Insert Markov pun here

A talk on Probabilistic Automata and Markov Chains

Aniruddha Deb

IIT Delhi

June 9, 2021

Stochastic vectors and stochastic matrices

Stochastic vectors and stochastic matrices

A **Stochastic Vector** $\vec{a}$ is a vector in $\mathbb{R}^n$ which satisfies the following conditions:

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- the sum of all the elements of $\vec{a}$ is 1

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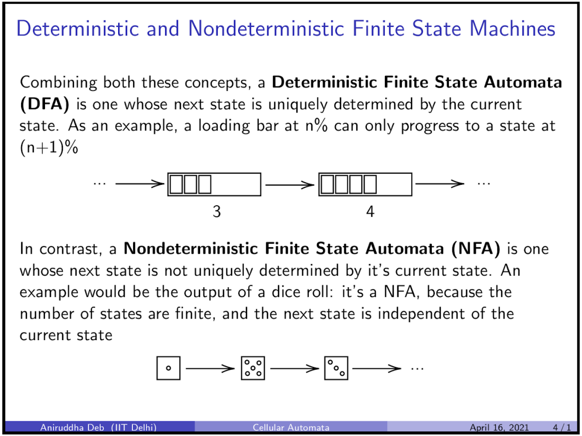
Stochastic vectors can be thought of as a form of **discrete probability distribution**: given a finite number of possible states, $\vec{a}$ gives the probability of transitioning to the ith state.

A short recap of NFA's



Deterministic and Nondeterministic Finite State Machines

Combining both these concepts, a **Deterministic Finite State Automata (DFA)** is one whose next state is uniquely determined by the current state. As an example, a loading bar at n% can only progress to a state at (n+1)%

In contrast, a **Nondeterministic Finite State Automata (NFA)** is one whose next state is not uniquely determined by it's current state. An example would be the output of a dice roll: it's a NFA, because the number of states are finite, and the next state is independent of the current state







Probabilistic Automata

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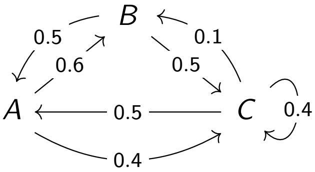

Modeling PA's: Markov Processes

PA's lend themselves to being modeled by markov matrices very well. As an example, the process in the previous slide can be modeled as follows:

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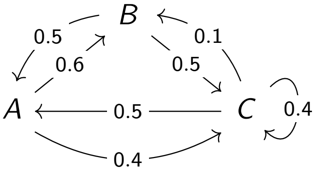

$$\vec{a}_A = \begin{bmatrix} 0 \\ 0.6 \\ 0.4 \end{bmatrix} \; \vec{a}_B = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix} \; \vec{a}_C = \begin{bmatrix} 0.5 \\ 0.1 \\ 0.4 \end{bmatrix}$$

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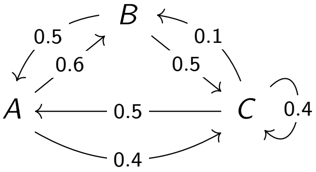

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Now, given an initial state a, $\mathcal{S}a$ would give the final state after one transition. By induction, $\mathcal{S}^n a$ would give the final state after n transitions.

More on Markov Matrices

- A markov matrix $\mathcal{S}$ multiplied by a stochastic vector $\vec{a}$ gives a stochastic vector

Proof: Consider the row vector $T = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$. This acts like a map from $\mathbb{R}^n \to \mathbb{R}$ defined as $T\vec{a} = a_1 + a_2 + \dots + a_n$. For a stochastic vector $\vec{a}$, $T\vec{a} = 1$. If we apply this map on $\mathcal{S}\vec{a}$, then we get

$$\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} \mathcal{S}\vec{a} = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} \vec{a} = 1$$

(Since every column vector of a stochastic matrix is a stochastic vector, $T\mathcal{S} = T$)

Hence, $\vec{a^1} = \mathcal{S}\vec{a}$ is a stochastic vector. By induction, $\vec{a^n} = \mathcal{S}^n\vec{a}$ is also a stochastic vector.

Eigenvalues and Eigenvectors of $\mathcal{S}$

Can you find an eigenvalue of $\mathcal{S}$ by inspection?

Analysing the steady state