Precalculus Workbook Solutions

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POLAR COORDINATES

1. Find the rectangular point that's equivalent to the polar point.

$$(r, \theta) = \left(-14, \frac{5\pi}{6}\right)$$

Solution:

Plugging the polar point into the conversion formulas gives

$$x = r\cos\theta$$

$$x = -14\cos\left(\frac{5\pi}{6}\right)$$

$$x = -14\left(-\frac{\sqrt{3}}{2}\right)$$

$$x = 7\sqrt{3}$$

and

$$y = r\sin\theta$$

$$y = -14\sin\left(\frac{5\pi}{6}\right)$$

$$y = -14\left(\frac{1}{2}\right)$$

$$y = -7$$

The polar point $(r, \theta) = (-14, 5\pi/6)$ is located in the fourth quadrant, and the rectangular point $(x, y) = (7\sqrt{3}, -7)$ is located in the fourth quadrant, so $(x, y) = (7\sqrt{3}, -7)$ is the equivalent rectangular point.

■ 2. Convert the rectangular point $(x, y) = (5\sqrt{2}, 5\sqrt{2})$ to polar coordinates.

Solution:

The value of r for this point will be

$$r = \sqrt{x^2 + y^2} = \sqrt{(5\sqrt{2})^2 + (5\sqrt{2})^2} = \sqrt{50 + 50} = \sqrt{100} = 10$$

and the value for θ will be

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{5\sqrt{2}}{5\sqrt{2}}\right) \approx \tan^{-1}(1) = \frac{\pi}{4}$$

So we could rewrite the rectangular point $(x, y) = (5\sqrt{2}, 5\sqrt{2})$ in polar coordinates as

$$(r, \theta) = \left(10, \frac{\pi}{4}\right)$$

■ 3. Convert the rectangular point to polar coordinates.

$$(x, y) = \left(\cos\left(\frac{2\pi}{7}\right), \sin\left(\frac{2\pi}{7}\right)\right)$$

Solution:

The value of r for this point will be

$$r = \sqrt{x^2 + y^2} = \sqrt{\cos^2\left(\frac{2\pi}{7}\right) + \sin^2\left(\frac{2\pi}{7}\right)} = \sqrt{1} = 1$$

and the value for θ will be

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sin\left(\frac{2\pi}{7}\right)}{\cos\left(\frac{2\pi}{7}\right)}\right) = \tan^{-1}\left(\tan\left(\frac{2\pi}{7}\right)\right) = \frac{2\pi}{7}$$

So we could rewrite the rectangular point in polar coordinates as

$$(r, \theta) = \left(1, \frac{2\pi}{7}\right)$$

■ 4. Convert the polar point into rectangular coordinates.

$$(r, \theta) = \left(6, \frac{\pi}{6}\right)$$

Solution:

To rewrite the polar point in rectangular coordinates, we'll plug $r = 6$ and $\theta = \pi/6$ into the conversion equations to get

$$x = r\cos\theta$$

$$x = 6\cos\left(\frac{\pi}{6}\right) = 6\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$$

and

$$y = r\sin\theta$$

$$y = 6\sin\left(\frac{\pi}{6}\right) = 6\left(\frac{\sqrt{1}}{2}\right) = 3$$

So the polar point $(r, \theta) = (6, \pi/6)$ is equivalent to the rectangular point $(x, y) = (3\sqrt{3}, 3)$.

■ 5. Convert the rectangular point $(x, y) = (2\sqrt{3}, -6)$ to polar coordinates.

Solution:

The value of r for this point will be

$$r = \sqrt{x^2 + y^2} = \sqrt{(2\sqrt{3})^2 + (-6)^2} = \sqrt{12 + 36} = \sqrt{48} = 4\sqrt{3}$$

and the value for θ will be

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-6}{2\sqrt{3}}\right) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

So we could rewrite the rectangular point $(x, y) = (2\sqrt{3}, -6)$ in polar coordinates as

$$(r, \theta) = \left(4\sqrt{3}, -\frac{\pi}{3}\right)$$

■ 6. Convert the polar point $(r, \theta) = (8, 2\pi)$ into rectangular coordinates.

Solution:

To rewrite the polar point in rectangular coordinates, we'll plug $r = 8$ and $\theta = 2\pi$ into the conversion equations to get

$$x = r\cos\theta$$

$$x = 8\cos(2\pi) = 8(1) = 8$$

and

$$y = r\sin\theta$$

$$y = 8\sin(2\pi) = 0$$

So the polar point $(r,\theta)=(8,2\pi)$ is equivalent to the rectangular point $(x,y)=(8,0)$.

MULTIPLE WAYS TO EXPRESS POLAR POINTS

■ 1. Find five polar points that are equivalent to $(6,-\pi/4)$, where the angle θ lies within $(-3\pi,3\pi)$.

Solution:

We can find equivalent points by keeping the value of r the same but adding or subtracting any multiple of 2π from θ, so

$$\left(6,-\frac{\pi}{4}+2\pi\right)=\left(6,\frac{7\pi}{4}\right)$$

and

$$\left(6,-\frac{\pi}{4}-2\pi\right)=\left(6,-\frac{9\pi}{4}\right)$$

We can also find equivalent points by changing the value of r to $-r$ while we add or subtract any odd integer multiple of π from θ, so

$$\left(-6,-\frac{\pi}{4}+\pi\right)=\left(-6,\frac{3\pi}{4}\right)$$

From this new point, we can add or subtract any multiple of 2π from θ, so

$$\left(-6,\frac{3\pi}{4}+2\pi\right)=\left(-6,\frac{11\pi}{4}\right)$$

and

$$\left(-6,\frac{3\pi}{4}-2\pi\right)=\left(-6,-\frac{5\pi}{4}\right)$$

So we can say that all of these are polar points equivalent to $(6,-\pi/4)$, where the angle θ lies within $(-3\pi,3\pi)$.

$$\left(6,-\frac{9\pi}{4}\right) \qquad \left(-6,-\frac{5\pi}{4}\right)$$

$$\left(6,\frac{7\pi}{4}\right) \qquad \left(-6,\frac{3\pi}{4}\right)$$

$$\left(-6,\frac{11\pi}{4}\right)$$

■ 2. Find a point that's equivalent to $(3,101\pi/2)$, with a positive radius r and a polar angle θ that lies within $(0,2\pi]$.

Solution:

The radius is already positive, so we only need to subtract some multiple of 2π from the angle.

If we realize that $50\pi=100\pi/2$, we could subtract $100\pi/2$ to find the equivalent point $(3,\pi/2)$.

■ 3. How many points are equivalent to the point $(1,\pi/7)$ and have a polar angle in the interval $0<\theta<5\pi$?

Solution:

We can find equivalent points by keeping the value of r the same but adding or subtracting any multiple of 2π from θ, so the points around $(1,\pi/7)$ are

$$\left(1,\frac{\pi}{7}-2\pi\right)=\left(1,-\frac{13\pi}{7}\right)$$

$$\left(1,\frac{\pi}{7}\right)$$

$$\left(1,\frac{\pi}{7}+2\pi\right)=\left(1,\frac{15\pi}{7}\right)$$

$$\left(1,\frac{\pi}{7}+4\pi\right)=\left(1,\frac{29\pi}{7}\right)$$

$$\left(1,\frac{\pi}{7}+6\pi\right)=\left(1,\frac{43\pi}{7}\right)$$

But the interval $0<\theta<5\pi$ is equivalent to $0<\theta<35\pi/7$, which means only the two points at $(1,15\pi/7)$ and $(1,29\pi/7)$ fall in the interval.

If we use the opposite value of r, then we can find points around $(1,\pi/7)$,

$$\left(-1,\frac{\pi}{7}-\pi\right)=\left(-1,-\frac{6\pi}{7}\right)$$

$$\left(-1,\frac{\pi}{7}+\pi\right)=\left(-1,\frac{8\pi}{7}\right)$$

The point $(-1,-6\pi/7)$ is outside the interval $0<\theta<35\pi/7$. Subtracting multiples of 2π from the angle in that point will take us even further outside the interval, and adding 2π to that point will get us to the other point we just found, $(-1,8\pi/7)$.

So we'll add multiples of 2π to $(-1,8\pi/7)$ until we find an angle outside the interval $0<\theta<35\pi/7$.

$$\left(-1,\frac{8\pi}{7}+2\pi\right)=\left(-1,\frac{22\pi}{7}\right)$$

$$\left(-1,\frac{8\pi}{7}+4\pi\right)=\left(-1,\frac{36\pi}{7}\right)$$

Because $(-1,36\pi/7)$ is outside the interval $0<\theta<35\pi/7$, we know that all other points with larger angles will also be outside the interval. So the full set of polar points equivalent to $(1,\pi/7)$, with an angle in the interval $0<\theta<5\pi$, is

$$\left(1,\frac{15\pi}{7}\right) \qquad \left(-1,\frac{8\pi}{7}\right)$$

$$\left(1,\frac{29\pi}{7}\right) \qquad \left(-1,\frac{22\pi}{7}\right)$$

■ 4. Find four polar points that are equivalent to the point $(-10,16\pi/7)$ and have a polar angle in the interval $(-2\pi,2\pi)$.

Solution:

We can find equivalent points by keeping the value of r the same but adding or subtracting any multiple of 2π from θ. Because $16\pi/7$ is greater than 2π, the upper edge of the interval $(-2\pi,2\pi)$, we'll subtract multiples of 2π from $16\pi/7$ to find equivalent polar points in the interval.

$$\left(-10,\frac{16\pi}{7}-2\pi\right)=\left(-10,\frac{2\pi}{7}\right)$$

$$\left(-10,\frac{16\pi}{7}-4\pi\right)=\left(-10,-\frac{12\pi}{7}\right)$$

We can also find equivalent points by changing the value of r to $-r$ while we add any any odd integer multiple of π to, or subtract any odd integer multiple of π from, the angle θ.

$$\left(10,\frac{16\pi}{7}+\pi\right)=\left(10,\frac{23\pi}{7}\right)$$

Then subtracting multiples of 2π from $23\pi/7$ gives two more polar points in the interval.

$$\left(10,\frac{23\pi}{7}-2\pi\right)=\left(10,\frac{9\pi}{7}\right)$$

$$\left(10,\frac{23\pi}{7}-4\pi\right)=\left(10,-\frac{5\pi}{7}\right)$$

■ 5. How many points in $0<\theta<3\pi/2$ are equivalent to the point $(2,-2\pi/3)$?

Solution:

We can find equivalent points by keeping the value of r the same but adding or subtracting any multiple of 2π from θ.

$$\left(2,-\frac{2\pi}{3}+2\pi\right)=\left(2,\frac{4\pi}{3}\right)$$

Changing the sign on r while adding an odd-integer multiple of π gives

$$\left(-2,-\frac{2\pi}{3}+\pi\right)=\left(-2,\frac{\pi}{3}\right)$$

So there are two points equivalent to $(2,-2\pi/3)$ in the interval $0<\theta<3\pi/2$.

■ 6. Find a point equivalent to $(-1,53\pi/6)$ with positive radius r and a polar angle θ within $(0,2\pi]$.

Solution:

First we'll find an equivalent point with a positive radius by changing the sign on r while adding π to the angle θ.

$$\left(-1,\frac{53\pi}{6}+\pi\right)=\left(1,\frac{59\pi}{6}\right)$$

To get an angle from $59\pi/6$ that's within $(0,2\pi]$, we'll subtract multiples of 2π.

$$\left(1,\frac{59\pi}{6}-8\pi\right)=\left(1,\frac{11\pi}{6}\right)$$

CONVERTING EQUATIONS

■ 1. Convert the rectangular equation into polar coordinates.

$$(x+9)^2+(y-13)^2=64$$

Solution:

Substitute $x=r\cos\theta$ and $y=r\sin\theta$ into the rectangular equation.

$$(x+9)^2+(y-13)^2=64$$

$$(r\cos\theta+9)^2+(r\sin\theta-13)^2=64$$

$$(r^2\cos^2\theta+18r\cos\theta+81)+(r^2\sin^2\theta-26r\sin\theta+169)=64$$

$$(r^2\cos^2\theta+r^2\sin^2\theta)+(18r\cos\theta-26r\sin\theta)+(81+169)=64$$

$$r^2(\cos^2\theta+\sin^2\theta)+(18r\cos\theta-26r\sin\theta)+(81+169)=64$$

Using the Pythagorean identity $\cos^2\theta+\sin^2\theta=1$, simplify the equation.

$$r^2(1)+(18r\cos\theta-26r\sin\theta)+(81+169)=64$$

$$r^2+18r\cos\theta-26r\sin\theta+250=64$$

$$r^2-26r\sin\theta+18r\cos\theta+186=0$$

■ 2. Convert the rectangular equation into polar coordinates.

$$\frac{(x+5)^2}{9} + \frac{(y-7)^2}{4} = 1$$

Solution:

Substitute $x = r\cos\theta$ and $y = r\sin\theta$ into the rectangular equation.

$$\frac{(x+5)^2}{9} + \frac{(y-7)^2}{4} = 1$$

$$\frac{(r\cos\theta+5)^2}{9} + \frac{(r\sin\theta-7)^2}{4} = 1$$

Multiply both sides of the equation by 36 to clear the fractions.

$$4(r\cos\theta+5)^2 + 9(r\sin\theta-7)^2 = 36$$

$$4(r^2\cos^2\theta + 10r\cos\theta + 25) + 9(r^2\sin^2\theta - 14r\sin\theta + 49) = 36$$

$$4r^2\cos^2\theta + 40r\cos\theta + 100 + 9r^2\sin^2\theta - 126r\sin\theta + 441 = 36$$

$$9r^2\sin^2\theta + 4r^2\cos^2\theta - 126r\sin\theta + 40r\cos\theta + 505 = 0$$

■ 3. Convert the rectangular equation into polar coordinates.

$$x = \frac{1}{y+3}$$

Solution:

Substitute $x = r\cos\theta$ and $y = r\sin\theta$ into the rectangular equation.

$$x = \frac{1}{y+3}$$

$$r\cos\theta = \frac{1}{r\sin\theta + 3}$$

Multiply both sides of the equation by $r\sin\theta + 3$ to clear the fractions.

$$(r\cos\theta)(r\sin\theta + 3) = 1$$

$$r^2\cos\theta\sin\theta + 3r\cos\theta = 1$$

Use the double angle identity

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\frac{1}{2}\sin(2\theta) = \sin\theta\cos\theta$$

to rewrite the equation.

$$r^2\left(\frac{1}{2}\sin(2\theta)\right) + 3r\cos\theta = 1$$

$$r^2\left(\frac{1}{2}\sin(2\theta)\right) + 3r\cos\theta = 1$$

Multiply both sides of the equation by 2 to clear the fractions.

$$r^2\sin(2\theta) + 6r\cos\theta = 2$$

■ 4. Convert the polar equation to rectangular coordinates.

$$r^2 = 4\sin^2\theta$$

Solution:

Multiply both sides of the equation by r^2,

$$r^2 = 4\sin^2\theta$$

$$r^4 = 4r^2\sin^2\theta$$

$$r^4 = 4(r\sin\theta)^2$$

then substitute $y = r\sin\theta$ into the right side and $x^2 + y^2 = r^2$ into the left side.

$$(x^2 + y^2)^2 = 4y^2$$

$$(x^2 + y^2)^2 - 4y^2 = 0$$

■ 5. Convert the polar equation to rectangular coordinates.

$$\theta = r^2 + 1$$

Solution:

Use the conversion equations

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$r^2 = x^2 + y^2$$

so make substitutions into the polar equation.

$$\theta = r^2 + 1$$

$$\tan^{-1}\left(\frac{y}{x}\right) = x^2 + y^2 + 1$$

$$\frac{y}{x} = \tan(x^2 + y^2 + 1)$$

$$y = x\tan(x^2 + y^2 + 1)$$

■ 6. Convert the polar equation to rectangular coordinates.

$$r = e^{\theta}$$

Solution:

Square both sides of the equation,

$$r = e^{\theta}$$

$$r^2 = e^{2\theta}$$

then use

$$r^2 = x^2 + y^2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

to substitute into the equation.

$$x^2 + y^2 = e^{2\tan^{-1}\left(\frac{y}{x}\right)}$$

$$\ln(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{1}{2}\ln(x^2 + y^2) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\tan\left(\frac{1}{2}\ln(x^2 + y^2)\right) = \frac{y}{x}$$

$$y = x\tan\left(\frac{1}{2}\ln(x^2 + y^2)\right)$$

GRAPHING POLAR CURVES IN A RECTANGULAR SYSTEM

■ 1. Sketch the graph of $r = 3\sin\theta - 3$ in a rectangular coordinate system.

Solution:

We set the argument of the trig function $\sin\theta$ equal to $\pi/2$, so $\theta = \pi/2$ and we'll evaluate the polar equation at the first few multiples of $\theta = \pi/2$.

$\theta = 0$ $\qquad r = 3\sin(0) - 3 = -3$

$\theta = \pi/2$ $\qquad r = 3\sin(\pi/2) - 3 = 0$

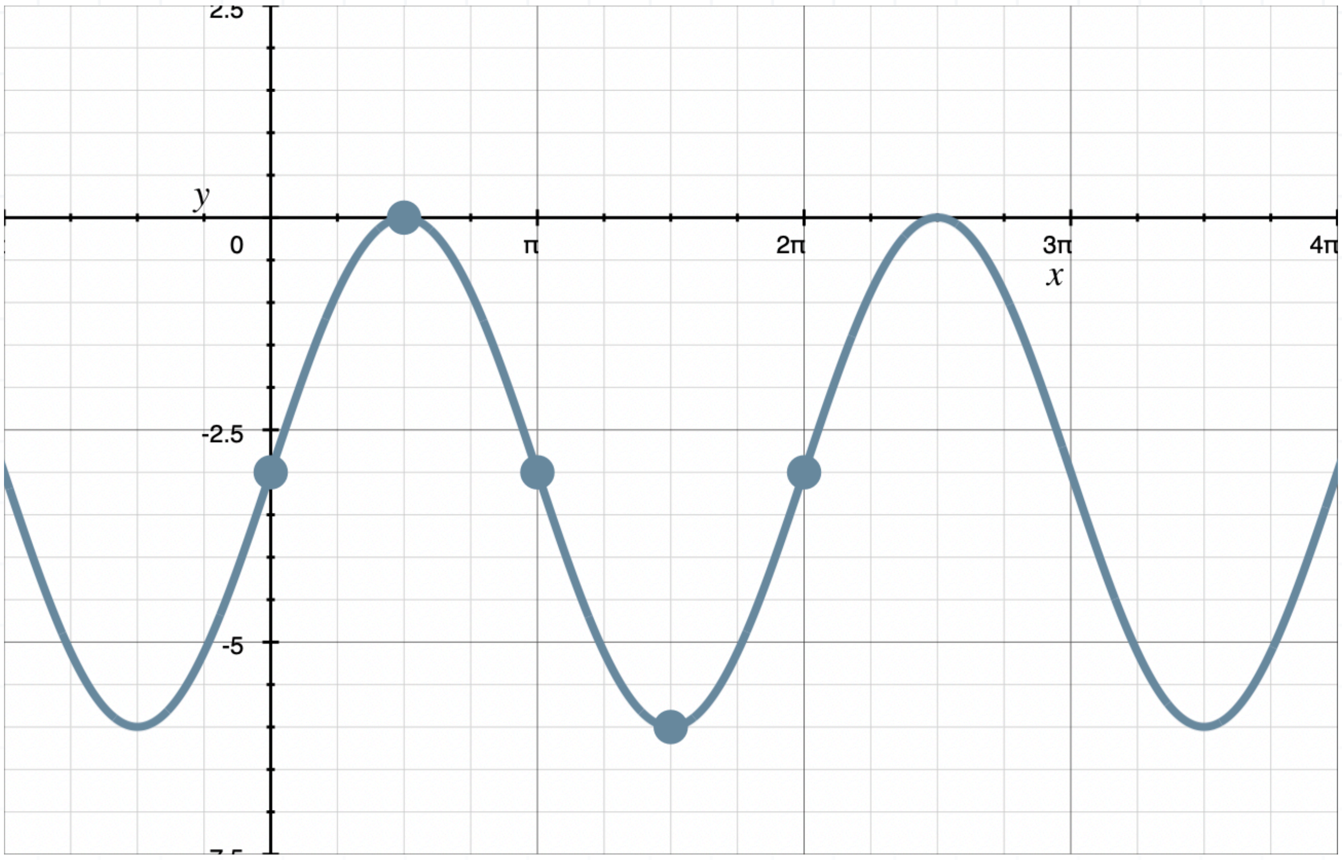
$\theta = \pi$ $\qquad r = 3\sin(\pi) - 3 = -3$

$\theta = 3\pi/2$ $\qquad r = 3\sin(3\pi/2) - 3 = -6$

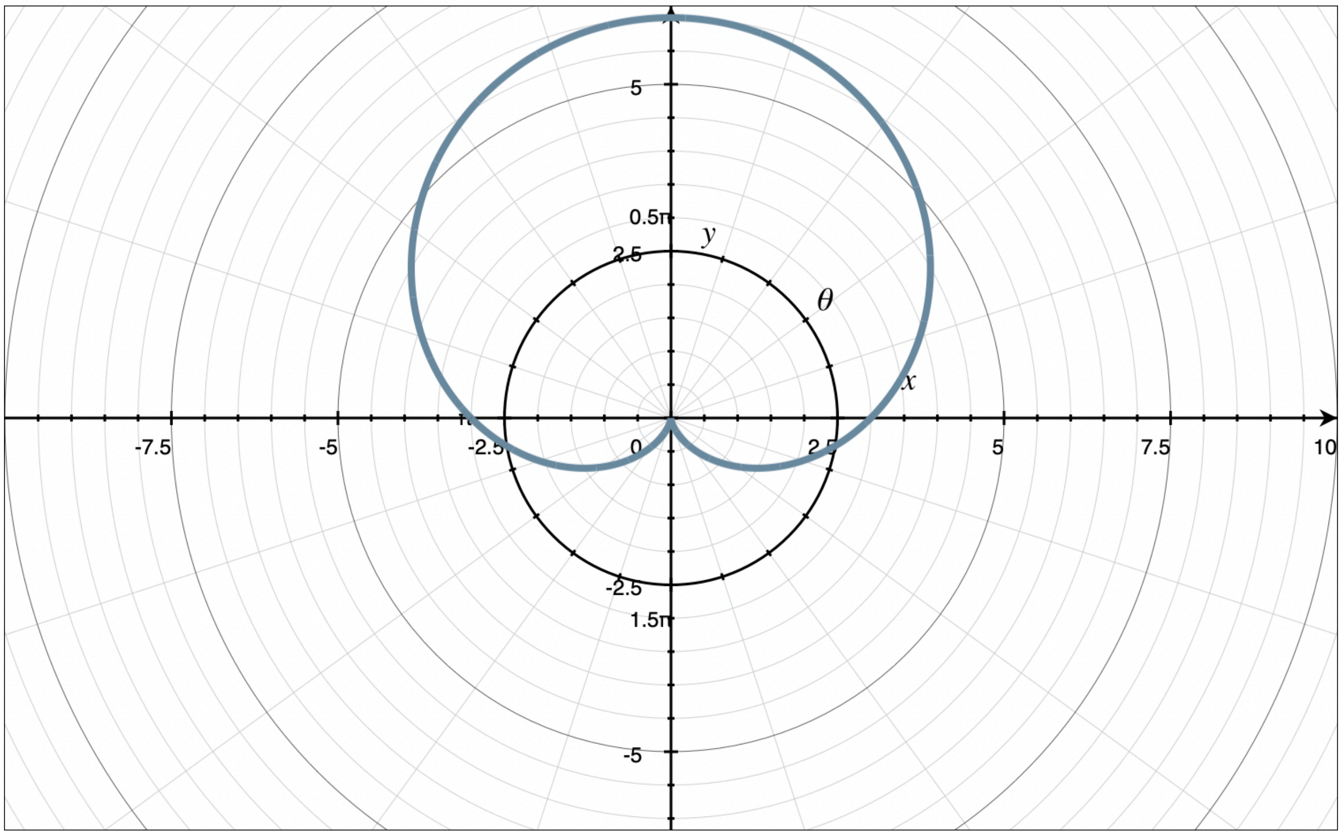
$\theta = 2\pi$ $\qquad r = 3\sin(2\pi) - 3 = -3$

... $\qquad$ ...

Plotting these points in a rectangular system and connecting them with a smooth curve, we get



If we transfer these points into polar coordinates, we get



■ 2. Convert $r = 4\cos\theta$ to rectangular coordinates and then sketch the graph of the resulting equation.

Solution:

Multiply both sides of the equation by r,

$$r = 4\cos\theta$$

$$r^2 = 4r\cos\theta$$

then use $r^2 = x^2 + y^2$ and $x = r\cos\theta$ to convert the equation to rectangular coordinates.

$$x^2 + y^2 = 4x$$

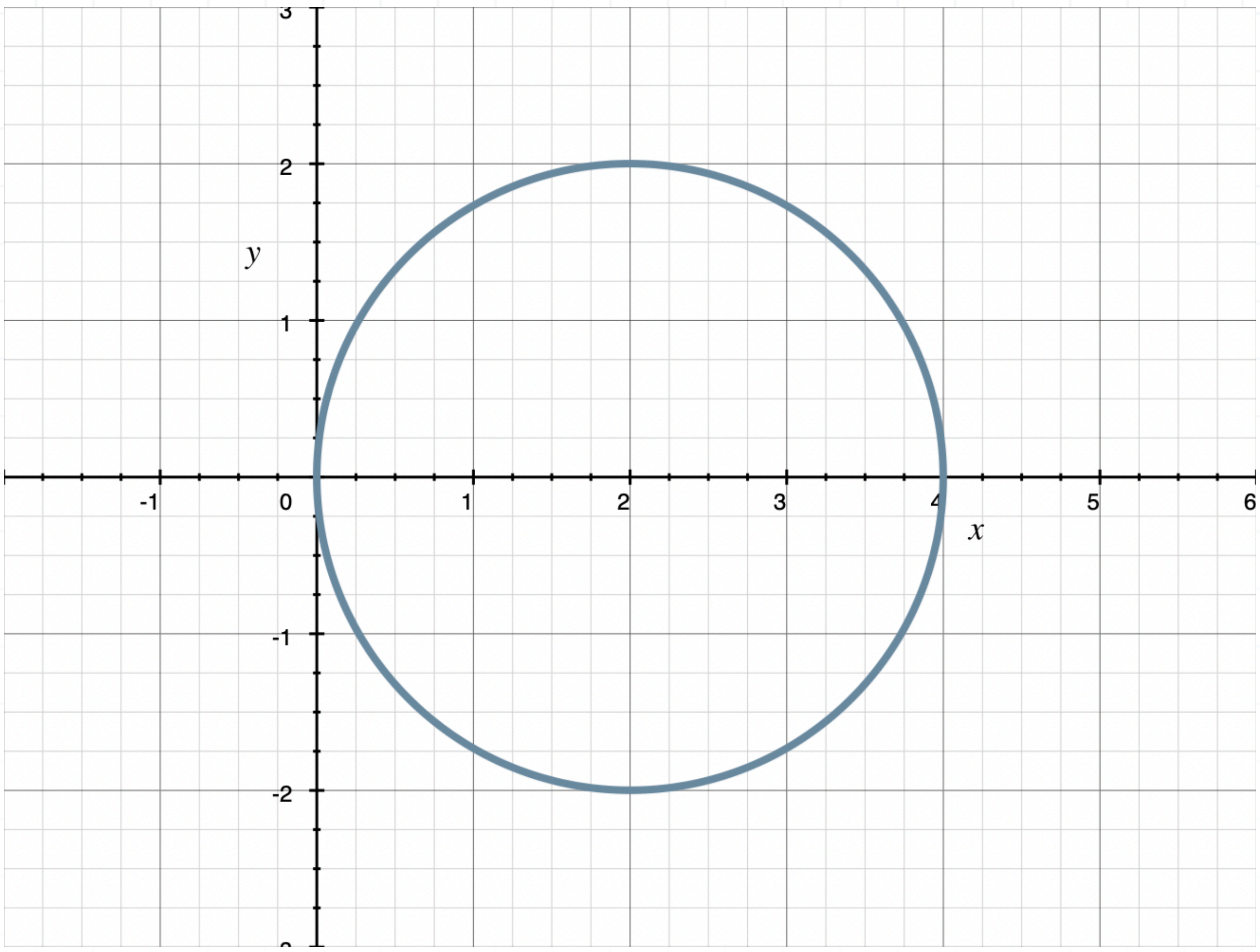
$$x^2 - 4x + y^2 = 0$$

Completing the square with respect to x gives

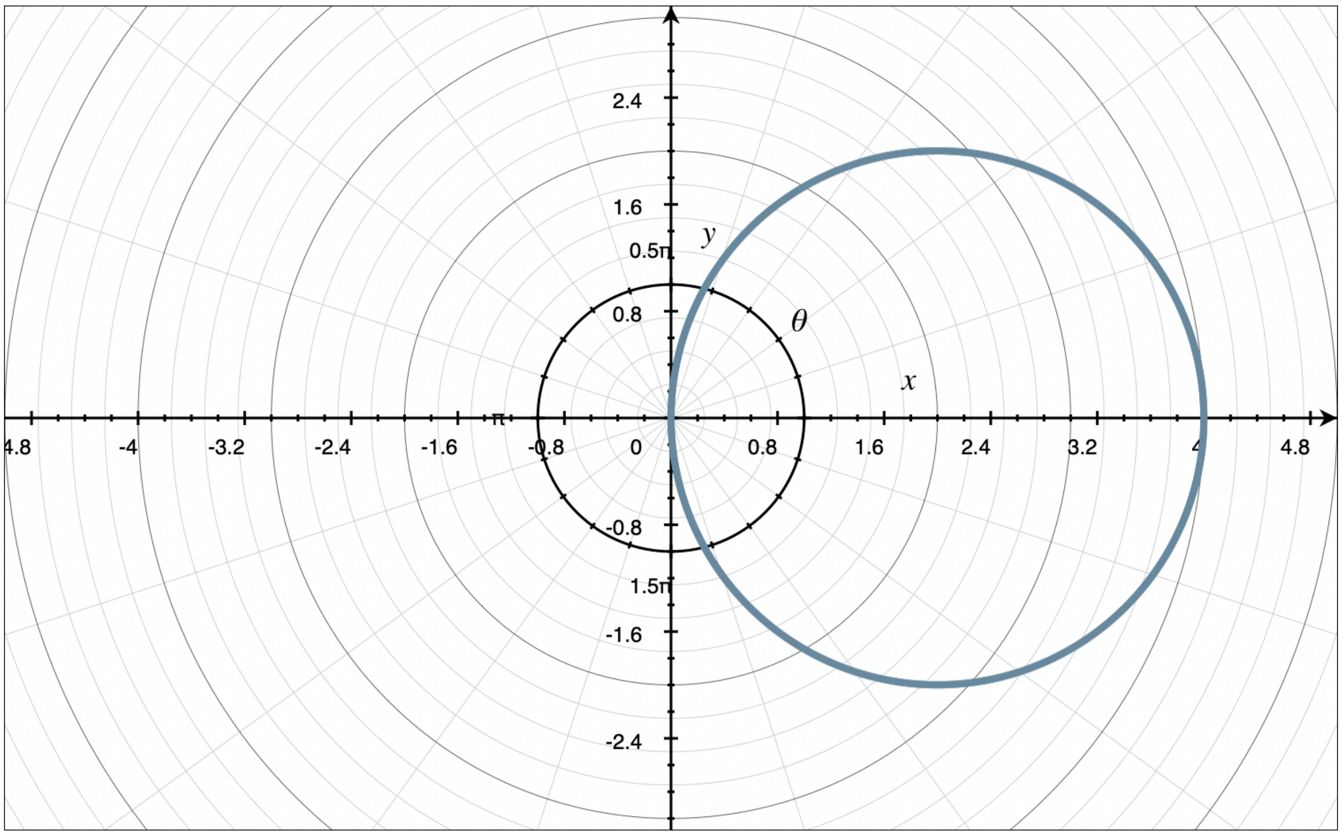
$$x^2 - 4x + 4 + y^2 = 0 + 4$$

$$(x - 2)^2 + y^2 = 4$$

Therefore, we get a circle centered at $(x, y) = (2,0)$ with radius 2.



In polar coordinates, the sketch of the curve would be



■ 3. Convert $r = 6\sin\theta - 2\cos\theta$ into rectangular coordinates and then sketch the graph of the resulting equation.

Solution:

Multiply both sides of the equation by r,

$$r = 6\sin\theta - 2\cos\theta$$

$$r^2 = 6r\sin\theta - 2r\cos\theta$$

then use $r^2 = x^2 + y^2$ and $y = r\sin\theta$ to convert the equation to rectangular coordinates.

$$x^2 + y^2 = 6y - 2x$$

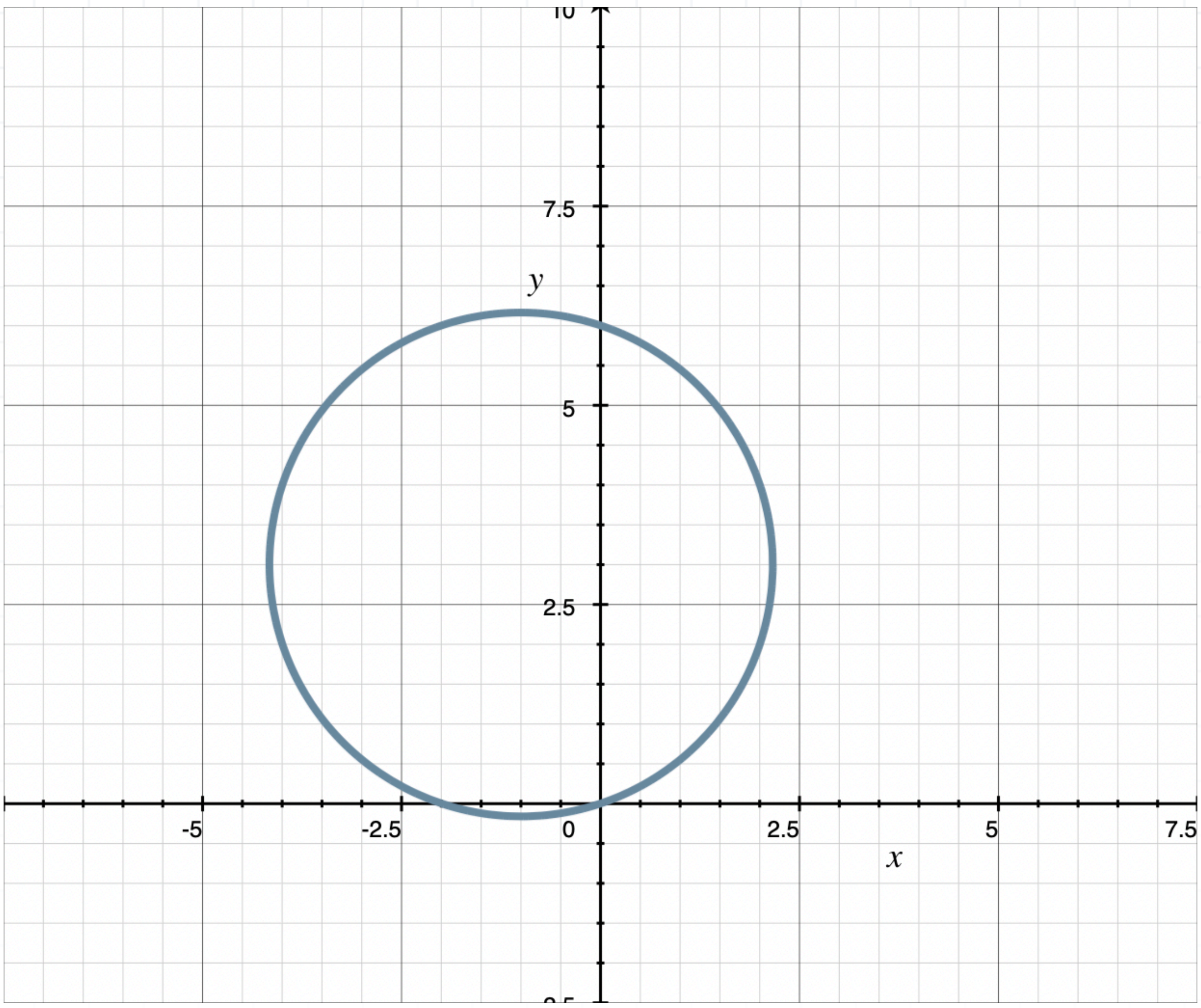
$$x^2 + 2x + y^2 - 6y = 0$$

Completing the square with respect to both x and y gives

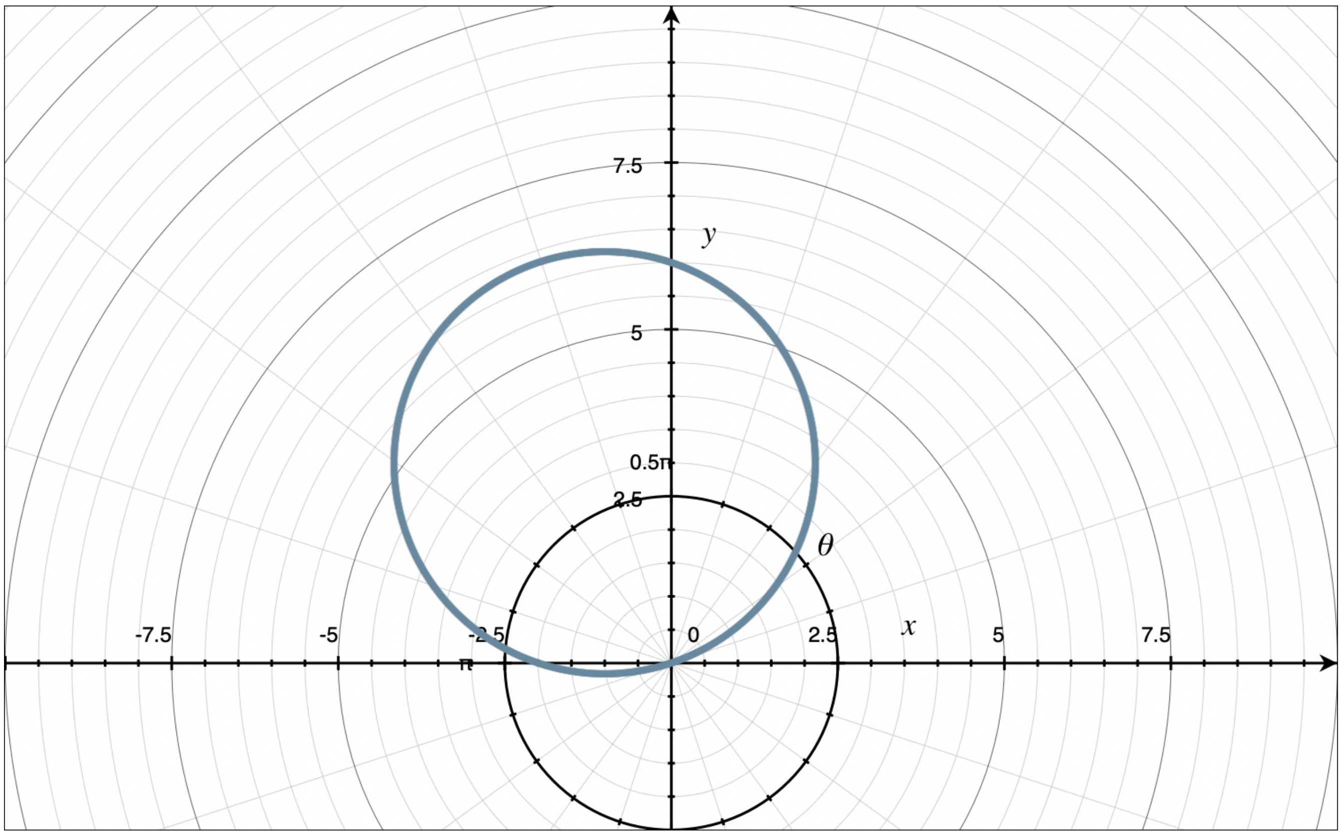
$$x^2 + 2x + 1 + y^2 - 6y + 9 = 0 + 1 + 9$$

$$(x+1)^2 + (y-3)^2 = 10$$

Therefore, we get a circle centered at $(x, y) = (-1,3)$ with radius $\sqrt{10}$.



In polar coordinates, the sketch of the curve would be



■ 4. Convert $r^2 \sin(2\theta) = 1$ into rectangular coordinates and then sketch the graph of the resulting equation. Hint: Apply the trigonometric identity $\sin(2\theta) = 2 \sin \theta \cos \theta$.

Solution:

Use the trigonometric identity $\sin(2\theta) = 2 \sin \theta \cos \theta$ to substitute into the equation.

$$r^2 \sin(2\theta) = 1$$

$$2r^2 \sin \theta \cos \theta = 1$$

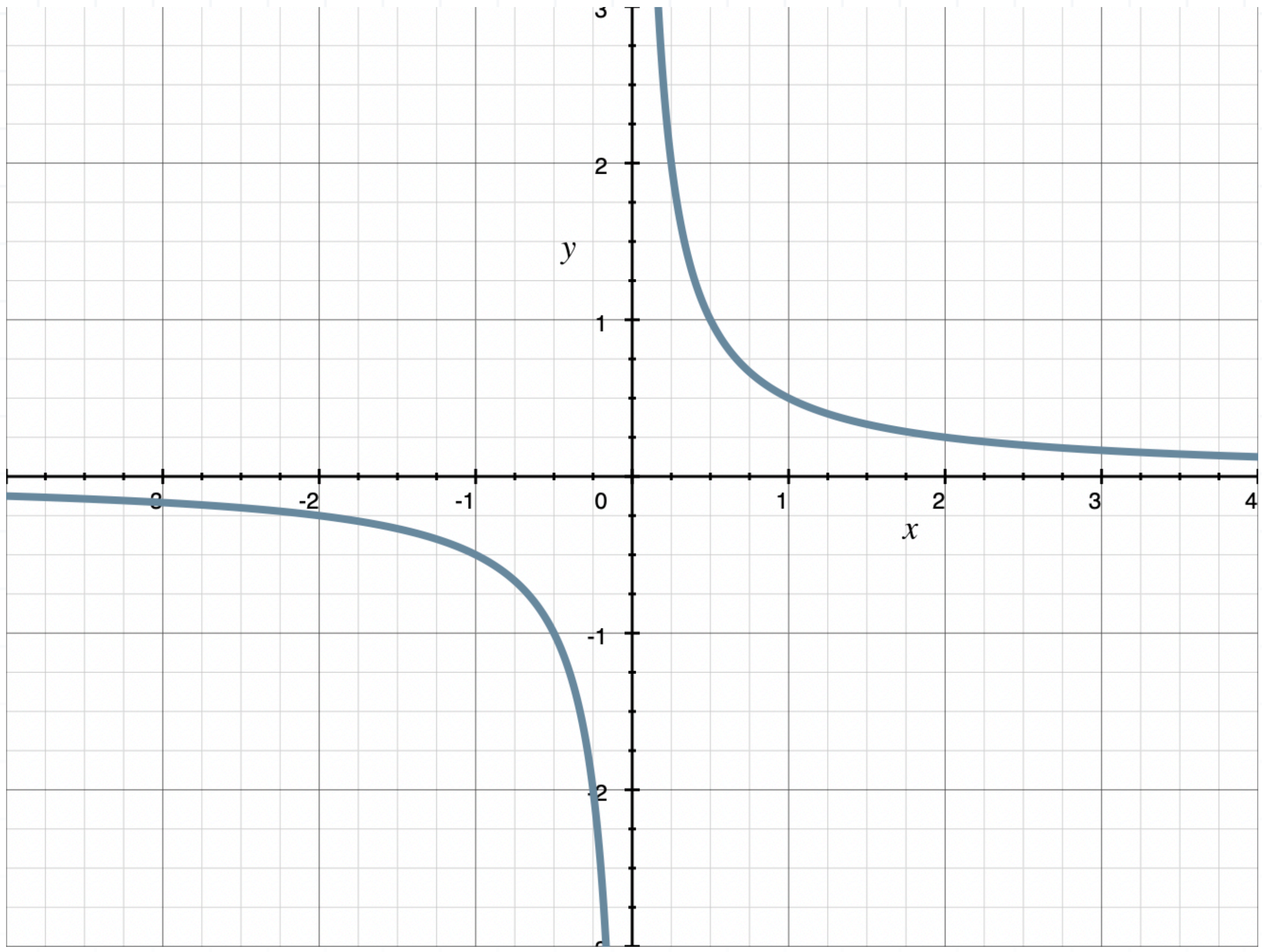
$$2(r \sin \theta)(r \cos \theta) = 1$$

Substitute $x = r \cos \theta$ and $y = r \sin \theta$.

$$2xy = 1$$

$$y = \frac{1}{2x}$$

The sketch of this equation in rectangular coordinates is



■ 5. Convert $r\cos\theta - \tan\theta + 2 = 0$ into rectangular coordinates and then sketch the graph of the resulting equation.

Solution:

Use $x = r\cos\theta$ and $\tan\theta = y/x$ to make substitutions into the equation.

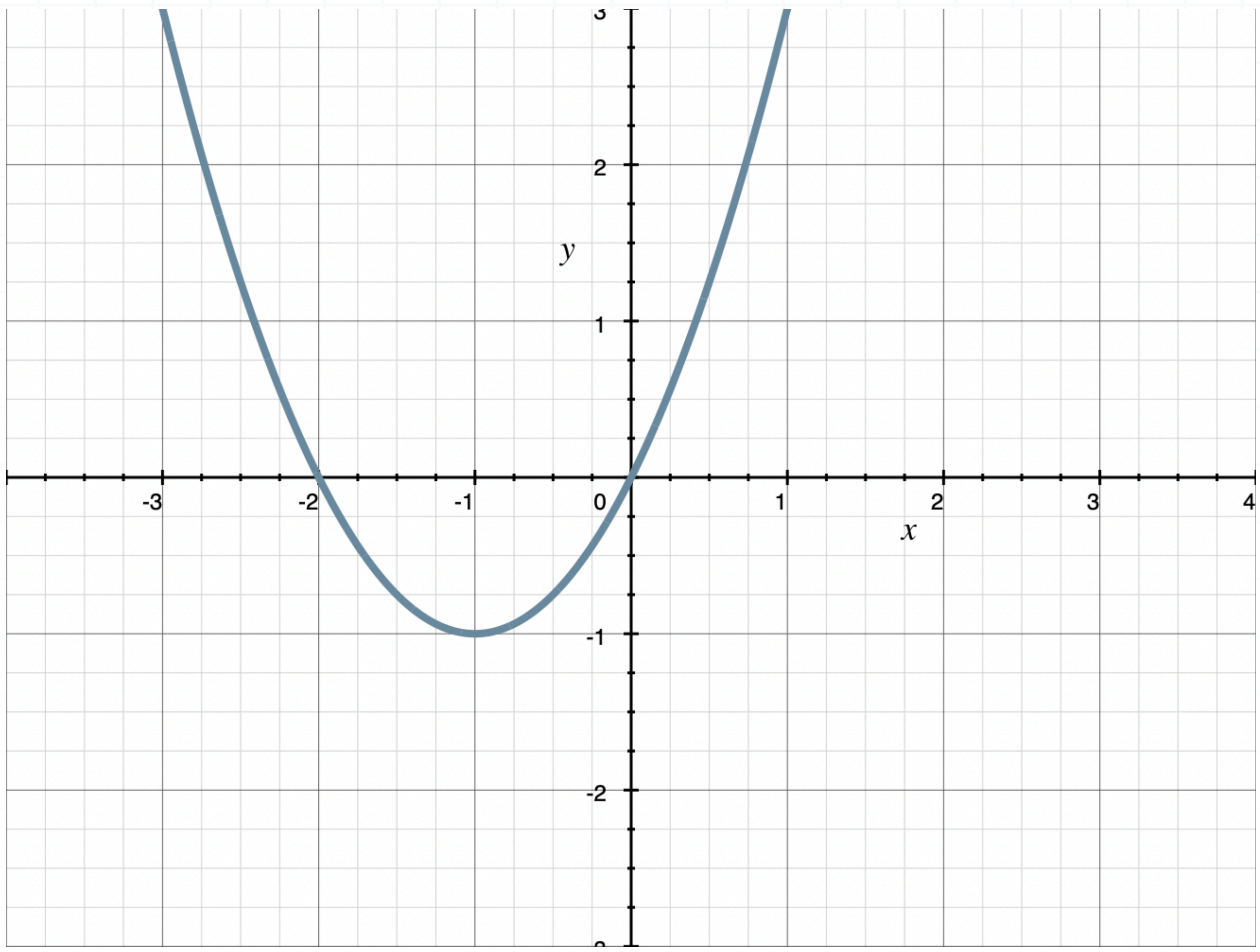
$$r\cos\theta - \tan\theta + 2 = 0$$

$$x - \frac{y}{x} + 2 = 0$$

$$\frac{y}{x} = x + 2$$

$$y = x(x+2)$$

Therefore, we get a parabola with vertex $(x, y) = (-1, -1)$.



■ 6. Convert the polar equation into rectangular coordinates and the sketch the graph of the resulting equation.

$$r^2 = \frac{\sin\theta}{\cos^3\theta}$$

Solution:

Multiply both sides of th equations by $\cos^2\theta$.

$$r^2 = \frac{\sin\theta}{\cos^3\theta}$$

$$r^2\cos^2\theta = \frac{\sin\theta}{\cos\theta}$$

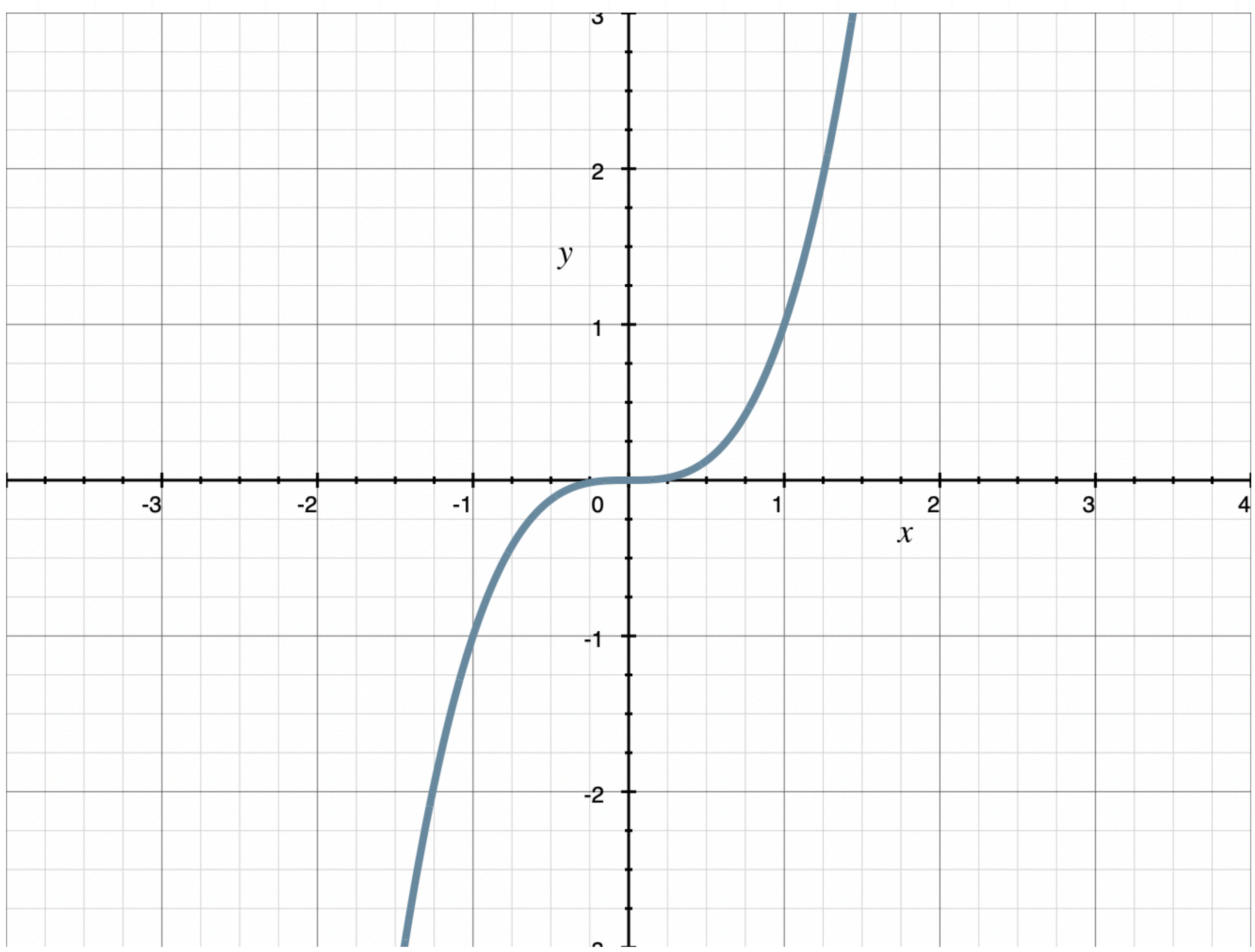
$$(r\cos\theta)^2 = \tan\theta$$

Use $x = r\cos\theta$ and $\tan\theta = y/x$ to make substitutions into the equation.

$$x^2 = \frac{y}{x}$$

$$y = x^3$$

Therefore, the sketch of the cubic function is

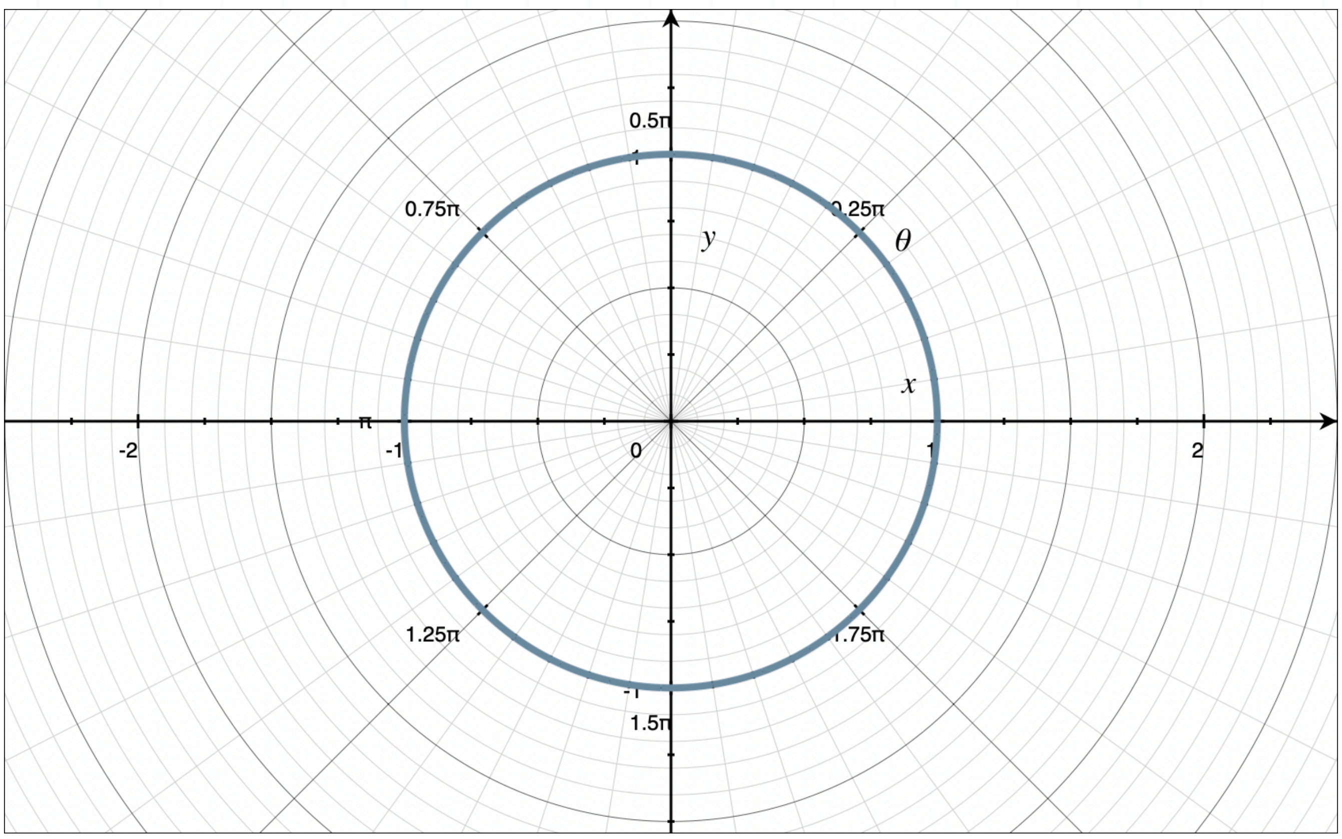


GRAPHING CIRCLES

■ 1. Sketch the graph of $r = -1$.

Solution:

The equation $r = -1$ represents a circle centered at the pole with radius 1, so the sketch of its graph is



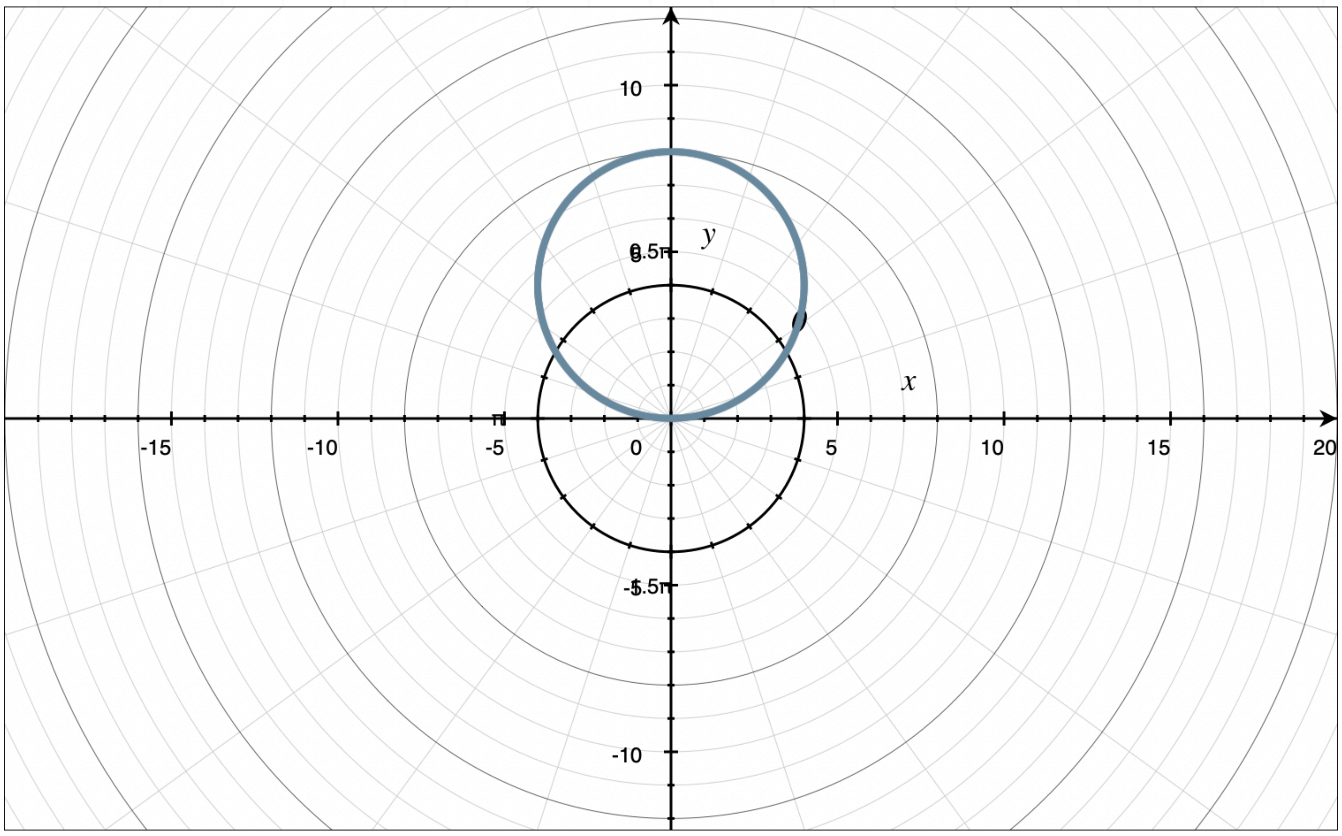
■ 2. Sketch the graph of $r = 8\sin\theta$.

Solution:

The argument of the trig function $\sin\theta$ is θ, so we'll set $\theta = \pi/2$, then evaluate the polar curve at multiples of $\theta = \pi/2$, starting with $\theta = 0$.

theta	0	π/2	π	3π/2	2π
r	0	8	0	-8	0

Plotting these points in polar coordinates and connecting them with a smooth curve gives us a sketch of the circle.



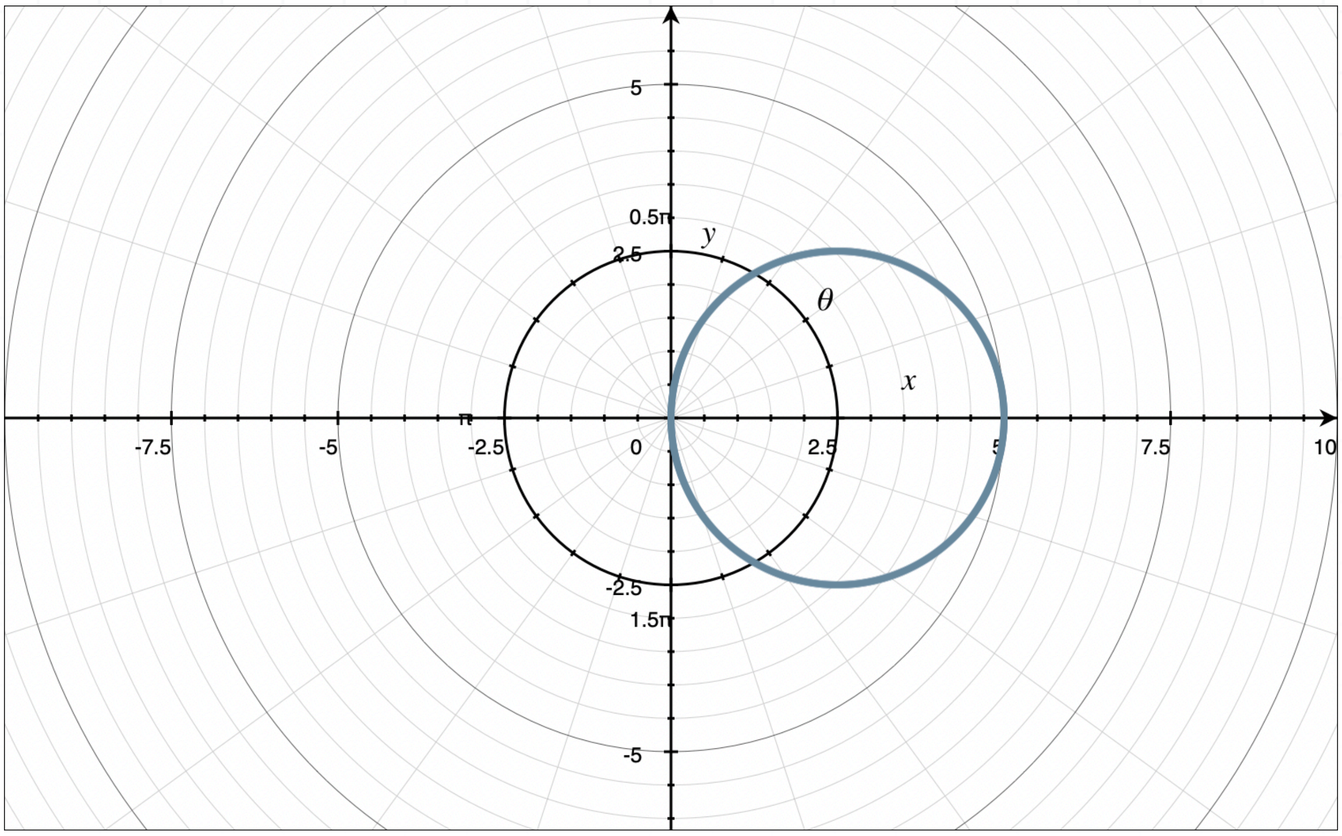
■ 3. Sketch the graph of $r = 5\cos\theta$.

Solution:

The argument of the trig function $\cos\theta$ is θ, so we'll set $\theta = \pi/2$, then evaluate the polar curve at multiples of $\theta = \pi/2$, starting with $\theta = 0$.

theta	0	π/2	π	3π/2	2π
r	5	0	-5	0	5

Plotting these points in polar coordinates and connecting them with a smooth curve gives us a sketch of the circle.



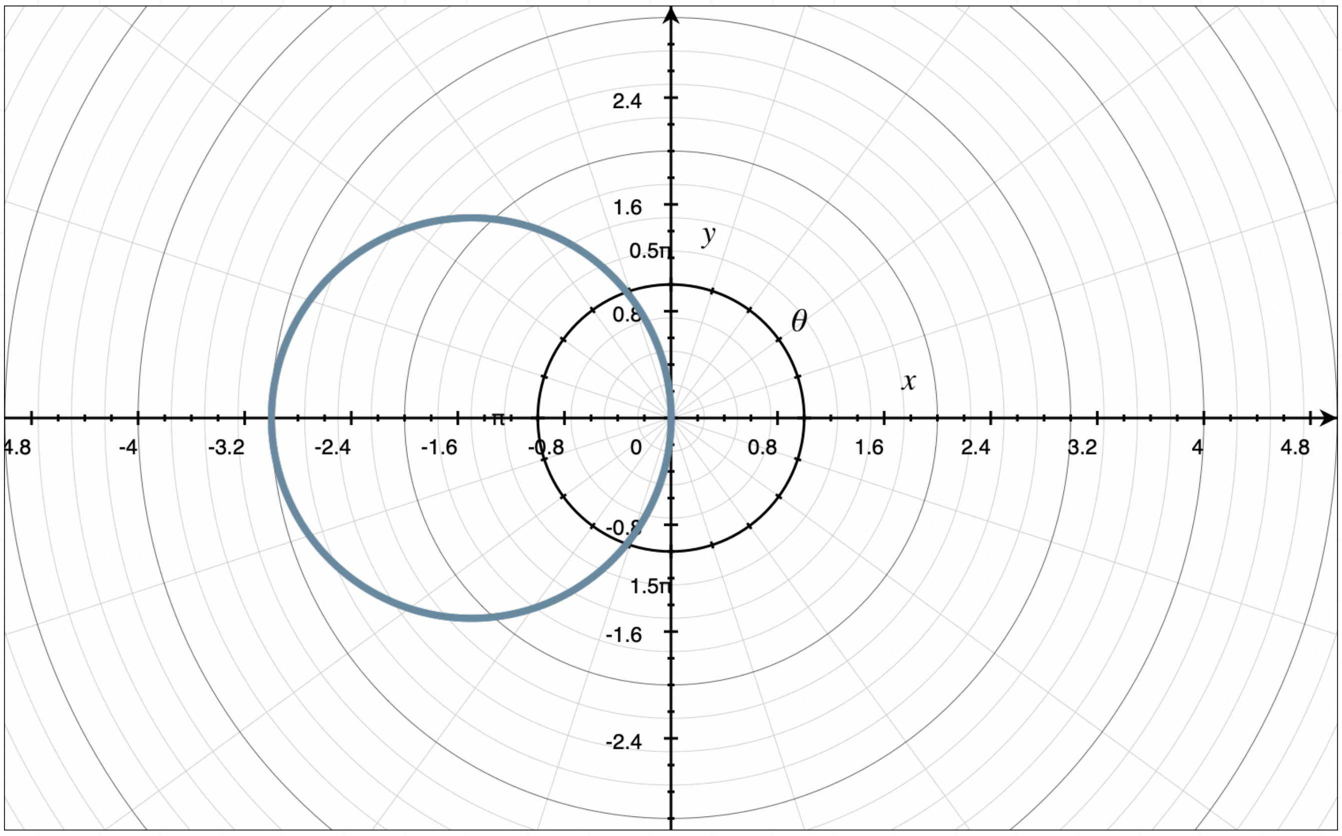
■ 4. Sketch the graph of $r = -3\cos\theta$.

Solution:

The argument of the trig function $\cos\theta$ is θ, so we'll set $\theta = \pi/2$, then evaluate the polar curve at multiples of $\theta = \pi/2$, starting with $\theta = 0$.

theta	0	π/2	π	3π/2	2π
r	-3	0	3	0	-3

Plotting these points in polar coordinates and connecting them with a smooth curve gives us a sketch of the circle.



