UNIT - 1

Matrix

It is transformation from one space to another space

Eigen Value & Eigen Vector of Matrix

Let A be a square matrix. The λ is called the Eigen value of A if $AX = \lambda X$ Then X is called the corresponding Eigen Vector to the eigen Value 'λ'

$$|A - \lambda I| = 0$$

If A is [3X3] square matrix then λ (λ = Eigen value of matrix A) is root of equation

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where

S_1 = Sum of Principle diagonal elements

S_2 = Sum of ~~matrix~~ minors containing diagonal elements (m_{11}, m_{22}, m_{33})

S_3 = determinate of A.

Properties of Eigen Values

① If $\lambda_1, \lambda_2 ---- \lambda_n$ are the Eigen values of ~~values~~ A then $\lambda_1^m, \lambda_2^m ---- \lambda_n^m$ are the Eigen Values of A^m

② If $\lambda_1, \lambda_2, ---- \lambda_n$ are the Eigen Values of A, then $K\lambda_1, K\lambda_2 ----- K\lambda_n$ are the Eigen Values of KA where K is a constant

③ The sum of the Eigen values of A is equal to the diagonal entries of A (trace of A)

④ The Product of the Eigen of A is equal to $|A|$

⑤ A square matrix A and its transpose A^T have the same Eigen value

⑥ If $\lambda_1, \lambda_2 ---- \lambda_n$ are Eigen values of square matrix A of order m Then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ---- \frac{1}{\lambda_n}$ are eigen values of A^{-1} where $(\lambda_i \neq 0)$

⑦ If A is an upper triangular or a lower triangular matrix, then Eigen Values of A are its diagonal entries.

Cayley-Hamilton Theorem

Every square matrix satisfy its own characteristic Equation

$A_{2\times2} \Rightarrow \quad A^2 - S_1A - S_2 = 0$

$S_1 =$ trace of A

$S_2 = |A|$

$A_{3\times3} \Rightarrow \quad A^3 - S_1A^2 + S_2A - S_3 = 0$

$S_1 =$ trace of A

$S_2 =$ sum of minors containg diagonal elements (m_{11}, m_{22}, m_{33})

$S_3 = |A|$

Similar matrix

Let A and B be two square matrices. The matrices A and B are said to be similar if there exists a non singular matrix P such that

$$A = P^{-1}BP$$

Similar matrices have the same Eigen values

Diagonal matrix

A square matrix A is said to be diagonal if $a_{ij} = 0$, $i \neq j$

Diagonalisation of a square matrix

Step I - obtain eigen values of matrix

Step II obtain Eigen vectors

Step III form a modal matrix (M) with X_1 as first
X_2 as second, X_3 as third column

Step IV Find M^{-1}

~~Step~~ Step V The diagonal matrix D is given by
$D = M^{-1} \cdot A \cdot M$

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

d_1, d_2, d_3 are Eigenvalues of A

Matrix A and D have same Eigen values

If A has N-distinct Eigen values then there must be 'n' linearly independent Eigen vectors $X_1, X_2, X_3 \ldots X_n$

Then $M = [X_1, X_2, X_3 \ldots X_n]$

Orthogonal matrix

A real square matrix A is said to be an orthogonal matrix if $AA^T = A^TA = I$

$$A^T = A^{-1}$$

Since A is a orthogonal matrix $A^T = A^{-1}$

Symmetric matrix

A real square matrix A is said to be symmetric if $A^T = A$

Some properties of orthogonal and symmetric matrices

1. If A is orthogonal then A^T is also orthogonal
2. If A is orthogonal then $|A| = \pm 1$
3. If λ is an eigen value of the orthogonal matrix A then $1/\lambda$ is also Eigen value of A
4. If A and B orthogonal matrix then AB is also orthogonal
5. If A is orthogonal matrix then the magnitude of the Eigen values of A is 1

(6) Eigen values of real symmetric matrix are real

(7) Eigen vectors corresponds to distinct Eigenvalues of a symmetric matrix are orthogonal vectors

