COL202 Homework 1

Aniruddha Deb, Prabal Singh

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1. An IPL tournament is played between n cricket teams, where each team plays exactly one match with every other team. How many matches are played? (This is easy.) Assume that no match ends in a tie. We say that a subset S of teams is *consistent* if it is possible to order teams in S as $T_1, \ldots, T_{|S|}$ (think of this as the strongest to weakest ordering) such that for every i, j with $1 \leq i < j \leq |S|$, T_i beats T_j. Prove that irrespective of the outcomes of the matches, there always exists a consistent subset S with $|S| \geq \log_2 n$.

   Solution: We shall prove by induction that there exists a consistent subset of size m in a tournament with 2^m nodes.

   Proof:

   Base case: in a tournament of size 2^1, there are two nodes, either of which form a consistent subset

   Induction hypothesis: In a tournament of size 2^{m-1}, there exists a consistent subset of $m-1$ nodes

   Now, consider a tournament of size 2^m. Select any team T from this tournament. By pigeonhole principle, T either wins against at least 2^{m-1} teams or loses to at least 2^{m-1} teams.

   - Case 1: T wins against 2^{m-1} teams. There exists a consistent subset of $m-1$ nodes in the 2^{m-1} teams T wins against, hence there is a consistent subset in the 2^m teams which includes T such that T wins against all other $m-1$ teams.
   - Case 2: T loses to 2^{m-1} teams. There exists a consistent subset of $m-1$ nodes in the 2^{m-1} teams T loses to, hence there is a consistent subset in the 2^m teams which includes T such that T loses to all other $m-1$ teams.

   Hence, there exists a consistent subset of size atleast $\lfloor \log_2 n \rfloor$ in a tournament of size n. $\square$

2. Call a non-empty subset S of integers *nice* if, for every $x, y \in S$ and every two integers a, b, we have $ax + by \in S$. Observe that the set of multiples of any integer is a nice set. Prove that, in fact, these are the only nice sets. In other words, prove that for every nice set S there exists an integer x such that $S = \{ax \mid a \in \mathbb{Z}\}$. (You might find one of the results from the tutorial useful.)

   Solution: From Bézout's identity (Q8 of the tutorial) we get that if $\gcd(a, b) = d$, then there exist $x, y \in \mathbb{Z}$ such that $ax + by = d$. Now, consider a set $S = \{x_1, x_2, ...\}$ that is nice. .

   - Case 1: there exists $x_i, x_j \in S$ such that $\gcd(x_i, x_j) = 1$. we have $1 \in S$. Hence, every element can be expressed as $a + bx_i$, $x_i \in S, a, b \in \mathbb{Z}$. We can express any integer n in this form, as a is the remainder and b is the quotient when n is divided by x_i, hence $S = \mathbb{Z}$. Therefore, $S = \{a \cdot 1 | a \in \mathbb{Z}\}$

- Case 2: there does not exist $x_i, x_j \in S$ such that $\gcd(x_i, x_j) = 1$: Then, suppose $\min\limits_{i,j \in \mathbb{N}} \gcd(x_i, x_j) = d$. Every number can be expressed as $ax_i + bx_j = d(ax'_i + bx'_j)$. Hence, $S = \{ad \mid a \in \mathbb{Z}\}$.

This proves our claim. $\square$

3. Prove "Claim 2" from the proof of Schröder-Bernstein Theorem discussed in Lecture 5. Here is the statement of the claim. Let A and B be infinite sets, f be an injection from A to B, and g be an injection from B to A. Let $B' = \{b \in B \mid \exists b^* \in B \setminus \operatorname{Im}(f)\ \exists k \in \mathbb{N} \cup \{0\} : \ b = (f \circ g)^k(b^*)\}$, and $A' = \{g(b) \mid b \in B'\}$. Then for every $b \in B$, the following statements are equivalent.

   1. $b \in B'$.
   2. If $f^{-1}(b)$ exists, then it is in A'.
   3. $g(b) \in A'$.