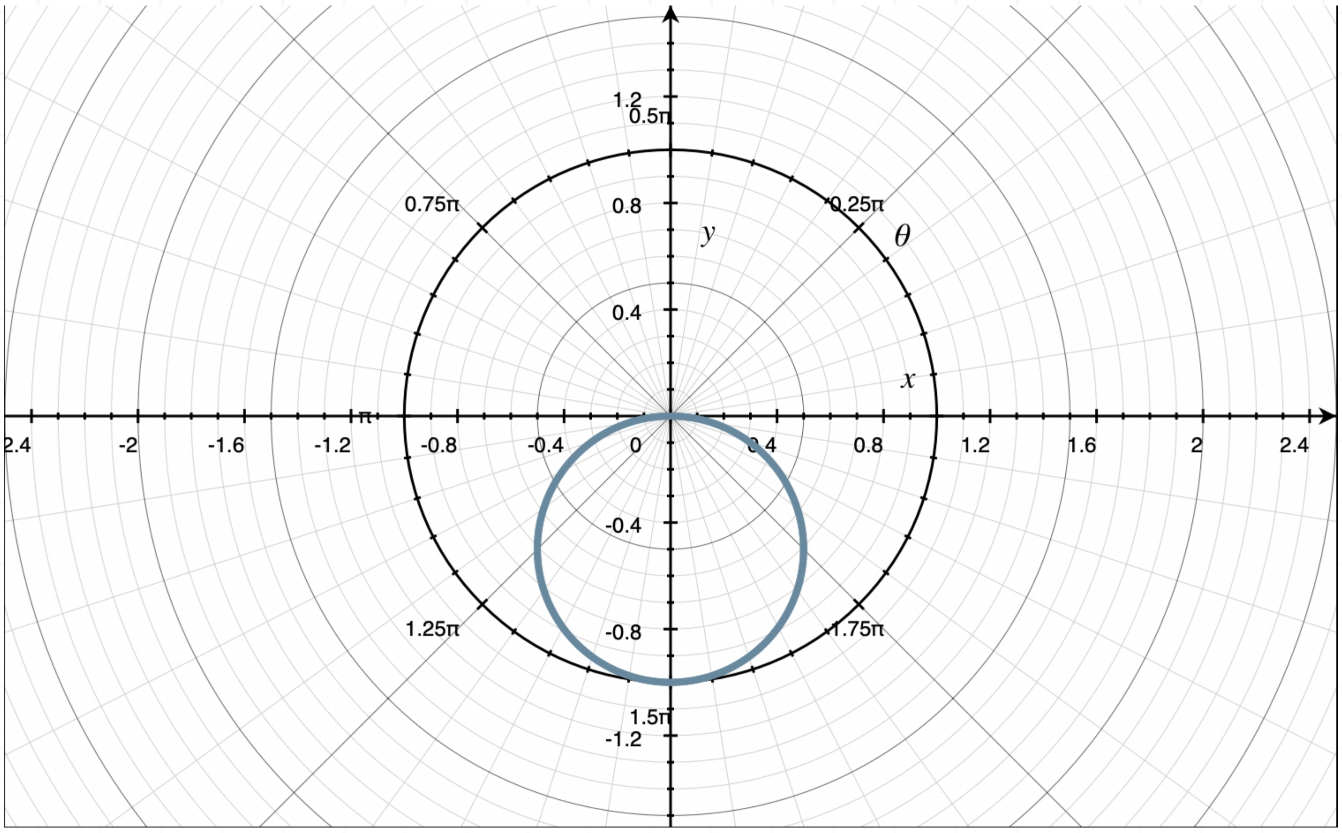
■ 5. Sketch the graph of $r = -\sin\theta$.

Solution:

The argument of the trig function $\cos\theta$ is θ, so we'll set $\theta = \pi/2$, then evaluate the polar curve at multiples of $\theta = \pi/2$, starting with $\theta = 0$.

theta	0	π/2	π	3π/2	2π
r	0	-1	0	1	0

Plotting these points in polar coordinates and connecting them with a smooth curve gives us a sketch of the circle.



■ 6. Sketch the graph of $r^2 = 4\sin^2\theta$.

Solution:

Rewrite, then factor the equation.

$$r^2 - 4\sin^2\theta = 0$$

$$(r - 2\sin\theta)(r + 2\sin\theta) = 0$$

Solve each binomial equation separately.

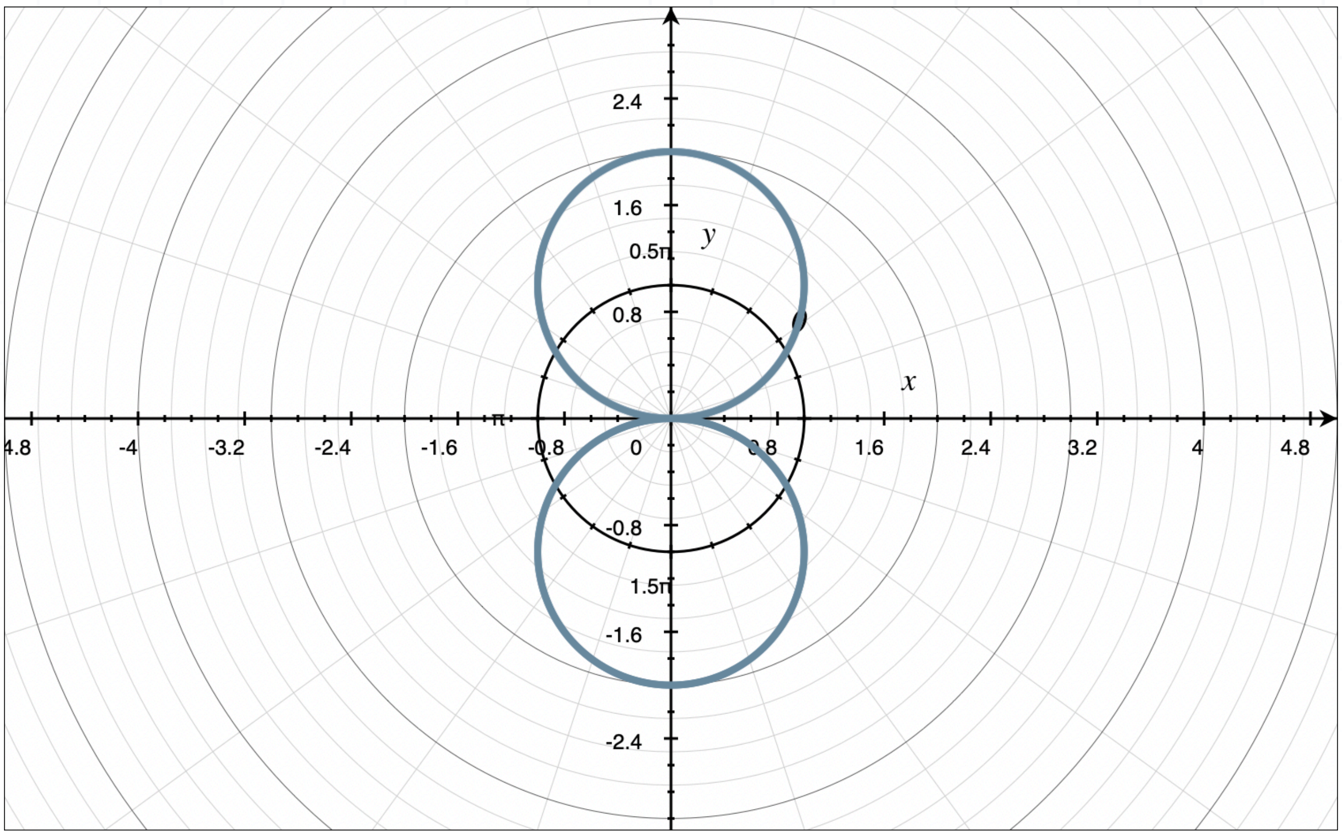
$$r - 2\sin\theta = 0 \qquad r + 2\sin\theta = 0$$

$$r_1 = 2\sin\theta \qquad r_2 = -2\sin\theta$$

Both equations give circles which we can sketch separately.

theta	0	π/2	π	3π/2	2π
r_1	0	2	0	-2	0
r_2	0	-2	0	2	0

Plotting these points in polar coordinates and connecting them with a smooth curve gives us a sketch of the circle.



GRAPHING ROSES

■ 1. Sketch the graph of the rose $r = 2\sin(5\theta)$.

Solution:

Since $c = 2$ and $n = 5$, this rose will have 5 petals that extend out to a distance of $r = 2$ from the pole. We'll set

$$5\theta = \frac{\pi}{2}$$

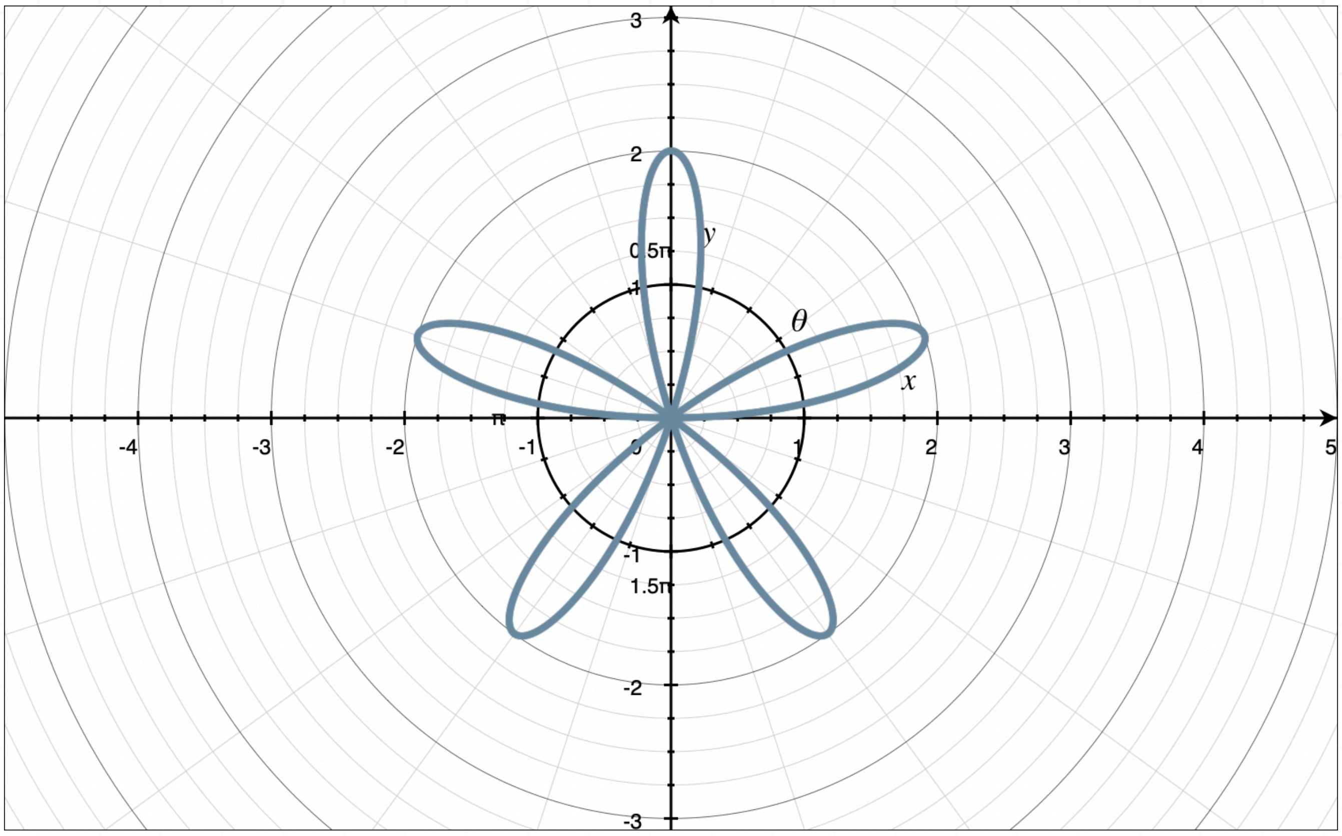
$$\theta = \frac{\pi}{10}$$

and then evaluate the polar curve at multiples of $\theta = \pi/10$, starting with $\theta = 0$.

theta	0	π/10	π/5	3π/10	2π/5	π/2	3π/5	7π/10	4π/5	9π/10	π
r	0	2	0	-2	0	2	0	-2	0	2	0

11π/10	6π/5	13π/10	7π/5	3π/2	8π/5	17π/10	9π/5	2π
-2	0	2	0	-2	0	2	0	-2

Plotting these points in polar coordinates and connecting them with a smooth curve gives us a sketch of the rose.



■ 2. Sketch the graph of the rose $r = -4\cos(3\theta)$.

Solution:

Since $c = -4$ and $n = 3$, this rose will have 3 petals that extend out to a distance of $r = 4$ from the pole. We'll set

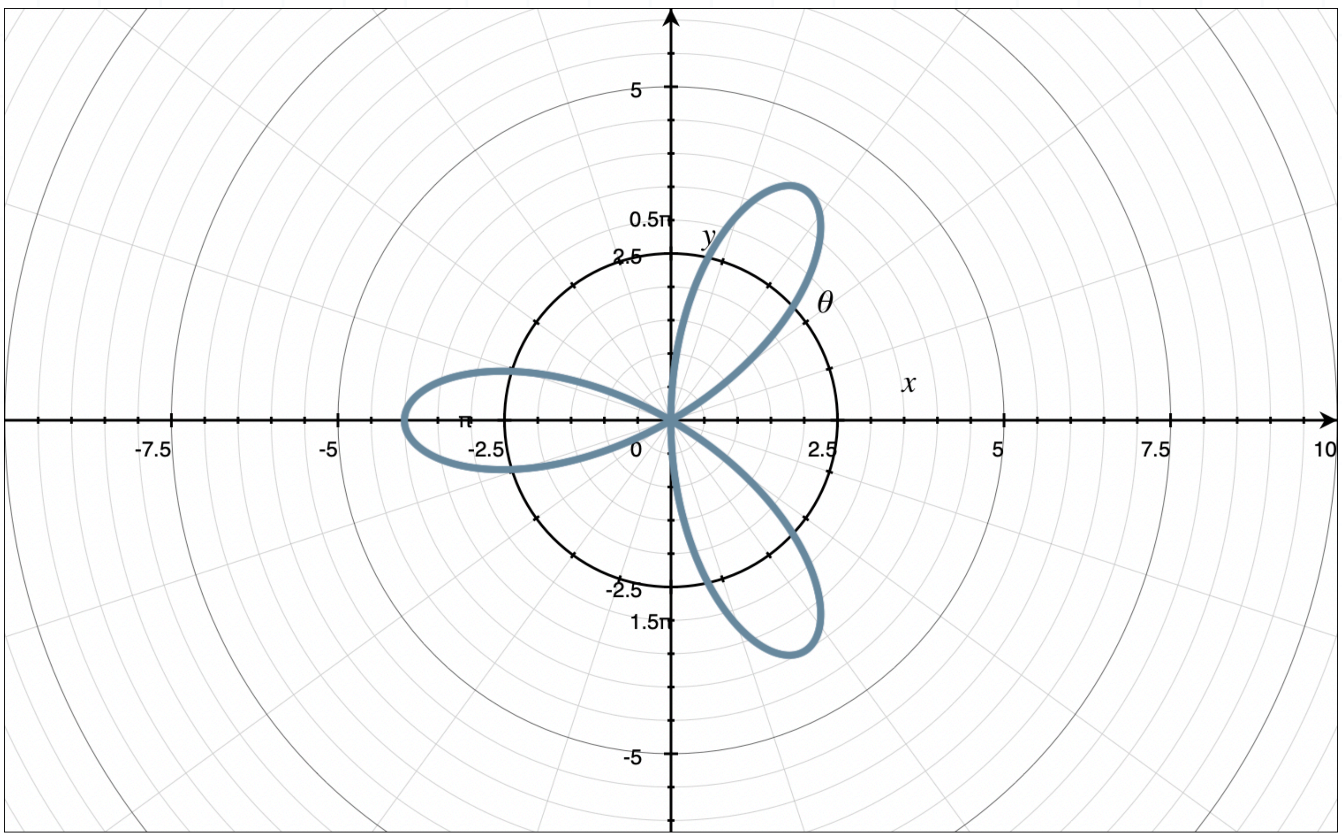
$$3\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{6}$$

and then evaluate the polar curve at multiples of $\theta = \pi/6$, starting with $\theta = 0$.

theta	0	π/6	π/3	π/2	2π/3	5π/6	π	7π/6	4π/3	3π/2	5π/3	11π/6	2π
r	-4	0	4	0	-4	0	4	0	-4	0	4	0	-4

Plotting these points in polar coordinates and connecting them with a smooth curve gives us a sketch of the rose.



■ 3. Sketch the graph of the rose $r = -3\sin(2\theta)$.

Solution:

Since $c = -3$ and $n = 2$, this rose will have 4 petals that extend out to a distance of $r = 3$ from the pole. We'll set

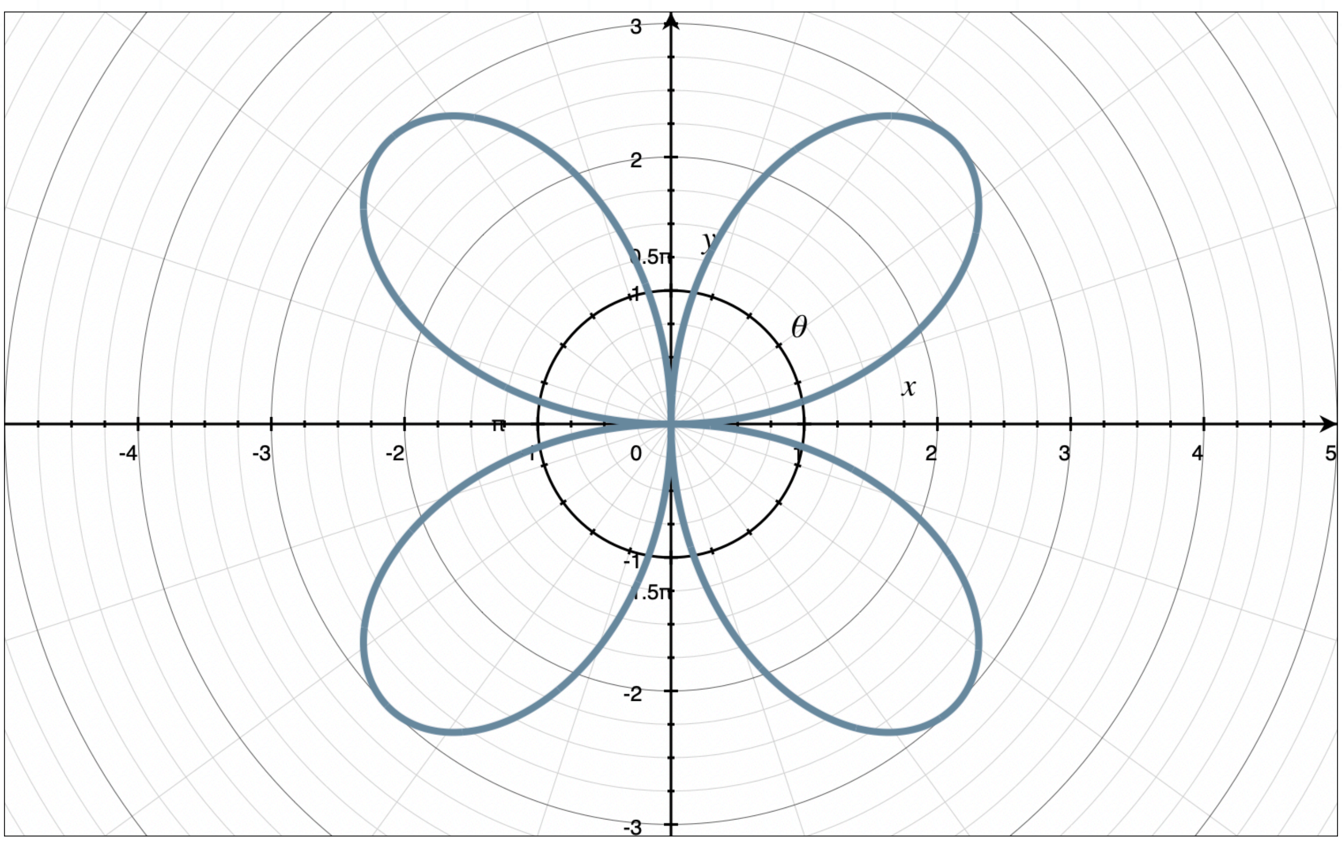
$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

and then evaluate the polar curve at multiples of $\theta = \pi/4$, starting with $\theta = 0$.

theta	0	π/4	π/2	3π/4	π	5π/4	3π/2	7π/4	2π
r	0	-3	0	3	0	-3	0	3	0

Plotting these points in polar coordinates and connecting them with a smooth curve gives us a sketch of the rose.



■ 4. Sketch the graph of the rose $r = 7\cos(4\theta)$.

Solution:

Since $c = 7$ and $n = 4$, this rose will have 8 petals that extend out to a distance of $r = 7$ from the pole. We'll set

$$4\theta = \frac{\pi}{2}$$

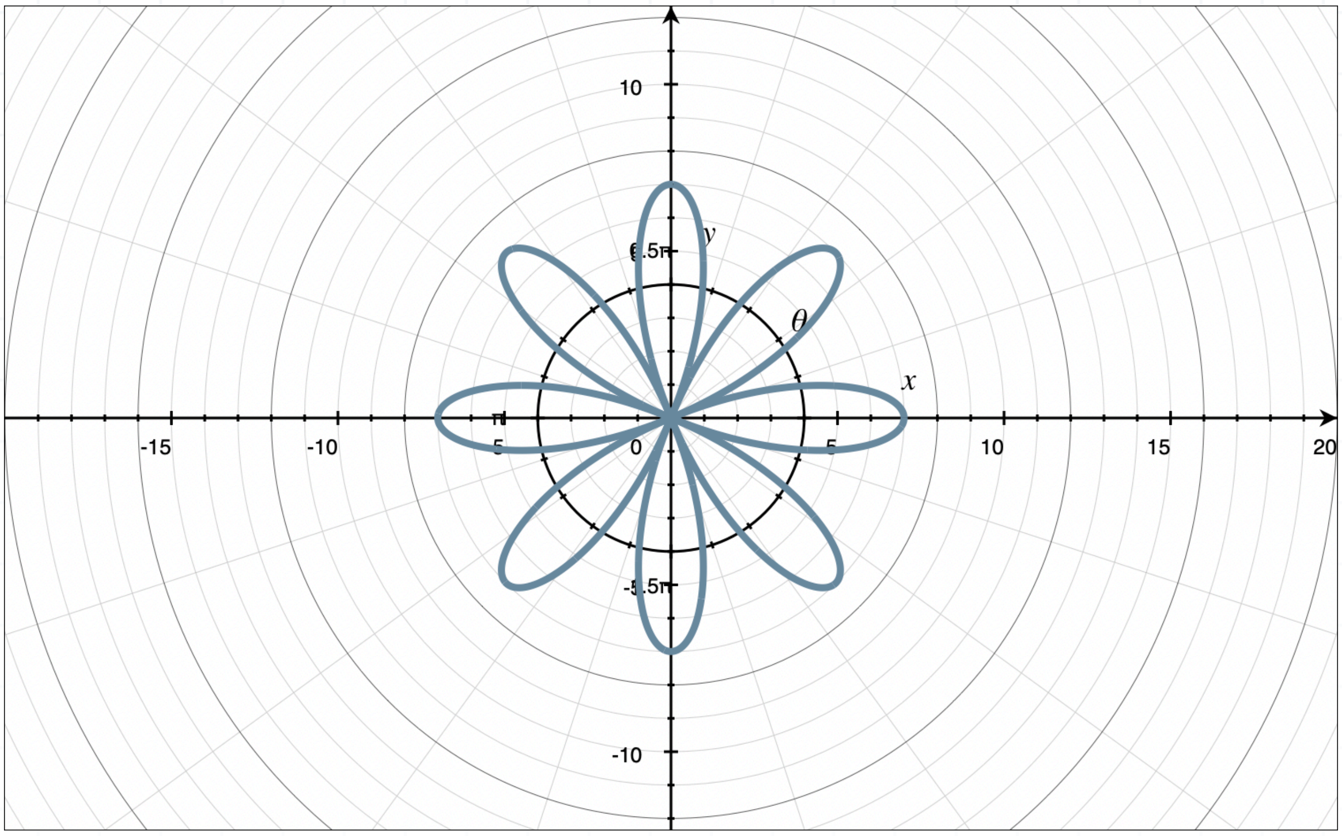
$$\theta = \frac{\pi}{8}$$

and then evaluate the polar curve at multiples of $\theta = \pi/8$, starting with $\theta = 0$.

theta	0	π/8	π/4	3π/8	π/2	5π/8	3π/4	7π/8
r	7	0	-7	0	7	0	-7	0

π	9π/8	5π/4	11π/8	3π/2	13π/8	7π/4	15π/8	2π
7	0	-7	0	7	0	-7	0	7

Plotting these points in polar coordinates and connecting them with a smooth curve gives us a sketch of the rose.



■ 5. How many petals will the rose have?

$$r = 5\cos(12\theta)$$

Solution:

The equation $r = 5\cos(12\theta)$ is in the form $r = c\cos(n\theta)$ with $c = 5$ and $n = 12$. Since n is even, the rose will have

$$|2n| = |2 \cdot 12| = 24 \text{ petals}$$

■ 6. Write an equation of a rose that has 9 petals and passes through the point $(6,3\pi/2)$.

Solution:

Since the rose has 9 petals, its equation is $r = c\cos(9\theta)$ or $r = c\sin(9\theta)$. Plugging $\theta = 3\pi/2$ into $r = c\cos(9\theta)$ gives

$$6 = c\cos\left(9\left(\frac{3\pi}{2}\right)\right)$$

$$6 = c\cos\left(\frac{27\pi}{2}\right)$$

$$6 = c(0)$$

$$6 \neq 0$$

So the cosine rose can't pass through $(6,3\pi/2)$. But the sine rose will pass through this point.

$$6 = c\sin\left(9\left(\frac{3\pi}{2}\right)\right)$$

$$6 = c\sin\left(\frac{27\pi}{2}\right)$$

$$6 = c(-1)$$

$$-6 = c$$

Therefore, the equation of the rose must be $r = -6\sin(9\theta)$.

GRAPHING CARDIOIDS

■ 1. Create a table of values for $r = 5 - 5\cos\theta$ over the interval $0 \le \theta \le 2\pi$, then use it to sketch the graph.

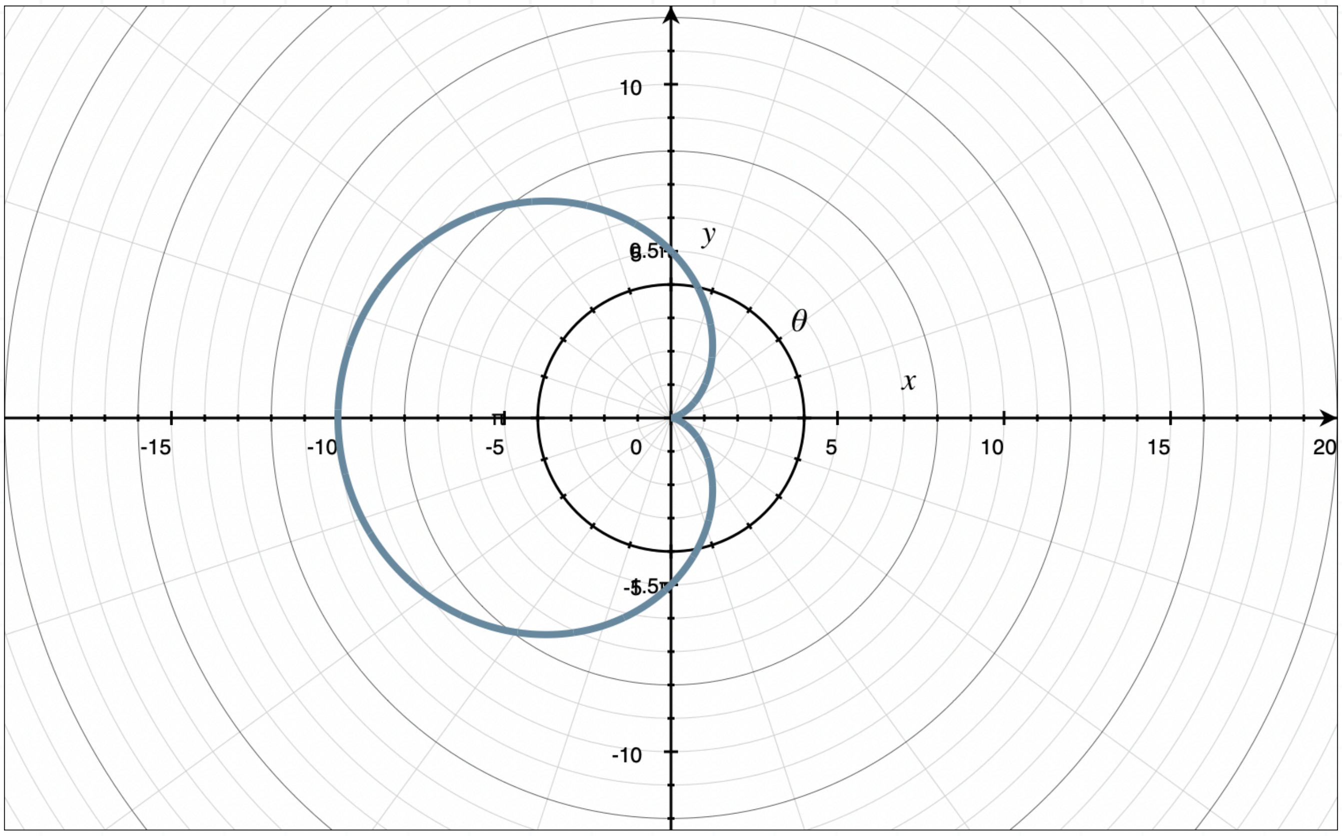
Solution:

Because $c = 5$, this cardioid will extend out to a distance of $2c = 2(5) = 10$ from the pole. Because it's a cosine cardioid where the sign between the terms is negative, the graph will sit mostly to the left of the vertical axis, with symmetry across the horizontal axis.

Now we'll make a table with for $\theta = 0$, $\pi/2$, π, $3\pi/2$, and 2π, and include the values of r that correspond to each of these θ-values.

theta	0	π/2	π	3π/2	2π
r	0	5	10	5	0

Plotting these points on the polar graph gives



■ 2. Create a table of values for $r = 3 - 3\sin\theta$ over the interval $0 \le \theta \le 2\pi$, then use it to sketch the graph.

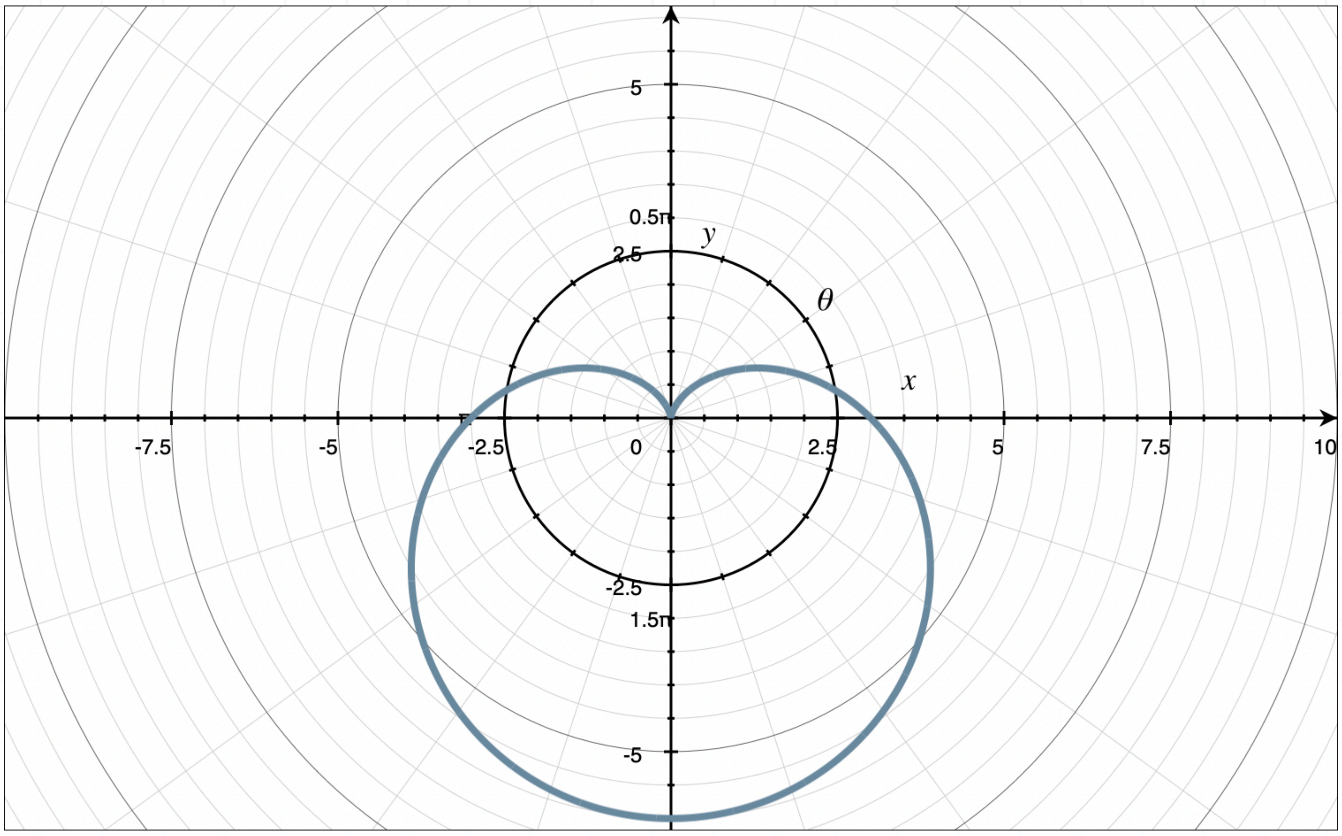
Solution:

Because $c = 3$, this cardioid will extend out to a distance of $2c = 2(3) = 6$ from the pole. Because it's a sine cardioid where the sign between the terms if negative, the graph will sit mostly below the horizontal axis, with symmetry across the vertical axis.

Now we'll make a table with for $\theta = 0$, $\pi/2$, π, $3\pi/2$, and 2π, and include the values of r that correspond to each of these θ-values.

theta	0	π/2	π	3π/2	2π
r	3	0	3	6	3

Plotting these points on the polar graph gives



■ 3. Create a table of values for $r = -2 - 2\sin\theta$ over the interval $0 \leq \theta \leq 2\pi$, then use it to sketch the graph.

Solution:

The value of c needs to be positive. Because this equation begins with $c = -2$, we need to factor out a negative sign.

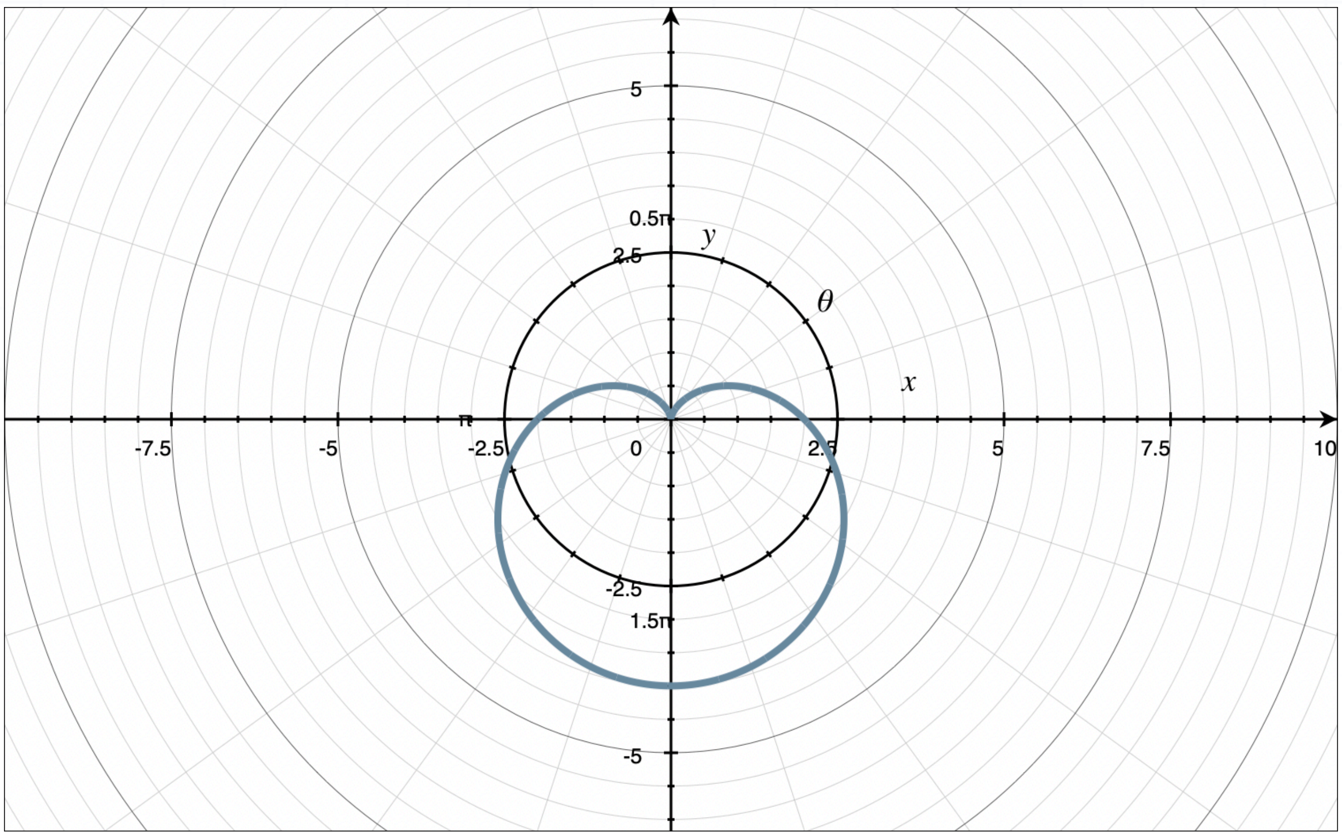
$$r = -(2 + 2\sin\theta)$$

Now we have the cardioid equation $r = 2 + 2\sin\theta$, and we'll just need to apply the negative sign to each of our r-values.

We'll make a table with for $\theta = 0$, $\pi/2$, π, $3\pi/2$, and 2π, and include the values of r that correspond to each of these θ-values.

theta	0	π/2	π	3π/2	2π
r	-2	-4	-2	0	-2

Plotting these points on the polar graph gives



■ 4. Create a table of values for $r = -3 - 3\cos\theta$ over the interval $0 \le \theta \le 2\pi$, then use it to sketch the graph.

Solution:

The value of c needs to be positive. Because this equation begins with $c = -3$, we need to factor out a negative sign.

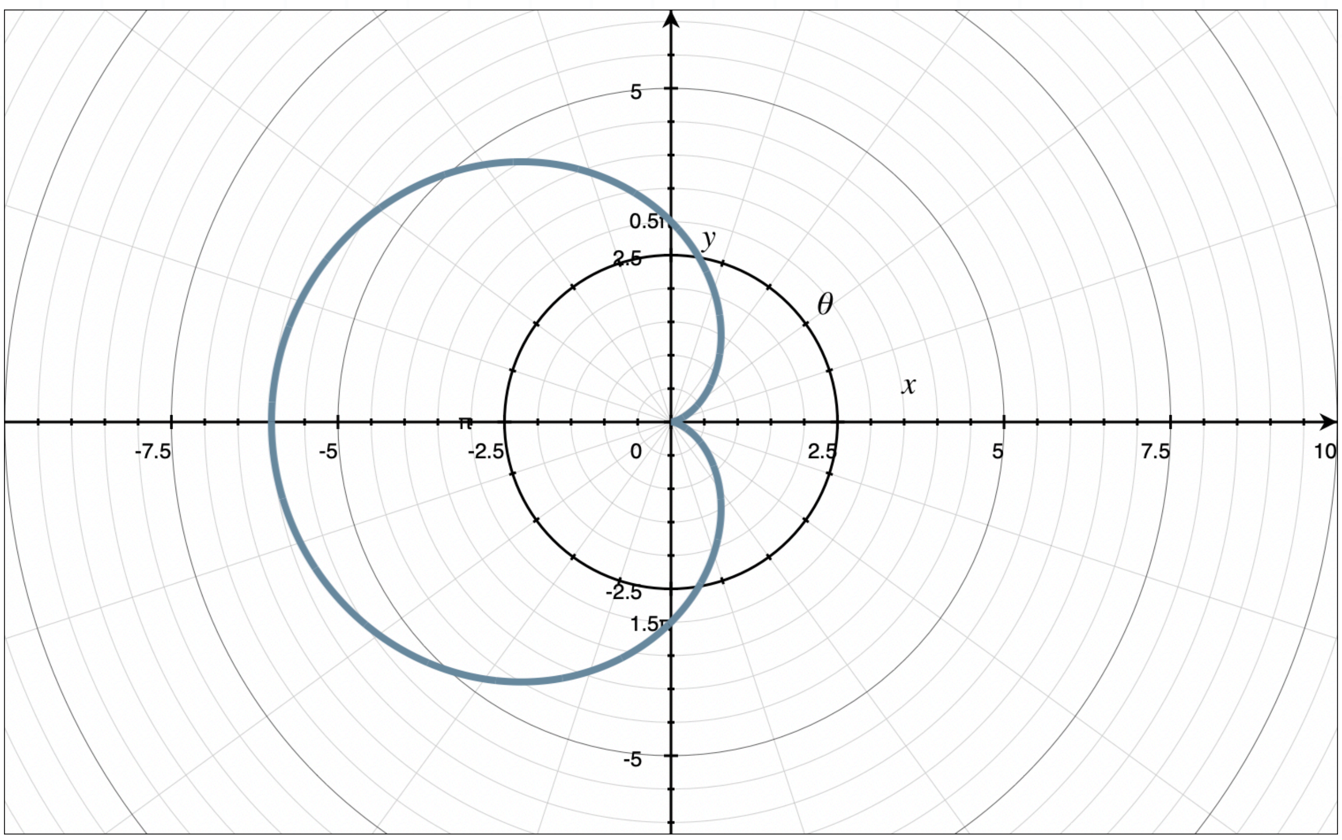
$$r = -(3 + 3\cos\theta)$$

Now we have the cardioid equation $r = 3 + 3\cos\theta$, and we'll just need to apply the negative sign to each of our r-values.

We'll make a table with for $\theta = 0$, $\pi/2$, π, $3\pi/2$, and 2π, and include the values of r that correspond to each of these θ-values.

theta	0	π/2	π	3π/2	2π
r	0	3	6	3	0

Plotting these points on the polar graph gives



■ 5. Sketch the graph of the cardioid $r = 2 + 2\cos\theta$.

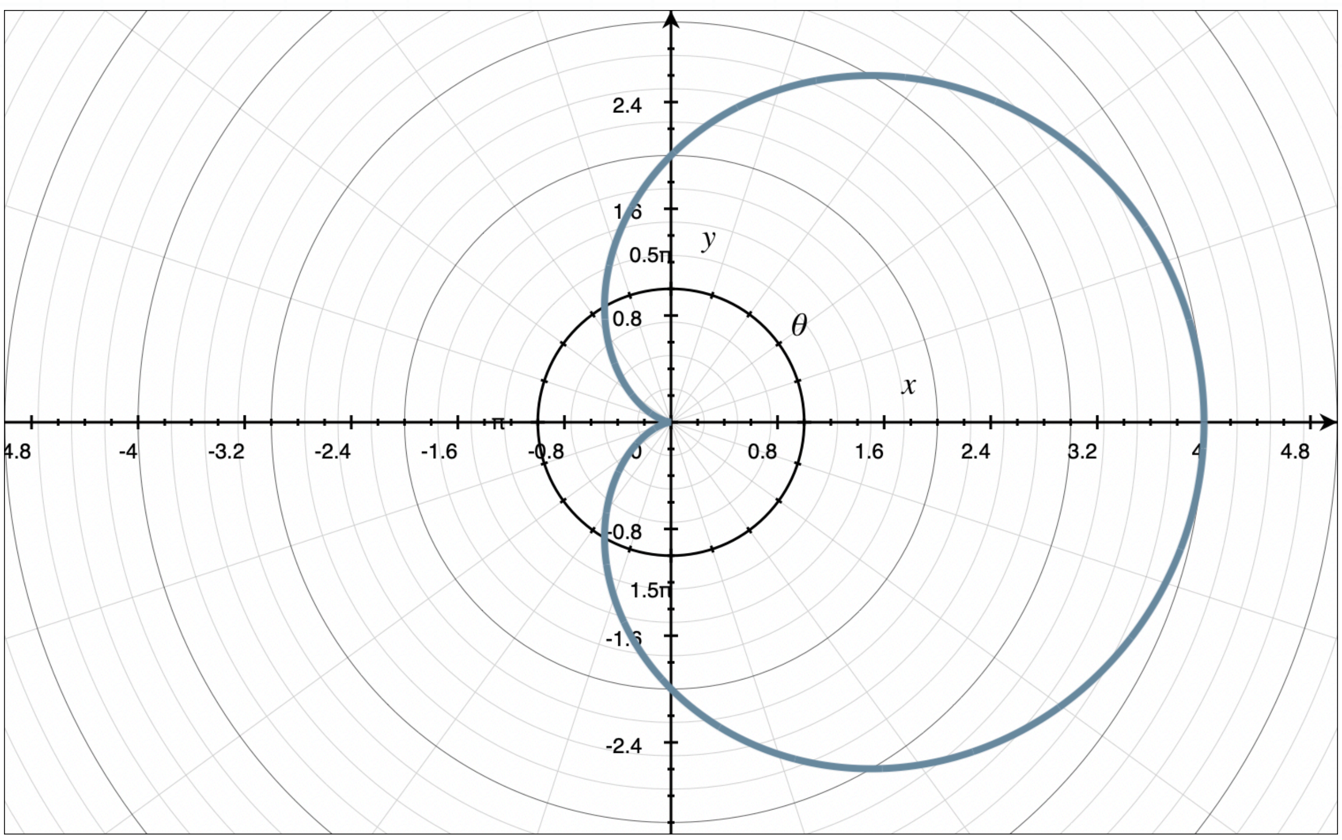
Solution:

Because $c = 2$, this cardioid will extend out to a distance of $2c = 2(2) = 4$ from the pole. Because it's a cosine cardioid where the sign between the terms is positive, the graph will sit mostly to the right of the vertical axis, with symmetry across the horizontal axis.

Now we'll make a table with for $\theta = 0$, $\pi/2$, π, $3\pi/2$, and 2π, and include the values of r that correspond to each of these θ-values.

theta	0	π/2	π	3π/2	2π
r	4	2	0	2	4

Plotting these points on the polar graph gives

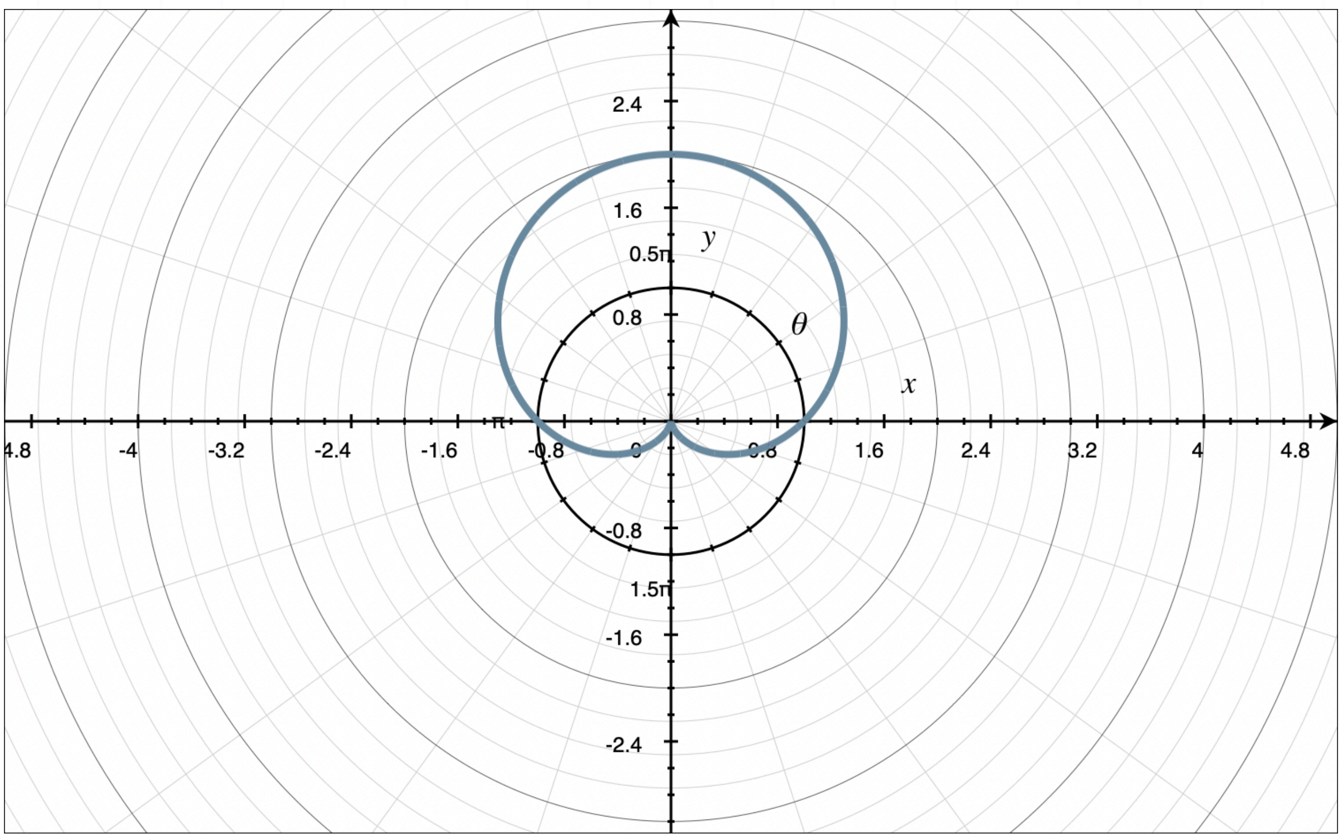


■ 6. Find the equation of the cardioid that extends out a distance of 2 from the pole, is symmetric across the vertical axis, and mostly sits above of the horizontal axis. Then sketch its graph.

Solution:

Since the cardioid is symmetric across the vertical axis, it's a sine cardioid that can be represented by $r = c \pm c \sin \theta$. We know that $2c = 2$, and therefore that $c = 1$.

Which means the cardioid's equation must be $r = 1 \pm \sin \theta$. But the cardioid sits mostly above the horizontal axis, so the equation must be $r = 1 + \sin \theta$ and the sketch of its graph is



GRAPHING LIMACONS

1. Sketch the graph of $r = 2 + 5\cos\theta$.

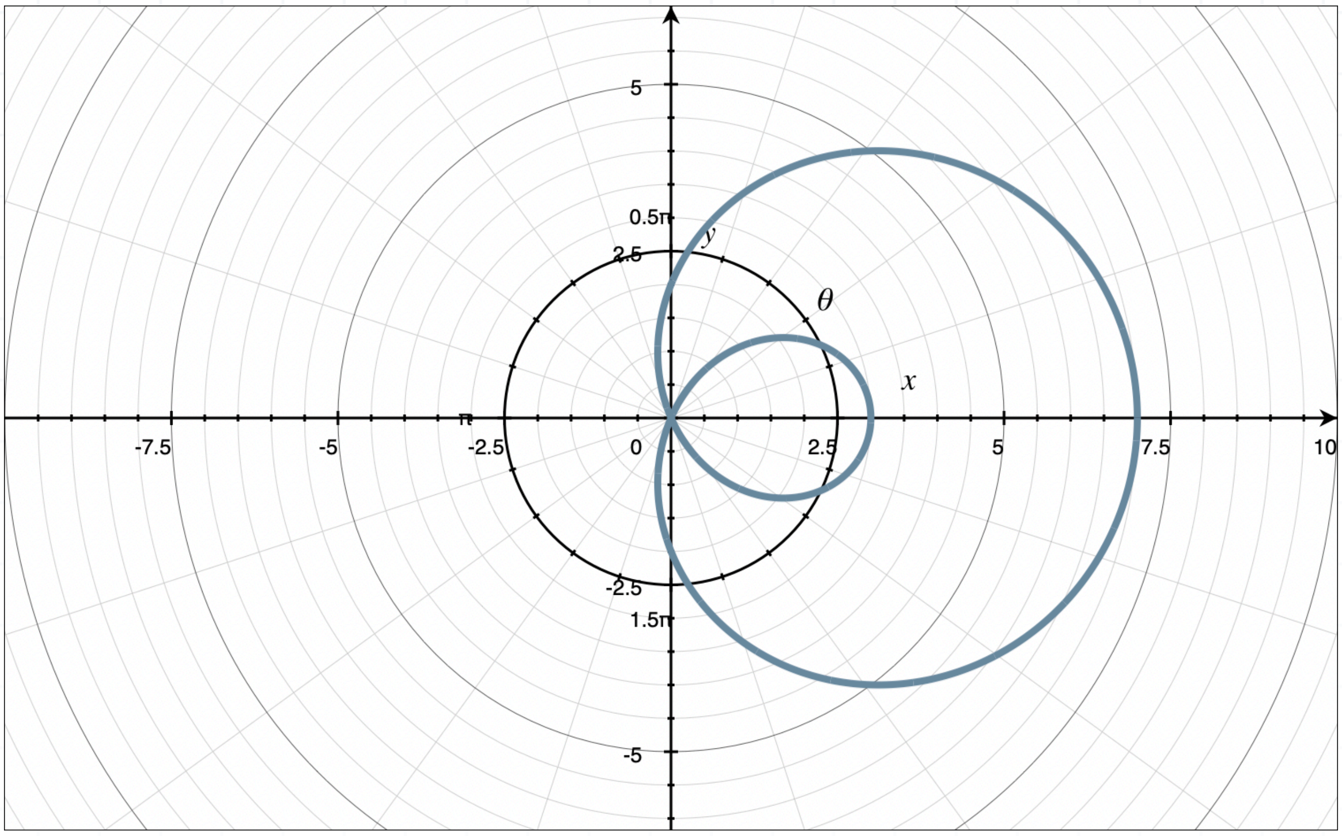
Solution:

With $a = 2$ and $b = 5$, we get $a/b = 2/5 < 1$, which means the graph of this limaçon will include a small loop. And because this is a cosine limaçon with a positive sign separating the terms, it'll be symmetric about the horizontal axis, and will sit mostly to the right of the vertical axis.

Now we'll make a table for $\theta = 0$, $\pi/2$, π, $3\pi/2$, and 2π, and include the values of r that correspond to each of these θ-values.

theta	0	π/2	π	3π/2	2π
r	7	2	-3	2	7

Plotting these points on the polar graph gives



■ 2. Sketch the graph of $r = 4 - \sin\theta$.

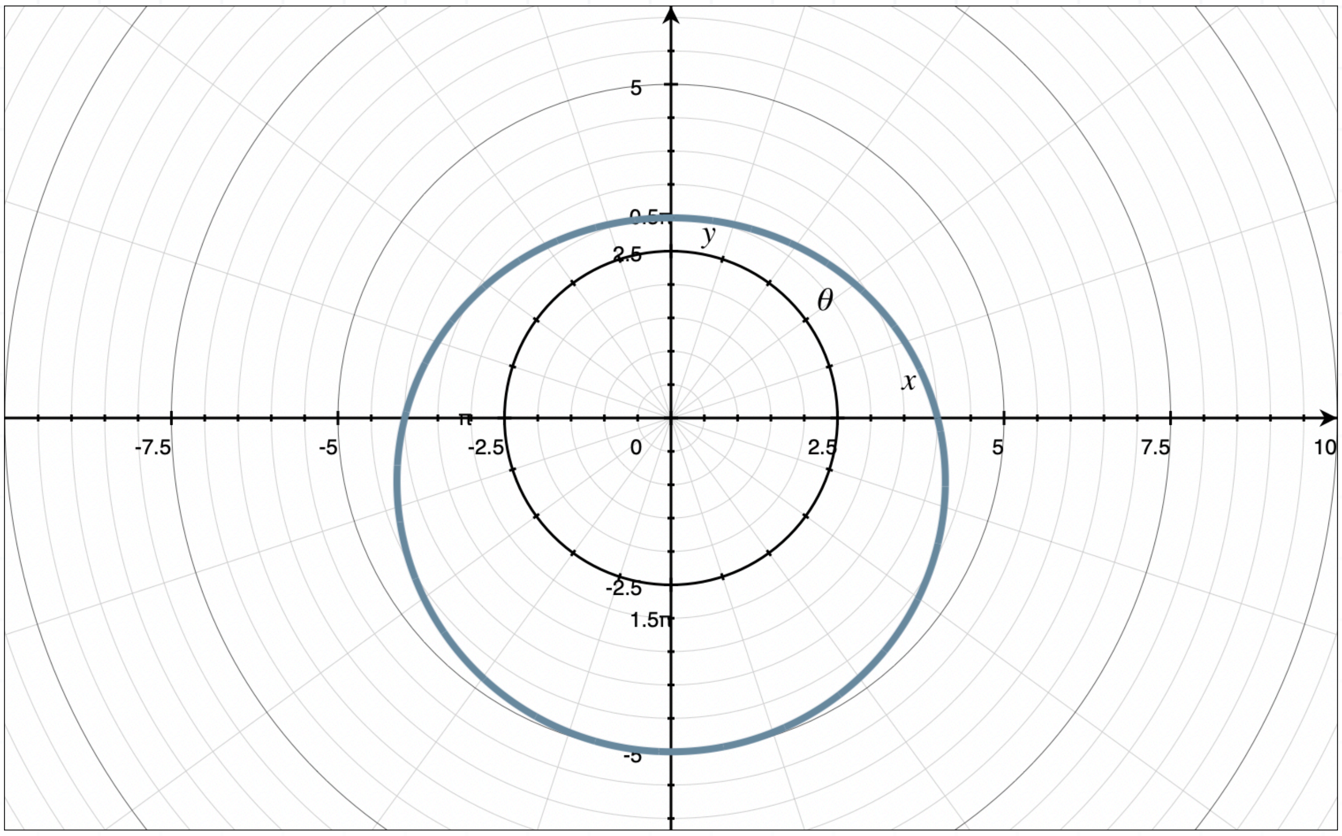
Solution:

With $a = 4$ and $b = 1$, we get $a/b = 4 > 2$, which means the graph of this limaçon will include a small dip, but the shape of the curve is close to a perfect circle. And because this is a sine limaçon with a negative sign separating the terms, it'll be symmetric about the vertical axis, and will sit mostly below the horizontal axis.

Now we'll make a table for $\theta = 0$, $\pi/2$, π, $3\pi/2$, and 2π, and include the values of r that correspond to each of these θ-values.

theta	0	π/2	π	3π/2	2π
r	4	3	4	5	4

Plotting these points on the polar graph gives



■ 3. Sketch the graph of $r = 7 - 6\cos\theta$.

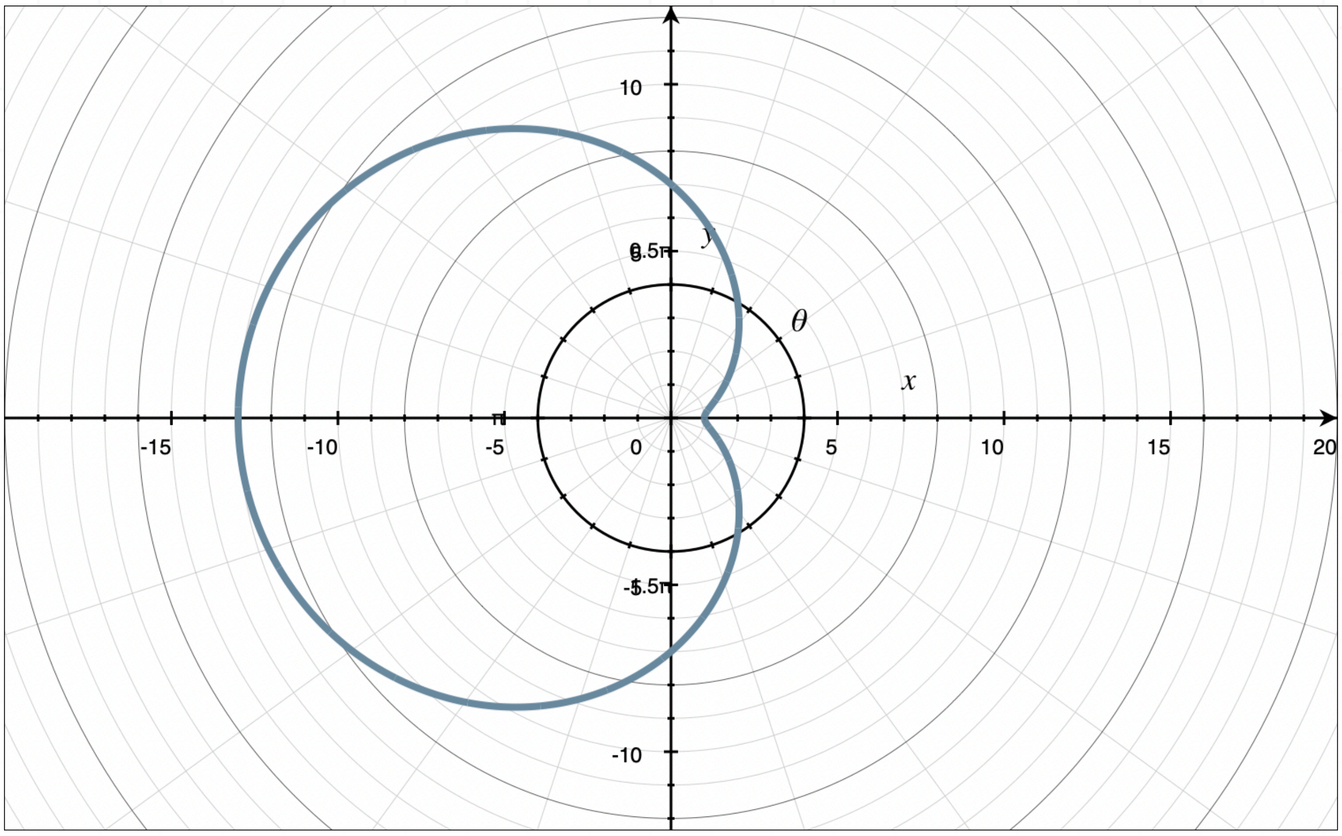
Solution:

With $a = 7$ and $b = 6$, we get $a/b = 7/6$ with $1 < 7/6 < 2$, which means the graph of this limaçon will include a small dip. And because this is a cosine limaçon with a negative sign separating the terms, it'll be symmetric about the horizontal axis, and will sit mostly to the left of the vertical axis.

Now we'll make a table for $\theta = 0$, $\pi/2$, π, $3\pi/2$, and 2π, and include the values of r that correspond to each of these θ-values.

theta	0	π/2	π	3π/2	2π
r	1	7	13	7	1

Plotting these points on the polar graph gives



■ 4. Sketch the graph of $r = 3 + 2\sin\theta$.

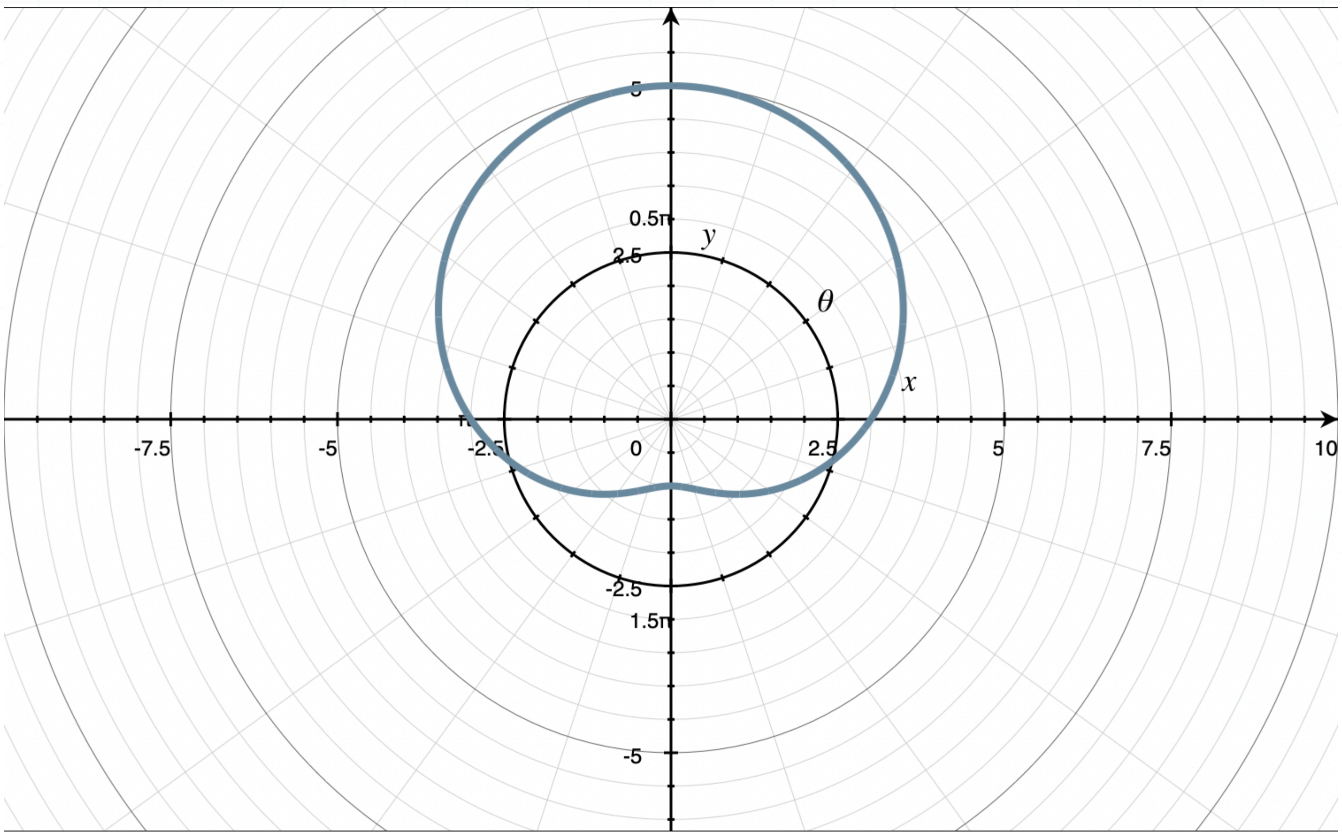
Solution:

With $a = 3$ and $b = 2$, we get $a/b = 3/2$, and $1 < 3/2 < 2$, which means the graph of this limaçon will include a small dip. And because this is a sine limaçon with a positive sign separating the terms, it'll be symmetric about the vertical axis, and will sit mostly above the horizontal axis.

Now we'll make a table for $\theta = 0$, $\pi/2$, π, $3\pi/2$, and 2π, and include the values of r that correspond to each of these θ-values.

theta	0	π/2	π	3π/2	2π
r	3	5	3	1	3

Plotting these points on the polar graph gives



■ 5. Sketch the graph of $r = 1 - 20\sin\theta$.

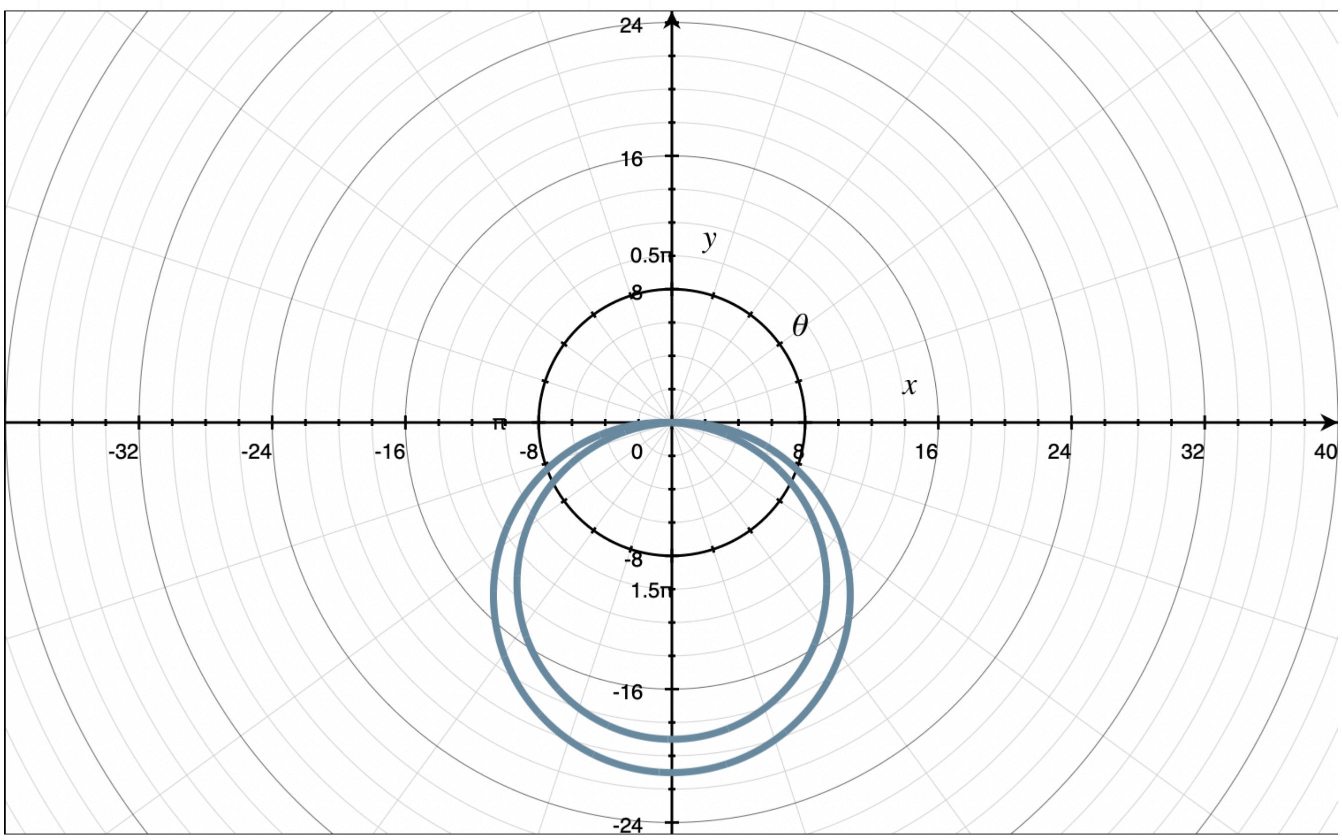
Solution:

With $a = 1$ and $b = 20$, we get $a/b = 1/20 < 1$, which means the graph of this limaçon will include a small loop. And because this is a sine limaçon with a negative sign separating the terms, it'll be symmetric about the vertical axis, and will sit mostly below the horizontal axis.

