ \left(\frac{\pi}{180^\circ}\right)$$

$$\frac{2\pi}{3}$$

■ 4. Convert the angle $-9\pi/2$ to degrees.

Solution:

Convert the radian angle measure to degrees.

$$-\frac{9\pi}{2}\left(\frac{180^\circ}{\pi}\right)$$

$$-\frac{9(180^\circ)}{2}$$

$$-9(90^\circ)$$

$$-810^\circ$$

5. What is the measure of 152.34° in DMS?

Solution:

Break apart the degree angle.

$$152.34^\circ$$

$$152^\circ + 0.34^\circ$$

$$152^\circ + \left(\frac{34}{100}\right)^\circ$$

Since 152° is already in degrees, we only need to convert the $(34/100)^\circ$ into DMS.

$$\left(\frac{34}{100}\right)^\circ\left(\frac{60'}{1^\circ}\right)$$

$$\left(\frac{(34)(60)}{100}\right)'$$

$$\left(\frac{204}{10}\right)'$$

$$20.4'$$

The minutes part will be $20'$, and we'll convert $0.4'$ into seconds.

$$0.4'\left(\frac{60''}{1'}\right)$$

$$(0.4(60))''$$

$$24''$$

Therefore, the angle 152.34° in DMS is $152^\circ 20' 24''$.

■ 6. What is the measure of 0.2565π in DMS?

Solution:

To convert from radians to DMS, we'll first convert from radians to degrees, and then to DMS. We'll start by multiplying by $180^\circ/\pi$ to get an approximation of the radian angle in degrees.

$$0.2565\pi\left(\frac{180^\circ}{\pi}\right)$$

$$46.17^\circ$$

Break the degree angle apart.

$$46.17^\circ$$

$$46^\circ + 0.17^\circ$$

Since 46° is already in degrees, we only need to convert the 0.17° into DMS.

$$0.17^\circ\left(\frac{60'}{1^\circ}\right)$$

$$10.2'$$

The minutes part will be $10'$, and we'll convert $0.2'$ into seconds.

$$0.2'\left(\frac{60''}{1'}\right)$$

$$12''$$

Therefore, the angle 0.2565π in DMS is $46^\circ 10' 12''$.

COTERMINAL ANGLES

1. If we start at -270° and move three full rotations clockwise around the origin, at which angle will we arrive?

Solution:

A clockwise rotation is a negative rotation, so if we call the new angle α, then the measure of the new angle will be

$$\alpha = -270^\circ - 3(360^\circ)$$

$$\alpha = -270^\circ - 1{,}080^\circ$$

$$\alpha = -1{,}350^\circ$$

2. If we start at $5\pi/6$ and move two full rotations counterclockwise around the origin, at which angle will we arrive?

Solution:

A counterclockwise rotation is a positive rotation, so if we call the new angle α, then the measure of the new angle will be

$$\alpha = \frac{5\pi}{6} + 2(2\pi)$$

$$\alpha = \frac{5\pi}{6} + 4\pi$$

Find a common denominator.

$$\alpha = \frac{5\pi}{6} + 4\pi\left(\frac{6}{6}\right)$$

$$\alpha = \frac{5\pi}{6} + \frac{24\pi}{6}$$

$$\alpha = \frac{29\pi}{6}$$

■ 3. Find the negative and positive coterminal angles that are one full rotation away from $\theta = 200^\circ$.

Solution:

To find angles that are coterminal with $\theta = 200^\circ$, we need to add and subtract 360°. So the coterminal angles that are one full rotation from $\theta = 200^\circ$ are

$$200^\circ - 1(360^\circ) = -160^\circ$$

$$200 + 1(360^\circ) = 560^\circ$$

■ 4. Which angle in the interval $[0^\circ, 360^\circ)$ is coterminal with $-1{,}624^\circ$?

Solution:

We'll add 360° to the angle $-1,624^\circ$ until we get an angle in the interval $[0^\circ,360^\circ)$.

$$-1,624^\circ + 360^\circ = -1,264^\circ$$

$$-1,264^\circ + 360^\circ = -904^\circ$$

$$-904^\circ + 360^\circ = -544^\circ$$

$$-544^\circ + 360^\circ = -184^\circ$$

$$-184^\circ + 360^\circ = 176^\circ$$

The angle 176° is within the interval $[0^\circ,360^\circ)$.

■ 5. Find the angle in the interval $[0,2\pi)$ that's coterminal with $16\pi/3$.

Solution:

We'll subtract 2π from the angle $16\pi/3$ until we get an angle in the interval $[0,2\pi)$.

$$\frac{16\pi}{3} - 2\pi = \frac{16\pi}{3} - \frac{6\pi}{3} = \frac{10\pi}{3}$$

$$\frac{10\pi}{3} - 2\pi = \frac{10\pi}{3} - \frac{6\pi}{3} = \frac{4\pi}{3}$$

The angle $4\pi/3$ is within the interval $[0,2\pi)$.

■ 6. Which angle in $[-\pi,\pi)$ is coterminal with $-19\pi/2$?

Solution:

We'll add 2π to the angle $-19\pi/2$ until we get an angle in the interval $[-\pi,\pi)$.

$$-\frac{19\pi}{2}+2\pi=-\frac{19\pi}{2}+\frac{4\pi}{2}=-\frac{15\pi}{2}$$

$$-\frac{15\pi}{2}+2\pi=-\frac{15\pi}{2}+\frac{4\pi}{2}=-\frac{11\pi}{2}$$

$$-\frac{11\pi}{2}+2\pi=-\frac{11\pi}{2}+\frac{4\pi}{2}=-\frac{7\pi}{2}$$

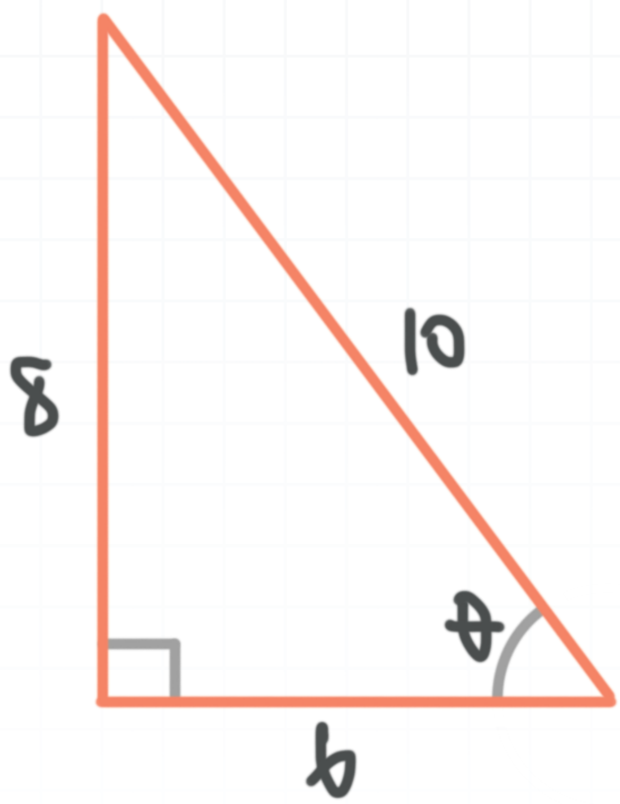
$$-\frac{7\pi}{2}+2\pi=-\frac{7\pi}{2}+\frac{4\pi}{2}=-\frac{3\pi}{2}$$

$$-\frac{3\pi}{2}+2\pi=-\frac{3\pi}{2}+\frac{4\pi}{2}=\frac{\pi}{2}$$

The angle $\pi/2$ is within the interval $[-\pi,\pi)$.

SINE, COSINE, AND TANGENT

1. Find cosine of the angle θ.



Solution:

Given the position of the angle θ in the right triangle, the length of the opposite side is 8, the length of the adjacent side is 6, and the length of the hypotenuse is 10.

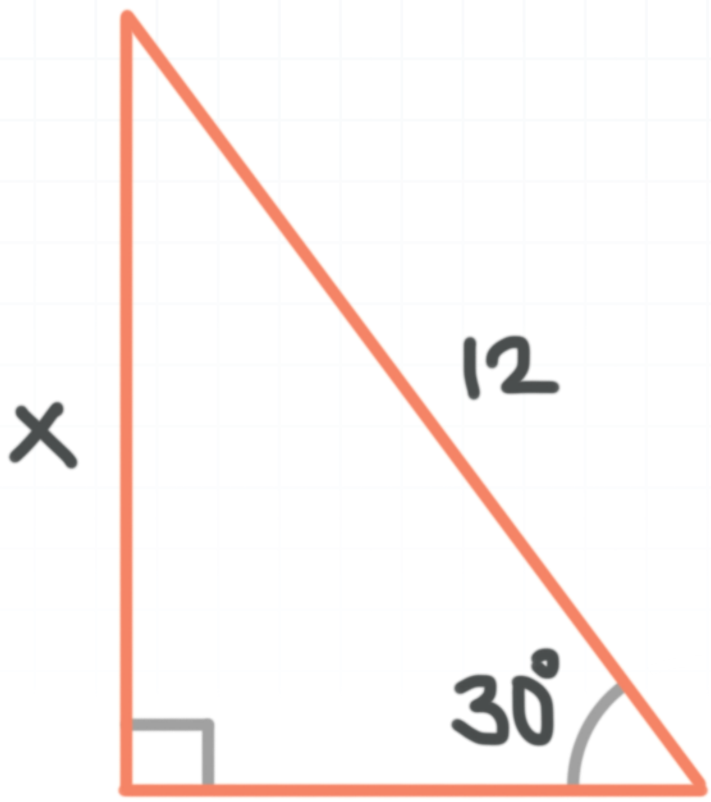
The cosine of that angle θ is equivalent to the length of the side adjacent to the angle θ, divided by the length of the hypotenuse.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Substitute, and we get

$$\cos\theta = \frac{6}{10} = \frac{3}{5}$$

2. Find the measure of the unknown angle of the triangle.



Solution:

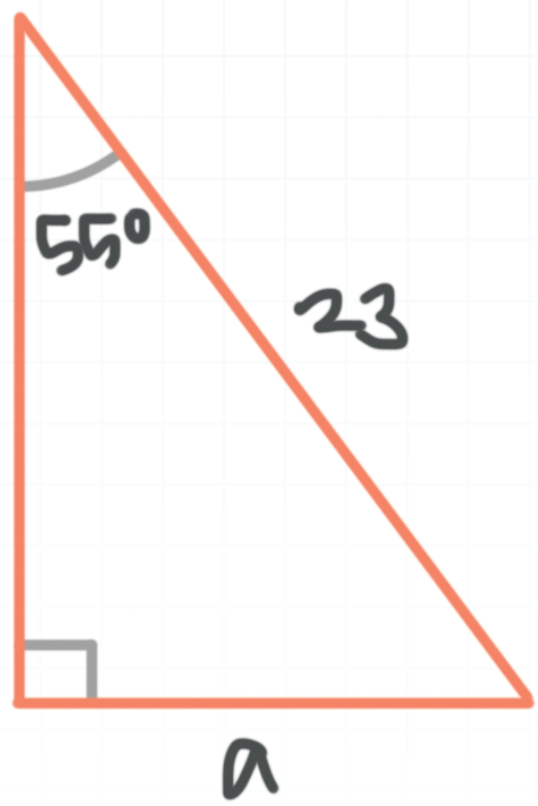
The figure shows a 90° angle and a 30° angle in the triangle. To find the measure of the unknown angle, we need to use the fact that the sum of the interior angles of a triangle is always 180°.

$$180^\circ - 90^\circ - 30^\circ$$

$$60^\circ$$

The measure of the third interior angle is 60°.

3. Find the equation that would be used to solve for a.



Solution:

Given the position of the angle $\theta = 55^\circ$ in the right triangle, the length of the opposite side is a and the length of the hypotenuse is 23.

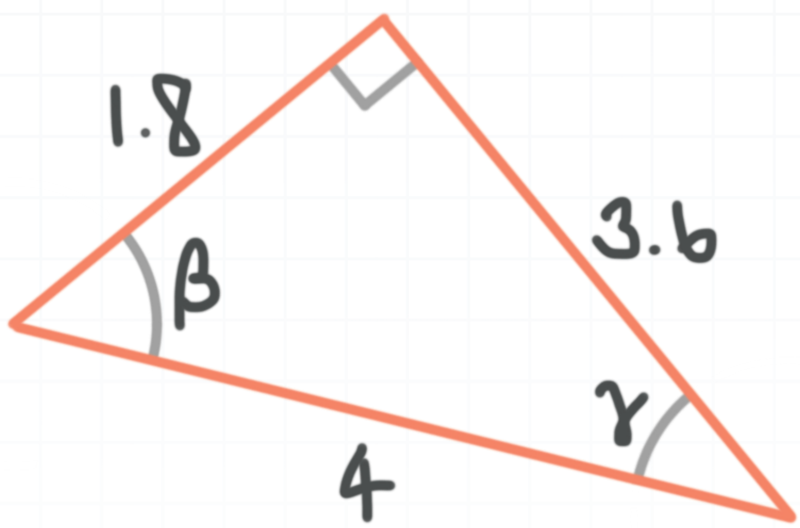
The sine of that angle θ is equivalent to the length of the side opposite the angle θ, divided by the length of the hypotenuse.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

Substitute and get

$$\sin 55^\circ = \frac{a}{23}$$

4. Find the sine, cosine, and tangent for β and γ.



Solution:

Given the position of the angle β in the right triangle, the length of the opposite side is 3.6, the length of the adjacent side is 1.8, and the length of the hypotenuse is 4.

Then the values of sine, cosine, and tangent for the angle β are

$$\sin\beta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3.6}{4} = 0.9$$

$$\cos\beta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1.8}{4} = 0.45$$

$$\tan\beta = \frac{\text{opposite}}{\text{adjacent}} = \frac{3.6}{1.8} = 2$$

Given the position of the angle γ in the right triangle, the length of the opposite side is 1.8, the length of the adjacent side is 3.6, and the length of the hypotenuse is 4.

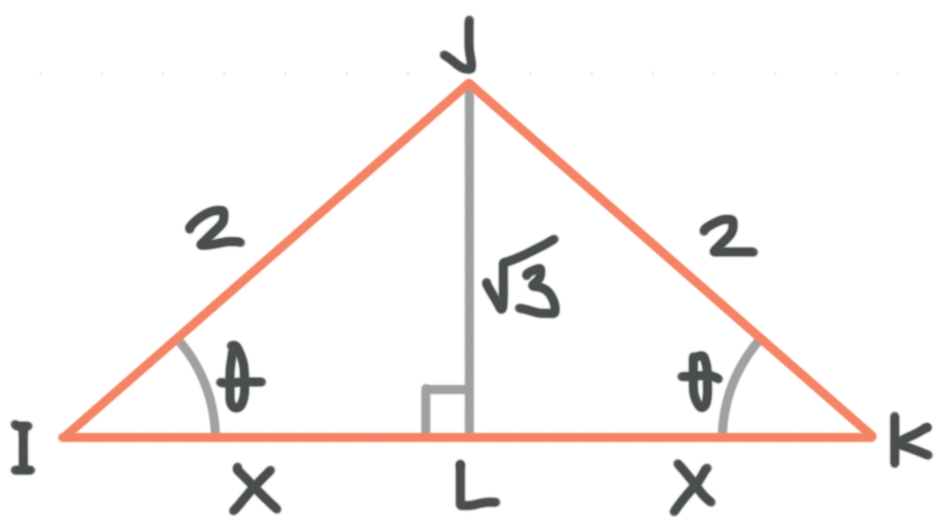
Then the values of sine, cosine, and tangent for the angle γ are

$$\sin\gamma = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1.8}{4} = 0.45$$

$$\cos\gamma = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3.6}{4} = 0.9$$

$$\tan\gamma = \frac{\text{opposite}}{\text{adjacent}} = \frac{1.8}{3.6} = \frac{1}{2}$$

■ 5. Find the value of sine of the angle θ, given that the triangle is isosceles (two of the sides have equal length, and the base angles are equal).



Solution:

The triangle IJK is isosceles, so, $IJ = JK = 2$.

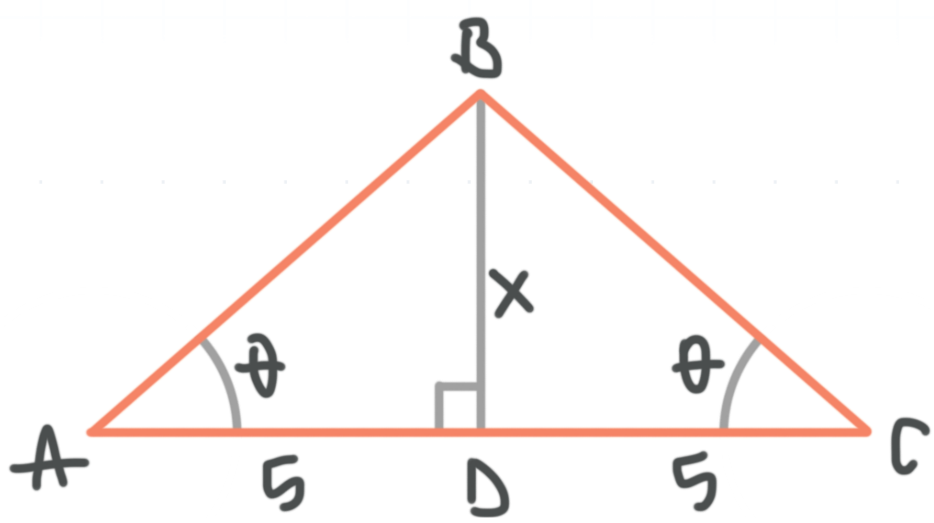
The sine of θ is equivalent to the length of the side opposite the angle θ, divided by the length of the hypotenuse.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

Given the position of the angle θ in the right triangle IJL, the length of the opposite side is $\sqrt{3}$, the length of the adjacent side is x, and the length of the hypotenuse is 2. Substitute and get

$$\sin\theta = \frac{\sqrt{3}}{2}$$

■ 6. Find the equation that would be used to solve for x, given $\overline{AB} = \overline{BC}$ and $\theta = 45^\circ$.



Solution:

Given the position of the angle $\theta = 45^\circ$ in the right triangle ABD, the length of the opposite side is x and the length of the adjacent side is 5.

The tangent of the angle θ is equivalent to the length of the side opposite the angle θ, divided by the length of the side adjacent to the angle θ.

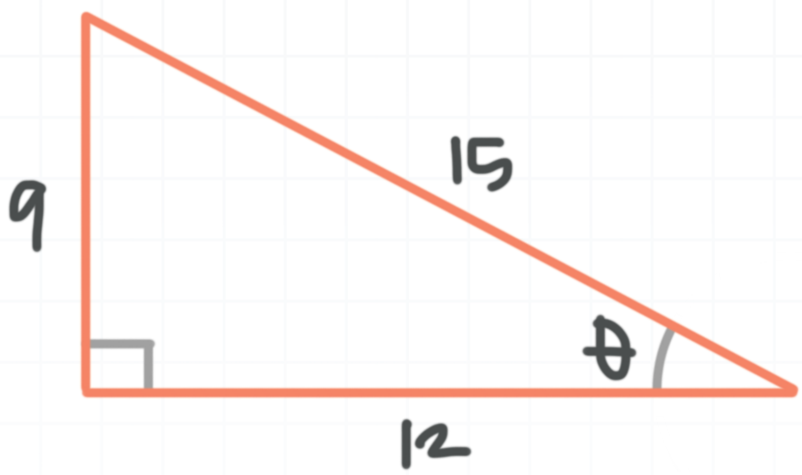
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Substitute to get

$$\tan 45^\circ = \frac{x}{5}$$

COSECANT, SECANT, COTANGENT, AND THE RECIPROCAL IDENTITIES

1. Find the value of secant of θ.



Solution:

Given the position of the angle θ in the right triangle, the length of the opposite side is 9, the length of the adjacent side is 12, and the length of the hypotenuse is 15.

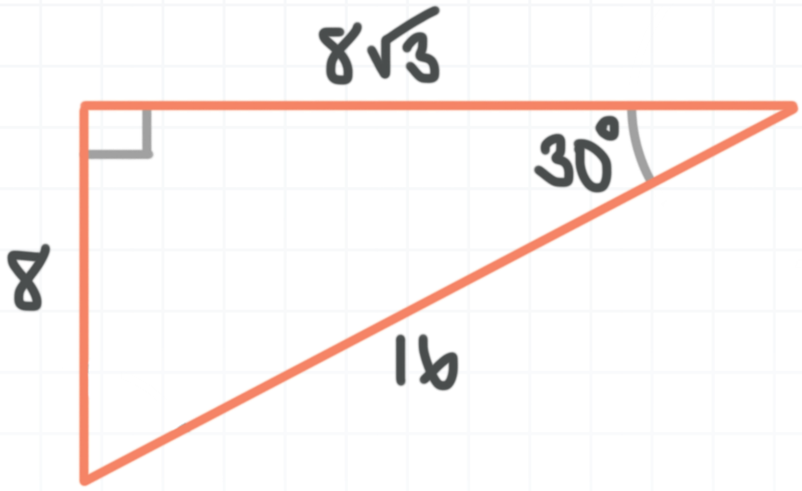
Then the value of secant for the angle is

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

Substitute, and we get

$$\sec\theta = \frac{15}{12} = \frac{5}{4}$$

2. Find the exact value of the six trigonometric functions for $\theta = 30^\circ$.



Solution:

Given the position of the angle θ in the right triangle, the length of the opposite side is 8, the length of the adjacent side is $8\sqrt{3}$, and the length of the hypotenuse is 16.

Then the values of the six trigonometric functions for $\theta = 30^\circ$ are

$$\sin 30^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{8}{16} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{8\sqrt{3}}{16} = \frac{\sqrt{3}}{2}$$

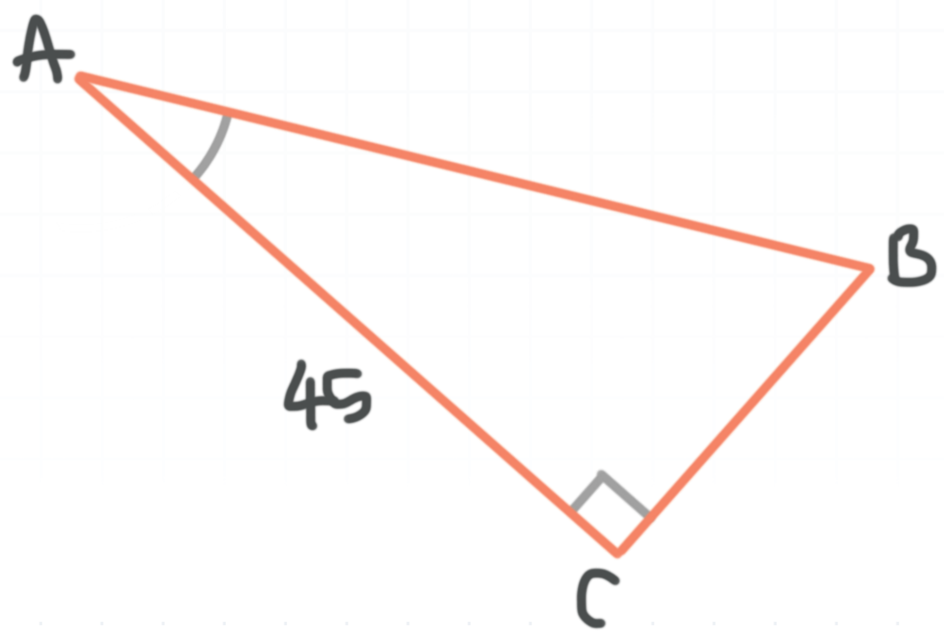
$$\tan 30^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{8}{8\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\csc 30^\circ = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{16}{8} = 2$$

$$\sec 30^\circ = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{16}{8\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot 30^\circ = \frac{\text{adjacent}}{\text{opposite}} = \frac{8\sqrt{3}}{8} = \sqrt{3}$$

■ 3. Given right triangle ABC, $\sin A = 28/53$. Find the exact value of secant, cosecant, and cotangent for the angle A.



Solution:

Given the position of the angle A in the right triangle, the length of the adjacent side is 45. Since $\sin A = 28/53$, we know the length of the opposite side is 28 and the length of the hypotenuse is 53.

Then the values of cosecant, secant, and cotangent of the angle A are

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{53}{28}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{53}{45}$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{45}{28}$$

■ 4. Find $\csc\theta$, if $\sin\theta = 8/17$.

Solution:

The sine and cosecant functions are related to each other by the reciprocal identities, so we can substitute the value of $\sin\theta$ into the reciprocal identity for $\csc\theta$.

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{8}{17}} = \frac{17}{8}$$

■ 5. If $\sec\theta = 61/60$ and $\tan\theta = 11/60$, determine the values of the other four trigonometric functions.

Solution:

We know that the secant of an angle is the ratio of the hypotenuse to the adjacent side, and that the tangent of an angle is the ratio of the opposite side to the adjacent side.

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Given the definitions of secant and tangent we've been given, the length of the opposite side is 11, the length of the adjacent side is 60, and the length of the hypotenuse is 61.

Then the values of sine, cosine, cosecant, and cotangent are

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{11}{61}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{60}{61}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{61}{11}$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{60}{11}$$

6. Given the value of $\cot\theta$, find the value of $\tan\theta$.

$$\cot\theta = \frac{63}{16}$$

Solution:

The cotangent and tangent functions are reciprocals of one another, so we can substitute the value of $\cot\theta$ into the reciprocal identity for $\tan\theta$.

$$\tan\theta = \frac{1}{\cot\theta} = \frac{1}{\frac{63}{16}} = \frac{16}{63}$$

THE QUOTIENT IDENTITIES

1. If $\sin\theta = 16/65$ and $\cos\theta = 63/65$, find $\cot\theta$.

Solution:

The cotangent of an angle is defined in terms of the sine and cosine by the quotient identity for cotangent.

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{63}{65}}{\frac{16}{65}} = \frac{63}{65}\left(\frac{65}{16}\right) = \frac{63}{16}$$

2. If $\tan\theta = 4/3$ and $\cos\theta = 3/5$, find $\sin\theta$.

Solution:

We can find sine of θ just by plugging these tangent and cosine values into the quotient identity for tangent.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Substitute, and we get

$$\frac{4}{3} = \frac{\sin\theta}{\frac{3}{5}}$$

$$\sin\theta = \frac{4}{3}\left(\frac{3}{5}\right) = \frac{4}{5}$$

■ 3. If $\cot\theta = \sqrt{13}/6$ and $\csc\theta = -7/6$, find $\cos\theta$.

Solution:

Use the reciprocal identity for sine, substituting the value of cosecant.

$$\sin\theta = \frac{1}{\csc\theta} = \frac{1}{-\frac{7}{6}} = -\frac{6}{7}$$

Now we can find cosine of θ just by plugging the cotangent and sine values into the quotient identity for cotangent.

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Substitute, and we get

$$\frac{\sqrt{13}}{6} = \frac{\cos\theta}{-\frac{6}{7}}$$

$$\cos\theta = \frac{\sqrt{13}}{6}\left(-\frac{6}{7}\right) = -\frac{\sqrt{13}}{7}$$

4. If $\cot\theta = -12/5$ and $\cos\theta = 12/13$, find $\sin\theta$.

Solution:

We can find sine of θ just by plugging these cotangent and cosine values into the quotient identity for cotangent.

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Substitute, and we get

$$-\frac{12}{5} = \frac{\frac{12}{13}}{\sin\theta}$$

$$\sin\theta = \frac{\frac{12}{13}}{-\frac{12}{5}} = \frac{12}{13}\left(-\frac{5}{12}\right) = -\frac{5}{13}$$

5. If $\sin\theta = 39/89$ and $\tan\theta = -39/80$, find $\cos\theta$.

Solution:

We can find cosine of θ just by plugging these tangent and sine values into the quotient identity for tangent.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Substitute, and we get

$$-\frac{39}{80} = \frac{\frac{39}{89}}{\cos\theta}$$

$$\cos\theta = \frac{\frac{39}{89}}{-\frac{39}{80}} = \frac{39}{89}\left(-\frac{80}{39}\right) = -\frac{80}{89}$$

■ 6. If $\tan\theta = 8/15$ and $\sec\theta = 17/15$, find $\sin\theta$.

Solution:

Use the reciprocal identity for cosine, substituting the value of secant.

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\frac{17}{15}} = \frac{15}{17}$$

We can find sine of θ just by plugging the tangent and cosine values into the quotient identity for tangent.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Substitute and get

$$\frac{8}{15} = \frac{\sin\theta}{\frac{15}{17}}$$

$$\sin\theta = \frac{8}{15}\left(\frac{15}{17}\right) = \frac{8}{17}$$

THE PYTHAGOREAN IDENTITIES

1. Find the positive value of $\cos(49.3^\circ)$ if $\sin(49.3^\circ) = 0.758$.

Solution:

We'll use a rewritten form of the Pythagorean identity with sine and cosine.

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos^2(49.3^\circ) = 1 - \sin^2(49.3^\circ)$$

$$\cos^2(49.3^\circ) = 1 - (0.758)^2$$

$$\cos^2(49.3^\circ) \approx 1 - 0.575$$

$$\cos^2(49.3^\circ) \approx 0.425$$

$$\cos(49.3^\circ) \approx \pm\sqrt{0.425}$$

Since we need to find the positive value of cosine, we ignore the negative value and say

$$\cos(49.3^\circ) \approx \sqrt{0.425}$$

$$\cos(49.3^\circ) \approx 0.652$$

2. In a right triangle, sine of the acute angle is $1/5$. What are the positive values of the cosine and cotangent of this angle?

We'll use a rewritten form of the Pythagorean identity with sine and cosine.

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos^2(\theta) = 1 - \left(\frac{1}{5}\right)^2$$

$$\cos^2\theta = 1 - \frac{1}{25}$$

$$\cos^2\theta = \frac{24}{25}$$

$$\cos\theta = \pm\sqrt{\frac{24}{25}}$$

Since we need to find the positive value of cosine, we ignore the negative value and say

$$\cos\theta = \sqrt{\frac{24}{25}}$$

$$\cos\theta = \frac{2\sqrt{6}}{5}$$

Now that we know the value of both sine and cosine of the angle, we'll use the quotient identity to find the value of cotangent.

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{2\sqrt{6}}{5}}{\frac{1}{5}} = \frac{2\sqrt{6}}{5}\left(\frac{5}{1}\right) = 2\sqrt{6}$$

■ 3. If $\sin\theta = 12/13$, what is the negative value of $\cot\theta$?

Solution:

We'll use a rewritten form of the Pythagorean identity with cotangent and cosecant.

$$\cot^2\theta = \csc^2\theta - 1$$

Use the reciprocal identity for cosecant to rewrite the Pythagorean identity as

$$\cot^2\theta = \left(\frac{1}{\sin\theta}\right)^2 - 1$$

Substitute, and we get

$$\cot^2\theta = \left(\frac{1}{\frac{12}{13}}\right)^2 - 1$$

$$\cot^2\theta = \left(\frac{13}{12}\right)^2 - 1$$

$$\cot^2\theta = \frac{169}{144} - 1$$

$$\cot^2\theta = \frac{25}{144}$$

$$\cot\theta = \pm\frac{5}{12}$$

Since we need to find the negative value of cotangent, we ignore the positive value and say

$$\cot\theta = -\frac{5}{12}$$

■ 4. If $\theta = 6\pi/5$ and $\sin\theta = -0.588$, what is the negative value of $\cos\theta$?

Solution:

We'll use a rewritten form of the Pythagorean identity with sine and cosine.

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos^2\left(\frac{6\pi}{5}\right) = 1 - \sin^2\left(\frac{6\pi}{5}\right)$$

$$\cos^2\left(\frac{6\pi}{5}\right) = 1 - (-0.588)^2$$

$$\cos^2\left(\frac{6\pi}{5}\right) \approx 1 - 0.346$$

$$\cos^2\left(\frac{6\pi}{5}\right) \approx 0.654$$

$$\cos\left(\frac{6\pi}{5}\right) \approx \pm\sqrt{0.654}$$

Since we need to find the negative value of cosine, we ignore the positive value and say

$$\cos\left(\frac{6\pi}{5}\right) \approx -\sqrt{0.654}$$

$$\cos\left(\frac{6\pi}{5}\right) \approx -0.809$$

■ 5. If θ is an angle in the second quadrant such that $\cos\theta = -0.412$, what is the negative value of $\tan\theta$?

Solution:

We'll use a rewritten form of the Pythagorean identity with tangent and secant.

$$\tan^2\theta = \sec^2\theta - 1$$

Use the reciprocal identity for secant to rewrite the Pythagorean identity as

$$\tan^2\theta = \left(\frac{1}{\cos\theta}\right)^2 - 1$$

Substitute, and we get

$$\tan^2\theta = \left(\frac{1}{-0.412}\right)^2 - 1$$

$$\tan^2\theta = 5.891 - 1$$

$$\tan^2\theta = 4.891$$

$$\tan\theta = \pm\, 2.212$$

Since we need to find the negative value of tangent, we ignore the positive value and say

$$\tan\theta = -\, 2.212$$

■ 6. Evaluate the expression if $\cos\theta = 1/\sqrt{3}$.

$$\tan^2\theta + \sin^2\theta + \sec^2\theta$$

Solution:

Use the Pythagorean identity with sine and cosine to find the value of sine.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\sin^2\theta = 1 - \left(\frac{1}{\sqrt{3}}\right)^2$$

$$\sin^2\theta = 1 - \frac{1}{3}$$

$$\sin^2\theta = \frac{2}{3}$$

If we use the quotient identity for tangent, then we get

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$$

$$\tan^2\theta = \frac{\frac{2}{3}}{\frac{1}{3}}$$

$$\tan^2\theta = \frac{2}{3}\left(\frac{3}{1}\right)$$

$$\tan^2\theta = 2$$

If we use the reciprocal identity for secant, we get

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\sec^2\theta = \frac{1}{\cos^2\theta}$$

$$\sec^2\theta = \frac{1}{\frac{1}{3}}$$

$$\sec^2\theta = 3$$

Then the value of the expression is

$$\tan^2\theta + \sin^2\theta + \sec^2\theta$$

$$2 + \frac{2}{3} + 3$$

$$\frac{6}{3} + \frac{2}{3} + \frac{9}{3}$$

$$\frac{17}{3}$$

SIGNS BY QUADRANT

1. Find $\sin\theta$ if the angle θ lies in the interval $[0^\circ, 180^\circ)$ and $\cos^2\theta - 0.36 = 0$.

Solution:

Rewrite the cosine equation.

$$\cos^2\theta - 0.36 = 0$$

$$\cos^2\theta = 0.36$$

Now that we have a value for $\cos^2\theta$, we can substitute it into a rewritten form of the Pythagorean identity for sine and cosine.

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\sin^2\theta = 1 - 0.36$$

$$\sin^2\theta = 0.64$$

$$\sin\theta = \pm\sqrt{0.64}$$

The interval $[0^\circ, 180^\circ)$ defines the first and second quadrants, which means θ must lie in one of those quadrants. All points in the first and second quadrants have a positive y-value, which means that the sine of any angle in those quadrants will be positive. So we ignore the negative value and say

$$\sin\theta = \sqrt{0.64}$$

$$\sin\theta = 0.8$$

■ 2. Find $\cot\theta$ if $\cos\theta = 0.6$ and the angle θ is in the interval $[5\pi,6\pi)$.

Solution:

We're told that θ is between 5π and 6π, so it lies in the third of fourth quadrants. We also know that cosine is positive, and since cosine is only positive in the first and fourth quadrants, θ must lie in the fourth quadrant.

In the fourth quadrant, $\sin\theta$ is negative, which means cotangent is also negative. So we'll find the value of sine using the Pythagorean identity with sine and cosine.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\sin^2\theta = 1 - 0.6^2$$

$$\sin^2\theta = 0.64$$

Because we know sine is negative,

$$\sin\theta = -\sqrt{0.64}$$

$$\sin\theta = -0.8$$

Using the quotient identity for cotangent, $\cot\theta$ must be

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\cot\theta = \frac{0.6}{-0.8}$$

$$\cot\theta = -\frac{3}{4}$$

■ 3. Find $\sin\theta$ if $\sec\theta = 3$ and $\cot\theta < 0$.

Solution:

Use the reciprocal identity to find $\cos\theta$.

$$\sec\theta = \frac{1}{\cos\theta}$$

$$3 = \frac{1}{\cos\theta}$$

$$\cos\theta = \frac{1}{3}$$

Since we were told that cotangent is negative, and cotangent is negative for angles that fall in the second and fourth quadrants, the angle must fall in the second or fourth quadrant.

We also know that cosine is positive, and since cosine is only positive for angles in the first and fourth quadrants, θ must be in first or fourth quadrant.

Comparing these results for cotangent and cosine tells us that the angle can only fall in the fourth quadrant.

In the fourth quadrant, $\sin\theta$ is negative, so we can find the sine of the angle using the Pythagorean identity with sine and cosine.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\sin^2\theta = 1 - \left(\frac{1}{3}\right)^2$$

$$\sin^2\theta = 1 - \frac{1}{9}$$

$$\sin^2\theta = \frac{8}{9}$$

$$\sin\theta = -\sqrt{\frac{8}{9}}$$

$$\sin\theta = -\frac{2\sqrt{2}}{3}$$

■ 4. At the angle -340°, what are the signs of sine and cosine.

Solution:

The angle -340° lies in the first quadrant. Since the x- and y-values of any point in the first quadrant are positive, we know that $\sin(-340^\circ)$ and $\cos(-340^\circ)$ must both be positive.

■ 5. In which quadrant does the angle θ lie, if $\tan\theta$ is positive and $\sec\theta$ is negative?

Solution:

If $\sec\theta$ is negative, then $\cos\theta$ is also negative, which means θ must lie in the second or third quadrants.

Since $\tan\theta$ is positive and cosine is negative, $\sin\theta$ must be negative, which means θ must lie in the third of fourth quadrants.

But since the sign of $\cos\theta$ restricts the angle to the second and third quadrants, the angle θ can only lie within the third quadrant.

■ 6. Find the largest among the values of the six trig functions of θ if $\cos\theta = -0.1$ and θ lies in the third quadrant.

Solution:

Since we're looking for the largest value among the six trig functions, we only need to consider the trig functions that will have a positive value. In the third quadrant, the only positive trig functions are tangent and cotangent.

To find the value of tangent, we'll use the Pythagorean identity with sine and cosine.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\sin^2\theta = 1 - (-0.1)^2$$

$$\sin^2\theta = 0.99$$

In the third quadrant, $\sin\theta$ is negative.

$$\sin\theta = -\sqrt{0.99}$$

With the values of sine and cosine, we can find the values of tangent and cotangent.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\sqrt{0.99}}{-0.1} = \sqrt{99} \approx 10$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{-0.1}{-\sqrt{0.99}} = \frac{1}{\sqrt{99}} \approx 0.1$$

Because $10 > 0.1$, of the six trig functions, tangent has the largest value at this particular angle θ.

WHEN THE TRIG FUNCTIONS ARE UNDEFINED

■ 1. For what angle is $\cot\theta$ undefined in the interval $(0,2\pi]$?

Solution:

We know by the quotient identity

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

that cotangent is undefined when sine is 0, and we know that sine is 0 when $y = 0$. We know we'll have $y = 0$ along the horizontal axis, which means that, in the interval

$$(0,2\pi]$$

cotangent will be undefined at $\theta = \pi$ and $\theta = 2\pi$.

■ 2. Determine whether or not $\cot(-43\pi/4)$ is defined.

Solution:

We know by the quotient identity

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

that cotangent is undefined when sine is 0, and we know that sine is 0 when $y=0$. We know we'll have $y=0$ along the horizontal axis, so we need to determine whether or not $-43\pi/4$ lies along the horizontal axis.

If we know one full rotation is $2\pi=8\pi/4$, then we can say $-43\pi/4$ is five full negative rotations ($-40\pi/4$) and then an additional $-3\pi/4$ rotations. The angle $-3\pi/4$ doesn't fall on the horizontal axis, which means $\cot(-43\pi/4)$ will be defined.

■ 3. Which trigonometric functions are undefined for $\theta=\pi/2$?

Solution:

The angle $\theta=\pi/2$ falls on the positive y-axis. So in a circle centered at the origin with radius 1,

$$y=\sin\left(\frac{\pi}{2}\right)=1$$

$$x=\cos\left(\frac{\pi}{2}\right)=0$$

Use the reciprocal identities to find cosecant and secant of the angle.

$$\csc\left(\frac{\pi}{2}\right)=\frac{1}{\sin\left(\frac{\pi}{2}\right)}=\frac{1}{1}=1$$

$$\sec\left(\frac{\pi}{2}\right)=\frac{1}{\cos\left(\frac{\pi}{2}\right)}=\frac{1}{0}$$

Use the quotient identities to find tangent and cotangent of the angle.

$$\tan\left(\frac{\pi}{2}\right)=\frac{\sin\left(\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}\right)}=\frac{1}{0}$$

$$\cot\left(\frac{\pi}{2}\right)=\frac{\cos\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)}=\frac{0}{1}=0$$

So to summarize, because we got a 0 value in the denominator of the secant and tangent functions, we know these two trig functions are undefined at $\theta=\pi/2$.

■ 4. Which of the six trigonometric functions are undefined along the y-axis (when $x=0$)?

Solution:

We know we'll have $x=0$ along the vertical axis. And we know that cosine is 0 when $x=0$. Therefore, we need to find the functions which are undefined when cosine is 0. These are the tangent and secant functions, because $\cos x$ is in the denominator in these functions.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

5. Find the angle where $\tan\theta$ is undefined in the given interval.

$$\left(\frac{7\pi}{3}, \frac{25\pi}{6}\right)$$

Solution:

We need to start by first identifying an interval that's coterminal with the given interval.

We know that one full rotation is given by 2π radians.

$$\frac{7\pi}{3} = \frac{6\pi}{3} + \frac{\pi}{3}$$

So we know that an angle of $7\pi/3$ is one full rotation, and then another $\pi/3$ rotation. So $7\pi/3$ is coterminal with $\pi/3$. To get from $7\pi/3$ to $\pi/3$, we're subtracting one full 2π rotation. So we need to also subtract a 2π rotation from $25\pi/6$.

$$\frac{25\pi}{6} - 2\pi = \frac{25\pi}{6} - \frac{12\pi}{6} = \frac{13\pi}{6}$$

Therefore, we would rewrite the given interval as

$$\left(\frac{\pi}{3},\frac{13\pi}{6}\right)$$

This is now an interval that spans from the first quadrant, through the second, third, and fourth quadrants, and back into the first quadrant, stopping $\pi/6$ rotations short of where we started.

We know by the quotient identity

$$\tan\theta=\frac{\sin\theta}{\cos\theta}$$

that $\tan\theta$ is undefined when $\cos\theta=0$, and we know that $\cos\theta=0$ along the vertical axis, which occurs in the interval $(\pi/3,13\pi/6)$ at

$$\theta=\frac{\pi}{2},\frac{3\pi}{2}$$

To find the angles that correspond to these in the original interval, we need to add 2π back to each angle, since we previously took 2π away from the original interval.

$$\theta=\frac{\pi}{2}+2\pi,\ \frac{3\pi}{2}+2\pi$$

$$\theta=\frac{\pi}{2}+\frac{4\pi}{2},\frac{3\pi}{2}+\frac{4\pi}{2}$$

$$\theta=\frac{5\pi}{2},\frac{7\pi}{2}$$

6. Find the values of all six trig functions at $\theta = \pi$, and say whether or not any of them are undefined at this angle.

Solution:

The angle $\theta = \pi$ falls on the negative x-axis. So in a circle centered at the origin with radius 1,

$$y = \sin \pi = 0$$

$$x = \cos \pi = -1$$

Use the reciprocal identities to find cosecant and secant of the angle.

$$\csc \pi = \frac{1}{\sin \pi} = \frac{1}{0}$$

$$\sec \pi = \frac{1}{\cos \pi} = \frac{1}{-1} = -1$$

Use the quotient identities to find tangent and cotangent of the angle.

$$\tan \pi = \frac{\sin \pi}{\cos \pi} = \frac{0}{-1} = 0$$

$$\cot \pi = \frac{\cos \pi}{\sin \pi} = \frac{-1}{0}$$

So to summarize, because we got a 0 value in the denominator of the cosecant and cotangent values, we know these two trig functions are undefined at $\theta = \pi$.

THE UNIT CIRCLE

■ 1. What is the coordinate point associated with $\theta = 2\pi/3$ along the unit circle?

Solution:

Looking at the unit circle shows that the coordinate point associated with $\theta = 2\pi/3$ in the second quadrant is

$$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

■ 2. The terminal side of the angle θ in $[0,2\pi)$ intersects the unit circle at the given point. Find the measure of θ in degrees.

$$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

Solution:

Because both the x- and y-values in the coordinate point are negative, we know the angle lies in the third quadrant. Comparing the coordinate point to points along the unit circle in the third quadrant, we see that $\theta = 240^\circ$.

3. Find $\sin\theta$ if $\theta \in [0,2\pi)$ and $\cos\theta = \sin\theta$.

Solution:

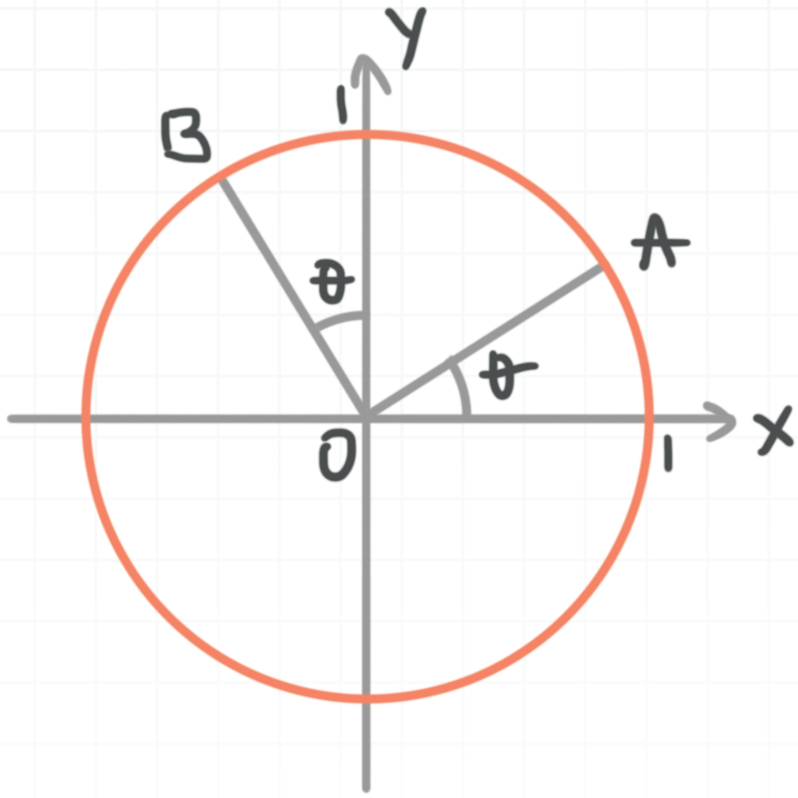
We know that $\sin\theta$ represents the y-coordinate, and $\cos\theta$ represents the x-coordinate. In the second and third quadrants, the signs of x and y are different, which means $\sin\theta$ and $\cos\theta$ cannot be equal.

If we look at the first and third quadrants of the unit circle, we can see that the x- and y-values are equal at $\theta = \pi/4$ and $\theta = 5\pi/4$.

At $\theta = \dfrac{\pi}{4}$, $\sin\theta = \dfrac{\sqrt{2}}{2}$

At $\theta = \dfrac{5\pi}{4}$, $\sin\theta = -\dfrac{\sqrt{2}}{2}$

4. The points A and B lie on the unit circle in quadrants I and II respectively. The angle between OA and the positive x-axis is θ. The angle between OB and the positive y-axis is θ. Find the sine of $\angle AOB$.



Solution:

The angle between OB and the positive x-axis is $90^\circ + \theta$. So $\angle AOB$ is $(90^\circ + \theta) - \theta = 90^\circ$. Because we know $\sin 90^\circ = 1$,

$$\sin AOB = \sin 90^\circ = 1$$

■ 5. Evaluate the expression.

$$2 \csc\left(\frac{49\pi}{6}\right) - 3\cos\left(\frac{13\pi}{3}\right) + \tan\left(\frac{25\pi}{4}\right)$$

Solution:

For each of the given angles, find a coterminal angle in $[0,2\pi)$.

$$\frac{49\pi}{6} = \frac{48\pi}{6} + \frac{\pi}{6} = 8\pi + \frac{\pi}{6}$$

$$\frac{13\pi}{3} = \frac{12\pi}{3} + \frac{\pi}{3} = 4\pi + \frac{\pi}{3}$$

$$\frac{25\pi}{4} = \frac{24\pi}{4} + \frac{\pi}{4} = 6\pi + \frac{\pi}{4}$$

Which means

$$\csc\left(\frac{49\pi}{6}\right) = \csc\left(\frac{\pi}{6}\right)$$

$$\cos\left(\frac{13\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$$

$$\tan\left(\frac{25\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right)$$

Rewrite the expression with these coterminal angles.

$$2\csc\left(\frac{\pi}{6}\right) - 3\cos\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{4}\right)$$

Apply reciprocal and quotient identities to put the expression in terms of only sine and cosine functions.

$$\frac{2}{\sin\left(\frac{\pi}{6}\right)} - 3\cos\left(\frac{\pi}{3}\right) + \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)}$$

Plug in the known values from the unit circle.

$$\frac{2}{\frac{1}{2}} - 3\left(\frac{1}{2}\right) + \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$

$$4 - \frac{3}{2} + 1$$

$$\frac{8}{2} - \frac{3}{2} + \frac{2}{2}$$

$$\frac{7}{2}$$

■ 6. Find the angle θ in the interval $[0,2\pi)$.

$$\sin\theta = \frac{1}{2} \text{ and } \cos\theta = -\frac{\sqrt{3}}{2}$$

Solution:

From the unit circle, we know the sine function takes the value $1/2$ at angles of $\pi/6$ and $5\pi/6$, so we need to evaluate cosine at both of these angles. We get

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \text{ and } \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

Therefore, the matching angle is $\theta = 5\pi/6$.

NEGATIVE ANGLES AND ANGLES MORE THAN ONE ROTATION

■ 1. θ has a measure of 42°. If it's rotated 6 full rotations in the negative direction, find its measure after the rotations.

Solution:

The new angle measure will be given by

$$42^\circ + (-6)(360^\circ)$$

$$42^\circ - 2{,}160^\circ$$

$$-2{,}118^\circ$$

■ 2. Find the values of the six trig functions at $\theta = -11\pi/3$.

Solution:

We know that one full rotation around the unit circle is a 2π rotation, which we can express in thirds as $6\pi/3$. Two full rotations would therefore be $12\pi/3$.

So we know that $-11\pi/3$ is more than one full negative (clockwise) rotation, but less than two full negative rotations. To get to $-11\pi/3$, we'll rotate one full negative rotation to get to $-6\pi/3$, and then continue on an additional $-5\pi/3$ to get to $-11\pi/3$. So we're looking for an angle that's coterminal with $-5\pi/3$.

To find the coterminal angle, we'll add 2π to $-5\pi/3$.

$$-\frac{5\pi}{3}+2\pi$$

$$-\frac{5\pi}{3}+\frac{6\pi}{3}$$

$$\frac{\pi}{3}$$

So $\pi/3$ is coterminal with $-11\pi/3$, and we can therefore find all six trig functions at $-11\pi/3$ by finding all six trig functions at $\pi/3$. Because the angles are coterminal, the values of the six trig functions are equivalent.

Pulling directly from the unit circle, we get

$$\sin\left(-\frac{11\pi}{3}\right)=\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

$$\cos\left(-\frac{11\pi}{3}\right)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$$

Using the quotient identity for tangent, we get

$$\tan\left(-\frac{11\pi}{3}\right)=\tan\left(\frac{\pi}{3}\right)=\frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)}=\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\frac{\sqrt{3}}{2}\cdot\frac{2}{1}=\sqrt{3}$$

Then, using the reciprocal identities for cosecant, secant, and cotangent, we get

$$\csc\left(-\frac{11\pi}{3}\right)=\csc\left(\frac{\pi}{3}\right)=\frac{1}{\sin\left(\frac{\pi}{3}\right)}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$$

$$\sec\left(-\frac{11\pi}{3}\right)=\sec\left(\frac{\pi}{3}\right)=\frac{1}{\cos\left(\frac{\pi}{3}\right)}=\frac{2}{1}=2$$

$$\cot\left(-\frac{11\pi}{3}\right)=\cot\left(\frac{\pi}{3}\right)=\frac{1}{\tan\left(\frac{\pi}{3}\right)}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$

■ 3. Find the angle in the interval $[0^\circ,360^\circ)$ that's coterminal with -427°.

Solution:

Let $\theta=-427^\circ$, and let α be the angle that lies in the interval $[0^\circ,360^\circ)$ and is coterminal with θ. To find α, let's add 360° to $\theta=-427^\circ$ until we get to an angle that lies in the interval $[0^\circ,360^\circ)$.

$$-427^\circ+360^\circ=-67^\circ$$

$$-67^\circ+360^\circ=293^\circ$$

This 293° angle lies within $[0^\circ,360^\circ)$ and is coterminal with $\theta=-427^\circ$.

4. Find the angle in the interval $[0,2\pi)$ that's coterminal with $\theta = -65\pi/6$.

Solution:

To find the number of full rotations included in $\theta = -65\pi/6$, we'll divide the angle by 2π.

$$\frac{-\frac{65\pi}{6}}{2\pi}$$

$$-\frac{65\pi}{6} \cdot \frac{1}{2\pi}$$

$$-\frac{65\pi}{12\pi}$$

$$-5.417$$

So $\theta = -65\pi/6$ is 5 full rotations in the negative direction, and then an additional 0.417 of one more rotation in the negative direction. So to find a coterminal angle, we'll get rid of the 5 full rotations by adding $5(2\pi)$ to the angle.

$$-\frac{65\pi}{6} + 5(2\pi)$$

$$-\frac{65\pi}{6} + 10\pi$$

$$-\frac{65\pi}{6}+\frac{60\pi}{6}$$

$$-\frac{5\pi}{6}$$

Now we have an angle that's less than one full rotation, but we'd still like to find a positive coterminal angle that's less than one full rotation. So we'll add 2π one more time.

$$-\frac{5\pi}{6}+2\pi$$

$$-\frac{5\pi}{6}+\frac{12\pi}{6}$$

$$\frac{7\pi}{6}$$

Therefore, we can say that $7\pi/6$ is coterminal with $\theta = -65\pi/6$.

■ 5. θ has a measure of $5\pi/3$. If it's rotated 8 full rotations in the negative direction, find its measure after the rotations.

Solution:

We know that one full rotation around the unit circle is a 2π rotation, which we can express in thirds as $6\pi/3$. Eight full rotations would therefore be $48\pi/3$.

The new angle measure will be given by

$$\frac{5\pi}{3} - \frac{48\pi}{3}$$

$$-\frac{43\pi}{3}$$

6. Find the value of all six trig functions at $\theta = -23\pi/6$.

Solution:

We know that one full rotation around the unit circle is a 2π rotation, which we can express in sixths as $12\pi/6$.

Let's add $12\pi/6$ to $\theta = -23\pi/6$ until we get an angle that lies in the interval $[0,2\pi)$.

$$-\frac{23\pi}{6} + \frac{12\pi}{6} = -\frac{11\pi}{6}$$

$$-\frac{11\pi}{6} + \frac{12\pi}{6} = \frac{\pi}{6}$$

So $\pi/6$ is coterminal with $\theta = -23\pi/6$, and we can therefore find all six trig functions at $-23\pi/6$ by finding all six trig functions at $\pi/6$. Because the angles are coterminal, the values of the six trig functions are equivalent at both angles.

Pulling directly from the unit circle, we get

$$\sin\left(-\frac{23\pi}{6}\right)=\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$$

$$\cos\left(-\frac{23\pi}{6}\right)=\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

Using the quotient identity for tangent, we get

$$\tan\left(-\frac{23\pi}{6}\right)=\tan\left(\frac{\pi}{6}\right)=\frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)}=\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{1}{2}\cdot\frac{2}{\sqrt{3}}==\frac{\sqrt{3}}{3}$$

Then, using the reciprocal identities for cosecant, secant, and cotangent, we get

$$\csc\left(-\frac{23\pi}{6}\right)=\csc\left(\frac{\pi}{6}\right)=\frac{1}{\sin\left(\frac{\pi}{6}\right)}=\frac{2}{1}=2$$

$$\sec\left(-\frac{23\pi}{6}\right)=\sec\left(\frac{\pi}{6}\right)=\frac{1}{\cos\left(\frac{\pi}{6}\right)}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$$

$$\cot\left(-\frac{23\pi}{6}\right)=\cot\left(\frac{\pi}{6}\right)=\frac{1}{\tan\left(\frac{\pi}{6}\right)}=\frac{3}{\sqrt{3}}=\sqrt{3}$$

COTERMINAL ANGLES IN A PARTICULAR INTERVAL

1. Find the angle in the interval $[-90^\circ, 90^\circ]$ that's coterminal with -748°.

Solution:

We need the angle -748° plus some multiple of 360° to lie inside $[-90^\circ, 90^\circ]$, so we can set up an inequality.

$$-90^\circ \leq -748^\circ + 360^\circ n \leq 90^\circ$$

$$-90^\circ + 748^\circ \leq 360^\circ n \leq 90^\circ + 748^\circ$$

$$658^\circ \leq 360^\circ n \leq 838^\circ$$

$$\frac{658^\circ}{360^\circ} \leq n \leq \frac{838^\circ}{360^\circ}$$

$$1.83 \leq n \leq 2.33$$

Since n has to be an integer, $n = 2$ and the coterminal angle we're looking for is

$$\theta = -748^\circ + 2(360^\circ)$$

$$\theta = -748^\circ + 720^\circ$$

$$\theta = -28^\circ$$

2. Find the angle in $[-4\pi, -2\pi]$ that's coterminal with $-21\pi/4$.

Solution:

We need the angle $-21\pi/4$ plus some multiple of 2π to lie inside $[-4\pi, -2\pi]$, so we can set up an inequality.

$$-4\pi \leq -\frac{21\pi}{4} + n(2\pi) \leq -2\pi$$

$$-4\pi + \frac{21\pi}{4} \leq n(2\pi) \leq -2\pi + \frac{21\pi}{4}$$

$$-\frac{16\pi}{4} + \frac{21\pi}{4} \leq n(2\pi) \leq -\frac{8\pi}{4} + \frac{21\pi}{4}$$

$$\frac{5\pi}{4} \leq n(2\pi) \leq \frac{13\pi}{4}$$

$$\frac{5\pi}{8\pi} \leq n \leq \frac{13\pi}{8\pi}$$

$$0.625 \leq n \leq 1.625$$

Since n has to be an integer, $n = 1$ and the coterminal angle we're looking for is

$$\theta = -\frac{21\pi}{4} + 1(2\pi)$$

$$\theta = -\frac{21\pi}{4} + \frac{8\pi}{4}$$

$$\theta = -\frac{13\pi}{4}$$

■ 3. Find the values of the six trig functions at the angle in the interval $[-\pi/6, 11\pi/6]$ that's coterminal with $43\pi/2$.

Solution:

We'll let $\theta = 43\pi/2$ and let α be the angle within $[-\pi/6, 11\pi/6]$ that's coterminal with θ. We'll use $\alpha = \theta + n(2\pi)$ and solve for the value of n that makes α lie in that interval.

$$-\frac{\pi}{6} \leq \alpha \leq \frac{11\pi}{6}$$

$$-\frac{\pi}{6} \leq \theta + n(2\pi) \leq \frac{11\pi}{6}$$

$$-\frac{\pi}{6} \leq \frac{43\pi}{2} + n(2\pi) \leq \frac{11\pi}{6}$$

$$-\frac{\pi}{6} - \frac{43\pi}{2} \leq n(2\pi) \leq \frac{11\pi}{6} - \frac{43\pi}{2}$$

$$-\frac{\pi}{6} - \frac{129\pi}{6} \leq n(2\pi) \leq \frac{11\pi}{6} - \frac{129\pi}{6}$$

$$-\frac{130\pi}{6} \leq n(2\pi) \leq -\frac{118\pi}{6}$$

$$-\frac{65\pi}{3} \le n(2\pi) \le -\frac{59\pi}{3}$$

$$-\frac{65\pi}{6\pi} \le n \le -\frac{59\pi}{6\pi}$$

$$-10.83 \le n \le -9.83$$

Because n has to be an integer, we know $n = -10$. To find α, we'll substitute $n = -10$ into $\alpha = \theta + n(2\pi)$.

$$\alpha = \frac{43\pi}{2} + (-10)(2\pi)$$

$$\alpha = \frac{43\pi}{2} - \frac{40\pi}{2}$$

$$\alpha = \frac{3\pi}{2}$$

Then the values of the six trig functions are

$$\sin\left(\frac{3\pi}{2}\right) = -1 \qquad \csc\left(\frac{3\pi}{2}\right) = \frac{1}{-1} = -1$$

$$\cos\left(\frac{3\pi}{2}\right) = 0 \qquad \sec\left(\frac{3\pi}{2}\right) = \frac{1}{0}$$

$$\tan\left(\frac{3\pi}{2}\right) = \frac{1}{0} \qquad \cot\left(\frac{3\pi}{2}\right) = \frac{0}{1} = 0$$

■ 4. Find the angle in the interval $[-550^\circ, -190^\circ]$ that's coterminal with 367°.

Solution:

We need the angle 367° plus some multiple of 360° to lie inside $[-550^\circ, -190^\circ]$, so we can set up an inequality.

$$-550^\circ \le 367^\circ + 360^\circ n \le -190^\circ$$

$$-550^\circ - 367^\circ \le 360^\circ n \le -190^\circ - 367^\circ$$

$$-917^\circ \le 360^\circ n \le -557^\circ$$

$$-\frac{917^\circ}{360^\circ} \le n \le -\frac{557^\circ}{360^\circ}$$

$$-2.55 \le n \le -1.55$$

Since n has to be an integer, $n = -2$ and the coterminal angle we're looking for is

$$\theta = 367^\circ + (-2)(360^\circ)$$

$$\theta = 367^\circ - 720^\circ$$

$$\theta = -353^\circ$$

■ 5. Find the angle in $[7\pi, 9\pi]$ that's coterminal with $7\pi/6$.

Solution:

We need the angle $7\pi/6$ plus some multiple of 2π to lie inside $[7\pi,9\pi]$, so we can set up an inequality.

$$7\pi \leq \frac{7\pi}{6} + n(2\pi) \leq 9\pi$$

$$7\pi - \frac{7\pi}{6} \leq n(2\pi) \leq 9\pi - \frac{7\pi}{6}$$

$$\frac{42\pi}{6} - \frac{7\pi}{6} \leq n(2\pi) \leq \frac{54\pi}{6} - \frac{7\pi}{6}$$

$$\frac{35\pi}{6} \leq n(2\pi) \leq \frac{47\pi}{6}$$

$$\frac{35\pi}{12\pi} \leq n \leq \frac{47\pi}{12\pi}$$

$$2.92 \leq n \leq 3.92$$

Since n has to be an integer, $n = 3$ and the coterminal angle we're looking for is

$$\theta = \frac{7\pi}{6} + 3(2\pi)$$

$$\theta = \frac{7\pi}{6} + \frac{36\pi}{6}$$

$$\theta = \frac{43\pi}{6}$$

6. Find the values of the six trig functions at the angle in the interval $[-\pi, \pi]$ that's coterminal with $-22\pi/3$.

Solution:

We'll let $\theta = -22\pi/3$ and α be the angle in $[-\pi, \pi]$ that's coterminal with θ. We'll use $\alpha = \theta + n(2\pi)$ and solve for the value of n that makes α lie in that interval.

$$-\pi \leq \alpha \leq \pi$$

$$-\pi \leq \theta + n(2\pi) \leq \pi$$

$$-\pi \leq -\frac{22\pi}{3} + n(2\pi) \leq \pi$$

$$-\pi + \frac{22\pi}{3} \leq n(2\pi) \leq \pi + \frac{22\pi}{3}$$

$$-\frac{3\pi}{3} + \frac{22\pi}{3} \leq n(2\pi) \leq \frac{3\pi}{3} + \frac{22\pi}{3}$$

$$\frac{19\pi}{3} \leq n(2\pi) \leq \frac{25\pi}{3}$$

$$3.17 \leq n \leq 4.17$$

Because n has to be an integer, we know $n = 4$. To find α, we'll substitute $n = 4$ into $\theta + n(2\pi)$.

$$\alpha = -\frac{22\pi}{3} + 4(2\pi)$$

$$\alpha = -\frac{22\pi}{3} + \frac{24\pi}{3}$$

$$\alpha = \frac{2\pi}{3}$$

Then the values of the six trig functions are

$$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$\tan\left(\frac{2\pi}{3}\right) = \frac{\sin\left(\frac{2\pi}{3}\right)}{\cos\left(\frac{2\pi}{3}\right)} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

$$\csc\left(\frac{2\pi}{3}\right) = \frac{1}{\sin\left(\frac{2\pi}{3}\right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2\sqrt{3}}{3}$$

$$\sec\left(\frac{2\pi}{3}\right) = \frac{1}{\cos\left(\frac{2\pi}{3}\right)} = \frac{1}{-\frac{1}{2}} = -2$$

$$\cot\left(\frac{2\pi}{3}\right) = \frac{1}{\tan\left(\frac{2\pi}{3}\right)} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

REFERENCE ANGLES

1. Find the reference angle for $\theta = -16\pi/3$.

Solution:

Find a coterminal angle in the interval $[0,2\pi)$ by adding 2π, or $6\pi/3$, until we find an angle in $[0,2\pi)$.

$$-\frac{16\pi}{3} + \frac{6\pi}{3} = -\frac{10\pi}{3}$$

$$-\frac{10\pi}{3} + \frac{6\pi}{3} = -\frac{4\pi}{3}$$

$$-\frac{4\pi}{3} + \frac{6\pi}{3} = \frac{2\pi}{3}$$

So $2\pi/3$ lies in quadrant II and is coterminal with $\theta = -16\pi/3$. Angles in the second quadrant have reference angles given by $\beta = \pi - \theta$, so the reference angle is

$$\beta = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$$

2. Which of the angles 19°, 119°, $1{,}019^\circ$, and $2{,}019^\circ$ has the smallest reference angle?

Solution:

First, find the coterminal angle inside $[0^\circ, 360^\circ)$ for $1{,}019^\circ$,

$$1{,}019^\circ - 360^\circ = 659^\circ$$

$$659^\circ - 360^\circ = 299^\circ$$

and for $2{,}019^\circ$.

$$2{,}019^\circ - 360^\circ = 1{,}659^\circ$$

$$1{,}659^\circ - 360^\circ = 1{,}299^\circ$$

$$1{,}299^\circ - 360^\circ = 939^\circ$$

$$939^\circ - 360^\circ = 579^\circ$$

$$579^\circ - 360^\circ = 219^\circ$$

The angle 19° is in the first quadrant, 119° is in the second quadrant, 219° is in the third quadrant, and 299° is in the fourth quadrant. Given the quadrant for each angle, we'll use the appropriate formula to calculate each reference angle.

Angle	**Reference angle**
19°	$\beta = \theta = 19^\circ$
119°	$\beta = 180^\circ - \theta = 180^\circ - 119^\circ = 61^\circ$
219°	$\beta = \theta - 180^\circ = 219^\circ - 180^\circ = 39^\circ$

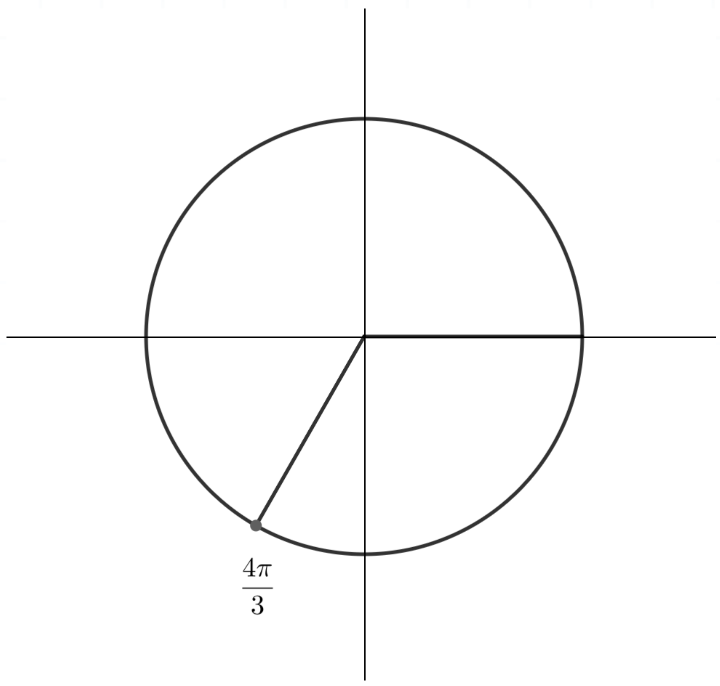
299° $\qquad\qquad\qquad\qquad$ $\beta = 360^\circ - \theta = 360^\circ - 299^\circ = 61^\circ$

Therefore, the angle 19° has the smallest reference angle.

■ 3. Sketch the angle $4\pi/3$ in standard position and find its reference angle.

Solution:

First sketch the angle.



So $4\pi/3$ lies in quadrant III. Angles in the third quadrant have reference angles given by $\beta = \theta - \pi$, so the reference angle is

$$\beta = \frac{4\pi}{3} - \pi = \frac{\pi}{3}$$

■ 4. Find the reference angle for $\theta = 438^\circ$.

Solution:

Find a coterminal angle in the interval $[0^\circ,360^\circ)$ by subtracting 360°.

$$438^\circ - 360^\circ = 78^\circ$$

So 78° lies in quadrant I and is coterminal with $\theta = 438^\circ$. Angles in the first quadrant have reference angles given by $\beta = \theta$, so the reference angle is

$$\beta = 78^\circ$$

■ 5. Which of the angles 68°, 168°, $1{,}068^\circ$, or $2{,}068^\circ$ has the largest reference angle?

Solution:

First, find the coterminal angle inside $[0^\circ,360^\circ)$ for $1{,}068^\circ$,

$$1{,}068^\circ - 360^\circ = 708^\circ$$

$$708^\circ - 360^\circ = 348^\circ$$

and for $2{,}068^\circ$.

$$2{,}068^\circ - 360^\circ = 1{,}708^\circ$$

$$1{,}708^\circ - 360^\circ = 1{,}348^\circ$$

$$1{,}348^\circ - 360^\circ = 988^\circ$$

$$988^\circ - 360^\circ = 628^\circ$$

$$628^\circ - 360^\circ = 268^\circ$$

The angle 68° is in the first quadrant, 168° is in the second quadrant, 268° is in the third quadrant, and 348° is in the fourth quadrant. Given the quadrant for each angle, we'll use the appropriate formula to calculate each reference angle.

Angle	**Reference angle**
68°	$\beta = \theta = 68^\circ$
168°	$\beta = 180^\circ - \theta = 180^\circ - 168^\circ = 12^\circ$
268°	$\beta = \theta - 180^\circ = 268^\circ - 180^\circ = 88^\circ$
348°	$\beta = 360^\circ - \theta = 360^\circ - 348^\circ = 12^\circ$

Therefore, the angle $2{,}068^\circ$ has the largest reference angle.

■ 6. Find the values of the six trig functions at the reference angle for $\theta = -27\pi/4$.

Solution:

Find a coterminal angle in the interval $[0,2\pi)$ by adding 2π, or $8\pi/4$, until we find an angle in $[0,2\pi)$.

$$-\frac{27\pi}{4}+\frac{8\pi}{4}=-\frac{19\pi}{4}$$

$$-\frac{19\pi}{4}+\frac{8\pi}{4}=-\frac{11\pi}{4}$$

$$-\frac{11\pi}{4}+\frac{8\pi}{4}=-\frac{3\pi}{4}$$

$$-\frac{3\pi}{4}+\frac{8\pi}{4}=\frac{5\pi}{4}$$

So $5\pi/4$ lies in quadrant III and is coterminal with $\theta=-27\pi/4$. Angles in the third quadrant have reference angles given by $\beta=\theta-\pi$, so the reference angle is

$$\beta=\frac{5\pi}{4}-\pi=\frac{\pi}{4}$$

Then the values of the six trig functions at the coterminal angle are

$$\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$\csc\left(\frac{\pi}{4}\right)=\frac{1}{\sin\left(\frac{\pi}{4}\right)}=\frac{1}{\frac{\sqrt{2}}{2}}=\sqrt{2}$$

$$\cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$$\sec\left(\frac{\pi}{4}\right)=\frac{1}{\cos\left(\frac{\pi}{4}\right)}=\frac{1}{\frac{\sqrt{2}}{2}}=\sqrt{2}$$

$$\tan\left(\frac{\pi}{4}\right)=\frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)}=\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}=1 \qquad \cot\left(\frac{\pi}{4}\right)=\frac{1}{\tan\left(\frac{\pi}{4}\right)}=\frac{1}{1}=1$$

SYMMETRY ACROSS AXES

1. Use the unit circle to find the angle that has the same cosine as the angle $\theta = 11\pi/6$.

Solution:

For any angle θ, the value of $\cos\theta$ is equal to the x-coordinate of the point at which the terminal side of θ intersects the unit circle.

Since $\theta = 11\pi/6$ lies in the fourth quadrant, its cosine will be positive. The only other quadrant in which the cosine is positive is the first quadrant. Because $\theta = 11\pi/6$ is

$$2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$$

below the positive x-axis, we need an angle that's $\pi/6$ above of the positive x-axis, which is the angle $\pi/6$.

2. If θ is an angle such that $\cos\theta = -0.567$, what is a possible value of $\cos(\theta + 900^\circ)$?

Solution:

Since $\cos\theta$ is negative, θ is in either the second or third quadrant.

We've been asked about the angle $\theta + 900^\circ$. If we realize that $900^\circ = 2(360^\circ) + 180^\circ$, we see that two full 360° rotations put us right back at the same angle θ, but that the additional 180° flips us across the x-axis and then across the y-axis, which means the signs on both x and y will change.

The angle θ in the second quadrant associated with $(-0.567, \sin\theta)$ will become the angle $\theta + 900^\circ$ associated with $(0.567, -\sin\theta)$. And the angle θ in the third quadrant associated with $(-0.567, \sin\theta)$ will become the angle $\theta + 900^\circ$ associated with $(0.567, -\sin\theta)$.

Therefore, for an angle $\theta + 900^\circ$, the cosine must be $\cos(\theta + 900^\circ) = 0.567$.

■ 3. If θ is an angle in the fourth quadrant such that $\cos\theta = 2/5$, find $\cos(\theta - 5\pi)$ and $\sin(\theta - 5\pi)$.

Solution:

First find $\sin\theta$ by plugging $\cos\theta = 2/5$ into the Pythagorean identity for sine and cosine.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta + \left(\frac{2}{5}\right)^2 = 1$$

$$\sin^2\theta = 1 - \left(\frac{2}{5}\right)^2$$

$$\sin^2\theta \approx 1 - \frac{4}{25}$$

$$\sin^2\theta = \frac{21}{25}$$

$$\sin\theta = \pm\sqrt{\frac{21}{25}} = \pm\frac{\sqrt{21}}{5}$$

We've been given that θ is in the fourth quadrant, so $\sin\theta$ is negative and $\sin\theta = -\sqrt{21}/5$. Since θ is in the fourth quadrant, we know x is positive and y is negative, so $\cos\theta = 2/5$ and $\sin\theta = -\sqrt{21}/5$.

When we subtract 5π from the angle θ, that takes us two full rotations of 2π in the negative direction, bringing us back to the fourth quadrant, and then takes us an additional $-\pi$ rotation into the second quadrant. In the second quadrant, and by symmetry, our positive x-value becomes a negative x-value, and our negative y-value becomes a positive y-value. Therefore

$$\cos(\theta - 5\pi) = -\frac{2}{5}$$

$$\sin(\theta - 5\pi) = \frac{\sqrt{21}}{5}$$

■ 4. Use the unit circle to find an angle that has the same sine as the angle $\theta = 225^\circ$.

