Precalculus Quizzes

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MATH

Topic: Polar coordinates

Question: What are the rectangular coordinates (x, y) of the point that has polar coordinates $(r, \theta) = (4, \pi)$?

Answer choices:

A $(x, y) = (4,0)$

B $(x, y) = (0, -4)$

C $(x, y) = (0,4)$

D $(x, y) = (-4,0)$

Solution: D

To rewrite the polar point in terms of rectangular coordinates, we'll plug $r = 4$ and $\theta = \pi$ into the conversion equations.

$$x = r\cos\theta$$

$$x = 4\cos\pi$$

$$x = 4(-1)$$

$$x = -4$$

and

$$y = r\sin\theta$$

$$y = 4\sin\pi$$

$$y = 4(0)$$

$$y = 0$$

So the polar point $(r, \theta) = (4, \pi)$ is equivalent to the rectangular point $(x, y) = (-4, 0)$.

Topic: Polar coordinates

Question: Convert the rectangular coordinate point $(x, y) = (0, -14)$ into polar coordinates.

Answer choices:

A $(r, \theta) = \left(-14, -\frac{\pi}{2}\right)$

B $(r, \theta) = \left(14, \frac{3\pi}{2}\right)$

C $(r, \theta) = \left(\sqrt{14}, -\frac{\pi}{2}\right)$

D $(r, \theta) = (14, \pi)$

Solution: B

Plugging $(0, -14)$ into the conversion formula $r^2 = x^2 + y^2$ gives

$$r^2 = 0^2 + (-14)^2$$

$$r^2 = 196$$

$$r = 14$$

To find θ, we'll plug into the inverse tangent conversion formula.

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\theta = \tan^{-1}\left(\frac{-14}{0}\right)$$

$$\tan\theta = \tan\left(\tan^{-1}\left(\frac{-14}{0}\right)\right)$$

$$\tan\theta = \frac{-14}{0}$$

$$\frac{\sin\theta}{\cos\theta} = \frac{-14}{0}$$

This equation is really asking us for the value(s) of θ where the cosine function is 0. We know that the cosine function has a zero value at angles along the vertical axis, like $\pi/2$, $3\pi/2$, etc. Since the rectangular point $(0, -14)$ is located on the negative side of the vertical axis, we'll choose the angle that lies along the negative side of the vertical axis, $\theta = 3\pi/2$.

So the given point in polar coordinates is

$$\left(14,\frac{3\pi}{2}\right)$$

Topic: Polar coordinates

Question: Convert the polar point $(r, \theta) = (5,11\pi/7)$ into rectangular coordinates.

Answer choices:

A $(x, y) \approx (-1.24, -2.38)$

B $(x, y) \approx (1.12, -4.88)$

C $(x, y) \approx (-4.87,1.11)$

D $(x, y) \approx (1.24, -2.38)$

Solution: B

To rewrite the polar point in terms of rectangular coordinates, we'll plug $r = 5$ and $\theta = 11\pi/7$ into the conversion equations.

$$x = r\cos\theta$$

$$x = 5\cos\left(\frac{11\pi}{7}\right)$$

$$x \approx 5(0.223) \approx 1.12$$

and

$$y = r\sin\theta$$

$$y = 5\sin\left(\frac{11\pi}{7}\right)$$

$$y \approx 5(-0.975) \approx -4.88$$

So the polar point $(r, \theta) = (5, 11\pi/7)$ is equivalent to the rectangular point $(x, y) \approx (1.12, -4.88)$.

Topic: Multiple ways to express polar points

Question: Identify the equivalent polar point.

$$\left(7,\frac{4\pi}{3}\right)$$

Answer choices:

A $\left(7,\frac{13\pi}{3}\right)$

B $\left(-7,\frac{\pi}{3}\right)$

C $\left(-7,\frac{10\pi}{3}\right)$

D $\left(\frac{4\pi}{3},7\right)$

Solution: B

Two points exist in the same location in polar space when they have the same value of r but their angles differ by a multiple of 2π, or when they have opposite values of r (r and $-r$) and their angles differ by some $2\pi n + \pi$, where n is any integer.

The points $(7,4\pi/3)$ and $(7,13\pi/3)$ have the same value of r, but their angles differ by $13\pi/3 - 4\pi/3 = 3\pi$, so they aren't equivalent.

The points $(7,4\pi/3)$ and $(-7,\pi/3)$ have opposite values of r, and their angles differ by $\pi/3 - 4\pi/3 = -\pi$, so they are equivalent.

The points $(7,4\pi/3)$ and $(-7,10\pi/3)$ have opposite values of r, but their angles differ by $10\pi/3 - 4\pi/3 = 2\pi$, so they aren't equivalent.

The points $(7,4\pi/3)$ and $(4\pi/3,7)$ don't have equivalent values of r or opposite values of r, so they aren't equivalent.

Topic: Multiple ways to express polar points

Question: Which point in polar coordinates is not equivalent to the others?

Answer choices:

A $\left(5,\frac{\pi}{2}\right)$

B $\left(-5,-\frac{\pi}{2}\right)$

C $\left(5,\frac{9\pi}{2}\right)$

D $\left(-5,-\frac{7\pi}{2}\right)$

Solution: D

The points in answer choices A and B and equivalent because they have opposite values of r and their angles differ by π.

The points in answer choices A and C are equivalent because they have equivalent values of r and their angles differ by 4π.

The points in answer choices B and D aren't equivalent because they have equivalent values of r but their angles differ by 3π. So the point in answer choice D isn't equivalent to others.

Topic: Multiple ways to express polar points

Question: Find an equivalent polar point that has a positive value of r and an angle in the interval $[0,2\pi)$.

$$\left(-15,-\frac{70\pi}{9}\right)$$

Answer choices:

A $\left(15,\frac{2\pi}{9}\right)$

B $\left(15,\frac{7\pi}{9}\right)$

C $\left(15,\frac{11\pi}{9}\right)$

D $\left(15,\frac{16\pi}{9}\right)$

Solution: C

Two points exist in the same location in polar space when they have the same value of r but their angles differ by a multiple of 2π, or when they have opposite values of r (r and $-r$) and their angles differ by some $2\pi n+\pi$, where n is any integer.

Because we're asked for a positive value of r, we know the points will have opposite values of r, which means their angles have to differ by $2\pi n+\pi$.

We'll add π to $-70\pi/9$ to get

$$-\frac{70\pi}{9}+\frac{9\pi}{9}=-\frac{61\pi}{9}$$

Then we'll add 2π repeatedly to the angle until we find an angle in the interval $[0,2\pi)$.

$$-\frac{61\pi}{9}+\frac{18\pi}{9}=-\frac{43\pi}{9}$$

$$-\frac{43\pi}{9}+\frac{18\pi}{9}=-\frac{25\pi}{9}$$

$$-\frac{25\pi}{9}+\frac{18\pi}{9}=-\frac{7\pi}{9}$$

$$-\frac{7\pi}{9}+\frac{18\pi}{9}=\frac{11\pi}{9}$$

So the equivalent polar point is

$$(r,\theta)=\left(15,\frac{11\pi}{9}\right)$$

Topic: Converting equations

Question: Convert the rectangular equation to polar coordinates.

$$y = -6x + 5$$

Answer choices:

A $\quad r(\sin\theta - 6\cos\theta) = 5$

B $\quad \tan\theta = -\frac{5}{6}$

C $\quad r(\sin\theta + 6\cos\theta) = 5$

D $\quad \tan\theta + 6 = -\frac{5}{\cos\theta}$

Solution: C

Substitute $x = r\cos\theta$ and $y = r\sin\theta$ to get

$$y = -6x + 5$$

$$r\sin\theta = -6r\cos\theta + 5$$

Rearrange the equation.

$$r\sin\theta + 6r\cos\theta = 5$$

$$r(\sin\theta + 6\cos\theta) = 5$$

Topic: Converting equations

Question: Convert the rectangular equation to polar coordinates.

$$x^2 + (y - 7)^2 = 49$$

Answer choices:

A $\quad r = -14 \sin \theta$

B $\quad r^2 = 14 \sin \theta$

C $\quad r = 14 \cos \theta$

D $\quad r = 14 \sin \theta$

Solution: D

Substitute $x = r\cos\theta$ and $y = r\sin\theta$ to get

$$x^2 + (y - 7)^2 = 49$$

$$(r\cos\theta)^2 + (r\sin\theta - 7)^2 = 49$$

$$r^2\cos^2\theta + (r^2\sin^2\theta - 14r\sin\theta + 49) = 49$$

$$r^2(\cos^2\theta + \sin^2\theta) - 14r\sin\theta + 49 = 49$$

Using the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$, simplify the equation.

$$r^2(1) - 14r\sin\theta + 49 = 49$$

$$r^2 - 14r\sin\theta + 49 = 49$$

$$r^2 - 14r\sin\theta = 0$$

$$r(r - 14\sin\theta) = 0$$

$$r = 0, \ r = 14\sin\theta$$

We know $x^2 + (y - 7)^2 = 49$ is a circle centered at $(0,7)$ with a radius of 7. The equation $r = 0$ is the circle centered at the pole with radius 0, so $r = 0$ can't represent the rectangular equation. Therefore, $r = 14\sin\theta$ must be the polar equation that's equivalent to $x^2 + (y - 7)^2 = 49$.

Topic: Converting equations

Question: Convert the polar equation to rectangular coordinates. Hint: use the trigonometric identity $\cos(2\theta) = \cos^2\theta - \sin^2\theta$.

$$r^4\cos(2\theta) = 1$$

Answer choices:

A $\quad x^2(x^2 + y^2) = 1$

B $\quad 2x(x^2 + y^2) = 1$

C $\quad x^4 - y^4 = 1$

D $\quad y^4 - x^4 = 1$

Solution: C

Use the trigonometric identity $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ to rewrite the equation as

$$r^4\cos(2\theta) = 1$$

$$r^4(\cos^2\theta - \sin^2\theta) = 1$$

$$r^2(r^2\cos^2\theta - r^2\sin^2\theta) = 1$$

$$r^2((r\cos\theta)^2 - (r\sin\theta)^2) = 1$$

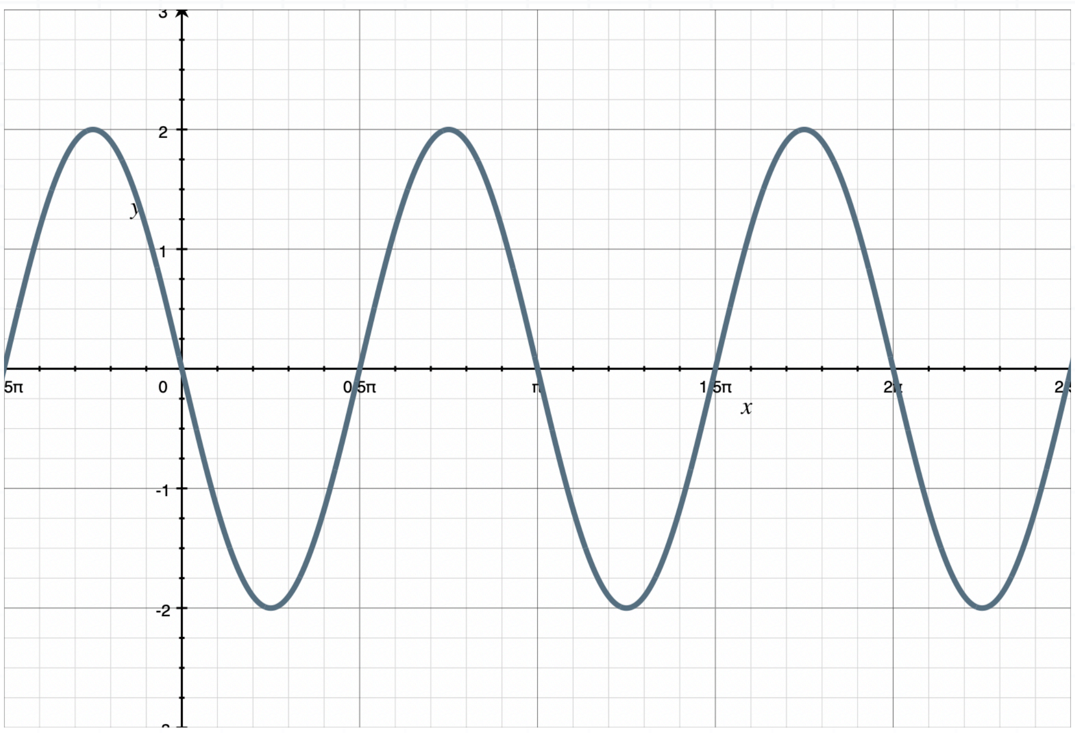
Substitute $r^2 = x^2 + y^2$, $x = r\cos\theta$, and $y = r\sin\theta$.

$$(x^2 + y^2)(x^2 - y^2) = 1$$

$$x^4 - y^4 = 1$$

Topic: Graphing polar curves in a rectangular system

Question: Which polar equation is represented by the graph?



Answer choices:

A $r = 1 + 2\sin\theta$

B $r = \cos(2\theta)$

C $r = 2 + \cos\theta$

D $r = -2\sin(2\theta)$

Solution: D

When $\theta = 0$, the four answer choices give

A $\quad r = 1 + 2\sin(0) = 1 + 2(0) = 1$

B $\quad r = \cos(2(0)) = 1$

C $\quad r = 2 + \cos(0) = 2 + 1 = 3$

D $\quad r = -2\sin(2(0)) = -2(0) = 0$

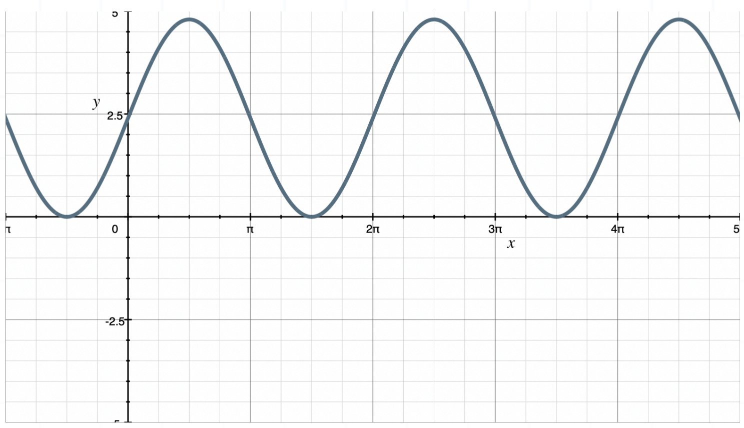
The graph intersects the origin $(0,0)$, and only answer choice D gives $r = 0$ when $\theta = 0$.

Topic: Graphing polar curves in a rectangular system

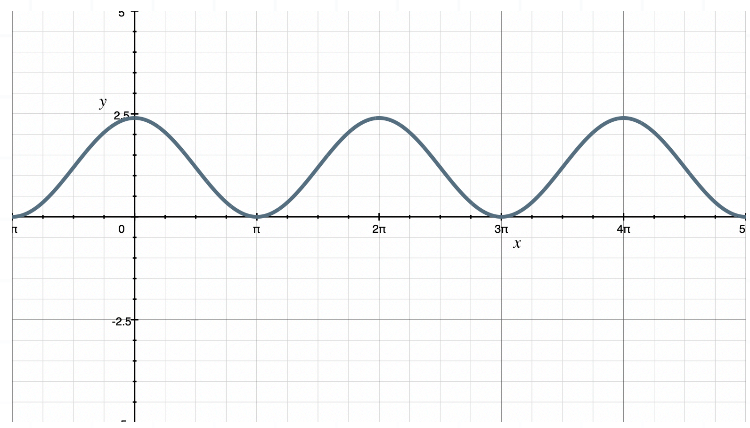
Question: Sketch the graph of $r = 1.2 + 1.2\cos\theta$ in a rectangular coordinate system.

Answer choices:

A



B



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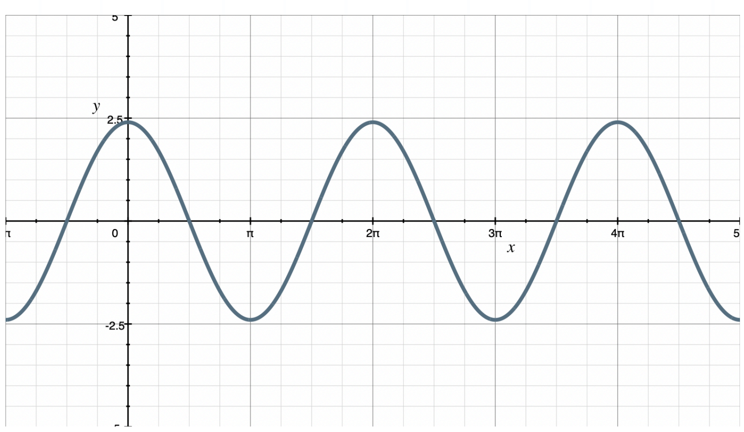
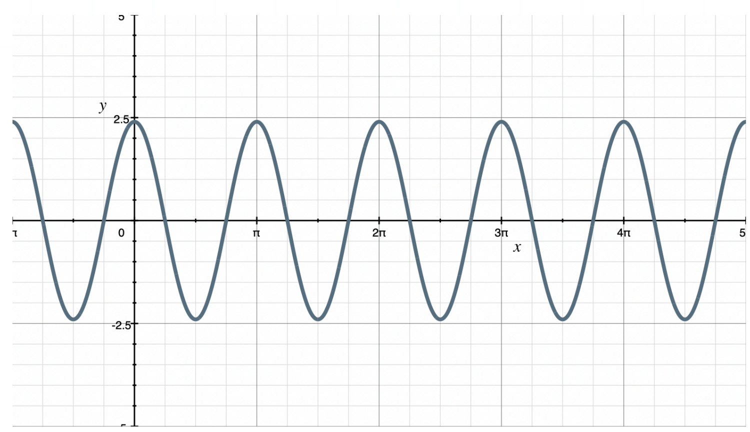


D



Solution: B

Evaluate the polar curve at $\theta = 0$, $\theta = \pi/2$, and $\theta = \pi$.

$$r(0) = 1.2 + 1.2\cos(0) = 1.2 + 1.2(1) = 2.4$$

$$r\left(\frac{\pi}{2}\right) = 1.2 + 1.2\cos\left(\frac{\pi}{2}\right) = 1.2 + 1.2(0) = 1.2$$

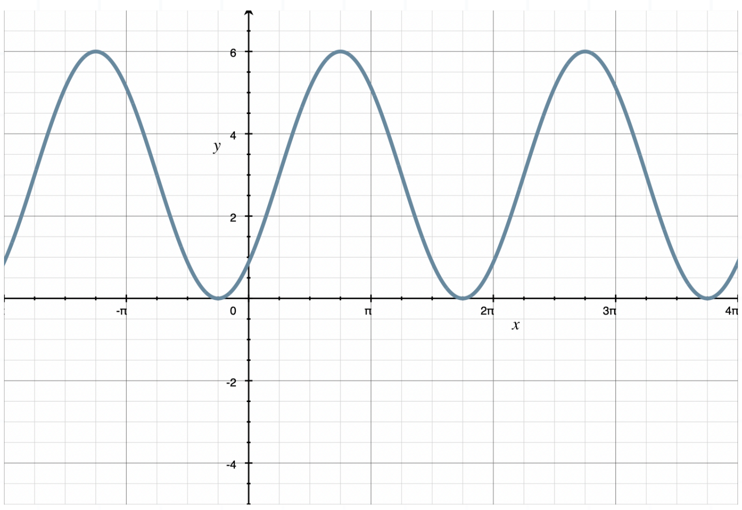
$$r(\pi) = 1.2 + 1.2\cos(\pi) = 1.2 + 1.2(-1) = 0$$

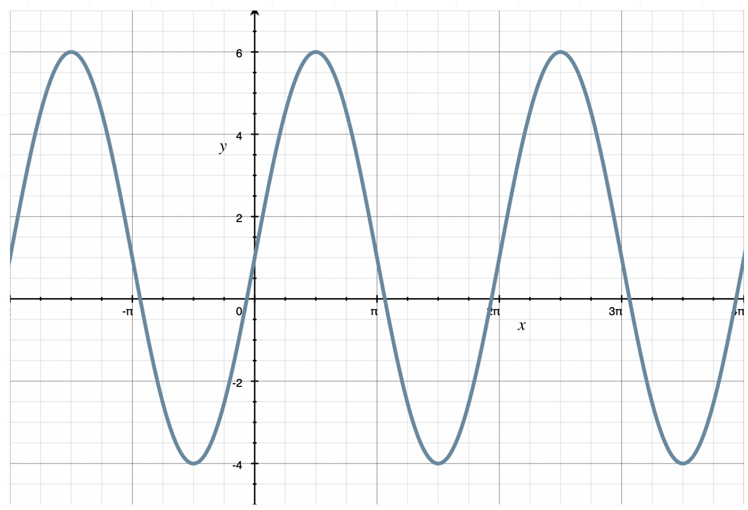
Answer choice B is the only curve that shows positive values at $\theta = 0$ and $\theta = \pi/2$ and a zero value at $\theta = \pi$.

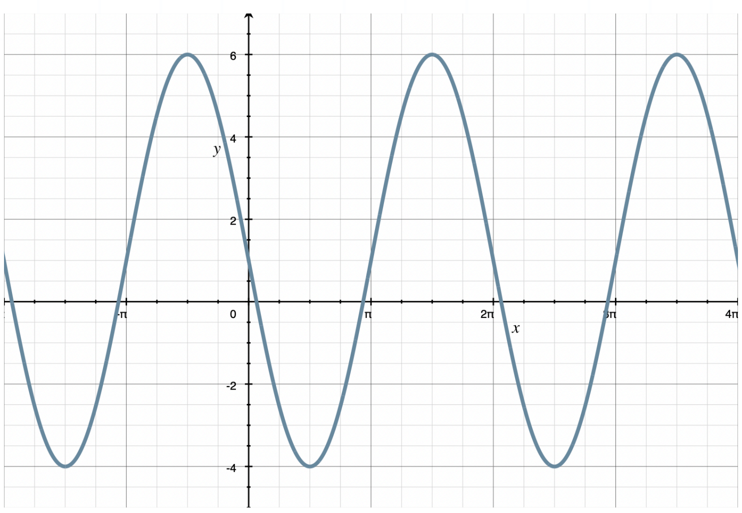
Topic: Graphing polar curves in a rectangular system

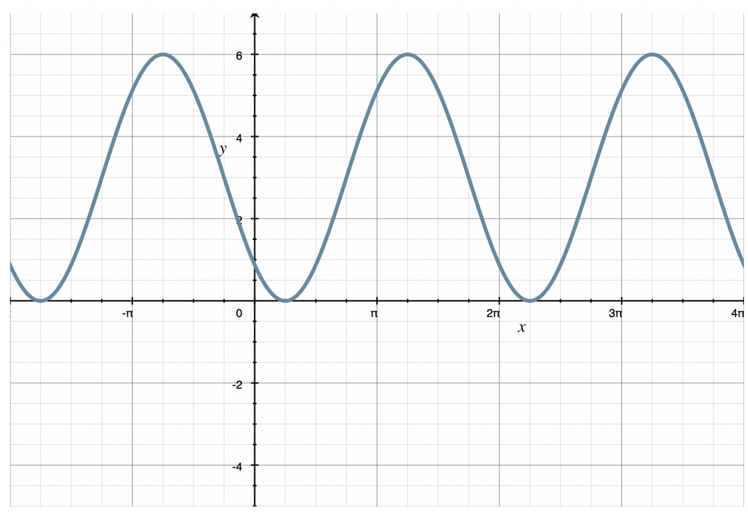
Question: Sketch the graph of $r = 5\sin\theta + 1$ in rectangular coordinates.

Answer choices:

A 

B 

C 

D 

Solution: B

Evaluate the curve at a few multiples of $\theta = \pi/2$, starting with $\theta = 0$.

$$\theta = 0 \qquad r(0) = 1$$

$$\theta = \frac{\pi}{2} \qquad r\left(\frac{\pi}{2}\right) = 5(1) + 1 = 6$$

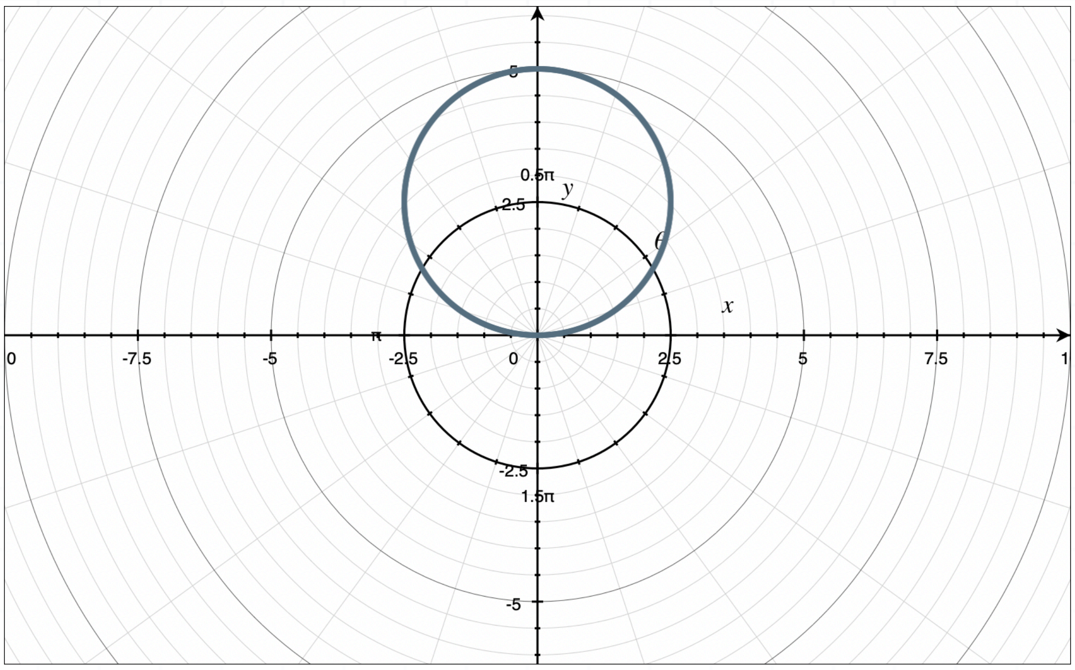
$$\theta = \pi \qquad r(\pi) = 0 + 1 = 1$$

$$\theta = \frac{3\pi}{2} \qquad r\left(\frac{3\pi}{2}\right) = 5(-1) + 1 = -4$$

Answer choice B is the only curve that has a positive value at both $\pi/2$ and $3\pi/2$.

Topic: Graphing circles

Question: Which polar equation is represented in the graph?



Answer choices:

A $\quad r = \frac{5}{2}\sin\theta$

B $\quad r = \frac{5}{2}\cos\theta$

C $\quad r = 5\sin\theta$

D $\quad r = -5\sin\theta$

Solution: C

Because the circle straddles the vertical axis, it's a sine circle, and we can rule out answer choice B.

Equations in the form $r = c \sin \theta$ extend out a distance of $|c|$ from the pole. This circle extends out to a distance of $r = 5$ from the pole, and since the circle is above the horizontal axis, which means the equation must be

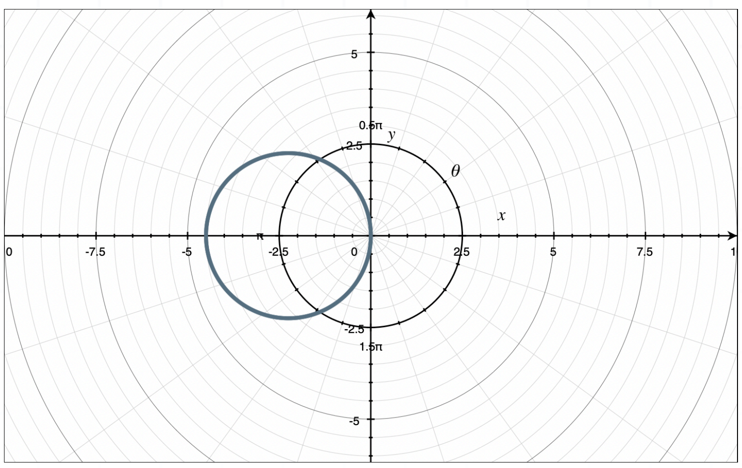
$$r = 5 \sin \theta$$

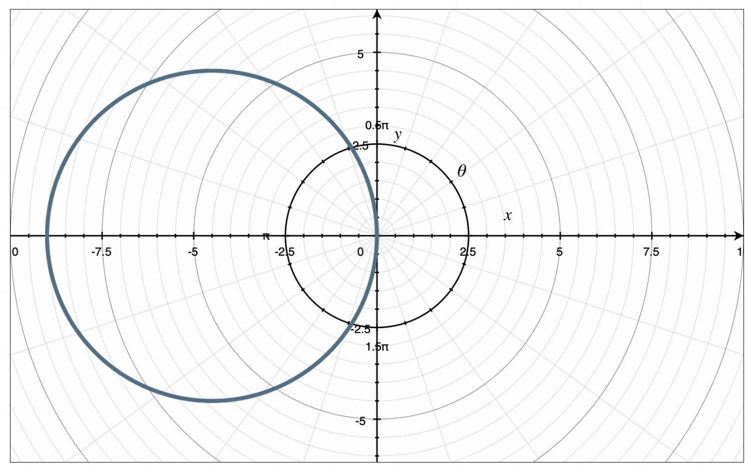
Topic: Graphing circles

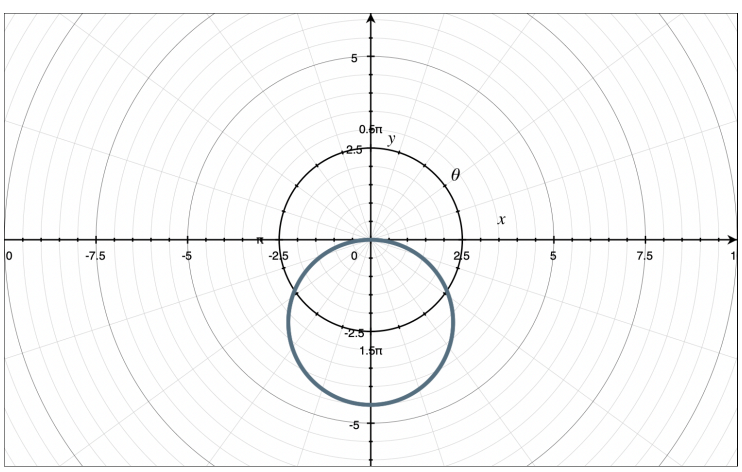
Question: Sketch the graph of the polar equation.

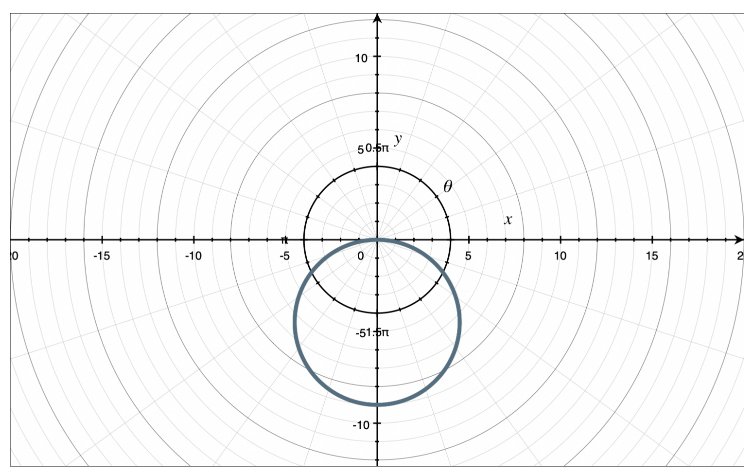
$$r = -9\cos\theta$$

Answer choices:

A 

B 

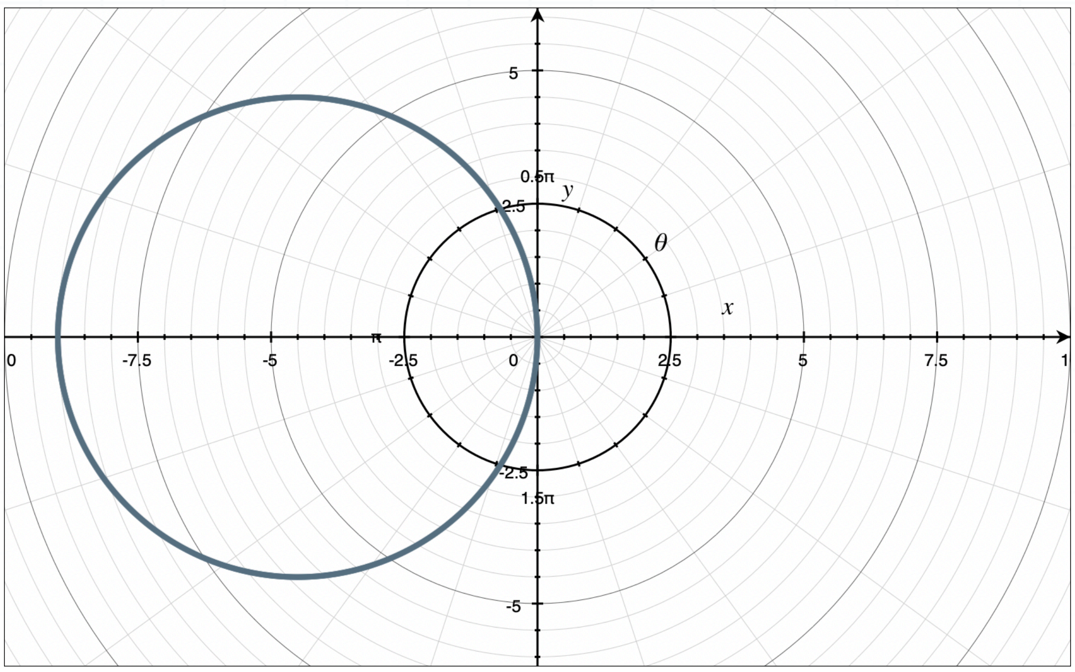
C 

D 

Solution: B

Cosine circles straddle the horizontal axis, so we can rule out answer choices C and D.

Equations in the form $r = c\cos\theta$ extend out a distance of $|c|$ from the pole, so the circle needs to extend out to $r = |-9| = 9$. Therefore, the sketch of the graph must be



Topic: Graphing circles

Question: Find the polar equation of the circle that passes through the pole and has its center at $(2,\pi/2)$.

Answer choices:

A $r = 2\sin\theta$

B $r = 2\cos\theta$

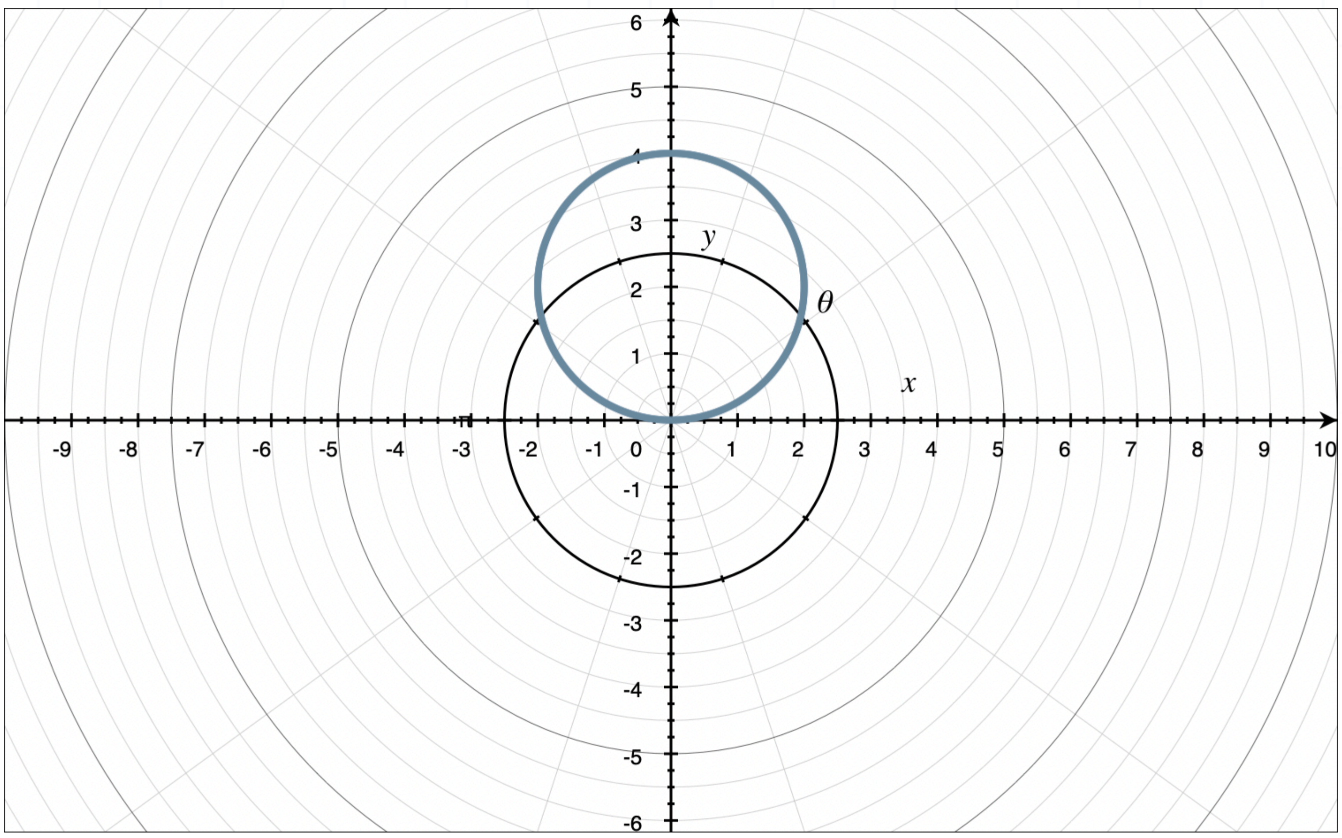
C $r = 4\sin\theta$

D $r = 4\cos\theta$

Solution: C

The center point $(2,\pi/2)$ sits on the positive side of the vertical axis, so we know the circle must be a sine circle, which eliminates answer choices B and D.

If the center of the circle is at $(2,\pi/2)$ and it passes through the pole, then the circle must extend out to a distance of 4 away from the pole. Therefore, the equation of the circle must be $r = 4\sin\theta$.



Topic: Graphing roses

Question: How many petals exist in the graph of the rose?

$$r = 4\sin(6\theta)$$

Answer choices:

A 6 petals

B 4 petals

C 8 petals

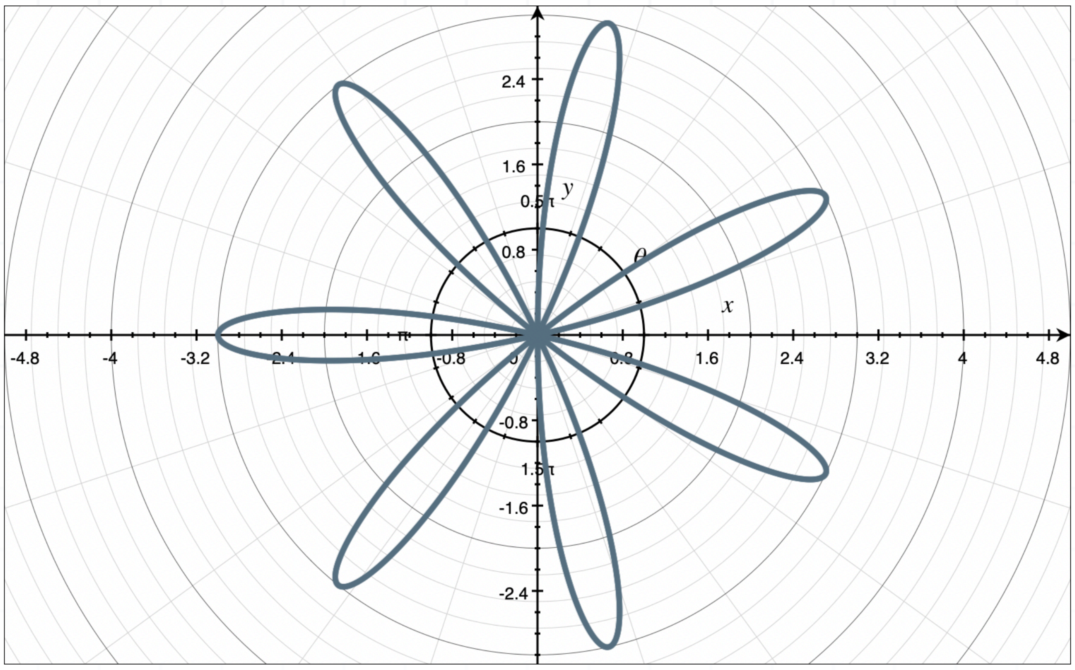
D 12 petals

Solution: D

The equation $r = 4\sin(6\theta)$ is in the form $r = c\sin(n\theta)$, with $c = 4$ and $n = 6$. Since n is even, the rose will have $|2n| = |2(6)| = |12| = 12$ petals.