Now we'll make a table for $\theta = 0$, $\pi/2$, π, $3\pi/2$, and 2π, and include the values of r that correspond to each of these θ-values.

theta	0	π/2	π	3π/2	2π
r	1	-19	1	21	1

Plotting these points on the polar graph gives



■ 6. Sketch the graph of $r = 9 + 4\cos\theta$.

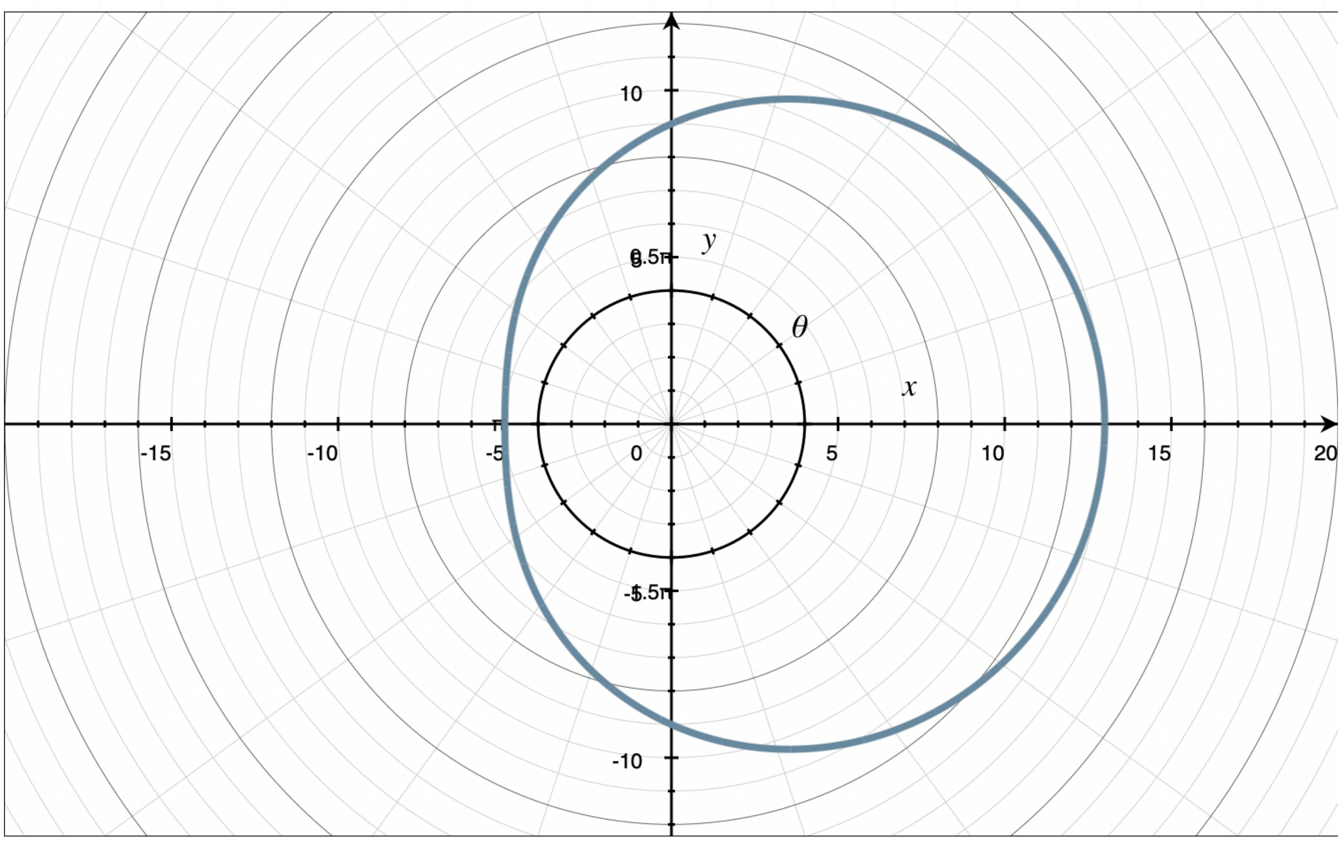
Solution:

With $a=9$ and $b=4$, we get $a/b=9/4>2$, which means the graph of this limaçon will include a small dip, but the shape of the curve is close to a perfect circle. And because this is a cosine limaçon with a positive sign separating the terms, it'll be symmetric about the horizontal axis, and will sit mostly to the right of the vertical axis.

Now we'll make a table for $\theta=0$, $\pi/2$, π, $3\pi/2$, and 2π, and include the values of r that correspond to each of these θ-values.

theta	0	π/2	π	3π/2	2π
r	13	9	5	9	13

Plotting these points on the polar graph gives

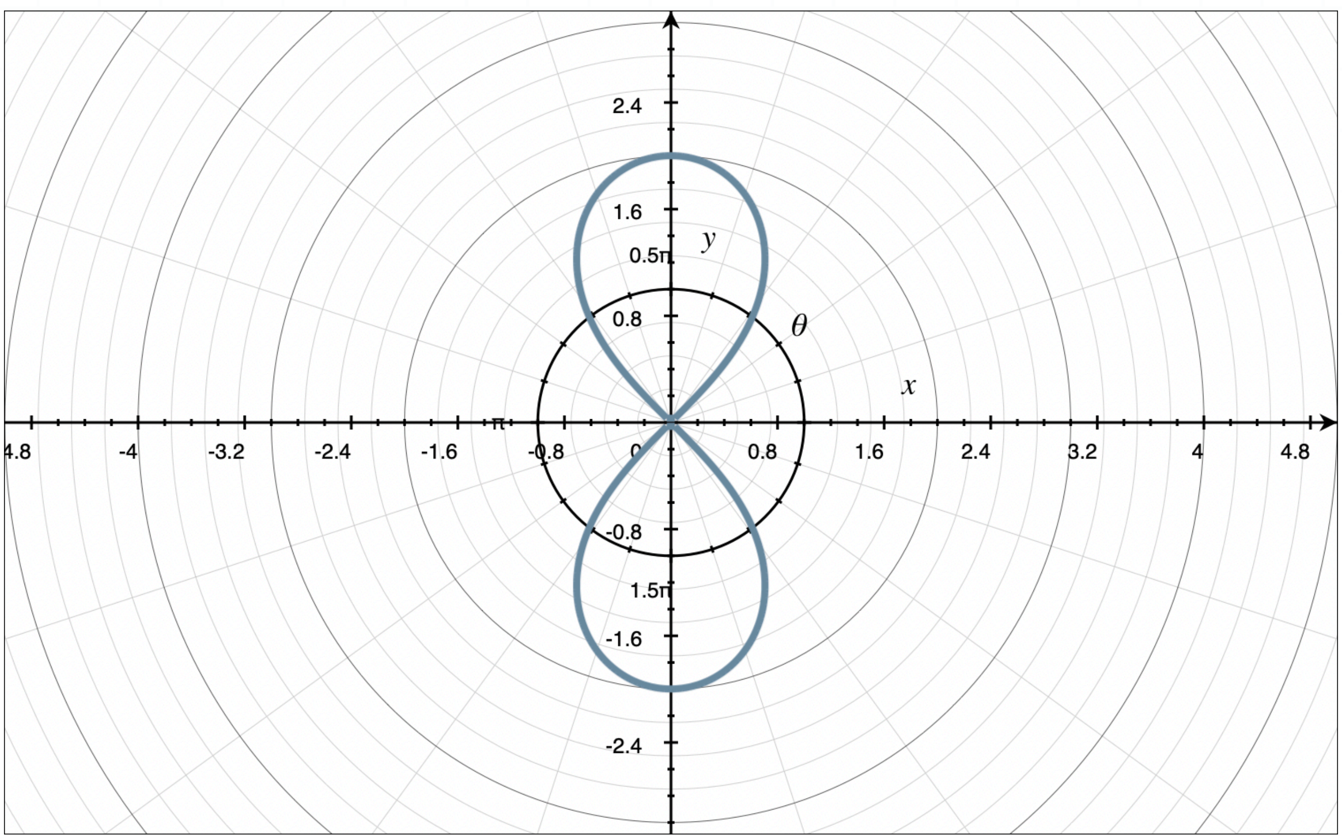


GRAPHING LEMNISCATES

■ 1. Sketch the graph of $r^2 = -4\cos(2\theta)$.

Solution:

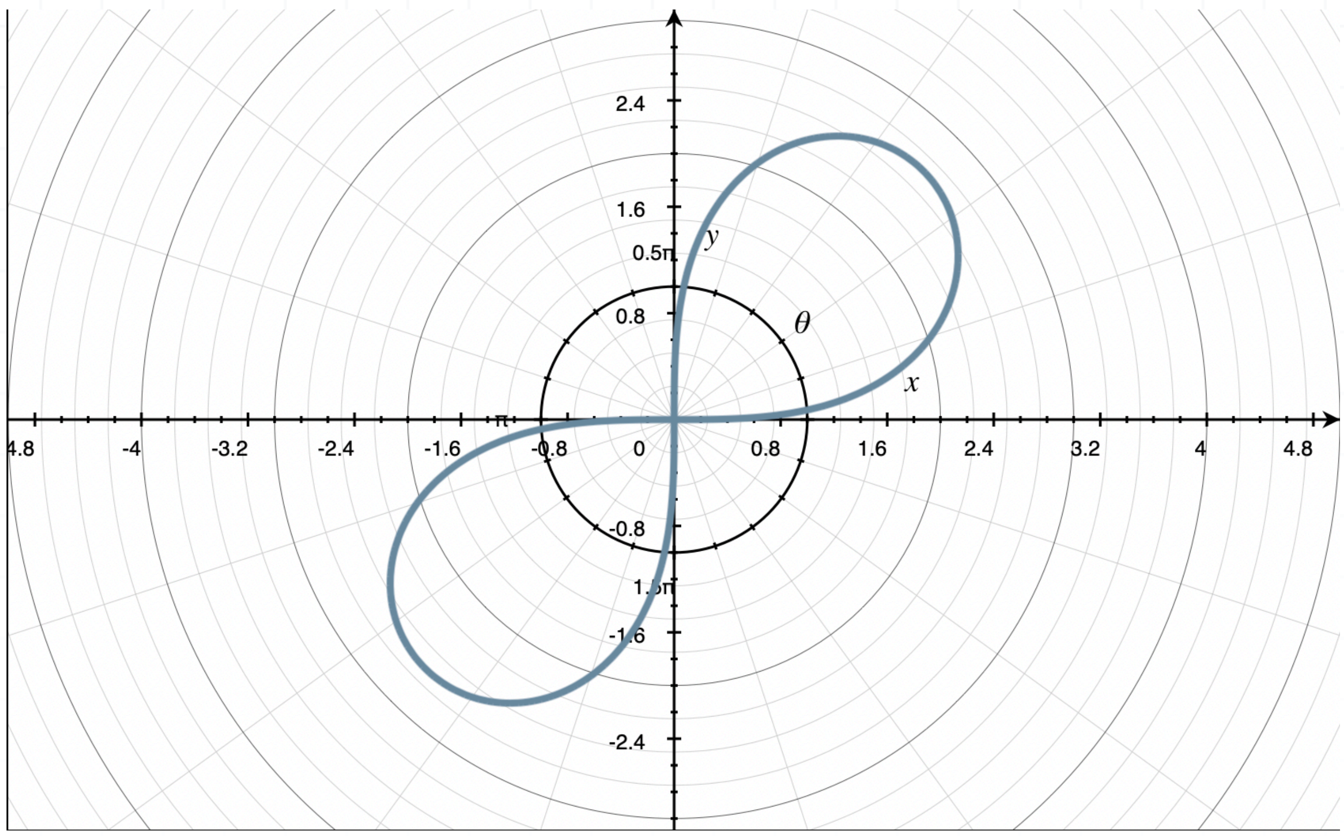
This is a negative cosine lemniscate, which means the two loops will sit along the vertical axis. And with $c = \sqrt{4} = 2$, the two loops will extend out a distance of $r = 2$. Therefore, the graph of the lemniscate will be



■ 2. Sketch the graph of $r^2 = 7\sin(2\theta)$.

Solution:

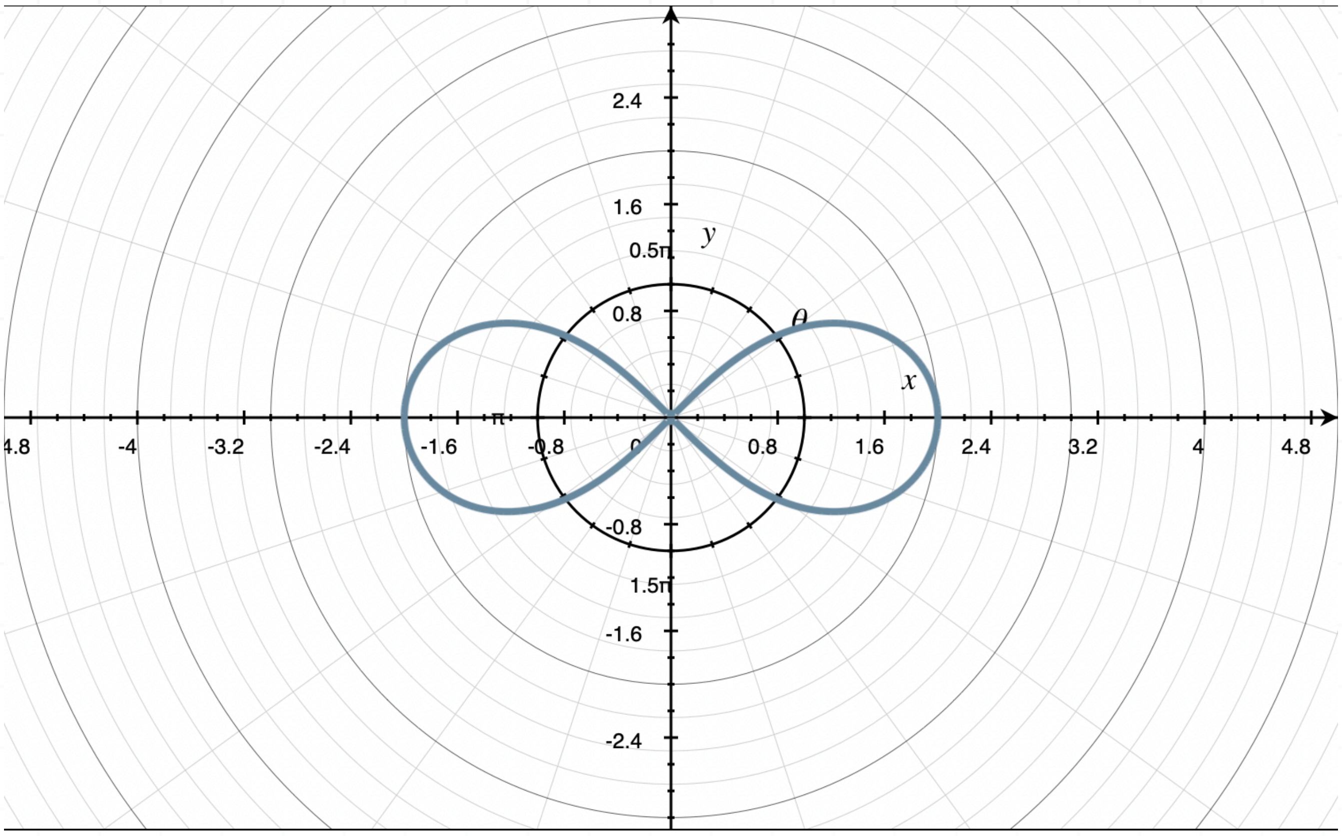
This is a positive sine lemniscate, which means the two loops will sit in the first and third quadrants. And with $c = \sqrt{7} \approx 2.65$, the loops extend out a distance of $r \approx 2.65$. Therefore, the graph of the lemniscate will be



■ 3. Sketch the graph of $r^2 = 4\cos(2\theta)$.

Solution:

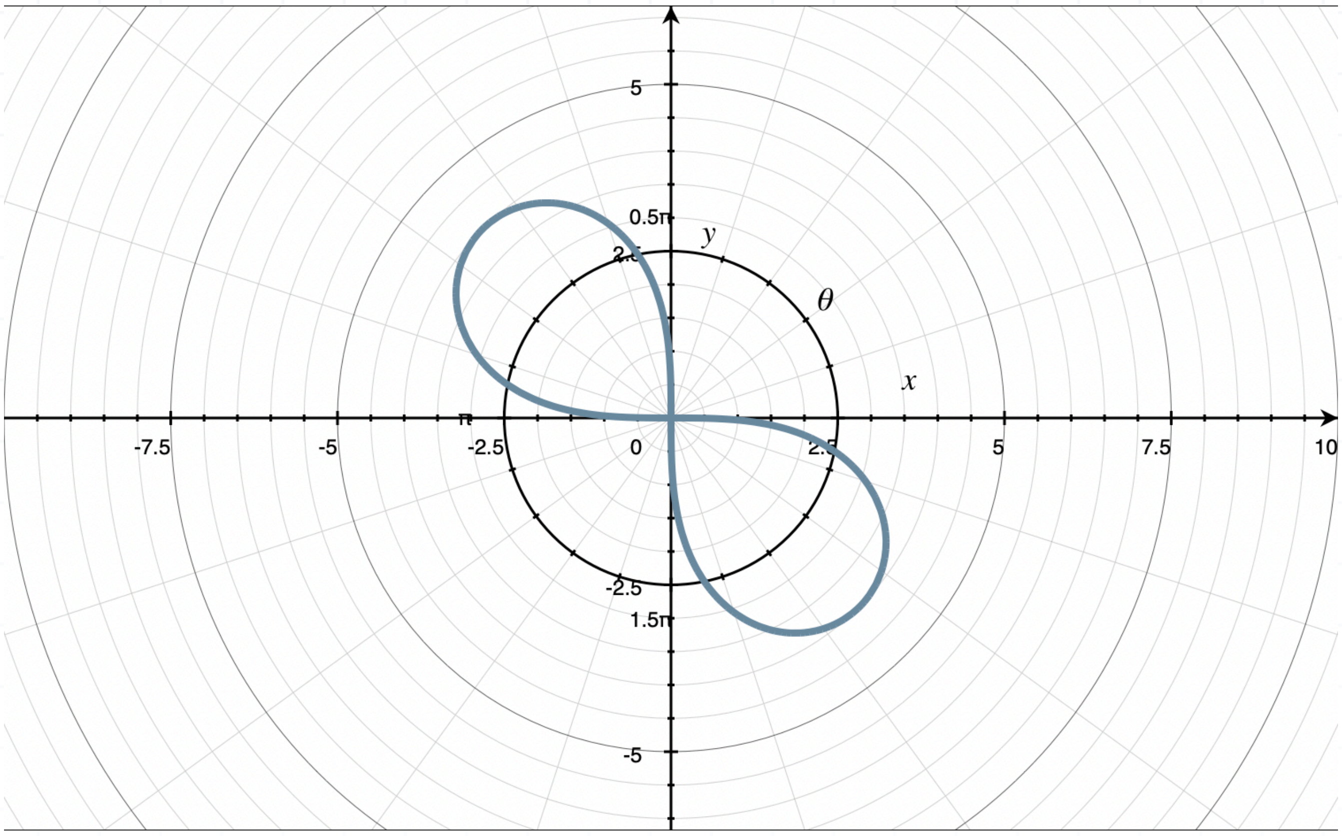
This is a positive cosine lemniscate, which means the two loops will sit along the horizontal axis. And with $c = \sqrt{4} = 2$, the two loops will extend out a distance of $r = 2$. Therefore, the graph of the lemniscate will be



■ 4. Sketch the graph of $r^2 = -16\sin(2\theta)$.

Solution:

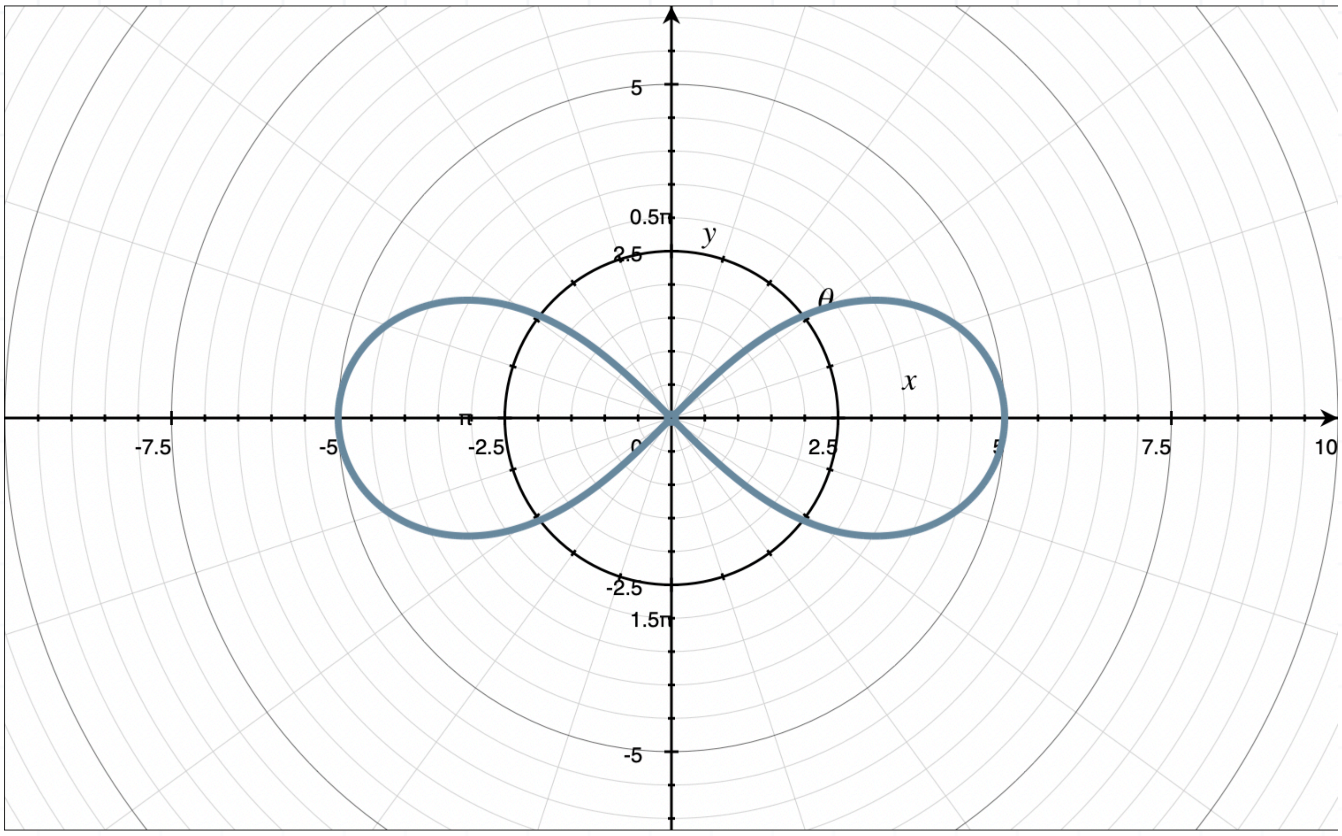
This is a negative sine lemniscate, which means the two loops will sit in the second and fourth quadrants. And with $c = \sqrt{16} = 4$, the loops extend out a distance of $r = 4$. Therefore, the graph of the lemniscate will be



■ 5. Find the equation of the lemniscate that has loops that sit along the horizontal axis and extend out to a distance of 5, then sketch its graph.

Solution:

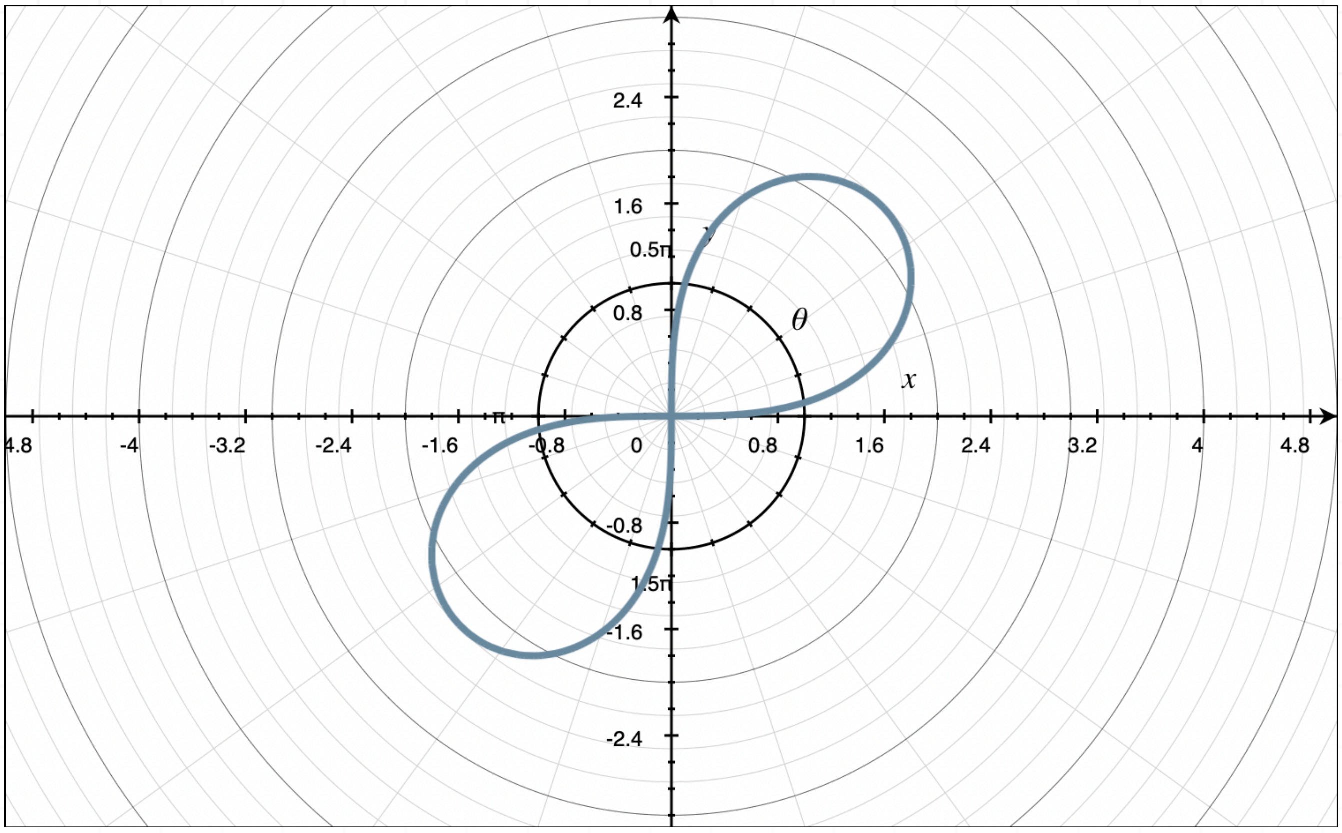
Since the loops sit along the horizontal axes, the lemniscate is a positive cosine lemniscate $r^2 = c^2\cos(2\theta)$. The loops extend out to a distance of 5, so we get $c = 5$, the equation will be $r^2 = 25\cos(2\theta)$, and the graph will be



■ 6. Find the equation of the lemniscate that has loops that sit in the first and third quadrants and extend out to a distance of $\sqrt{5}$, then sketch its graph.

Solution:

Since the loops sit in the first and third quadrants, the lemniscate is a positive sine lemniscate $r^2 = c^2\sin(2\theta)$. The loops extend out to a distance of $\sqrt{5}$, so we get $c = \sqrt{5}$, the equation will be $r^2 = 5\sin(2\theta)$, and the graph will be


2.4
1.6
0.5π
0.8
θ
x
π
-4.8
-4
-3.2
-2.4
-1.6
-0.8
0
0.8
1.6
2.4
3.2
4
4.8
-0.8
1.5π
-1.6
-2.4

INTERSECTION OF POLAR CURVES

■ 1. Find the points of intersection of $r = 4 - 4\cos\theta$ and $r = 4 + 4\sin\theta$.

Solution:

Set the curves equal to one another.

$$4 - 4\cos\theta = 4 + 4\sin\theta$$

$$-4\cos\theta = 4\sin\theta$$

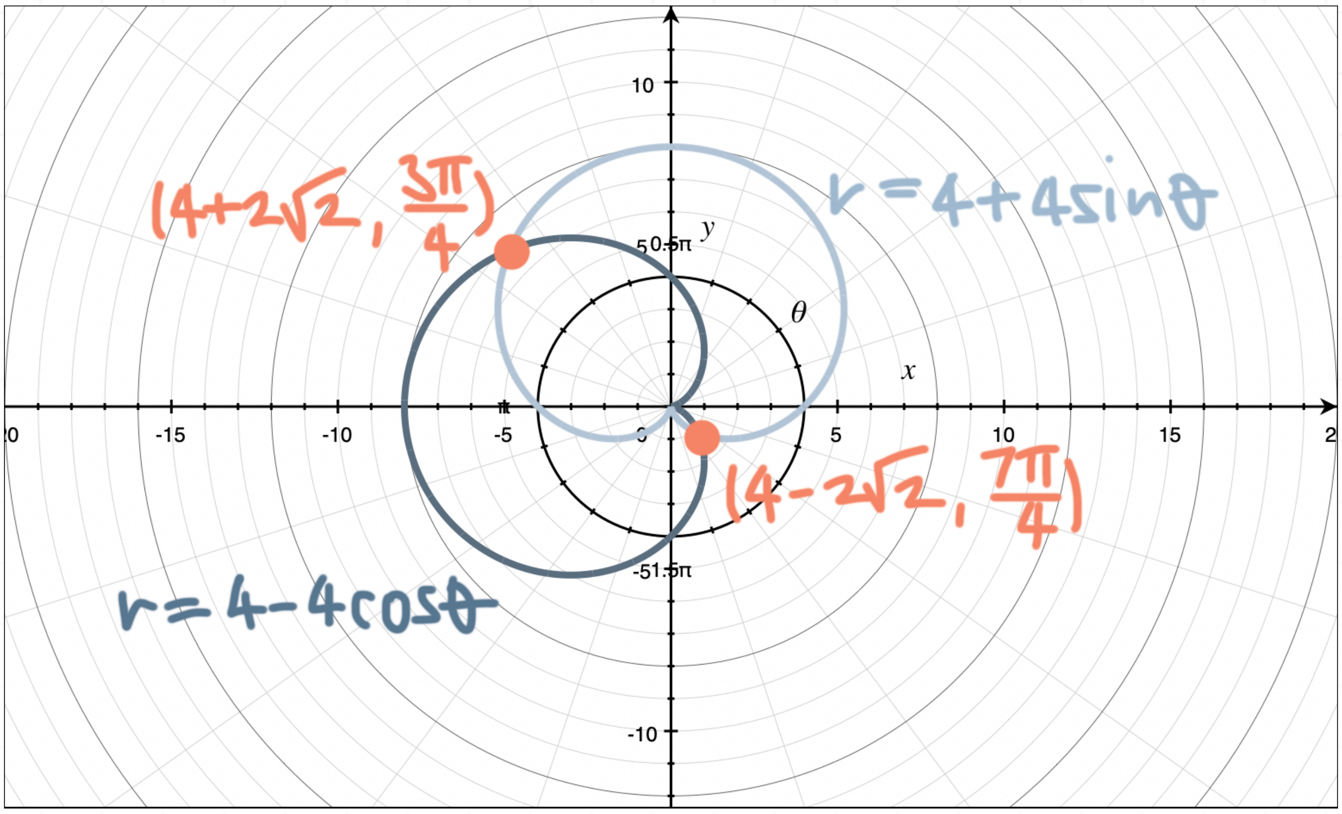
$$-\cos\theta = \sin\theta$$

From the unit circle from Trigonometry, we know that sine and cosine have equal values (except opposite signs) at $\theta = 3\pi/4$ and $\theta = 7\pi/4$. If we plug these angles back into either of the polar equations, we get $r = 4 + 2\sqrt{2}$ at $\theta = 3\pi/4$, and $r = 4 - 2\sqrt{2}$ at $\theta = 7\pi/4$. Which means the points of intersection for these curves are

$$(r, \theta) = \left(4 + 2\sqrt{2}, \frac{3\pi}{4}\right)$$

$$(r, \theta) = \left(4 - 2\sqrt{2}, \frac{7\pi}{4}\right)$$

If we sketch the curves we can see these points of intersection as well as a third point of intersection at the pole, $(0,0)$.



Therefore, the points of intersection of the curves are

$$(r,\theta)=\left(4+2\sqrt{2},\frac{3\pi}{4}\right)$$

$$(r,\theta)=\left(4-2\sqrt{2},\frac{7\pi}{4}\right)$$

$$(r,\theta)=(0,0)$$

■ 2. Find the points of intersection of $r=3-4\sin\theta$ and $r=-\sin\theta$.

Solution:

Set the curves equal to one another.

$$3-4\sin\theta=-\sin\theta$$

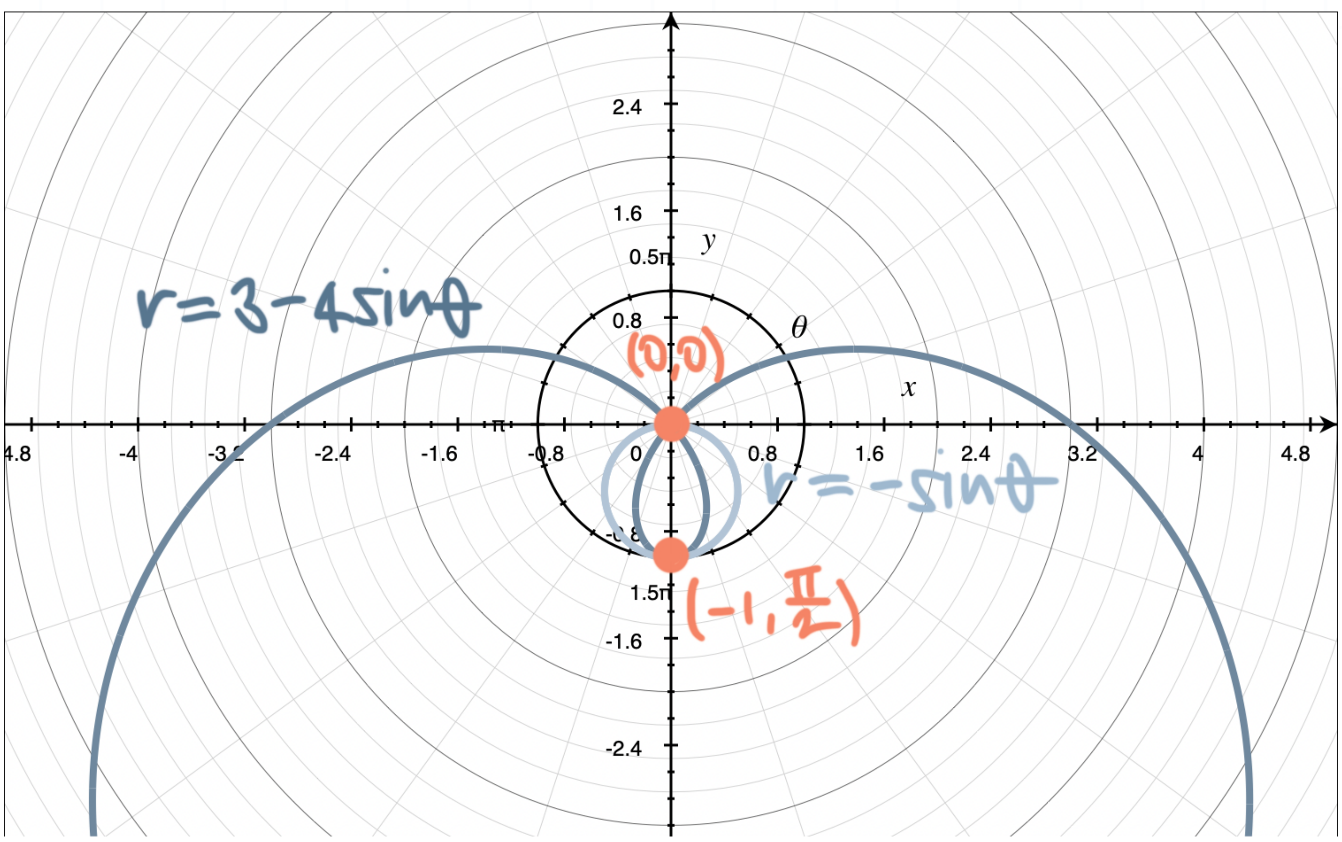
$$3\sin\theta = 3$$

$$\sin\theta = 1$$

From the unit circle from Trigonometry, we know that sine has a value of 1 at $\theta = \pi/2$ and $\theta = 3\pi/2$. If we plug these angles back into both polar equations, we get $r = -1$ from both equations at $\theta = \pi/2$. But we get $r = 7$ and $r = 1$ at $\theta = 3\pi/2$, so the curves only intersect at $\theta = \pi/2$.

$$(r,\theta) = \left(-1, \frac{\pi}{2}\right)$$

If we sketch the graph of both curves on the same set of axes, we can see this point of intersection.



What we notice when we sketch the curves is that we actually have a second “hidden” point of intersection at $(0,0)$ that we didn’t uncover when we set the curves equal to each other and solved for θ.

So, the points of intersection of the curves are

$$(r, \theta) = \left(-1, \frac{\pi}{2}\right) = \left(1, \frac{3\pi}{2}\right)$$

$$(r, \theta) = (0,0)$$

3. Find the points of intersection of $r = -3 - 3\sin\theta$ and $r = 3 - \sin\theta$.

Solution:

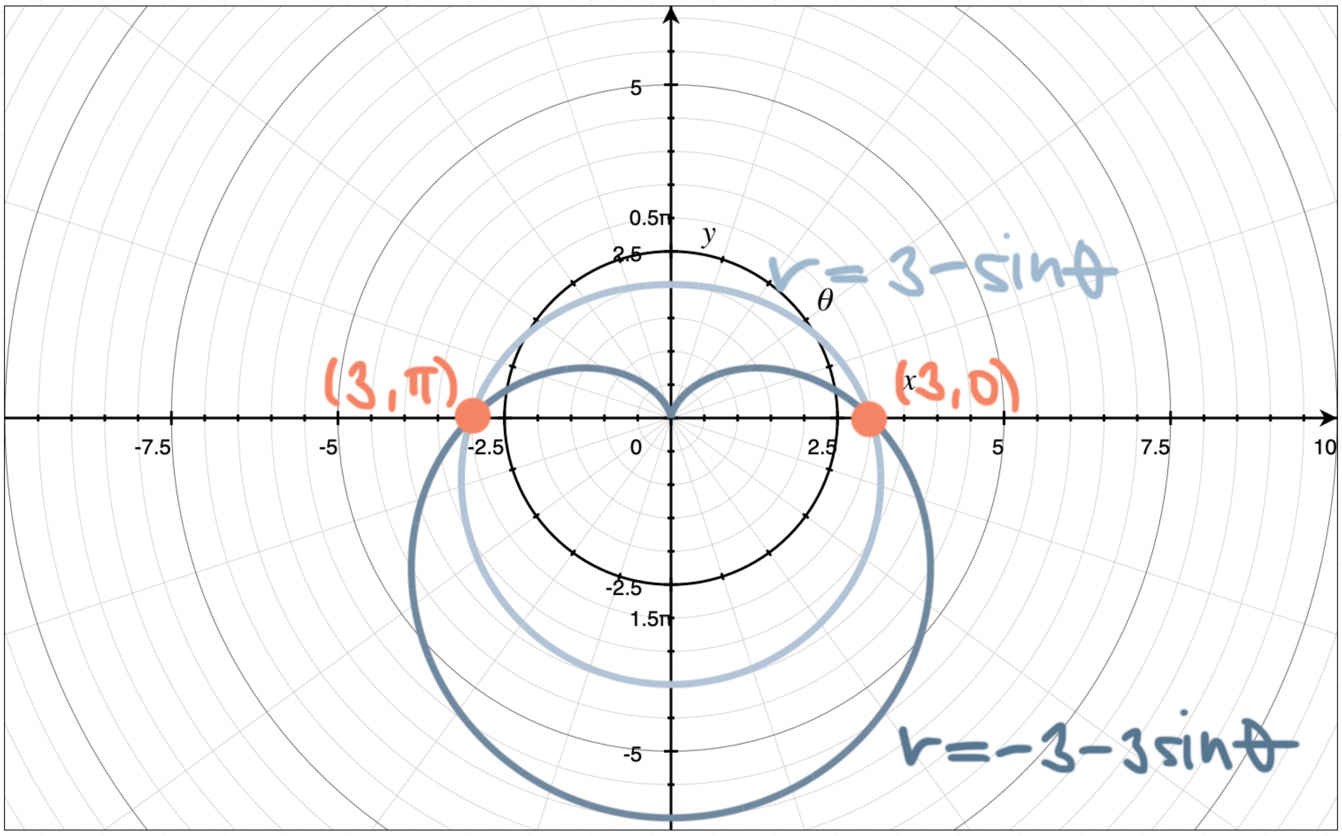
Set the curves equal to one another.

$$-3 - 3\sin\theta = 3 - \sin\theta$$

$$2\sin\theta = -6$$

$$\sin\theta = -3$$

This doesn't give us much information, but if we sketch the graph of both curves on the same set of axes, we can see two points of intersection.



Visually, we can see that the curves have two points of intersection.

$$(r,\theta) = (3,0)$$

$$(r,\theta) = (3,\pi)$$

■ 4. Find the points of intersection of $r = 1 + 2\sin\theta$ and $r = 1 + 2\cos\theta$.

Solution:

Set the curves equal to one another.

$$1 + 2\sin\theta = 1 + 2\cos\theta$$

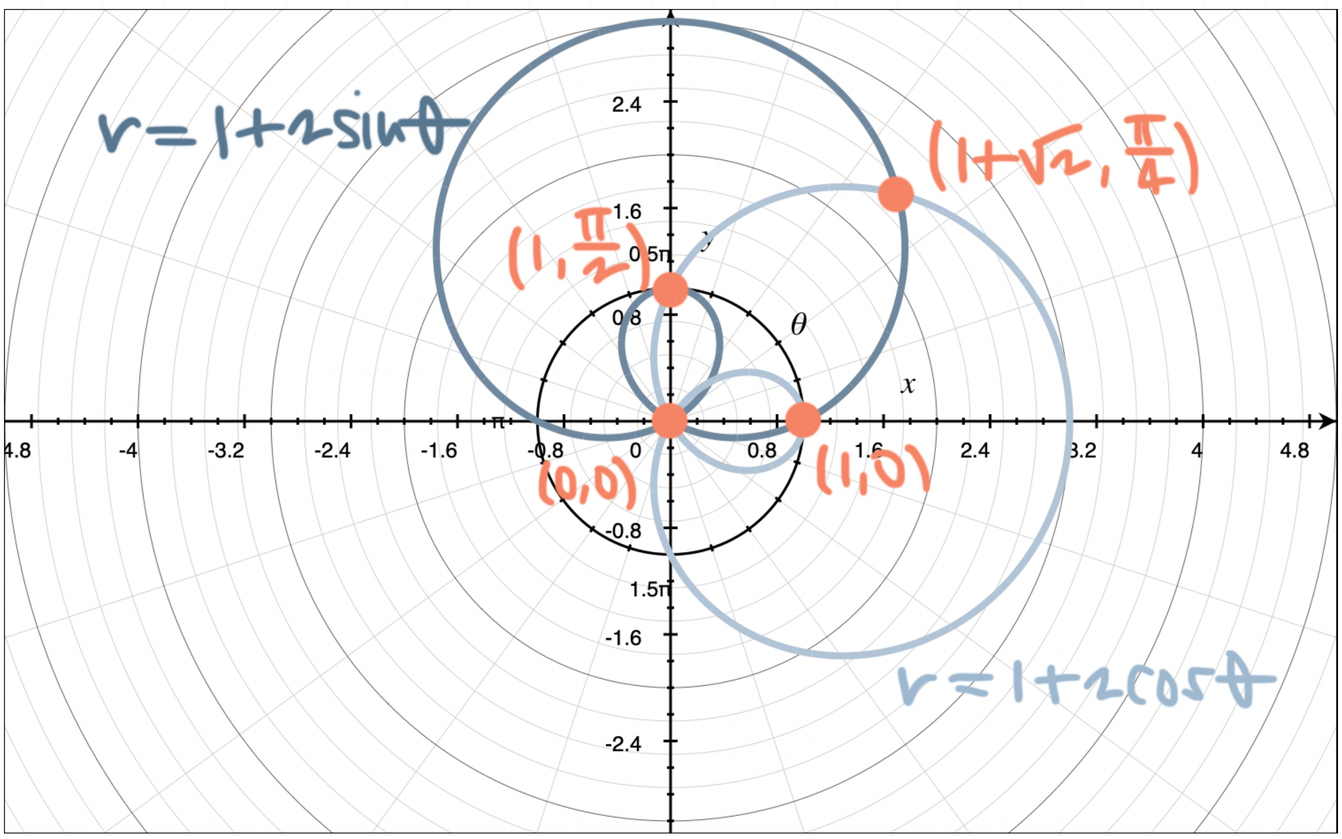
$$2\sin\theta = 2\cos\theta$$

$$\sin\theta = \cos\theta$$

From the unit circle from Trigonometry, we know that sine and cosine have equal values at $\theta = \pi/4$ and $\theta = 3\pi/4$. If we plug these angles back into both polar equations, we get $r = 1 + \sqrt{2}$ from both equation at $\theta = \pi/4$, and $r = 1 + \sqrt{2}$ and $r = 1 - \sqrt{2}$ at $\theta = 3\pi/4$. Which means the curves intersect at

$$(r, \theta) = \left(1 + \sqrt{2}, \frac{\pi}{4}\right)$$

If we sketch the graph of both curves on the same set of axes, we can see this points of intersection.



We can also see points of intersection at $(1,0)$, $(1,\pi/2)$, and at the pole $(0,0)$. Therefore, the points of intersection of the curves are

$$(r, \theta) = \left(1 + \sqrt{2}, \frac{\pi}{4}\right)$$

$$(r, \theta) = (1,0)$$

$$(r, \theta) = \left(1, \frac{\pi}{2}\right)$$

$$(r, \theta) = (0,0)$$

■ 5. How many points of intersection exist for the graphs of the polar equations? Hint: Think just about the shape of each curve, without trying to solve algebraically for the points of intersection.

$$r = 5\sin(3\theta)$$

$$r = 4$$

Solution:

The graph of $r = 5\sin(3\theta)$ is a three-petal rose, and the graph of $r = 4$ is the circle centered at the pole with radius 4.

Given these shapes, we know the curves will have either

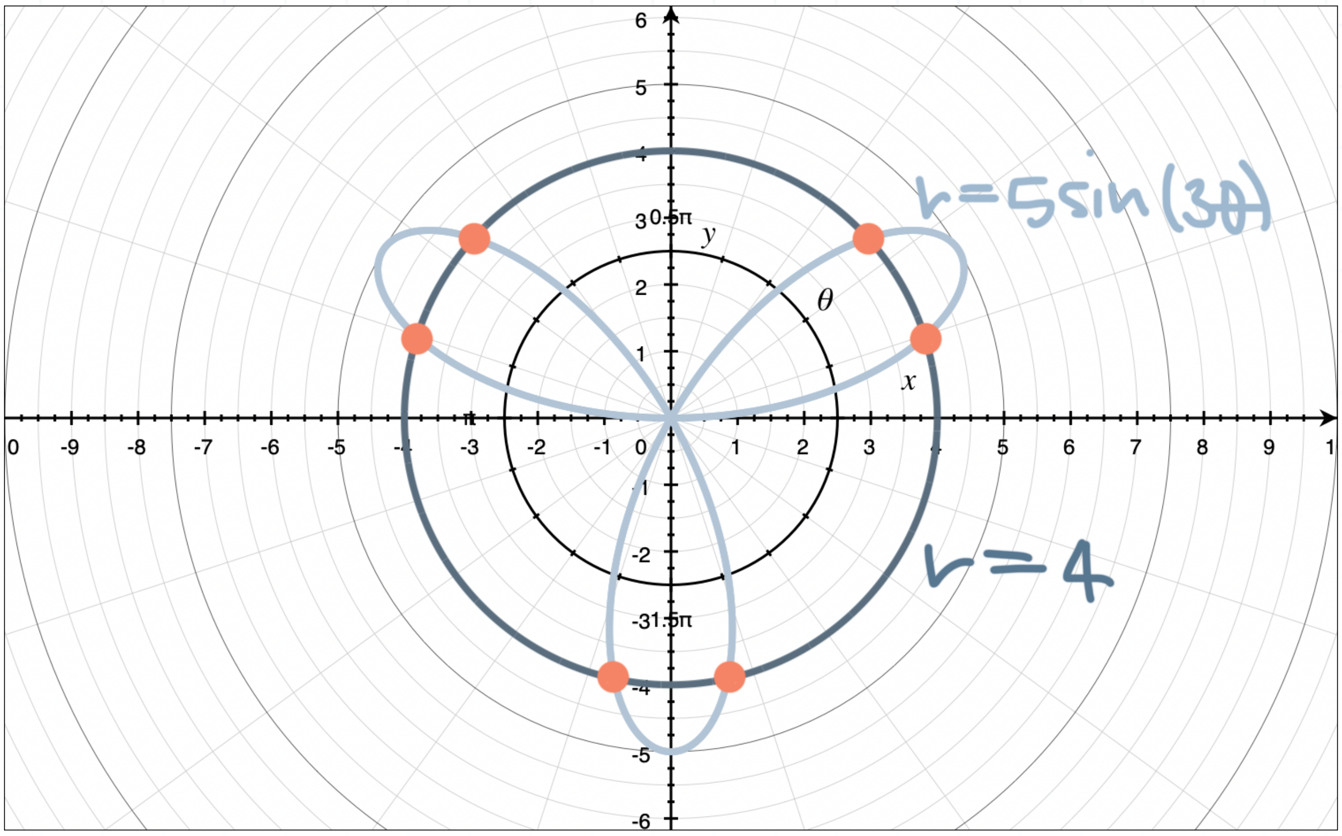
1) zero points of intersection if the rose petals remain inside the circle and therefore never touch it, or

2) three points of intersection if the tips of the rose petals sit on the circle exactly, or

3) six points of intersection if the rose petals extend past the circle, therefore creating two points of intersection per petal.

The tips of the petals of a rose given in the form $r = c\sin(n\theta)$ will extend out to a distance of $r = |c|$ away from the pole, which means these petals extend to a distance of $r = 5$, which is past the circle with radius 4.

Therefore, there are six points of intersection between the two curves.



■ 6. How many points of intersection exist for the graphs of the polar equations? Hint: Think just about the shape of each curve, without trying to solve algebraically for the points of intersection.

$$r = 2\sin(2\theta)$$

$$r^2 = 9\sin(2\theta)$$

Solution:

The graph of $r = 2\sin(2\theta)$ is a rose in the form $r = c\sin(n\theta)$, with petals that extend to a distance of $r = |2| = 2$ away from the pole. Because n is even, there are $|2n| = |2(2)| = |4| = 4$ petals.

The graph of $r^2 = 9\sin(2\theta)$ is a sine lemniscate in the form $r^2 = c^2\cos(2\theta)$, with loops in the first and third quadrants that extend to a distance of $r = \sqrt{9} = 3$ away from the pole.

Given the shapes of these curves, we know that at least the pole is a point of intersection.

To look for other points of intersection, let's investigate the position of the rose petals. Setting the argument of the rose's equation, 2θ, equal to $\pi/2$, we get

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}$$

At increments of $\theta = \pi/4$, the rose's equation gives

$$\theta = 0 \qquad r = 2\sin(2(0)) = 2(0) = 0$$

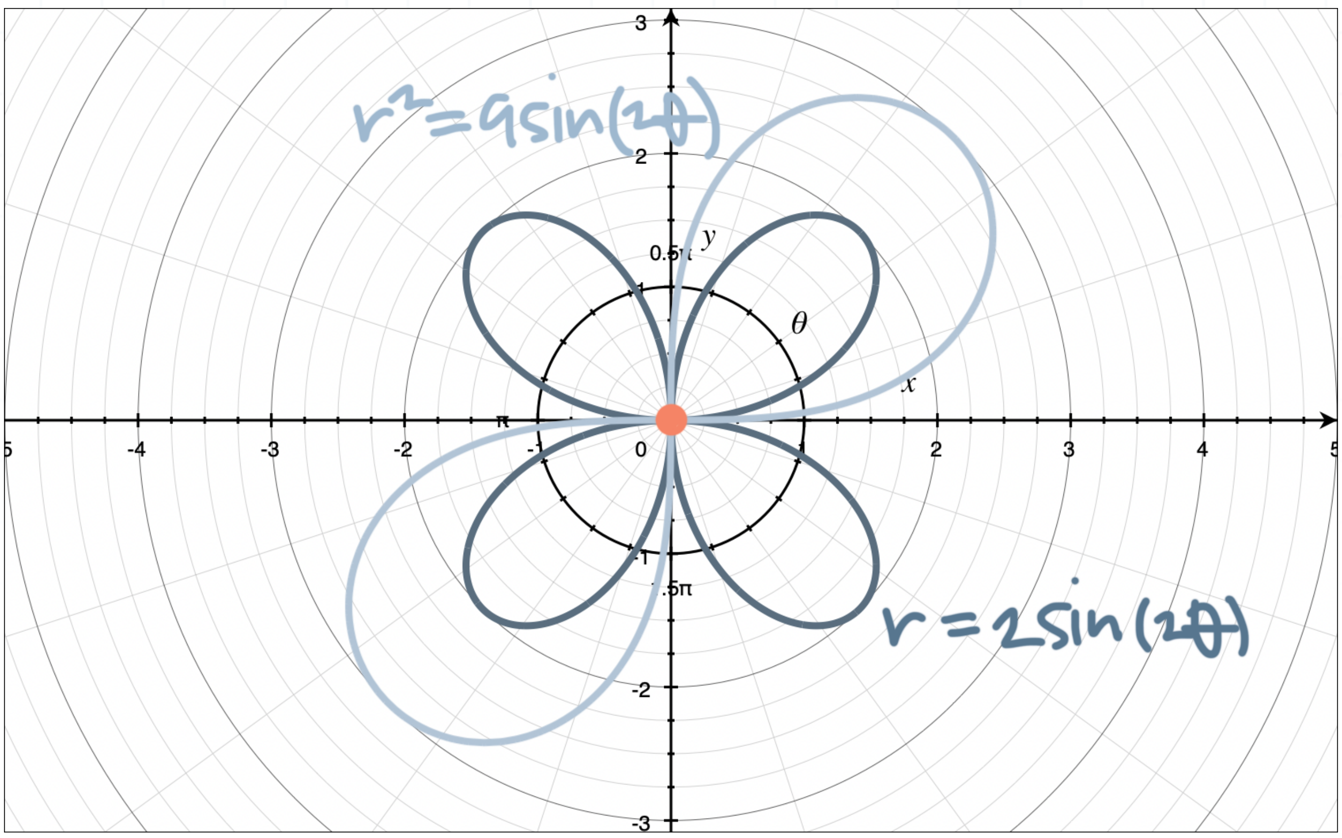
$$\theta = \pi/4 \qquad r = 2\sin(2(\pi/4)) = 2\sin(\pi/2) = 2(1) = 2$$

$$\theta = \pi/2 \qquad r = 2\sin(2(\pi/2)) = 2\sin(\pi) = 2(0) = 0$$

This shows us that the first rose petal begins at $(0,0)$, extends out to its tip at $(2,\pi/4)$, and ends by looping back to the pole at $(0,\pi/2)$.

The rose petal is symmetric around $\theta = \pi/4$, and so is the loop of the sine lemniscate in the first quadrant. The rose petal only extends out to $r = 2$, while the lemniscate loop extends out to $r = 3$. That means the rose petal is completely contained within the lemniscate loop. This will be true in the third quadrant as well.

Therefore, the pole is the only point of intersection between the two curves.



COMPLEX NUMBERS

1. Simplify the imaginary number.

$$i^{437}$$

Solution:

We need to look for the largest number less than or equal to 437 that's divisible by 4. 437 isn't divisible by 4, so we try 436. 436 is the first number we come to that's divisible by 4, so we separate the exponent.

$$i^{437}$$

$$i^{436+1}$$

$$i^{436}i^1$$

Rewrite i^{436} as a power of 4.

$$(i^4)^{109}i^1$$

We know that i^4 is always 1, so

$$(1)^{109}i^1$$

$$1i^1$$

$$i^1$$

i

2. Simplify the imaginary number.

$i^{2,314}$

Solution:

We need to look for the largest number less than or equal to $2{,}314$ that's divisible by 4. $2{,}314$ isn't divisible by 4, so we try $2{,}313$, then $2{,}312$. $2{,}312$ is the first number we come to that's divisible by 4, so we separate the exponent.

$$i^{2,314}$$

$$i^{2,312+2}$$

$$i^{2,312}i^2$$

Rewrite $i^{2,312}$ as a power of 4.

$$(i^4)^{578}i^2$$

We know that i^4 is always 1, so

$$(1)^{578}i^2$$

$$1i^2$$

$$i^2$$

-1

■ 3. Name the real and imaginary parts of the complex number.

$$z = -5 + 17i$$

Solution:

For a complex number in the form $z = a + bi$, a is always the real part and b is always the imaginary part. If b is negative, you have to remember to include the negative sign when you name the imaginary part. So in the complex number $z = -5 + 17i$, -5 is the real part, and 17 is the imaginary part.

■ 4. Name the real and imaginary parts of the complex number.

$$z = \sqrt{7} - 4\pi i$$

Solution:

For a complex number in the form $z = a + bi$, a is always the real part and b is always the imaginary part. If b is negative, you have to remember to include the negative sign when you name the imaginary part. So in the complex number $z = \sqrt{7} - 4\pi i$, $\sqrt{7}$ is the real part, and -4π is the imaginary part.

5. How can the numbers be classified?

$$z = -3 + 9i$$

$$z = 0 - 15i$$

$$z = 6 + 0i$$

Solution:

For the number $z = -3 + 9i$, both the real part and the imaginary part are non-zero. So, this is a complex number.

For the number $z = 0 - 15i$, the real part is 0 and the imaginary part is non-zero. Because the real-number part of the complex number is 0, we end up with

$$z = 0 - 15i$$

$$z = -15i$$

This is a pure imaginary number. Because every pure imaginary number is also a complex number, we can call this a complex number as well.

For the number $z = 6 + 0i$, the real part is non-zero and the imaginary part is 0. Because the imaginary part of the complex number is 0, we end up with

$$z = 6 + 0i$$

$$z = 6$$

This is a real number. Because every real number is also a complex number, we can call this a complex number as well.

■ 6. How can the numbers be classified?

$$z = 0 - \pi i$$

$$z = -\sqrt{5} + 0i$$

$$z = -11 + \frac{2}{3}i$$

Solution:

For the number $z = 0 - \pi i$, the real part is 0 and the imaginary part is non-zero. Because the real-number part of the complex number is 0, we end up with

$$z = 0 - \pi i$$

$$z = -\pi i$$

This is a pure imaginary number. Because every pure imaginary number is also a complex number, we can call this a complex number as well.

For the number $z = -\sqrt{5} + 0i$, the real part is non-zero and the imaginary part is 0. Because the imaginary part of a complex number is 0, we end up with

$$z = -\sqrt{5} + 0i$$

$$z = -\sqrt{5}$$

This is a real number. Because every real number is also a complex number, we can call this a complex number as well.

For the number $z = -11 + (2/3)i$, both the real part and the imaginary part are non-zero. So, this is a complex number.

COMPLEX NUMBER OPERATIONS

■ 1. Find the sum and difference of the complex numbers.

$$\frac{7}{5}-\frac{2}{3}i$$

$$\frac{7}{2}-\frac{8}{3}i$$

Solution:

The sum of the complex numbers is

$$\left(\frac{7}{5}-\frac{2}{3}i\right)+\left(\frac{7}{2}-\frac{8}{3}i\right)$$

$$\left(\frac{7}{5}+\frac{7}{2}\right)+\left(-\frac{2}{3}i-\frac{8}{3}i\right)$$

$$\left(\frac{14}{10}+\frac{35}{10}\right)+\left(-\frac{10}{3}i\right)$$

$$\frac{49}{10}-\frac{10}{3}i$$

The difference of the complex numbers is

$$\left(\frac{7}{5}-\frac{2}{3}i\right)-\left(\frac{7}{2}-\frac{8}{3}i\right)$$

$$\frac{7}{5}-\frac{2}{3}i-\frac{7}{2}+\frac{8}{3}i$$

$$\frac{7}{5}-\frac{7}{2}-\frac{2}{3}i+\frac{8}{3}i$$

$$\frac{14}{10}-\frac{35}{10}-\frac{2}{3}i+\frac{8}{3}i$$

$$-\frac{21}{10}+\frac{6}{3}i$$

$$-\frac{21}{10}+2i$$

■ 2. Find the product of the complex numbers.

$$-7i$$

$$-5+9i$$

Solution:

Use the distributive property to find the product of the complex numbers.

$$-7i(-5+9i)$$

$$(-7i)(-5)+(-7i)(9i)$$

$$35i-63i^2$$

Using $i^2 = -1$ in the last term, we get

$$35i - 63(-1)$$

$$35i + 63$$

$$63 + 35i$$

■ 3. Find the product of the complex numbers.

$$5 - 2i$$

$$6 - 11i$$

Solution:

Use FOIL to find the product of the complex numbers.

$$(5 - 2i)(6 - 11i)$$

$$(5)(6) + (5)(-11i) + (-2i)(6) + (-2i)(-11i)$$

$$30 - 55i - 12i + 22i^2$$

$$30 - 67i + 22i^2$$

Using $i^2 = -1$ in the last term, we get

$$30 - 67i + 22(-1)$$

$$30 - 67i - 22$$

$$8 - 67i$$

■ 4. Divide the complex number $-4 + 15i$ by the imaginary number $5i$.

Solution:

Set up the division.

$$\frac{-4 + 15i}{5i}$$

$$\frac{-4}{5i} + \frac{15i}{5i}$$

$$-\frac{4}{5}i^{-1} + 3$$

We know that i^{-1} is equal to $-i$.

$$-\frac{4}{5}(-i) + 3$$

$$\frac{4}{5}i + 3$$

$$3 + \frac{4}{5}i$$

■ 5. Find the complex conjugate of each complex number.

$9 - 9i$

$-3 + 13i$

$11 - 22i$

Solution:

For each of these, we keep the real part (9, -3, or 11) and change the sign of the imaginary part (from -9 to 9, from 13 to -13, or from -22 to 22). So the complex conjugates are:

The complex conjugate of $9 - 9i$ is $9 + 9i$.

The complex conjugate of $-3 + 13i$ is $-3 - 13i$.

The complex conjugate of $11 - 22i$ is $11 + 22i$.

■ 6. Express the fraction in the form $a + bi$ where a and b are real numbers.

$$\frac{-3 + 7i}{4 - 5i}$$

Solution:

Multiply by the conjugate of the denominator.

$$\left(\frac{-3+7i}{4-5i}\right)\left(\frac{4+5i}{4+5i}\right)$$

$$\frac{(-3+7i)(4+5i)}{(4-5i)(4+5i)}$$

Use FOIL to expand the numerator and denominator.

$$\frac{-12-15i+28i+35i^2}{16+20i-20i-25i^2}$$

$$\frac{-12+13i+35i^2}{16-25i^2}$$

Using $i^2 = -1$ gives

$$\frac{-12+13i+35(-1)}{16-25(-1)}$$

$$\frac{-12+13i-35}{16+25}$$

$$\frac{-47+13i}{41}$$

$$-\frac{47}{41}+\frac{13}{41}i$$

GRAPHING COMPLEX NUMBERS

■ 1. Graph $-3+5i$, $2-4i$, and 5 in the complex plane.

Solution:

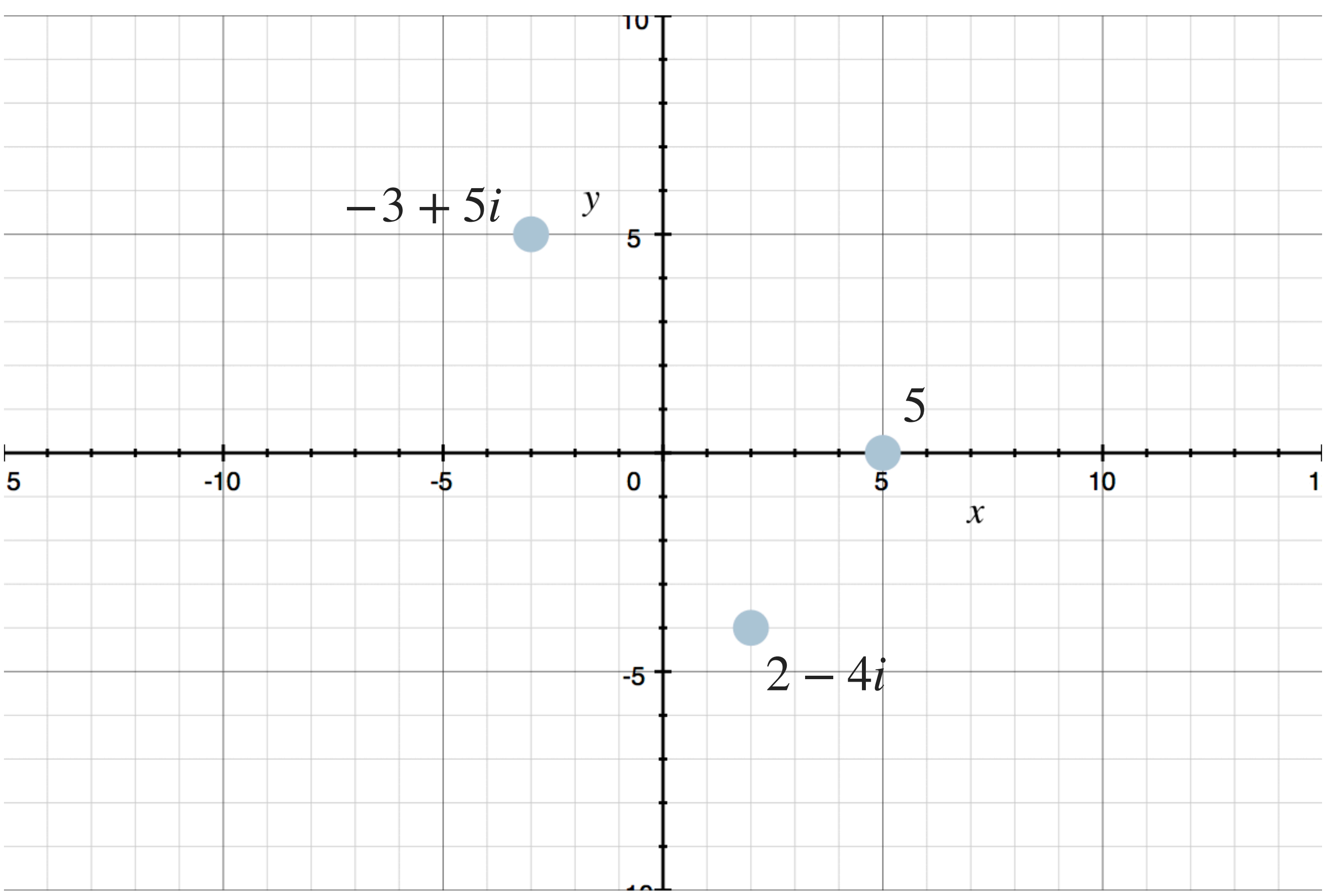
Let's break down each complex number.

$-3+5i$ has real part $a=-3$ and imaginary part $b=5$.