Solution:

For any angle θ, the value of $\sin\theta$ is equal to the y-coordinate of the point at which the terminal side of θ intersects the unit circle.

Since $\theta = 225^\circ$ lies in the third quadrant, the sine of the angle will be negative. The only other quadrant in which the sine is negative is the fourth quadrant. Because $\theta = 225^\circ$ is $225^\circ - 180^\circ = 45^\circ$ below the negative x-axis, we need an angle that's 45° below of the positive x-axis, which is the angle

$$360^\circ - 45^\circ = 315^\circ$$

■ 5. If θ is an angle such that $\cos\theta = 3/5$, what are two possible values of $\sin(\theta + 9\pi)$?

Solution:

Use the Pythagorean identity with sine and cosine, and the given value $\cos\theta = 3/5$ to find $\sin\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta + \left(\frac{3}{5}\right)^2 = 1$$

$$\sin^2\theta = 1 - \left(\frac{3}{5}\right)^2$$

$$\sin^2\theta \approx 1 - \frac{9}{25}$$

$$\sin^2\theta = \frac{16}{25}$$

$$\sin\theta = \pm\sqrt{\frac{16}{25}}$$

$$\sin\theta = \pm\frac{4}{5}$$

Since $\cos\theta$ is positive, θ is in either the first or fourth quadrant. In the first quadrant we'll have $\sin\theta = 4/5$, and in the fourth quadrant we'll have $\sin\theta = -4/5$.

We've been asked about the angle $\theta + 9\pi$. If we realize that $9\pi = 4(2\pi) + \pi$, we see that four full 2π rotations put us right back at the same angle θ, but that the additional π flips us across the y-axis and then across the x-axis, which means the signs on both x and y will change.

Which means that the angle θ in the first quadrant associated with $(3/5,4/5)$ will become the angle $\theta + 9\pi$ associated with $(-3/5,-4/5)$. And the angle θ in the first quadrant associated with $(3/5,-4/5)$ will become the angle $\theta + 9\pi$ associated with $(-3/5,4/5)$.

Therefore, for an angle $\theta + 9\pi$, the possible values of sine of the angle are the y-values from $(-3/5,-4/5)$ and $(-3/5,4/5)$, so

$$\sin(\theta + 9\pi) = \pm \frac{4}{5}$$

■ 6. If θ is an angle in the third quadrant such that $\sin\theta = -0.255$, what is the value of $\cos(\theta - 11\pi)$?

Solution:

Use the Pythagorean identity with sine and cosine, and the given value $\sin\theta = -0.255$, to find $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

$$(-0.255)^2 + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - (-0.255)^2$$

$$\cos^2\theta \approx 1 - 0.065$$

$$\cos^2\theta \approx 0.935$$

$$\cos\theta \approx \pm\sqrt{0.935}$$

Since θ is in the third quadrant, cosine of the angle must be negative.

$$\cos\theta \approx -\sqrt{0.935}$$

$$\cos\theta \approx -0.967$$

We've been asked about the angle $\theta - 11\pi$. If we realize that $-11\pi = -10\pi - \pi$, we see that a -10π rotation puts us right back at the same angle θ, but that the additional $-\pi$ flips us across the y-axis and then across the x-axis, which means the signs on both x and y will change.

Which means that the angle θ in the third quadrant associated with $(-0.967, -0.255)$ will become the angle $\theta - 11\pi$ associated with $(0.967, 0.255)$.

Therefore, for an angle $\theta - 11\pi$, the value of cosine of the angle is the x -value from $(0.967, 0.255)$.

$$\cos(\theta - 11\pi) = 0.967$$

EVEN-ODD IDENTITIES

■ 1. Which of the six trig functions are even?

Solution:

The cosine and secant functions are even because they're symmetric about the y-axis. The sine, cosecant, tangent, and cotangent functions are odd because they're symmetric about the origin.

■ 2. Evaluate the expression.

$$\sec(-45^\circ) \cdot \csc(-45^\circ) \cdot \tan(-45^\circ) \cdot \cot(-45^\circ)$$

Solution:

Use even-odd identities to get rid of the negative angles.

$$\tan(-\theta) = -\tan\theta$$

$$\cot(-\theta) = -\cot\theta$$

$$\sec(-\theta) = \sec\theta$$

$$\csc(-\theta) = -\csc\theta$$

So the expression becomes

$$\sec(-45^\circ)\cdot\csc(-45^\circ)\cdot\tan(-45^\circ)\cdot\cot(-45^\circ)$$

$$\sec 45^\circ\cdot(-\csc 45^\circ)\cdot(-\tan 45^\circ)\cdot(-\cot 45^\circ)$$

$$\sqrt{2}\cdot(-\sqrt{2})\cdot(-1)\cdot(-1)$$

$$-2\cdot 1$$

$$-2$$

3. List the values in order from smallest to largest.

$$\sin\left(-\frac{\pi}{6}\right),\ \cos\left(-\frac{\pi}{6}\right),\ \tan\left(-\frac{\pi}{6}\right)$$

Solution:

Use even-odd identities to get rid of the negative angles.

$$\sin(-\theta)=-\sin(\theta)$$

$$\cos(-\theta)=\cos(\theta)$$

$$\tan(-\theta)=-\tan(\theta)$$

Then the given values are

$$\sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2} = -0.5$$

$$\cos\left(-\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \approx 0.9$$

$$\tan\left(-\frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{\sqrt{3}}{3} \approx -0.6$$

Now we can order the values from smallest to largest.

$$\tan\left(-\frac{\pi}{6}\right), \sin\left(-\frac{\pi}{6}\right), \cos\left(-\frac{\pi}{6}\right)$$

■ 4. Evaluate the expression.

$$\cos(-30^\circ) + \cos(-45^\circ) + \sin(-45^\circ) + \sin(-60^\circ)$$

Solution:

Use even-odd identities to get rid of the negative angles.

$$\sin(-\theta) = -\sin(\theta)$$

$$\cos(-\theta) = \cos(\theta)$$

So the expression

$$\cos(-30^\circ) + \cos(-45^\circ) + \sin(-45^\circ) + \sin(-60^\circ)$$

becomes

$$\cos 30^\circ + \cos 45^\circ - \sin 45^\circ - \sin 60^\circ$$

$$\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}$$

$$0$$

■ 5. Find the values of cotangent and cosecant at $\theta = -29\pi/2$.

Solution:

Since the cosine function is even and the sine function is odd, we can say

$$\cos\left(-\frac{29\pi}{2}\right) = \cos\left(\frac{29\pi}{2}\right)$$

$$\sin\left(-\frac{29\pi}{2}\right) = -\sin\left(\frac{29\pi}{2}\right)$$

Then to find the value of cosine of this positive angle, we'll get the coterminal angle for $29\pi/2$ by dividing it by 2π.

$$\frac{\frac{29\pi}{2}}{2\pi} = \frac{29\pi}{2(2\pi)} = \frac{29}{4} \approx 7.25$$

So $\theta = 29\pi/2$ is 7 full positive rotations, plus an additional 0.25 rotation. If we take away 7 full rotations, or 14π, from $\theta = 29\pi/2$, we're left with

$$\frac{29\pi}{2} - 14\pi = \frac{29\pi}{2} - \frac{28\pi}{2} = \frac{\pi}{2}$$

So $\theta = 29\pi/2$ is coterminal with $\alpha = \pi/2$, the sines of these angles are equal, and the cosines of these angles are equal.

$$\sin\left(\frac{29\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\cos\left(\frac{29\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

Apply even-odd identities and reciprocal identities to find cosecant and cotangent of $\theta = -29\pi/2$.

$$\csc\left(-\frac{29\pi}{2}\right) = -\csc\left(\frac{29\pi}{2}\right) = -\frac{1}{\sin\left(\frac{29\pi}{2}\right)} = -\frac{1}{1} = -1$$

$$\cot\left(-\frac{29\pi}{2}\right) = -\cot\left(\frac{29\pi}{2}\right) = -\frac{\cos\left(\frac{29\pi}{2}\right)}{\sin\left(\frac{29\pi}{2}\right)} = -\frac{0}{1} = 0$$

■ 6. Which of the six trig functions has the largest value at $\theta = -\pi/3$?

Solution:

The angle $\theta = -\pi/3$ lies in quadrant IV. Only the cosine and secant functions of an angle in quadrant IV are positive. So we only need to determine which of these is larger than the other.

Use even-odd identities to get rid of the negative angles.

$$\cos(-\theta) = \cos\theta$$

$$\sec(-\theta) = \sec\theta$$

Then

$$\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\sec\left(-\frac{\pi}{3}\right) = \sec\left(\frac{\pi}{3}\right) = 2$$

So out of all the trig functions, the secant function has the largest value at $\theta = -\pi/3$.

THE SET OF ALL POSSIBLE ANGLES

1. A trigonometric equation has two solutions in $[0,2\pi)$, which are $\pi/4$ and $5\pi/4$. Give the complete solution set.

Solution:

Extend each of the two solutions to include coterminal angles.

$$x_1 = \frac{\pi}{4} + 2\pi k \text{ where } k \text{ is any integer}$$

$$x_2 = \frac{5\pi}{4} + 2\pi k \text{ where } k \text{ is any integer}$$

If we expand these solutions into a list of angles, we get

$$x_1 = \ldots, \frac{\pi}{4}, \frac{\pi}{4} + 2\pi, \frac{\pi}{4} + 4\pi, \ldots$$

$$x_2 = \ldots, \frac{5\pi}{4}, \frac{5\pi}{4} + 2\pi, \frac{5\pi}{4} + 4\pi, \ldots = \ldots, \frac{\pi}{4} + \pi, \frac{\pi}{4} + 3\pi, \frac{\pi}{4} + 5\pi, \ldots$$

Combine the lists into one ordered list.

$$x = \ldots, \frac{\pi}{4}, \frac{\pi}{4} + \pi, \frac{\pi}{4} + 2\pi, \frac{\pi}{4} + 3\pi, \frac{\pi}{4} + 4\pi, \frac{\pi}{4} + 5\pi, \ldots$$

From the combined list, we see that we can express the pattern as

$$x = \frac{\pi}{4} + \pi k, \text{ where } k \text{ is any integer}$$

$$x = \left\{ \frac{\pi}{4} + \pi k : \ k \in \mathbb{Z} \right\}$$

2. Solve the trigonometric equation.

$$\cos(3x + 5\pi) = 0$$

Solution:

We know that the cosine of any angle will be 0 at $\pi/2$, $3\pi/2$, and any angles coterminal with those.

Extend each of the two solutions to include coterminal angles.

$\theta_1 = \frac{\pi}{2} + 2\pi k$, where k is any integer

$\theta_2 = \frac{3\pi}{2} + 2\pi k$, where k is any integer

If we expand these solutions into a list of angles, we get

$$\theta_1 = \ldots, \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 4\pi, \ldots$$

$$\theta_2 = \ldots, \frac{3\pi}{2}, \frac{3\pi}{2} + 2\pi, \frac{3\pi}{2} + 4\pi, \ldots = \ldots, \frac{\pi}{2} + \pi, \frac{\pi}{2} + 3\pi, \frac{\pi}{2} + 5\pi, \ldots$$

Combine the lists into one ordered list.

$$\theta = \ldots, \frac{\pi}{2}, \frac{\pi}{2} + \pi, \frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 3\pi, \frac{\pi}{2} + 4\pi, \frac{\pi}{2} + 5\pi, \ldots$$

From the combined list, we see that we can express the pattern as

$$\theta = \frac{\pi}{2} + \pi k \text{, where } k \text{ is any integer}$$

Because $\theta = 3x + 5\pi$, we get

$$3x + 5\pi = \frac{\pi}{2} + \pi k$$

$$3x = \frac{\pi}{2} + \pi k - 5\pi$$

$$3x = -\frac{9\pi}{2} + \pi k$$

$$x = -\frac{3\pi}{2} + \frac{\pi k}{3}$$

$$x = \left\{-\frac{3\pi}{2} + \frac{\pi k}{3} : \ k \in \mathbb{Z}\right\}$$

■ 3. Find the set of coordinate points in the xy-plane that satisfy the equation.

$$\sin x \cos y = 0$$

Solution:

A product is equal to 0 if one or both of the factors is equal to 0. So instead of one equation in terms of x and y, we can consider two simple equations, each in terms of one variable.

$$\sin x = 0$$

$$\cos y = 0$$

Consider the first equation. From the unit circle we know that the value of sine is 0 at

$$\ldots -\pi, 0, \pi, 2\pi, 3\pi, \ldots$$

$$\{\pi k : \ k \in \mathbb{Z}\}$$

Consider the second equation. From the unit circle we know that the value of cosine is 0 at

$$\ldots -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$$

$$\left\{\frac{\pi}{2} + \pi n : \ n \in \mathbb{Z}\right\}$$

Combine the solution sets.

$$\left\{(x, y) \ \middle|\ x = \pi k \quad \text{or} \quad y = \frac{\pi}{2} + \pi n : \ k, n \in \mathbb{Z}\right\}$$

■ 4. Solve the equation $\cos\theta + \sqrt{2} = -\cos\theta$ for all possible values of θ.

Solution:

We can rewrite the equation as

$$2\cos\theta = -\sqrt{2}$$

$$\cos\theta = -\frac{\sqrt{2}}{2}$$

From the unit circle, we know that

$$\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

So $3\pi/4$ is one solution, but we need to include all other angles that are coterminal with $3\pi/4$.

$$\left\{\frac{3\pi}{4} + 2\pi k: \ k \in \mathbb{Z}\right\}$$

We have to realize that $\cos\theta = -\sqrt{2}/2$ is also true at $\theta = 5\pi/4$, since cosine represents the x-value from the coordinate point, and x is negative in both the second and third quadrants.

The complete set of angles that are coterminal with $\theta = 5\pi/4$ is

$$\left\{\frac{5\pi}{4} + 2\pi k: \ k \in \mathbb{Z}\right\}$$

Combining these results, we say that the complete set of solutions to $\cos\theta = -\sqrt{2}/2$ is

$$\left\{\frac{3\pi}{4}+2\pi k:\ k\in\mathbb{Z}\right\}\cup\left\{\frac{5\pi}{4}+2\pi k:\ k\in\mathbb{Z}\right\}$$

■ 5. Solve the trigonometric equation.

$$\sin\left(\frac{x}{2}\right)=-\frac{1}{2}$$

Solution:

We know that the sine of the angles $7\pi/6$ and $11\pi/6$ (and any angles coterminal with these) is $-1/2$.

Extend each of the two solutions to include coterminal angles.

$$\theta_1=\frac{7\pi}{6}+2\pi k\text{, where } k \text{ is any integer}$$

$$\theta_2=\frac{11\pi}{6}+2\pi k\text{, where } k \text{ is any integer}$$

Because $\theta=x/2$, we get

$$\frac{x}{2}=\frac{7\pi}{6}+2\pi k$$

$$x=\frac{7\pi}{3}+4\pi k$$

and

$$\frac{x}{2} = \frac{11\pi}{6} + 2\pi k$$

$$x = \frac{11\pi}{3} + 4\pi k$$

Combining our results, we say that the complete set of solutions to $\sin(x/2) = -1/2$ is

$$\left\{\frac{7\pi}{3} + 4\pi k:\ k \in \mathbb{Z}\right\} \cup \left\{\frac{11\pi}{3} + 4\pi k:\ k \in \mathbb{Z}\right\}$$

■ 6. Solve the trigonometric equation.

$$4\cos^2\theta = 1$$

Solution:

We can rewrite the equation as

$$\cos^2\theta = \frac{1}{4}$$

$$\cos\theta = \pm\frac{1}{2}$$

We know that the cosine of any angle will be $1/2$ at $\pi/3$, $5\pi/3$, and any angles coterminal with these, so extend each of these solutions to include coterminal angles.

$$\theta_1 = \frac{\pi}{3} + 2\pi k, \text{ where } k \text{ is any integer}$$

$$\theta_2 = \frac{5\pi}{3} + 2\pi k, \text{ where } k \text{ is any integer}$$

We know that the cosine of any angle will be $-1/2$ at $2\pi/3$, $4\pi/3$, and any angles coterminal with these, so extend each of these solutions to include coterminal angles.

$$\theta_3 = \frac{2\pi}{3} + 2\pi k, \text{ where } k \text{ is any integer}$$

$$\theta_4 = \frac{4\pi}{3} + 2\pi k, \text{ where } k \text{ is any integer}$$

Combining the results by grouping θ_1 with θ_4 and θ_2 with θ_3, we say that the complete set of possible solutions to $\cos\theta = \pm 1/2$ is

$$\left\{\frac{\pi}{3} + \pi k: \ k \in \mathbb{Z}\right\} \cup \left\{\frac{2\pi}{3} + \pi k: \ k \in \mathbb{Z}\right\}$$

POINTS NOT ON THE UNIT CIRCLE

1. On the circle with center at the origin and radius 5, find the point(s) where $\cos\theta = -0.6$.

Solution:

We'll use the formula for cosine.

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

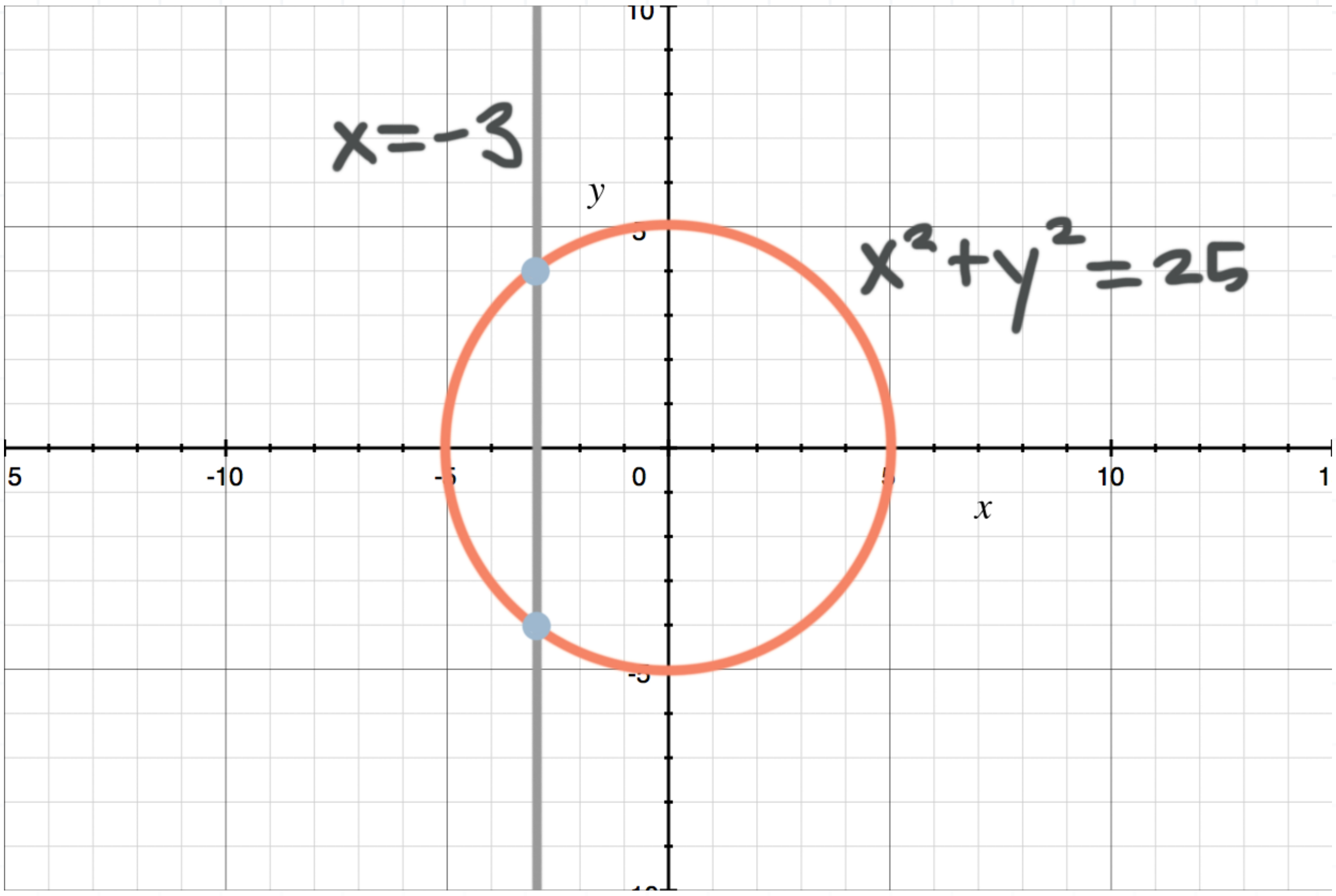
$$-0.6 = \frac{x}{\sqrt{x^2+y^2}}$$

Because the circle has radius 5, we know $\sqrt{x^2+y^2} = 5$, so

$$-0.6 = \frac{x}{5}$$

$$x = -3$$

So the points we're looking for are the points where the vertical line $x = -3$ intersects the circle centered at the origin with radius 5.



Substitute $x = -3$ into the equation of the circle.

$$x^2 + y^2 = 25$$

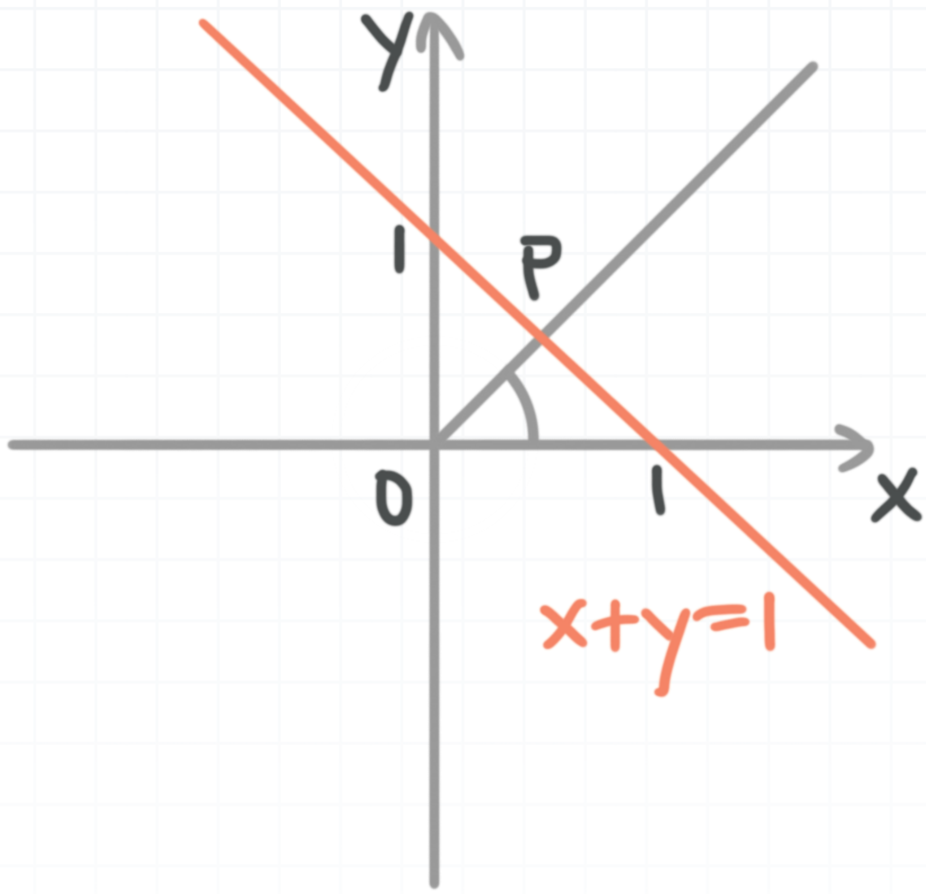
$$(-3)^2 + y^2 = 25$$

$$y^2 = 16$$

$$y = \pm 4$$

So the points along a circle of radius 5 at which $\cos\theta = -0.6$ are $(-3, -4)$ and $(-3,4)$.

■ 2. Find the point P on the line $x + y = 1$ where sine of the angle between P and the positive direction of the x-axis is $\sqrt{2}/2$.



Solution:

We know that the sine of an angle is given by

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\sqrt{2}}{2} = \frac{y}{\sqrt{x^2+y^2}}$$

Given $x+y=1$, we get $x = 1-y$, so

$$\frac{\sqrt{2}}{2} = \frac{y}{\sqrt{(1-y)^2+y^2}}$$

Square both sides of the equation.

$$\frac{2}{4} = \frac{y^2}{(1-y)^2 + y^2}$$

$$2((1-y)^2 + y^2) = 4y^2$$

$$2(1 - 2y + y^2 + y^2) = 4y^2$$

$$2 - 4y + 2y^2 + 2y^2 = 4y^2$$

$$2 - 4y = 0$$

$$4y = 2$$

$$y = \frac{1}{2}$$

Substitute $y = 1/2$ into $x = 1 - y$.

$$x = 1 - \frac{1}{2}$$

$$x = \frac{1}{2}$$

The point P has coordinates $(1/2,1/2)$.

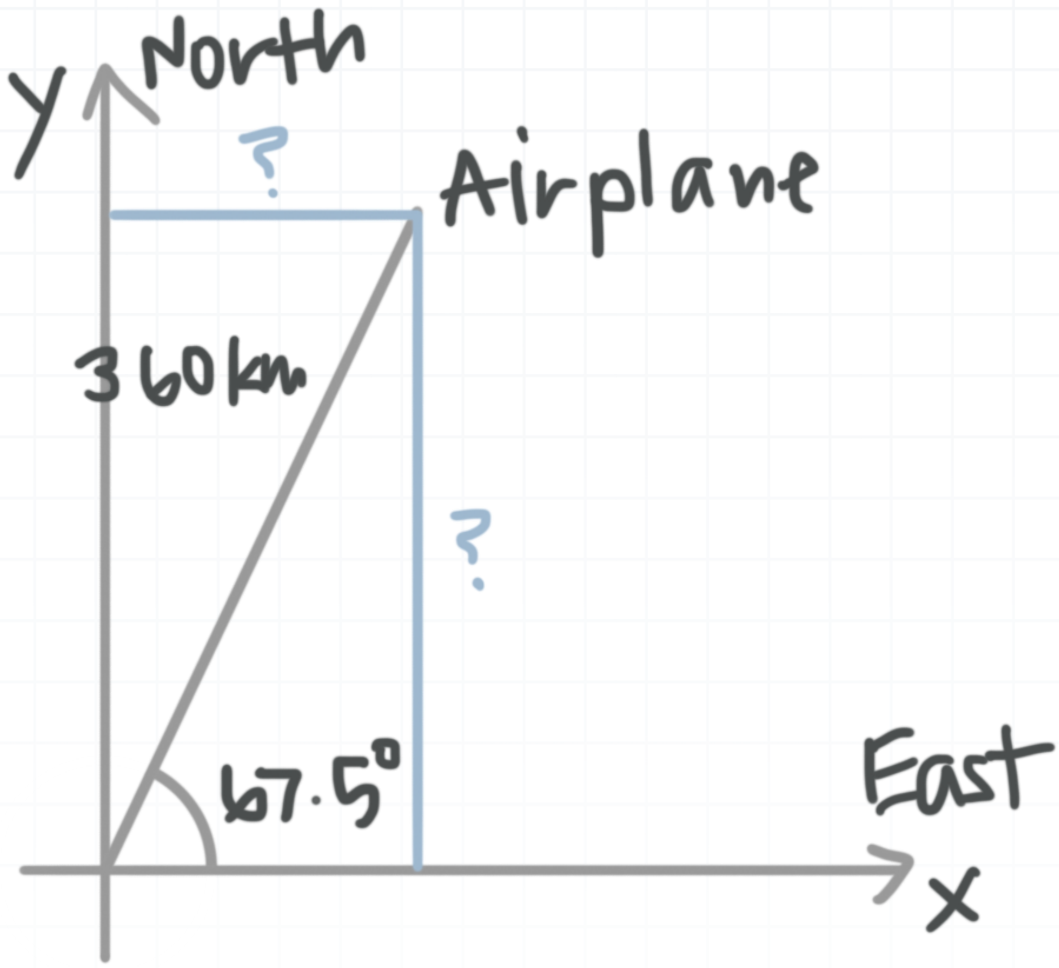
■ 3. An airplane flew 3 hours North-North-East (NNE) with a constant velocity of 120 km/h. How far north is the plane from the airport and how far east is the plane from the airport?

Solution:

A direction of NNE is the angle

$$\frac{3}{4} \cdot 90^\circ = 67.5^\circ$$

The airplane flew $3 \cdot 120 = 360$ km from the airport, so we'll model the problem using the coordinate system. The airport is at the origin, the x-axis is directly east, the y-axis is directly north, and one unit is 1 km.



We need to find the coordinates (x, y) of the airplane, given the angle $\theta = 67.5^\circ$ and the distance from the origin $D = 360$ km. We'll use the formulas for sine and cosine.

$$\sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\sin 67.5^\circ = \frac{y}{360}$$

$$y = 360 \sin 67.5^\circ \approx 333 \text{ km}$$

and

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\cos 67.5^\circ = \frac{x}{360}$$

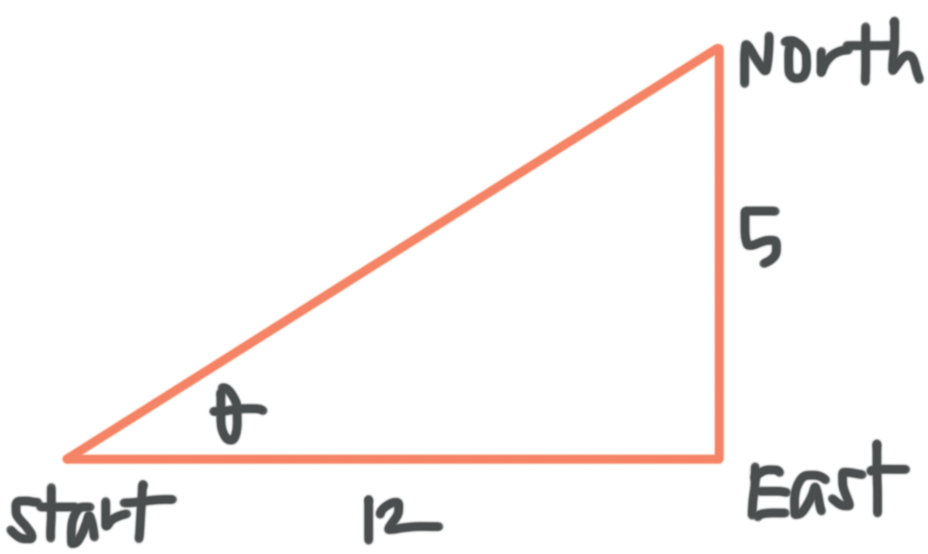
$$x = 360 \cos 67.5^\circ \approx 138 \text{ km}$$

So the airplane is 333 km north of the airport and 138 km east of the airport.

■ 4. An airplane flew due East with a constant velocity and reached its destination in 12 minutes. Then the plane turned North and flew with the same velocity an additional 5 minutes. Find the cosine of the angle between the line passing through the initial and current positions of the plane, and the direction of due East.

Solution:

Sketch the situation.



If the plane starts at $(0,0)$, and since the airplane flew with a constant velocity, its current position is $(12,5)$ so we need to find the value of $\cos\theta$ for the angle θ whose terminal side contains the point $(12,5)$.

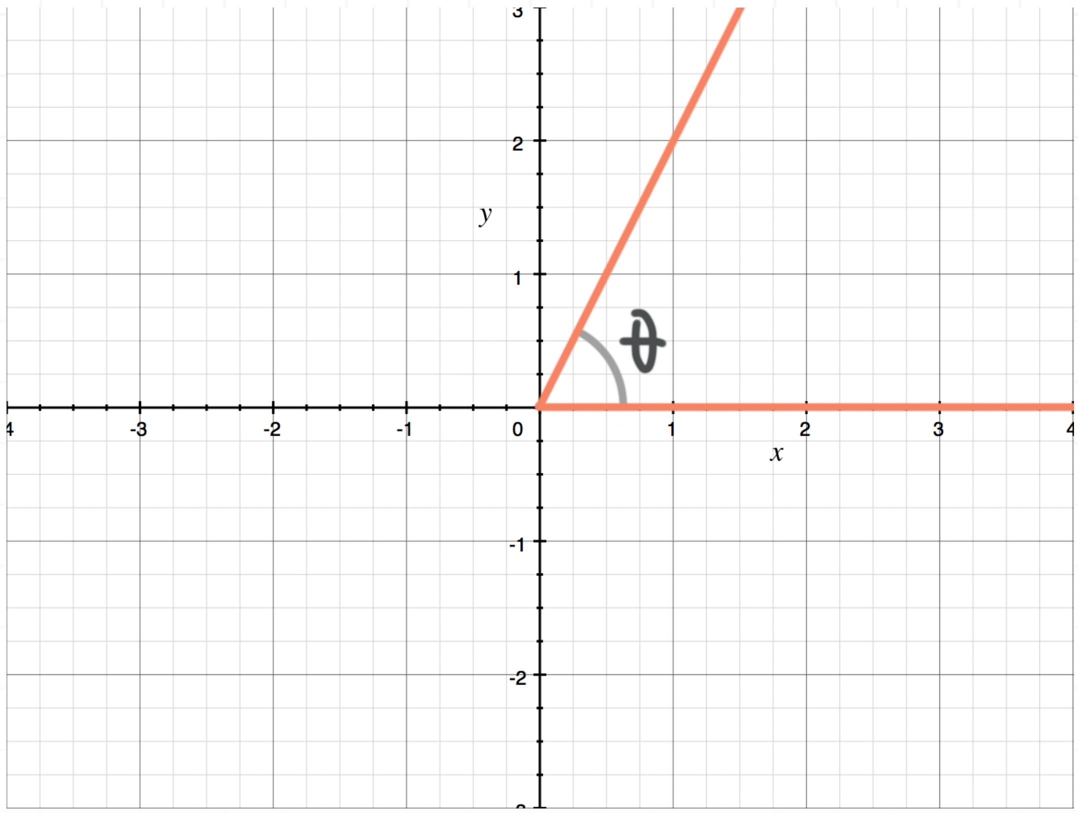
$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\cos\theta = \frac{12}{\sqrt{12^2+5^2}}$$

$$\cos\theta = \frac{12}{\sqrt{169}}$$

$$\cos\theta = \frac{12}{13}$$

■ 5. Find the sine of the acute angle between the line $y = 2x$ and the positive direction of the x-axis.



Solution:

Choose a point on the line $y = 2x$, like $(1,2)$., then find $\sin\theta$ for the angle θ whose terminal side contains the point $(1,2)$.

$$\sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\sin\theta = \frac{2}{\sqrt{1^2 + 2^2}}$$

$$\sin\theta = \frac{2}{\sqrt{5}}$$

$$\sin\theta = \frac{2\sqrt{5}}{5}$$

6. Find the set of all possible angles θ that pass through the point $P(-3, -3\sqrt{3})$.

Solution:

We can use either the sine or cosine formula, but let's use cosine.

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\cos\theta = \frac{-3}{\sqrt{(-3)^2 + (-3\sqrt{3})^2}}$$

$$\cos\theta = \frac{-3}{\sqrt{9 + 27}}$$

$$\cos\theta = \frac{-3}{6}$$

$$\cos\theta = -\frac{1}{2}$$

We know $\cos 60^\circ = 1/2$, and $\cos\theta = -1/2$, so the reference angles of θ and 60° are equal. Since P lies in the third quadrant (because both x and y coordinate are negative), we need to find the angle in the third quadrant that has A reference angle of 60°. In the third quadrant, the reference angle β is $\beta = \theta - 180^\circ$. Substitute and get

$$60^\circ = \theta - 180^\circ$$

$$\theta = 240^\circ$$

Then the complete solution set includes every angle that's coterminal with $\theta = 240^\circ$.

$\theta = 240^\circ + 360^\circ k$ where k is any integer

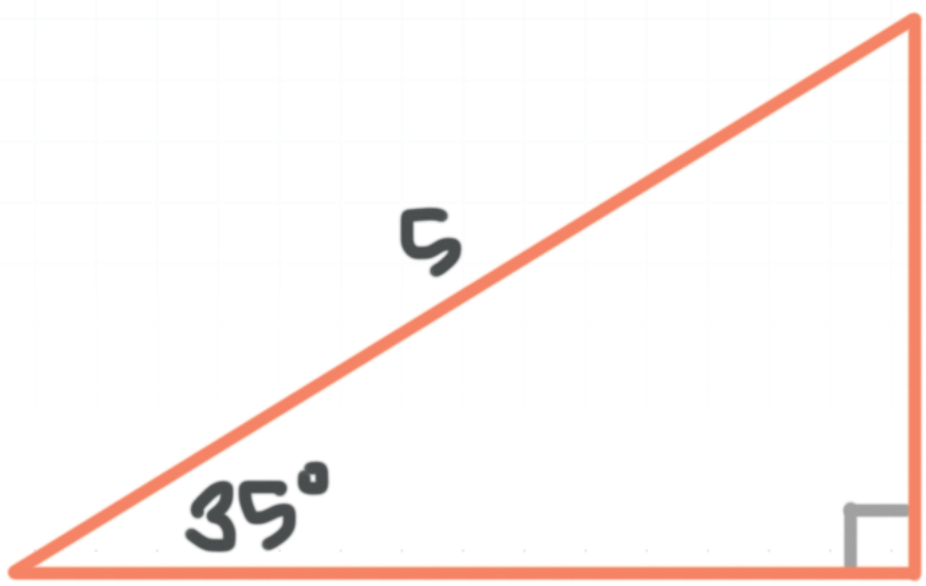
$$\{240^\circ + 360^\circ k : \ k \in \mathbb{Z}\}$$

SOLVING RIGHT TRIANGLES

■ 1. A right triangle has a hypotenuse with length 5 and one 35° interior angle. Solve the triangle.

Solution:

Sketch the triangle.



Because the triangle is right and therefore includes a 90° angle, and the three interior angles of the triangle sum to 180°, the third interior angle has a measure of

$$180^\circ - 90^\circ - 35^\circ$$

$$55^\circ$$

The angles and side lengths have equal ratios, so we can set up the equation

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\frac{a}{\sin 35^\circ}=\frac{b}{\sin 55^\circ}=\frac{5}{\sin 90^\circ}$$

Use the first and third parts of this three-part equation to solve for a.

$$\frac{a}{\sin 35^\circ}=\frac{5}{\sin 90^\circ}$$

$$a=\frac{5\sin 35^\circ}{\sin 90^\circ}\approx\frac{5(0.5736)}{1}\approx 2.9$$

Use the second and third parts of the three-part equation to solve for b.

$$\frac{b}{\sin 55^\circ}=\frac{5}{\sin 90^\circ}$$

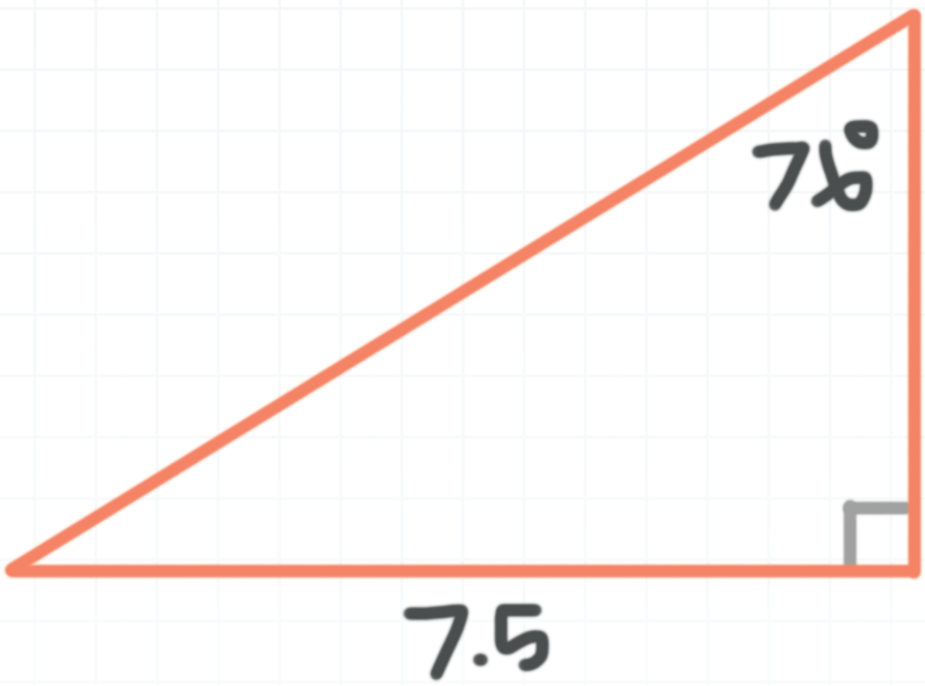
$$b=\frac{5\sin 55^\circ}{\sin 90^\circ}\approx\frac{5(0.8192)}{1}\approx 4.1$$

Therefore, the triangle has side lengths $a\approx 2.9$, $b\approx 4.1$, and $c=5$, and angle measures $A=35^\circ$, $B=55^\circ$, and $C=90^\circ$.

■ 2. A right triangle has a leg with length 7.5 which is opposite of an angle of 76°. Solve the triangle.

Solution:

Sketch the triangle.



Because the triangle is right and therefore includes a 90° angle, and the three interior angles of the triangle sum to 180°, the third interior angle has a measure of

$$180^\circ - 90^\circ - 76^\circ$$

$$14^\circ$$

The angles and side lengths have equal ratios, so we can set up the equation

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 14^\circ} = \frac{7.5}{\sin 76^\circ} = \frac{c}{\sin 90^\circ}$$

Use the first and second parts of this three-part equation to solve for a.

$$\frac{a}{\sin 14^\circ} = \frac{7.5}{\sin 76^\circ}$$

$$a = \frac{7.5 \sin 14^\circ}{\sin 76^\circ} \approx \frac{7.5(0.2419)}{0.9703} \approx 1.9$$

Use the second and third parts of the three-part equation to solve for c.

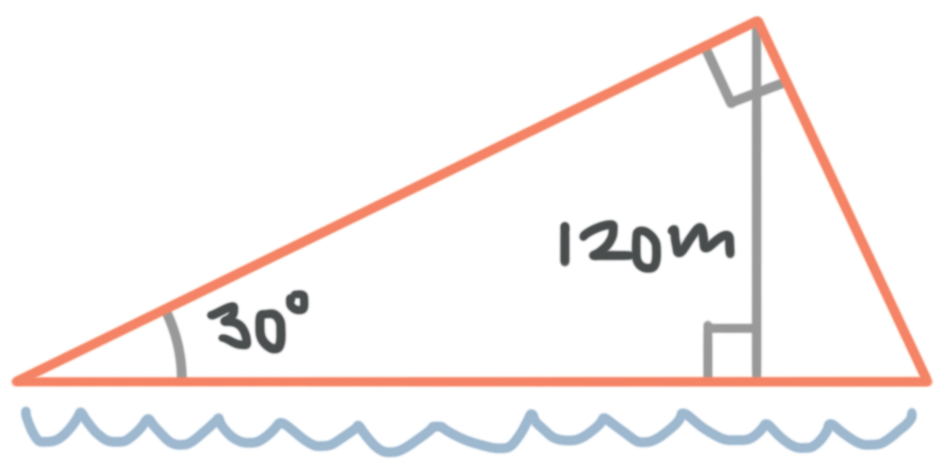
$$\frac{7.5}{\sin 76^\circ} = \frac{c}{\sin 90^\circ}$$

$$c = \frac{7.5 \sin 90^\circ}{\sin 76^\circ} \approx \frac{7.5(1)}{0.9703} \approx 7.7$$

Therefore, the triangle has side lengths $a \approx 1.9$, $b = 7.5$, and $c \approx 7.7$, and angle measures $A = 14^\circ$, $B = 76^\circ$, and $C = 90^\circ$.

■ 3. A plot of land shaped like a right triangle has its hypotenuse along a river. The angle between one of the legs and the river is 30°, and the distance from the river to the furthest point of the plot is 120 m.

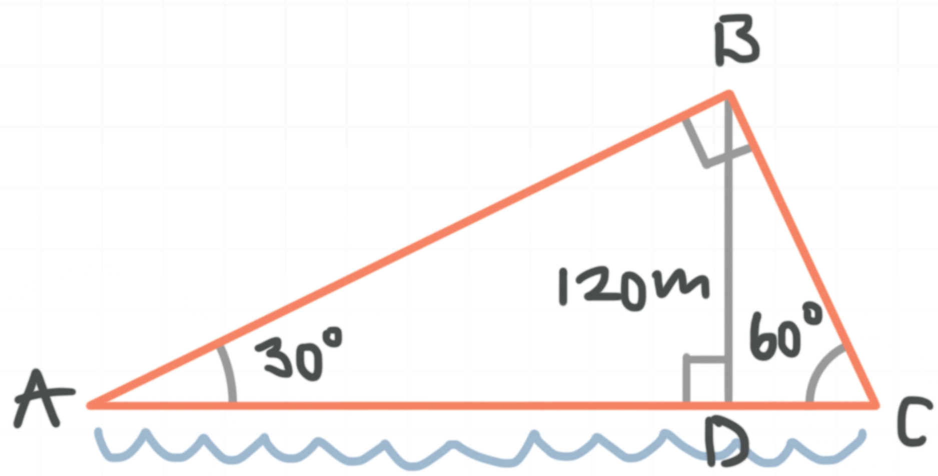
The land owner decides to build a fence around the plot and needs to find the total length of the fence. Since the fence is sold by sections of 10 meters, round up to the nearest 10 meters.



Solution:

Call the right triangle ABC, where $\overline{AC}$ is the hypotenuse along the river, and B is the right angle. Then $m\angle BAC = 30^\circ$. And since the sum of the two acute angles in a right triangle is 90°, we can say $m\angle BCA = 90^\circ - 30^\circ = 60^\circ$.

Let D be the point on side $\overline{AC}$ where $\overline{BD}$ is perpendicular to $\overline{AC}$. Then $\overline{BD}$ is the distance from B to $\overline{AC}$.



We'll solve the problem in four steps.

(1) Find $\overline{AB}$ from the triangle ABD,

(2) Find $\overline{BC}$ from the triangle BCD,

(3) Find $\overline{AC}$ using the Pythagorean Theorem, and

(4) Find the perimeter as the sum of these three side lengths.

In triangle ABD, we know the acute angle $BAD = 30^\circ$, the opposite leg is $\overline{BD}$, and we need to find the hypotenuse $\overline{AB}$.

$$\sin A = \frac{\overline{BD}}{\overline{AB}}$$

$$\overline{AB} = \frac{\overline{BD}}{\sin A} = \frac{120}{\sin 30^\circ} = \frac{120}{0.5} = 240 \text{ m}$$

In triangle CBD, we know the $\angle C = 60^\circ$, the opposite leg is $\overline{BD}$, and we need to find the hypotenuse $\overline{BC}$.

$$\overline{BC} = \frac{\overline{BD}}{\sin C} = \frac{120}{\sin 60^\circ} \approx 139 \text{ m}$$

Use the Pythagorean Theorem to find the hypotenuse $\overline{AC}$.

$$\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$$

$$240^2 + 139^2 = \overline{AC}^2$$

$$\overline{AC} = \sqrt{240^2 + 139^2} \approx 277 \text{ m}$$

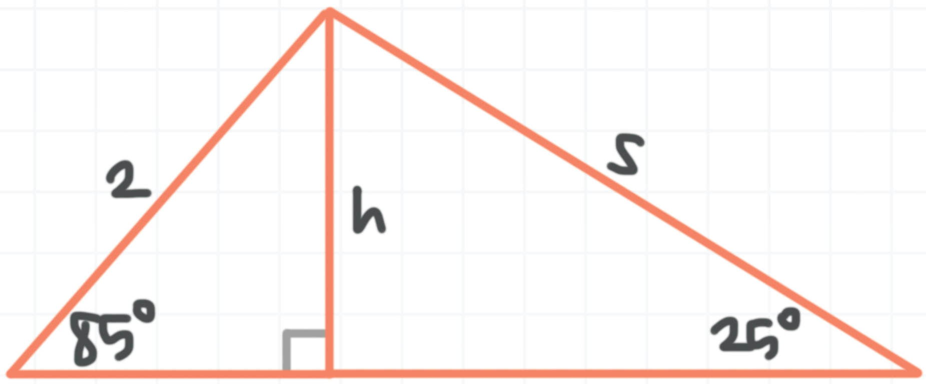
The length of the plot's complete perimeter is

$$240 \text{ m} + 139 \text{ m} + 277 \text{ m}$$

$$656 \text{ m}$$

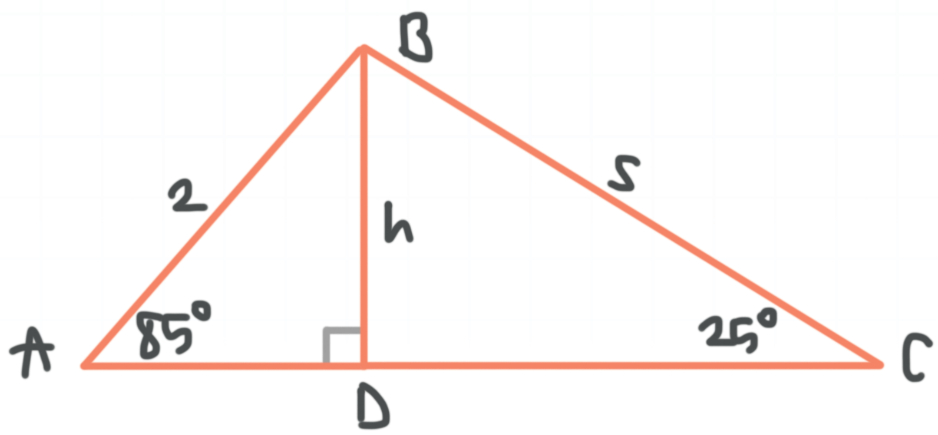
Rounding up to the nearest 10 meters, the owner needs to buy 660 m of fence.

4. Find the values of h and s, rounded to the nearest whole number.



Solution:

Call the triangle ABC, and let D be the point on side $\overline{AC}$ where $\overline{BD}$ is perpendicular to $\overline{AC}$.



In triangle ABD, we know the acute angle $BAD = 85^\circ$, the hypotenuse is $\overline{AB} = 2$, and we need to find the opposite leg $\overline{BD}$.

$$\sin A = \frac{\overline{BD}}{\overline{AB}}$$

$$\overline{BD} = \overline{AB} \sin A$$

$$\overline{BD} = 2 \sin 85^\circ \approx 2(0.996) \approx 2$$

In triangle CBD, we know the acute angle $BCD = 25^\circ$, the opposite leg is $\overline{BD} \approx 2$, and we need to find the hypotenuse $\overline{BC}$.

$$\sin C = \frac{\overline{BD}}{\overline{BC}}$$

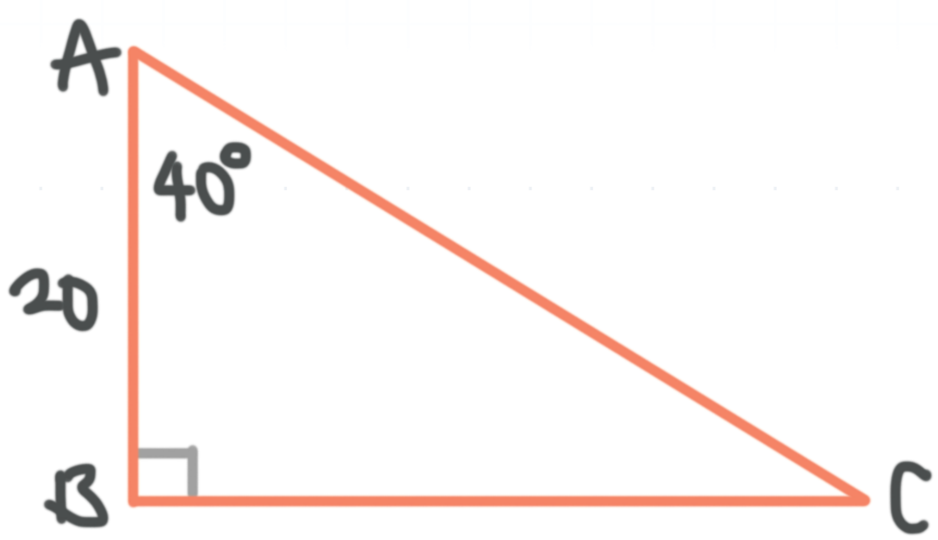
$$\overline{BC} = \frac{\overline{BD}}{\sin C} = \frac{2}{\sin 25^\circ} \approx \frac{2}{0.423} \approx 5$$

So $h = \overline{BD} \approx 2$ and $s = \overline{BC} \approx 5$.

■ 5. The length of one leg of a right triangle is 20, and the angle opposite the other leg has a measure of 40°. What is the length of the other leg?

Solution:

Sketch the triangle.



Then we can say

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\overline{AB}}{\overline{AC}}$$

$$\cos 40^\circ = \frac{20}{\overline{AC}}$$

$$\overline{AC} = \frac{20}{\cos 40^\circ} \approx \frac{20}{0.766} \approx 26.1$$

Now use the Pythagorean Theorem.

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$$

$$\overline{BC}^2 = \overline{AC}^2 - \overline{AB}^2$$

$$\overline{BC}^2 \approx 26.1^2 - 20^2$$

$$\overline{BC}^2 \approx 681.21 - 400$$

$$\overline{BC}^2 \approx 281.21$$

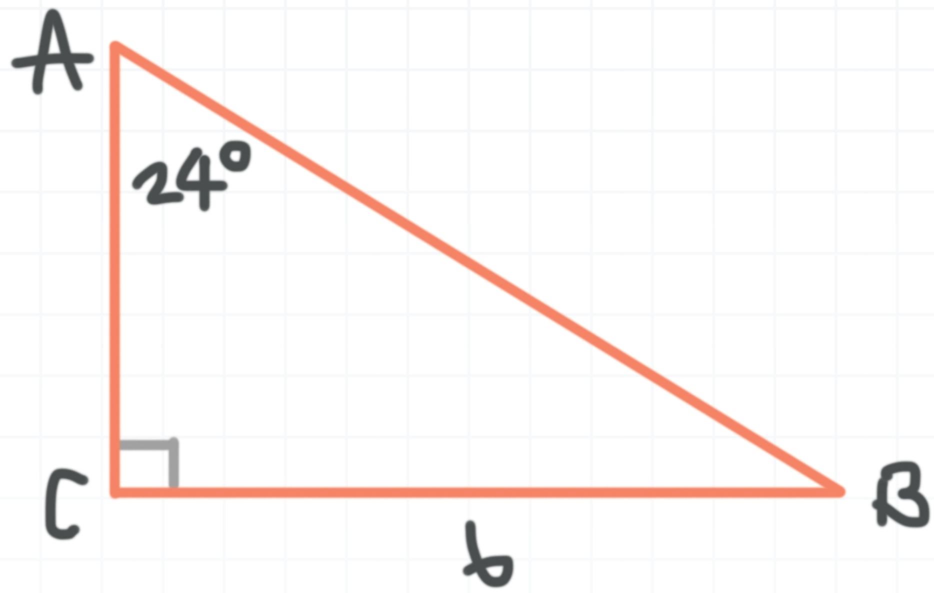
$$\overline{BC} \approx \sqrt{281.21}$$

$$\overline{BC} \approx 16.8$$

■ 6. A right triangle has a leg with length 6. The angle opposite that leg is 24°. Find the measures of all three interior angles and the lengths of all three sides.

Solution:

It's a good idea to draw a picture of the triangle before we start. Based on what we know, we have



Because we're dealing with a right triangle, we know we have a 90° angle and the 24° angle we were given. So the angle B must be

$$B = 180^\circ - 90^\circ - 24^\circ$$

$$B = 66^\circ$$

We know the length of side a is $a = 6$, so all we need now is the length of the other two sides.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$$

Substitute the values, and we get

$$\sin 24^\circ = \frac{6}{c}$$

$$c = \frac{6}{\sin 24^\circ} \approx \frac{6}{0.407} \approx 14.7$$

Now use the Pythagorean Theorem.

$$c^2 = a^2 + b^2$$

$$b^2 = c^2 - a^2$$

$$b^2 \approx 14.7^2 - 6^2$$

$$b^2 \approx 216.09 - 36$$

$$b^2 \approx 180.09$$

$$b \approx \sqrt{180.09}$$

$$b \approx 13.4$$

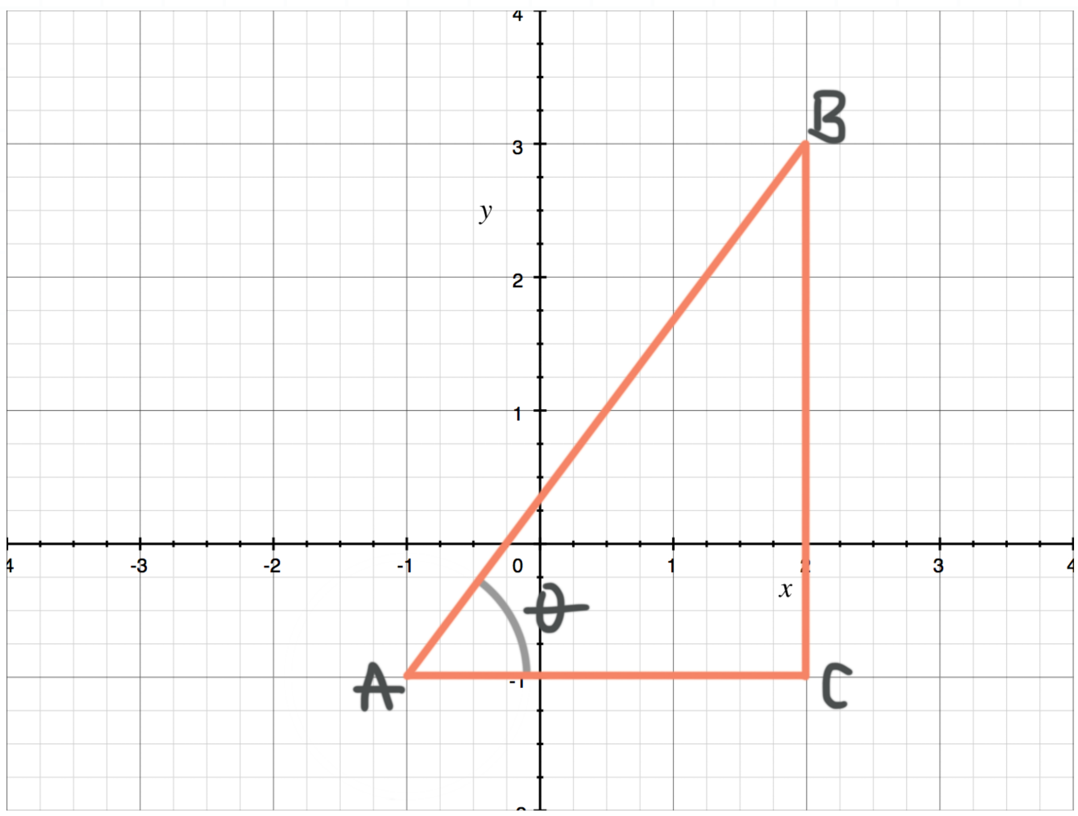
So the triangle has side lengths $a = 6$, $b \approx 13.4$, and $c \approx 14.7$, and interior angle measures $A = 24^\circ$, $B = 66^\circ$, and $C = 90^\circ$.

ANGLES OF ELEVATION AND DEPRESSION

■ 1. Find $\cos\theta$, where θ is the angle of elevation of the point $(2,3)$ with respect to the point $(-1,-1)$.

Solution:

Consider the right triangle ABC such that $A(-1,-1)$, $B(2,3)$, and $C(2,-1)$. Then we'll say $\theta = \angle CAB$.



In triangle ABC, the length of AC is 3 and the length of BC is 4. In order to find the length of AB, use the Pythagorean Theorem.

$$AC^2 + BC^2 = AB^2$$

$$3^2 + 4^2 = AB^2$$

$$9 + 16 = AB^2$$

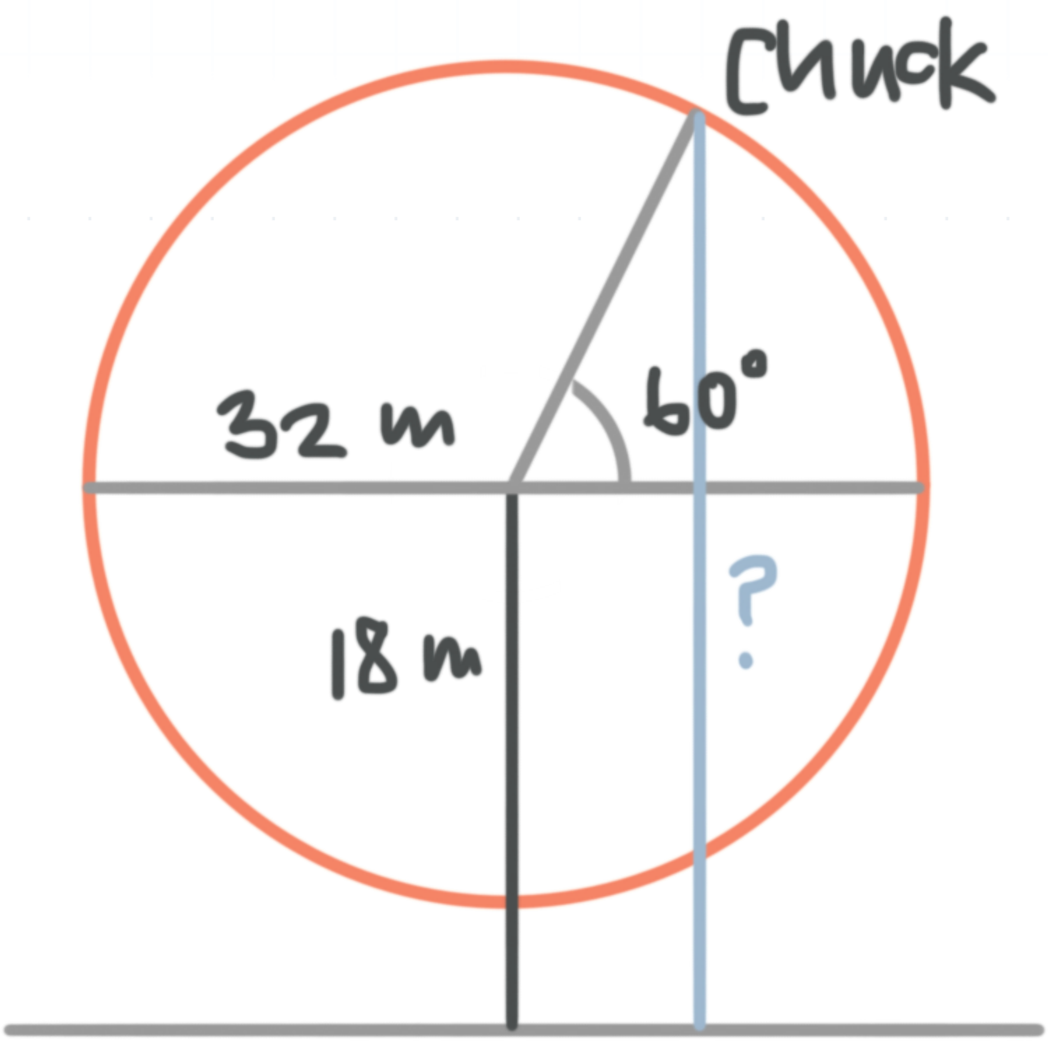
$$25 = AB^2$$

$$AB = 5$$

Then we can use the side lengths of the triangle to find $\cos\theta$.

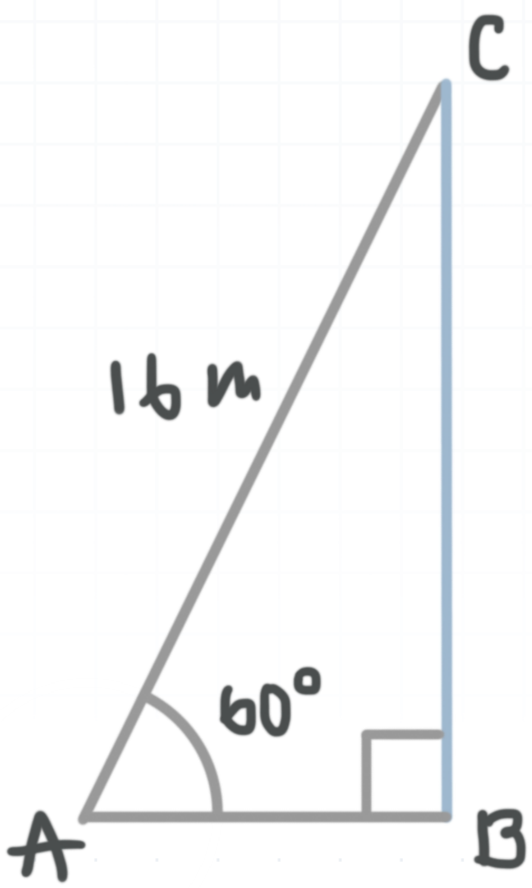
$$\cos\theta = \frac{AC}{AB} = \frac{3}{5}$$

■ 2. Chuck rides a ferris wheel at a carnival. Find the distance from Chuck to the ground (rounded to the nearest meter), if the center of the wheel is 18 m from the ground, the diameter of the wheel is 32 m, and the angle of elevation of Chuck with respect to the center of the wheel is 60°.



Solution:

Find the vertical distance from Chuck to the center of the wheel by creating a right triangle. If A is the center of the wheel and Chuck is at C, then



The angle $CAB = 60^\circ$ is the angle of elevation, BC is the vertical distance we're interested in, and AC is the radius of the wheel. Since the radius is the half of the diameter,

$$AC = \frac{32}{2} = 16 \text{ m}$$

We can use the sine function to find BC.

$$\sin A = \frac{BC}{AC}$$

$$BC = AC \sin A$$

$$BC = 16 \sin 60^\circ$$

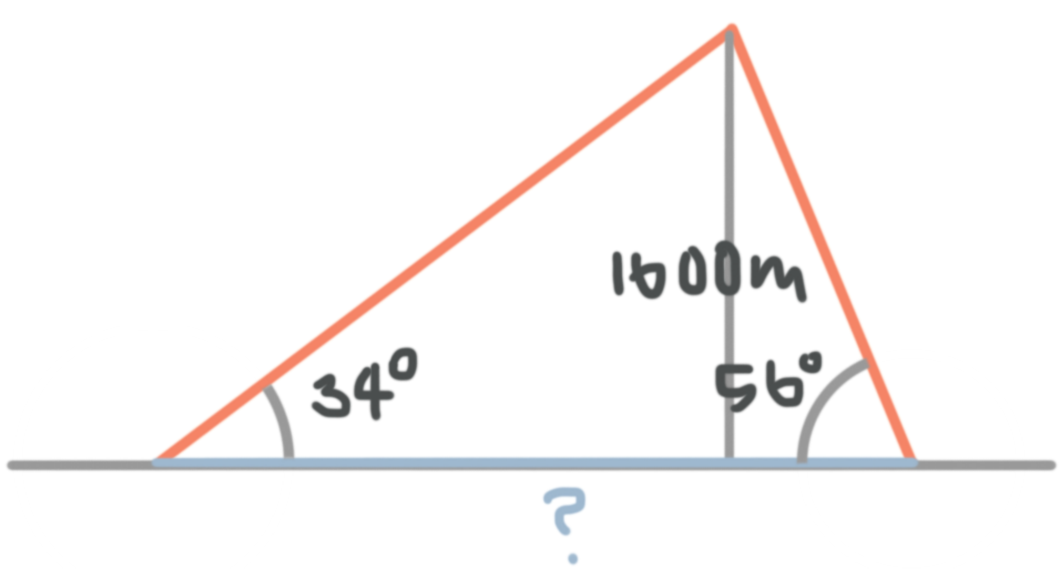
$$BC = 16\frac{\sqrt{3}}{2} = 8\sqrt{3} \approx 14 \text{ m}$$

Since the vertical distance from Chuck to the center of the wheel is 14 m, and the distance from the center of the wheel to the ground is 18 m, then the distance from Chuck to the ground is

$$18 \text{ m} + 14 \text{ m}$$

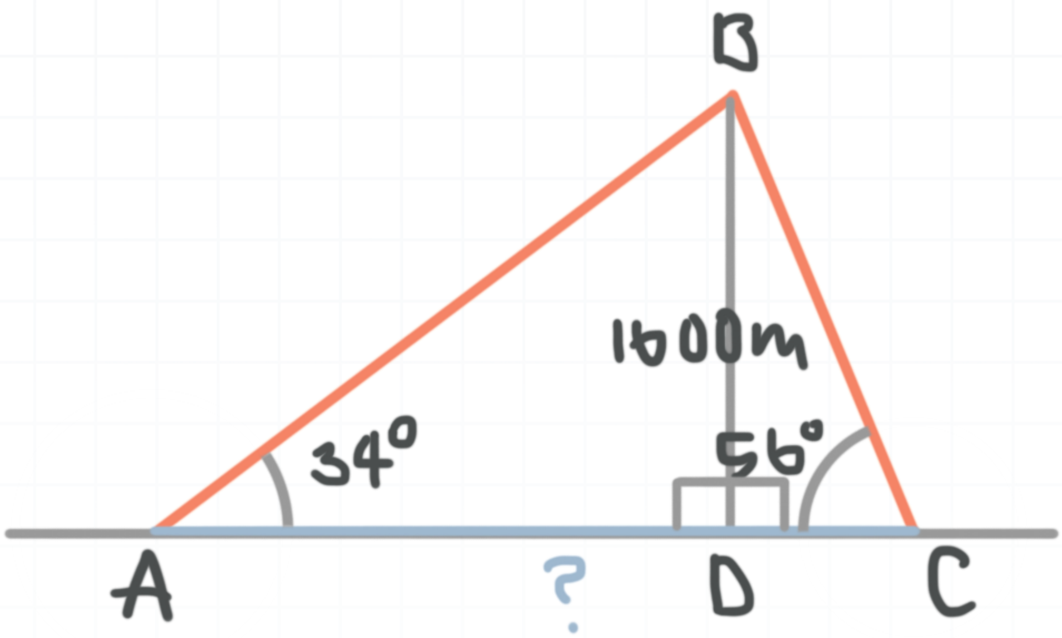
$$32 \text{ m}$$

■ 3. A company considers building a tunnel through a mountain. The height of the mountain is $1{,}600$ m above the ground, and the angles of elevation of the peak with respect to the left and right endpoints of the tunnel are 34° and 56° respectively. Find the length of the tunnel to the nearest meter.



Solution:

Let's set up the triangle with some extra information.



Now if we just look at triangle ADB, the angle of elevation at the left edge of the tunnel is $A = 34^\circ$, the length of BD is $1{,}600$ m, and the length of AD is the distance we need to find (we'll find CD later).

We can set up an equation using the tangent function.

$$\tan A = \frac{BD}{AD}$$

$$AD \tan A = BD$$

$$AD = \frac{BD}{\tan A} = \frac{1{,}600}{\tan 34^\circ} \approx 2{,}372 \text{ m}$$

Now look at triangle CDB.

$$\tan C = \frac{BD}{CD}$$

$$CD \tan C = BD$$

$$CD = \frac{BD}{\tan C} = \frac{1{,}600}{\tan 56^\circ} \approx 1{,}079 \text{ m}$$

Then the length of the tunnel is

$$AD + CD$$

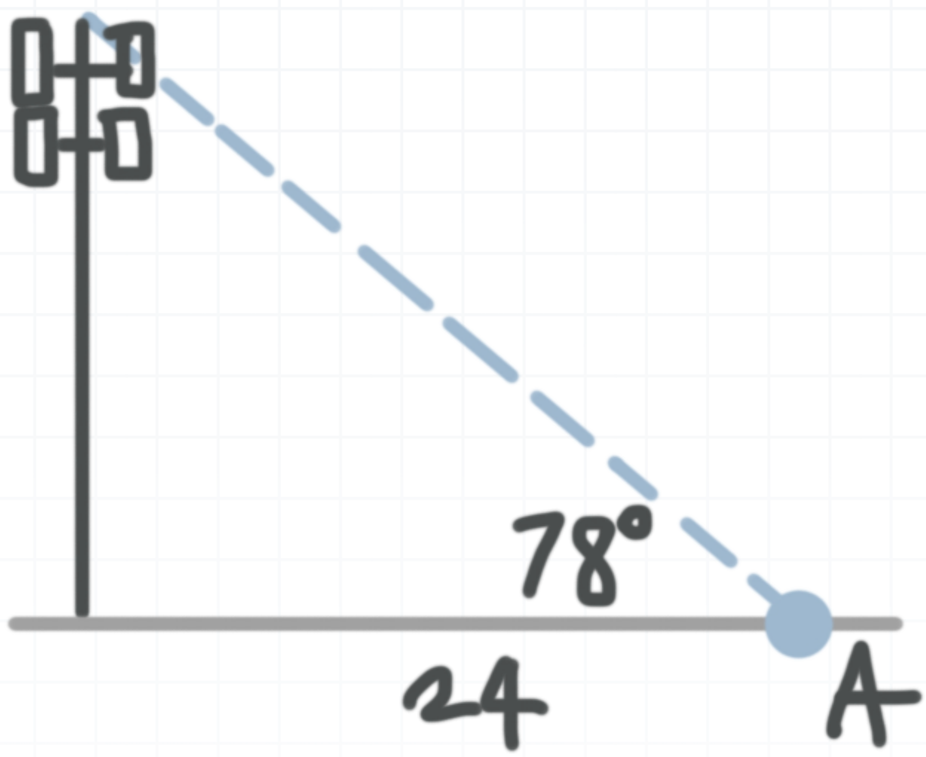
$$2{,}372 \text{ m} + 1{,}079 \text{ m}$$

$$3{,}451 \text{ m}$$

4. Suppose you measure the horizontal distance of a cell-phone tower from a nearby point on the ground A to be 24 feet. If the angle of elevation of the top of the tower with respect to A is 78°, how far is the top of the tower from A, and what is the height of the tower?

Solution:

Sketch the situation.