Topic: Graphing roses

Question: Identify the polar equation represented by the graph.



Answer choices:

A $r = 3\cos(7\theta)$

B $r = 7\sin(3\theta)$

C $r = 3\sin(7\theta)$

D $r = -3\cos(7\theta)$

Solution: D

The rose in the graph has 7 petals that extend out to a distance of $r = 3$ from the pole, which means the equation represented by the graph must take one of these forms:

$$r = 3\cos(7\theta) \qquad r = -3\cos(7\theta)$$

$$r = 3\sin(7\theta) \qquad r = -3\sin(7\theta)$$

The graph has a petal that straddles the horizontal axis and none that straddle the vertical axis, so it must be a cosine rose, so we can narrow down the equation to either $r = 3\cos(7\theta)$ or $r = -3\cos(7\theta)$.

When $\theta = 0$, these curves give

$$r = 3\cos(7(0)) = 3(1) = 3 \qquad r = -3\cos(7(0)) = -3(1) = -3$$

At $\theta = 0$, the graph shows either $r = 0$ or $r = -3$, but not $r = 3$, which means the equation represented by the graph must be $r = -3\cos(7\theta)$.

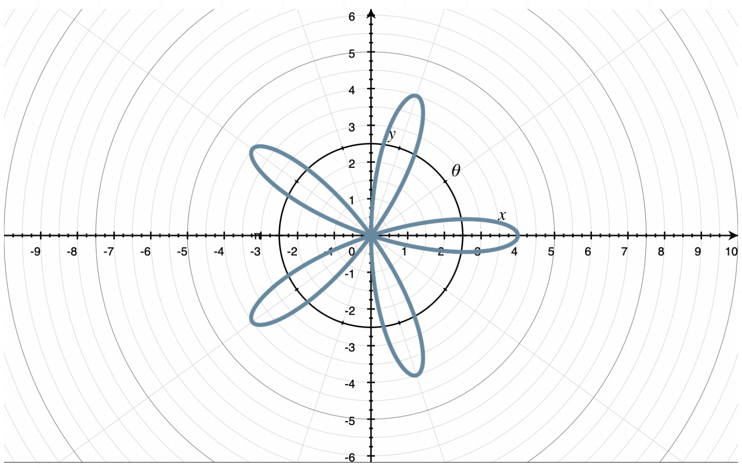
Topic: Graphing roses

Question: Sketch the graph of the polar equation.

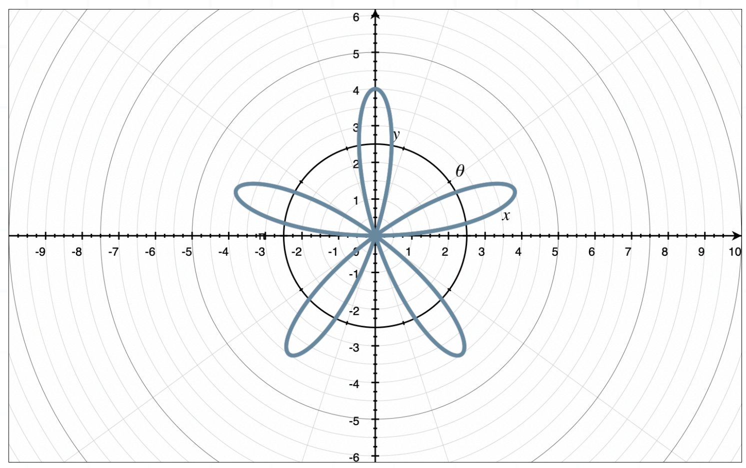
$$r = 4\sin(5\theta)$$

Answer choices:

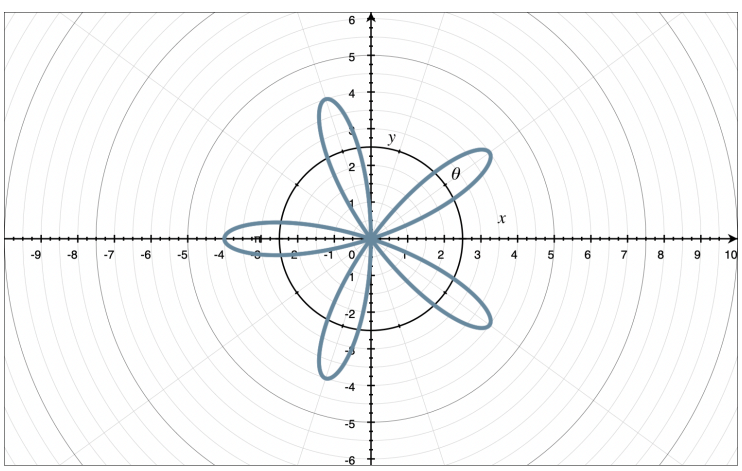
A



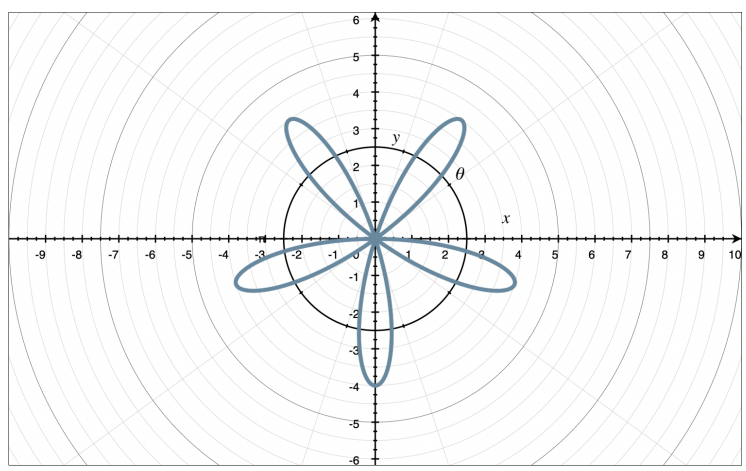
B



C



D



Solution: B

Given the equation of the rose, we know it has 5 petals that extend out a distance of 4 from the pole.

Because the roses in each answer choice each have exactly one petal sitting on one of the major axes, we know the roses with a petal on the horizontal axis are cosine roses, while the roses with a petal on the vertical axis are sine roses. So we can eliminate answer choices A and C.

The polar equation at $\theta = 0$ gives $r = 0$ and at $\theta = \pi/10$ gives 4. The curve in answer choice B is the only curve that matches these points $(0,0)$ and $(4,\pi/10)$.

Topic: Graphing cardioids

Question: Which polar coordinate point (r, θ) lies on the cardioid?

$$r = 6 - 6\sin\theta$$

Answer choices:

A $(r, \theta) = \left(6 + 3\sqrt{3}, \frac{\pi}{3}\right)$

B $(r, \theta) = \left(6 - 2\sqrt{2}, \frac{\pi}{6}\right)$

C $(r, \theta) = \left(6 + 3\sqrt{3}, \frac{5\pi}{3}\right)$

D $(r, \theta) = \left(6 + 2\sqrt{2}, -\frac{\pi}{4}\right)$

Solution: C

If we substitute $\theta = 5\pi/3$ from answer choice C into the equation of the cardioid, we get

$$r = 6 - 6\sin\theta$$

$$r = 6 - 6\sin\left(\frac{5\pi}{3}\right)$$

$$r = 6 - 6\left(-\frac{\sqrt{3}}{2}\right)$$

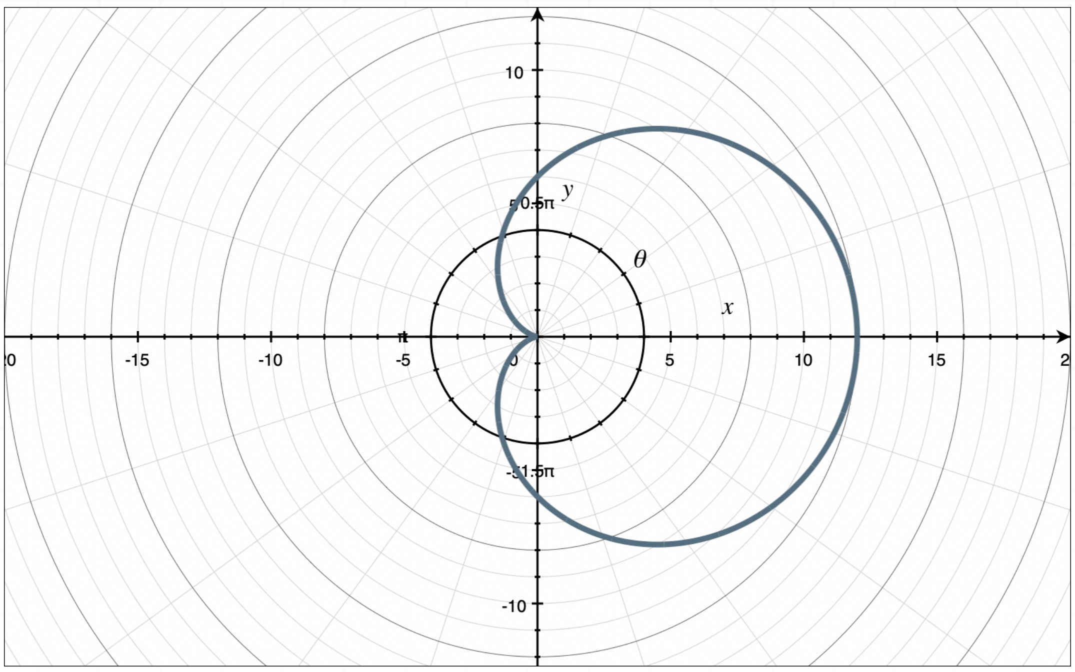
$$r = 6 + 3\sqrt{3}$$

So the cardioid $r = 6 - 6\sin\theta$ passes through the polar coordinate point

$$(r, \theta) = \left(6 + 3\sqrt{3}, \frac{5\pi}{3}\right)$$

Topic: Graphing cardioids

Question: Identify the polar equation represented by the graph.



Answer choices:

A $r = 6 + 6\cos\theta$

B $r = 12 + 12\cos\theta$

C $r = 6 - 6\sin\theta$

D $r = 12 + 12\sin\theta$

Solution: A

This cardioid has symmetry around the horizontal axis, which means it's a cosine cardioid. And it sits mostly to the right of the vertical axis, which means the equation represented by the graph must take the form

$$r = c + c\cos\theta$$

The cardioid's furthest distance from the pole will occur at $2c$. This cardioid reaches $2c = 12$, which means $c = 6$, and therefore that the equation of the cardioid must be

$$r = 6 + 6\cos\theta$$

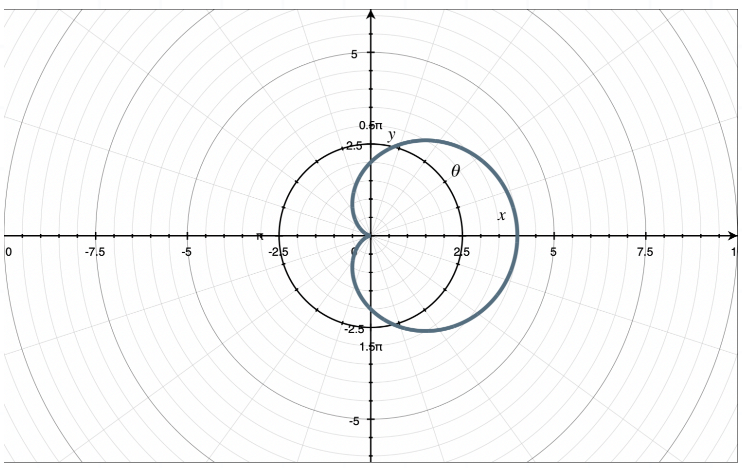
Topic: Graphing cardioids

Question: Sketch the graph of the cardioid.

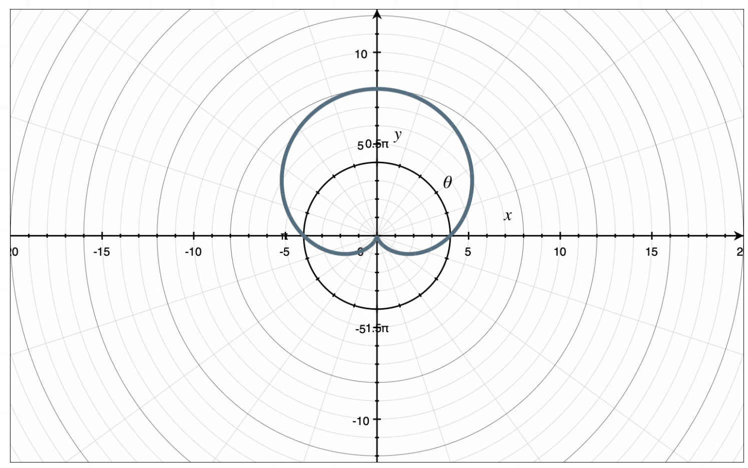
$$r = 4 + 4\sin\theta$$

Answer choices:

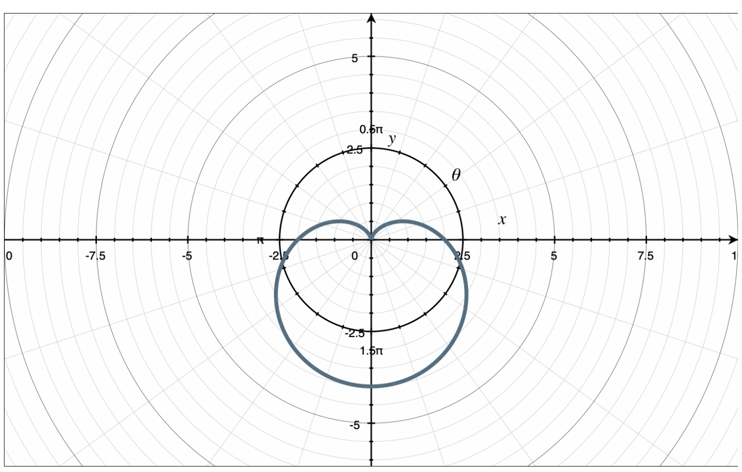
A



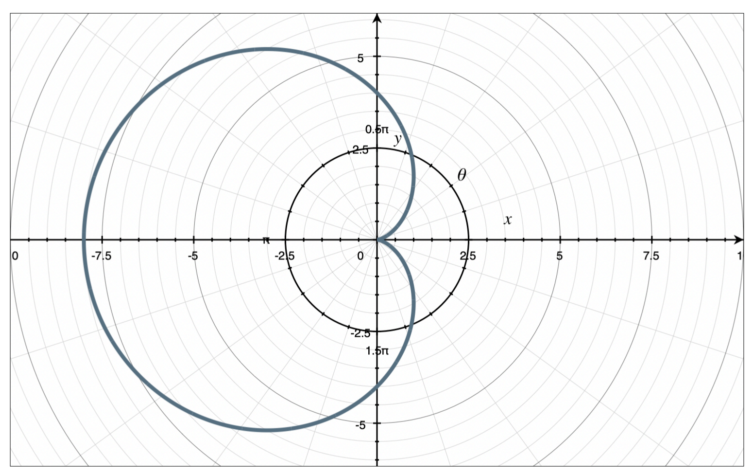
B



C



D

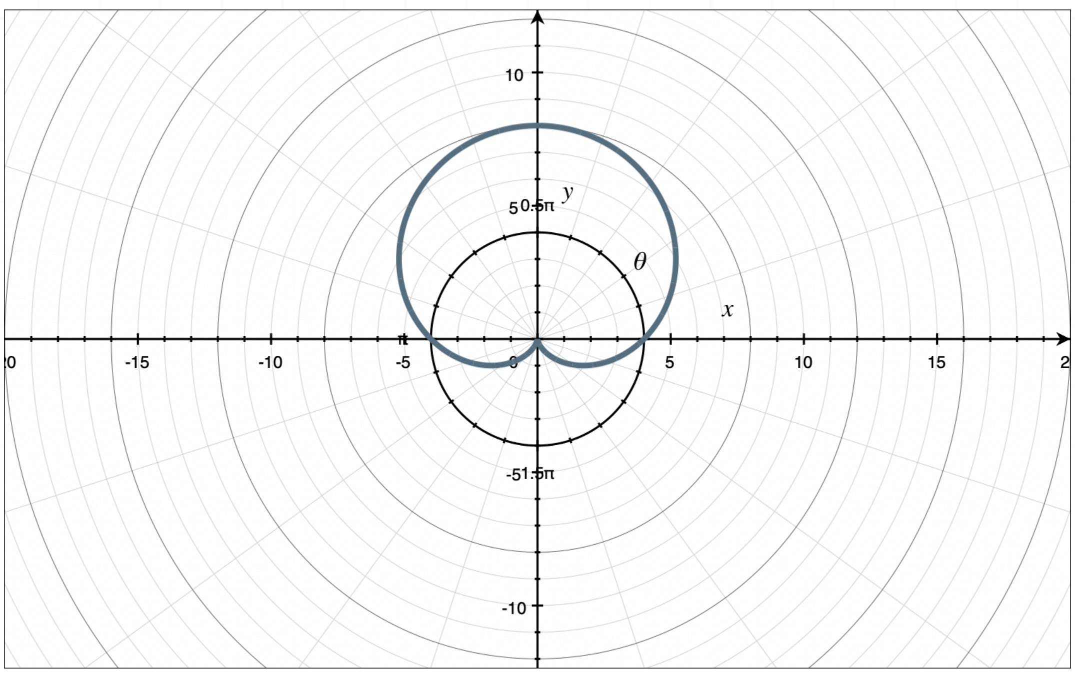


Solution: B

The equation $r=4+4\sin\theta$ is a sine cardioid, which means its graph must be symmetric around the vertical axis.

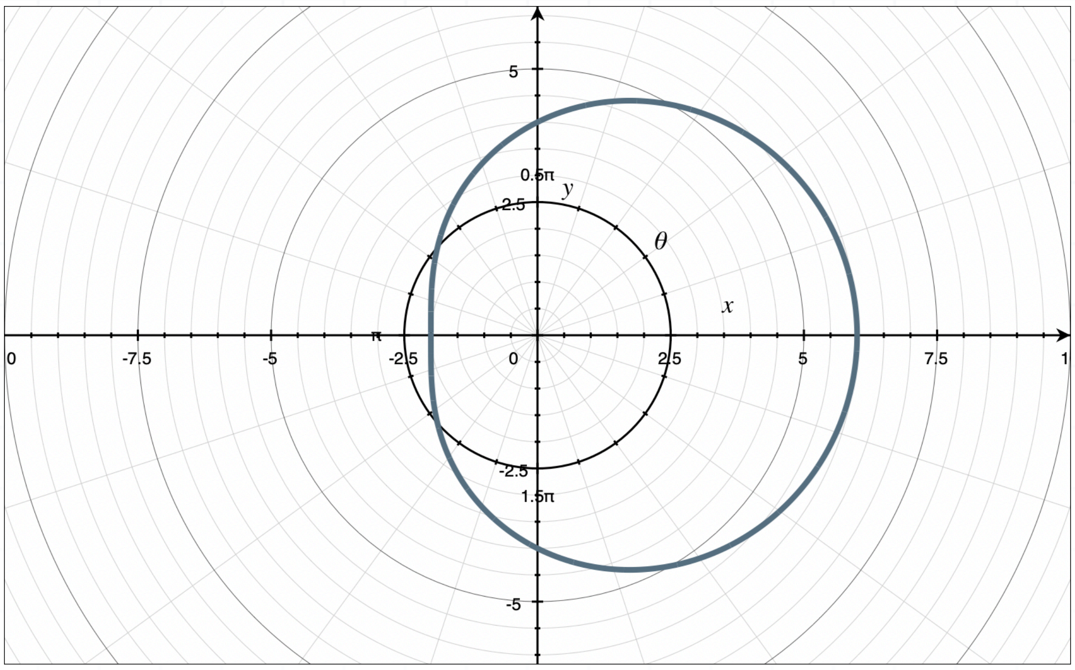
And because of the positive sign between the terms, the graph has to sit mostly above the horizontal axis.

Therefore, the graph of the cardioid must be



Topic: Graphing limaçons

Question: Identify the polar equation represented by the graph.



Answer choices:

A $r = 2 - 4\cos\theta$

B $r = 5 + \sin\theta$

C $r = 1 - 4\sin\theta$

D $r = 4 + 2\cos\theta$

Solution: D

Because the limaçon is symmetric around the horizontal axis, it's a cosine limaçon.

And since the limaçon sits mostly to the right of the vertical axis, the sign between the terms in the limaçon's equation must be positive.

Therefore, the equation of the limaçon must be

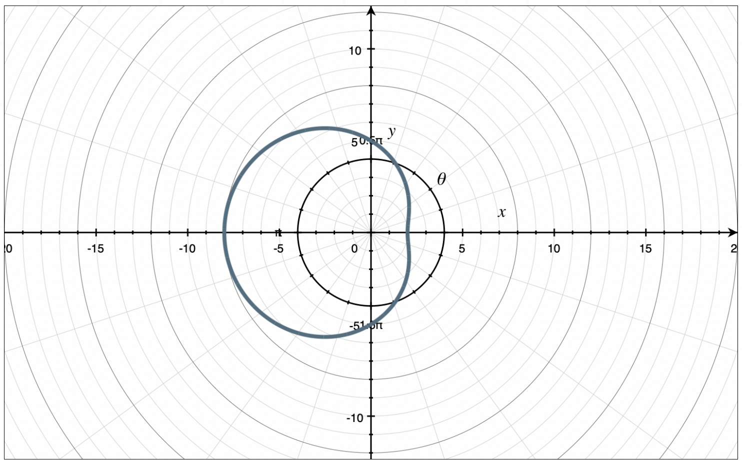
$$r = 4 + 2\cos\theta$$

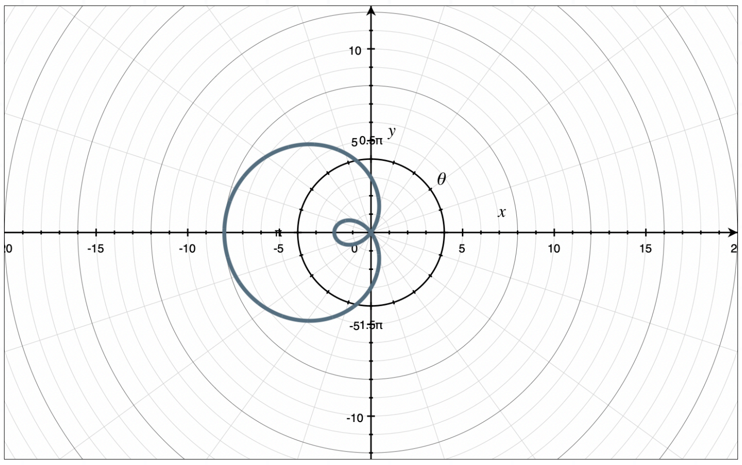
Topic: Graphing limaçons

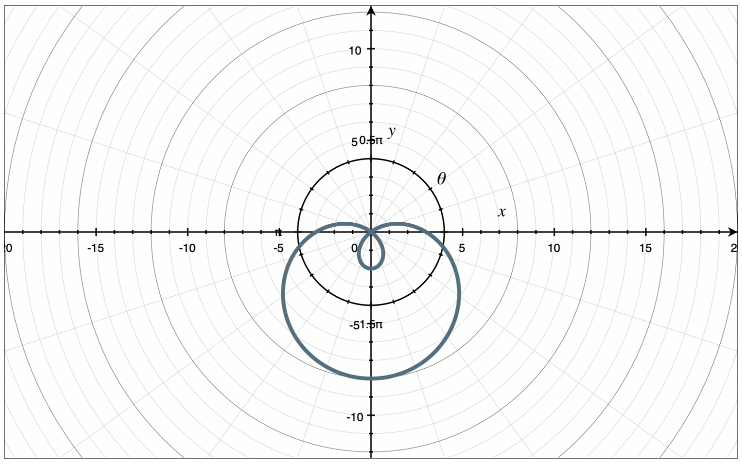
Question: Sketch the graph of the limaçon.

$$r = 3 - 5\cos\theta$$

Answer choices:

A


B


C


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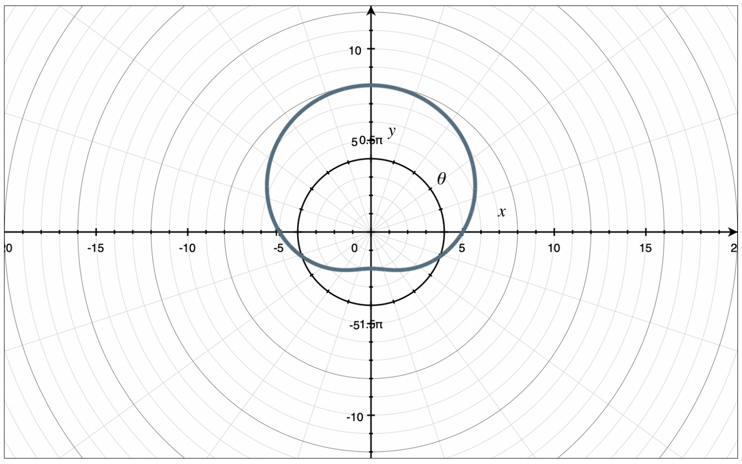
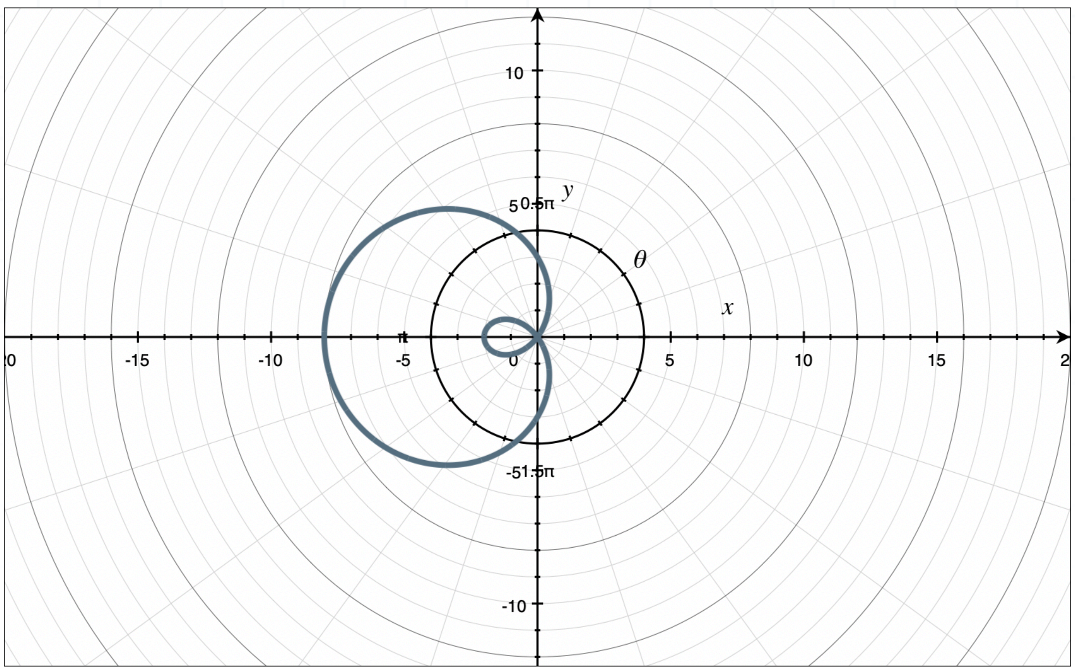


Solution: B

The equation $r = 3 - 5\cos\theta$ is a cosine limaçon, which means its graph must be symmetric around the horizontal axis.

And because of the negative sign between the terms, the graph has to sit mostly to the left of the vertical axis.

Therefore, the graph of the cardioid must be

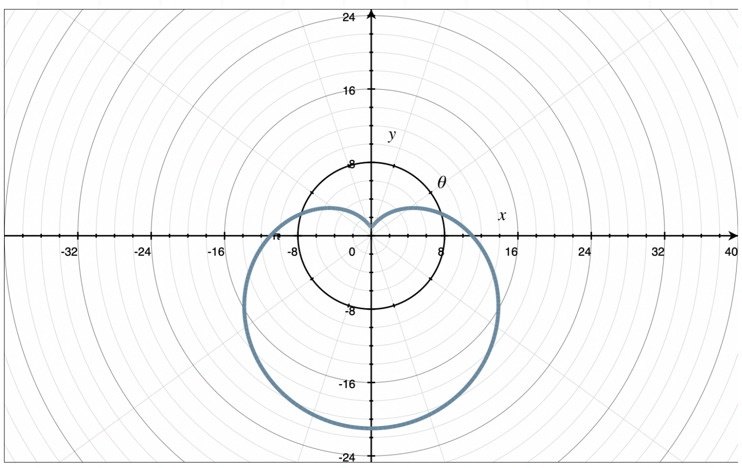


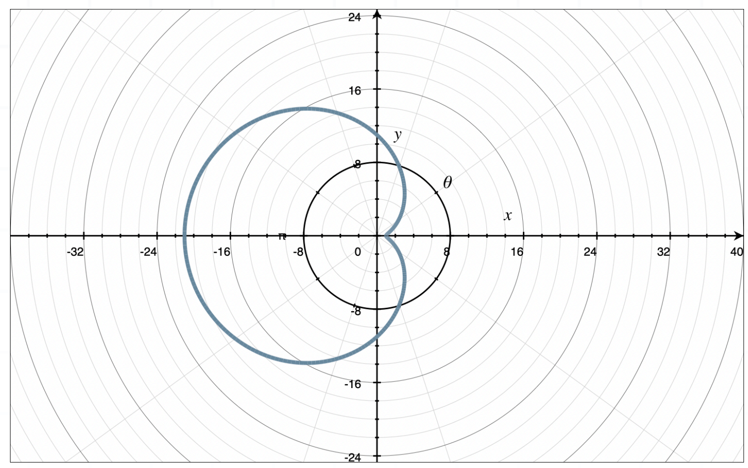
Topic: Graphing limaçons

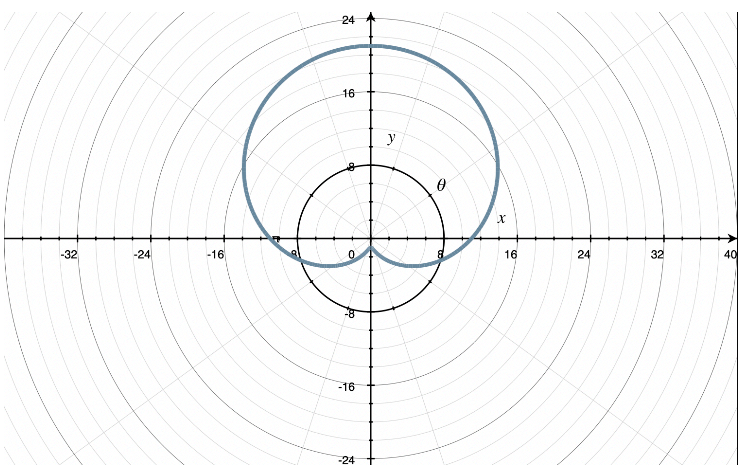
Question: Sketch the graph of the equation in polar coordinates.

$$r = 11 - 10\sin\theta$$

Answer choices:

A


B


C


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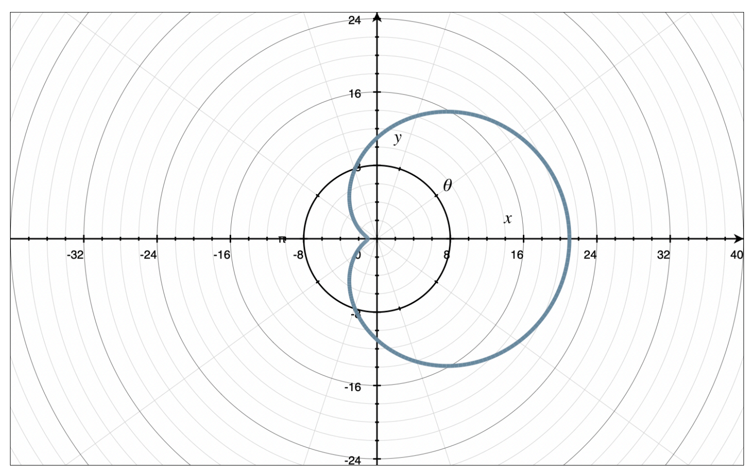
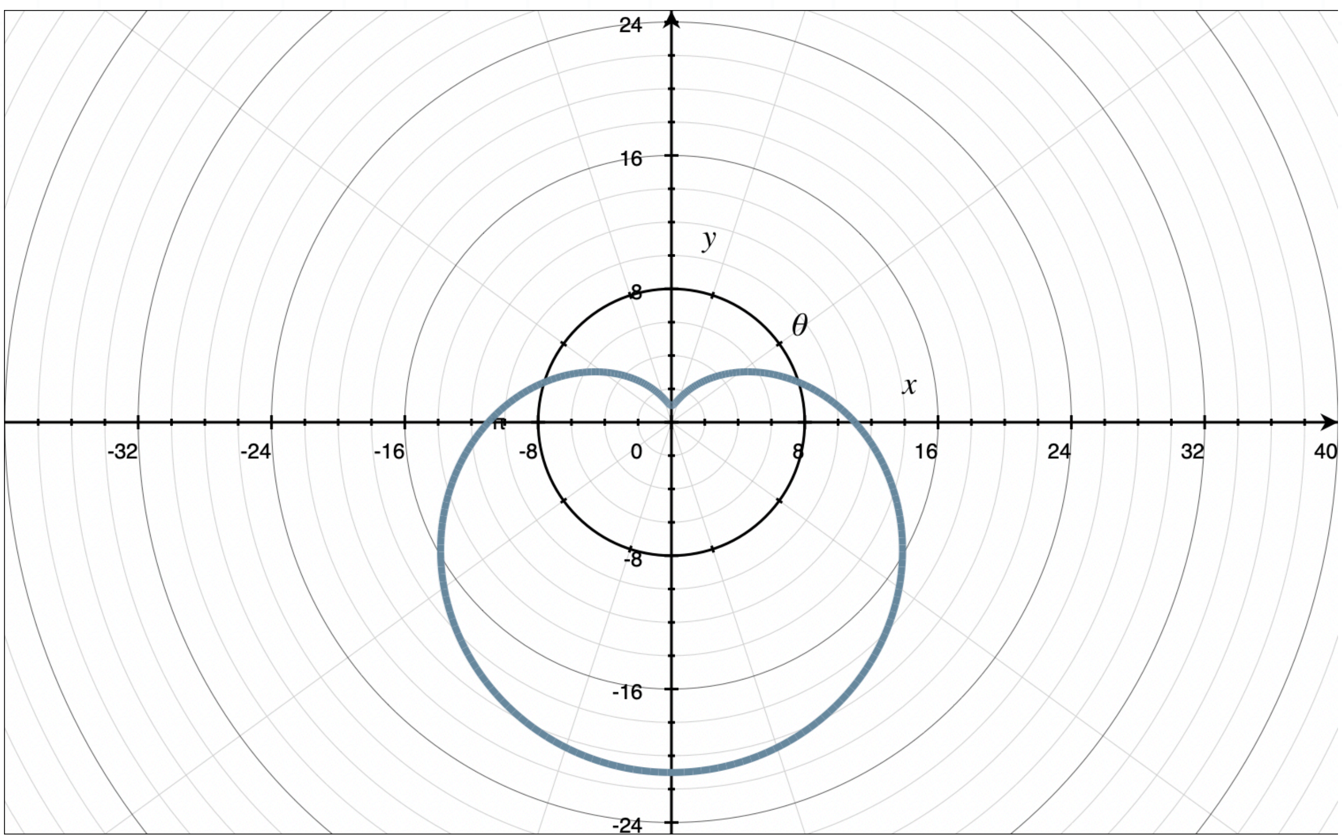


Solution: A

The equation $r = 11 - 10\sin\theta$ is a sine limaçon, which means its graph must be symmetric around the vertical axis, which rules out answer choices B and D.

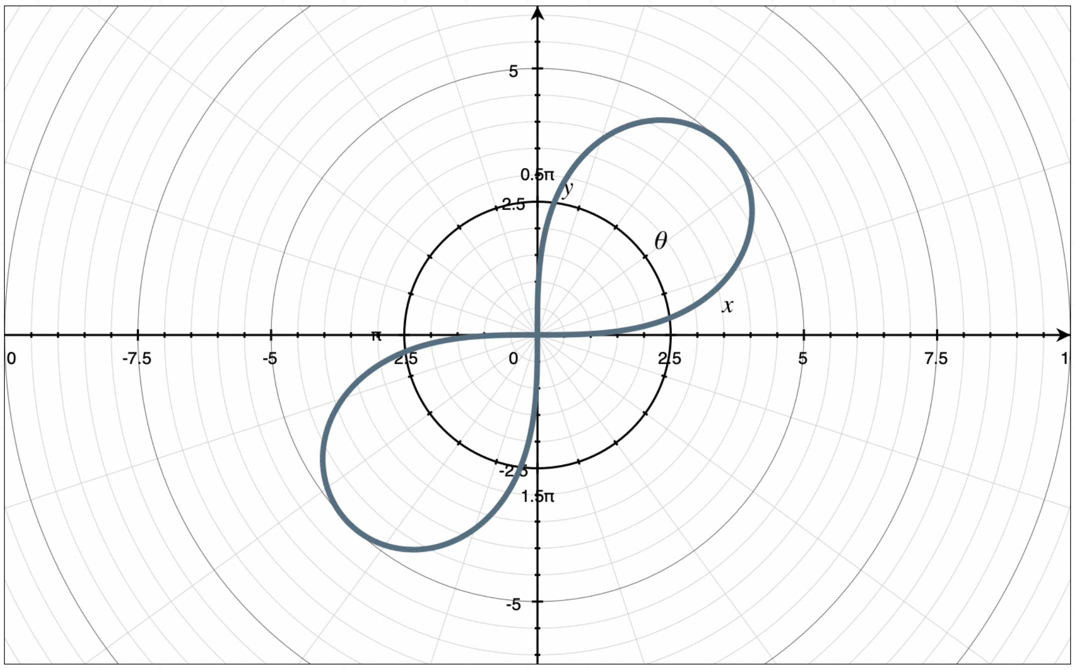
Because the equation includes a the negative sign between the terms, the graph has to sit mostly below of the vertical axis, and answer choice A is the only graph that matches these characteristics.

We also know that, since $1 < 11/10 < 2$, there's a small dip in the graph of the limaçon.



Topic: Graphing lemniscates

Question: Identify the polar equation represented by the graph.



Answer choices:

A $r^2 = 5\cos(2\theta)$

B $r^2 = 25\sin(2\theta)$

C $r^2 = 5\sin(2\theta)$

D $r^2 = 25\cos(2\theta)$

Solution: B

Because the loops of the lemniscate lie within quadrants, instead of along axes, it must be a sine lemniscate, not a cosine lemniscate.

And since the loops of the lemniscate lie in the first and third quadrants, it must be a sine lemniscate with c^2.

These two facts narrow down the equation represented by the graph to and equation in the form

$$r^2 = c^2 \sin(2\theta)$$

The loops of the lemniscate in the graph extend out from the pole to a distance of $r = 5$, which means $c = 5$ and the equation represented by the graph is

$$r^2 = 5^2 \sin(2\theta)$$

$$r^2 = 25 \sin(2\theta)$$

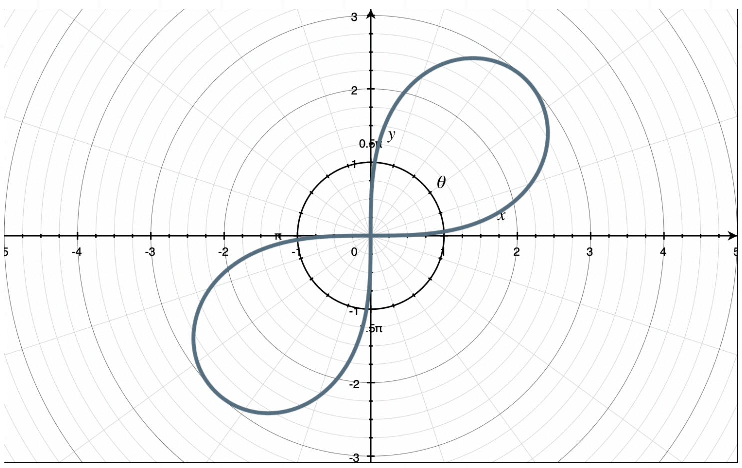
Topic: Graphing lemniscates

Question: Sketch the graph of the lemniscate.