   Solution:

   (1)⇒(2): Since $b = (f \circ g)^k(b^*)$ exists, $k \geq 1$. Therefore,

   $$f^{-1}(b) = f^{-1}(f \circ g)^k(b^*) = g(f \circ g)^{k-1}(b^*)$$

   but since

   $$(f \circ g)^{k-1}(b^*) \in B' \implies g((f \circ g)^{k-1}(b^*)) \in A$$

   Therefore $f^{-1}(b) \in A$

   (2)⇒(3): If $f^{-1}(b)$ exists, then it is in A', this implies $f^{-1}(b) = g(b')$ for some $b' \in B'$. This means

   $$(f \circ g)(b') \in B' \text{ or } b \in B'$$

   Then by definition of A', $g(b) \in A'$

   (3)⇒(1): $g(b) \in A' \Rightarrow b \in A'$. Since $g(b) \in A'$, there exists b such that $b \in B'$ $\square$

4. Given a set A, the set of finite length strings over A is denoted by A^*. Prove that if A is a finite set, then A^* is necessarily countable. What can you say about the cardinality of A^* if A is countably infinite instead?

   Solution: Denote by A^*_i the set of finite length strings having length equal to i. If A is finite, then so is A^*_i, as $|A^*_i| = |A|^i$. Since the union of countably many finite sets is countable, we have

   $$A^* = \bigcup_{i \in \mathbb{N}} A^*_i$$

   is countable.

   If A is countably infinite instead, then A^*_i is countable, by the same logic (There is a bijection between A^*_i and $A \times A \times \cdots \times A$, i times). Since $A \times A \times \cdots \times A$ is countable, A^*_i is also countable, and the countable union of countable sets is countable. Therefore, A^*_i is also countable. $\square$

5. Prove by mathematical induction that every graph has at least two vertices having equal degree.

Solution:

Base case: in a graph with 2 nodes, only two cases arise: either the nodes are connected to each other or they are not. In either case, both nodes have equal degree.

Induction hypothesis: If $P(k)$ is the proposition that a graph with k vertices has at least two vertices having equal degree, then $P(n-1)$ is true.

Consider a graph with $n-1$ vertices. Let $a, b \in V$ be the two vertices having equal degree. Assume that all other vertices have distinct number of edges. The n-1 vertices will have no. of edges from among $\{0, 1, \ldots, d, \ldots, n-2\}$, where d is the degree which corresponds to 2 vertices.

Observation: All the numbers in $\{0, 1, \ldots, d, \ldots, n-2\} \setminus \{d\}$ corresponds to 1 vertex exactly, (Except of course only one of $\{0, n-2\}$ will have a corresponding vertex). So we have a one-one correspondence between $n-3$ vertices and $\{0, 1, \ldots, d, \ldots, n-2\} \setminus \{d\}$.

Add a new vertex c to this graph. Two cases arise:

- Case 1: c is connected to both a, b or neither a nor b: the degree of a and b doesn't change, and hence they still have the same degree.
- Case 2: We join the first edge from c to a.
  - Case 2.1: A vertex corresponding to $d+1$, say i, already exists; Then i and a are vertices of same degree now
  - Case 2.2: A vertex corresponding to $d+1$, say i, does not exist; Then a vertex corresponding to 1, say i, definitely exists. (Since there is a one to one correspondence from $n-3$ vertices to $D = \{0, 1, \ldots, d, \ldots, n-2\} \setminus \{d\}$, with $n-2$ elements, only 1 element in D to which no vertex corresponds.) So, i and c have same degree.

  The same cases as above arise when you add the next edge (For the second edge, either vertex with degree 2 exists, or if the edge is drawn to one of two vertices with same degree d, a vertex with degree $d+1$ exists). Hence, we can go on adding edges between c and rest of the vertices, and in every stage we have a pair of vertices of same degree. This process ends for some $k \leq n-1$ edges.

If there are more than 2 (say q) vertices with same degree, say d, new vertex c forms edge with $0 \leq k \leq q$ of the vertices, either $k \geq 2$ or $q-k \geq 2$. We either get a pair with degree d or with a degree $d+1$

Hence $P(n-1) \Rightarrow P(n)$, and by induction, $P(n)$ holds for all n. $\square$