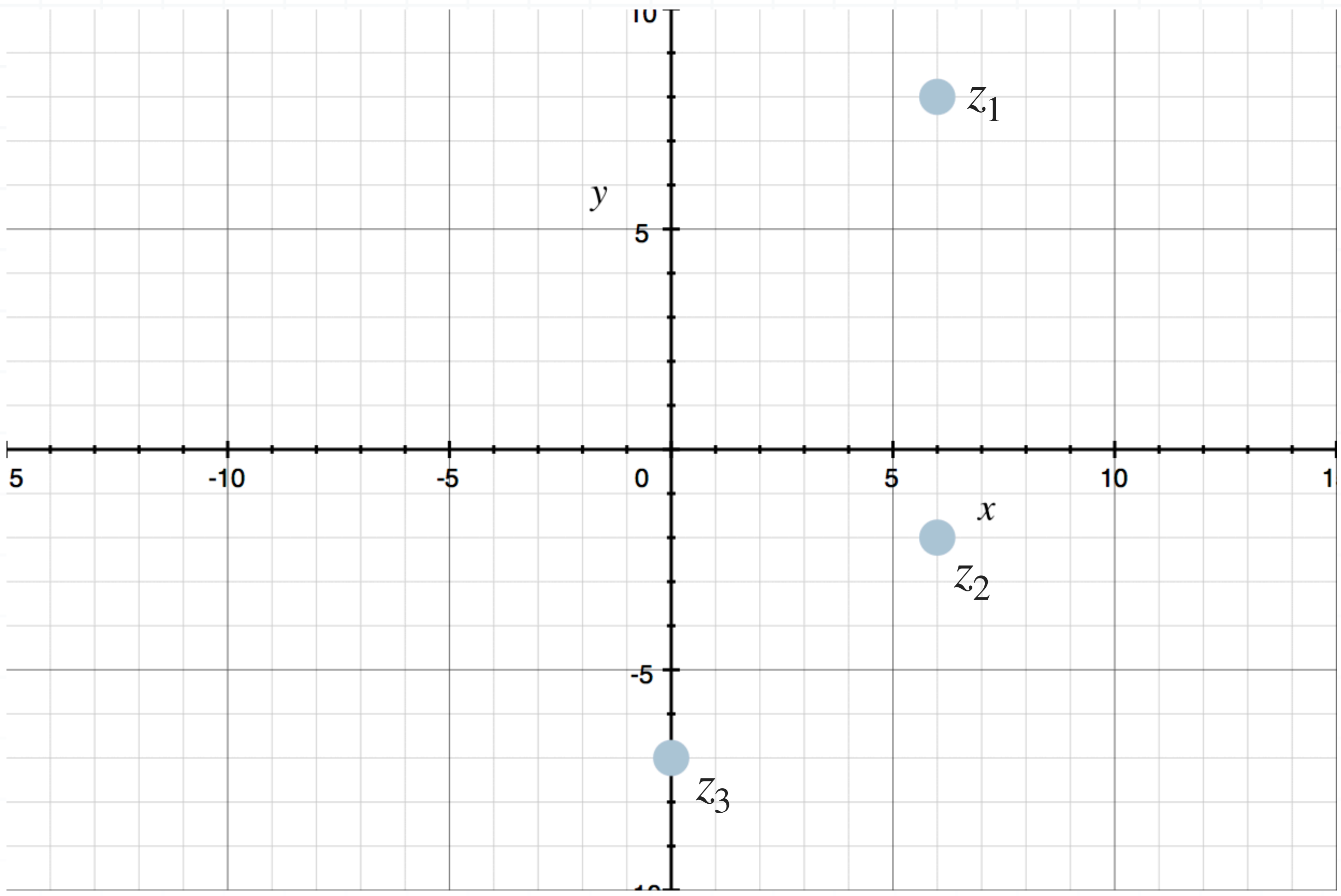
$2-4i$ has real part $a=2$ and imaginary part $b=-4$.

5 has real part $a=5$ and imaginary part $b=0$.

Now we can graph all of them together.



■ 2. Which three complex numbers are represented in the graph?



Solution:

The point z_1 is 6 units to the right of the vertical axis and 8 units above the horizontal axis, so it's the complex number $6 + 8i$.

The point z_2 is 6 units to the right of the vertical axis and 2 units below the horizontal axis, so it's the complex number $6 - 2i$.

The point z_3 is 0 units to the left or right of the vertical axis and 7 units below the horizontal axis, so it's the complex number $-7i$.

■ 3. Graph the sum of the complex numbers $5 - 4i$ and $-1 + 10i$.

Solution:

First, find the sum.

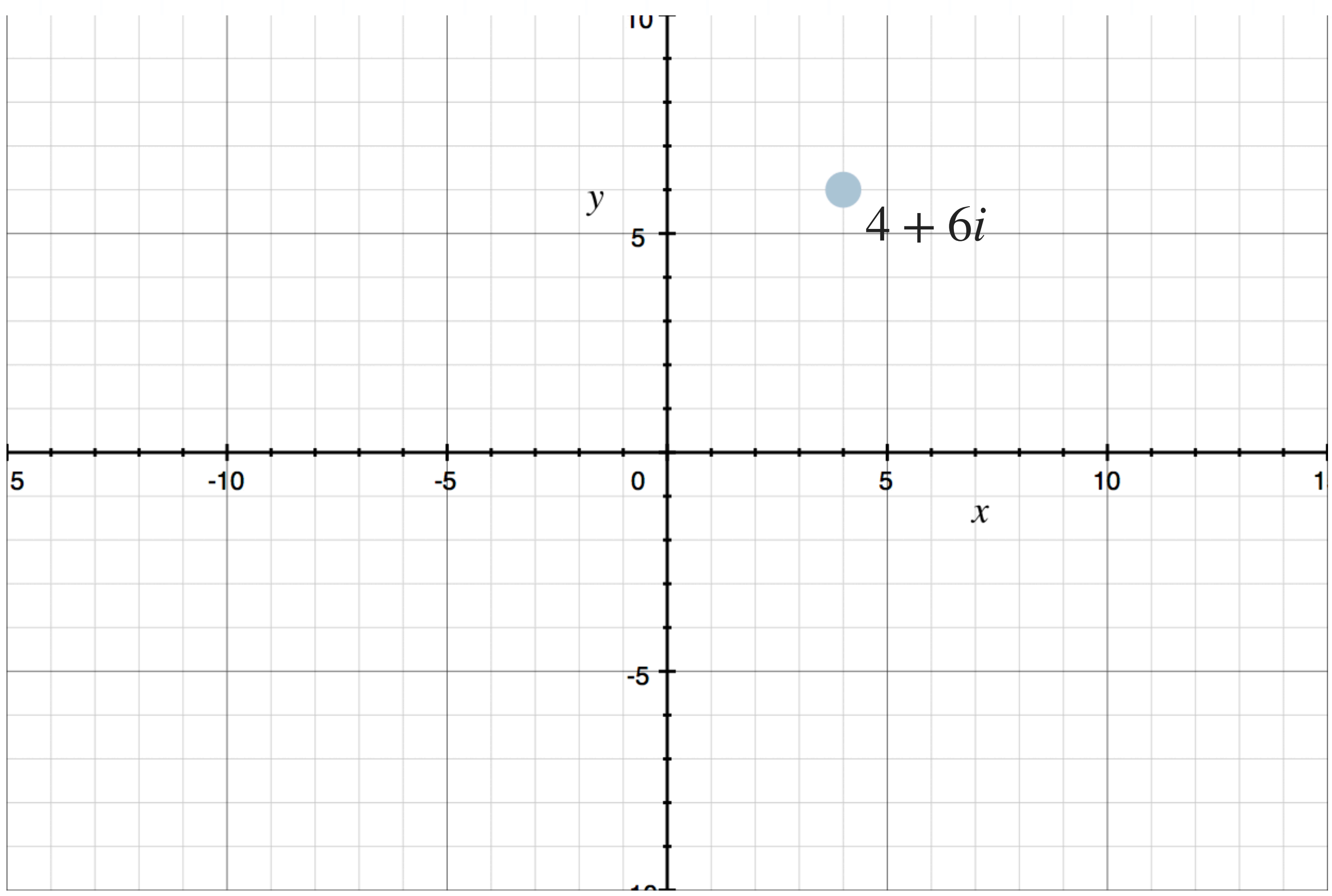
$$(5 - 4i) + (-1 + 10i)$$

$$(5 + (-1)) + (-4 + 10)i$$

$$(5 - 1) + (-4 + 10)i$$

$$4 + 6i$$

Now graph the complex number $4 + 6i$, which has a real part 4 and an imaginary part 6.



■ 4. Graph the difference of the complex numbers $8-7i$ and $13-4i$.

Solution:

First, find the difference.

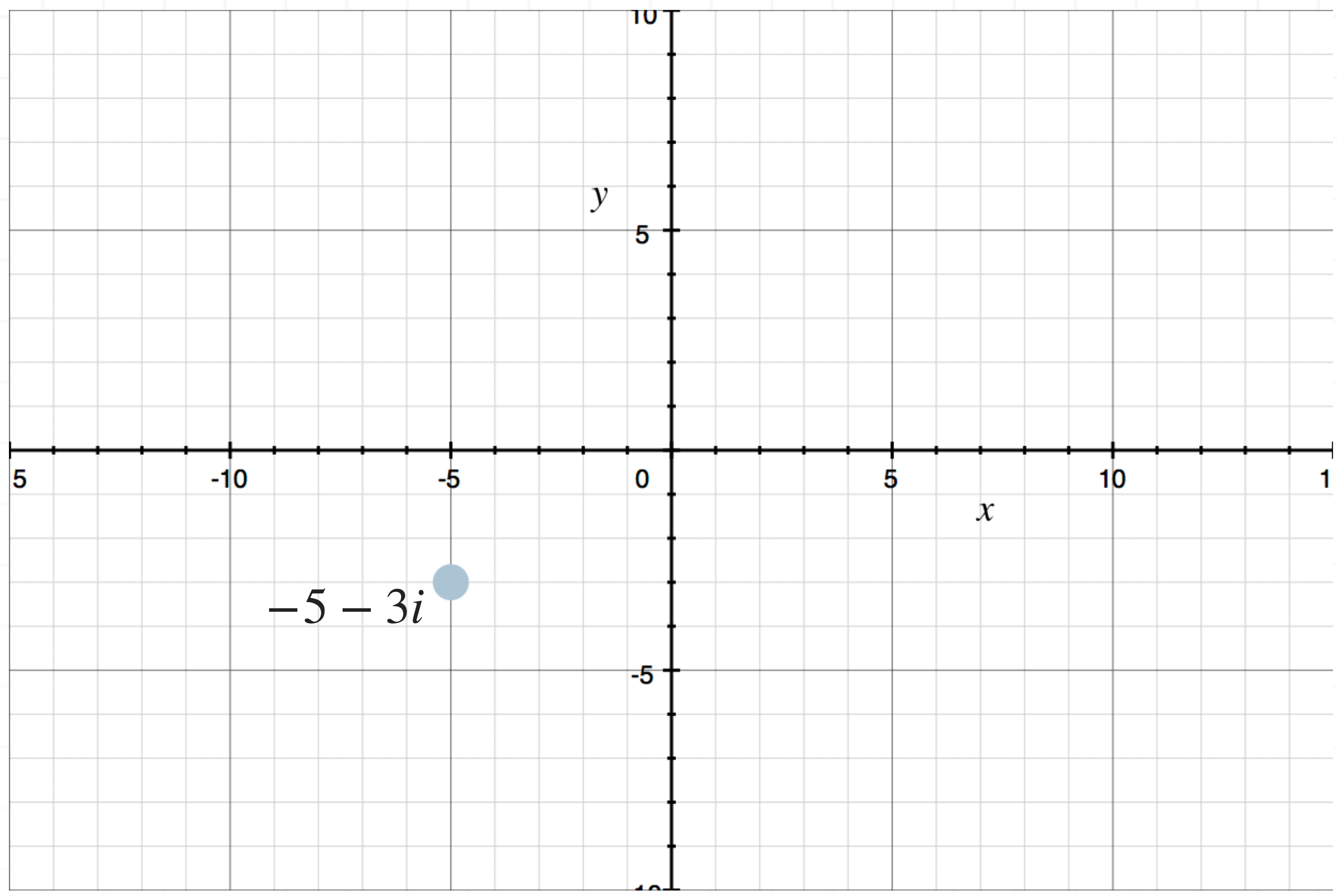
$$(8-7i)-(13-4i)$$

$$(8-13)+(-7-(-4))i$$

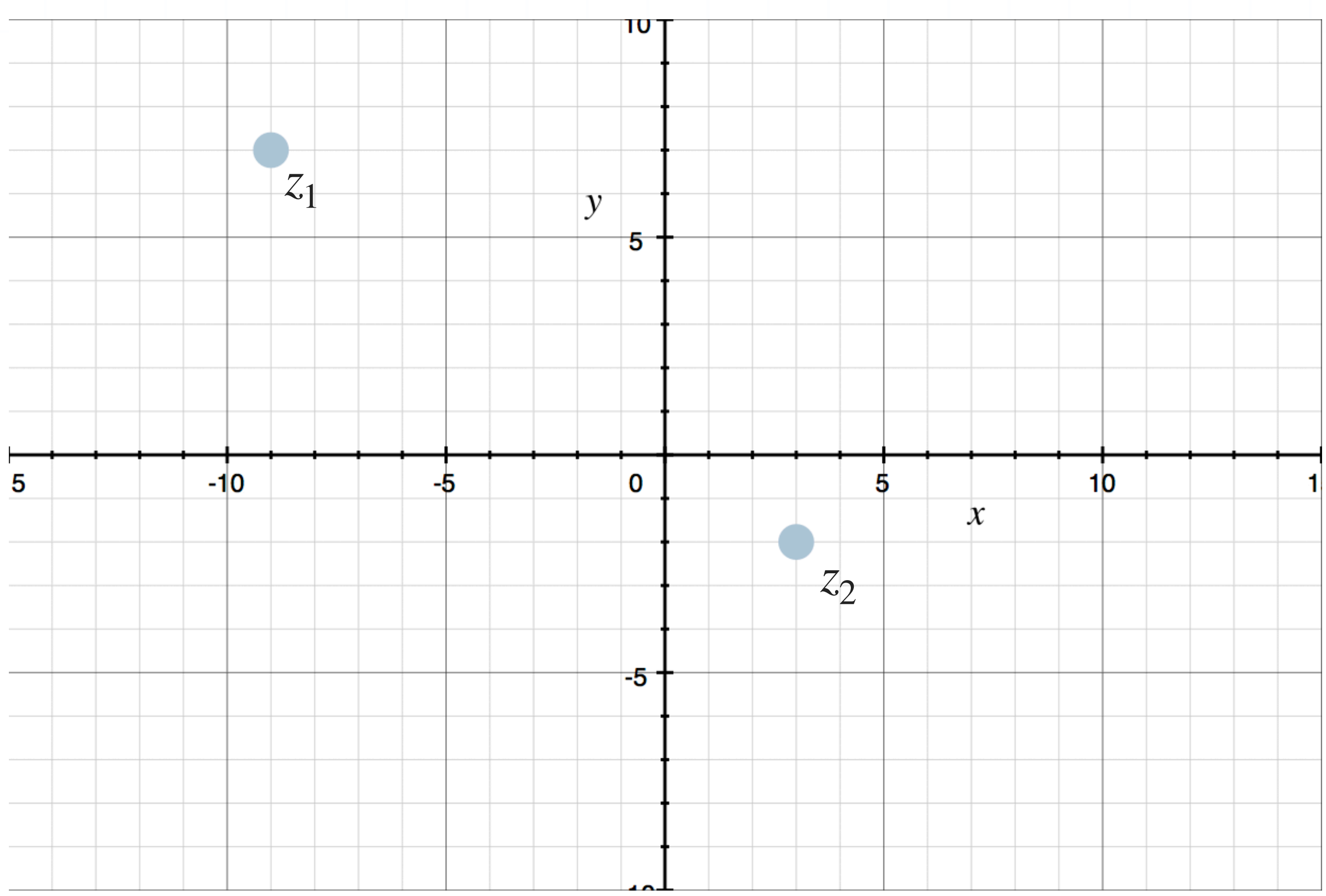
$$(8-13)+(-7+4)i$$

$$-5-3i$$

Now graph the complex number $-5-3i$, which has a real part -5 and an imaginary part -3.



■ 5. Graph the sum of the complex numbers z_1 and z_2.



Solution:

The point z_1 is 9 units to the left of the vertical axis and 7 units above the horizontal axis, which means that complex number is $z_1 = -9 + 7i$.

The point z_2 is 3 units to the right of the vertical axis and 2 units below the horizontal axis, which means that complex number is $z_2 = 3 - 2i$.

The sum of z_1 and z_2 is

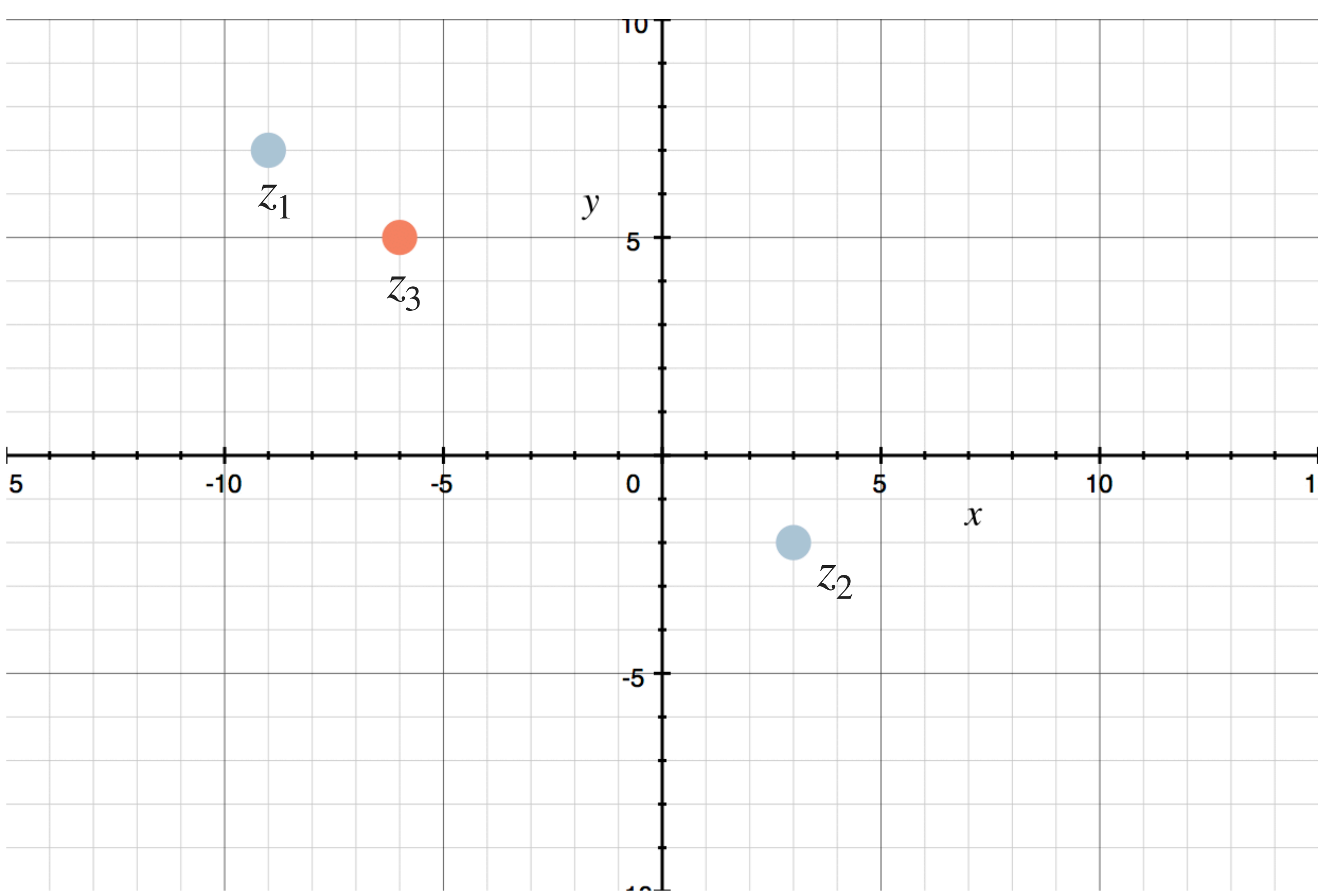
$$z_1 + z_2 = (-9 + 7i) + (3 - 2i)$$

$$z_1 + z_2 = (-9 + 3) + (7 + (-2))i$$

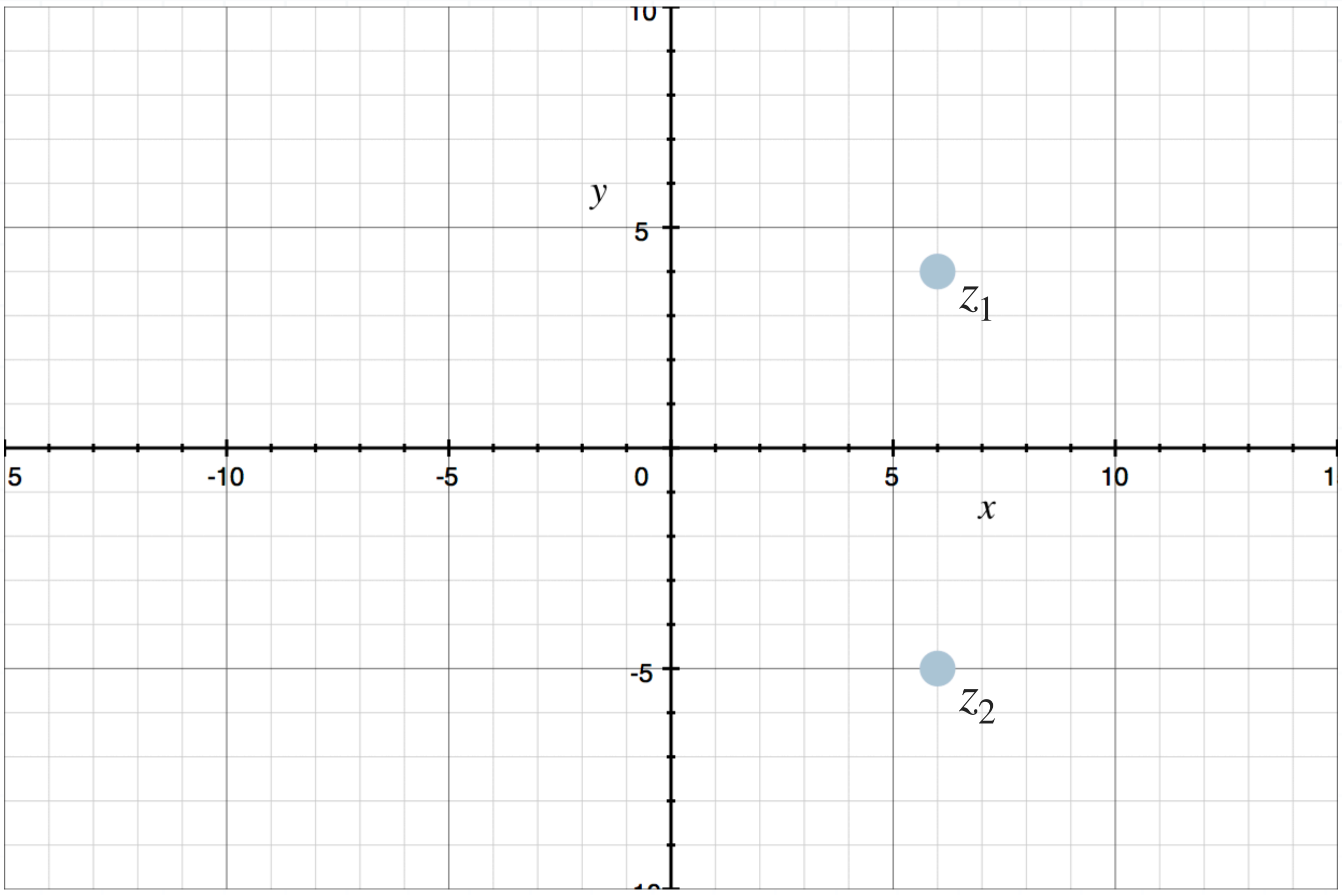
$$z_1 + z_2 = (-9 + 3) + (7 - 2)i$$

$$z_1 + z_2 = -6 + 5i$$

So if we plot the sum on the same set of axes, we get



■ 6. Graph the difference of the complex numbers z_1 and z_2.



Solution:

The point z_1 is 6 units to the right of the vertical axis and 4 units above the horizontal axis, which means that complex number is $z_1 = 6 + 4i$.

The point z_2 is 6 units to the right of the vertical axis and 5 units below the horizontal axis, which means that complex number is $z_2 = 6 - 5i$.

The difference of z_1 and z_2 is

$$z_1 - z_2 = (6 + 4i) - (6 - 5i)$$

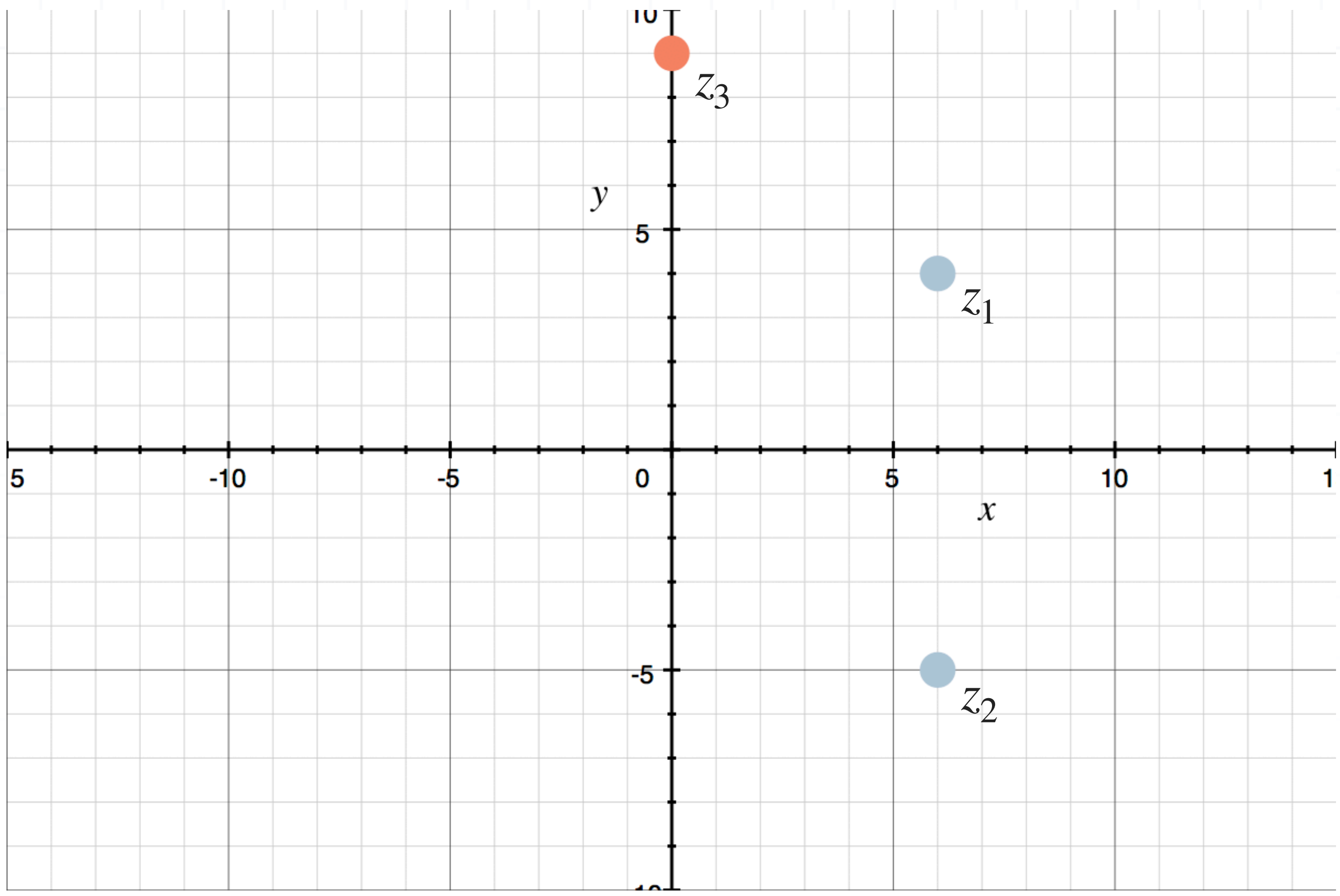
$$z_1 - z_2 = (6 - 6) + (4 - (-5))i$$

$$z_1 - z_2 = (6 - 6) + (4 + 5)i$$

$$z_1 - z_2 = 0 + 9i$$

$$z_1 - z_2 = 9i$$

So if we plot the difference on the same set of axes, we get



DISTANCES AND MIDPOINTS

■ 1. Find the distance between $s = 5 + 3i$ and $t = 1 - i$.

Solution:

Using the distance formula we have,

$$d = \sqrt{(5-1)^2 + (3-(-1))^2}$$

$$d = \sqrt{4^2 + 4^2}$$

$$d = \sqrt{16 + 16}$$

$$d = \sqrt{32}$$

$$d = 4\sqrt{2}$$

■ 2. Find the distance between $u = -5 - 3i$ and $v = 4 + 2i$.

Solution:

Using the distance formula we have,

$$d = \sqrt{(-5-4)^2 + (-3-2)^2}$$

$$d = \sqrt{(-9)^2 + (-5)^2}$$

$$d = \sqrt{81 + 25}$$

$$d = \sqrt{106}$$

■ 3. Find the distance between $w = 2 + 6i$ and $z = -2 - 6i$.

Solution:

Using the distance formula we have,

$$d = \sqrt{(2 - (-2))^2 + (6 - (-6))^2}$$

$$d = \sqrt{(2 + 2)^2 + (6 + 6)^2}$$

$$d = \sqrt{4^2 + 12^2}$$

$$d = \sqrt{16 + 144}$$

$$d = \sqrt{160}$$

$$d = 4\sqrt{10}$$

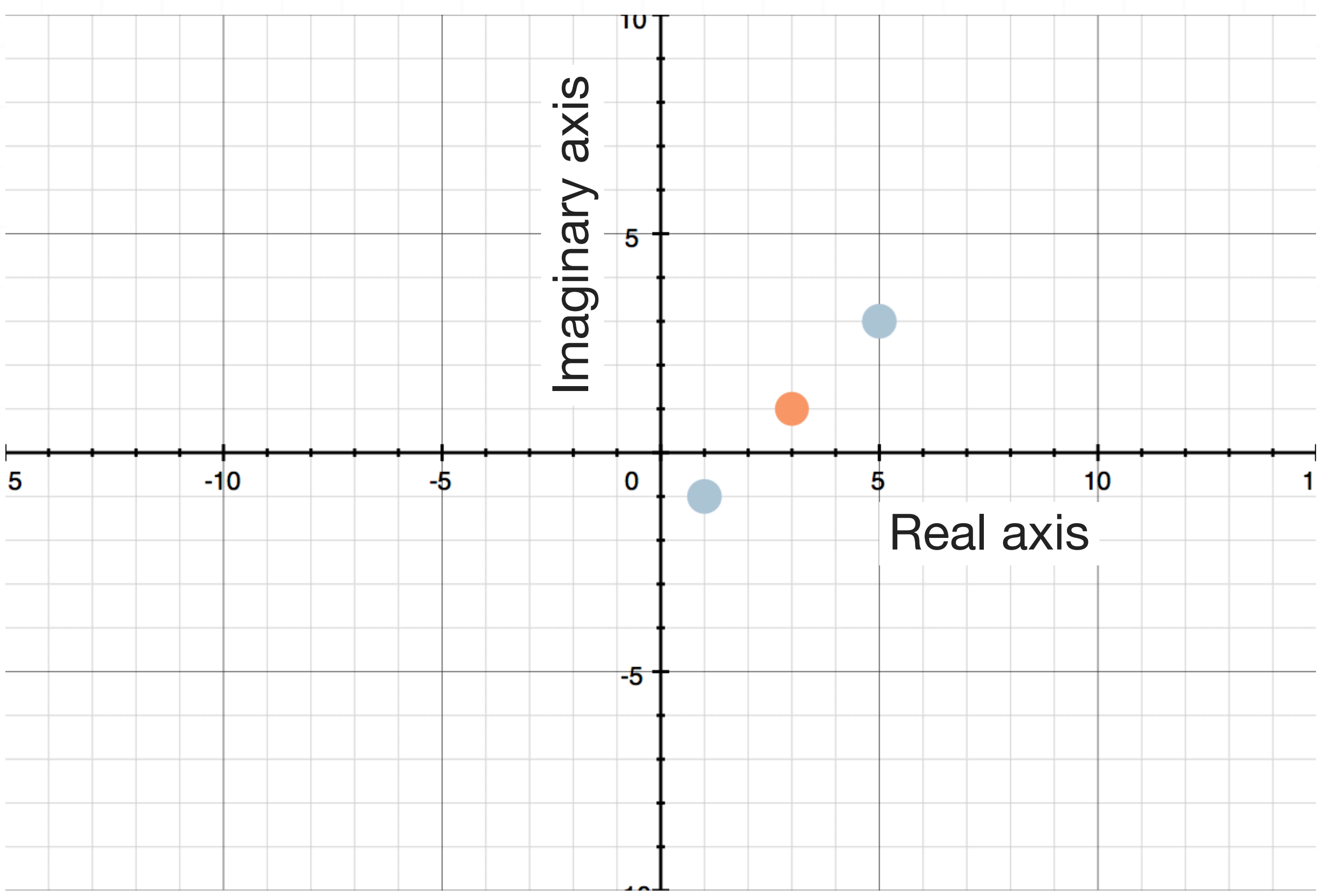
■ 4. Find the midpoint between $s = 5 + 3i$ and $t = 1 - i$.

Solution:

The distance between the real parts is $5 - 1 = 4$, and half of that distance is $4/2 = 2$. The value that's 2 units from 5 and 2 units from 2 is 3. The midpoint of the real parts is 3.

The distance between the imaginary parts is $3 - (-1) = 3 + 1 = 4$, and half of that distance is $4/2 = 2$. The value that's 2 units from 3 and 2 units from -1 is 1. The midpoint of the imaginary parts is 1.

The midpoint between $s = 5 + 3i$ and $t = 1 - i$ is $m = 3 + i$.



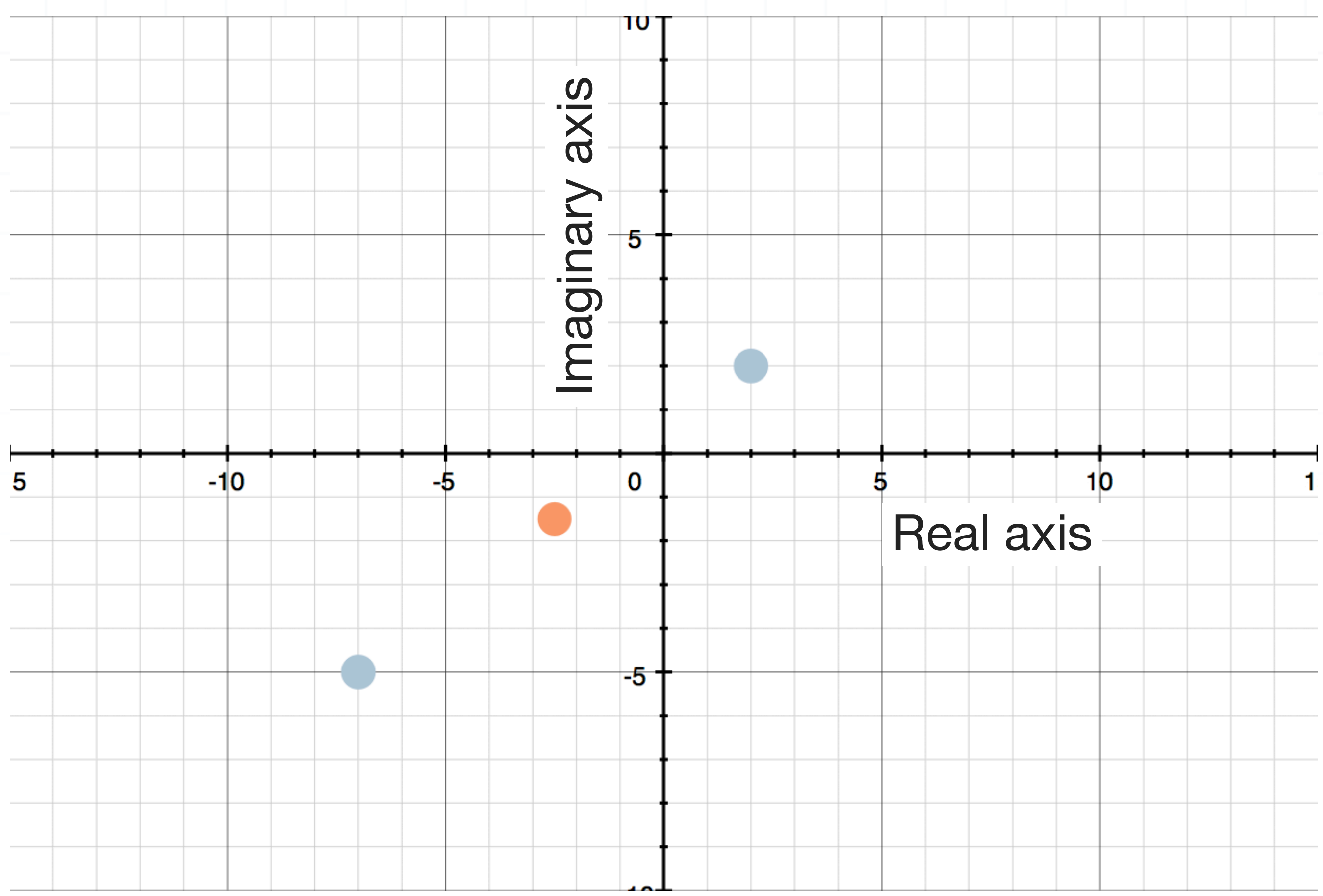
■ 5. Find the midpoint between $u = -7 - 5i$ and $z = 2 + 2i$.

Solution:

The distance between the real parts is $-7-2=-9$, and half of that distance is $-9/2=-4.5$. The value that's -4.5 units from -7 and -4.5 units from 2 is -2.5. The midpoint of the real parts is -2.5.

The distance between the imaginary parts is $-5-2=-7$, and half of that distance is $-7/2=-3.5$. The value that's -3.5 units from -7 and -3.5 units from 2 is -1.5. The midpoint of the imaginary parts is -1.5.

The midpoint between $u=-7-5i$ and $z=2+2i$ is $z=-2.5-1.5i$.



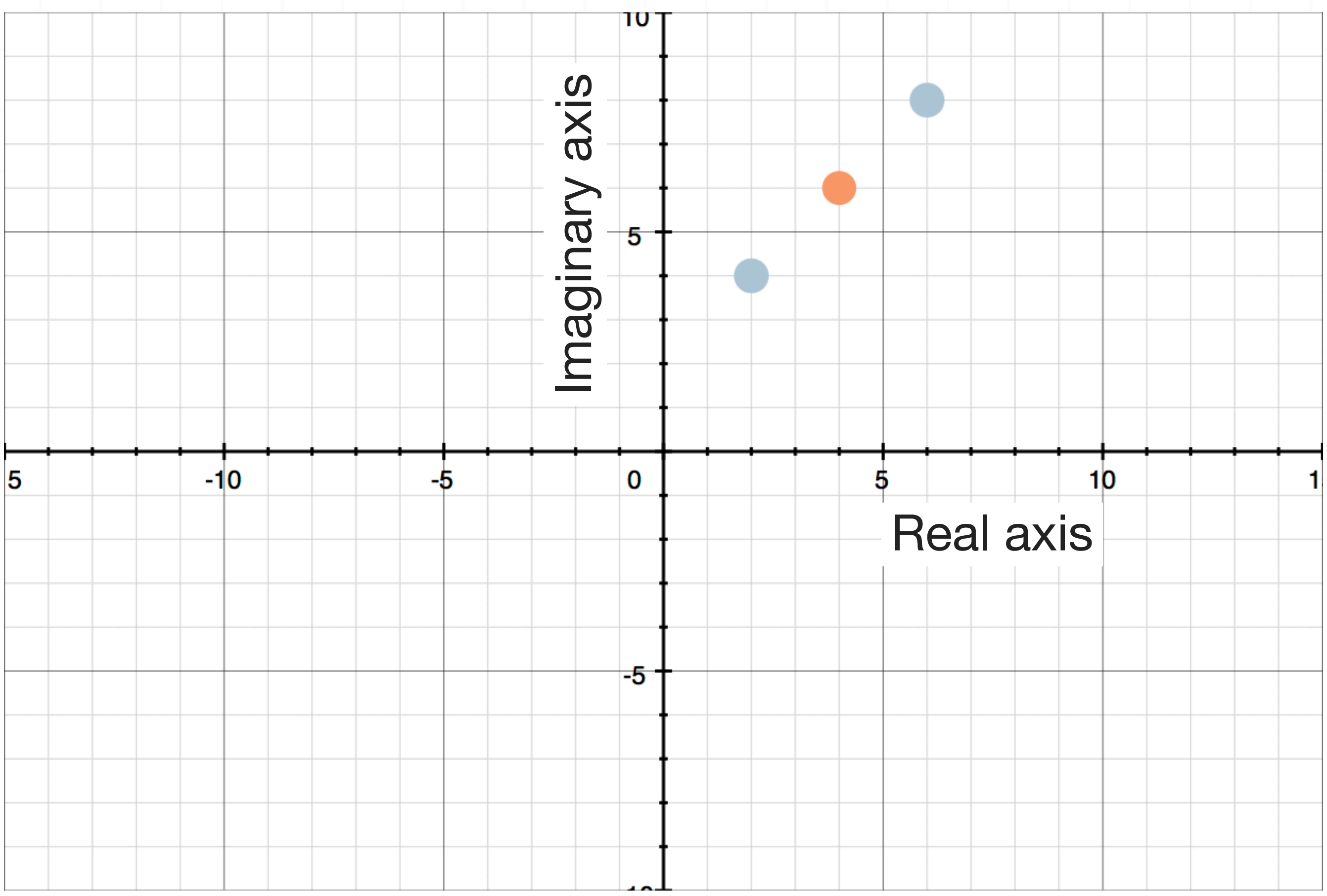
■ 6. Graph the midpoint between $w=6+8i$ and $z=2+4i$.

Solution:

The distance between the real parts is $6 - 2 = 4$, and half of that distance is $4/2 = 2$. The value that's 2 units from 6 and 2 units from 2 is 4. The midpoint of the real parts is 4.

The distance between the imaginary parts is $8 - 4 = 4$, and half of that distance is $4/2 = 2$. The value that's 2 units from 8 and 2 units from 4 is 6. The midpoint of the imaginary parts is 6.

The midpoint between $w = 6 + 8i$ and $z = 2 + 4i$ is $m = 4 + 6i$.



COMPLEX NUMBERS IN POLAR FORM

■ 1. If the complex number $6-2i$ is expressed in polar form, which quadrant contains the angle θ?

Solution:

If we set the complex number equal to its polar form, we get

$$6-2i=r(\cos\theta+i\sin\theta)$$

$$6-2i=r\cos\theta+ri\sin\theta$$

From this equation, we know that

$$6=r\cos\theta$$

$$\cos\theta=\frac{6}{r}$$

The value of r is always positive, since r represents a distance, so $6/r$ has to be greater than 0, which means $\cos\theta$ has to be positive.

We also know from $6-2i=r\cos\theta+ri\sin\theta$ that

$$-2=r\sin\theta$$

$$\sin\theta=-\frac{2}{r}$$

Because the value of r is always positive, $-2/r$ has to be less than 0, which means $\sin\theta$ has to be negative.

Angles with a positive cosine and negative sine are always in the fourth quadrant.

■ 2. Find r for the complex number.

$$-9-3i$$

Solution:

In the complex number, the real part is $a=-9$ and the imaginary part is $b=-3$, so the value of r will be

$$r=\sqrt{a^2+b^2}$$

$$r=\sqrt{(-9)^2+(-3)^2}$$

$$r=\sqrt{81+9}$$

$$r=\sqrt{90}$$

$$r=3\sqrt{10}$$

■ 3. What is the polar form of the complex number?

$$5 + 12i$$

Solution:

If we match up $5 + 12i$ with the standard form $a + bi$, we get $a = 5$ and $b = 12$, so

$$r = \sqrt{a^2 + b^2} = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

The value of θ is

$$\tan\theta = \frac{b}{a}$$

$$\tan\theta = \frac{12}{5}$$

$$\arctan(\tan\theta) = \arctan\frac{12}{5}$$

$$\theta = \arctan\frac{12}{5}$$

$$\theta \approx 1.18$$

Then the complex number written in polar form is

$$z = r(\cos\theta + i\sin\theta)$$

$$z \approx 13[\cos(1.18) + i\sin(1.18)]$$

■ 4. Write the complex number in polar form.

$$11i$$

Solution:

The complex number $11i$ can be written as $0 + 11i$, so its real part is 0, which means the number is located on the imaginary axis. Because $a = 0$ and $b = 11$, the distance of $0 + 11i$ from the origin is

$$r = \sqrt{a^2 + b^2} = \sqrt{0^2 + 11^2} = \sqrt{0 + 121} = \sqrt{121} = 11$$

Since the imaginary part of $0 + 11i$ is 11, which is positive, $0 + 11i$ is located on the positive imaginary axis, so $\theta = \pi/2$. In polar form, we get

$$r(\cos\theta + i\sin\theta)$$

$$11\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

■ 5. What is the polar form of the complex number?

$$z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

Solution:

In the complex number, the real part is $a = -\sqrt{3}/2$ and the imaginary part is $b = -1/2$, so the value of r will be

$$r = \sqrt{a^2 + b^2}$$

$$r = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}$$

$$r = \sqrt{\frac{3}{4} + \frac{1}{4}}$$

$$r = \sqrt{1}$$

$$r = 1$$

The value of θ is

$$\tan\theta = \frac{b}{a}$$

$$\tan\theta = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \left(-\frac{1}{2}\right)\left(-\frac{2}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\arctan(\tan\theta) = \arctan\frac{\sqrt{3}}{3}$$

$$\theta = \arctan\frac{\sqrt{3}}{3}$$

$\theta = \dfrac{\pi}{6}$ or $\theta = \dfrac{7\pi}{6}$

Because the complex number is in quadrant III, we use $\theta = 7\pi/6$. Then the complex number written in polar form is

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 1\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

■ 6. Write the complex number in polar form.

$$-5$$

Solution:

The complex number -5 can be written as $-5 + 0i$, so its imaginary part is 0, which means the number is located on the real axis. Because $a = -5$ and $b = 0$, the distance of $-5 + 0i$ from the origin is

$$r = \sqrt{a^2 + b^2} = \sqrt{(-5)^2 + 0^2} = \sqrt{25 + 0} = \sqrt{25} = 5$$

Since the real part of $-5 + 0i$ is -5, which is negative, $-5 + 0i$ is located on the negative real axis, so $\theta = \pi$. In polar form, we get

$$r(\cos\theta + i\sin\theta)$$

$$5(\cos\pi + i\sin\pi)$$

MULTIPLYING AND DIVIDING POLAR FORMS

■ 1. What is the product z_1z_2 of the complex numbers in polar form?

$$z_1 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$z_2 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Solution:

Plug the complex numbers into the formula for the product of complex numbers.

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$z_1z_2 = (5\sqrt{2})\left[\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)\right]$$

Simplify.

$$z_1z_2 = 5\sqrt{2}\left[\cos\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right) + i\sin\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right)\right]$$

$$z_1z_2 = 5\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$$

■ 2. What is the product z_1z_2 of the complex numbers in polar form?

$$z_1 = \sqrt{3}\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$$

$$z_2 = \frac{\sqrt{5}}{3}\left(\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right)$$

Solution:

Plug the complex numbers into the formula for the product of complex numbers.

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$z_1z_2 = \left(\sqrt{3}\frac{\sqrt{5}}{3}\right)\left[\cos\left(\frac{4\pi}{5} + \frac{11\pi}{8}\right) + i\sin\left(\frac{4\pi}{5} + \frac{11\pi}{8}\right)\right]$$

Simplify.

$$z_1z_2 = \frac{\sqrt{15}}{3}\left[\cos\left(\frac{32\pi}{40} + \frac{55\pi}{40}\right) + i\sin\left(\frac{32\pi}{40} + \frac{55\pi}{40}\right)\right]$$

$$z_1z_2 = \frac{\sqrt{15}}{3}\left(\cos\frac{87\pi}{40} + i\sin\frac{87\pi}{40}\right)$$

You could leave the answer this way, but ideally we'd like to reduce the angle to one that's coterminal, but in the interval $[0,2\pi)$. If we subtract 2π from the angle, we get

$$z_1z_2 = \frac{\sqrt{15}}{3}\left[\cos\left(\frac{87\pi}{40} - 2\pi\right) + i\sin\left(\frac{87\pi}{40} - 2\pi\right)\right]$$

$$z_1z_2 = \frac{\sqrt{15}}{3}\left[\cos\left(\frac{87\pi}{40} - \frac{80\pi}{40}\right) + i\sin\left(\frac{87\pi}{40} - \frac{80\pi}{40}\right)\right]$$

$$z_1z_2 = \frac{\sqrt{15}}{3}\left(\cos\frac{7\pi}{40} + i\sin\frac{7\pi}{40}\right)$$

■ 3. What is the quotient z_1/z_2 of the complex numbers in polar form?

$$z_1 = 12\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$$z_2 = 15\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

Solution:

Plug the complex numbers into the formula for the quotient of complex numbers.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{12}{15}\left[\cos\left(\frac{7\pi}{6} - \frac{\pi}{2}\right) + i\sin\left(\frac{7\pi}{6} - \frac{\pi}{2}\right)\right]$$

Simplify.

$$\frac{z_1}{z_2} = \frac{4}{5}\left[\cos\left(\frac{7\pi}{6} - \frac{3\pi}{6}\right) + i\sin\left(\frac{7\pi}{6} - \frac{3\pi}{6}\right)\right]$$

$$\frac{z_1}{z_2} = \frac{4}{5}\left(\cos\frac{4\pi}{6} + i\sin\frac{4\pi}{6}\right)$$

$$\frac{z_1}{z_2} = \frac{4}{5}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

■ 4. What is the quotient z_1/z_2 of the complex numbers in polar form?

$$z_1 = \sqrt{7}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

$$z_2 = \frac{1}{\sqrt{2}}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Solution:

Plug the complex numbers into the formula for the quotient of complex numbers.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$\frac{z_1}{z_2}=\frac{\sqrt{7}}{\frac{1}{\sqrt{2}}}\left[\cos\left(\frac{\pi}{12}-\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{12}-\frac{2\pi}{3}\right)\right]$$

Simplify.

$$\frac{z_1}{z_2}=\sqrt{14}\left[\cos\left(\frac{\pi}{12}-\frac{8\pi}{12}\right)+i\sin\left(\frac{\pi}{12}-\frac{8\pi}{12}\right)\right]$$

$$\frac{z_1}{z_2}=\sqrt{14}\left(\cos\frac{-7\pi}{12}+i\sin\frac{-7\pi}{12}\right)$$

You could leave the answer this way, but ideally we'd like to reduce the angle to one that's coterminal, but in the interval $[0,2\pi)$. If we add 2π to the angle, we get

$$\frac{z_1}{z_2}=\sqrt{14}\left[\cos\left(\frac{-7\pi}{12}+2\pi\right)+i\sin\left(\frac{-7\pi}{12}+2\pi\right)\right]$$

$$\frac{z_1}{z_2}=\sqrt{14}\left[\cos\left(\frac{-7\pi}{12}+\frac{24\pi}{12}\right)+i\sin\left(\frac{-7\pi}{12}+\frac{24\pi}{12}\right)\right]$$

$$z_1z_2=\sqrt{14}\left(\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}\right)$$

■ 5. What is the product z_1z_2 of the complex numbers in polar form?

$$z_1=\frac{\sqrt{15}}{4}\left(\cos\frac{7\pi}{2}+i\sin\frac{7\pi}{2}\right)$$

$$z_2 = \frac{1}{\sqrt{5}}\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right)$$

Solution:

Plug the complex numbers into the formula for the product of complex numbers.

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$z_1z_2 = \left(\frac{\sqrt{15}}{4} \cdot \frac{1}{\sqrt{5}}\right)\left[\cos\left(\frac{7\pi}{2} + \frac{6\pi}{5}\right) + i\sin\left(\frac{7\pi}{2} + \frac{6\pi}{5}\right)\right]$$

Simplify.

$$z_1z_2 = \frac{\sqrt{15}}{4\sqrt{5}}\left[\cos\left(\frac{35\pi}{10} + \frac{12\pi}{10}\right) + i\sin\left(\frac{35\pi}{10} + \frac{12\pi}{10}\right)\right]$$

$$z_1z_2 = \frac{\sqrt{3}}{4}\left(\cos\frac{47\pi}{10} + i\sin\frac{47\pi}{10}\right)$$

You could leave the answer this way, but ideally we'd like to reduce the angle to one that's coterminal, but in the interval $[0,2\pi)$. If we subtract $2 \cdot 2\pi = 4\pi$ from the angle, we get

$$z_1z_2 = \frac{\sqrt{3}}{4}\left[\cos\left(\frac{47\pi}{10} - 4\pi\right) + i\sin\left(\frac{47\pi}{10} - 4\pi\right)\right]$$

$$z_1z_2 = \frac{\sqrt{3}}{4}\left[\cos\left(\frac{47\pi}{10} - \frac{40\pi}{10}\right) + i\sin\left(\frac{47\pi}{10} - \frac{40\pi}{10}\right)\right]$$

$$z_1z_2 = \frac{\sqrt{3}}{4}\left(\cos\frac{7\pi}{10} + i\sin\frac{7\pi}{10}\right)$$

■ 6. Suppose that a complex number z is the product $z_1 \cdot z_2$ of the given complex numbers. If z is expressed in polar form, $r(\cos\theta + i\sin\theta)$, where is θ located?

$$z_1 = 3\sqrt{5}\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$$

$$z_2 = 6\left(\cos\frac{7\pi}{10} + i\sin\frac{7\pi}{10}\right)$$

Solution:

Plug the complex numbers into the formula for the product of complex numbers.

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$z_1z_2 = (3\sqrt{5} \cdot 6)\left[\cos\left(\frac{2\pi}{5} + \frac{7\pi}{10}\right) + i\sin\left(\frac{2\pi}{5} + \frac{7\pi}{10}\right)\right]$$

Simplify.

$$z_1z_2 = 18\sqrt{5}\left[\cos\left(\frac{4\pi}{10}+\frac{7\pi}{10}\right)+i\sin\left(\frac{4\pi}{10}+\frac{7\pi}{10}\right)\right]$$

$$z_1z_2 = 18\sqrt{5}\left(\cos\frac{11\pi}{10}+i\sin\frac{11\pi}{10}\right)$$

The fraction $11/10$ is equal to 1.1, so the angle is 1.1π, which is in the third quadrant.

POWERS OF COMPLEX NUMBERS AND DE MOIVRE'S THEOREM

■ 1. Find z^5 in polar form.

$$z = 2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

Solution:

Plug $r = 2$, $\theta = \pi/12$, and $n = 5$ into De Moivre's Theorem.

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^5 = 2^5\left[\cos\left(5\cdot\frac{\pi}{12}\right) + i\sin\left(5\cdot\frac{\pi}{12}\right)\right]$$

$$z^5 = 32\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

■ 2. Find z^7 in polar form.

$$z = \sqrt{5}\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$$

Solution:

Plug $r=\sqrt{5}$, $\theta=2\pi/5$, and $n=7$ into De Moivre's Theorem.

$$z^n=r^n[\cos(n\theta)+i\sin(n\theta)]$$

$$z^7=(\sqrt{5})^7\left[\cos\left(7\cdot\frac{2\pi}{5}\right)+i\sin\left(7\cdot\frac{2\pi}{5}\right)\right]$$

$$z^7=125\sqrt{5}\left(\cos\frac{14\pi}{5}+i\sin\frac{14\pi}{5}\right)$$

We could leave the answer this way, but the angle $14\pi/5$ is larger than 2π, so we can reduce the angle to one that's coterminal with $14\pi/5$, but within the interval $[0,2\pi)$.

$$\frac{14\pi}{5}-2\pi=\frac{14\pi}{5}-2\pi\left(\frac{5}{5}\right)=\frac{14\pi}{5}-\frac{10\pi}{5}=\frac{4\pi}{5}$$

So the complex number z^4 in polar form can be written as

$$z^7=125\sqrt{5}\left(\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}\right)$$

■ 3. Find z^6 in rectangular form $a+bi$.

$$z=\frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)$$

Solution:

Plug $r = \sqrt{2}/2$, $\theta = \pi/8$, and $n = 6$ into De Moivre's Theorem.

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^6 = \left(\frac{\sqrt{2}}{2}\right)^6 \left[\cos\left(6 \cdot \frac{\pi}{8}\right) + i\sin\left(6 \cdot \frac{\pi}{8}\right)\right]$$

$$z^6 = \frac{1}{8}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$z^6 = \frac{1}{8}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

$$z^6 = \frac{1}{8}\left(-\frac{\sqrt{2}}{2}\right) + \frac{1}{8}\left(\frac{\sqrt{2}}{2}i\right)$$

$$z^6 = -\frac{\sqrt{2}}{16} + \frac{\sqrt{2}}{16}i$$

■ 4. Find z^3 in rectangular form $a + bi$.

$$z = 2\sqrt{6}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

Solution:

Plug $r = 2\sqrt{6}$, $\theta = 5\pi/3$, and $n = 3$ into De Moivre's Theorem.

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^3 = (2\sqrt{6})^3\left[\cos\left(3\cdot\frac{5\pi}{3}\right) + i\sin\left(3\cdot\frac{5\pi}{3}\right)\right]$$

$$z^3 = 8\cdot 6\sqrt{6}(\cos 5\pi + i\sin 5\pi)$$

$$z^3 = 48\sqrt{6}(\cos 5\pi + i\sin 5\pi)$$

$$z^3 = 48\sqrt{6}(-1 + i(0))$$

$$z^3 = -48\sqrt{6}$$

■ 5. Find z^5 in polar form.

$$z = -4 - 4i$$

Solution:

First, convert $z = -4 - 4i$ to polar form by finding the modulus $|z|$ and the angle θ. The distance from the origin is

$$|z| = r = \sqrt{a^2 + b^2}$$

$$|z| = r = \sqrt{(-4)^2 + (-4)^2}$$

$$|z| = r = \sqrt{16 + 16}$$

$$|z| = r = \sqrt{32}$$

$$|z| = r = 4\sqrt{2}$$

and the angle is

$$\theta = \arctan\frac{b}{a} = \arctan\frac{-4}{-4} = \arctan(1) = \frac{\pi}{4} \quad \text{or} \quad \frac{5\pi}{4}$$

Because the complex number $z = -4 - 4i$ is in quadrant III, we use $\theta = 5\pi/4$. Then $z = -4 - 4i$ in polar form is

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

To find z^5, plug $r = 4\sqrt{2}$, $\theta = 5\pi/4$, and $n = 5$ into De Moivre's Theorem.

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^5 = (4\sqrt{2})^5\left[\cos\left(5\cdot\frac{5\pi}{4}\right) + i\sin\left(5\cdot\frac{5\pi}{4}\right)\right]$$

$$z^5 = 4{,}096\sqrt{2}\left(\cos\frac{25\pi}{4} + i\sin\frac{25\pi}{4}\right)$$

We could leave the answer this way, but the angle $25\pi/4$ is larger than 2π, so we can reduce the angle to one that's coterminal with $25\pi/4$, but within the interval $[0,2\pi)$.

$$\frac{25\pi}{4} - 3(2\pi) = \frac{25\pi}{4} - 6\pi\left(\frac{4}{4}\right) = \frac{25\pi}{4} - \frac{24\pi}{4} = \frac{\pi}{4}$$

So the complex number z^5 in polar form can be written as

$$z^5 = 4{,}096\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

■ 6. Find z^4 in rectangular form $a + bi$.

$$z = \sqrt{6} - \sqrt{2}i$$

Solution:

First, convert $z = \sqrt{6} - \sqrt{2}i$ to polar form by finding the modulus $|z|$ and the angle θ. The distance from the origin is

$$|z| = r = \sqrt{a^2 + b^2}$$

$$|z| = r = \sqrt{(\sqrt{6})^2 + (-\sqrt{2})^2}$$

$$|z| = r = \sqrt{6 + 2}$$

$$|z| = r = \sqrt{8}$$

$$|z| = r = 2\sqrt{2}$$

and the angle is

$$\theta = \arctan\frac{b}{a} = \arctan\frac{-\sqrt{2}}{\sqrt{6}} = \arctan\frac{-\sqrt{3}}{3} = -\frac{\pi}{6}$$

Then $z = \sqrt{6} - \sqrt{2}i$ in polar form is

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 2\sqrt{2}\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

To find z^4, plug $r = 2\sqrt{2}$, $\theta = -\pi/6$, and $n = 4$ into De Moivre's Theorem.

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^4 = (2\sqrt{2})^4\left[\cos\left(4 \cdot -\frac{\pi}{6}\right) + i\sin\left(4 \cdot -\frac{\pi}{6}\right)\right]$$

$$z^4 = 64\left[\cos\left(-\frac{4\pi}{6}\right) + i\sin\left(-\frac{4\pi}{6}\right)\right]$$

$$z^4 = 64\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$$

$$z^4 = 64\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right]$$

$$z^4 = 64\left(-\frac{1}{2}\right) + 64\left(-\frac{\sqrt{3}}{2}i\right)$$

$$z^4 = -32 - 32\sqrt{3}i$$

COMPLEX NUMBER EQUATIONS

■ 1. Find the solutions of the complex equation.

$$z^2 = 49$$

Solution:

Rewrite z^2 as

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^2 = r^2[\cos(2\theta) + i\sin(2\theta)]$$

Rewrite 49 as the complex number $49 + 0i$. The modulus and angle of $49 + 0i$ are

$$r = \sqrt{49^2 + 0^2}$$

$$r = \sqrt{49^2}$$

$$r = 49$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{49} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 49[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)]$$

$$z = 49[\cos(2\pi k) + i\sin(2\pi k)]$$

We could also write this in degrees instead of radians as

$$z = 49[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Starting again with $z^2 = 49$, we can start making substitutions.

$$z^2 = 49$$

$$r^2[\cos(2\theta) + i\sin(2\theta)] = 49$$

$$r^2[\cos(2\theta) + i\sin(2\theta)] = 49[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get a new system of equations.

$$r^2 = 49$$

$$2\theta = 360^\circ k$$

From these equations, we get

$r^2 = 49$, so $r = 7$

$2\theta = 360^\circ k$, so $\theta = 180^\circ k$

To $\theta = 180^\circ k$, if we plug in $k = 0,\ 1,\ \dots$, we get

For $k = 0$, $\theta = 180^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 180^\circ(1) = 180^\circ$

...

We could keep going for $k = 2, 3, 4, 5, \ldots$, but $k = 2$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 180^\circ$.

Plugging these two angles and $r = 7$ into the formula for polar form of a complex number, we'll get the solutions to $z^2 = 49$.

$$z_1 = 7[\cos(0^\circ) + i\sin(0^\circ)] = 7[1 + i(0)] = 7$$

$$z_2 = 7[\cos(180^\circ) + i\sin(180^\circ)] = 7[-1 + i(0)] = -7$$

■ 2. Find the solution of the complex equation that lies in the third quadrant.

$$z^3 = 216$$

Solution:

Rewrite z^3 as

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^3 = r^3[\cos(3\theta) + i\sin(3\theta)]$$

Rewrite 216 as the complex number $216 + 0i$. The modulus and angle of $216 + 0i$ are

$$r = \sqrt{216^2 + 0^2}$$

$$r = \sqrt{216^2}$$

$$r = 216$$

and

$$\theta = \arctan \frac{b}{a} = \arctan \frac{0}{216} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 216[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)]$$

$$z = 216[\cos(2\pi k) + i\sin(2\pi k)]$$

We could also write this in degrees instead of radians as

$$z = 216[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Starting again with $z^3 = 216$, we can start making substitutions.

$$z^3 = 216$$

$$r^3[\cos(3\theta) + i\sin(3\theta)] = 216$$

$$r^3[\cos(3\theta) + i\sin(3\theta)] = 216[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get a new system of equations.

$$r^3 = 216$$

$$3\theta = 360^\circ k$$

From these equations, we get

$r^3 = 216$, so $r = 6$

$3\theta = 360^\circ k$, so $\theta = 120^\circ k$

To $\theta = 120^\circ k$, if we plug in $k = 0, 1, 2, \dots$, we get

For $k = 0$, $\theta = 120^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 120^\circ(1) = 120^\circ$

For $k = 2$, $\theta = 120^\circ(2) = 240^\circ$

...

We could keep going for $k = 3, 4, 5, 6, \dots$, but $k = 3$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 120^\circ, 240^\circ$.

Plugging these three angles and $r = 6$ into the formula for polar form of a complex number, we'll get the solutions to $z^3 = 216$.

$$z_1 = 6[\cos(0^\circ) + i\sin(0^\circ)] = 6[1 + i(0)] = 6$$

$$z_2 = 6[\cos(120^\circ) + i\sin(120^\circ)] = 6\left[-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = -3 + 3\sqrt{3}i$$

$$z_3 = 6[\cos(240^\circ) + i\sin(240^\circ)] = 6\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = -3 - 3\sqrt{3}i$$

Roots in the third quadrant will have a negative real part and a negative imaginary part. So, z_3 is the solution in the third quadrant.

■ 3. Find the solutions of the complex equation.

$$z^4 = 256$$

Solution:

Rewrite z^4 as

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^4 = r^4[\cos(4\theta) + i\sin(4\theta)]$$

Rewrite 256 as the complex number $256 + 0i$. The modulus and angle of $256 + 0i$ are

$$r = \sqrt{256^2 + 0^2}$$

$$r = \sqrt{256^2}$$

$$r = 256$$

and

$$\theta = \arctan \frac{b}{a} = \arctan \frac{0}{256} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 256[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)]$$

$$z = 256[\cos(2\pi k) + i\sin(2\pi k)]$$

We could also write this in degrees instead of radians as

$$z = 256[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Starting again with $z^4 = 256$, we can start making substitutions.

$$z^4 = 256$$

$$r^4[\cos(4\theta) + i\sin(4\theta)] = 256$$

$$r^4\left[\cos(4\theta) + i\sin(4\theta)\right] = 256[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get a new system of equations.

$$r^4 = 256$$

$$4\theta = 360^\circ k$$

From these equations, we get

$r^4 = 256$, so $r = 4$

$4\theta = 360^\circ k$, so $\theta = 90^\circ k$

To $\theta = 90^\circ k$, if we plug in $k = 0, 1, 2, 3, \ldots,$ we get

For $k = 0$, $\theta = 90^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 90^\circ(1) = 90^\circ$

For $k = 2$, $\theta = 90^\circ(2) = 180^\circ$

For $k = 3$, $\theta = 90^\circ(3) = 270^\circ$

...

We could keep going for $k = 4, 5, 6, 7, \ldots,$ but $k = 4$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 90^\circ, 180^\circ, 270^\circ$.

Plugging these four angles and $r = 4$ into the formula for polar form of a complex number, we'll get the solutions to $z^4 = 256$.

$$z_1 = 4[\cos(0^\circ) + i\sin(0^\circ)] = 4[1 + i(0)] = 4$$

$$z_2 = 4[\cos(90^\circ) + i\sin(90^\circ)] = 4[0 + i(1)] = 4i$$

$$z_3 = 4[\cos(180^\circ) + i\sin(180^\circ)] = 4[-1 + i(0)] = -4$$

$$z_4 = 4[\cos(270^\circ) + i\sin(270^\circ)] = 4[0 + i(-1)] = -4i$$

■ 4. Find the solutions of the complex equation.

$$z^6 = 729$$

Solution:

Rewrite z^6 as

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^6 = r^6[\cos(6\theta) + i\sin(6\theta)]$$

Rewrite 729 as the complex number $729 + 0i$. The modulus and angle of $729 + 0i$ are

$$r = \sqrt{729^2 + 0^2}$$

$$r = \sqrt{729^2}$$

$$r = 729$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{729} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 729[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)]$$

$$z = 729[\cos(2\pi k) + i\sin(2\pi k)]$$

We could also write this in degrees instead of radians as

$$z = 729[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Starting again with $z^6 = 729$, we can start making substitutions.

$$z^6 = 729$$

$$r^6[\cos(6\theta) + i\sin(6\theta)] = 729$$

$$r^6[\cos(6\theta) + i\sin(6\theta)] = 729[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get a new system of equations.

$$r^6 = 729$$

$$6\theta = 360^\circ k$$

From these equations, we get

$r^6 = 729$, so $r = 3$

$6\theta = 360^\circ k$, so $\theta = 60^\circ k$

To $\theta = 60^\circ k$, if we plug in $k = 0, 1, 2, 3, 4, 5, \ldots$, we get

For $k = 0$, $\theta = 60^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 60^\circ(1) = 60^\circ$

For $k = 2$, $\theta = 60^\circ(2) = 120^\circ$

For $k = 3$, $\theta = 60^\circ(3) = 180^\circ$

For $k = 4$, $\theta = 60^\circ(4) = 240^\circ$

For $k = 5$, $\theta = 60^\circ(5) = 300^\circ$

...

We could keep going for $k = 6, 7, 8, 9, \dots$, but $k = 6$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$.

Plugging these six angles and $r = 3$ into the formula for polar form of a complex number, we'll get the solutions to $z^6 = 729$.

$$z_1 = 3[\cos(0^\circ) + i\sin(0^\circ)] = 3[1 + i(0)] = 3$$

$$z_2 = 3[\cos(60^\circ) + i\sin(60^\circ)] = 3\left[\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

$$z_3 = 3[\cos(120^\circ) + i\sin(120^\circ)] = 3\left[-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

$$z_4 = 3[\cos(180^\circ) + i\sin(180^\circ)] = 3\left[-1 + i(0)\right] = -3$$

$$z_5 = 3[\cos(240^\circ) + i\sin(240^\circ)] = 3\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$$

$$z_6 = 3[\cos(300^\circ) + i\sin(300^\circ)] = 3\left[\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = \frac{3}{2} - \frac{3\sqrt{3}}{2}i$$

■ 5. Find the solutions of the complex equation.

$$z^5 = 32$$

Solution:

Rewrite z^5 as

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^5 = r^5[\cos(5\theta) + i\sin(5\theta)]$$

Rewrite 32 as the complex number $32 + 0i$. The modulus and angle of $32 + 0i$ are

$$r = \sqrt{32^2 + 0^2}$$

$$r = \sqrt{32^2}$$

$$r = 32$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{32} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 32[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \ldots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \ldots)]$$

$$z = 32[\cos(2\pi k) + i\sin(2\pi k)]$$

We could also write this in degrees instead of radians as

$$z = 32[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Starting again with $z^5 = 32$, we can start making substitutions.

$$z^5 = 32$$

$$r^5[\cos(5\theta) + i\sin(5\theta)] = 32$$

$$r^5[\cos(5\theta) + i\sin(5\theta)] = 32[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get a new system of equations.

$$r^5 = 32$$

$$5\theta = 360^\circ k$$

From these equations, we get

$r^5 = 32$, so $r = 2$

$5\theta = 360^\circ k$, so $\theta = 72^\circ k$

To $\theta = 72^\circ k$, if we plug in $k = 0, 1, 2, 3, 4, \ldots$, we get

For $k = 0$, $\theta = 72^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 72^\circ(1) = 72^\circ$

For $k = 2$, $\theta = 72^\circ(2) = 144^\circ$

For $k = 3$, $\theta = 72^\circ(3) = 216^\circ$

For $k = 4$, $\theta = 72^\circ(4) = 288^\circ$

...