In this case, $\theta = 78^\circ$ and $b = 24$ feet. Also, the distance from A to the top of the tower is c, and the height of the tower is a. We can use this information to find c.

$$\cos\theta = \frac{b}{c}$$

$$c = \frac{b}{\cos\theta} = \frac{24}{\cos 78^\circ} \approx \frac{24}{0.208} \approx 115 \text{ ft}$$

Then to find a, we'll use

$$\frac{a}{\sin\theta} = \frac{b}{\cos\theta}$$

$$a = \frac{b\sin\theta}{\cos\theta} \approx \frac{24(0.978)}{0.208} \approx 113 \text{ feet}$$

We also can use

$$\tan\theta = \frac{a}{b}$$

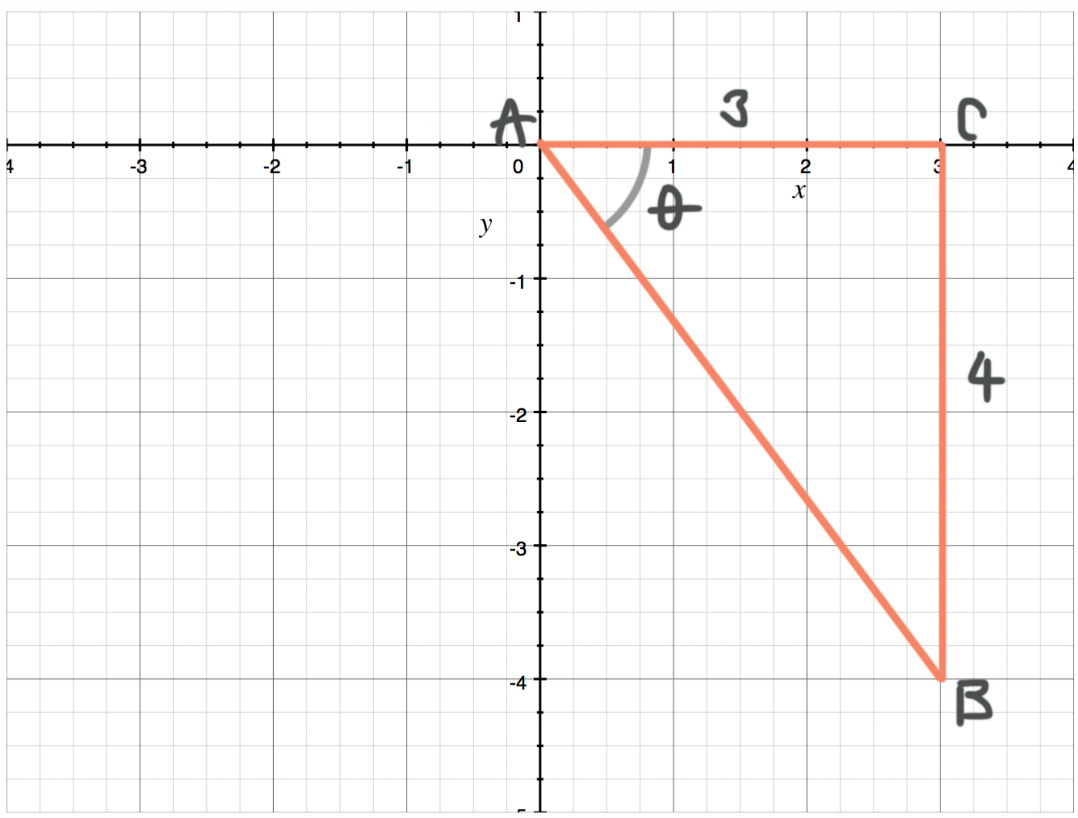
$$a = b \tan \theta$$

$$a = 24 \tan 78^\circ = 24(4.705) \approx 113 \text{ feet}$$

■ 5. Find $\sin \theta$, where θ is the angle of depression of the point $(3, -4)$ with respect to origin.

Solution:

Make a right triangle from the two points, where A is at the origin, and B is at $(3, -4)$.



Use the Pythagorean Theorem to find the length of $\overline{AB}$.

$$\overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2$$

$$3^2 + 4^2 = \overline{AB}^2$$

$$9 + 16 = \overline{AB}^2$$

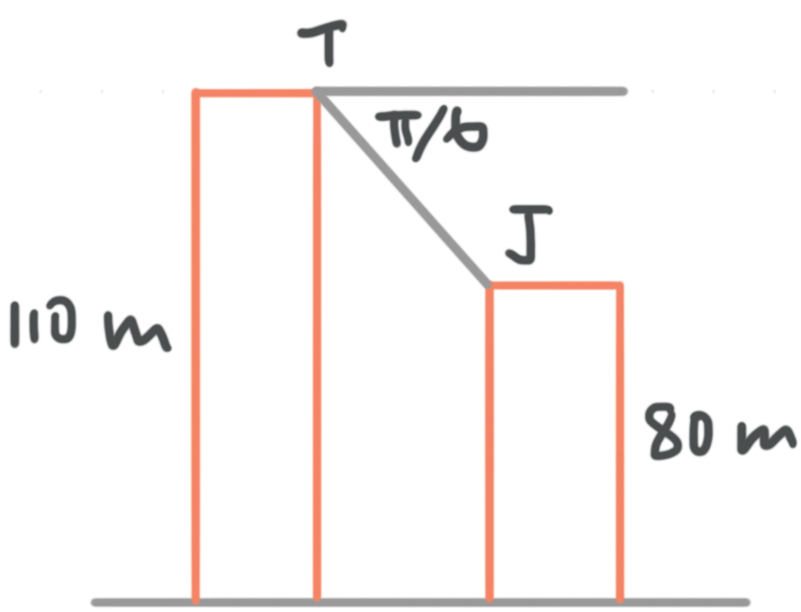
$$25 = \overline{AB}^2$$

$$\overline{AB} = 5$$

Then we can find $\sin\theta$.

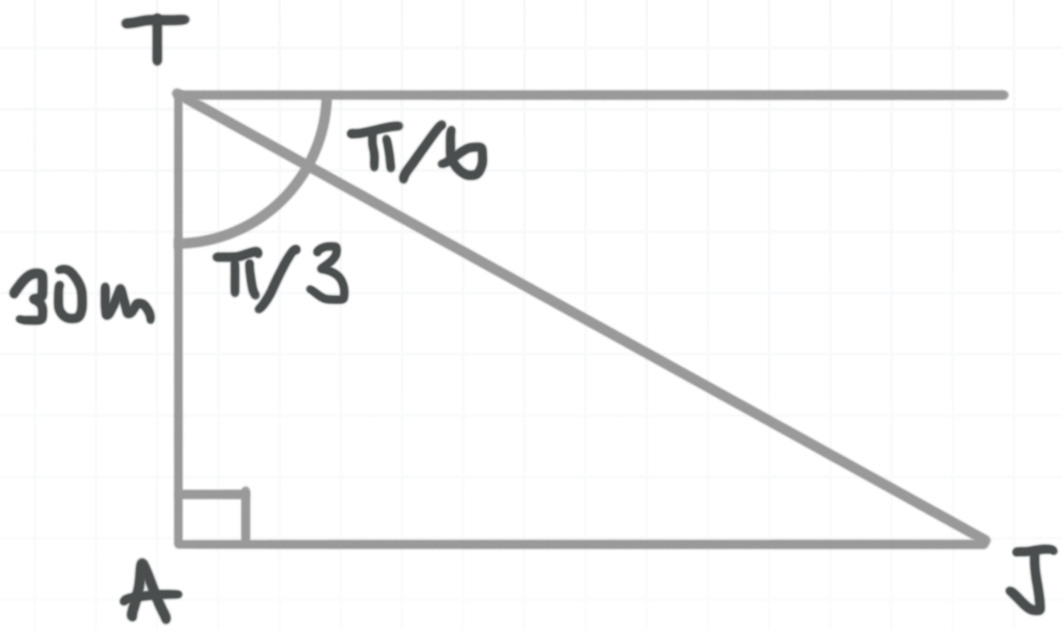
$$\sin\theta = \frac{\overline{BC}}{\overline{AB}} = \frac{4}{5}$$

■ 6. Tom stands on the roof of a 110 m high building, and he looks down at Jerry, who's standing on the roof of a neighboring 80 m high building. The angle of depression of Jerry with respect to Tom is $\pi/6$. Find the distance between the buildings to the nearest meter.



Solution:

Sketch a triangle that includes the points where Tom and Jerry are standing.



Because there's a 90° angle ($\pi/2$ radian angle) at T and we know the angle of depression is $\pi/6$, angle ATJ must be

$$\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

The length of side $\overline{AT}$ is the difference between building heights.

$$\overline{AT} = 110 - 80 = 30 \text{ m}$$

Then we can set up a tangent equation.

$$\tan(\angle ATJ) = \frac{\overline{AJ}}{\overline{AT}}$$

$$\overline{AJ} = \overline{AT}\tan(\angle ATJ)$$

$$\overline{AJ} = 30\tan\left(\frac{\pi}{3}\right) = 30\sqrt{3} \approx 52 \text{ m}$$

RADIANS AND ARC LENGTH

■ 1. Find the degree measure of the central angle if the length of an arc carved out by this central angle is 9.42 and the radius of the circle is $r = 6$.

Solution:

Use the arc length formula

$$s = r\theta$$

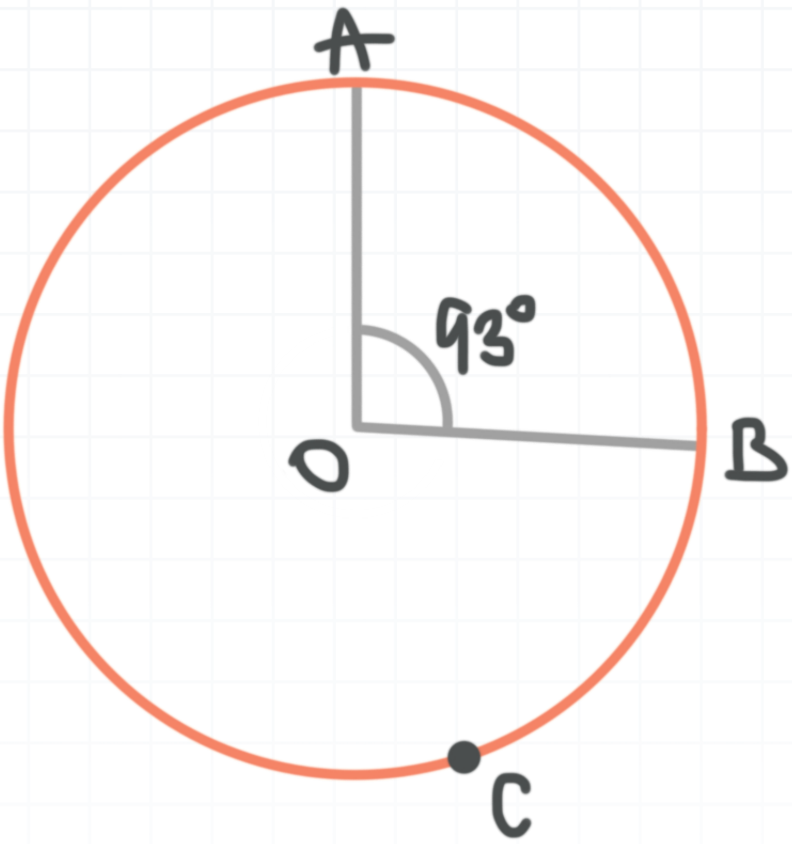
$$9.42 = 6\theta$$

$$\theta = \frac{9.42}{6} = 1.57$$

Now remember that 1 radian is approximately 57.32°, so we get

$$\theta = 1.57(57.32^\circ) = 90^\circ$$

■ 2. In circle O, the diameter is 30 cm, and the measure of arc AB is 93°. Find the length of arc ACB.



Solution:

To find the length of the arc ACB, first we need to find the central angle $m\angle ACB$. We remember that a full circle sweeps out 360°, so

$$m\angle ACB = 360^\circ - m\angle AOB$$

$$m\angle ACB = 360^\circ - 93^\circ$$

$$m\angle ACB = 267^\circ$$

We can only use an angle defined in radians in the arc length formula, so we'll need to convert 267° to radians.

$$267^\circ \left(\frac{\pi \text{ radians}}{180^\circ}\right) \approx 1.483\pi \text{ radians}$$

Since the diameter is 30 cm, the radius will be $30/2 = 15$ cm. Now we'll plug what we know into the arc length formula.

$$s = r\theta$$

$$s = 15(1.483\pi)$$

$$s \approx 69.85$$

■ 3. A circle has a central angle of $35^\circ 23' 6''$ which subtends an arc of length 3π cm. Find the diameter of the circle to the nearest centimeter.

Solution:

We can only use an angle defined in radians in the arc length formula, so we'll need to convert $35^\circ 23' 6''$ to radians.

We'll convert the seconds part first. We need to convert $6''$ from seconds to minutes. We know that $1' = 60''$, so we'll multiply $6''$ by $1'/60''$ in order to cancel the seconds and be left with just minutes.

$$6''\left(\frac{1'}{60''}\right)$$

$$\left(\frac{1}{10}\right)' = 0.1'$$

Then the total minutes in $35^\circ 23' 6''$ is

$$23.1'$$

To convert this value for minutes into degrees, we'll multiply by $1^\circ/60'$ in order to cancel the minutes and be left with just degrees.

$$23.1'\left(\frac{1^\circ}{60'}\right)$$

$$0.385^\circ$$

Putting this together with the 35° from the original angle, we get approximately

$$35.385^\circ$$

$$35.385^\circ\left(\frac{\pi \text{ radians}}{180^\circ}\right) \approx 0.197\pi \text{ radians}$$

Now we'll plug what we know into the arc length formula.

$$s = r\theta$$

$$3\pi = r(0.197\pi)$$

$$r = \frac{3\pi}{0.197\pi} \approx 15$$

$$d = 2r \approx 2(15) \approx 30$$

■ 4. A circle has a radius of 19 cm. Find the central angle that subtends an arc of length 47.5 cm, rounding the answer to the nearest second.

Solution:

Use the arc length formula

$$s = r\theta$$

$$47.5 = 19\theta$$

$$\theta = \frac{47.5}{19} = 2.5$$

Now remember that 1 radian is approximately 57.32°, so we get

$$\theta = 2.5(57.32^\circ) = 143.3^\circ$$

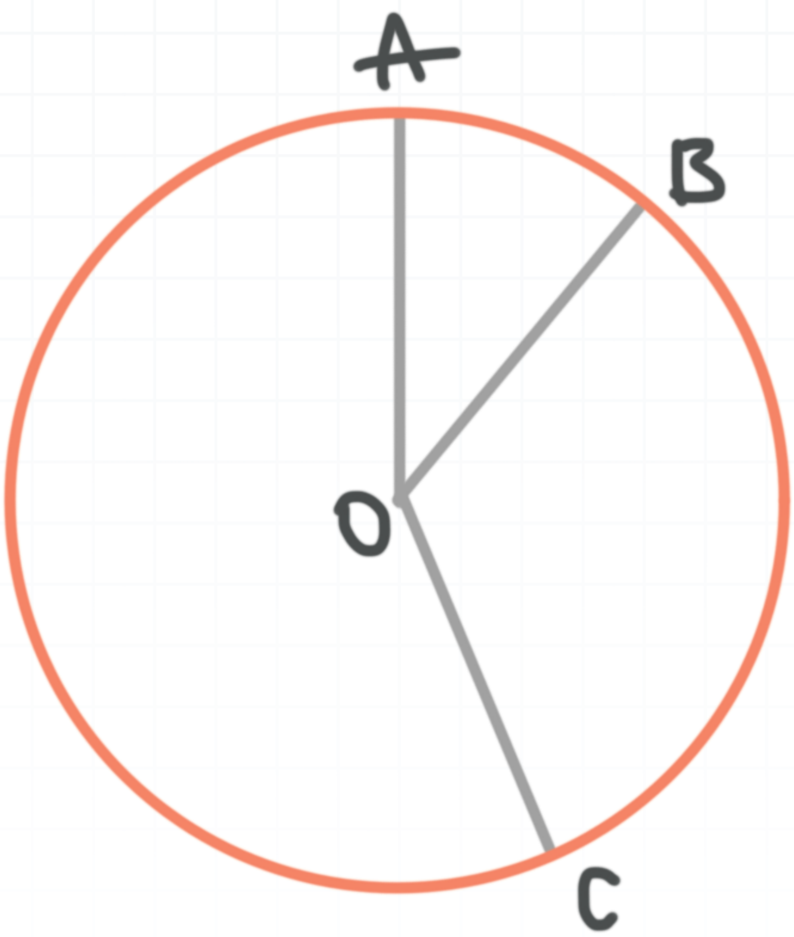
Now we need to convert degrees into DMS. The angle in degrees is 143.3°, so the degrees part in DMS is 143°. All we have to do is convert 0.3° to minutes and seconds. First, we'll convert 0.3° to minutes, and then if we get a decimal value for the minutes, we'll convert the remaining part to seconds.

$$0.3^\circ \left(\frac{60'}{1^\circ}\right)$$

$$18'$$

We've found that 0.3° converts to $18'$. Since 18 is an integer, there's nothing left to convert to seconds, so the angle in DMS is $143^\circ 18'$.

5. If AOB is a central angle of 53°, the angle $BOC = 122^\circ$, and the radius is 9 cm, then find the length of the arc ABC. Use $\pi = 3.14$ and round the answer to one decimal place.



Solution:

To find the length of the arc ABC, first we need to find the central angle $m\angle ABC$.

$$m\angle ABC = m\angle AOB + m\angle BOC$$

$$m\angle ABC = 53^\circ + 122^\circ$$

$$m\angle ABC = 175^\circ$$

We can only use an angle defined in radians in the arc length formula, so we'll need to convert 175° to radians.

$$175^\circ\left(\frac{\pi \text{ radians}}{180^\circ}\right)=\frac{35\pi}{36} \text{ radians}$$

Now we'll plug what we know into the arc length formula.

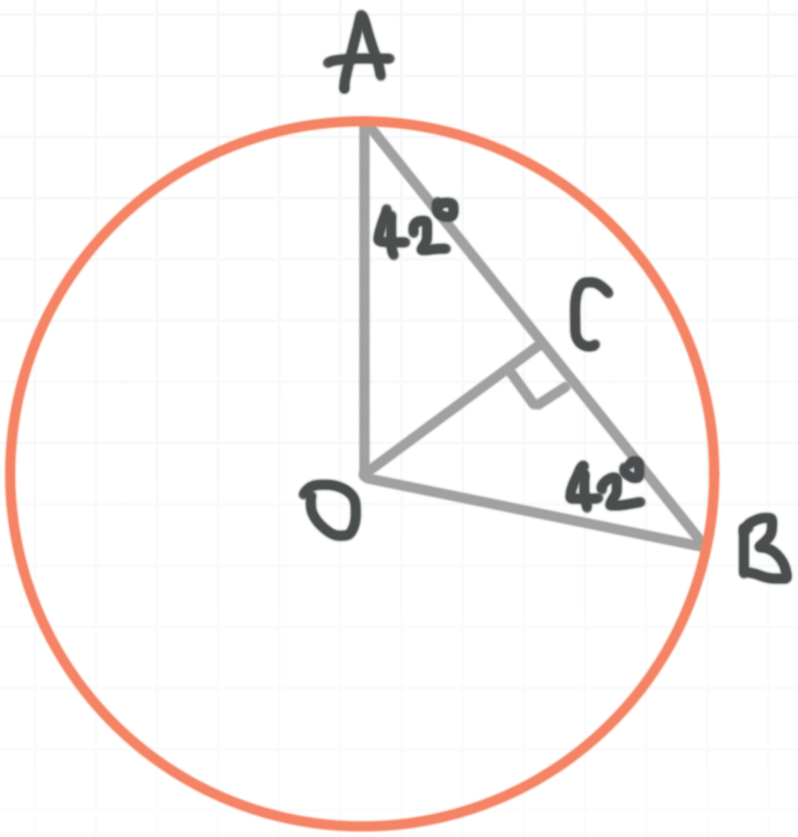
$$s=r\theta$$

$$s=9\left(\frac{35\pi}{36}\right)$$

$$s=\frac{35\pi}{4}$$

$$s\approx 27.5$$

■ 6. Find the length of the arc AB given that the radius of the circle is 12 cm. Round the answer to one decimal place.



Solution:

The sum of the interior angles of a triangle is 180°. So the central angle will be

$$m\angle AOB = 180^\circ - 42^\circ - 42^\circ$$

$$m\angle AOB = 96^\circ$$

We can only use an angle defined in radians in the arc length formula, so we'll need to convert 96° to radians.

$$96^\circ\left(\frac{\pi \text{ radians}}{180^\circ}\right) = \frac{24\pi}{45} \text{ radians}$$

Now we'll plug what we know into the arc length formula.

$$s = r\theta$$

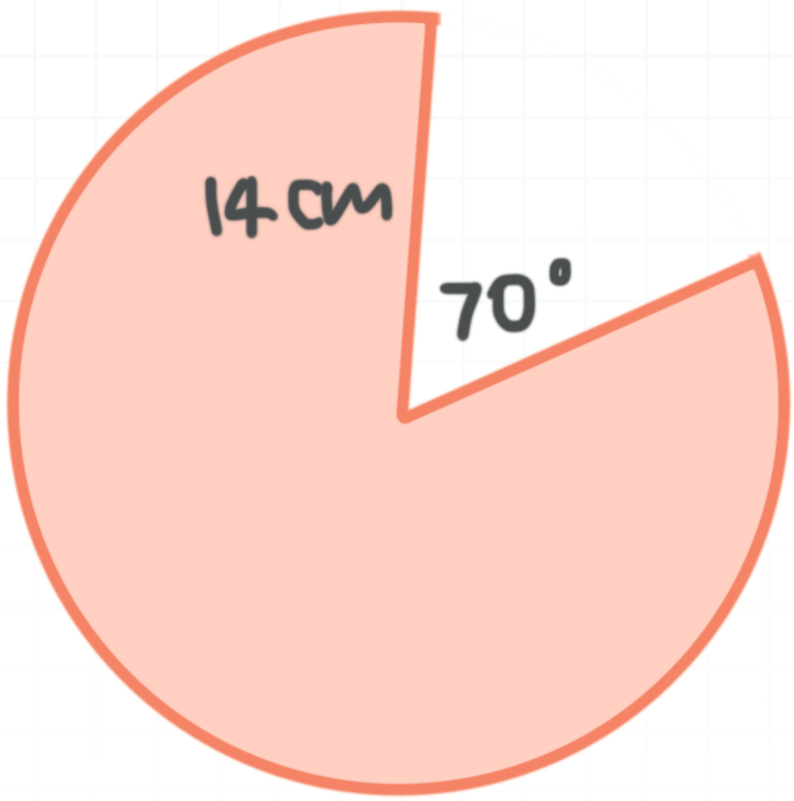
$$s = 12\left(\frac{24\pi}{45}\right)$$

$$s = \frac{288\pi}{45}$$

$$s = \frac{32\pi}{5}$$

AREA OF A CIRCULAR SECTOR

1. Find the area of the shaded region.



Solution:

The angle of the circular sector is

$$\theta = 360^\circ - 70^\circ$$

$$\theta = 290^\circ$$

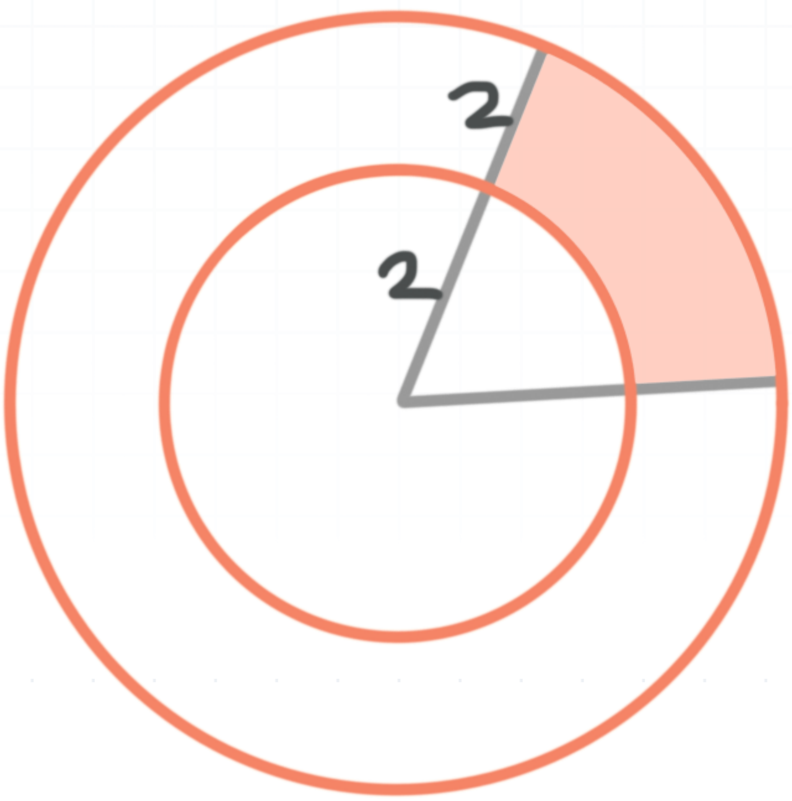
Plugging this angle and the radius into the formula for the area of a circular sector (in degrees) gives

$$A = \left(\frac{\pi}{360}\right) r^2\theta$$

$$A=\left(\frac{\pi}{360}\right)(14)^2(290)$$

$$A=\frac{1{,}421\pi}{9}$$

2. Find the area of the shaded region between the concentric circles, if the angle that subtends the arc is 80°.



Solution:

If we find the area of the sector for the larger circle, given that its interior angle measure is 80° and its radius is 4, we get

$$A=\left(\frac{\pi}{360}\right)r^2\theta$$

$$A = \left(\frac{\pi}{360}\right)(4)^2(80)$$

$$A = \frac{32\pi}{9}$$

The area of the sector for the smaller circle, given that its interior angle measure is 80° and its radius is 2, is

$$A = \left(\frac{\pi}{360}\right)r^2\theta$$

$$A = \left(\frac{\pi}{360}\right)(2)^2(80)$$

$$A = \frac{8\pi}{9}$$

So the area of the shaded region is

$$A = \frac{32\pi}{9} - \frac{8\pi}{9}$$

$$A = \frac{24\pi}{9}$$

$$A = \frac{8\pi}{3}$$

■ 3. A circle has radius 13. Find the area A of a sector of the circle that has a central angle of $2\pi/5$.

Solution:

The area of the circular sector with radius $r = 13$ and central angle $2\pi/5$ is

$$A = \frac{1}{2}r^2\theta$$

$$A = \frac{1}{2}(13)^2\left(\frac{2\pi}{5}\right)$$

$$A = \frac{169\pi}{5}$$

4. A pizza with 16 inch diameter is sliced into 8 equal slices. Find the area of one of the pizza slices.

Solution:

Let's think of θ in degrees. The central angle associated with each slice is $360^\circ/8 = 45^\circ$. Then the area of each slice is

$$A = \left(\frac{\pi}{360}\right)r^2\theta$$

$$A = \left(\frac{\pi}{360}\right)(8)^2(45)$$

$$A = 8\pi$$

5. Find the area of a sector of a circle that has diameter $\overline{GH}$ with $G(-1,-1)$ and $H(5,7)$ if the arc which bounds that sector subtends a central angle of $4\pi/9$. Use the distance formula for d to find the length of the diameter.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Solution:

Start by finding the length of the diameter.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(7 - (-1))^2 + (5 - (-1))^2}$$

$$d = \sqrt{8^2 + 6^2}$$

$$d = \sqrt{100}$$

$$d = 10$$

The length of the diameter is 10 units, so the radius is 5 units.

$$A = \frac{1}{2}r^2\theta$$

$$A = \frac{1}{2}(5)^2\left(\frac{4\pi}{9}\right)$$

$$A = \frac{50\pi}{9}$$

■ 6. The area of a sector of a circle is formed with a central angle $3\pi/4$ and has area 54π. Find the diameter of the circle.

Solution:

Substitute the area and central angle into the circular sector area formula,

$$A = \frac{1}{2}r^2\theta$$

$$54\pi = \frac{1}{2}r^2\left(\frac{3\pi}{4}\right)$$

then solve for the radius.

$$54\pi = r^2\left(\frac{3\pi}{8}\right)$$

$$54\pi\left(\frac{8}{3\pi}\right) = r^2$$

$$18(8) = r^2$$

$$r^2 = 144$$

$$r = 12$$

Then the diameter of the circle is double the radius, so $d = 2r = 2(12) = 24$.

TRIG FUNCTIONS OF REAL NUMBERS

■ 1. Find $\sec 1.56$ using a calculator to evaluate only cosine. Round the result to three decimal places.

Solution:

We'll use a calculator to evaluate $\cos 1.56$, making sure the calculator is set to radian mode.

$$\cos 1.56 \approx 0.011$$

Then the value of $\sec 1.56$ is

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\sec 1.56 \approx \frac{1}{0.011}$$

$$\sec 1.56 \approx 90.909$$

■ 2. Find $\cot 0.567$ using a calculator to evaluate only sine and cosine. Round the result to four decimal places.

$$\cot 0.567$$

Solution:

We'll use a calculator to evaluate $\sin 0.567$ and $\cos 0.567$, making sure the calculator is set to radian mode.

$$\sin 0.567 \approx 0.5371$$

$$\cos 0.567 \approx 0.8435$$

Then the value of $\cot 0.567$ is

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\cot 0.567 \approx \frac{0.8435}{0.5371}$$

$$\cot 0.567 \approx 1.5705$$

■ 3. Find the value of all six circular functions at $a = 1.273$ using a calculator to evaluate only sine and cosine.

Solution:

We'll use a calculator to evaluate $\sin 1.273$ and $\cos 1.273$, making sure the calculator is set to radian mode.

$$\sin 1.273 \approx 0.9560$$

$$\cos 1.273 \approx 0.2934$$

Now we can find $\tan 1.273$ using the quotient identity for tangent.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan 1.273 \approx \frac{0.9560}{0.2934}$$

$$\tan 1.273 \approx 3.2584$$

Then the reciprocal identities give the values of cosecant, secant, and cotangent.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \csc 1.273 \approx \frac{1}{0.9560} \approx 1.0460$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \sec 1.273 \approx \frac{1}{0.2934} \approx 3.4083$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \cot 1.273 \approx \frac{1}{3.2584} \approx 0.3069$$

■ 4. Find the value of all six circular functions at $t = -0.2489$.

Solution:

We'll use a calculator to evaluate the circular functions at -0.2489, making sure the calculator is set to radian mode.

$$\sin(-0.2489) \approx -0.2463 \qquad \csc(-0.2489) \approx -4.0601$$

$$\cos(-0.2489) \approx 0.9692 \qquad \sec(-0.2489) \approx 1.0318$$

$$\tan(-0.2489) \approx -0.2542 \qquad \cot(-0.2489) \approx -3.9339$$

■ 5. Find $\tan 3.49$ using a calculator to evaluate only sine and cosine. Round the result to two decimal places.

Solution:

We'll use a calculator to evaluate $\sin 3.49$ and $\cos 3.49$, making sure the calculator is set to radian mode.

$$\sin 3.49 \approx -0.3414$$

$$\cos 3.49 \approx -0.9399$$

Then the value of $\tan 3.49$ is

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan 3.49 \approx \frac{-0.3414}{-0.9399}$$

$$\tan 3.49 \approx 0.36$$

■ 6. Find the value of all six circular functions at $s = -4.5$, using a calculator to evaluate only sine and cosine.

Solution:

We'll use a calculator to evaluate the circular functions at -4.5, making sure the calculator is set to radian mode.

$$\sin(-4.5) \approx 0.9775$$

$$\cos(-4.5) \approx -0.2108$$

Now we can find $\tan(-4.5)$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan(-4.5) \approx \frac{0.9775}{-0.2108}$$

$$\tan(-4.5) \approx -4.6371$$

Then use the reciprocal identities to find cosecant, secant, and cotangent.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \csc(-4.5) \approx \frac{1}{0.9775} \approx 1.0230$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \sec(-4.5) \approx \frac{1}{-0.2108} \approx -4.7438$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \cot(-4.5) \approx \frac{1}{-4.6371} \approx -0.2157$$

LINEAR AND ANGULAR VELOCITY

■ 1. What is the angular velocity, in radians per second, of a wheel that rotates at a constant rate and sweeps out an angle of $33\pi/4$ radians in 0.6 seconds?

Solution:

The angular velocity is

$$\omega = \frac{\theta}{t}$$

$$\omega = \frac{\frac{33\pi}{4}}{0.6}$$

$\omega = 13.75\pi$ radians per second

■ 2. The wind turbine has a circular blade with diameter 154 meters that rotates at 18 rotations per minute. Find the angular velocity of the blade in degrees per second.

Solution:

Find angular velocity in revolutions per minute.

$$\omega = 18\ \frac{\text{rev}}{\text{min}}$$

Convert revolutions per minute to radians per second.

$$\omega = 18\ \frac{\text{rev}}{\text{min}} \times 2\pi\ \frac{\text{rad}}{\text{rev}} \times \frac{1\ \text{min}}{60\ \text{sec}}$$

$$\omega = \frac{36\pi}{60}\ \frac{\text{rad}}{\text{sec}}$$

$$\omega = \frac{3\pi}{5}\ \frac{\text{rad}}{\text{sec}}$$

Convert from radians per second to degrees per second.

$$\frac{3\pi}{5}\ \frac{\text{rad}}{\text{sec}} \times \frac{180^\circ}{\pi\ \text{rad}}$$

108° per second

■ 3. What is the angular velocity, in radians per second, of a wheel that rotates at a constant rate and sweeps out an angle of $21\pi/5$ radians in 0.85 seconds?

Solution:

The angular velocity is

$$\omega = \frac{\theta}{t}$$

$$\omega = \frac{\frac{21\pi}{5}}{0.85}$$

$\omega \approx 4.94\pi$ radians per second

■ 4. Suppose a frisbee rotates at a constant rate of 105 revolutions per minute. What is its angular velocity ω in radians per second?

Solution:

To convert from revolutions per minute to radians per second, we'll set up a conversion equation.

$$\omega = \left(105\frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi\text{ rad}}{1\text{ rev}}\right)\left(\frac{1\text{ min}}{60\text{ sec}}\right)$$

$\omega = 3.5\pi$ radians per second

■ 5. Find angular velocity, in radians per minute, of an object that rotates at a constant rate and sweeps out an angle of 985° in 8.4 seconds.

Solution:

We'll first convert the angle 985° to radians.

$$\theta = 985^\circ \left(\frac{\pi \text{ radians}}{180^\circ} \right)$$

$$\theta = \frac{197\pi}{36} \text{ radians}$$

Now we'll find angular velocity.

$$\omega = \frac{\theta}{t}$$

$$\omega = \frac{\frac{197\pi}{36} \text{ radians}}{8.4 \text{ seconds}}$$

$$\omega = \frac{197\pi}{36(8.4)} \text{ radians per second}$$

$$\omega \approx 0.651\pi \text{ radians per second}$$

Convert this value to radians per minute.

$$\omega \approx \frac{0.651\pi \text{ radians}}{1 \text{ second}} \left(\frac{60 \text{ seconds}}{1 \text{ minute}} \right)$$

$$\omega \approx 39.06\pi \text{ radians per minute}$$

■ 6. A cylinder with a 3.4 ft radius is rotating at 150 rpm. Give the angular velocity in rad/sec and in degrees per second.

Solution:

We already know angular velocity, we just need to convert the units.

$$\omega = 150\ \frac{\text{revolutions}}{\text{minute}}$$

$$\omega = \left(150\ \frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi\ \text{rad}}{1\ \text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{sec}}\right)$$

$$\omega = \frac{150(2\pi)}{60}\ \text{radians per second}$$

$$\omega \approx 5\pi\ \text{radians per second}$$

Convert from radians to degrees.

$$\omega \approx 5\pi\ \frac{\text{rad}}{\text{sec}} \times \frac{180^\circ}{\pi\ \text{rad}}$$

$$\omega \approx 900^\circ\ \text{per second}$$

RELATING LINEAR AND ANGULAR VELOCITY

1. A saw has a circular blade with diameter 10 inches that rotates at $5{,}000$ revolutions per minute. Find the approximate linear velocity of the saw teeth (in ft/sec) as they contact the wood being cut.

Solution:

Find angular velocity.

$$\omega = 5{,}000 \frac{\text{rev}}{\text{min}}$$

$$\omega = 5{,}000 \frac{\text{rev}}{\text{min}} \times 2\pi \frac{\text{rad}}{\text{rev}} \times \frac{1 \text{ min}}{60 \text{ sec}}$$

$$\omega = 5{,}000 \times 2\pi \times \frac{1}{60} \times \frac{\text{rad}}{\text{sec}}$$

$$\omega = 166.67\pi \text{ rad/sec}$$

Because the diameter of the blade is 10 inches, its radius is 5 inches, so linear velocity is

$$v = \omega r$$

$$v = \left(166.67\pi \ \frac{\text{rad}}{\text{sec}}\right)\left(5 \text{ in } \frac{1 \text{ ft}}{12 \text{ in}}\right)$$

$$v = \left(166.67\pi \frac{\text{rad}}{\text{sec}}\right)\left(\frac{5 \text{ ft}}{12}\right)$$

$$v \approx 218.2 \text{ ft/sec}$$

■ 2. A car's tire has a radius of 12.5 inches and turns with an angular velocity of 84.5 radians per second. Find the approximate linear velocity of the car in miles per hour. (Use the fact that there are 12 inches in 1 foot, and approximately $5{,}280$ feet in 1 mile.)

Solution:

The radius is given as $r = 12.5$ in, and angular velocity is given as $\omega = 84.5$ rad/sec. We want the velocity of the car in miles per hour, so we'll start by finding inches traveled in an hour.

Substitute $\theta = \omega t$ into $s = r\theta$ to get

$$s = r\omega t$$

$$s = 12.5(84.5)(3{,}600)$$

$$s = 3{,}802{,}500 \text{ inches}$$

The wheel travelled $3{,}802{,}500$ inches in one hour, or

$$\frac{3{,}802{,}500}{12} = 316{,}875 \text{ ft}$$

or

$$\frac{316{,}875}{5{,}280} \approx 60 \text{ miles}$$

So the wheel was traveling at a speed of approximately 60 miles per hour.

■ 3. A bicycle tire with a diameter of 26 inches turns with an angular velocity of 2 radians per seconds. Find the distance traveled in 5 minutes by a point on the tire.

Solution:

Because the diameter of the tire is 26 inches, its radius is 13 inches, so linear velocity is

$$v = \omega r$$

$$v = \left(2\ \frac{\text{rad}}{\text{sec}}\right)(13 \text{ in})$$

$$v = 26 \text{ in/sec}$$

Find distance.

$$s = vt$$

$$s = \left(26\ \frac{\text{in}}{\text{sec}}\right)(5 \text{ min})\left(60\ \frac{\text{sec}}{\text{min}}\right)$$

$$s = 7{,}800 \text{ in}$$

4. A tire with a radius of 0.75 feet is rotating at 36 miles per hour. Find the angular velocity of a point on its rim, expressed in revolutions per minute.

Solution:

Since the radius is given in feet, we need to convert miles per hour to feet per hour.

$$v = \left(36\ \frac{\text{mi}}{\text{hr}}\right)\left(5{,}280\ \frac{\text{ft}}{\text{mi}}\right)$$

$$v = 190{,}080\ \frac{\text{ft}}{\text{hr}}$$

Linear velocity is $v = \omega r$, so

$$\omega = \frac{v}{r}$$

$$\omega = \frac{190{,}080\ \frac{\text{ft}}{\text{hr}}}{0.75\ \text{ft}}$$

$$\omega = 253{,}440\ \frac{\text{rad}}{\text{hr}}$$

Now we need to convert radians per hour to revolutions per minute.

$$\omega = \left(253{,}440\ \frac{\text{rad}}{\text{hr}}\right)\left(\frac{1\text{ rev}}{2\pi\text{ rad}}\right)\left(\frac{\text{hr}}{60\text{ min}}\right)$$

$$\omega = \frac{253{,}440}{2\pi(60)}\text{ revolutions per minute}$$

If we say $\pi \approx 3.14$, we get

$$\omega \approx 672.6\text{ revolutions per minute}$$

■ 5. The carousel at the county fair makes 3.5 revolutions per minute. The linear speed of a person riding inside the carousel is 2.9 ft/sec. How far is this person from the carousel's center?

Solution:

First we need to convert revolutions per minute to radians per second.

$$\omega = \left(3.5\ \frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi\text{ rad}}{1\text{ rev}}\right)\left(\frac{1\text{ min}}{60\text{ sec}}\right)$$

$$\omega = \frac{3.5(2\pi)}{60}\text{ radians per second}$$

$$\omega \approx 0.117\pi\text{ radians per second}$$

Linear velocity is $v = \omega r$, so

$$r = \frac{v}{\omega}$$

$$r = \frac{2.9}{0.117\pi}$$

$$r \approx 7.89 \text{ ft}$$

6. A disk is spinning at 27 rpm. If a fly is sitting 9 cm from the center of the disk, what is the angular velocity of the fly in radians/sec? What is the speed of the fly in cm/sec? After 2 min, how far has the fly traveled?

Solution:

First we need to convert the angular velocity from revolutions per minute to radians per second.

$$\omega = 27\frac{\text{rev}}{\text{min}}$$

$$\omega = \left(27\frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ sec}}\right)$$

$$\omega = \frac{27(2\pi)}{60} \text{ radians per second}$$

$$\omega \approx 0.9\pi \text{ radians per second}$$

Now we need to find the linear velocity of the fly.

$$v = r\omega$$

$$v \approx (9 \text{ cm})\left(\frac{0.9\pi}{\text{sec}}\right)$$

$$v \approx 9(0.9\pi) \text{ centimeters per second}$$

$$v \approx 25 \text{ centimeters per second}$$

To find how far the fly travels in 2 min, we use

$$s = vt$$

$$s = \left(25 \ \frac{\text{cm}}{\text{sec}}\right)(2 \text{ min})\left(60 \ \frac{\text{sec}}{\text{min}}\right)$$

$$s = 3{,}000 \text{ cm}$$

SKETCHING SINE AND COSINE

1. Sketch the graph of $y = 3\sin(\theta/2)$.

Solution:

Four points on the sine function are

$$(0,0) \qquad \left(\frac{\pi}{2},1\right) \qquad (\pi,0) \qquad \left(\frac{3\pi}{2},-1\right)$$

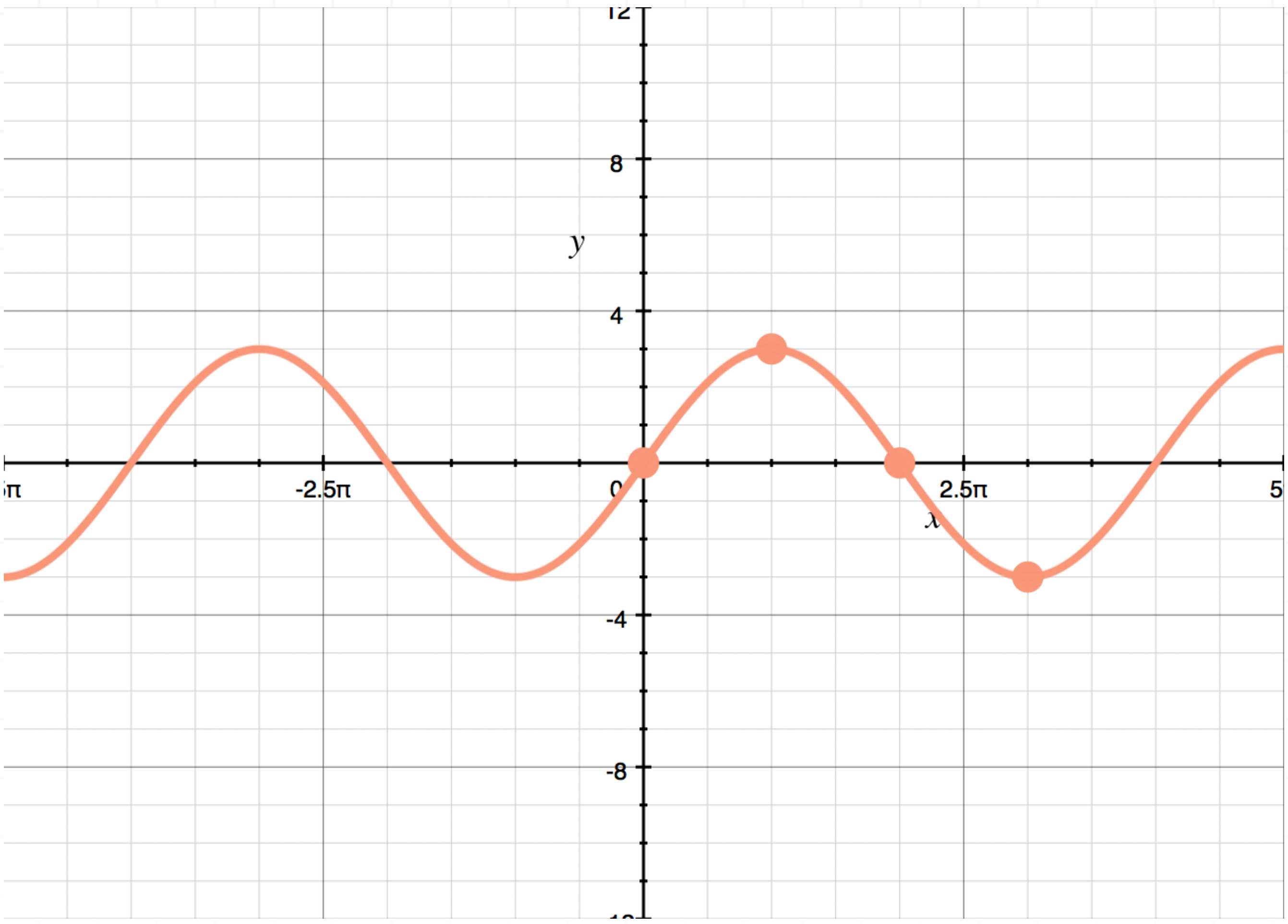
Because $b = 1/2$, the sine function gets stretched horizontally by a factor of 2, which means we can double the x-value in each point.

$$(0,0) \qquad (\pi,1) \qquad (2\pi,0) \qquad (3\pi,-1)$$

Then because $a = 3$, the function gets stretched vertically by a factor of 3, which means we multiply the y-value in each point by 3.

$$(0,0) \qquad (\pi,3) \qquad (2\pi,0) \qquad (3\pi,-3)$$

Then the graph with these four points is



■ 2. Sketch the graph of $y = 2.6\cos(3\theta)$.

Solution:

Four points on the cosine function are

$$(0,1) \qquad \left(\frac{\pi}{2},0\right) \qquad (\pi,-1) \qquad \left(\frac{3\pi}{2},0\right)$$

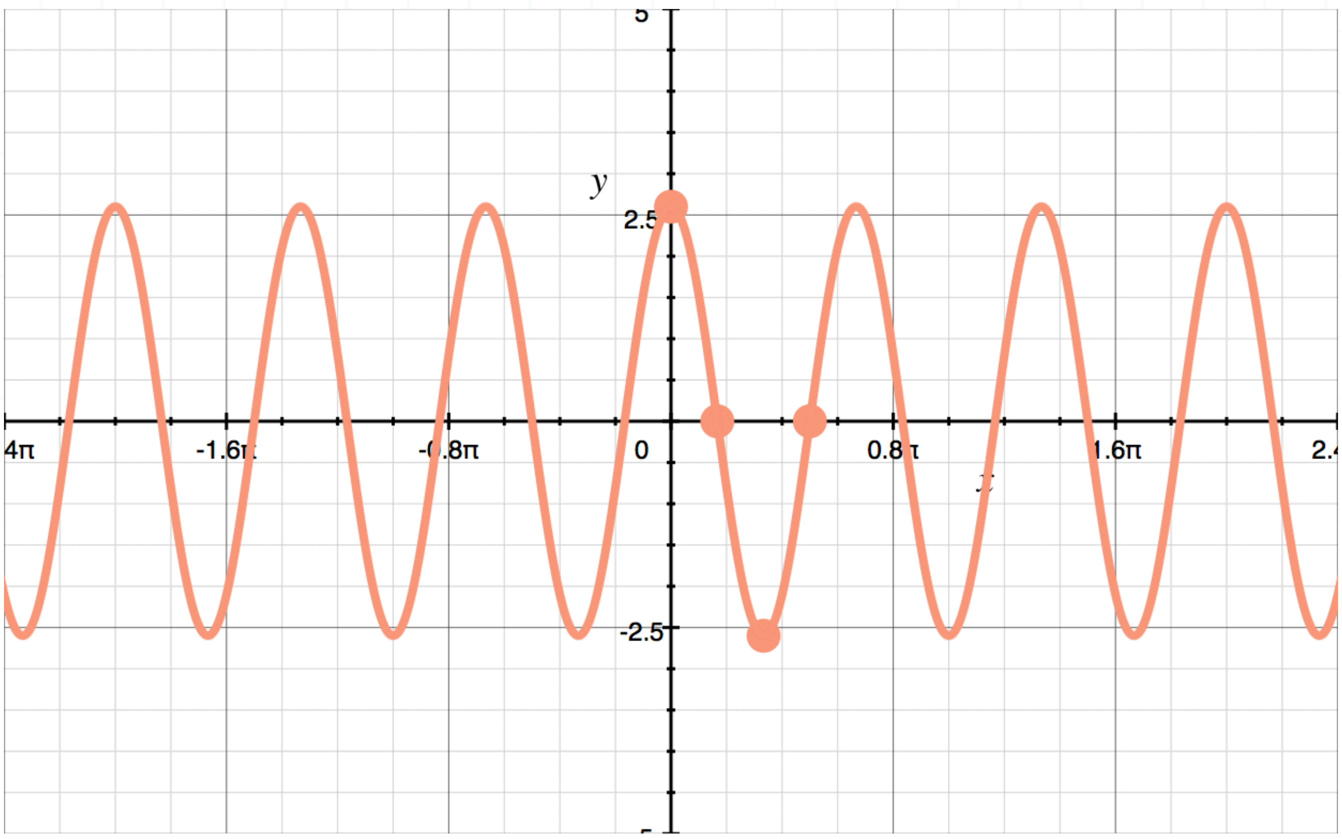
Because $b = 3$, the cosine function gets compressed horizontally by a factor of 3, which means we can divide the x-value in each point by 3.

$$(0,1) \qquad \left(\frac{\pi}{6},0\right) \qquad \left(\frac{\pi}{3},-1\right) \qquad \left(\frac{\pi}{2},0\right)$$

Then because $a = 2.6$, the function gets stretched vertically by a factor of 2.6, which means we multiply the y-value in each point by 2.6.

$$(0,2.6) \qquad \left(\frac{\pi}{6},0\right) \qquad \left(\frac{\pi}{3},-2.6\right) \qquad \left(\frac{\pi}{2},0\right)$$

Then the graph with these four points is



■ 3. Sketch the graph of $y = -4\cos(\theta/3)$.

Solution:

Four points on the cosine function are

$$(0,1) \qquad \left(\frac{\pi}{2},0\right) \qquad (\pi,-1) \qquad \left(\frac{3\pi}{2},0\right)$$

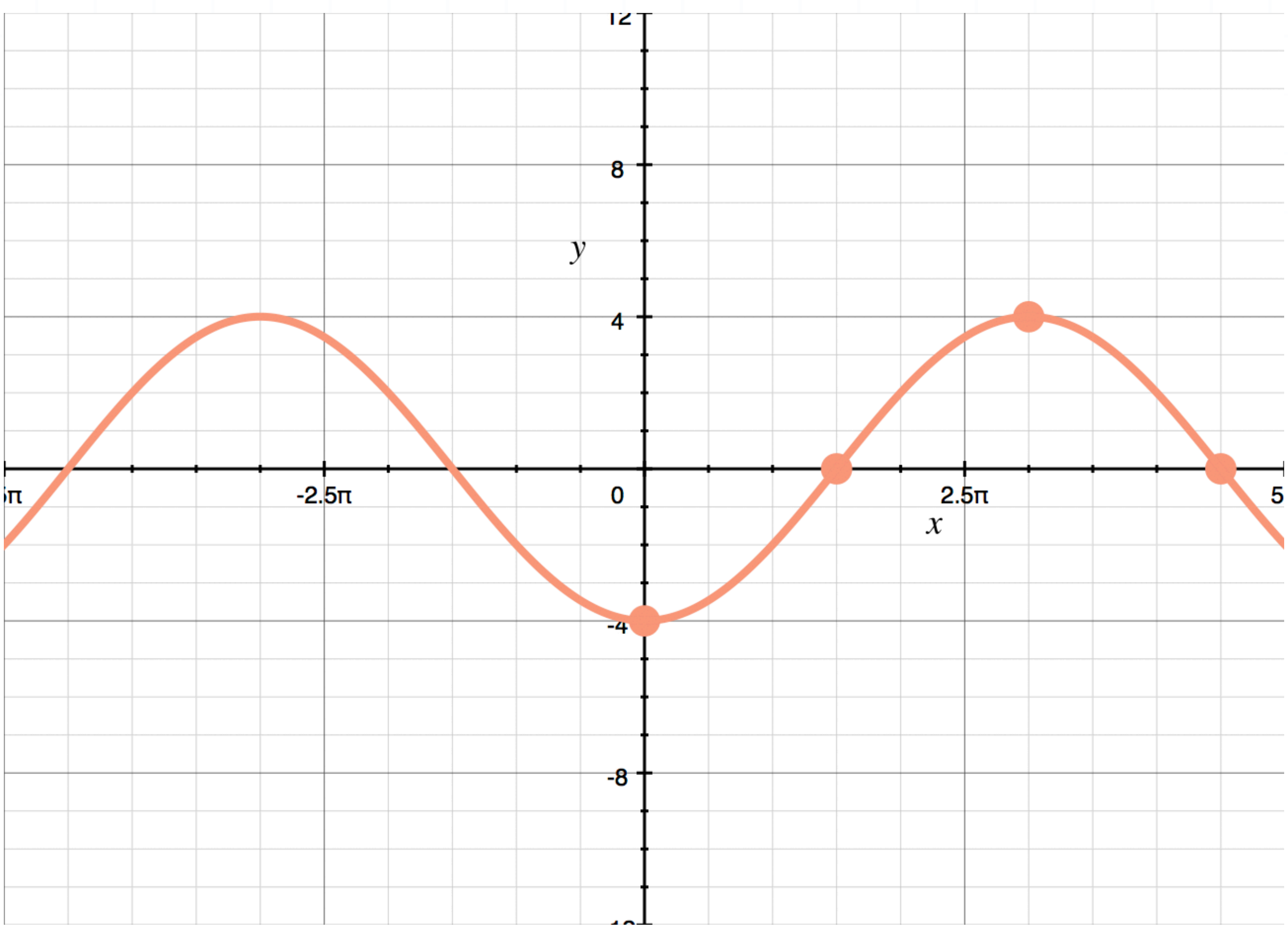
Because $b = 1/3$, the cosine function gets stretched horizontally by a factor of 3, which means we can triple the x-value in each point.

$$(0,1) \qquad \left(\frac{3\pi}{2},0\right) \qquad (3\pi,-1) \qquad \left(\frac{9\pi}{2},0\right)$$

Then because $a = -4$, the function gets stretched vertically by a factor of 4, but then flipped over the x-axis, which means we multiply the y-value in each point by -4.

$$(0,-4) \qquad \left(\frac{3\pi}{2},0\right) \qquad (3\pi,4) \qquad \left(\frac{9\pi}{2},0\right)$$

Then the graph with these four points is



■ 4. On the same set of axes, graph $y = 2\cos\theta$ and $y = \sin 2\theta$.

Solution:

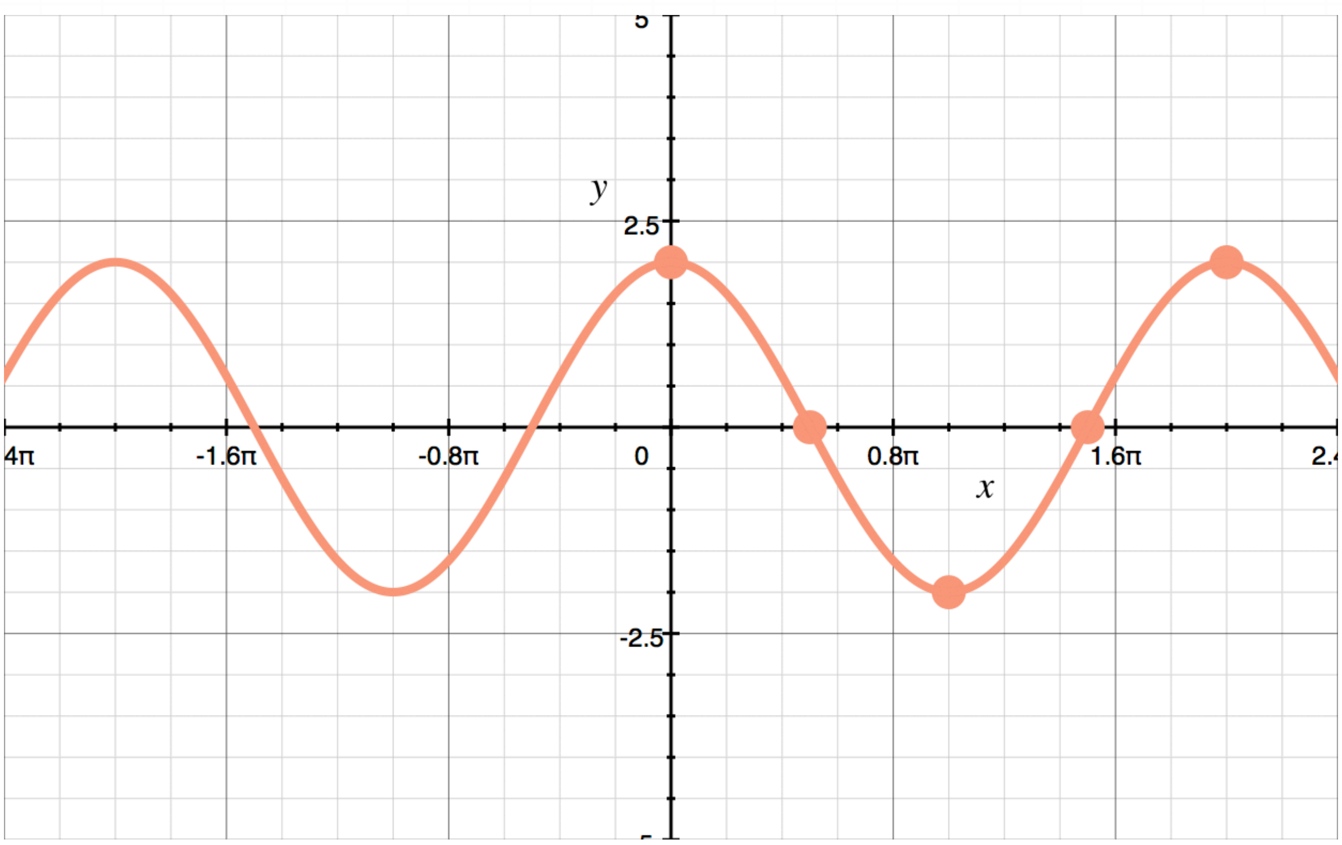
Five points on the cosine function are

$$(0,1) \qquad \left(\frac{\pi}{2},0\right) \qquad (\pi,-1) \qquad \left(\frac{3\pi}{2},0\right) \qquad (2\pi,1)$$

Because $a=2$, the function gets stretched vertically by a factor of 2, which means we multiply the y-value in each point by 2.

$$(0,2) \qquad \left(\frac{\pi}{2},0\right) \qquad (\pi,-2) \qquad \left(\frac{3\pi}{2},0\right) \qquad (2\pi,2)$$

Then the graph with these five points is



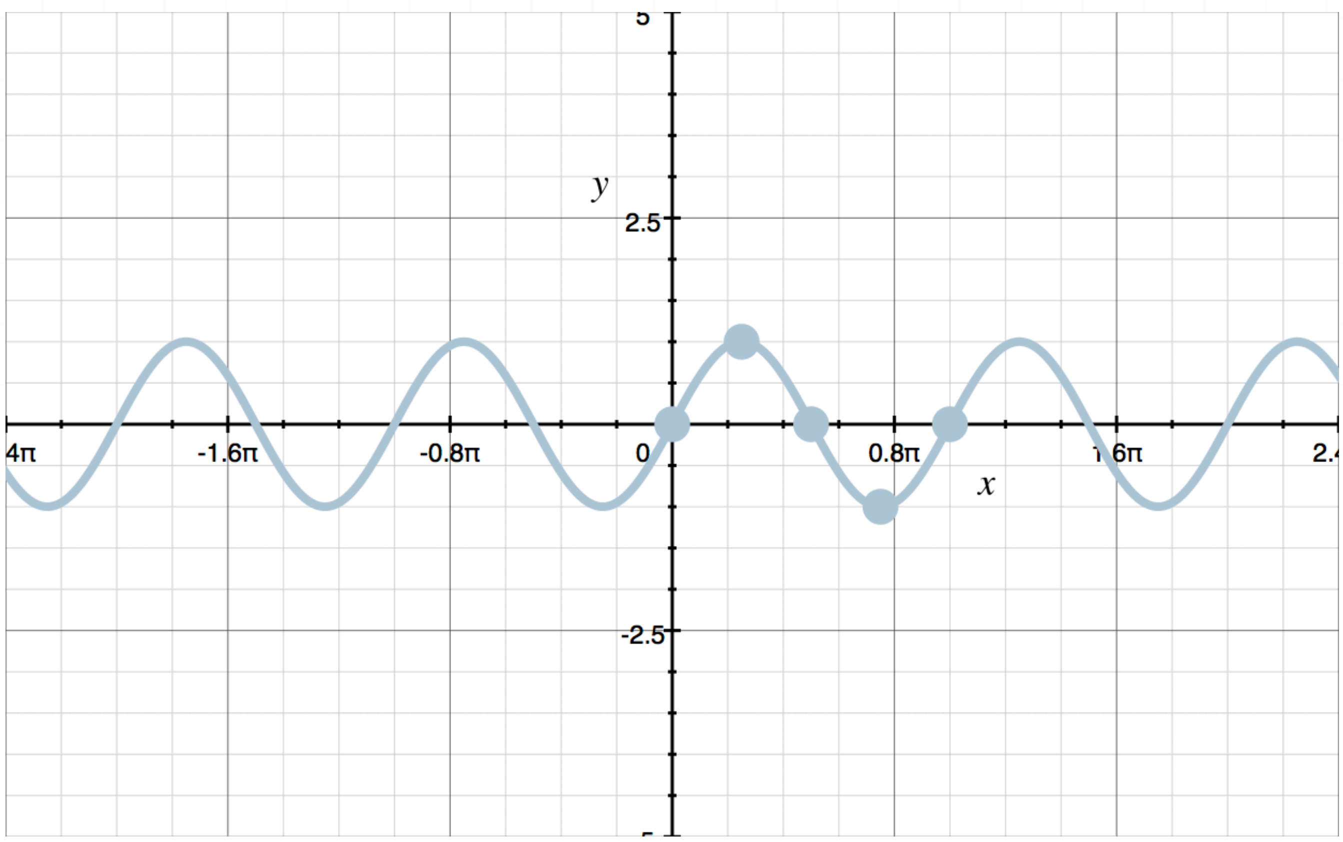
Five points on the sine function are

$$(0,0) \qquad \left(\frac{\pi}{2},1\right) \qquad (\pi,0) \qquad \left(\frac{3\pi}{2},-1\right) \qquad (2\pi,0)$$

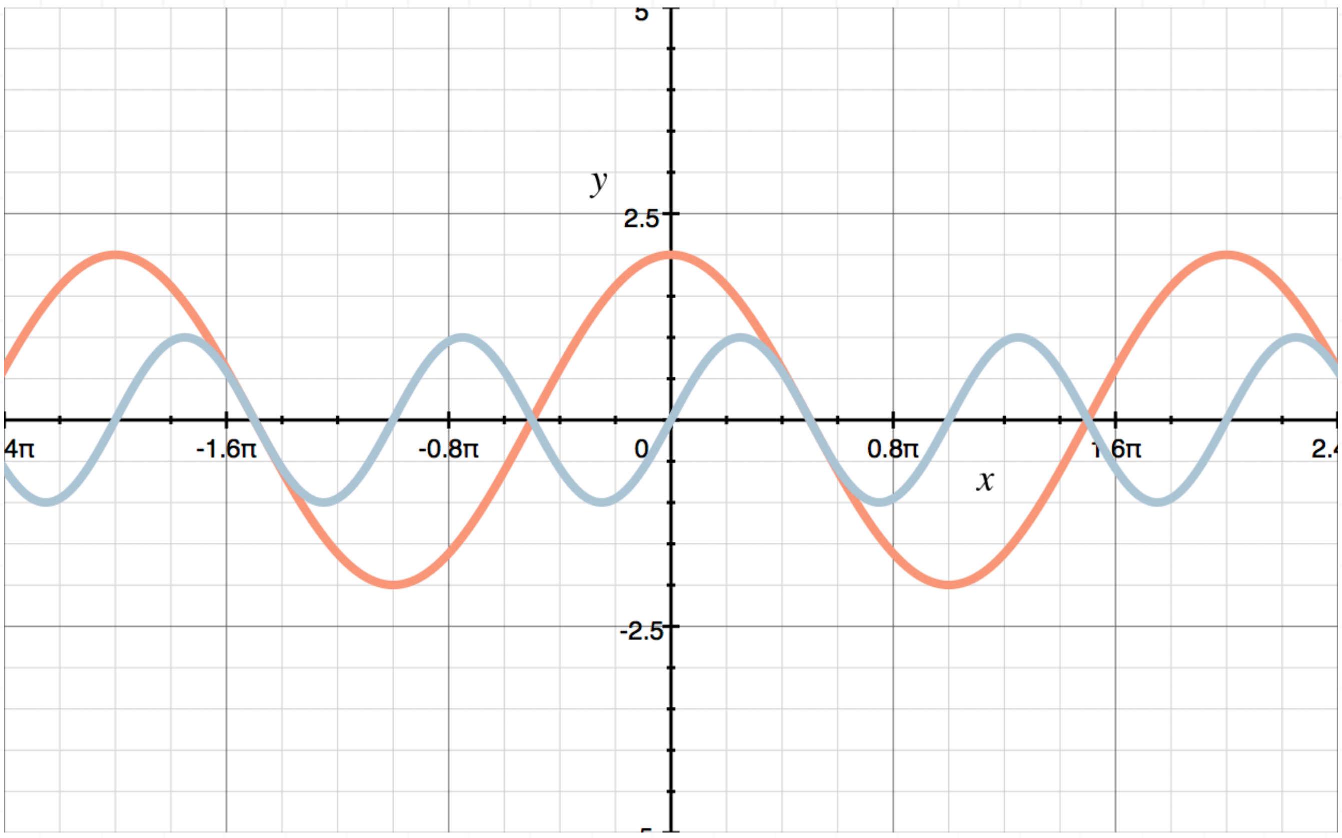
Because $b=2$, the sine function gets compressed horizontally by a factor of 2, which means we can divide the x-value in each point by 2.

$$(0,0) \qquad \left(\frac{\pi}{4},1\right) \qquad \left(\frac{\pi}{2},0\right) \qquad \left(\frac{3\pi}{4},-1\right) \qquad (\pi,0)$$

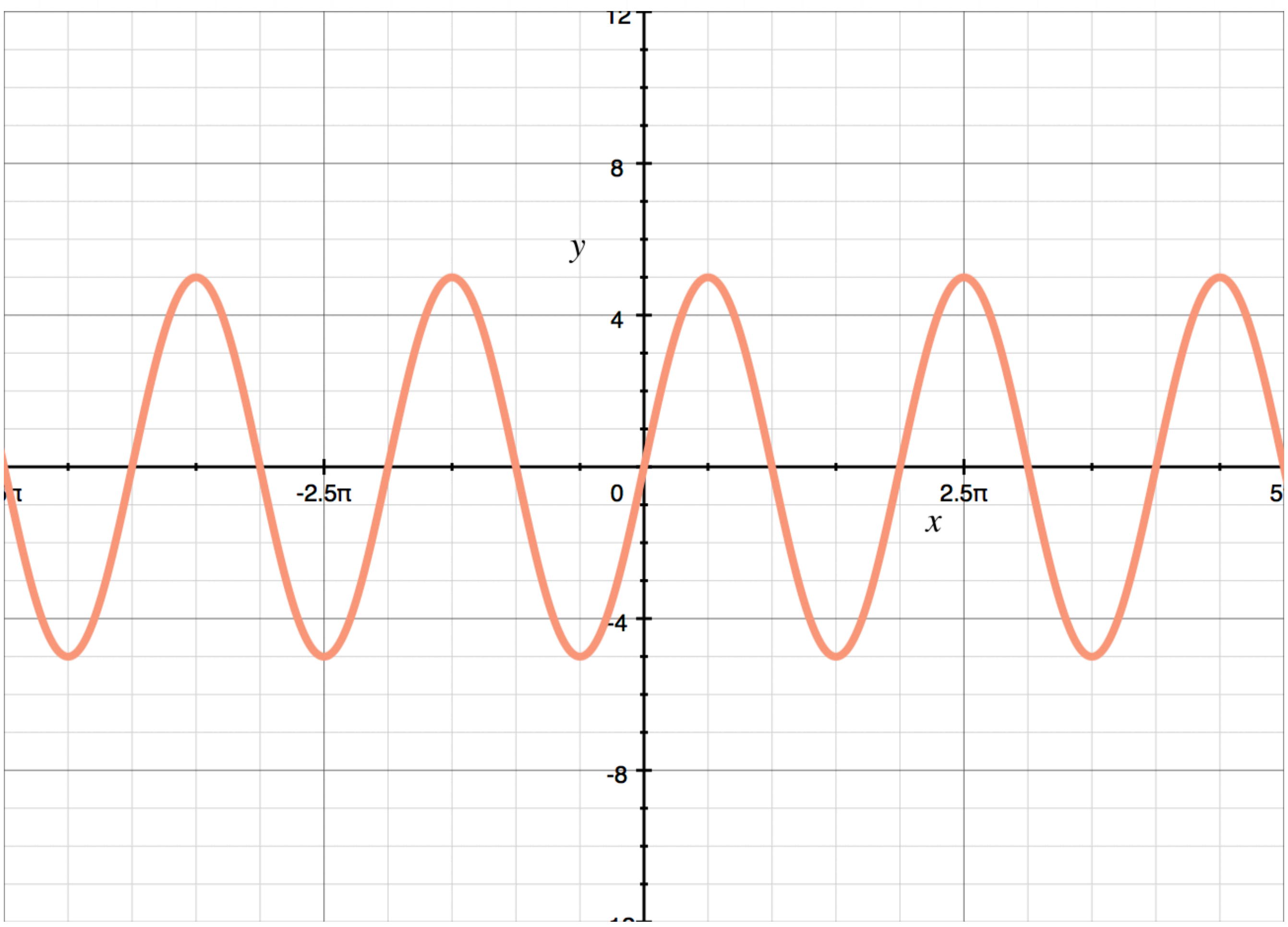
Then the graph with these five points is



Then on the same set of axes, the graphs $y=2\cos\theta$ and $y=\sin 2\theta$ will be



5. Which function is represented by the curve?



Solution:

Because the value of the function at $\theta = 0$ is 0, the curve can't be the graph of a cosine function. So we'll only need to determine which sine function the graph represents.

Five points on the sine function are

$$(0,0) \qquad \left(\frac{\pi}{2},1\right) \qquad (\pi,0) \qquad \left(\frac{3\pi}{2},-1\right) \qquad (2\pi,0)$$

From the graph we see the points

$$(0,0) \qquad \left(\frac{\pi}{2},5\right) \qquad (\pi,0) \qquad \left(\frac{3\pi}{2},-5\right) \qquad (2\pi,0)$$

Each point on the graph has its y-value multiplied by 5, in comparison to the points on the standard sine function. So the function in the graph is the standard sine function, stretched vertically by a factor of 5, therefore $a = 5$.

So the function $y = 5\sin\theta$ is the only one that could be represented by the graph.