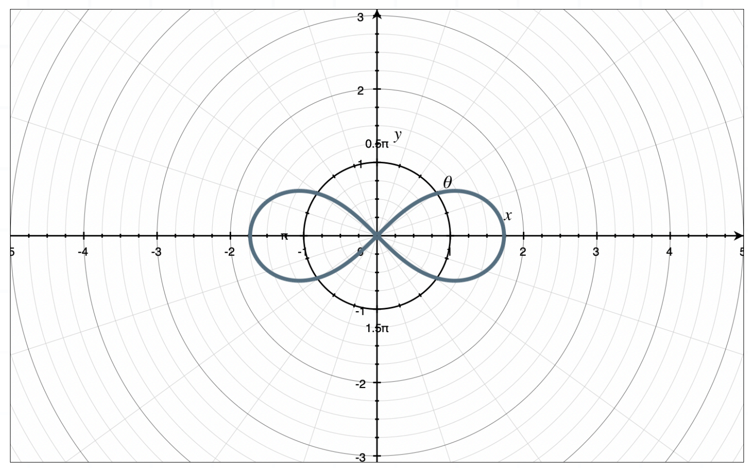
$$r^2 = 9\sin(2\theta)$$

Answer choices:

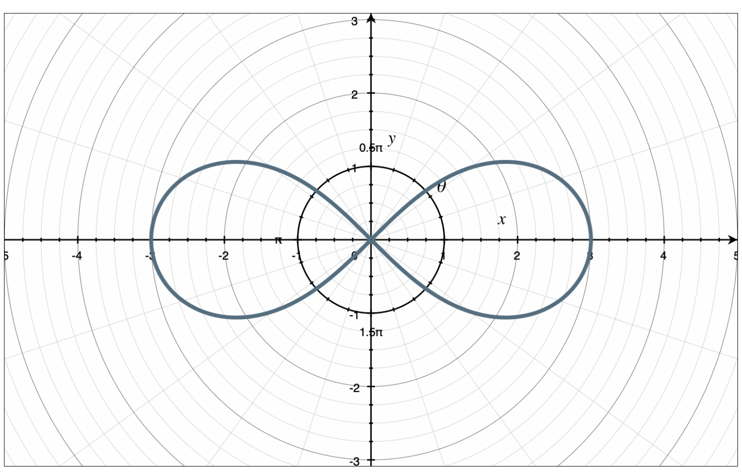
A



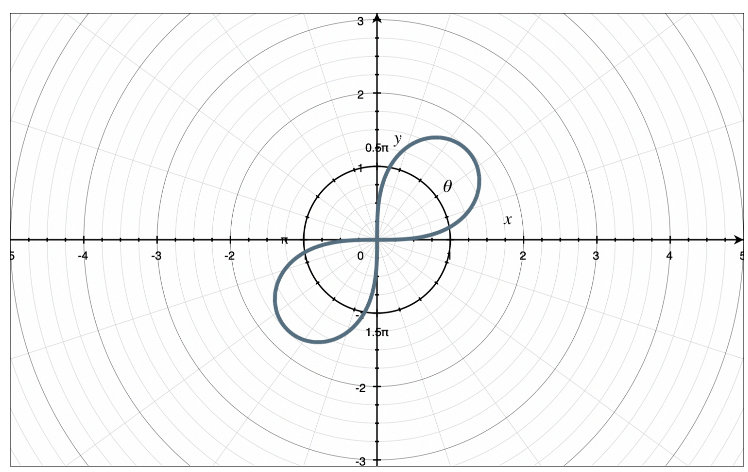
B



C



D



Solution: A

Answer choices B and C show cosine lemniscates, since their loops are symmetric around the horizontal axis. But answer choices A and D show sine lemniscates because their loops are contained within individual quadrants, and not across axes.

The given lemniscate equation takes the form

$$r^2 = c^2 \sin(2\theta)$$

We know that the loops of the lemniscate extend out away from the pole to a distance of $r = c$, so the loops of this lemniscate should extend to $r = \sqrt{9} = 3$.

Answer choice A is the only sine lemniscate with loops that extend to $r = 3$.

Topic: Graphing lemniscates

Question: Which polar coordinate point (r,θ) lies on the lemniscate?

$$r^2 = 6\sin(2\theta)$$

Answer choices:

A $(r,\theta) = \left(\sqrt[4]{27}, \frac{\pi}{6}\right)$

B $(r,\theta) = \left(\sqrt{3}, \frac{\pi}{6}\right)$

C $(r,\theta) = \left(3\sqrt{3}, \frac{\pi}{3}\right)$

D $(r,\theta) = \left(\sqrt{3}, \frac{\pi}{3}\right)$

Solution: A

If we substitute $\theta = \pi/6$ from answer choice A into the equation of the lemniscate, we get

$$r^2 = 6\sin(2\theta)$$

$$r^2 = 6\sin\left(2 \cdot \frac{\pi}{6}\right)$$

$$r^2 = 6\sin\left(\frac{\pi}{3}\right)$$

$$r^2 = 6\left(\frac{\sqrt{3}}{2}\right)$$

$$r^2 = 3\sqrt{3}$$

$$r = \sqrt[4]{27}$$

So the lemniscate $r^2 = 6\sin(2\theta)$ passes through the polar coordinate point

$$(r, \theta) = \left(\sqrt[4]{27}, \frac{\pi}{6}\right)$$

Topic: Intersection of polar curves

Question: Find any points of intersection of the polar curves.

$$r = 2\cos\theta$$

$$r = 3 - 4\cos\theta$$

Answer choices:

A $\left(\sqrt{3}, \frac{5\pi}{6}\right)$ $\left(\sqrt{3}, \frac{7\pi}{6}\right)$ $(0,0)$

B $\left(5, \frac{2\pi}{3}\right)$ $\left(5, \frac{4\pi}{3}\right)$ $(0,0)$

C $\left(1, \frac{\pi}{3}\right)$ $\left(1, \frac{5\pi}{3}\right)$ $(0,0)$

D $\left(3 - 2\sqrt{3}, \frac{\pi}{6}\right)$ $\left(3 - 2\sqrt{3}, \frac{11\pi}{6}\right)$ $(0,0)$

Solution: C

Set the curves equal to each other.

$$2\cos\theta = 3 - 4\cos\theta$$

$$6\cos\theta = 3$$

$$\cos\theta = \frac{1}{2}$$

We find $\cos\theta = 1/2$ at $\theta = \pi/3$ and $\theta = 5\pi/3$. Plugging these angles back into the equation of the circle, we get

$$r\left(\frac{\pi}{3}\right) = 2\cos\left(\frac{\pi}{3}\right) = 2\left(\frac{1}{2}\right) = 1$$

$$r\left(\frac{5\pi}{3}\right) = 2\cos\left(\frac{5\pi}{3}\right) = 2\left(\frac{1}{2}\right) = 1$$

Plugging the same angles into the equation of the limaçon, we get

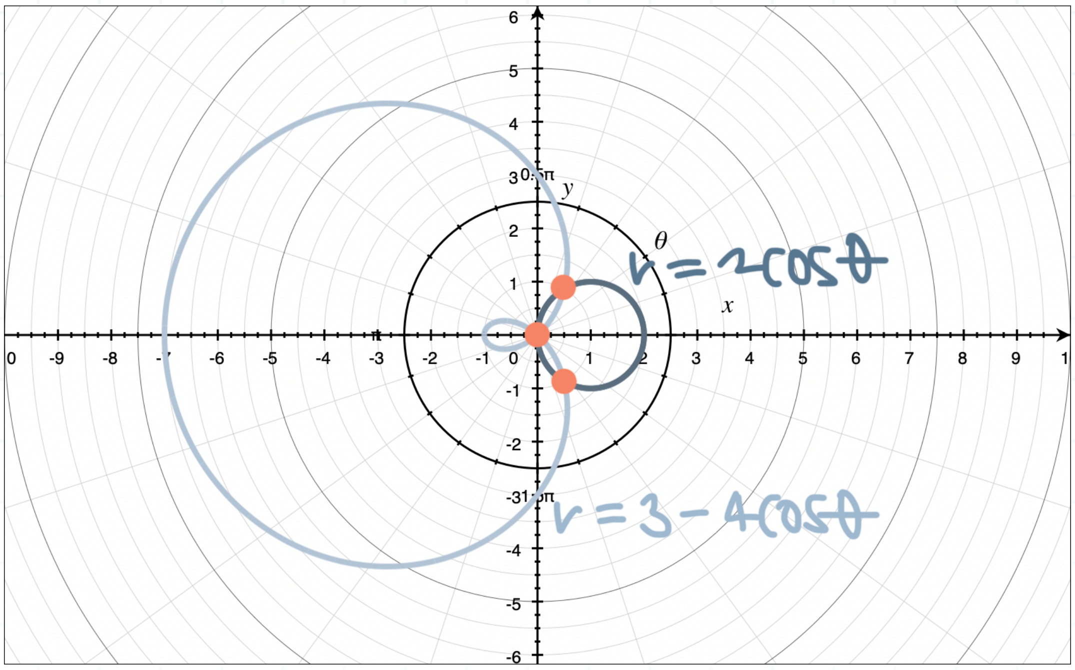
$$r\left(\frac{\pi}{3}\right) = 3 - 4\cos\left(\frac{\pi}{3}\right) = 3 - 4\left(\frac{1}{2}\right) = 3 - 2 = 1$$

$$r\left(\frac{5\pi}{3}\right) = 3 - 4\cos\left(\frac{5\pi}{3}\right) = 3 - 4\left(\frac{1}{2}\right) = 3 - 2 = 1$$

So we have two points of intersection,

$$(r,\theta) = \left(1,\frac{\pi}{3}\right) \qquad (r,\theta) = \left(1,\frac{5\pi}{3}\right)$$

If we sketch the curves,



we see the two points of intersection we found, as well as intersection at the pole. So the curves have three points of intersection.

Topic: Intersection of polar curves

Question: Find the number of points of intersection of the polar curves.

$$r = 2 + \sqrt{3}\sin\theta$$

$$r = 2 + \sqrt{3}\cos\theta$$

Answer choices:

A $\quad 0$

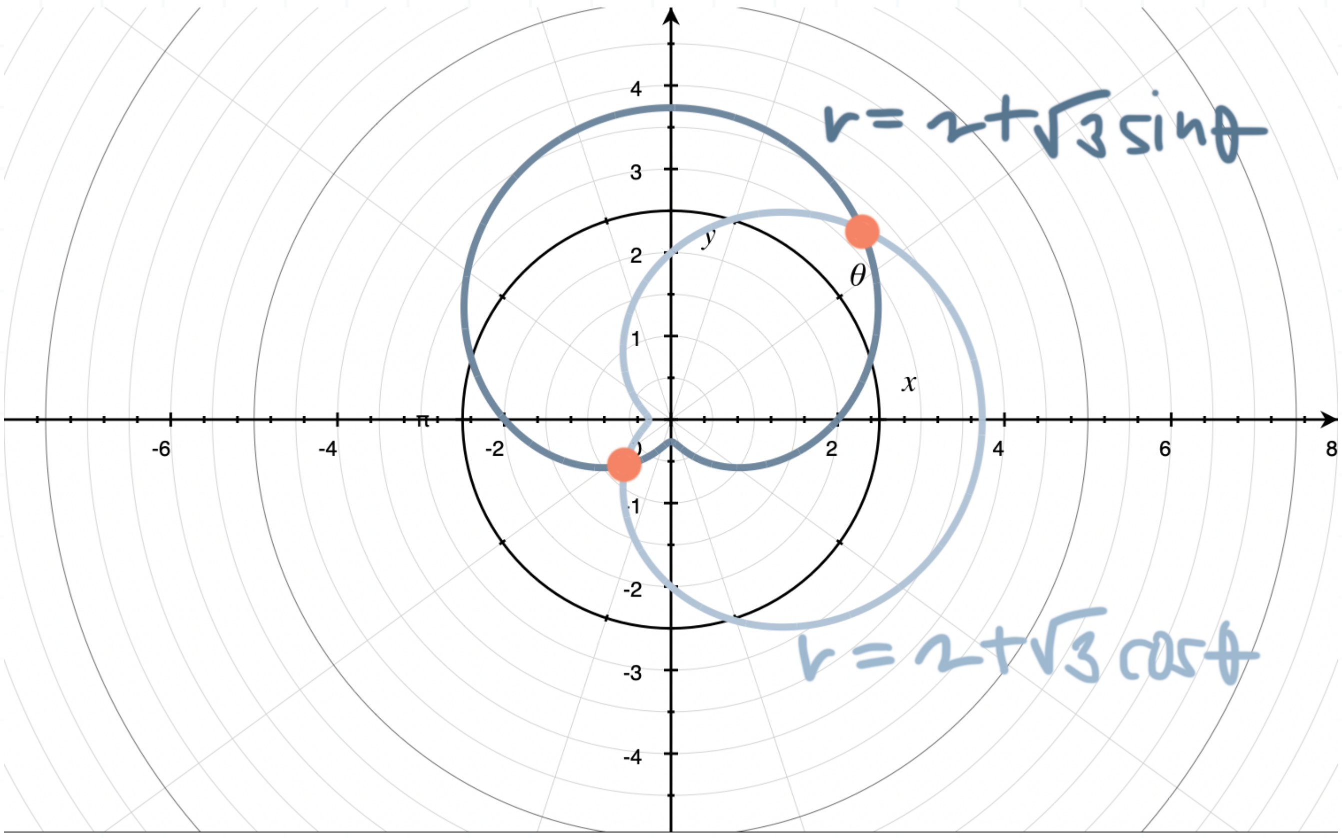
B $\quad 1$

C $\quad 2$

D $\quad 3$

Solution: C

If we sketch the curves, we see that they're both cardioids, and the curves have two points of intersection.



Topic: Intersection of polar curves

Question: Find all the points of intersection of the polar curves.

$$r = -1 + 3\sin\theta$$

$$r = 2 + 3\sin\theta$$

Answer choices:

A $(-1,0)$, $(2,0)$

B $(-1,0)$, $\left(-1, -\frac{\pi}{2}\right)$

C $(0,0)$

D No intersection points

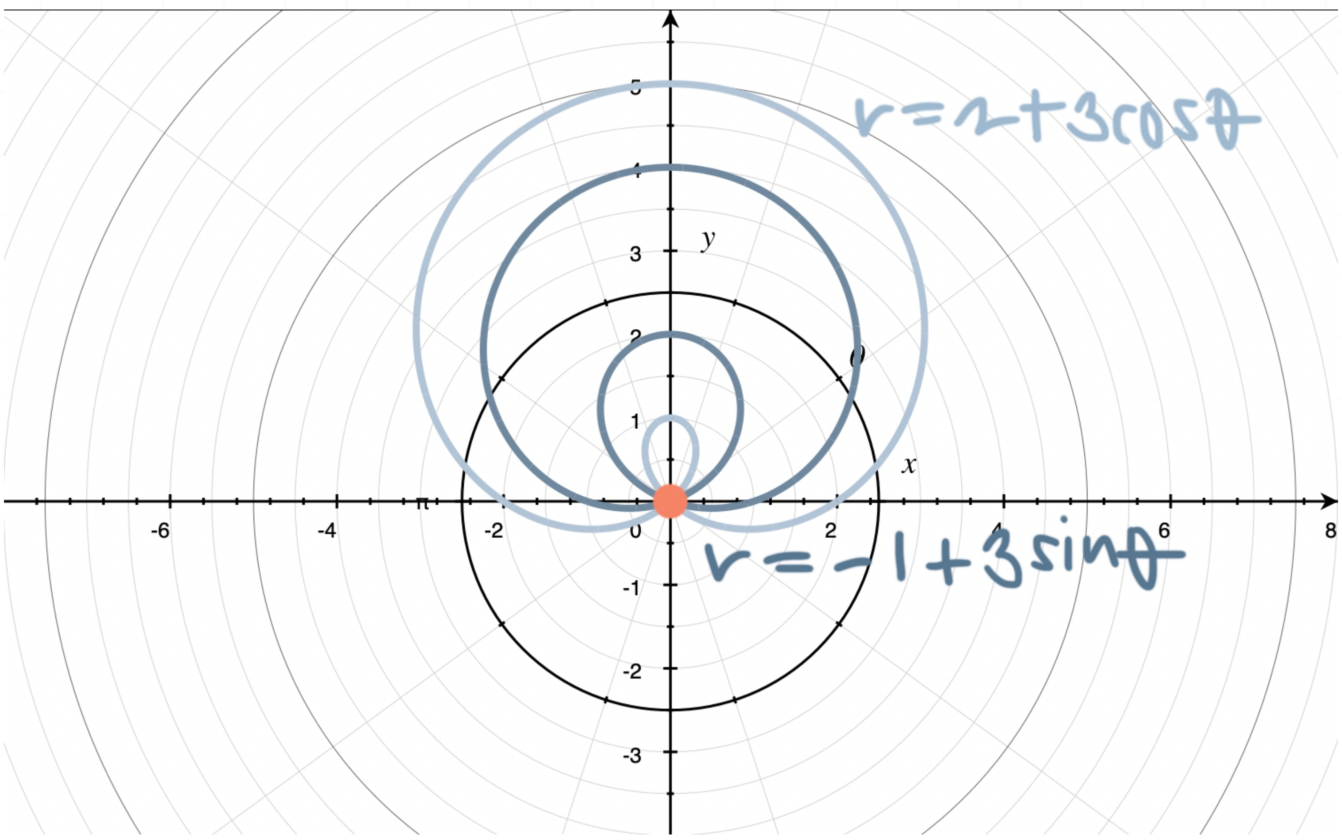
Solution: C

Set the curves equal to each other.

$$-1 + 3\sin\theta = 2 + 3\sin\theta$$

$$-1 = 2$$

There are no polar points that satisfy both equations. But if we sketch the curves,



we see that they intersect one another at the pole.

Topic: Complex numbers

Question: Simplify the imaginary number.

$$i^{1,343}$$

Answer choices:

A i

B -1

C $-i$

D 1

Solution: C

We need to look for the largest number less than or equal to $1{,}343$ that's divisible by 4. $1{,}343$ isn't divisible by 4, so we try $1{,}342$, then $1{,}341$, then $1{,}340$. $1{,}340$ is the first number we come to that's divisible by 4, so we separate the exponent.

$$i^{1,343}$$

$$i^{1,340+3}$$

$$i^{1,340}i^3$$

Rewrite $1{,}340$ as a power of 4.

$$(i^4)^{335}i^3$$

We know that i^4 is always 1, so

$$(1)^{335}i^3$$

$$1i^3$$

$$i^3$$

We know that i^3 is equal to $-i$, so

$$i^{1,343} = -i$$

Topic: Complex numbers

Question: Name the imaginary part of the complex number.

$$z = 2 - 11i$$

Answer choices:

A $\quad 2$

B $\quad 11$

C $\quad -11$

D $\quad -11i$

Solution: C

For a complex number in the form $z = a + bi$, a is always the real part and b is always the imaginary part. If b is negative, you have to remember to include the negative sign when you name the imaginary part. So in the complex number $z = 2 - 11i$, 2 is the real part, and -11 is the imaginary part.

Topic: Complex numbers

Question: How can the number be classified?

$$z = 0 - 4i$$

Answer choices:

A Complex number

B Real number

C Pure imaginary number

D Both A and C

Solution: D

Every real number and every imaginary number is also a complex number. Because the number simplifies as

$$z = 0 - 4i$$

$$z = -4i$$

and the real part disappears, it can be classified as a pure imaginary number. But as a pure imaginary number, it's also automatically a complex number.

Topic: Complex number operations

Question: What are the sum and difference of the complex numbers?

$$2\frac{5}{6} - \frac{1}{3}i$$

$$-4\frac{1}{6} + \frac{1}{2}i$$

Answer choices:

A The sum is $-(11/6) + (1/6)i$

The difference is $(7/2) + (-1/3)i$

B The sum is $-(4/3) + (1/6)i$

The difference is $7 + (-5/6)i$

C The sum is $-(7/6) + (1/3)i$

The difference is $5 + (-2/3)i$

D The sum is $-1 + (1/3)i$

The difference is $(20/3) + (-1/6)i$

Solution: B

The sum of the complex numbers is

$$\left(\frac{17}{6}-\frac{1}{3}i\right)+\left(-\frac{25}{6}+\frac{1}{2}i\right)$$

$$\left(\frac{17}{6}-\frac{25}{6}\right)+\left(-\frac{1}{3}i+\frac{1}{2}i\right)$$

$$\left(\frac{17}{6}-\frac{25}{6}\right)+\left(-\frac{1}{3}+\frac{1}{2}\right)i$$

$$-\frac{8}{6}+\frac{1}{6}i$$

$$-\frac{4}{3}+\frac{1}{6}i$$

The difference of the complex numbers is

$$\left(\frac{17}{6}-\frac{1}{3}i\right)-\left(-\frac{25}{6}+\frac{1}{2}i\right)$$

$$\frac{17}{6}-\frac{1}{3}i+\frac{25}{6}-\frac{1}{2}i$$

$$\frac{17}{6}+\frac{25}{6}-\frac{1}{3}i-\frac{1}{2}i$$

$$\frac{42}{6}-\frac{5}{6}i$$

$$7-\frac{5}{6}i$$

Topic: Complex number operations

Question: What is the product of the complex numbers?

$-9-5i$

$7+13i$

Answer choices:

A $-63-65i$

B $48+i$

C $-128-82i$

D $2-152i$

Solution: D

Use FOIL to find the product of the complex numbers.

$$(-9-5i)(7+13i)$$

$$(-9)(7)+(-9)(13i)+(-5i)(7)+(-5i)(13i)$$

$$-63+(-9)(13)i+(-5)(7)i+(-5)(13)(i^2)$$

$$-63-117i-35i+(-65)(i^2)$$

Using $i^2=-1$ in the last term, we get

$$-63-117i-35i+(-65)(-1)$$

$$-63-117i-35i+65$$

$$(-63+65)+(-117i-35i)$$

$$2+(-117-35)i$$

$$2-152i$$

Topic: Complex number operations

Question: Express the fraction in the form $a + bi$ where a and b are real numbers.

$$\frac{5+2i}{1+3i}$$

Answer choices:

A $-\frac{5}{4}+\frac{7}{4}i$

B $\frac{3}{5}-\frac{9}{10}i$

C $\frac{11}{10}-\frac{13}{10}i$

D $\frac{7}{8}+\frac{1}{8}i$

Solution: C

Multiply by the conjugate of the denominator.

$$\left(\frac{5+2i}{1+3i}\right)\left(\frac{1-3i}{1-3i}\right)$$

$$\frac{(5+2i)(1-3i)}{(1+3i)(1-3i)}$$

Use FOIL to expand the numerator and denominator.

$$\frac{5-15i+2i-6i^2}{1-3i+3i-9i^2}$$

$$\frac{5-13i-6i^2}{1-9i^2}$$

Using $i^2=-1$ gives

$$\frac{5-13i-6(-1)}{1-9(-1)}$$

$$\frac{5-13i+6}{1+9}$$

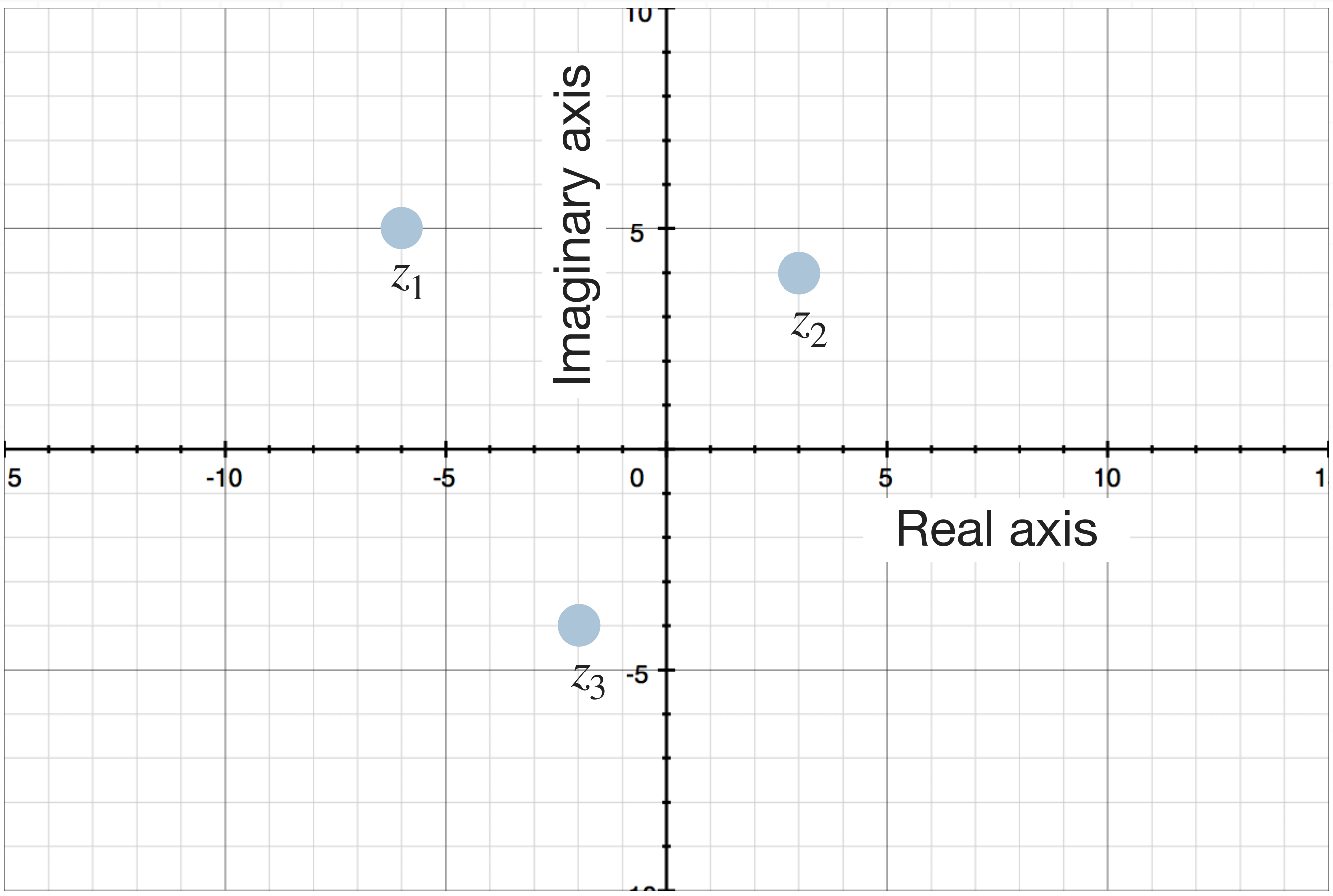
$$\frac{11-13i}{10}$$

Split the fraction.

$$\frac{11}{10}-\frac{13}{10}i$$

Topic: Graphing complex numbers

Question: Which three complex numbers are represented in the graph?



Answer choices:

A $3 - 4i$, $2 + 4i$, and $-5 + 6i$

B $-2 - 4i$, $3 + 4i$, and $-6 + 5i$

C $5 + 6i$, $2 + 4i$, and $-3 - 4i$

D $3 + 4i$, $-6 - 5i$, and $-4 - 2i$

Solution: B

The point z_1 is 6 units to the left of the vertical axis and 5 units above the horizontal axis, so it's the complex number $-6 + 5i$.

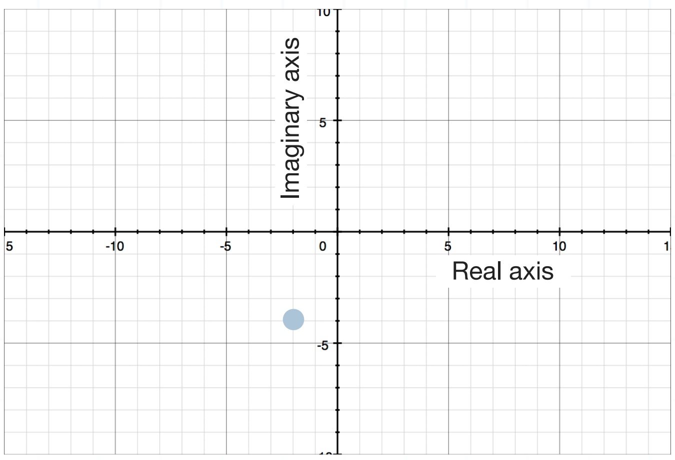
The point z_2 is 3 units to the right of the vertical axis and 4 units above the horizontal axis, so it's the complex number $3 + 4i$.

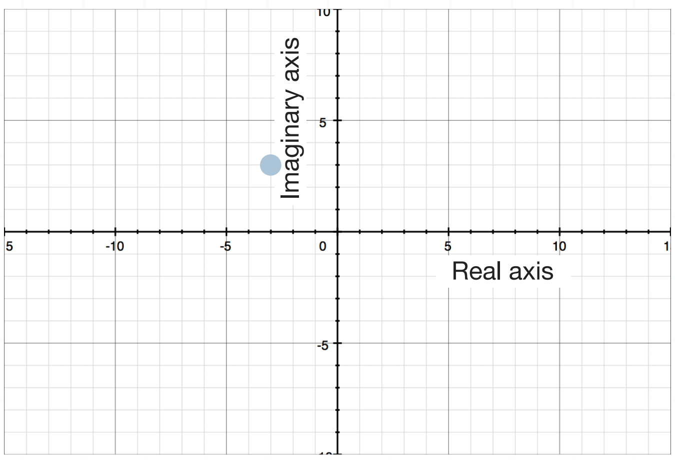
The point z_3 is 2 units to the left of the vertical axis and 4 units below the horizontal axis, so it's the complex number $-2 - 4i$.

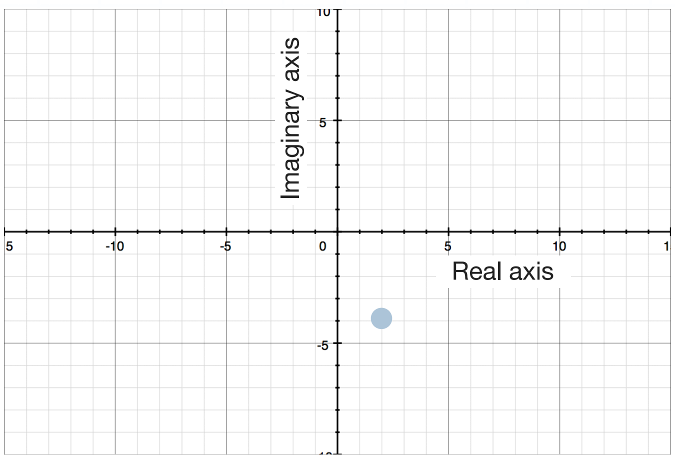
Topic: Graphing complex numbers

Question: Which graph shows the difference of $-8+5i$ and $-6+9i$ ($(-8+5i)-(-6+9i)$)?

Answer choices:

A


B


C


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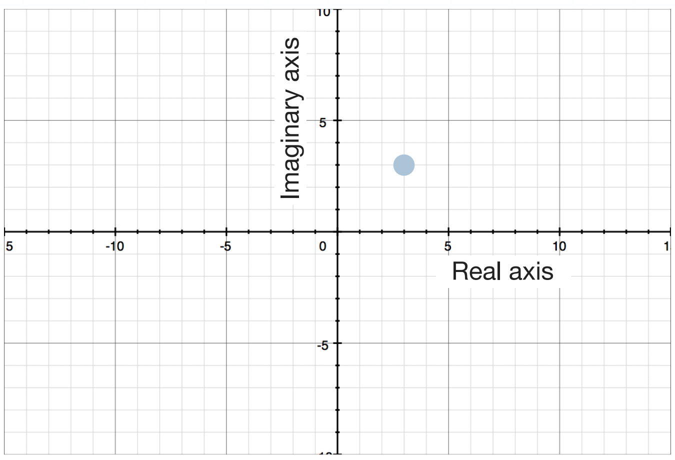


Solution: A

First, we'll compute the difference of the complex numbers.

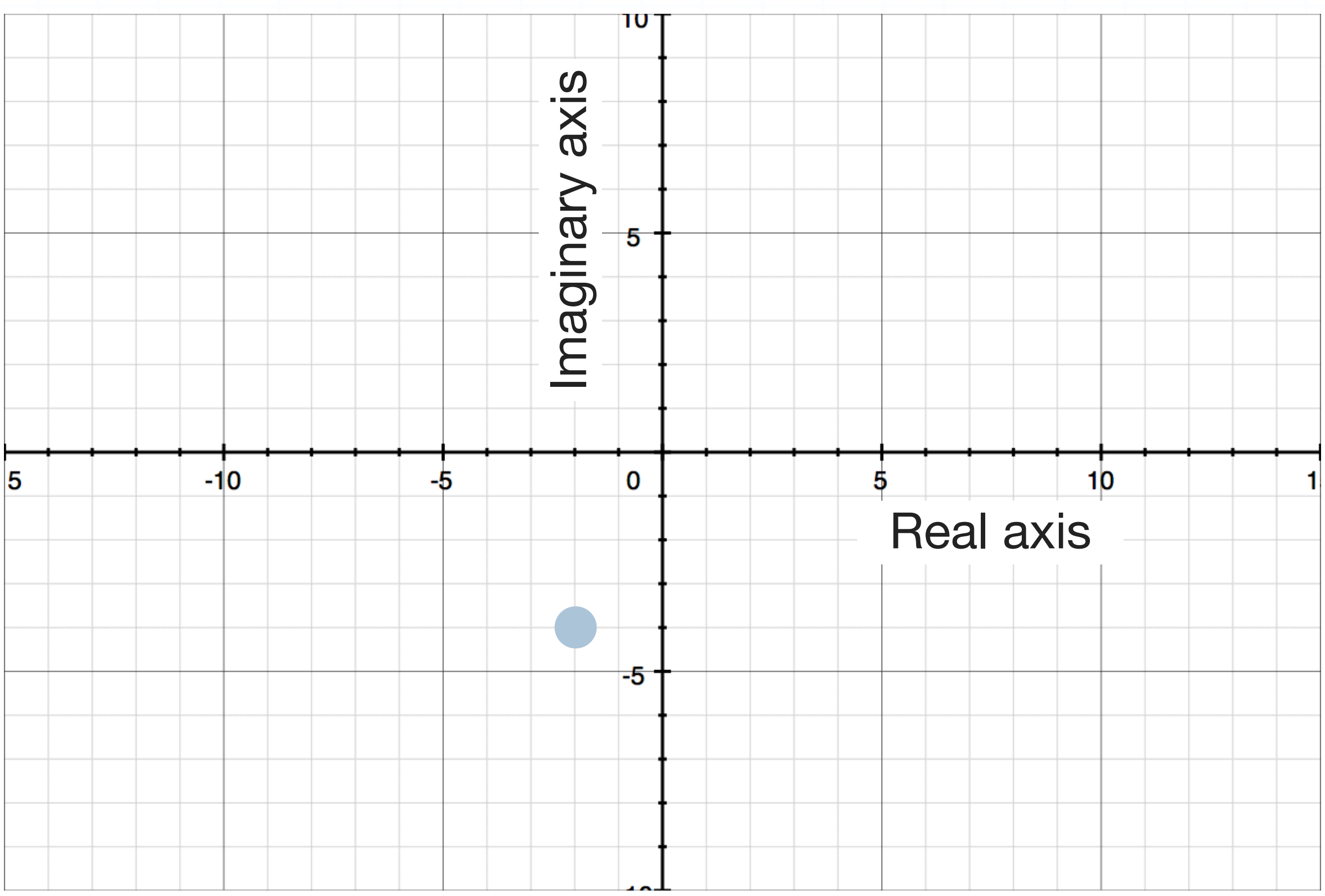
$$(-8+5i)-(-6+9i)$$

$$(-8-(-6))+(5-9)i$$

$$(-8+6)+(5-9)i$$

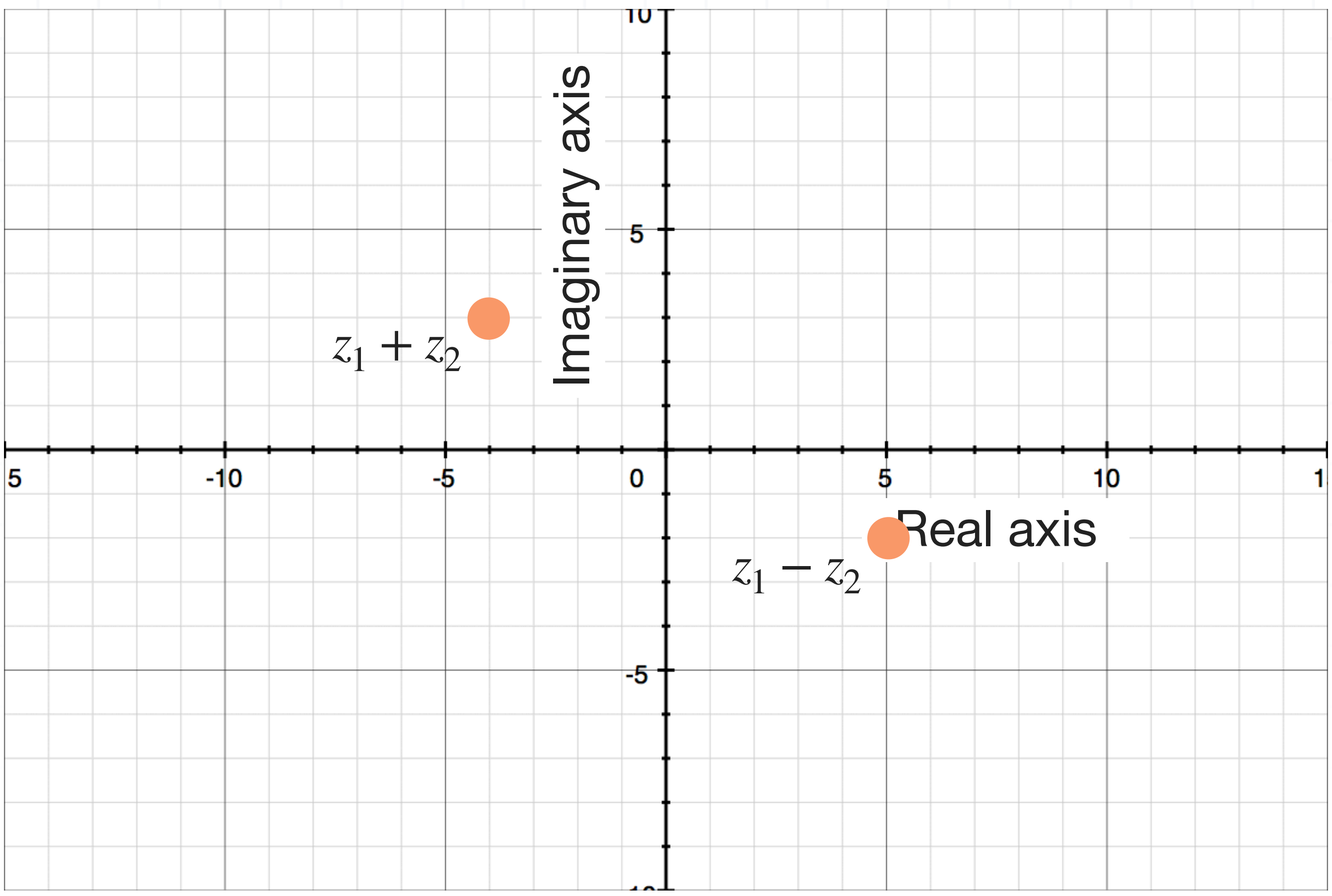
$$-2-4i$$

The real part of their difference is -2, and the imaginary part is -4. This means that the difference should be graphed 2 units to the left of the vertical axis and 4 units below the horizontal axis.



Topic: Graphing complex numbers

Question: The points on the graph are the sum $z_1 + z_2$ and difference $z_1 - z_2$ of two z_1 and z_2. Use a system of equations to find z_1 and z_2.



Answer choices:

A $z_1 = (5/2) - (1/2)i$ and $z_2 = (1/2) + (1/2)i$

B $z_1 = (3/2) + (7/2)i$ and $z_2 = -(5/2) - (3/2)i$

C $z_1 = (1/2) + (1/2)i$ and $z_2 = -(9/2) + (5/2)i$

D $z_1 = (7/2) - (3/2)i$ and $z_2 = (5/2) + (1/2)i$

Solution: C

The points on the graph are $z_1 + z_2 = -4 + 3i$ and $z_1 - z_2 = 5 - 2i$. We'll set up this system of equations:

$$z_1 + z_2 = -4 + 3i$$

$$z_1 - z_2 = 5 - 2i$$

Add the two equations together to eliminate z_2.

$$z_1 + z_2 = -4 + 3i$$

$$z_1 + z_2 + (z_1 - z_2) = -4 + 3i + (5 - 2i)$$

$$z_1 + z_2 + z_1 - z_2 = -4 + 3i + 5 - 2i$$

$$2z_1 = 1 + i$$

$$z_1 = \frac{1}{2} + \frac{1}{2}i$$

Substitute z_1 back into one of the other equations to find z_2.

$$z_1 + z_2 = -4 + 3i$$

$$\frac{1}{2} + \frac{1}{2}i + z_2 = -4 + 3i$$

$$z_2 = -4 + 3i - \frac{1}{2} - \frac{1}{2}i$$

$$z_2 = -\frac{9}{2} + \frac{5}{2}i$$

Topic: Distances and midpoints

Question: Use the distance formula to find the distance between the complex numbers $u = 3 + 4i$ and $z = 2 - 3i$.