We could keep going for $k = 5, 6, 7, 8, 9, \dots$, but $k = 5$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 72^\circ, 144^\circ, 216^\circ, 288^\circ$.

Plugging these five angles and $r = 2$ into the formula for polar form of a complex number, we'll get the solutions to $z^5 = 32$. Because these angles are not on the unit circle, we will need to use decimal approximations.

$$z_1 = 2[\cos(0^\circ) + i\sin(0^\circ)] = 2[1 + i(0)] = 2$$

$$z_2 \approx 2[\cos(72^\circ) + i\sin(72^\circ)] \approx 2[0.309 + 0.951i] \approx 0.618 + 1.902i$$

$$z_3 \approx 2[\cos(144^\circ) + i\sin(144^\circ)] \approx 2[-0.809 + 0.588i] \approx -1.618 + 1.176i$$

$$z_4 \approx 2[\cos(216^\circ) + i\sin(216^\circ)] \approx 2[-0.809 - 0.588i] \approx -1.618 - 1.176i$$

$$z_5 \approx 2[\cos(288^\circ) + i\sin(288^\circ)] \approx 2[0.309 - 0.951i] \approx 0.618 - 1.902i$$

■ 6. How many solutions of the complex equation lie in the second quadrant?

$$z^8 = 256$$

Solution:

Rewrite z^8 as

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^8 = r^8[\cos(8\theta) + i\sin(8\theta)]$$

Rewrite 256 as the complex number $256 + 0i$. The modulus and angle of $256 + 0i$ are

$$r = \sqrt{256^2 + 0^2}$$

$$r = \sqrt{256^2}$$

$$r = 256$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{256} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 256[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)]$$

$$z = 256[\cos(2\pi k) + i\sin(2\pi k)]$$

We could also write this in degrees instead of radians as

$$z = 256[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Starting again with $z^8 = 256$, we can start making substitutions.

$$z^8 = 256$$

$$r^8[\cos(8\theta) + i\sin(8\theta)] = 256$$

$$r^8[\cos(8\theta) + i\sin(8\theta)] = 256[\cos(360^\circ k) + i\sin(360^\circ k)]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get a new system of equations.

$$r^8 = 256$$

$$8\theta = 360^\circ k$$

From these equations, we get

$r^8 = 256$, so $r = 2$

$8\theta = 360^\circ k$, so $\theta = 45^\circ k$

To $\theta = 45^\circ k$, if we plug in $k = 0, 1, 2, 3, 4, 5, 6, 7, ...$, we get

For $k = 0$, $\theta = 45^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 45^\circ(1) = 45^\circ$

For $k = 2$, $\theta = 45^\circ(2) = 90^\circ$

For $k = 3$, $\theta = 45^\circ(3) = 135^\circ$

For $k = 4$, $\theta = 45^\circ(4) = 180^\circ$

For $k = 5$, $\theta = 45^\circ(5) = 225^\circ$

For $k = 6$, $\theta = 45^\circ(6) = 270^\circ$

For $k = 7$, $\theta = 45^\circ(7) = 315^\circ$

...

We could keep going for $k = 8, 9, 10, 11, \dots$, but $k = 8$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 45^\circ, 90^\circ, 135^\circ, 180^\circ, 225^\circ, 270^\circ, 315^\circ$.

Plugging these eight angles and $r = 2$ into the formula for polar form of a complex number, we'll get the solutions to $z^8 = 256$.

$$z_1 = 2[\cos(0^\circ) + i\sin(0^\circ)] = 2[1 + i(0)] = 2$$

$$z_2 = 2[\cos(45^\circ) + i\sin(45^\circ)] = 2\left[\frac{\sqrt{2}}{2} + i\left(\frac{\sqrt{2}}{2}\right)\right] = \sqrt{2} + \sqrt{2}i$$

$$z_3 = 2[\cos(90^\circ) + i\sin(90^\circ)] = 2\left[0 + i(1)\right] = 2i$$

$$z_4 = 2[\cos(135^\circ) + i\sin(135^\circ)] = 2\left[-\frac{\sqrt{2}}{2} + i\left(\frac{\sqrt{2}}{2}\right)\right] = -\sqrt{2} + \sqrt{2}i$$

$$z_5 = 2[\cos(180^\circ) + i\sin(180^\circ)] = 2\left[-1 + i(0)\right] = -2$$

$$z_6 = 2[\cos(225^\circ) + i\sin(225^\circ)] = 2\left[-\frac{\sqrt{2}}{2} - i\left(\frac{\sqrt{2}}{2}\right)\right] = -\sqrt{2} - \sqrt{2}i$$

$$z_7 = 2[\cos(270^\circ) + i\sin(270^\circ)] = 2\left[0 + i(-1)\right] = -2i$$

$$z_8 = 2[\cos(315^\circ) + i\sin(315^\circ)] = 2\left[\frac{\sqrt{2}}{2} - i\left(\frac{\sqrt{2}}{2}\right)\right] = \sqrt{2} - \sqrt{2}i$$

Roots in the second quadrant will have a negative real part and a positive imaginary part. That's only z_4, so there's one solution in the second quadrant.

ROOTS OF COMPLEX NUMBERS

1. Find the cube roots of the complex number.

$$z = 27\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Solution:

We're looking for the third (or cube) roots of z, which means there will be 3 of them, given by $k = 0,\ 1,\ 2$. And since the complex number is given in radians, we'll plug $n = 3$ into the formula for nth roots in radians.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

$$\sqrt[3]{z} = \sqrt[3]{r}\left[\cos\left(\frac{\theta + 2\pi k}{3}\right) + i\sin\left(\frac{\theta + 2\pi k}{3}\right)\right]$$

With $r = 27$ and $\theta = \pi/4$ from the complex number, we get

$$\sqrt[3]{z} = \sqrt[3]{27}\left[\cos\left(\frac{\frac{\pi}{4} + 2\pi k}{3}\right) + i\sin\left(\frac{\frac{\pi}{4} + 2\pi k}{3}\right)\right]$$

Now we'll find values for $k = 0,\ 1,\ 2$.

For $k = 0$:

$$\sqrt[3]{z}_{k=0} = \sqrt[3]{27}\left[\cos\left(\frac{\frac{\pi}{4}+2\pi(0)}{3}\right) + i\sin\left(\frac{\frac{\pi}{4}+2\pi(0)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=0} = 3\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

For $k = 1$:

$$\sqrt[3]{z}_{k=1} = \sqrt[3]{27}\left[\cos\left(\frac{\frac{\pi}{4}+2\pi(1)}{3}\right) + i\sin\left(\frac{\frac{\pi}{4}+2\pi(1)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=1} = 3\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

For $k = 2$:

$$\sqrt[3]{z}_{k=2} = \sqrt[3]{27}\left[\cos\left(\frac{\frac{\pi}{4}+2\pi(2)}{3}\right) + i\sin\left(\frac{\frac{\pi}{4}+2\pi(2)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=2} = 3\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$$

The roots are

$$\sqrt[3]{z}_{k=0} = 3\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

$$\sqrt[3]{z}_{k=1} = 3\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$\sqrt[3]{z}_{k=2} = 3\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$$

■ 2. Find the 4th roots of the complex number.

$$z = 256(\cos 60^\circ + i \sin 60^\circ)$$

Solution:

We're looking for the 4th roots of z, which means there will be 4 of them, given by $k = 0, 1, 2, 3$. And since the complex number is given in degrees, we'll plug $n = 4$ into the formula for nth roots in degrees.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i \sin\left(\frac{\theta + 360^\circ k}{n}\right)\right]$$

$$\sqrt[4]{z} = \sqrt[4]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{4}\right) + i \sin\left(\frac{\theta + 360^\circ k}{4}\right)\right]$$

With $r = 256$ and $\theta = 60^\circ$ from the complex number, we get

$$\sqrt[4]{z} = \sqrt[4]{256}\left[\cos\left(\frac{60^\circ + 360^\circ k}{4}\right) + i \sin\left(\frac{60^\circ + 360^\circ k}{4}\right)\right]$$

Now we'll find values for $k = 0, 1, 2, 3$.

For $k = 0$:

$$\sqrt[4]{z}_{k=0} = \sqrt[4]{256}\left[\cos\left(\frac{60^\circ + 360^\circ(0)}{4}\right) + i \sin\left(\frac{60^\circ + 360^\circ(0)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=0} = 4[\cos(15^\circ) + i\sin(15^\circ)]$$

For $k = 1$:

$$\sqrt[4]{z}_{k=1} = \sqrt[4]{256}\left[\cos\left(\frac{60^\circ + 360^\circ(1)}{4}\right) + i\sin\left(\frac{60^\circ + 360^\circ(1)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=1} = 4[\cos(105^\circ) + i\sin(105^\circ)]$$

For $k = 2$:

$$\sqrt[4]{z}_{k=2} = \sqrt[4]{256}\left[\cos\left(\frac{60^\circ + 360^\circ(2)}{4}\right) + i\sin\left(\frac{60^\circ + 360^\circ(2)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=2} = 4[\cos(195^\circ) + i\sin(195^\circ)]$$

For $k = 3$:

$$\sqrt[4]{z}_{k=3} = \sqrt[4]{256}\left[\cos\left(\frac{60^\circ + 360^\circ(3)}{4}\right) + i\sin\left(\frac{60^\circ + 360^\circ(3)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=3} = 4[\cos(285^\circ) + i\sin(285^\circ)]$$

The roots are

$$\sqrt[4]{z}_{k=0} = 4[\cos(15^\circ) + i\sin(15^\circ)]$$

$$\sqrt[4]{z}_{k=1} = 4[\cos(105^\circ) + i\sin(105^\circ)]$$

$$\sqrt[4]{z}_{k=2} = 4[\cos(195^\circ) + i\sin(195^\circ)]$$

$$\sqrt[4]{z}_{k=3} = 4[\cos(285^\circ) + i\sin(285^\circ)]$$

■ 3. Find the 5th roots of the complex number that lies in the first quadrant of the complex plane.

$$z = 25(\cos 80^\circ + i\sin 80^\circ)$$

Solution:

We're looking for the 5th roots of z, which means there will be 5 of them, given by $k = 0,\ 1,\ 2,\ 3,\ 4$. And since the complex number is given in degrees, we'll plug $n = 5$ into the formula for nth roots in degrees.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i\sin\left(\frac{\theta + 360^\circ k}{n}\right)\right]$$

$$\sqrt[5]{z} = \sqrt[5]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{5}\right) + i\sin\left(\frac{\theta + 360^\circ k}{5}\right)\right]$$

With $r = 25$ and $\theta = 80^\circ$ from the complex number, we get

$$\sqrt[5]{z} = \sqrt[5]{25}\left[\cos\left(\frac{80^\circ + 360^\circ k}{5}\right) + i\sin\left(\frac{80^\circ + 360^\circ k}{5}\right)\right]$$

Now we'll find values for $k = 0,\ 1,\ 2,\ 3,\ 4$.

For $k = 0$:

$$\sqrt[5]{z}_{k=0}=\sqrt[5]{25}\left[\cos\left(\frac{80^\circ+360^\circ(0)}{5}\right)+i\sin\left(\frac{80^\circ+360^\circ(0)}{5}\right)\right]$$

$$\sqrt[5]{z}_{k=0}=\sqrt[5]{25}[\cos(16^\circ)+i\sin(16^\circ)]$$

For $k=1$:

$$\sqrt[5]{z}_{k=1}=\sqrt[5]{25}\left[\cos\left(\frac{80^\circ+360^\circ(1)}{5}\right)+i\sin\left(\frac{80^\circ+360^\circ(1)}{5}\right)\right]$$

$$\sqrt[5]{z}_{k=1}=\sqrt[5]{25}[\cos(88^\circ)+i\sin(88^\circ)]$$

For $k=2$:

$$\sqrt[5]{z}_{k=2}=\sqrt[5]{25}\left[\cos\left(\frac{80^\circ+360^\circ(2)}{5}\right)+i\sin\left(\frac{80^\circ+360^\circ(2)}{5}\right)\right]$$

$$\sqrt[5]{z}_{k=2}=\sqrt[5]{25}[\cos(160^\circ)+i\sin(160^\circ)]$$

For $k=3$:

$$\sqrt[5]{z}_{k=3}=\sqrt[5]{25}\left[\cos\left(\frac{80^\circ+360^\circ(3)}{5}\right)+i\sin\left(\frac{80^\circ+360^\circ(3)}{5}\right)\right]$$

$$\sqrt[5]{z}_{k=3}=\sqrt[5]{25}[\cos(232^\circ)+i\sin(232^\circ)]$$

For $k=4$:

$$\sqrt[5]{z}_{k=4}=\sqrt[5]{25}\left[\cos\left(\frac{80^\circ+360^\circ(4)}{5}\right)+i\sin\left(\frac{80^\circ+360^\circ(4)}{5}\right)\right]$$

$$\sqrt[5]{z}_{k=4} = \sqrt[5]{25}[\cos(304^\circ) + i\sin(304^\circ)]$$

The roots are

$$\sqrt[5]{z}_{k=0} = \sqrt[5]{25}[\cos(16^\circ) + i\sin(16^\circ)]$$

$$\sqrt[5]{z}_{k=1} = \sqrt[5]{25}[\cos(88^\circ) + i\sin(88^\circ)]$$

$$\sqrt[5]{z}_{k=2} = \sqrt[5]{25}[\cos(160^\circ) + i\sin(160^\circ)]$$

$$\sqrt[5]{z}_{k=3} = \sqrt[5]{25}[\cos(232^\circ) + i\sin(232^\circ)]$$

$$\sqrt[5]{z}_{k=4} = \sqrt[5]{25}[\cos(304^\circ) + i\sin(304^\circ)]$$

Anything in the first quadrant will fall in the interval $(0^\circ, 90^\circ)$. In this case, the angles for $k = 0$ and $k = 1$ are in the first quadrant.

■ 4. Find the 4th roots of the complex number.

$$z = 34\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$$

Solution:

We're looking for the 4th roots of z, which means there will be 4 of them, given by $k = 0, 1, 2, 3$. And since the complex number is given in radians, we'll plug $n = 4$ into the formula for nth roots in radians.

$$\sqrt[n]{z}=\sqrt[n]{r}\left[\cos\left(\frac{\theta+2\pi k}{n}\right)+i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$$

$$\sqrt[4]{z}=\sqrt[4]{r}\left[\cos\left(\frac{\theta+2\pi k}{4}\right)+i\sin\left(\frac{\theta+2\pi k}{4}\right)\right]$$

With $r=34$ and $\theta=3\pi/5$ from the complex number, we get

$$\sqrt[4]{z}=\sqrt[4]{34}\left[\cos\left(\frac{\frac{3\pi}{5}+2\pi k}{4}\right)+i\sin\left(\frac{\frac{3\pi}{5}+2\pi k}{4}\right)\right]$$

Now we'll find values for $k=0, 1, 2, 3$.

For $k=0$:

$$\sqrt[4]{z}_{k=0}=\sqrt[4]{34}\left[\cos\left(\frac{\frac{3\pi}{5}+2\pi(0)}{4}\right)+i\sin\left(\frac{\frac{3\pi}{5}+2\pi(0)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=0}=\sqrt[4]{34}\left(\cos\frac{3\pi}{20}+i\sin\frac{3\pi}{20}\right)$$

For $k=1$:

$$\sqrt[4]{z}_{k=1}=\sqrt[4]{34}\left[\cos\left(\frac{\frac{3\pi}{5}+2\pi(1)}{4}\right)+i\sin\left(\frac{\frac{3\pi}{5}+2\pi(1)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=1}=\sqrt[4]{34}\left(\cos\frac{13\pi}{20}+i\sin\frac{13\pi}{20}\right)$$

For $k = 2$:

$$\sqrt[4]{z}_{k=2} = \sqrt[4]{34}\left[\cos\left(\frac{\frac{3\pi}{5} + 2\pi(2)}{4}\right) + i\sin\left(\frac{\frac{3\pi}{5} + 2\pi(2)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=2} = \sqrt[4]{34}\left(\cos\frac{23\pi}{20} + i\sin\frac{23\pi}{20}\right)$$

For $k = 3$:

$$\sqrt[4]{z}_{k=3} = \sqrt[4]{34}\left[\cos\left(\frac{\frac{3\pi}{5} + 2\pi(3)}{4}\right) + i\sin\left(\frac{\frac{3\pi}{5} + 2\pi(3)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=3} = \sqrt[4]{34}\left(\cos\frac{33\pi}{20} + i\sin\frac{33\pi}{20}\right)$$

The roots are

$$\sqrt[4]{z}_{k=0} = \sqrt[4]{34}\left(\cos\frac{3\pi}{20} + i\sin\frac{3\pi}{20}\right)$$

$$\sqrt[4]{z}_{k=1} = \sqrt[4]{34}\left(\cos\frac{13\pi}{20} + i\sin\frac{13\pi}{20}\right)$$

$$\sqrt[4]{z}_{k=2} = \sqrt[4]{34}\left(\cos\frac{23\pi}{20} + i\sin\frac{23\pi}{20}\right)$$

$$\sqrt[4]{z}_{k=3} = \sqrt[4]{34}\left(\cos\frac{33\pi}{20} + i\sin\frac{33\pi}{20}\right)$$

■ 5. Find the 6th roots of the complex number that lies in the second quadrant of the complex plane.

$$z = 11\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

Solution:

We're looking for the 6th roots of z, which means there will be 6 of them, given by $k = 0, 1, 2, 3, 4, 5$. And since the complex number is given in radians, we'll plug $n = 6$ into the formula for nth roots in radians.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

$$\sqrt[6]{z} = \sqrt[6]{r}\left[\cos\left(\frac{\theta + 2\pi k}{6}\right) + i\sin\left(\frac{\theta + 2\pi k}{6}\right)\right]$$

With $r = 11$ and $\theta = 5\pi/6$ from the complex number, we get

$$\sqrt[6]{z} = \sqrt[6]{11}\left[\cos\left(\frac{\frac{5\pi}{6} + 2\pi k}{6}\right) + i\sin\left(\frac{\frac{5\pi}{6} + 2\pi k}{6}\right)\right]$$

Now we'll find values for $k = 0, 1, 2, 3, 4, 5$.

For $k = 0$:

$$\sqrt[6]{z}_{k=0}=\sqrt[6]{11}\left[\cos\left(\frac{\frac{5\pi}{6}+2\pi(0)}{6}\right)+i\sin\left(\frac{\frac{5\pi}{6}+2\pi(0)}{6}\right)\right]$$

$$\sqrt[6]{z}_{k=0}=\sqrt[6]{11}\left(\cos\frac{5\pi}{36}+i\sin\frac{5\pi}{36}\right)$$

For $k=1$:

$$\sqrt[6]{z}_{k=1}=\sqrt[6]{11}\left[\cos\left(\frac{\frac{5\pi}{6}+2\pi(1)}{6}\right)+i\sin\left(\frac{\frac{5\pi}{6}+2\pi(1)}{6}\right)\right]$$

$$\sqrt[6]{z}_{k=1}=\sqrt[6]{11}\left(\cos\frac{17\pi}{36}+i\sin\frac{17\pi}{36}\right)$$

For $k=2$:

$$\sqrt[6]{z}_{k=2}=\sqrt[6]{11}\left[\cos\left(\frac{\frac{5\pi}{6}+2\pi(2)}{6}\right)+i\sin\left(\frac{\frac{5\pi}{6}+2\pi(2)}{6}\right)\right]$$

$$\sqrt[6]{z}_{k=2}=\sqrt[6]{11}\left(\cos\frac{29\pi}{36}+i\sin\frac{29\pi}{36}\right)$$

We can start to see how we're just adding $12\pi/36$ to the angle each time we find a new k-value, so we can list the roots as

$$\sqrt[6]{z}_{k=0}=\sqrt[6]{11}\left(\cos\frac{5\pi}{36}+i\sin\frac{5\pi}{36}\right)$$

$$\sqrt[6]{z}_{k=1} = \sqrt[6]{11}\left(\cos\frac{17\pi}{36} + i\sin\frac{17\pi}{36}\right)$$

$$\sqrt[6]{z}_{k=2} = \sqrt[6]{11}\left(\cos\frac{29\pi}{36} + i\sin\frac{29\pi}{36}\right)$$

$$\sqrt[6]{z}_{k=3} = \sqrt[6]{11}\left(\cos\frac{41\pi}{36} + i\sin\frac{41\pi}{36}\right)$$

$$\sqrt[6]{z}_{k=4} = \sqrt[6]{11}\left(\cos\frac{53\pi}{36} + i\sin\frac{53\pi}{36}\right)$$

$$\sqrt[6]{z}_{k=5} = \sqrt[6]{11}\left(\cos\frac{65\pi}{36} + i\sin\frac{65\pi}{36}\right)$$

If we find the decimal approximations of these angles, we get

For $k = 0$, $(5/36)\pi \approx 0.14\pi$

For $k = 1$, $(17/36)\pi \approx 0.47\pi$

For $k = 2$, $(29/36)\pi \approx 0.81\pi$

For $k = 3$, $(41/36)\pi \approx 1.14\pi$

For $k = 4$, $(53/36)\pi \approx 1.47\pi$

For $k = 5$, $(65/36)\pi \approx 1.81\pi$

Anything in the second quadrant will fall in the interval $(0.5\pi, 1.0\pi)$, which in this case is the angle for $k = 2$.

■ 6. Find the 7th roots of the complex number.

$$z = 20(\cos 120^\circ + i \sin 120^\circ)$$

Solution:

We're looking for the 7th roots of z, which means there will be 7 of them, given by $k = 0, 1, 2, 3, 4, 5, 6$. And since the complex number is given in degrees, we'll plug $n = 7$ into the formula for nth roots in degrees.

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i \sin\left(\frac{\theta + 360^\circ k}{n}\right)\right]$$

$$\sqrt[7]{z} = \sqrt[7]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{7}\right) + i \sin\left(\frac{\theta + 360^\circ k}{7}\right)\right]$$

With $r = 20$ and $\theta = 120^\circ$ from the complex number, we get

$$\sqrt[7]{z} = \sqrt[7]{20}\left[\cos\left(\frac{120^\circ + 360^\circ k}{7}\right) + i \sin\left(\frac{120^\circ + 360^\circ k}{7}\right)\right]$$

Now we'll find values for $k = 0, 1, 2, 3, 4, 5, 6$.

For $k = 0$:

$$\sqrt[7]{z}_{k=0} = \sqrt[7]{20}\left[\cos\left(\frac{120^\circ + 360^\circ(0)}{7}\right) + i \sin\left(\frac{120^\circ + 360^\circ(0)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=0} = \sqrt[7]{20}\left[\cos\left(\frac{120}{7}\right)^{\circ} + i\sin\left(\frac{120}{7}\right)^{\circ}\right]$$

For $k = 1$:

$$\sqrt[7]{z}_{k=1} = \sqrt[7]{20}\left[\cos\left(\frac{120^{\circ} + 360^{\circ}(1)}{7}\right) + i\sin\left(\frac{120^{\circ} + 360^{\circ}(1)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=1} = \sqrt[7]{20}\left[\cos\left(\frac{480}{7}\right)^{\circ} + i\sin\left(\frac{480}{7}\right)^{\circ}\right]$$

For $k = 2$:

$$\sqrt[7]{z}_{k=2} = \sqrt[7]{20}\left[\cos\left(\frac{120^{\circ} + 360^{\circ}(2)}{7}\right) + i\sin\left(\frac{120^{\circ} + 360^{\circ}(2)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=2} = \sqrt[7]{20}\left[\cos\left(\frac{840}{7}\right)^{\circ} + i\sin\left(\frac{840}{7}\right)^{\circ}\right]$$

We can start to see how we're just adding $360/7$ to the angle each time we find a new k-value, so we can list the roots as

$$\sqrt[7]{z}_{k=0} = \sqrt[7]{20}\left[\cos\left(\frac{120}{7}\right)^{\circ} + i\sin\left(\frac{120}{7}\right)^{\circ}\right]$$

$$\sqrt[7]{z}_{k=1} = \sqrt[7]{20}\left[\cos\left(\frac{480}{7}\right)^{\circ} + i\sin\left(\frac{480}{7}\right)^{\circ}\right]$$

$$\sqrt[7]{z}_{k=2} = \sqrt[7]{20}\left[\cos\left(\frac{840}{7}\right)^{\circ} + i\sin\left(\frac{840}{7}\right)^{\circ}\right]$$

$$= \sqrt[7]{20}[\cos(120)^{\circ} + i\sin(120)^{\circ}]$$

$$\sqrt[7]{z}_{k=3} = \sqrt[7]{20}\left[\cos\left(\frac{1{,}200}{7}\right)^{\circ} + i\sin\left(\frac{1{,}200}{7}\right)^{\circ}\right]$$

$$\sqrt[7]{z}_{k=4} = \sqrt[7]{20}\left[\cos\left(\frac{1{,}560}{7}\right)^{\circ} + i\sin\left(\frac{1{,}560}{7}\right)^{\circ}\right]$$

$$\sqrt[7]{z}_{k=5} = \sqrt[7]{20}\left[\cos\left(\frac{1{,}920}{7}\right)^{\circ} + i\sin\left(\frac{1{,}920}{7}\right)^{\circ}\right]$$

$$\sqrt[7]{z}_{k=6} = \sqrt[7]{20}\left[\cos\left(\frac{2{,}280}{7}\right)^{\circ} + i\sin\left(\frac{2{,}280}{7}\right)^{\circ}\right]$$

MATRIX DIMENSIONS AND ENTRIES

■ 1. Give the dimensions of the matrix.

$$D = \begin{bmatrix} 11 & 9 \\ -4 & 8 \end{bmatrix}$$

Solution:

We always give the dimensions of a matrix as rows $\times$ columns. Matrix D has 2 rows and 2 columns, so D is a 2×2 matrix.

■ 2. Give the dimensions of the matrix.

$$A = \begin{bmatrix} 3 & 5 & -2 & 1 & 8 \end{bmatrix}$$

Solution:

We always give the dimensions of a matrix as rows $\times$ columns. Matrix A has 1 row and 5 columns, so A is a 1×5 matrix.

■ 3. Given matrix J, find $J_{4,1}$.

$$J = \begin{bmatrix} 6 \\ 2 \\ 7 \\ 1 \end{bmatrix}$$

Solution:

The value of $J_{4,1}$ is the entry in the fourth row, first column of matrix J, which is 1, so $J_{4,1} = 1$.

■ 4. Given matrix C, find $C_{1,2}$.

$$C = \begin{bmatrix} 3 & 12 \\ 1 & 4 \\ 9 & 5 \\ -3 & 2 \end{bmatrix}$$

Solution:

The value of $C_{1,2}$ is the entry in the first row, second column of matrix C, which is 12, so $C_{1,2} = 12$.

■ 5. Given matrix N, state the dimensions and find $N_{1,3}$.

$$N = \begin{bmatrix} 1 & 5 & 9 \\ 14 & -8 & 6 \end{bmatrix}$$

Solution:

We always give the dimensions of a matrix as rows $\times$ columns. Matrix N has 2 rows and 3 columns, so N is a 2×3 matrix.

The value of $N_{1,3}$ is the entry in the first row, third column of matrix N, which is 9, so $N_{1,3} = 9$.

■ 6. Given matrix S, state the dimensions and find $S_{3,4}$.

$$S = \begin{bmatrix} 3 & 6 & -7 & 1 & 0 \\ 0 & 9 & 15 & 3 & 4 \\ 4 & 0 & 2 & 11 & 8 \\ -5 & 8 & 7 & 9 & 2 \end{bmatrix}$$

Solution:

We always give the dimensions of a matrix as rows $\times$ columns. Matrix S has 4 rows and 5 columns, so S is a 4×5 matrix.

The value of $S_{3,4}$ is the entry in the third row, fourth column of matrix S, which is 11, so $S_{3,4} = 11$.

REPRESENTING SYSTEMS WITH MATRICES

■ 1. Represent the system with an augmented matrix called A.

$$-2x + 5y = 12$$

$$6x - 2y = 4$$

Solution:

The system contains the variables x and y along with a constant. Which means the augmented matrix will have two columns, one for each variable, plus a column for the constants, so three columns in total. Because there are two equations in the system, the matrix will have two rows. Plugging the coefficients and constants into an augmented matrix gives

$$A = \begin{bmatrix} -2 & 5 & 12 \\ 6 & -2 & 4 \end{bmatrix}$$

■ 2. Represent the system with an augmented matrix called D.

$$9y - 3x + 12 = 0$$

$$8 - 4x = 11y$$

Solution:

This system can be reorganized by putting each equation in order, with x and y on the left side, and the constant on the right side.

$$-3x + 9y = -12$$

$$4x + 11y = 8$$

The system contains the variables x and y along with a constant. Which means the augmented matrix will have two columns, one for each variable, plus a column for the constants, so three columns in total. Because there are two equations in the system, the matrix will have two rows. Plugging the coefficients and constants into an augmented matrix gives

$$D = \begin{bmatrix} -3 & 9 & -12 \\ 4 & 11 & 8 \end{bmatrix}$$

■ 3. Represent the system with an augmented matrix called H.

$$4a + 7b - 5c + 13d = 6$$

$$3a - 8b = -2c + 1$$

Solution:

The second equation can be reorganized by putting a, b, and c on the left side, and the constant on the right side. We also recognize that there is no d-term in the second equation, so we add in a 0 "filler" term.

$$4a + 7b - 5c + 13d = 6$$

$$3a - 8b + 2c + 0d = 1$$

The system contains the variables a, b, c, and d, along with a constant. Which means the augmented matrix will have four columns, one for each variable, plus a column for the constants, so five columns in total. Because there are two equations in the system, the matrix will have two rows. Plugging the coefficients and constants into an augmented matrix gives

$$H = \begin{bmatrix} 4 & 7 & -5 & 13 & 6 \\ 3 & -8 & 2 & 0 & 1 \end{bmatrix}$$

■ 4. Represent the system with an augmented matrix called M.

$$-2x + 4y = 9 - 6z$$

$$7y + 2z - 3 = -3t - 9x$$

Solution:

Both equations can be reorganized by putting x, y, z, and t on the left side, and the constant on the right side. We also recognize that there is no t -term in the first equation, so we add in a 0 “filler” term.

$$-2x + 4y + 6z + 0t = 9$$

$$9x + 7y + 2z + 3t = 3$$

The system contains the variables x, y, z, and t, along with a constant. Which means the augmented matrix will have four columns, one for each variable, plus a column for the constants, so five columns in total. Because there are two equations in the system, the matrix will have two rows. Plugging the coefficients and constants into an augmented matrix gives

$$M = \begin{bmatrix} -2 & 4 & 6 & 0 & 9 \\ 9 & 7 & 2 & 3 & 3 \end{bmatrix}$$

■ 5. Represent the system with an augmented matrix called A.

$$3x - 8y + z = 7$$

$$2z = 3y - 2x + 4$$

$$5y = 12 - 9x$$

Solution:

The second and third equations can be reorganized by putting x, y, and z on the left side, and the constant on the right side. We also recognize that there is no z-term in the third equation, so we add in a 0 "filler" term.

$$3x - 8y + z = 7$$

$$2x - 3y + 2z = 4$$

$$9x + 5y + 0z = 12$$

The system contains the variables x, y, and z, along with a constant. Which means the augmented matrix will have three columns, one for each variable, plus a column for the constants, so four columns in total. Because there are three equations in the system, the matrix will have three rows. Plugging the coefficients and constants into an augmented matrix gives

$$A = \begin{bmatrix} 3 & -8 & 1 & 7 \\ 2 & -3 & 2 & 4 \\ 9 & 5 & 0 & 12 \end{bmatrix}$$

■ 6. Represent the system with an augmented matrix called K.

$$-4b + 2c = 3 - 7a$$

$$9c = 4 - 2b$$

$$8a - 2c = 5b$$

Solution:

All three of these equations can be reorganized by putting a, b, and c on the left side, and the constant on the right side. We also recognize that there is no a-term in the second equation, and no constant in the third equation, so we add in 0 “filler” terms.

$$7a - 4b + 2c = 3$$

$$0a + 2b + 9c = 4$$

$$8a - 5b - 2c = 0$$

The system contains the variables a, b, and c, along with a constant. Which means the augmented matrix will have three columns, one for each variable, plus a column for the constants, so four columns in total. Because there are three equations in the system, the matrix will have three rows. Plugging the coefficients and constants into an augmented matrix gives

$$K = \begin{bmatrix} 7 & -4 & 2 & 3 \\ 0 & 2 & 9 & 4 \\ 8 & -5 & -2 & 0 \end{bmatrix}$$

SIMPLE ROW OPERATIONS

■ 1. Write the new matrix after $R_1 \leftrightarrow R_2$.

$$\begin{bmatrix} 2 & 6 & -4 & 1 \\ 8 & 2 & 1 & -5 \end{bmatrix}$$

Solution:

The operation described by $R_1 \leftrightarrow R_2$ is switching row 1 with row 2. The matrix after $R_1 \leftrightarrow R_2$ is

$$\begin{bmatrix} 8 & 2 & 1 & -5 \\ 2 & 6 & -4 & 1 \end{bmatrix}$$

■ 2. Write the new matrix after $R_2 \leftrightarrow R_4$.

$$\begin{bmatrix} 1 & 2 & 7 & -3 \\ 6 & 1 & 5 & -4 \\ -7 & 7 & 0 & 3 \\ 9 & 2 & 8 & 3 \end{bmatrix}$$

Solution:

The operation described by $R_2 \leftrightarrow R_4$ is switching row 2 with row 4. Nothing will happen to rows 1 and 3. The matrix after $R_2 \leftrightarrow R_4$ is

$$\begin{bmatrix} 1 & 2 & 7 & -3 \\ 9 & 2 & 8 & 3 \\ -7 & 7 & 0 & 3 \\ 6 & 1 & 5 & -4 \end{bmatrix}$$

■ 3. Write the new matrix after $R_1 \leftrightarrow 3R_2$.

$$\begin{bmatrix} 9 & 2 & -7 \\ 1 & 6 & 4 \end{bmatrix}$$

Solution:

The operation described by $R_1 \leftrightarrow 3R_2$ is multiplying row 2 by a constant of 3 and then switching those two rows. The matrix after $3R_2$ is

$$\begin{bmatrix} 9 & 2 & -7 \\ 3 & 18 & 12 \end{bmatrix}$$

The matrix after $R_1 \leftrightarrow 3R_2$ is

$$\begin{bmatrix} 3 & 18 & 12 \\ 9 & 2 & -7 \end{bmatrix}$$

■ 4. Write the new matrix after $3R_2 \leftrightarrow 3R_4$.

$$\begin{bmatrix} 0 & 11 & 6 \\ 7 & -3 & 9 \\ 8 & 8 & 1 \\ 6 & 2 & 4 \end{bmatrix}$$

Solution:

The operation described by $3R_2 \leftrightarrow 3R_4$ is multiplying row 2 by a constant of 3, multiplying row 4 by a constant of 3, and then switching those two rows. Nothing will happen to rows 1 and 3. The matrix after $3R_2$ is

$$\begin{bmatrix} 0 & 11 & 6 \\ 21 & -9 & 27 \\ 8 & 8 & 1 \\ 6 & 2 & 4 \end{bmatrix}$$

The matrix after $3R_4$ is

$$\begin{bmatrix} 0 & 11 & 6 \\ 21 & -9 & 27 \\ 8 & 8 & 1 \\ 18 & 6 & 12 \end{bmatrix}$$

The matrix after $3R_2 \leftrightarrow 3R_4$ is

$$\begin{bmatrix} 0 & 11 & 6 \\ 18 & 6 & 12 \\ 8 & 8 & 1 \\ 21 & -9 & 27 \end{bmatrix}$$

■ 5. Write the new matrix after $R_1 + 2R_2 \to R_1$.

$$\begin{bmatrix} 6 & 2 & 7 \\ 1 & -5 & 15 \end{bmatrix}$$

Solution:

The operation described by $R_1 + 2R_2 \to R_1$ is multiplying row 2 by a constant of 2, adding that resulting row to row 1, and using that result to replace row 1. $2R_2$ is

$$\begin{bmatrix} 2(1) & 2(-5) & 2(15) \end{bmatrix}$$

$$\begin{bmatrix} 2 & -10 & 30 \end{bmatrix}$$

The sum $R_1 + 2R_2$ is

$$\begin{bmatrix} 6+2 & 2-10 & 7+30 \end{bmatrix}$$

$$\begin{bmatrix} 8 & -8 & 37 \end{bmatrix}$$

The matrix after $R_1 + 2R_2 \to R_1$, which is replacing row 1 with this row we just found, is

$$\begin{bmatrix} 8 & -8 & 37 \\ 1 & -5 & 15 \end{bmatrix}$$

■ 6. Write the new matrix after $4R_2 + R_3 \to R_3$.

$$\begin{bmatrix} 13 & 5 & -2 & 9 \\ 8 & 2 & 0 & 6 \\ 4 & 1 & 7 & -3 \end{bmatrix}$$

Solution:

The operation described by $4R_2 + R_3 \to R_3$ is multiplying row 2 by a constant of 4, adding that resulting row to row 3, and using that result to replace row 3. $4R_2$ is

$$\begin{bmatrix} 4(8) & 4(2) & 4(0) & 4(6) \end{bmatrix}$$

$$\begin{bmatrix} 32 & 8 & 0 & 24 \end{bmatrix}$$

The sum $4R_2 + R_3$ is

$$\begin{bmatrix} 32+4 & 8+1 & 0+7 & 24-3 \end{bmatrix}$$

$$\begin{bmatrix} 36 & 9 & 7 & 21 \end{bmatrix}$$

The matrix after $4R_2 + R_3 \to R_3$, which is replacing row 3 with this row we just found, is

$$\begin{bmatrix} 13 & 5 & -2 & 9 \\ 8 & 2 & 0 & 6 \\ 36 & 9 & 7 & 21 \end{bmatrix}$$

GAUSS-JORDAN ELIMINATION AND REDUCED ROW-ECHELON FORM

■ 1. Use Gauss-Jordan elimination to solve the system.

$$x+2y=-2$$

$$3x+2y=6$$

Solution:

The augmented matrix for the system is

$$\begin{bmatrix}1 & 2 & = & -2\\ 3 & 2 & = & 6\end{bmatrix}$$

After $3R_1-R_2\to R_2$, the matrix is

$$\begin{bmatrix}1 & 2 & = & -2\\ 0 & 4 & = & -12\end{bmatrix}$$

The first column is done. After $(1/4)R_2\to R_2$, the matrix is

$$\begin{bmatrix}1 & 2 & = & -2\\ 0 & 1 & = & -3\end{bmatrix}$$

After $R_1-2R_2\to R_1$, the matrix is

$$\begin{bmatrix}1 & 0 & = & 4\\ 0 & 1 & = & -3\end{bmatrix}$$

The second column is done, so we get the solution set

$$x = 4$$

$$y = -3$$

■ 2. Use Gauss-Jordan elimination to solve the system.

$$2x + 4y = 22$$

$$3x + 3y = 15$$

Solution:

The augmented matrix for the system is

$$\begin{bmatrix} 2 & 4 & = & 22 \\ 3 & 3 & = & 15 \end{bmatrix}$$

After $(1/2)R_1 \to R_1$ and $(1/3)R_2 \to R_2$the matrix is

$$\begin{bmatrix} 1 & 2 & = & 11 \\ 1 & 1 & = & 5 \end{bmatrix}$$

After $R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & = & 11 \\ 0 & 1 & = & 6 \end{bmatrix}$$

The first column is done. After $R_1 - 2R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & = & -1 \\ 0 & 1 & = & 6 \end{bmatrix}$$

The second column is done, so we get the solution set

$$x = -1$$

$$y = 6$$

■ 3. Use Gauss-Jordan elimination to solve the system.

$$x - 3y - 6z = 4$$

$$y + 2z = -2$$

$$-4x + 12y + 21z = -4$$

Solution:

The augmented matrix for the system is

$$\begin{bmatrix} 1 & -3 & -6 & = & 4 \\ 0 & 1 & 2 & = & -2 \\ -4 & 12 & 21 & = & -4 \end{bmatrix}$$

After $4R_1 + R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & -3 & -6 & = & 4 \\ 0 & 1 & 2 & = & -2 \\ 0 & 0 & -3 & = & 12 \end{bmatrix}$$

The first column is done. After $3R_2 + R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & -2 \\ 0 & 1 & 2 & = & -2 \\ 0 & 0 & -3 & = & 12 \end{bmatrix}$$

The second column is done. After $(-1/3)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & -2 \\ 0 & 1 & 2 & = & -2 \\ 0 & 0 & 1 & = & -4 \end{bmatrix}$$

After $R_2 - 2R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & -2 \\ 0 & 1 & 0 & = & 6 \\ 0 & 0 & 1 & = & -4 \end{bmatrix}$$

The third column is done, so we get the solution set

$$x = -2$$

$$y = 6$$

$$z = -4$$

■ 4. Use Gauss-Jordan elimination to solve the system.

$$2y + 4z = 4$$

$$x + 3y + 3z = 5$$

$$2x + 7y + 6z = 10$$

Solution:

The augmented matrix for the system is

$$\begin{bmatrix} 0 & 2 & 4 & = & 4 \\ 1 & 3 & 3 & = & 5 \\ 2 & 7 & 6 & = & 10 \end{bmatrix}$$

After $(1/2)R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 0 & 1 & 2 & = & 2 \\ 1 & 3 & 3 & = & 5 \\ 2 & 7 & 6 & = & 10 \end{bmatrix}$$

Because the first entry in the first row is 0, swap it with the second row to get

$$\begin{bmatrix} 1 & 3 & 3 & = & 5 \\ 0 & 1 & 2 & = & 2 \\ 2 & 7 & 6 & = & 10 \end{bmatrix}$$

After $R_3 - 2R_1 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 3 & 3 & = & 5 \\ 0 & 1 & 2 & = & 2 \\ 0 & 1 & 0 & = & 0 \end{bmatrix}$$

The first column is done. After $R_1 - 3R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & = & -1 \\ 0 & 1 & 2 & = & 2 \\ 0 & 1 & 0 & = & 0 \end{bmatrix}$$

After $R_2 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & = & -1 \\ 0 & 1 & 2 & = & 2 \\ 0 & 0 & 2 & = & 2 \end{bmatrix}$$

The second column is done. After $(1/2)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & = & -1 \\ 0 & 1 & 2 & = & 2 \\ 0 & 0 & 1 & = & 1 \end{bmatrix}$$

After $R_1 + 3R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & 2 \\ 0 & 1 & 2 & = & 2 \\ 0 & 0 & 1 & = & 1 \end{bmatrix}$$

After $R_2 - 2R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & 2 \\ 0 & 1 & 0 & = & 0 \\ 0 & 0 & 1 & = & 1 \end{bmatrix}$$

The third column is done, so we get the solution set

$$x = 2$$

$$y = 0$$

$$z = 1$$

■ 5. Use Gauss-Jordan elimination to solve the system.

$$3x + 12y + 42z = -27$$

$$x + 2y + 8z = -5$$

$$2x + 5y + 16z = -6$$

Solution:

The augmented matrix for the system is

$$\begin{bmatrix} 3 & 12 & 42 & = & -27 \\ 1 & 2 & 8 & = & -5 \\ 2 & 5 & 16 & = & -6 \end{bmatrix}$$

After $(1/3)R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 4 & 14 & = & -9 \\ 1 & 2 & 8 & = & -5 \\ 2 & 5 & 16 & = & -6 \end{bmatrix}$$

After $R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 4 & 14 & = & -9 \\ 0 & 2 & 6 & = & -4 \\ 2 & 5 & 16 & = & -6 \end{bmatrix}$$

After $2R_1 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 4 & 14 & = & -9 \\ 0 & 2 & 6 & = & -4 \\ 0 & 3 & 12 & = & -12 \end{bmatrix}$$

The first column is done. After $(1/2)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 4 & 14 & = & -9 \\ 0 & 1 & 3 & = & -2 \\ 0 & 3 & 12 & = & -12 \end{bmatrix}$$

After $R_1 - 4R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 2 & = & -1 \\ 0 & 1 & 3 & = & -2 \\ 0 & 3 & 12 & = & -12 \end{bmatrix}$$

After $R_3 - 3R_2 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 2 & = & -1 \\ 0 & 1 & 3 & = & -2 \\ 0 & 0 & 3 & = & -6 \end{bmatrix}$$

The second column is done. After $(1/3)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 2 & = & -1 \\ 0 & 1 & 3 & = & -2 \\ 0 & 0 & 1 & = & -2 \end{bmatrix}$$

After $R_1 - 2R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & 3 \\ 0 & 1 & 3 & = & -2 \\ 0 & 0 & 1 & = & -2 \end{bmatrix}$$

After $R_2 - 3R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & 3 \\ 0 & 1 & 0 & = & 4 \\ 0 & 0 & 1 & = & -2 \end{bmatrix}$$

The third column is done, so we get the solution set

$$x = 3$$

$$y = 4$$

$$z = -2$$

■ 6. Use Gauss-Jordan elimination to solve the system.

$$4x + 8y + 4z = 20$$

$$4x + 6y = 4$$

$$3x + 3y - z = 1$$

Solution:

The augmented matrix for the system is

$$\begin{bmatrix} 4 & 8 & 4 & = & 20 \\ 4 & 6 & 0 & = & 4 \\ 3 & 3 & -1 & = & 1 \end{bmatrix}$$

After $(1/4)R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & = & 5 \\ 4 & 6 & 0 & = & 4 \\ 3 & 3 & -1 & = & 1 \end{bmatrix}$$

After $4R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & = & 5 \\ 0 & 2 & 4 & = & 16 \\ 3 & 3 & -1 & = & 1 \end{bmatrix}$$

After $3R_1 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & = & 5 \\ 0 & 2 & 4 & = & 16 \\ 0 & 3 & 4 & = & 14 \end{bmatrix}$$

The first column is done. After $(1/2)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & = & 5 \\ 0 & 1 & 2 & = & 8 \\ 0 & 3 & 4 & = & 14 \end{bmatrix}$$

After $R_1 - 2R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & = & -11 \\ 0 & 1 & 2 & = & 8 \\ 0 & 3 & 4 & = & 14 \end{bmatrix}$$

After $3R_2 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & = & -11 \\ 0 & 1 & 2 & = & 8 \\ 0 & 0 & 2 & = & 10 \end{bmatrix}$$

The second column is done. After $(1/2)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & = & -11 \\ 0 & 1 & 2 & = & 8 \\ 0 & 0 & 1 & = & 5 \end{bmatrix}$$

After $R_1 + 3R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & 4 \\ 0 & 1 & 2 & = & 8 \\ 0 & 0 & 1 & = & 5 \end{bmatrix}$$

After $R_2 - 2R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & 4 \\ 0 & 1 & 0 & = & -2 \\ 0 & 0 & 1 & = & 5 \end{bmatrix}$$

The third column is done, so we get the solution set

$$x = 4$$

$$y = -2$$

$$z = 5$$

MATRIX ADDITION AND SUBTRACTION

■ 1. Add the matrices.

$$\begin{vmatrix} 7 & 6 \\ 17 & 9 \end{vmatrix} + \begin{vmatrix} 0 & 8 \\ -2 & 5 \end{vmatrix}$$

Solution:

To add matrices, you simply add together entries from corresponding positions in each matrix.

$$\begin{vmatrix} 7 & 6 \\ 17 & 9 \end{vmatrix} + \begin{vmatrix} 0 & 8 \\ -2 & 5 \end{vmatrix}$$

$$\begin{vmatrix} 7+0 & 6+8 \\ 17+(-2) & 9+5 \end{vmatrix}$$

$$\begin{vmatrix} 7 & 14 \\ 15 & 14 \end{vmatrix}$$

■ 2. Add the matrices.

$$\begin{vmatrix} 8 & 3 \\ -4 & 7 \\ 6 & 0 \\ 1 & 13 \end{vmatrix} + \begin{vmatrix} 6 & 7 \\ 2 & -3 \\ 9 & 11 \\ 7 & -2 \end{vmatrix}$$

Solution:

To add matrices, you simply add together entries from corresponding positions in each matrix.

$$\begin{vmatrix} 8 & 3 \\ -4 & 7 \\ 6 & 0 \\ 1 & 13 \end{vmatrix} + \begin{vmatrix} 6 & 7 \\ 2 & -3 \\ 9 & 11 \\ 7 & -2 \end{vmatrix}$$

$$\begin{vmatrix} 8+6 & 3+7 \\ -4+2 & 7+(-3) \\ 6+9 & 0+11 \\ 1+7 & 13+(-2) \end{vmatrix}$$

$$\begin{vmatrix} 14 & 10 \\ -2 & 4 \\ 15 & 11 \\ 8 & 11 \end{vmatrix}$$

■ 3. Subtract the matrices.

$$\begin{vmatrix} 7 & 9 \\ 4 & -1 \end{vmatrix} - \begin{vmatrix} 3 & 8 \\ 12 & -3 \end{vmatrix}$$

Solution:

To subtract matrices, you simply subtract entries from corresponding positions in each matrix.

$$\begin{vmatrix} 7 & 9 \\ 4 & -1 \end{vmatrix} - \begin{vmatrix} 3 & 8 \\ 12 & -3 \end{vmatrix}$$

$$\begin{vmatrix} 7-3 & 9-8 \\ 4-12 & -1-(-3) \end{vmatrix}$$

$$\begin{vmatrix} 4 & 1 \\ -8 & 2 \end{vmatrix}$$

■ 4. Subtract the matrices.

$$\begin{vmatrix} 8 & 11 & 2 & 9 \\ 6 & 3 & 16 & 8 \end{vmatrix} - \begin{vmatrix} 6 & 11 & 7 & -4 \\ 5 & 8 & 1 & 15 \end{vmatrix}$$

Solution:

To subtract matrices, you simply subtract entries from corresponding positions in each matrix.

$$\begin{vmatrix} 8 & 11 & 2 & 9 \\ 6 & 3 & 16 & 8 \end{vmatrix} - \begin{vmatrix} 6 & 11 & 7 & -4 \\ 5 & 8 & 1 & 15 \end{vmatrix}$$

$$\begin{vmatrix} 8-6 & 11-11 & 2-7 & 9-(-4) \\ 6-5 & 3-8 & 16-1 & 8-15 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 0 & -5 & 13 \\ 1 & -5 & 15 & -7 \end{vmatrix}$$

5. Solve for m.

$$\begin{vmatrix} 6 & 5 \\ 9 & -9 \end{vmatrix} + \begin{vmatrix} 3 & 7 \\ 1 & 6 \end{vmatrix} = m + \begin{vmatrix} 7 & 12 \\ -3 & -1 \end{vmatrix} - \begin{vmatrix} 1 & 8 \\ 4 & -7 \end{vmatrix}$$

Solution:

Let's start with the matrix addition on the left side of the equation and the matrix subtraction on the right side of the equation.

$$\begin{vmatrix} 6 & 5 \\ 9 & -9 \end{vmatrix} + \begin{vmatrix} 3 & 7 \\ 1 & 6 \end{vmatrix} = m + \begin{vmatrix} 7 & 12 \\ -3 & -1 \end{vmatrix} - \begin{vmatrix} 1 & 8 \\ 4 & -7 \end{vmatrix}$$

$$\begin{vmatrix} 6+3 & 5+7 \\ 9+1 & -9+6 \end{vmatrix} = m + \begin{vmatrix} 7-1 & 12-8 \\ -3-4 & -1-(-7) \end{vmatrix}$$

$$\begin{vmatrix} 9 & 12 \\ 10 & -3 \end{vmatrix} = m + \begin{vmatrix} 6 & 4 \\ -7 & 6 \end{vmatrix}$$

To isolate m, we'll subtract the matrix on the right from both sides in order to move it to the left.

$$\begin{vmatrix} 9 & 12 \\ 10 & -3 \end{vmatrix} - \begin{vmatrix} 6 & 4 \\ -7 & 6 \end{vmatrix} = m$$

$$\begin{vmatrix} 9-6 & 12-4 \\ 10-(-7) & -3-6 \end{vmatrix} = m$$

$$\begin{vmatrix} 3 & 8 \\ 17 & -9 \end{vmatrix} = m$$

The conclusion is that the value of m that makes the equation true is this matrix:

$$m = \begin{vmatrix} 3 & 8 \\ 17 & -9 \end{vmatrix}$$

■ 6. Solve for n.

$$\begin{vmatrix} 4 & 12 \\ 9 & 8 \end{vmatrix} - \begin{vmatrix} 0 & 3 \\ 9 & 9 \end{vmatrix} = n - \begin{vmatrix} 6 & 3 \\ 5 & 11 \end{vmatrix} + \begin{vmatrix} 7 & -4 \\ -18 & 1 \end{vmatrix}$$

Solution:

Let's start with the matrix subtraction on the left side of the equation and the matrix addition on the right side of the equation.

$$\begin{vmatrix} 4 & 12 \\ 9 & 8 \end{vmatrix} - \begin{vmatrix} 0 & 3 \\ 9 & 9 \end{vmatrix} = n - \begin{vmatrix} 6 & 3 \\ 5 & 11 \end{vmatrix} + \begin{vmatrix} 7 & -4 \\ -18 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 4-0 & 12-3 \\ 9-9 & 8-9 \end{vmatrix} = n - \begin{vmatrix} 6+7 & 3+(-4) \\ 5+(-18) & 11+1 \end{vmatrix}$$

$$\begin{vmatrix} 4 & 9 \\ 0 & -1 \end{vmatrix} = n - \begin{vmatrix} 13 & -1 \\ -13 & 12 \end{vmatrix}$$

To isolate n, we'll add the matrix on the right to both sides in order to move it to the left.

$$\begin{vmatrix} 4 & 9 \\ 0 & -1 \end{vmatrix} + \begin{vmatrix} 13 & -1 \\ -13 & 12 \end{vmatrix} = n$$

$$\begin{vmatrix} 4+13 & 9+(-1) \\ 0+(-13) & -1+12 \end{vmatrix} = n$$

$$\begin{vmatrix} 17 & 8 \\ -13 & 11 \end{vmatrix} = n$$

The conclusion is that the value of n that makes the equation true is this matrix:

$$n = \begin{vmatrix} 17 & 8 \\ -13 & 11 \end{vmatrix}$$

SCALAR MULTIPLICATION AND ZERO MATRICES

■ 1. Use scalar multiplication to simplify the expression.

$$\frac{1}{4}\begin{vmatrix} 12 & 8 & 3 \\ 2 & -16 & 0 \\ 1 & 5 & 7 \end{vmatrix}$$

Solution:

The scalar $1/4$ is being multiplied by the matrix. Distribute the scalar across every entry in the matrix.

$$\frac{1}{4}\begin{vmatrix} 12 & 8 & 3 \\ 2 & -16 & 0 \\ 1 & 5 & 7 \end{vmatrix}$$

$$\begin{vmatrix} \frac{1}{4}(12) & \frac{1}{4}(8) & \frac{1}{4}(3) \\ \frac{1}{4}(2) & \frac{1}{4}(-16) & \frac{1}{4}(0) \\ \frac{1}{4}(1) & \frac{1}{4}(5) & \frac{1}{4}(7) \end{vmatrix}$$

$$\begin{vmatrix} 3 & 2 & \frac{3}{4} \\ \frac{1}{2} & -4 & 0 \\ \frac{1}{4} & \frac{5}{4} & \frac{7}{4} \end{vmatrix}$$

■ 2. Solve for y.

$$4\begin{vmatrix} 2 & 9 \\ -5 & 0 \end{vmatrix} + y = 5\begin{vmatrix} 1 & -3 \\ 6 & 8 \end{vmatrix}$$

Solution:

Apply the scalars to the matrices.

$$\begin{vmatrix} 4(2) & 4(9) \\ 4(-5) & 4(0) \end{vmatrix} + y = \begin{vmatrix} 5(1) & 5(-3) \\ 5(6) & 5(8) \end{vmatrix}$$

$$\begin{vmatrix} 8 & 36 \\ -20 & 0 \end{vmatrix} + y = \begin{vmatrix} 5 & -15 \\ 30 & 40 \end{vmatrix}$$

Subtract the matrix on the left from both sides of the equation in order to isolate y.

$$y = \begin{vmatrix} 5 & -15 \\ 30 & 40 \end{vmatrix} - \begin{vmatrix} 8 & 36 \\ -20 & 0 \end{vmatrix}$$

$$y = \begin{vmatrix} 5-8 & -15-36 \\ 30-(-20) & 40-0 \end{vmatrix}$$

$$y = \begin{vmatrix} -3 & -51 \\ 50 & 40 \end{vmatrix}$$

■ 3. Solve for n.

$$-2\begin{bmatrix} 6 & 5 \\ 0 & 11 \end{bmatrix} = n - 4\begin{bmatrix} 2 & 4 \\ -1 & 9 \end{bmatrix}$$

Solution:

Apply the scalars to the matrices.

$$\begin{bmatrix} -2(6) & -2(5) \\ -2(0) & -2(11) \end{bmatrix} = n - \begin{bmatrix} 4(2) & 4(4) \\ 4(-1) & 4(9) \end{bmatrix}$$

$$\begin{bmatrix} -12 & -10 \\ 0 & -22 \end{bmatrix} = n - \begin{bmatrix} 8 & 16 \\ -4 & 36 \end{bmatrix}$$

Add the matrix on the right to both sides of the equation in order to isolate n.

$$\begin{bmatrix} -12 & -10 \\ 0 & -22 \end{bmatrix} + \begin{bmatrix} 8 & 16 \\ -4 & 36 \end{bmatrix} = n$$

$$\begin{bmatrix} -12+8 & -10+16 \\ 0+(-4) & -22+36 \end{bmatrix} = n$$

$$\begin{bmatrix} -4 & 6 \\ -4 & 14 \end{bmatrix} = n$$

$$n = \begin{bmatrix} -4 & 6 \\ -4 & 14 \end{bmatrix}$$

■ 4. Add the zero matrix to the given matrix.

$$\begin{vmatrix} 8 & 17 \\ -6 & 0 \end{vmatrix}$$

Solution:

Adding the zero matrix to any other matrix doesn't change the value of the matrix, so

$$\begin{vmatrix} 8 & 17 \\ -6 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} 8 & 17 \\ -6 & 0 \end{vmatrix}$$

■ 5. Find the opposite matrix.

$$\begin{vmatrix} 6 & 8 & 0 \\ 2 & -3 & 11 \\ 4 & 12 & 9 \end{vmatrix}$$

Solution:

To get the opposite of a matrix, multiply it by a scalar of -1. Then the opposite of the given matrix is

$$(-1)\begin{vmatrix} (-1)6 & (-1)8 & (-1)0 \\ (-1)2 & (-1)(-3) & (-1)11 \\ (-1)4 & (-1)12 & (-1)9 \end{vmatrix}$$

$$\begin{vmatrix} -6 & -8 & 0 \\ -2 & 3 & -11 \\ -4 & -12 & -9 \end{vmatrix}$$

■ 6. Multiply the matrix by a scalar of 0.

$$\begin{vmatrix} 14 & -1 & 7 & 5 \\ 3 & 7 & 18 & -4 \end{vmatrix}$$

Solution:

Multiplying any matrix by a scalar of 0 results in a zero matrix.

$$(0)\begin{vmatrix} 14(0) & -1(0) & 7(0) & 5(0) \\ 3(0) & 7(0) & 18(0) & -4(0) \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

MATRIX MULTIPLICATION

■ 1. If matrix A is 3×3 and matrix B is 3×4, say whether AB or BA is defined, and give the dimensions of any product that's defined.

Solution:

Line up the dimensions for the products AB and BA, and compare the middle terms, which represent the columns from the first matrix and the rows from the second matrix.

$$AB: \quad 3 \times \boxed{3 \qquad 3} \times 4$$

$$BA: \quad 3 \times \boxed{4 \qquad 3} \times 3$$

The middle numbers match for AB, so that product is defined. For BA, the middle numbers don't match, so that product isn't defined.

The dimensions of AB are given by the outside numbers, which are the rows from the first matrix and the columns from the second matrix.

$$AB: \quad \boxed{3} \times 3 \qquad 3 \times \boxed{4}$$

So the dimensions of AB will be 3×4.

■ 2. Find the product of matrices A and B.

$$A = \begin{bmatrix} 2 & 6 \\ -3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 0 \\ 5 & -4 \end{bmatrix}$$

Solution:

Multiply matrix A by matrix B.

$$A \cdot B = \begin{bmatrix} 2 & 6 \\ -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 \\ 5 & -4 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 2(-2) + 6(5) & 2(0) + 6(-4) \\ -3(-2) + 1(5) & -3(0) + 1(-4) \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 26 & -24 \\ 11 & -4 \end{bmatrix}$$

■ 3. Find the product of matrices A and B.

$$A = \begin{bmatrix} 5 & -1 \\ 0 & 11 \\ 7 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 1 & 8 \\ -3 & 0 & 4 \end{bmatrix}$$

Solution:

Multiply matrix A by matrix B.

$$A \cdot B = \begin{bmatrix} 5 & -1 \\ 0 & 11 \\ 7 & -2 \end{bmatrix} \cdot \begin{bmatrix} 6 & 1 & 8 \\ -3 & 0 & 4 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 5(6)+(-1)(-3) & 5(1)+(-1)(0) & 5(8)+(-1)(4) \\ 0(6)+11(-3) & 0(1)+11(0) & 0(8)+11(4) \\ 7(6)+(-2)(-3) & 7(1)+(-2)(0) & 7(8)+(-2)(4) \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 33 & 5 & 36 \\ -33 & 0 & 44 \\ 48 & 7 & 48 \end{bmatrix}$$

■ 4. Find the product of matrices A and B.

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 8 \\ 0 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 2 \\ 4 & 8 \end{bmatrix}$$

Solution:

Multiply matrix A by matrix B.

$$A \cdot B = \begin{bmatrix} 3 & -2 \\ 1 & 8 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 & 2 \\ 4 & 8 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 3(5) + (-2)(4) & 3(2) + (-2)(8) \\ 1(5) + 8(4) & 1(2) + 8(8) \\ 0(5) + 3(4) & 0(2) + 3(8) \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 7 & -10 \\ 37 & 66 \\ 12 & 24 \end{bmatrix}$$

■ 5. Use the distributive property to find $A(B+C)$.

$$A = \begin{bmatrix} 2 & 0 \\ 4 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 1 \\ 3 & -1 \end{bmatrix}$$

Solution:

Applying the distributive property to the initial expression, we get

$$A(B+C) = AB + AC$$

Use matrix multiplication to find $AB + AC$.

$$AB + AC = \begin{bmatrix} 2 & 0 \\ 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 3 & -1 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 2(3) + (0)(5) & 2(1) + 0(4) \\ 4(3) + (-2)(5) & 4(1) + (-2)(4) \end{bmatrix}$$

$$+ \begin{bmatrix} 2(6) + 0(3) & 2(1) + 0(-1) \\ 4(6) + (-2)(3) & 4(1) + (-2)(-1) \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 6 & 2 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} 12 & 2 \\ 18 & 6 \end{bmatrix}$$

Now use matrix addition.

$$AB + AC = \begin{bmatrix} 6 + 12 & 2 + 2 \\ 2 + 18 & -4 + 6 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 18 & 4 \\ 20 & 2 \end{bmatrix}$$

So the value of the original expression is

$$A(B + C) = \begin{bmatrix} 18 & 4 \\ 20 & 2 \end{bmatrix}$$

■ 6. Find the product of matrices A and B.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & -2 & 8 & 1 \\ 7 & 3 & 5 & 2 \end{bmatrix}$$

Solution:

Multiply matrix A by matrix B.

$$A \cdot B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 6 & -2 & 8 & 1 \\ 7 & 3 & 5 & 2 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 0(6)+0(7) & 0(-2)+0(3) & 0(8)+0(5) & 0(1)+0(2) \\ 0(6)+0(7) & 0(-2)+0(3) & 0(8)+0(5) & 0(1)+0(2) \\ 0(6)+0(7) & 0(-2)+0(3) & 0(8)+0(5) & 0(1)+0(2) \\ 0(6)+0(7) & 0(-2)+0(3) & 0(8)+0(5) & 0(1)+0(2) \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

IDENTITY MATRICES

■ 1. Write the identity matrix I_4.

Solution:

We always call the identity matrix I, and it's always a square matrix, like 2×2, 3×3, 4×4, etc. For that reason, it's common to abbreviate I_{2x2} as just I_2, or I_{3x3} as just I_3, etc. So, I_4 is the 4×4 identity matrix.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

■ 2. If we want to find the product IA, where I is the identity matrix and A is a 4×2, then what are the dimensions of I?

Solution:

Start by setting up the equation $I \cdot A = A$, then substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 4 \times 2 = 4 \times 2$$

Break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 4 \times 2 = 4 \times 2$$

For matrix multiplication to be defined, you need the same number of columns in the first matrix as rows in the second matrix.

$$R \times \boxed{4 \cdot 4} \times 2 = 4 \times 2$$

The dimensions of the product come from the rows in the first matrix and the columns in the second matrix, so

$$\boxed{4} \times 4 \cdot 4 \times 2 = \boxed{4} \times 2$$

Therefore, the identity matrix in this case is I_4.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

■ 3. If we want to find the product IA, where I is the identity matrix and A is a 3×4, then what are the dimensions of I?

Solution:

Start by setting up the equation $I \cdot A = A$, then substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 3 \times 4 = 3 \times 4$$

Break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 3 \times 4 = 3 \times 4$$

For matrix multiplication to be defined, you need the same number of columns in the first matrix as rows in the second matrix.

$$R \times \boxed{3 \cdot 3} \times 4 = 3 \times 4$$

The dimensions of the product come from the rows in the first matrix and the columns in the second matrix, so

$$\boxed{3} \times 3 \cdot 3 \times 4 = \boxed{3} \times 4$$

Therefore, the identity matrix in this case is I_3.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

■ 4. If we want to find the product IA, where I is the identity matrix and A is given, then what are the dimensions of I? What is the product IA?

$$A = \begin{bmatrix} 2 & 8 \\ -2 & 7 \\ 3 & 5 \end{bmatrix}$$

Solution:

Start by setting up the equation $I \cdot A = A$, then substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 3 \times 2 = 3 \times 2$$

Break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 3 \times 2 = 3 \times 2$$

For matrix multiplication to be defined, you need the same number of columns in the first matrix as rows in the second matrix.

$$R \times \boxed{3 \cdot 3} \times 2 = 3 \times 2$$

The dimensions of the product come from the rows in the first matrix and the columns in the second matrix, so

$$\boxed{3} \times 3 \cdot 3 \times 2 = \boxed{3} \times 2$$

Therefore, the identity matrix in this case is I_3.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The product IA is

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 8 \\ -2 & 7 \\ 3 & 5 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1(2) + 0(-2) + 0(3) & 1(8) + 0(7) + 0(5) \\ 0(2) + 1(-2) + 0(3) & 0(8) + 1(7) + 0(5) \\ 0(2) + 0(-2) + 1(3) & 0(8) + 0(7) + 1(5) \end{bmatrix}$$

$$IA = \begin{bmatrix} 2 & 8 \\ -2 & 7 \\ 3 & 5 \end{bmatrix}$$

As we expected, we get back to matrix A after multiplying it by the identity matrix I_3.

■ 5. If we want to find the product IA, where I is the identity matrix and A is given, then what are the dimensions of I? What is the product IA?

$$A = \begin{bmatrix} 7 & 1 & 3 & -2 \\ 5 & 5 & 2 & 9 \end{bmatrix}$$

Solution:

Start by setting up the equation $I \cdot A = A$, then substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 2 \times 4 = 2 \times 4$$

Break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 2 \times 4 = 2 \times 4$$

For matrix multiplication to be defined, you need the same number of columns in the first matrix as rows in the second matrix.

$$R \times \boxed{2 \cdot 2} \times 4 = 2 \times 4$$

The dimensions of the product come from the rows in the first matrix and the columns in the second matrix, so

$$\boxed{2} \times 2 \cdot 2 \times 4 = \boxed{2} \times 4$$

Therefore, the identity matrix in this case is I_2.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The product IA is

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 & 1 & 3 & -2 \\ 5 & 5 & 2 & 9 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1(7)+0(5) & 1(1)+0(5) & 1(3)+0(2) & 1(-2)+0(9) \\ 0(7)+1(5) & 0(1)+1(5) & 0(3)+1(2) & 0(-2)+1(9) \end{bmatrix}$$

$$IA = \begin{bmatrix} 7 & 1 & 3 & -2 \\ 5 & 5 & 2 & 9 \end{bmatrix}$$

As we expected, we get back to matrix A after multiplying it by the identity matrix I_2.