6. Graph $y = -4\cos(\theta/2)$ and $y = 3\sin\theta$ on the same set of axes.

Solution:

Five points on the cosine function are

$$(-\pi,-1) \qquad \left(-\frac{\pi}{2},0\right) \qquad (0,1) \qquad \left(\frac{\pi}{2},0\right) \qquad (\pi,-1)$$

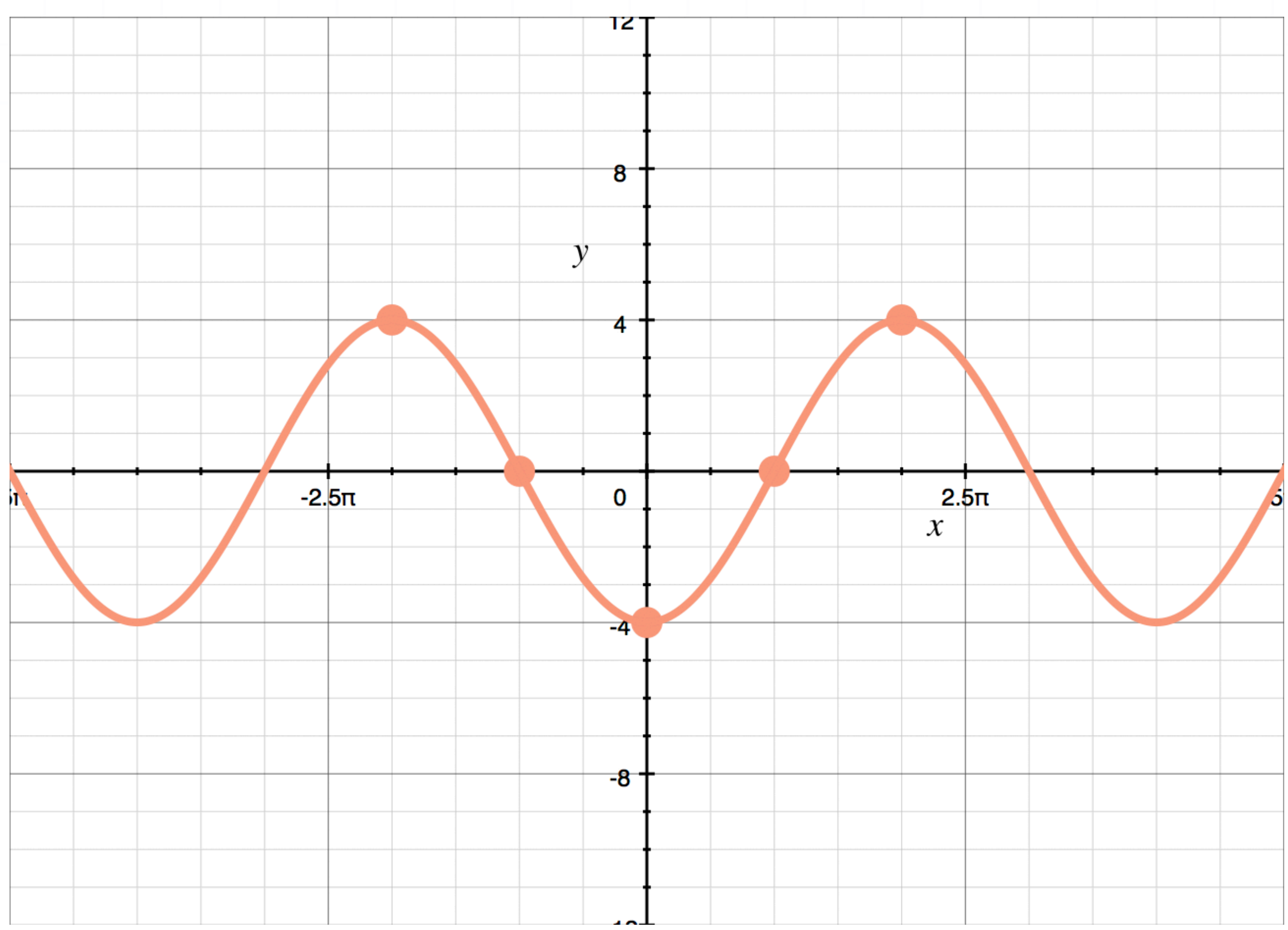
Because $b=1/2$, the cosine function gets stretched horizontally by a factor of 2, which means we can double the x-value in each point.

$$(-2\pi,-1) \qquad (-\pi,0) \qquad (0,1) \qquad (\pi,0) \qquad (2\pi,-1)$$

Then because $a=-4$, the function gets stretched vertically by a factor of 4, but then flipped over the x-axis, which means we multiply the y-value in each point by -4.

$$(-2\pi,4) \qquad (-\pi,0) \qquad (0,-4) \qquad (\pi,0) \qquad (2\pi,4)$$

Then the graph with these five points is



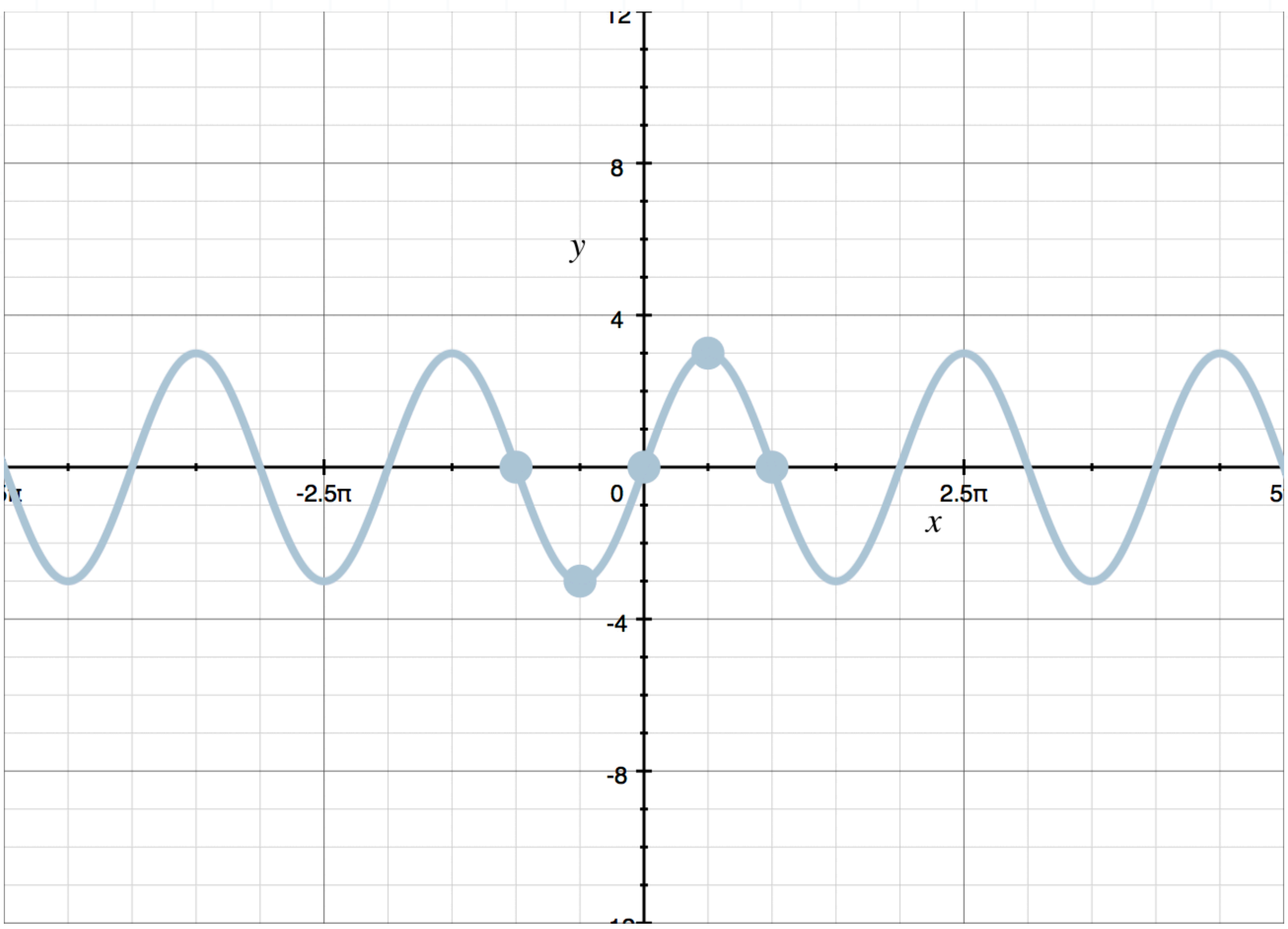
Five points on the sine function are

$$(-\pi,0) \qquad \left(-\frac{\pi}{2},-1\right) \qquad (0,0) \qquad \left(\frac{\pi}{2},1\right) \qquad (\pi,0)$$

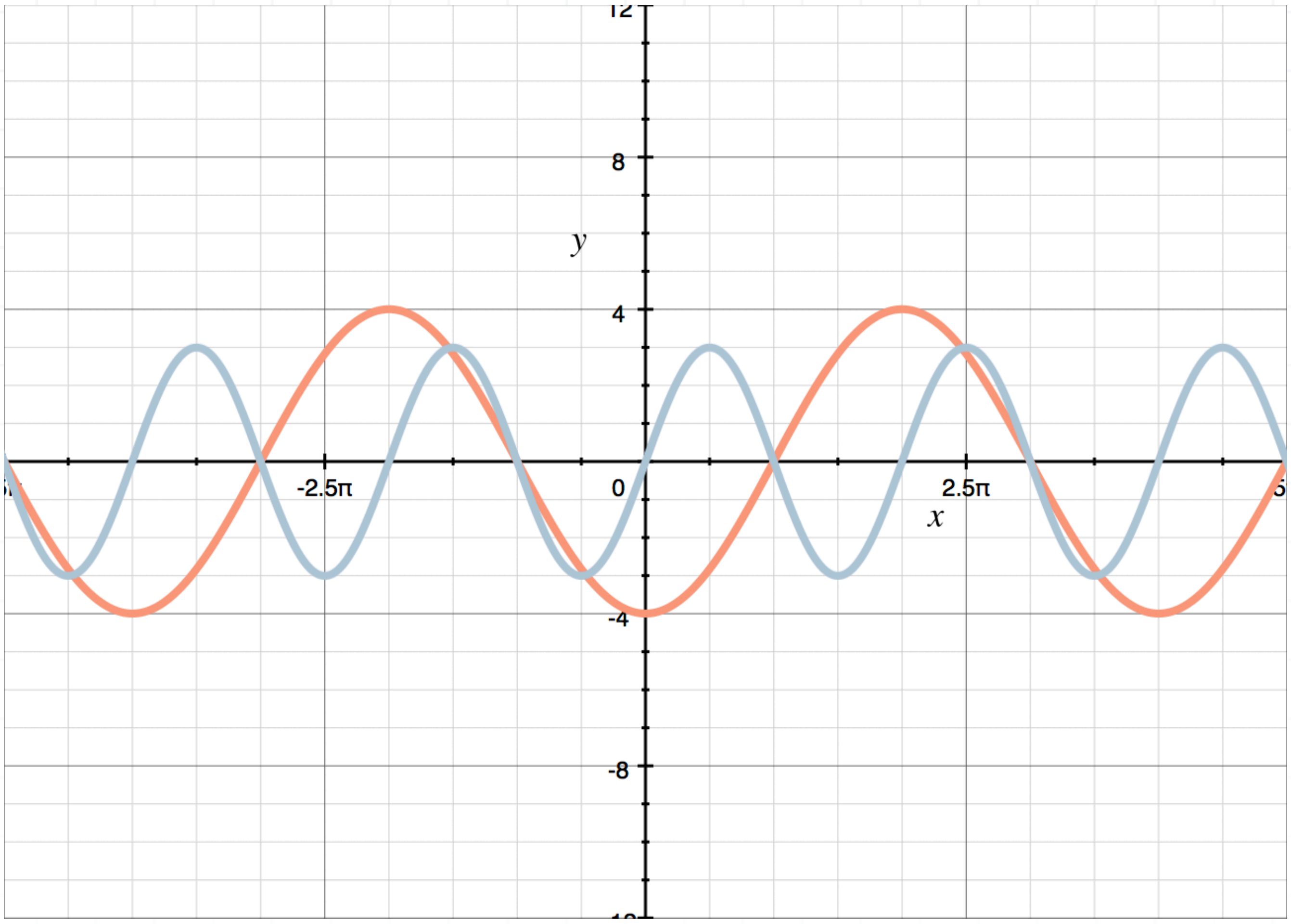
Because $a=3$, the function gets stretched vertically by a factor of 3, which means we multiply the y-value in each point by 3.

$$(-\pi,0) \qquad \left(-\frac{\pi}{2},-3\right) \qquad (0,0) \qquad \left(\frac{\pi}{2},3\right) \qquad (\pi,0)$$

Then the graph with these five points is



Then on the same set of axes, the graphs $y=-4\cos(\theta/2)$ and $y=3\sin\theta$ will be


12
8
y
4
-2.5π
0
2.5π
x
-4
-8

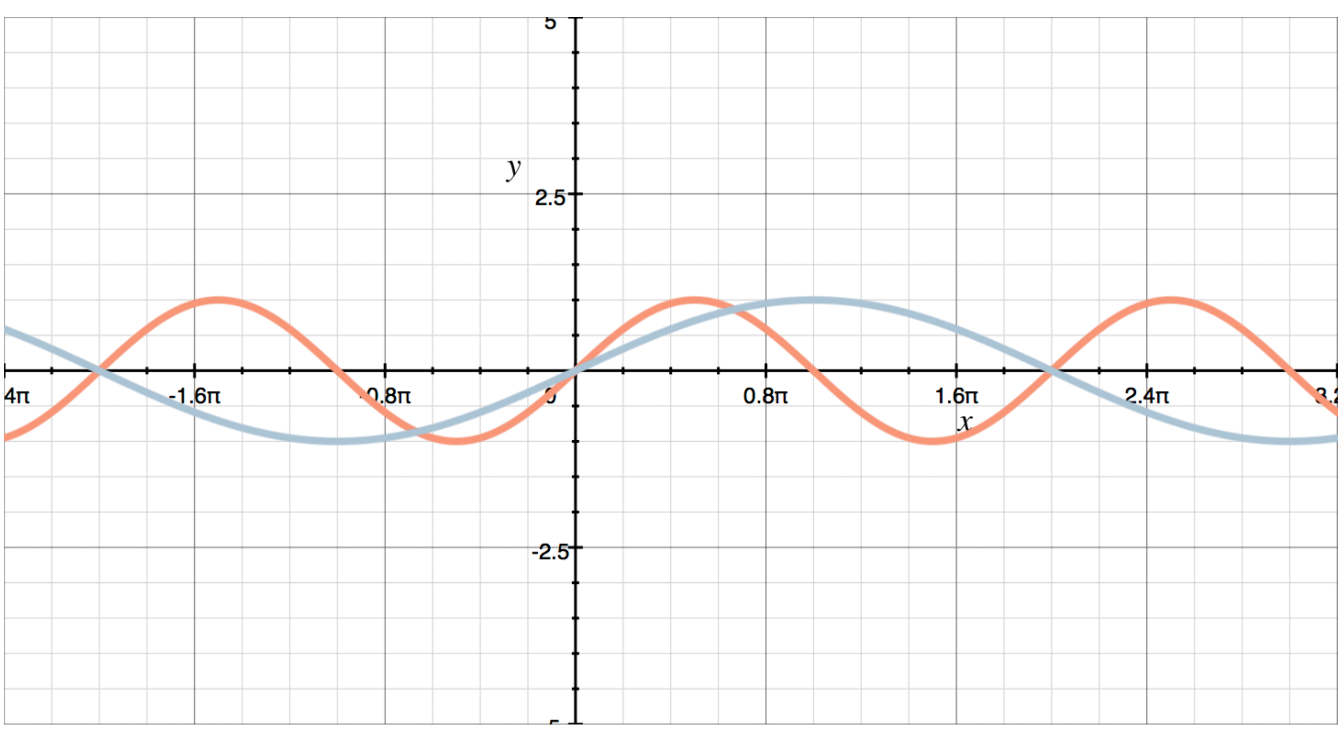
SKETCHING COSECANT AND SECANT

■ 1. Sketch the graph of $y = \csc(\theta/2)$.

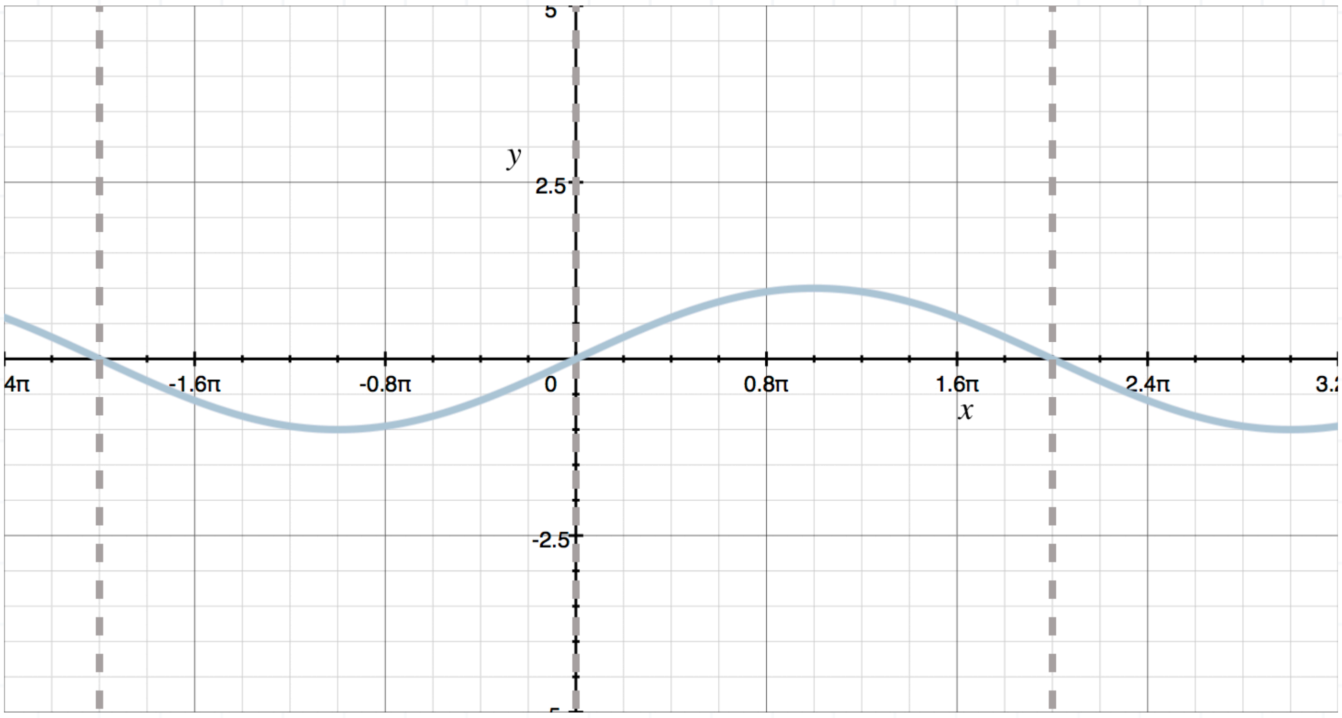
Solution:

Because the reciprocal function of cosecant is sine, we want to start by replacing cosecant with sine in the function we've been given. In other words, the corresponding function is $y = \sin(\theta/2)$.

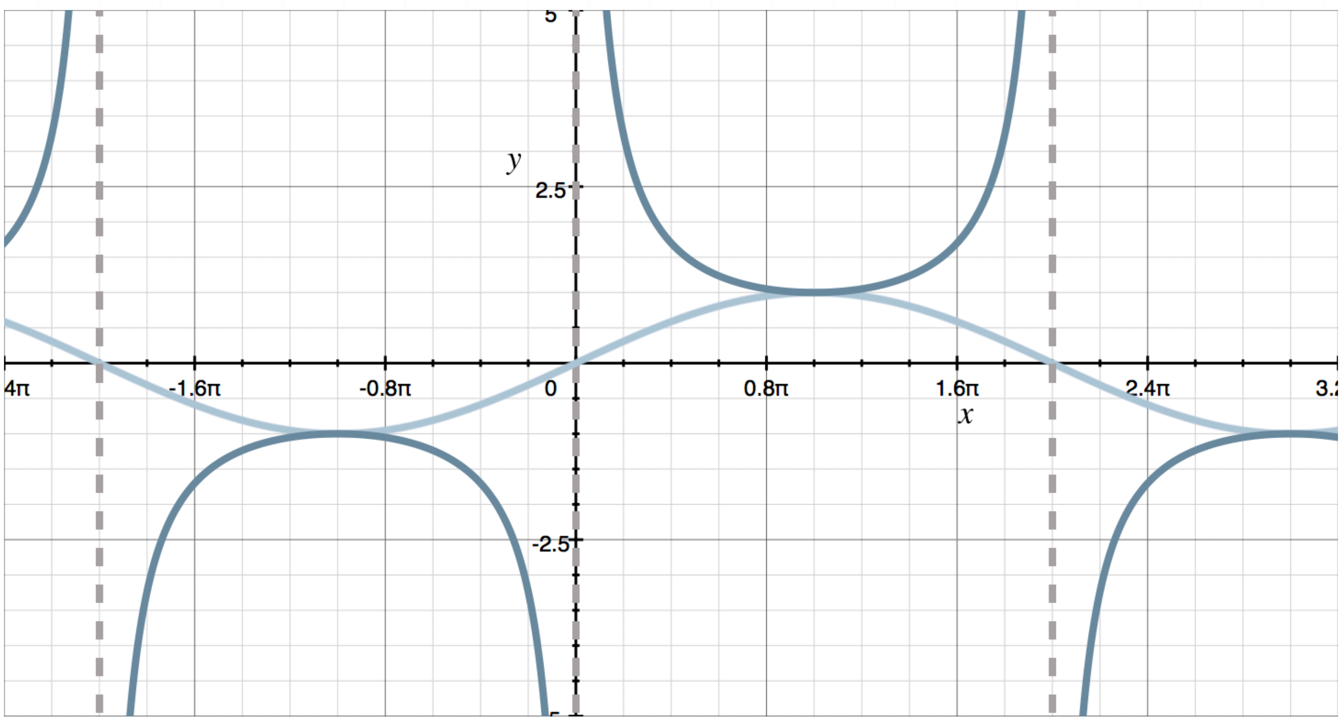
To graph $y = \sin(\theta/2)$, we see that $b = 1/2$, so we'll stretch $y = \sin\theta$ horizontally by a factor of 2, which means we can double the x-values in the coordinate points along $y = \sin\theta$. If we sketch $y = \sin\theta$ in red and $y = \sin(\theta/2)$ in blue, we get



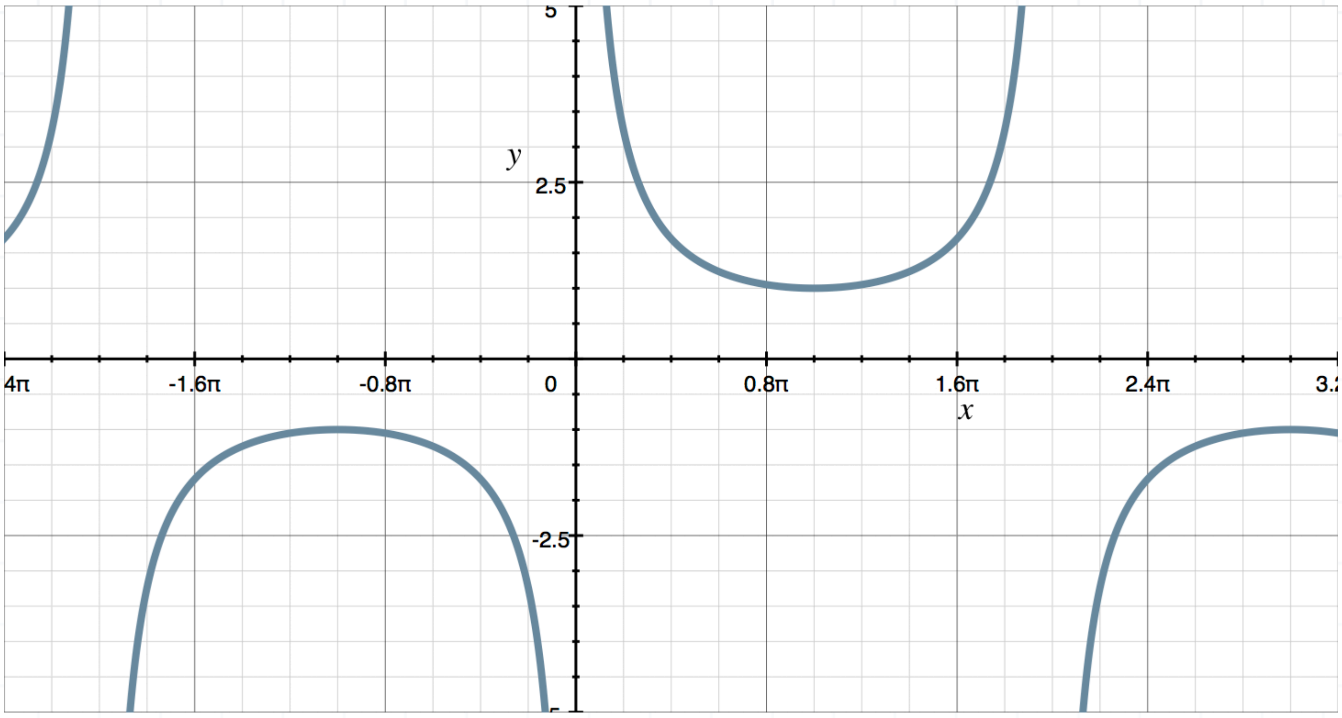
Then we'll sketch in vertical asymptotes at the midline of $y = \sin(\theta/2)$.



Sketch in the U-shapes for $y = \csc(\theta/2)$.



Finally, we'll take away the sine curve and the vertical asymptotes to get the sketch of $y = \csc(\theta/2)$.

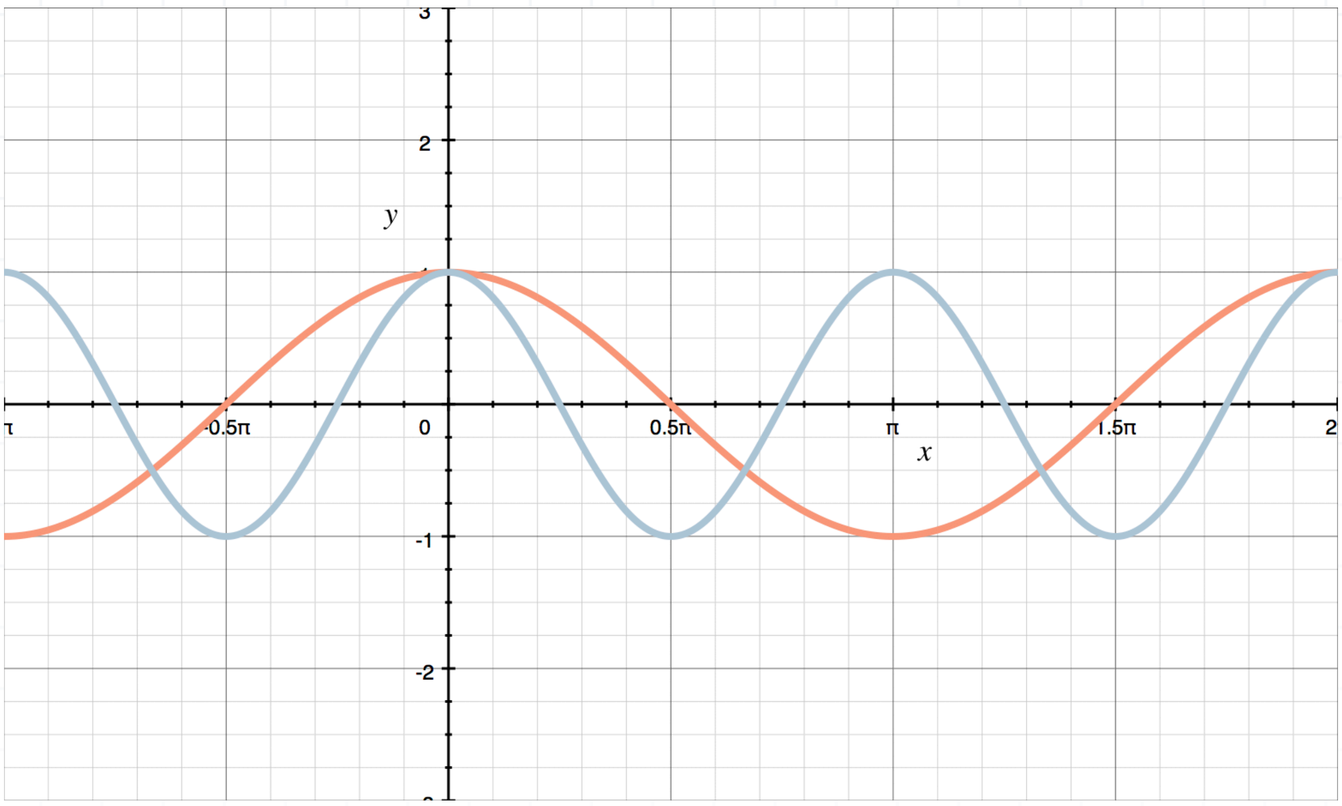


2. Sketch the graph of $y = -\sec(2\theta)$.

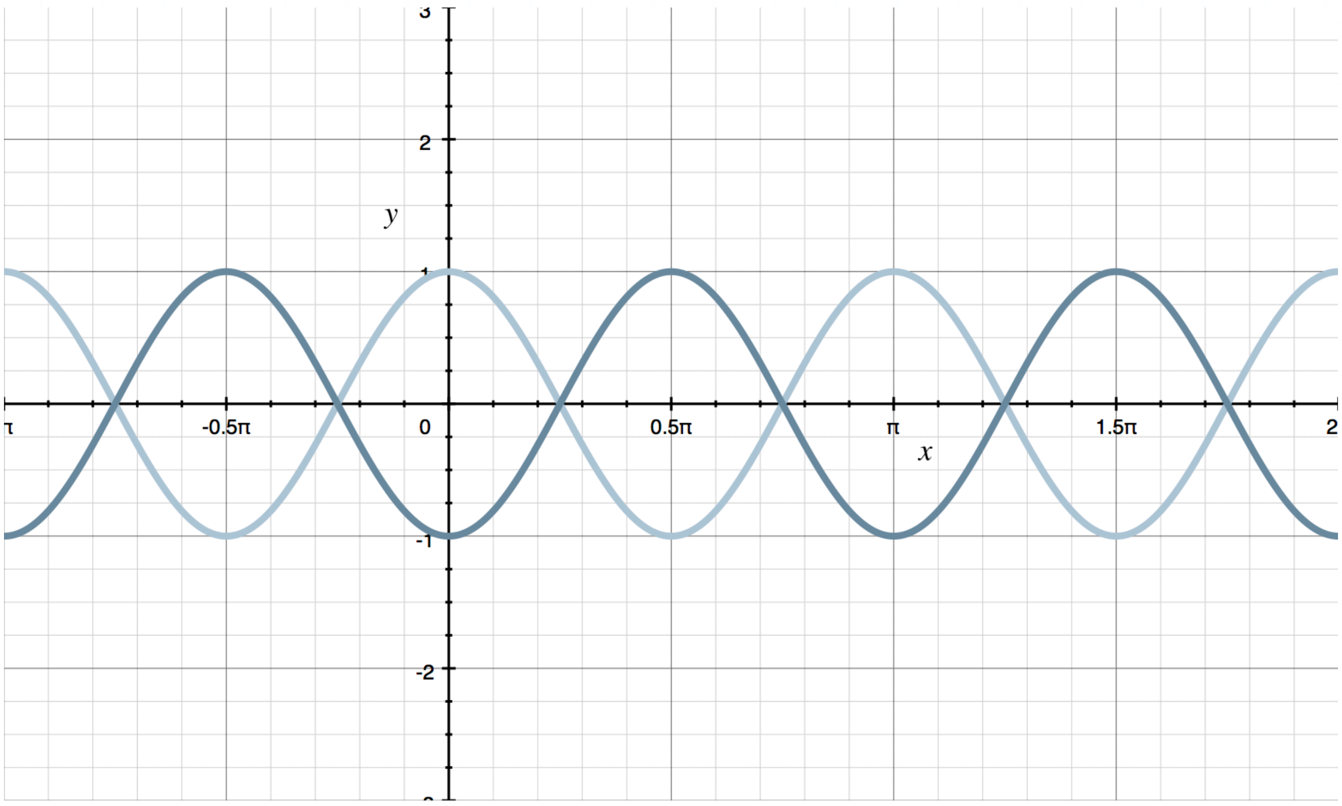
Solution:

Because the reciprocal function of secant is cosine, we want to start by replacing secant with cosine in the function we've been given. In other words, the corresponding function is $y = -\cos(2\theta)$.

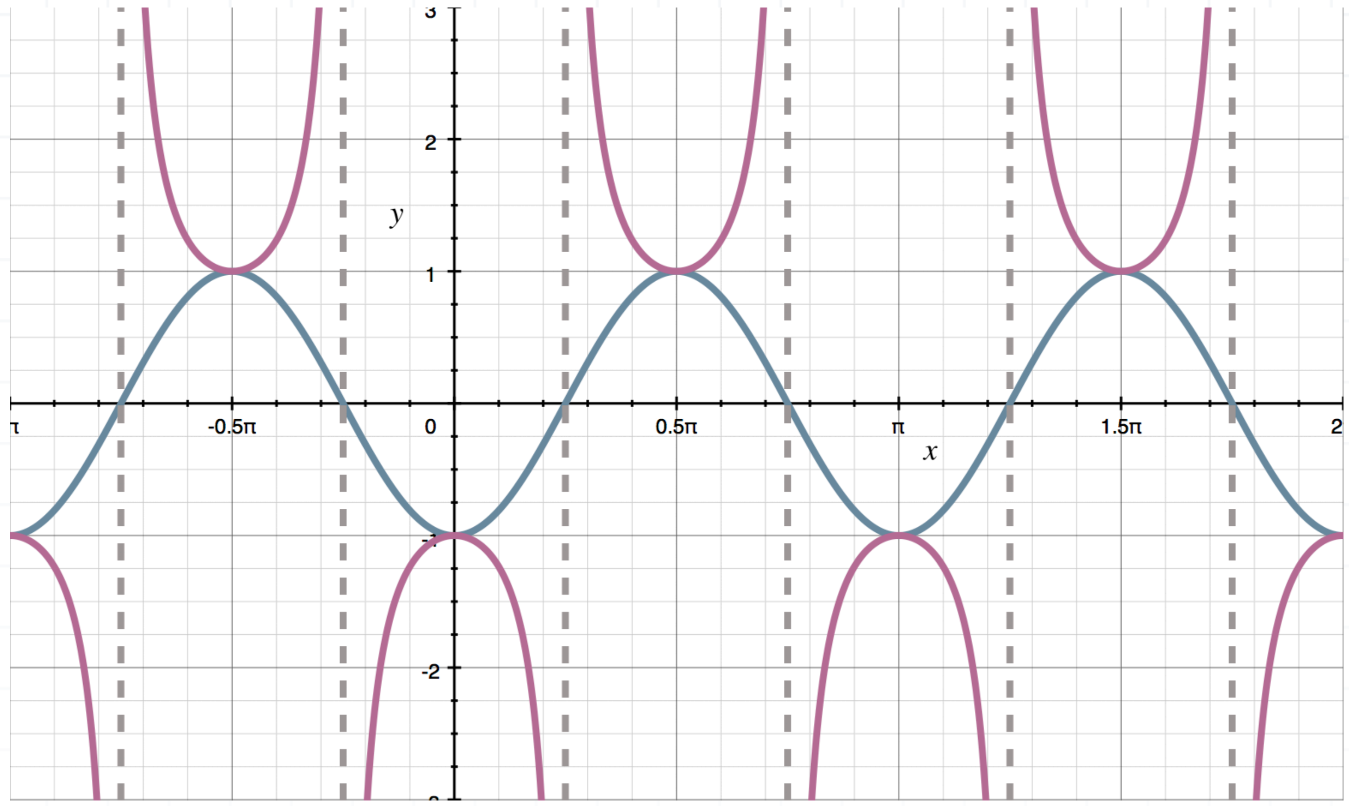
To graph $y = -\cos(2\theta)$, we see that $b = 2$, so we'll compress $y = \cos\theta$ horizontally by a factor of 2, which means we can halve the x-values in the coordinate points along $y = \cos\theta$. If we sketch $y = \cos\theta$ in red and $y = \cos 2\theta$ in blue, we get



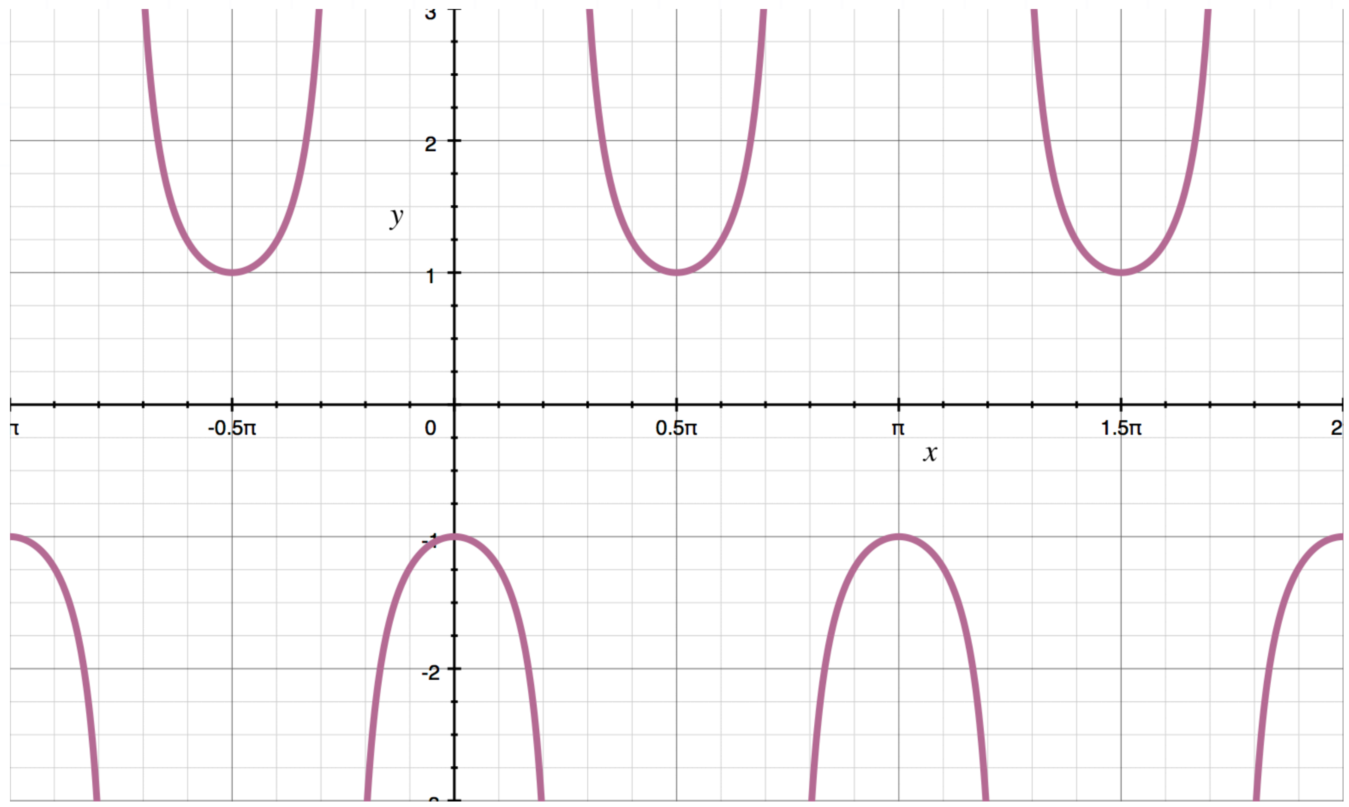
Then because $a = -1$, we'll flip the graph over the x-axis, which means we multiply the y-value in each point by -1. So if we sketch $y = \cos 2\theta$ in blue and $y = -\cos(2\theta)$ in dark blue, we get



Then we'll sketch in vertical asymptotes at the midline of $y = -\sec(2\theta)$ and add the U-shapes for $y = -\sec(2\theta)$.



Finally, we'll take away the sine curve and the vertical asymptotes to get the sketch of $y = -\sec(2\theta)$.

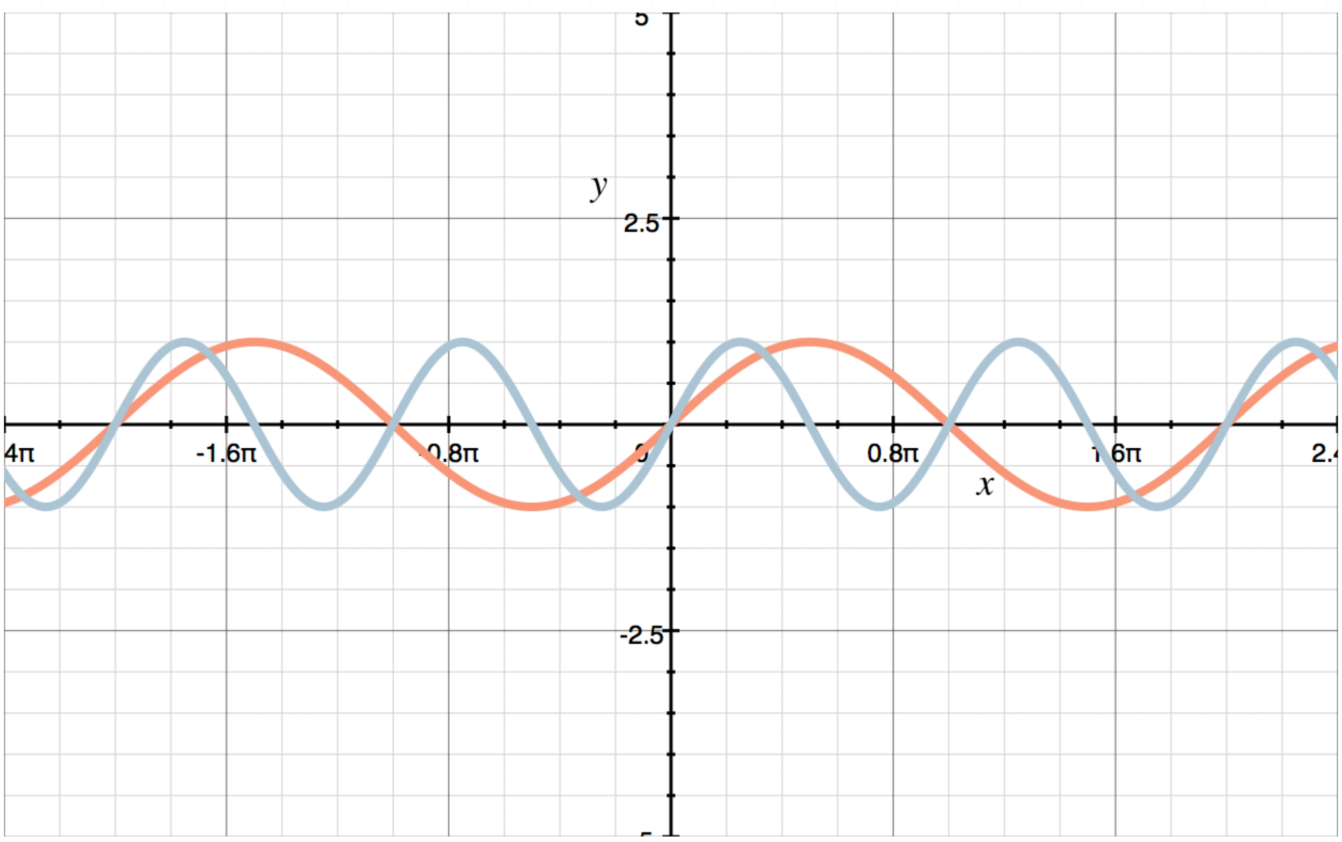


■ 3. Sketch the graph of $y = 5\csc(2\theta)$.

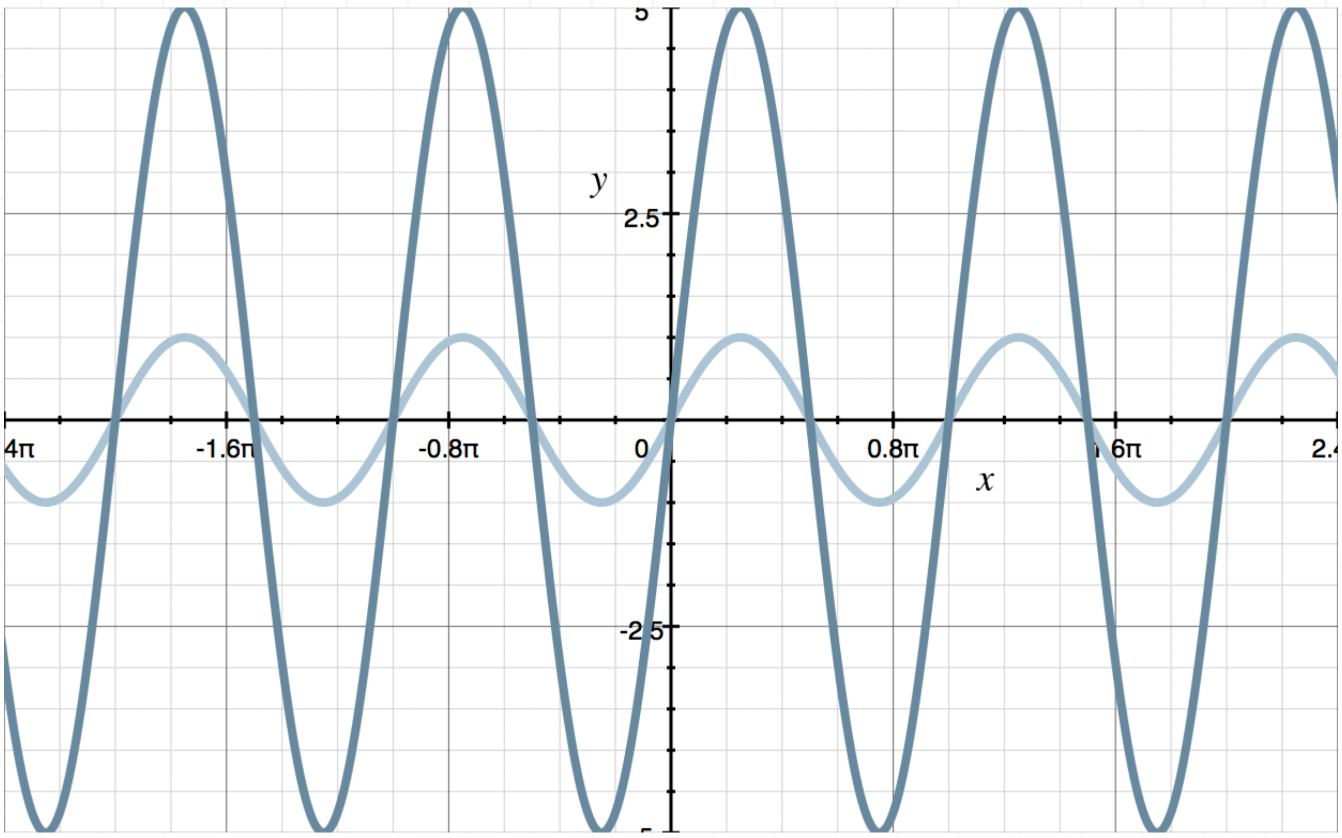
Solution:

Because the reciprocal function of cosecant is sine, we want to start by replacing cosecant with sine in the function we've been given. In other words, the corresponding function is $y = 5\sin(2\theta)$.

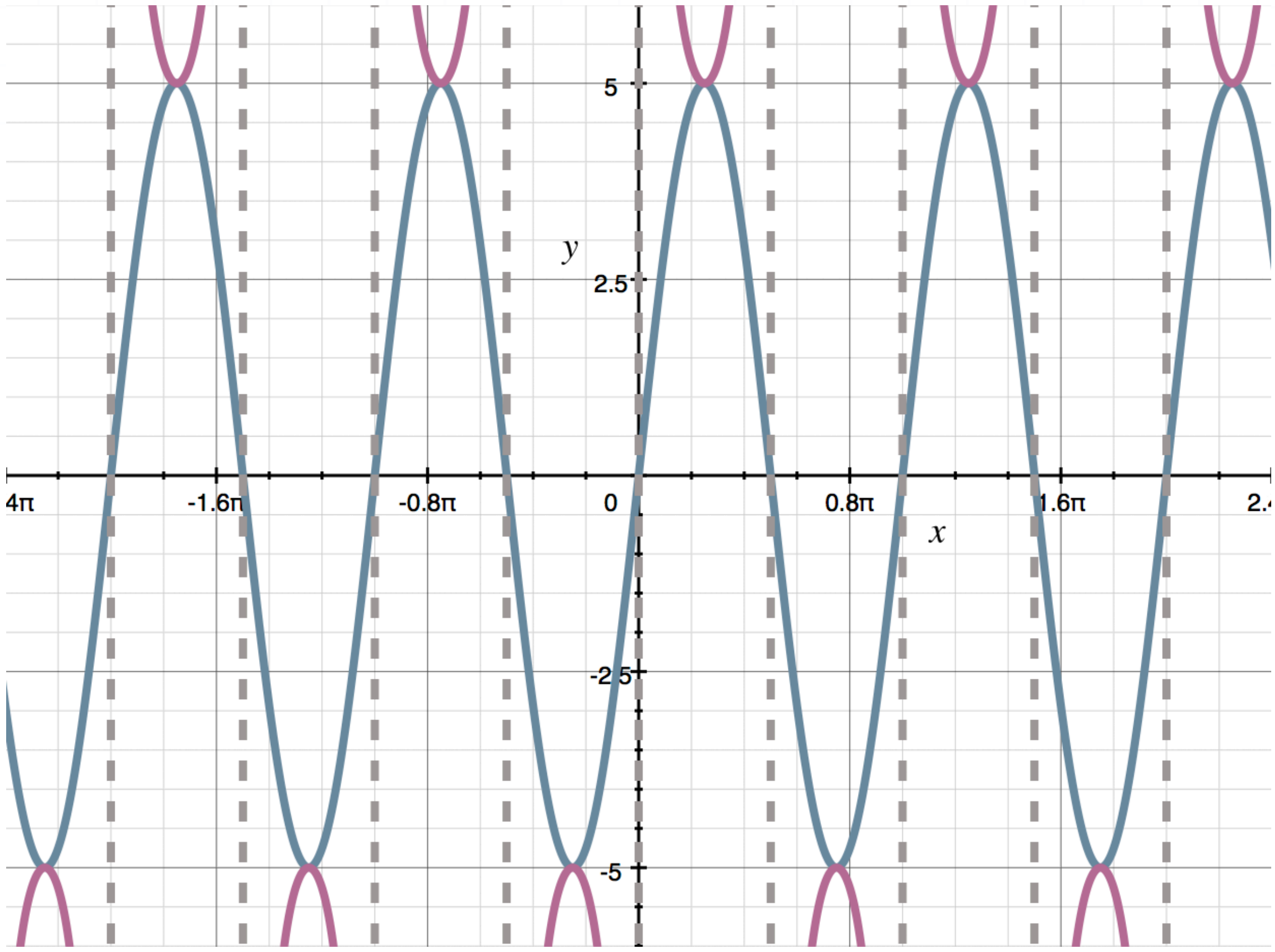
To graph $y = 5\sin(2\theta)$, we see that $b = 2$, so we'll compress $y = \sin\theta$ horizontally by a factor of 2, which means we can halve the x-values in the coordinate points along $y = \sin\theta$. If we sketch $y = \sin\theta$ in red and $y = \sin 2\theta$ in blue, we get



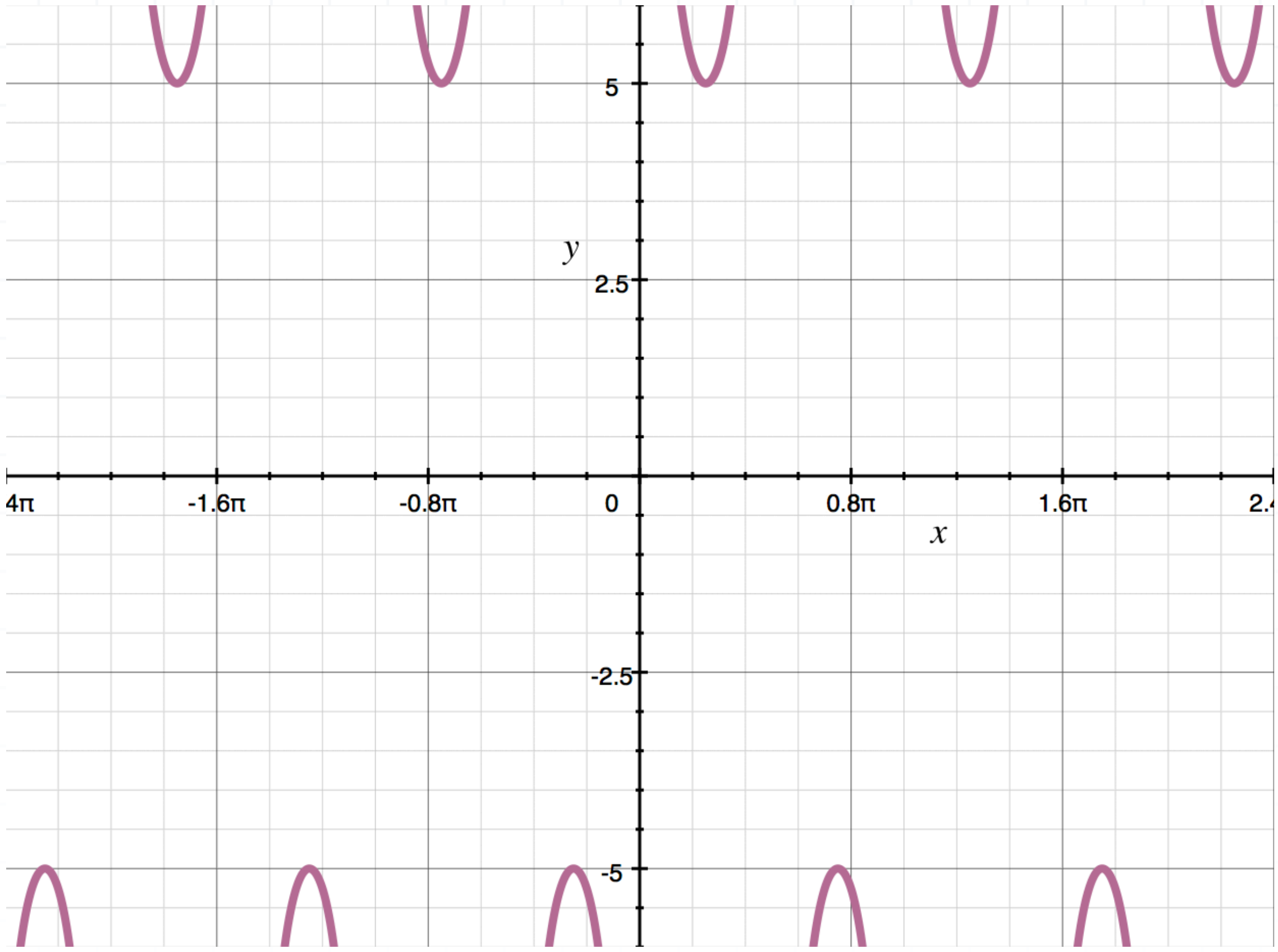
Then because $a = 5$, we'll vertically stretch the sine curve by multiplying all the y-values by 5. If we sketch $y = \sin 2\theta$ in blue and $y = 5\sin 2\theta$ in dark blue, we get



Then we'll sketch in vertical asymptotes at the midline of $y = 5\sin 2\theta$ and add the U-shapes for $y = 5\csc(2\theta)$.



Finally, we'll take away the sine curve and the vertical asymptotes to get the final sketch of $y = 5\csc(2\theta)$.

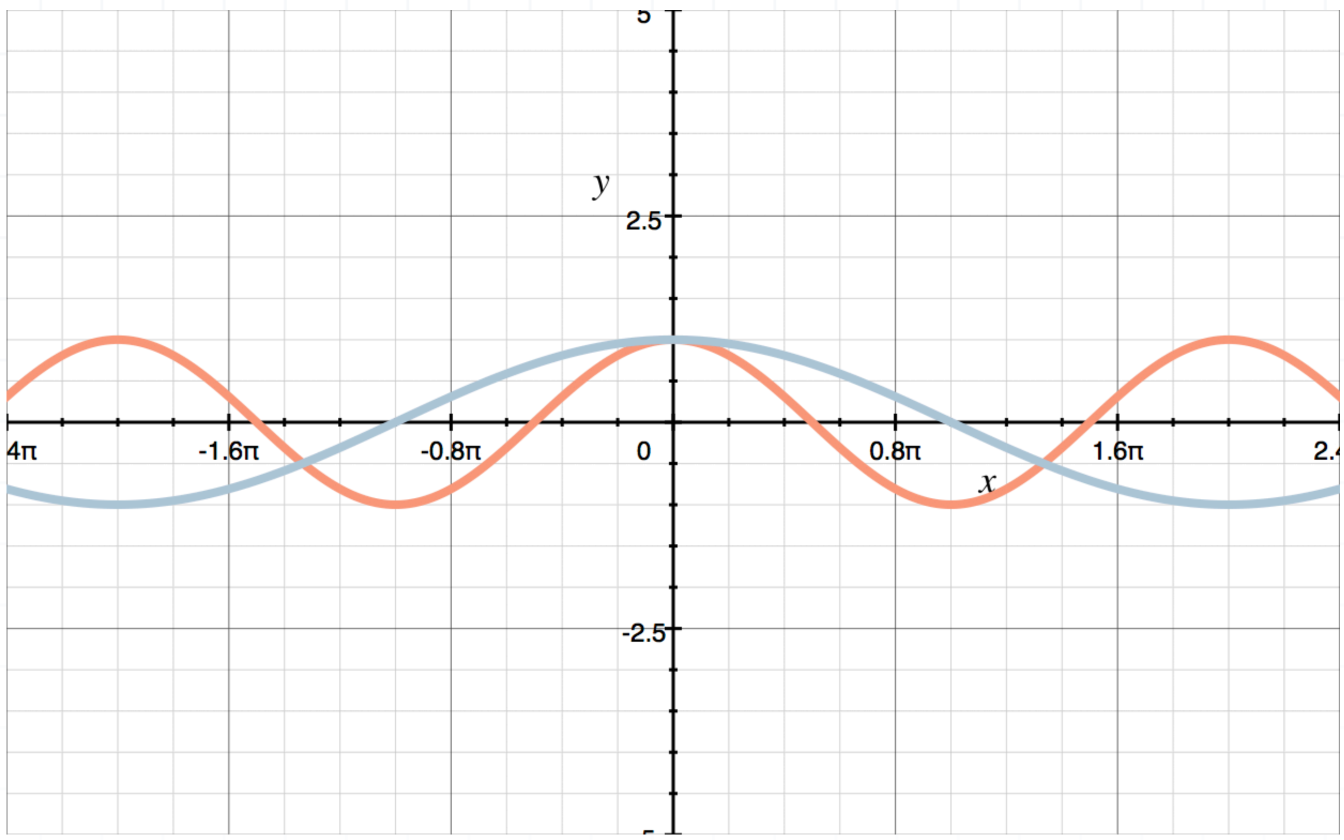


■ 4. Sketch the graph of $y = (1/4)\sec(\theta/2)$.

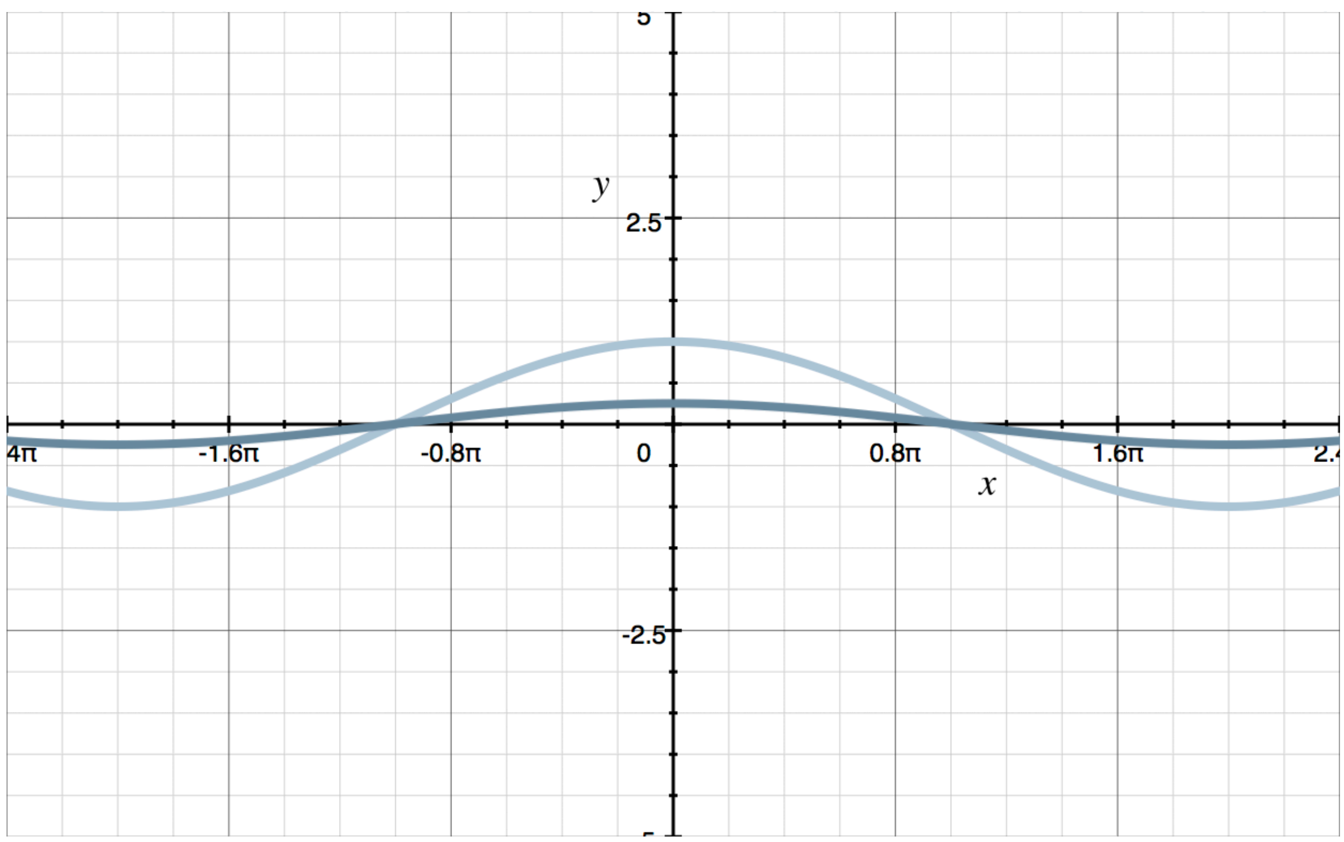
Solution:

Because the reciprocal function of secant is cosine, we want to start by replacing secant with cosine in the function we've been given. In other words, the corresponding function is $y = (1/4)\cos(\theta/2)$.

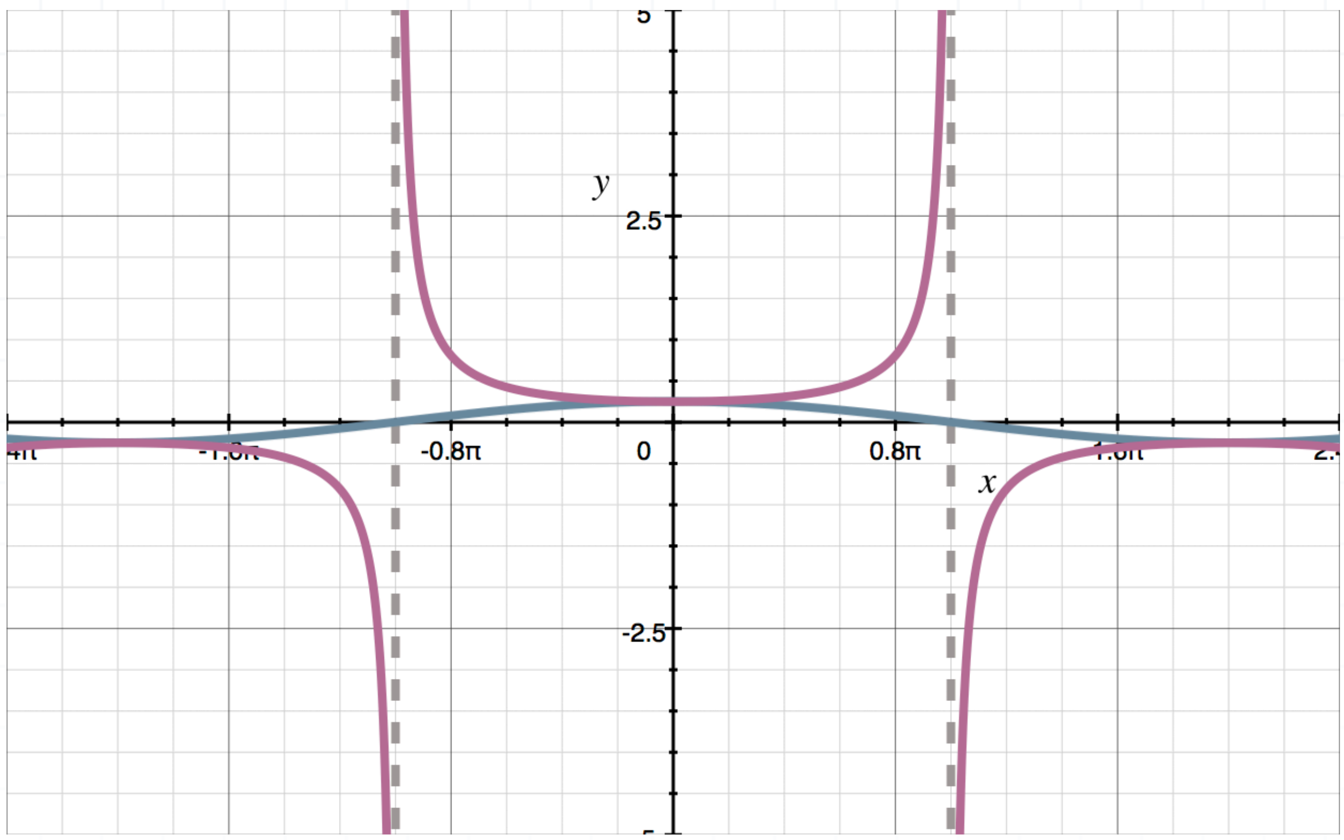
To graph $y = (1/4)\cos(\theta/2)$, we see that $b = 1/2$, so we'll stretch $y = \cos\theta$ horizontally by a factor of 2, which means we can double the x-values in the coordinate points along $y = \cos\theta$. If we sketch $y = \cos\theta$ in red and $y = \cos(\theta/2)$ in blue, we get



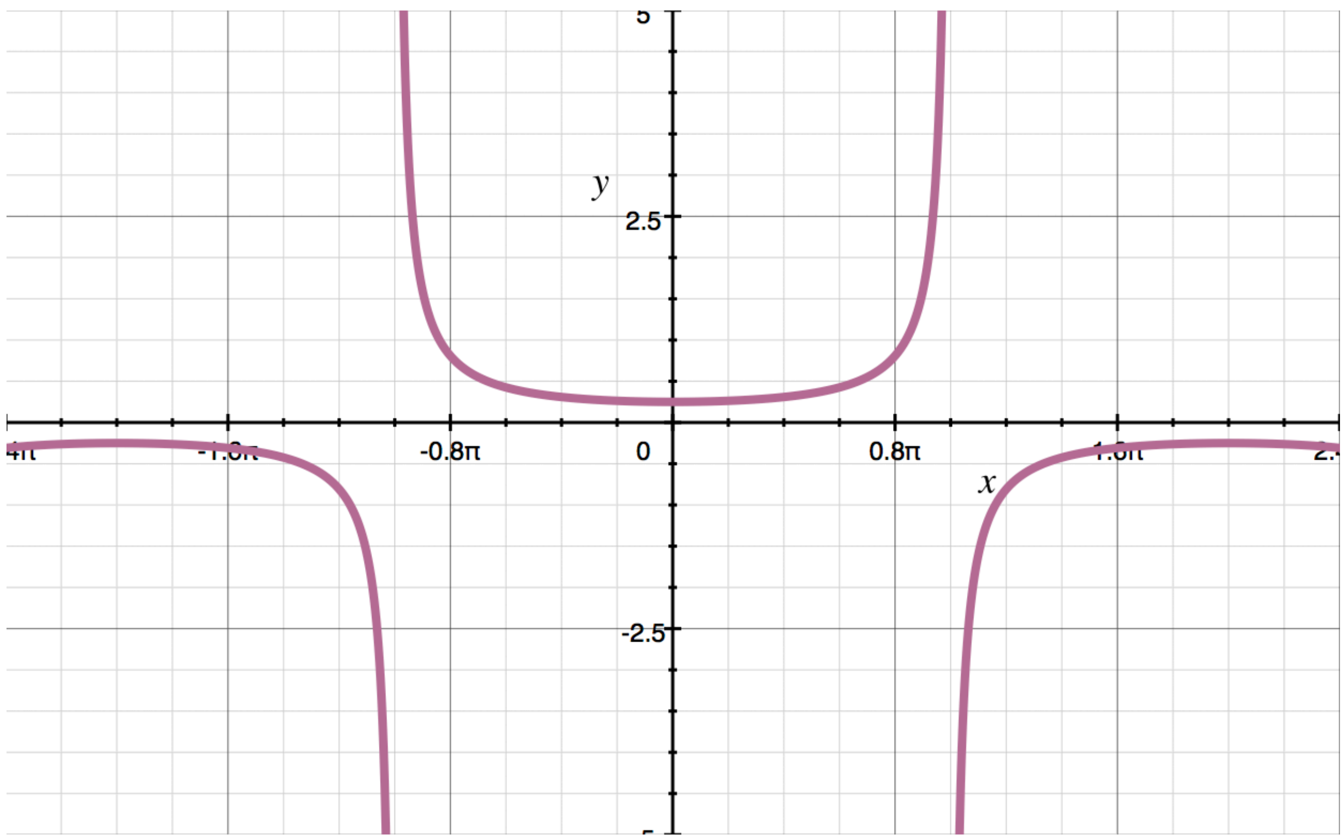
Then because $a = 1/4$, we'll compress the graph vertically by dividing all the y-values by 4. If we sketch $y = \cos(\theta/2)$ in blue and $y = (1/4)\cos(\theta/2)$ in dark blue, we get



Then we'll sketch in vertical asymptotes at the midline of $y = (1/4)\cos(\theta/2)$ and add the U-shapes for $y = (1/4)\sec(\theta/2)$.



Finally, we'll take away the sine curve and the vertical asymptotes to get the sketch of $y = (1/4)\sec(\theta/2)$.

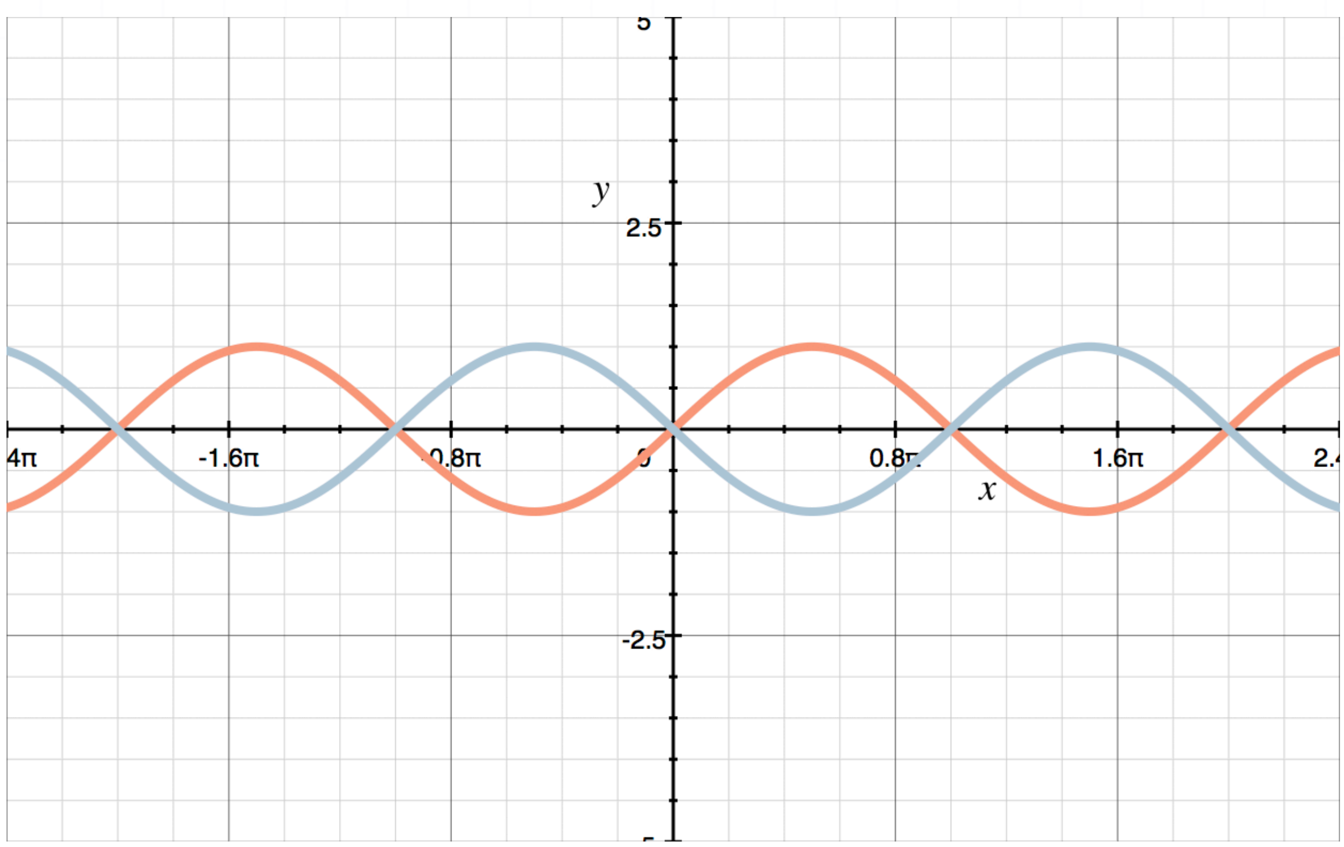


■ 5. Sketch the graph of $y = (1/2)\csc(-\theta)$.

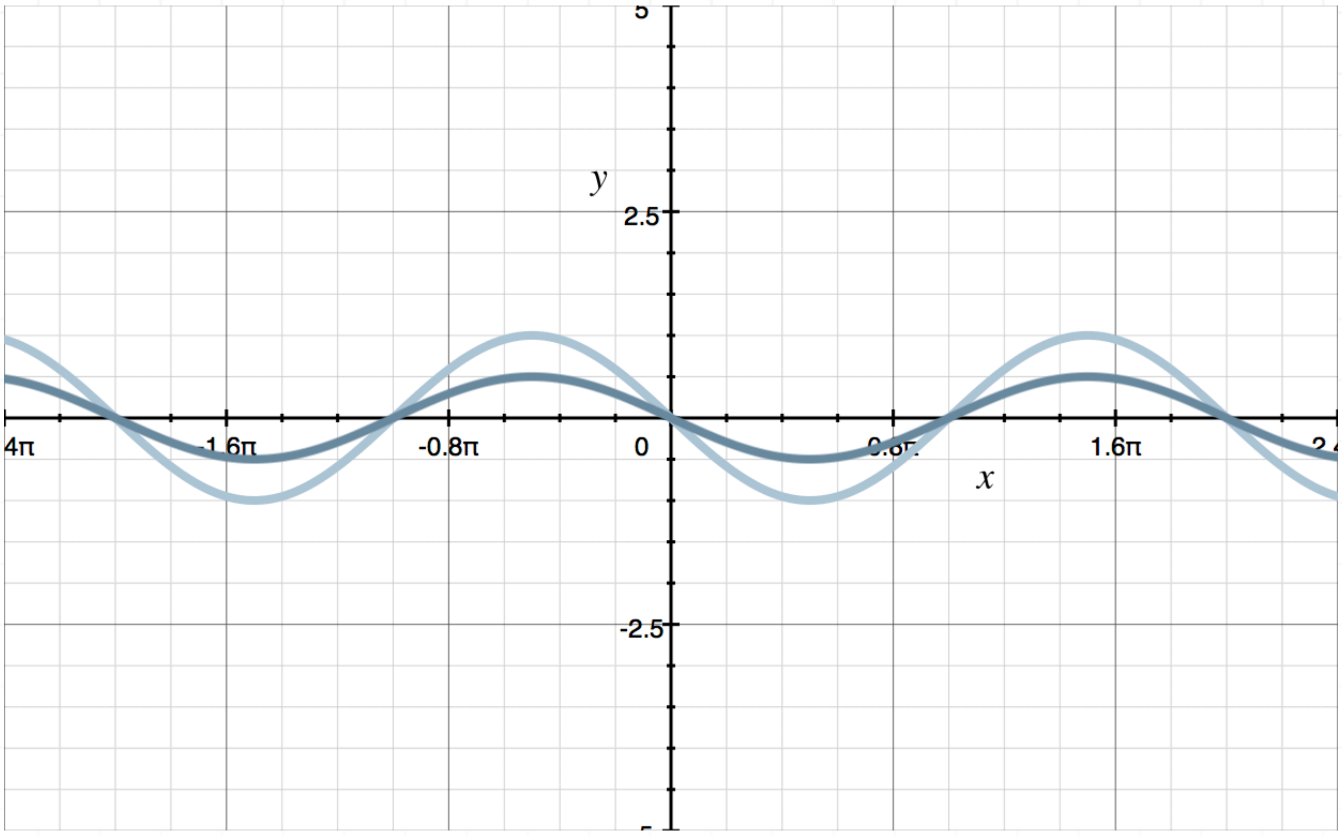
Solution:

Because the reciprocal function of cosecant is sine, we want to start by replacing cosecant with sine in the function we've been given. In other words, the corresponding function is $y = 1/2\sin(-\theta)$.