Answer choices:

A $d = \sqrt{15}$

B $d = 5\sqrt{2}$

C $d = \sqrt{26}$

D $d = 4$

Solution: B

The distance formula is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The x-coordinates are the constants of the complex numbers and the y-coordinates are the coefficients of the imaginary numbers. Substitute the values into the distance formula and evaluate.

$$d = \sqrt{(3 - 2)^2 + (4 - (-3))^2}$$

$$d = \sqrt{(3 - 2)^2 + (4 + 3)^2}$$

$$d = \sqrt{1^2 + 7^2}$$

$$d = \sqrt{1 + 49}$$

$$d = \sqrt{50}$$

$$d = \sqrt{25 \cdot 2}$$

$$d = 5\sqrt{2}$$

Topic: Distances and midpoints

Question: Find the distance between the two complex numbers, $u = -3 - 3i$ and $z = -4 + 6i$, by graphing and using the Pythagorean theorem.

Answer choices:

A $c = \sqrt{10}$

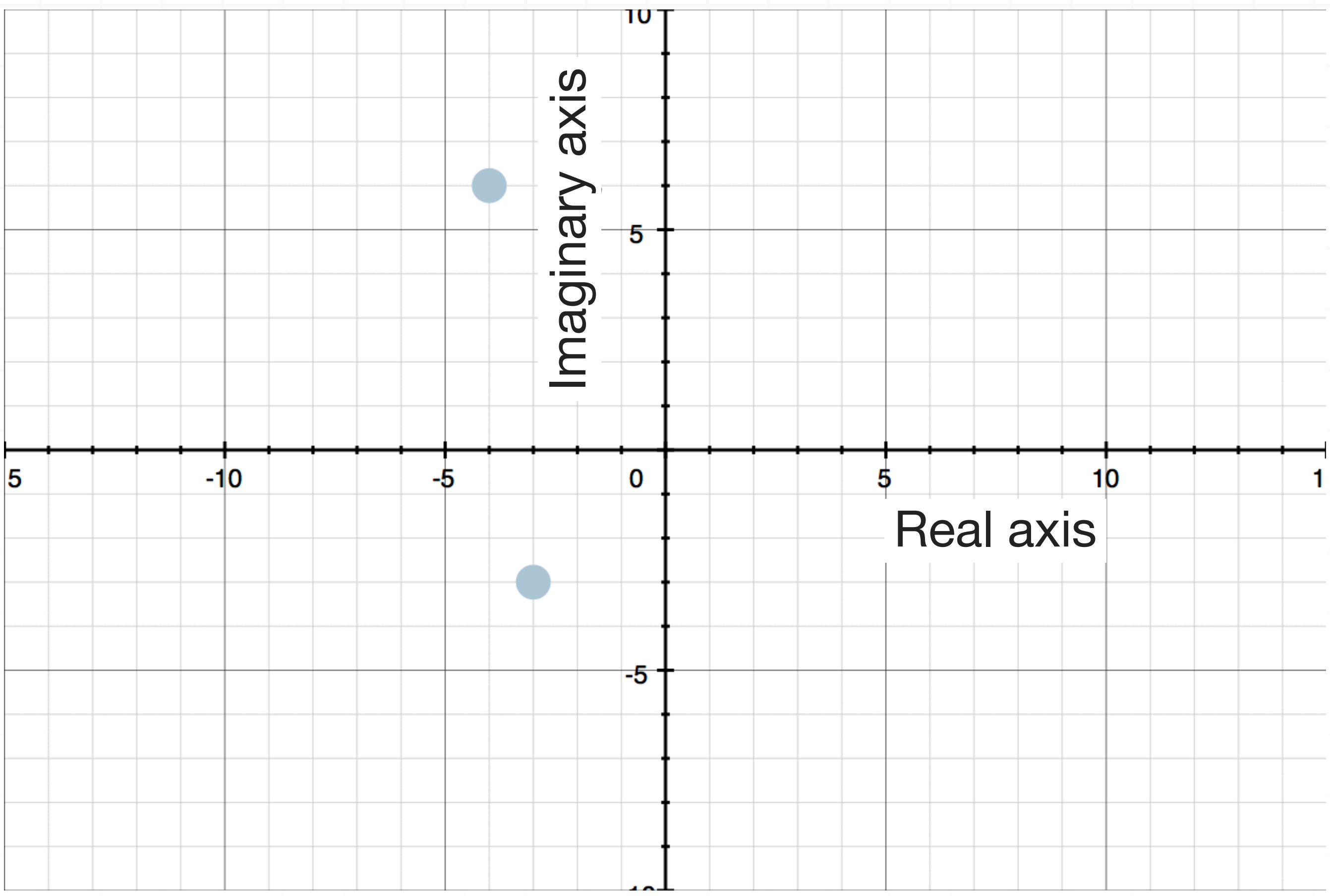
B $c = \sqrt{58}$

C $c = \sqrt{82}$

D $c = \sqrt{130}$

Solution: C

Graph $u = -3 - 3i$ and $z = -4 + 6i$ in the complex plane.



To find the distance between $u = -3 - 3i$ and $z = -4 + 6i$, start by finding the difference between the real parts and the imaginary parts.

The distance between the real parts is $-3 - (-4) = -3 + 4 = 1$, and the distance between the imaginary parts is $-3 - 6 = -9$. Then by the Pythagorean theorem, the distance between $u = -3 - 3i$ and $z = -4 + 6i$ is

$$1^2 + (-9)^2 = c^2$$

$$1 + 81 = c^2$$

$$82 = c^2$$

$$c = \sqrt{82}$$

Topic: Distances and midpoints

Question: Find the midpoint between $u = -3 - 3i$ and $z = -4 + 6i$.

Answer choices:

A $m = -3.5 + 1.5i$

B $m = -2 + i$

C $m = -1.5 + 1.5i$

D $m = -1 - 2i$

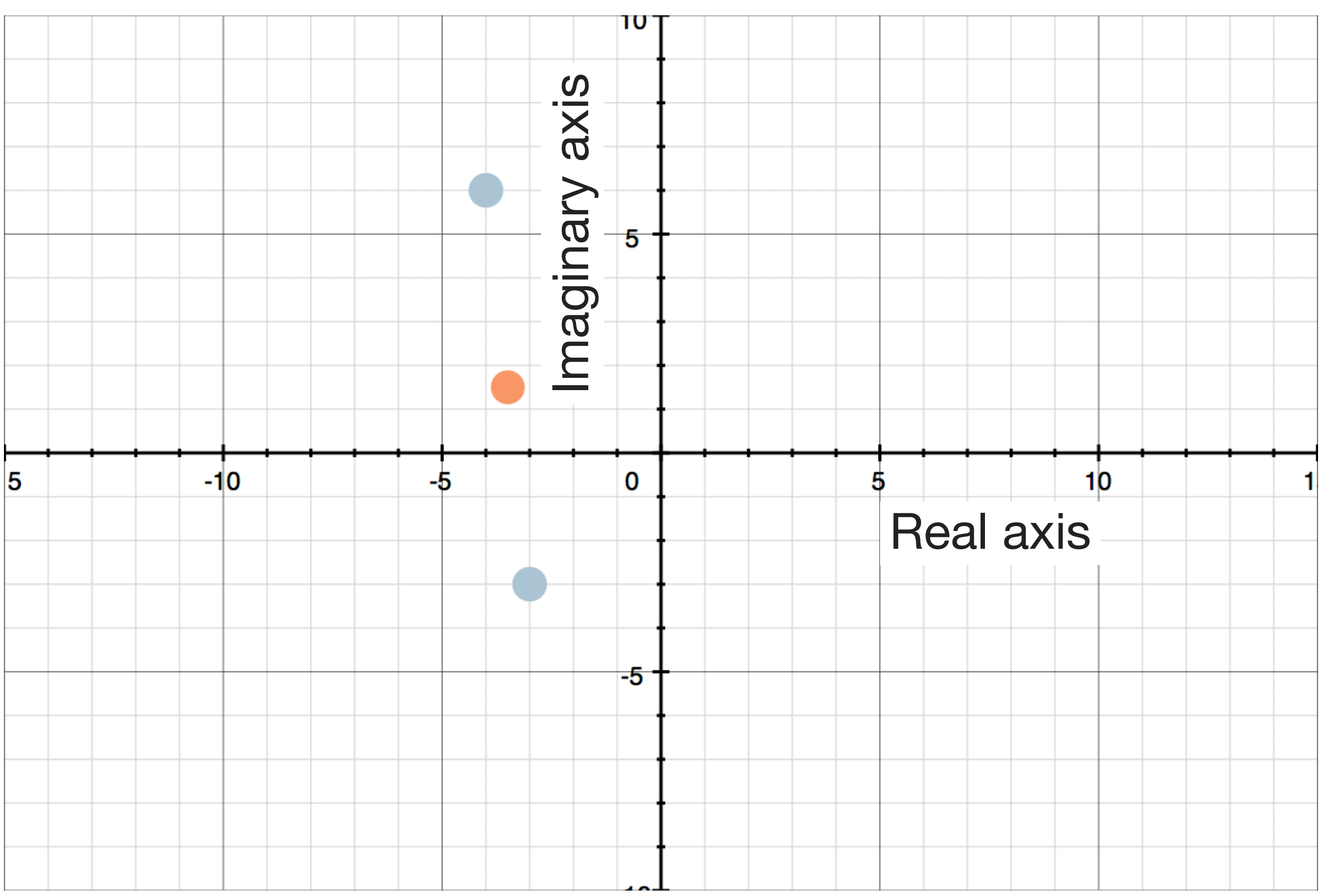
Solution: A

To find the midpoint between complex numbers, we find the midpoint of the real parts, and separately the midpoint of the imaginary parts.

The distance between the real parts of $u = -3 - 3i$ and $z = -4 + 6i$ is $-3 - (-4) = -3 + 4 = 1$. Half of that distance is $1/2 = 0.5$, so we look for the value that's 0.5 units from -3 and 0.5 units from -4, so the midpoint between those real parts must be -3.5.

The distance between the imaginary parts of $u = -3 - 3i$ and $z = -4 + 6i$ is $-3 - 6 = -9$. Half of that distance is $-9/2 = -4.5$, so we look for the value that's -4.5 units from -3 and -4.5 units from -6, so the midpoint between those imaginary parts must be 1.5.

So the midpoint between $u = -3 - 3i$ and $z = -4 + 6i$ is $m = -3.5 + 1.5i$. If we graph all three of these in the complex plane, we get



Topic: Complex numbers in polar form

Question: If the complex number $-3-7i$ is expressed in polar form, which quadrant contains the angle θ?

Answer choices:

A In the first quadrant

B On the negative vertical axis

C In the third quadrant

D On the positive horizontal axis

Solution: C

If we set the complex number equal to its polar form, we get

$$-3 - 7i = r(\cos\theta + i\sin\theta)$$

$$-3 - 7i = r\cos\theta + ri\sin\theta$$

From this equation, we know that

$$-3 = r\cos\theta$$

$$\cos\theta = -\frac{3}{r}$$

The value of r is always positive, since r represents a distance, so $-3/r$ has to be less than 0, which means $\cos\theta$ has to be negative.

We also know from $-3 - 7i = r\cos\theta + ri\sin\theta$ that

$$-7 = r\sin\theta$$

$$\sin\theta = -\frac{7}{r}$$

Because the value of r is always positive, $-7/r$ has to be less than 0, which means $\sin\theta$ has to be negative.

The values of $\cos\theta$ and $\sin\theta$ are negative in the third quadrant.

Topic: Complex numbers in polar form

Question: What is the polar form of the complex number?

$$-4+6i$$

Answer choices:

A $2\sqrt{5}\left[\cos(0.98)+i\sin(0.98)\right]$

B $2\sqrt{13}\left[\cos(2.16)+i\sin(2.16)\right]$

C $2\sqrt{13}\left[\cos(5.30)+i\sin(5.30)\right]$

D $2\sqrt{5}\left[\cos(3.14)+i\sin(3.14)\right]$

Solution: B

If we write the complex number $-4+6i$ as $a+bi$, we get $a=-4$ and $b=6$, so

$$r=\sqrt{a^2+b^2}=\sqrt{(-4)^2+6^2}=\sqrt{16+36}=\sqrt{52}=\sqrt{4(13)}=2\sqrt{13}$$

and

$$\tan\theta=\frac{b}{a}=\frac{6}{-4}=-\frac{3}{2}$$

Using the trigonometric identity $\sec^2\theta=1+\tan^2\theta$, we get

$$\sec^2\theta=1+\left(-\frac{3}{2}\right)^2=1+\frac{9}{4}=\frac{4(1)+1(9)}{4}=\frac{13}{4}$$

$$\cos^2\theta=(\cos\theta)^2=\left(\frac{1}{\sec\theta}\right)^2=\frac{1}{\sec^2\theta}=\frac{1}{\left(\frac{13}{4}\right)}=\frac{4}{13}$$

So

$$\cos\theta=\pm\sqrt{\frac{4}{13}}=\pm\frac{2}{\sqrt{13}}$$

The real part of z is $a=-4$, and the imaginary part is $b=6$, which puts the complex number in the second quadrant. Since the cosine of every angle in the second quadrant is negative, we get

$$\cos\theta=-\frac{2}{\sqrt{13}}$$

$$\arccos(\cos\theta) = \arccos\left(-\frac{2}{\sqrt{13}}\right)$$

$$\theta = \arccos\left(-\frac{2}{\sqrt{13}}\right)$$

$\theta \approx 2.16$ radians

Substituting the values of r and θ into the polar form for a complex number, we get

$$r(\cos\theta + i\sin\theta)$$

$$2\sqrt{13}\left[\cos(2.16) + i\sin(2.16)\right]$$

Topic: Complex numbers in polar form

Question: Write the complex number in polar form.

$$-14i$$

Answer choices:

A $\quad -14\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

B $\quad -14\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

C $\quad 14\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

D $\quad 14\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

Solution: D

The complex number $-14i$ can be written as $0 - 14i$, so its real part is 0, which means the number is located on the imaginary axis. Because $a = 0$ and $b = -14$, the distance of $0 - 14i$ from the origin is

$$r = \sqrt{a^2 + b^2} = \sqrt{0^2 + (-14)^2} = \sqrt{0 + 196} = \sqrt{196} = 14$$

Since the imaginary part of $0 - 14i$ is -14, which is negative, $0 - 14i$ is located on the negative imaginary axis, so $\theta = 3\pi/2$. In polar form, we get

$$r(\cos\theta + i\sin\theta)$$

$$14\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

Topic: Multiplying and dividing polar forms

Question: What is the product z_1z_2 of the complex numbers in polar form?

$$z_1 = \sqrt{2}\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right)$$

$$z_2 = \frac{7}{3\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Answer choices:

A $\quad \dfrac{7}{\sqrt{6}}\left(\cos\dfrac{21\pi}{20} + i\sin\dfrac{21\pi}{20}\right)$

B $\quad \dfrac{7}{3}\left(\cos\dfrac{13\pi}{10} + i\sin\dfrac{13\pi}{10}\right)$

C $\quad \dfrac{3\sqrt{2}}{7}\left(\cos\dfrac{23\pi}{20} + i\sin\dfrac{23\pi}{20}\right)$

D $\quad \dfrac{7}{3}\left(\cos\dfrac{3\pi}{20} + i\sin\dfrac{3\pi}{20}\right)$

Solution: D

Plug the complex numbers into the formula for the product of complex numbers.

$$z_1z_2 = r_1r_2\left[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\right]$$

$$z_1z_2 = \left(\sqrt{2}\cdot\frac{7}{3\sqrt{2}}\right)\left[\cos\left(\frac{7\pi}{5}+\frac{3\pi}{4}\right)+i\sin\left(\frac{7\pi}{5}+\frac{3\pi}{4}\right)\right]$$

Simplify.

$$z_1z_2 = \frac{7}{3}\left[\cos\left(\frac{28\pi}{20}+\frac{15\pi}{20}\right)+i\sin\left(\frac{28\pi}{20}+\frac{15\pi}{20}\right)\right]$$

$$z_1z_2 = \frac{7}{3}\left(\cos\frac{43\pi}{20}+i\sin\frac{43\pi}{20}\right)$$

You could leave the answer this way, but ideally we'd like to reduce the angle to one that's coterminal, but in the interval $[0,2\pi)$. If we subtract 2π from the angle, we get

$$z_1z_2 = \frac{7}{3}\left[\cos\left(\frac{43\pi}{20}-\frac{40\pi}{20}\right)+i\sin\left(\frac{43\pi}{20}-\frac{40\pi}{20}\right)\right]$$

$$z_1z_2 = \frac{7}{3}\left(\cos\frac{3\pi}{20}+i\sin\frac{3\pi}{20}\right)$$

Topic: Multiplying and dividing polar forms

Question: What is the quotient z_1/z_2 of the complex numbers in polar form?

$$z_1 = 16\left(\cos\frac{9\pi}{13} + i\sin\frac{9\pi}{13}\right)$$

$$z_2 = \frac{5}{\sqrt{3}}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

Answer choices:

A $\quad \dfrac{16\sqrt{3}}{5}\left(\cos\dfrac{41\pi}{78} + i\sin\dfrac{41\pi}{78}\right)$

B $\quad \dfrac{80}{\sqrt{3}}\left(\cos\dfrac{17\pi}{39} + i\sin\dfrac{17\pi}{39}\right)$

C $\quad \dfrac{16}{5\sqrt{3}}\left(\cos\dfrac{9\pi}{78} + i\sin\dfrac{9\pi}{78}\right)$

D $\quad \dfrac{16\sqrt{3}}{5}\left(\cos\dfrac{17\pi}{39} + i\sin\dfrac{17\pi}{39}\right)$

Solution: A

Plug the complex numbers into the formula for the quotient of complex numbers.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{16}{\frac{5}{\sqrt{3}}}\left[\cos\left(\frac{9\pi}{13} - \frac{\pi}{6}\right) + i\sin\left(\frac{9\pi}{13} - \frac{\pi}{6}\right)\right]$$

Simplify.

$$\frac{z_1}{z_2} = 16 \cdot \frac{\sqrt{3}}{5}\left[\cos\left(\frac{54\pi}{78} - \frac{13\pi}{78}\right) + i\sin\left(\frac{54\pi}{78} - \frac{13\pi}{78}\right)\right]$$

$$\frac{z_1}{z_2} = \frac{16\sqrt{3}}{5}\left(\cos\frac{41\pi}{78} + i\sin\frac{41\pi}{78}\right)$$

Topic: Multiplying and dividing polar forms

Question: Suppose that a complex number z is the quotient z_1/z_2 of the given complex numbers. If z is expressed in polar form, $r(\cos\theta + i\sin\theta)$, where is θ located?

$$z_1 = 4\left(\cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right)$$

$$z_2 = \frac{17}{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

Answer choices:

A In the first quadrant

B On the negative horizontal axis

C In the second quadrant

D On the positive vertical axis

Solution: C

Plug the complex numbers into the formula for the quotient of complex numbers.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{4}{\frac{17}{3}}\left[\cos\left(\frac{13\pi}{9} - \frac{5\pi}{6}\right) + i\sin\left(\frac{13\pi}{9} - \frac{5\pi}{6}\right)\right]$$

Simplify.

$$\frac{z_1}{z_2} = 4 \cdot \frac{3}{17}\left[\cos\left(\frac{78\pi}{54} - \frac{45\pi}{54}\right) + i\sin\left(\frac{78\pi}{54} - \frac{45\pi}{54}\right)\right]$$

$$\frac{z_1}{z_2} = \frac{12}{17}\left(\cos\frac{33\pi}{54} + i\sin\frac{33\pi}{54}\right)$$

The fraction $33/54$ is approximately equal to 0.6, so the angle is about 0.6π, which is in the second quadrant.

Topic: Powers of complex numbers and De Moivre's Theorem

Question: Find z^5 in rectangular form $a+bi$?

$$z = 3\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Answer choices:

A $\quad 75\sqrt{2} + 75\sqrt{2}i$

B $\quad 972 - 972i$

C $\quad 243\sqrt{2} + 243\sqrt{2}i$

D $\quad 75 - 225i$

Solution: B

Plug $r=3\sqrt{2}$, $\theta=3\pi/4$, and $n=5$ into De Moivre's theorem.

$$z^n=r^n\left[\cos(n\theta)+i\sin(n\theta)\right]$$

$$z^5=\left(3\sqrt{2}\right)^5\left[\cos\left(5\cdot\frac{3\pi}{4}\right)+i\sin\left(5\cdot\frac{3\pi}{4}\right)\right]$$

Then simplify.

$$z^5=3^5\left(\sqrt{2}\right)^5\left(\cos\frac{15\pi}{4}+i\sin\frac{15\pi}{4}\right)$$

$$z^5=243\left(4\sqrt{2}\right)\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)$$

$$z^5=243\left(2\sqrt{2}\right)\left(\sqrt{2}-\sqrt{2}i\right)$$

$$z^5=486\sqrt{2}\left(\sqrt{2}-\sqrt{2}i\right)$$

$$z^5=486\sqrt{2}\sqrt{2}-486\sqrt{2}\sqrt{2}i$$

$$z^5=486(2)-486(2)i$$

$$z^5=972-972i$$

Topic: Powers of complex numbers and De Moivre's Theorem

Question: Find z^4 in polar form?

$$z = \sqrt{3}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Answer choices:

A $\quad 6\left(\cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3}\right)$

B $\quad 18\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

C $\quad 9\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

D $\quad 3\left(\cos\frac{8\pi}{3} - i\sin\frac{8\pi}{3}\right)$

Solution: C

Plug $r = \sqrt{3}$, $\theta = 2\pi/3$, and $n = 4$ into De Moivre's theorem.

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^4 = \left(\sqrt{3}\right)^4 \left[\cos\left(4 \cdot \frac{2\pi}{3}\right) + i\sin\left(4 \cdot \frac{2\pi}{3}\right)\right]$$

Then simplify.

$$z^4 = 9\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right)$$

We could leave the answer this way, but the angle $8\pi/3$ is larger than 2π, so we can reduce the angle to one that's coterminal with $8\pi/3$, but within the interval $[0,2\pi)$.

$$\frac{8\pi}{3} - 2\pi = \frac{8\pi}{3} - 2\pi\left(\frac{3}{3}\right) = \frac{8\pi}{3} - \frac{6\pi}{3} = \frac{2\pi}{3}$$

So the complex number z^4 in polar form can be written as

$$z^4 = 9\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Topic: Powers of complex numbers and De Moivre's Theorem

Question: In the complex number z, a is a positive real number and k is a nonnegative integer. Where in the complex plane is z^6 located?

$$z = a\left[\cos\left(\frac{(2k+3)\pi}{2}\right) + i\sin\left(\frac{(2k+3)\pi}{2}\right)\right]$$

Answer choices:

A In the first quadrant

B On the positive vertical axis

C In the third quadrant

D On the negative horizontal axis

Solution: D

From the given complex number, we have

$$r = a$$

$$\theta = \frac{(2k+3)\pi}{2}$$

Because we're looking for z^6, we know $n = 6$, and we plug everything into De Moivre's theorem.

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^6 = a^6\left[\cos\left(6 \cdot \frac{(2k+3)\pi}{2}\right) + i\sin\left(6 \cdot \frac{2k+3)\pi}{2}\right)\right]$$

Then simplify.

$$z^6 = a^6\left[\cos\left(\frac{(12k+18)\pi}{2}\right) + i\sin\left(\frac{(12k+18)\pi}{2}\right)\right]$$

$$z^6 = a^6\left[\cos(6k+9)\pi + i\sin(6k+9)\pi\right]$$

If we test a couple of k-values, we get

For $k = 0$:

$$z^6 = a^6\left[\cos(6(0)+9)\pi + i\sin(6(0)+9)\pi\right]$$

$$z^6 = a^6(\cos 9\pi + i\sin 9\pi)$$

$$z^6 = a^6(-1 + 0i)$$

For $k = 1$:

$$z^6 = a^6 \left[\cos(6(1) + 9)\pi + i \sin(6(1) + 9)\pi\right]$$

$$z^6 = a^6(\cos 15\pi + i \sin 15\pi)$$

$$z^6 = a^6(-1 + 0i)$$

For $k = 2$:

$$z^6 = a^6 \left[\cos(6(2) + 9)\pi + i \sin(6(2) + 9)\pi\right]$$

$$z^6 = a^6(\cos 21\pi + i \sin 21\pi)$$

$$z^6 = a^6(-1 + 0i)$$

We could keep going, but we realize that we're getting the same value each time, which is $z^6 = a^6(-1 + 0i)$.

The problem told us that a is a positive real number, which means that when we distribute it across the $(-1 + 0i)$, we'll still have a negative real part and a zero imaginary part. In the case when the real part is negative, and the imaginary part is 0, that always put us somewhere on the negative half of the horizontal (real) axis.

Topic: Complex number equations

Question: Find the solution of the complex equation that lies in the third quadrant.

$$z^3 = 125$$

Answer choices:

A $\quad z = -\frac{5}{2} + \frac{5\sqrt{3}}{2}i$

B $\quad z = -\frac{5}{2} - \frac{5\sqrt{3}}{2}i$

C $\quad z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

D $\quad z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Solution: B

Rewrite z^3 as

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^3 = r^3\left[\cos(3\theta) + i\sin(3\theta)\right]$$

Rewrite 125 as the complex number $125 + 0i$. The modulus and angle of $125 + 0i$ are

$$r = \sqrt{125^2 + 0^2}$$

$$r = \sqrt{125^2}$$

$$r = 125$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{125} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 125\left[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)\right]$$

$$z = 125\left[\cos(2\pi k) + i\sin(2\pi k)\right]$$

We could also write this in degrees instead of radians as

$$z = 125\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Starting again with $z^3 = 125$, we can start making substitutions.

$$z^3 = 125$$

$$r^3 \left[\cos(3\theta) + i\sin(3\theta)\right] = 125$$

$$r^3 \left[\cos(3\theta) + i\sin(3\theta)\right] = 125 \left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get this system of equations:

$$r^3 = 125$$

$$3\theta = 360^\circ k$$

From these equations, we get

$r^3 = 125$, so $r = 5$

$3\theta = 360^\circ k$, so $\theta = 120^\circ k$

To $\theta = 120^\circ k$, if we plug in $k = 0,\ 1,\ 2,\ \ldots$, we get

For $k = 0$, $\theta = 120^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 120^\circ(1) = 120^\circ$

For $k = 2$, $\theta = 120^\circ(2) = 240^\circ$

...

We could keep going for $k = 3,\ 4,\ 5,\ 6,\ \ldots$, but $k = 3$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ,\ 120^\circ,\ 240^\circ$.

Plugging these three angles and $r = 5$ into the formula for polar form of a complex number, we'll get the solutions to $z^3 = 125$.

$$z_1 = 5\left[\cos(0^\circ) + i\sin(0^\circ)\right] = 5\left[1 + i(0)\right] = 5$$

$$z_2 = 5\left[\cos(120^\circ) + i\sin(120^\circ)\right] = 5\left[-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = -\frac{5}{2} + \frac{5\sqrt{3}}{2}i$$

$$z_3 = 5\left[\cos(240^\circ) + i\sin(240^\circ)\right] = 5\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = -\frac{5}{2} - \frac{5\sqrt{3}}{2}i$$

Topic: Complex number equations

Question: Find the solutions of the complex equation.

$$z^2 = 81$$

Answer choices:

A $z = 3$ and $z = -3$

B $z = 3i$ and $z = -3i$

C $z = 9$ and $z = -9$

D $z = 9i$ and $z = -9i$

Solution: C

Rewrite z^2 as

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^2 = r^2 \left[\cos(2\theta) + i\sin(2\theta)\right]$$

Rewrite 81 as the complex number $81 + 0i$. The modulus and angle of $81 + 0i$ are

$$r = \sqrt{81^2 + 0^2}$$

$$r = \sqrt{81^2}$$

$$r = 81$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{81} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 81\left[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)\right]$$

$$z = 81\left[\cos(2\pi k) + i\sin(2\pi k)\right]$$

We could also write this in degrees instead of radians as

$$z = 81\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Starting again with $z^2 = 81$, we can start making substitutions.

$$z^2 = 81$$

$$r^2\left[\cos(2\theta) + i\sin(2\theta)\right] = 81$$

$$r^2\left[\cos(2\theta) + i\sin(2\theta)\right] = 81\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get this system of equations:

$$r^2 = 81$$

$$2\theta = 360^\circ k$$

From these equations, we get

$r^2 = 81$, so $r = 9$

$2\theta = 360^\circ k$, so $\theta = 180^\circ k$

To $\theta = 180^\circ k$, if we plug in $k = 0,\ 1,\ \dots$, we get

For $k = 0$, $\theta = 180^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 180^\circ(1) = 180^\circ$

...

We could keep going for $k = 2,\ 3,\ 4,\ 5,\ \dots$, but $k = 2$ gives 360°, which is coterminal with the 0° value we already found for $k = 0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ,\ 180^\circ$.

Plugging these two angles and $r = 9$ into the formula for polar form of a complex number, we'll get the solutions to $z^2 = 81$.

$$z_1 = 9\left[\cos(0^\circ) + i\sin(0^\circ)\right] = 9\left[1 + i(0)\right] = 9$$

$$z_2 = 9\left[\cos(180^\circ) + i\sin(180^\circ)\right] = 9\left[-1 + i(0)\right] = -9$$

Topic: Complex number equations

Question: How many solutions of the complex equation lie in the fourth quadrant?

$$z^6 = 64$$

Answer choices:

A 1

B 2

C 3

D 4

Solution: A

Rewrite z^6 as

$$z^n = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$z^6 = r^6 \left[\cos(6\theta) + i\sin(6\theta)\right]$$

Rewrite 64 as the complex number $64 + 0i$. The modulus and angle of $64 + 0i$ are

$$r = \sqrt{64^2 + 0^2}$$

$$r = \sqrt{64^2}$$

$$r = 64$$

and

$$\theta = \arctan\frac{b}{a} = \arctan\frac{0}{64} = \arctan 0 = 0$$

This arctan equation is true at $\theta = 0$, but also at 2π, 4π, 6π, 8π, etc. So if we put this into polar form, we get

$$z = 64\left[\cos(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots) + i\sin(0,\ 2\pi,\ 4\pi,\ 8\pi,\ \dots)\right]$$

$$z = 64\left[\cos(2\pi k) + i\sin(2\pi k)\right]$$

We could also write this in degrees instead of radians as

$$z = 64\left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Starting again with $z^6 = 64$, we can start making substitutions.

$$z^6 = 64$$

$$r^6 \left[\cos(6\theta) + i\sin(6\theta)\right] = 64$$

$$r^6 \left[\cos(6\theta) + i\sin(6\theta)\right] = 64 \left[\cos(360^\circ k) + i\sin(360^\circ k)\right]$$

Because we've got the same polar form on both sides of the equation, we can equate associated values and get this system of equations:

$$r^6 = 64$$

$$6\theta = 360^\circ k$$

From these equations, we get

$r^6 = 64$, so $r = 2$

$6\theta = 360^\circ k$, so $\theta = 60^\circ k$

To $\theta = 60^\circ k$, if we plug in $k = 0, 1, 2, 3, 4, 5, \ldots$, we get

For $k = 0$, $\theta = 60^\circ(0) = 0^\circ$

For $k = 1$, $\theta = 60^\circ(1) = 60^\circ$

For $k = 2$, $\theta = 60^\circ(2) = 120^\circ$

For $k = 3$, $\theta = 60^\circ(3) = 180^\circ$

For $k = 4$, $\theta = 60^\circ(4) = 240^\circ$

For $k = 5$, $\theta = 60^\circ(5) = 300^\circ$

...

We could keep going for $k=6, 7, 8, 9, \dots$, but $k=6$ gives 360°, which is coterminal with the 0° value we already found for $k=0$, so we realize that we'll just be starting to repeat the same solutions. So the only solutions we need to consider are $\theta = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$.

Plugging these six angles and $r=2$ into the formula for polar form of a complex number, we'll get the solutions to $z^6=64$.

$$z_1 = 2\left[\cos(0^\circ) + i\sin(0^\circ)\right] = 2\left[1 + i(0)\right] = 2$$

$$z_2 = 2\left[\cos(60^\circ) + i\sin(60^\circ)\right] = 2\left[\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = 1 + \sqrt{3}i$$

$$z_3 = 2\left[\cos(120^\circ) + i\sin(120^\circ)\right] = 2\left[-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = -1 + \sqrt{3}i$$

$$z_4 = 2\left[\cos(180^\circ) + i\sin(180^\circ)\right] = 2\left[-1 + i(0)\right] = -2$$

$$z_5 = 2\left[\cos(240^\circ) + i\sin(240^\circ)\right] = 2\left[-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = -1 - \sqrt{3}i$$

$$z_6 = 2\left[\cos(300^\circ) + i\sin(300^\circ)\right] = 2\left[\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = 1 - \sqrt{3}i$$

Roots in the fourth quadrant will have a positive real part and a negative imaginary part. That's only z_6, so there's one solution in the fourth quadrant.

Topic: Roots of complex numbers

Question: Which of the following is a cube root of z?

$$z = 64\left(\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right)$$

Answer choices:

A $\quad 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

B $\quad 6\left(\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right)$

C $\quad 4\left(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right)$

D $\quad 6\left(\cos\frac{11\pi}{2} + i\sin\frac{11\pi}{2}\right)$

Solution: C

We're looking for the third (or cube) roots of z, which means there will be 3 of them, given by $k=0,\ 1,\ 2$. And since the complex number is given in radians, we'll plug $n=3$ into the formula for nth roots in radians.

$$\sqrt[n]{z}=\sqrt[n]{r}\left[\cos\left(\frac{\theta+2\pi k}{n}\right)+i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$$

$$\sqrt[3]{z}=\sqrt[3]{r}\left[\cos\left(\frac{\theta+2\pi k}{3}\right)+i\sin\left(\frac{\theta+2\pi k}{3}\right)\right]$$

With $r=64$ and $\theta=11\pi/8$ from the complex number, we get

$$\sqrt[3]{z}=\sqrt[3]{64}\left[\cos\left(\frac{\frac{11\pi}{8}+2\pi k}{3}\right)+i\sin\left(\frac{\frac{11\pi}{8}+2\pi k}{3}\right)\right]$$

Now we'll find values for $k=0,\ 1,\ 2$.

For $k=0$:

$$\sqrt[3]{z}_{k=0}=\sqrt[3]{64}\left[\cos\left(\frac{\frac{11\pi}{8}+2\pi(0)}{3}\right)+i\sin\left(\frac{\frac{11\pi}{8}+2\pi(0)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=0}=4\left(\cos\frac{11\pi}{24}+i\sin\frac{11\pi}{24}\right)$$

For $k=1$:

$$\sqrt[3]{z}_{k=1} = \sqrt[3]{64}\left[\cos\left(\frac{\frac{11\pi}{8}+2\pi(1)}{3}\right)+i\sin\left(\frac{\frac{11\pi}{8}+2\pi(1)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=1} = 4\left(\cos\frac{27\pi}{24}+i\sin\frac{27\pi}{24}\right)$$

$$\sqrt[3]{z}_{k=1} = 4\left(\cos\frac{9\pi}{8}+i\sin\frac{9\pi}{8}\right)$$

For $k=2$:

$$\sqrt[3]{z}_{k=2} = \sqrt[3]{64}\left[\cos\left(\frac{\frac{11\pi}{8}+2\pi(2)}{3}\right)+i\sin\left(\frac{\frac{11\pi}{8}+2\pi(2)}{3}\right)\right]$$

$$\sqrt[3]{z}_{k=2} = 4\left(\cos\frac{43\pi}{24}+i\sin\frac{43\pi}{24}\right)$$

The roots are

$$\sqrt[3]{z}_{k=0} = 4\left(\cos\frac{11\pi}{24}+i\sin\frac{11\pi}{24}\right)$$

$$\sqrt[3]{z}_{k=1} = 4\left(\cos\frac{9\pi}{8}+i\sin\frac{9\pi}{8}\right)$$

$$\sqrt[3]{z}_{k=2} = 4\left(\cos\frac{43\pi}{24}+i\sin\frac{43\pi}{24}\right)$$

The matching root is from $k=1$.

Topic: Roots of complex numbers

Question: How many of the seventh roots of z lie in the third quadrant of the complex plane?

$$z = 15\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$$

Answer choices:

A One

B Two

C Three

D None

Solution: B

We're looking for the seventh roots of z, which means there will be 7 of them, given by $k=0, 1, 2, 3, 4, 5, 6$. And since the complex number is given in radians, we'll plug $n=7$ into the formula for nth roots in radians.

$$\sqrt[n]{z}=\sqrt[n]{r}\left[\cos\left(\frac{\theta+2\pi k}{n}\right)+i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$$

$$\sqrt[7]{z}=\sqrt[7]{r}\left[\cos\left(\frac{\theta+2\pi k}{7}\right)+i\sin\left(\frac{\theta+2\pi k}{7}\right)\right]$$

With $r=15$ and $\theta=\pi/10$ from the complex number, we get

$$\sqrt[7]{z}=\sqrt[7]{15}\left[\cos\left(\frac{\frac{\pi}{10}+2\pi k}{7}\right)+i\sin\left(\frac{\frac{\pi}{10}+2\pi k}{7}\right)\right]$$

Now we'll find values for $k=0, 1, 2, 3, 4, 5, 6$.

For $k=0$:

$$\sqrt[7]{z}_{k=0}=\sqrt[7]{15}\left[\cos\left(\frac{\frac{\pi}{10}+2\pi(0)}{7}\right)+i\sin\left(\frac{\frac{\pi}{10}+2\pi(0)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=0}=\sqrt[7]{15}\left(\cos\frac{\pi}{70}+i\sin\frac{\pi}{70}\right)$$

For $k=1$:

$$\sqrt[7]{z}_{k=1}=\sqrt[7]{15}\left[\cos\left(\frac{\frac{\pi}{10}+2\pi(1)}{7}\right)+i\sin\left(\frac{\frac{\pi}{10}+2\pi(1)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=1}=\sqrt[7]{15}\left(\cos\frac{21\pi}{70}+i\sin\frac{21\pi}{70}\right)$$

For $k=2$:

$$\sqrt[7]{z}_{k=2}=\sqrt[7]{15}\left[\cos\left(\frac{\frac{\pi}{10}+2\pi(2)}{7}\right)+i\sin\left(\frac{\frac{\pi}{10}+2\pi(2)}{7}\right)\right]$$

$$\sqrt[7]{z}_{k=2}=\sqrt[7]{15}\left(\cos\frac{41\pi}{70}+i\sin\frac{41\pi}{70}\right)$$

We can start to see how we're just adding $20\pi/70$ to the angle each time we find a new k-value, so we can list the roots as

$$\sqrt[7]{z}_{k=0}=\sqrt[7]{15}\left(\cos\frac{\pi}{70}+i\sin\frac{\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=1}=\sqrt[7]{15}\left(\cos\frac{21\pi}{70}+i\sin\frac{21\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=2}=\sqrt[7]{15}\left(\cos\frac{41\pi}{70}+i\sin\frac{41\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=3}=\sqrt[7]{15}\left(\cos\frac{61\pi}{70}+i\sin\frac{61\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=4}=\sqrt[7]{15}\left(\cos\frac{81\pi}{70}+i\sin\frac{81\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=5}=\sqrt[7]{15}\left(\cos\frac{101\pi}{70}+i\sin\frac{101\pi}{70}\right)$$

$$\sqrt[7]{z}_{k=6} = \sqrt[7]{15}\left(\cos\frac{121\pi}{70} + i\sin\frac{121\pi}{70}\right)$$

If we find the decimal approximations of these angles, we get

For $k = 0$, $(1/70)\pi \approx 0.01\pi$

For $k = 1$, $(21/70)\pi \approx 0.3\pi$

For $k = 2$, $(41/70)\pi \approx 0.59\pi$

For $k = 3$, $(61/70)\pi \approx 0.87\pi$

For $k = 4$, $(81/70)\pi \approx 1.16\pi$

For $k = 5$, $(101/70)\pi \approx 1.44\pi$

For $k = 6$, $(121/70)\pi \approx 1.73\pi$

Anything in the third quadrant will fall in the interval $(1\pi, 1.5\pi)$, which in this case are the angles for $k = 4$ and $k = 5$, so two of the seventh roots fall in the third quadrant.

Topic: Roots of complex numbers

Question: Find the 4th root of the complex number that lies in the fourth quadrant of the complex plane.

$$z = 16\left(\cos 30^\circ + i\sin 30^\circ\right)$$

Answer choices:

A $\quad 2\left[\cos(277.5^\circ) + i\sin(277.5^\circ)\right]$

B $\quad 2\left[\cos(297.5^\circ) + i\sin(297.5^\circ)\right]$

C $\quad 2\left[\cos(317.5^\circ) + i\sin(317.5^\circ)\right]$

D $\quad 2\left[\cos(337.5^\circ) + i\sin(337.5^\circ)\right]$

Solution: A

We're looking for the 4th roots of z, which means there will be 4 of them, given by $k=0, 1, 2, 3$. And since the complex number is given in degrees, we'll plug $n=4$ into the formula for nth roots in degrees.

$$\sqrt[n]{z}=\sqrt[n]{r}\left[\cos\left(\frac{\theta+360^\circ k}{n}\right)+i\sin\left(\frac{\theta+360^\circ k}{n}\right)\right]$$

$$\sqrt[4]{z}=\sqrt[4]{r}\left[\cos\left(\frac{\theta+360^\circ k}{4}\right)+i\sin\left(\frac{\theta+360^\circ k}{4}\right)\right]$$

With $r=16$ and $\theta=30^\circ$ from the complex number, we get

$$\sqrt[4]{z}=\sqrt[4]{16}\left[\cos\left(\frac{30^\circ+360^\circ k}{4}\right)+i\sin\left(\frac{30^\circ+360^\circ k}{4}\right)\right]$$

Now we'll find values for $k=0, 1, 2, 3$.