■ 6. If A is a 2×4 matrix what are the dimensions of the identity matrix that make the equation true?

$$A \cdot I = A$$

Solution:

Set up the equation $A \cdot I = A$, then substitute the dimensions for A into the equation.

$$A \cdot I = A$$

$$2 \times 4 \cdot I = 2 \times 4$$

Break up the dimensions of I as $R \times C$.

$$2 \times 4 \cdot R \times C = 2 \times 4$$

The number of rows in the second matrix must be equal to the number of columns from the first matrix.

$$2 \times \boxed{4 \cdot 4} \times C = 2 \times 4$$

The dimensions of the product come from the rows of the first matrix and the columns of the second matrix, so

$$2 \times 4 \cdot 4 \times \boxed{4} = 2 \times \boxed{4}$$

So the identity matrix is I_4, the 4×4 identity matrix.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

TRANSFORMATIONS

■ 1. Find the resulting vector $\overrightarrow{b}$ after $\overrightarrow{a} = (1,6)$ undergoes a transformation by matrix M.

$$M = \begin{bmatrix} -7 & 1 \\ 0 & -2 \end{bmatrix}$$

Solution:

To apply a transformation matrix to vector $\overrightarrow{a}$, we'll multiply the matrix by the vector.

$$\overrightarrow{b} = M\overrightarrow{a} = \begin{bmatrix} -7 & 1 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$$\overrightarrow{b} = M\overrightarrow{a} = \begin{bmatrix} -7(1) + 1(6) \\ 0(1) - 2(6) \end{bmatrix}$$

$$\overrightarrow{b} = M\overrightarrow{a} = \begin{bmatrix} -7 + 6 \\ 0 - 12 \end{bmatrix}$$

$$\overrightarrow{b} = M\overrightarrow{a} = \begin{bmatrix} -1 \\ -12 \end{bmatrix}$$

■ 2. Sketch triangle $\triangle ABC$ with vertices $(2,3)$, $(-3,-1)$, and $(1,-4)$, and the transformation of $\triangle ABC$ after it's transformed by matrix L.

$$L = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Solution:

Put the vertices of $\triangle ABC$ into a matrix.

$$\begin{bmatrix} 2 & -3 & 1 \\ 3 & -1 & -4 \end{bmatrix}$$

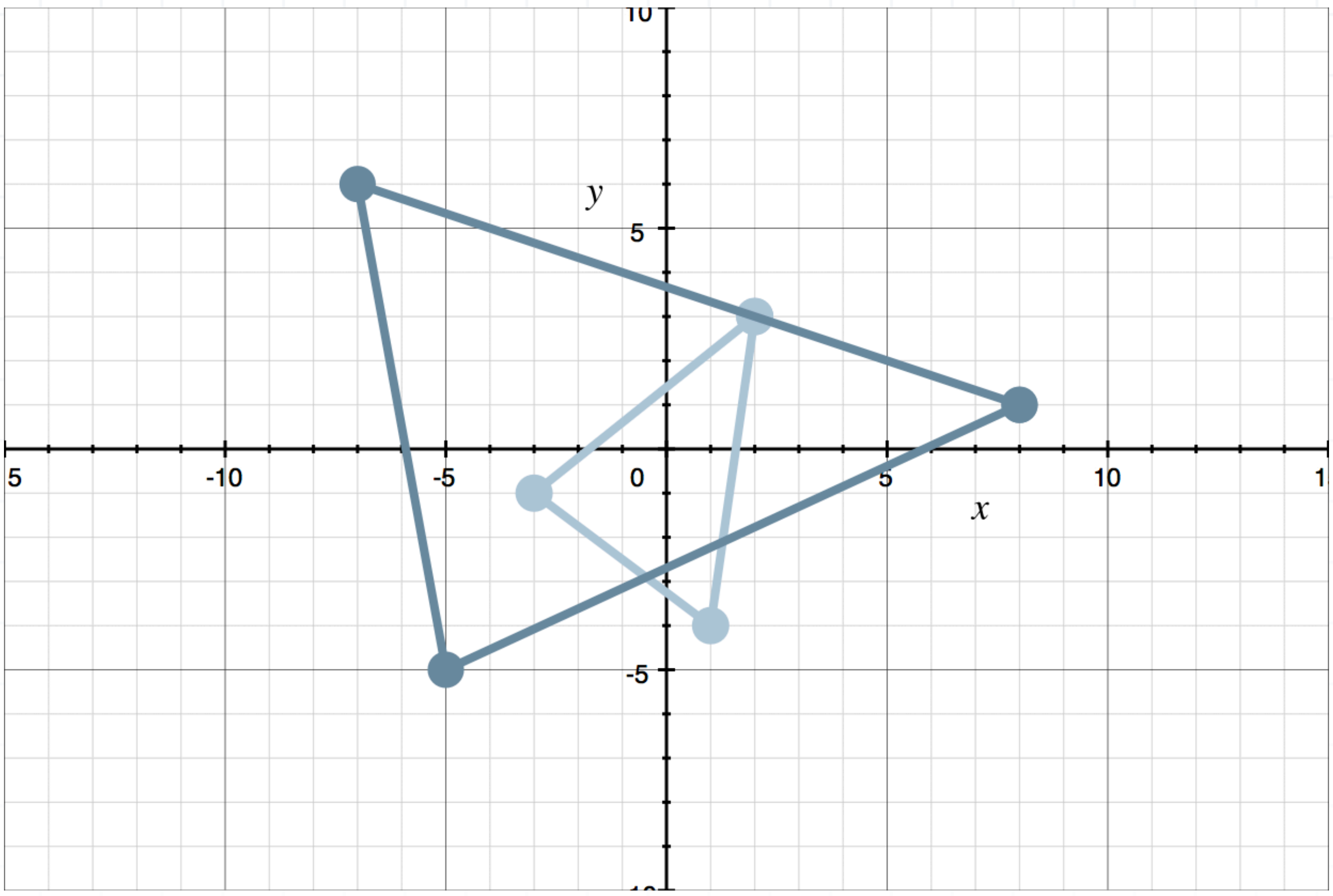
Apply the transformation of L to the vertex matrix.

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 & 1 \\ 3 & -1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1(2)+2(3) & 1(-3)+2(-1) & 1(1)+2(-4) \\ 2(2)-1(3) & 2(-3)-1(-1) & 2(1)-1(-4) \end{bmatrix}$$

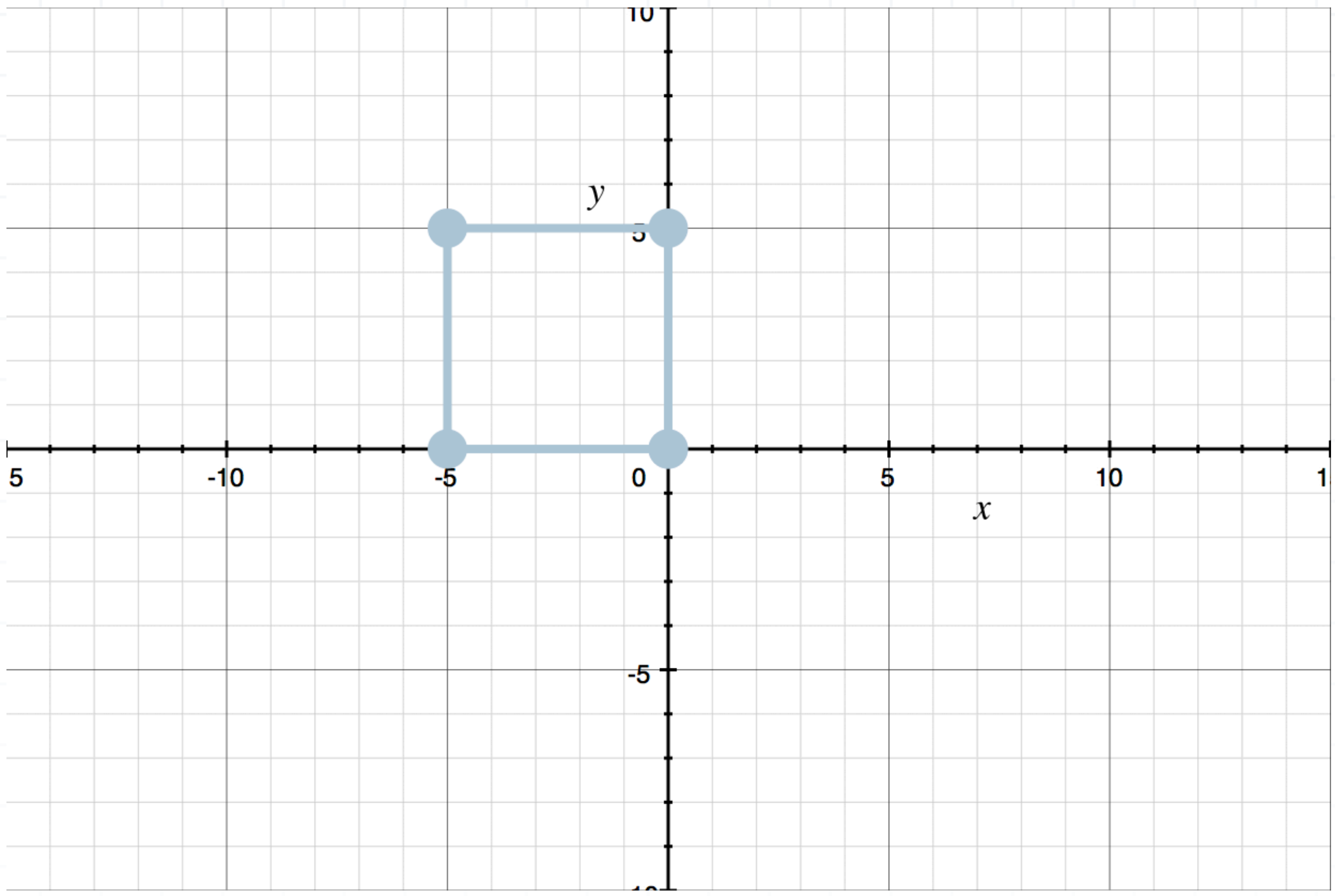
$$\begin{bmatrix} 8 & -5 & -7 \\ 1 & -5 & 6 \end{bmatrix}$$

The original triangle $\triangle ABC$ is sketched in light blue, and its transformation after L is in dark blue.



■ 3. Sketch the transformation of the square in the graph after it's transformed by matrix Z.

$$Z = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$



Solution:

Put the vertices of the square into a matrix.

$$\begin{bmatrix} 0 & -5 & -5 & 0 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

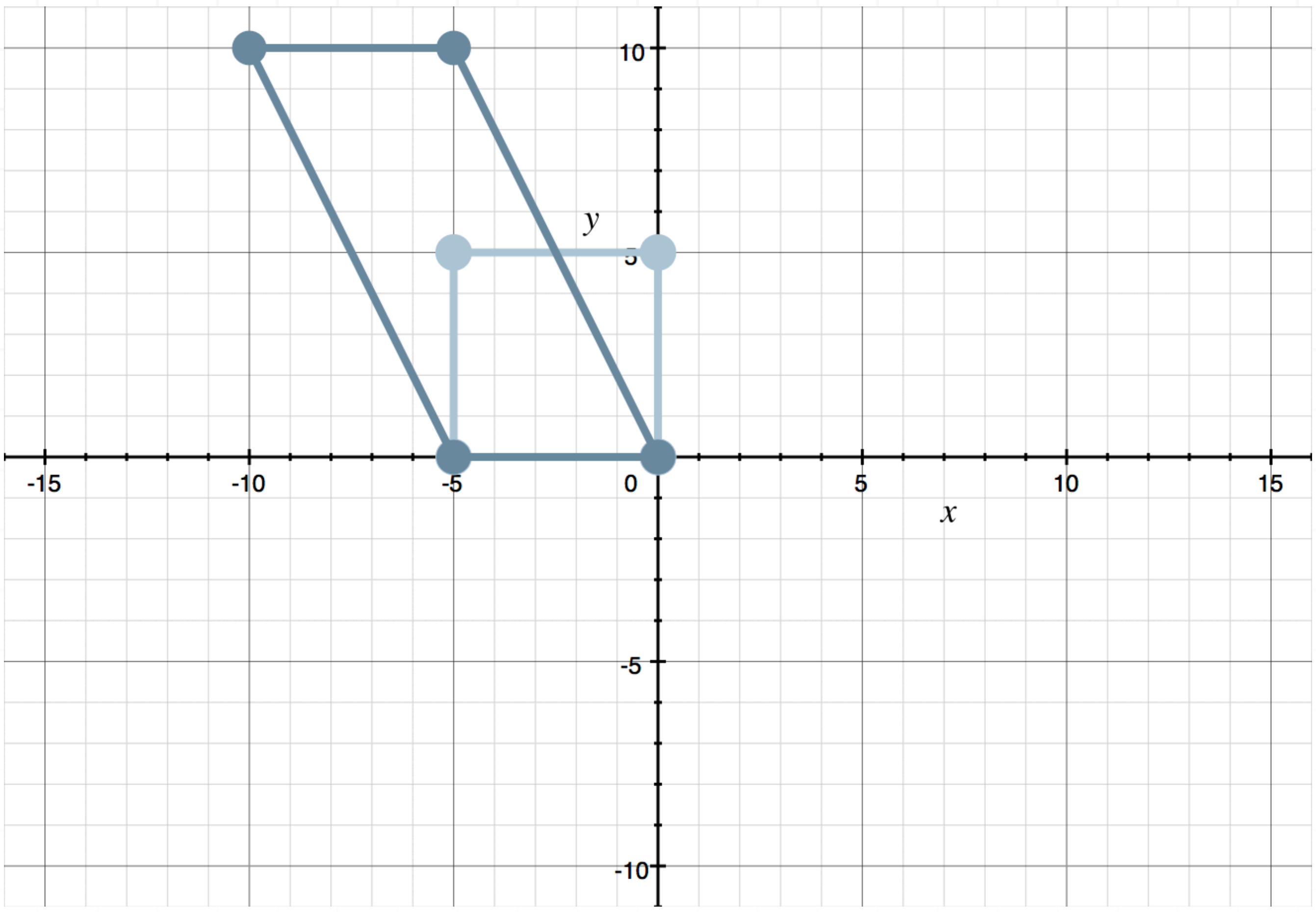
Apply the transformation of Z to the vertex matrix.

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & -5 & -5 & 0 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1(0) - 1(0) & 1(-5) - 1(0) & 1(-5) - 1(5) & 1(0) - 1(5) \\ 0(0) + 2(0) & 0(-5) + 2(0) & 0(-5) + 2(5) & 0(0) + 2(5) \end{bmatrix}$$

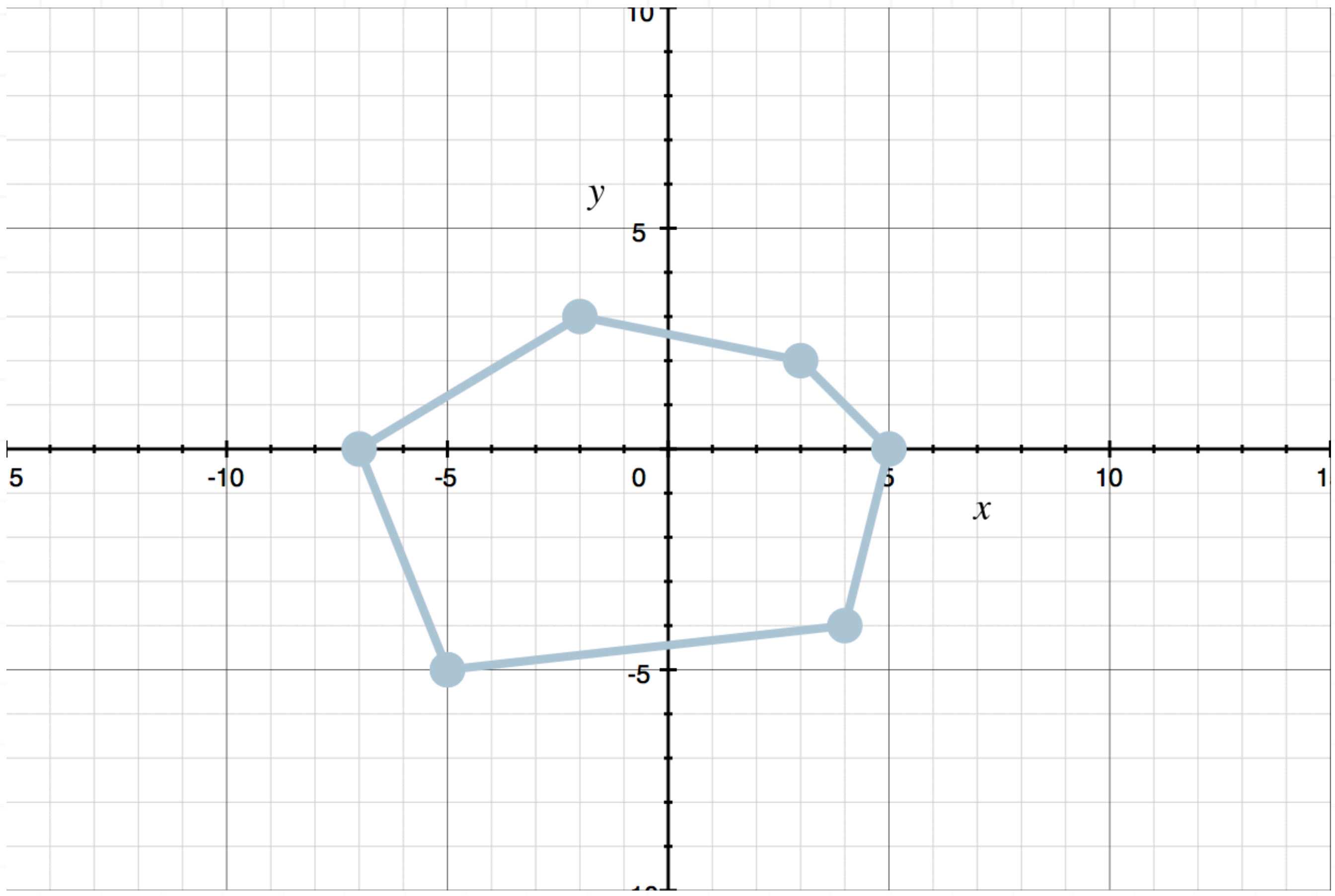
$$\begin{bmatrix} 0 & -5 & -10 & -5 \\ 0 & 0 & 10 & 10 \end{bmatrix}$$

The original square is sketched in light blue, and its transformation after Z is in dark blue.



■ 4. Sketch the transformation of the hexagon after it's transformed by matrix Y.

$$Y = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$



Solution:

Put the vertices of the hexagon into a matrix.

$$\begin{bmatrix} -5 & 4 & 5 & 3 & -2 & -7 & -5 \\ -5 & -4 & 0 & 2 & 3 & 0 & -5 \end{bmatrix}$$

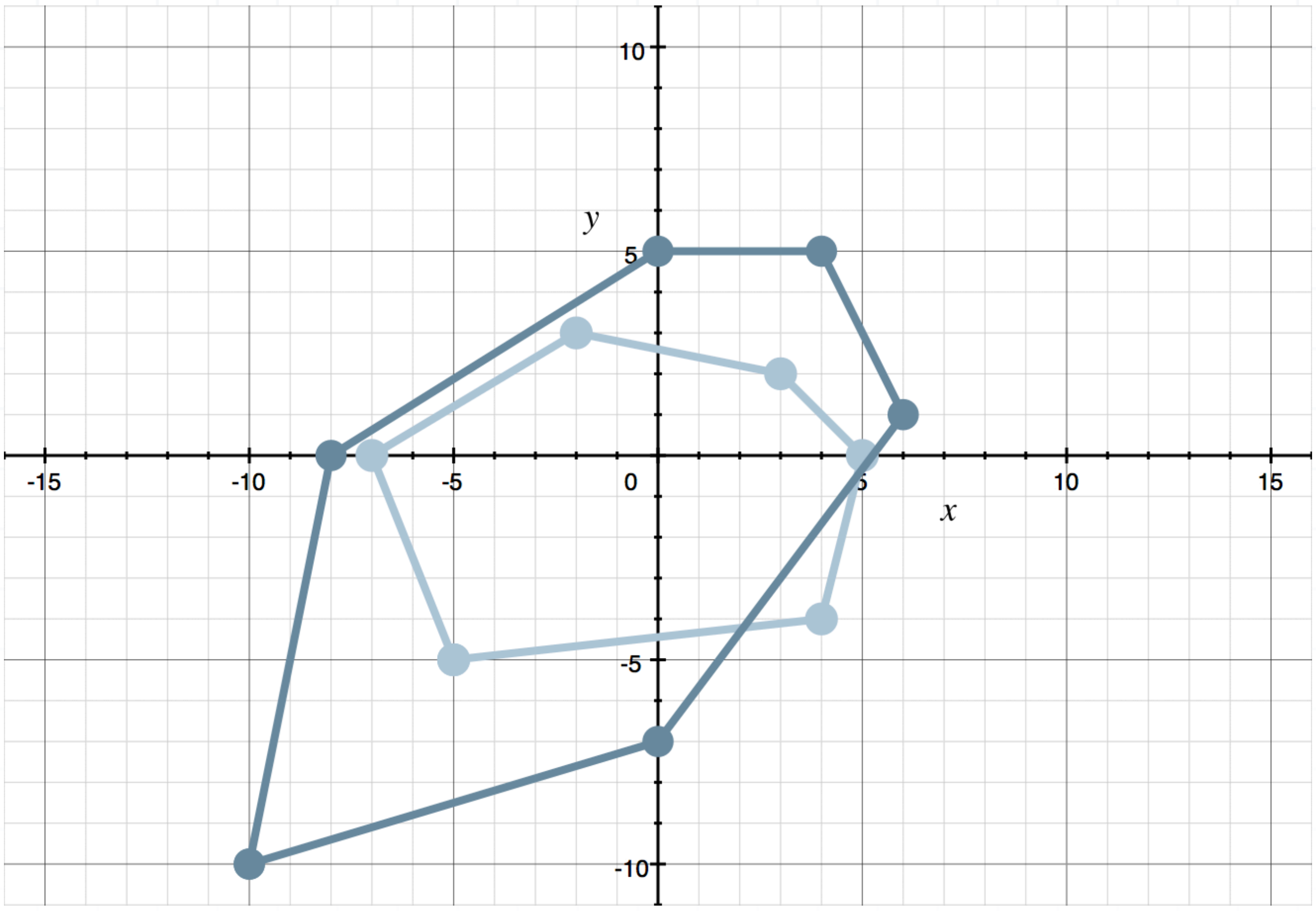
Apply the transformation of Z to the vertex matrix.

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -5 & 4 & 5 & 3 & -2 & -7 \\ -5 & -4 & 0 & 2 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -5(0) - 5(2) & 4(0) - 4(2) & 5(0) + 0(2) & 3(0) + 2(2) & -2(0) + 3(2) & -7(0) + 0(2) \\ -5(1) - 5(1) & 4(1) - 4(1) & 5(1) + 0(1) & 3(1) + 2(1) & -2(1) + 3(1) & -7(1) + 0(1) \end{bmatrix}$$

$$\begin{bmatrix} -10 & -8 & 0 & 4 & 6 & 0 \\ -10 & 0 & 5 & 5 & 1 & -7 \end{bmatrix}$$

The original hexagon is sketched in light blue, and its transformation after Y is in dark blue.



■ 5. What happens to the unit vector $\vec{a} = (1,0)$ after the transformation given by matrix K.

$$K = \begin{bmatrix} 3 & -5 \\ -1 & 0 \end{bmatrix}$$

Solution:

Where $\vec{a} = (1,0)$ lands is given by the first column of the transformation matrix. So $\vec{a}$ will land on $(3, -1)$ after the transformation by K.

■ 6. What happens to the unit vector $\overrightarrow{b}=(0,1)$ after the transformation given by matrix K.

$$K=\begin{bmatrix} 3 & -5 \\ -1 & 0 \end{bmatrix}$$

Solution:

Where $\overrightarrow{b}=(0,1)$ lands is given by the second column of the transformation matrix. So $\overrightarrow{b}$ will land on $(-5,0)$ after the transformation by K.

MATRIX INVERSES, AND INVERTIBLE AND SINGULAR MATRICES

■ 1. Find the determinant of the matrix.

$$B = \begin{bmatrix} -3 & 8 \\ 0 & -2 \end{bmatrix}$$

Solution:

The determinant is

$$B = \begin{vmatrix} -3 & 8 \\ 0 & -2 \end{vmatrix}$$

$$B = (-3)(-2) - (8)(0)$$

$$B = 6 - 0$$

$$B = 6$$

■ 2. Find the determinant of the matrix.

$$B = \begin{bmatrix} 1 & -6 \\ 5 & 5 \end{bmatrix}$$

Solution:

The determinant is

$$B = \begin{vmatrix} 1 & -6 \\ 5 & 5 \end{vmatrix}$$

$$B = (1)(5) - (-6)(5)$$

$$B = 5 + 30$$

$$B = 35$$

■ 3. Find the inverse of matrix G.

$$G = \begin{bmatrix} -3 & 8 \\ 0 & -2 \end{bmatrix}$$

Solution:

Plug into the formula for the inverse matrix.

$$G^{-1} = \frac{1}{|G|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$G^{-1} = \frac{1}{\begin{vmatrix} -3 & 8 \\ 0 & -2 \end{vmatrix}} \begin{bmatrix} -2 & -8 \\ 0 & -3 \end{bmatrix}$$

$$G^{-1} = \frac{1}{(-3)(-2) - (8)(0)} \begin{bmatrix} -2 & -8 \\ 0 & -3 \end{bmatrix}$$

$$G^{-1} = \frac{1}{6}\begin{bmatrix} -2 & -8 \\ 0 & -3 \end{bmatrix}$$

$$G^{-1} = \begin{bmatrix} -\frac{1}{3} & -\frac{4}{3} \\ 0 & -\frac{1}{2} \end{bmatrix}$$

■ 4. Find the inverse of matrix N.

$$N = \begin{bmatrix} 11 & -4 \\ 5 & -3 \end{bmatrix}$$

Solution:

Plug into the formula for the inverse matrix.

$$N^{-1} = \frac{1}{|N|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$N^{-1} = \frac{1}{\begin{vmatrix} 11 & -4 \\ 5 & -3 \end{vmatrix}}\begin{bmatrix} -3 & 4 \\ -5 & 11 \end{bmatrix}$$

$$N^{-1} = \frac{1}{(11)(-3) - (-4)(5)}\begin{bmatrix} -3 & 4 \\ -5 & 11 \end{bmatrix}$$

$$N^{-1} = \frac{1}{-33 + 20}\begin{bmatrix} -3 & 4 \\ -5 & 11 \end{bmatrix}$$

$$N^{-1} = -\frac{1}{13}\begin{bmatrix} -3 & 4 \\ -5 & 11 \end{bmatrix}$$

$$N^{-1} = \begin{bmatrix} \frac{3}{13} & -\frac{4}{13} \\ \frac{5}{13} & -\frac{11}{13} \end{bmatrix}$$

■ 5. Is the matrix invertible or singular?

$$Z = \begin{bmatrix} 4 & 2 \\ -2 & -1 \end{bmatrix}$$

Solution:

Find the determinant of the matrix.

$$\begin{vmatrix} 4 & 2 \\ -2 & -1 \end{vmatrix}$$

$$(4)(-1) - (2)(-2)$$

$$-4 + 4$$

$$0$$

Because the determinant is 0, Z is a singular matrix that has no inverse.

■ 6. Is the matrix invertible or singular?

$$Y = \begin{bmatrix} 0 & 6 \\ 2 & -1 \end{bmatrix}$$

Solution:

Find the determinant of the matrix.

$$\begin{vmatrix} 0 & 6 \\ 2 & -1 \end{vmatrix}$$

$$(0)(-1) - (6)(2)$$

$$0 - 12$$

$$-12$$

Because the determinant is non-zero, Y is an invertible matrix with a defined inverse.

SOLVING SYSTEMS WITH INVERSE MATRICES

■ 1. Use an inverse matrix to solve the system.

$$-4x + 3y = -14$$

$$7x - 4y = 32$$

Solution:

Transfer the system into a matrix equation.

$$\begin{bmatrix} -4 & 3 \\ 7 & -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -14 \\ 32 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M^{-1} = \frac{1}{(-4)(-4) - (3)(7)} \begin{bmatrix} -4 & -3 \\ -7 & -4 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{5} \begin{bmatrix} -4 & -3 \\ -7 & -4 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{7}{5} & \frac{4}{5} \end{bmatrix}$$

The solution to the system is

$$\vec{a} = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{7}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} -14 \\ 32 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{4}{5}(-14) + \frac{3}{5}(32) \\ \frac{7}{5}(-14) + \frac{4}{5}(32) \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{56}{5} + \frac{96}{5} \\ -\frac{98}{5} + \frac{128}{5} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{40}{5} \\ \frac{30}{5} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

■ 2. Use an inverse matrix to solve the system.

$$6x - 11y = 2$$

$$-10x + 7y = -26$$

Solution:

Transfer the system into a matrix equation.

$$\begin{bmatrix} 6 & -11 \\ -10 & 7 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -26 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M^{-1} = \frac{1}{(6)(7) - (-11)(-10)} \begin{bmatrix} 7 & 11 \\ 10 & 6 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{68} \begin{bmatrix} 7 & 11 \\ 10 & 6 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -\frac{7}{68} & -\frac{11}{68} \\ -\frac{10}{68} & -\frac{6}{68} \end{bmatrix}$$

The solution to the system is

$$\vec{a} = \begin{bmatrix} -\frac{7}{68} & -\frac{11}{8} \\ -\frac{10}{68} & -\frac{6}{68} \end{bmatrix} \begin{bmatrix} 2 \\ -26 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{7}{68}(2) - \frac{11}{68}(-26) \\ -\frac{10}{68}(2) - \frac{6}{68}(-26) \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{14}{68} + \frac{286}{68} \\ -\frac{20}{68} + \frac{156}{68} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{272}{68} \\ \frac{136}{68} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

■ 3. Use an inverse matrix to solve the system.

$$13y - 6x = -81$$

$$7x + 17 = -22y$$

Solution:

Transfer the system into a matrix equation.

$$\begin{bmatrix} -6 & 13 \\ 7 & 22 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -81 \\ -17 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M^{-1} = \frac{1}{(-6)(22) - (13)(7)} \begin{bmatrix} 22 & -13 \\ -7 & -6 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{223} \begin{bmatrix} 22 & -13 \\ -7 & -6 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -\frac{22}{223} & \frac{13}{223} \\ \frac{7}{223} & \frac{6}{223} \end{bmatrix}$$

The solution to the system is

$$\vec{a} = \begin{bmatrix} -\frac{22}{223} & \frac{13}{223} \\ \frac{7}{223} & \frac{6}{223} \end{bmatrix} \begin{bmatrix} -81 \\ -17 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{22}{223}(-81) + \frac{13}{223}(-17) \\ \frac{7}{223}(-81) + \frac{6}{223}(-17) \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{1{,}782}{223} - \frac{221}{223} \\ -\frac{567}{223} - \frac{102}{223} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{1{,}561}{223} \\ -\frac{669}{223} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

■ 4. Sketch a graph of vectors to visually find the solution to the system.

$$3x = 3$$

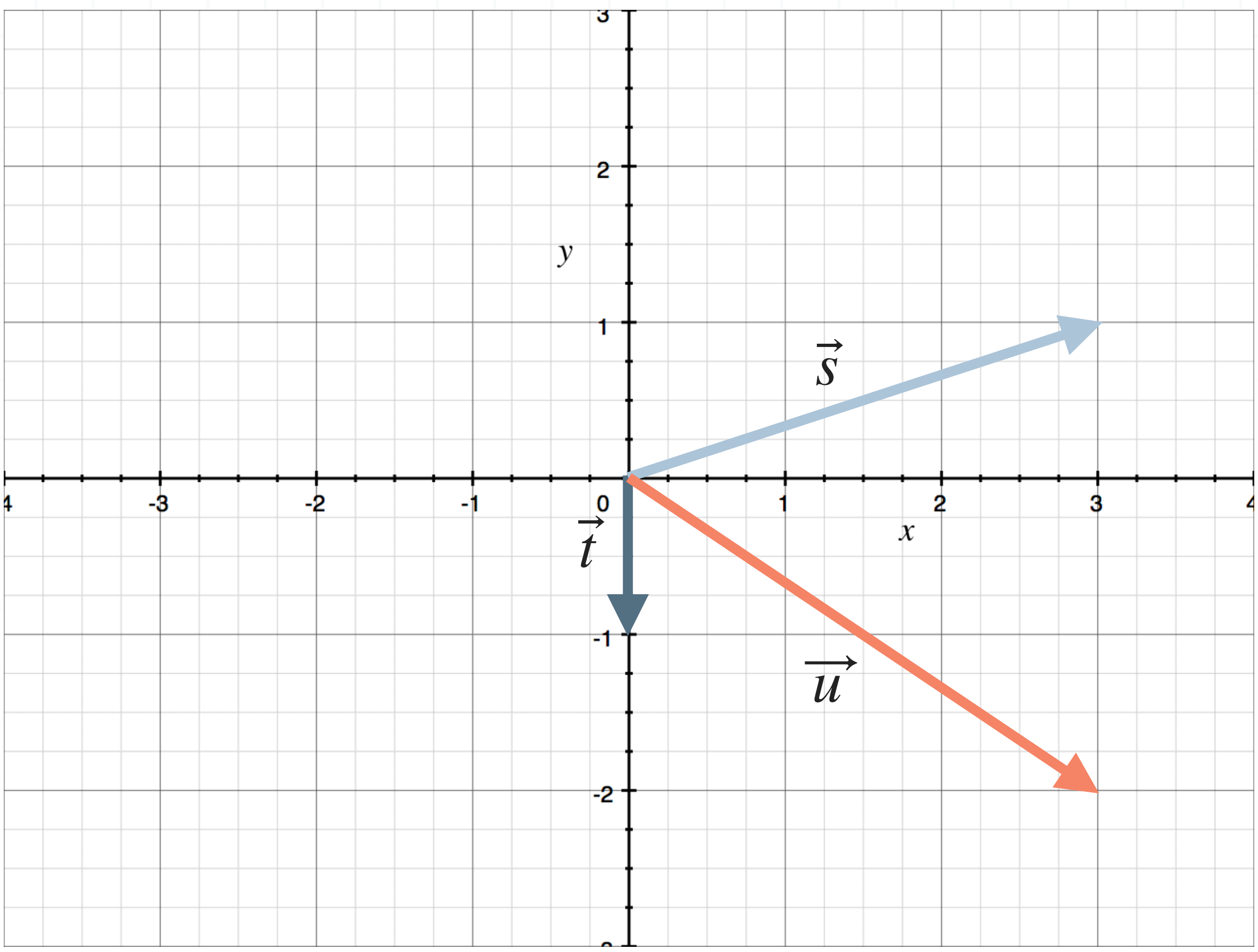
$$x - y = -2$$

Solution:

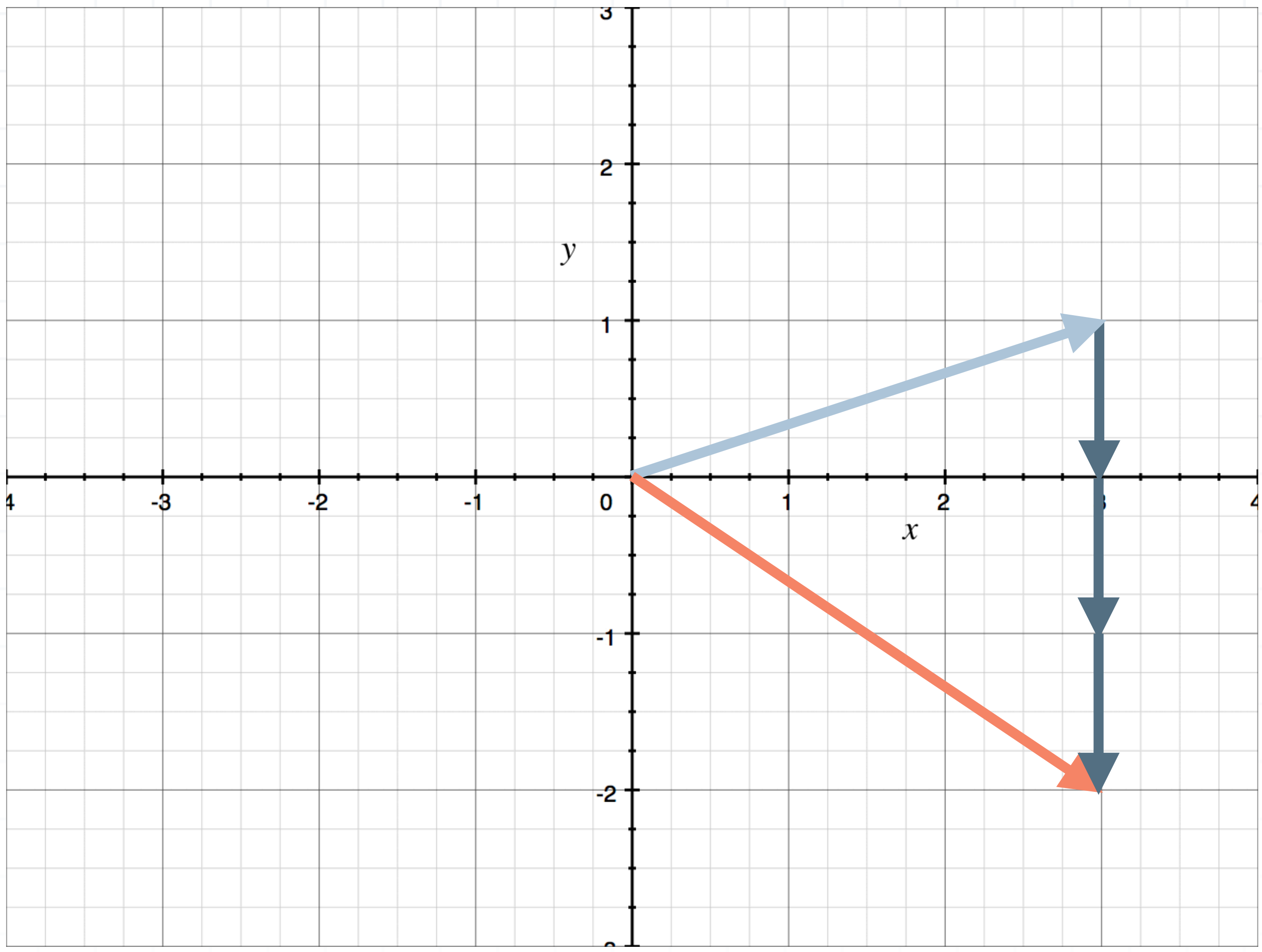
Put the system into a matrix equation.

$$\begin{bmatrix}3\\1\end{bmatrix}x+\begin{bmatrix}0\\-1\end{bmatrix}y=\begin{bmatrix}3\\-2\end{bmatrix}$$

From the vector equation, we can sketch the vectors $\vec{s}=(3,1)$ for x, $\vec{t}=(0,-1)$ for y, and the resulting vector $\overrightarrow{u}=(3,-2)$.



If we play around a little bit with the vectors in the graph, we can see that putting one $\vec{s}$ and three $\vec{t}$s together will get us back to the terminal point of $\overrightarrow{u}$, so $x=1$ and $y=3$.



■ 5. Sketch a graph of vectors to visually find the solution to the system.

$$-y = -4$$

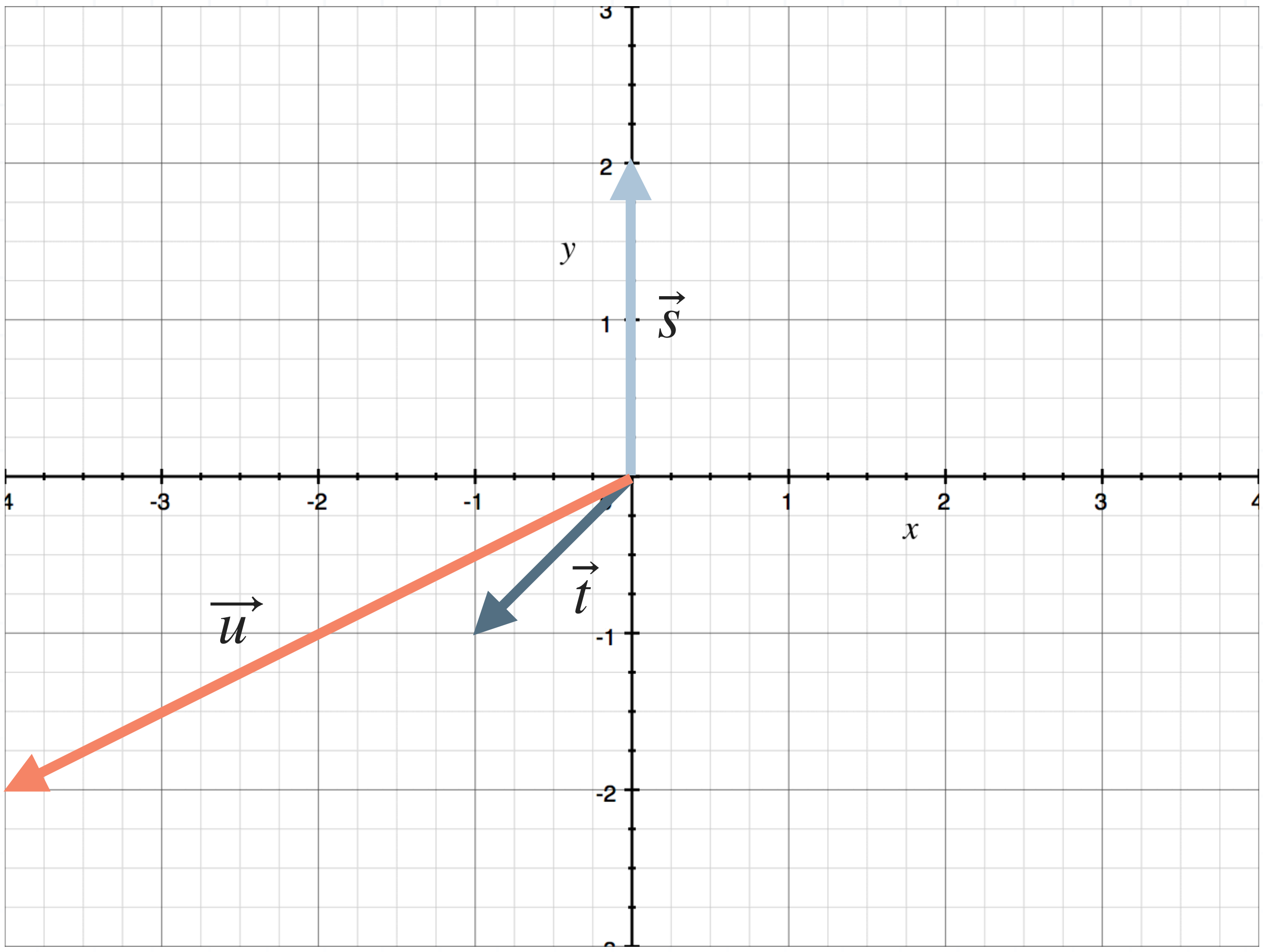
$$2x - y = -2$$

Solution:

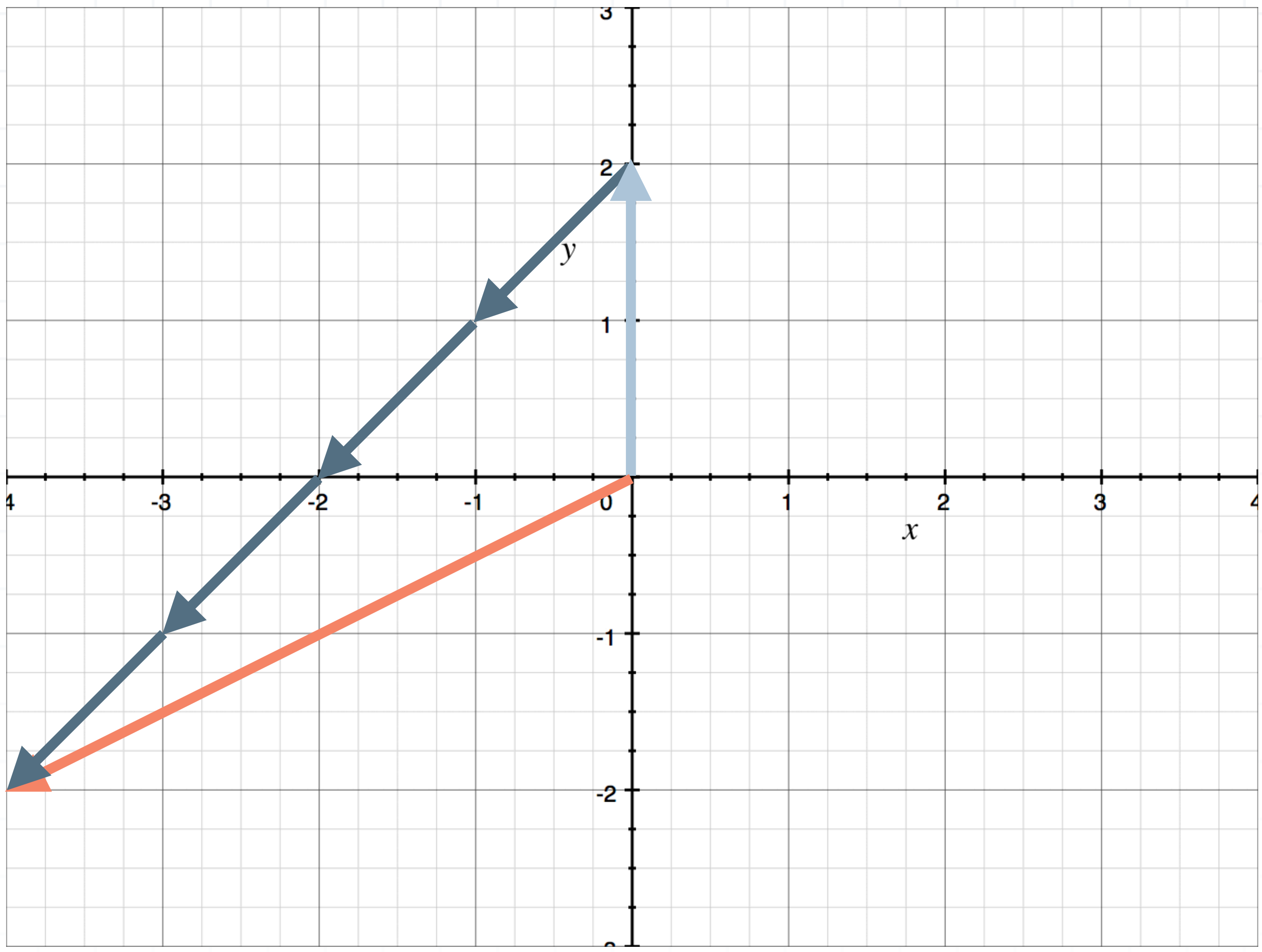
Put the system into a matrix equation.

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} x + \begin{bmatrix} -1 \\ -1 \end{bmatrix} y = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

From the vector equation, we can sketch the vectors $\vec{s}=(0,2)$ for x, $\vec{t}=(-1,-1)$ for y, and the resulting vector $\vec{u}=(-4,-2)$.



If we play around a little bit with the vectors in the graph, we can see that putting one $\vec{s}$ and four $\vec{t}$s together will get us back to the terminal point of $\vec{u}$, so $x=1$ and $y=4$.



■ 6. Sketch a graph of vectors to visually find the solution to the system.

$$x - y = 0$$

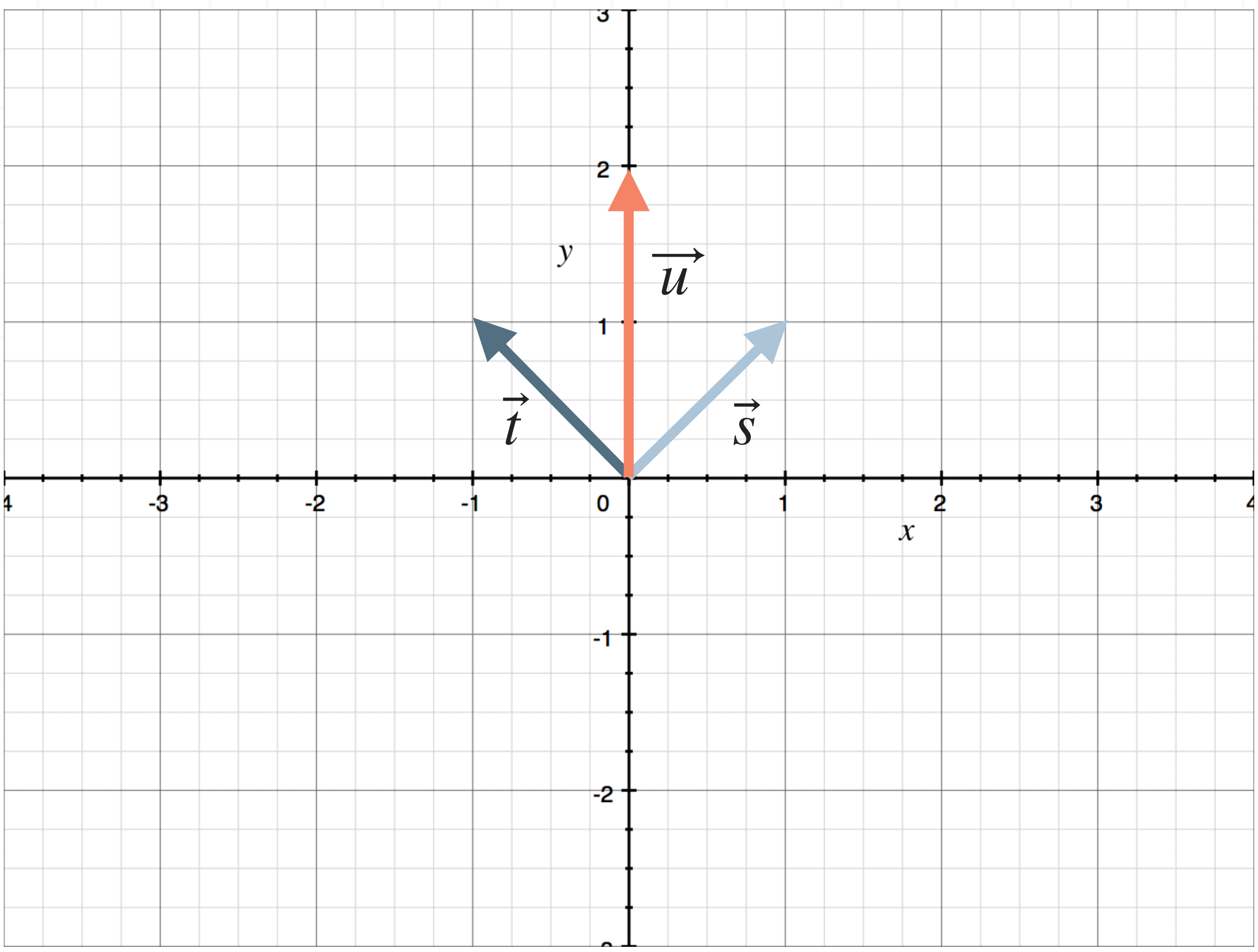
$$x + y = 2$$

Solution:

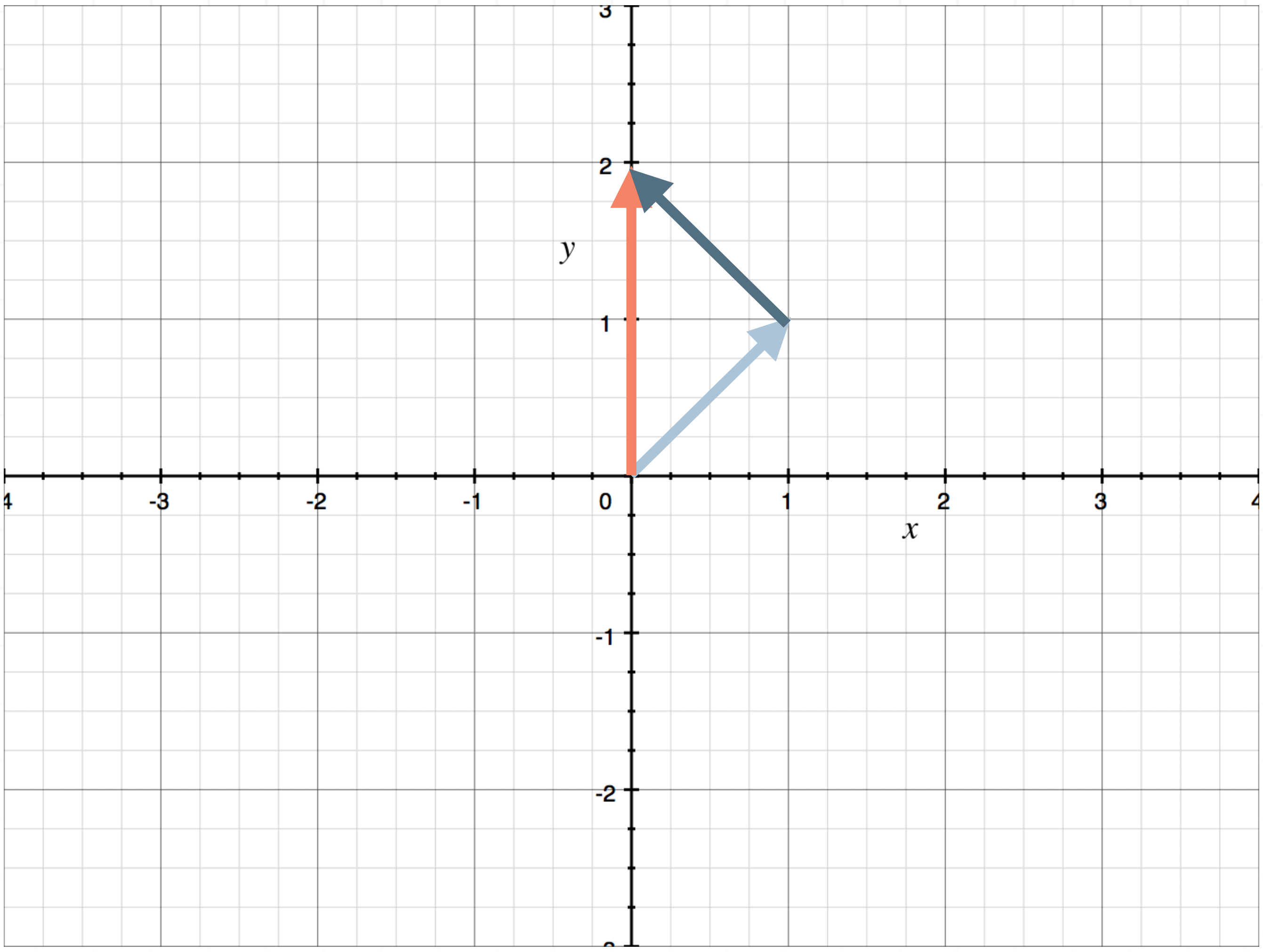
Put the system into a matrix equation.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} y = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

From the vector equation, we can sketch the vectors $\vec{s}=(1,1)$ for x, $\vec{t}=(-1,1)$ for y, and the resulting vector $\vec{u}=(0,2)$.



If we play around a little bit with the vectors in the graph, we can see that putting one $\vec{s}$ and one $\vec{t}$ together will get us back to the terminal point of $\vec{u}$, so $x=1$ and $y=1$.



y
x
3
2
1
0
-1
-2
-3
-4
1
2
3
4

SOLVING SYSTEMS WITH CRAMER'S RULE

■ 1. Use Cramer's Rule to find the expression that would give the solution for x. You do not need to solve the system.

$$2x - y = 5$$

$$x + 3y = 15$$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}$$

To find D_x, replace the first column of the coefficient matrix with the answer column.

$$D_x = \begin{vmatrix} 5 & -1 \\ 15 & 3 \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_x/D.

$$\frac{\begin{vmatrix} 5 & -1 \\ 15 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}}$$

■ 2. Use Cramer's Rule to find the expression that would give the solution for x. You do not need to solve the system.

$$ax + by = e$$

$$cx + dy = f$$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

To find D_x, replace the first column of the coefficient matrix with the answer column.

$$D_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_x/D.

$$\frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

■ 3. Use Cramer's Rule to find the expression that would give the solution for y. You do not need to solve the system.

$$3x + 4y = 11$$

$$2x - 3y = -4$$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix}$$

To find D_y, replace the second column of the coefficient matrix with the answer column.

$$D_y = \begin{vmatrix} 3 & 11 \\ 2 & -4 \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_y/D.

$$\frac{\begin{vmatrix} 3 & 11 \\ 2 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix}}$$

■ 4. Use Cramer's Rule to find the expression that would give the solution for y. You do not need to solve the system.

$$ax + by = e$$

$$cx + dy = f$$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

To find D_y, replace the second column of the coefficient matrix with the answer column.

$$D_y = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_y/D.

$$\frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

■ 5. Use Cramer's Rule to solve for x.

$$3x + 2y = 1$$

$$6x + 5y = 4$$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}$$

To find D_x, replace the first column of the coefficient matrix with the answer column.

$$D_x = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_x/D.

$$\frac{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}$$

Calculate the value of x.

$$x = \frac{1(5) - 2(4)}{3(5) - 2(6)}$$

$$x = \frac{5 - 8}{15 - 12}$$

$$x = \frac{-3}{3}$$

$$x = -1$$

■ 6. Use Cramer's Rule to solve for y.

$$3x + 2y = 1$$

$$6x + 5y = 4$$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}$$

To find D_y, replace the second column of the coefficient matrix with the answer column.

$$D_y = \begin{vmatrix} 3 & 1 \\ 6 & 4 \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_y/D.

$$\frac{\begin{vmatrix} 3 & 1 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}$$

Calculate the value of y.

$$y = \frac{3(4) - 1(6)}{3(5) - 2(6)}$$

$$y = \frac{12 - 6}{15 - 12}$$

$$y = \frac{6}{3}$$

$$y = 2$$

FRACTION DECOMPOSITION

■ 1. Find the form of partial fractions decomposition of the rational function.

$$f(x) = \frac{6x + 16}{x^2 + 10x + 21}$$

Solution:

Factor the denominator of the rational function.

$$f(x) = \frac{6x + 16}{(x + 3)(x + 7)}$$

The denominator is now factored to distinct linear factors, so there should be two fractions in the decomposition that have denominators from the original denominator and numerators that are constants.

$$f(x) = \frac{A}{x + 3} + \frac{B}{x + 7}$$

■ 2. Identify the repeated factors in the denominator of rational function.

$$f(x) = \frac{3x + 7}{(x - 1)(x^2 - 1)(x^2 + 1)^3(x^2 - 2x - 3)}$$

Solution:

The denominator of $f(x)$ can be factored further.

$$f(x) = \frac{3x+7}{(x-1)(x-1)(x+1)(x^2+1)^3(x+1)(x-3)}$$

$$f(x) = \frac{3x+7}{(x-1)^2(x+1)^2(x^2+1)^3(x-3)}$$

So the $x-1$, $x+1$, and x^2+1 factors are repeated.

■ 3. How many fractions will exist in the partial fractions decomposition of the function?

$$f(x) = \frac{1}{(x^2+1)(x^4+5x^2+6)}$$

Solution:

The denominator of the rational function can be factored further.

$$f(x) = \frac{1}{(x^2+1)(x^2+2)(x^2+3)}$$

The denominator is factored as far as it can be, so there are three fractions in the decomposition of $f(x)$.

$$f(x) = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{x^2+3}$$

■ 4. Find the form of the partial fractions decomposition of the function, without solving for the constants.

$$f(x) = \frac{1}{(x^2+1)(x^4-1)}$$

Solution:

The denominator of the function $f(x)$ can be factored further.

$$f(x) = \frac{1}{(x^2+1)(x^2+1)(x^2-1)}$$

$$f(x) = \frac{1}{(x^2+1)^2(x+1)(x-1)}$$

So the decomposition will be

$$f(x) = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{(x+1)} + \frac{F}{(x-1)}$$

■ 5. Find the form of the partial fractions decomposition of the function, without solving for the constants.

$$f(x) = \frac{x^2}{(x^2+2x)(x^2+2x+2)}$$

Solution:

The denominator of the function $f(x)$ can be factored further.

$$f(x) = \frac{x^2}{x(x+2)(x^2+2x+2)}$$

$$f(x) = \frac{x}{(x+2)(x^2+2x+2)}$$

So the decomposition will be

$$f(x) = \frac{A}{x+2} + \frac{Bx+C}{x^2+2x+2}$$

■ 6. Find the form of the partial fractions decomposition of the function, without solving for the constants.

$$f(x) = \frac{x^2+2}{(1-x)(1-2x)(1-3x)}$$

Solution:

The denominator of the function $f(x)$ is already factored with three distinct linear factors, so the decomposition of $f(x)$ should include three fractions.

