SoCP Question 3

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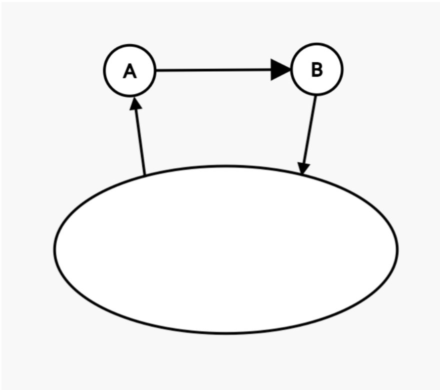
Proof for Original question: Since every player wins and loses atleast one match, consider any two players A and B, where A wins against B



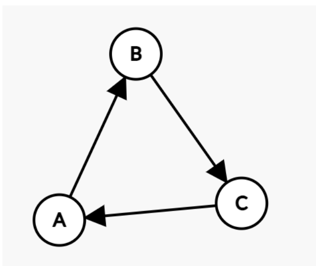
We now need to prove that there exists a player C such that C loses to B and C wins against A. We proceed by contradiction. Assume that such a C does not exist. Three cases arise:

- Every node in the graph wins against both A and B. Not possible, as then B would have no victories
- Every node in the graph loses to both A and B. Not possible, as then A would have no losses
- Every node in the graph loses to A and wins against B. Not possible, as then A would have no losses and B would have no victories

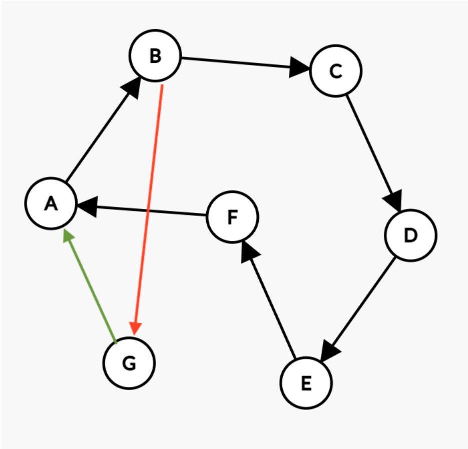
Therefore, there exists at least one C such that C loses to B and C wins against A. Hence, we can find A,B,C such that A wins against B, B wins against C and C wins against A. ■

Proof for Bonus Question: This is the famous Tournament Theorem.We need to prove that in the tournament graph, a hamiltonian cycle exists.

Consider the base case, when $n = 3$. There are only three players, and since every player wins atleast one match, every player would also lose atleast one match. Since every player plays only two matches, one with each other player, they lose one match and win one. Hence, a hamiltonian cycle exists in the tournament graph



For the induction step, assume that a hamiltonian cycle exists in a graph with n players. On adding the $n + 1$th player (node G), they would win one match against some player A and lose one match against some player B since they win and lose atleast one match.



we can then clearly see than a hamiltonian cycle exists if we proceed in the path $\cdots \to A \to G \to B \to \cdots$ ■