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Subject: Statistical Methods

Tutorial: Tutorial 8

Multivariate regression

Q1.

$$Y_i = \beta_0 + \beta_i z_{i1} + \varepsilon_i$$

z_1	0	1	2	3	4
Y	1	4	3	8	9

$$Z' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

$$Z'Z = \begin{bmatrix} 5 & 10 \\ 10 & 30 \end{bmatrix}$$

$$(Z'Z)^{-1} = \begin{bmatrix} 0.6 & -0.2 \\ -0.2 & 0.1 \end{bmatrix}$$

$$Z'Y = \begin{bmatrix} 25 \\ 70 \end{bmatrix}$$

$$\hat{\beta} = \begin{bmatrix} \hat{\beta_0} \\ \hat{\beta_1} \end{bmatrix} = (Z'Z)^{-1} \cdot Z'Y = \begin{bmatrix} 0.6 & -0.2 \\ -0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 25 \\ 70 \end{bmatrix}$$

$$\hat{\beta} = \begin{bmatrix} \hat{\beta_0} \\ \hat{\beta_1} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Fitted equation : $\hat{Y} = 1 + 2z_1$

$$\hat{Y} = z\hat{\beta} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{bmatrix}$$

For $z_1 = 5$;

$$\hat{y}_i = 1 + 2(5)$$
$$\hat{y}_i = 11$$

$$\varepsilon' = Y - \hat{Y} = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 8 \\ 9 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$$S^2 = \frac{SSE}{n-k-1}$$

$$SSE = Y'Y - \hat{\beta} z'Y$$
$$= 171 - 165$$
$$= 6$$

$$\sigma^2 = \frac{6}{5-1-1} = 2$$

Residual sum of square is $\hat{\varepsilon}' \cdot \hat{\varepsilon} = 0^2 + 1^2 + (-2)^2 + (1)^2 + (0)^2$
$= 6$

$$\text{Var}(\hat{\beta}) = \sigma^2 (Z'Z)^{-1}$$

$$= \begin{bmatrix} 1.2 & -0.4 \\ -0.4 & 0.2 \end{bmatrix}$$

$$P = Z(Z'Z)^{-1} Z'$$

$$= \begin{bmatrix} 0.6 & 0.4 & 0.2 & 0 & -0.2 \\ 0.4 & 0.3 & 0.2 & 0.1 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0.1 & 0.2 & 0.3 & 0.4 \\ -0.2 & 0 & 0.2 & 0.4 & 0.6 \end{bmatrix}$$

$$\text{Var}(\hat{Y}) = P\sigma^2$$

$$= \begin{bmatrix} 1.2 & 0.8 & 0.4 & 0 & -0.4 \\ 0.8 & 0.6 & 0.4 & 0.2 & 0 \\ 0.4 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0.2 & 0.4 & 0.6 & 0.8 \\ -0.4 & 0 & 0.4 & 0.8 & 1.2 \end{bmatrix}$$

$$\text{Var}(\varepsilon) = (I - P)\sigma^2$$

$$= 2 \begin{bmatrix} 0.4 & -0.4 & -0.2 & 0 & 0.2 \\ -0.4 & 0.7 & -0.2 & 0.1 & 0 \\ -0.2 & -0.2 & 0.8 & -0.2 & -0.2 \\ 0 & -0.1 & 0.2 & 0.7 & -0.4 \\ 0.2 & 0 & -0.2 & -0.4 & 0.4 \end{bmatrix}$$

$$\text{Var}(e) = \begin{bmatrix} 0.8 & -0.8 & -0.4 & 0 & 0.4 \\ -0.8 & 1.4 & -0.4 & -0.2 & 0 \\ -0.4 & -0.4 & 1.6 & -0.4 & -0.4 \\ 0 & -0.2 & -0.4 & 1.4 & -0.8 \\ 0.4 & 0 & -0.4 & -0.8 & 0.8 \end{bmatrix}$$

Q2.

$$Y_i = \beta_0 + \beta_1 x_{i1} + \varepsilon_i$$

x_1	0	1	2
Y	6	0	0

$$X' = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$Y = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$X'X = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$(X'X)^{-1} = \begin{bmatrix} 0.833 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

$$X'Y = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = (X'X)^{-1} \cdot X'Y = \begin{bmatrix} 4.998 \\ -3 \end{bmatrix}$$