The denominator of each of the three fractions will be one of the factors from the original denominator, and the numerator of each of the three fractions will be a distinct constant, A, B, C, etc. So the decomposition will be

$$f(x) = \frac{A}{1-x} + \frac{B}{1-2x} + \frac{C}{1-3x}$$

DISTINCT LINEAR FACTORS

■ 1. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{4}{(3x-1)(x+1)}$$

Solution:

These are distinct linear factors.

$$\frac{4}{(3x-1)(x+1)} = \frac{A}{3x-1} + \frac{B}{x+1}$$

To solve for A, remove $3x - 1$ and then evaluate the resulting left side at $x = 1/3$.

$$\frac{4}{x+1} \to \frac{4}{\frac{1}{3}+1} \to 3$$

To solve for B, remove $x + 1$ and then evaluate the resulting left side at $x = -1$.

$$\frac{4}{3x-1} \to \frac{4}{3(-1)-1} \to -1$$

Plugging $A = 3$ and $B = -1$ back into the partial fractions decomposition gives

$$f(x) = \frac{3}{3x-1} - \frac{1}{x+1}$$

■ 2. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{24}{x(x+4)(x-2)}$$

Solution:

These are distinct linear factors.

$$\frac{24}{x(x+4)(x-2)} = \frac{A}{x} + \frac{B}{x+4} + \frac{C}{x-2}$$

To solve for A, remove x and then evaluate the resulting left side at $x = 0$.

$$\frac{24}{(x+4)(x-2)} \to \frac{24}{(4)(-2)} \to -3$$

To solve for B, remove $x+4$ and then evaluate the resulting left side at $x = -4$.

$$\frac{24}{x(x-2)} \to \frac{24}{-4(-4-2)} \to 1$$

To solve for C, remove $x-2$ and then evaluate the resulting left side at $x = 2$.

$$\frac{24}{x(x+4)} \to \frac{24}{2(2+4)} \to 2$$

Plugging $A = -3$, $B = 1$, and $C = 2$ back into the partial fractions decomposition gives

$$f(x) = -\frac{3}{x} + \frac{1}{x+4} + \frac{2}{x-2}$$

■ 3. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{3x}{(x+2)(x+5)}$$

Solution:

These are distinct linear factors.

$$\frac{3x}{(x+2)(x+5)} = \frac{A}{x+2} + \frac{B}{x+5}$$

To solve for A, remove $x+2$ and then evaluate the resulting left side at $x = -2$.

$$\frac{3x}{x+5} \to \frac{3(-2)}{-2+5} \to -2$$

To solve for B, remove $x+5$ and then evaluate the resulting left side at $x = -5$.

$$\frac{3x}{x+2} \to \frac{3(-5)}{-5+2} \to 5$$

Plugging $A = -2$ and $B = 5$ back into the partial fractions decomposition gives

$$f(x) = -\frac{2}{x+2} + \frac{5}{x+5}$$

■ 4. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{6x}{(x^2-1)(x-2)}$$

Solution:

We have to start by factoring the denominator.

$$f(x) = \frac{6x}{(x+1)(x-1)(x-2)}$$

These are distinct linear factors.

$$\frac{6x}{(x+1)(x-1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2}$$

To solve for A, remove $x+1$ and then evaluate the resulting left side at $x = -1$.

$$\frac{6x}{(x-1)(x-2)} \to \frac{6(-1)}{(-1-1)(-1-2)} \to -1$$

To solve for B, remove $x-1$ and then evaluate the resulting left side at $x=1$.

$$\frac{6x}{(x+1)(x-2)} \to \frac{6(1)}{(1+1)(1-2)} \to -3$$

To solve for C, remove $x-2$ and then evaluate the resulting left side at $x=2$.

$$\frac{6x}{(x+1)(x-1)} \to \frac{6(2)}{(2+1)(2-1)} \to 4$$

Plugging $A=-1$, $B=-3$, and $C=4$ back into the partial fractions decomposition gives

$$f(x) = -\frac{1}{x+1} - \frac{3}{x-1} + \frac{4}{x-2}$$

■ 5. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x+1}{9x^3 - x}$$

Solution:

We have to start by factoring the denominator.

$$f(x) = \frac{x+1}{x(3x+1)(3x-1)}$$

These are distinct linear factors.

$$\frac{x+1}{x(3x+1)(3x-1)} = \frac{A}{x} + \frac{B}{3x+1} + \frac{C}{3x-1}$$

To solve for A, remove x and then evaluate the resulting left side at $x = 0$.

$$\frac{x+1}{(3x+1)(3x-1)} \to \frac{0+1}{(0+1)(0-1)} \to -1$$

To solve for B, remove $3x+1$ and then evaluate the resulting left side at $x = -1/3$.

$$\frac{x+1}{x(3x-1)} \to \frac{-\frac{1}{3}+1}{-\frac{1}{3}\left(3\left(-\frac{1}{3}\right)-1\right)} \to 1$$

To solve for C, remove $3x-1$ and then evaluate the resulting left side at $x = 1/3$.

$$\frac{x+1}{x(3x+1)} \to \frac{\frac{1}{3}+1}{\frac{1}{3}\left(3\left(\frac{1}{3}\right)+1\right)} \to 2$$

Plugging $A = -1$, $B = 1$, and $C = 2$ back into the partial fractions decomposition gives

$$f(x) = -\frac{1}{x} + \frac{1}{3x+1} + \frac{2}{3x-1}$$

■ 6. Rewrite the function as its partial fractions decomposition.

$$f(x)=\frac{6x-24}{(x^2-1)(x^2-4)}$$

Solution:

We have to start by factoring the denominator.

$$f(x)=\frac{6x-24}{(x+1)(x-1)(x+2)(x-2)}$$

These are distinct linear factors.

$$\frac{6x-24}{(x+1)(x-1)(x+2)(x-2)}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{x+2}+\frac{D}{x-2}$$

To solve for A, remove $x+1$ and then evaluate the resulting left side at $x=-1$.

$$\frac{6x-24}{(x-1)(x+2)(x-2)}\to\frac{6(-1)-24}{(-1-1)(-1+2)(-1-2)}\to-5$$

To solve for B, remove $x-1$ and then evaluate the resulting left side at $x=1$.

$$\frac{6x-24}{(x+1)(x+2)(x-2)}\to\frac{6(1)-24}{(1+1)(1+2)(1-2)}\to 3$$

To solve for C, remove $x+2$ and then evaluate the resulting left side at $x=-2$.

$$\frac{6x-24}{(x+1)(x-1)(x-2)}\to\frac{6(-2)-24}{(-2+1)(-2-1)(-2-2)}\to 3$$

To solve for D, remove $x-2$ and then evaluate the resulting left side at $x=2$.

$$\frac{6x-24}{(x+1)(x-1)(x+2)} \to \frac{6(2)-24}{(2+1)(2-1)(2+2)} \to -1$$

Plugging $A=-5$, $B=3$, $C=3$, and $D=-1$ back into the partial fractions decomposition gives

$$f(x)=-\frac{5}{x+1}+\frac{3}{x-1}+\frac{3}{x+2}-\frac{1}{x-2}$$

REPEATED LINEAR FACTORS

■ 1. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 - 2x^2 + 2x - 3}{(x-2)^4}$$

Solution:

These are repeated linear factors.

$$\frac{x^3 - 2x^2 + 2x - 3}{(x-2)^4} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{(x-2)^4}$$

Combine the fractions by finding the common denominator.

$$\frac{x^3 - 2x^2 + 2x - 3}{(x-2)^4} = \frac{A(x-2)^3 + B(x-2)^2 + C(x-2) + D}{(x-2)^4}$$

Because the denominators are equivalent, the numerators must also be equivalent, so we'll set the numerators equal to one another.

$$x^3 - 2x^2 + 2x - 3 = A(x-2)^3 + B(x-2)^2 + C(x-2) + D$$

The linear factor $x - 2$ is equal to 0 when $x = 2$, so we'll evaluate this equation at $x = 2$ in order to solve for D.

$$2^3 - 2 \cdot 2^2 + 2(2) - 3 = A(2-2)^3 + B(2-2)^2 + C(2-2) + D$$

$$8 - 2 \cdot 4 + 2 \cdot 2 - 3 = D$$

$$D = 1$$

We'll plug this back into the numerator equation,

$$x^3 - 2x^2 + 2x - 3 = A(x-2)^3 + B(x-2)^2 + C(x-2) + 1$$

and then simplify the right side, collecting like terms.

$$x^3 - 2x^2 + 2x - 3 = A(x^3 - 6x^2 + 12x - 8) + B(x^2 - 4x + 4) + C(x-2) + 1$$

$$x^3 - 2x^2 + 2x - 3 = Ax^3 + (-6A + B)x^2 + (12A - 4B + C)x - 8A + 4B - 2C + 1$$

So we get the system of equations

$$A = 1$$

$$-6A + B = -2$$

$$12A - 4B + C = 2$$

$$-8A + 4B - 2C + 1 = -3$$

or $A = 1$ with

$$B = 4$$

$$-4B + C = -10$$

$$4B - 2C = 4$$

or $A = 1$ and $B = 4$ with

$$-16 + C = -10$$

$$16 - 2C = 4$$

or $A = 1$, $B = 4$, $C = 6$, and $D = 1$. Plugging these values back into the partial fractions decomposition gives

$$f(x) = \frac{1}{x-2} + \frac{4}{(x-2)^2} + \frac{6}{(x-2)^3} + \frac{1}{(x-2)^4}$$

■ 2. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 + 3x^2 + 3x + 3}{(x+1)^4}$$

Solution:

These are repeated linear factors.

$$\frac{x^3 + 3x^2 + 3x + 3}{(x+1)^4} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{(x+1)^4}$$

Combine the fractions by finding the common denominator.

$$\frac{x^3 + 3x^2 + 3x + 3}{(x+1)^4} = \frac{A(x+1)^3 + B(x+1)^2 + C(x+1) + D}{(x+1)^4}$$

Because the denominators are equivalent, the numerators must also be equivalent, so we'll set the numerators equal to one another.

$$x^3 + 3x^2 + 3x + 3 = A(x+1)^3 + B(x+1)^2 + C(x+1) + D$$

The linear factor $x+1$ is equal to 0 when $x=-1$, so we'll evaluate this equation at $x=-1$ in order to solve for D.

$$(-1)^3+3(-1)^2+3(-1)+3=A(-1+1)^3+B(-1+1)^2+C(-1+1)+D$$

$$-1+3-3+3=D$$

$$D=2$$

We'll plug this back into the numerator equation,

$$x^3+3x^2+3x+3=A(x+1)^3+B(x+1)^2+C(x+1)+2$$

and then simplify the right side, collecting like terms.

$$x^3+3x^2+3x+3=A(x^3+3x^2+3x+1)+B(x^2+2x+1)+C(x+1)+2$$

$$x^3+3x^2+3x+3=Ax^3+(3A+B)x^2+(3A+2B+C)x+A+B+C+2$$

So we get the system of equations

$$A=1$$

$$3A+B=3$$

$$3A+2B+C=3$$

$$A+B+C+2=3$$

or $A=1$ with

$$B=0$$

$$2B+C=0$$

$$B + C = 0$$

or $A = 1$, $B = 0$, $C = 0$, and $D = 2$. Plugging these values back into the partial fractions decomposition gives

$$f(x) = \frac{1}{x+1} + \frac{2}{(x+1)^4}$$

■ 3. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x}{(x+2)^3}$$

Solution:

These are repeated linear factors.

$$\frac{x}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

Combine the fractions by finding the common denominator.

$$\frac{x}{(x+2)^3} = \frac{A(x+2)^2 + B(x+2) + C}{(x+2)^3}$$

Because the denominators are equivalent, the numerators must also be equivalent, so we'll set the numerators equal to one another.

$$x = A(x+2)^2 + B(x+2) + C$$

The linear factor $x+2$ is equal to 0 when $x=-2$, so we'll evaluate this equation at $x=-2$ in order to solve for C.

$$-2=A(-2+2)^2+B(-2+2)+C$$

$$C=-2$$

We'll plug this back into the numerator equation,

$$x=A(x+2)^2+B(x+2)-2$$

and then simplify the right side, collecting like terms.

$$x=A(x^2+4x+4)+B(x+2)-2$$

$$x=Ax^2+(4A+B)x+4A+2B-2$$

So we get the system of equations

$$A=0$$

$$4A+B=1$$

$$4A+2B-2=0$$

or $A=0$, $B=1$, and $C=-2$. Plugging these values back into the partial fractions decomposition gives

$$f(x)=\frac{1}{(x+2)^2}-\frac{2}{(x+2)^3}$$

■ 4. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{2x+5}{x^2-2x+1}$$

Solution:

We can rewrite $f(x)$ as

$$\frac{2x+5}{x^2-2x+1} = \frac{2x+5}{(x-1)^2}$$

These are repeated linear factors.

$$\frac{2x+5}{x^2-2x+1} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

Combine the fractions by finding the common denominator.

$$\frac{2x+5}{x^2-2x+1} = \frac{A(x-1)+B}{(x-1)^2}$$

Because the denominators are equivalent, the numerators must also be equivalent, so we'll set the numerators equal to one another.

$$2x+5 = Ax - A + B$$

Then we have

$$A = 2$$

$$-A + B = 5$$

Plug $A = 2$ and $B = 7$ into the partial fractions decomposition.

$$f(x) = \frac{2}{x-1} + \frac{7}{(x-1)^2}$$

■ 5. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^2 - 21x + 100}{(x-10)^3}$$

Solution:

These are repeated linear factors.

$$\frac{x^2 - 21x + 100}{(x-10)^3} = \frac{A}{x-10} + \frac{B}{(x-10)^2} + \frac{C}{(x-10)^3}$$

Combine the fractions by finding the common denominator.

$$\frac{x^2 - 21x + 100}{(x-10)^3} = \frac{A(x-10)^2 + B(x-10) + C}{(x-10)^3}$$

Because the denominators are equivalent, the numerators must also be equivalent, so we'll set the numerators equal to one another.

$$x^2 - 21x + 100 = A(x-10)^2 + B(x-10) + C$$

The linear factor $x - 10$ is equal to 0 when $x = 10$, so we'll evaluate this equation at $x = 10$ in order to solve for C.

$$10^2 - 21(10) + 100 = A(10-10)^2 + B(10-10) + C$$

$$100 - 210 + 100 = C$$

$$C = -10$$

We'll plug this back into the numerator equation,

$$x^2 - 21x + 100 = Ax^2 - 20Ax + 100 + Bx - 10B - 10$$

Then we have

$$A = 1$$

$$-21 = -20A + B$$

$$100 = 100 - 10B - 10$$

Plug $A = 1$, $B = -1$, and $C = -10$ into the partial fractions decomposition.

$$f(x) = \frac{1}{x - 10} - \frac{1}{(x - 10)^2} - \frac{10}{(x - 10)^3}$$

6. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^2 + 8x}{(x + 8)^3}$$

Solution:

We can rewrite $f(x)$ as

$$\frac{x^2+8x}{(x+8)^3}=\frac{x(x+8)}{(x+8)^3}=\frac{x}{(x+8)^2}$$

These are repeated linear factors.

$$\frac{x}{(x+8)^2}=\frac{A}{x+8}+\frac{B}{(x+8)^2}$$

Combine the fractions by finding the common denominator.

$$\frac{x}{(x+8)^2}=\frac{A(x+8)+B}{(x+8)^2}$$

Because the denominators are equivalent, the numerators must also be equivalent, so we'll set the numerators equal to one another.

$$x=Ax+8A+B$$

Then we have

$$A=1$$

$$8A+B=0$$

Plug $A=1$ and $B=-8$ into the partial fractions decomposition.

$$f(x)=\frac{1}{x+8}-\frac{8}{(x+8)^2}$$

DISTINCT QUADRATIC FACTORS

■ 1. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{3x^3 + 3}{(x^2 + 2)(x^2 + 5)}$$

Solution:

These are distinct quadratic factors, so we'll set up the decomposition as

$$\frac{3x^3 + 3}{(x^2 + 2)(x^2 + 5)} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{x^2 + 5}$$

Combine the fractions by finding a common denominator.

$$\frac{3x^3 + 3}{(x^2 + 2)(x^2 + 5)} = \frac{(Ax + B)(x^2 + 5) + (Cx + D)(x^2 + 2)}{(x^2 + 2)(x^2 + 5)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$3x^3 + 3 = (Ax + B)(x^2 + 5) + (Cx + D)(x^2 + 2)$$

Multiply out the right side of this numerator equation,

$$3x^3 + 3 = Ax^3 + 5Ax + Bx^2 + 5B + Cx^3 + 2Cx + Dx^2 + 2D$$

then group like terms and factor.

$$3x^3 + 3 = (A + C)x^3 + (B + D)x^2 + (5A + 2C)x + 5B + 2D$$

Then we get

$$A + C = 3 \qquad B + D = 0$$

$$5A + 2C = 0 \qquad 5B + 2D = 3$$

Solving the two equations on the left as a system gives $A = -2$ and $C = 5$, while solving the two equations on the right as a system gives $B = 1$ and $D = -1$. Plugging these back into the partial fractions decomposition gives

$$f(x) = \frac{-2x + 1}{x^2 + 2} + \frac{5x - 1}{x^2 + 5}$$

■ 2. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{4x^2 + 4x}{(2x^2 + 2)(3x^2 + 1)}$$

Solution:

Rewrite the denominator by factoring a 2 out of $2x^2 + 2$, then canceling that 2 against the numerator.

$$f(x) = \frac{4x^2 + 4x}{(2x^2 + 2)(3x^2 + 1)} = \frac{2x^2 + 2x}{(x^2 + 1)(3x^2 + 1)}$$

These are distinct quadratic factors, so we'll set up the decomposition as

$$\frac{2x^2+2x}{(x^2+1)(3x^2+1)}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{x^2+5}$$

Combine the fractions by finding a common denominator.

$$\frac{2x^2+2x}{(x^2+1)(3x^2+1)}=\frac{(Ax+B)(3x^2+1)+(Cx+D)(x^2+1)}{(x^2+1)(3x^2+1)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$2x^2+2x=(Ax+B)(3x^2+1)+(Cx+D)(x^2+1)$$

Multiply out the right side of this numerator equation,

$$2x^2+2x=3Ax^3+Ax+3Bx^2+B+Cx^3+Cx+Dx^2+D$$

then group like terms and factor.

$$2x^2+2x=(3A+C)x^3+(3B+D)x^2+(A+C)x+B+D$$

Then we get

$$3A+C=0 \qquad 3B+D=2$$

$$A+C=2 \qquad B+D=0$$

Solving the two equations on the left as a system gives $A=-1$ and $C=3$, while solving the two equations on the right as a system gives $B=1$ and $D=-1$. Plugging these back into the partial fractions decomposition gives

$$f(x)=\frac{-x+1}{x^2+1}+\frac{3x-1}{3x^2+1}$$

3. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{6x^3 - 13x^2 + 4x - 3}{(4x^2 + 1)(x^2 + x + 1)}$$

Solution:

These are distinct quadratic factors, so we'll set up the decomposition as

$$\frac{6x^3 - 13x^2 + 4x - 3}{(4x^2 + 1)(x^2 + x + 1)} = \frac{Ax + B}{4x^2 + 1} + \frac{Cx + D}{x^2 + x + 1}$$

Combine the fractions by finding a common denominator.

$$\frac{6x^3 - 13x^2 + 4x - 3}{(4x^2 + 1)(x^2 + x + 1)} = \frac{(Ax + B)(x^2 + x + 1) + (Cx + D)(4x^2 + 1)}{(4x^2 + 1)(x^2 + x + 1)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$6x^3 - 13x^2 + 4x - 3 = (Ax + B)(x^2 + x + 1) + (Cx + D)(4x^2 + 1)$$

Multiply out the right side of this numerator equation,

$$6x^3 - 13x^2 + 4x - 3 = Ax^3 + Ax^2 + Ax + Bx^2 + Bx + B + 4Cx^3 + Cx + 4Dx^2 + D$$

then group like terms and factor.

$$6x^3 - 13x^2 + 4x - 3 = (A + 4C)x^3 + (A + B + 4D)x^2 + (A + B + C)x + B + D$$

Then we get

$$A + 4C = 6$$

$$A + B + 4D = -13$$

$$A + B + C = 4$$

$$B + D = -3$$

Solving these as a system of equations gives $A = 2$, $B = 1$, $C = 1$, and $D = -4$. Plugging the values back into the partial fractions decomposition gives

$$f(x) = \frac{2x + 1}{4x^2 + 1} + \frac{x - 4}{x^2 + x + 1}$$

■ 4. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 + 1}{(x^2 + 1)(x^2 + x + 2)}$$

Solution:

These are distinct quadratic factors, so we'll set up the decomposition as

$$\frac{x^3 + 1}{(x^2 + 1)(x^2 + x + 2)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + x + 2}$$

Combine the fractions by finding a common denominator.

$$\frac{x^3+1}{(x^2+1)(x^2+x+2)}=\frac{(Ax+B)(x^2+x+2)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+x+2)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$x^3+1=(Ax+B)(x^2+x+2)+(Cx+D)(x^2+1)$$

Multiply out the right side of this numerator equation,

$$x^3+1=Ax^3+Ax^2+2Ax+Bx^2+Bx+2B+Cx^3+Cx+Dx^2+D$$

then group like terms and factor.

$$x^3+1=(A+C)x^3+(A+B+D)x^2+(2A+B+C)x+2B+D$$

Then we get

$$A+C=1$$

$$A+B+D=0$$

$$2A+B+C=0$$

$$2B+D=1$$

Solving these as a system of equations gives $A=-1$, $B=0$, $C=2$, and $D=1$. Plugging the values back into the partial fractions decomposition gives

$$f(x)=-\frac{x}{x^2+1}+\frac{2x+1}{x^2+x+2}$$

5. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x+3}{(x^2+x+1)(x^2+2x+2)}$$

Solution:

These are distinct quadratic factors, so we'll set up the decomposition as

$$\frac{x+3}{(x^2+x+1)(x^2+2x+2)} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2+2x+2}$$

We'll combine the fractions on the right side of the equation by finding a common denominator.

$$\frac{x+3}{(x^2+x+1)(x^2+2x+2)} = \frac{(Ax+B)(x^2+2x+2)+(Cx+D)(x^2+x+1)}{(x^2+x+1)(x^2+2x+2)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$x+3 = (Ax+B)(x^2+2x+2)+(Cx+D)(x^2+x+1)$$

Multiply out the right side of this numerator equation,

$$x+3 = Ax^3+2Ax^2+2Ax+Bx^2+2Bx+2B+Cx^3+Cx^2+Cx+Dx^2+Dx+D$$

then group like terms and factor.

$$x+3 = (A+C)x^3+(2A+B+C+D)x^2+(2A+2B+C+D)x+(2B+D)$$

Then we can equate coefficients to build a system of equations.

$$A + C = 0 \qquad 2A + B + C + D = 0$$

$$2B + D = 3 \qquad 2A + 2B + C + D = 1$$

Solving the two equations on the right as a system gives $B = 1$, while solving the other equations gives $A = -2$, $C = 2$, and $D = 1$. Plugging these back into the partial fractions decomposition gives

$$f(x) = \frac{-2x + 1}{x^2 + x + 1} + \frac{2x + 1}{x^2 + 2x + 2}$$

■ 6. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{3x^4 + 3x^3 + 3x^2}{(x^2 + 1)(2x^2 + 5)(x^2 + x + 1)}$$

Solution:

We can rewrite $f(x)$ as

$$f(x) = \frac{3x^4 + 3x^3 + 3x^2}{(x^2 + 1)(2x^2 + 5)(x^2 + x + 1)} = \frac{3x^2(x^2 + x + 1)}{(x^2 + 1)(2x^2 + 5)(x^2 + x + 1)}$$

$$f(x) = = \frac{3x^2}{(x^2 + 1)(2x^2 + 5)}$$

These are distinct quadratic factors, so we'll set up the decomposition as

$$\frac{3x^2}{(x^2 + 1)(2x^2 + 5)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{2x^2 + 5}$$

We'll combine the fractions on the right side of the equation by finding a common denominator.

$$\frac{3x^2}{(x^2+1)(2x^2+5)} = \frac{(Ax+B)(2x^2+5)+(Cx+D)(x^2+1)}{x^2+1}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$3x^2 = (Ax+B)(2x^2+5)+(Cx+D)(x^2+1)$$

Multiply out the right side of this numerator equation,

$$3x^2 = 2Ax^3+5Ax+2Bx^2+5B+Cx^3+Cx+Dx^2+D$$

then group like terms and factor.

$$3x^2 = (2A+C)x^3+(2B+D)x^2+(5A+C)x+5B+D$$

Then we can equate coefficients to build a system of equations.

$$2A+C=0 \qquad\qquad 2B+D=3$$

$$5A+C=0 \qquad\qquad 5B+D=0$$

Solving the two equations on the left as a system gives $A=0$ and $C=0$, while solving the two equations on the right as a system gives $B=-1$ and $D=5$. Plugging these back into the partial fractions decomposition gives

$$f(x) = -\frac{1}{x^2+1}+\frac{5}{2x^2+5}$$

REPEATED QUADRATIC FACTORS

■ 1. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{10x^3 - 7}{(5x^2 + 3)^2}$$

Solution:

These are repeated quadratic factors, so we'll set up the decomposition as

$$\frac{10x^3 - 7}{(5x^2 + 3)^2} = \frac{Ax + B}{5x^2 + 3} + \frac{Cx + D}{(5x^2 + 3)^2}$$

Combine the fractions by finding a common denominator.

$$\frac{10x^3 - 7}{(5x^2 + 3)^2} = \frac{(Ax + B)(5x^2 + 3) + Cx + D}{(5x^2 + 3)^2}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$10x^3 - 7 = (Ax + B)(5x^2 + 3) + Cx + D$$

Multiply out the right side of this numerator equation,

$$10x^3 - 7 = 5Ax^3 + 3Ax + 5Bx^2 + 3B + Cx + D$$

then group like terms and factor.

$$10x^3 - 7 = 5Ax^3 + 5Bx^2 + (3A + C)x + 3B + D$$

Then we get

$$5A = 10 \qquad 3A + C = 0$$

$$5B = 0 \qquad 3B + D = -7$$

Solving the two equations on the left as a system gives $A = 2$ and $B = 0$, while solving the two equations on the right as a system gives $C = -6$ and $D = -7$. Plugging these back into the partial fractions decomposition gives

$$f(x) = \frac{2x}{5x^2 + 3} - \frac{6x + 7}{(5x^2 + 3)^2}$$

■ 2. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{3x^3 + 2x^2 + 30x + 16}{(x^2 + 9)^2}$$

Solution:

These are repeated quadratic factors, so we'll set up the decomposition as

$$\frac{3x^3 + 2x^2 + 30x + 16}{(x^2 + 9)^2} = \frac{Ax + B}{x^2 + 9} + \frac{Cx + D}{(x^2 + 9)^2}$$

Combine the fractions by finding a common denominator.

$$\frac{3x^3 + 2x^2 + 30x + 16}{(x^2 + 9)^2} = \frac{(Ax + B)(x^2 + 9) + Cx + D}{(x^2 + 9)^2}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$3x^3 + 2x^2 + 30x + 16 = (Ax + B)(x^2 + 9) + Cx + D$$

Multiply out the right side of this numerator equation,

$$3x^3 + 2x^2 + 30x + 16 = Ax^3 + 9Ax + Bx^2 + 9B + Cx + D$$

then group like terms and factor.

$$3x^3 + 2x^2 + 30x + 16 = Ax^3 + Bx^2 + (9A + C)x + 9B + D$$

Then we get

$$A = 3 \qquad\qquad 9A + C = 30$$

$$B = 2 \qquad\qquad 9B + D = 16$$

Solving the two equations on the right as a system gives $C = 3$ and $D = -2$. Plugging these back into the partial fractions decomposition gives

$$f(x) = \frac{3x + 2}{x^2 + 9} + \frac{3x - 2}{(x^2 + 9)^2}$$

■ 3. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 - 3x^2 - 11}{(x^2 + x + 2)^2}$$

Solution:

These are repeated quadratic factors, so we'll set up the decomposition as

$$\frac{x^3 - 3x^2 - 11}{(x^2 + x + 2)^2} = \frac{Ax + B}{x^2 + x + 2} + \frac{Cx + D}{(x^2 + x + 2)^2}$$

Combine the fractions by finding a common denominator.

$$\frac{x^3 - 3x^2 - 11}{(x^2 + x + 2)^2} = \frac{(Ax + B)(x^2 + x + 2) + Cx + D}{(x^2 + x + 2)^2}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$x^3 - 3x^2 - 11 = (Ax + B)(x^2 + x + 2) + Cx + D$$

Multiply out the right side of this numerator equation,

$$x^3 - 3x^2 - 11 = Ax^3 + Ax^2 + 2Ax + Bx^2 + Bx + 2B + Cx + D$$

then group like terms and factor.

$$x^3 - 3x^2 - 11 = Ax^3 + (A + B)x^2 + (2A + B + C)x + 2B + D$$

Then we get

$$A = 1 \qquad\qquad 2A + B + C = 0$$

$$A + B = -3 \qquad\qquad 2B + D = -11$$

Solving the two equations on the left as a system gives $B=-4$, while solving the two equations on the right as a system gives $C=2$ and $D=-3$. Plugging these back into the partial fractions decomposition gives

$$f(x)=\frac{x-4}{x^2+x+2}+\frac{2x-3}{(x^2+x+2)^2}$$

■ 4. Rewrite the function as its partial fractions decomposition.

$$f(x)=\frac{9x^4+6x^2+x+3}{(3x^2+1)^3}$$

Solution:

These are repeated quadratic factors, so we'll set up the decomposition as

$$\frac{9x^4+6x^2+x+3}{(3x^2+1)^3}=\frac{Ax+B}{3x^2+1}+\frac{Cx+D}{(3x^2+1)^2}+\frac{Ex+F}{(3x^2+1)^3}$$

Combine the fractions by finding a common denominator.

$$\frac{9x^4+6x^2+x+3}{(3x^2+1)^3}=\frac{(Ax+B)(9x^4+6x^2+1)+(Cx+D)(3x^2+1)+Ex+F}{(3x^2+1)^3}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$9x^4+6x^2+x+3=(Ax+B)(9x^4+6x^2+1)+(Cx+D)(3x^2+1)+Ex+F$$

Multiply out the right side of this numerator equation,

$$9x^4 + 6x^2 + x + 3 = 9Ax^5 + 6Ax^3 + Ax + 9Bx^4 + 6Bx^2 + B$$

$$+3Cx^3 + Cx + 3Dx^2 + D + Ex + F$$

then group like terms and factor.

$$9x^4 + 6x^2 + x + 3 = 9Ax^5 + 9Bx^4 + (6A + 3C)x^3$$

$$+(6B + 3D)x^2 + (A + C + E)x + B + D + F$$

Then we get

$$9A = 0 \qquad 6A + 3C = 0 \qquad A + C + E = 1$$

$$9B = 9 \qquad 6B + 3D = 6 \qquad B + D + F = 3$$

Solving the two equations on the left as a system gives $A = 0$ and $B = 1$, solving the two equations in the middle as a system gives $C = 0$ and $D = 0$, while solving the two equations on the right as a system gives $E = 1$ and $F = 2$. Plugging these back into the partial fractions decomposition gives

$$f(x) = \frac{1}{3x^2 + 1} + \frac{x + 2}{(3x^2 + 1)^3}$$

■ 5. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 + 5x^2 + 13x + 2}{(x^2 + 4x + 6)^2}$$

Solution:

These are repeated quadratic factors, so we'll set up the decomposition as

$$\frac{x^3+5x^2+13x+2}{(x^2+4x+6)^2}=\frac{Ax+B}{x^2+4x+6}+\frac{Cx+D}{(x^2+4x+6)^2}$$

Combine the fractions by finding a common denominator.

$$\frac{x^3+5x^2+13x+2}{(x^2+4x+6)^2}=\frac{(Ax+B)(x^2+4x+6)+Cx+D}{(x^2+4x+6)^2}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$x^3+5x^2+13x+2=(Ax+B)(x^2+4x+6)+Cx+D$$

Multiply out the right side of this numerator equation,

$$x^3+5x^2+13x+2=Ax^3+4Ax^2+6Ax+Bx^2+4Bx+6B+Cx+D$$

then group like terms and factor.

$$x^3+5x^2+13x+2=Ax^3+(4A+B)x^2+(6A+4B+C)x+6B+D$$

Then we get

$$A=1 \qquad\qquad 6A+4B+C=13$$

$$4A+B=5 \qquad\qquad 6B+D=2$$

Solving the two equations on the left as a system gives $B=1$. Solving the two equations on the right as a system gives $C=3$ and $D=-4$.

Plugging these back into the partial fractions decomposition gives

$$f(x) = \frac{x+1}{x^2+4x+6} + \frac{3x-4}{(x^2+4x+6)^2}$$

■ 6. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^7}{(x^2+1)^4}$$

Solution:

These are repeated quadratic factors, so we'll set up the decomposition as

$$\frac{x^7}{(x^2+1)^4} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{(x^2+1)^3} + \frac{Gx+H}{(x^2+1)^4}$$

Combine the fractions by finding a common denominator.

$$\frac{x^7}{(x^2+1)^4} = \frac{(Ax+B)(x^2+1)^3 + (Cx+D)(x^2+1)^2 + (Ex+F)(x^2+1) + Gx+H}{(x^2+1)^4}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$x^7 = (Ax+B)(x^2+1)^3 + (Cx+D)(x^2+1)^2 + (Ex+F)(x^2+1) + Gx+H$$

Multiply out the right side of this numerator equation,

$$x^7 = (Ax+B)(x^6+3x^4+3x^2+1) + (Cx+D)(x^4+2x^2+1)$$

$$+(Ex+F)(x^2+1)+Gx+H$$

then group like terms and factor.

$$x^7=Ax^7+3Ax^5+3Ax^3+Ax+Bx^6+3Bx^4+3Bx^2+B+Cx^5+2Cx^3+Cx$$

$$+Dx^4+2Dx^2+D+Ex^3+Ex+Fx^2+F+Gx+H$$

$$x^7=Ax^7+Bx^6+(3A+C)x^5+(3B+D)x^4+(3A+2C+E)x^3+(3B+2D+F)x^2$$

$$+(A+C+E+G)x+B+D+F+H$$

Then we get

$$A=1$$

$$B=0$$

$$3A+C=0 \qquad C=-3$$

$$3B+D=0 \qquad D=0$$

$$3A+2C+E=0 \qquad E=3$$

$$3B+2D+F=0 \qquad F=0$$

$$A+C+E+G=0 \qquad G=-1$$

$$B+D+F+H=0 \qquad H=0$$

Plugging these back into the partial fractions decomposition gives

$$f(x)=\frac{x}{x^2+1}-\frac{3x}{(x^2+1)^2}+\frac{3x}{(x^2+1)^3}-\frac{x}{(x^2+1)^4}$$

MIXED FACTORS

■ 1. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{2x^4 + 16}{x(x^2 + 2)^2}$$

Solution:

We'll set up the decomposition as

$$\frac{2x^4 + 16}{x(x^2 + 2)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2} + \frac{Dx + E}{(x^2 + 2)^2}$$

To solve for A, we'll remove the x factor from the left side, then evaluate what remains at $x = 0$.

$$\frac{0 + 16}{(0 + 2)^2} \to \frac{16}{4} \to 4 = A$$

To solve for the other constants, we'll find a common denominator on the right side, then combine the fractions.

$$\frac{2x^4 + 16}{x(x^2 + 2)^2} = \frac{4(x^4 + 4x^2 + 4) + (Bx + C)x(x^2 + 2) + (Dx + E)x}{x(x^2 + 2)^2}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$2x^4+16=4(x^4+4x^2+4)+(Bx+C)x(x^2+2)+(Dx+E)x$$

Multiply out the right side of this numerator equation,

$$2x^4+16=4x^4+16x^2+16+Bx^4+2Bx^2+Cx^3+2Cx+Dx^2+Ex$$

then group like terms and factor.

$$2x^4+16=(4+B)x^4+Cx^3+(16+2B+D)x^2+(2C+E)x+16$$

We can equate coefficients on either side of the equation to build a system of equations.

$$2=4+B \qquad\qquad 16+2B+D=0$$

$$C=0 \qquad\qquad 2C+E=0$$

Solving the two equations on the left as a system gives $B=-2$ and $C=0$, while solving the two equations on the right as a system gives $D=-12$ and $E=0$. Plugging these back into the partial fractions decomposition gives

$$f(x)=\frac{4}{x}-\frac{2x}{x^2+2}-\frac{12x}{(x^2+2)^2}$$

■ 2. Rewrite the function as its partial fractions decomposition.

$$f(x)=\frac{4x+8}{(x^2-1)(2x+2)(2x+1)}$$

Solution:

We can rewrite $f(x)$ as

$$\frac{4x+8}{(x^2-1)(2x+2)(2x+1)}=\frac{4x+8}{2(x-1)(x+1)(x+1)(2x+1)}=\frac{2x+4}{(x-1)(2x+1)(x+1)^2}$$

We'll set up the decomposition as

$$\frac{2x+4}{(x-1)(2x+1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{2x+1}+\frac{C}{x+1}+\frac{D}{(x+1)^2}$$

To solve for A, we'll remove the $x-1$ factor from the left side, then evaluate what remains on the left side at $x=1$.

$$\frac{2+4}{(2+1)(1+1)^2}\to\frac{6}{12}\to\frac{1}{2}=A$$

To solve for B, we'll remove the $2x+1$ factor from the left side, then evaluate what remains at $x=-1/2$.

$$\frac{-1+4}{-\frac{3}{2}\cdot\frac{1}{4}}\to\frac{3}{-\frac{3}{8}}\to-8=B$$

To solve for D, we'll remove the $(x+1)^2$ factor from the left side, then evaluate what remains at $x=-1$.

$$\frac{-2+4}{(-2)(-2+1)}=\frac{2}{2}=1=D$$

Now set $x=0$.

$$\frac{4}{-1 \cdot 1 \cdot 1^2} = \frac{\frac{1}{2}}{-1} + \frac{-8}{1} + \frac{C}{1} + \frac{1}{1}$$

$$-4 = \frac{-15}{2} + C$$

$$C = \frac{7}{2}$$

Plugging the values back into the partial fractions decomposition gives

$$f(x) = \frac{1}{2(x-1)} - \frac{8}{2x+1} + \frac{7}{2(x+1)} + \frac{1}{(x+1)^2}$$

■ 3. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{2x^3 + 7x^2 - 2x + 5}{x^4 - 1}$$

Solution:

We'll set up the decomposition as

$$\frac{2x^3 + 7x^2 - 2x + 5}{(x^2+1)(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

To solve for A, we'll remove the $x - 1$ factor from the left side, then evaluate what remains at $x = 1$.

$$\frac{2+7-2+5}{2 \cdot 2} \to \frac{12}{4} \to 3 = A$$

To solve for B, we'll remove the $x+1$ factor from the left side, then evaluate what remains at $x=-1$.

$$\frac{-2+7+2+5}{2(-2)} \to \frac{12}{-4} \to -3 = B$$

Combine the fractions by finding a common denominator.

$$\frac{2x^3+7x^2-2x+5}{(x^2+1)(x-1)(x+1)} = \frac{3(x+1)(x^2+1)-3(x-1)(x^2+1)+(Cx+D)(x^2-1)}{(x^2+1)(x-1)(x+1)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$2x^3+7x^2-2x+5 = 3(x+1)(x^2+1)-3(x-1)(x^2+1)+(Cx+D)(x^2-1)$$

Multiply out the right side of this numerator equation,

$$2x^3+7x^2-2x+5 = 3x^3+3x^2+3x+3-3x^3+3x^2-3x+3+Cx^3-Cx+Dx^2-D$$

then group like terms and factor.

$$2x^3+7x^2-2x+5 = 6x^2+6+Cx^3-Cx+Dx^2-D$$

$$2x^3+7x^2-2x+5 = Cx^3+(6+D)x^2-Cx+6-D$$

Then we get

$$2 = C$$

$$5 = 6-D$$

Plugging $C=2$ and $D=1$ back into the partial fractions decomposition gives

$$f(x) = \frac{3}{x-1} - \frac{3}{x+1} + \frac{2x+1}{x^2+1}$$

■ 4. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{36}{(x+2)(x^2-1)^2}$$

Solution:

We'll set up the decomposition as

$$\frac{36}{(x+2)(x+1)^2(x-1)^2} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{x-1} + \frac{E}{(x-1)^2}$$

To solve for A, we'll remove the $x+2$ factor from the left side, then evaluate what remains at $x = -2$.

$$\frac{36}{1(-3)^2} \to 4 = A$$

To solve for C, we'll remove the $(x+1)^2$ factor from the left side, then evaluate what remains at $x = -1$.

$$\frac{36}{1(-2)^2} \to 9 = C$$

To solve for E, we'll remove the $(x-1)^2$ factor from the left side, then evaluate what remains at $x = 1$.

$$\frac{36}{3 \cdot 2^2} \to 3 = E$$

Now set $x = 0$,

$$\frac{36}{2 \cdot 1 \cdot 1} = \frac{4}{2} + B + 9 - D + 3$$

$$B - D = 4$$

And then set $x = 2$.

$$\frac{36}{4 \cdot 3^2 \cdot 1} = \frac{4}{4} + \frac{B}{3} + \frac{9}{9} + D + 3$$

$$\frac{B}{3} + D = -4$$

Then we get

$$B - D = 4$$

$$\frac{B}{3} + D = -4$$

Solving these as a system of equations, we get $B = 0$ and $D = -4$. Plugging these values back into the partial fractions decomposition gives

$$f(x) = \frac{4}{x+2} + \frac{9}{(x+1)^2} - \frac{4}{x-1} + \frac{3}{(x-1)^2}$$

■ 5. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{4x^4 + 7x^3 + 4x^2 + 3x - 2}{x^3(x^2 + x + 1)}$$

Solution:

We'll set up the decomposition as

$$\frac{4x^4 + 7x^3 + 4x^2 + 3x - 2}{x^3(x^2 + x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx + E}{x^2 + x + 1}$$

To solve for C, we'll remove the x^3 factor from the left side, then evaluate what remains at $x = 0$.

$$\frac{-2}{1} \to -2 = C$$

Combine the fractions by finding a common denominator.

$$\frac{4x^4 + 7x^3 + 4x^2 + 3x - 2}{x^3(x^2 + x + 1)}$$

$$= \frac{Ax^2(x^2 + x + 1) + Bx(x^2 + x + 1) + C(x^2 + x + 1) + x^3(Dx + E)}{x^3(x^2 + x + 1)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$4x^4 + 7x^3 + 4x^2 + 3x - 2 = Ax^2(x^2 + x + 1) + Bx(x^2 + x + 1)$$

$$-2(x^2 + x + 1) + x^3(Dx + E)$$

Multiply out the right side of this numerator equation,

$$4x^4 + 7x^3 + 4x^2 + 3x - 2 = Ax^4 + Ax^3 + Ax^2 + Bx^3 + Bx^2 + Bx$$

$$-2x^2 - 2x - 2 + Dx^4 + Ex^3$$

then group like terms and factor.

$$4x^4 + 7x^3 + 4x^2 + 3x - 2 = (A + D)x^4 + (A + B + E)x^3$$

$$+(A + B - 2)x^2 + (B - 2)x - 2$$

Then we get

$$A + D = 4 \qquad\qquad A + B - 2 = 4$$

$$A + B + E = 7 \qquad\qquad B - 2 = 3$$

Solving these as a system of equations gives $A = 1$, $B = 5$, $D = 3$, and $E = 1$. Plugging these back into the partial fractions decomposition gives

$$f(x) = \frac{1}{x} + \frac{5}{x^2} - \frac{2}{x^3} + \frac{3x + 1}{x^2 + x + 1}$$

■ 6. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x + 1}{(x^2 + 1)(x^2 + x + 1)}$$

Solution:

We'll set up the decomposition as

$$\frac{x+1}{(x^2+1)(x^2+x+1)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+x+1}$$

Combine the fractions by finding a common denominator.

$$\frac{x+1}{(x^2+1)(x^2+x+1)}=\frac{(Ax+B)(x^2+x+1)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+x+1)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$x+1=(Ax+B)(x^2+x+1)+(Cx+D)(x^2+1)$$

Multiply out the right side of this numerator equation,

$$x+1=Ax^3+Ax^2+Ax+Bx^2+Bx+B+Cx^3+Cx+Dx^2+D$$

then group like terms and factor.

$$x+1=(A+C)x^3+(A+B+D)x^2+(A+B+C)x+B+D$$

Then we get

$$A+C=0 \qquad A+B+C=1$$

$$A+B+D=0 \qquad B+D=1$$

Solving these as a system of equations gives $A=-1$, $B=1$, $C=1$, and $D=0$. Plugging these back into the partial fractions decomposition gives

$$f(x)=\frac{-x+1}{x^2+1}+\frac{x}{x^2+x+1}$$

IDENTIFYING CONIC SECTIONS

■ 1. Identify the equation as a circle, ellipse, parabola, or hyperbola.

$$5y^2 - 2 = x + 3y + 6$$

Solution:

This equation has a y^2 term and an x term. Because one variable is squared and the other is not, the equation represents a parabola.

■ 2. Identify the equation as a circle, ellipse, parabola, or hyperbola.

$$x^2 - 5x + 2y = 1 - y^2$$

Solution:

This equation has both an x^2 and y^2 term, so it's either a circle, an ellipse, or a hyperbola. If we move everything but the constant to the same side of the equation,

$$x^2 - 5x + 2y = 1 - y^2$$

$$x^2 + y^2 - 5x + 2y = 1$$

we see that the x^2 and y^2 terms have the same sign when they're on the same side of the equation (they're both positive), so the equation represents either an ellipse or a circle.

Because the coefficients on x^2 and y^2 are equal (they're both 1), the equation represents a circle.

■ 3. Identify the equation as a circle, ellipse, parabola, or hyperbola.

$$8y^2 - 9x + 2y = -2x^2 + 6$$

Solution:

This equation has both an x^2 and y^2 term, so it's either a circle, an ellipse, or a hyperbola. If we move everything but the constant to the same side of the equation,

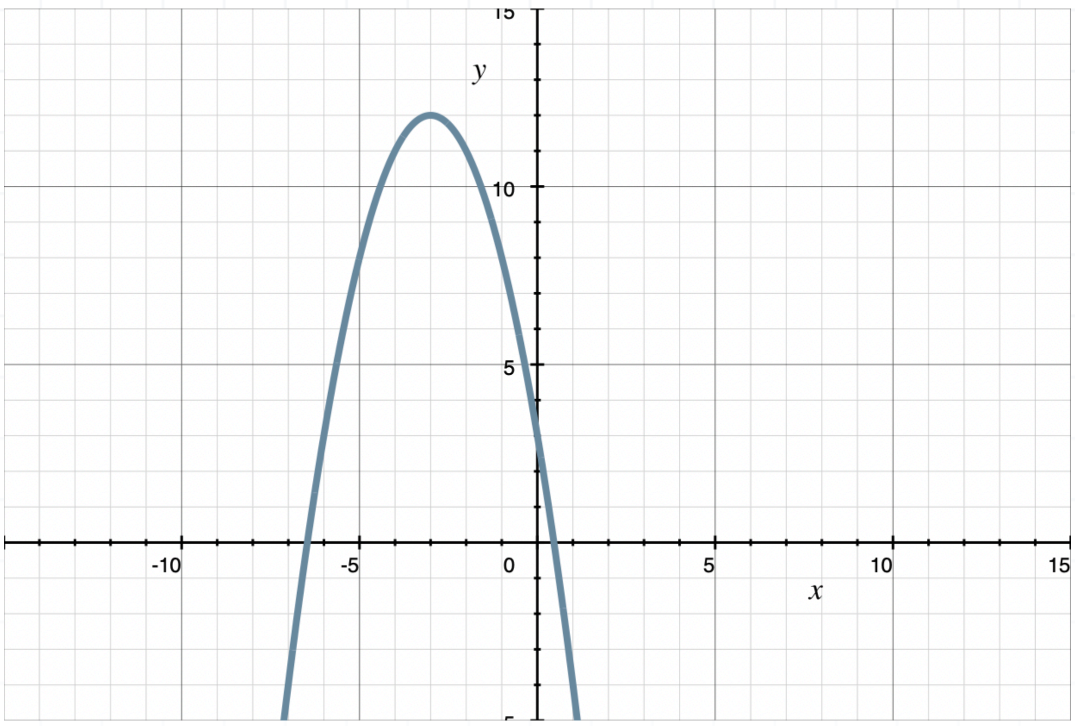
$$8y^2 - 9x + 2y = -2x^2 + 6$$

$$8y^2 + 2x^2 - 9x + 2y = 6$$

we see that the x^2 and y^2 terms have the same sign when they're on the same side of the equation (they're both positive), so the equation represents either an ellipse or a circle.

Because the coefficients on x^2 and y^2 are unequal (they're 8 and 2), the equation represents an ellipse.

■ 4. Identify the graph as a circle, ellipse, parabola, or hyperbola.



Solution:

This is the graph of a parabola.

■ 5. Identify the equation as a circle, ellipse, parabola, or hyperbola.

$$11x + 12y^2 - 2 = 9y - 12x^2 + 15$$

Solution:

This equation has both an x^2 and y^2 term, so it's either a circle, an ellipse, or a hyperbola. If we move everything but the constant to the same side of the equation,

$$11x + 12y^2 - 2 = 9y - 12x^2 + 15$$

$$12x^2 + 12y^2 + 11x - 9y = 17$$

we see that the x^2 and y^2 terms have the same sign when they're on the same side of the equation (they're both positive), so the equation represents an ellipse or a circle.

Because the coefficients on x^2 and y^2 are equal (they're both 12), the equation represents a circle.

■ 6. Identify the equation as a circle, ellipse, parabola, or hyperbola.

$$-5x + 14y - 4x^2 = 25 - 2y^2$$

Solution:

This equation has both an x^2 and y^2 term, so it's either a circle, an ellipse, or a hyperbola. If we move everything but the constant to the same side of the equation,

$$-5x + 14y - 4x^2 = 25 - 2y^2$$

$$-4x^2 + 2y^2 - 5x + 14y = 25$$

we see that the x^2 and y^2 terms have different signs when they're on the same side of the equation (the y^2 term is positive and the x^2 term is negative), so the equation represents a hyperbola.

CIRCLES

1. If the center of a circle is $(-4,1)$ and a point on the circle is $(0,-2)$, find the equation of the circle.

Solution:

We need to find the radius. The horizontal distance between these points is $0-(-4)=4$, and the vertical distance between the points is $-2-1=-3$. Plugging these into the Pythagorean Theorem, we get

$$a^2+b^2=c^2$$

$$4^2+(-3)^2=r^2$$

$$r^2=16+9$$

$$r=\sqrt{25}$$

$$r=5$$

Now plug the center and radius into the equation of the circle.

$$(x-h)^2+(y-k)^2=r^2$$

$$(x-(-4))^2+(y-1)^2=5^2$$

$$(x+4)^2+(y-1)^2=25$$

■ 2. If the center of a circle is $(7, -2)$ and a point on the circle is $(10, -4)$, find the equation of the circle.

Solution:

We need to find the radius. The horizontal distance between these points is $10 - 7 = 3$, and the vertical distance between the points is $-4 - (-2) = -2$. Plugging these into the Pythagorean Theorem, we get

$$a^2 + b^2 = c^2$$

$$3^2 + (-2)^2 = r^2$$

$$r^2 = 9 + 4$$

$$r = \sqrt{13}$$

Now plug the center and radius into the equation of the circle.

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - 7)^2 + (y - (-2))^2 = (\sqrt{13})^2$$

$$(x - 7)^2 + (y + 2)^2 = 13$$

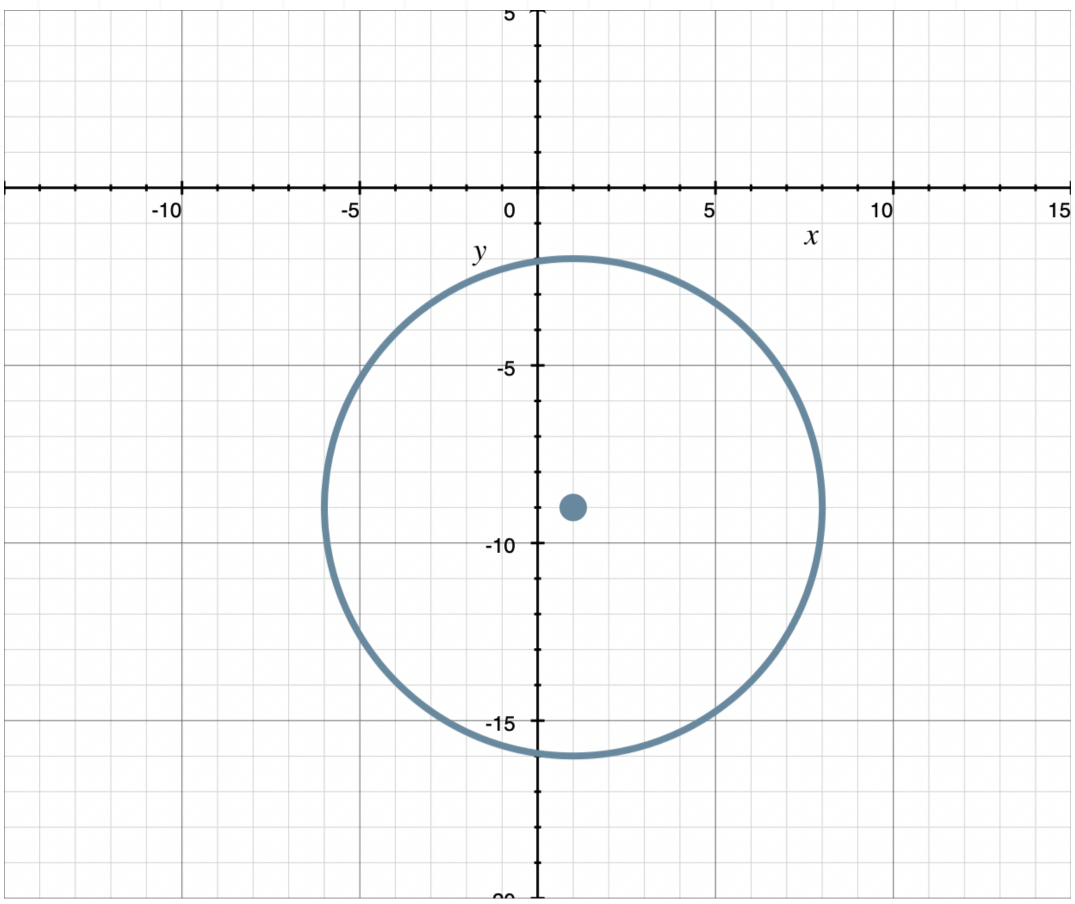
■ 3. Graph the circle.

$$(x - 1)^2 + (y + 9)^2 = 49$$

Solution:

The standard form for the equation of a circle is $(x-h)^2+(y-k)^2=r^2$ where the center of the circle is (h,k) and the radius is r.

The equation $(x-1)^2+(y+9)^2=49$ is in standard form, so we can see that the center of the circle is at $(h,k)=(1,-9)$ and the radius is $r=\sqrt{49}=7$.



■ 4. Find the center and radius of the circle.

$$(x-7)^2+(y+11)^2=18$$

Solution:

The standard form for the equation of a circle is $(x-h)^2+(y-k)^2=r^2$ where the center of the circle is (h,k) and the radius is r.

The equation $(x-7)^2+(y+11)^2=18$ is in standard form, so we can see that the center of the circle is at $(h,k)=(7,-11)$, and the radius is $r=\sqrt{18}=3\sqrt{2}$.

■ 5. Find the center and radius of the circle.

$$x^2+y^2-4x-2y+1=0$$

Solution:

Group together the x and y terms.

$$(x^2-4x)+(y^2-2y)+1=0$$

Complete the square with respect to x. Take the coefficient of -4 on $-4x$ and divide it by 2 to get $-4/2=-2$. Square this value to get $(-2)^2=4$. Then add this value to both sides of the equation.

$$(x^2-4x+4)+(y^2-2y)+1=0+4$$

Complete the square with respect to y. Take the coefficient of -2 on $-2y$ and divide it by 2 to get $-2/2=-1$. Square this value to get $(-1)^2=1$. Then add this value to both sides of the equation.

$$(x^2-4x+4)+(y^2-2y+1)+1=0+4+1$$

$$(x^2-4x+4)+(y^2-2y+1)=4$$

We're left with two perfect squares on the left that can be factored.

$$(x-2)^2+(y-1)^2=4$$

Now that the equation is in standard form, we can see that the center of the circle is at $(h,k)=(2,1)$, and the radius is $r=\sqrt{4}=2$.

■ 6. Find the center and radius of the circle.

$$x^2+y^2+12x-26y+173=0$$

Solution:

Group together the x and y terms.

$$(x^2+12x)+(y^2-26y)+173=0$$

Complete the square with respect to x. Take the coefficient of 12 on $12x$ and divide it by 2 to get $12/2=6$. Square this value to get $6^2=36$. Then add this value to both sides of the equation.

$$(x^2+12x+36)+(y^2-26y)+173=0+36$$

Complete the square with respect to y. Take the coefficient of -26 on $-26y$ and divide it by 2 to get $-26/2=-13$. Square this value to get $(-13)^2=169$. Then add this value to both sides of the equation.

$$(x^2+12x+36)+(y^2-26y+169)+173=0+36+169$$

$$(x^2 + 12x + 36) + (y^2 - 26y + 169) = 32$$

We're left with two perfect squares on the left that can be factored.

$$(x + 6)^2 + (y - 13)^2 = 32$$

Now that the equation is in standard form, we can see that the center of the circle is at $(h, k) = (-6,13)$, and the radius is $r = \sqrt{32} = 4\sqrt{2}$.

ELLIPSES

■ 1. Sketch the graph of the ellipse by finding its center and major and minor radii.

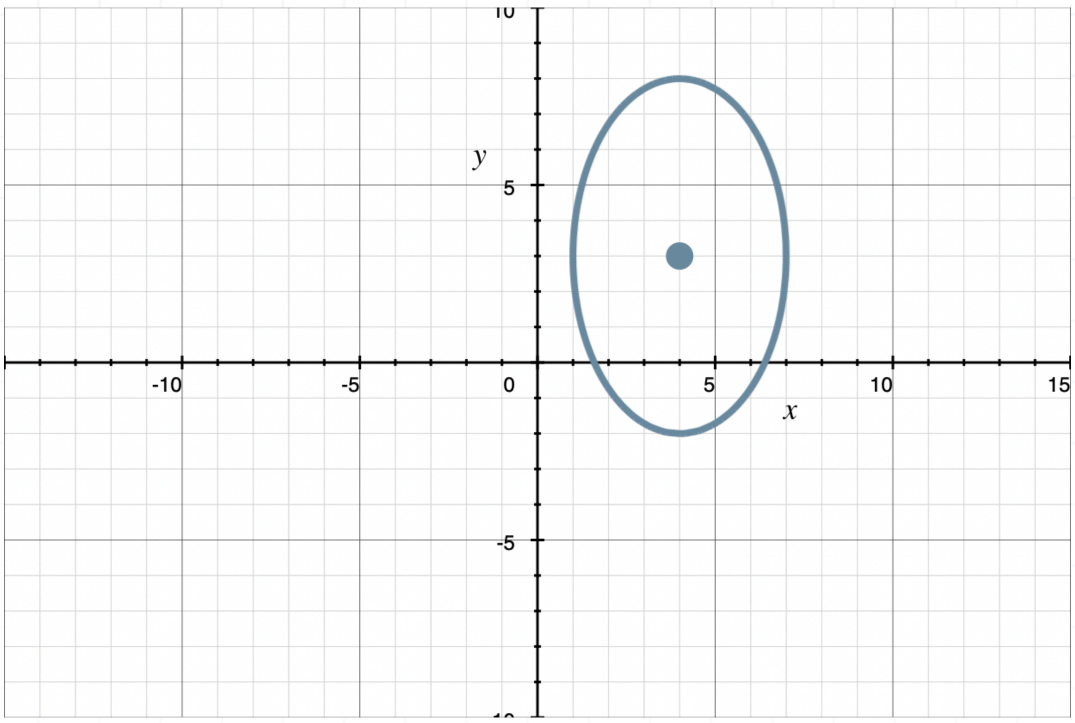
$$\frac{(x-4)^2}{9}+\frac{(y-3)^2}{25}=1$$

Solution:

To graph an ellipse, we need to find the center and the length of each axis. Since $25>9$, this is a tall ellipse, and the standard form of the equation of a tall ellipse is

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

The center of the ellipse is at $(4,3)$. The length of the horizontal radius is $\sqrt{9}=3$ and the length of the vertical radius is $\sqrt{25}=5$. So the graph of the ellipse is



■ 2. Sketch the graph of the ellipse by finding its center and major and minor radii.

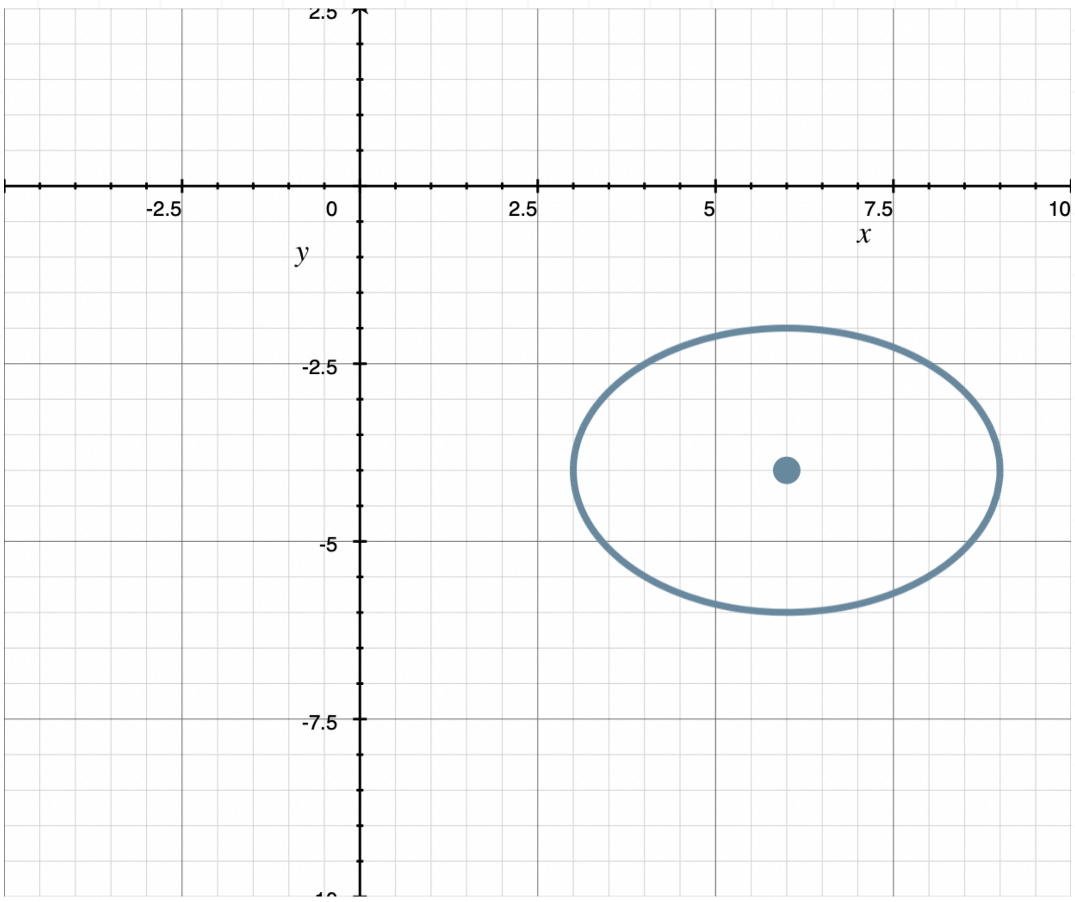
$$\frac{(x-6)^2}{9}+\frac{(y+4)^2}{4}=1$$

Solution:

To graph an ellipse, we need to find the center and the length of each axis. Since $9>4$, this is a wide ellipse, and the standard form of the equation of a wide ellipse is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

The center of the ellipse is at $(6,-4)$. The length of the horizontal radius is $\sqrt{9}=3$ and the length of the vertical radius is $\sqrt{4}=2$. So the graph of the ellipse is



■ 3. Find the coordinates of the foci of the ellipse.

$$\frac{(x+7)^2}{4}+\frac{(y+6)^2}{20}=1$$

Solution:

To find the coordinates of the foci, add the focal length to the center along the major axis. The focal length is given by $c=\sqrt{a^2-b^2}$. Since $20>4$, this is a tall ellipse, and the standard form of the equation of a tall ellipse is

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

The focal length of the ellipse is

$$c=\sqrt{a^2-b^2}=\sqrt{20-4}=\sqrt{16}=4$$

The major axis is the vertical axis because $20>4$, which means we need to add and subtract 4 from the y-value of the center $(-7,-6)$ in order to find the foci. The coordinates of the foci are

$(-7,-6+4)$ and $(-7,-6-4)$

$(-7,-2)$ and $(-7,-10)$

■ 4. Find the coordinates of the foci of the ellipse.

$$\frac{(x-3)^2}{8}+\frac{(y-6)^2}{5}=1$$

Solution:

To find the coordinates of the foci, add the focal length to the center along the major axis. The focal length is given by $c=\sqrt{a^2-b^2}$. Since $8>5$, this is a wide ellipse, and the standard form of the equation of a wide ellipse is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

The focal length of the ellipse is

$$c = \sqrt{a^2 - b^2} = \sqrt{8 - 5} = \sqrt{3}$$

The major axis is the horizontal axis because $8 > 5$, which means we need to add and subtract $\sqrt{3}$ from the x-value of the center $(3,6)$ in order to find the foci. The coordinates of the foci are

$(3 + \sqrt{3},6)$ and $(3 - \sqrt{3},6)$

■ 5. Sketch the graph of the ellipse.

$$x^2 - 12y + 37 = 6 - 3y^2 - 10x$$

Solution:

First, collect all the non-constant terms onto one side of the equation, leaving the constant term alone on the other side.

$$x^2 - 12y + 37 = 6 - 3y^2 - 10x$$

$$x^2 + 10x + 3y^2 - 12y = -31$$

Because the x^2 and y^2 terms have the same sign when they're on the same side of the equation (they're both positive), we know equation represents an ellipse.

We need to complete the square with respect to both variables. We'll start with x.

$$x^2 + 10x + 3y^2 - 12y = -31$$

$$x^2 + 10x + 25 + 3y^2 - 12y = -31 + 25$$

$$(x + 5)^2 + 3y^2 - 12y = -6$$

Now we'll complete the square with respect to y.

$$\frac{1}{3}(x + 5)^2 + y^2 - 4y = -2$$

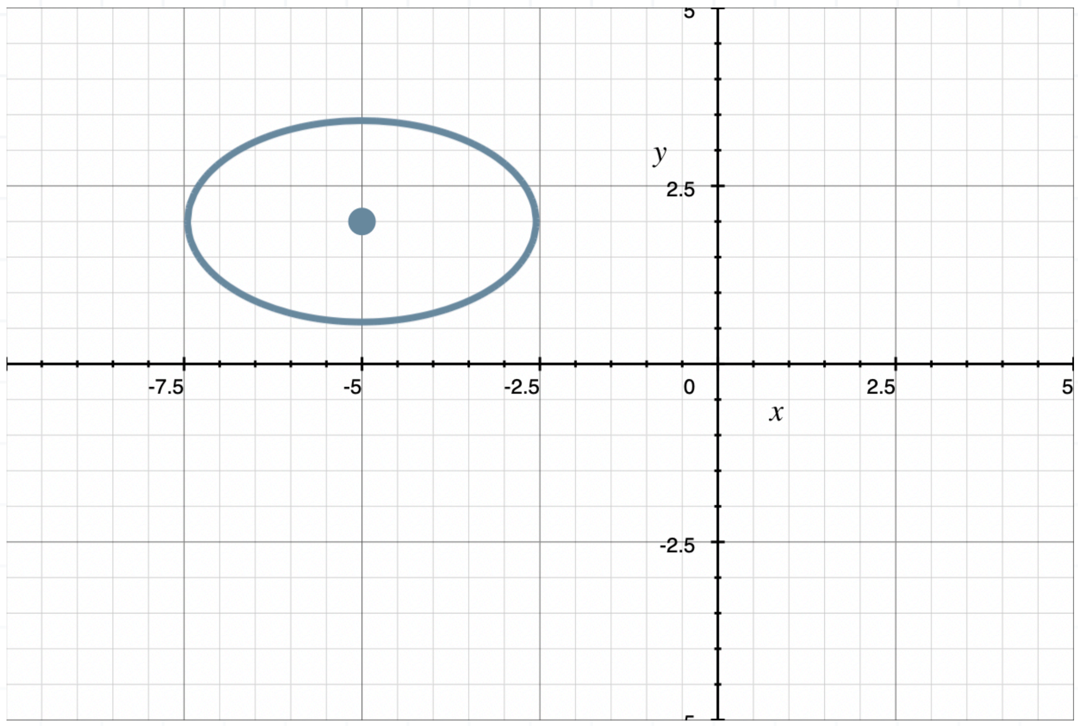
$$\frac{1}{3}(x + 5)^2 + y^2 - 4y + 4 = -2 + 4$$

$$\frac{1}{3}(x + 5)^2 + (y - 2)^2 = 2$$

Last, we'll divide through by 2 to put the ellipse into standard form.

$$\frac{(x + 5)^2}{6} + \frac{(y - 2)^2}{2} = 1$$

With the ellipse in standard form, we can see that the center is at $(-5,2)$. The length of the major radius is $\sqrt{6}$, and the length of the minor radius is $\sqrt{2}$, so it's a wide ellipse and its graph is



■ 6. Sketch the graph of the ellipse.

$$14y - 24x + 85 = 16 - 4x^2 - y^2$$

Solution:

First, collect all the non-constant terms onto one side of the equation, leaving the constant term alone on the other side.

$$14y - 24x + 85 = 16 - 4x^2 - y^2$$

$$4x^2 - 24x + y^2 + 14y = -69$$

Because the x^2 and y^2 terms have the same sign when they're on the same side of the equation (they're both positive), we know the equation represents an ellipse.

We need to complete the square with respect to both variables. We'll start with x.

$$x^2 - 6x + \frac{1}{4}y^2 + \frac{7}{2}y = \frac{-69}{4}$$

$$x^2 - 6x + 9 + \frac{1}{4}y^2 + \frac{7}{2}y = \frac{-69}{4} + 9$$

$$(x-3)^2 + \frac{1}{4}y^2 + \frac{7}{2}y = \frac{-33}{4}$$

Now we'll complete the square with respect to y.

$$4(x-3)^2 + y^2 + 14y = -33$$

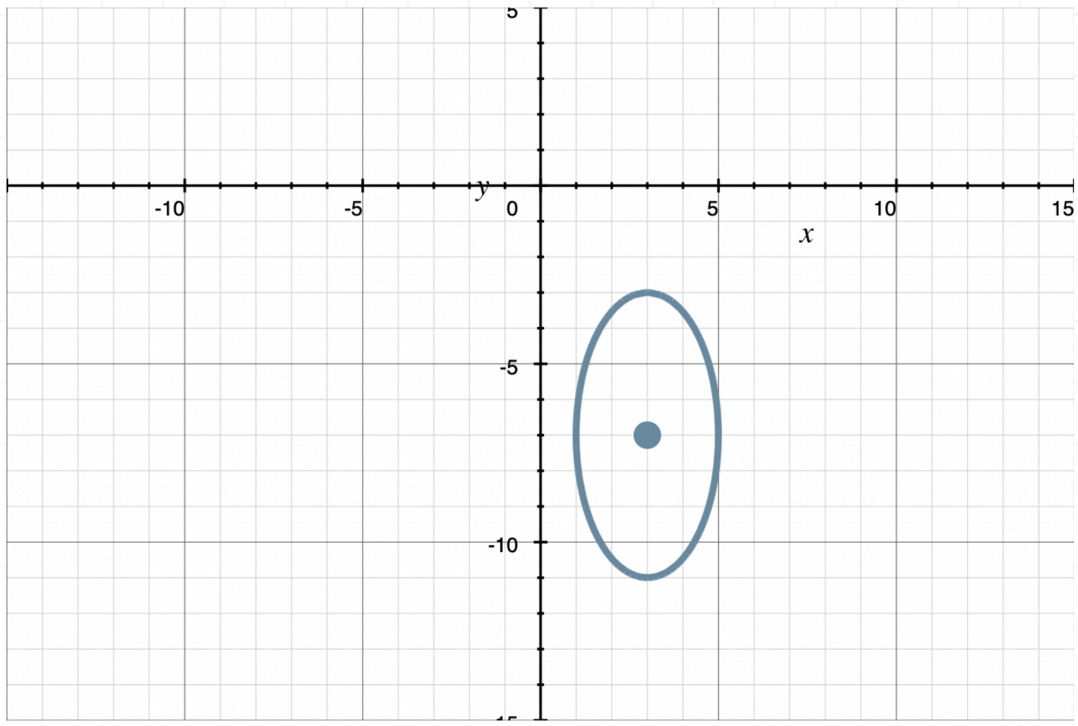
$$4(x-3)^2 + y^2 + 14y + 49 = -33 + 49$$

$$4(x-3)^2 + (y+7)^2 = 16$$

Last, we'll divide through by 16 to put the ellipse into standard form.

$$\frac{(x-3)^2}{4} + \frac{(y+7)^2}{16} = 1$$

With the ellipse in standard form, we can see that the center is at $(3,-7)$. The length of the major radius is $\sqrt{16} = 4$, and the length of the minor radius is $\sqrt{4} = 2$, so it's a tall ellipse and its graph is


y
x
-10
-5
0
5
10
15
-5
-10

PARABOLAS

■ 1. Find the equation of the parabola with a focus at $(-1,9)$ and a directrix at $y = 7$.

Solution:

The directrix is horizontal, so the equation of the parabola has the form $4p(y-k) = (x-h)^2$. Its focus is $(h, k+p) = (-1,9)$, and the directrix is $y = k - p = 7$, so we get the system of equations

$$h = -1$$

$$k + p = 9$$

$$k - p = 7$$

Solving the system gives $k = 8$ and $p = 1$, so the equation of the parabola is

$$4(y-8) = (x+1)^2$$

■ 2. Find the equation of the parabola with a focus at $(3, -7)$ and a directrix at $y = -3$.

Solution:

The directrix is horizontal, so the equation of the parabola has the form $4p(y-k)=(x-h)^2$. Its focus is $(h,k+p)=(3,-7)$, and the directrix is $y=k-p=-3$, so we get the system of equations

$$h=3$$

$$k+p=-7$$

$$k-p=-3$$

Solving the system gives $k=-5$ and $p=-2$, so the equation of the parabola is $-8(y+5)=(x-3)^2$.

■ 3. Find the focus and directrix of the parabola.

$$y=x^2-3$$

Solution:

Rewrite the equation in vertex form.

$$y=1(x-0)^2-3$$

The equation is in the vertex form with $a=1$, $h=0$, and $k=-3$. The focus is at

$$\left(h,k+\frac{1}{4a}\right)=\left(0,-3+\frac{1}{4}\right)=\left(0,-\frac{11}{4}\right)$$

and the directrix is

$$y = k - \frac{1}{4a}$$

$$y = -3 - \frac{1}{4} = -\frac{13}{4}$$

■ 4. Find the focus and directrix of the parabola.

$$y = -\frac{1}{3}(x-1)^2 + 2$$

Solution:

The equation is in the vertex form with $a = -1/3$, $h = 1$, and $k = 2$. The focus is at

$$\left(h, k + \frac{1}{4a}\right) = \left(1, 2 - \frac{3}{4}\right) = \left(1, \frac{5}{4}\right)$$

and the directrix is

$$y = k - \frac{1}{4a}$$

$$y = 2 + \frac{3}{4} = \frac{11}{4}$$

■ 5. Find each piece of the parabola from its equation.

$$y = \frac{1}{2}x^2 + 4$$

Solution:

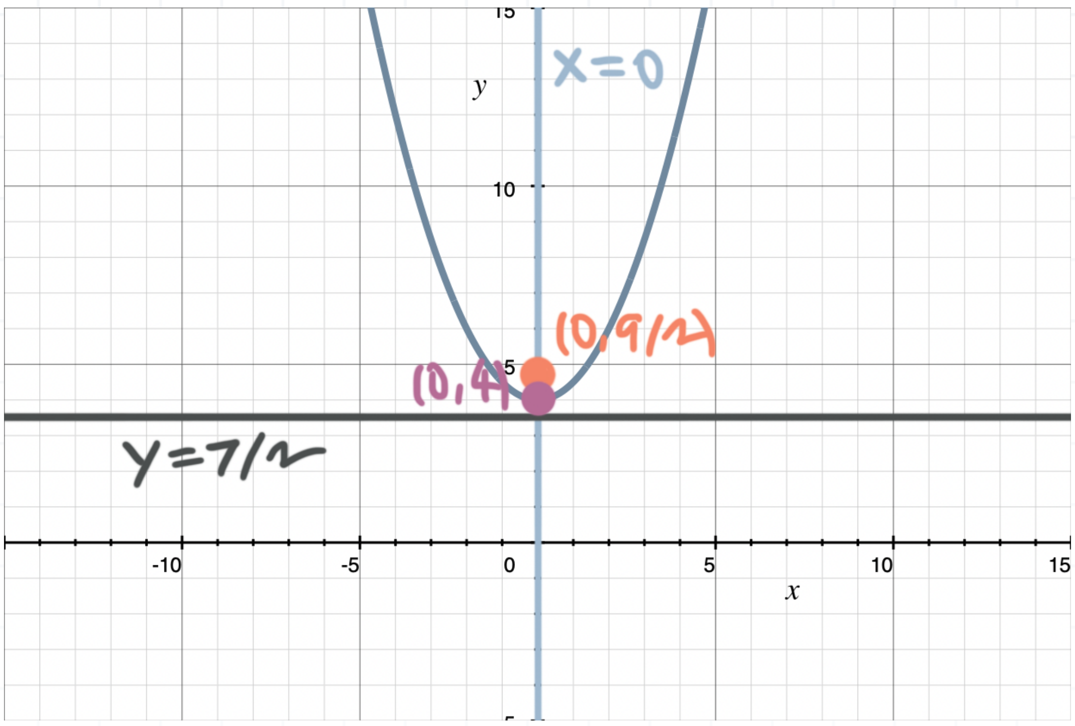
Rewrite the equation in vertex form.

$$y = \frac{1}{2}(x - 0)^2 + 4$$

From vertex form, we can pull all the components of the parabola.