For $k=0$:

$$\sqrt[4]{z}_{k=0}=\sqrt[4]{16}\left[\cos\left(\frac{30^\circ+360^\circ(0)}{4}\right)+i\sin\left(\frac{30^\circ+360^\circ(0)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=0}=2\left[\cos(7.5^\circ)+i\sin(7.5^\circ)\right]$$

For $k=1$:

$$\sqrt[4]{z}_{k=1}=\sqrt[4]{16}\left[\cos\left(\frac{30^\circ+360^\circ(1)}{4}\right)+i\sin\left(\frac{30^\circ+360^\circ(1)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=1} = 2\left[\cos(97.5^\circ) + i\sin(97.5^\circ)\right]$$

For $k = 2$:

$$\sqrt[4]{z}_{k=2} = \sqrt[4]{16}\left[\cos\left(\frac{30^\circ + 360^\circ(2)}{4}\right) + i\sin\left(\frac{30^\circ + 360^\circ(2)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=2} = 2\left[\cos(187.5^\circ) + i\sin(187.5^\circ)\right]$$

For $k = 3$:

$$\sqrt[4]{z}_{k=3} = \sqrt[4]{16}\left[\cos\left(\frac{30^\circ + 360^\circ(3)}{4}\right) + i\sin\left(\frac{30^\circ + 360^\circ(3)}{4}\right)\right]$$

$$\sqrt[4]{z}_{k=3} = 2\left[\cos(277.5^\circ) + i\sin(277.5^\circ)\right]$$

The roots are

$$\sqrt[4]{z}_{k=0} = 2\left[\cos(7.5^\circ) + i\sin(7.5^\circ)\right] \approx 1.982 + 0.262i$$

$$\sqrt[4]{z}_{k=1} = 2\left[\cos(97.5^\circ) + i\sin(97.5^\circ)\right] \approx -0.262 + 1.982i$$

$$\sqrt[4]{z}_{k=2} = 2\left[\cos(187.5^\circ) + i\sin(187.5^\circ)\right] \approx -1.982 - 0.262i$$

$$\sqrt[4]{z}_{k=3} = 2\left[\cos(277.5^\circ) + i\sin(277.5^\circ)\right] \approx 0.262 - 1.982i$$

The root in the fourth quadrant will have a positive real part and a negative imaginary part, which is the root for $k = 3$.

Topic: Matrix dimensions and entries

Question: Give the dimensions of the matrix.

$$K = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 2 & 5 & 6 & -2 \end{bmatrix}$$

Answer choices:

A The dimensions are 4×2

B The dimensions are 1×8

C The dimensions are 2×4

D The dimensions are 3×3

Solution: C

We always give the dimensions of a matrix as rows $\times$ columns. Matrix K has 2 rows and 4 columns, so K is a 2×4 matrix.

Topic: Matrix dimensions and entries

Question: Given matrix B, find $B_{2,1}$.

$$B = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$$

Answer choices:

A -1

B 0

C -1

D 3

Solution: B

The value of $B_{2,1}$ is the entry in the second row, first column of matrix B, which is 0, so $B_{2,1} = 0$.

Topic: Matrix dimensions and entries

Question: Give the dimensions of matrix M and find $M_{3,2}$.

$$M = \begin{bmatrix} 1 & 3 & 7 \\ 0 & -1 & 2 \\ 9 & 4 & 6 \end{bmatrix}$$

Answer choices:

A The dimensions are 3×3 and $M_{3,2} = 4$

B The dimensions are 2×3 and $M_{3,2} = 2$

C The dimensions are 3×3 and $M_{3,2} = 2$

D The dimensions are 3×1 and $M_{3,2} = 4$

Solution: A

We always give the dimensions of a matrix as rows $\times$ columns. Matrix M has 3 rows and 3 columns, so M is a 3×3 matrix.

The value of $M_{3,2}$ is the entry in the third row, second column of matrix M, which is 4, so $M_{3,2} = 4$.

Topic: Representing systems with matrices

Question: Represent the system with an augmented matrix called B.

$$4x + 2y = 8$$

$$-2x + 7y = 11$$

Answer choices:

A $\quad B = \left[\begin{array}{cc|c} 8 & 2 & 4 \\ 11 & -2 & 7 \end{array}\right]$

B $\quad B = \left[\begin{array}{cc|c} 4 & 2 & 8 \\ 7 & -2 & 11 \end{array}\right]$

C $\quad B = \left[\begin{array}{cc|c} 2 & 4 & 8 \\ -2 & 7 & 11 \end{array}\right]$

D $\quad B = \left[\begin{array}{cc|c} 4 & 2 & 8 \\ -2 & 7 & 11 \end{array}\right]$

Solution: D

As you're building out an augmented matrix, you want to be sure that you have all the variables in the same order, and all your constants grouped together on the same side of the equation. That way, with everything lined up, it'll be easy to make sure that each entry in a column represents the same variable or constant, and that each row in the matrix captures the entire equation.

This problem is straightforward because the system is set up correctly with all variables in both equations.

$$4x + 2y = 8$$

$$-2x + 7y = 11$$

The system contains the variables x and y along with a constant. Which means the augmented matrix will have two columns, one for each variable, plus a column for the constants, so three columns in total. Because there are two equations in the system, the matrix will have two rows.

Plugging the coefficients and constants into an augmented matrix gives

$$B = \begin{bmatrix} 4 & 2 & | & 8 \\ -2 & 7 & | & 11 \end{bmatrix}$$

Topic: Representing systems with matrices

Question: Represent the system with an augmented matrix called G.

$$a - 3b + 9c + 6d = 4$$

$$8a + 6c = 9d + 15$$

Answer choices:

A $G = \begin{bmatrix} 1 & -3 & 9 & 6 & 4 \\ 8 & 0 & 6 & -9 & 15 \end{bmatrix}$

B $G = \begin{bmatrix} 1 & 9 & 6 & 4 \\ 8 & 6 & -9 & 15 \end{bmatrix}$

C $G = \begin{bmatrix} 1 & 3 & 9 & 6 & 4 \\ 8 & 0 & 6 & 9 & 15 \end{bmatrix}$

D $G = \begin{bmatrix} 1 & -3 & 9 & 6 & 4 \\ 15 & 6 & 0 & 5 & 8 \end{bmatrix}$

Solution: A

As you're building out an augmented matrix, you want to be sure that you have all the variables in the same order, and all your constants grouped together on the same side of the equation. That way, with everything lined up, it'll be easy to make sure that each entry in a column represents the same variable or constant, and that each row in the matrix captures the entire equation.

To do this, the second equation can be reorganized by putting a, c, and d on the left side, and the constant on the right side. We also recognize that there is no b-term in the second equation, so we add in a 0 "filler" term.

$$a - 3b + 9c + 6d = 4$$

$$8a + 0b + 6c - 9d = 15$$

The system contains the variables a, b, c, and d, along with a constant. Which means the augmented matrix will have four columns, one for each variable, plus a column for the constants, so five columns in total. Because there are two equations in the system, the matrix will have two rows.

Plugging the coefficients and constants into an augmented matrix gives

$$G = \begin{bmatrix} 1 & -3 & 9 & 6 & 4 \\ 8 & 0 & 6 & -9 & 15 \end{bmatrix}$$

Topic: Representing systems with matrices

Question: Represent the system with an augmented matrix called N.

$$6a + 4b - c = 9$$

$$5b = -6a + 7c - 6$$

$$3c = 14 - 2a$$

Answer choices:

A $\quad N = \begin{bmatrix} 6 & 4 & -1 & 9 \\ 5 & -6 & 7 & -6 \\ 3 & 14 & -2 & 0 \end{bmatrix}$

B $\quad N = \begin{bmatrix} 6 & 4 & -1 & 9 \\ -6 & 5 & 7 & -6 \\ -2 & 3 & -14 & 0 \end{bmatrix}$

C $\quad N = \begin{bmatrix} 6 & 4 & -1 & 9 \\ 6 & 5 & -7 & -6 \\ 2 & 0 & 3 & 14 \end{bmatrix}$

D $\quad N = \begin{bmatrix} -2 & 3 & 0 & -14 \\ 6 & 4 & 1 & 9 \\ 6 & 5 & 7 & 6 \end{bmatrix}$

Solution: C

As you're building out an augmented matrix, you want to be sure that you have all the variables in the same order, and all your constants grouped together on the same side of the equation. That way, with everything lined up, it'll be easy to make sure that each entry in a column represents the same variable or constant, and that each row in the matrix captures the entire equation.

To do this, the second two equations can be reorganized by putting a, b, and c on the left side, and the constant on the right side. We also recognize that there is no b-term in the third equation, so we add in a 0 "filler" term.

$$6a + 4b - c = 9$$

$$6a + 5b - 7c = -6$$

$$2a + 0b + 3c = 14$$

The system contains the variables a, b, and c, along with a constant. Which means the augmented matrix will have three columns, one for each variable, plus a column for the constants, so four columns in total. Because there are three equations in the system, the matrix will have three rows. Plugging the coefficients and constants into an augmented matrix gives

$$N = \begin{bmatrix} 6 & 4 & -1 & 9 \\ 6 & 5 & -7 & -6 \\ 2 & 0 & 3 & 14 \end{bmatrix}$$

Topic: Simple row operations

Question: Write the new matrix after $R_1 \leftrightarrow R_2$.

$$\begin{bmatrix} 1 & -2 & 5 \\ 6 & 7 & 0 \\ 7 & 4 & 9 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 1 & -2 & 5 \\ 7 & 4 & 9 \\ 6 & 7 & 0 \end{bmatrix}$

B $\begin{bmatrix} 7 & 4 & 9 \\ 6 & 7 & 0 \\ 1 & -2 & 5 \end{bmatrix}$

C $\begin{bmatrix} 6 & 7 & 0 \\ 1 & -2 & 5 \\ 7 & 4 & 9 \end{bmatrix}$

D $\begin{bmatrix} 7 & 4 & 9 \\ 1 & -2 & 5 \\ 6 & 7 & 0 \end{bmatrix}$

Solution: C

The operation described by $R_1 \leftrightarrow R_2$ is switching row 1 with row 2. Nothing will happen to row 3. The matrix after $R_1 \leftrightarrow R_2$ is

$$\begin{bmatrix} 6 & 7 & 0 \\ 1 & -2 & 5 \\ 7 & 4 & 9 \end{bmatrix}$$

Topic: Simple row operations

Question: Write the new matrix after $2R_2 \leftrightarrow 4R_3$.

$$\begin{bmatrix} 6 & 1 & 5 & -8 \\ -2 & 3 & 7 & 9 \\ 5 & -2 & 0 & 1 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 6 & 1 & 5 & -8 \\ 20 & -8 & 0 & 4 \\ -4 & 6 & 14 & 18 \end{bmatrix}$

B $\begin{bmatrix} 6 & 1 & 5 & -8 \\ -4 & 6 & 14 & 18 \\ 5 & -2 & 0 & 1 \end{bmatrix}$

C $\begin{bmatrix} 6 & 1 & 5 & -8 \\ -2 & 3 & 7 & 9 \\ 20 & -8 & 0 & 4 \end{bmatrix}$

D $\begin{bmatrix} 6 & 1 & 5 & -8 \\ -4 & 6 & 14 & 18 \\ 20 & -8 & 0 & 4 \end{bmatrix}$

Solution: A

The operation described by $2R_2 \leftrightarrow 4R_3$ is multiplying row 2 by a constant of 2, multiplying row 3 by a constant of 4, and then switching those two rows. Nothing will happen to row 1. The matrix after $2R_2$ is

$$\begin{bmatrix} 6 & 1 & 5 & -8 \\ -4 & 6 & 14 & 18 \\ 5 & -2 & 0 & 1 \end{bmatrix}$$

The matrix after $4R_3$ is

$$\begin{bmatrix} 6 & 1 & 5 & -8 \\ -4 & 6 & 14 & 18 \\ 20 & -8 & 0 & 4 \end{bmatrix}$$

The matrix after $2R_2 \leftrightarrow 4R_3$ is

$$\begin{bmatrix} 6 & 1 & 5 & -8 \\ 20 & -8 & 0 & 4 \\ -4 & 6 & 14 & 18 \end{bmatrix}$$

Topic: Simple row operations

Question: Write the new matrix after $3R_1 + R_3 \to R_1$.

$$\begin{bmatrix} 7 & 8 & -2 & 0 \\ 5 & 1 & 6 & 13 \\ 4 & -7 & 3 & 9 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 21 & 24 & -6 & 0 \\ 5 & 1 & 6 & 13 \\ 4 & -7 & 3 & 9 \end{bmatrix}$

B $\begin{bmatrix} 7 & 8 & -2 & 0 \\ 5 & 1 & 6 & 13 \\ 25 & 17 & -3 & 9 \end{bmatrix}$

C $\begin{bmatrix} 7 & 8 & -2 & 0 \\ 25 & 17 & -3 & 9 \\ 4 & -7 & 3 & 9 \end{bmatrix}$

D $\begin{bmatrix} 25 & 17 & -3 & 9 \\ 5 & 1 & 6 & 13 \\ 4 & -7 & 3 & 9 \end{bmatrix}$

Solution: D

The operation described by $3R_1 + R_3 \to R_1$ is multiplying row 1 by a constant of 3, adding that resulting row to row 3, and using that result to replace row 1. The matrix after $3R_1$ is

$$\begin{bmatrix} 21 & 24 & -6 & 0 \\ 5 & 1 & 6 & 13 \\ 4 & -7 & 3 & 9 \end{bmatrix}$$

The sum $3R_1 + R_3$ is

$$\begin{bmatrix} 25 & 17 & -3 & 9 \end{bmatrix}$$

The matrix after $3R_1 + R_3 \to R_1$, which is replacing row 1 with this row we just found, is

$$\begin{bmatrix} 25 & 17 & -3 & 9 \\ 5 & 1 & 6 & 13 \\ 4 & -7 & 3 & 9 \end{bmatrix}$$

Topic: Gauss-Jordan elimination and reduced row-echelon form

Question: Use Gauss-Jordan elimination to solve the system.

$$x + 3y = 13$$

$$2x + 4y = 16$$

Answer choices:

A $x = 5,\ y = -2$

B $x = 3,\ y = -1$

C $x = -1,\ y = 3$

D $x = -2,\ y = 5$

Solution: D

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & = & 13 \\ 2 & 4 & = & 16 \end{bmatrix}$$

The first row already has a leading 1. After $2R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 3 & = & 13 \\ 0 & 2 & = & 10 \end{bmatrix}$$

The first column is done. After $(1/2)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 3 & = & 13 \\ 0 & 1 & = & 5 \end{bmatrix}$$

After $R_1 - 3R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & = & -2 \\ 0 & 1 & = & 5 \end{bmatrix}$$

The second column is done, and we get the solution set

$$x = -2$$

$$y = 5$$

Topic: Gauss-Jordan elimination and reduced row-echelon form

Question: Use Gauss-Jordan elimination to solve the system.

$$x + 4z = 11$$

$$x - y + 4z = 6$$

$$2x + 9z = 25$$

Answer choices:

A $x = -1$, $y = 5$, $z = 3$

B $x = 11$, $y = 6$, $z = 25$

C $x = 1$, $y = 0$, $z = 12$

D $x = -3$, $y = 8$, $z = 3$

Solution: A

The augmented matrix is

$$\begin{bmatrix} 1 & 0 & 4 & = & 11 \\ 1 & -1 & 4 & = & 6 \\ 2 & 0 & 9 & = & 25 \end{bmatrix}$$

The first row already has a leading 1. After $R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 4 & = & 11 \\ 0 & 1 & 0 & = & 5 \\ 2 & 0 & 9 & = & 25 \end{bmatrix}$$

After $2R_1 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 4 & = & 11 \\ 0 & 1 & 0 & = & 5 \\ 0 & 0 & -1 & = & -3 \end{bmatrix}$$

The first and second columns are done. After $(-1)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 4 & = & 11 \\ 0 & 1 & 0 & = & 5 \\ 0 & 0 & 1 & = & 3 \end{bmatrix}$$

After $R_1 - 4R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & -1 \\ 0 & 1 & 0 & = & 5 \\ 0 & 0 & 1 & = & 3 \end{bmatrix}$$

The third column is done, and we get the solution set

$x = -1$

$y = 5$

$z = 3$

Topic: Gauss-Jordan elimination and reduced row-echelon form

Question: Use Gauss-Jordan elimination to solve the system.

$$2x + 4y + 10z = 30$$

$$x + y + 3z = 10$$

$$2x + y + 2z = 9$$

Answer choices:

A $x = 7,\ y = -3,\ z = 5$

B $x = -4,\ y = 1,\ z = 0$

C $x = 2,\ y = -1,\ z = 3$

D $x = 30,\ y = 10,\ z = 9$

Solution: C

The augmented matrix is

$$\begin{bmatrix} 2 & 4 & 10 & = & 30 \\ 1 & 1 & 3 & = & 10 \\ 2 & 1 & 2 & = & 9 \end{bmatrix}$$

After $(1/2)R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 2 & 5 & = & 15 \\ 1 & 1 & 3 & = & 10 \\ 2 & 1 & 2 & = & 9 \end{bmatrix}$$

After $R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & 5 & = & 15 \\ 0 & 1 & 2 & = & 5 \\ 2 & 1 & 2 & = & 9 \end{bmatrix}$$

After $2R_1 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 2 & 5 & = & 15 \\ 0 & 1 & 2 & = & 5 \\ 0 & 3 & 8 & = & 21 \end{bmatrix}$$

The first column is done. After $R_1 - 2R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 1 & = & 5 \\ 0 & 1 & 2 & = & 5 \\ 0 & 3 & 8 & = & 21 \end{bmatrix}$$

After $R_3 - 3R_2 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 1 & = & 5 \\ 0 & 1 & 2 & = & 5 \\ 0 & 0 & 2 & = & 6 \end{bmatrix}$$

The second column is done. After $(1/2)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 1 & = & 5 \\ 0 & 1 & 2 & = & 5 \\ 0 & 0 & 1 & = & 3 \end{bmatrix}$$

After $R_1 - R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & 2 \\ 0 & 1 & 2 & = & 5 \\ 0 & 0 & 1 & = & 3 \end{bmatrix}$$

After $R_2 - 2R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & = & 2 \\ 0 & 1 & 0 & = & -1 \\ 0 & 0 & 1 & = & 3 \end{bmatrix}$$

The third column is done, and we get the solution set

$$x = 2$$

$$y = -1$$

$$z = 3$$

Topic: Matrix addition and subtraction

Question: Add the matrices.

$$\begin{bmatrix} 4 & -3 & 6 \\ 8 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 1 \\ 11 & 4 & -9 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 4 & -3 & 6 \\ 19 & 6 & -8 \end{bmatrix}$

B $\begin{bmatrix} 7 & -3 & 7 \\ 19 & 6 & -8 \end{bmatrix}$

C $\begin{bmatrix} 7 & -3 & 7 \\ 8 & 2 & 1 \end{bmatrix}$

D $\begin{bmatrix} 7 & 3 & 7 \\ 19 & 6 & 8 \end{bmatrix}$

Solution: B

To add matrices, you simply add together entries from corresponding positions in each matrix.

$$\begin{bmatrix} 4 & -3 & 6 \\ 8 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 1 \\ 11 & 4 & -9 \end{bmatrix}$$

$$\begin{bmatrix} 4+3 & -3+0 & 6+1 \\ 8+11 & 2+4 & 1+(-9) \end{bmatrix}$$

$$\begin{bmatrix} 7 & -3 & 7 \\ 19 & 6 & -8 \end{bmatrix}$$

Topic: Matrix addition and subtraction

Question: Subtract the matrices.

$$\begin{bmatrix} 8 & 1 & 3 \\ 6 & -4 & 5 \\ 0 & 1 & 9 \end{bmatrix} - \begin{bmatrix} 6 & 12 & 5 \\ 5 & 1 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 14 & 13 & 8 \\ 1 & 5 & 5 \\ 2 & 6 & 7 \end{bmatrix}$

B $\begin{bmatrix} -2 & 11 & 2 \\ -1 & 5 & -5 \\ -2 & 6 & -7 \end{bmatrix}$

C $\begin{bmatrix} 14 & 13 & 8 \\ 11 & 7 & 5 \\ -2 & 8 & 11 \end{bmatrix}$

D $\begin{bmatrix} 2 & -11 & -2 \\ 1 & -5 & 5 \\ 2 & -6 & 7 \end{bmatrix}$

Solution: D

To subtract matrices, you simply subtract entries from corresponding positions in each matrix.

$$\begin{bmatrix} 8 & 1 & 3 \\ 6 & -4 & 5 \\ 0 & 1 & 9 \end{bmatrix} - \begin{bmatrix} 6 & 12 & 5 \\ 5 & 1 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 8-6 & 1-12 & 3-5 \\ 6-5 & -4-1 & 5-0 \\ 0-(-2) & 1-7 & 9-2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -11 & -2 \\ 1 & -5 & 5 \\ 2 & -6 & 7 \end{bmatrix}$$

Topic: Matrix addition and subtraction

Question: Solve for x.

$$\begin{bmatrix} 8 & 2 \\ 7 & 9 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = x + \begin{bmatrix} 5 & 7 \\ -5 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 6 & -4 \end{bmatrix}$$

Answer choices:

A $\quad x = \begin{bmatrix} 13 & 6 \\ 5 & 13 \end{bmatrix}$

B $\quad x = \begin{bmatrix} -13 & -6 \\ -5 & -13 \end{bmatrix}$

C $\quad x = \begin{bmatrix} -1 & -8 \\ 3 & 3 \end{bmatrix}$

D $\quad x = \begin{bmatrix} 1 & 8 \\ -3 & -3 \end{bmatrix}$

Solution: C

Let's start with the matrix subtraction on the left side of the equation and the matrix addition on the right side of the equation.

$$\begin{bmatrix} 8 & 2 \\ 7 & 9 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = x + \begin{bmatrix} 5 & 7 \\ -5 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 6 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 8-2 & 2-3 \\ 7-3 & 9-1 \end{bmatrix} = x + \begin{bmatrix} 5+2 & 7+0 \\ -5+6 & 9+(-4) \end{bmatrix}$$

$$\begin{bmatrix} 6 & -1 \\ 4 & 8 \end{bmatrix} = x + \begin{bmatrix} 7 & 7 \\ 1 & 5 \end{bmatrix}$$

To isolate x, we'll subtract the matrix on the right from both sides in order to move it to the left.

$$\begin{bmatrix} 6 & -1 \\ 4 & 8 \end{bmatrix} - \begin{bmatrix} 7 & 7 \\ 1 & 5 \end{bmatrix} = x$$

$$\begin{bmatrix} 6-7 & -1-7 \\ 4-1 & 8-5 \end{bmatrix} = x$$

$$\begin{bmatrix} -1 & -8 \\ 3 & 3 \end{bmatrix} = x$$

The conclusion is that the value of x that makes the equation true is this matrix:

$$x = \begin{bmatrix} -1 & -8 \\ 3 & 3 \end{bmatrix}$$

Topic: Scalar multiplication and zero matrices

Question: Use scalar multiplication to simplify the expression.

$$4\begin{bmatrix} 5 & 2 & 1 \\ -2 & 4 & 7 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 9 & 6 & 5 \\ 2 & 8 & 11 \end{bmatrix}$

B $\begin{bmatrix} 20 & 8 & 4 \\ -2 & 4 & 7 \end{bmatrix}$

C $\begin{bmatrix} 5 & 2 & 1 \\ -8 & 16 & 28 \end{bmatrix}$

D $\begin{bmatrix} 20 & 8 & 4 \\ -8 & 16 & 28 \end{bmatrix}$

Solution: D

In this problem, 4 is the scalar. We distribute the scalar across every entry in the matrix, and the result of the scalar multiplication is

$$4\begin{bmatrix}5 & 2 & 1\\ -2 & 4 & 7\end{bmatrix}$$

$$\begin{bmatrix}4(5) & 4(2) & 4(1)\\ 4(-2) & 4(4) & 4(7)\end{bmatrix}$$

$$\begin{bmatrix}20 & 8 & 4\\ -8 & 16 & 28\end{bmatrix}$$

Topic: Scalar multiplication and zero matrices

Question: Solve for x.

$$3\begin{bmatrix} 7 & 1 \\ 8 & 3 \end{bmatrix} + x = -4\begin{bmatrix} 0 & -5 \\ -2 & 3 \end{bmatrix}$$

Answer choices:

A $\quad x = \begin{bmatrix} -21 & 17 \\ -16 & -21 \end{bmatrix}$

B $\quad x = \begin{bmatrix} 21 & 23 \\ 32 & -3 \end{bmatrix}$

C $\quad x = \begin{bmatrix} 21 & -17 \\ 16 & 21 \end{bmatrix}$

D $\quad x = \begin{bmatrix} -21 & -23 \\ -32 & 3 \end{bmatrix}$

Solution: A

Apply the scalars to the matrices.

$$\begin{bmatrix}3(7) & 3(1)\\ 3(8) & 3(3)\end{bmatrix}+x=\begin{bmatrix}-4(0) & -4(-5)\\ -4(-2) & -4(3)\end{bmatrix}$$

$$\begin{bmatrix}21 & 3\\ 24 & 9\end{bmatrix}+x=\begin{bmatrix}0 & 20\\ 8 & -12\end{bmatrix}$$

Subtract the matrix on the left from both sides of the equation in order to isolate x.

$$x=\begin{bmatrix}0 & 20\\ 8 & -12\end{bmatrix}-\begin{bmatrix}21 & 3\\ 24 & 9\end{bmatrix}$$

$$x=\begin{bmatrix}0-21 & 20-3\\ 8-24 & -12-9\end{bmatrix}$$

$$x=\begin{bmatrix}-21 & 17\\ -16 & -21\end{bmatrix}$$

Topic: Scalar multiplication and zero matrices

Question: Choose the O_{4x2} matrix.

Answer choices:

A $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

B $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

C $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

D $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution: B

We always name the zero matrix with a capital O. And optionally, you can add a subscript with the dimensions of the zero matrix. Since the values in a zero matrix are all zeros, just having the dimensions of the zero matrix tells you what the entire matrix looks like. So O_{4x2} is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Topic: Matrix multiplication

Question: Find the product of matrices A and B.

$$A = \begin{bmatrix} 5 & 2 \\ 0 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 9 & 1 \\ 6 & -1 \end{bmatrix}$$

Answer choices:

A $\quad A \cdot B = \begin{bmatrix} 43 & 3 \\ -5 & 6 \end{bmatrix}$

B $\quad A \cdot B = \begin{bmatrix} 0 & -14 \\ 23 & 4 \end{bmatrix}$

C $\quad A \cdot B = \begin{bmatrix} 57 & 3 \\ -12 & 2 \end{bmatrix}$

D $\quad A \cdot B = \begin{bmatrix} 9 & 6 \\ 23 & -8 \end{bmatrix}$

Solution: C

Multiply matrix A by matrix B.

$$A \cdot B = \begin{bmatrix} 5 & 2 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} 9 & 1 \\ 6 & -1 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 5(9) + 2(6) & 5(1) + 2(-1) \\ 0(9) + (-2)(6) & 0(1) + (-2)(-1) \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 57 & 3 \\ -12 & 2 \end{bmatrix}$$

Topic: Matrix multiplication

Question: Find the product of matrices A and B.

$$A = \begin{bmatrix} 7 & 2 & -4 \\ -5 & 10 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 7 & 1 \\ 7 & 2 \\ -2 & 6 \end{bmatrix}$$

Answer choices:

A $\quad A \cdot B = \begin{bmatrix} 71 & -13 \\ 29 & 33 \end{bmatrix}$

B $\quad A \cdot B = \begin{bmatrix} 45 & -30 \\ -16 & 52 \end{bmatrix}$

C $\quad A \cdot B = \begin{bmatrix} -41 & 56 \\ 29 & -16 \end{bmatrix}$

D $\quad A \cdot B = \begin{bmatrix} 43 & 33 \\ 82 & 19 \end{bmatrix}$

Solution: A

Multiply matrix A by matrix B.

$$A \cdot B = \begin{bmatrix} 7 & 2 & -4 \\ -5 & 10 & 3 \end{bmatrix} \cdot \begin{bmatrix} 7 & 1 \\ 7 & 2 \\ -2 & 6 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 7(7) + 2(7) + (-4)(-2) & 7(1) + 2(2) + (-4)(6) \\ (-5)(7) + 10(7) + 3(-2) & (-5)(1) + 10(2) + 3(6) \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 71 & -13 \\ 29 & 33 \end{bmatrix}$$

Topic: Matrix multiplication

Question: Use the distributive property to find $A(B+C)$.

$$A=\begin{bmatrix}3 & -1\\ 1 & 4\end{bmatrix}$$

$$B=\begin{bmatrix}5 & 2\\ -2 & 3\end{bmatrix}$$

$$C=\begin{bmatrix}2 & 0\\ 6 & 2\end{bmatrix}$$

Answer choices:

A $\quad A(B+C)=\begin{bmatrix}-2 & 15\\ 3 & 32\end{bmatrix}$

B $\quad A(B+C)=\begin{bmatrix}17 & 1\\ 23 & 22\end{bmatrix}$

C $\quad A(B+C)=\begin{bmatrix}3 & -14\\ 27 & 1\end{bmatrix}$

D $\quad A(B+C)=\begin{bmatrix}8 & 9\\ -14 & 17\end{bmatrix}$

Solution: B

Applying the distributive property to the initial expression, we get

$$A(B+C)=AB+AC$$

Now use matrix multiplication.

$$AB+AC=\begin{bmatrix}3 & -1\\ 1 & 4\end{bmatrix}\cdot\begin{bmatrix}5 & 2\\ -2 & 3\end{bmatrix}+\begin{bmatrix}3 & -1\\ 1 & 4\end{bmatrix}\begin{bmatrix}2 & 0\\ 6 & 2\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}3(5)+(-1)(-2) & 3(2)+(-1)(3)\\ 1(5)+4(-2) & 1(2)+4(3)\end{bmatrix}+\begin{bmatrix}3(2)+(-1)(6) & 3(0)+(-1)(2)\\ 1(2)+4(6) & 1(0)+4(2)\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}17 & 3\\ -3 & 14\end{bmatrix}+\begin{bmatrix}0 & -2\\ 26 & 8\end{bmatrix}$$

Adding the matrices gives

$$AB+AC=\begin{bmatrix}17+0 & 3+(-2)\\ -3+26 & 14+8\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}17 & 1\\ 23 & 22\end{bmatrix}$$

So the value of the original expression is

$$A(B+C)=\begin{bmatrix}17 & 1\\ 23 & 22\end{bmatrix}$$

Topic: Identity matrices

Question: Which identity matrix is I_3?

Answer choices:

A $I_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B $I_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

C $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

D $I_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Solution: C

We always call the identity matrix I, and it's always a square matrix, like 2×2, 3×3, 4×4, etc. For that reason, it's common to abbreviate I_{2x2} as just I_2, or I_{3x3} as just I_3, etc. So, I_3 is the 3×3 identity matrix.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Topic: Identity matrices

Question: Which identity matrix can be multiplied by A (in other words, $I \cdot A$), if A is a 2×4 matrix?

Answer choices:

A $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B $\quad I_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

C $\quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

D $\quad I_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Solution: A

Start by setting up the equation $I \cdot A = A$. Next, substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 2 \times 4 = 2 \times 4$$

Break down the dimensions of the identity matrix as rows $\times$ columns.

$$R \times C \cdot 2 \times 4 = 2 \times 4$$

In order to be able to multiply matrices, we need the same number of columns in the first matrix as we have rows in the second matrix. So the identity matrix must have 2 columns.

$$R \times \boxed{2 \cdot 2} \times 4 = 2 \times 4$$

And the dimensions of the resulting matrix come from the rows of the first matrix and the columns of the second matrix. So the identity matrix must have 2 rows.

$$\boxed{2} \times 2 \cdot 2 \times 4 = \boxed{2} \times 4$$

Therefore, the identity matrix we need is I_2.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Topic: Identity matrices

Question: If we want to find IA, which identity matrix should we use, and what is the product?

$$A = \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$$

Answer choices:

A Use $I_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and the product is $IA = \begin{bmatrix} 2 & 8 & 4 \\ -3 & 7 & 1 \end{bmatrix}$

B Use $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and the product is $IA = \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$

C Use $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and the product is $IA = \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$

D Use $I_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and the product is $IA = \begin{bmatrix} 2 & 8 & 4 \\ -3 & 7 & 1 \end{bmatrix}$

Solution: B

A is a 2×3 matrix, and we need to find IA. We also know that IA will be 2×3. So we'll set up an equation of dimensions.

$$I \cdot A = A$$

$$I \cdot 2 \times 3 = 2 \times 3$$

$$R \times C \cdot 2 \times 3 = 2 \times 3$$

For matrix multiplication to be valid, we need the same number of columns in the first matrix as we have rows in the second matrix.

$$R \times \boxed{2 \cdot 2} \times 3 = 2 \times 3$$

The dimensions of the result are given by the rows from the first matrix, and columns from the second matrix.

$$\boxed{2} \times 2 \cdot 2 \times 3 = \boxed{2} \times 3$$

So the identity matrix is 2×2, which means it's I_2.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then the product of I_2 and matrix A is

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1(-3) + 0(2) & 1(7) + 0(8) & 1(1) + 0(4) \\ 0(-3) + 1(2) & 0(7) + 1(8) & 0(1) + 1(4) \end{bmatrix}$$

$$IA = \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$$

Topic: Transformations

Question: Find the resulting vector $\vec{b}$ after $\vec{a}=(-4,2)$ undergoes a transformation by matrix Q.

$$Q=\begin{bmatrix}11 & 1\\ 0 & -6\end{bmatrix}$$

Answer choices:

A $\quad \vec{b}=\begin{bmatrix}42\\ -12\end{bmatrix}$

B $\quad \vec{b}=\begin{bmatrix}-42\\ 12\end{bmatrix}$

C $\quad \vec{b}=\begin{bmatrix}-42\\ -12\end{bmatrix}$

D $\quad \vec{b}=\begin{bmatrix}42\\ 12\end{bmatrix}$

Solution: C

To apply a transformation matrix to vector $\overrightarrow{a}$, we'll multiply the matrix by the vector.

$$\overrightarrow{b} = M\overrightarrow{a} = \begin{bmatrix} 11 & 1 \\ 0 & -6 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$\overrightarrow{b} = M\overrightarrow{a} = \begin{bmatrix} 11(-4) + 1(2) \\ 0(-4) - 6(2) \end{bmatrix}$$

$$\overrightarrow{b} = M\overrightarrow{a} = \begin{bmatrix} -44 + 2 \\ 0 - 12 \end{bmatrix}$$

$$\overrightarrow{b} = M\overrightarrow{a} = \begin{bmatrix} -42 \\ -12 \end{bmatrix}$$

Topic: Transformations

Question: What are the vertices of the transformation of the polygon with vertices $(-2,1)$, $(1,3)$, $(2,-2)$, and $(-3,-1)$ after it's transformed by matrix P.

$$P = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$$

Answer choices:

A $(2,2)$, $(-3,7)$, $(-3,2)$, and $(4,-3)$

B $(2,2)$, $(-3,7)$, $(-4,-4)$, and $(6,-6)$

C $(4,1)$, $(-2,10)$, $(-3,2)$, and $(4,-3)$

D $(4,1)$, $(-2,10)$, $(-4,-4)$, and $(6,-6)$

Solution: D

Put the vertices of the polygon into a matrix.

$$\begin{bmatrix} -2 & 1 & 2 & -3 \\ 1 & 3 & -2 & -1 \end{bmatrix}$$

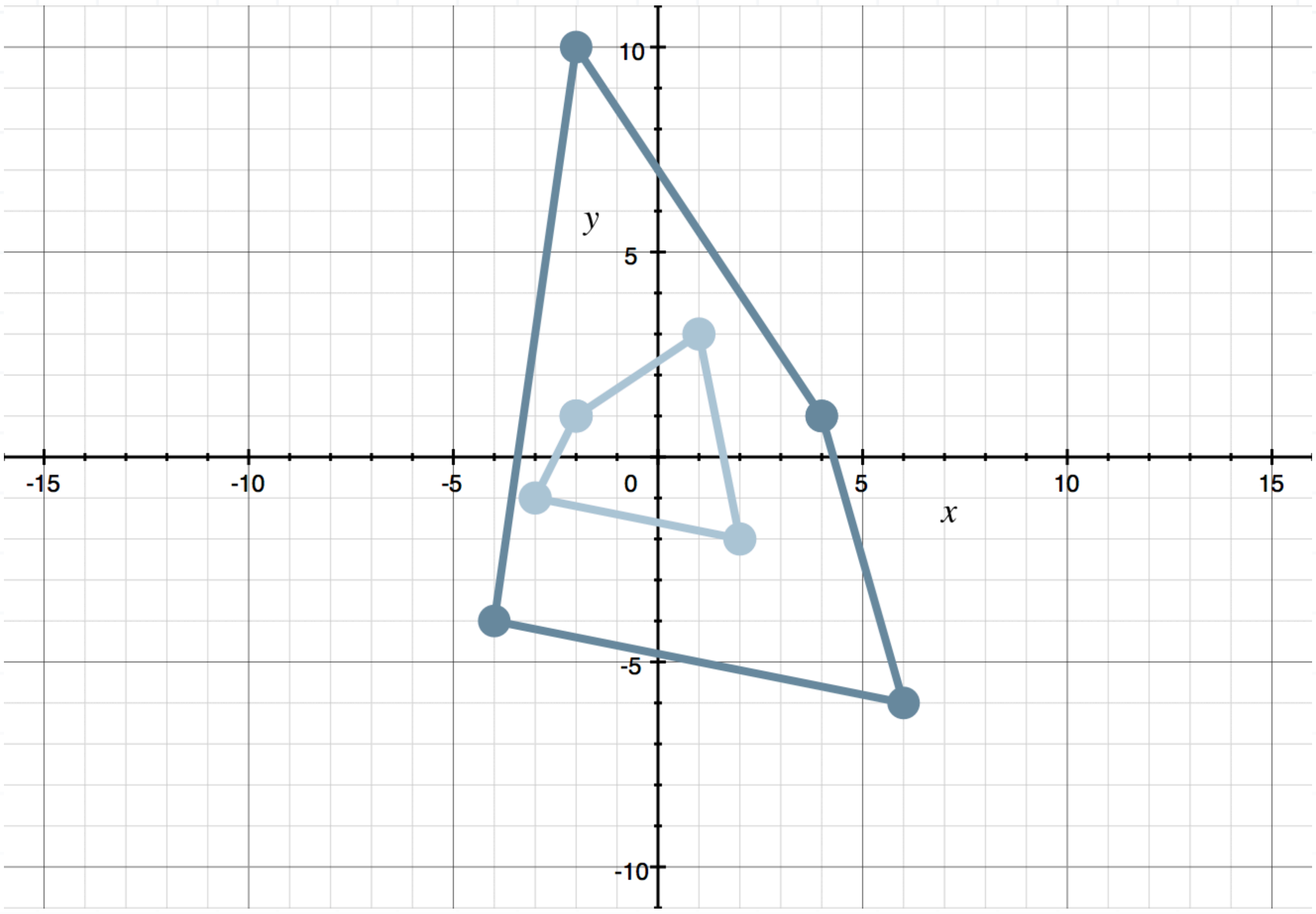
Apply the transformation of Z to the vertex matrix.

$$\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 & 2 & -3 \\ 1 & 3 & -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2(-2)+0(1) & -2(1)+0(3) & -2(2)+0(-2) & -2(-3)+0(-1) \\ 1(-2)+3(1) & 1(1)+3(3) & 1(2)+3(-2) & 1(-3)+3(-1) \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & -4 & 6 \\ 1 & 10 & -4 & -6 \end{bmatrix}$$

The original polygon is sketched in light blue, and its transformation after P is in dark blue.