Fitted eqn $\Rightarrow \hat{Y} = 4.998 - 3X_1$

For $X_1 = 1.5$, $Y = 0.498$

For $x_1 = 3$, $Y = -4.002$

$$\hat{Y} = X\hat{\beta} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4.998 \\ -3 \end{bmatrix} = \begin{bmatrix} 4.998 \\ 1.998 \\ -1.002 \end{bmatrix}$$

$$\varepsilon' = Y - \hat{Y} = \begin{bmatrix} 1.002 \\ -1.998 \\ 1.002 \end{bmatrix}$$

$$\begin{aligned} SSE &= Y'Y - \hat{\beta} X'Y \\ &= 36 - 29.988 \\ &= 6.012 \\ &\approx 6 \end{aligned}$$

$$\sigma^2 = \frac{SSE}{n-k-1} = \frac{6.012}{3-1-1} = 6.012 \approx 6$$

$$\begin{aligned} \text{Residual sum of squares} &= (1.002)^2 + (-1.998)^2 + (1.002)^2 \\ &= 6.000012 \approx 6 \end{aligned}$$

$$\text{Var}(\hat{\beta}) = \sigma^2 (X'X)^{-1}$$

$$= \begin{bmatrix} 4.998 & -3 \\ -3 & 3 \end{bmatrix}$$

$$P = X (X'X)^{-1} X'$$

$$P = \begin{bmatrix} 5/6 & 1/3 & -1/6 \\ 1/3 & 1/3 & 1/3 \\ -1/6 & 1/3 & 5/6 \end{bmatrix}$$

$$\text{Var}(\hat{Y}) = P\sigma^2$$

$$= \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$$

$$\text{Var}(e) = (I - P)\sigma^2$$

$$= 6 \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

Q3.

$$Y_i = \beta_0 + \beta_1 z_{i1} + \varepsilon_i$$

z_1	2	4	6
Y	3	6	7

$$Z' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \end{bmatrix}$$

$$Z'Z = \begin{bmatrix} 3 & 12 \\ 12 & 56 \end{bmatrix} ; (Z'Z)^{-1} = \begin{bmatrix} 7/3 & -1/2 \\ -1/2 & 1/8 \end{bmatrix}$$

$$Z'Y = \begin{bmatrix} 16 \\ 72 \end{bmatrix}$$

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = (Z'Z)^{-1} \cdot Z'Y = \begin{bmatrix} 7/3 & -1/2 \\ -1/2 & 1/8 \end{bmatrix} \cdot \begin{bmatrix} 16 \\ 72 \end{bmatrix}$$

$$= \begin{bmatrix} 1.28 \\ 1 \end{bmatrix}$$

Fitted Eqn $\Rightarrow \hat{Y} = 1.28 + z_1$

For $z = 2.5$, $\hat{Y} = 3.78$

For $z = 7$, $\hat{Y} = 8.28$

$\hat{Y} = z\hat{\beta} =$

$$\hat{Y} = Z\hat{\beta} = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1.28 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.28 \\ 5.28 \\ 7.28 \end{bmatrix}$$

$$\varepsilon' = Y - \hat{Y} = \begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix} - \begin{bmatrix} 3.28 \\ 5.28 \\ 7.28 \end{bmatrix} = \begin{bmatrix} -0.28 \\ 0.72 \\ -0.28 \end{bmatrix}$$

Residual sum of squares is :

$$= (-0.28)^2 + (0.72)^2 + (-0.28)^2$$
$$= 0.6752$$

$$SSE = Y'Y - \hat{\beta} Z'Y$$
$$= 94 - 92.48$$
$$= 1.52$$

$$\sigma^2 = \frac{1.52}{3-1-1} = 1.52$$

$$\text{Var}(\hat{\beta}) = \sigma^2 (Z'Z)^{-1}$$

$$= \begin{bmatrix} 3.55 & -0.76 \\ -0.76 & 0.19 \end{bmatrix}$$

$$P = Z(Z'Z)^{-1}Z'$$

$$= \begin{bmatrix} 0.83 & 0.33 & -0.17 \\ 0.33 & 0.33 & 0.33 \\ -0.17 & 0.33 & 0.83 \end{bmatrix}$$

$$\text{Var}(\hat{Y}) = P\sigma^2$$

$$= \begin{bmatrix} 1.2616 & 0.5016 & -0.2584 \\ 0.5016 & 0.5016 & 0.5016 \\ -0.2584 & 0.5016 & 1.2616 \end{bmatrix}$$

$$\text{Var}(e) = (I-P)\sigma^2$$

$$= \begin{bmatrix} 0.17 & -0.33 & 0.17 \\ -0.33 & 0.67 & -0.33 \\ 0.17 & -0.33 & 0.17 \end{bmatrix} \times 1.52$$

$$= \begin{bmatrix} 0.2584 & -0.5016 & 0.2584 \\ 0.5016 & 1.0184 & 0.5016 \\ 0.2584 & -0.5016 & 0.2584 \end{bmatrix}$$