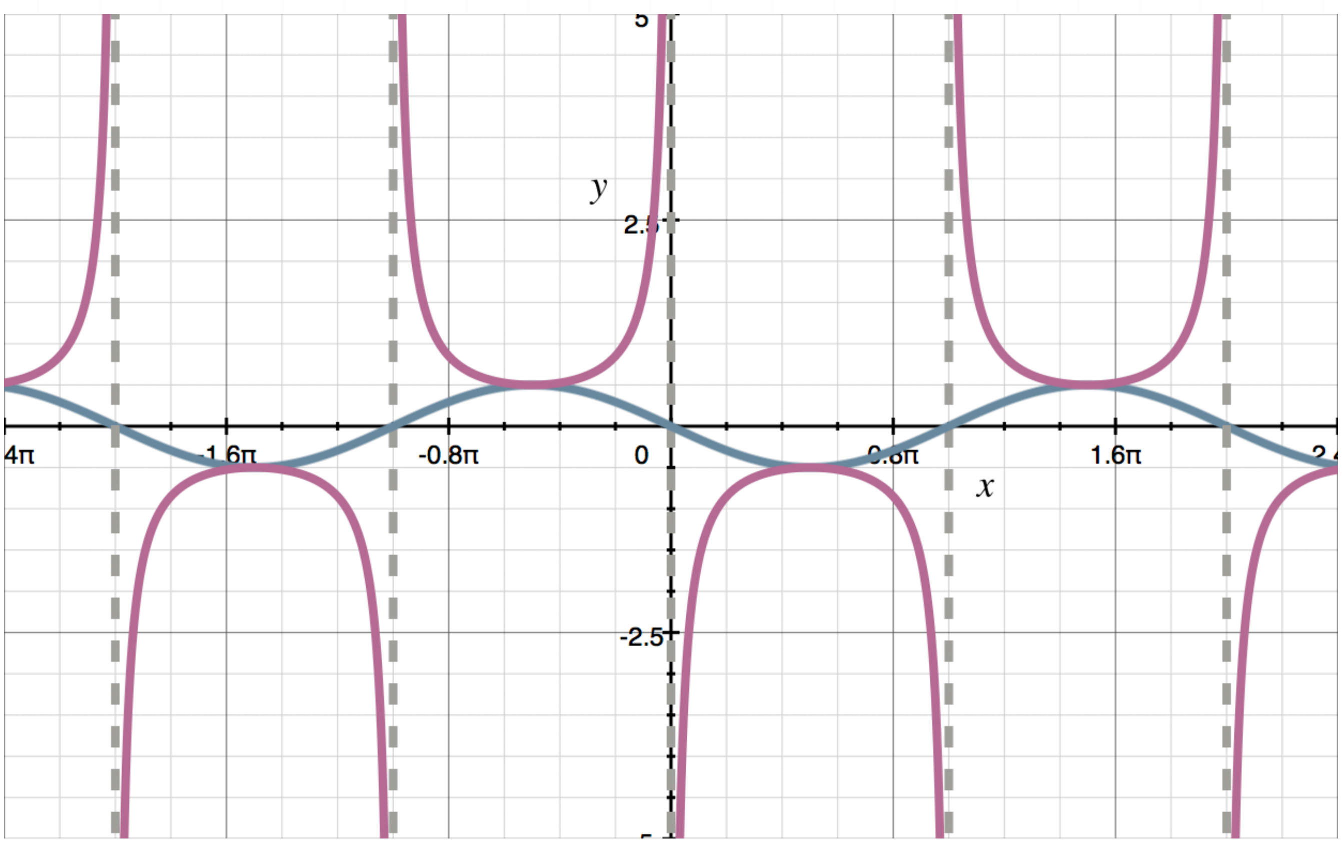
To graph $y = 1/2\sin(-\theta)$, we see that $b = -1$, so we'll flip $y = \sin\theta$ over the y-axis, which means we multiply the x-values in the coordinate points along $y = \sin\theta$ by -1. So if we sketch $y = \sin\theta$ in red and $y = \sin(-\theta)$ in blue, we get



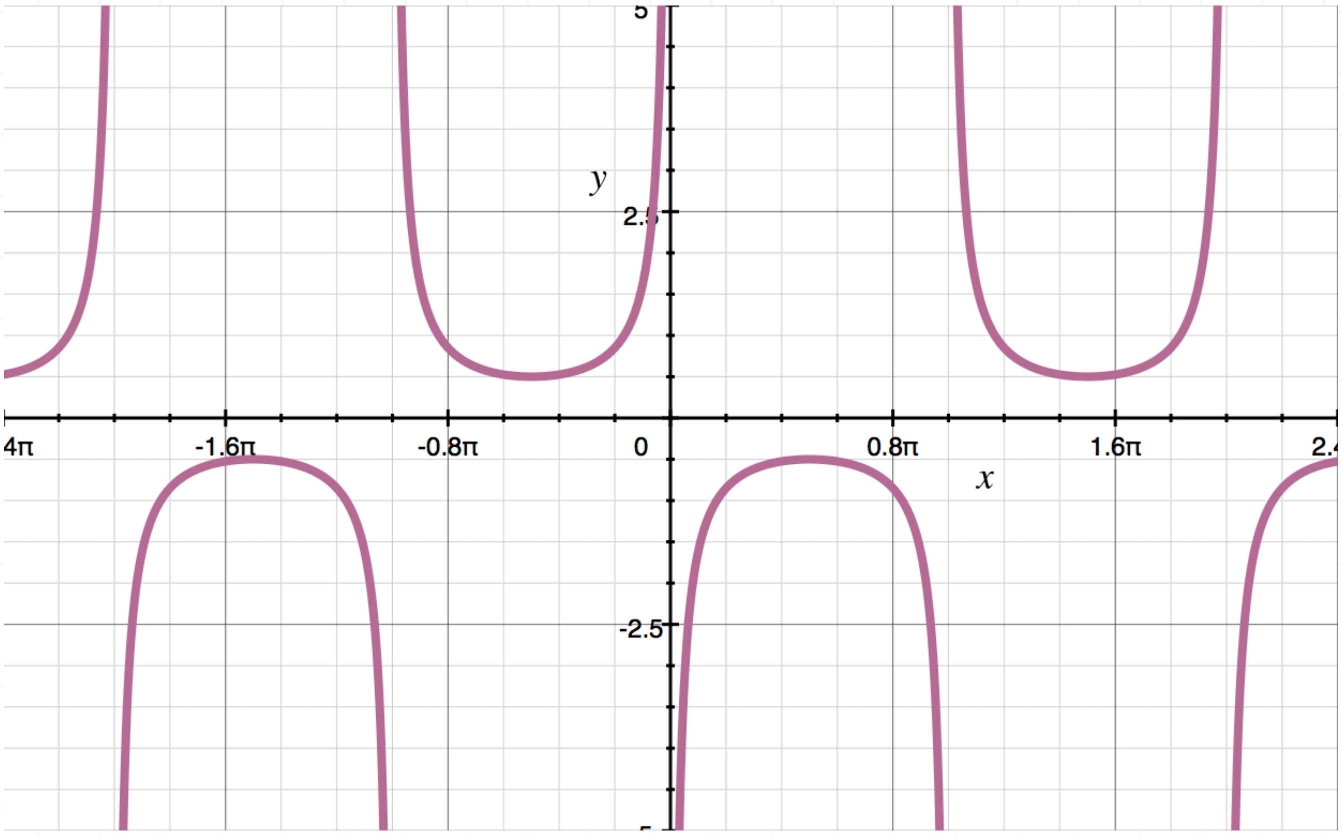
Then because $a = 1/2$, we'll compress the graph vertically by dividing all the y-values by 2. So if we sketch $y = \sin(-\theta)$ in blue and $y = (1/2)\sin(-\theta)$ in dark blue, we get



Then we'll sketch in vertical asymptotes at the midline of $y = (1/2)\sin(-\theta)$ and add the U-shapes for $y = (1/2)\csc(-\theta)$.



Finally, we'll take away the sine curve and the vertical asymptotes to get the sketch of $y = (1/2)\csc(-\theta)$.

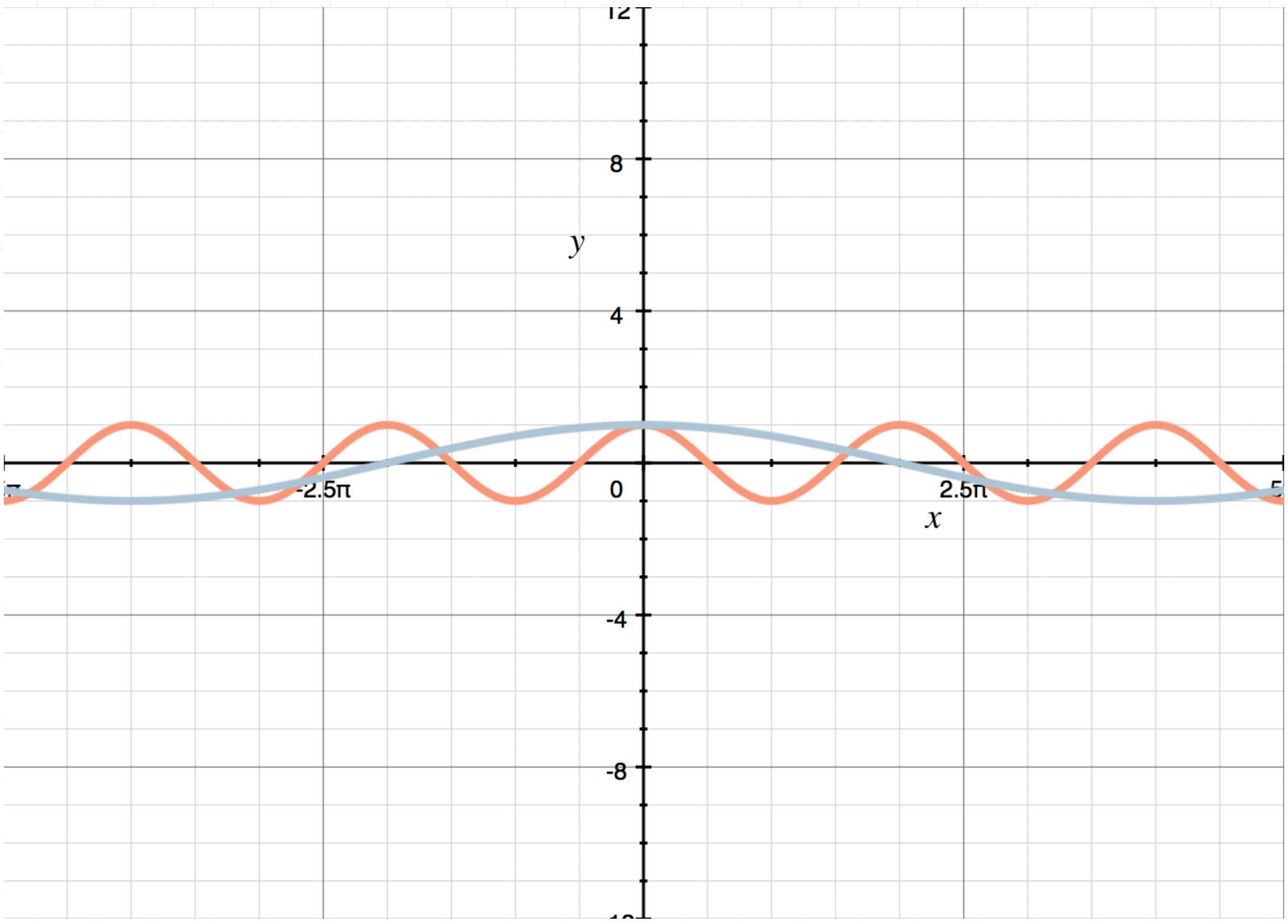


■ 6. Sketch the graph of $y = -2\sec(\theta/4)$.

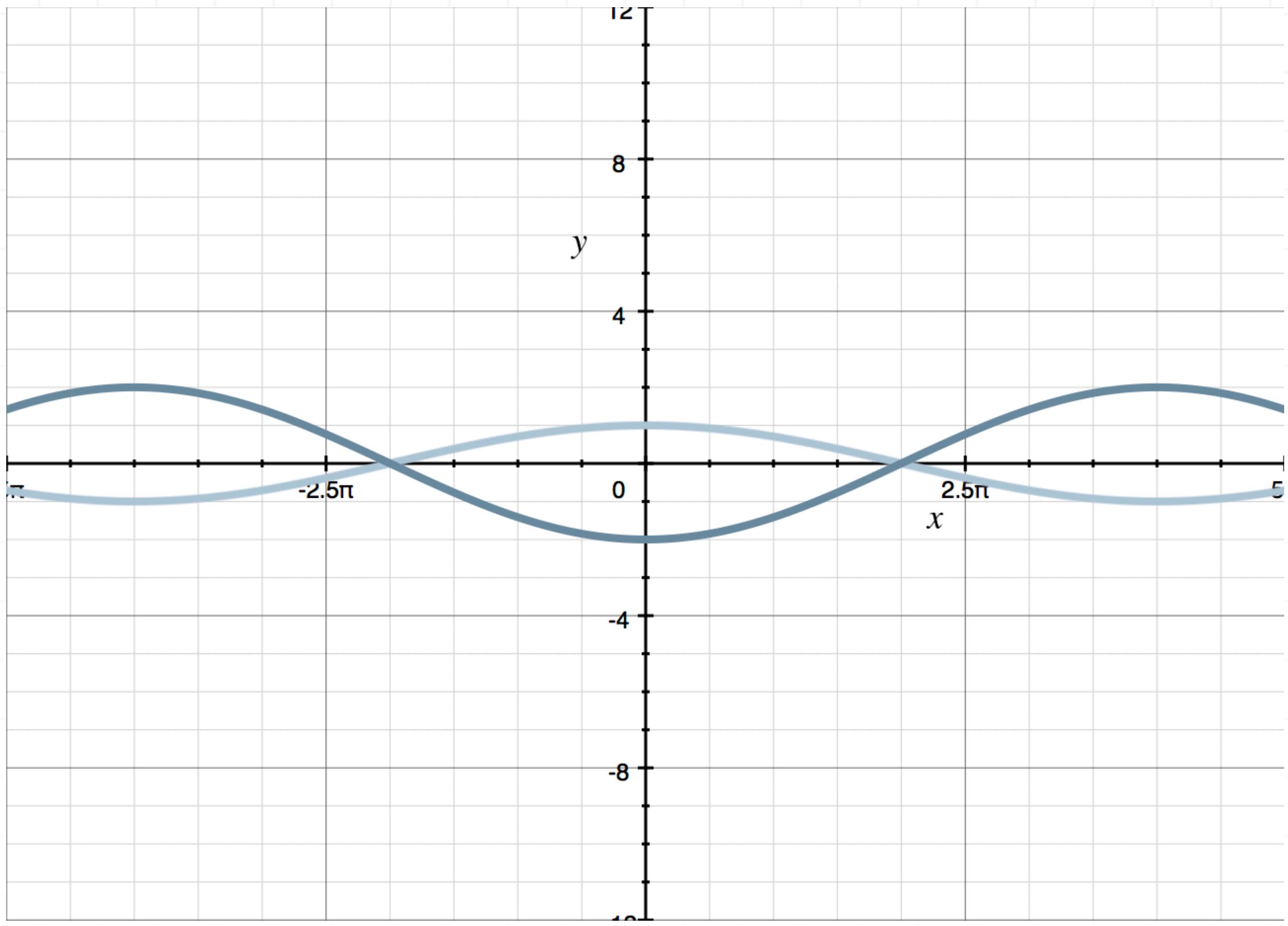
Solution:

Because the reciprocal function of secant is cosine, we want to start by replacing secant with cosine in the function we've been given. In other words, the corresponding function is $y = -2\cos(\theta/4)$.

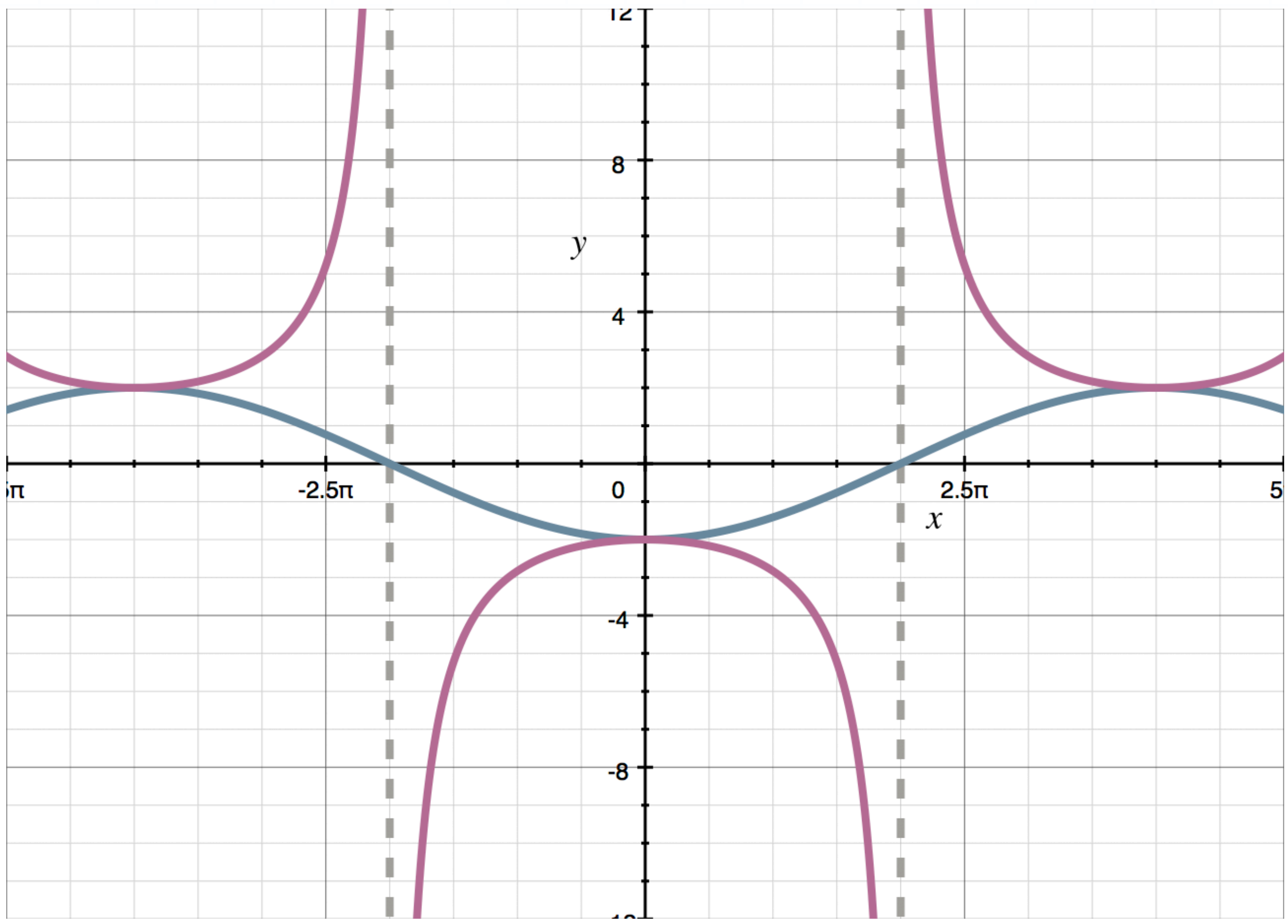
To graph $y = -2\cos(\theta/4)$, we see that $b = 1/4$, so we'll stretch $y = \cos\theta$ horizontally by a factor of 4, which means we can quadruple the x-values in the coordinate points along $y = \cos\theta$. So if we sketch $y = \cos\theta$ in red and $y = \cos(\theta/4)$ in blue, we get



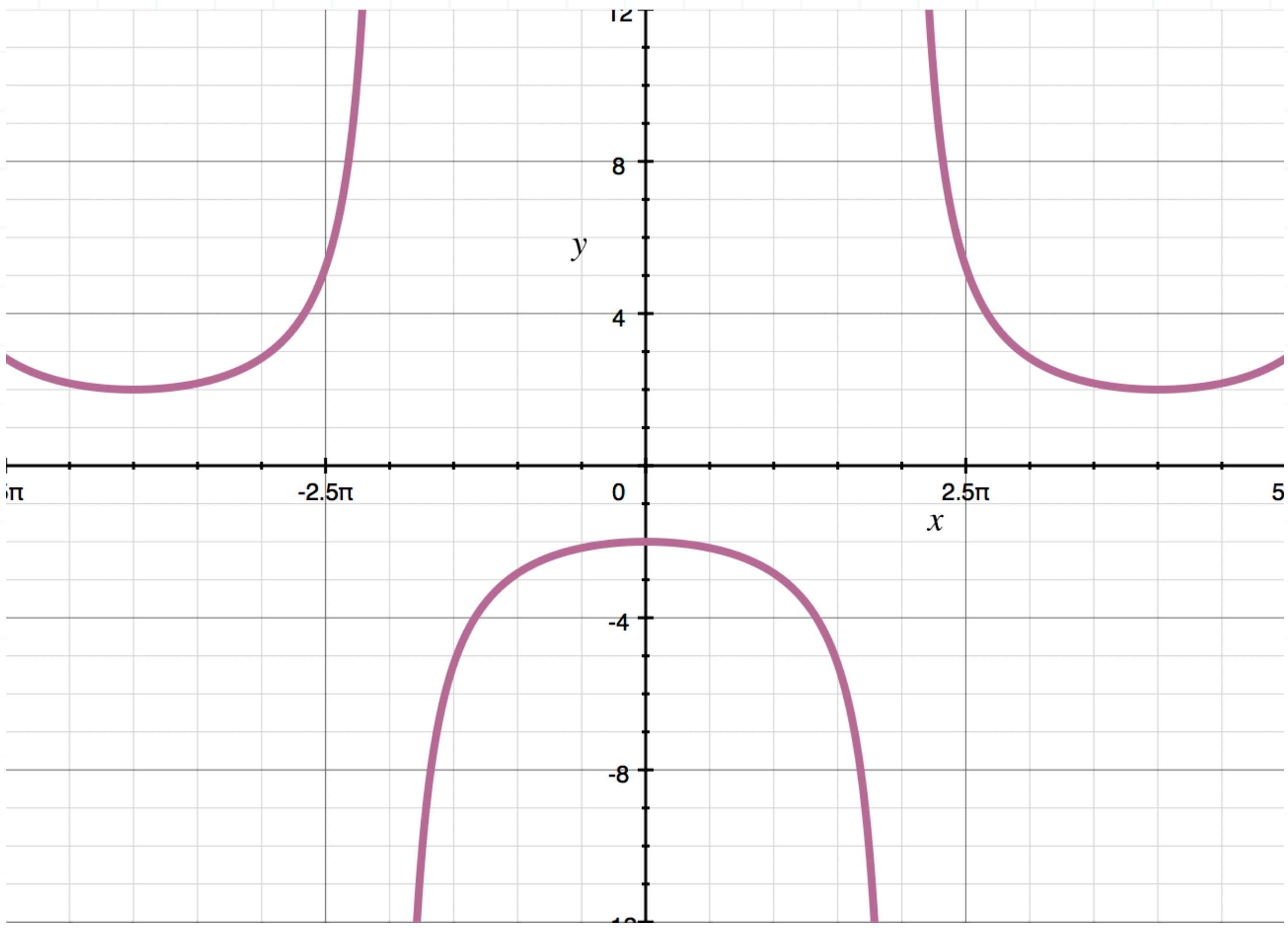
Then because $a = -2$, we'll stretch the graph vertically by doubling all the y-values and then flipping the curve over the x-axis, which means we multiply the y-value in each point by -1. So if we sketch $y = \cos(\theta/4)$ in blue and $y = -2\cos(\theta/4)$ in dark blue, we get



Then we'll sketch in vertical asymptotes at the midline of $y = -2\cos(\theta/4)$ and add the U-shapes for $y = -2\sec(\theta/4)$.

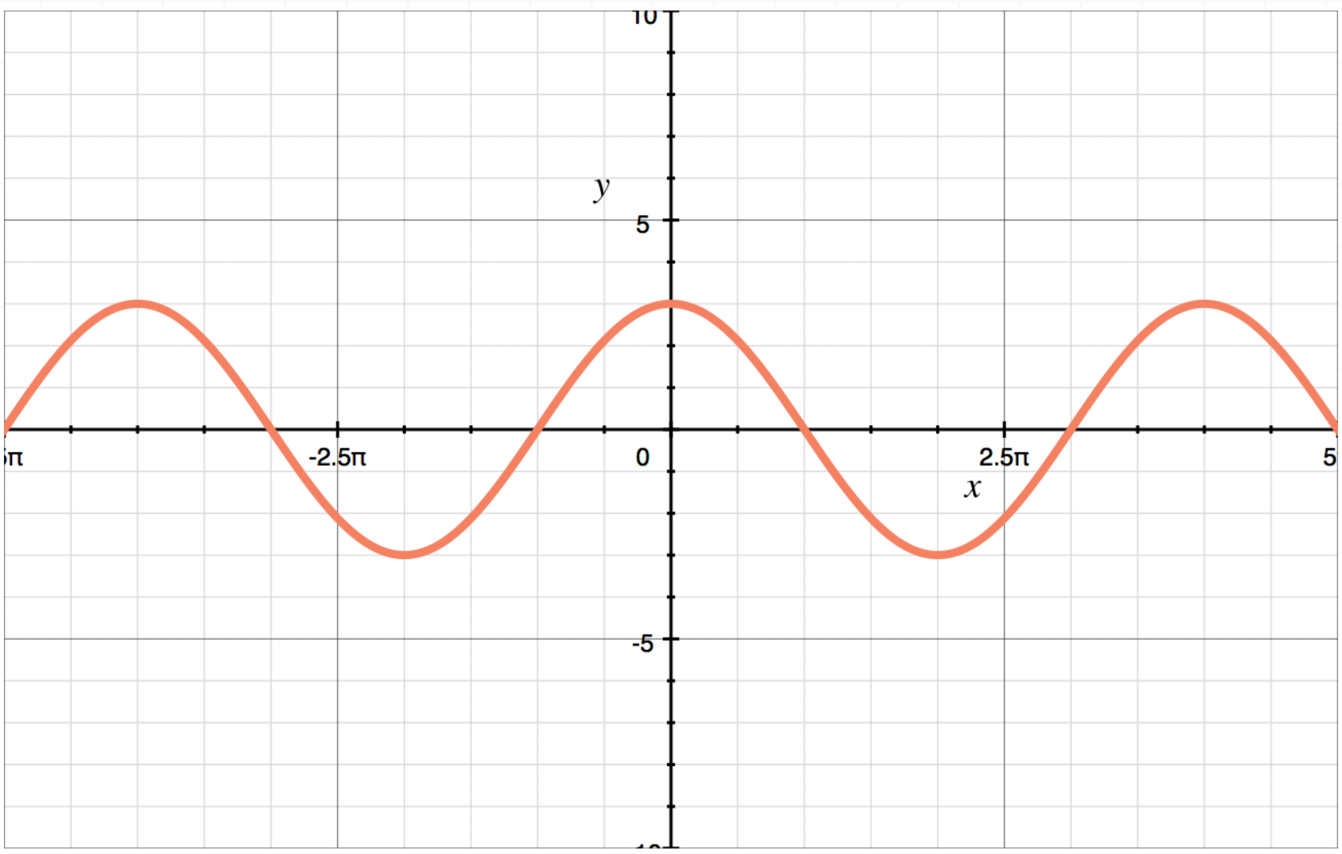


Finally, we'll take away the sine curve and the vertical asymptotes to get the sketch of $y = -2\sec(\theta/4)$.



PERIOD AND AMPLITUDE

■ 1. Find all possible cosine functions that could represent the graph.



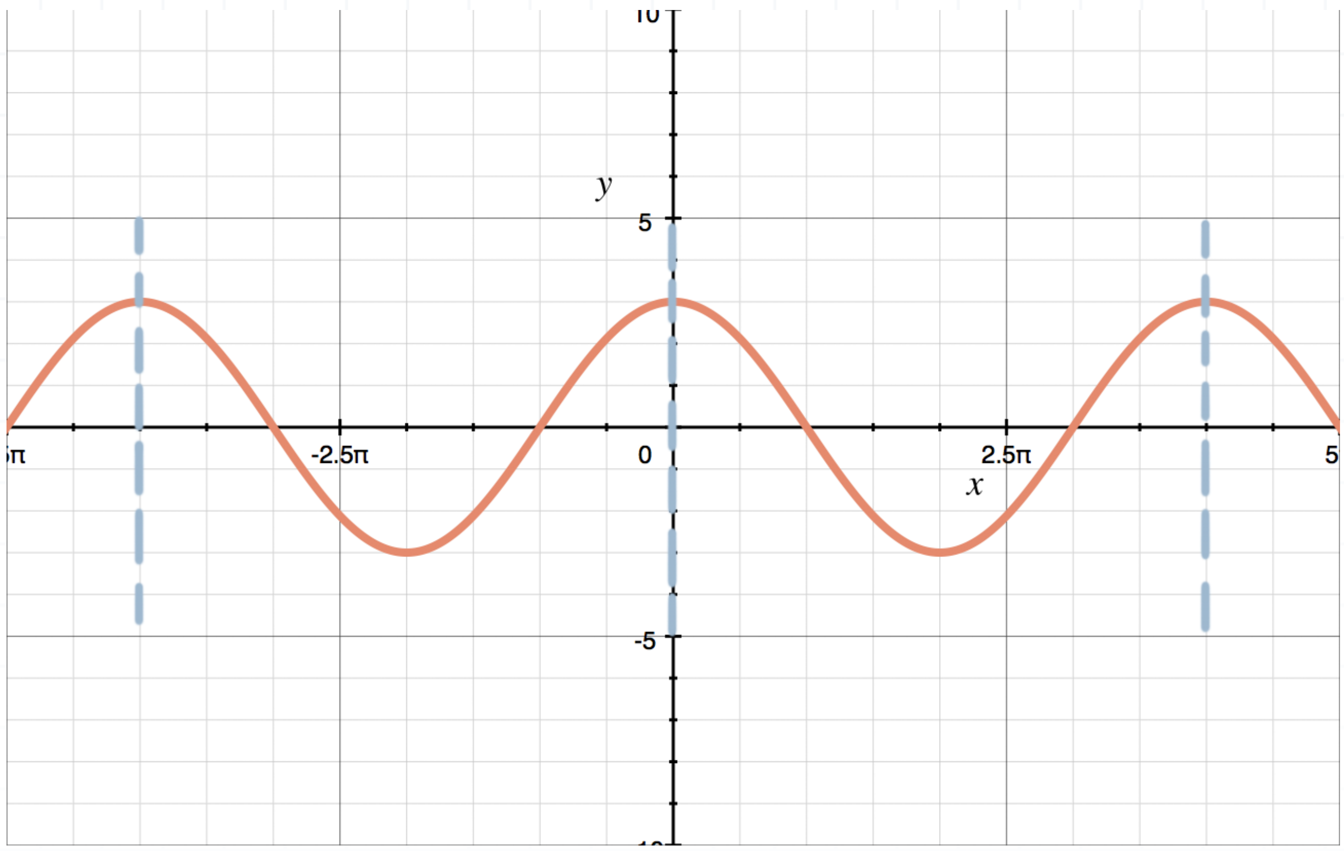
Solution:

Since the graph has maximum value 3 and minimum value -3, the amplitude will be

$$|a| = \frac{3 - (-3)}{2} = 3$$

which means $a = -3$ or $a = 3$. Since the function at $x = 0$ is positive, a is positive, which means $a = 3$.

We can see from the graph that one period is defined on $[0,4\pi]$, so the period is 4π,



and b must be

$$\frac{2\pi}{|b|} = 4\pi$$

$$|b| = \frac{2\pi}{4\pi} = \frac{1}{2}$$

$$b = \pm\frac{1}{2}$$

With $a = 3$ and $b = \pm 1/2$, the equation of the cosine function could be

$$y = 3\cos\left(\frac{x}{2}\right) \text{ or } y = 3\cos\left(-\frac{x}{2}\right)$$

■ 2. Modify the basic sine function so that it has a period of 60° and an amplitude of 3.

Solution:

Set the period for the basic sine function, $360^\circ / |b|$, equal to the 60° value we want.

$$\frac{360^\circ}{|b|} = 60^\circ$$

$$|b| = \frac{360^\circ}{60^\circ} = 6$$

$$b = \pm 6$$

For the amplitude to be 3, the value of a will be $a = \pm 3$. Therefore, since b can be $b = \pm 6$ and a can be $a = \pm 3$, there are four possible sine functions with a period of 60° and an amplitude of 3.

$$y = 3\sin(6\theta) \qquad y = 3\sin(-6\theta)$$

$$y = -3\sin(6\theta) \qquad y = -3\sin(-6\theta)$$

■ 3. Which one of these functions does not have a period of 3π?

$$y = -7\tan\left(\frac{x}{3}\right) \qquad y = -7\sec\left(\frac{2x}{3}\right)$$

$$y = 7\tan\left(\frac{2x}{3}\right) \qquad\qquad y = 7\sec\left(\frac{2x}{3}\right)$$

Solution:

Set the period for the basic tangent function, $\pi/|b|$, equal to the 3π value we want.

$$\frac{\pi}{|b|} = 3\pi$$

$$|b| = \frac{\pi}{3\pi} = \frac{1}{3}$$

$$b = \pm\frac{1}{3}$$

Therefore, the tangent function $y = 7\tan(2x/3)$ is the function whose period is not equal to 3π.

■ 4. Find all possible sine and cosine functions $y = a\sin(bx)$ and $y = a\cos(bx)$ which have a period of 135° and an amplitude of 10.

Solution:

Because amplitude is given by $|a|$, we know $a = \pm 10$. And because the period of sine and cosine is $360^\circ/|b|$,

$$\frac{360^\circ}{|b|} = 135^\circ$$

$$|b| = \frac{360^\circ}{135^\circ} = \frac{8}{3}$$

$$b = \pm\frac{8}{3}$$

Putting these values together, there are eight possible functions that satisfy the given conditions, including four sine functions,

$$y = 10\sin\left(\frac{8x}{3}\right) \qquad y = -10\sin\left(\frac{8x}{3}\right)$$

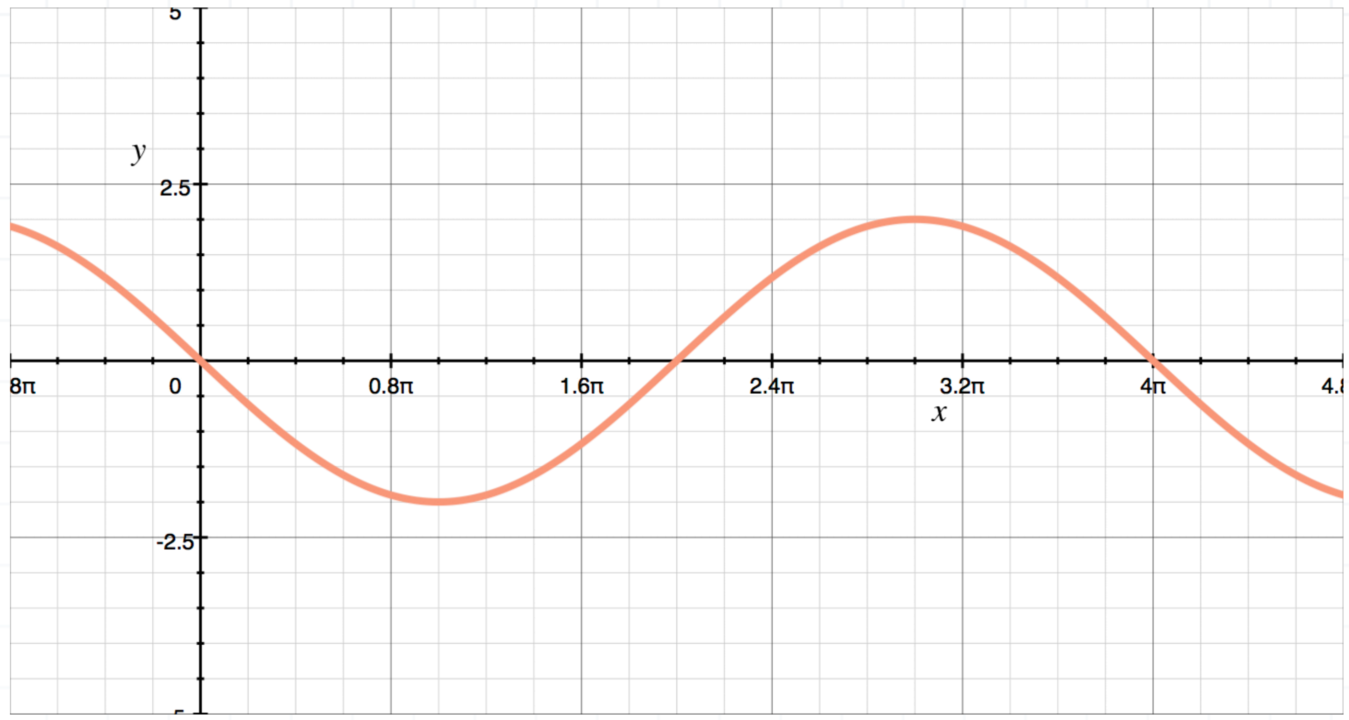
$$y = 10\sin\left(-\frac{8x}{3}\right) \qquad y = -10\sin\left(-\frac{8x}{3}\right)$$

and four cosine functions.

$$y = 10\cos\left(\frac{8x}{3}\right) \qquad y = -10\cos\left(\frac{8x}{3}\right)$$

$$y = 10\cos\left(-\frac{8x}{3}\right) \qquad y = -10\cos\left(-\frac{8x}{3}\right)$$

■ 5. Give the amplitude and period of the function in the graph, then write an equation for the curve.



Solution:

Since the graph has maximum value 2 and minimum value -2, the amplitude will be

$$|a| = \frac{2-(-2)}{2} = 2$$

which means $a = -2$ or $a = 2$. Since the function at $x = 0$ is 0, the graph represents a sine function.

We can see from the graph that one period is defined on $[0,4\pi]$, so the period is 4π, and b must be

$$\frac{2\pi}{|b|} = 4\pi$$

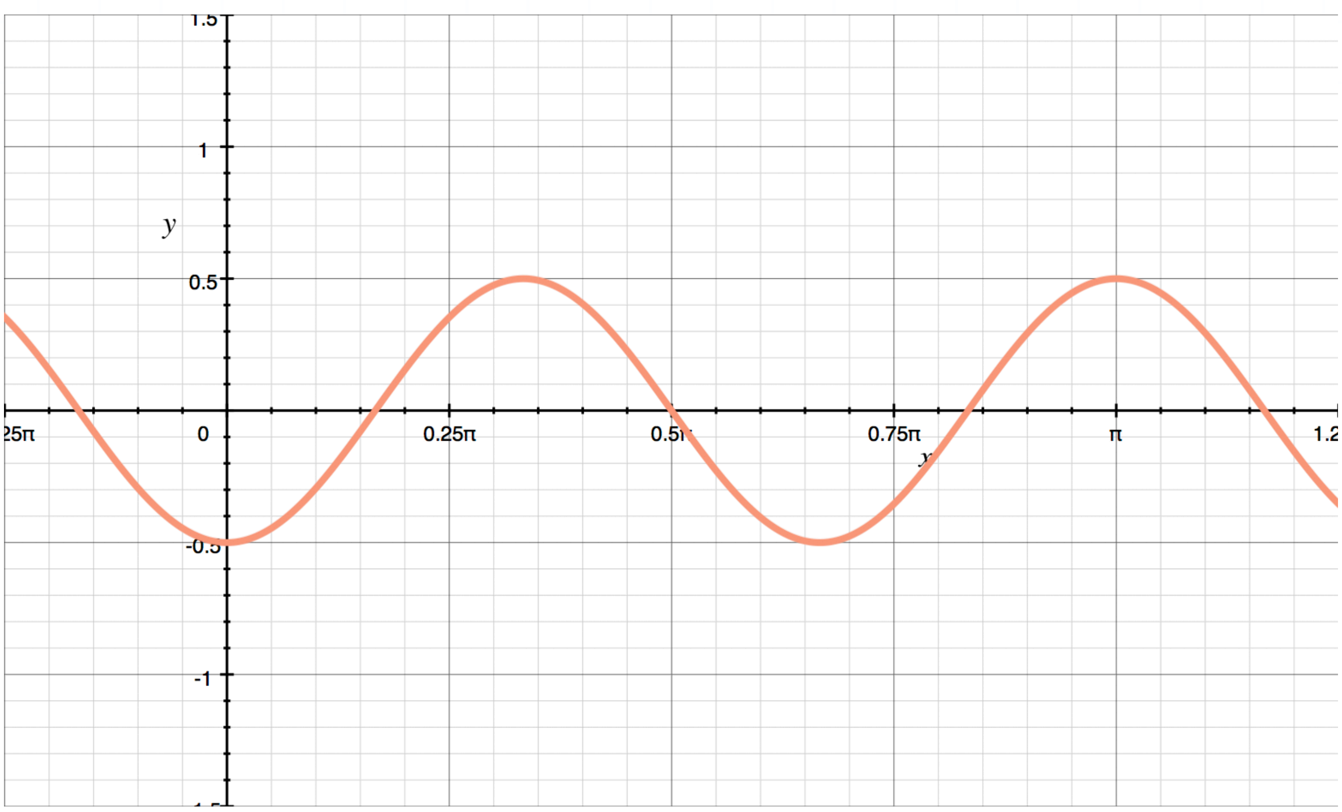
$$|b| = \frac{2\pi}{4\pi} = \frac{1}{2}$$

$$b = \pm \frac{1}{2}$$

Since the function at $x = \pi$ is negative, a is negative and b is positive, which means $a = -2$ and $b = 1/2$, or a is positive and b is negative, which means $a = 2$ and $b = -1/2$. With $a = -2$ and $b = \pm 1/2$, the equation of the curve could be

$$y = -2\sin\left(\frac{x}{2}\right) \text{ or } y = 2\sin\left(-\frac{x}{2}\right)$$

■ 6. Give the amplitude and period of function in the graph, then write the equation of the graph if we know that the function is <u>not</u> flipped across the y-axis.



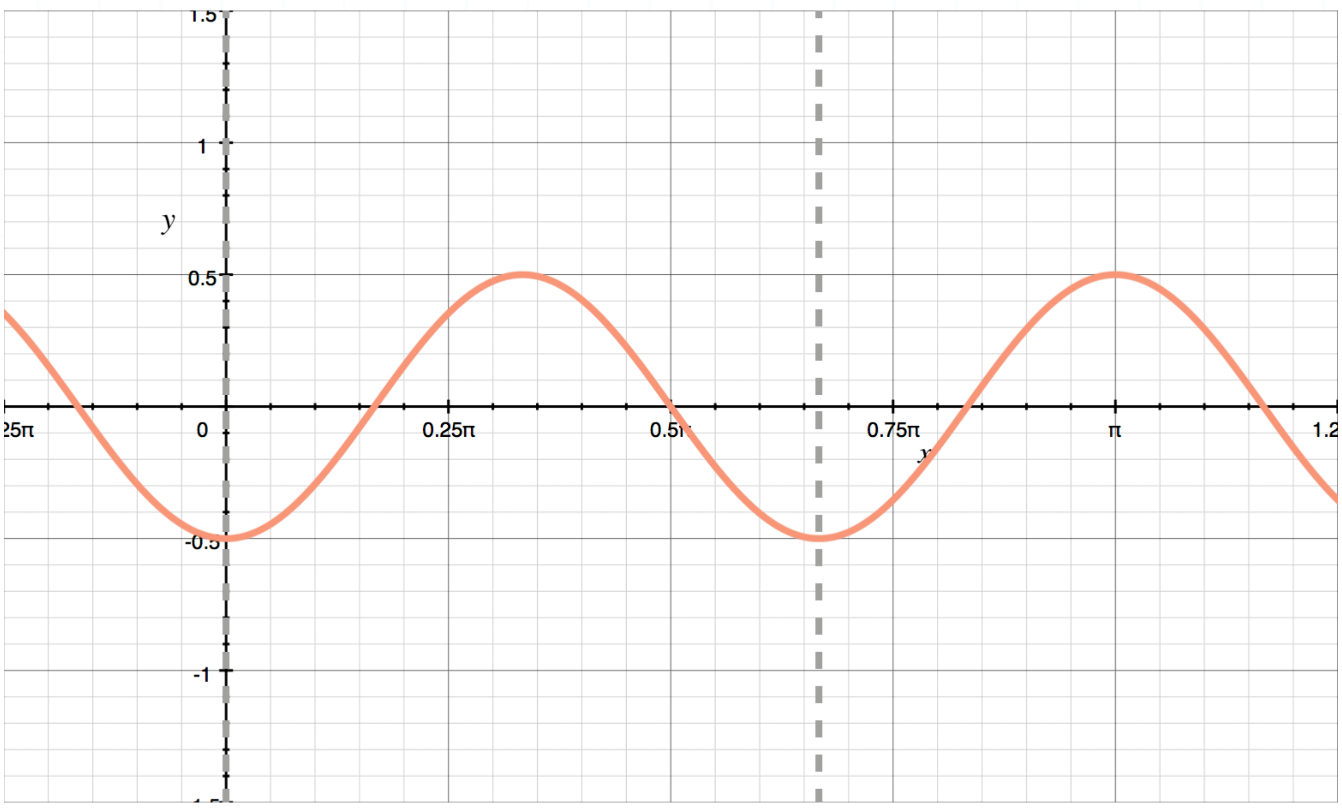
Solution:

Since the graph has maximum value $1/2$ and minimum value $-1/2$, the amplitude will be

$$|a| = \frac{1/2 - (-1/2)}{2} = \frac{1}{2}$$

which means $a = -1/2$ or $a = 1/2$. Since the function at $x = 0$ is negative, a is negative, which means $a = -1/2$. Since the function at $x = 0$ is not 0, the graph represents a cosine function.

We can see from the graph that one period is defined on $[0,2\pi/3]$, so the period is $2\pi/3$,



and b must be

$$\frac{2\pi}{|b|} = \frac{2\pi}{3}$$

$$|b| = \frac{2\pi}{\frac{2\pi}{3}} = 3$$

$$b = \pm 3$$

Since the function isn't flipped over the y-axis, we know $b = 3$. With $a = -1/2$ and $b = 3$, the equation of the curve could be

$$y = -\frac{1}{2}\cos(3x)$$

SKETCHING TANGENT AND COTANGENT

■ 1. What are the vertical asymptotes of $y = -2\cot(3x)$?

Solution:

To find adjacent vertical asymptotes for $y = a\cot(bx)$, solve $bx = 0$ and $bx = \pi$. The function $y = -2\cot(3x)$ is in the form $y = a\cot(bx)$ where $a = -2$ and $b = 3$.

$$3x = 0$$

$$x = 0$$

and

$$3x = \pi$$

$$x = \frac{\pi}{3}$$

To find another asymptotes, we need to find the period. Since b is positive and the period of $\cot\theta$ is π, the period of $y = -2\cot(3x)$ is

$$\frac{\pi}{|b|} = \frac{\pi}{|3|} = \frac{\pi}{3}$$

So every vertical asymptote will be separated by $\pi/3$ units. So some other neighboring asymptotes are at

$$0 - \frac{\pi}{3} = -\frac{\pi}{3}$$

$$\frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\frac{2\pi}{3} + \frac{\pi}{3} = \frac{3\pi}{3} = \pi$$

Therefore the asymptotes for $y = -2\cot(3x)$ are

$$\ldots, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \ldots$$

■ 2. Sketch the graph of $y = -3\tan(2x)$.

Solution:

We'll find two adjacent vertical asymptotes by solving $bx = -\pi/2$ and $bx = \pi/2$ for x. With $b = 2$, we get

$$2x = -\frac{\pi}{2}$$

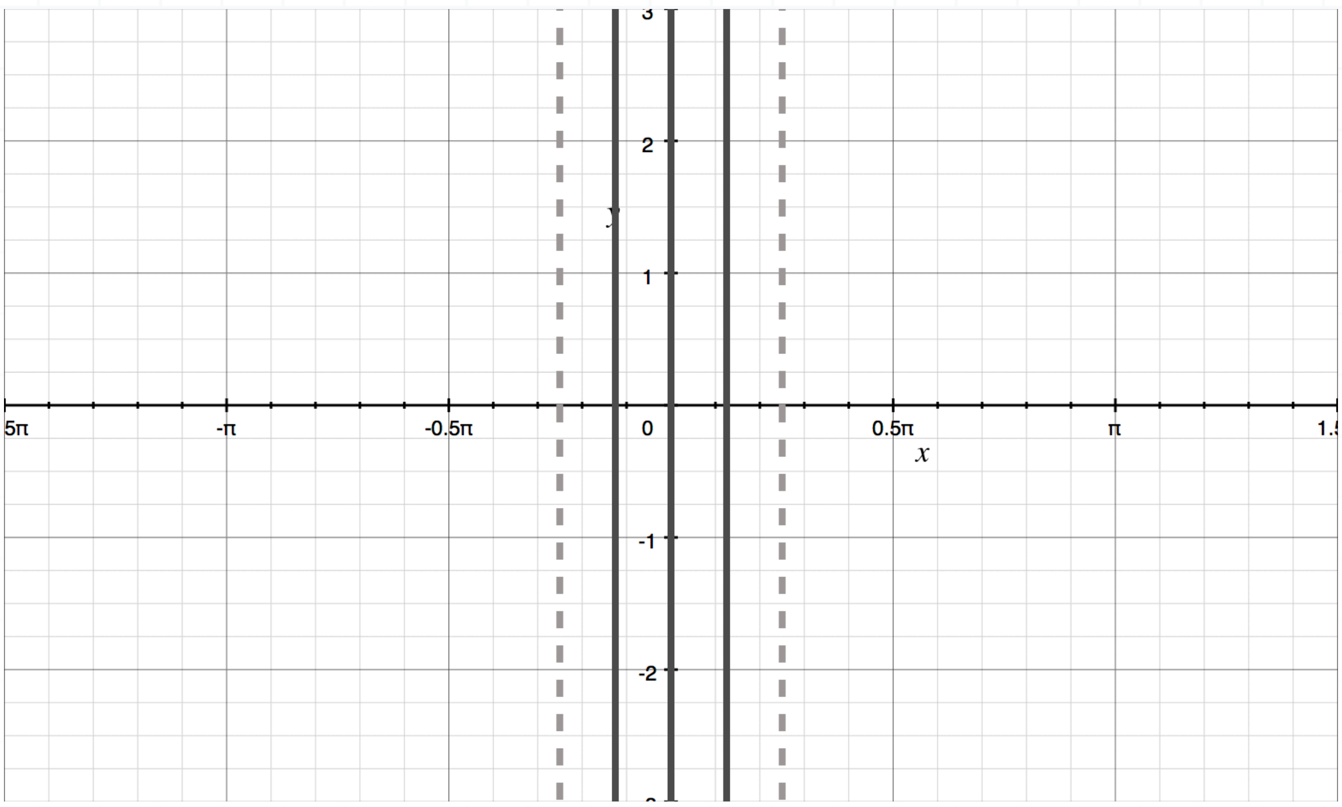
$$x = -\frac{\pi}{4}$$

and

$$2x = \frac{\pi}{2}$$

$$x = \frac{\pi}{4}$$

We'll sketch in the vertical asymptotes $x = -\pi/4$ and $x = \pi/4$, and then divide the interval between $x = -\pi/4$ and $x = \pi/4$ into four equal parts.



The dividing lines of each of these four sub-intervals are $x = -\pi/8$, $x = 0$, and $x = \pi/8$, so we'll evaluate $y = -3\tan(2x)$ at those three values. We'll get

$$y = -3\tan\left(-\frac{2\pi}{8}\right)$$

$$y = -3\tan\left(-\frac{\pi}{4}\right)$$

$$y = -3(-1)$$

$$y = 3$$

and

$$y = -3\tan(2(0))$$

$$y = -3\tan(0)$$

$$y = -3(0)$$

$$y = 0$$

and

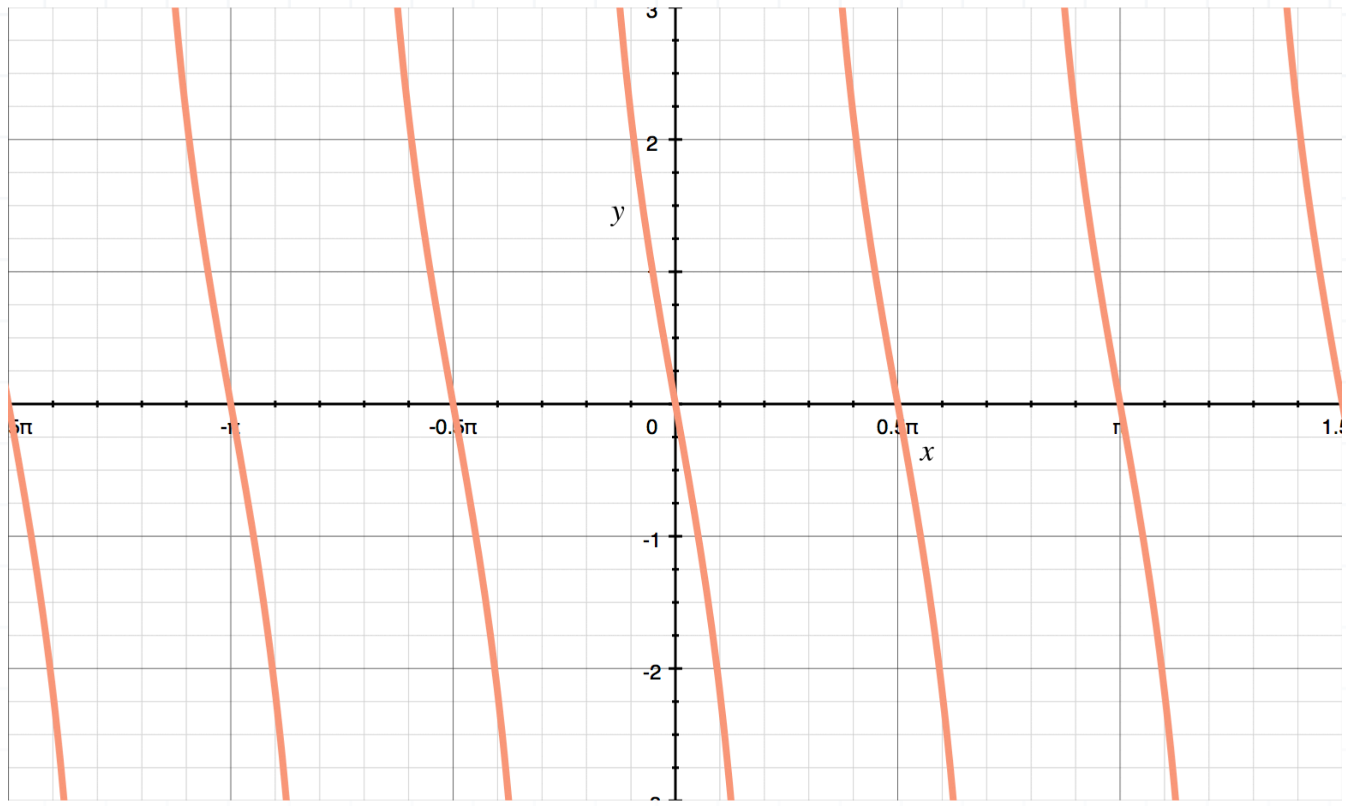
$$y = -3\tan\left(\frac{2\pi}{8}\right)$$

$$y = -3\tan\left(\frac{\pi}{4}\right)$$

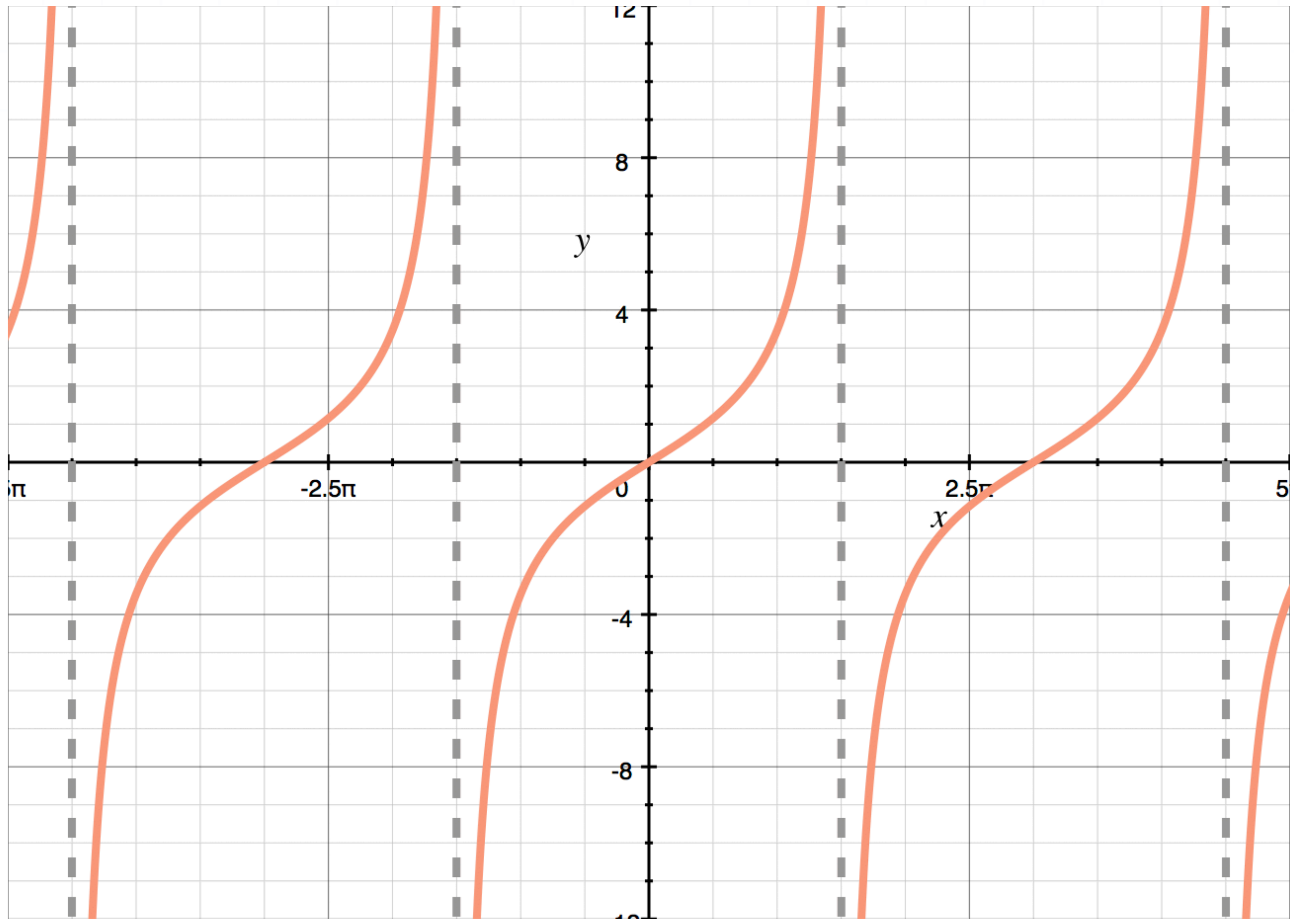
$$y = -3(1)$$

$$y = -3$$

We'll plot those points, and then connect them with a smooth curve, respecting the asymptotes. Finally, we'll repeat that pattern both to the left and right, and take away the asymptotes and other guiding lines that we sketched, and we'll get the final graph of $y = -3\tan(2x)$.



■ 3. Which function is represented by the curve if $a=2$?



Solution:

Because the value of the function at $\theta = 0$ is 0, the curve will be the graph of a tangent function. So we'll only need to determine which tangent function the graph represents.

We can see from the graph that one period is defined on $[-3\pi/2,3\pi/2]$, so the period is 3π, and b must be

$$\frac{\pi}{|b|} = 3\pi$$

$$|b| = \frac{\pi}{3\pi} = \frac{1}{3}$$

$$b = \pm\frac{1}{3}$$

Since the function is increasing, a and b are either both positive or both negative. We know that $a = 2$, so b is positive, which means $b = 1/3$. With $a = 2$ and $b = 1/3$, the equation of the tangent function could be

$$y = 2\tan\left(\frac{x}{3}\right)$$

■ 4. Sketch the graph of $y = 2\cot(-x/2)$.

Solution:

We'll find two adjacent vertical asymptotes by solving $bx = 0$ and $bx = \pi$ for x. With $b = -1/2$, we get

$$-\frac{x}{2} = -0$$

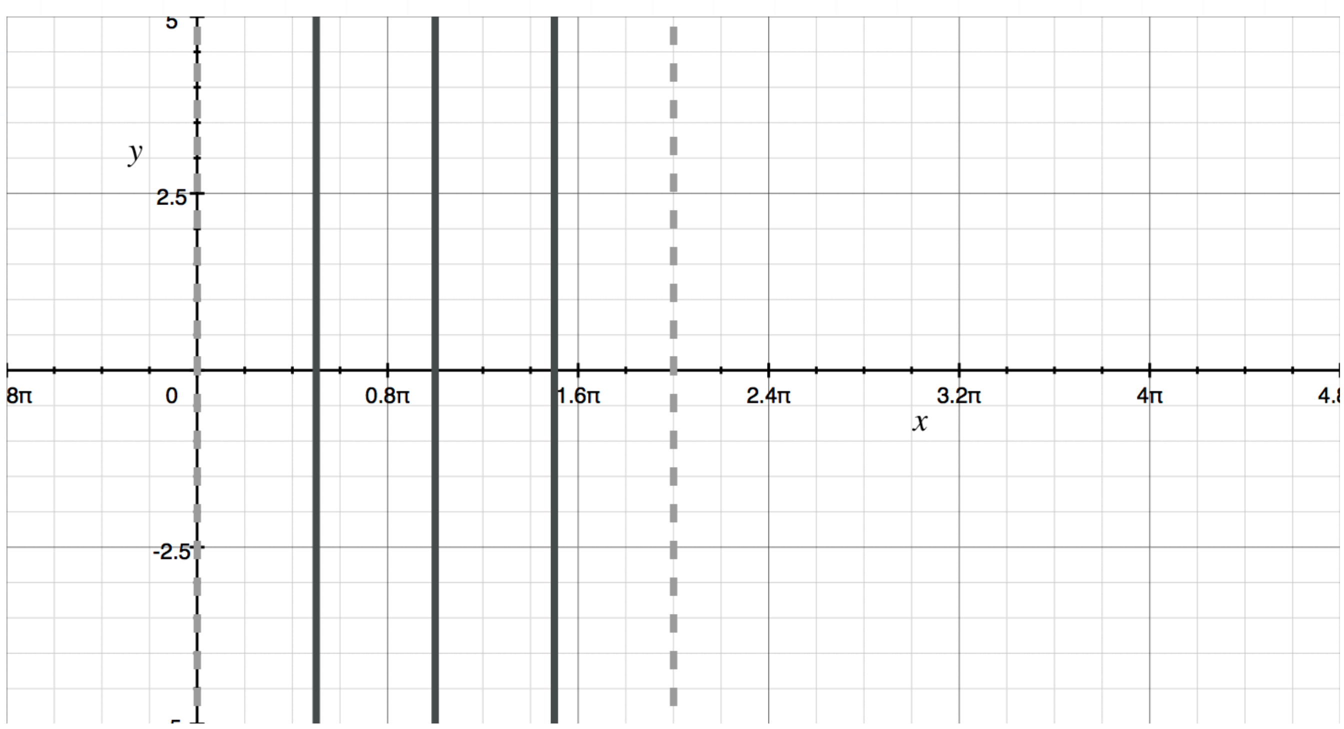
$$x = 0$$

and

$$-\frac{x}{2} = \pi$$

$$x = -2\pi$$

We'll sketch in the vertical asymptotes $x = 0$ and $x = 2\pi$, and then divide the interval between $x = 0$ and $x = 2\pi$ into four equal parts.



The dividing lines of each of these four sub-intervals are $x = \pi/2$, $x = \pi$, and $x = 3\pi/2$, so we'll evaluate $y = 2\cot(-x/2)$ at those three values. We'll get

$$y = 2\cot\left(-\frac{1}{2} \cdot \frac{\pi}{2}\right)$$

$$y = 2\cot\left(-\frac{\pi}{4}\right)$$

$$y = 2(-1)$$

$$y = -2$$

and

$$y = 2\cot\left(-\frac{1}{2} \cdot \pi\right)$$

$$y = 2\cot\left(-\frac{\pi}{2}\right)$$

$$y = 2(0)$$

$$y = 0$$

and

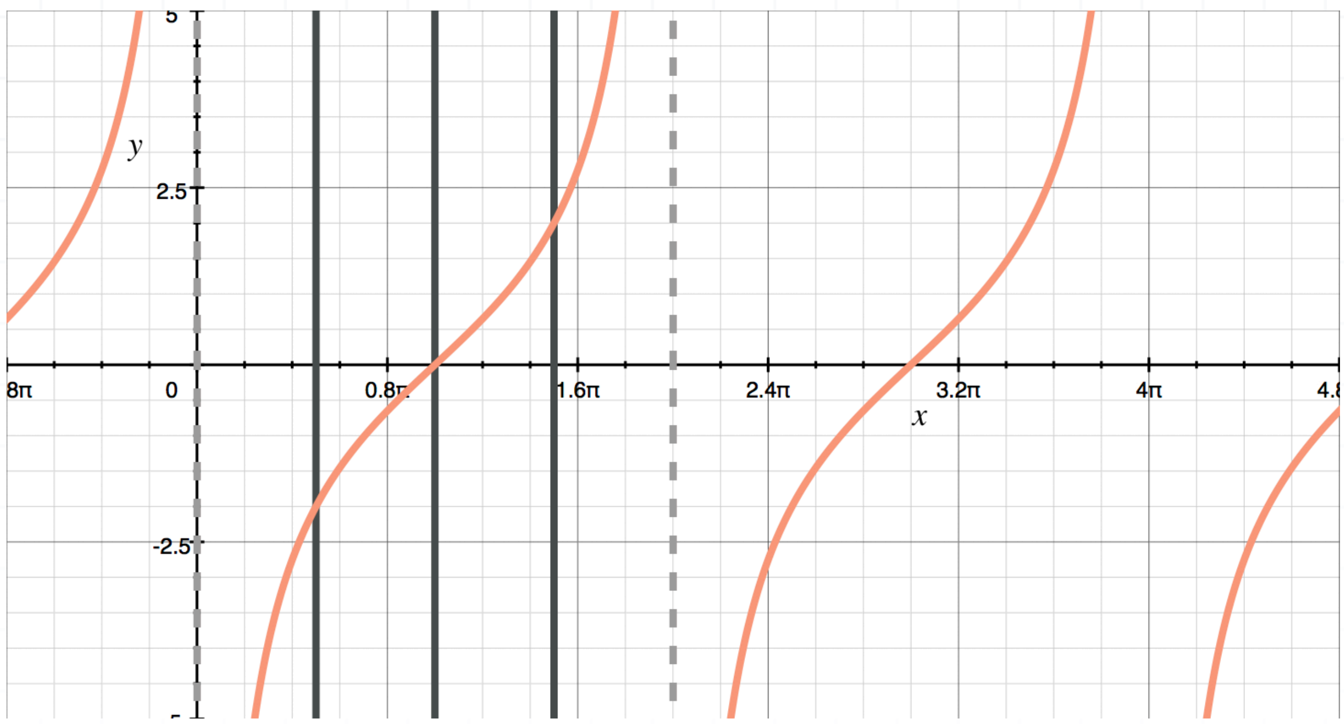
$$y = 2\cot\left(-\frac{1}{2} \cdot \frac{3\pi}{2}\right)$$

$$y = 2\cot\left(-\frac{3\pi}{4}\right)$$

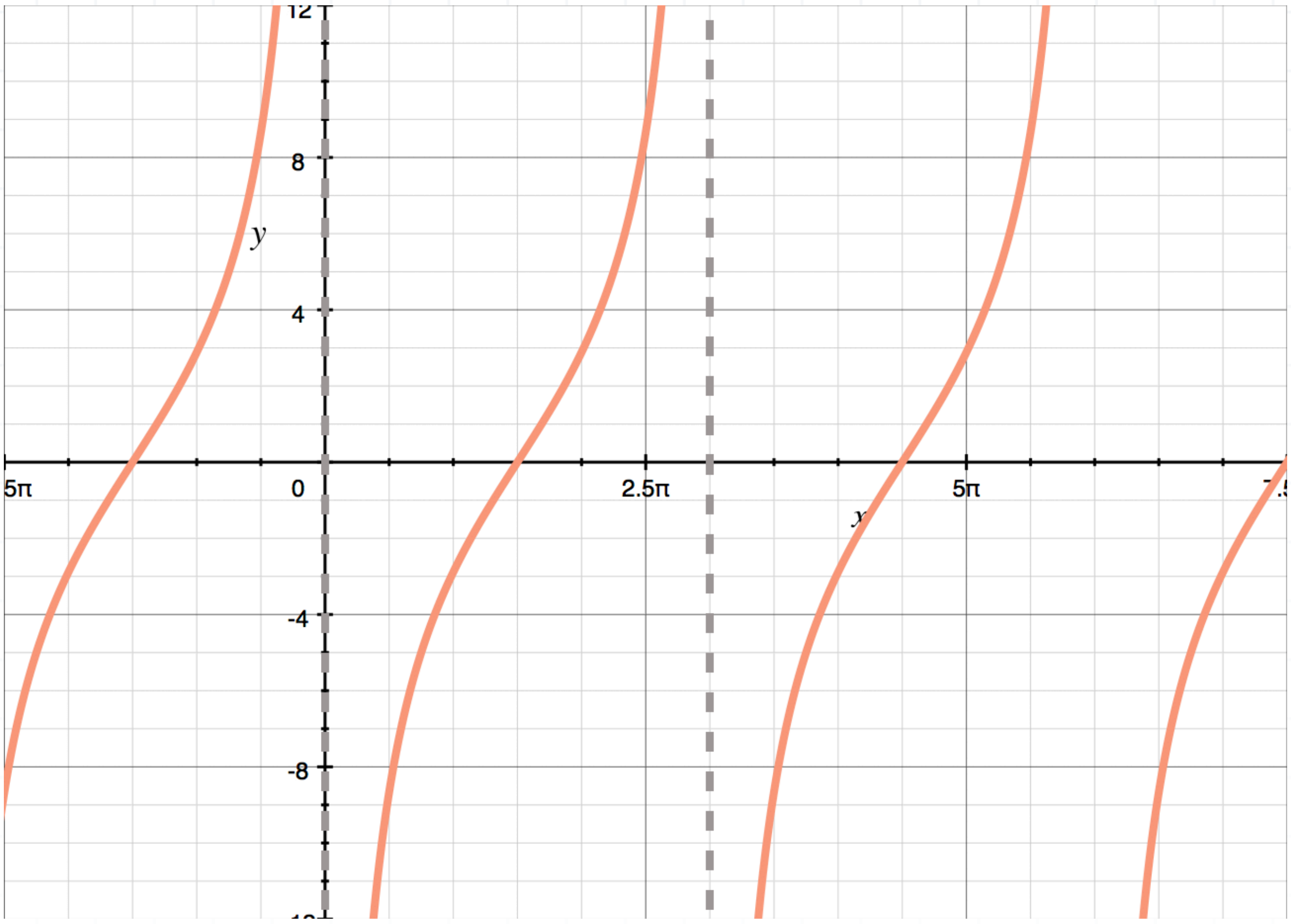
$$y = 2(1)$$

$$y = 2$$

We'll plot those points, and then connect them with a smooth curve, respecting the asymptotes. Finally, we'll repeat that pattern both to the left and right, and take away the asymptotes and other guiding lines that we sketched, and we'll get the final graph of $y = 2\cot(-x/2)$.



■ 5. Which function is represented by the curve if $a = -5$?



Solution:

Because the value of the function at $\theta = 0$ isn't 0, the curve will be the graph of a cotangent function. So we'll only need to determine which cotangent function the graph represents.