10
5
-5
-10
-15
-10
-5
0
5
10
15
x
y

Topic: Transformations

Question: What are the vertices of the transformation of the triangle with vertices $(-3,0)$, $(1,2)$, and $(1,-2)$ after it's transformed by matrix S.

$$S = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$$

Answer choices:

A $(0,-6)$, $(-2,4)$, and $(2,0)$

B $(0,-4)$, $(-1,3)$, and $(2,2)$

C $(1,-3)$, $(-1,6)$, and $(3,1)$

D $(2,-1)$, $(0,2)$, and $(1,4)$

Solution: A

Put the vertices of the triangle into a matrix.

$$\begin{bmatrix} -3 & 1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

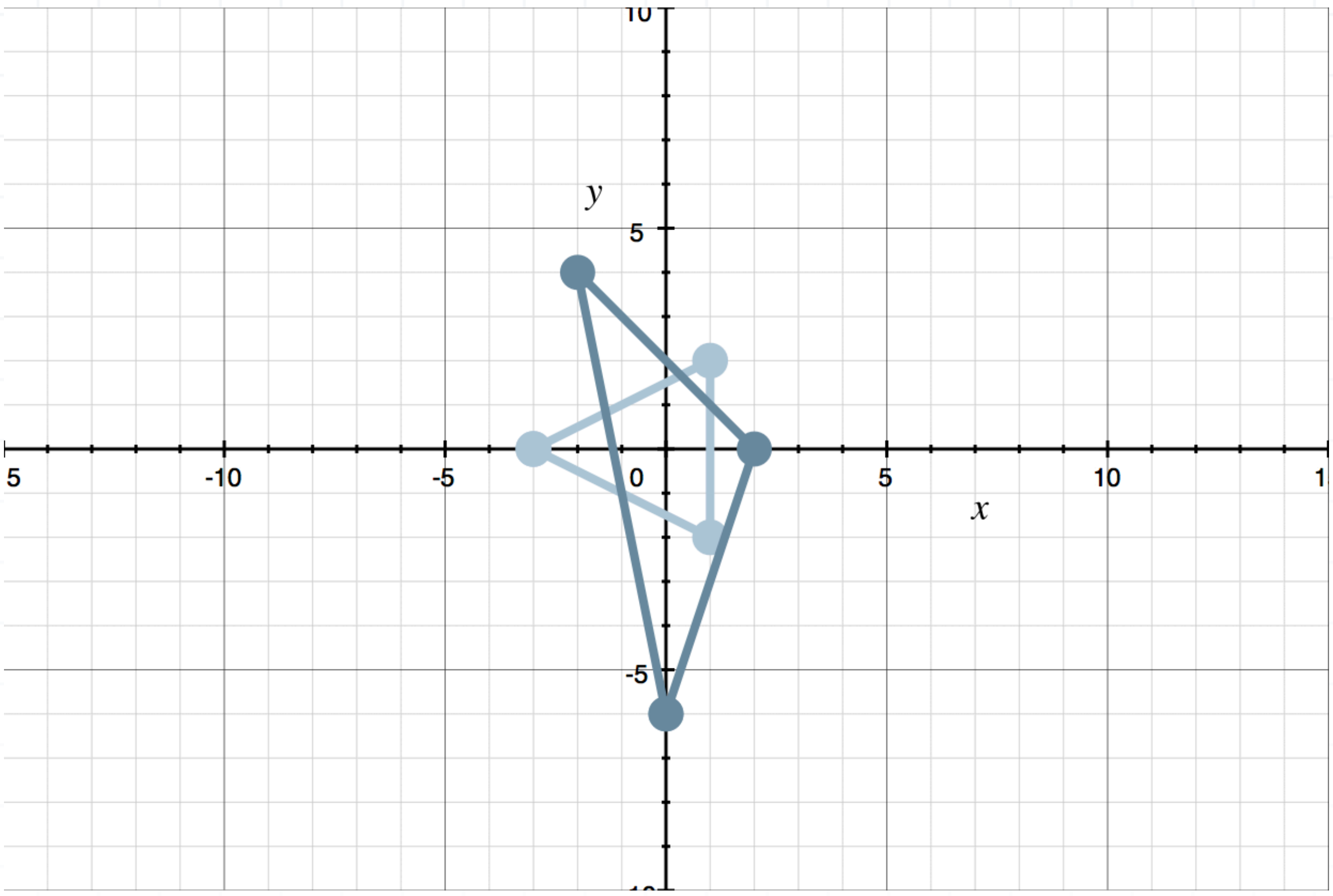
Apply the transformation of S to the vertex matrix.

$$\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 0(-3) - 1(0) & 0(1) - 1(2) & 0(1) - 1(-2) \\ 2(-3) + 1(0) & 2(1) + 1(2) & 2(1) + 1(-2) \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & 2 \\ -6 & 4 & 0 \end{bmatrix}$$

The original triangle is sketched in light blue, and its transformation after S is in dark blue.


10
y
5
-5
-10
0
5
10
x
-5

Topic: Matrix inverses, and invertible and singular matrices

Question: Are the matrices inverses of one another?

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{1}{13} & \frac{5}{13} \\ \frac{3}{13} & -\frac{2}{13} \end{bmatrix}$$

Answer choices:

A Yes

B No

C There's not enough information to know

Solution: A

To find the inverse of matrix A, plug it into the formula for the inverse matrix.

$$A^{-1}=\frac{1}{|A|}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

$$A^{-1}=\frac{1}{\begin{vmatrix}2 & 5\\ 3 & 1\end{vmatrix}}\begin{bmatrix}1 & -5\\ -3 & 2\end{bmatrix}$$

Find the determinant in the denominator of the fraction.

$$A^{-1}=\frac{1}{(2)(1)-(5)(3)}\begin{bmatrix}1 & -5\\ -3 & 2\end{bmatrix}$$

$$A^{-1}=\frac{1}{2-15}\begin{bmatrix}1 & -5\\ -3 & 2\end{bmatrix}$$

$$A^{-1}=-\frac{1}{13}\begin{bmatrix}1 & -5\\ -3 & 2\end{bmatrix}$$

Then distribute the scalar across the matrix.

$$A^{-1}=\begin{bmatrix}-\frac{1}{13}(1) & -\frac{1}{13}(-5)\\ -\frac{1}{13}(-3) & -\frac{1}{13}(2)\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}-\frac{1}{13} & \frac{5}{13}\\ \frac{3}{13} & -\frac{2}{13}\end{bmatrix}$$

Because the value we found matches matrix B, it means that matrices A and B are inverses of one another.

Topic: Matrix inverses, and invertible and singular matrices

Question: Find the inverse of matrix M.

$$M = \begin{bmatrix} 0 & -2 \\ -4 & 5 \end{bmatrix}$$

Answer choices:

A $\quad M^{-1} = \begin{bmatrix} 0 & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{5}{8} \end{bmatrix}$

B $\quad M^{-1} = \begin{bmatrix} -\frac{5}{8} & -\frac{1}{4} \\ -\frac{1}{2} & 0 \end{bmatrix}$

C $\quad M^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{2} & -\frac{5}{8} \end{bmatrix}$

D $\quad M^{-1} = \begin{bmatrix} -\frac{5}{8} & \frac{1}{4} \\ \frac{1}{2} & 0 \end{bmatrix}$

Solution: B

Plug the values from the matrix into the formula for the inverse matrix.

$$M^{-1} = \frac{1}{|M|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$M^{-1} = \frac{1}{\begin{vmatrix} 0 & -2 \\ -4 & 5 \end{vmatrix}}\begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix}$$

Find the determinant in the denominator of the fraction.

$$M^{-1} = \frac{1}{(0)(5) - (-2)(-4)}\begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix}$$

$$M^{-1} = \frac{1}{0 - 8}\begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{8}\begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix}$$

Then distribute the scalar across the matrix.

$$M^{-1} = \begin{bmatrix} -\frac{1}{8}(5) & -\frac{1}{8}(2) \\ -\frac{1}{8}(4) & -\frac{1}{8}(0) \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -\frac{5}{8} & -\frac{1}{4} \\ -\frac{1}{2} & 0 \end{bmatrix}$$

Topic: Matrix inverses, and invertible and singular matrices

Question: Classify the matrix.

$$L = \begin{bmatrix} 3 & 7 \\ 0 & -1 \end{bmatrix}$$

Answer choices:

A The matrix is invertible

B The matrix is singular

C The matrix is invertible and singular

D The matrix is neither invertible nor singular

Solution: A

A matrix is either invertible or singular, it can never be both. To determine whether a matrix in the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible or singular, we need to look at the ratio of a to b, compared to the ratio of c to d.

For the given matrix L,

$$\frac{a}{b} = \frac{3}{7}$$

$$\frac{c}{d} = \frac{0}{-1} = 0$$

Because these ratios aren't equivalent, that means matrix L is invertible. If the ratios had been equivalent, the matrix would have been singular.

Topic: Solving systems with inverse matrices

Question: Use an inverse matrix to find the solution to the system.

$$3x + 12y = 51$$

$$-2x + 6y = -6$$

Answer choices:

A $x = -9$ and $y = -2$

B $x = -9$ and $y = 2$

C $x = 9$ and $y = -2$

D $x = 9$ and $y = 2$

Solution: D

Start by transferring the system into a matrix equation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

$$\begin{bmatrix} 3 & 12 \\ -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 51 \\ -6 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M = \begin{bmatrix} 3 & 12 \\ -2 & 6 \end{bmatrix}$$

$$M^{-1} = \frac{1}{(3)(6) - (12)(-2)} \begin{bmatrix} 6 & -12 \\ 2 & 3 \end{bmatrix}$$

$$M^{-1} = \frac{1}{42} \begin{bmatrix} 6 & -12 \\ 2 & 3 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} \frac{1}{7} & -\frac{2}{7} \\ \frac{1}{21} & \frac{1}{14} \end{bmatrix}$$

Then we can say that the solution to the system is

$$\overrightarrow{a} = M^{-1}\overrightarrow{b}$$

$$\overrightarrow{a} = \begin{bmatrix} \frac{1}{7} & -\frac{2}{7} \\ \frac{1}{21} & \frac{1}{14} \end{bmatrix} \begin{bmatrix} 51 \\ -6 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{1}{7}(51) - \frac{2}{7}(-6) \\ \frac{1}{21}(51) + \frac{1}{14}(-6) \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{51}{7} + \frac{12}{7} \\ \frac{51}{21} - \frac{6}{14} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{63}{7} \\ \frac{17}{7} - \frac{3}{7} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{63}{7} \\ \frac{14}{7} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$$

Using this process with the inverse matrix, we conclude that $x = 9$ and $y = 2$.

Topic: Solving systems with inverse matrices

Question: Use an inverse matrix to find the solution to the system.

$$y - 5x = -15$$

$$3x + 8y = 95$$

Answer choices:

A $x = 5$ and $y = 10$

B $x = -5$ and $y = 10$

C $x = 5$ and $y = -10$

D $x = -5$ and $y = -10$

Solution: A

Start by transferring the system into a matrix equation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

$$\begin{bmatrix} -5 & 1 \\ 3 & 8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -15 \\ 95 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M = \begin{bmatrix} -5 & 1 \\ 3 & 8 \end{bmatrix}$$

$$M^{-1} = \frac{1}{(-5)(8) - (1)(3)} \begin{bmatrix} 8 & -1 \\ -3 & -5 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{43} \begin{bmatrix} 8 & -1 \\ -3 & -5 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -\frac{8}{43} & \frac{1}{43} \\ \frac{3}{43} & \frac{5}{43} \end{bmatrix}$$

Then we can say that the solution to the system is

$$\overrightarrow{a} = M^{-1}\overrightarrow{b}$$

$$\overrightarrow{a} = \begin{bmatrix} -\frac{8}{43} & \frac{1}{43} \\ \frac{3}{43} & \frac{5}{43} \end{bmatrix} \begin{bmatrix} -15 \\ 95 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{8}{43}(-15) + \frac{1}{43}(95) \\ \frac{3}{43}(-15) + \frac{5}{43}(95) \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{120}{43} + \frac{95}{43} \\ -\frac{45}{43} + \frac{475}{43} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{215}{43} \\ \frac{430}{43} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

Using this process with the inverse matrix, we conclude that $x = 5$ and $y = 10$.

Topic: Solving systems with inverse matrices

Question: Use an inverse matrix to find the solution to the system.

$$4x + 8y = -20$$

$$-12x - 3y = -66$$

Answer choices:

A $x = 7$ and $y = -6$

B $x = -7$ and $y = 6$

C $x = 7$ and $y = 6$

D $x = -7$ and $y = -6$

Solution: A

We could divide through both equations in the system to reduce them. The first equation $4x + 8y = -20$ becomes

$$\frac{4}{4}x + \frac{8}{4}y = -\frac{20}{4}$$

$$x + 2y = -5$$

And the equation $-12x - 3y = -66$ becomes

$$\frac{-12}{-3}x + \frac{-3}{-3}y = \frac{-66}{-3}$$

$$4x + y = 22$$

Then transfer the system into a matrix equation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 22 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$$

$$M^{-1} = \frac{1}{(1)(1) - (2)(4)} \begin{bmatrix} 1 & -2 \\ -4 & 1 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{7}\begin{bmatrix} 1 & -2 \\ -4 & 1 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -\frac{1}{7} & \frac{2}{7} \\ \frac{4}{7} & -\frac{1}{7} \end{bmatrix}$$

Then we can say that the solution to the system is

$$\vec{a} = M^{-1}\vec{b}$$

$$\vec{a} = \begin{bmatrix} -\frac{1}{7} & \frac{2}{7} \\ \frac{4}{7} & -\frac{1}{7} \end{bmatrix}\begin{bmatrix} -5 \\ 22 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{1}{7}(-5) + \frac{2}{7}(22) \\ \frac{4}{7}(-5) - \frac{1}{7}(22) \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{5}{7} + \frac{44}{7} \\ -\frac{20}{7} - \frac{22}{7} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{49}{7} \\ -\frac{42}{7} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$$

Using this process with the inverse matrix, we conclude that $x = 7$ and $y = -6$.

Topic: Solving systems with Cramer's Rule

Question: Which expression would give the solution for y in this system?

$$3x - 2y = 21$$

$$-6x - 5y = 12$$

Answer choices:

A $\dfrac{\begin{vmatrix} 3 & -2 \\ -6 & -5 \end{vmatrix}}{\begin{vmatrix} 21 & -2 \\ 12 & -5 \end{vmatrix}}$

B $\dfrac{\begin{vmatrix} 3 & 21 \\ -6 & 12 \end{vmatrix}}{\begin{vmatrix} 3 & 6 \\ -2 & 1 \end{vmatrix}}$

C $\dfrac{\begin{vmatrix} 21 & -2 \\ 12 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -6 & -5 \end{vmatrix}}$

D $\dfrac{\begin{vmatrix} 3 & 21 \\ -6 & 12 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -6 & -5 \end{vmatrix}}$

Solution: D

Using the given system

$$3x - 2y = 21$$

$$-6x - 5y = 12$$

we can say

$$D = \begin{vmatrix} 3 & -2 \\ -6 & -5 \end{vmatrix}$$

and

$$D_y = \begin{vmatrix} 3 & 21 \\ -6 & 12 \end{vmatrix}$$

We can put those together to solve for the value of y.

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 3 & 21 \\ -6 & 12 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -6 & -5 \end{vmatrix}}$$

Topic: Solving systems with Cramer's Rule

Question: Which expression below would give the solution for x in this system?

$$3x + 3y = 9$$

$$2x - y = -9$$

Answer choices:

A $\dfrac{\begin{vmatrix} 3 & 9 \\ 2 & -9 \end{vmatrix}}{\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}}$

B $\dfrac{\begin{vmatrix} 9 & 3 \\ -9 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}}$

C $\dfrac{\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 9 & 3 \\ -9 & -1 \end{vmatrix}}$

D $\dfrac{\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 9 \\ 2 & -9 \end{vmatrix}}$

Solution: B

Using the given system

$$3x + 3y = 9$$

$$2x - y = -9$$

we can say

$$D = \begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}$$

and

$$D_x = \begin{vmatrix} 9 & 3 \\ -9 & -1 \end{vmatrix}$$

We can put those together to solve for the value of x.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 9 & 3 \\ -9 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}}$$

Topic: Solving systems with Cramer's Rule

Question: Which system below would give this value?

$$\frac{D_x}{D} = \frac{\begin{vmatrix} 1 & -5 \\ 15 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 1 & 2 \end{vmatrix}}$$

Answer choices:

A $\quad 3x - 5y = 1$ and $x + 2y = 15$

B $\quad x - 5y = 3$ and $15x - 2y = 1$

C $\quad 3x + y = -5$ and $x - 15y = 2$

D $\quad x - 2y = 1$ and $3x + 15y = 2$

Solution: A

One way to start this is to figure out the D for each answer choice and see which one(s) match the given expression.

For answer choice A we get

$$D = \begin{vmatrix} 3 & -5 \\ 2 & -1 \end{vmatrix}$$

For answer choice B we get

$$D = \begin{vmatrix} 1 & -5 \\ 15 & -2 \end{vmatrix}$$

For answer choice C we get

$$D = \begin{vmatrix} 3 & 1 \\ 1 & -15 \end{vmatrix}$$

For answer choice D we get

$$D = \begin{vmatrix} 1 & -2 \\ 3 & 15 \end{vmatrix}$$

Only answer choice A matched the D in the given expression, so there's no need to check the D_x determinant.

Topic: Fraction decomposition

Question: Rewrite the function as a single fraction.

$$f(x) = \frac{3}{x+1} + \frac{2}{(x+1)^2} - \frac{3}{x-2}$$

Answer choices:

A $\quad f(x) = \dfrac{-7x-13}{(x+1)(x-2)}$

B $\quad f(x) = \dfrac{7x+13}{(x+1)(x-2)}$

C $\quad f(x) = \dfrac{-7x-13}{(x+1)^2(x-2)}$

D $\quad f(x) = \dfrac{7x+13}{(x+1)^2(x-2)}$

Solution: C

We can work backwards to find the original function associated with this decomposition by finding a common denominator,

$$f(x) = \frac{3(x+1)(x-2)}{(x+1)^2(x-2)} + \frac{2(x-2)}{(x+1)^2(x-2)} - \frac{3(x+1)^2}{(x+1)^2(x-2)}$$

combining the fractions,

$$f(x) = \frac{3(x+1)(x-2) + 2(x-2) - 3(x+1)^2}{(x+1)^2(x-2)}$$

and then simplifying.

$$f(x) = \frac{3x^2 - 3x - 6 + 2x - 4 - 3x^2 - 6x - 3}{(x+1)^2(x-2)}$$

$$f(x) = \frac{-7x - 13}{(x+1)^2(x-2)}$$

Topic: Fraction decomposition

Question: Identify the irreducible quadratic factors of the denominator of the rational function.

$$f(x) = \frac{2x+1}{x^2(x^2+1)(x^2-1)(x+1)^2}$$

Answer choices:

A $\quad x^2+1$

B $\quad x^2, x^2+1$

C $\quad x^2, x^2+1, x^2-1$

D $\quad x^2, x^2+1, x^2-1, (x+1)^2$

Solution: A

The factors in the denominator can be factored further as

$x^2 = x \cdot x$	repeated linear factors
$x^2 - 1 = (x + 1)(x - 1)$	distinct linear factors
$(x + 1)^2 = (x + 1)(x + 1)$	repeated linear factors

Therefore, only $x^2 + 1$ is an irreducible quadratic factor.

Topic: Fraction decomposition

Question: Find the form of the partial fractions decomposition of the rational function.

$$f(x) = \frac{x^2 + 4x + 1}{16x^4 - 1}$$

Answer choices:

A $\quad f(x) = \dfrac{A}{4x^2 + 1} + \dfrac{B}{4x^2 - 1}$

B $\quad f(x) = \dfrac{Ax + B}{4x^2 + 1} + \dfrac{Cx + D}{4x^2 - 1}$

C $\quad f(x) = \dfrac{A}{4x^2 + 1} + \dfrac{B}{2x + 1} + \dfrac{C}{2x - 1}$

D $\quad f(x) = \dfrac{A}{2x + 1} + \dfrac{B}{2x - 1} + \dfrac{Cx + D}{4x^2 + 1}$

Solution: D

We need to factor the denominator first.

$$16x^4 - 1$$

$$(4x^2 - 1)(4x^2 + 1)$$

$$(2x - 1)(2x + 1)(4x^2 + 1)$$

Which means we'll rewrite the rational function as

$$f(x) = \frac{x^2 + 4x + 1}{(2x - 1)(2x + 1)(4x^2 + 1)}$$

The denominator of $f(x)$ can't be factored any further, and it's now the product of three distinct factors. Therefore, the decomposition of $f(x)$ should include three fractions.

The numerators of the linear factors will be the constants A and B, and the numerator of the quadratic will be the expression $Cx + D$.

$$f(x) = \frac{A}{2x + 1} + \frac{B}{2x - 1} + \frac{Cx + D}{4x^2 + 1}$$

Topic: Distinct linear factors

Question: Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{5x + 3}{(x + 3)(x - 3)}$$

Answer choices:

A $\quad f(x) = -\frac{3}{x - 3} - \frac{2}{x + 3}$

B $\quad f(x) = \frac{3}{x - 3} - \frac{2}{x + 3}$

C $\quad f(x) = \frac{2}{x + 3} - \frac{3}{x - 3}$

D $\quad f(x) = \frac{2}{x + 3} + \frac{3}{x - 3}$

Solution: D

These are distinct linear factors.

$$\frac{5x+3}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3}$$

To solve for A, remove the $x+3$ factor and set $x = -3$ to find the value of the left side of the decomposition equation.

$$\frac{5x+3}{x-3} \to \frac{5(-3)+3}{-3-3} \to \frac{-12}{-6} \to 2$$

To solve for B, we'll remove the $x-3$ factor and set $x=3$.

$$\frac{5x+3}{x+3} \to \frac{5(3)+3}{3+3} \to \frac{18}{6} \to 3$$

Plugging $A=2$ and $B=3$ back into the partial fractions decomposition gives

$$f(x) = \frac{A}{x+3} + \frac{B}{x-3}$$

$$f(x) = \frac{2}{x+3} + \frac{3}{x-3}$$

Topic: Distinct linear factors

Question: Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{4x-6}{x(x-1)(x-2)}$$

Answer choices:

A $f(x) = \frac{3}{x} - \frac{2}{x-1} - \frac{1}{x-2}$

B $f(x) = -\frac{3}{x} + \frac{2}{x-1} + \frac{1}{x-2}$

C $f(x) = -\frac{3}{x} + \frac{6}{x-1} - \frac{3}{x-2}$

D $f(x) = \frac{3}{x} - \frac{6}{x-1} + \frac{3}{x-2}$

Solution: B

These are distinct linear factors, so we'll set up the decomposition equation as

$$\frac{4x-6}{x(x-1)(x-2)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-2}$$

To solve for A, remove x and set $x=0$ to find the value of the left side of the decomposition equation.

$$\frac{4x-6}{(x-1)(x-2)} \to \frac{4(0)-6}{(0-1)(0-2)} \to \frac{-6}{2}=-3$$

To solve for B, remove $x-1$ and set $x=1$ to find the value of the left side of the decomposition equation.

$$\frac{4x-6}{x(x-2)} \to \frac{4(1)-6}{1(1-2)} \to \frac{-2}{-1}=2$$

To solve for C, remove $x-2$ and set $x=2$ to find the value of the left side of the decomposition equation.

$$\frac{4x-6}{x(x-1)} \to \frac{4(2)-6}{2(2-1)} \to \frac{2}{2}=1$$

Therefore, the partial fractions decomposition is

$$f(x)=-\frac{3}{x}+\frac{2}{x-1}+\frac{1}{x-2}$$

Topic: Distinct linear factors

Question: Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{2}{(x-1)(x+1)}$$

Answer choices:

A $\quad f(x) = -\frac{1}{x-1} - \frac{1}{x+1}$

B $\quad f(x) = -\frac{1}{x-1} + \frac{1}{x+1}$

C $\quad f(x) = \frac{1}{x-1} + \frac{1}{x+1}$

D $\quad f(x) = \frac{1}{x-1} - \frac{1}{x+1}$

Solution: D

These are distinct linear factors, so we'll set up the decomposition equation as

$$\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

To solve for A, remove $x-1$ and set $x=1$ to find the value of the left side of the decomposition equation.

$$\frac{2}{x+1} \to \frac{2}{1+1} \to 1$$

To solve for B, remove $x+1$ and set $x=-1$ to find the value of the left side of the decomposition equation.

$$\frac{2}{x-1} \to \frac{2}{-1-1} \to -1$$

Plugging $A=1$ and $B=-1$ back into the partial fractions decomposition gives

$$f(x) = \frac{1}{x-1} - \frac{1}{x+1}$$

Topic: Repeated linear factors

Question: Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{4x^2 + 10x + 8}{(x+1)^3}$$

Answer choices:

A $\quad f(x) = \dfrac{4}{x+1} + \dfrac{2}{(x+1)^2} + \dfrac{2}{(x+1)^3}$

B $\quad f(x) = \dfrac{4}{x+1} - \dfrac{2}{(x+1)^2} + \dfrac{2}{(x+1)^3}$

C $\quad f(x) = \dfrac{4}{x+1} + \dfrac{2}{(x+1)^2} - \dfrac{2}{(x+1)^3}$

D $\quad f(x) = \dfrac{4}{x+1} - \dfrac{2}{(x+1)^2} - \dfrac{2}{(x+1)^3}$

Solution: A

The denominator of this original rational function is $(x+1)^3$. Because the linear factor $x+1$ is raised to the power of 3, we'll need to include the third, second, and first powers of the factor in the decomposition.

$$\frac{4x^2+10x+8}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

Now that we have the decomposition equation, we'll combine the fractions on the right side by finding a common denominator.

$$\frac{4x^2+10x+8}{(x+1)^3}=\frac{A(x+1)^2}{(x+1)(x+1)^2}+\frac{B(x+1)}{(x+1)^2(x+1)}+\frac{C}{(x+1)^3}$$

$$\frac{4x^2+10x+8}{(x+1)^3}=\frac{A(x+1)^2}{(x+1)^3}+\frac{B(x+1)}{(x+1)^3}+\frac{C}{(x+1)^3}$$

$$\frac{4x^2+10x+8}{(x+1)^3}=\frac{A(x+1)^2+B(x+1)+C}{(x+1)^3}$$

Because the denominators are equivalent, that means the numerators must also be equivalent, so we'll set the numerators equal to one another.

$$4x^2+10x+8=A(x+1)^2+B(x+1)+C$$

The linear factor $x+1$ is equal to 0 when $x=-1$, so we'll evaluate this equation at $x=-1$ in order to solve for C.

$$4(-1)^2+10(-1)+8=A(-1+1)^2+B(-1+1)+C$$

$$4-10+8=C$$

$$C = 2$$

We'll plug this back into the numerator equation,

$$4x^2 + 10x + 8 = A(x+1)^2 + B(x+1) + 2$$

and then simplify the right side, collecting like terms.

$$4x^2 + 10x + 8 = A(x^2 + 2x + 1) + B(x+1) + 2$$

$$4x^2 + 10x + 8 = Ax^2 + 2Ax + A + Bx + B + 2$$

$$4x^2 + 10x + 8 = Ax^2 + (2Ax + Bx) + (A + B + 2)$$

$$4x^2 + 10x + 8 = Ax^2 + (2A + B)x + (A + B + 2)$$

The coefficients on x^2 are 4 and A, the coefficients on x are 10 and $2A + B$, and the constants are 8 and $A + B + 2$, so we get the system of equations

$$A = 4$$

$$2A + B = 10$$

$$A + B + 2 = 8$$

or $A = 4$ with

$$8 + B = 10$$

$$4 + B + 2 = 8$$

or $A = 4$ and $B = 2$. Plugging $A = 4$, $B = 2$, and $C = 2$ back into the partial fractions decomposition gives

$$f(x) = \frac{4}{x+1} + \frac{2}{(x+1)^2} + \frac{2}{(x+1)^3}$$

Topic: Repeated linear factors

Question: Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x+2}{(x-1)^2}$$

Answer choices:

A $\quad f(x) = \dfrac{1}{(x+1)^2} + \dfrac{3}{(x-1)}$

B $\quad f(x) = \dfrac{3}{(x-1)^2} + \dfrac{1}{(x-1)}$

C $\quad f(x) = \dfrac{3}{(x-1)^2} - \dfrac{1}{(x-1)}$

D $\quad f(x) = \dfrac{1}{(x-1)^2} - \dfrac{3}{(x-1)}$

Solution: B

Because the linear factor $x-1$ is raised to the power of 2, we'll need to include the second and first powers of the factor in the decomposition equation.

$$\frac{x+2}{(x-1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}$$

Now that we have the decomposition equation, we'll combine the fractions on the right side by finding a common denominator.

$$\frac{x+2}{(x-1)^2}=\frac{A(x-1)+B}{(x-1)^2}$$

$$\frac{x+2}{(x-1)^2}=\frac{Ax-A+B}{(x-1)^2}$$

Because the denominators are equivalent, the numerators must also be equivalent, so we'll set the numerators equal to one another.

$$x+2=Ax-A+B$$

Therefore,

$$A=1$$

$$-A+B=2$$

So $A=1$ and $B=3$. Plugging these values back into the partial fractions decomposition gives

$$f(x) = \frac{3}{(x-1)^2} + \frac{1}{x-1}$$

Topic: Repeated linear factors

Question: Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^2 - 6x + 9}{(x-2)^3}$$

Answer choices:

A $\quad f(x) = \frac{1}{(x-2)} + \frac{2}{(x-2)^2} + \frac{1}{(x-2)^3}$

B $\quad f(x) = \frac{1}{(x-2)} - \frac{2}{(x-2)^2} + \frac{1}{(x-2)^3}$

C $\quad f(x) = \frac{5}{(x-2)} + \frac{10}{(x-2)^2} + \frac{9}{(x-2)^3}$

D $\quad f(x) = \frac{5}{(x-2)} - \frac{10}{(x-2)^2} + \frac{9}{(x-2)^3}$

Solution: B

Because the linear factor $x-2$ is raised to the power of 3, we'll need to include the third, second, and first powers of the factor in the decomposition equation.

$$\frac{x^2-6x+9}{(x-2)^3}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3}$$

Now that we have the decomposition equation, we'll combine the fractions on the right side by finding a common denominator.

$$\frac{x^2-6x+9}{(x-2)^3}=\frac{A(x-2)^2}{(x-2)(x-2)^2}+\frac{B(x-2)}{(x-2)^2(x-2)}+\frac{C}{(x-2)^3}$$

$$\frac{x^2-6x+9}{(x-2)^3}=\frac{A(x-2)^2+B(x-2)+C}{(x-2)^3}$$

Because the denominators are equivalent, the numerators must also be equivalent, so we'll set the numerators equal to one another.

$$x^2-6x+9=A(x-2)^2+B(x-2)+C$$

The linear factor $x-2$ is equal to 0 when $x=2$, so we'll evaluate this equation at $x=2$ in order to solve for C.

$$2^2-6(2)+9=A(2-2)^2+B(2-2)+C$$

$$4-12+9=C$$

$$C=1$$

We'll plug this back into the numerator equation,

$$x^2 - 6x + 9 = A(x-2)^2 + B(x-2) + 1$$

and then simplify the right side, collecting like terms.

$$x^2 - 6x + 9 = A(x^2 - 4x + 4) + B(x-2) + 1$$

$$x^2 - 6x + 9 = Ax^2 + (-4A + B)x + 4A - 2B + 1$$

So we get the system of equations

$$A = 1$$

$$-4A + B = -6$$

$$4A - 2B + 1 = 9$$

Solving the system gives $A = 1$, $B = -2$, and $C = 1$, and plugging these values back into the partial fractions decomposition gives

$$f(x) = \frac{1}{(x-2)} - \frac{2}{(x-2)^2} + \frac{1}{(x-2)^3}$$

Topic: Distinct quadratic factors

Question: Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{6x^3 + 2x^2 - x + 12}{(2x^2 + 1)(x^2 + 1)}$$

Answer choices:

A $\quad f(x) = \dfrac{-6x - 5}{2x^2 + 1} + \dfrac{5x + 14}{x^2 + 1}$

B $\quad f(x) = \dfrac{-8x - 5}{2x^2 + 1} + \dfrac{7x + 14}{x^2 + 1}$

C $\quad f(x) = \dfrac{-8x + 22}{2x^2 + 1} + \dfrac{7x - 10}{x^2 + 1}$

D $\quad f(x) = \dfrac{-6x + 22}{2x^2 + 1} + \dfrac{5x - 10}{x^2 + 1}$

Solution: C

These are distinct quadratic factors, so we'll set up the decomposition equation as

$$\frac{6x^3+2x^2-x+12}{(2x^2+1)(x^2+1)}=\frac{Ax+B}{2x^2+1}+\frac{Cx+D}{x^2+1}$$

Combine the fractions on the right by finding a common denominator.

$$\frac{6x^3+2x^2-x+12}{(2x^2-1)(x^2+1)}=\frac{(Ax+B)(x^2+1)+(Cx+D)(2x^2+1)}{(2x^2+1)(x^2+1)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$6x^3+2x^2-x+12=(Ax+B)(x^2+1)+(Cx+D)(2x^2+1)$$

$$6x^3+2x^2-x+12=Ax^3+Ax+Bx^2+B+2Cx^3+Cx+2Dx^2+D$$

$$6x^3+2x^2-x+12=(A+2C)x^3+(B+2D)x^2+(A+C)x+B+D$$

Equating coefficients gives us a system of equations that we can break into two smaller systems.

$$A+2C=6 \qquad\qquad B+2D=2$$

$$A+C=-1 \qquad\qquad B+D=12$$

Solving the two equations on the left as a system gives $A=-8$ and $C=7$, while solving the two equations on the right as a system gives $B=22$ and $D=-10$. Plugging these back into the partial fractions decomposition gives

$$f(x) = \frac{-8x + 22}{2x^2 + 1} + \frac{7x - 10}{x^2 + 1}$$

Topic: Distinct quadratic factors

Question: Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 - 2x^2 + 6x - 4}{(4x^2 + 2)(2x^2 + 6)}$$

Answer choices:

A $\quad f(x) = \dfrac{11x - 6}{10(4x^2 + 2)} - \dfrac{3x + 2}{10(2x^2 + 6)}$

B $\quad f(x) = \dfrac{13x - 6}{10(4x^2 + 2)} - \dfrac{4x + 2}{10(2x^2 + 6)}$

C $\quad f(x) = \dfrac{11x - 10}{10(4x^2 + 2)} - \dfrac{3x}{10(2x^2 + 6)}$

D $\quad f(x) = \dfrac{13x - 10}{10(4x^2 + 2)} - \dfrac{4x}{10(2x^2 + 6)}$

Solution: A

These are distinct quadratic factors, so we'll set up the decomposition equation as

$$\frac{x^3-2x^2+6x-4}{(4x^2+2)(2x^2+6)}=\frac{Ax+B}{4x^2+2}+\frac{Cx+D}{2x^2+6}$$

Combine the fractions on the right by finding a common denominator.

$$\frac{x^3-2x^2+6x-4}{(4x^2+2)(2x^2+6)}=\frac{(Ax+B)(2x^2+6)+(Cx+D)(4x^2+2)}{(4x^2+2)(2x^2+6)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$x^3-2x^2+6x-4=(Ax+B)(2x^2+6)+(Cx+D)(4x^2+2)$$

$$x^3-2x^2+6x-4=2Ax^3+6Ax+2Bx^2+6B+4Cx^3+2Cx+4Dx^2+2D$$

$$x^3-2x^2+6x-4=(2A+4C)x^3+(2B+4D)x^2+(6A+2C)x+6B+2D$$

Equating coefficients gives us a system of equations that we can break into two smaller systems.

$$2A+4C=1 \qquad 2B+4D=-2$$

$$6A+2C=6 \qquad 6B+2D=-4$$

Solving the two equations on the left as a system gives $A=11/10$ and $C=-3/10$, while solving the two equations on the right as a system gives $B=-6/10$ and $D=-2/10$. Plugging these back into the partial fractions decomposition gives

$$f(x) = \frac{\frac{11}{10}x - \frac{6}{10}}{4x^2 + 2} + \frac{-\frac{3}{10}x - \frac{2}{10}}{2x^2 + 6}$$

$$f(x) = \frac{11x - 6}{10(4x^2 + 2)} - \frac{3x + 2}{10(2x^2 + 6)}$$

Topic: Distinct quadratic factors

Question: Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{4x^3 + 2x}{(x^2 + 2)(x^2 + 2x + 2)}$$

Answer choices:

A $\quad f(x) = \frac{5}{x^2 + 2x + 2} - \frac{4x + 5}{x^2 + 2}$

B $\quad f(x) = \frac{4x + 3}{x^2 + 2x + 2} - \frac{3}{x^2 + 2}$

C $\quad f(x) = \frac{4x + 5}{x^2 + 2x + 2} - \frac{5}{x^2 + 2}$

D $\quad f(x) = \frac{4x + 3}{x^2 + 2} - \frac{3}{x^2 + 2x + 2}$

Solution: B

These are distinct quadratic factors, so we'll set up the decomposition equation as

$$\frac{4x^3+2x}{(x^2+2)(x^2+2x+2)}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{x^2+2x+2}$$

Combine the fractions on the right by finding a common denominator.

$$\frac{4x^3+2x}{(x^2+2)(x^2+2x+2)}=\frac{(Ax+B)(x^2+2x+2)+(Cx+D)(x^2+2)}{(x^2+2)(x^2+2x+2)}$$

Once the denominators on both sides are equivalent, we'll set the numerators equal to each other.

$$4x^3+2x=(Ax+B)(x^2+2x+2)+(Cx+D)(x^2+2)$$

$$4x^3+2x=Ax^3+2Ax^2+2Ax+Bx^2+2Bx+2B+Cx^3+2Cx+Dx^2+2D$$

$$4x^3+2x=(A+C)x^3+(2A+B+D)x^2+(2A+2B+2C)x+2B+2D$$

Equating coefficients gives us a system of equations,

$$A+C=4 \qquad 2A+2B+2C=2$$

$$2A+B+D=0 \qquad 2B+2D=0$$

which we can solve to get $A=0$, $B=-3$, $C=4$, and $D=3$. Plugging these back into the partial fractions decomposition gives

$$f(x)=-\frac{3}{x^2+2}+\frac{4x+3}{x^2+2x+2}$$

Topic: Repeated quadratic factors

Question: Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 + 2x - 1}{(x^2 + 1)^2}$$

Answer choices:

A $\quad f(x) = \frac{x - 1}{(x^2 + 1)^2} + \frac{x}{(x^2 + 1)}$

B $\quad f(x) = \frac{x + 1}{(x^2 + 1)^2} + \frac{x}{(x^2 + 1)}$

C $\quad f(x) = \frac{x - 1}{(x^2 + 1)^2} - \frac{x}{(x^2 + 1)}$

D $\quad f(x) = \frac{x - 1}{(x^2 + 1)^2} + \frac{x}{(x^2 + 1)^2}$

Solution: A

These are repeated quadratic factors, so we'll set up the decomposition equation as

$$\frac{x^3+2x-1}{(x^2+1)^2}=\frac{Ax+B}{(x^2+1)^2}+\frac{Cx+D}{x^2+1}$$

Combine the fractions on the right by finding a common denominator.

$$\frac{x^3+2x-1}{(x^2+1)^2}=\frac{Ax+B+(Cx+D)(x^2+1)}{(x^2+1)^2}$$

Now that the denominators are equivalent, we can set the numerators equal to one another.

$$x^3+2x-1=Ax+B+Cx^3+Cx+Dx^2+D$$

$$x^3+2x-1=Cx^3+Dx^2+(A+C)x+B+D$$

If we equate coefficients from the left and right sides, we get a system of equations.

$$C=1$$

$$D=0$$

$$A+C=2$$

$$B+D=-1$$

Solving the system gives $A=1$, $B=-1$, $C=1$, and $D=0$. Plugging the values back into the partial fractions decomposition gives

$$f(x) = \frac{x-1}{(x^2+1)^2} + \frac{x}{(x^2+1)}$$

Topic: Repeated quadratic factors

Question: Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{2x^2 - 2x + 6}{(2x^2 + 3)^3}$$

Answer choices:

A $\quad f(x) = \frac{3 - 2x}{2x^2 + 3} + \frac{1}{(2x^2 + 3)^2}$

B $\quad f(x) = \frac{1}{2x^2 + 3} - \frac{2}{(2x^2 + 3)^2} + \frac{3}{(2x^2 + 3)^3}$

C $\quad f(x) = \frac{1}{2x^2 + 3} + \frac{3 - 2x}{(2x^2 + 3)^3}$

D $\quad f(x) = \frac{1}{(2x^2 + 3)^2} + \frac{3 - 2x}{(2x^2 + 3)^3}$

Solution: D

These are repeated quadratic factors, so we'll set up the decomposition equation as

$$\frac{2x^2-2x+6}{(2x^2+3)^3}=\frac{Ax+B}{(2x^2+3)}+\frac{Cx+D}{(2x^2+3)^2}+\frac{Ex+F}{(2x^2+3)^3}$$

Combine the fractions on the right by finding a common denominator.

$$\frac{2x^2-2x+6}{(2x^2+3)^3}=\frac{(Ax+B)(2x^2+3)^2+(Cx+D)(2x^2+3)+Ex+F}{(2x^2+3)^3}$$

Now that the denominators are equivalent, we can set the numerators equal to one another.

$$2x^2-2x+6=(Ax+B)(4x^4+12x^2+9)+(Cx+D)(2x^2+3)+Ex+F$$

$$2x^2-2x+6=4Ax^5+12Ax^3+9Ax+4Bx^4+12Bx^2+9B$$

$$+2Cx^3+3Cx+2Dx^2+3D+Ex+F$$

$$2x^2-2x+6=4Ax^5+4Bx^4+(12A+2C)x^3+(12B+2D)x^2$$

$$+(9A+3C+E)x+9B+3D+F$$

If we equate coefficients from the left and right sides, we get a system of equations.