Unit Vector

Let X be a non zero vector, then its unit vector is

$X = (a, b, c)$

$$\hat{X}_1 = \frac{X_1}{\sqrt{a^2+b^2+c^2}}$$

Diagonalisation by orthogonal Transformation

Step I :- Find Eigen Values $\lambda_1, \lambda_2, \ldots \lambda_n$

Step II Find the linearly independent Eigen vectors they are pairwise orthogonal i.e $X_j \perp X_i$ $i \neq j$

Step III Calculate unit vectors

Step IV Form modal matrix

Step V $N^T = N^{-1}$

Step 6 $N^{-1} A N = D$

Quadratic forms

A homogeneous polynomial of second degree in any number of variables is called Quadratic form

Any quadratic form can be expressed in matrix form considering a quadratic

$$ax^2+by^2+cxy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Reduction to connanical form

A quadratic equation which only contains only lies in squared terms is called connanical form

Step I : Find A

Step II :- Find Eigen Values & Eigen Vectors of A

Step III - check whether the Eigen Vectors are orthogonal to each other

Step IV : Construct modal matrix (P) & find normalised model matrix N

Step V find $N^T A N = D$

Step VI find $Y D Y^T$ where $Y = [y_1, y_2, y_3]$

Nature of the Canonical form

The number of non-zero terms = r (rank of A)

The number of positive terms = s (index of Q.F)

The difference of number of positive & negative square terms = $2s - r$
(Signature of the Q.f)

The number of positive Eigen Values of A = Index (P)

Definite & Indefinite quadratic forms

Let $Q = X^T A X$ be a quadratic form

① Q is said to be positive definite if all Eigenvalue of A are positive

② Q is said to be negative definite if all Eigenvalues are negative

③ Q is said to be positive semidefinite if all Eigenvalues are non negative and atleast one of them is zero

④ Q is said to be negative semidefinite if all Eigenvalues are non-positive and atleast one of them is zero

⑤ Q is said to indefinite in all other cases

The Quadratic form is said to be

(i) Positive definite if $r = n$, $s = n$

(ii) Positive semidefinite if $r < n$, $s = n$ $\{$ n = no. of variables in Q.F

(iii) negative definite if $r = n$, $s = 0$

(iv) negative semidefinite if $r < n$, $s = 0$

(v) Indefinite in all other cases

Principal minor is denoted by D

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & - - & a_{1n} \\ a_{21} & a_{22} & a_{23} & - - & a_{2n} \\ \vdots & & & & \\ & & & & a_{2n} \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} \end{bmatrix}$$

$D_1 = |a_{11}| = a_{11}$

$D_2 = (2 \times 2) \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$D_{33} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{32} & a_{33} \end{vmatrix}$

1. A Q.F is Positive definate if $D_i > 0$
2. A Q.F is negative definate if $(-1)^i D_i > 0$
3. Q.F is Positive semi definat if $D_i \geq 0$ and atleast one $D_i = 0$
4. Q.F is negative semi definate if $(-1)^i D_i \geq 0$ & atleast one $D_i = 0$
5. In all other cases, it is indefinite

UNIT-2

Suppose $u = f(u,v)$

Then $\partial u/\partial x$ = Partial derivative of function 'x' with respect x alone by treating other variable as constant

$$\frac{\partial^2 u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right).$$

Homogenous function

A function $f(x,y,z)$ is called a homogenous function of degree if

$$f(tx, ty, tz) = t^n f(x,y,z)$$

Euler's theorem

If $f(x,y)$ is a homogenous function of degree n

then $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial x} = nf$

Total differentiation

Suppose

$$u = u(x, y, z) \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

Then total differention

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z} \times \frac{\partial z}{\partial t}$$

Maxima & Minima

$y = f(x)$ $\quad \frac{dy}{dx} = 0$

Then condition for maxima & minima

If (a) is a extreme point

$$\left.\frac{d^2y}{dx^2}\right|_{(a)} > 0 \quad \text{minima}$$

$$\left.\frac{d^2y}{dx^2}\right|_{a} < 0 \quad \text{maxima}$$

Let $P = f(x,y)$

The condition for maxima or minima are

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial x} = 0$$

If (a,b) is an extreme point then

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = 0, \quad \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = 0$$

$$r = \frac{\partial^2 f}{\partial x^2}, \quad t = \frac{\partial^2 f}{\partial y^2}, \quad s = \frac{\partial^2 f}{\partial x \partial y}$$