We can see from the graph that one period is defined on $[0,3\pi]$, so the period is 3π, and b must be

$$\frac{\pi}{|b|} = 3\pi$$

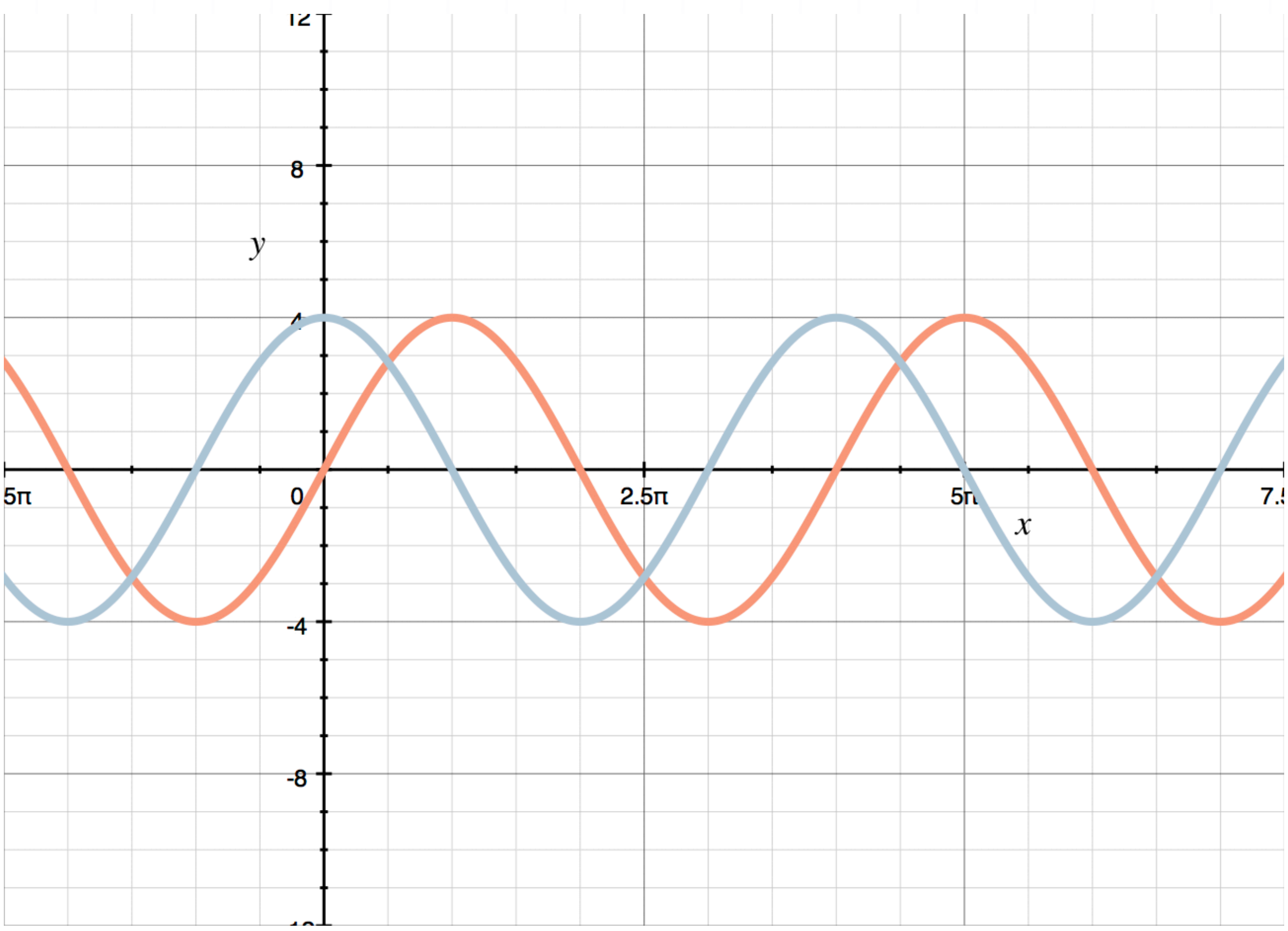
$$|b| = \frac{\pi}{3\pi} = \frac{1}{3}$$

$$b = \pm \frac{1}{3}$$

Since the function is increasing, a or b must be negative, because the cotangent function decreases. We know that $a = -5$, so b will be positive, which means $b = 1/3$. With $a = -5$ and $b = 1/3$, the equation of the cotangent function could be

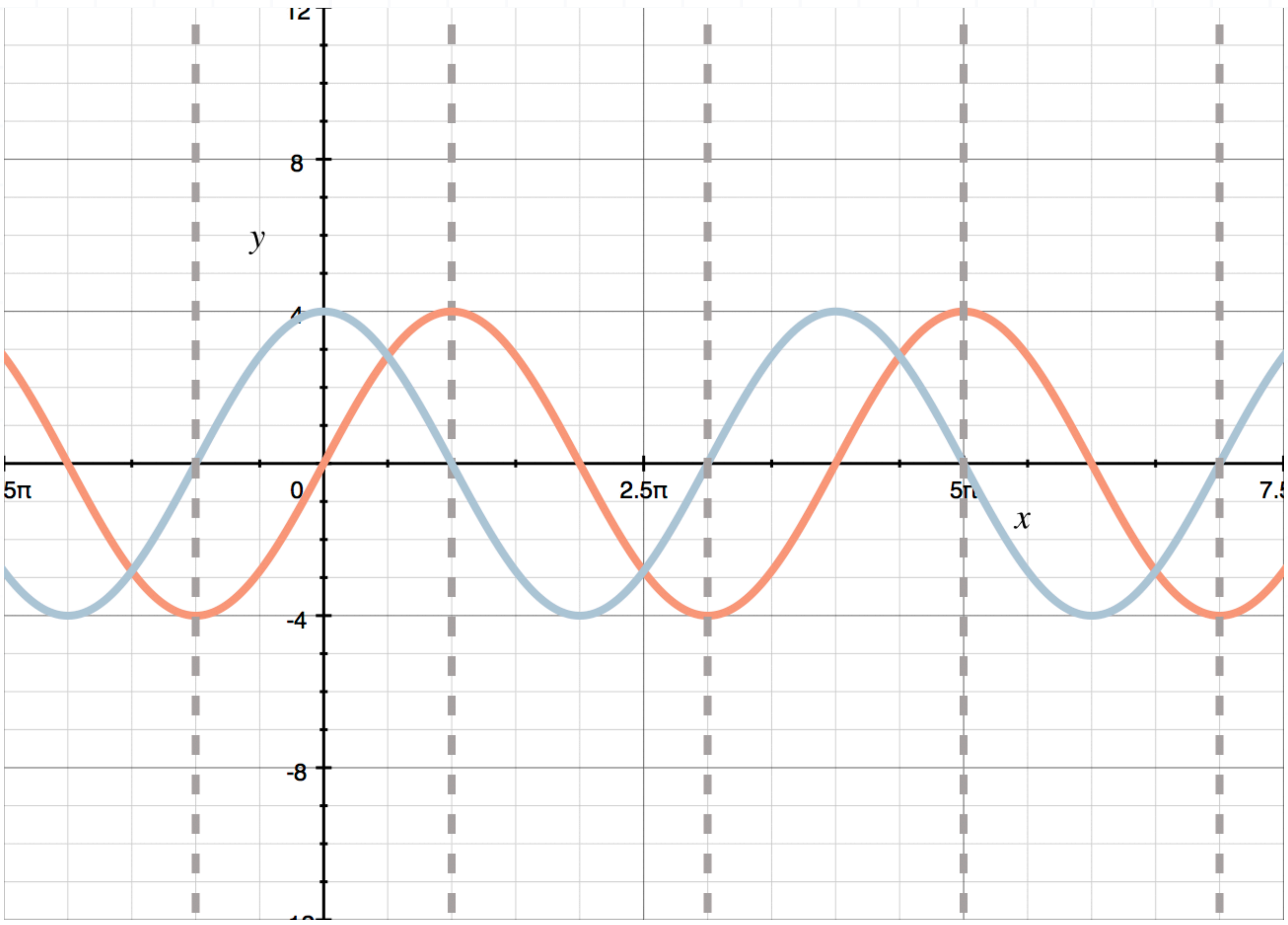
$$y = -5 \cot \left(\frac{x}{3} \right)$$

6. Sketch the graph of $y = 4 \tan(x/2)$, using the graph of $y = 4 \sin(x/2)$ in red and $y = 4 \cos(x/2)$ in blue.

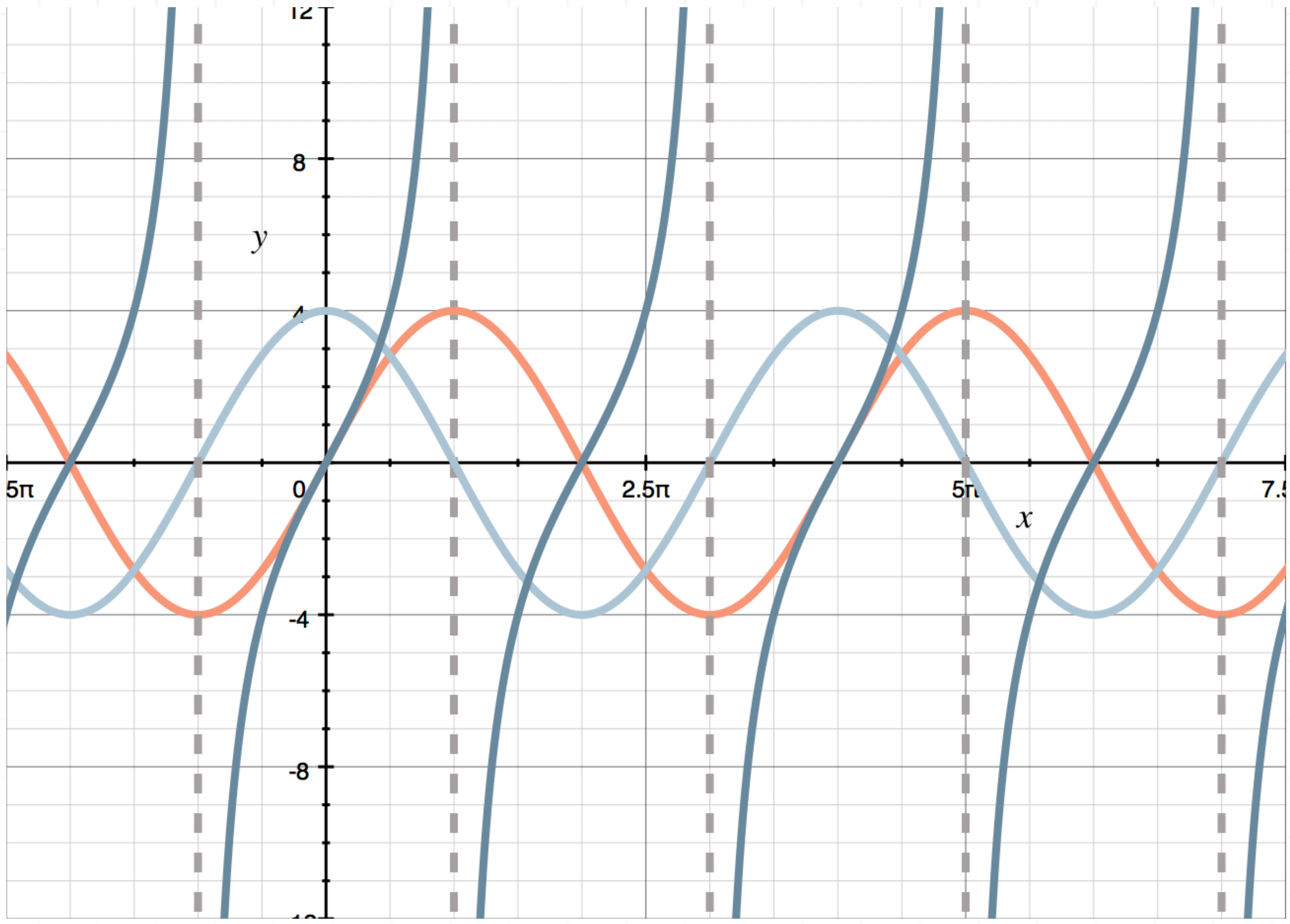


Solution:

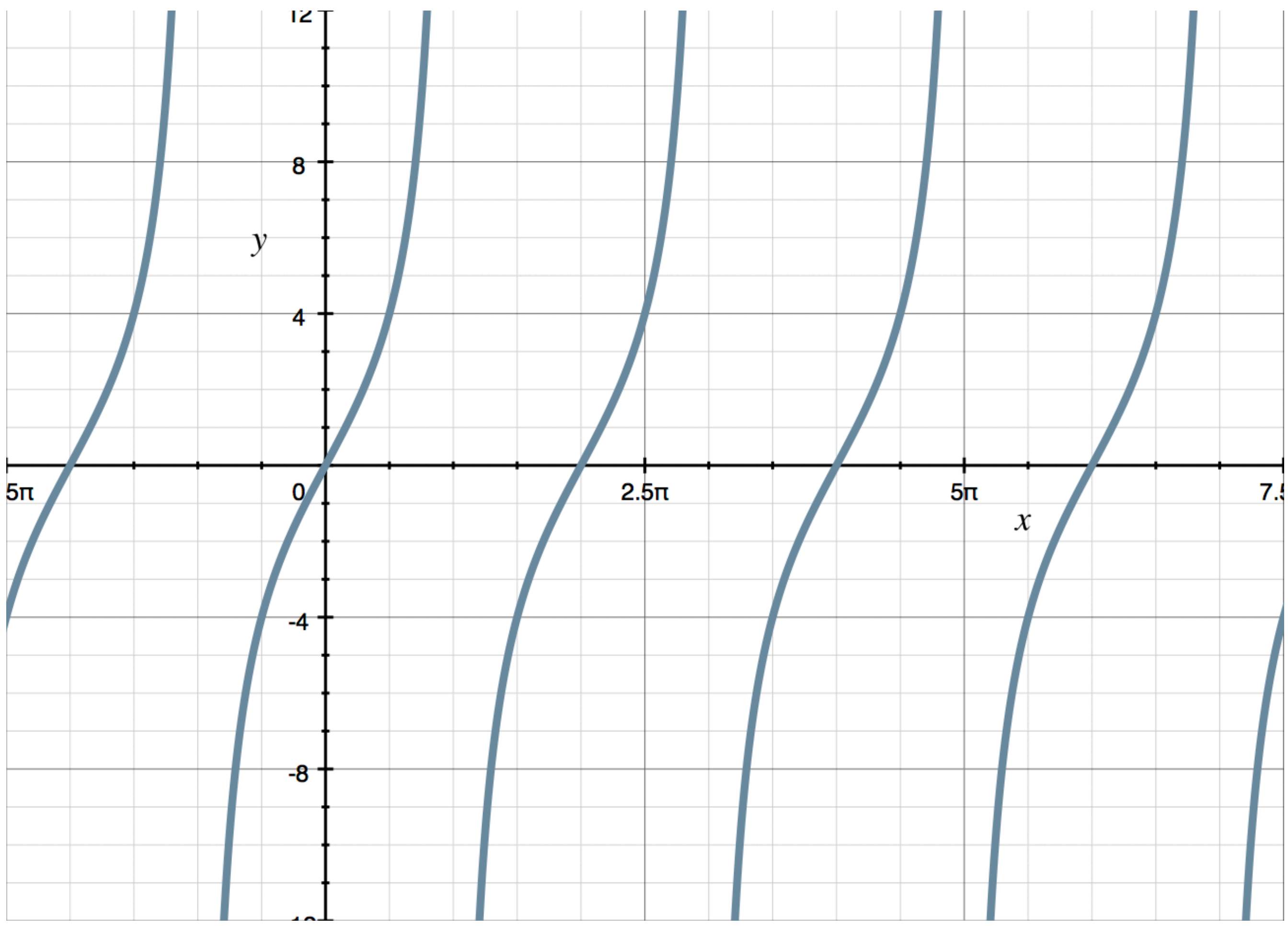
We know that $\tan x = \sin x / \cos x$, so we can say that all the zeros of the $\sin x$ are also zeros of $\tan x$ (where $\sin x = 0$, $\tan x = 0$ as well), and all zeros of $\cos x$ are vertical asymptotes of $\tan x$ (where $\cos x = 0$, $\tan x$ has vertical asymptotes).



Therefore we get vertical asymptotes at $\ldots, -\pi, \pi, 3\pi, \ldots$ and we know that tangent will be zero at $\ldots, 0, 2\pi, 4\pi, \ldots$. Since the tangent function is increasing, we get the graph of $y = 4\tan(x/2)$.

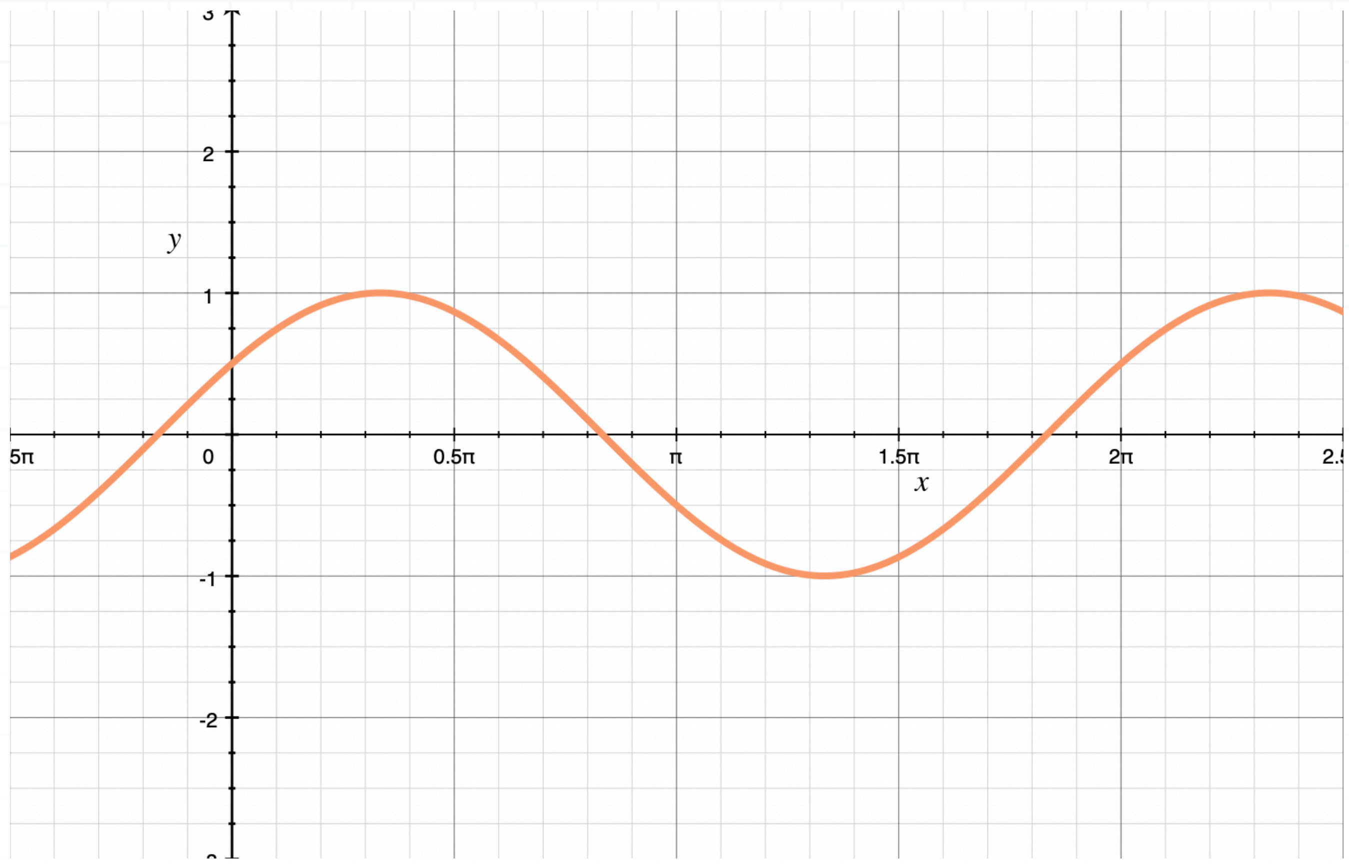


Taking away the sine and cosine functions, the graph of $y = 4\tan(x/2)$ is



HORIZONTAL AND VERTICAL SHIFTS

■ 1. Determine the equation of the cosine function shown in the graph.



Solution:

The graph looks like it could be the cosine function, shifted to the right by a little less than $\pi/2$, so we need to find the equation of the function in the form $y = \cos(x - c)$. To find c, we need to solve the equation

$$\cos(0 - c) = \frac{1}{2}$$

$$\cos(-c) = \frac{1}{2}$$

From the unit circle, we know that $\cos(-\pi/3) = 1/2$, so $c = \pi/3$. Therefore, the equation of the function is

$$y = \cos\left(x - \frac{\pi}{3}\right)$$

2. Determine the phase shift and the vertical shift of the sine function.

$$y = 2\sin\left(x + \frac{\pi}{6}\right) - 2$$

Solution:

The general form of the sine function is

$$y = a\sin(b(x + c)) + d$$

where c represents the horizontal shift and d represents the vertical shift. When c is positive, then the sine function gets shifted c units to the left horizontally. When d is negative, then the curve is shifted d units downward. So this particular sine function gets shifted $\pi/6$ units to the left and 2 units downward.

3. Sketch the graph of $y = \sin(\theta - \pi)$.

Solution:

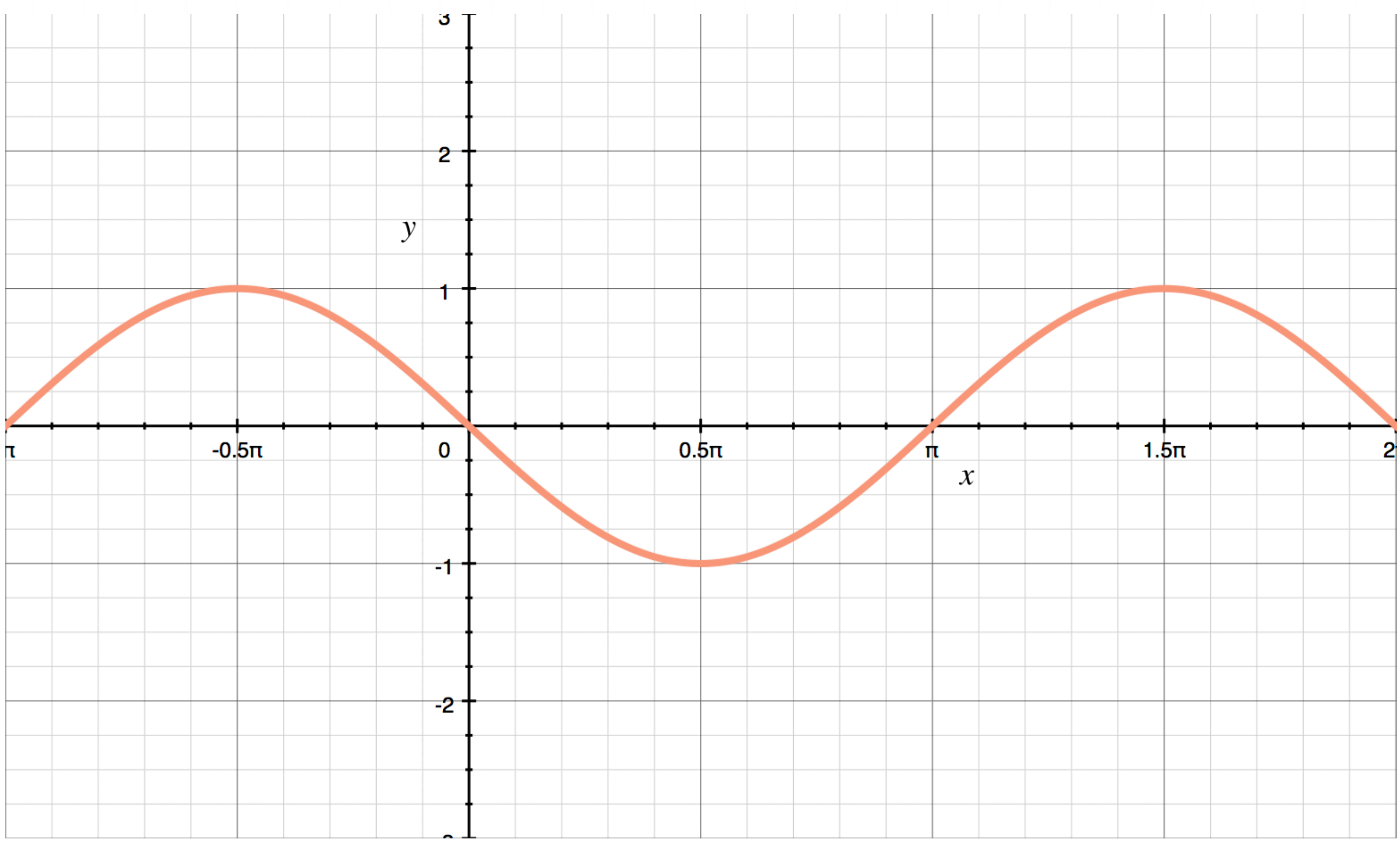
We know that a set of points along $y=\sin\theta$ is

$$(0,0) \qquad \left(\frac{\pi}{2},1\right) \qquad (\pi,0) \qquad \left(\frac{3\pi}{2},-1\right) \qquad (2\pi,0)$$

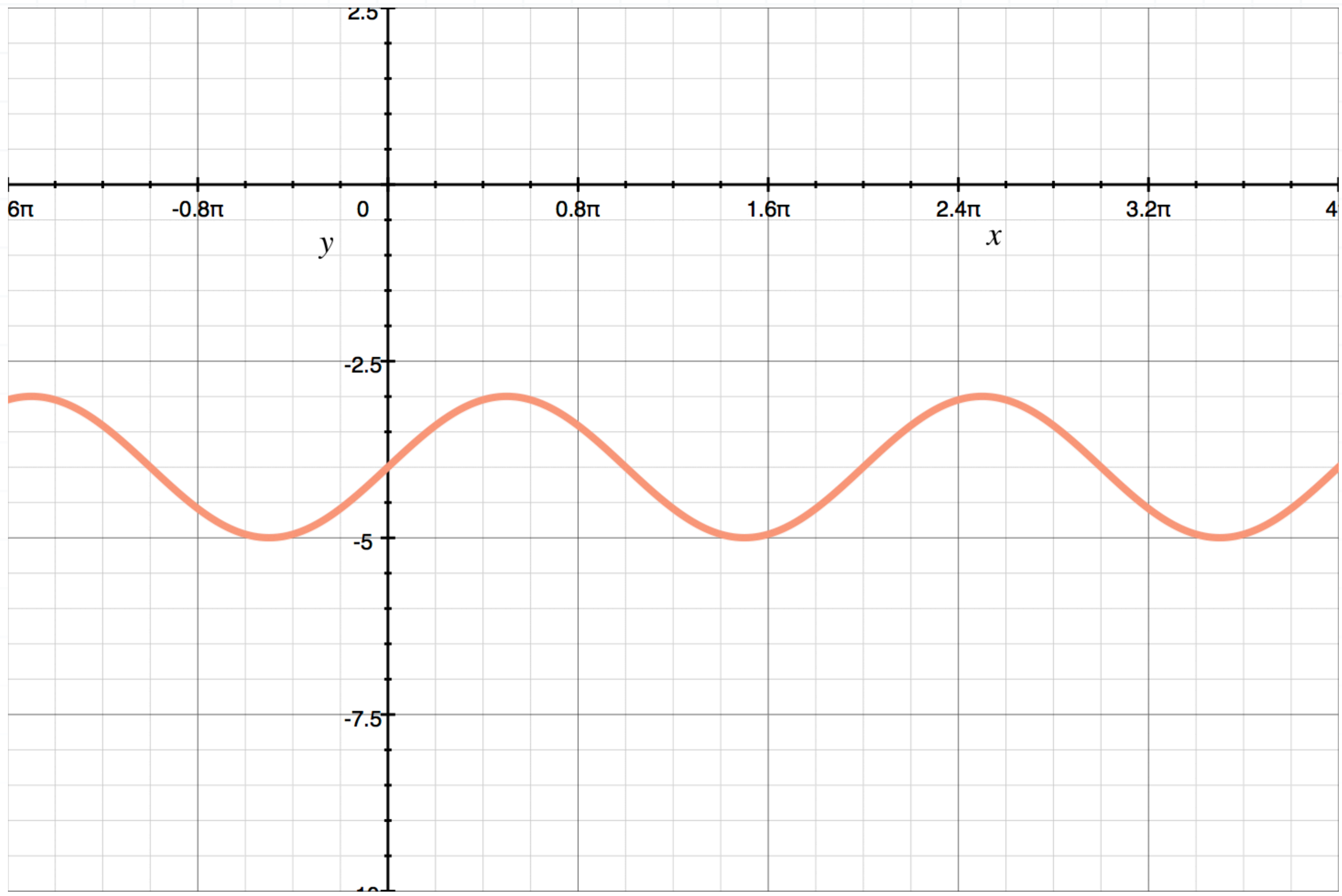
Because $c=-\pi$, which means c is negative, the graph of $y=\sin(\theta-\pi)$ is shifted to the right, which means we'll add π to the x-value from each coordinate point.

$$(\pi,0) \qquad \left(\frac{3\pi}{2},1\right) \qquad (2\pi,0) \qquad \left(\frac{5\pi}{2},-1\right) \qquad (3\pi,0)$$

If we plot the points and connect them with a smooth curve, we get the graph of $y=\sin(\theta-\pi)$.



4. Determine the equation of the sine function shown in the graph.



Solution:

The graph looks like it could be the sine function, shifted down. So we need to find the function in the form $y = \sin x + d$.

To find d, we need to solve the equation

$$\sin(0) - d = 4$$

$$-d = 4$$

$$d = -4$$

Therefore, we get the function $y = \sin x - 4$.

■ 5. A trigonometric function has an amplitude of 3 units, a horizontal shift to the left by $\pi/4$, a vertical shift down by 7 units, and no reflections. Represent the curve with a cosine function.

Solution:

The general form of the cosine function is

$$y = a\cos(b(x + c)) + d$$

Since the trigonometric function has an amplitude of 3 units, we know $|a| = 3$. If the function doesn't have a reflection, then a can't be negative, which means $a = 3$.

$$y = 3\cos(b(x + c)) + d$$

Since we don't have information about the period, we'll assume that the period is $b = 1$.

$$y = 3\cos(x + c) + d$$

A horizontal shift to the left by $\pi/4$ means we add $\pi/4$ to the argument to get

$$y = 3\cos\left(x + \frac{\pi}{4}\right) + d$$

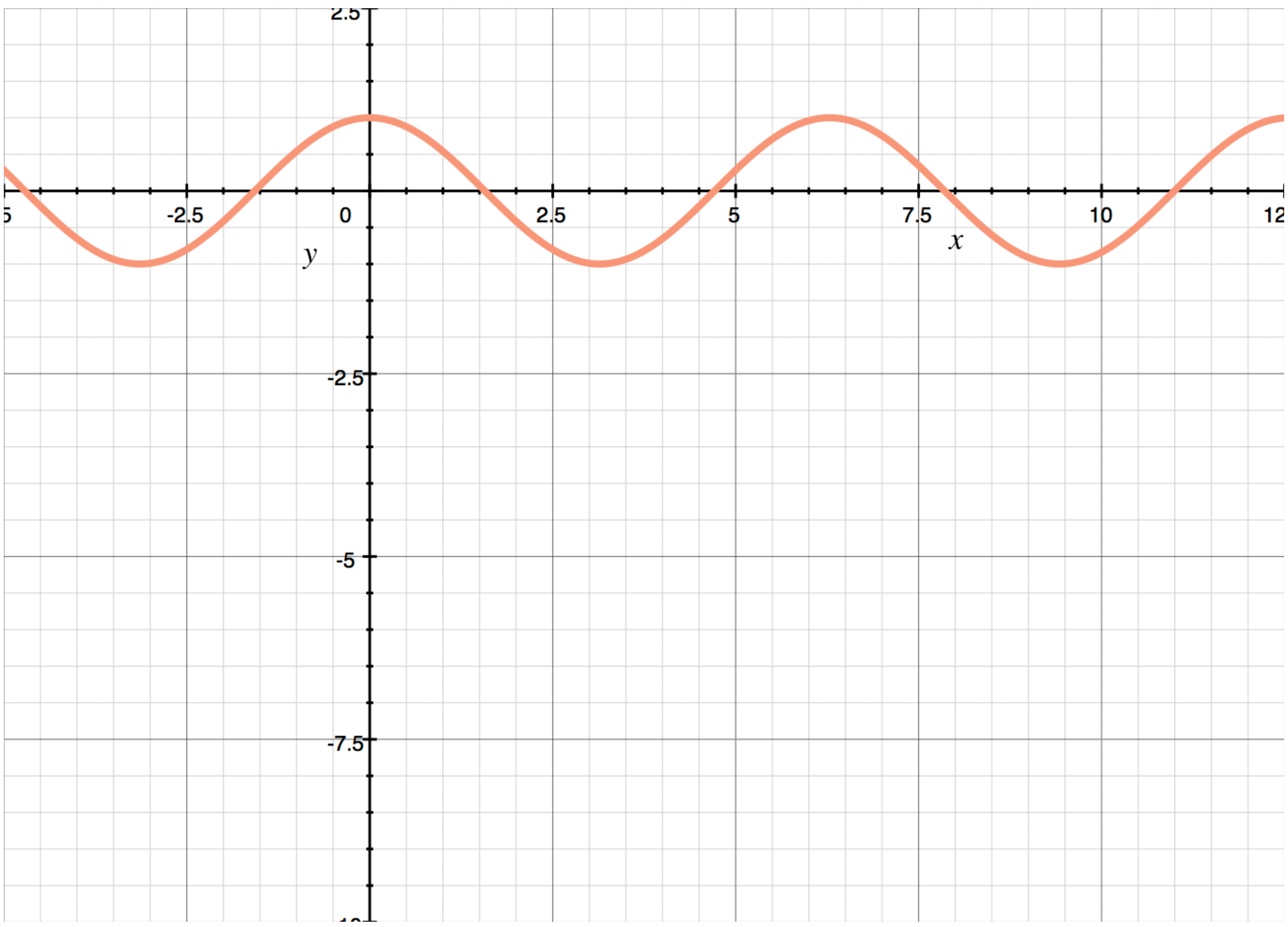
A vertical shift down by 7 means we subtract 7 from the function to get

$$y = 3\cos\left(x + \frac{\pi}{4}\right) - 7$$

■ 6. Sketch the graph of $y = \cos\theta - 5$.

Solution:

We can start by sketching the graph of $y = \cos\theta$.

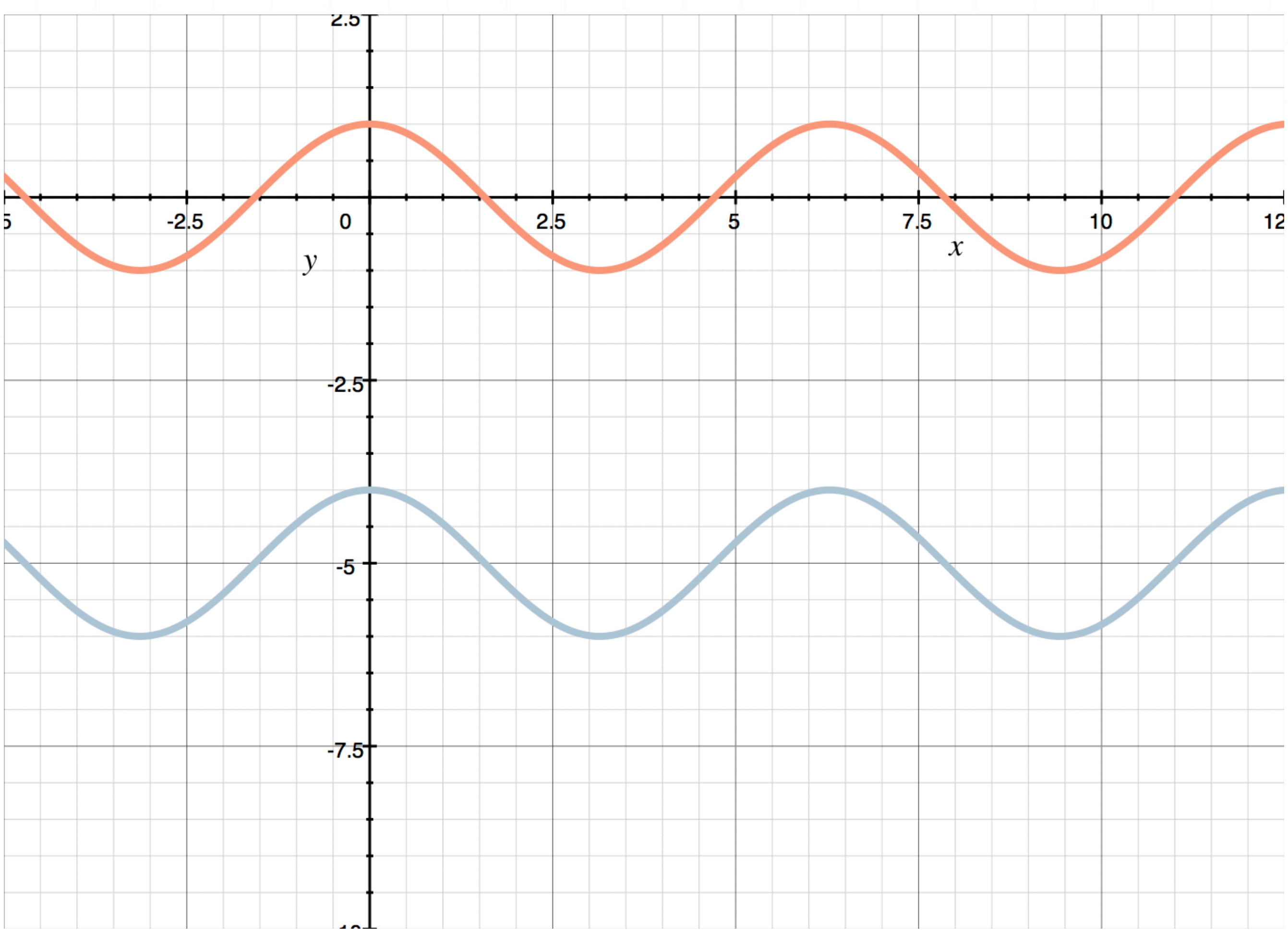


Then the graph of $y = \cos\theta - 5$ will be identical to this one, but shifted 5 units downward. We could also find points on $y = \cos\theta - 5$ algebraically by subtracting 5 from the y-values in the coordinate points along $y = \cos\theta$.

$$y = \cos\theta: \quad (0,1) \quad \left(\frac{\pi}{2},0\right) \quad (\pi,-1) \quad \left(\frac{3\pi}{2},0\right) \quad (2\pi,1)$$

$$y = \cos\theta - 5: \quad (0,-4) \quad \left(\frac{\pi}{2},-5\right) \quad (\pi,-6) \quad \left(\frac{3\pi}{2},-5\right) \quad (2\pi,-4)$$

If we keep the graph of $y = \cos\theta$ in red and add the graph of $y = \cos\theta - 5$ in blue, we get



GRAPHING TRANSFORMATIONS

■ 1. What are the period, amplitude, and range of the function?

$$y = -3\cos(2\pi x - 1) + 4$$

Solution:

Rewrite the function in the form $y = a\sin(b(\theta + c)) + d$.

$$y = -3\cos\left(2\pi\left(x - \frac{1}{2\pi}\right)\right) + 4$$

Because this is a cosine function, the period is $2\pi/|b|$, so the period of this function is

$$\frac{2\pi}{|b|} = \frac{2\pi}{|2\pi|} = \frac{2\pi}{2\pi} = 1$$

The amplitude of the function is given by $|a| = |-3| = 3$.

The basic cosine function has a range of $[-1,1]$, so the range of a cosine function with an amplitude of 3 will be $[-3,3]$. But this particular cosine function has a vertical shift up of 4 units, so the range will be

$$[-3 + 4,\ 3 + 4]$$

$$[1,\ 7]$$

To summarize, the period is 1, the amplitude is 3, and the range is $[1, 7]$.

■ 2. Find the equation of the curve that's the result of applying the following sequence of transformations to $f(x) = \sin(x - \pi)$.

1. A horizontal compression by a factor of 2

2. A horizontal shift to the right by 3π

3. A vertical stretch by a factor of 5

4. A reflection over the x-axis

5. A vertical shift down by 2

Solution:

Starting with $f(x) = \sin(x - \pi)$,

1. A horizontal compression by a factor of 2 means that we need to multiply the argument by 2 to get $f(x) = \sin(2(x - \pi))$, or $f(x) = \sin(2x - 2\pi)$.

2. A horizontal shift to the right by 3π means we subtract 3π from the argument to get

$$f(x) = \sin(2(x - 3\pi) - 2\pi)$$

$$f(x) = \sin(2x - 6\pi - 2\pi)$$

$$f(x) = \sin(2x - 8\pi)$$

3. A vertical stretch by a factor of 5 means we multiply the sine function by 5 to get $f(x) = 5\sin(2x - 8\pi)$.

4. A reflection over the x-axis means we multiply the function by -1 to get $f(x) = -5\sin(2x - 8\pi)$.

5. A vertical shift down by 2 means we subtract 2 from the function to get $f(x) = -5\sin(2x - 8\pi) - 2$.

Now we can rewrite the function in the form $y = a\sin(b(\theta + c)) + d$.

$$f(x) = -5\sin(2(x - 4\pi)) - 2$$

■ 3. Which function has an amplitude of 2 and a range of $[-3,1]$?

$-2\sin(5x - 3\pi) - 2$	$2\sin(3x - 3\pi) - 1$
$-4\cos(2x + 3\pi) - 2$	$4\cos(2x + 3\pi) - 1$

Solution:

The sine functions have values $a = -2$ and $a = 2$, both of which give an amplitude of 2, whereas the cosine functions both have an amplitude of 4.

Now we need to check the vertical shifts in order to check the range. For any function $\pm 2\sin x$, the range is $[-2,2]$. A vertical shift down by 2 units in $-2\sin(5x - 3\pi) - 2$ would shift that range to $[-2 - 2,\ 2 - 2] = [-4,0]$. Whereas a vertical shift down by 1 unit in $2\sin(3x - 3\pi) - 1$ would shift the range $[-2,2]$ to $[-2-1, 2-1] = [-3,1]$.

Therefore, $2\sin(3x - 3\pi) - 1$ is the only curve with an amplitude of 2 and a range of $[-3,1]$.

■ 4. Find the equation of the curve that's the result of applying the following sequence of transformations to $f(x) = 2\sin(3x)$.

1. A horizontal shift to the left by $\pi/12$

2. A reflection over the y-axis

3. A reflection over the x-axis

Solution:

Starting with $f(x) = 2\sin(3x)$,

1. A horizontal shift to the left by $\pi/12$ means we add $\pi/12$ to the x-value to get

$$f(x) = 2\sin\left(3\left(x + \frac{\pi}{12}\right)\right)$$

$$f(x) = 2\sin\left(3x + \frac{3\pi}{12}\right)$$

$$f(x) = 2\sin\left(3x + \frac{\pi}{4}\right)$$

2. A reflection over the y-axis means we replace x with $-x$ to get

$$f(x) = 2\sin\left(3(-x) + \frac{\pi}{4}\right)$$

$$f(x) = 2\sin\left(-3x + \frac{\pi}{4}\right)$$

3. A reflection over the x-axis means we multiply the function by -1 to get

$$f(x) = -2\sin\left(-3x + \frac{\pi}{4}\right)$$

Now we can rewrite the function in the form $y = a\sin(b(\theta + c)) + d$.

$$f(x) = -2\sin\left(-3\left(x - \frac{\pi}{12}\right)\right)$$

■ 5. What will be the zeros of the function $f(x) = \cos x$ after the following sequence of transformations?

1. A horizontal compression by a factor of 2

2. A horizontal shift to the right by $\pi/6$

3. A vertical stretch by a factor of 5

Solution:

Starting with $f(x) = \cos x$,

1. A horizontal compression by a factor of 2 means that we need to replace x in the argument with $2x$ to get $f(x) = \cos(2x)$.

2. A horizontal shift to the right by $\pi/6$ means that we need to subtract $\pi/6$ from the x-value to get

$$f(x) = \cos\left(2\left(x - \frac{\pi}{6}\right)\right)$$

3. A vertical stretch by a factor of 5 means that we need to multiply the function by 5 to get

$$f(x) = 5\cos\left(2\left(x - \frac{\pi}{6}\right)\right)$$

Now we need to look for the zeros of this function. The cosine function will be 0 at $x = \pi/2$, $x = 3\pi/2$, and all angles coterminal with these.

$$2\left(x - \frac{\pi}{6}\right) = \frac{\pi}{2}$$

$$x - \frac{\pi}{6} = \frac{\pi}{4}$$

$$x = \frac{\pi}{4} + \frac{\pi}{6} = \frac{3\pi}{12} + \frac{2\pi}{12} = \frac{5\pi}{12}$$

and

$$2\left(x-\frac{\pi}{6}\right)=\frac{3\pi}{2}$$

$$x-\frac{\pi}{6}=\frac{3\pi}{4}$$

$$x=\frac{3\pi}{4}+\frac{\pi}{6}=\frac{9\pi}{12}+\frac{2\pi}{12}=\frac{11\pi}{12}$$

So the function will have zeros at

$$x=\left\{\frac{5\pi}{12}+2\pi k \ \mid \ k\in\mathbb{Z}\right\}\cup\left\{\frac{11\pi}{12}+2\pi k \ \mid \ k\in\mathbb{Z}\right\}$$

These to values of x are both angles less than π ($5\pi/12<\pi$ and $11\pi/12<\pi$). This suggests that we might have two more zeros somewhere on $\pi<x<2\pi$. Let's set the argument of the transformed function equal to the next two angles at which $\cos x=0$, which are $5\pi/2$ and $7\pi/2$.

$$2\left(x-\frac{\pi}{6}\right)=\frac{5\pi}{2}$$

$$x-\frac{\pi}{6}=\frac{5\pi}{4}$$

$$x=\frac{5\pi}{4}+\frac{\pi}{6}=\frac{15\pi}{12}+\frac{2\pi}{12}=\frac{17\pi}{12}$$

and

$$2\left(x-\frac{\pi}{6}\right)=\frac{7\pi}{2}$$

$$x - \frac{\pi}{6} = \frac{7\pi}{4}$$

$$x = \frac{7\pi}{4} + \frac{\pi}{6} = \frac{21\pi}{12} + \frac{2\pi}{12} = \frac{23\pi}{12}$$

All four of these values lie in the principal interval $[0,2\pi)$, so we want to include all of them in our expression for the zeros of the transformed functions.

$$x = \left\{ \frac{5\pi}{12} + 2\pi k \ \mid \ k \in \mathbb{Z} \right\} \cup \left\{ \frac{11\pi}{12} + 2\pi k \ \mid \ k \in \mathbb{Z} \right\}$$

$$x = \left\{ \frac{17\pi}{12} + 2\pi k \ \mid \ k \in \mathbb{Z} \right\} \cup \left\{ \frac{23\pi}{12} + 2\pi k \ \mid \ k \in \mathbb{Z} \right\}$$

However, we should notice that the four angles $5\pi/12$, $11\pi/12$, $17\pi/12$, and $23\pi/12$ all differ by $\pi/2$, which means we can actually rewrite the combination of the four sets as just

$$x = \left\{ \frac{5\pi}{12} + \frac{k\pi}{2} \ \mid \ k \in \mathbb{Z} \right\}$$

■ 6. What transformations are applied to transform $y = \sin\theta$ into the given function?

$$y = 3\sin\left(3\theta + \frac{3\pi}{2}\right)$$

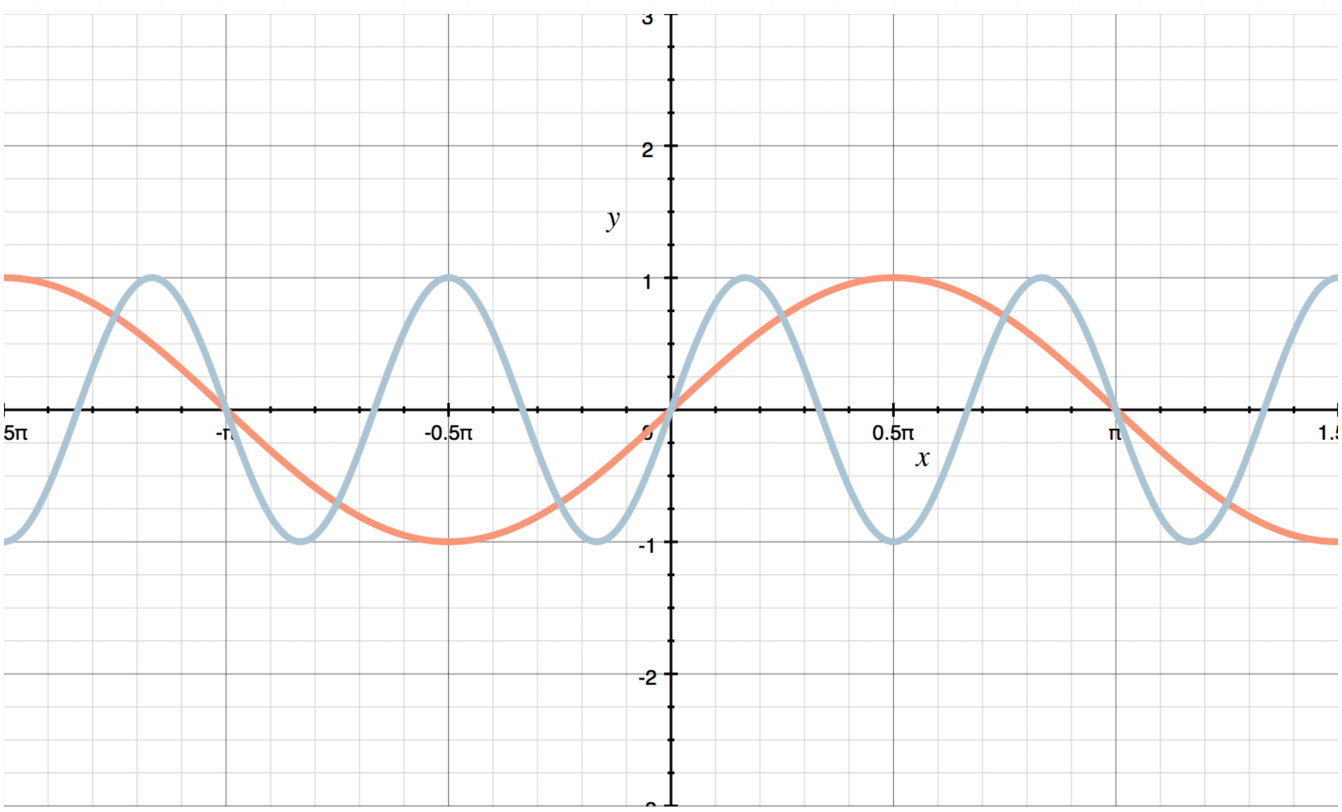
Solution:

To put the given sine function into the form $a \sin(b(\theta + c)) + d$, we need to factor 3 out of the argument.

$$y = 3 \sin \left(3 \left(\theta + \frac{\pi}{2}\right)\right)$$

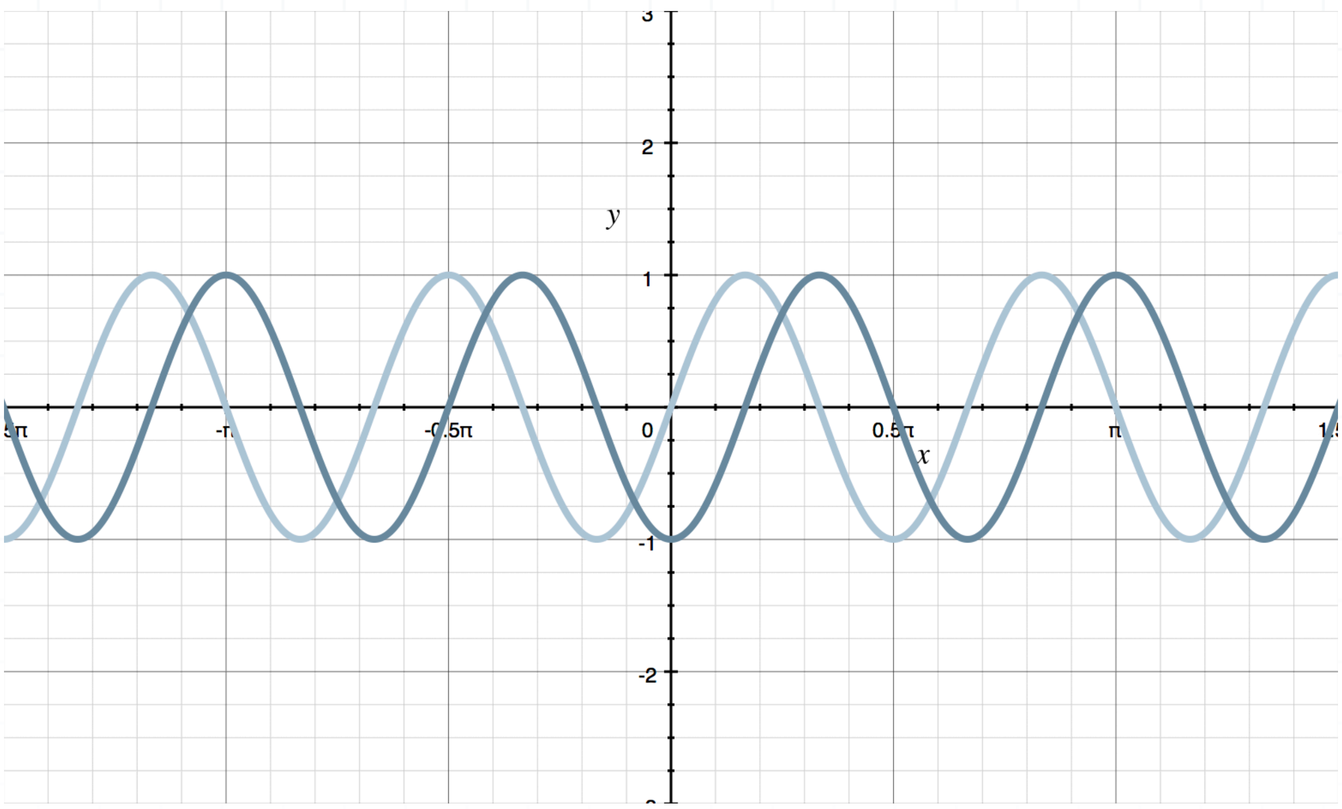
With the function now in the form $a \sin(b(\theta + c)) + d$, we can see that we have $a = 3$, $b = 3$, $c = \pi/2$, and $d = 0$.

Since $b = 3$, the function gets horizontally compressed by a factor of 3, which means all the x-values in the coordinate points get divided by 3, while the y-values stay the same. If we graph $y = \sin \theta$ in red and $y = \sin(3\theta)$ in blue, we get

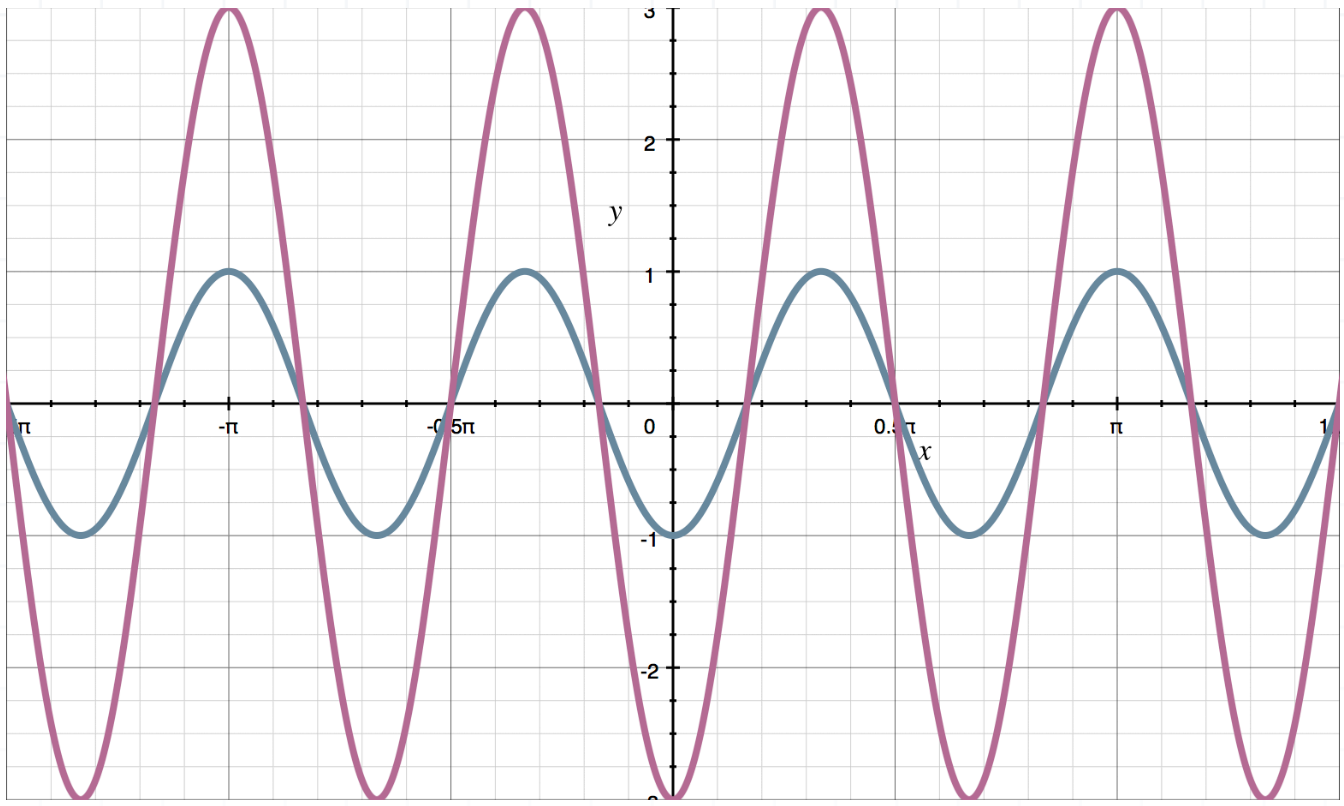


The horizontal shift is given by $c = \pi/2$. Compared with the graph of $y = \sin(3\theta)$, the graph of $y = \sin(3(\theta + \pi/2))$ will be shifted $\pi/2$ units to the left, which means we'll subtract $\pi/2$ from the x-values in the coordinate points along $y = \sin(3\theta)$, while keeping the y-values the same.

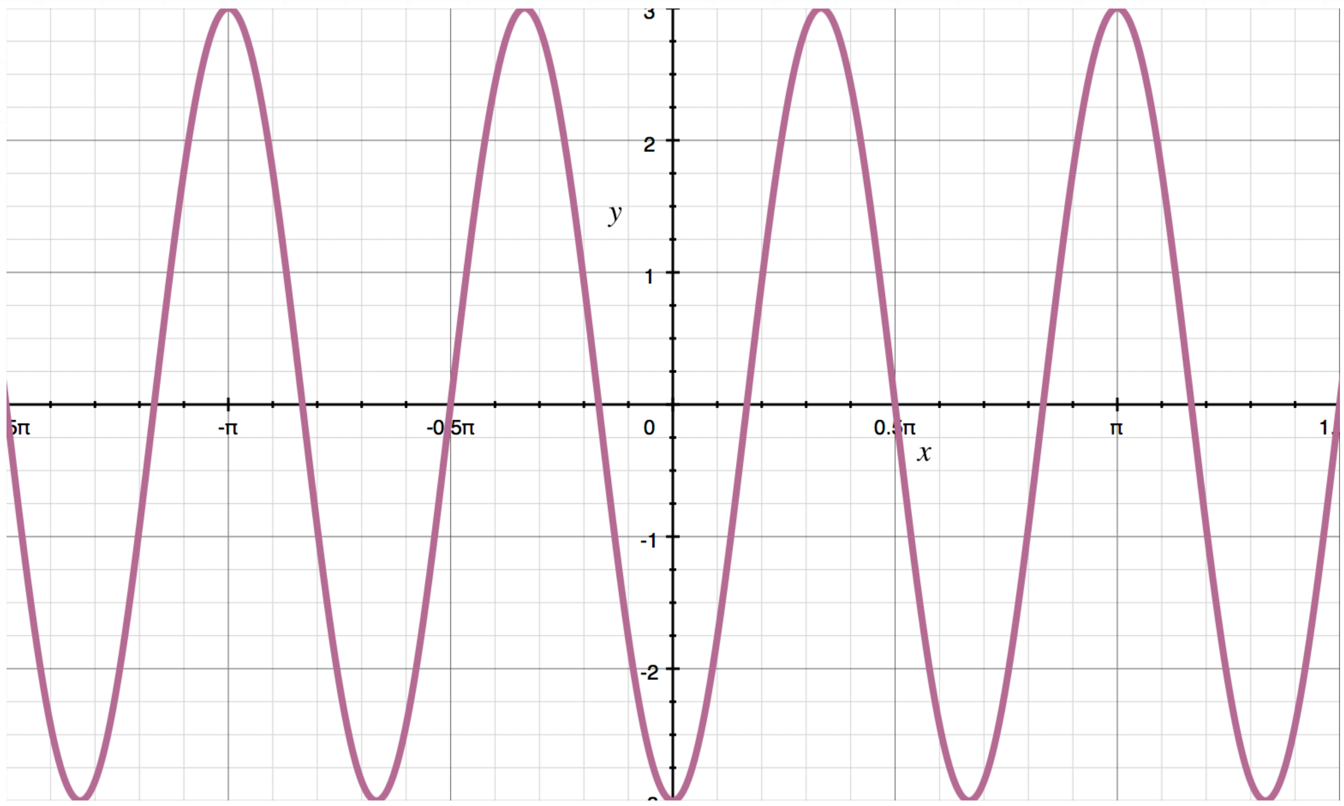
If we graph $y = \sin(3\theta)$ in blue and $y = \sin(3(\theta + \pi/2))$ in dark blue, we get



Because $a = 3$, the last transformation is a vertical stretch by a factor of 3, which means the y-value of every point gets multiplied by 3. If we graph $y = \sin(3(\theta + \pi/2))$ in dark blue and $y = 3\sin(3(\theta + \pi/2))$ in purple, we get



Taking the previous graph away, the graph of $y = 3\sin(3(\theta + \pi/2))$ is



GRAPHING COMBINATIONS

■ 1. Find the period of the function.

$$\tan(3\theta - \pi) - \sin(6\theta)$$

Solution:

The period of $\tan(3\theta - \pi)$ is $\pi/3$, and the period of $\sin(6\theta)$ is $\pi/3$. When the combination function is a sum or difference like it is in this problem, we take the least common multiple of the two periods of the original functions as the period of the combination. The least common multiple of $\pi/3$ and $\pi/3$ is $\pi/3$, so the period of the combination is $\pi/3$.

■ 2. Find the period of the function.

$$\frac{\sin\left(5\theta - \frac{\pi}{2}\right)}{\cos(2\theta)}$$

Solution:

The period of $\sin(5\theta - (\pi/2))$ is $2\pi/5$, and the period of $\cos(2\theta)$ is π. When the combination function is a product or quotient like it is in this problem, we take the least common multiple of the original functions as the period of the combination. So since the least common multiple of π and $2\pi/5$ is 2π, the period of the combination is 2π.

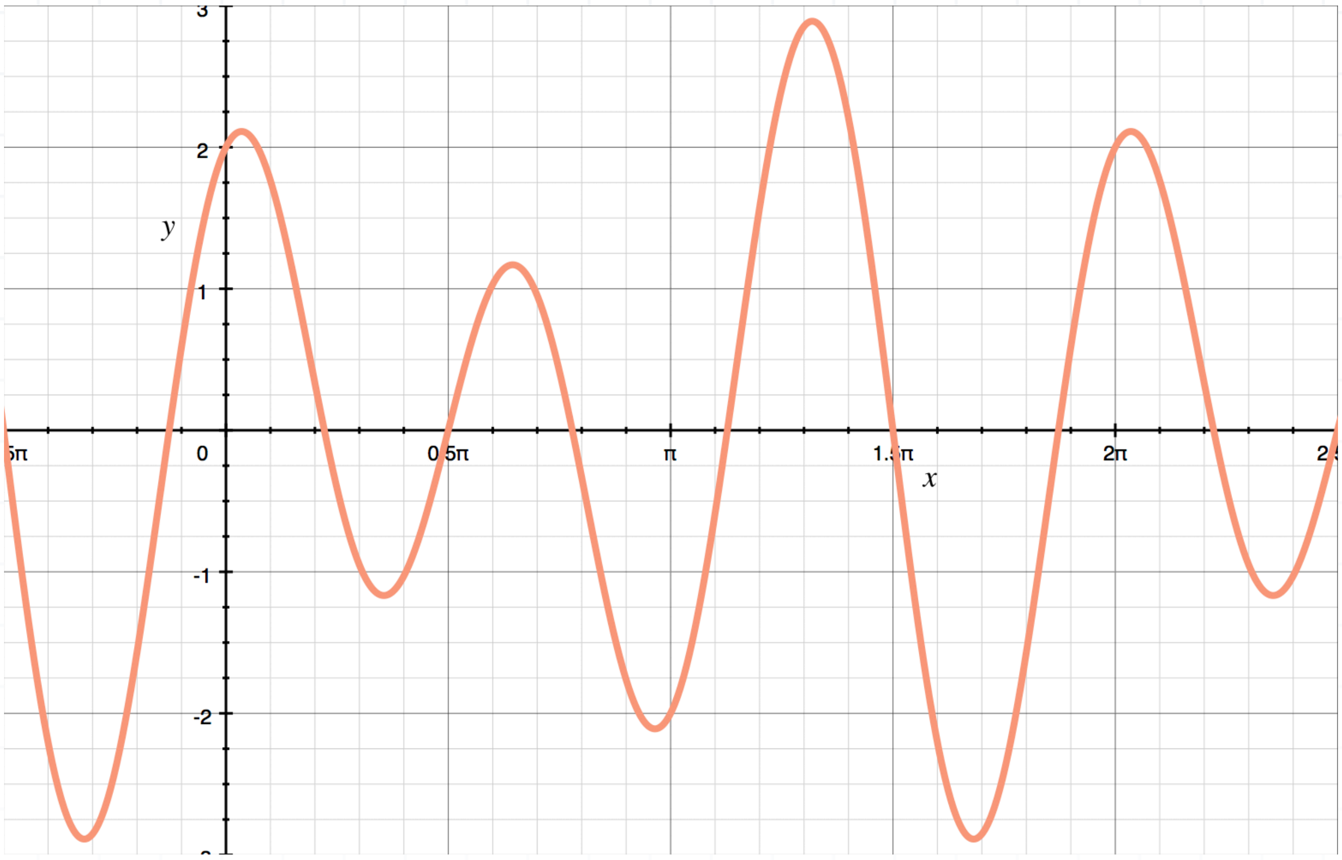
■ 3. Graph the combination function $2\cos(3\theta) + \sin(2\theta)$.

Solution:

We want to recognize that $2\cos(3\theta) + \sin(2\theta)$ is really just the sum of $2\cos(3\theta)$ and $\sin(2\theta)$.

The period of $2\cos(3\theta)$ is $2\pi/3$, and the period of $\sin(2\theta)$ is π. When the combination function is a sum or difference like it is in this problem, we take the least common multiple of the two periods of the original functions as the period of the combination. So since the least common multiple of $2\pi/3$ and π is 2π, the period of the combination is 2π.

We can sketch the graph of the combination simply by adding together the values of the other two functions at each point. For example, at $\theta = 0$, $2\cos(3\theta)$ has a value of 2, and $\sin(2\theta)$ has a value of 0, which means $2\cos(3\theta) + \sin(2\theta)$ will have a value of $2 + 0 = 2$, and we can plot that point. We'll continue on like this until we have a rough sketch of the graph over one full 2π period, and then we'll repeat the sketch of one period in both directions.



4. Graph the combination function.

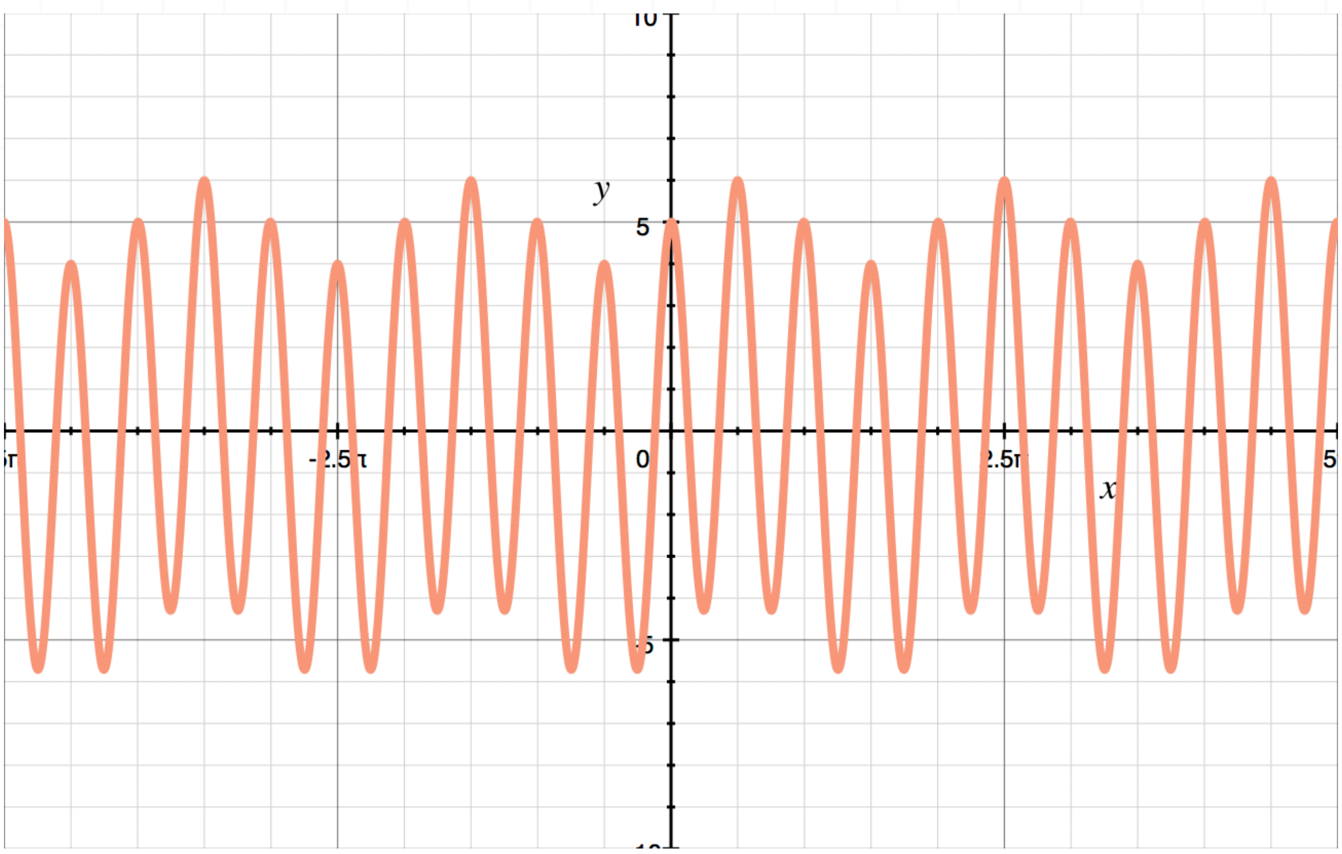
$$\cos\left(\theta - \frac{\pi}{2}\right) - 5\sin\left(4\theta + \frac{3\pi}{2}\right)$$

Solution:

We want to recognize that the function is really just the difference of $\cos(\theta - (\pi/2))$ and $5\sin(4\theta + (3\pi/2))$.

The period of $\cos(\theta - (\pi/2))$ is 2π, and the period of $5\sin(4\theta + (3\pi/2))$ is $\pi/2$. When the combination function is a sum or difference like it is in this problem, we take the least common multiple of the two periods of the original functions as the period of the combination. So since the least common multiple of $\pi/2$ and 2π is 2π, the period of the combination is 2π.

We can sketch the graph of the combination simply by subtracting the values of the other two functions at each point. For example, at $\theta = 0$, $\cos(\theta - (\pi/2))$ has a value of 0, and $5\sin(4\theta + (3\pi/2))$ has a value of -5, which means the original function will have a value of $0 - (-5) = 5$, and we can plot that point. We'll continue on like this until we have a rough sketch of the graph over one full 2π period, and then we'll repeat the sketch of one period in both directions.



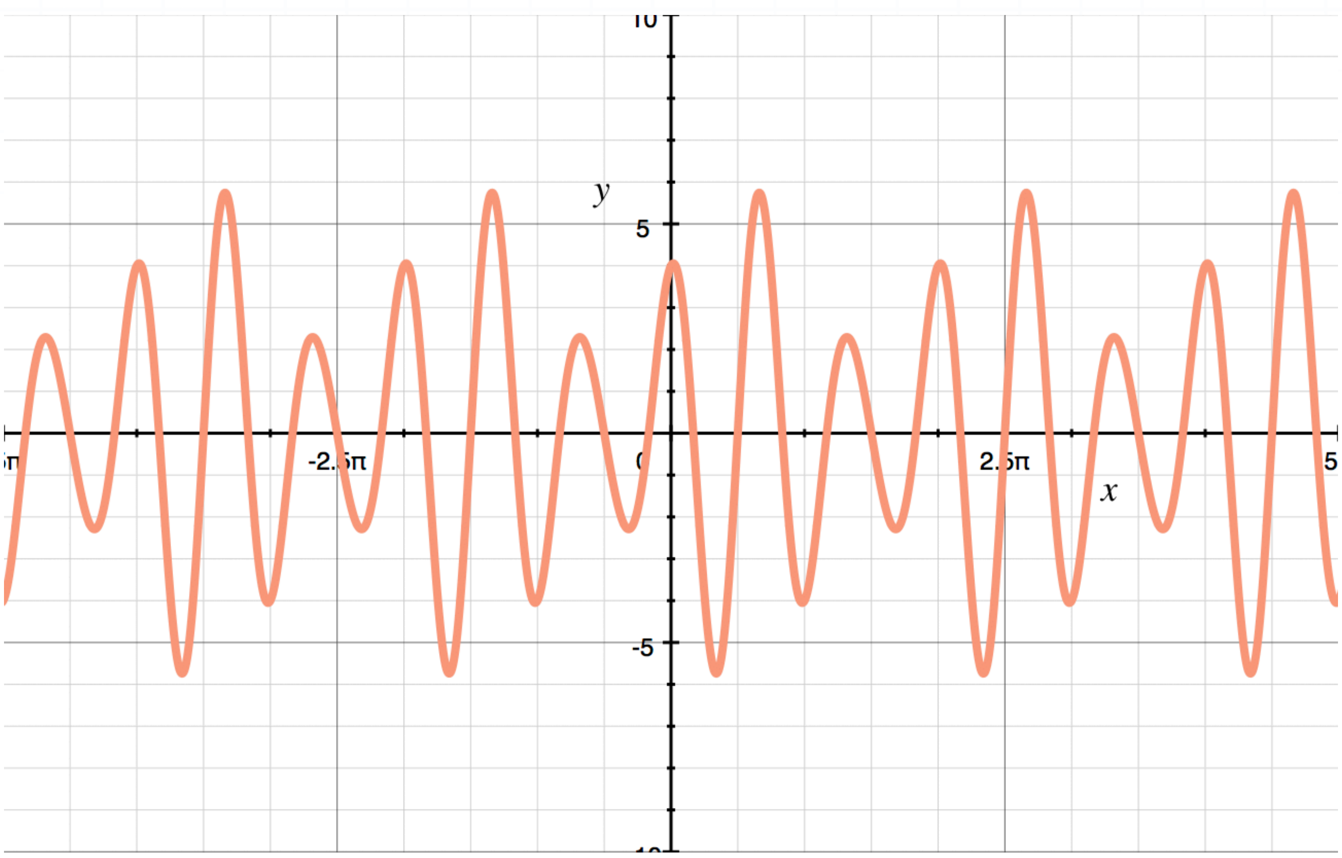
5. Graph the combination function $(2\cos(3\theta - 2\pi))(\sin\theta + 2)$.

Solution:

We want to recognize that $(2\cos(3\theta - 2\pi))(\sin\theta + 2)$ is really just the product of $(2\cos(3\theta - 2\pi))$ and $(\sin\theta + 2)$.

The period of $(2\cos(3\theta - 2\pi))$ is $2\pi/3$, and the period of $(\sin\theta + 2)$ is 2π. When the combination function is a product or quotient like it is in this problem, we take the least common multiple of the original functions as the period of the combination. So since the least common multiple of 2π and $2\pi/3$ is 2π, the period of the combination is 2π.

We can sketch the graph of the combination simply by multiplying together the values of the other two functions at each point. For example, at $\theta = 0$, $(2\cos(3\theta - 2\pi))$ has a value of 2, and $(\sin\theta + 2)$ has a value of 2, which means $(2\cos(3\theta - 2\pi))(\sin\theta + 2)$ will have a value of $2(2) = 4$, and we can plot that point. We'll continue on like this until we have a rough sketch of the graph over one full 2π period, and then we'll repeat the sketch of one period in both directions.



■ 6. Graph the combination function.

$$\frac{\cos\left(3\theta + \frac{3\pi}{4}\right)}{3\sin\left(\theta - \frac{\pi}{2}\right)}$$

Solution:

We want to recognize that the given quotient is really just the quotient of $\cos(3\theta + (3\pi/4))$ and $3\sin(\theta - (\pi/2))$.

We expect that the behavior of the quotient combination will be a little bizarre near any values where $3\sin(\theta - (\pi/2))$ gets close to 0, since $3\sin(\theta - (\pi/2))$ is the denominator of the quotient combination, and a function is undefined whenever its denominator is 0.

Let's look at where $3\sin(\theta - (\pi/2))$ will get close to 0 by setting $3\sin(\theta - (\pi/2))$ equal to 0 and solving for θ.

$$3\sin\left(\theta - \frac{\pi}{2}\right) = 0$$

The sine function is equal to 0 at angles of $0, \pi, 2\pi, 3\pi, 4\pi, \ldots$, in other words, at multiples of π, or any value $n\pi$ where n is an integer. Therefore we can solve an equation for θ.

$$\theta - \frac{\pi}{2} = n\pi$$

$$\theta = n\pi + \frac{\pi}{2}$$

$$\theta = \frac{2n\pi}{2} + \frac{\pi}{2}$$

$$\theta = \frac{(2n+1)\pi}{2}$$

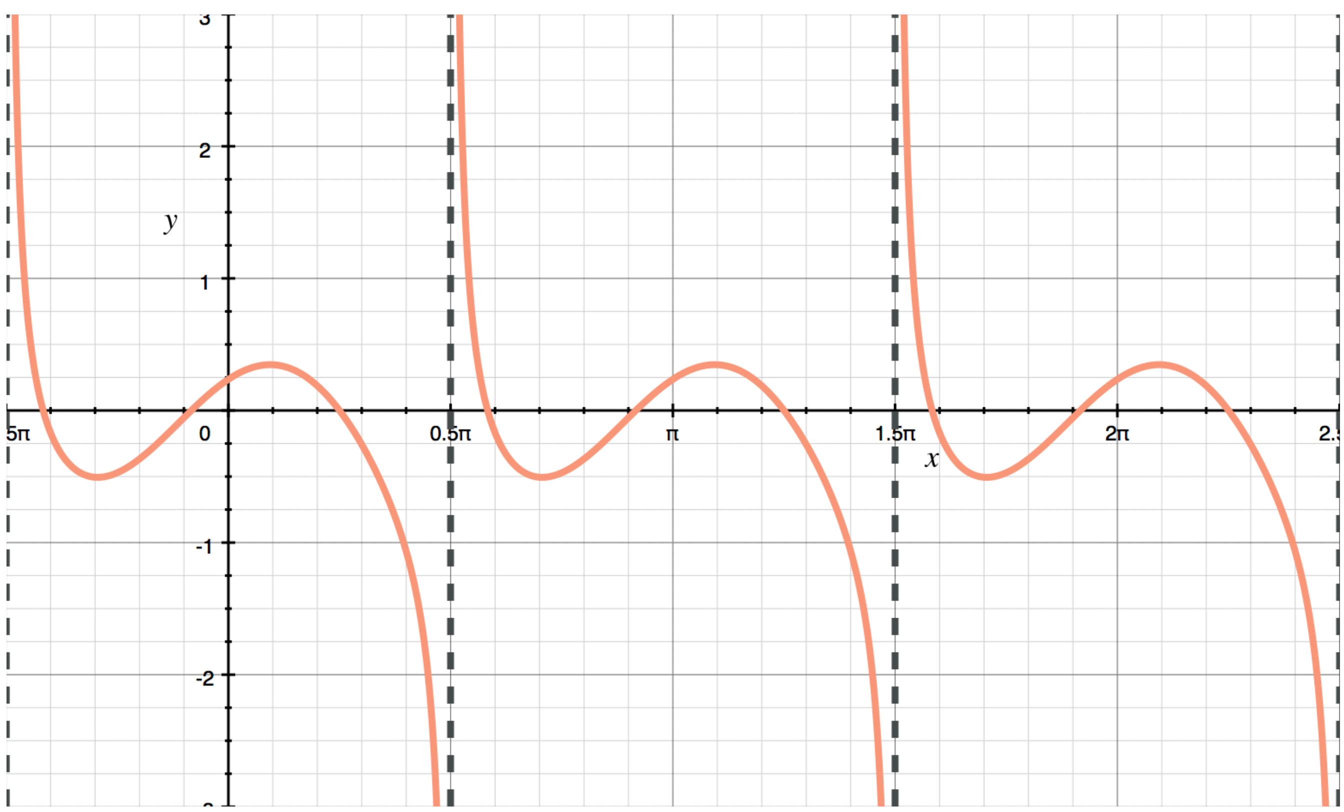
Let's plug in some integers for n, to see which values we get for θ.

$n = 0$ $\qquad \theta = [2(0) + 1]\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

$n = 1$ $\qquad \theta = [2(1) + 1]\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$

$n = 2$ $\qquad \theta = [2(2) + 1]\left(\frac{\pi}{2}\right) = \frac{5\pi}{2}$

These will all be vertical asymptotes of the quotient combination, and of course we could continue finding more evenly-spaced asymptotes to the left and right.