$$4A=0$$

$$4B=0$$

$$12A+2C=0$$

$$12B + 2D = 2$$

$$9A + 3C + E = -2$$

$$9B + 3D + F = 6$$

Solving the system gives $A = 0$, $B = 0$, $C = 0$, $D = 1$, $E = -2$, and $F = 3$. Plugging these values back into the partial fractions decomposition gives us

$$f(x) = \frac{0x + 0}{(2x^2 + 3)} + \frac{0x + 1}{(2x^2 + 3)^2} + \frac{-2x + 3}{(2x^2 + 3)^3}$$

$$f(x) = \frac{1}{(2x^2 + 3)^2} + \frac{3 - 2x}{(2x^2 + 3)^3}$$

Topic: Repeated quadratic factors

Question: Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3}{(x^2 - 2x + 3)^2}$$

Answer choices:

A $\quad f(x) = \dfrac{x-6}{x^2 - 2x + 3} + \dfrac{x+2}{(x^2 - 2x + 3)^2}$

B $\quad f(x) = \dfrac{x+6}{x^2 - 2x + 3} + \dfrac{x-2}{(x^2 - 2x + 3)^2}$

C $\quad f(x) = \dfrac{x+2}{x^2 - 2x + 3} + \dfrac{x-6}{(x^2 - 2x + 3)^2}$

D $\quad f(x) = \dfrac{x-2}{x^2 - 2x + 3} + \dfrac{x+6}{(x^2 - 2x + 3)^2}$

Solution: C

These are repeated quadratic factors, so we'll set up the decomposition equation as

$$\frac{x^3}{(x^2-2x+3)^2}=\frac{Ax+B}{x^2-2x+3}+\frac{Cx+D}{(x^2-2x+3)^2}$$

Combine the fractions on the right by finding a common denominator.

$$\frac{x^3}{(x^2-2x+3)^2}=\frac{(Ax+B)(x^2-2x+3)+Cx+D}{(x^2-2x+3)^2}$$

Now that the denominators are equivalent, we can set the numerators equal to one another.

$$x^3=Ax^3-2Ax^2+3Ax+Bx^2-2Bx+3B+Cx+D$$

$$x^3=Ax^3+(-2A+B)x^2+(3A-2B+C)x+3B+D$$

If we equate coefficients from the left and right sides, we get a system of equations.

$$A=1$$

$$-2A+B=0$$

$$3A-2B+C=0$$

$$3B+D=0$$

Solving the system gives $A=1$, $B=2$, $C=1$, and $D=-6$. Plugging these values back into the partial fractions decomposition gives

$$f(x) = \frac{x+2}{x^2-2x+3} + \frac{x-6}{(x^2-2x+3)^2}$$

Topic: Mixed factors

Question: Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 - 8x^2 - 1}{(x+3)(x-2)(x^2+1)}$$

Answer choices:

A $\quad f(x) = \dfrac{2}{x+3} + \dfrac{1}{x-2} + \dfrac{1}{x^2+1}$

B $\quad f(x) = \dfrac{2}{x+3} - \dfrac{1}{x-2} - \dfrac{1}{x^2+1}$

C $\quad f(x) = \dfrac{2}{x+3} - \dfrac{1}{x-2} + \dfrac{1}{x^2+1}$

D $\quad f(x) = \dfrac{2}{x+3} + \dfrac{1}{x-2} - \dfrac{1}{x^2+1}$

Solution: B

The denominator of the rational function is already factored as completely as possible, and it's the product of distinct linear and distinct quadratic factors. Which means we can set up the decomposition equation as

$$\frac{x^3-8x^2-1}{(x+3)(x-2)(x^2+1)}=\frac{A}{x+3}+\frac{B}{x-2}+\frac{Cx+D}{x^2+1}$$

To solve for A, remove the $x+3$ factor from the left side of the equation and set $x=-3$.

$$\frac{x^3-8x^2-1}{(x-2)(x^2+1)}\to\frac{(-3)^3-8(-3)^2-1}{(-3-2)((-3)^2+1)}\to\frac{-100}{-50}\to 2=A$$

To solve for B, we'll remove the $x-2$ factor from the left side of the equation and set $x=2$.

$$\frac{x^3-8x^2-1}{(x+3)(x^2+1)}\to\frac{2^3-8(2)^2-1}{(2+3)(2^2+1)}\to\frac{-25}{25}\to -1=B$$

To solve for C and D, we'll find a common denominator on the right side in order to combine fractions.

$$\frac{x^3-8x^2-1}{(x+3)(x-2)(x^2+1)}=\frac{A(x-2)(x^2+1)}{(x+3)(x-2)(x^2+1)}+\frac{B(x+3)(x^2+1)}{(x+3)(x-2)(x^2+1)}$$

$$+\frac{(Cx+D)(x+3)(x-2)}{(x+3)(x-2)(x^2+1)}$$

$$\frac{x^3-8x^2-1}{(x+3)(x-2)(x^2+1)}=\frac{A(x-2)(x^2+1)+B(x+3)(x^2+1)+(Cx+D)(x+3)(x-2)}{(x+3)(x-2)(x^2+1)}$$

Now that the denominators are equivalent, we can set the numerators equal to one another,

$$x^3 - 8x^2 - 1 = A(x-2)(x^2+1) + B(x+3)(x^2+1) + (Cx+D)(x+3)(x-2)$$

then expand the right side and collect like terms.

$$x^3 - 8x^2 - 1 = A(x^3 + x - 2x^2 - 2) + B(x^3 + x + 3x^2 + 3) + (Cx+D)(x^2 + x - 6)$$

$$x^3 - 8x^2 - 1 = (A + B + C)x^3 + (-2A + 3B + C + D)x^2$$

$$+(A + B - 6C + D)x + (-2A + 3B - 6D)$$

Equating coefficients from the left and right sides gives us a system of equations.

$$A + B + C = 1$$

$$-2A + 3B + C + D = -8$$

$$A + B - 6C + D = 0$$

$$-2A + 3B - 6D = -1$$

We know the values of A and B, so the system simplifies to

$$C = 0$$

$$C + D = -1$$

$$6C - D = 1$$

$$6D = -6$$

We can see from what we have left that $C=0$ and $D=-1$. Plugging $A=2$, $B=-1$, $C=0$, and $D=-1$ back into the partial fractions decomposition gives

$$f(x)=\frac{A}{x+3}+\frac{B}{x-2}+\frac{Cx+D}{x^2+1}$$

$$f(x)=\frac{2}{x+3}+\frac{-1}{x-2}+\frac{0x+(-1)}{x^2+1}$$

$$f(x)=\frac{2}{x+3}-\frac{1}{x-2}-\frac{1}{x^2+1}$$

Topic: Mixed factors

Question: Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^2 + 2x + 4}{x^2(x^2 + 4)}$$

Answer choices:

A $\quad f(x) = \frac{1}{x} + \frac{1}{2x^2} - \frac{2x - 1}{2(x^2 + 4)}$

B $\quad f(x) = \frac{1}{2x} - \frac{1}{x^2} - \frac{2x - 1}{2(x^2 + 4)}$

C $\quad f(x) = \frac{1}{x} + \frac{1}{2x^2} - \frac{x}{2(x^2 + 4)}$

D $\quad f(x) = \frac{1}{2x} + \frac{1}{x^2} - \frac{x}{2(x^2 + 4)}$

Solution: D

The denominator of the rational function is already factored as completely as possible, and it's the product of repeated linear factors and a distinct quadratic factor. Which means we can set up the decomposition equation as

$$\frac{x^2+2x+4}{x^2(x^2+4)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+4}$$

To solve for B, we'll remove x^2 from the left side, then evaluate what remains on the left side at $x=0$.

$$\frac{0^2+2(0)+4}{(0^2+4)} \to 1=B$$

To solve for the other constants, we'll find a common denominator on the right side, then combine the fractions.

$$\frac{x^2+2x+4}{x^2(x^2+4)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+4}$$

$$\frac{x^2+2x+4}{x^2(x^2+4)}=\frac{Ax(x^2+4)+B(x^2+4)+(Cx+D)x^2}{x^2(x^2+4)}$$

Because the denominators are equivalent, we can set the numerators equal to one another.

$$x^2+2x+4=Ax(x^2+4)+B(x^2+4)+(Cx+D)x^2$$

$$x^2+2x+4=Ax^3+4Ax+Bx^2+4B+Cx^3+Dx^2$$

$$x^2+2x+4=(A+C)x^3+(B+D)x^2+4Ax+4B$$

We can equate coefficients on either side of the equation to build a system of equations.

$$A + C = 0$$

$$B + D = 1$$

$$4A = 2$$

$$4B = 4$$

Solving this systems gives $A = 1/2$, $B = 1$, $C = -1/2$, and $D = 0$. If we plug these values into the partial fractions decomposition equation, we get

$$f(x) = \frac{\frac{1}{2}}{x} + \frac{1}{x^2} - \frac{\frac{1}{2}x}{x^2 + 4}$$

$$f(x) = \frac{1}{2x} + \frac{1}{x^2} - \frac{x}{2x^2 + 8}$$

Topic: Mixed factors

Question: Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 + 4x^2 - 10}{x^2(x^2 - 1)}$$

Answer choices:

A $\quad f(x) = \frac{1}{x^2} + \frac{7}{x+1} - \frac{5}{x-1}$

B $\quad f(x) = \frac{1}{x^2} - \frac{7}{x+1} + \frac{5}{x-1}$

C $\quad f(x) = \frac{10}{x^2} + \frac{\frac{7}{2}}{x+1} - \frac{\frac{5}{2}}{x-1}$

D $\quad f(x) = \frac{10}{x^2} - \frac{\frac{7}{2}}{x+1} + \frac{\frac{5}{2}}{x-1}$

Solution: C

The denominator of the rational function can be factored further as

$$f(x) = \frac{x^3 + 4x^2 - 10}{x^2(x^2 - 1)}$$

$$f(x) = \frac{x^3 + 4x^2 - 10}{x^2(x + 1)(x - 1)}$$

Now that the denominator is factored as completely as possible, we have repeated linear factors and distinct linear factors. So we'll set up the partial fractions decomposition equation as

$$\frac{x^3 + 4x^2 - 10}{x^2(x^2 - 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1} + \frac{D}{x - 1}$$

To solve for B, we'll remove x^2 from the left side, then evaluate what remains on the left side at $x = 0$.

$$\frac{0^3 + 4(0)^2 - 10}{(0^2 - 1)} \to 10 = B$$

To solve for C, we'll remove $x + 1$ from the left side, then evaluate what remains on the left side at $x = -1$.

$$\frac{(-1)^3 + 4(-1)^2 - 10}{(-1)^2(-1 - 1)} \to \frac{-1 + 4 - 10}{1(-2)} \to \frac{7}{2} = C$$

To solve for D, we'll remove $x - 1$ from the left side, then evaluate what remains on the left side at $x = 1$.

$$\frac{(1)^3+4(1)^2-10}{(1)^2(1+1)} \to \frac{1+4-10}{1(2)} \to -\frac{5}{2}=D$$

To solve for A, we'll find a common denominator on the right side, then combine the fractions.

$$\frac{x^3+4x^2-10}{x^2(x^2-1)}=\frac{A}{x}+\frac{10}{x^2}+\frac{\frac{7}{2}}{x+1}+\frac{-\frac{5}{2}}{x-1}$$

$$\frac{x^3+4x^2-10}{x^2(x^2-1)}=\frac{Ax(x+1)(x-1)+10(x+1)(x-1)+\frac{7}{2}x^2(x-1)-\frac{5}{2}x^2(x+1)}{x^2(x^2-1)}$$

Because the denominators are equivalent, we can set the numerators equal to one another.

$$x^3+4x^2-10=Ax(x^2-1)+10(x^2-1)+\frac{7}{2}x^2(x-1)-\frac{5}{2}x^2(x+1)$$

$$x^3+4x^2-10=Ax^3-Ax+10x^2-10+\frac{7}{2}x^3-\frac{7}{2}x^2-\frac{5}{2}x^3-\frac{5}{2}x^2$$

$$x^3+4x^2-10=\left(A+\frac{7}{2}-\frac{5}{2}\right)x^3+\left(10-\frac{7}{2}-\frac{5}{2}\right)x^2-Ax-10$$

$$x^3+4x^2-10=(A+1)x^3+4x^2-Ax-10$$

We can equate coefficients on either side of the equation to build a system of equations.

$$A+1=1$$

$$A=0$$

So we'll plug $A = 0$, $B = 10$, $C = 7/2$, and $D = -5/2$ into the partial fractions decomposition equation to get

$$f(x) = \frac{0}{x} + \frac{10}{x^2} + \frac{\frac{7}{2}}{x+1} + \frac{-\frac{5}{2}}{x-1}$$

$$f(x) = \frac{10}{x^2} + \frac{7}{2(x+1)} - \frac{5}{2(x-1)}$$

Topic: Identifying conic sections

Question: Identify the equation as a circle, ellipse, parabola, or hyperbola.

$$4x^2 + x = 4y^2 + y + 3$$

Answer choices:

A Circle

B Ellipse

C Parabola

D Hyperbola

Solution: D

The given equation has both an x^2 term and a y^2 term, so it's either a circle, an ellipse, or a hyperbola. If we move everything but the constant to the same side of the equation, we get

$$4x^2 - 4y^2 + x - y = 3$$

The x^2 and y^2 terms have different signs, so the equation represents a hyperbola.

Topic: Identifying conic sections

Question: Identify the equation as a circle, ellipse, parabola, or hyperbola.

$$x^2 + 2y^2 + 3x = 5y + x^2 + 4$$

Answer choices:

A Circle

B Ellipse

C Parabola

D Hyperbola

Solution: C

If we move all non-constant terms to the left side, and all the constant terms to the right side, we get

$$x^2 + 2y^2 + 3x = 5y + x^2 + 4$$

$$2y^2 + 3x - 5y = 4$$

This equation has a y^2 term but no x^2 term, so it represents a parabola.

Topic: Identifying conic sections

Question: Which of the following equations represents an ellipse?

Answer choices:

A $x^2 + 2x + 5y = 2y^2 + 1$

B $x^2 - 3x - 6y - 6y^2 = 12$

C $-4x^2 + 10 - 2y^2 = 4x + 4y$

D $3x^2 + 5 = -2y^2$

Solution: C

If we rewrite all of the answer choices by collecting all the non-constant terms on the left side of the equations, and moving all the constant terms to the right side of the equations, we get

A $\quad x^2 - 2y^2 + 2x + 5y = 1$

B $\quad x^2 - 3x - 6y - 6y^2 = 12$

C $\quad 4x^2 + 2y^2 + 4x + 4y = 10$

D $\quad 3x^2 + 2y^2 = -5$

We see that equations A and B both have different signs on the x^2 and y^2 terms, so they represent hyperbolas.

Equation C has the same sign on the x^2 and y^2 terms, but the coefficients on those terms aren't equal, so equation C represents an ellipse.

Equation D has no solution, so it represents an empty set.

Topic: Circles

Question: The center of a circle is $(3, -2)$ and a point on the circle is $(1,1)$. Find the equation of the circle.

Answer choices:

A $(x-3)^2+(y+2)^2=13$

B $(x-3)^2+(y+2)^2=17$

C $(x+3)^2+(y-2)^2=13$

D $(x+3)^2+(y-2)^2=17$

Solution: A

Start by finding the radius of the circle. The horizontal distance between the center $(3,-2)$ and $(1,1)$ is $3-1=2$, and the vertical distance between the center $(3,-2)$ and $(1,1)$ is $1-(-2)=3$.

Using the Pythagorean Theorem, the distance between the points, and therefore the length of the radius, is

$$r^2 = 2^2 + 3^2$$

$$r^2 = 4 + 9$$

$$r^2 = 13$$

$$r = \sqrt{13}$$

Now plug the center and radius into the equation of the circle.

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-3)^2 + (y-(-2))^2 = 13$$

$$(x-3)^2 + (y+2)^2 = 13$$

Topic: Circles

Question: Find the center and the radius of the circle.

$$4x^2 + 4y^2 + 4x - 8y = 11$$

Answer choices:

A The center is $(-1,2)$ and the radius is 16

B The center is $(-1,2)$ and the radius is 4

C The center is $(-1/2,1)$ and the radius is 4

D The center is $(-1/2,1)$ and the radius is 2

Solution: D

Group x and y terms together.

$$(4x^2 + 4x) + (4y^2 - 8y) = 11$$

$$4(x^2 + x) + 4(y^2 - 2y) = 11$$

$$(x^2 + x) + (y^2 - 2y) = \frac{11}{4}$$

To complete the square with respect to x, we'll take the coefficient of 1 on x and divide it by 2 to get $1/2$. We'll square this value to get $(1/2)^2 = 1/4$, and then we'll add this result to both sides of the equation.

$$\left(x^2 + x + \frac{1}{4}\right) + (y^2 - 2y) = \frac{11}{4} + \frac{1}{4}$$

$$\left(x + \frac{1}{2}\right)^2 + (y^2 - 2y) = \frac{12}{4}$$

$$\left(x + \frac{1}{2}\right)^2 + (y^2 - 2y) = 3$$

To complete the square with respect to y, we'll take the coefficient of -2 on y and divide it by 2 to get $-2/2 = -1$. We'll square this value to get $(-1)^2 = 1$, and then we'll add this value to both sides of the equation.

$$\left(x + \frac{1}{2}\right)^2 + (y^2 - 2y + 1) = 3 + 1$$

$$\left(x + \frac{1}{2}\right)^2 + (y - 1)^2 = 4$$

Now the equation is in standard form and we can identify that the center is $(-1/2,1)$ and the radius is $\sqrt{4}=2$.

Topic: Circles

Question: Find the equation of the circle with center $(-3,2)$ and radius 4.

Answer choices:

A $x^2 + y^2 + 6x - 4y = 9$

B $x^2 + y^2 + 6x - 4y = 3$

C $x^2 + y^2 - 6x + 4y = 9$

D $x^2 + y^2 - 6x + 4y = 3$

Solution: B

Plug the center and the radius into the equation of the circle.

$$(x-h)^2+(y-k)^2=r^2$$

$$(x-(-3))^2+(y-2)^2=4^2$$

$$(x+3)^2+(y-2)^2=16$$

$$x^2+6x+9+y^2-4y+4=16$$

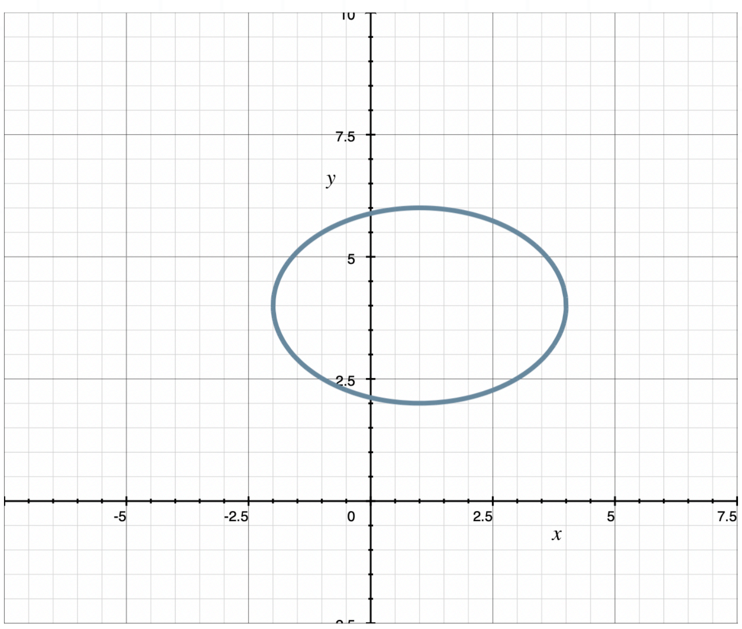
$$x^2+y^2+6x-4y=3$$

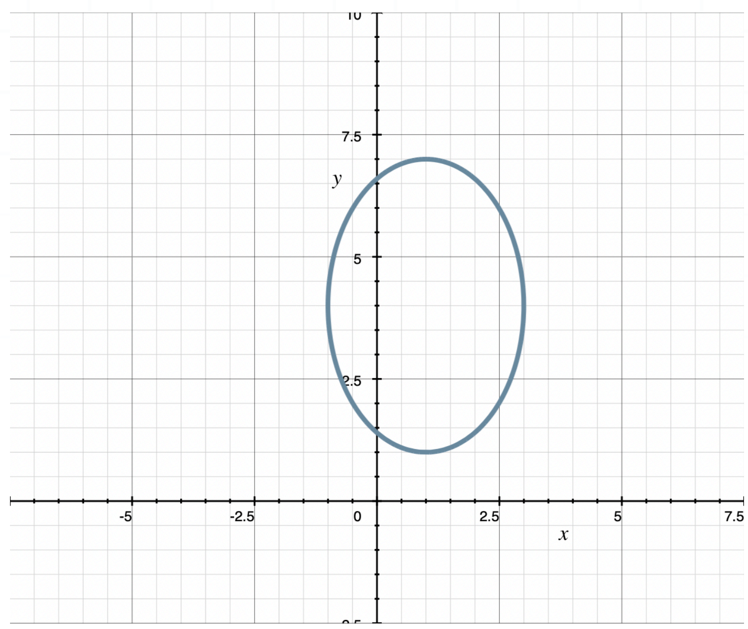
Topic: Ellipses

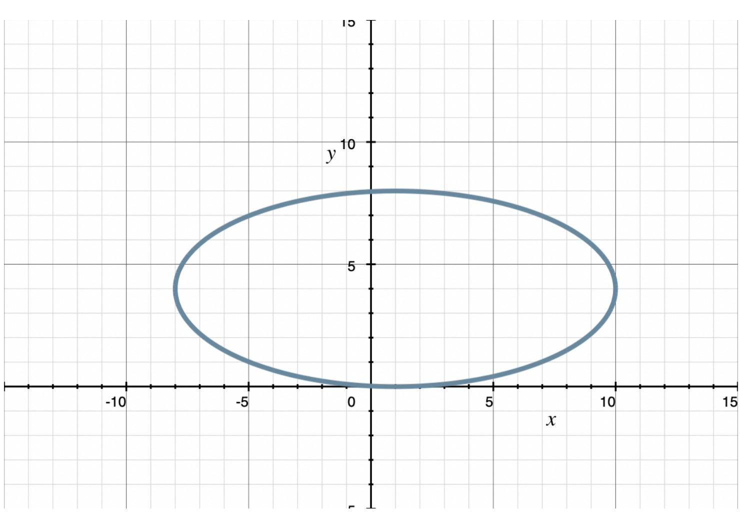
Question: Sketch the graph of the ellipse by finding its center and major and minor radii.

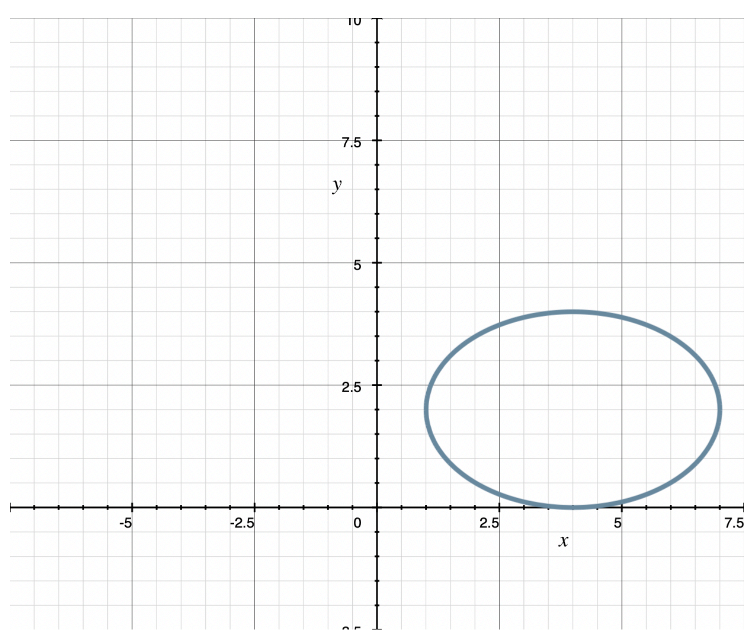
$$\frac{(x-1)^2}{9}+\frac{(y-4)^2}{4}=1$$

Answer choices:

A


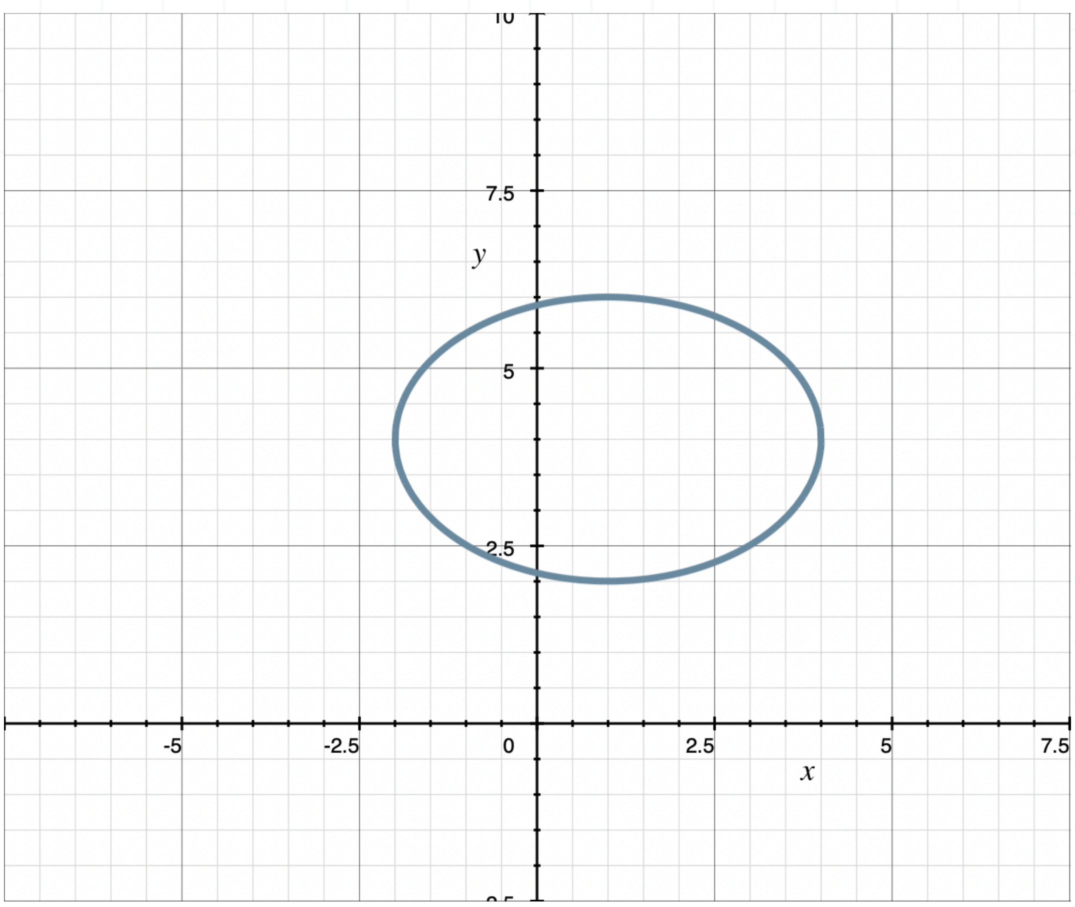
B


C


D


Solution: A

Based on the equation, the center is at $(1,4)$. The length of the horizontal radius is $\sqrt{9}=3$ and the length of the vertical radius is $\sqrt{4}=2$. So, the graph of the the ellipse is represented by A.



Topic: Ellipses

Question: Find the coordinates of the foci of the ellipse.

$$(x-2)^2+4y^2=3$$

Answer choices:

A $\left(2,-\frac{3}{2}\right)$ and $\left(2,\frac{3}{2}\right)$

B $\left(\frac{1}{2},0\right)$ and $\left(\frac{7}{2},0\right)$

C $\left(2-\frac{\sqrt{3}}{2},0\right)$ and $\left(2+\frac{\sqrt{3}}{2},0\right)$

D $(2,-1)$ and $(2,1)$

Solution: B

The standard form of the ellipse is

$$\frac{(x-2)^2}{3}+\frac{4y^2}{3}=1$$

Therefore, $a^2=3$, $b^2=3/4$, and the major axis is vertical. The focal length is

$$c=\sqrt{a^2-b^2}=\sqrt{3-\frac{3}{4}}=\sqrt{\frac{9}{4}}=\frac{3}{2}$$

We need to add and subtract $3/2$ from x-coordinate of the center $(2,0)$ to find the foci.

$$\left(2-\frac{3}{2},0\right) \text{ and } \left(2+\frac{3}{2},0\right)$$

$$\left(\frac{1}{2},0\right) \text{ and } \left(\frac{7}{2},0\right)$$

Topic: Ellipses

Question: Find the directrices of the ellipse.

$$\frac{(x+1)^2}{6}+\frac{(y+3)^2}{10}=1$$

Answer choices:

A $y=-2$ and $y=8$

B $y=-8$ and $y=2$

C $x=-2$ and $x=4$

D $x=-6$ and $x=4$

Solution: B

Because $10 > 6$, this is a tall ellipse, and $a^2 = 10$ and $b^2 = 6$. The focal length is

$$c = \sqrt{a^2 - b^2} = \sqrt{10 - 6} = \sqrt{4} = 2$$

The center of the ellipse is $(-1, -3)$, so the directrices of the tall ellipse are

$$y = k \pm \frac{a^2}{c}$$

$$y = -3 \pm \frac{10}{2}$$

$y = -8$ and $y = 2$

Topic: Parabolas

Question: Find the vertex, focus, and directrix of the parabola.

$$2y^2 - 5x + 3y - 7 = 0$$

Answer choices:

A Vertex $\left(-\frac{13}{8}, -\frac{3}{4}\right)$ Focus $\left(-1, -\frac{3}{4}\right)$ Directrix $x = -\frac{9}{4}$

B Vertex $\left(\frac{13}{8}, \frac{3}{4}\right)$ Focus $\left(-1, -\frac{3}{4}\right)$ Directrix $x = \frac{9}{4}$

C Vertex $\left(-\frac{13}{8}, -\frac{3}{4}\right)$ Focus $\left(-\frac{9}{4}, -\frac{3}{4}\right)$ Directrix $x = -\frac{9}{4}$

D Vertex $\left(-\frac{3}{4}, -\frac{13}{8}\right)$ Focus $\left(-1, -\frac{3}{4}\right)$ Directrix $x = \frac{9}{4}$

Solution: A

First, rewrite the equation of the parabola,

$$2y^2 - 5x + 3y - 7 = 0$$

$$2y^2 + 3y = 5x + 7$$

complete the square with respect to y,

$$y^2 + \frac{3}{2}y = \frac{5}{2}x + \frac{7}{2}$$

$$y^2 + \frac{3}{2}y + \frac{9}{16} = \frac{5}{2}x + \frac{7}{2} + \frac{9}{16}$$

then simplify and factor.

$$\left(y + \frac{3}{4}\right)^2 = \frac{5}{2}x + \frac{65}{16}$$

$$\left(y + \frac{3}{4}\right)^2 = \frac{5}{2}\left(x + \frac{13}{8}\right)$$

Rewriting the coefficient on the right side by factoring out a 4 will put the equation in conics form.

$$\left(y + \frac{3}{4}\right)^2 = 4\left(\frac{5}{8}\right)\left(x + \frac{13}{8}\right)$$

The parabola has a horizontal axis at $y = -3/4$ and opens to the right. Its vertex is $(-13/8, -3/4)$. Since $p = 5/8$, its focus is

$$\left(-\frac{13}{8}+\frac{5}{8}, -\frac{3}{4}\right)$$

$$\left(-1, -\frac{3}{4}\right)$$

The equation of the directrix is

$$x = -\frac{13}{8} - \frac{5}{8}$$

$$x = -\frac{9}{4}$$

Topic: Parabolas

Question: Find the equation of a parabola with focus $(1,4)$ and directrix $x = -3$.

Answer choices:

A $14(x-1) = \left(y - \frac{1}{2}\right)^2$

B $14\left(y - \frac{1}{2}\right) = (x-1)^2$

C $8(x+1) = (y-4)^2$

D $8(y-4) = (x+1)^2$

Solution: C

Because the directrix is vertical, the parabola has the form

$$4p(x-h)=(y-k)^2$$

The focus of the parabola is $(h+p,k)$, and the directrix is $x=h-p$. Given

$$k=4$$

$$h+p=1$$

$$h-p=-3$$

we solve these as a system of equations to find $h=-1$ and $p=2$. Then the equation of the parabola is

$$4(2)(x-(-1))=(y-4)^2$$

$$8(x+1)=(y-4)^2$$

Topic: Parabolas

Question: Find the focus and directrix of the parabola.

$$4y = x^2 + 2x + 3$$

Answer choices:

A The focus is $(-1,3/2)$ and the directrix is $y = -1/2$

B The focus is $(-1,3)$ and the directrix is $y = 1$

C The focus is $(0,1/2)$ and the directrix is $x = -2$

D The focus is $(1,1/2)$ and the directrix is $y = -3/2$

Solution: A

Rewrite the equation in conics form.

$$4y = x^2 + 2x + 1 - 1 + 3$$

$$4y = (x^2 + 2x + 1) - 1 + 3$$

$$4y = (x + 1)^2 + 2$$

$$4y - 2 = (x + 1)^2$$

$$4\left(y - \frac{1}{2}\right) = (x + 1)^2$$

Therefore, $p = 1$, $k = 1/2$, and $h = -1$. Using these values, we can say that the focus is

$$(h, k + p) = \left(-1, \frac{3}{2}\right)$$

and the directrix is

$$y = k - p = -\frac{1}{2}$$

Topic: Hyperbolas

Question: Find the asymptotes and foci of the hyperbola.

$$\frac{(x+2)^2}{16}-\frac{(y-4)^2}{9}=1$$

Answer choices:

A The asymptotes are $y-4=\pm(3/4)(x+2)$; the foci are $(-7,4)$, $(3,4)$

B The asymptotes are $y-4=\pm(4/3)(x+2)$; the foci are $(-2,-1)$, $(-2,9)$

C The asymptotes are $y=\pm(3/4)x$; the foci are $(-7,4)$, $(3,4)$

D The asymptotes are $y-4=\pm(16/9)(x+2)$; the foci are $(-2,-1)$, $(-2,9)$

Solution: A

Because the x^2 term is positive, the hyperbola opens left and right. We know $a=\sqrt{16}=4$ and $b=\sqrt{9}=3$, and the center is $(h,k)=(-2,4)$.

Therefore, the asymptotes are

$$y-k=\pm\frac{b}{a}(x-h)$$

$$y-4=\pm\frac{3}{4}(x+2)$$

The focal length is,

$$c=\sqrt{a^2+b^2}=\sqrt{16+9}=\sqrt{25}=5$$

so the foci $(h\pm c,k)$ are

$$(h\pm c,k)=(-2\pm 5,4)$$

$$(h\pm c,k)=(-7,4),\ (3,4)$$

Topic: Hyperbolas

Question: Find the vertices and directrices of the hyperbola.

$$\frac{(y+1)^2}{36}-\frac{(x-3)^2}{64}=1$$

Answer choices:

A The vertices are $(-3,-1)$ and $(9,-1)$ and the directrices are $y=-23/5$ and $y=13/5$

B The vertices are $(3,-7)$ and $(3,5)$ and the directrices are $x=-3/5$ and $x=33/5$

C The vertices are $(-1,-3)$ and $(-1,9)$ and the directrices are $x=-3/5$ and $x=33/5$

D The vertices are $(3,-7)$ and $(3,5)$ and the directrices are $y=-23/5$ and $y=13/5$

Solution: D

Because the y^2 term is positive, the hyperbola opens up and down. We know $a = \sqrt{36} = 6$ and $b = \sqrt{64} = 8$, and the center is $(h, k) = (3, -1)$.

Therefore, the vertices are

$$(h, k \pm a) = (3, -1 \pm 6)$$

$$(h, k \pm a) = (3, -7),\ (3,5)$$

The focal length is,

$$c = \sqrt{a^2 + b^2} = \sqrt{36 + 64} = \sqrt{100} = 10$$

so the directrices are

$$y = k \pm \frac{a^2}{c} = -1 \pm \frac{36}{10}$$

$$y = -\frac{23}{5} \text{ and } y = \frac{13}{5}$$

Topic: Hyperbolas

Question: Find the equation of a hyperbola with foci at $(0,1)$ and $(4,1)$, and one vertex at $(1,1)$.

Answer choices:

A $(x-1)^2 - \frac{(y-2)^2}{3} = 1$

B $(x-2)^2 - \frac{(y-1)^2}{3} = 1$

C $\frac{(x-2)^2}{3} - (y-1)^2 = 1$

D $\frac{(y-1)^2}{3} - (x-2)^2 = 1$

Solution: B

Because the foci have the same y-coordinate, the major axis is horizontal. Therefore, the hyperbola has the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

The foci are at

$(h-c, k) = (0,1)$ and $(h+c, k) = (4,1)$

which means we can set up the system of equations $h - c = 0$ and $h + c = 4$, and solve it to find $h = 2$, $c = 2$, and $k = 1$.

The vertex $(1,1)$ is closer to the left focus, so it's the left vertex.

$$(h-a, k) = (1,1)$$

$h - a = 1$, so $a = 1$

Then we can say

$$c = \sqrt{a^2 + b^2}$$

$$4 = 1 + b^2$$

$$b^2 = 3$$

So, the equation of the hyperbola is

$$(x-2)^2 - \frac{(y-1)^2}{3} = 1$$

Topic: Rotating axes

Question: Identify the shape of the conic.

$$3x^2 - xy + 5y^2 - 2x - 18 = 0$$

Answer choices:

A Ellipse

B Parabola

C Hyperbola

D Straight line

Solution: A

From standard form, we can identify $A=3$, $B=-1$, and $C=5$, so the value of the discriminant is

$$B^2-4AC=(-1)^2-4(3)(5)=1-60=-59<0$$

Since the discriminant is less than zero, the conic is an ellipse.

Topic: Rotating axes

Question: Which of the following is the equation of a parabola?

Answer choices:

A $4x^2 + xy + 2x + 10y = 0$

B $2x^2 + 2y^2 - 3x + 7y - 4 = 0$

C $12xy - x^2 - 3y^2 + 2x + 3 = 0$

D $x^2 + 9y^2 - 6xy - x - y = 0$

Solution: D

Find the value of the discriminant for each answer choice.

A $\quad B^2 - 4AC = 1 - 0 = 1 \neq 0$

B $\quad B^2 - 4AC = 0 - 4(4) = -16 \neq 0$

C $\quad B^2 - 4AC = 144 - 4(3) = 132 \neq 0$

D $\quad B^2 - 4AC = 36 - 4(9) = 0$

So the equation in answer choice D represents a parabola.