Equation	$y = a(x - h)^2 + k$	$y = \frac{1}{2}(x - 0)^2 + 4$
Vertex	(h, k)	$(0,4)$
Axis	$x = h$	$x = 0$
Focus	$\left(h, k + \frac{1}{4a}\right)$	$\left(0, \frac{9}{2}\right)$
Directrix	$y = k - \frac{1}{4a}$	$y = \frac{7}{2}$

A sketch of the parabola is



■ 6. Find each piece of the parabola from its equation.

$$x = -\frac{2}{3}(y+2)^2 + 1$$

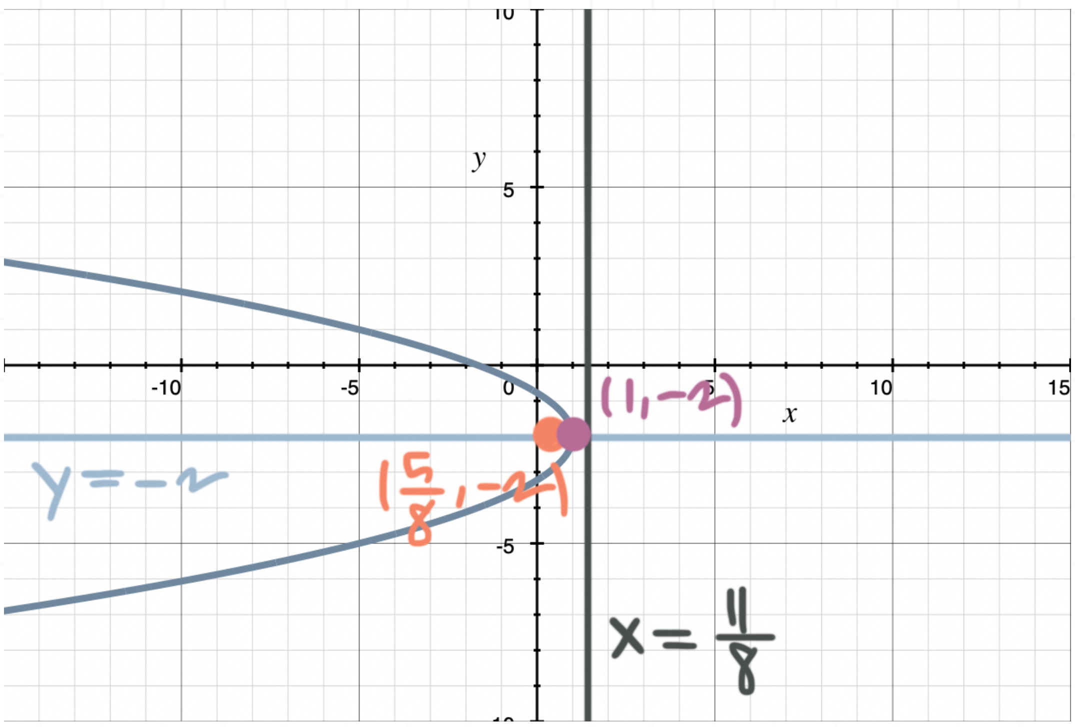
Solution:

From vertex form, we can pull all the components of the parabola.

Equation	$x = a(y-k)^2 + h$	$x = -\frac{2}{3}(y+2)^2 + 1$
Vertex	(h, k)	$(1, -2)$
Axis	$y = k$	$y = -2$
Focus	$\left(h + \frac{1}{4a}, k\right)$	$\left(\frac{5}{8}, -2\right)$

Directrix	$x = h - \dfrac{1}{4a}$	$x = \dfrac{11}{8}$

A sketch of the parabola is



HYPERBOLAS

■ 1. Find the asymptotes, foci, vertices, and directrices of the hyperbola.

$$\frac{y^2}{4} - \frac{x^2}{25} = 1$$

Solution:

This hyperbola is in the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Comparing this to the hyperbola's equation, we know that $a = \sqrt{4} = 2$ and $b = \sqrt{25} = 5$, which means the asymptotes of the hyperbola are

$$y = \pm \frac{a}{b}x = \pm \frac{2}{5}x$$

Because the hyperbola is centered at the origin and opens up and down, we know $(0,2)$ and $(0, -2)$ are the vertices of the hyperbola.

We can find the foci of the hyperbola as

$$c = \sqrt{a^2 + b^2}$$

$$c = \sqrt{4 + 25}$$

$$c = \sqrt{29}$$

This is the distance of the foci from the center $(0,0)$, so the foci are at $(0,\pm\sqrt{29})$, and the directrices are at

$$y=\pm\frac{a^2}{c}=\pm\frac{4}{\sqrt{29}}=\pm\frac{4\sqrt{29}}{29}$$

■ 2. Find the asymptotes, foci, vertices, and directrices of the hyperbola.

$$\frac{x^2}{4}-\frac{y^2}{81}=1$$

Solution:

This hyperbola is in the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Comparing this to the hyperbola's equation, we know that $a=\sqrt{4}=2$ and $b=\sqrt{81}=9$, which means the asymptotes of the hyperbola are

$$y=\pm\frac{b}{a}x=\pm\frac{9}{2}x$$

Because the hyperbola is centered at the origin and opens left and right, we know $(2,0)$ and $(-2,0)$ are the vertices of the hyperbola.

We can find the foci of the hyperbola as

$$c = \sqrt{a^2 + b^2}$$

$$c = \sqrt{4 + 81}$$

$$c = \sqrt{85}$$

This is the distance of the foci from the center $(0,0)$, so the foci are at $(\pm\sqrt{85},0)$, and the directrices are at

$$y = \pm\frac{a^2}{c} = \pm\frac{4}{\sqrt{85}} = \pm\frac{4\sqrt{85}}{85}$$

■ 3. Find the asymptotes, foci, vertices, and directrices of the hyperbola.

$$\frac{(y-3)^2}{36} - \frac{(x+2)^2}{9} = 1$$

Solution:

This hyperbola is in the form

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Comparing this to the hyperbola's equation, we know that $a = \sqrt{36} = 6$ and $b = \sqrt{9} = 3$, which means the asymptotes of the hyperbola are

$$y = \pm\frac{a}{b}(x-h) + k$$

$$y = \pm \frac{6}{3}(x+2)+3$$

$$y = \pm 2(x+2)+3$$

We can find the focal length of the hyperbola as

$$c = \sqrt{a^2+b^2}$$

$$c = \sqrt{36+9}$$

$$c = \sqrt{45}$$

$$c = 3\sqrt{5}$$

so the foci are at $(h, k \pm c) = (-2, 3 \pm 3\sqrt{5})$, and the directrices are at

$$y = k \pm \frac{a^2}{c} = 3 \pm \frac{36}{3\sqrt{5}} = 3 \pm \frac{12}{\sqrt{5}}$$

■ 4. Find the asymptotes, foci, vertices, and directrices of the hyperbola.

$$\frac{(x+1)^2}{25} - \frac{(y+4)^2}{144} = 1$$

Solution:

This hyperbola is in the form

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Comparing this to the hyperbola's equation, we know that $a=\sqrt{25}=5$ and $b=\sqrt{144}=12$, which means the asymptotes of the hyperbola are

$$y=\pm\frac{b}{a}(x-h)+k$$

$$y=\pm\frac{12}{5}(x+1)-4$$

We can find the focal length of the hyperbola as

$$c=\sqrt{a^2+b^2}$$

$$c=\sqrt{25+144}$$

$$c=\sqrt{169}$$

$$c=13$$

so the foci are at $(h\pm c,k)=(-1\pm 13,-4)$, or $(-14,-4)$ and $(12,-4)$, and the directrices are at

$$x=h\pm\frac{a^2}{c}=-1\pm\frac{25}{13}$$

$x=-\dfrac{38}{13}$ and $x=\dfrac{12}{13}$

■ 5. Sketch the graph of the hyperbola.

$$\frac{y^2}{16} - \frac{x^2}{4} = 1$$

Solution:

This hyperbola is in the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Comparing this to the hyperbola's equation, we know that $a = \sqrt{16} = 4$ and $b = \sqrt{4} = 2$, which means the asymptotes of the hyperbola are

$$y = \pm \frac{a}{b}x = \pm \frac{4}{2}x = \pm 2x$$

We can find the focal length of the hyperbola as

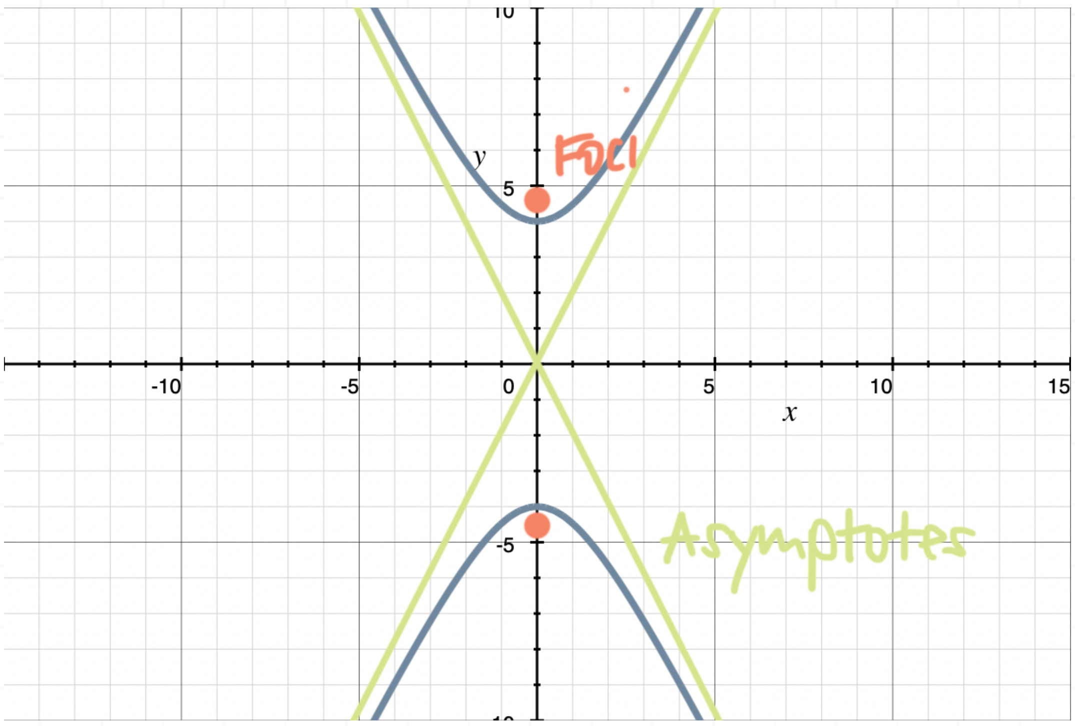
$$c = \sqrt{a^2 + b^2}$$

$$c = \sqrt{16 + 4}$$

$$c = \sqrt{20}$$

$$c = 2\sqrt{5}$$

so the foci are at $(0, \pm 2\sqrt{5})$, and the graph of the hyperbola is



■ 6. Sketch the graph of the hyperbola.

$$\frac{(x+1)^2}{2} - \frac{(y-1)^2}{12} = 1$$

Solution:

This hyperbola is in the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Comparing this to the hyperbola's equation, we know that $a = \sqrt{2}$ and $b = \sqrt{12} = 2\sqrt{3}$, which means the asymptotes of the hyperbola are

$$y = \pm \frac{b}{a}(x-h) + k$$

$$y = \pm \frac{2\sqrt{3}}{\sqrt{2}}(x+1)+1$$

$$y = \pm \sqrt{6}(x+1)+1$$

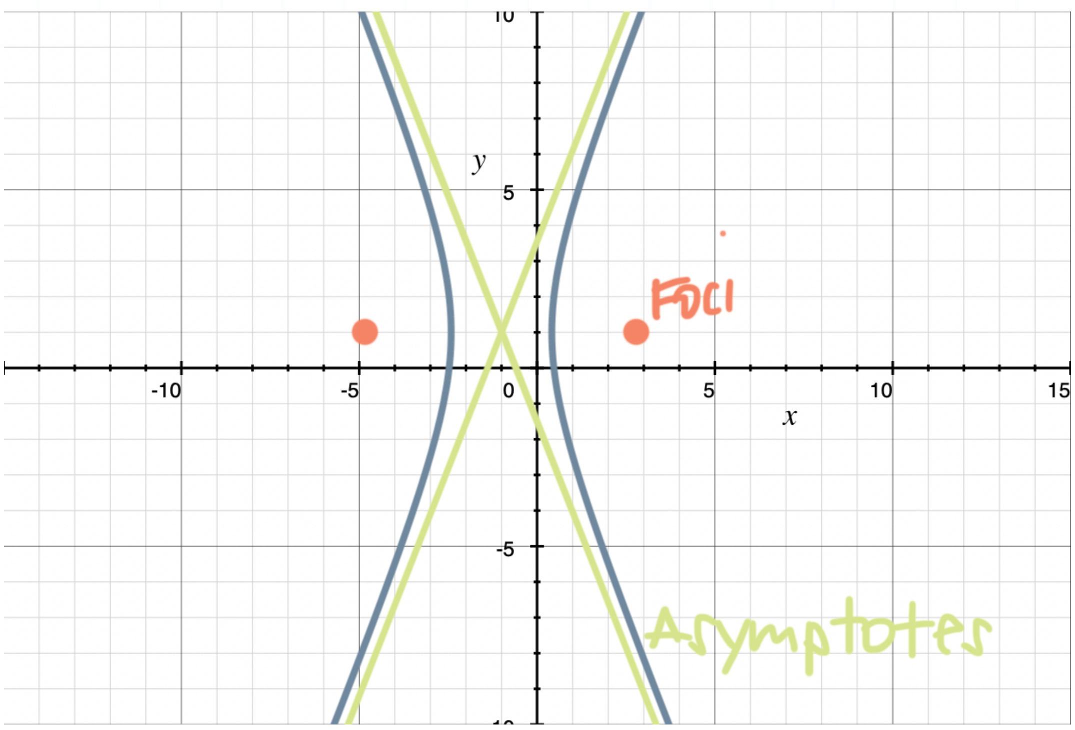
We can find the focal length of the hyperbola as

$$c = \sqrt{a^2+b^2}$$

$$c = \sqrt{2+12}$$

$$c = \sqrt{14}$$

so the foci are at $(h \pm c, k) = (-1 \pm \sqrt{14}, 1)$, and the graph of the hyperbola is



ROTATING AXES

■ 1. Find the angle of rotation of the conic.

$$3x^2 + 2xy + y^2 - y - 12 = 0$$

Solution:

Comparing the conic equation to the standard form of a conic, we can identify $A = 3$, $B = 2$, and $C = 1$.

$$\cot(2\phi) = \frac{A - C}{B} = \frac{3 - 1}{2} = 1$$

This equation can only be true when

$$\cos(2\phi) = 1$$

$$2\phi = \frac{\pi}{4} + n\pi$$

$$\phi = \frac{\pi}{8} + \frac{n\pi}{2}$$

$$\phi = \frac{\pi}{8}$$

■ 2. Find the vertex of the parabola.

$$x^2 + 2xy + y^2 = 2x - 2y + 4$$

Solution:

Rewrite the equation in the standard form.

$$x^2 + 2xy + y^2 - 2x + 2y - 4 = 0$$

Comparing the conic equation to the standard form of a conic, we can identify $A = 1$, $B = 2$, and $C = 1$.

$$\cot(2\phi) = \frac{A - C}{B} = \frac{1 - 1}{2} = 0$$

This equation can only be true when

$$\cos(2\phi) = 0$$

$$2\phi = \frac{\pi}{2} + n\pi$$

$$\phi = \frac{\pi}{4} + \frac{n\pi}{2}$$

$$\phi = \frac{\pi}{4}$$

Then

$$x = X\cos\left(\frac{\pi}{4}\right) - Y\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(X - Y)$$

$$y = X\sin\left(\frac{\pi}{4}\right) + Y\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(X + Y)$$

so the equation $(x+y)^2 = 2(x-y+2)$ transforms to

$$\left(\frac{\sqrt{2}}{2}(X-Y)+\frac{\sqrt{2}}{2}(X+Y)\right)^2 = 2\left(\frac{\sqrt{2}}{2}(X-Y)-\frac{\sqrt{2}}{2}(X+Y)+2\right)$$

$$\left(\frac{\sqrt{2}}{2}(X-Y+X+Y)\right)^2 = 2\left(\frac{\sqrt{2}}{2}(X-Y-X-Y)+2\right)$$

$$\left(\frac{\sqrt{2}}{2}(2X)\right)^2 = 2\left(\frac{\sqrt{2}}{2}(-2Y)+2\right)$$

$$(X\sqrt{2})^2 = 2(-Y\sqrt{2}+2)$$

$$2X^2 = -2\sqrt{2}Y+4$$

$$X^2 = -\sqrt{2}Y+2$$

$$Y = -\frac{1}{\sqrt{2}}X^2+\sqrt{2}$$

The vertex of the parabola is $(X, Y) = (h, k) = (0, \sqrt{2})$, so

$$x = \frac{\sqrt{2}}{2}(X-Y) = \frac{\sqrt{2}}{2}(0-\sqrt{2}) = -1$$

$$y = \frac{\sqrt{2}}{2}(X+Y) = \frac{\sqrt{2}}{2}(0+\sqrt{2}) = 1$$

and the vertex is $(-1,1)$.

■ 3. Sketch the graph of $x^2 + \sqrt{3}xy = 1$.

Solution:

Comparing the conic equation to the standard form of a conic, we can identify $A = 1$, $B = \sqrt{3}$, and $C = 0$.

$$\cot(2\phi) = \frac{A - C}{B} = \frac{1 - 0}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

This equation can only be true when

$$2\phi = \frac{\pi}{3} + n\pi$$

$$\phi = \frac{\pi}{6} + \frac{n\pi}{2}$$

$$\phi = \frac{\pi}{6}$$

Then

$$x = X\cos\left(\frac{\pi}{6}\right) - Y\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}(X\sqrt{3} - Y)$$

$$y = X\sin\left(\frac{\pi}{6}\right) + Y\cos\left(\frac{\pi}{6}\right) = \frac{1}{2}(X + \sqrt{3}Y)$$

so the equation $x^2 + \sqrt{3}xy = 1$ transforms to

$$x(x + \sqrt{3}y) = 1$$

$$\frac{1}{2}(X\sqrt{3}-Y)\left(\frac{1}{2}(X\sqrt{3}-Y)+\frac{\sqrt{3}}{2}(X+\sqrt{3}Y)\right)=1$$

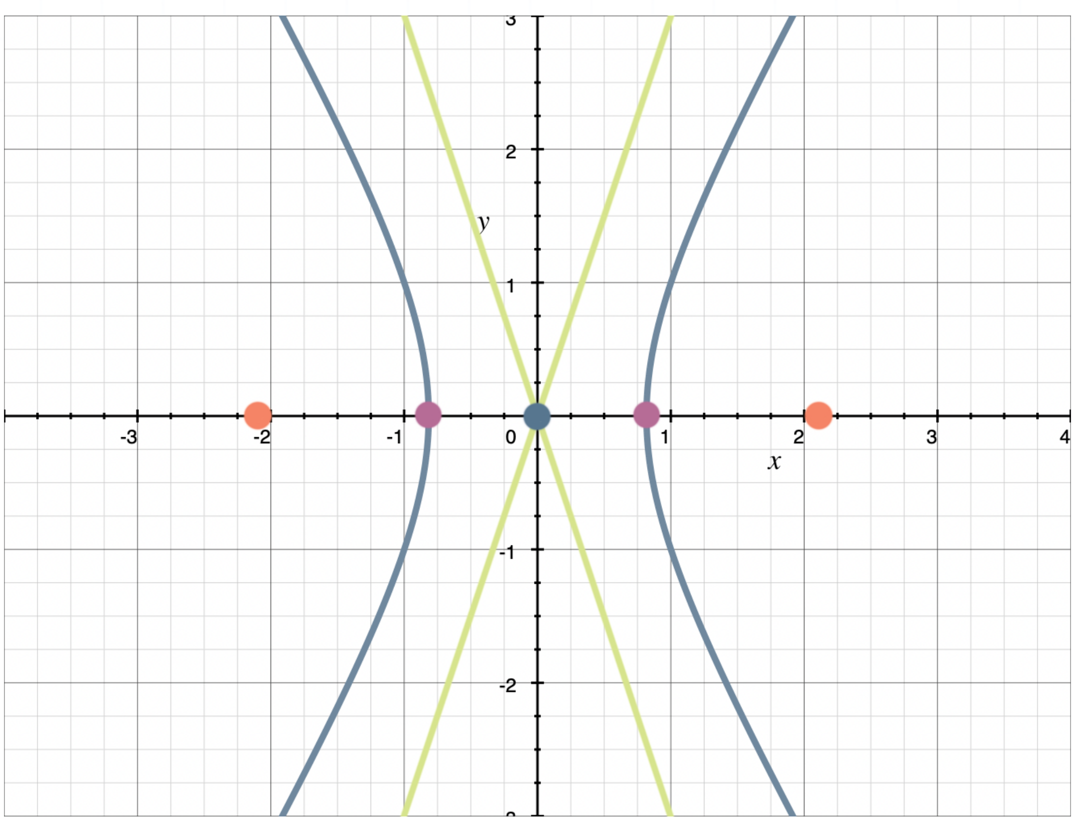
$$\frac{1}{2}(X\sqrt{3}-Y)\left(\frac{\sqrt{3}}{2}X-\frac{1}{2}Y+\frac{\sqrt{3}}{2}X+\frac{3}{2}Y\right)=1$$

$$\frac{1}{2}(X\sqrt{3}-Y)(\sqrt{3}X+Y)=1$$

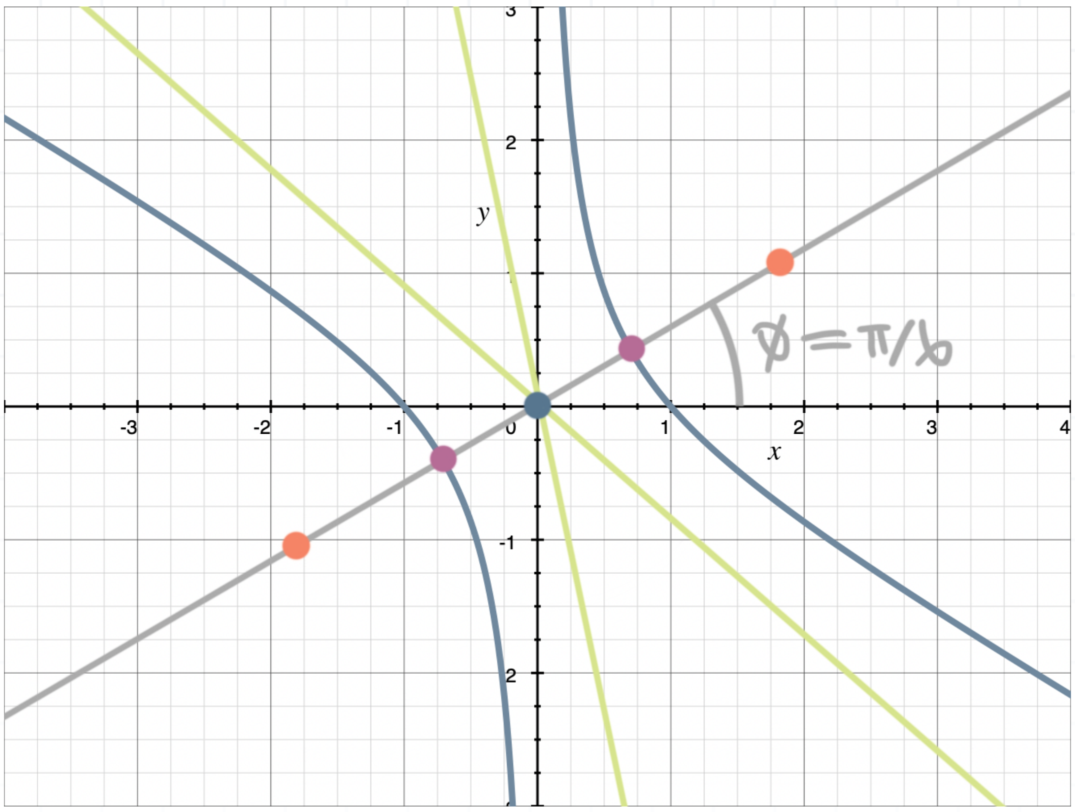
$$\frac{1}{2}(3X^2-Y^2)=1$$

$$\frac{3X^2}{2}-\frac{Y^2}{2}=1$$

So a sketch of this hyperbola, without rotation, is



But the center of the hyperbola is $(0,0)$, the asymptotes are $Y=\pm 3X$, the vertices are $(\pm\sqrt{2/3},0)$, and the foci are $(\pm 2\sqrt{10}/3,0)$. The axes X and Y are rotated by $\phi = \pi/6$ around the origin, which means a sketch of the rotated hyperbola is



■ 4. Find foci of the conic.

$$2x^2 - 4xy + 5y^2 - 4x - 8y + 8 = 0$$

Solution:

Comparing the conic equation to the standard form of a conic, we can identify $A = 2$, $B = -4$, and $C = 5$.

$$\cot(2\phi) = \frac{A - C}{B} = \frac{2 - 5}{-4} = \frac{3}{4}$$

Remember that cotangent is equivalent to adjacent/opposite, so the length of the adjacent leg is 3 and the length of the opposite leg is 4, and the length of the hypotenuse of the right triangle must be

$$a^2 + b^2 = c^2$$

$$3^2 + 4^2 = c^2$$

$$9 + 16 = c^2$$

$$25 = c^2$$

$$c = 5$$

We need the value of cosine for the half-angle formulas, and cosine is equivalent to adjacent/hypotenuse, so $\cos(2\phi) = 3/5$. The half-angle formulas give

$$\cos\phi = \sqrt{\frac{1 + \cos 2\phi}{2}} = \sqrt{\frac{1 + \frac{3}{5}}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

$$\sin\phi = \sqrt{\frac{1 - \cos 2\phi}{2}} = \sqrt{\frac{1 - \frac{3}{5}}{2}} = \frac{1}{\sqrt{5}}$$

The rotation of axes formulas give

$$x = X\cos\phi - Y\sin\phi = \frac{1}{\sqrt{5}}(2X - Y)$$

$$y = X\sin\phi + Y\cos\phi = \frac{1}{\sqrt{5}}(X + 2Y)$$

Then the equation $2x^2-4xy+5y^2-4x-8y+8=0$ transforms into

$$\frac{2}{5}(2X-Y)^2-\frac{4}{5}(2X-Y)(X+2Y)+(X+2Y)^2$$

$$-\frac{4}{\sqrt{5}}(2X-Y)-\frac{8}{\sqrt{5}}(X+2Y)+8=0$$

$$\frac{2}{5}(4X^2-4XY+Y^2)-\frac{4}{5}(2X^2+3XY-2Y^2)+(X^2+4XY+4Y^2)$$

$$-\frac{8}{\sqrt{5}}X+\frac{4}{\sqrt{5}}Y-\frac{8}{\sqrt{5}}X-\frac{16}{\sqrt{5}}Y+8=0$$

$$\frac{8}{5}X^2-\frac{8}{5}XY+\frac{2}{5}Y^2-\frac{8}{5}X^2-\frac{12}{5}XY+\frac{8}{5}Y^2+X^2+4XY+4Y^2-\frac{16}{\sqrt{5}}X-\frac{12}{\sqrt{5}}Y+8=0$$

$$X^2+6Y^2-\frac{16}{\sqrt{5}}X-\frac{12}{\sqrt{5}}Y+8=0$$

$$X^2-\frac{16}{\sqrt{5}}X+6Y^2-\frac{12}{\sqrt{5}}Y+8=0$$

$$\left(X-\frac{8}{\sqrt{5}}\right)^2+6\left(Y-\frac{1}{\sqrt{5}}\right)^2=\frac{64}{5}+\frac{6}{5}-8=6$$

$$\frac{\left(X-\frac{8}{\sqrt{5}}\right)^2}{6}+\left(Y-\frac{1}{\sqrt{5}}\right)^2=1$$

The conic is an ellipse, and its foci are

$$(h \pm c, k) = \left(\frac{8}{\sqrt{5}} \pm \sqrt{5}, \frac{1}{\sqrt{5}}\right)$$

$$(h \pm c, k) = \left(\frac{13}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \text{ and } \left(\frac{3}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

In the xy-coordinate plane, the foci are

$$(x, y) = \left(\frac{1}{\sqrt{5}}(2X - Y), \frac{1}{\sqrt{5}}(X + 2Y)\right)$$

so the foci are $(5,3)$ and $(1,1)$.

■ 5. Use the discriminant to determine the shape of the conic.

$$-2x^2 - xy - y^2 + 4x + y + 3 = 0$$

Solution:

Comparing the conic equation to the standard form of a conic, we can identify $A = -2$, $B = -1$, and $C = -1$. So the value of the discriminant is

$$B^2 - 4AC = (-1)^2 - 4(-2)(-1) = -7$$

Since the discriminant is negative, we know the conic is an ellipse.

■ 6. Use the discriminant to determine the shape of the conic.

$$25x^2 + 30xy + 9y^2 - 12x - 8 = 0$$

Solution:

Comparing the conic equation to the standard form of a conic, we can identify $A = 25$, $B = 30$, and $C = 9$. So the value of the discriminant is

$$B^2 - 4AC = 30^2 - 4(25)(9) = 0$$

Since the discriminant is zero, we know the conic is a parabola.

POLAR EQUATIONS OF CONICS

■ 1. A hyperbola has vertices at $(2,-1)$ and $(2,3)$, directrices $y=0$ and $y=2$, and foci at $(2,5)$ and $(2,-3)$. Find the eccentricity of hyperbola.

Solution:

Plugging the values we've been given into the equation for eccentricity gives

$$e=\frac{d(P,F)}{d(P,l)}=\frac{d((2,-1),(2,5))}{d((2,-1),y=2)}=\frac{5+1}{2+1}=\frac{6}{3}=2$$

■ 2. Find the eccentricity of the conic.

$$\frac{(x+1)^2}{4}-\frac{(y-1)^2}{32}=1$$

Solution:

Because the x^2 term is the positive term, we know the hyperbola opens left and right, that $a=\sqrt{4}=2$ and that $b=\sqrt{32}=4\sqrt{2}$. We can also see that the center is at $(h,k)=(-1,1)$. The foci are

$$(h\pm c,k)=(-1\pm\sqrt{4+32},1)=(-1\pm 6,1)=(-7,1),\ (5,1)$$

The directrices are

$$x = h \pm \frac{a^2}{c} = -1 \pm \frac{4}{6}$$

$$x = -\frac{5}{3} \text{ and } x = -\frac{1}{3}$$

Let's choose the point $(1,1)$ on the conic to find the eccentricity.

$$e = \frac{d(P, F)}{d(P, l)}$$

$$e = \frac{d((5,1), (1,1))}{d(x = -\frac{1}{3}, (1,1))} = \frac{4}{\frac{4}{3}} = 3$$

■ 3. Find the foci of the ellipse.

$$r = \frac{5}{3 - 2\cos\theta}$$

Solution:

First, we rewrite the equation in the standard form by multiplying the numerator and denominator by the reciprocal of 3, which is $1/3$.

$$r = \frac{5}{3 - 2\cos\theta} = \frac{5\left(\frac{1}{3}\right)}{3\left(\frac{1}{3}\right) - 2\left(\frac{1}{3}\right)\cos\theta}$$

$$r = \frac{5}{3 - 2\cos\theta} = \frac{\frac{5}{3}}{1 - \frac{2}{3}\cos\theta}$$

Because $e = 2/3$, this is a wide ellipse. The equation includes $\cos\theta$, so the directrix is $x = -d$. Its vertices lie on the x-axis and their coordinates are

$$r = \frac{5}{3 - 2\cos 0} = \frac{5}{3 - 2} = 5$$

$$(x, y) = (5,0)$$

and

$$r = \frac{5}{3 - 2\cos\pi} = \frac{5}{3 + 2} = 1$$

$$(x, y) = (-1,0)$$

When we use these polar equations, we know that one focus is positioned exactly at the origin, so $(0,0)$. The distance between $(0,0)$ and the first vertex is same as the distance between the second focus and second vertex.

$$d((-1,0),(0,0)) = d((x,0),(5,0))$$

$$1 = 5 - x$$

$$x = 4$$

Therefore, the second focus is at $(4,0)$.

■ 4. A conic has a focus at $(0,0)$ with a corresponding directrix of $y = -5$ that passes thought the point $(5,0)$. Write the conic equation in polar coordinates.

Solution:

Given the directrix $y = -5$, the polar equation will be

$$r = \frac{ed}{1 - e\sin\theta}$$

Substitute $r = 5$, $\theta = 0$, and $d = 5$ to find the eccentricity.

$$5 = \frac{5e}{1 - e\sin 0}$$

$$5 = \frac{5e}{1 - 0}$$

$$5 = 5e$$

$$e = 1$$

So the equation of this conic is

$$r = \frac{5}{1 - \sin\theta}$$

■ 5. Determine the shape of the conic section.

$$r = \frac{10}{6 + 4\cos\theta}$$

Solution:

Rewrite the conic in standard form by multiplying the numerator and denominator by the reciprocal of 6, which is $1/6$.

$$r = \frac{10}{6 + 4\cos\theta} = \frac{10\left(\frac{1}{6}\right)}{6\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right)\cos\theta}$$

$$r = \frac{\frac{10}{6}}{1 + \frac{4}{6}\cos\theta} = \frac{\frac{5}{3}}{1 + \frac{2}{3}\cos\theta}$$

So, the eccentricity is $e = 2/3 < 1$, which means the conic is an ellipse.

■ 6. Find the equation of the conic section that has eccentricity $e = 5/4$, directrix $x = -2$, and is rotated by $\alpha = \pi/3$.

Solution:

Given the directrix $x = -2$, the polar equation will be

$$r = \frac{ed}{1 - e\cos(\theta - \alpha)}$$

$$r = \frac{\frac{5}{4}(2)}{1 + \frac{5}{4}\sin\left(\theta - \frac{\pi}{3}\right)}$$

$$r = \frac{\frac{5}{2}}{1 + \frac{5}{4}\sin\left(\theta - \frac{\pi}{3}\right)}$$

Multiplying by $4/4$ to clear the fractions, we get

$$r = \frac{10}{4 + 5\sin\left(\theta - \frac{\pi}{3}\right)}$$

PARAMETRIC CURVES AND ELIMINATING THE PARAMETER

■ 1. Sketch the curve defined by the parametric equations.

$$x = \arcsin t$$

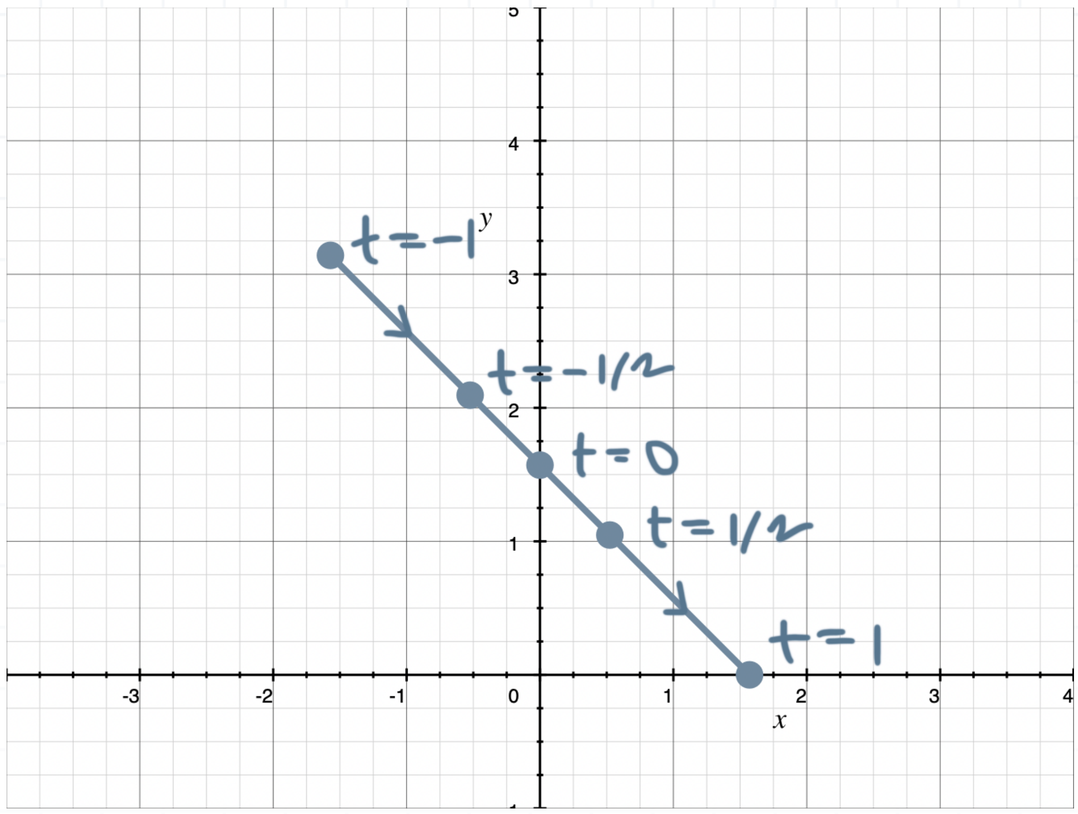
$$y = \arccos t$$

Solution:

Let's explore what the curve looks like for values of the parameter, starting at $t = -1$. If we plug integer values of t into the parametric equations for both x and y, we get

t	-1	-1/2	0	1/2	1
x	-π/2	-π/6	0	π/6	π/2
y	π	2π/3	π/2	π/3	0

If we plot these (x, y) coordinate points, we see the shape of the curve starting to emerge.



■ 2. Sketch the curve defined by the parametric equations.

$$x = 1 + 2\cos t$$

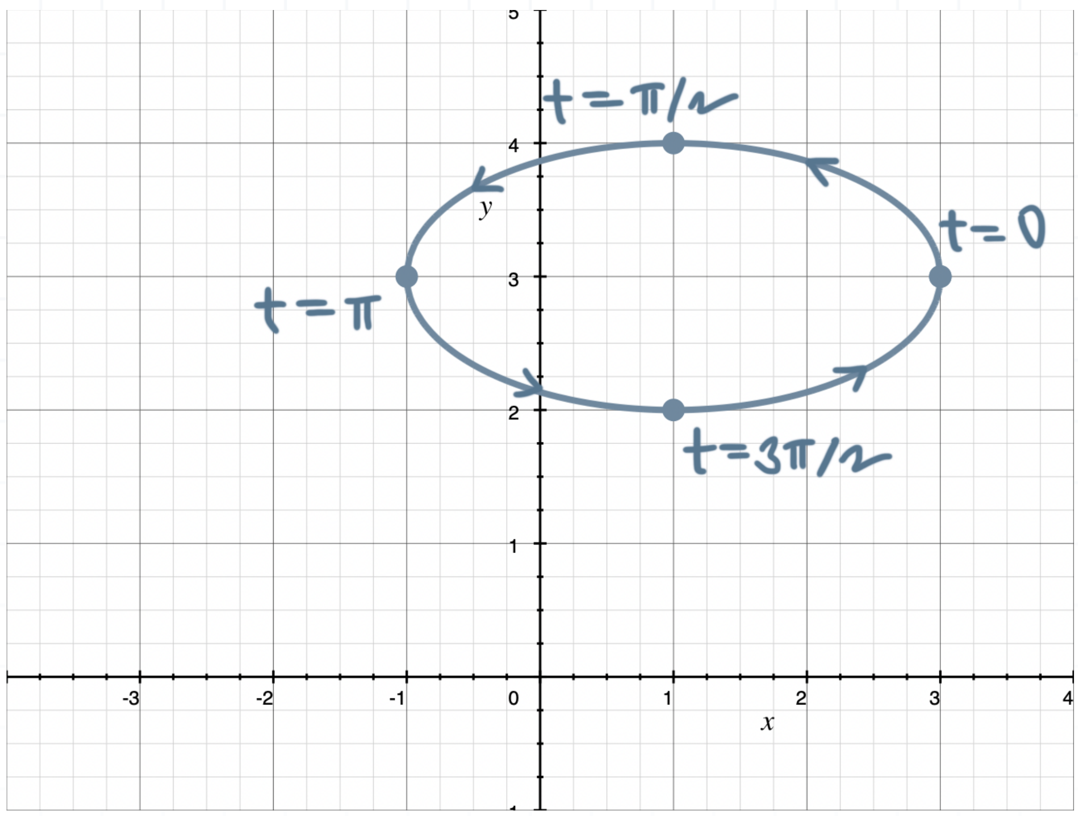
$$y = 3 + \sin t$$

Solution:

Let's explore what the curve looks like for values of the parameter, starting at $t = 0$. If we plug integer values of t into the parametric equations for both x and y, we get

t	0	π/2	π	3π/2	2π
x	3	1	-1	1	3
y	3	4	3	2	3

If we plot these (x, y) coordinate points, we see the shape of the curve starting to emerge.



■ 3. Eliminate the parameter.

$$x = t^2 + 3t - 4$$

$$y = \sqrt[3]{t}$$

Solution:

If we solve $y = \sqrt[3]{t}$ for t, we get $t = y^3$. Then we can substitute $t = y^3$ into $x = t^2 + 3t - 4$.

$$x = t^2 + 3t - 4$$

$$x = (y^3)^2 + 3y^3 - 4$$

$$x = y^6 + 3y^3 - 4$$

■ 4. Eliminate the parameter.

$$x = t + \frac{1}{t}$$

$$y = t^2 + \frac{1}{t^2}$$

Solution:

Square both sides of the equation for x.

$$x^2 = \left(t + \frac{1}{t}\right)^2$$

$$x^2 = t^2 + 2 + \frac{1}{t^2}$$

$$x^2 = t^2 + \frac{1}{t^2} + 2$$

Now we can substitute $y = t^2 + 1/t^2$ into this equation to get

$$x^2 = y + 2$$

$$y = x^2 - 2$$

■ 5. Eliminate the parameter.

$$x = 2 + 3\cos t$$

$$y = 4 - \cos t$$

Solution:

If we solve $y = 4 - \cos t$ for $\cos t$, we get $\cos t = 4 - y$. Then we can substitute $\cos t = 4 - y$ into $x = 2 + 3\cos t$.

$$x = 2 + 3\cos t$$

$$x = 2 + 3(4 - y)$$

$$x = 2 + 12 - 3y$$

$$x = 14 - 3y$$

We know that $-1 \leq \cos t \leq 1$, which means that $-1 \leq 4 - y \leq 1$, and therefore that the equation $x = 14 - 3y$ is only defined for $3 \leq y \leq 5$.

■ 6. Eliminate the parameter.

$$x = 1 + 2\cos t$$

$$y = 5 + 3\sin t$$

Solution:

To solve either parametric equation for t would require us to introduce an inverse trig function, which gets a little messy. Instead, let's realize that we can use the Pythagorean identity $\sin^2 t + \cos^2 t = 1$.

$$\frac{x-1}{2} = \cos t$$

$$\frac{y-5}{3} = \sin t$$

If we square both parametric equations, they become

$$\left(\frac{x-1}{2}\right)^2 = \cos^2 t$$

$$\left(\frac{y-5}{3}\right)^2 = \sin^2 t$$

Now that the equations are in this form, we can make substitutions into the Pythagorean identity.

$$\sin^2 t + \cos^2 t = 1$$

$$\left(\frac{y-5}{3}\right)^2 + \left(\frac{x-1}{2}\right)^2 = 1$$

$$\frac{(y-5)^2}{9} + \frac{(x-1)^2}{4} = 1$$

This is the equation of a tall ellipse centered at $(1,5)$.

DIRECTION OF THE PARAMETER

■ 1. Sketch the graph of the parametric curve and indicate the direction of increasing t for $t > 0$.

$$x = t^2$$

$$y = t - 2$$

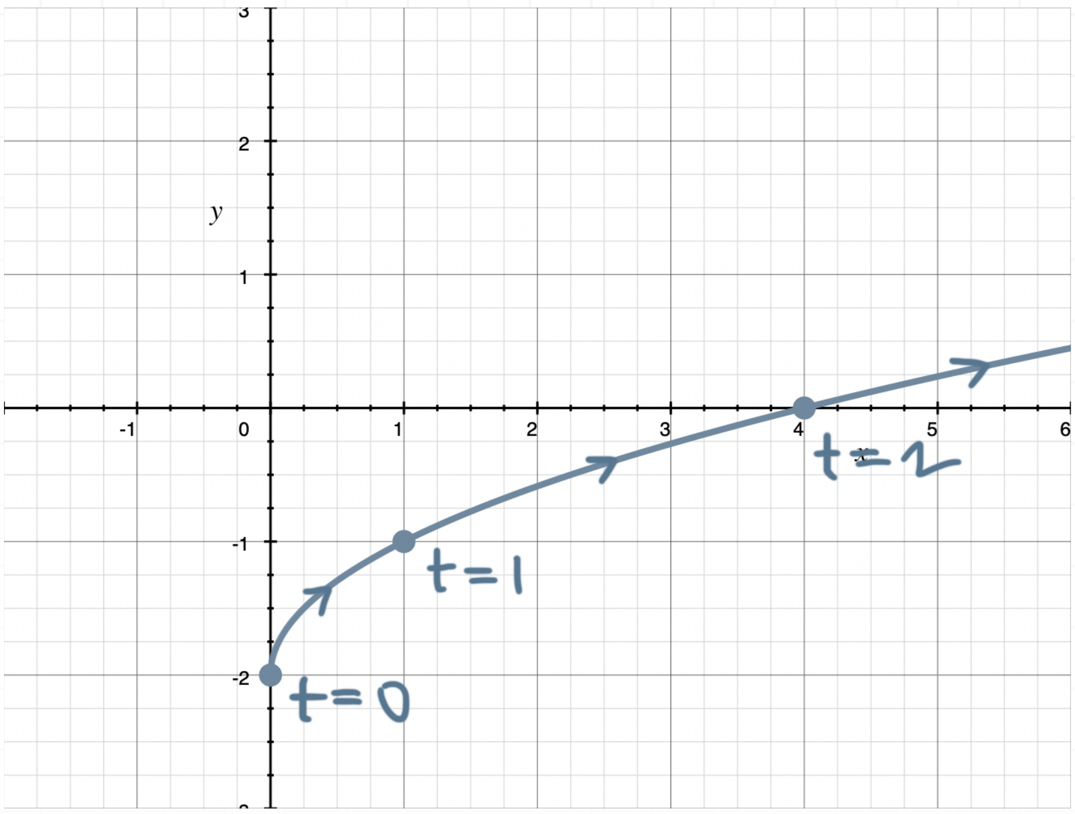
Solution:

Solve $y = t - 2$ for t to get $t = y + 2$, then substitute this value into $x = t^2$ to get $x = (y + 2)^2$. So the rectangular equation is the parabola with vertex at $(0, -2)$ that opens to the right.

If we choose a few values of t and find their corresponding x and y values,

t	0	1	2	3	4
x	0	1	4	9	16
y	-2	-1	0	1	2

then we can use these points to sketch the parabola, including the direction of increasing t.



■ 2. Sketch the graph of the parametric curve and indicate the direction of increasing t.

$$x = 1 + \cos t$$

$$y = 2 + \sin t$$

Solution:

We'll solve both equations for the trig function,

$$\cos t = x - 1$$

$$\sin t = y - 2$$

and then square each equation.

$$\cos^2 t = (x-1)^2$$

$$\sin^2 t = (y-2)^2$$

Then we can plug these results into the Pythagorean identity with sine and cosine.

$$\sin^2 t + \cos^2 t = 1$$

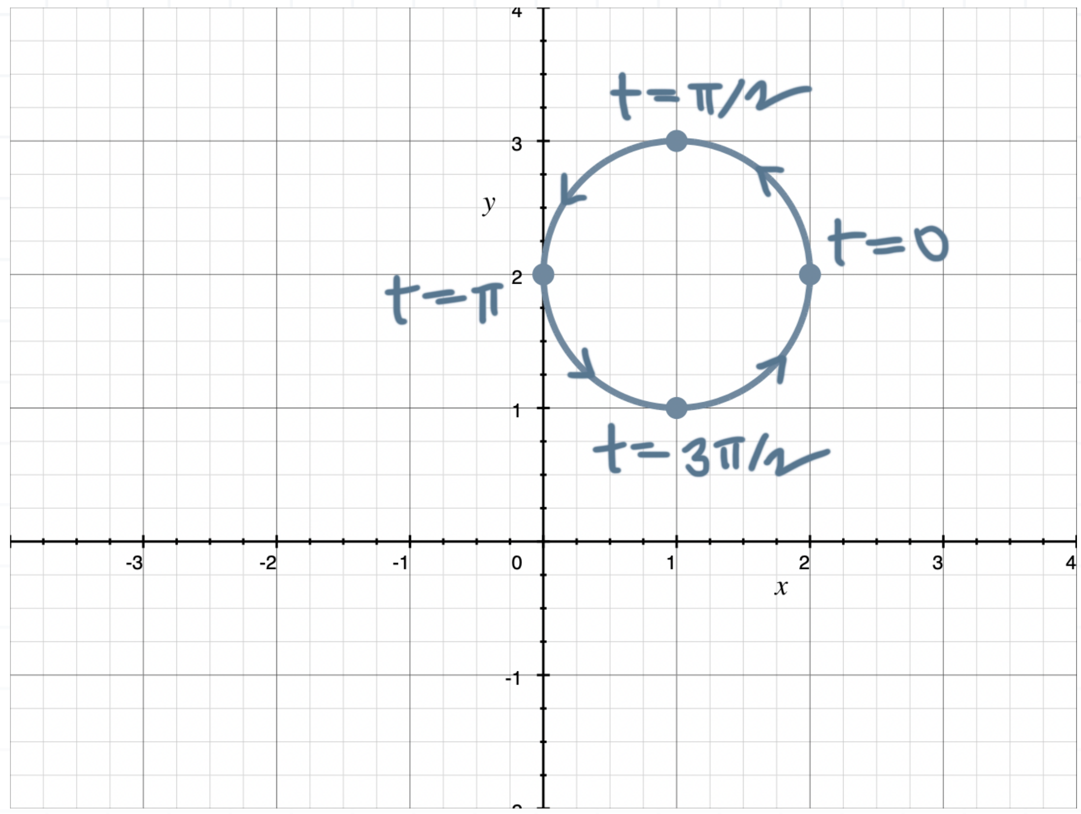
$$(y-2)^2 + (x-1)^2 = 1$$

$$(x-1)^2 + (y-2)^2 = 1$$

We know this is the equation of a circle centered at $(1,2)$ with radius 1. If we choose a few values of t and find their corresponding x and y values,

t	0	π/2	π	3π/2	2π
x	2	1	0	1	2
y	2	3	2	1	2

then we can use these points to sketch the circle, including the direction of increasing t.



■ 3. Sketch the graph of the parametric curve and indicate the direction of increasing t.

$$x = t^2$$

$$y = t^3$$

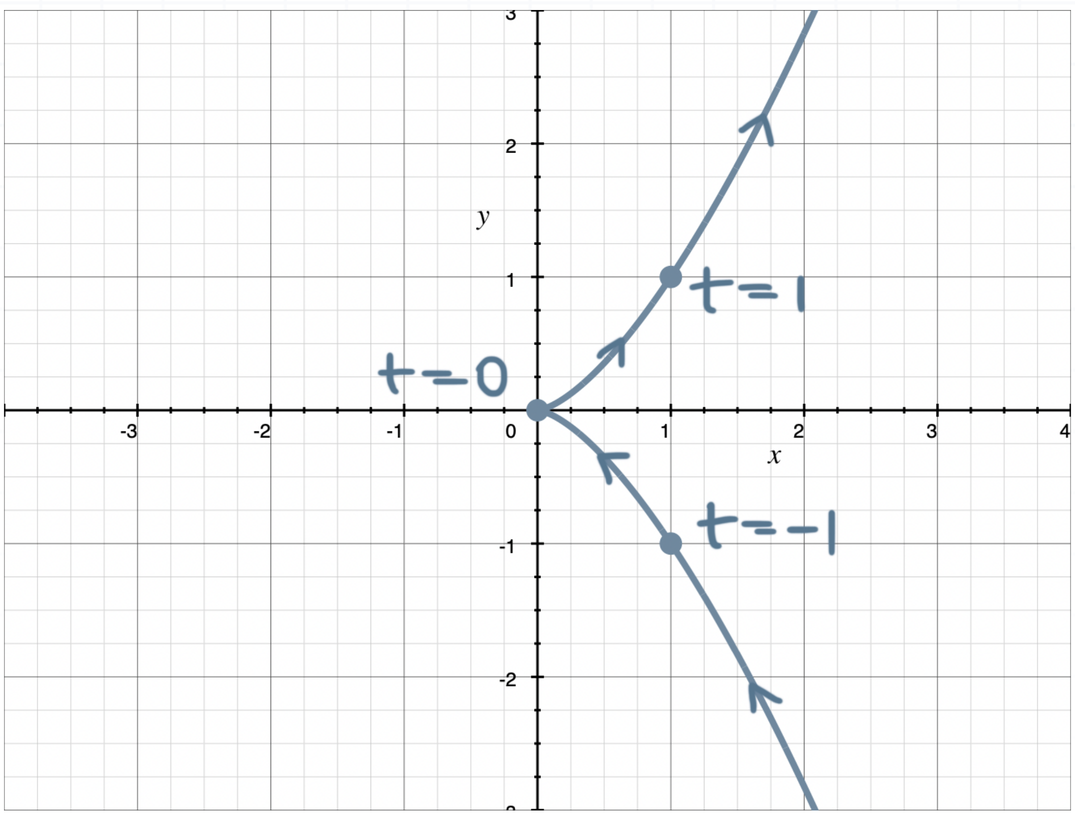
Solution:

Solve $y = t^3$ for t to get $t = \sqrt[3]{y}$, then substitute this value into $x = t^2$ to get $x = (\sqrt[3]{y})^2 = y^{\frac{2}{3}}$.

If we choose a few values of t and find their corresponding x and y values,

t	-2	-1	0	1	2
x	4	1	0	1	4
y	-8	-1	0	1	8

then we can use these points to sketch the graph, including the direction of increasing t.



■ 4. Sketch the graph of the parametric curve and indicate the direction of increasing t on the interval $\pi/6 < t < \pi/3$.

$$x = \tan t$$

$$y = \cot t$$

Solution:

We can rewrite the equation for y as $y = 1/\tan t$, then substitute $x = \tan t$ into the rewritten equation for y to get

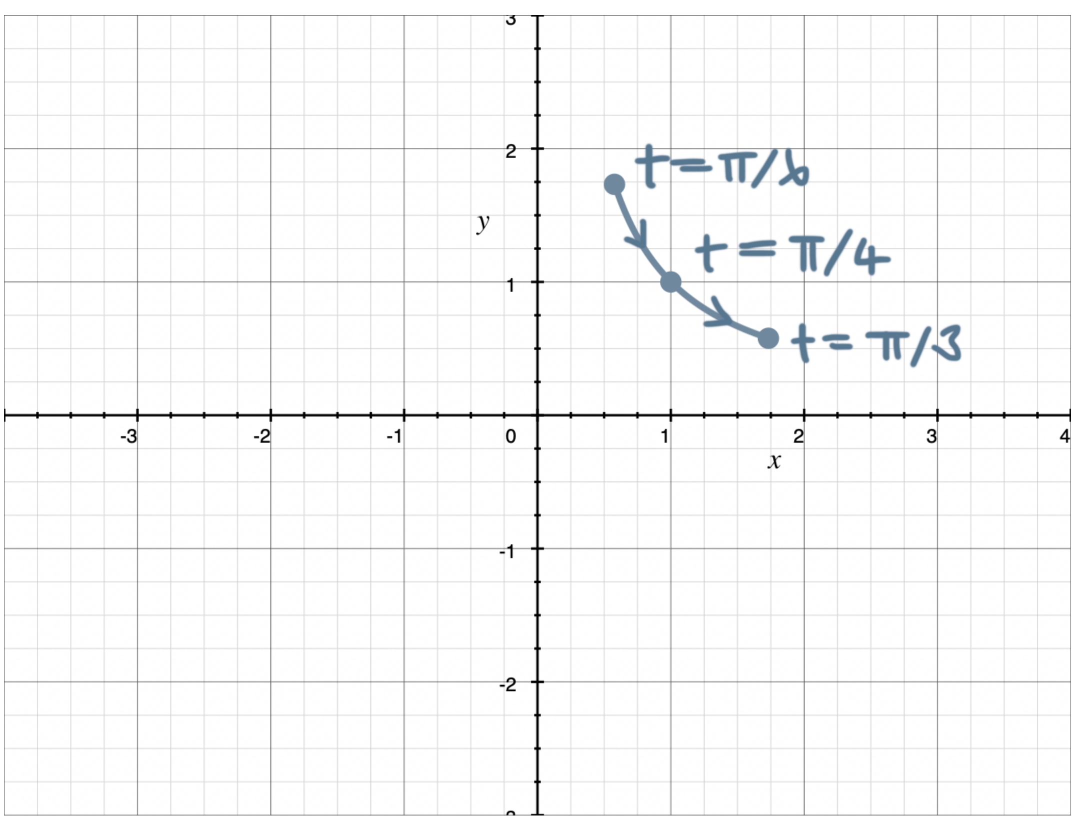
$$y = \frac{1}{\tan t}$$

$$y = \frac{1}{x}$$

We know this is the equation of a hyperbola. If we choose a few values of t and find their corresponding x and y values,

t	π/6	π/4	π/3
x	1/√3	1	√3
y	√3	1	1/√3

then we can use these points to sketch the hyperbola, including the direction of increasing t.



■ 5. Sketch the graph of the parametric curve and indicate the direction of increasing t.

$$x = \sin t + \cos t$$

$$y = \sin t - \cos t$$

Solution:

If we add the two equations, we get

$$x + y = \sin t + \cos t + \sin t - \cos t$$

$$x + y = 2\sin t$$

and if we subtract the second equation from the first we get

$$x - y = \sin t + \cos t - (\sin t - \cos t)$$

$$x - y = 2\cos t$$

Square the sum equation and the difference equation.

$$(x + y)^2 = 4\sin^2 t$$

$$(x - y)^2 = 4\cos^2 t$$

Then we can plug these results into the Pythagorean identity with sine and cosine.

$$\sin^2 t + \cos^2 t = 1$$

$$(x+y)^2+(x-y)^2=4$$

$$x^2+2xy+y^2+x^2-2xy+y^2=4$$

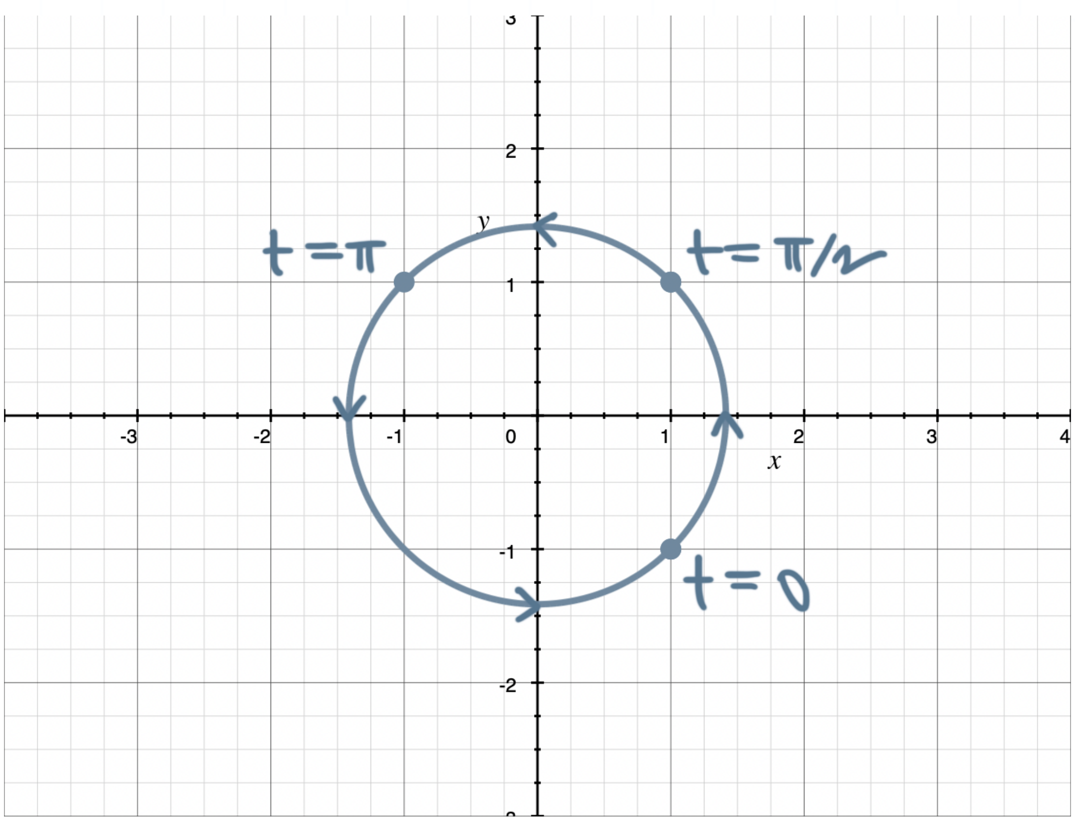
$$2x^2+2y^2=4$$

$$x^2+y^2=2$$

We know this is the equation of a circle centered at $(0,0)$ with radius $\sqrt{2}$. If we choose a few values of t and find their corresponding x and y values,

t	0	π/2	π	3π/2	2π
x	1	1	-1	1	1
y	-1	1	1	1	-1

then we can use these points to sketch the circle, including the direction of increasing t.



6. Sketch the graph of the parametric curve and indicate the direction of increasing t.

$$x = t^2 + t$$

$$y = 1 - t$$

Solution:

Solve $y = 1 - t$ for t to get $t = 1 - y$, then substitute this value into $x = t^2 + t$ to get

$$x = (1 - y)^2 + 1 - y$$

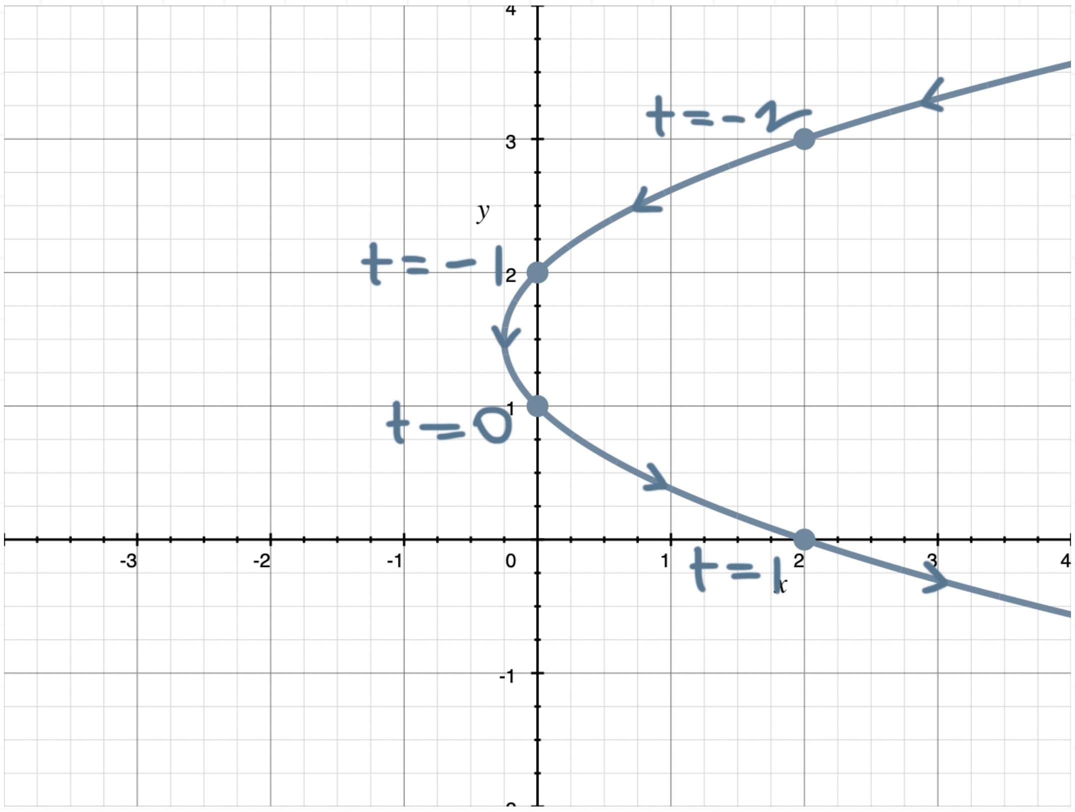
$$x = 1 - 2y + y^2 + 1 - y$$

$$x = y^2 - 3y + 2$$

We know this is the equation of parabola. If we choose a few values of t and find their corresponding x and y values,

t	-2	-1	0	1	2
x	2	0	0	2	6
y	3	2	1	0	-1

then we can use these points to sketch the parabola, including the direction of increasing t.


t = -2
t = -1
t = 0
t = 1
x
y
-3
-2
-1
0
1
2
3
4
-1
1
2
3
4

FINDING THE PARAMETRIC REPRESENTATION

■ 1. Express the rectangular equation in parametric form.

$$(x-3)^2-(y-4)^2=4$$

Solution:

Notice that $(x-3)^2-(y-4)^2=4$ is the equation of the hyperbola that's centered at $(3,4)$. If we rewrite the equation as,

$$\frac{(x-3)^2}{4}-\frac{(y-4)^2}{4}=1$$

$$\left(\frac{x-3}{2}\right)^2-\left(\frac{y-4}{2}\right)^2=1$$

then we can match this equation to the Pythagorean identity $\sec^2 t-\tan^2 t=1$. We'll set

$$\sec t=\frac{x-3}{2} \qquad \tan t=\frac{y-4}{2}$$

$$2\sec t=x-3 \qquad 2\tan t=y-4$$

to get the parametric equations

$$x=3+2\sec t$$

$$y=4+2\tan t$$

■ 2. Express the rectangular equation in parametric form.

$$x^2 + 2xy + x + 4y - 5 = 0$$

Solution:

We can rewrite the equation by solving it for y.

$$x^2 + 2xy + x + 4y - 5 = 0$$

$$2xy + 4y = 5 - x^2 - x$$

$$y(2x + 4) = 5 - x^2 - x$$

$$y = \frac{5 - x - x^2}{2x + 4}$$

For every real number x, the equation gives a unique value of y, so y is a function of x, and we can substitute $x = t$, which gives the parametric equations

$$x = t$$

$$y = \frac{5 - t - t^2}{2t + 4}$$

■ 3. Express the rectangular equation in parametric form.

$$x^2 + 4y^2 - 6x + 8y = 0$$

Solution:

We can rewrite the equation by completing the square with respect to both variables.

$$x^2 - 6x + 4y^2 + 8y = 0$$

$$(x^2 - 6x + 9) + 4(y^2 + 2y + 1) = 9 + 4$$

$$(x - 3)^2 + 4(y + 1)^2 = 13$$

Notice that $(x - 3)^2 + 4(y + 1)^2 = 13$ is the equation of an ellipse that's centered at $(3, -1)$. If we rewrite the equation as,

$$\frac{(x - 3)^2}{13} + \frac{4(y + 1)^2}{13} = 1$$

$$\left(\frac{x - 3}{\sqrt{13}}\right)^2 - \left(\frac{2(y + 1)}{\sqrt{13}}\right)^2 = 1$$

then we can match this equation to the Pythagorean identity $\sin^2 t + \cos^2 t = 1$. We'll set

$$\sin t = \frac{x - 3}{\sqrt{13}} \qquad \cos t = \frac{2(y + 1)}{\sqrt{13}}$$

$$\sqrt{13} \sin t = x - 3 \qquad \frac{\sqrt{13}}{2} \cos t = y + 1$$

to get the parametric equations

$$x = 3 + \sqrt{13}\sin t$$

$$y = -1 + \frac{\sqrt{13}}{2}\cos t$$

■ 4. Express the rectangular equation in parametric form.

$$x^2 + 4xy + 4y^2 = 0$$

Solution:

We can rewrite the equation as

$$x^2 + 4xy + 4y^2 = 0$$

$$(x + 2y)^2 = 0$$

For every real number x, the equation gives a unique value of y, so x is a function of y, and we can substitute $y = t$, which gives the parametric equations

$$y = t$$

$$x = -2t$$

■ 5. Express the rectangular equation in parametric form.

$$x^2y^2 - y^2 + 1 = 0$$

Solution:

We can rewrite the equation as

$$x^2y^2 - y^2 + 1 = 0$$

$$x^2 - 1 + \frac{1}{y^2} = 0 \qquad y \neq 0$$

$$x^2 + \frac{1}{y^2} = 1 \qquad y \neq 0$$

$$x^2 + \left(\frac{1}{y}\right)^2 = 1 \qquad y \neq 0$$

Then we can match this equation to the Pythagorean identity $\sin^2 t + \cos^2 t = 1$. We'll set

$$\sin t = \frac{1}{y} \qquad \cos t = x$$

$$y = \frac{1}{\sin t}$$

$$y = \csc t$$

to get the parametric equations

$$x = \cos t$$

$$y = \csc t$$

■ 6. Express the rectangular equation in parametric form.

$$xy + \sin y - 1 = 0$$

Solution:

We can rewrite the equation by solving it for x.

$$xy + \sin y - 1 = 0$$

$$xy = 1 - \sin y$$

$$x = \frac{1 - \sin y}{y}$$

For every real number y, the equation gives a unique value of x, so x is a function of y, and we can substitute $y = t$, which gives the parametric equations

$$y = t$$

$$x = \frac{1 - \sin t}{t}$$

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