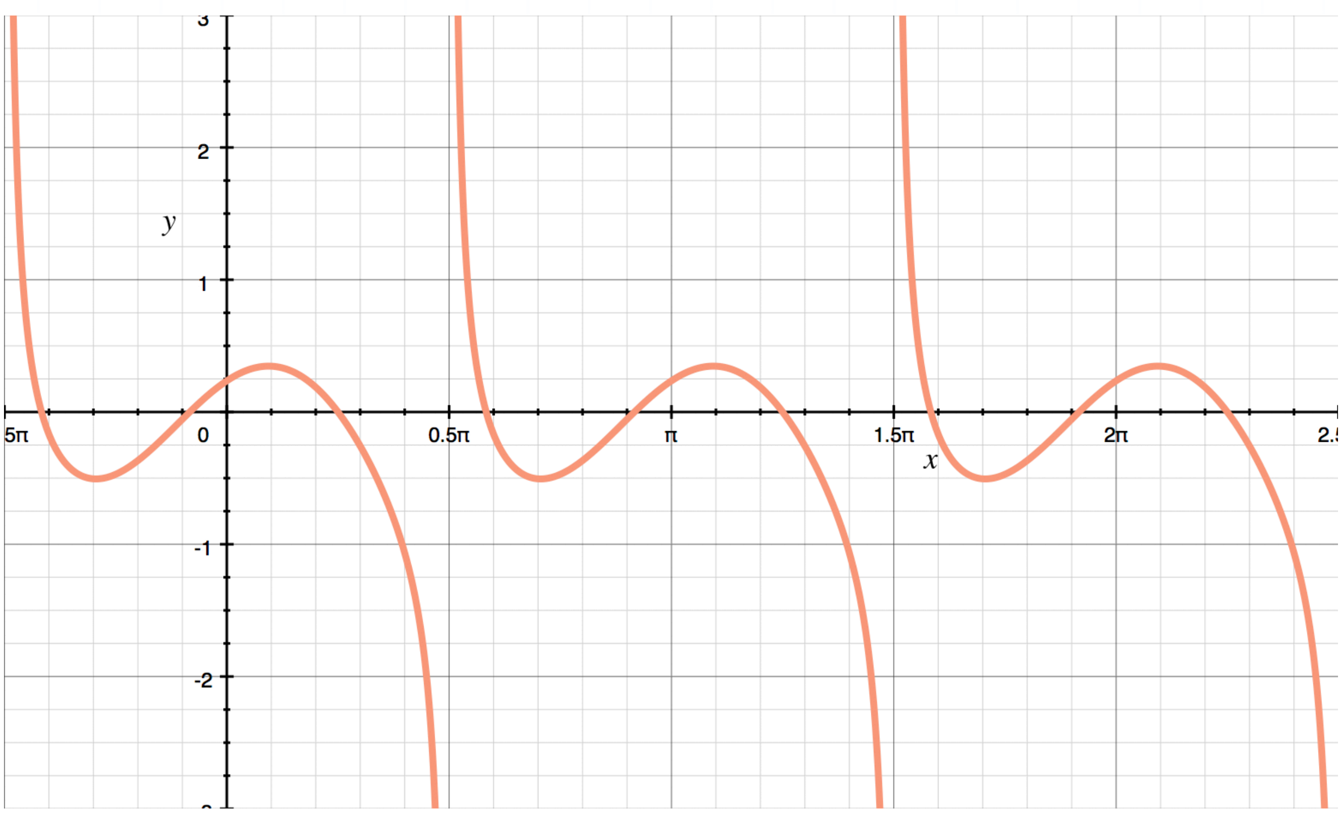


The period of $\cos(3\theta + (3\pi/4))$ is $2\pi/3$, and the period of $3\sin(\theta - (\pi/2))$ is 2π. When the combination function is a product or quotient like it is in this problem, we take the least common multiple of the original functions as the period of the combination. So since the least common multiple of 2π and $2\pi/3$ is 2π, the period of the combination is 2π.

We can sketch the graph of the combination simply by dividing the values of the other two functions at each point. For example, at $\theta = 0$, $\cos(3\theta + (3\pi/4))$ has a value of $-\sqrt{2}/2$, and $3\sin(\theta - (\pi/2))$ has a value of -3, which means the combination quotient will have a value of

$$\frac{-\frac{\sqrt{2}}{2}}{-3} = \frac{\sqrt{2}}{6}$$

and we can plot that point. We'll continue on like this until we have a rough sketch of the graph, making sure to respect the asymptotes as we go.



INVERSE TRIG RELATIONS

■ 1. In degrees, use the unit circle to find the set of angles whose cosine is $-\sqrt{2}/2$.

Solution:

On the unit circle, we know that the x-value in the coordinate point is the value that gives us the cosine of the angle. Therefore, because we're told that cosine of the angle is $-\sqrt{2}/2$, we need to find the angles in the unit circle where the corresponding coordinate point has an x-value equal to $-\sqrt{2}/2$.

Those angles are 135° and 225°. To give the full set of angles, we have to give all of the angles that are coterminal with these two.

$$\theta = 135^\circ + n(360^\circ) \text{ and } \theta = 225^\circ + n(360^\circ)$$

■ 2. In both radians and degrees, use the unit circle to find the set of angles whose sine is -1.

Solution:

On the unit circle, we know that the y-value in the coordinate point is the value that gives us the sine of the angle. Therefore, because we're told that sine of the angle is -1, we need to find the angles in the unit circle where the corresponding coordinate point has a y-value equal to -1.

The only angle that does this is the angle $3\pi/2$. To give the full set of radian angles, we have to give all of the angles that are coterminal with $3\pi/2$.

$$\theta = \frac{3\pi}{2} + 2n\pi$$

We know that $3\pi/2 = 270^\circ$, so the set of angles in degrees will be

$$\theta = 270^\circ + n(360^\circ)$$

■ 3. In both radians and degrees, use the unit circle to find the set of angles whose secant is 2.

Solution:

We remember that

$$\sec\theta = \frac{1}{\cos\theta}$$

So if secant is 2, then cosine will be $1/2$. So to find the set of angles whose secant is 2 we can find the set of angles whose cosine is $1/2$.

On the unit circle, we know that the x-value in the coordinate point is the value that gives us the cosine of the angle. Therefore, because we're told that cosine of the angle is $1/2$, we need to find the angles in the unit circle where the corresponding coordinate point has an x-value equal to $1/2$.

Those angles are $\pi/3$ and $5\pi/3$. To give the full set of radian angles, we have to give all of the angles that are coterminal with these two.

$$\theta = \frac{\pi}{3} + 2n\pi \text{ and } \theta = \frac{5\pi}{3} + 2n\pi$$

We know that $\pi/3 = 60^\circ$ and $5\pi/3 = 300^\circ$, so the set of angles in degrees will be

$$\theta = 60^\circ + n(360^\circ) \text{ and } \theta = 300^\circ + n(360^\circ)$$

4. In both radians and degrees, use the unit circle to find the set of angles whose cosecant is 1.

Solution:

We remember that

$$\csc\theta = \frac{1}{\sin\theta}$$

So, if cosecant is 1, then sine will be 1. So to find the set of angles whose cosecant is 1 we can find the set of angles whose sine is 1.

On the unit circle, we know that the y-value in the coordinate point is the value that gives us the sine of the angle. Therefore, because we're told that sine of the angle is 1, we need to find the angles in the unit circle where the corresponding coordinate point has a y-value equal to 1.

The only angle that does this is the angle $\pi/2$. To give the full set of radian angles, we have to give all of the angles that are coterminal $\pi/2$.

$$\theta = \frac{\pi}{2} + 2n\pi$$

We know that $\pi/2 = 90^\circ$, so the set of angles in degrees will be

$$\theta = 90^\circ + n(360^\circ)$$

5. In both radians and degrees, use the unit circle to find the set of angles whose tangent is 1.

Solution:

We remember that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

So if tangent is 1, then sine will be equal to cosine. So to find the set of angles whose tangent is 1 we can find the set of angles whose sine is equal to cosine.

On the unit circle, we know that the y-value in the coordinate point is the value that gives us the sine of the angle and the x-value in the coordinate point is the value that gives us the cosine of the angle. Therefore, because we're told that sine of the angle is equal to cosine of the angle, we need to find the angles in the unit circle where the corresponding coordinate point has a y-value equal to the x-value.

Those angles are $\pi/4$ and $5\pi/4$. To give the full set of radian angles, we have to give all of the angles that are coterminal with these two.

$$\theta = \frac{\pi}{4} + 2n\pi \text{ and } \theta = \frac{5\pi}{4} + 2n\pi$$

We can combine this set of angles into just one expression.

$$\theta = \frac{\pi}{4} + n\pi$$

We know that $\pi/4 = 45^\circ$ and $5\pi/4 = 225^\circ$, so the set of angles in degrees will be

$$\theta = 45^\circ + n(360^\circ) \text{ and } \theta = 225^\circ + n(360^\circ)$$

We can combine this set of angles into just one expression.

$$\theta = 45^\circ + n(180^\circ)$$

■ 6. In both radians and degrees, use the unit circle to find the set of angles whose cotangent is 0.

Solution:

We remember that

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

So if cotangent is 0, then cosine will be 0. Therefore, to find the set of angles whose cotangent is 0 we can find the set of angles whose cosine is 0.

On the unit circle, we know that the x-value in the coordinate point is the value that gives us the cosine of the angle. Therefore, because we're told that cosine of the angle is 0, we need to find the angles in the unit circle where the corresponding coordinate point has an x-value equal to 0.

Those angles are $\pi/2$ and $3\pi/2$. To give the full set of radian angles, we have to give all of the angles that are coterminal with these two.

$$\theta = \frac{\pi}{2} + 2n\pi \text{ and } \theta = \frac{3\pi}{2} + 2n\pi$$

We can combine this set of angles into just one expression.

$$\theta = \frac{\pi}{2} + n\pi$$

We know that $\pi/2 = 90^\circ$ and $3\pi/2 = 270^\circ$, so the set of angles in degrees will be

$$\theta = 90^\circ + n(360^\circ) \text{ and } \theta = 270^\circ + n(360^\circ)$$

We can combine this set of angles into just one expression.

$$\theta = 90^\circ + n(180^\circ)$$

INVERSE TRIG FUNCTIONS

1. Find the value of the inverse tangent function.

$$\tan^{-1}(0)$$

Solution:

We remember that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

So if tangent is 0, then sine will be 0. If we look at the unit circle, we can see that the sine function is 0 when $\theta = 0$ and when $\theta = \pi$. But because we're dealing with the inverse tangent function, we only want an angle in the interval $[-\pi/2,\pi/2]$.

The angle that works is $\theta = 0$, so we'll say

$$\tan^{-1}(0) = 0$$

2. Find the value of the inverse cotangent function.

$$\cot^{-1}(-1)$$

Solution:

We remember that

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

So if cotangent is -1, then sine will be equal to cosine, but have the opposite sign. If we look at the unit circle, we can see that $\sin\theta = -\cos\theta$ when $\theta = 3\pi/4$ and when $\theta = 7\pi/4$. But because we're dealing with the inverse cotangent function, we only want an angle in the interval $(0,\pi)$.

Remember that the interval $(0,\pi)$ spans the first and second quadrants. The angle $\theta = 3\pi/4$ is in the second quadrant, and the angle $\theta = 7\pi/4$ is in the fourth quadrant. The angle that works is $\theta = 3\pi/4$, so we'll say

$$\cot^{-1}(-1) = \frac{3\pi}{4}$$

3. Find the value of the inverse sine function.

$$\sin^{-1}\left(-\frac{1}{2}\right)$$

Solution:

If we look at the unit circle, we can see that the sine function is $-1/2$ when $\theta = 7\pi/6$ and when $\theta = 11\pi/6$. But because we're dealing with the inverse sine function, we only want an angle in the interval $[-\pi/2,\pi/2]$.

Both $\theta = 7\pi/6$ and $\theta = 11\pi/6$ fall outside the interval $[-\pi/2,\pi/2]$, which means we'll need to find an angle coterminal with either $\theta = 7\pi/6$ or $\theta = 11\pi/6$ that falls within $[-\pi/2,\pi/2]$.

Remember that the interval $[-\pi/2,\pi/2]$ spans the fourth and first quadrants. The angle $\theta = 7\pi/6$ is in the third quadrant, and the angle $\theta = 11\pi/6$ is in the fourth quadrant. Which means the angle we need is one that's coterminal with $\theta = 11\pi/6$, but in the interval $[-\pi/2,\pi/2]$. The angle that works is $\theta = -\pi/6$, so we'll say

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

■ 4. Find the value of the inverse secant function.

$$\sec^{-1}(-2)$$

Solution:

We remember that

$$\sec\theta = \frac{1}{\cos\theta}$$

So, if secant is -2, then cosine will be $-1/2$. If we look at the unit circle, we can see that the cosine function is $-1/2$ when $\theta = 2\pi/3$ and when $\theta = 4\pi/3$. But because we're dealing with the inverse secant function, we only want an angle in the interval $[0,\pi/2) \cup (\pi/2,\pi]$.

Remember that the interval $[0,\pi/2) \cup (\pi/2,\pi]$ spans the first and second quadrants. The angle $\theta = 2\pi/3$ is in the second quadrant, and the angle $\theta = 4\pi/3$ is in the third quadrant. The angle that works is $\theta = 2\pi/3$, so we'll say

$$\sec^{-1}(-2) = \frac{2\pi}{3}$$

■ 5. Find the value of the inverse cosine function.

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

Solution:

If we look at the unit circle, we can see that the cosine function is $\sqrt{3}/2$ when $\theta = \pi/6$ and when $\theta = 11\pi/6$. But because we're dealing with the inverse cosine function, we only want an angle in the interval $[0,\pi]$.

The angle $\theta = \pi/6$ is the only angle in $[0,\pi]$, so

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

■ 6. Find the value of the inverse cosecant function.

$$\csc^{-1}(-\sqrt{2})$$

Solution:

We remember that

$$\csc\theta = \frac{1}{\sin\theta}$$

So if cosecant is $-\sqrt{2}$, then sine will be $-\sqrt{2}/2$. If we look at the unit circle, we can see that the sine function is $-\sqrt{2}/2$ when $\theta = 5\pi/4$ and when $\theta = 7\pi/4$. But because we're dealing with the inverse cosecant function, we only want an angle in the interval $[-\pi/2,0) \cup (0,\pi/2]$.

Both $\theta = 5\pi/4$ and $\theta = 7\pi/4$ fall outside the interval $[-\pi/2,0) \cup (0,\pi/2]$, which means we'll need to find an angle coterminal with either $\theta = 5\pi/4$ or $\theta = 7\pi/4$ that falls within $[-\pi/2,0) \cup (0,\pi/2]$.

Remember that the interval $[-\pi/2,0) \cup (0,\pi/2]$ spans the fourth and first quadrants. The angle $\theta = 5\pi/4$ is in the third quadrant, and the angle $\theta = 7\pi/4$ is in the fourth quadrant. Which means the angle we need is one that's coterminal with $\theta = 7\pi/4$, but in the interval $[-\pi/2,0) \cup (0,\pi/2]$. The angle that works is $\theta = -\pi/4$, so we'll say

$$\csc^{-1}(-\sqrt{2}) = -\frac{\pi}{4}$$

TRIG FUNCTIONS OF INVERSE TRIG FUNCTIONS

1. Find the value of the expression.

$$\sin\left(\tan^{-1}\left(\frac{1}{3}\right)\right)$$

Solution:

Let θ represent the angle in $(-\pi/2,\pi/2)$ whose tangent is $1/3$. Then we can say

$$\theta = \tan^{-1}\left(\frac{1}{3}\right)$$

$$\tan\theta = \frac{1}{3}$$

Because $\tan\theta$ is positive, θ must be an angle in $(0,\pi/2)$, so θ is a positive angle that lies in quadrant I.

$$\tan\theta = \frac{1 = \text{opposite}}{3 = \text{adjacent}}$$

Given a triangle with opposite leg 1 and adjacent leg 3, the hypotenuse must be

$$a^2 + b^2 = c^2$$

$$1^2 + 3^2 = c^2$$

$$c^2 = 1 + 9$$

$$c^2 = 10$$

$$c = \sqrt{10}$$

Because sine is equivalent to opposite/hypotenuse, we get

$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{10}}$$

$$\sin\left(\tan^{-1}\left(\frac{1}{3}\right)\right) = \sin\theta = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

■ 2. Find the value of $\tan^{-1}(\sin\pi)$.

Solution:

We first need to find $\sin\pi$.

$$\sin\pi = 0$$

The angle in $(-\pi/2, \pi/2)$ whose tangent is 0 is 0. Therefore,

$$\tan^{-1}(\sin\pi) = \tan^{-1}(0) = 0$$

3. Find the value of the expression.

$$\csc\left(\cot^{-1}\left(\frac{1}{x}\right)\right)$$

Solution:

Set $\theta = \cot^{-1}(1/x)$. Then we can say

$$\theta = \cot^{-1}\left(\frac{1}{x}\right)$$

$$\theta = \cot^{-1}\left(\frac{1 = \text{adjacent}}{x = \text{opposite}}\right)$$

Given a triangle with adjacent leg 1 and opposite leg x, the hypotenuse must be

$$a^2 + b^2 = c^2$$

$$1^2 + x^2 = c^2$$

$$c^2 = 1 + x^2$$

$$c = \sqrt{1 + x^2}$$

Now that we know that the triangle we're describing has adjacent leg 1, opposite leg x, and hypotenuse $\sqrt{1 + x^2}$, we can find the cosecant of the interior angle of that triangle. Because cosecant is equivalent to hypotenuse/opposite, we get

$$\frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{1+x^2}}{x}$$

This expression represents $\csc(\cot^{-1}(1/x))$.

$$\csc\left(\cot^{-1}\frac{1}{x}\right) = \frac{\sqrt{1+x^2}}{x}$$

4. Find the value of the expression.

$$\cos\left(\sec^{-1}\left(-\frac{9}{2}\right)\right)$$

Solution:

Let θ represent the angle in $[0,\pi/2)$ or $(\pi/2,\pi]$ whose secant is $-9/2$. Then we can say

$$\theta = \sec^{-1}\left(-\frac{9}{2}\right)$$

$$\sec\theta = -\frac{9}{2}$$

Because $\sec\theta$ is negative, θ must be an angle in $(\pi/2,\pi]$, so θ is a negative angle that lies in quadrant II.

$$\sec\theta = \frac{9 = \text{hypotenuse}}{-2 = \text{adjancent}}$$

Because cosine is equivalent to adjacent/hypotenuse, we get

$$\frac{\text{adjacent}}{\text{hypotenuse}}=\frac{-2}{9}=-\frac{2}{9}$$

$$\cos\left(\sec^{-1}\left(-\frac{9}{2}\right)\right)=\cos\theta=-\frac{2}{9}$$

■ 5. Find the value of the expression.

$$\cot\left(\cos^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)\right)$$

Solution:

We first need to find $\sin(\pi/4)$.

$$\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

The angle in $[0,\pi]$ whose cosine is $\sqrt{2}/2$ is $\pi/4$, so

$$\cos^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)=\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)=\frac{\pi}{4}$$

Then

$$\cot\left(\frac{\pi}{4}\right) = 1$$

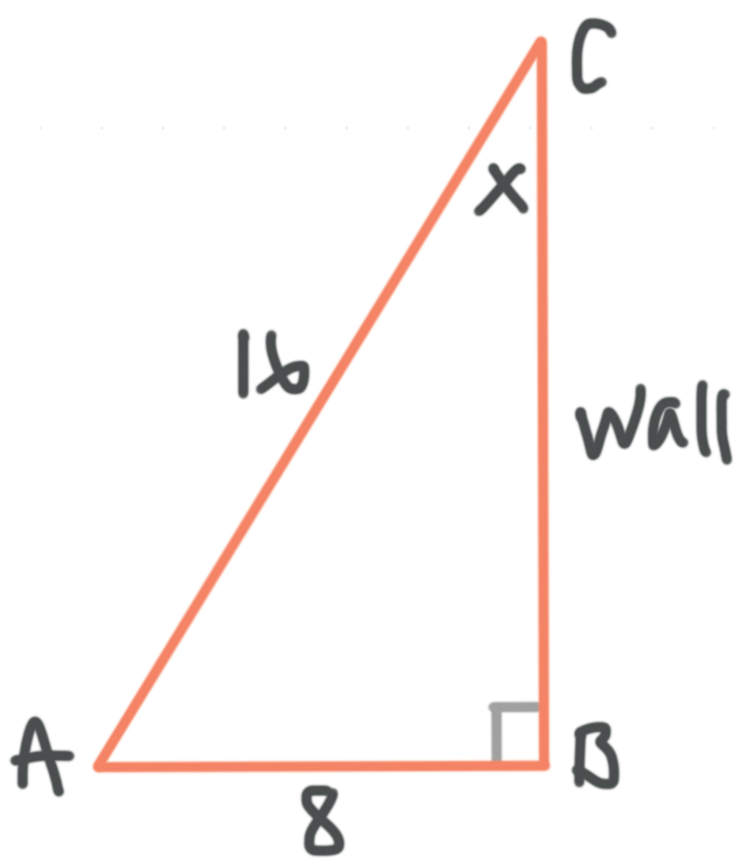
Therefore,

$$\cot\left(\cos^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)\right) = 1$$

■ 6. A 16-foot ladder leans against a brick wall. The base of the ladder is 8 feet from the wall. Find the angle the ladder makes with the wall.

Solution:

Let's sketch the scenario.



We can set up an equation using the sine function.

$$\sin x = \frac{\overline{AB}}{\overline{AC}}$$

$$\sin x = \frac{8}{16}$$

$$\sin x = \frac{1}{2}$$

$$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{1}{2}\right)$$

Using the inverse property $\sin^{-1}(\sin x) = x$, we get

$$x = \sin^{-1}\left(\frac{1}{2}\right)$$

The angle in $[-\pi/2,\pi/2]$ whose sine is $1/2$ is $\pi/6$, so $x = \pi/6$.

SUM-DIFFERENCE IDENTITIES FOR SINE AND COSINE

1. Evaluate the expression.

$$\cos\left(\frac{13\pi}{12}\right)$$

Solution:

From just the unit circle, we wouldn't know the values of sine and cosine at $13\pi/12$, but we can rewrite $13\pi/12$ as

$$\frac{13\pi}{12} = \frac{(4+9)\pi}{12} = \frac{4\pi}{12} + \frac{9\pi}{12} = \frac{\pi}{3} + \frac{3\pi}{4}$$

Therefore, by the sum identity for the cosine function,

$$\cos\left(\frac{13\pi}{12}\right) = \cos\left(\frac{\pi}{3} + \frac{3\pi}{4}\right)$$

$$\cos\left(\frac{13\pi}{12}\right) = \left(\cos\frac{\pi}{3}\right)\left(\cos\frac{3\pi}{4}\right) - \left(\sin\frac{\pi}{3}\right)\left(\sin\frac{3\pi}{4}\right)$$

$$\cos\left(\frac{13\pi}{12}\right) = \left(\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$\cos\left(\frac{13\pi}{12}\right) = -\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$

$$\cos\left(\frac{13\pi}{12}\right) = -\frac{\sqrt{2}+\sqrt{6}}{4}$$

■ 2. Find $\sin 75^\circ$.

Solution:

Rewrite $\sin 75^\circ$ as $\sin(45^\circ + 30^\circ)$, then apply the sum identity for sine.

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$\sin(45^\circ + 30^\circ) = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$\sin(45^\circ + 30^\circ) = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$\sin(45^\circ + 30^\circ) = \frac{\sqrt{6}+\sqrt{2}}{4}$$

■ 3. Simplify the expressions.

$$\cos\left(\frac{\pi}{2} + \theta\right) \text{ and } \cos\left(\frac{\pi}{2} - \theta\right)$$

Solution:

To find cosine of the sum, use the sum identity for cosine.

$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$$

$$\cos\left(\frac{\pi}{2}+\theta\right)=\cos\frac{\pi}{2}\cos\theta-\sin\frac{\pi}{2}\sin\theta$$

$$\cos\left(\frac{\pi}{2}+\theta\right)=0\cdot\cos\theta-1\cdot\sin\theta$$

$$\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$$

To find cosine of the difference, use a difference identity.

$$\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi$$

$$\cos\left(\frac{\pi}{2}-\theta\right)=\cos\frac{\pi}{2}\cos\theta+\sin\frac{\pi}{2}\sin\theta$$

$$\cos\left(\frac{\pi}{2}-\theta\right)=0\cdot\cos\theta+1\cdot\sin\theta$$

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$

■ 4. Find the value of $a-2b$, if a and b are real numbers.

$$\sin(\theta - \alpha) = a \sin\theta \cos\alpha + b \cos\theta \sin\alpha$$

Solution:

For two angles θ and α, the difference identity for sine is

$$\sin(\theta - \alpha) = \sin\theta \cos\alpha - \cos\theta \sin\alpha$$

Therefore $a = 1$ and $b = -1$. Substitute and evaluate the expression.

$$a - 2b$$

$$1 - 2(-1)$$

$$1 + 2$$

$$3$$

■ 5. Find the exact value of the expression.

$$\cos\left(\sin^{-1}\frac{\sqrt{3}}{2} - \cos^{-1}\frac{4}{5}\right)$$

Solution:

To find the exact value of the expression, we need to use the difference identity for the cosine function.

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Define α and β.

$$\alpha = \sin^{-1}\frac{\sqrt{3}}{2}$$

$$\beta = \cos^{-1}\frac{4}{5}$$

Using the properties of inverses we can write

$$\sin\alpha = \frac{\sqrt{3}}{2}, \quad -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$$

$$\cos\beta = \frac{4}{5}, \ 0 \le \beta \le \pi$$

Then using the Pythagorean identities we can find $\cos\alpha$ and $\sin\beta$.

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \sin^2\alpha$$

Substitute $\sin\alpha = \sqrt{3}/2$.

$$\cos^2\alpha = 1 - \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\cos^2\alpha = 1 - \frac{3}{4}$$

$$\cos^2\alpha = \frac{1}{4}$$

$$\cos\alpha = \pm\sqrt{\frac{1}{4}}$$

Since $-\pi/2 \leq \alpha \leq \pi/2$, we can say $\cos\alpha = 1/2$.

$$\sin^2\beta + \cos^2\beta = 1$$

$$\sin^2\beta = 1 - \cos^2\beta$$

Substitute $\cos\beta = 4/5$.

$$\sin^2\beta = 1 - \left(\frac{4}{5}\right)^2$$

$$\sin^2\beta = 1 - \frac{16}{25}$$

$$\sin^2\beta = \frac{9}{25}$$

$$\sin\beta = \pm\sqrt{\frac{9}{25}}$$

Since $0 \leq \beta \leq \pi$, we can say $\sin\beta = 3/5$. Then using the difference formula for cosine, we get

$$\cos\left(\sin^{-1}\frac{\sqrt{3}}{2} - \cos^{-1}\frac{4}{5}\right) = \cos(\alpha - \beta)$$

$$\cos\left(\sin^{-1}\frac{\sqrt{3}}{2} - \cos^{-1}\frac{4}{5}\right) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}-\cos^{-1}\frac{4}{5}\right)=\frac{1}{2}\left(\frac{4}{5}\right)+\frac{\sqrt{3}}{2}\left(\frac{3}{5}\right)$$

$$\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}-\cos^{-1}\frac{4}{5}\right)=\frac{4}{10}+\frac{3\sqrt{3}}{10}$$

$$\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}-\cos^{-1}\frac{4}{5}\right)=\frac{4+3\sqrt{3}}{10}$$

■ 6. Find the solutions to the equation in the interval $[0,\pi)$.

$$\cos\left(\theta-\frac{\pi}{2}\right)+\sin\left(\theta-\frac{3\pi}{2}\right)=0$$

Solution:

Use the sum-difference identities for cosine,

$$\cos(\theta-\alpha)=\cos\theta\cos\alpha+\sin\theta\sin\alpha$$

$$\cos\left(\theta-\frac{\pi}{2}\right)=\cos\theta\cos\frac{\pi}{2}+\sin\theta\sin\frac{\pi}{2}$$

$$\cos\left(\theta-\frac{\pi}{2}\right)=\cos\theta(0)+\sin\theta(1)$$

$$\cos\left(\theta-\frac{\pi}{2}\right)=\sin\theta$$

and sine.

$$\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha$$

$$\sin\left(\theta - \frac{3\pi}{2}\right) = \sin\theta\cos\frac{3\pi}{2} - \cos\theta\sin\frac{3\pi}{2}$$

$$\sin\left(\theta - \frac{3\pi}{2}\right) = \sin\theta(0) - \cos\theta(-1)$$

$$\sin\left(\theta - \frac{3\pi}{2}\right) = \cos\theta$$

Therefore we get

$$\sin\theta + \cos\theta = 0$$

$$\sin\theta = -\cos\theta$$

From the unit circle we know that $\sin\theta = -\cos\theta$ at $\theta = 3\pi/4$ and $\theta = 7\pi/4$. We're looking for solutions over the interval $[0,\pi)$, so the only solution is $\theta = 3\pi/4$.

COFUNCTION IDENTITIES

1. Find an angle θ that satisfies the equation.

$$\tan\left(-\frac{3\pi}{4}\right) = \cot\theta$$

Solution:

The equation we're given tells us that the cotangent of some angle is equivalent to tangent of $-3\pi/4$. Tangent and cotangent are cofunctions, which means we can plug into the cofunction identity for tangent that relates them.

$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$$

$$\tan\left(-\frac{3\pi}{4}\right) = \cot\left(\frac{\pi}{2} - \left(-\frac{3\pi}{4}\right)\right)$$

$$\tan\left(-\frac{3\pi}{4}\right) = \cot\left(\frac{\pi}{2} + \frac{3\pi}{4}\right)$$

Find a common denominator.

$$\tan\left(-\frac{3\pi}{4}\right) = \cot\left(\frac{\pi}{2}\left(\frac{2}{2}\right) + \frac{3\pi}{4}\right)$$

$$\tan\left(-\frac{3\pi}{4}\right) = \cot\left(\frac{2\pi}{4} + \frac{3\pi}{4}\right)$$

$$\tan\left(-\frac{3\pi}{4}\right) = \cot\left(\frac{5\pi}{4}\right)$$

So the angle θ that satisfies the equation is $\theta = 5\pi/4$. And this result tells us that tangent of the angle $-3\pi/4$ has the same value as cotangent of the angle $5\pi/4$.

■ 2. Find an acute angle that satisfies the equation.

$$\sin\left(2\alpha - \frac{5\pi}{6}\right) = \cos\left(4\alpha - \frac{\pi}{3}\right)$$

Solution:

Use the cofunction identity for sine.

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

Let $\theta = 2\alpha - (5\pi/6)$. Then

$$4\alpha - \frac{\pi}{3} = \frac{\pi}{2} - \left(2\alpha - \frac{5\pi}{6}\right)$$

$$4\alpha - \frac{\pi}{3} = \frac{\pi}{2} - 2\alpha + \frac{5\pi}{6}$$

$$4\alpha + 2\alpha = \frac{\pi}{2} + \frac{5\pi}{6} + \frac{\pi}{3}$$

Find a common denominator.

$$6\alpha = \frac{\pi}{2}\left(\frac{3}{3}\right) + \frac{5\pi}{6} + \frac{\pi}{3}\left(\frac{2}{2}\right)$$

$$6\alpha = \frac{3\pi}{6} + \frac{5\pi}{6} + \frac{2\pi}{6}$$

$$6\alpha = \frac{10\pi}{6}$$

$$\alpha = \frac{10\pi}{36}$$

$$\alpha = \frac{5\pi}{18}$$

■ 3. What is the value of θ?

$$\tan\left(\frac{\pi}{6} - \theta\right) = \cot\left(\frac{\pi}{6}\right)$$

Solution:

Rewrite the left side of the equation as

$$\tan\left(\frac{\pi}{6} - \theta\right) = \tan\left(\frac{\pi}{2} - \frac{\pi}{3} - \theta\right)$$

From the cofunction identity for tangent,

$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$$

we get

$$\tan\left(\frac{\pi}{6} - \theta\right) = \tan\left(\frac{\pi}{2} - \frac{\pi}{3} - \theta\right)$$

$$\tan\left(\frac{\pi}{6} - \theta\right) = \tan\left(\frac{\pi}{2} - \left(\frac{\pi}{3} + \theta\right)\right)$$

$$\tan\left(\frac{\pi}{6} - \theta\right) = \cot\left(\frac{\pi}{3} + \theta\right)$$

$$\tan\left(\frac{\pi}{6} - \theta\right) = \cot\left(\frac{\pi}{6}\right)$$

So we get

$$\frac{\pi}{3} + \theta = \frac{\pi}{6}$$

$$\theta = \frac{\pi}{6} - \frac{\pi}{3}$$

$$\theta = -\frac{\pi}{6}$$

■ 4. Find the value of $\cos\theta$.

$$\sin\left(\frac{\pi}{2}-\theta\right)+\frac{1}{4}\csc\left(\frac{\pi}{2}-\theta\right)=1$$

Solution:

From the cofunction identities, we know that

$$\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)$$

$$\sec\theta=\csc\left(\frac{\pi}{2}-\theta\right)$$

So we can rewrite the equation as

$$\sin\left(\frac{\pi}{2}-\theta\right)+\frac{1}{4}\csc\left(\frac{\pi}{2}-\theta\right)=1$$

$$\cos\theta+\frac{1}{4}\sec\theta=1$$

$$\cos\theta+\frac{1}{4\cos\theta}=1$$

$$4\cos^2\theta+1=4\cos\theta$$

$$4\cos^2\theta-4\cos\theta+1=0$$

$$(2\cos\theta-1)^2=0$$

$$2\cos\theta-1=0$$

$$\cos\theta = \frac{1}{2}$$

5. Rewrite the expression as the cosine of an angle in terms of α and β.

$$\sin\left(\frac{\pi}{2} - \alpha - \beta\right)$$

Solution:

Rewrite the expression as

$$\sin\left(\frac{\pi}{2} - (\alpha + \beta)\right)$$

Apply the cofunction identity for cosine,

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

to get

$$\sin\left(\frac{\pi}{2} - (\alpha + \beta)\right) = \cos(\alpha + \beta)$$

Now apply the sum identity for cosine to get

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

6. Find an angle θ that satisfies the equation.

$$\csc\left(\frac{\pi}{5}\right) = \sec\theta$$

Solution:

The equation we're given tells us that the secant of some angle is equivalent to cosecant of $\pi/5$. Secant and cosecant are cofunctions, which means we can plug into the cofunction identity for cosecant that relates them.

$$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$$

$$\csc\left(\frac{\pi}{5}\right) = \sec\left(\frac{\pi}{2} - \frac{\pi}{5}\right)$$

Find a common denominator.

$$\csc\left(\frac{\pi}{5}\right) = \sec\left(\frac{\pi}{2}\left(\frac{5}{5}\right) - \frac{\pi}{5}\left(\frac{2}{2}\right)\right)$$

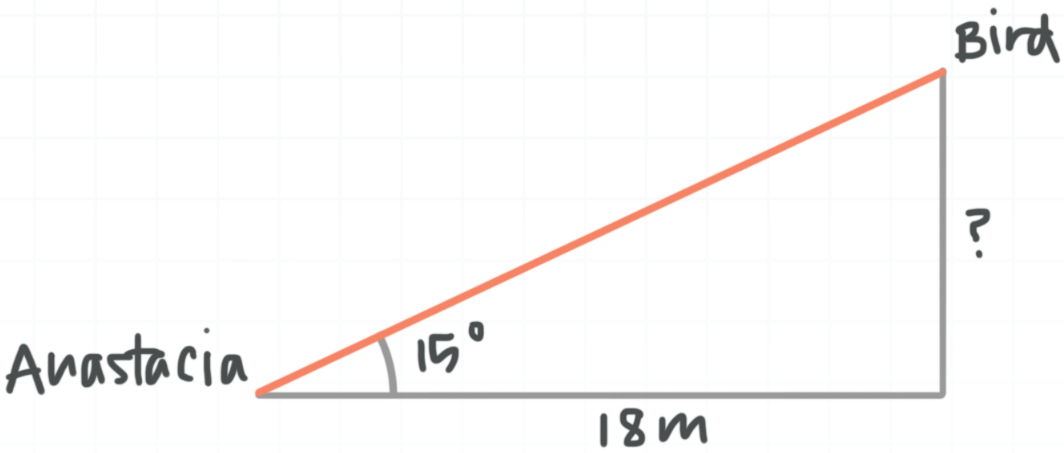
$$\csc\left(\frac{\pi}{5}\right) = \sec\left(\frac{5\pi}{10} - \frac{2\pi}{10}\right)$$

$$\csc\left(\frac{\pi}{5}\right) = \sec\left(\frac{3\pi}{10}\right)$$

So the angle θ that satisfies the equation is $\theta = 3\pi/10$. And this result tells us that cosecant of the angle $\pi/5$ has the same value as secant of the angle $3\pi/10$.

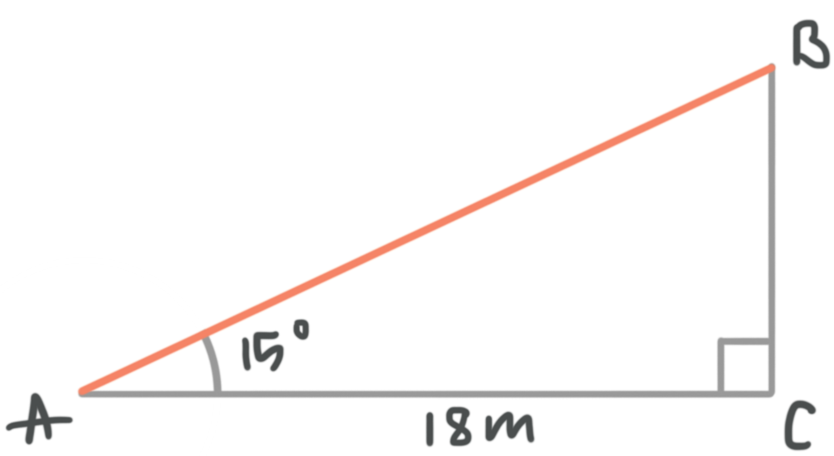
SUM-DIFFERENCE IDENTITIES FOR TANGENT

■ 1. Cara is watching a bird on a tree. She measured the angle of elevation of the bird as 15°, and the distance to the tree as 18 meters. Find the exact altitude of the bird above the ground.



Solution:

Call the right triangle ABC where A is Cara's position, B is the bird's position, and C is the point on the ground just below the bird.



$\overline{BC}$ is the side we're interested in, and by the definition of tangent, we can say

$$\tan A = \frac{\overline{BC}}{\overline{AC}}$$

$$\overline{BC} = \overline{AC} \tan A$$

$$\overline{BC} = 18 \tan 15^\circ$$

To find the exact value of $\tan 15^\circ$, rewrite the angle as the difference $45^\circ - 30^\circ$.

$$\overline{BC} = 18 \tan(45^\circ - 30^\circ)$$

Then apply the difference identity for tangent.

$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta \tan\phi}$$

$$\overline{BC} = 18 \cdot \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$$

$$\overline{BC} = 18 \cdot \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}}$$

$$\overline{BC} = 18 \cdot \frac{3 - \sqrt{3}}{3 + \sqrt{3}}$$

Rationalize the denominator using conjugate method.

$$\overline{BC} = 18 \cdot \frac{(3 - \sqrt{3})^2}{(3 + \sqrt{3})(3 - \sqrt{3})}$$

$$\overline{BC} = 18 \cdot \frac{9 - 6\sqrt{3} + 3}{9 - 3}$$

$$\overline{BC} = 18 \cdot \frac{12 - 6\sqrt{3}}{6}$$

$$\overline{BC} = 3(12 - 6\sqrt{3})$$

$$\overline{BC} = 36 - 18\sqrt{3} \text{ m}$$

■ 2. Find the exact value of $\tan 105^\circ$.

Solution:

From just the unit circle, we wouldn't know the value of tangent at 105°, but we can rewrite 105° as

$$105^\circ = 45^\circ + 60^\circ$$

Therefore, by the sum identity for tangent, $\tan 105^\circ = \tan(45^\circ + 60^\circ)$, and

$$\tan(\theta + \alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha}$$

$$\tan(45^\circ + 60^\circ) = \frac{\tan 45^\circ + \tan 60^\circ}{1 - \tan 45^\circ \tan 60^\circ}$$

$$\tan(45^\circ + 60^\circ) = \frac{1 + \sqrt{3}}{1 - (1)(\sqrt{3})}$$

$$\tan(45^\circ + 60^\circ) = \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

Rationalize the denominator using conjugate method.

$$\tan(45^\circ + 60^\circ) = \frac{1+\sqrt{3}}{1-\sqrt{3}} \cdot \frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$\tan(45^\circ + 60^\circ) = \frac{(1+\sqrt{3})^2}{(1-\sqrt{3})(1+\sqrt{3})}$$

$$\tan(45^\circ + 60^\circ) = \frac{1+2\sqrt{3}+3}{1-3}$$

$$\tan(45^\circ + 60^\circ) = -\frac{4+2\sqrt{3}}{2}$$

$$\tan(45^\circ + 60^\circ) = -2-\sqrt{3}$$

■ 3. Find the exact values of $\tan(\theta - \alpha)$ if θ is an angle in the first quadrant whose cosine is $3/5$ and α is an angle in the fourth quadrant whose sine is $-5/13$.

Solution:

Before we can find the values of $\tan(\theta - \alpha)$, we need to find the values of $\sin\theta$ and $\cos\alpha$. To find the sine from the cosine, we'll rewrite the Pythagorean identity with sine and cosine,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

and then substitute $\cos\theta = 3/5$.

$$\sin^2\theta = 1 - \left(\frac{3}{5}\right)^2$$

$$\sin^2\theta = 1 - \frac{9}{25}$$

$$\sin^2\theta = \frac{16}{25}$$

$$\sin\theta = \pm\sqrt{\frac{16}{25}}$$

Since θ is in the first quadrant, we know that $\sin\theta$ is positive. So we can ignore the negative value and say

$$\sin\theta = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

Now we need to find the value of $\cos\alpha$. Again by the Pythagorean identity,

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \sin^2\alpha$$

Substituting $\sin\alpha = -5/13$, we get

$$\cos^2\alpha = 1 - \left(-\frac{5}{13}\right)^2$$

$$\cos^2\alpha = 1 - \frac{25}{169}$$

$$\cos^2\alpha = \frac{144}{169}$$

$$\cos\alpha = \pm\sqrt{\frac{144}{169}}$$

Since α is in the fourth quadrant, we know that $\cos\alpha$ is positive. So we can ignore the negative value and say

$$\cos\alpha = \sqrt{\frac{144}{169}} = \frac{12}{13}$$

Now we can find the values of $\tan\theta$ and $\tan\alpha$. Using the definition of the tangent function, we get

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{5}\left(\frac{5}{3}\right) = \frac{4}{3}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{-\frac{5}{13}}{\frac{12}{13}} = -\frac{5}{13}\left(\frac{13}{12}\right) = -\frac{5}{12}$$

By the difference identity for the tangent function,

$$\tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha}$$

$$\tan(\theta - \alpha) = \frac{\frac{4}{3} - \left(-\frac{5}{12}\right)}{1 + \left(\frac{4}{3}\right)\left(-\frac{5}{12}\right)}$$

$$\tan(\theta - \alpha) = \frac{\frac{16+5}{12}}{1 - \frac{20}{36}}$$

$$\tan(\theta - \alpha) = \frac{\frac{21}{12}}{\frac{36-20}{36}}$$

$$\tan(\theta - \alpha) = \frac{\frac{21}{12}}{\frac{16}{36}}$$

$$\tan(\theta - \alpha) = \frac{\frac{7}{4}}{\frac{4}{9}}$$

$$\tan(\theta - \alpha) = \frac{7}{4}\left(\frac{9}{4}\right)$$

$$\tan(\theta - \alpha) = \frac{63}{16}$$

■ 4. Simplify the expressions $\tan(\pi + \theta)$ and $\tan(\pi - \theta)$.

Solution:

To find tangent of the sum, use the sum identity for tangent.

$$\tan(\theta+\alpha)=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$$

$$\tan(\pi+\theta)=\frac{\tan\pi+\tan\theta}{1-\tan\pi\tan\theta}$$

$$\tan(\pi+\theta)=\frac{0+\tan\theta}{1-(0)\tan\theta}$$

$$\tan(\pi+\theta)=\frac{\tan\theta}{1}$$

$$\tan(\pi+\theta)=\tan\theta$$

To find tangent of the difference, use a difference identity.

$$\tan(\theta-\alpha)=\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}$$

$$\tan(\pi-\theta)=\frac{\tan\pi-\tan\theta}{1+\tan\pi\tan\theta}$$

$$\tan(\pi-\theta)=\frac{0-\tan\theta}{1+(0)\tan\theta}$$

$$\tan(\pi-\theta)=\frac{-\tan\theta}{1}$$

$$\tan(\pi-\theta)=-\tan\theta$$

5. Find the exact values of $\tan(\theta+\alpha)$ if θ is an angle in the second quadrant whose cosine is $-4/7$ and α is an angle in the third quadrant whose cosine is $-9/10$.

Solution:

Before we can find the value of $\tan(\theta+\alpha)$, we need to find the values of $\sin\theta$ and $\cos\alpha$. To find the sines from the cosines, we'll rewrite the Pythagorean identity with sine and cosine,

$$\sin^2\theta+\cos^2\theta=1$$

$$\sin^2\theta=1-\cos^2\theta$$

and then substitute $\cos\theta=-4/7$.

$$\sin^2\theta=1-\left(-\frac{4}{7}\right)^2$$

$$\sin^2\theta=1-\frac{16}{49}$$

$$\sin^2\theta=\frac{33}{49}$$

$$\sin\theta=\pm\sqrt{\frac{33}{49}}$$

Since θ is in the second quadrant, we know that $\sin\theta$ is positive. So we can ignore the negative value and say

$$\sin\theta = \sqrt{\frac{33}{49}} = \frac{\sqrt{33}}{7}$$

Now we need to find the value of $\sin\alpha$. Again by the Pythagorean identity,

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin^2\alpha = 1 - \cos^2\alpha$$

Substituting $\cos\alpha = -9/10$, we get

$$\sin^2\alpha = 1 - \left(-\frac{9}{10}\right)^2$$

$$\sin^2\alpha = 1 - \frac{81}{100}$$

$$\sin^2\alpha = \frac{19}{100}$$

$$\sin\alpha = \pm\sqrt{\frac{19}{100}}$$

Since α is in the third quadrant, we know that $\sin\alpha$ is negative. So we can ignore the positive value and say

$$\sin\alpha = -\sqrt{\frac{19}{100}} = -\frac{\sqrt{19}}{10}$$

Now we can find the values of $\tan\theta$ and $\tan\alpha$. Using the definition of the tangent function, we get

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{33}}{7}}{-\frac{4}{7}} = \frac{\sqrt{33}}{7}\left(-\frac{7}{4}\right) = -\frac{\sqrt{33}}{4}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{-\frac{\sqrt{19}}{10}}{-\frac{9}{10}} = -\frac{\sqrt{19}}{10}\left(-\frac{10}{9}\right) = \frac{\sqrt{19}}{9}$$

By the sum identity for the tangent function,

$$\tan(\theta+\alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha}$$

$$\tan(\theta+\alpha) = \frac{-\frac{\sqrt{33}}{4} + \frac{\sqrt{19}}{9}}{1 - \left(-\frac{\sqrt{33}}{4}\right)\left(\frac{\sqrt{19}}{9}\right)}$$

$$\tan(\theta+\alpha) = \frac{-\frac{9\sqrt{33}}{36} + \frac{4\sqrt{19}}{36}}{1 + \frac{\sqrt{33}\sqrt{19}}{36}}$$

$$\tan(\theta+\alpha) = \frac{\frac{4\sqrt{19} - 9\sqrt{33}}{36}}{\frac{36 + \sqrt{33}\sqrt{19}}{36}}$$

$$\tan(\theta+\alpha) = \frac{4\sqrt{19} - 9\sqrt{33}}{36}\left(\frac{36}{36 + \sqrt{33}\sqrt{19}}\right)$$

$$\tan(\theta+\alpha) = \frac{4\sqrt{19} - 9\sqrt{33}}{36 + \sqrt{627}}$$

6. Find the exact value of the expression.

$$\tan\left(\sin^{-1}\frac{1}{2}-\cos^{-1}\frac{1}{2}\right)$$

Solution:

To find the exact value of the expression, we need to use the difference identity for the tangent function.

$$\tan(\theta-\alpha)=\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}$$

Define α and β.

$$\alpha=\sin^{-1}\frac{1}{2}$$

$$\beta=\cos^{-1}\frac{1}{2}$$

Using the properties of inverses, we can write

$$\sin\alpha=\frac{1}{2},\quad -\frac{\pi}{2}\leq\alpha\leq\frac{\pi}{2}$$

$$\cos\beta=\frac{1}{2},\ 0\leq\beta\leq\pi$$

Use the Pythagorean identity with sine and cosine to find $\cos\alpha$ and $\sin\beta$.

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \sin^2\alpha$$

Substitute $\sin\alpha = 1/2$.

$$\cos^2\alpha = 1 - \left(\frac{1}{2}\right)^2$$

$$\cos^2\alpha = 1 - \frac{1}{4}$$

$$\cos^2\alpha = \frac{3}{4}$$

$$\cos\alpha = \pm\sqrt{\frac{3}{4}}$$

Since $-\pi/2 \le \alpha \le \pi/2$, we can say $\cos\alpha = \sqrt{3}/2$.

$$\sin^2\beta + \cos^2\beta = 1$$

$$\sin^2\beta = 1 - \cos^2\beta$$

Substitute $\cos\beta = 1/2$.

$$\sin^2\beta = 1 - \left(\frac{1}{2}\right)^2$$

$$\sin^2\beta = 1 - \frac{1}{4}$$

$$\sin^2\beta = \frac{3}{4}$$

$$\sin\beta = \pm\sqrt{\frac{3}{4}}$$

Since $0 \le \beta \le \pi$, we can say $\sin\beta = \sqrt{3}/2$. Then using the difference formula for cosine, we get

$$\tan\left(\sin^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{2}\right) = \tan(\alpha - \beta)$$

$$\tan\left(\sin^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{2}\right) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\tan\left(\sin^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{2}\right) = \frac{\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right) - \frac{\sqrt{3}}{2}\left(\frac{2}{1}\right)}{1 + \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{2}{1}\right)}$$

$$\tan\left(\sin^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{2}\right) = \frac{\frac{1}{\sqrt{3}} - \sqrt{3}}{1 + 1}$$

$$\tan\left(\sin^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{2}\right) = \frac{\frac{\sqrt{3}}{3} - \frac{3\sqrt{3}}{3}}{2}$$

$$\tan\left(\sin^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{2}\right) = \frac{-\frac{2\sqrt{3}}{3}}{2}$$

$$\tan\left(\sin^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{2}\right) = -\frac{\sqrt{3}}{3}$$

DOUBLE-ANGLE IDENTITIES

1. If θ is an angle in the fourth quadrant whose sine is $-3/5$, what are the values of $\tan 2\theta$?

Solution:

By the double-angle identity for tangent,

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

In order to use the double-angle identity, we first need to find $\cos\theta$ and then $\tan\theta$. By the Pythagorean identity with sine and cosine,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

Since $\sin\theta = -3/5$, we get

$$\cos^2\theta = 1 - \left(-\frac{3}{5}\right)^2$$

$$\cos^2\theta = 1 - \frac{9}{25}$$

$$\cos^2\theta = \frac{16}{25}$$

Since θ is in the fourth quadrant, we know that $\cos\theta$ is positive, so we can ignore the negative value and say

$$\cos\theta = \frac{4}{5}$$

Now, substituting $\cos\theta = 4/5$ and $\sin\theta = -3/5$ into

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

we get

$$\tan\theta = \frac{-\frac{3}{5}}{\frac{4}{5}} = -\frac{3}{5}\left(\frac{5}{4}\right) = -\frac{3}{4}$$

Now to find $\tan 2\theta$, we'll substitute into the double-angle identity for tangent.

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\tan 2\theta = \frac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2}$$

$$\tan 2\theta = \frac{-\frac{3}{2}}{1-\frac{9}{16}}$$

$$\tan 2\theta = \frac{-\frac{3}{2}}{\frac{16-9}{16}}$$

$$\tan 2\theta = \frac{-\frac{3}{2}}{\frac{7}{16}}$$

$$\tan 2\theta = -\frac{3}{2}\left(\frac{16}{7}\right)$$

$$\tan 2\theta = -\frac{24}{7}$$

■ 2. If θ is an angle in the third quadrant whose tangent is $3/4$, what are the values of $\cos 2\theta$?

Solution:

By the double-angle identity for cosine,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

or

$$\cos 2\theta = 2\cos^2\theta - 1$$

In order to use the double-angle identity, we first need to find $\cos\theta$. We remember that

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{c}$$

We need to find the hypotenuse using the Pythagorean Theorem.

$$c^2 = a^2 + b^2$$

$$c^2 = 3^2 + 4^2$$

$$c^2 = 9 + 16$$

$$c^2 = 25$$

$$c = 5$$

Therefore

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5}$$

To find $\cos 2\theta$, we'll substitute into the double-angle identity for cosine.

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = 2\left(\frac{4}{5}\right)^2 - 1$$

$$\cos 2\theta = 2\left(\frac{16}{25}\right) - 1$$

$$\cos 2\theta = \frac{32}{25} - 1$$

$$\cos 2\theta = \frac{7}{25}$$

■ 3. Use a double-angle identity to rewrite the expression.

$$(\sin x + \cos x)^2$$

Solution:

We'll rewrite the expression as

$$(\sin x + \cos x)^2$$

$$(\sin x + \cos x)(\sin x + \cos x)$$

$$\sin^2 x + \sin x \cos x + \cos x \sin x + \cos^2 x$$

$$\sin^2 x + 2 \sin x \cos x + \cos^2 x$$

$$\sin^2 x + \cos^2 x + 2 \sin x \cos x$$

Remember that $\sin^2 x + \cos^2 x = 1$.

$$1 + 2 \sin x \cos x$$

Now use the double-angle identity for sine.

$$1 + \sin 2x$$

■ 4. If θ is an angle in the third quadrant whose sine is $-1/\sqrt{5}$, what is the value of $\sin 2\theta$?

Solution:

In order to use the double-angle identity $\sin 2\theta = 2\sin\theta\cos\theta$, we first need to find $\cos\theta$. By the basic Pythagorean identity with sine and cosine,

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

and since $\sin\theta = -1/\sqrt{5}$, we get

$$\cos^2\theta = 1 - \left(-\frac{1}{\sqrt{5}}\right)^2$$

$$\cos^2\theta = 1 - \frac{1}{5}$$

$$\cos^2\theta = \frac{4}{5}$$

Since θ is in the third quadrant, we know that $\cos\theta$ is negative, so we can ignore the positive value and say

$$\cos\theta = -\frac{2}{\sqrt{5}}$$

Substituting $\cos\theta = -2/\sqrt{5}$ and $\sin\theta = -1/\sqrt{5}$ into the double-angle identity for sine, we get

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin 2\theta = 2\left(-\frac{1}{\sqrt{5}}\right)\left(-\frac{2}{\sqrt{5}}\right)$$

$$\sin 2\theta = \frac{4}{5}$$

■ 5. If θ is an angle in the third quadrant whose tangent is $7/24$, what is the value of $\tan 2\theta$?

Solution:

To find $\tan 2\theta$, we'll substitute into the double-angle identity for tangent.

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\tan 2\theta = \frac{2\left(\frac{7}{24}\right)}{1-\left(\frac{7}{24}\right)^2}$$

$$\tan 2\theta = \frac{\frac{7}{12}}{1-\frac{49}{576}}$$

$$\tan 2\theta = \frac{\frac{7}{12}}{\frac{527}{576}}$$

$$\tan 2\theta = \frac{7}{12}\left(\frac{576}{527}\right)$$

$$\tan 2\theta = \frac{7}{1}\left(\frac{48}{527}\right)$$

$$\tan 2\theta = \frac{336}{527}$$

■ 6. Use a double-angle formula to rewrite the expression.

$$12\sin(4x)\cos(4x)$$

Solution:

Rewrite the expression as

$$12\sin(4x)\cos(4x)$$

$$6(2\sin(4x)\cos(4x))$$

Then the double-angle formula for sine lets us simplify to

$$6\sin(8x)$$

HALF-ANGLE IDENTITIES

1. Use a half-angle identity to find the exact value of the expression.

$$\sin 15^\circ$$

Solution:

Use the the half-angle identity for sine

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

We know that $\theta/2=15^\circ$, so $\theta=2(15^\circ)=30^\circ$. Now substitute to get

$$\sin 15^\circ=\pm\sqrt{\frac{1-\cos 30^\circ}{2}}$$

$$\sin 15^\circ=\pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$\sin 15^\circ=\pm\sqrt{\frac{\frac{2}{2}-\frac{\sqrt{3}}{2}}{2}}$$

$$\sin 15^\circ=\pm\sqrt{\frac{2-\sqrt{3}}{4}}$$

$$\sin 15^\circ = \pm \frac{\sqrt{2-\sqrt{3}}}{2}$$

Since 15° is in the first quadrant, sine is positive.

$$\sin 15^\circ = \frac{\sqrt{2-\sqrt{3}}}{2}$$

■ 2. If θ is the angle in Quadrant II with $\sin\theta = 7/25$, what are the values of $\sin(\theta/2)$ and $\cos(\theta/2)$?

Solution:

We'll start by using $\sin\theta = 7/25$ and the Pythagorean identity with sine and cosine to find the corresponding value of $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos^2\theta = 1 - \left(\frac{7}{25}\right)^2$$

$$\cos^2\theta = 1 - \frac{49}{625}$$

$$\cos^2\theta = \frac{576}{625}$$

$$\cos\theta = \pm\sqrt{\frac{576}{625}}$$

Since θ is in the second quadrant, the cosine is negative, so

$$\cos\theta = -\sqrt{\frac{576}{625}} = -\frac{\sqrt{576}}{\sqrt{625}} = -\frac{24}{25}$$

Then by the half-angle identity for cosine, we get

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\left(-\frac{24}{25}\right)}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{\frac{25-24}{25}}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1}{50}}$$

$$\cos\frac{\theta}{2} = \pm\frac{1}{5\sqrt{2}}$$

$$\cos\frac{\theta}{2} = \pm\frac{\sqrt{2}}{10}$$

By the half-angle identity for sine, we get

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\left(-\frac{24}{25}\right)}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{\frac{25+24}{25}}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{49}{50}}$$

$$\sin\frac{\theta}{2} = \pm\frac{7}{5\sqrt{2}}$$

$$\sin\frac{\theta}{2} = \pm\frac{7\sqrt{2}}{10}$$

To find the quadrant of the angle $\theta/2$, we'll divide through the inequality we were given by 2.

$$\frac{\pi}{2} < \theta < \pi$$

$$\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}$$

The angle $\pi/4$ is halfway through the first quadrant, and the angle $\pi/2$ is along the positive side of the y-axis, so $\theta/2$ has to be in the first quadrant. Both the cosine function and the sine function are positive for all angles in the first quadrant, so

$$\cos\frac{\theta}{2}=\frac{\sqrt{2}}{10}$$

$$\sin\frac{\theta}{2}=\frac{7\sqrt{2}}{10}$$

■ 3. If θ is the angle in the interval $(0,\pi/2)$ with $\tan\theta=2$, what are the values of $\sin(\theta/2)$ and $\cos(\theta/2)$?

Solution:

Since θ lies in first quadrant, we know that the values of sine, cosine, and tangent for the angle will all be positive.

Because tangent is equivalent to opposite/adjacent, we get

$$\frac{\text{opposite}}{\text{adjacent}}=\frac{2}{1}$$

Given a triangle with adjacent leg 1 and opposite leg 2, the hypotenuse must be

$$a^2+b^2=c^2$$

$$1^2+2^2=c^2$$

$$c^2=1+4$$

$$c^2=5$$

$$c = \sqrt{5}$$

Because cosine is equivalent to adjacent/hypotenuse, we get

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

If we substitute $\cos\theta = \sqrt{5}/5$ into the half-angle identity for cosine, we get

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\frac{\sqrt{5}}{5}}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{\frac{5+\sqrt{5}}{5}}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{5+\sqrt{5}}{10}}$$

If we also substitute $\cos\theta = \sqrt{5}/5$ into the half-angle identity for sine, we get

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\frac{\sqrt{5}}{5}}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{\frac{5-\sqrt{5}}{5}}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{5-\sqrt{5}}{10}}$$

To figure out the quadrant of $\theta/2$, we'll start with the fact that we were told θ is in the first quadrant, $0 < \theta < \pi/2$. We'll divide through the inequality by 2 to change θ into $\theta/2$.

$$0 < \theta < \frac{\pi}{2}$$

$$0 < \frac{\theta}{2} < \frac{\pi}{4}$$

This inequality tells us that the angle $\theta/2$ falls between 0 (which is along the positive direction of the x-axis) and $\pi/4$ (which is halfway through the first quadrant). Therefore, the bounds $\theta/2 = [0,\pi/4]$ define angles only in the first quadrant, so $\theta/2$ must be in the first quadrant.

For any angle in the first quadrant, sine and cosine are positive.

$$\cos\frac{\theta}{2} = \sqrt{\frac{5+\sqrt{5}}{10}}$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{5-\sqrt{5}}{10}}$$

4. If θ is the angle in the interval $(3\pi/2, 2\pi)$ with $\sin\theta = -15/17$, what are the values of $\tan(\theta/2)$ and $\cot(\theta/2)$?

Solution:

We'll start by using $\sin\theta = -15/17$ and the Pythagorean identity with sine and cosine to find the corresponding value of $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos^2\theta = 1 - \left(-\frac{15}{17}\right)^2$$

$$\cos^2\theta = 1 - \frac{225}{289}$$

$$\cos^2\theta = \frac{64}{289}$$

$$\cos\theta = \pm\sqrt{\frac{64}{289}}$$

Since θ is in the fourth quadrant, the cosine of every angle in the fourth quadrant is positive, so

$$\cos\theta = \sqrt{\frac{64}{289}} = \frac{\sqrt{64}}{\sqrt{289}} = \frac{8}{17}$$

Then by the half-angle identity for tangent, we get

$$\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$$

$$\tan\frac{\theta}{2}=\frac{-\frac{15}{17}}{1+\frac{8}{17}}$$

$$\tan\frac{\theta}{2}=\frac{-\frac{15}{17}}{\frac{17+8}{17}}$$

$$\tan\frac{\theta}{2}=\frac{-\frac{15}{17}}{\frac{25}{17}}$$

$$\tan\frac{\theta}{2}=-\frac{15}{17}\left(\frac{17}{25}\right)$$

$$\tan\frac{\theta}{2}=-\frac{3}{5}$$

We remember that

$$\cot\theta=\frac{1}{\tan\theta}$$

$$\cot\frac{\theta}{2}=\frac{1}{\tan\frac{\theta}{2}}$$

$$\cot\frac{\theta}{2}=\frac{1}{-\frac{3}{5}}$$

$$\cot\frac{\theta}{2}=-\frac{5}{3}$$

■ 5. Use a half-angle identity to find the exact value of the expression.

$$\sec\left(\frac{7\pi}{8}\right)$$

Solution:

First we need to find the cosine of $7\pi/8$. We know that $\theta/2 = 7\pi/8$, so

$$\theta = 2\left(\frac{7\pi}{8}\right) = \frac{7\pi}{4}$$

Using the half-angle identity for cosine, we get

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\cos\left(\frac{7\pi}{8}\right) = \pm\sqrt{\frac{1+\cos\left(\frac{7\pi}{4}\right)}{2}}$$

$$\cos\left(\frac{7\pi}{8}\right) = \pm\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}$$

$$\cos\left(\frac{7\pi}{8}\right) = \pm\sqrt{\frac{\frac{2}{2}+\frac{\sqrt{2}}{2}}{2}}$$

$$\cos\left(\frac{7\pi}{8}\right) = \pm\sqrt{\frac{2+\sqrt{2}}{4}}$$

$$\cos\left(\frac{7\pi}{8}\right) = \pm\frac{\sqrt{2+\sqrt{2}}}{2}$$

Since $7\pi/8$ is in the second quadrant, cosine is negative.

$$\cos\left(\frac{7\pi}{8}\right) = -\frac{\sqrt{2+\sqrt{2}}}{2}$$

Then by the reciprocal identity for secant,

$$\sec\left(\frac{7\pi}{8}\right) = \frac{1}{\cos\left(\frac{7\pi}{8}\right)}$$

$$\sec\left(\frac{7\pi}{8}\right) = \frac{1}{-\frac{\sqrt{2+\sqrt{2}}}{2}}$$

$$\sec\left(\frac{7\pi}{8}\right) = -\frac{2}{\sqrt{2+\sqrt{2}}}$$

$$\sec\left(\frac{7\pi}{8}\right) = -\frac{2}{\sqrt{2+\sqrt{2}}}\cdot\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2-\sqrt{2}}}$$

$$\sec\left(\frac{7\pi}{8}\right) = -\frac{2\sqrt{2-\sqrt{2}}}{\sqrt{2}}$$

$$\sec\left(\frac{7\pi}{8}\right) = -\frac{2\sqrt{2-\sqrt{2}}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$\sec\left(\frac{7\pi}{8}\right) = -\frac{2\sqrt{4-2\sqrt{2}}}{2}$$

$$\sec\left(\frac{7\pi}{8}\right) = -\sqrt{4-2\sqrt{2}}$$

6. Prove the identity.

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$$

Solution:

We'll rewrite the right side of the half-angle identity for tangent.

$$\tan\frac{\theta}{2} = \pm\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}$$

$$\tan\frac{\theta}{2} = \pm\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}} \cdot \frac{\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}}$$

$$\tan\frac{\theta}{2}=\frac{\sqrt{1-\cos^2\theta}}{\sqrt{(1+\cos\theta)^2}}$$

Use $\sin^2\theta+\cos^2\theta=1$, or $\sin^2\theta=1-\cos^2\theta$ to simplify the numerator.

$$\tan\frac{\theta}{2}=\frac{\sqrt{\sin^2\theta}}{1+\cos\theta}$$

$$\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$$

PRODUCT-TO-SUM IDENTITIES

■ 1. Rewrite $\cos(x-y)\cos(x+y)$ as a sum.

Solution:

Using the product-to-sum identity,

$$\cos\theta\cos\alpha = \frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

set $\theta = x - y$ and $\alpha = x + y$ to get

$$\frac{1}{2}\left[\cos(x-y+x+y)+\cos(x-y-(x+y))\right]$$

$$\frac{1}{2}\left[\cos(2x)+\cos(x-y-x-y)\right]$$

$$\frac{1}{2}\left[\cos(2x)+\cos(-2y)\right]$$

Using the even-odd identity $\cos(-\theta)=\cos\theta$, we get

$$\frac{1}{2}\left[\cos(2x)+\cos(2y)\right]$$

■ 2. Rewrite $\cos(x-15^\circ)\sin(x+15^\circ)$ as a sum.

Solution:

Using the product-to-sum identity,

$$\cos\theta\sin\alpha = \frac{1}{2}\left[\sin(\theta+\alpha) - \sin(\theta-\alpha)\right]$$

set $\theta = x - 15^\circ$ and $\alpha = x + 15^\circ$ to get

$$\frac{1}{2}\left[\sin(x - 15^\circ + x + 15^\circ) - \sin(x - 15^\circ - (x + 15^\circ))\right]$$

$$\frac{1}{2}\left[\sin(2x) - \sin(x - 15^\circ - x - 15^\circ)\right]$$

$$\frac{1}{2}\left[\sin(2x) - \sin(-30^\circ)\right]$$

Using the even-odd identity $\sin(-\theta) = -\sin\theta$, we get

$$\frac{1}{2}\left[\sin(2x) + \sin(30^\circ)\right]$$

■ 3. Find a sum equivalent to $\cos^3 x$.

Solution:

Rewrite $\cos^3 x$ as

$$\cos^3 x = \cos x \cdot \cos x \cdot \cos x$$

Consider just the first two factors, $\cos x \cdot \cos x$, and apply the product-to-sum identity

$$\cos\theta\cos\alpha = \frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

With $\theta = x$ and $\alpha = x$, we get

$$\frac{1}{2}\left[\cos(x+x)+\cos(x-x)\right]$$

$$\frac{1}{2}\left[\cos(2x)+\cos(0)\right]$$

$$\frac{1}{2}\left[\cos(2x)+1\right]$$

So $\cos^3 x$ is

$$\cos^3 x = \frac{1}{2}\left[\cos(2x)+1\right]\cos x$$

$$\cos^3 x = \left[\frac{1}{2}\cos(2x)+\frac{1}{2}\right]\cos x$$

$$\cos^3 x = \frac{1}{2}\cos(2x)\cos x + \frac{1}{2}\cos x$$

To rewrite $\cos(2x)\cos x$, we'll again use

$$\cos\theta\cos\alpha = \frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

with $\theta = 2x$ and $\alpha = x$.

$$\cos(2x)\cos x = \frac{1}{2}\left[\cos(2x + x) + \cos(2x - x)\right]$$

$$\cos(2x)\cos x = \frac{1}{2}\left[\cos(3x) + \cos x\right]$$

Substitute this into the expression for $\cos^3 x$.

$$\cos^3 x = \frac{1}{2}\left[\frac{1}{2}\left[\cos(3x) + \cos x\right]\right] + \frac{1}{2}\cos x$$

$$\cos^3 x = \frac{1}{4}\left[\cos(3x) + \cos x\right] + \frac{1}{2}\cos x$$

$$\cos^3 x = \frac{1}{4}\cos(3x) + \frac{1}{4}\cos x + \frac{1}{2}\cos x$$

$$\cos^3 x = \frac{1}{4}\cos(3x) + \frac{1}{4}\cos x + \frac{2}{4}\cos x$$

$$\cos^3 x = \frac{1}{4}\cos(3x) + \frac{3}{4}\cos x$$

4. Find the exact value of each expression.

$$\left(\sin\frac{3\pi}{8}\right)\left(\cos\frac{3\pi}{8}\right)$$

$$\sin^2\left(\frac{3\pi}{8}\right)$$

$$\cos^2\left(\frac{3\pi}{8}\right)$$

Solution:

To compute $(\sin(3\pi/8))(\cos(3\pi/8))$, we can use the product-to-sum identity for the product of sine and cosine.

$$\sin\theta\cos\alpha = \frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

$$\left(\sin\frac{3\pi}{8}\right)\left(\cos\frac{3\pi}{8}\right) = \frac{1}{2}\left[\sin\left(\frac{3\pi}{8}+\frac{3\pi}{8}\right)+\sin\left(\frac{3\pi}{8}-\frac{3\pi}{8}\right)\right]$$

$$\left(\sin\frac{3\pi}{8}\right)\left(\cos\frac{3\pi}{8}\right) = \frac{1}{2}\left(\sin\frac{3\pi}{4}+\sin 0\right)$$

Pulling the values of sine on the right side from the unit circle, we get

$$\left(\sin\frac{3\pi}{8}\right)\left(\cos\frac{3\pi}{8}\right) = \frac{1}{2}\left(\frac{\sqrt{2}}{2}+0\right)$$

$$\left(\sin\frac{3\pi}{8}\right)\left(\cos\frac{3\pi}{8}\right) = \frac{\sqrt{2}}{4}$$

To find $\sin^2(3\pi/8)$, we'll use the product-to-sum identity for the product of two sine functions.

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

$$\left(\sin\frac{3\pi}{8}\right)\left(\sin\frac{3\pi}{8}\right) = \frac{1}{2}\left[\cos\left(\frac{3\pi}{8}-\frac{3\pi}{8}\right)-\cos\left(\frac{3\pi}{8}+\frac{3\pi}{8}\right)\right]$$

$$\sin^2\left(\frac{3\pi}{8}\right)=\frac{1}{2}\left(\cos 0-\cos\frac{3\pi}{4}\right)$$

Pulling the values of cosine on the right side from the unit circle, we get

$$\sin^2\left(\frac{3\pi}{8}\right)=\frac{1}{2}\left(1-\left(-\frac{\sqrt{2}}{2}\right)\right)$$

$$\sin^2\left(\frac{3\pi}{8}\right)=\frac{1}{2}+\frac{\sqrt{2}}{4}$$

$$\sin^2\left(\frac{3\pi}{8}\right)=\frac{2}{4}+\frac{\sqrt{2}}{4}$$

$$\sin^2\left(\frac{3\pi}{8}\right)=\frac{2+\sqrt{2}}{4}$$

To get the value of $\cos^2(3\pi/8)$, we'll use the product-to-sum identity for the product of two cosine functions.

$$\cos\theta\cos\alpha=\frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

$$\left(\cos\frac{3\pi}{8}\right)\left(\cos\frac{3\pi}{8}\right)=\frac{1}{2}\left[\cos\left(\frac{3\pi}{8}+\frac{3\pi}{8}\right)+\cos\left(\frac{3\pi}{8}-\frac{3\pi}{8}\right)\right]$$

$$\cos^2\left(\frac{3\pi}{8}\right)=\frac{1}{2}\left(\cos\frac{3\pi}{4}+\cos 0\right)$$

Pulling the values of cosine on the right side from the unit circle, we get

$$\cos^2\left(\frac{3\pi}{8}\right)=\frac{1}{2}\left(-\frac{\sqrt{2}}{2}+1\right)$$

$$\cos^2\left(\frac{3\pi}{8}\right)=-\frac{\sqrt{2}}{4}+\frac{1}{2}$$

$$\cos^2\left(\frac{3\pi}{8}\right)=-\frac{\sqrt{2}}{4}+\frac{2}{4}$$

$$\cos^2\left(\frac{3\pi}{8}\right)=\frac{2-\sqrt{2}}{4}$$

■ 5. Simplify the expression.

$$\sin(x-y)\cos y+\cos(x-y)\sin y$$

Solution:

Consider just the first two factors, $\sin(x-y)\cos y$, and apply the product-to-sum identity

$$\sin\theta\cos\alpha=\frac{1}{2}\left[\sin(\theta+\alpha)+\sin(\theta-\alpha)\right]$$

With $\theta=x-y$ and $\alpha=y$, we get

$$\sin(x-y)\cos y=\frac{1}{2}\left[\sin(x-y+y)+\sin(x-y-y)\right]$$

$$\sin(x-y)\cos y = \frac{1}{2}\left[\sin x + \sin(x-2y)\right]$$

$$\sin(x-y)\cos y = \frac{1}{2}\sin x + \frac{1}{2}\sin(x-2y)$$

To rewrite $\cos(x-y)\sin y$, we'll use

$$\cos\theta\sin\alpha = \frac{1}{2}\left[\sin(\theta+\alpha) - \sin(\theta-\alpha)\right]$$

With $\theta = x - y$ and $\alpha = y$, we get

$$\cos(x-y)\sin y = \frac{1}{2}\left[\sin(x-y+y) - \sin(x-y-y)\right]$$

$$\cos(x-y)\sin y = \frac{1}{2}\left[\sin x - \sin(x-2y)\right]$$

$$\cos(x-y)\sin y = \frac{1}{2}\sin x - \frac{1}{2}\sin(x-2y)$$

Then the original expression can be written as

$$\frac{1}{2}\sin x + \frac{1}{2}\sin(x-2y) + \frac{1}{2}\sin x - \frac{1}{2}\sin(x-2y)$$

$$\sin x$$

■ 6. Find the value of the expression.

$$\sin^2\left(\frac{\pi}{12}\right) + \sin^2\left(\frac{3\pi}{12}\right) + \sin^2\left(\frac{5\pi}{12}\right)$$

Solution:

To find $\sin^2(\pi/12)$, we'll use the product-to-sum identity for the product of two sine functions.

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha)-\cos(\theta+\alpha)\right]$$

With $\theta = \pi/12$ and $\alpha = \pi/12$, we get

$$\left(\sin\frac{\pi}{12}\right)\left(\sin\frac{\pi}{12}\right) = \frac{1}{2}\left[\cos\left(\frac{\pi}{12}-\frac{\pi}{12}\right)-\cos\left(\frac{\pi}{12}+\frac{\pi}{12}\right)\right]$$

$$\sin^2\left(\frac{\pi}{12}\right) = \frac{1}{2}\left(\cos 0-\cos\frac{\pi}{6}\right)$$

Pulling the values of cosine on the right side from the unit circle, we get

$$\sin^2\left(\frac{\pi}{12}\right) = \frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)$$

$$\sin^2\left(\frac{\pi}{12}\right) = \frac{1}{2}-\frac{\sqrt{3}}{4}$$

$$\sin^2\left(\frac{\pi}{12}\right) = \frac{2}{4}-\frac{\sqrt{3}}{4}$$

$$\sin^2\left(\frac{\pi}{12}\right) = \frac{2-\sqrt{3}}{4}$$

To find $\sin^2(3\pi/12)$, we'll use the product-to-sum identity for the product of two sine functions.