Topic: Rotating axes

Question: Identify the conic section represented by $4xy + 4y^2 = 1$ and determine its angle of rotation. Hint: use the trigonometric identities below to simplify the equation of the conic in X and Y.

$$\sin\theta\cos\alpha = (1/2)[\sin(\theta+\alpha) + \sin(\theta-\alpha)]$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$2\sin^2\theta = 1 - \cos(2\theta)$$

$$2\cos^2\theta = 1 + \cos(2\theta)$$

Answer choices:

A It's the ellipse $2(1+\sqrt{2})X^2 + 2(1-\sqrt{2})Y^2 = 1$ rotated by $\phi = 3\pi/8$

B It's the hyperbola $2(\sqrt{2}+1)X^2 - 2(\sqrt{2}-1)Y^2 = 1$ rotated by $\phi = 3\pi/8$

C It's the ellipse $4X^2 + Y^2 = 1$ rotated by $\phi = \pi/4$

D It's the hyperbola $4X^2 - Y^2 = 1$ rotated by $\phi = \pi/4$

Solution: B

From the equation $4xy + 4y^2 = 1$ of the conic, identify $A = 0$, $B = 4$, and $C = 4$. So the angle of rotation is

$$\cot(2\phi) = \frac{A - C}{B} = -\frac{4}{4} = -1$$

$$2\phi = \frac{3\pi}{4} + n\pi$$

$$\phi = \frac{3\pi}{8} + \frac{n\pi}{2}$$

Choosing $\phi = 3\pi/8$, we plug into the formulas for x and y in terms of X and Y to get

$$x = X\cos\left(\frac{3\pi}{8}\right) - Y\sin\left(\frac{3\pi}{8}\right)$$

$$y = X\sin\left(\frac{3\pi}{8}\right) + Y\cos\left(\frac{3\pi}{8}\right)$$

Plug these values back into the equation of the conic.

$$1 = 4xy + 4y^2$$

$$1 = 4\left[X\cos\left(\frac{3\pi}{8}\right) - Y\sin\left(\frac{3\pi}{8}\right)\right]\left[X\sin\left(\frac{3\pi}{8}\right) + Y\cos\left(\frac{3\pi}{8}\right)\right]$$

$$+4\left[X\sin\left(\frac{3\pi}{8}\right) + Y\cos\left(\frac{3\pi}{8}\right)\right]^2$$

$$1=4\left[X^2\sin\left(\frac{3\pi}{8}\right)\cos\left(\frac{3\pi}{8}\right)+XY\cos^2\left(\frac{3\pi}{8}\right)\right.$$

$$\left.-XY\sin^2\left(\frac{3\pi}{8}\right)-Y^2\sin\left(\frac{3\pi}{8}\right)\cos\left(\frac{3\pi}{8}\right)\right]$$

$$+4\left[X^2\sin^2\left(\frac{3\pi}{8}\right)+2XY\sin\left(\frac{3\pi}{8}\right)\cos\left(\frac{3\pi}{8}\right)+Y^2\cos^2\left(\frac{3\pi}{8}\right)\right]$$

$$1=4X^2\sin\left(\frac{3\pi}{8}\right)\cos\left(\frac{3\pi}{8}\right)+4XY\cos^2\left(\frac{3\pi}{8}\right)$$

$$-4XY\sin^2\left(\frac{3\pi}{8}\right)-4Y^2\sin\left(\frac{3\pi}{8}\right)\cos\left(\frac{3\pi}{8}\right)$$

$$+4X^2\sin^2\left(\frac{3\pi}{8}\right)+8XY\sin\left(\frac{3\pi}{8}\right)\cos\left(\frac{3\pi}{8}\right)+4Y^2\cos^2\left(\frac{3\pi}{8}\right)$$

$$1=4X^2\left[\sin^2\left(\frac{3\pi}{8}\right)+\sin\left(\frac{3\pi}{8}\right)\cos\left(\frac{3\pi}{8}\right)\right]$$

$$+4XY\left[2\sin\left(\frac{3\pi}{8}\right)\cos\left(\frac{3\pi}{8}\right)+\cos^2\left(\frac{3\pi}{8}\right)-\sin^2\left(\frac{3\pi}{8}\right)\right]$$

$$+4Y^2\left[\cos^2\left(\frac{3\pi}{8}\right)-\sin\left(\frac{3\pi}{8}\right)\cos\left(\frac{3\pi}{8}\right)\right]$$

Apply the product-to-sum identity $\sin\theta\cos\alpha=(1/2)[\sin(\theta+\alpha)+\sin(\theta-\alpha)]$.

$$1=4X^2\left[\sin^2\left(\frac{3\pi}{8}\right)+\frac{1}{2}\left[\sin\left(\frac{3\pi}{8}+\frac{3\pi}{8}\right)+\sin\left(\frac{3\pi}{8}-\frac{3\pi}{8}\right)\right]\right]$$

$$+4XY\left[2\frac{1}{2}\left[\sin\left(\frac{3\pi}{8}+\frac{3\pi}{8}\right)+\sin\left(\frac{3\pi}{8}-\frac{3\pi}{8}\right)\right]+\cos^2\left(\frac{3\pi}{8}\right)-\sin^2\left(\frac{3\pi}{8}\right)\right]$$

$$+4Y^2\left[\cos^2\left(\frac{3\pi}{8}\right)-\frac{1}{2}\left[\sin\left(\frac{3\pi}{8}+\frac{3\pi}{8}\right)+\sin\left(\frac{3\pi}{8}-\frac{3\pi}{8}\right)\right]\right]$$

$$1=4X^2\left[\sin^2\left(\frac{3\pi}{8}\right)+\frac{1}{2}\sin\left(\frac{3\pi}{4}\right)+\frac{1}{2}\sin(0)\right]$$

$$+4XY\left[\sin\left(\frac{3\pi}{4}\right)+\sin(0)+\cos^2\left(\frac{3\pi}{8}\right)-\sin^2\left(\frac{3\pi}{8}\right)\right]$$

$$+4Y^2\left[\cos^2\left(\frac{3\pi}{8}\right)-\frac{1}{2}\sin\left(\frac{3\pi}{4}\right)-\frac{1}{2}\sin(0)\right]$$

$$1=4X^2\left[\sin^2\left(\frac{3\pi}{8}\right)+\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\right]+4XY\left[\frac{\sqrt{2}}{2}+\cos^2\left(\frac{3\pi}{8}\right)-\sin^2\left(\frac{3\pi}{8}\right)\right]$$

$$+4Y^2\left[\cos^2\left(\frac{3\pi}{8}\right)-\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\right]$$

$$1=4X^2\left[\sin^2\left(\frac{3\pi}{8}\right)+\frac{\sqrt{2}}{4}\right]+4XY\left[\frac{\sqrt{2}}{2}+\cos^2\left(\frac{3\pi}{8}\right)-\sin^2\left(\frac{3\pi}{8}\right)\right]$$

$$+4Y^2\left[\cos^2\left(\frac{3\pi}{8}\right)-\frac{\sqrt{2}}{4}\right]$$

Apply the double-angle identity $\cos(2\theta)=\cos^2\theta-\sin^2\theta$.

$$1=4X^2\left[\sin^2\left(\frac{3\pi}{8}\right)+\frac{\sqrt{2}}{4}\right]+4XY\left[\frac{\sqrt{2}}{2}+\cos\left(2\cdot\frac{3\pi}{8}\right)\right]$$

$$+4Y^2\left[\cos^2\left(\frac{3\pi}{8}\right)-\frac{\sqrt{2}}{4}\right]$$

$$1=4X^2\left[\sin^2\left(\frac{3\pi}{8}\right)+\frac{\sqrt{2}}{4}\right]+4XY\left[\frac{\sqrt{2}}{2}+\cos\left(\frac{3\pi}{4}\right)\right]$$

$$+4Y^2\left[\cos^2\left(\frac{3\pi}{8}\right)-\frac{\sqrt{2}}{4}\right]$$

$$1=4X^2\left[\sin^2\left(\frac{3\pi}{8}\right)+\frac{\sqrt{2}}{4}\right]+4XY\left[\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right]+4Y^2\left[\cos^2\left(\frac{3\pi}{8}\right)-\frac{\sqrt{2}}{4}\right]$$

$$1=4X^2\left[\sin^2\left(\frac{3\pi}{8}\right)+\frac{\sqrt{2}}{4}\right]+4Y^2\left[\cos^2\left(\frac{3\pi}{8}\right)-\frac{\sqrt{2}}{4}\right]$$

$$1=2X^2\left[2\sin^2\left(\frac{3\pi}{8}\right)+\frac{\sqrt{2}}{2}\right]+2Y^2\left[2\cos^2\left(\frac{3\pi}{8}\right)-\frac{\sqrt{2}}{2}\right]$$

Apply the double-angle identities $2\sin^2\theta=1-\cos(2\theta)$ and $2\cos^2\theta=1+\cos(2\theta)$.

$$1 = 2X^2\left[1 - \cos\left(2 \cdot \frac{3\pi}{8}\right) + \frac{\sqrt{2}}{2}\right] + 2Y^2\left[1 + \cos\left(2 \cdot \frac{3\pi}{8}\right) - \frac{\sqrt{2}}{2}\right]$$

$$1 = 2X^2\left[1 - \cos\left(\frac{3\pi}{4}\right) + \frac{\sqrt{2}}{2}\right] + 2Y^2\left[1 + \cos\left(\frac{3\pi}{4}\right) - \frac{\sqrt{2}}{2}\right]$$

$$1 = 2X^2\left[1 - \left(-\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}\right] + 2Y^2\left[1 + \left(-\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}\right]$$

$$1 = 2X^2(1 + \sqrt{2}) + 2Y^2(1 - \sqrt{2})$$

$$2(\sqrt{2} + 1)X^2 - 2(\sqrt{2} - 1)Y^2 = 1$$

Therefore, the equation represents a rotated hyperbola.

Topic: Polar equations of conics

Question: Find the eccentricity of the conic section.

$$\frac{x^2}{6}+\frac{y^2}{2}=1$$

Answer choices:

A $\frac{1}{3}$

B $\frac{2}{3}$

C $\frac{1}{\sqrt{3}}$

D $\sqrt{\frac{2}{3}}$

Solution: D

The equation is an ellipse, and its foci are located at

$$(\pm c,0) = (\pm\sqrt{6-2},0) = (\pm 2,0)$$

The directrices are

$$x = \pm\frac{a^2}{c}$$

$$x = \pm\frac{6}{2} = \pm 3$$

If we choose any point on the ellipse, like $(x, y) = (\sqrt{6},0)$, then we can calculate the eccentricity as

$$e = \frac{d((\sqrt{6},0),(2,0))}{d((\sqrt{6},0),(3,0))} = \frac{\sqrt{6}-2}{\sqrt{6}-3} = \frac{\sqrt{2}(\sqrt{3}-\sqrt{2})}{\sqrt{3}(\sqrt{2}-\sqrt{3})} = -\frac{\sqrt{2}(\sqrt{3}-\sqrt{2})}{\sqrt{3}(\sqrt{3}-\sqrt{2})} = -\sqrt{\frac{2}{3}}$$

But the eccentricity has to be a positive constant, so $e = \sqrt{2/3}$.

Topic: Polar equations of conics

Question: A parabola's focus is $(0,0)$ and its vertex is $(2,0)$. Write the equation of the parabola in polar coordinates.

Answer choices:

A $\quad r = \dfrac{2}{1 + \cos\theta}$

B $\quad r = \dfrac{4}{1 + \cos\theta}$

C $\quad r = \dfrac{2}{1 - \cos\theta}$

D $\quad r = \dfrac{4}{1 - \cos\theta}$

Solution: B

The focus and the vertex of the parabola both fall on the horizontal line $y=0$, so the parabola is a shifted left/right parabola.

Therefore, we can say the focus is $(h+p,k)=(0,0)$ and the vertex is $(h,k)=(2,0)$, which means $h=2$ and $p=-2$. Then the directrix is

$$d=x=h-p$$

$$d=x=4$$

The eccentricity of a parabola is always $e=1$, which means that the polar equation of the conic is

$$r=\frac{ed}{1+e\cos\theta}$$

$$r=\frac{4}{1+\cos\theta}$$

Topic: Polar equations of conics

Question: Find the equation of the conic section that has eccentricity $e = 3/2$, directrix $y = -5$, and that's rotated by $\alpha = 2\pi/3$.

Answer choices:

A $\quad r = \dfrac{15}{2 + 3\sin\left(\theta - \frac{2\pi}{3}\right)}$

B $\quad r = \dfrac{5}{1 + 3\cos\left(\theta - \frac{2\pi}{3}\right)}$

C $\quad r = \dfrac{15}{2 - 3\sin\left(\theta - \frac{2\pi}{3}\right)}$

D $\quad r = \dfrac{10}{2 - 3\cos\left(\theta - \frac{2\pi}{3}\right)}$

Solution: C

Given the directrix $y = -5$, the polar equation will be

$$r = \frac{ed}{1 - e\sin(\theta - \alpha)}$$

$$r = \frac{\frac{3}{2}(5)}{1 - \frac{3}{2}\sin\left(\theta - \frac{2\pi}{3}\right)}$$

$$r = \frac{\frac{15}{2}}{1 - \frac{3}{2}\sin\left(\theta - \frac{2\pi}{3}\right)}$$

Multiplying by $2/2$ to clear the fractions, we get

$$r = \frac{15}{2 - 3\sin\left(\theta - \frac{2\pi}{3}\right)}$$

Topic: Parametric curves and eliminating the parameter

Question: Eliminate the parameter. Hint: Use the law of logarithms $\ln(AB) = \ln A + \ln B$ to rewrite y.

$$x = t^3 + t$$

$$y = \ln|t| + \ln(t^2 + 1)$$

Answer choices:

A $\quad x = e^y$

B $\quad x = e^{|y|}$

C $\quad y = \ln x$

D $\quad y = \ln|x|$

Solution: D

Apply $\ln(AB) = \ln A + \ln B$.

$$y = \ln|t| + \ln(t^2 + 1)$$

$$y = \ln(|t| \cdot (t^2 + 1))$$

$$y = \ln|t^3 + t|$$

Substitute $x = t^3 + t$ to eliminate the parameter.

$$y = \ln|x|$$

Topic: Parametric curves and eliminating the parameter

Question: Eliminate the parameter. Hint: Use the hyperbolic identity $\cosh^2(t) - \sinh^2(t) = 1$.

$$x = 2\sinh(t)$$

$$y = 3\cosh(t)$$

Answer choices:

A $\dfrac{x^2}{2} - \dfrac{y^2}{3} = 1$

B $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

C $\dfrac{y^2}{3} - \dfrac{x^2}{2} = 1$

D $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$

Solution: D

Rewrite the parametric equations by solving them for the hyperbolic functions. We get

$$x = 2\sinh(t)$$

$$\sinh(t) = \frac{x}{2}$$

$$\sinh^2(t) = \frac{x^2}{4}$$

and

$$y = 3\cosh(t)$$

$$\cosh(t) = \frac{y}{3}$$

$$\cosh^2(t) = \frac{y^2}{9}$$

Plugging these values into the hyperbolic identity, we get

$$\cosh^2(t) - \sinh^2(t) = 1$$

$$\frac{y^2}{9} - \frac{x^2}{4} = 1$$

Topic: Parametric curves and eliminating the parameter

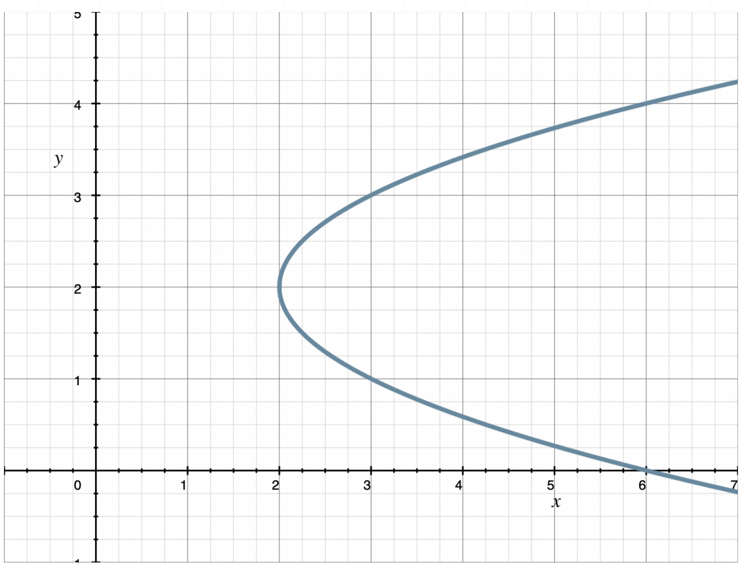
Question: Sketch the curve defined by the parametric equations.

$$x = \frac{t^2}{2}$$

$$y = \frac{1}{t^2}$$

Answer choices:

A



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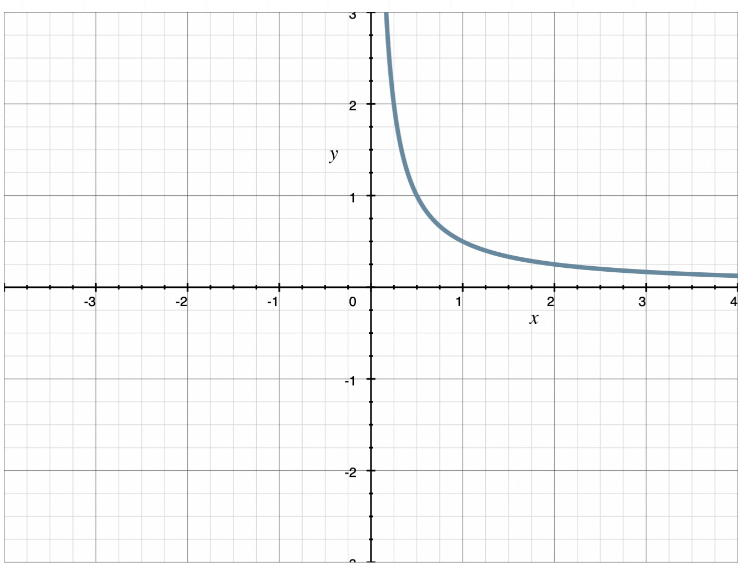

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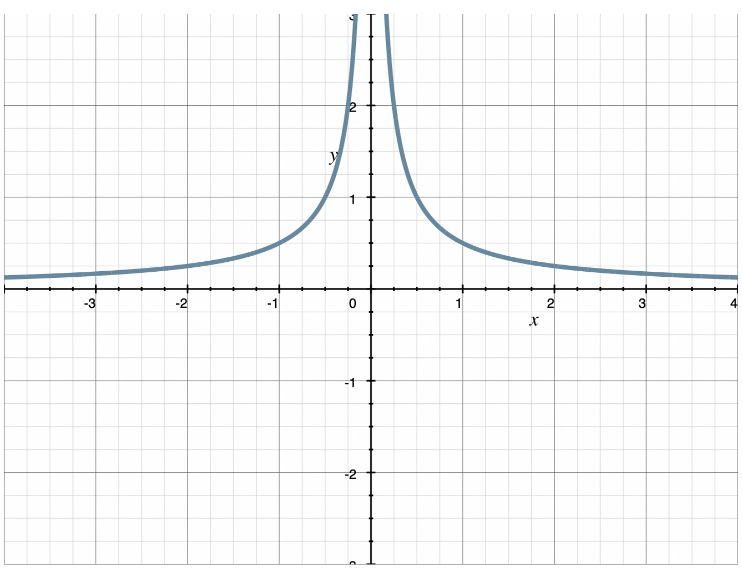

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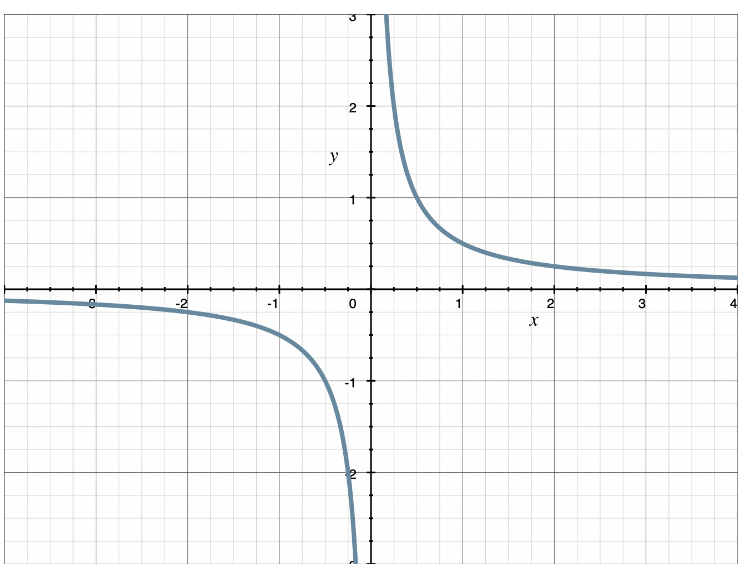
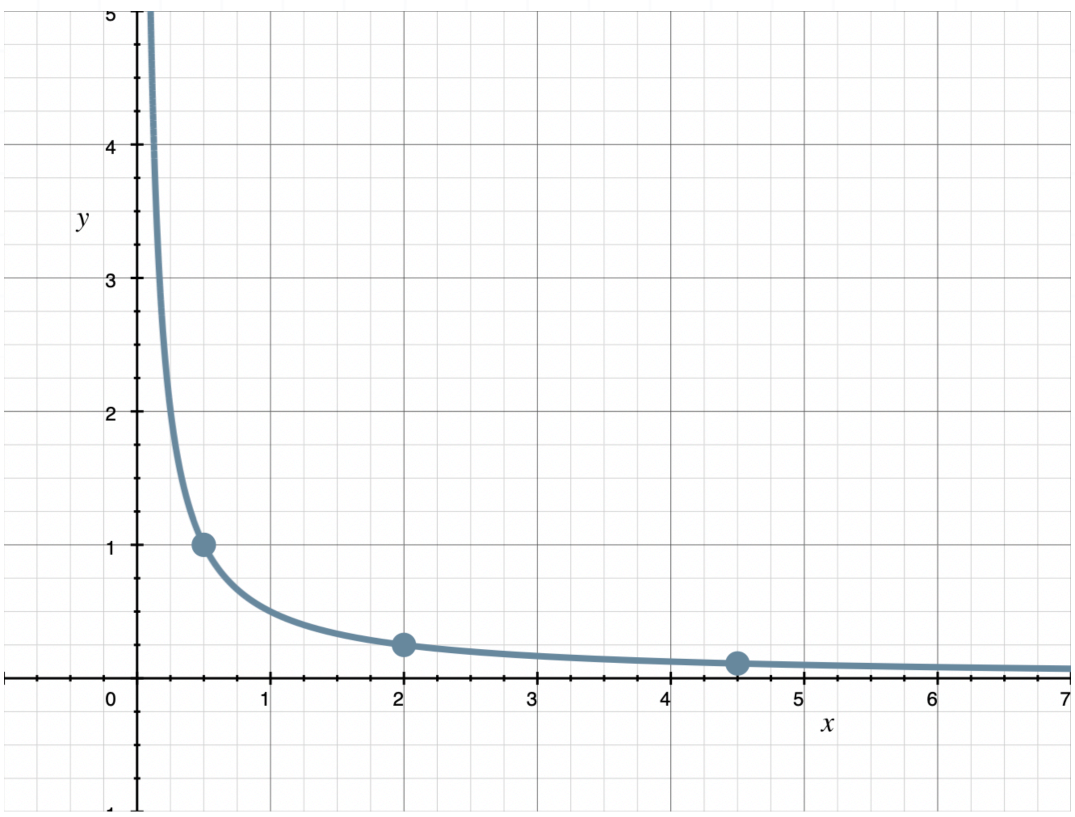


Solution: B

If we plug various values of t into the equations for both x and y, we get

t	-4	-3	-2	-1	0	1	2	3	4
x	8	9/2	2	1/2	0	1/2	2	9/2	8
y	1/16	1/9	1/4	1	-	1	1/4	1/9	1/16

If we plot and connect these (x, y) coordinate points, we can see a sketch of the parametric equation.



Topic: Direction of the parameter

Question: Consider the parametric equations $x = 3 + 2\sin t$ and $y = 2 - \cos t$, where t is defined on $[-\pi/2, \pi/2]$. What is the shape of the parametric curve?

Answer choices:

A The lower half of an ellipse traced out counterclockwise

B The upper half of a circle traced out clockwise

C The right half of an ellipse traced out clockwise

D The left half of a circle traced out counterclockwise

Solution: A

If we solve the parametric equations for $\sin t$ and $\cos t$, we get

$$x = 3 + 2\sin t$$

$$\sin t = \frac{x-3}{2}$$

and

$$y = 2 - \cos t$$

$$\cos t = 2 - y$$

Substituting these values into the Pythagorean identity for sine and cosine,

$$\sin^2 t + \cos^2 t = 1$$

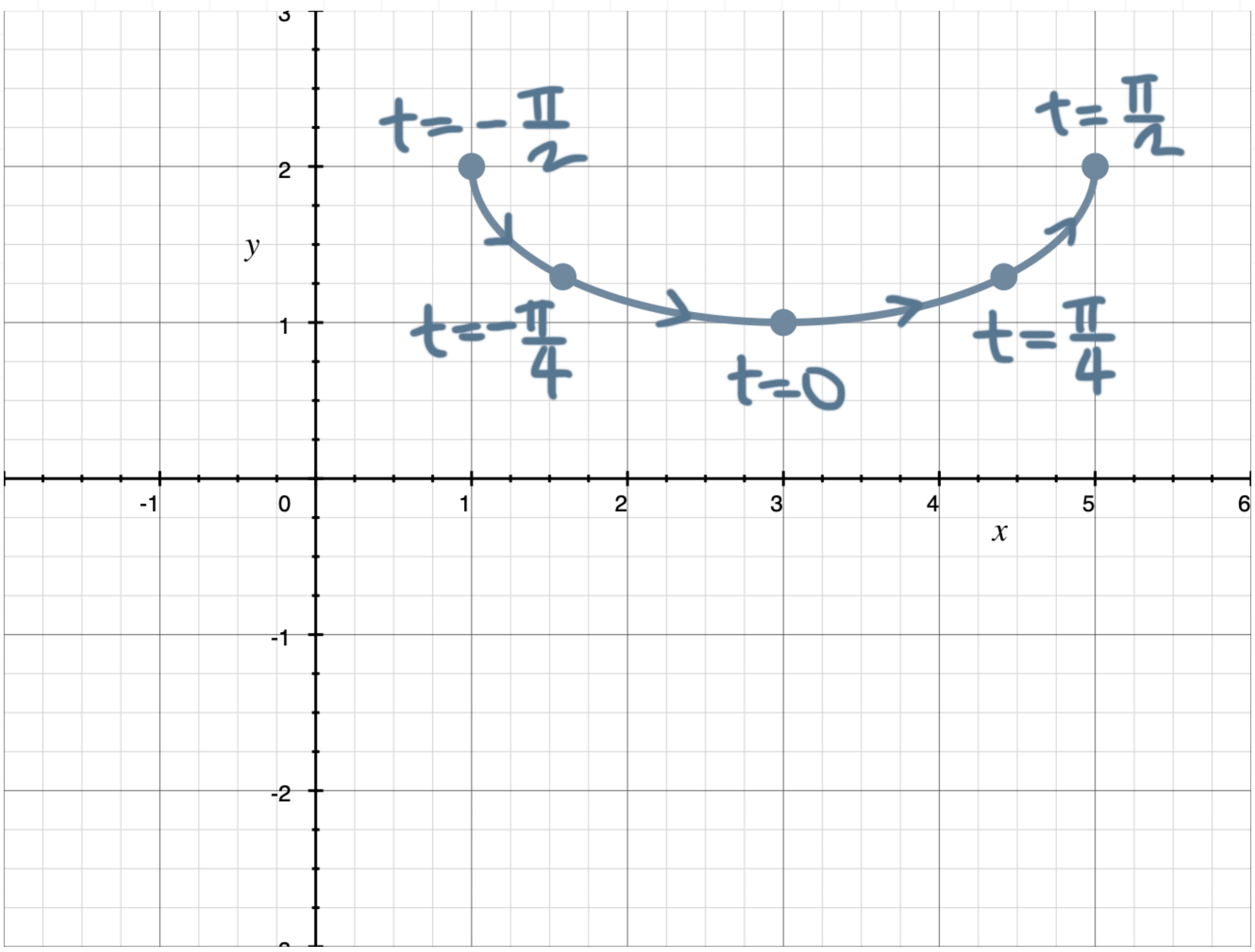
$$\left(\frac{x-3}{2}\right)^2 + (2-y)^2 = 1$$

$$\frac{(x-3)^2}{4} + (2-y)^2 = 1$$

we see that the parametric equations represent an ellipse. Given the interval $[-\pi/2,\pi/2]$, let's find a few points in the interval along the curve.

t	-π/2	-π/4	0	π/4	π/2
x	1	3-√2	3	3+√2	5
y	2	2-√2/2	1	2-√2/2	2

If we plot these points, connect them with a smooth curve, and indicate the direction of increasing t based on the values in our table, we get the lower half of the ellipse as it's traced out in a counterclockwise direction.



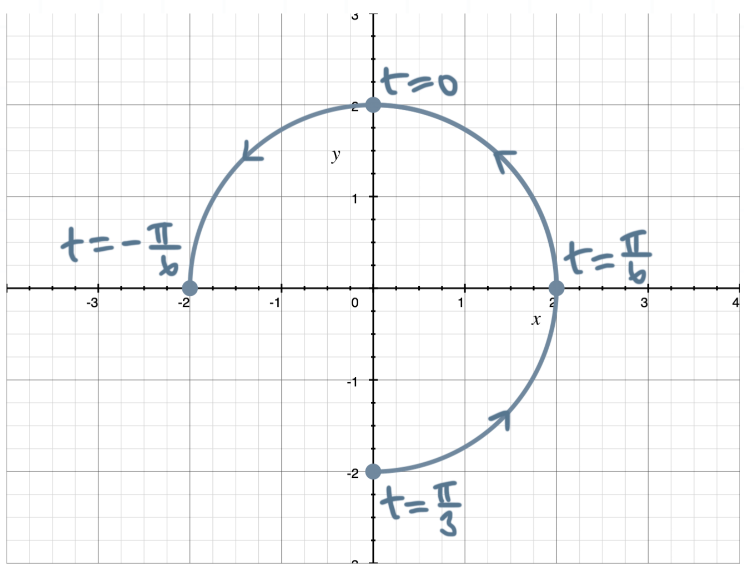
Topic: Direction of the parameter

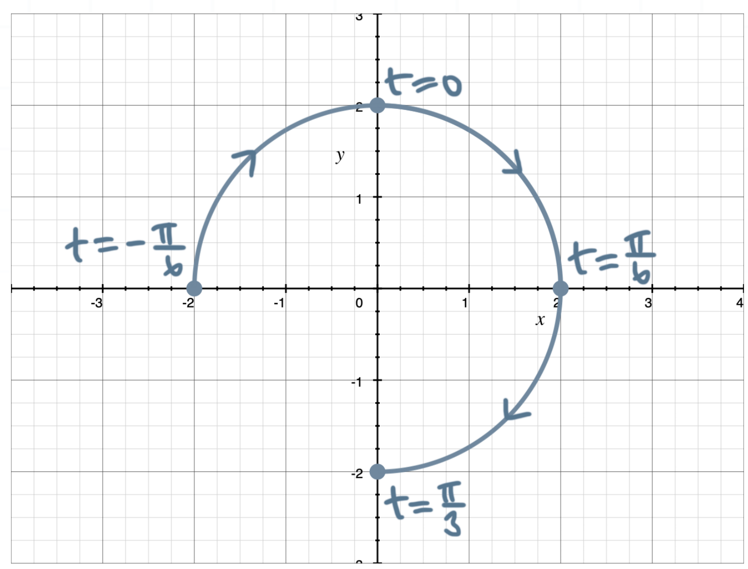
Question: Sketch the graph of the parametric curve on the interval $-\pi/6 \leq t \leq \pi/3$.

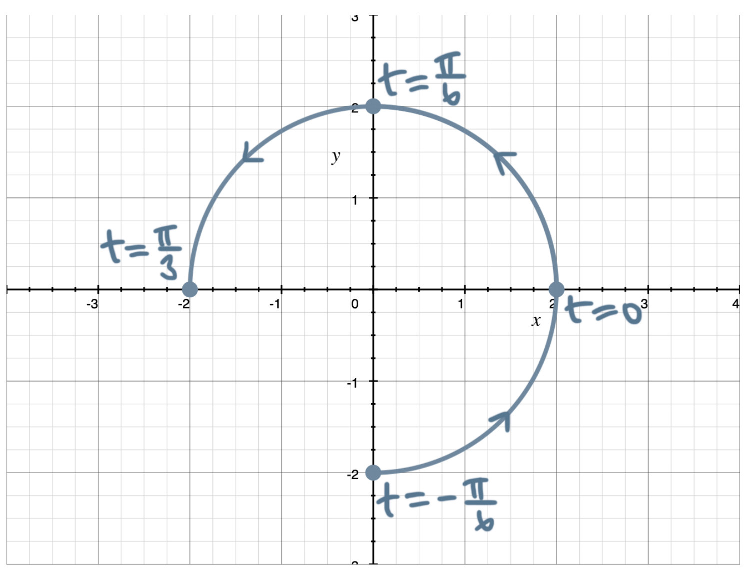
$$x = 2\cos(3t)$$

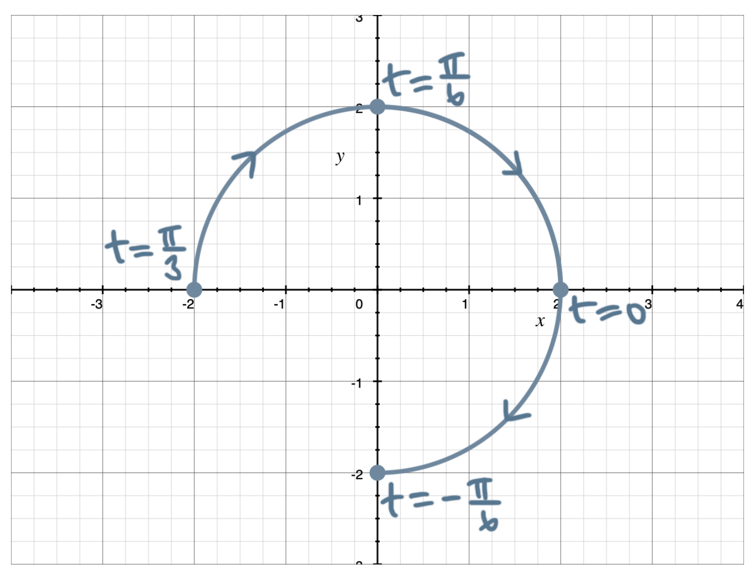
$$y = 2\sin(3t)$$

Answer choices:

A 

B 

C 

D 

Solution: C

If we solve the parametric equations for $\sin(3t)$ and $\cos(3t)$, we get

$$x = 2\cos(3t)$$

$$\cos(3t) = \frac{x}{2}$$

and

$$y = 2\sin(3t)$$

$$\sin(3t) = \frac{y}{2}$$

Substituting these values into the Pythagorean identity for sine and cosine,

$$\sin^2 t + \cos^2 t = 1$$

$$\sin^2(3t) + \cos^2(3t) = 1$$

$$\left(\frac{y}{2}\right)^2 + \left(\frac{x}{2}\right)^2 = 1$$

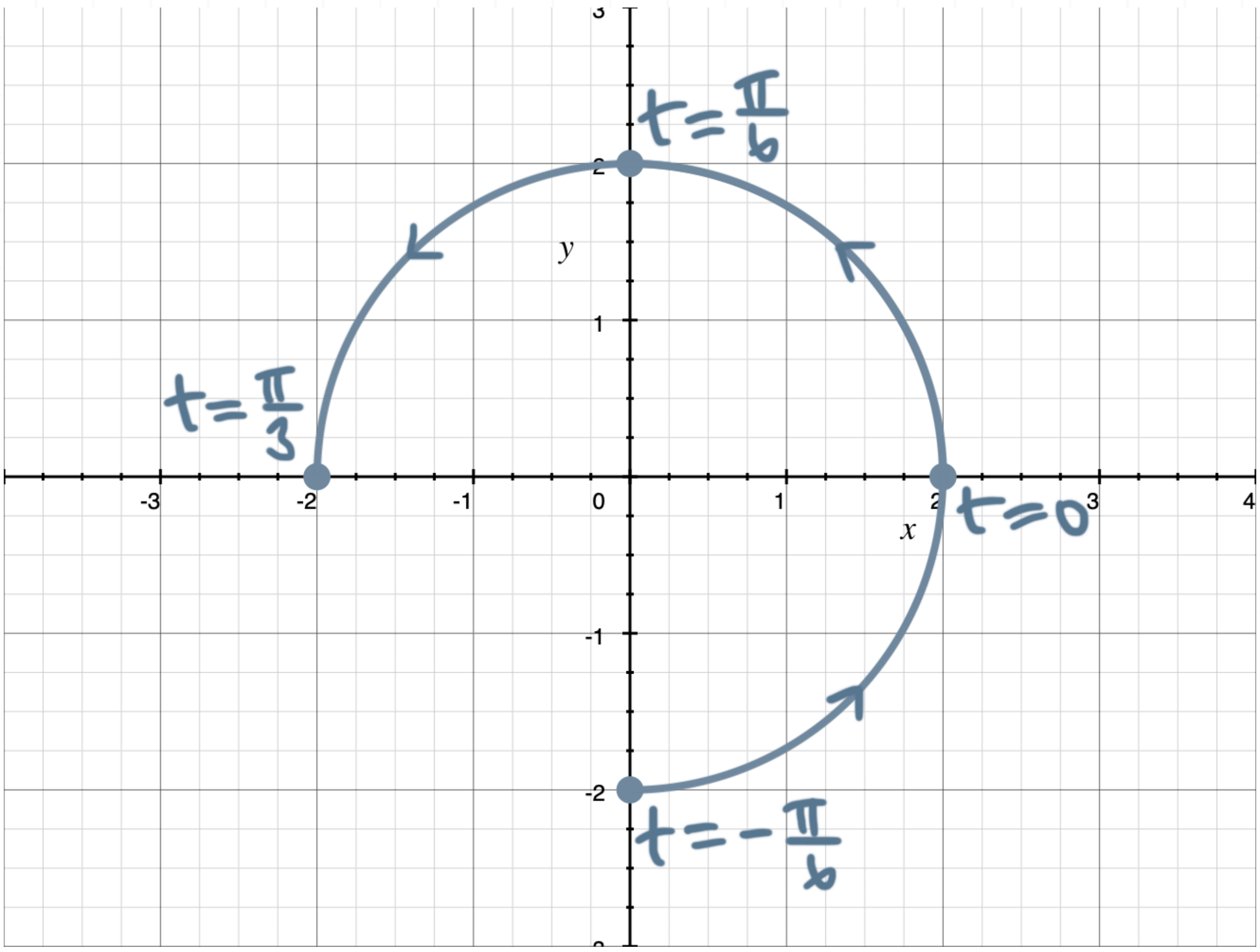
$$\frac{x^2}{4} + \frac{y^2}{4} = 1$$

$$x^2 + y^2 = 4$$

we see that the parametric equations represent a circle centered at the origin with radius 2. Given the interval $-\pi/6 \le t \le \pi/3$, let's find a few points in the interval along the curve.

t	-π/6	0	π/6	π/3
x	0	2	0	-2
y	-2	0	2	0

If we plot these points, connect them with a smooth curve, and indicate the direction of increasing t based on the values in our table, we get three-quarters of a circle as it's traced out in a counterclockwise direction.



Topic: Direction of the parameter

Question: Which of the parameterizations of a circle has a clockwise direction for increasing t?

Answer choices:

A $x = \cos t$ and $y = \sin t$

B $x = \cos(t - (\pi/2))$ and $y = \sin(t - (\pi/2))$

C $x = -\cos(t + \pi)$ and $y = \sin(t + \pi)$

D $x = -\cos(\pi - t)$ and $y = \sin(\pi - t)$

Solution: C

The direction of increasing t for the parametric curve in answer choice A is counterclockwise.

t	0	π/2	π
x	1	0	-1
y	0	1	0

The direction of increasing t for the parametric curve in answer choice B is counterclockwise.

t	0	π/2	π
x	0	1	0
y	-1	0	1

The direction of increasing t for the parametric curve in answer choice B is clockwise.

t	0	π/2	π
x	1	0	-1
y	0	-1	0

The direction of increasing t for the parametric curve in answer choice D is counterclockwise.

t	0	π/2	π
x	1	0	-1
y	0	1	0

Topic: Finding the parametric representation

Question: Express the rectangular equation in parametric form.

$$x^2 - 2(y - 1) = 0$$

Answer choices:

A $\quad x = t$ and $y = \dfrac{t^2}{2}$

B $\quad x = t$ and $y = \dfrac{t^2}{2} + 1$

C $\quad x = t$ and $y = \dfrac{t^2}{2} + \dfrac{1}{2}$

D $\quad x = t$ and $y = t^2 + 2$

Solution: B

Solve the equation for y in terms of x.

$$x^2 - 2(y - 1) = 0$$

$$x^2 - 2y + 2 = 0$$

$$x^2 + 2 = 2y$$

$$y = \frac{x^2}{2} + 1$$

Setting $x = t$, the parametric equations are

$$x = t$$

$$y = \frac{t^2}{2} + 1$$

Topic: Finding the parametric representation

Question: Express the rectangular equation in parametric form.

$$\frac{(x+4)^2}{16}+\frac{(y-2)^2}{4}=1$$

Answer choices:

A $x=-4+4\cos t$ and $y=2+2\sin t$

B $x=-4+\cos(4t)$ and $y=2+\sin(2t)$

C $x=-4+16\cos t$ and $y=2+4\sin t$

D $x=16t-4$ and $y=-4t-2$

Solution: A

Notice that the given equation is the equation of an ellipse that's centered at $(-4,2)$ with radius 1. If we rewrite the equation as,

$$\frac{(x+4)^2}{16}+\frac{(y-2)^2}{4}=1$$

$$\left(\frac{x+4}{4}\right)^2+\left(\frac{y-2}{2}\right)^2=1$$

then we can match this equation to the Pythagorean identity $\sin^2 t+\cos^2 t=1$. We'll set

$$\cos t=\frac{x+4}{4} \qquad \sin t=\frac{y-2}{2}$$

$$4\cos t=x+4 \qquad 2\sin t=y-2$$

to get the parametric equations

$$x=-4+4\cos t$$

$$y=2+2\sin t$$

Topic: Finding the parametric representation

Question: Express the rectangular equation in parametric form.

$$(x+1)^2-(y-3)^2=1$$

Answer choices:

A $x=-1+t$ and $y=3+(1/t)$

B $x=-1+\tan t$ and $y=3t\cot t$

C $x=-1+\cosh t$ and $y=3+\sinh t$

D $x=\sqrt{t}-1$ and $y=\sqrt{t}+3$

Solution: C

We can match the equation of the hyperbola $(x+1)^2-(y-3)^2=1$ with the trigonometric identity,

$$\cosh^2 t - \sinh^2 t = 1$$

to get equations for x and y.

$$x+1=\cosh t \qquad\qquad y-3=\sinh t$$

$$x=-1+\cosh t \qquad\qquad y=3+\sinh t$$

W W W . K R I S T A K I N G M A T H . C O M