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha) - \cos(\theta+\alpha)\right]$$

With $\theta = 3\pi/12$ and $\alpha = 3\pi/12$, we get

$$\left(\sin\frac{3\pi}{12}\right)\left(\sin\frac{3\pi}{12}\right) = \frac{1}{2}\left[\cos\left(\frac{3\pi}{12}-\frac{3\pi}{12}\right) - \cos\left(\frac{3\pi}{12}+\frac{3\pi}{12}\right)\right]$$

$$\sin^2\left(\frac{3\pi}{12}\right) = \frac{1}{2}\left(\cos 0 - \cos\frac{\pi}{2}\right)$$

Pulling the values of cosine on the right side from the unit circle, we get

$$\sin^2\left(\frac{3\pi}{12}\right) = \frac{1}{2}(1-0)$$

$$\sin^2\left(\frac{3\pi}{12}\right) = \frac{1}{2}$$

To find $\sin^2(5\pi/12)$, we'll use the product-to-sum identity for the product of two sine functions.

$$\sin\theta\sin\alpha = \frac{1}{2}\left[\cos(\theta-\alpha) - \cos(\theta+\alpha)\right]$$

With $\theta = 5\pi/12$ and $\alpha = 5\pi/12$, we get

$$\left(\sin\frac{5\pi}{12}\right)\left(\sin\frac{5\pi}{12}\right) = \frac{1}{2}\left[\cos\left(\frac{5\pi}{12}-\frac{5\pi}{12}\right) - \cos\left(\frac{5\pi}{12}+\frac{5\pi}{12}\right)\right]$$

$$\sin^2\left(\frac{5\pi}{12}\right)=\frac{1}{2}\left(\cos 0-\cos\frac{5\pi}{6}\right)$$

Pulling the values of cosine on the right side from the unit circle, we get

$$\sin^2\left(\frac{5\pi}{12}\right)=\frac{1}{2}\left(1-\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$\sin^2\left(\frac{5\pi}{12}\right)=\frac{1}{2}+\frac{\sqrt{3}}{4}$$

$$\sin^2\left(\frac{5\pi}{12}\right)=\frac{2}{4}+\frac{\sqrt{3}}{4}$$

$$\sin^2\left(\frac{5\pi}{12}\right)=\frac{2+\sqrt{3}}{4}$$

Then the original expression can be rewritten as

$$\frac{2-\sqrt{3}}{4}+\frac{1}{2}+\frac{2+\sqrt{3}}{4}$$

$$\frac{2-\sqrt{3}}{4}+\frac{2}{4}+\frac{2+\sqrt{3}}{4}$$

$$\frac{6}{4}$$

$$\frac{3}{2}$$

SUM-TO-PRODUCT IDENTITIES

1. Rewrite the function as a product.

$$f(x)=\sin\left(x+\frac{\pi}{6}\right)-\sin\left(x-\frac{\pi}{6}\right)$$

Solution:

Apply the sum-to-product identity

$$\sin\theta-\sin\alpha=2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)$$

to get

$$f(x)=2\cos\left(\frac{x+\frac{\pi}{6}+x-\frac{\pi}{6}}{2}\right)\sin\left(\frac{x+\frac{\pi}{6}-\left(x-\frac{\pi}{6}\right)}{2}\right)$$

$$f(x)=2\cos\left(\frac{2x}{2}\right)\sin\left(\frac{x+\frac{\pi}{6}-x+\frac{\pi}{6}}{2}\right)$$

$$f(x)=2\cos x\sin\left(\frac{\frac{2\pi}{6}}{2}\right)$$

$$f(x) = 2\cos x \sin\left(\frac{2\pi}{12}\right)$$

$$f(x) = 2\cos x \sin\left(\frac{\pi}{6}\right)$$

From the unit circle, the value of $\sin(\pi/6)$ is $1/2$, so

$$f(x) = 2\cos x\left(\frac{1}{2}\right)$$

$$f(x) = \cos x$$

2. Find a product equal to $\sin(x+y)+\sin(x-y)$.

Solution:

Apply the sum-to-product identity

$$\sin\theta + \sin\alpha = 2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

to get

$$2\sin\left(\frac{x+y+x-y}{2}\right)\cos\left(\frac{x+y-(x-y)}{2}\right)$$

$$2\sin\left(\frac{2x}{2}\right)\cos\left(\frac{x+y-x+y}{2}\right)$$

$$2\sin x\cos\left(\frac{2y}{2}\right)$$

$$2\sin x\cos y$$

■ 3. Find the exact value of the expression.

$$\frac{\cos 93^\circ + \cos 27^\circ}{\cos 33^\circ}$$

Solution:

To the numerator, apply the sum-to-product identity

$$\cos\theta + \cos\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

to get

$$2\cos\left(\frac{93^\circ + 27^\circ}{2}\right)\cos\left(\frac{93^\circ - 27^\circ}{2}\right)$$

$$2\cos\left(\frac{120^\circ}{2}\right)\cos\left(\frac{66^\circ}{2}\right)$$

$$2\cos 60^\circ \cos 33^\circ$$

Substitute this value into the original fraction.

$$\frac{2\cos 60^\circ \cos 33^\circ}{\cos 33^\circ}$$

$$2\cos 60^\circ$$

$$2\left(\frac{1}{2}\right)$$

$$1$$

4. Simplify the expression.

$$\frac{\sin(7\theta)+\sin(3\theta)}{\cos(7\theta)+\cos(3\theta)}$$

Solution:

Apply the sum-to-product identity

$$\sin\theta+\sin\alpha=2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

to the numerator to get

$$2\sin\left(\frac{7\theta+3\theta}{2}\right)\cos\left(\frac{7\theta-3\theta}{2}\right)$$

$$2\sin\left(\frac{10\theta}{2}\right)\cos\left(\frac{4\theta}{2}\right)$$

$$2\sin(5\theta)\cos(2\theta)$$

Apply the sum-to-product identity

$$\cos\theta + \cos\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

to the denominator to get

$$2\cos\left(\frac{7\theta+3\theta}{2}\right)\cos\left(\frac{7\theta-3\theta}{2}\right)$$

$$2\cos\left(\frac{10\theta}{2}\right)\cos\left(\frac{4\theta}{2}\right)$$

$$2\cos(5\theta)\cos(2\theta)$$

Then the original expression becomes

$$\frac{2\sin(5\theta)\cos(2\theta)}{2\cos(5\theta)\cos(2\theta)}$$

$$\frac{\sin(5\theta)}{\cos(5\theta)}$$

$$\tan(5\theta)$$

■ 5. Find a product equal to $\cos(3\theta) + \cos(5\theta) - 2\cos(\theta)\cos(8\theta)$.

Solution:

Apply the sum-to-product identity

$$\cos\theta+\cos\alpha=2\cos\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

to rewrite $\cos(3\theta)+\cos(5\theta)$. We can set $\theta=3\theta$ and $\alpha=5\theta$ and rewrite the product as

$$\cos(3\theta)+\cos(5\theta)=2\cos\left(\frac{3\theta+5\theta}{2}\right)\cos\left(\frac{3\theta-5\theta}{2}\right)$$

$$\cos(3\theta)+\cos(5\theta)=2\cos\left(\frac{8\theta}{2}\right)\cos\left(\frac{-2\theta}{2}\right)$$

$$\cos(3\theta)+\cos(5\theta)=2\cos(4\theta)\cos(-\theta)$$

Using the even-odd identity $\cos(-\theta)=\cos\theta$, we get

$$\cos(3\theta)+\cos(5\theta)=2\cos(4\theta)\cos(\theta)$$

Substitute into the expression, then simplify.

$$\cos(3\theta)+\cos(5\theta)-2\cos(\theta)\cos(8\theta)=2\cos(4\theta)\cos(\theta)-2\cos(\theta)\cos(8\theta)$$

$$\cos(3\theta)+\cos(5\theta)-2\cos(\theta)\cos(8\theta)=2\cos(\theta)(\cos(4\theta)-\cos(8\theta))$$

Apply the sum-to-product identity

$$\cos\theta-\cos\alpha=-2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)$$

to rewrite $\cos(4\theta)-\cos(8\theta)$. We can set $\theta=4\theta$ and $\alpha=8\theta$ and rewrite the product as

$$\cos(4\theta)-\cos(8\theta)=-2\sin\left(\frac{4\theta+8\theta}{2}\right)\sin\left(\frac{4\theta-8\theta}{2}\right)$$

$$\cos(4\theta) - \cos(8\theta) = -2\sin\left(\frac{12\theta}{2}\right)\sin\left(\frac{-4\theta}{2}\right)$$

$$\cos(4\theta) - \cos(8\theta) = -2\sin(6\theta)\sin(-2\theta)$$

Using the odd-even identity $\sin(-\theta) = -\sin\theta$, we get

$$\cos(4\theta) - \cos(8\theta) = 2\sin(6\theta)\sin(2\theta)$$

And finally

$$\cos(3\theta) + \cos(5\theta) - 2\cos(\theta)\cos(8\theta) = 2\cos(4\theta)\cos(\theta) - 2\cos(\theta)\cos(8\theta)$$

$$\cos(3\theta) + \cos(5\theta) - 2\cos(\theta)\cos(8\theta) = 2\cos(\theta)(\cos(4\theta) - \cos(8\theta))$$

$$\cos(3\theta) + \cos(5\theta) - 2\cos(\theta)\cos(8\theta) = 2\cos\theta(2\sin(6\theta)\sin(2\theta))$$

$$\cos(3\theta) + \cos(5\theta) - 2\cos(\theta)\cos(8\theta) = 4\cos\theta\sin(6\theta)\sin(2\theta)$$

6. Find the exact value of the expression.

$$16\sin 390^\circ + 22\sin 240^\circ + 16\sin 150^\circ - 22\sin 120^\circ$$

Solution:

First we need to rewrite the expression as

$$16\sin 390^\circ + 16\sin 150^\circ + 22\sin 240^\circ - 22\sin 120^\circ$$

$$16(\sin 390^\circ + \sin 150^\circ) + 22(\sin 240^\circ - \sin 120^\circ)$$

Using the sum-to-product identity,

$$\sin\theta + \sin\alpha = 2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

we can set $\theta = 390^\circ$ and $\alpha = 150^\circ$ and rewrite the product as

$$\sin 390^\circ + \sin 150^\circ = 2\sin\left(\frac{390^\circ + 150^\circ}{2}\right)\cos\left(\frac{390^\circ - 150^\circ}{2}\right)$$

$$\sin 390^\circ + \sin 150^\circ = 2\sin\left(\frac{540^\circ}{2}\right)\cos\left(\frac{240}{2}\right)$$

$$\sin 390^\circ + \sin 150^\circ = 2\sin 270^\circ \cos 120^\circ$$

$$\sin 390^\circ + \sin 150^\circ = 2(-1)\left(-\frac{1}{2}\right)$$

$$\sin 390^\circ + \sin 150^\circ = 1$$

Now using the sum-to-product identity,

$$\sin\theta - \sin\alpha = 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)$$

we can set $\theta = 240^\circ$ and $\alpha = 120^\circ$ and rewrite the product as

$$\sin 240^\circ - \sin 120^\circ = 2\cos\left(\frac{240^\circ + 120^\circ}{2}\right)\sin\left(\frac{240^\circ - 120^\circ}{2}\right)$$

$$\sin 240^\circ - \sin 120^\circ = 2\cos\left(\frac{360^\circ}{2}\right)\sin\left(\frac{120^\circ}{2}\right)$$

$$\sin 240^\circ - \sin 120^\circ = 2\cos 180^\circ \sin 60^\circ$$

$$\sin 240^\circ - \sin 120^\circ = 2(-1)\left(\frac{\sqrt{3}}{2}\right)$$

$$\sin 240^\circ - \sin 120^\circ = -\sqrt{3}$$

Then the value of the original expression is

$$16(1) + 22(-\sqrt{3})$$

$$16 - 22\sqrt{3}$$

PROVING THE TRIG EQUATION

1. Prove the trig equation.

$$\tan\left(\frac{x}{2}\right)=\frac{1-\cos x}{\sin x}$$

Solution:

Start by working on the numerator $1-\cos x$, using a half-angle identity.

$$\sin^2\left(\frac{x}{2}\right)=\frac{1-\cos x}{2}$$

$$2\sin^2\left(\frac{x}{2}\right)=1-\cos x$$

Set this aside for a moment, and work on the denominator $\sin x$, using the double-angle identity $\sin 2\theta=2\sin\theta\cos\theta$. Make a substitution of $\theta=x/2$.

$$\sin x=2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

Substitute the two values we've just found into the original expression.

$$\tan\left(\frac{x}{2}\right)=\frac{2\sin^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$$

$$\tan\left(\frac{x}{2}\right)=\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}$$

$$\tan\left(\frac{x}{2}\right)=\tan\left(\frac{x}{2}\right)$$

■ 2. Prove the trigonometric equation.

$$\frac{\sin(5x)-\sin x}{\cos(5x)+\cos x}=\tan(2x)$$

Solution:

Apply a sum-to-product identity to the numerator on the left side.

$$\sin(5x)-\sin x=2\cos\left(\frac{5x+x}{2}\right)\sin\left(\frac{5x-x}{2}\right)$$

$$\sin(5x)-\sin x=2\cos\left(\frac{6x}{2}\right)\sin\left(\frac{4x}{2}\right)$$

$$\sin(5x)-\sin x=2\cos(3x)\sin(2x)$$

Set this aside for a moment, and apply a sum-to-product identity to the denominator.

$$\cos(5x)+\cos x=2\cos\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right)$$

$$\cos(5x) + \cos x = 2\cos\left(\frac{6x}{2}\right)\cos\left(\frac{4x}{2}\right)$$

$$\cos(5x) + \cos x = 2\cos(3x)\cos(2x)$$

Substitute the two values we've just found into the original expression.

$$\frac{2\cos(3x)\sin(2x)}{2\cos(3x)\cos(2x)} = \tan(2x)$$

$$\frac{\sin(2x)}{\cos(2x)} = \tan(2x)$$

$$\tan(2x) = \tan(2x)$$

■ 3. Prove the trigonometric equation.

$$\sin(x - \pi)\sin(x + \pi) = \sin^2 x$$

Solution:

Apply a product-to-sum identity to the left side of the equation.

$$\frac{1}{2}\left[\cos(x - \pi - (x + \pi)) - \cos(x - \pi + x + \pi)\right] = \sin^2 x$$

$$\frac{1}{2}\left[\cos(-2\pi) - \cos(2x)\right] = \sin^2 x$$

$$\frac{1}{2}\left[1 - \cos(2x)\right] = \sin^2 x$$

$$\frac{1-\cos(2x)}{2}=\sin^2 x$$

Set this aside for a moment, and then starting with the half-angle identity

$$\sin^2\left(\frac{\theta}{2}\right)=\frac{1-\cos\theta}{2}$$

substitute $x=\theta/2$ to get

$$\sin^2 x=\frac{1-\cos(2x)}{2}$$

Substitute this value into the left side of the equation we left off with.

$$\frac{1-\cos(2x)}{2}=\sin^2 x$$

$$\sin^2 x=\sin^2 x$$

■ 4. Prove the trigonometric equation.

$$\sin(-x)\cos(-x)\tan(-x)\csc(-x)=-\sin x$$

Solution:

Rewrite the tangent and cosecant functions in terms of sine and cosine, then cancel factors.

$$\sin(-x)\cos(-x)\left(\frac{\sin(-x)}{\cos(-x)}\right)\left(\frac{1}{\sin(-x)}\right)=-\sin x$$

$$\sin(-x)\sin(-x)\left(\frac{1}{\sin(-x)}\right) = -\sin x$$

$$\sin(-x) = -\sin x$$

Apply the even-odd identity $\sin(-x) = -\sin x$.

$$-\sin x = -\sin x$$

■ 5. Prove the trigonometric equation.

$$(\sin t + \cos t)^2 - 1 = \sin(2t)$$

Solution:

Expand the expression.

$$\sin^2 t + 2\sin t\cos t + \cos^2 t - 1 = \sin(2t)$$

$$(\sin^2 t + \cos^2 t) + 2\sin t\cos t - 1 = \sin(2t)$$

Apply the Pythagorean identity $\sin^2 t + \cos^2 t = 1$ to get

$$1 + 2\sin t\cos t - 1 = \sin(2t)$$

$$2\sin t\cos t = \sin(2t)$$

Apply the double-angle identity $\sin(2\theta) = 2\sin\theta\cos\theta$.

$$\sin(2t) = \sin(2t)$$

6. Prove the trigonometric equation.

$$\frac{\cos(270^\circ + x)}{\sin(180^\circ - x)} = 1$$

Solution:

Apply the sum-difference identity $\cos(\theta + \alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$ to the numerator, setting $\theta = 270^\circ$ and $\alpha = x$.

$$\cos(270^\circ + x) = \cos 270^\circ \cos x - \sin 270^\circ \sin x$$

$$\cos(270^\circ + x) = (0)\cos x - (-1)\sin x$$

$$\cos(270^\circ + x) = \sin x$$

Apply the sum-difference identity $\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha$ to the denominator, setting $\theta = 180^\circ$ and $\alpha = x$.

$$\sin(180^\circ - x) = \sin 180^\circ \cos x - \cos 180^\circ \sin x$$

$$\sin(180^\circ - x) = (0)\cos x - (-1)\sin x$$

$$\sin(180^\circ - x) = \sin x$$

Substituting the values we've found into the given expression, we get

$$\frac{\cos(270^\circ + x)}{\sin(180^\circ - x)} = 1$$

$$\frac{\sin x}{\sin x} = 1$$

$$1 = 1$$

COMPLETE SOLUTION SET OF THE EQUATION

■ 1. Find the complete solution set of the equation $\cos^2 x - 3\cos x + 2 = 0$.

Solution:

The equation is a quadratic equation in terms of $\cos x$, so we can factor it as

$$(\cos x - 1)(\cos x - 2) = 0$$

$\cos x - 1 = 0$ or $\cos x - 2 = 0$

$\cos x = 1$ or $\cos x = 2$

The equation $\cos x = 2$ has no solutions since the cosine function is only defined on the range $[-1,1]$ and therefore can't be equal to 2. The first equation is true at $x = 0$, and every angle coterminal with 0.

$$x = 0,\ 2\pi,\ 4\pi,\ 6\pi,\ \dots$$

$x = 2\pi n$ where n is any integer

■ 2. Find all the solutions of the trig equation, then list only the solutions that lie in the interval $[0,2\pi)$.

$$3\csc^2\theta - 2\cot^2\theta - 4 = 0$$

Solution:

If we use the Pythagorean identity for cotangent, $\cot^2\theta = \csc^2\theta - 1$, we can rewrite the equation as

$$3\csc^2\theta - 2(\csc^2\theta - 1) - 4 = 0$$

$$3\csc^2\theta - 2\csc^2\theta + 2 - 4 = 0$$

$$\csc^2\theta - 2 = 0$$

$$\csc^2\theta = 2$$

$$\csc\theta = \pm\sqrt{2}$$

$$\frac{1}{\sin\theta} = \pm\sqrt{2}$$

$$\sin\theta = \pm\frac{1}{\sqrt{2}}$$

$$\sin\theta = \pm\frac{\sqrt{2}}{2}$$

This is the value of sine at $\theta = \pi/4$, $\theta = 3\pi/4$, $\theta = 5\pi/4$, $\theta = 7\pi/4$, and every angle coterminal with those.

$$\theta = \left\{\frac{\pi}{4} + \pi k \ \mid \ k \in \mathbb{Z}\right\} \cup \left\{\frac{3\pi}{4} + \pi k \ \mid \ k \in \mathbb{Z}\right\}$$

In the interval $[0,2\pi)$, only $\theta = \pi/4$, $\theta = 3\pi/4$, $\theta = 5\pi/4$, and $\theta = 7\pi/4$ will satisfy the equation.

■ 3. Find the complete solution set of the equation.

$$4\cos^3\theta - 2\cos^2\theta - 2\cos\theta + 1 = 0$$

Solution:

We can rewrite the equation as

$$2\cos^2\theta(2\cos\theta - 1) - (2\cos\theta - 1) = 0$$

$$(2\cos\theta - 1)(2\cos^2\theta - 1) = 0$$

The only way the left side of the equation is 0 is if $2\cos\theta - 1 = 0$, $2\cos^2\theta - 1 = 0$, or both. So we need to solve these equations individually to find the values of θ that satisfy the equation. We get

$$2\cos\theta - 1 = 0$$

$$2\cos\theta = 1$$

$$\cos\theta = \frac{1}{2}$$

and

$$2\cos^2\theta - 1 = 0$$

$$2\cos^2\theta = 1$$

$$\cos^2\theta = \frac{1}{2}$$

$$\cos\theta = \pm\frac{1}{\sqrt{2}}$$

$$\cos\theta = \pm\frac{\sqrt{2}}{2}$$

The equation $\cos\theta = 1/2$ is true at $\theta = \pi/3$ and $\theta = 5\pi/3$. But the set of all angles coterminal with these two angles is

$$\theta = \frac{\pi}{3} + 2n\pi \text{ and } \theta = \frac{5\pi}{3} + 2n\pi$$

The equation $\cos\theta = \pm\sqrt{2}/2$ is true at $\theta = \pi/4$, $\theta = 3\pi/4$, $\theta = 5\pi/4$, and $\theta = 7\pi/4$. But the set of all angles coterminal with these four angles is

$$\theta = \frac{\pi}{4} + n\pi \text{ and } \theta = \frac{3\pi}{4} + n\pi$$

Putting all these sets together, we can say that the complete solution set of $4\cos^3\theta - 2\cos^2\theta - 2\cos\theta + 1 = 0$ includes all of these, where n is any integer:

$$\theta = \frac{\pi}{4} + n\pi$$

$$\theta = \frac{3\pi}{4} + n\pi$$

$$\theta = \frac{\pi}{3} + 2n\pi$$

$$\theta = \frac{5\pi}{3} + 2n\pi$$

■ 4. Find all the solutions of the trig equation, then list only the solutions that lie in the interval $[0,2\pi)$.

$$\cos\theta + 1 = \sin\theta$$

Solution:

First we need to square both sides.

$$(\cos\theta + 1)^2 = \sin^2\theta$$

If we use the Pythagorean identity for sine, $\sin^2\theta = 1 - \cos^2\theta$, we can rewrite the equation as

$$(\cos\theta + 1)^2 = 1 - \cos^2\theta$$

$$\cos^2\theta + 2\cos\theta + 1 = 1 - \cos^2\theta$$

$$2\cos^2\theta + 2\cos\theta = 0$$

$$2\cos\theta(\cos\theta + 1) = 0$$

The only way the left side of the equation is 0 is if $2\cos\theta = 0$, $\cos\theta + 1 = 0$, or both. So we need to solve these equations individually to find the values of θ that satisfy the equation. We get

$$2\cos\theta = 0$$

$$\cos\theta = 0$$

and

$$\cos\theta + 1 = 0$$

$$\cos\theta = -1$$

The equation $\cos\theta = 0$ is true at $\theta = \pi/2$ and $\theta = 3\pi/2$, and the equation $\cos\theta = -1$ is true at $\theta = \pi$. The equation will also be true at all angles coterminal with these.

$$\theta = \left\{\frac{\pi}{2} + \pi k \ \mid \ k \in \mathbb{Z}\right\} \cup \left\{\pi + 2\pi k \ \mid \ k \in \mathbb{Z}\right\}$$

Of this set, the only solutions in the interval $[0,2\pi)$ are $\theta = \pi/2$, $\theta = \pi$, and $\theta = 3\pi/2$. However, if we plug each of these three solutions back into the original equation, we get

$$\cos\left(\frac{\pi}{2}\right) + 1 = \sin\left(\frac{\pi}{2}\right) \quad \to \quad 0 + 1 = 1 \quad \to \quad 1 = 1$$

$$\cos\pi + 1 = \sin\pi \quad \to \quad -1 + 1 = 0 \quad \to \quad 0 = 0$$

$$\cos\left(\frac{3\pi}{2}\right) + 1 = \sin\left(\frac{3\pi}{2}\right) \quad \to \quad 0 + 1 = -1 \quad \to \quad 1 \neq -1$$

and we can see that only $\theta = \pi/2$ and $\theta = \pi$ are actually valid solutions. The false $\theta = 3\pi/2$ solution gets introduced when we square both sides of the original equation.

5. Find all the solutions of the trig equation, then list only the solutions that lie in the interval $[0,2\pi)$.

$$2(\sin^2\theta - \cos^2\theta) = \sqrt{3}$$

Solution:

We can rewrite the equation as

$$\sin^2\theta - \cos^2\theta = \frac{\sqrt{3}}{2}$$

$$\cos^2\theta - \sin^2\theta = -\frac{\sqrt{3}}{2}$$

Use the double-angle identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$ to rewrite the left side of the equation.

$$\cos(2\theta) = -\frac{\sqrt{3}}{2}$$

From the unit circle, we know that cosine is $-\sqrt{3}/2$ at $5\pi/6$ and $7\pi/6$. Therefore, we need to solve two equations:

$$2\theta = \frac{5\pi}{6} + 2n\pi$$

$$\theta_1 = \frac{5\pi}{12} + n\pi$$

and

$$2\theta = \frac{7\pi}{6} + 2n\pi$$

$$\theta_2 = \frac{7\pi}{12} + n\pi$$

The full solution set is

$$\theta = \left\{\frac{5\pi}{12} + n\pi \ \middle|\ n \in \mathbb{Z}\right\} \cup \left\{\frac{7\pi}{12} + n\pi \ \middle|\ n \in \mathbb{Z}\right\}$$

and the solution set in the interval $[0,2\pi)$ is $\theta = 5\pi/12$, $7\pi/12$, $17\pi/12$, and $19\pi/12$.

6. Find the complete solution set of the equation.

$$4\sin\left(\theta - \frac{\pi}{3}\right)\cos\left(\theta - \frac{\pi}{3}\right) = \sqrt{3}$$

Solution:

Divide through both sides of the equation by 2.

$$2\sin\left(\theta - \frac{\pi}{3}\right)\cos\left(\theta - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Use the double-angle identity $\sin(2\theta) = 2\sin\theta\cos\theta$ to rewrite the left side of the equation.

$$\sin\left[2\left(\theta - \frac{\pi}{3}\right)\right] = \frac{\sqrt{3}}{2}$$

From the unit circle, we know that sine is $\sqrt{3}/2$ at $\pi/3$ and $2\pi/3$. Therefore, we need to solve two equations:

$$2\left(\theta - \frac{\pi}{3}\right) = \frac{\pi}{3} + 2n\pi$$

$$\theta - \frac{\pi}{3} = \frac{\pi}{6} + n\pi$$

$$\theta = \frac{\pi}{6} + \frac{\pi}{3} + n\pi$$

$$\theta = \frac{\pi}{2} + n\pi$$

and

$$2\left(\theta - \frac{\pi}{3}\right) = \frac{2\pi}{3} + 2n\pi$$

$$\theta - \frac{\pi}{3} = \frac{\pi}{3} + n\pi$$

$$\theta = \frac{2\pi}{3} + n\pi$$

Putting all these sets together, we can say that the complete solution set of the equation includes both angle sets.

$$\theta = \left\{\frac{\pi}{2} + n\pi \ \mid \ n \in \mathbb{Z}\right\} \cup \left\{\frac{2\pi}{3} + n\pi \ \mid \ n \in \mathbb{Z}\right\}$$

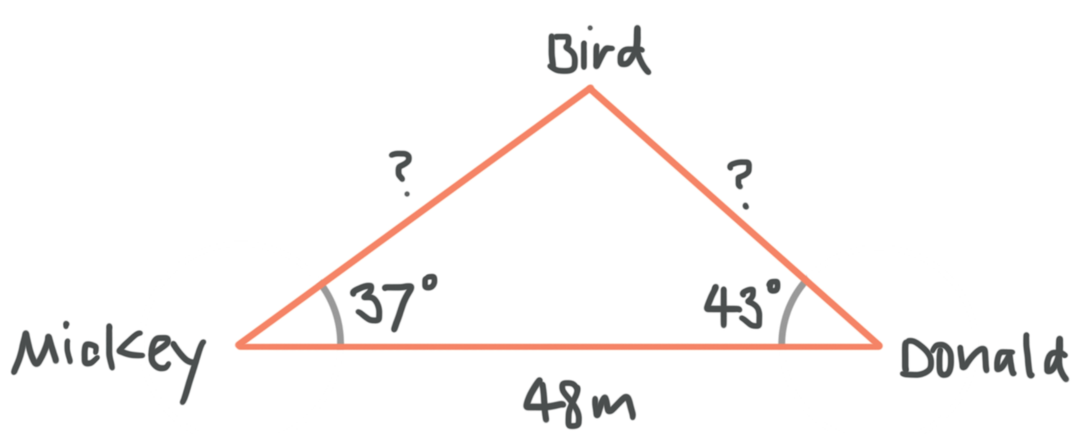
LAW OF SINES

■ 1. The interior angle measures of a triangle are 97°, 43°, and 40°. How many triangles can be made with these measurements?

Solution:

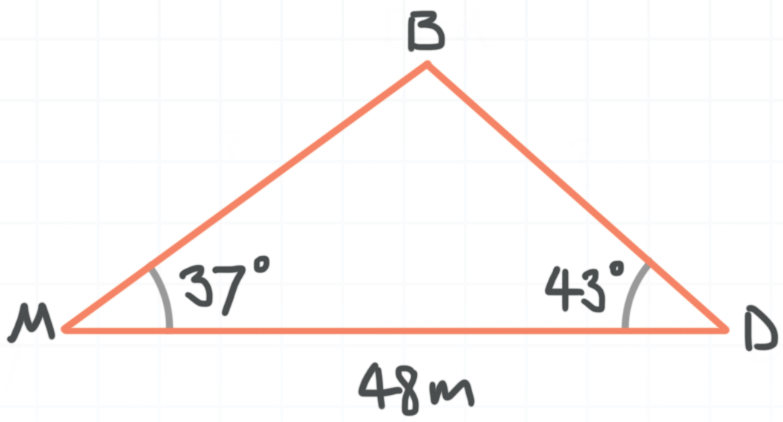
Given only the three interior angles measures of the triangle, it's impossible to determine the length of any side of the triangle. In other words, the interior angle measures determine the shape of the triangle, but not the size. Therefore, given only the angle measures, we could make an infinite number of triangles.

■ 2. Mickey and Donald stand on different sides of a tree. Each of them sees the same bird in the tree. They measure the angles of elevation from themselves to the bird, and get 37° and 43° respectively. If Mickey and Donald are 48 m apart, find the distances from Mickey and Donald to the bird.



Solution:

If M is Mickey, B is the bird, and D is Donald, then we have the triangle MBD.



The third interior angle of the triangle is

$$180^\circ - 37^\circ - 43^\circ$$

$$100^\circ$$

Then, plugging everything into the law of sines, we get

$$\frac{\overline{MD}}{\sin B} = \frac{\overline{MB}}{\sin D} = \frac{\overline{DB}}{\sin M}$$

$$\frac{48}{\sin 100^\circ} = \frac{\overline{MB}}{\sin 43^\circ} = \frac{\overline{DB}}{\sin 37^\circ}$$

Find the length of side MB using the first and second parts of this three-part equation.

$$\frac{48}{\sin 100^\circ} = \frac{\overline{MB}}{\sin 43^\circ}$$

$$\overline{MB} = \frac{48 \sin 43^\circ}{\sin 100^\circ} \approx 33 \text{ m}$$

Find the length of side DB using the first and third parts of the three-part equation.

$$\frac{48}{\sin 100^\circ} = \frac{\overline{DB}}{\sin 37^\circ}$$

$$\overline{DB} = \frac{48 \sin 37^\circ}{\sin 100^\circ} \approx 29 \text{ m}$$

Therefore, Mickey is about 33 m from the bird, and Donald is about 29 m from the bird.

■ 3. If the measures of two interior angles of a triangle are 53° and 44°, and the length of the side opposite the 44° angle is 7, find the length b of the side opposite the 53° angle and the length c of the third side.

Solution:

We know the third interior angle has measure

$$180^\circ - 44^\circ - 53^\circ$$

$$83^\circ$$

Then the law of sines gives

$$\frac{7}{\sin 44^\circ}=\frac{b}{\sin 53^\circ}=\frac{c}{\sin 83^\circ}$$

Find b using the first two parts of this three-part equation.

$$\frac{7}{\sin 44^\circ}=\frac{b}{\sin 53^\circ}$$

$$b=\frac{7\sin 53^\circ}{\sin 44^\circ}\approx\frac{7(0.799)}{0.695}\approx 8$$

Find c using the first and third parts of the three-part equation.

$$\frac{7}{\sin 44^\circ}=\frac{c}{\sin 83^\circ}$$

$$c=\frac{7\sin 83^\circ}{\sin 44^\circ}\approx\frac{7(0.993)}{0.695}\approx 10$$

So the other two sides of the triangle have lengths $b\approx 8$ and $c\approx 10$.

■ 4. Solve the triangle with angle measures $A=30^\circ$ and $C=90^\circ$ and side length $c=13$.

Solution:

The remaining angle of the triangle is

$$180^\circ-30^\circ-60^\circ$$

$$90^\circ$$

Plug all the values we have from the triangle into the law of sines.

$$\frac{a}{\sin 30^\circ} = \frac{b}{\sin 60^\circ} = \frac{13}{\sin 90^\circ}$$

Use values from the unit circle to simplify the equation.

$$\frac{a}{\frac{1}{2}} = \frac{b}{\frac{\sqrt{3}}{2}} = \frac{13}{1}$$

$$2a = \frac{2b}{\sqrt{3}} = 13$$

$$2\sqrt{3}a = 2b = 13\sqrt{3}$$

Use the first and third parts of this three-part equation to find a.

$$2\sqrt{3}a = 13\sqrt{3}$$

$$2a = 13$$

$$a = \frac{13}{2}$$

Use the second and third parts of the three-part equation to find b.

$$2b = 13\sqrt{3}$$

$$b = \frac{13\sqrt{3}}{2}$$

Then the triangle has side lengths $a = 13/2$, $b = 13\sqrt{3}/2$, and $c = 13$, and angle measures $A = 30^\circ$, $B = 60^\circ$, and $C = 90^\circ$.

5. Solve the triangle with angle measures $A = 45^\circ$ and $B = 45^\circ$ and side length $c = 10$.

Solution:

The remaining angle of the triangle is

$$180^\circ - 45^\circ - 45^\circ$$

$$90^\circ$$

Plug all the values we have from the triangle into the law of sines.

$$\frac{a}{\sin 45^\circ} = \frac{b}{\sin 45^\circ} = \frac{10}{\sin 90^\circ}$$

Use values from the unit circle to simplify the equation.

$$\frac{a}{\frac{\sqrt{2}}{2}} = \frac{b}{\frac{\sqrt{2}}{2}} = \frac{10}{1}$$

$$\frac{2a}{\sqrt{2}} = \frac{2b}{\sqrt{2}} = 10$$

$$2a = 2b = 10\sqrt{2}$$

$$a = b = 5\sqrt{2}$$

Then the triangle has side lengths $a = 5\sqrt{2}$, $b = 5\sqrt{2}$, and $c = 10$, and angle measures $A = 45^\circ$, $B = 45^\circ$, and $C = 90^\circ$.

■ 6. Find the lengths of the two unknown sides of a triangle with angle measures $A = 58^\circ$ and $B = 42^\circ$ and side length $a = 12$.

Solution:

The third interior angle of the triangle has measure

$$180^\circ - 58^\circ - 42^\circ$$

$$80^\circ$$

Plug the values we already have into the law of sines.

$$\frac{12}{\sin 58^\circ} = \frac{b}{\sin 42^\circ} = \frac{c}{\sin 80^\circ}$$

Use the first two parts of this three-part equation to solve for b.

$$\frac{12}{\sin 58^\circ} = \frac{b}{\sin 42^\circ}$$

$$b = \frac{12 \sin 42^\circ}{\sin 58^\circ} \approx \frac{12(0.6691)}{0.8480} \approx 9.47$$

Use the first third parts of the three-part equation to solve for c.

$$\frac{12}{\sin 58^\circ} = \frac{c}{\sin 80^\circ}$$

$$c = \frac{12 \sin 80^\circ}{\sin 58^\circ} \approx \frac{12(0.9848)}{0.8480} \approx 13.94$$

So the two unknown sides have lengths $b \approx 9.47$ and $c \approx 13.94$.

THE AMBIGUOUS CASE OF THE LAW OF SINES

■ 1. A triangle has one side with length 15 and another with length 28. The angle opposite the side with length 15 is 128°. Complete the triangle.

Solution:

Let $a=15$ and $b=28$, and let $A=128^\circ$ be the angle opposite a. Substituting these values into the law of sines gives

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\frac{15}{\sin 128^\circ}=\frac{28}{\sin B}=\frac{c}{\sin C}$$

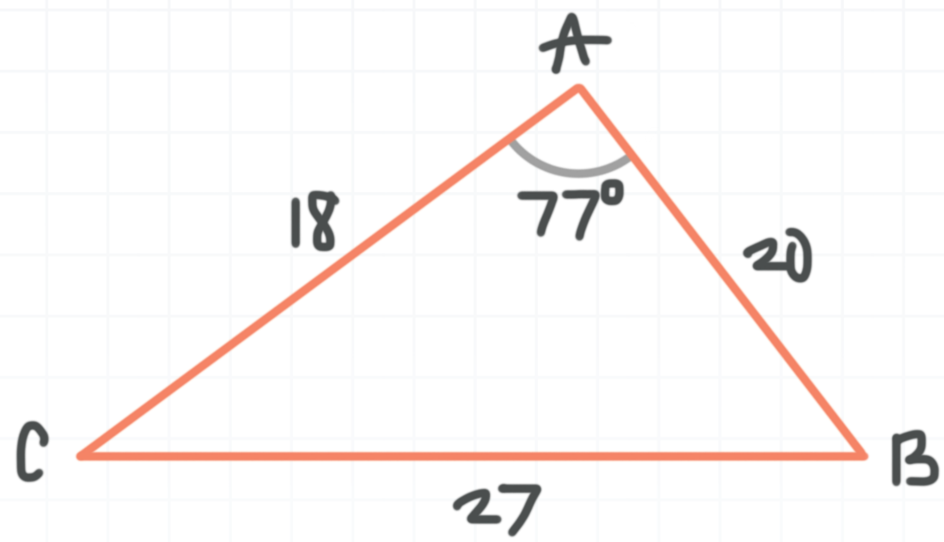
Use just the first two parts of this three-part equation in order to solve for B.

$$\frac{15}{\sin 128^\circ}=\frac{28}{\sin B}$$

$$\sin B=\frac{28\sin 128^\circ}{15}\approx\frac{28(0.788)}{15}\approx 1.47$$

Since the sine of an angle can't be greater than 1, it’s impossible to build a triangle with these properties.

■ 2. Find $\angle B$.



Solution:

Plugging what we know from the figure into the law of sines gives

$$\frac{27}{\sin 77^\circ} = \frac{18}{\sin B} = \frac{20}{\sin C}$$

Find B using the first two parts of this three-part equation.

$$\frac{27}{\sin 77^\circ} = \frac{18}{\sin B}$$

$$\sin B = \frac{18 \sin 77^\circ}{27} \approx 0.65$$

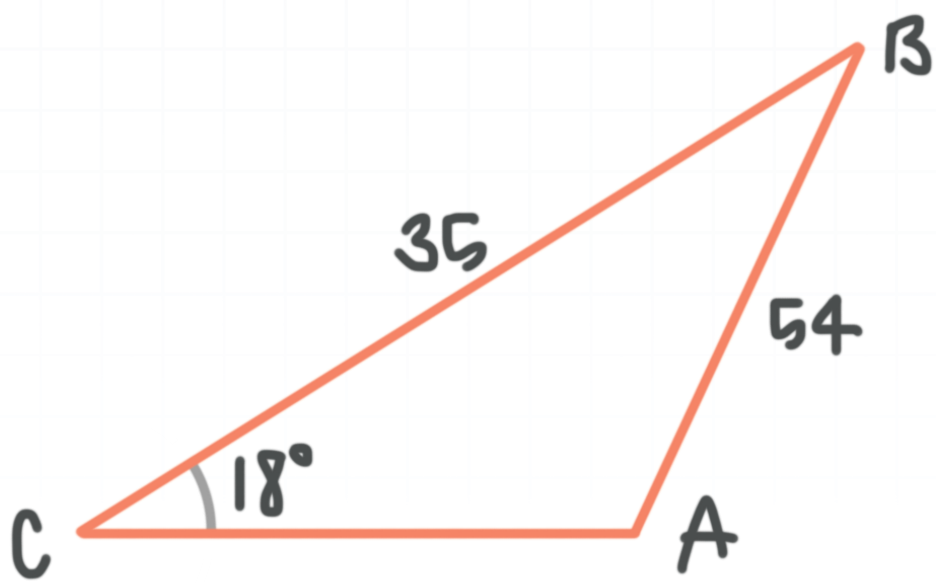
It's possible for $\sin B \approx 0.65$ in both the first and second quadrants. If we use a calculator to find $B \approx \arcsin 0.65$, we get the acute angle $B_1 \approx 40.5^\circ$. But to find $\sin B \approx 0.65$ in the second quadrant, we subtract $B_1 \approx 40.5^\circ$ from 180°.

$$B_2 \approx 180^\circ - 40.5^\circ$$

$$B_2 \approx 139.5^\circ$$

If $A = 77^\circ$ and $B_1 \approx 40.5^\circ$, then $C_1 \approx 180^\circ - 77^\circ - 40.5^\circ \approx 62.5^\circ$. If $A = 77^\circ$ and $B_2 \approx 139.5^\circ$, then $C_2 \approx 180^\circ - 77^\circ - 139.5^\circ \approx -36.5^\circ$. It's impossible to get a negative angle for C, so there's only one possible value for B, which is $B_1 \approx 40.5^\circ$.

3. Find $\angle A$.



Solution:

Plugging what we know into the law of sines gives

$$\frac{35}{\sin A} = \frac{B}{\sin B} = \frac{54}{\sin 18^\circ}$$

Find A using the first and third parts of this three-part equation.

$$\frac{35}{\sin A} = \frac{54}{\sin 18^\circ}$$

$$\sin A = \frac{35 \sin 18^\circ}{54} \approx 0.2$$

It's possible for $\sin A \approx 0.2$ in both the first and second quadrants. If we use a calculator to find $A \approx \arcsin 0.2$, we get the acute angle $A_1 \approx 11.5^\circ$. But to find $\sin A \approx 0.2$ in the second quadrant, we subtract $A_1 \approx 11.5^\circ$ from 180°.

$$A_2 \approx 180^\circ - 11.5^\circ$$

$$A_2 \approx 169.5^\circ$$

If $C = 18^\circ$ and $A_1 \approx 11.5^\circ$, then $B_1 \approx 180^\circ - 18^\circ - 11.5^\circ \approx 150.5^\circ$. If $C = 18^\circ$ and $A_2 \approx 169.5^\circ$, then $B_2 \approx 180^\circ - 18^\circ - 169.5^\circ \approx -7.5^\circ$. It's impossible to get a negative angle for B, so there's only one possible value for A, which is $A_1 \approx 11.5^\circ$.

■ 4. If the lengths of two sides of a triangle are 18 and 34, and the measure of the interior angle opposite the side of length 34 is $B = 127^\circ$, find the length of the third side and the measures of angles A and C, where A is opposite the side of length 18.

Solution:

Plugging what we know into the law of sines gives

$$\frac{18}{\sin A} = \frac{34}{\sin 127^\circ} = \frac{c}{\sin C}$$

Find A using the first two parts of this three-part equation.

$$\frac{18}{\sin A} = \frac{34}{\sin 127^\circ}$$

$$\sin A = \frac{18 \sin 127^\circ}{34} \approx \frac{14.375}{34} \approx 0.423$$

If A is acute, then $A = 25^\circ$, and if angle A is obtuse, then $A = 155^\circ$. But it's impossible to have $A = 155^\circ$ because the sum of the interior angles A and B would be $155^\circ + 127^\circ = 282^\circ$, which exceeds 180°, so $A = 25^\circ$.

Then the third interior angle has measure

$$C = 180^\circ - 127^\circ - 25^\circ$$

$$C = 28^\circ$$

Find c using the second and third parts of the three-part equation.

$$\frac{34}{\sin 127^\circ} \approx \frac{c}{\sin 28^\circ}$$

$$c \approx \frac{34 \sin 28^\circ}{\sin 127^\circ} \approx \frac{34(0.469)}{0.799} \approx 20$$

So $A = 25^\circ$, $B = 127^\circ$, $C = 28^\circ$, $a = 18$, $b = 34$, and $c \approx 20$.

■ 5. A triangle has side lengths $a = 27$ and $c = 15$ and interior angle $A = 55^\circ$. Find all possible measures of the angle C to the nearest degree.

Solution:

Plugging what we know into the law of sines gives

$$\frac{27}{\sin 55^\circ} = \frac{b}{\sin B} = \frac{15}{\sin C}$$

Find C using the first and third parts of this three-part equation.

$$\frac{27}{\sin 55^\circ} = \frac{15}{\sin C}$$

$$\sin C = \frac{15 \sin 55^\circ}{27} \approx \frac{15(0.819)}{27} \approx 0.455$$

If C is acute then $C = 27^\circ$, and if C is obtuse then $C = 153^\circ$. But it's impossible to have $C = 153^\circ$ because the sum of interior angles A and C would be $153^\circ + 55^\circ = 208^\circ$, which exceeds 180°, so $C = 27^\circ$ is the only possible value of C.

■ 6. How many triangles are possible with side lengths 5 and 24, where the angle opposite the side with length 24 is 95°?

Solution:

Let $a = 24$ and $b = 5$, and let angle $A = 95^\circ$. Then, plugging what we know into the law of sines gives

$$\frac{24}{\sin 95^\circ} = \frac{5}{\sin B} = \frac{c}{\sin C}$$

Find B using the first two parts of this three-part equation.

$$\frac{24}{\sin 95^\circ} = \frac{5}{\sin B}$$

$$\sin B = \frac{5 \sin 95^\circ}{24} \approx 0.208$$

If B is acute then $B = 12^\circ$, and if B is obtuse then $B = 168^\circ$. But it's impossible to have $B = 168^\circ$ because the sum of interior angles A and B would be $168^\circ + 95^\circ = 263^\circ$, which exceeds 180°, so $B = 12^\circ$, which means only one triangle is possible.

AREA FROM THE LAW OF SINES

1. Find the area of the triangle in which two of the sides have lengths 15 and 24 and the measure of the included angle is 47°.

Solution:

Let $a = 15$ and $b = 24$. Then the included angle is $C = 47^\circ$. Plugging what we know into the law of sines for the area of a triangle, we get

$$\text{Area} = \frac{1}{2}ab\sin C$$

$$\text{Area} = \frac{1}{2}(15)(24)\sin 47^\circ$$

$$\text{Area} \approx 180(0.731)$$

$$\text{Area} \approx 131.6$$

2. Find the area of the triangle with interior angles 101° and 25°, if the included side has length 23.

Solution:

The third angle in the triangle has measure

$$180^\circ - 101^\circ - 25^\circ = 54^\circ$$

Then, plugging everything we know into the law of sines, we get

$$\frac{a}{\sin 101^\circ} = \frac{b}{\sin 25^\circ} = \frac{23}{\sin 54^\circ}$$

Find a using the first and third parts of this three-part equation.

$$\frac{a}{\sin 101^\circ} = \frac{23}{\sin 54^\circ}$$

$$a = \frac{23 \sin 101^\circ}{\sin 54^\circ} \approx \frac{23(0.982)}{0.809} \approx 28$$

Use the law of sines for the area of a triangle.

$$\text{Area} = \frac{1}{2}ac \sin B$$

$$\text{Area} \approx \frac{1}{2}(28)(23)\sin 25^\circ$$

$$\text{Area} \approx \frac{644}{2}(0.423)$$

$$\text{Area} \approx 136$$

■ 3. Find the area of the triangle in which two of the sides have lengths 36 and 17 and the measure of the included angle is 90°.

Solution:

Let $a = 17$ and $b = 36$. Then the included angle is $C = 90^\circ$. Plugging what we know into the law of sines for the area of a triangle, we get

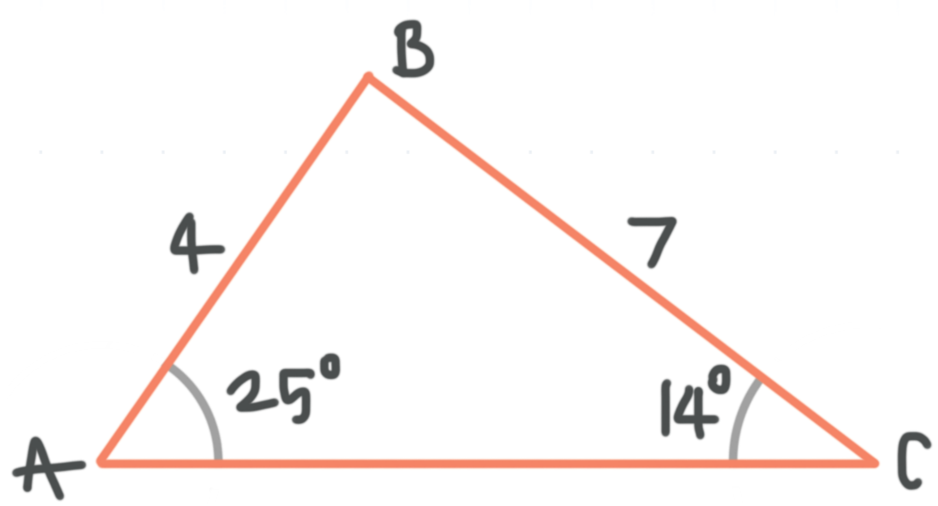
$$\text{Area} = \frac{1}{2}ab\sin C$$

$$\text{Area} = \frac{1}{2}(17)(36)\sin 90^\circ$$

$$\text{Area} = \frac{612}{2}(1)$$

$$\text{Area} = 306$$

■ 4. Find the area of the triangle.



Solution:

The third angle in the triangle has measure

$$180^\circ - 14^\circ - 25^\circ = 141^\circ$$

Let $a = 7$ and $c = 4$. Then the included angle is $B = 141^\circ$. Plugging what we know into the law of sines for the area of a triangle, we get

$$\text{Area} = \frac{1}{2}ac\sin B$$

$$\text{Area} = \frac{1}{2}(7)(4)\sin 141^\circ$$

$$\text{Area} \approx 14(0.629)$$

$$\text{Area} \approx 8.8$$

■ 5. Find the area of the triangle with interior angles 90° and 35°, if the included side has length 7.

Solution:

The third angle in the triangle has measure

$$180^\circ - 90^\circ - 35^\circ = 55^\circ$$

Then, plugging everything we know into the law of sines, we get

$$\frac{a}{\sin 35^\circ} = \frac{7}{\sin 55^\circ} = \frac{c}{\sin 90^\circ}$$

Find a using the first and second parts of this three-part equation.

$$\frac{a}{\sin 35^\circ} = \frac{7}{\sin 55^\circ}$$

$$a = \frac{7\sin 35^\circ}{\sin 55^\circ} \approx \frac{7(0.574)}{0.819} \approx 5$$

Use the law of sines for the area of a triangle.

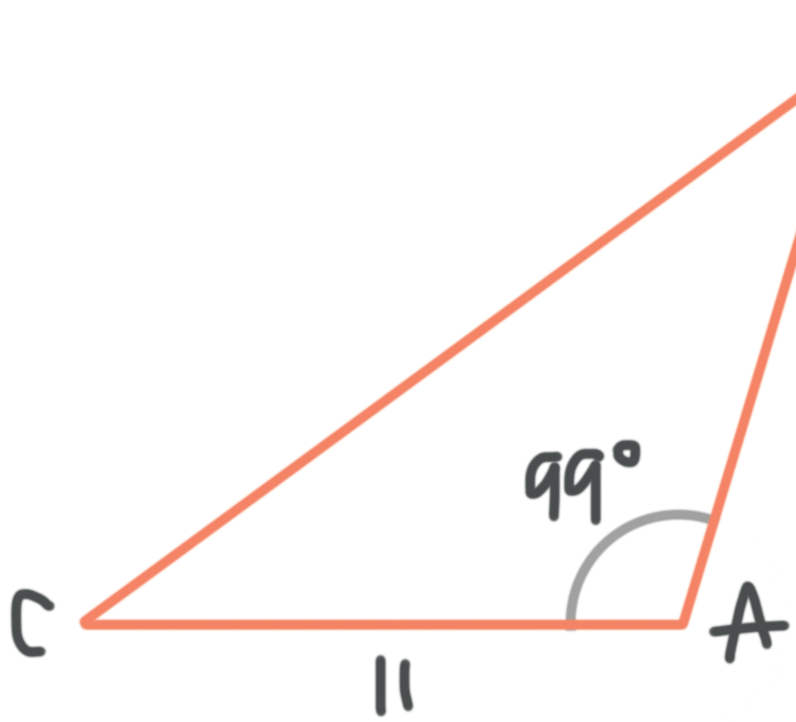
$$\text{Area} = \frac{1}{2}ab\sin C$$

$$\text{Area} \approx \frac{1}{2}(5)(7)\sin 90^\circ$$

$$\text{Area} \approx \frac{35}{2}(1)$$

$$\text{Area} \approx 17.5$$

■ 6. Find the area of a triangle.



Solution:

Let $b = 11$ and $c = 11$. Then the included angle is $A = 99^\circ$. Plugging what we know into the law of sines for the area of a triangle, we get

$$\text{Area} = \frac{1}{2}bc\sin A$$

$$\text{Area} = \frac{1}{2}(11)(11)\sin 99^\circ$$

$$\text{Area} \approx \frac{121}{2}(0.988)$$

$$\text{Area} \approx 60$$

LAW OF COSINES

■ 1. Solve the triangle where two of the sides are 18 and 13 and the measure of their included angle is 121°.

Solution:

Plugging what we know into the law of cosines with $\cos C$ gives

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 18^2 + 13^2 - 2(18)(13)\cos 121^\circ$$

$$c^2 = 324 + 169 - 26(18)\cos 121^\circ$$

$$c^2 = 493 - 468(-0.515)$$

$$c^2 \approx 734$$

$$c \approx 27$$

Rewrite the law of cosines with $\cos A$ by solving it for $\cos A$.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{a^2 - (b^2 + c^2)}{-2bc}$$

Plug in what we know to find A.

$$\cos A \approx \frac{18^2 - (13^2 + 27^2)}{-2(13)(27)}$$

$$\cos A \approx \frac{324 - (169 + 729)}{-26(27)}$$

$$\cos A \approx \frac{324 - 898}{-702}$$

$$\cos A \approx \frac{-574}{-702}$$

$$\cos A \approx 0.818$$

$$A \approx \arccos 0.818$$

$$A \approx 35.1^\circ$$

The third angle is therefore

$$B \approx 180^\circ - 121^\circ - 35.1^\circ$$

$$B \approx 23.9^\circ$$

■ 2. If the side lengths of a triangle are $a = 15$, $b = 9$, and $c = 21$, what are the measures of its three interior angles?

Solution:

Rewrite the law of cosines with $\cos A$ by solving it for $\cos A$.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{a^2 - (b^2 + c^2)}{-2bc}$$

Plugging in all three side lengths gives

$$\cos A \approx \frac{15^2 - (9^2 + 21^2)}{-2(9)(21)}$$

$$\cos A \approx \frac{225 - (81 + 441)}{-18(21)}$$

$$\cos A \approx \frac{225 - 522}{-378}$$

$$\cos A \approx \frac{-297}{-378}$$

$$\cos A = 0.786$$

$$A \approx \arccos 0.786$$

$$A \approx 38.2^\circ$$

Rewrite the law of cosines with $\cos B$ by solving it for $\cos B$.

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$\cos B = \frac{b^2 - (a^2 + c^2)}{-2ac}$$

Plugging in all three side lengths gives

$$\cos B \approx \frac{9^2 - (15^2 + 21^2)}{-2(15)(21)}$$

$$\cos B \approx \frac{81 - (225 + 441)}{-30(21)}$$

$$\cos B \approx \frac{81 - 666}{-630}$$

$$\cos B \approx \frac{-585}{-630}$$

$$\cos B \approx 0.929$$

$$B \approx \arccos 0.929$$

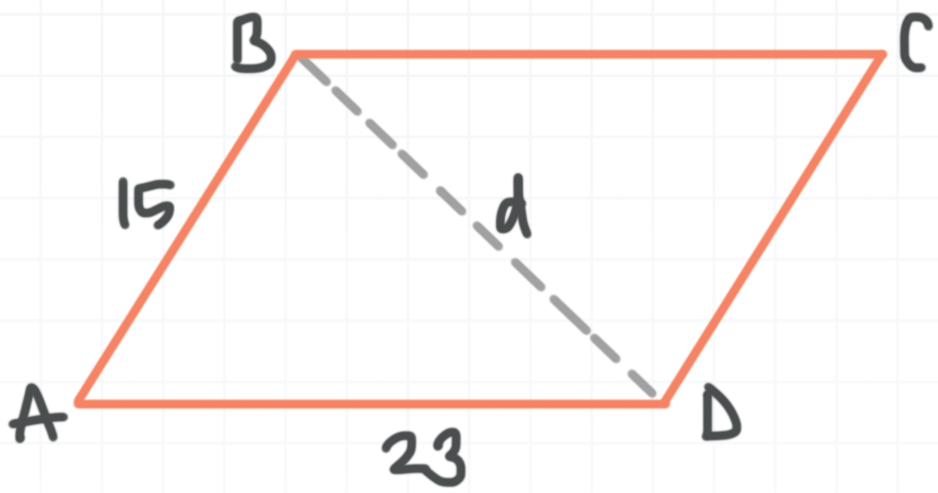
$$B \approx 21.7^\circ$$

Then the third angle is

$$C = 180^\circ - 38.2^\circ - 21.7^\circ$$

$$C = 120.3^\circ$$

■ 3. If the measure of the angle B is 56°, find the length of the parallelogram's diagonal, d, to the nearest centimeter. Hint: Consecutive angles of a parallelogram are supplementary, so $m\angle A + m\angle B = 180^\circ$.



Solution:

If consecutive angles of the parallelogram are supplementary, then

$$m\angle A + m\angle B = 180^\circ$$

$$m\angle A = 180^\circ - m\angle B$$

$$m\angle A = 180^\circ - 56^\circ$$

$$m\angle A = 124^\circ$$

Plugging what we know into the law of cosines with $\cos A$ gives

$$d^2 = \overline{AB}^2 + \overline{AD}^2 - 2(\overline{AB})(\overline{AD})\cos A$$

$$d^2 = 15^2 + 23^2 - 2(15)(23)\cos 124^\circ$$

$$d^2 = 225 + 529 - 30(23)\cos 124^\circ$$

$$d^2 = 754 - 690(-0.559)$$

$$d^2 \approx 1{,}140$$

$$d \approx 33.8$$

4. Solve the triangle where two of the sides are 27 and 14 and the measure of their included angle is 33°.

Solution:

Plugging what we know into the law of cosines with $\cos C$ gives

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 27^2 + 14^2 - 2(27)(14)\cos 33^\circ$$

$$c^2 = 729 + 196 - 54(14)\cos 33^\circ$$

$$c^2 = 925 - 756(0.839)$$

$$c^2 \approx 290.7$$

$$c \approx 17$$

Rewrite the law of cosines with $\cos A$ by solving it for $\cos A$.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{a^2 - (b^2 + c^2)}{-2bc}$$

Plug in what we know to find A.

$$\cos A \approx \frac{27^2 - (14^2 + 17^2)}{-2(14)(17)}$$

$$\cos A \approx \frac{729 - (196 + 291)}{-28(17)}$$

$$\cos A \approx \frac{729 - 487}{-476}$$

$$\cos A \approx \frac{242}{-476}$$

$$\cos A \approx -0.508$$

$$A \approx \arccos(-0.508)$$

$$A \approx 120.5^\circ$$

The third angle is therefore

$$B \approx 180^\circ - 33^\circ - 120.5^\circ$$

$$B \approx 26.5^\circ$$

■ 5. If the side lengths of a triangle are $a = 17$, $b = 25$, and $c = 28$, what are the measures of its three interior angles?

Solution:

Rewrite the law of cosines with $\cos C$ by solving it for $\cos C$.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Plugging in all three side lengths gives

$$\cos C = \frac{17^2 + 25^2 - 28^2}{2(17)(25)}$$

$$\cos C = \frac{289 + 625 - 784}{34(25)}$$

$$\cos C = \frac{130}{850}$$

$$\cos C \approx 0.153$$

$$C \approx \arccos 0.153$$

$$C \approx 81.2^\circ$$

Rewrite the law of cosines with $\cos B$ by solving it for $\cos B$.

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$\cos B = \frac{a^2 - b^2 + c^2}{2ac}$$

Plugging in all three side lengths gives

$$\cos B = \frac{17^2 - 25^2 + 28^2}{2(17)(28)}$$

$$\cos B = \frac{289 - 625 + 784}{34(28)}$$

$$\cos B = \frac{448}{952}$$

$$\cos B \approx 0.471$$

$$B \approx \arccos 0.471$$

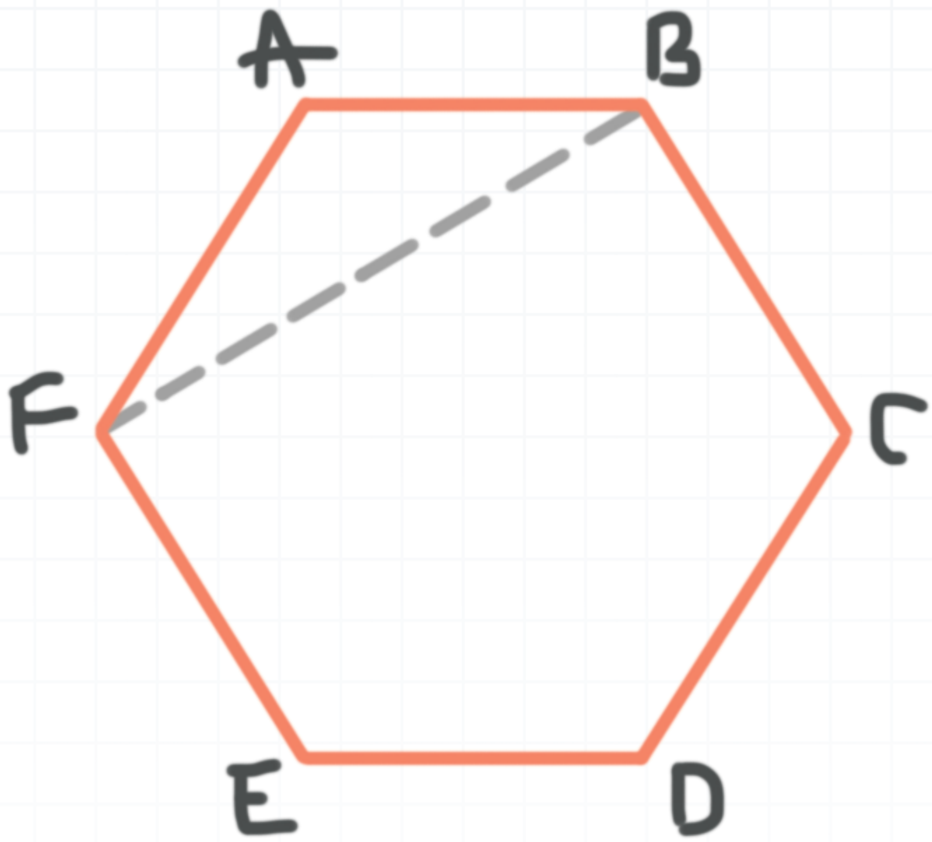
$$B \approx 61.9^\circ$$

Then the third angle is

$$C \approx 180^\circ - 81.2^\circ - 61.9^\circ$$

$$C \approx 36.9^\circ$$

■ 6. A regular hexagon (all side lengths are equal, and all interior angles are equal) has side lengths of 20 inches. Find $\overline{FB}$ to the nearest tenth. Hint: The sum of the interior angles of a hexagon is 720°.



Solution:

The sum of the interior angles of the hexagon is 720°. Since the hexagon is regular, all the interior angles are equal.

$$m\angle A = \frac{720^\circ}{6} = 120^\circ$$

Plugging what we know into the law of cosines with $\cos A$ gives

$$\overline{FB}^2 = \overline{FA}^2 + \overline{AB}^2 - 2(\overline{FA})(\overline{AB})\cos A$$

$$\overline{FB}^2 = 20^2 + 20^2 - 2(20)(20)\cos 120^\circ$$

$$\overline{FB}^2 = 400 + 400 - 40(20)\cos 120^\circ$$

$$\overline{FB}^2 = 800 - 800(-0.5)$$

$$\overline{FB}^2 \approx 1{,}200$$

$$\overline{FB} \approx 34.6$$

HERON'S FORMULA

1. The lengths of the sides of a triangle have a ratio of $5:8:12$, and the triangle's perimeter is 200 cm. Find the area of the triangle.

Solution:

Since the sides of the triangle have a ratio of $5:8:12$ and the perimeter is 200 cm, we can write

$$5x + 8x + 12x = 200$$

$$25x = 200$$

$$x = \frac{200}{25}$$

$$x = 8$$

Then the sides of the triangle will be 40 cm, 64 cm, and 96 cm.

Then find s, which is half the perimeter of the triangle.

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(p) = \frac{1}{2}(200) = 100$$

By Heron's formula, the area of the triangle is

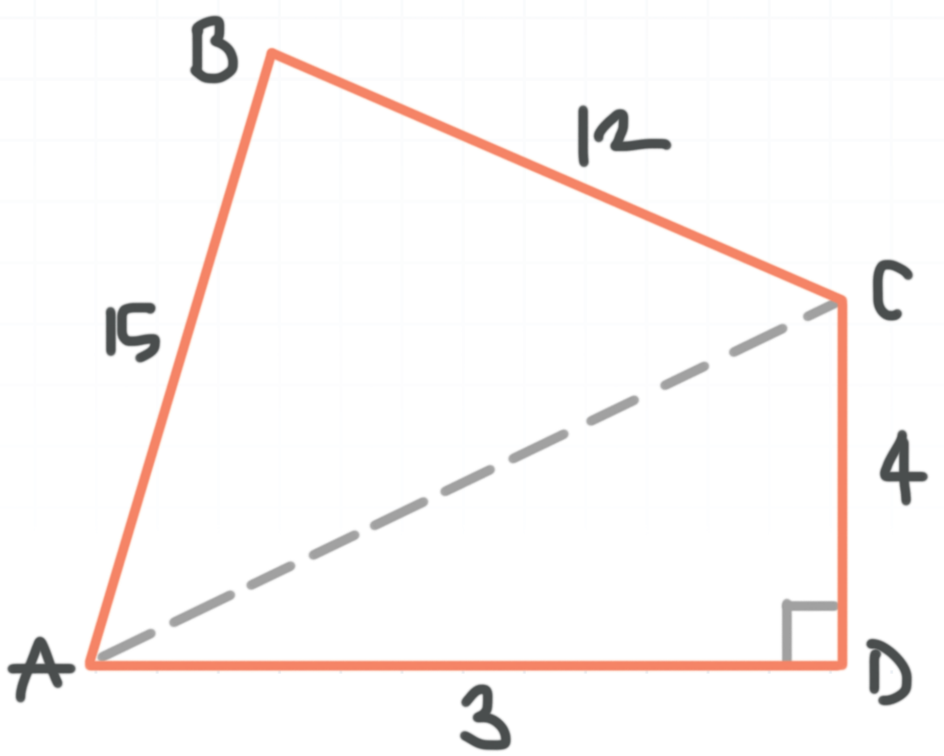
$$\text{Area} = \sqrt{100(100-40)(100-64)(100-96)}$$

$$\text{Area} = \sqrt{100(60)(36)(4)}$$

$$\text{Area} = \sqrt{864{,}000}$$

$$\text{Area} \approx 930 \text{ cm}^2$$

■ 2. Find the area of the quadrilateral, given that it's made of two separate triangles.



Solution:

To find the area of the quadrilateral we need to find the sum of the area of two triangles.

The area of the right triangle ACD is

$$A_1 = \frac{1}{2}(AD)(CD)$$

$$A_1 = \frac{1}{2}(3)(4)$$

$$A_1 = 6$$

Now we need to find the hypotenuse AC. Using the Pythagorean theorem, we get

$$AC^2 = AD^2 + CD^2$$

$$AC^2 = 3^2 + 4^2$$

$$AC^2 = 9 + 16$$

$$AC^2 = 25$$

$$AC = 5$$

Now we can use Heron's formula to find the area of triangle ABC, but first we'll have to find s, which is half the perimeter of the triangle.

$$s = \frac{1}{2}(AB + BC + AC) = \frac{1}{2}(15 + 12 + 5) = \frac{1}{2}(32) = 16$$

Then by Heron's formula, the area of the triangle is

$$A_2 = \sqrt{16(16-15)(16-12)(16-5)}$$

$$A_2 = \sqrt{16(1)(4)(11)}$$

$$A_2 = \sqrt{704}$$

$$A_2 \approx 26.5$$

So the area of the quadrilateral is

$$\text{Area} = A_1 + A_2$$

$$\text{Area} \approx 6 + 26.5$$

$$\text{Area} \approx 32.5$$

■ 3. A triangle and a parallelogram have the same base and the same area. If the sides of the triangle are 12 cm, 14 cm, and 16 cm, and the parallelogram has a base of 14 cm, find the height of the parallelogram. Hint: The area of a parallelogram is $A = bh$, where b is its base and h is its height.

Solution:

Find s, which is half the perimeter of the triangle.

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(12 + 14 + 16) = \frac{1}{2}(26 + 16) = \frac{1}{2}(42) = 21$$

Then by Heron's formula, the area of the triangle is

$$\text{Area} = \sqrt{21(21 - 12)(21 - 14)(21 - 16)}$$

$$\text{Area} = \sqrt{21(9)(7)(5)}$$

$$\text{Area} = \sqrt{6{,}615}$$

$$\text{Area} \approx 81$$

Since the triangle and the parallelogram have the same base and the same area, and the parallelogram has a base of 14 cm,

$$14 \times h = 81$$

$$h = \frac{81}{14}$$

$$h \approx 5.8 \text{ cm}$$

■ 4. Find the area of a triangle with side lengths 34 cm and 29 cm, if half its perimeter is 62 cm.

Solution:

Since half the perimeter of the triangle is 62 cm, we can find the length of third side.

$$s = \frac{1}{2}(a + b + c)$$

$$62 = \frac{1}{2}(34 + 29 + c)$$

$$124 = 63 + c$$

$$c = 124 - 63 = 61$$

Then by Heron's formula, the area of the triangle is

$$\text{Area} = \sqrt{62(62 - 34)(62 - 29)(62 - 61)}$$

$$\text{Area} = \sqrt{62(28)(33)(1)}$$

$$\text{Area} = \sqrt{57{,}288}$$

$$\text{Area} \approx 239 \text{ cm}^2$$

■ 5. An isosceles triangle (a triangle with two equal side lengths) has a half perimeter of 48 in. Its two equal sides measure 27 in each. Find the area of the triangle.

Solution:

Since half the perimeter of the triangle is 48 in, we can find the length of third side.

$$s = \frac{1}{2}(a + b + c)$$

$$48 = \frac{1}{2}(27 + 27 + c)$$

$$96 = 54 + c$$

$$c = 42$$

Then by Heron's formula, the area of the triangle is

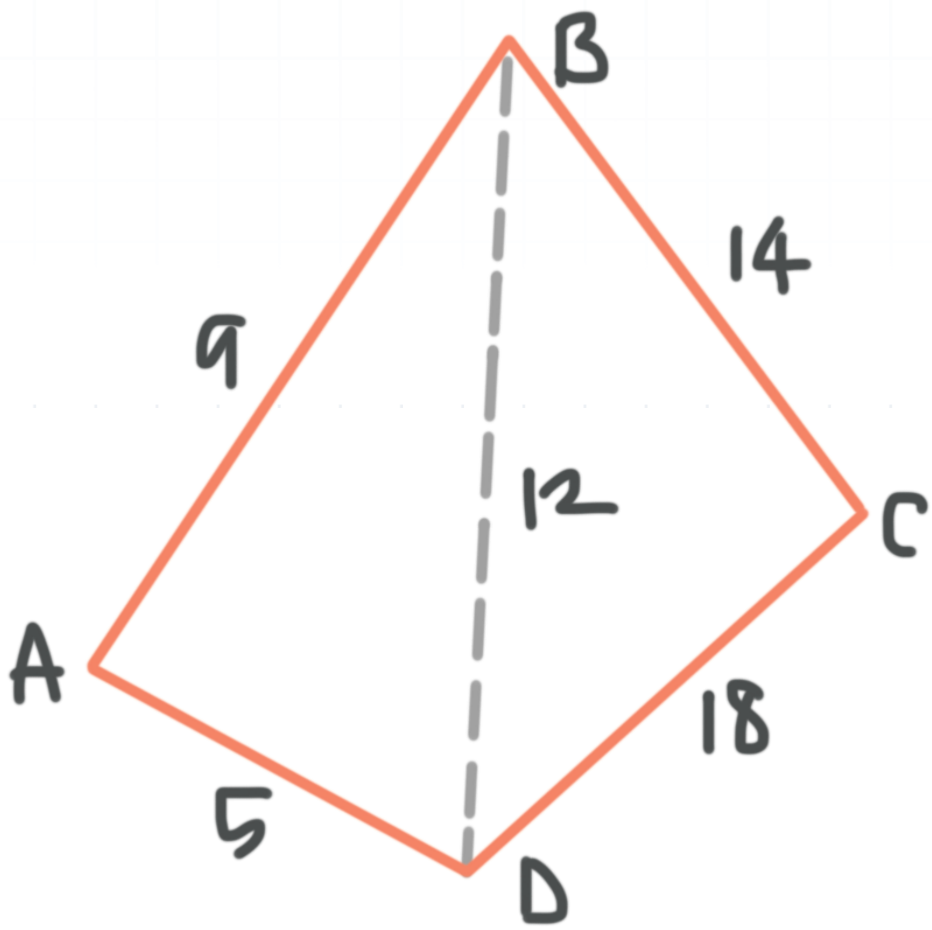
$$\text{Area} = \sqrt{48(48-27)(48-27)(48-42)}$$

$$\text{Area} = \sqrt{48(21)(21)(6)}$$

$$\text{Area} = \sqrt{127{,}008}$$

$$\text{Area} \approx 356 \text{ in}^2$$

■ 6. Find the area of the quadrilateral by finding the sum of the areas of the triangles.



Solution:

Find s_1, which is half the perimeter of triangle ABD.

$$s_1 = \frac{1}{2}(AB + BD + AD) = \frac{1}{2}(9 + 12 + 5) = \frac{1}{2}(26) = 13$$

Then by Heron's formula, the area of triangle ABD is

$$A_1 = \sqrt{13(13-9)(13-12)(13-5)}$$

$$A_1 = \sqrt{13(4)(1)(8)}$$

$$A_1 = \sqrt{416}$$

$$A_1 \approx 20$$

Find s_2, which is half the perimeter of triangle BCD.

$$s_2 = \frac{1}{2}(BC + CD + BD) = \frac{1}{2}(14 + 18 + 12) = \frac{1}{2}(44) = 22$$

Then by Heron's formula, the area of triangle BCD is

$$A_2 = \sqrt{22(22-14)(22-18)(22-12)}$$

$$A_2 = \sqrt{22(8)(4)(10)}$$

$$A_2 = \sqrt{7{,}040}$$

$$A_2 \approx 84$$

Then the area of the quadrilateral is

$$\text{Area} = A_1 + A_2$$

$$\text{Area} = 20 + 84$$

$$\text{Area} = 104$$

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