If $rt - s^2 > 0$ & $r < 0$
f attains maximum value at (a,b)

If $rt - s^2 > 0$ & $r > 0$
f attains minimum value at (a,b)

If $rt - s^2 < 0$
f ~~atten~~ does not attain any extreme value

Constrained maxima or minima using Lagrange's multiple method

Suppose $f = f(x, y)$ to be a function connected by the constraint

$$\phi(x, y, z) = c$$

Thus the auxiliary Eqn is

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial z} = 0$$

where $F = f + \lambda\phi$

Step 1: construct $F = f + \lambda\phi$

Step 2: obtain $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$

Step 3: solve $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$ for x, y, z

Taylor Series Expansion

Suppose $f = f(x, y)$

Then the taylor Series Expansion of f about (a, b)

is $f(x,y) = f(a,b) + \frac{1}{1!}\left[\frac{h\partial}{\partial x} + \frac{K\partial}{\partial y}\right] f(a,b)$

$+ \frac{1}{2!}\left[\frac{h\partial}{\partial x} + \frac{K\partial}{\partial y}\right]^2 f(a,b) + \frac{1}{3}\left[\frac{h\partial}{\partial x} + \frac{K\partial}{\partial y}\right]^3 f(a,b)$

$h = (x-a)$, $K = y-b$

Jacobains

If u and v are functions of two independent variables x & y, then the Jacobaine of u, v with respect to x, y is given by

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Properties of Jacobains

1) If J_1 is Jacobains of u,v with respect to x,y & J_2 is another Jacbbains of x,y with to u,v then

$$J_1 \times J_2 = 1$$

2 If u,v are function of r,s and r,s are function of x,y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

UNIT - 3

The general form of a differential Equation is

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{d^2y}{dx} + a_2 y = f(x)$$

Homogenous differential Equation

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$$

Complementory function

$$D = \frac{\partial}{\partial x} \qquad D^2 = \frac{d^2}{dx^2}$$

$$(a_0 D^2 + a_1 D + a_2)y = 0$$

The auxilarly Eq^n is

$$a_0 m^2 + a_1 m + a_2 = 0$$

Case(1) Roots are real but distinct $m_1 \neq m_2$

$$C.F = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

Case (II) Roots are real but Equal $m_1 = m_2 = m$

$C.F. = (C_1 + C_2 x) e^{mx}$

Case (III) Roots are complex

$m_1 = \alpha + i\beta \qquad m_2 = \alpha - i\beta$

$C.F = e^{\alpha x} (C_1 \cos \beta x + C_2 \cos \beta x)$

Heterogenous function

Type-I $(a_0 D^2 + a_1 D + a_2) y = e^{ax}$

$y = C.F + P.I$

$P.I = \dfrac{e^{ax}}{a_0 D^2 + a_1 D + a_2}$ Replace D by a

$f(D) = a_0 D^2 + a_1 D + a_2$

$= \dfrac{e^{ax}}{f(a)}$

If $f(a) = 0$ then

$$P.I = \frac{x e^{ax}}{f'(D)} = \frac{x e^{ax}}{f'(a)}$$

If $f'(a) = 0$ then

$$P.I = \frac{x^2 e^{ax}}{f''(a)}$$

Type-2

$$(a_0 D^2 + a_1 D + a_2) y = \sin(ax) \text{ or } \cos(ax)$$

$$\therefore P.I. = \frac{\sin(ax) / \cos(ax)}{a_0 D^2 + a_1 D + a_2} \qquad D^2 \text{ by } -a^2$$

If Denominator is 0 then multiply x to Numerator & differentiate the denominator the make D^2 terminator

Type-3

$$(a_0 D^2 + a_1 D + a_2) y = b_0 x^2 + b_1 x + b_2$$

$$P.I = \frac{b_0 x^2 + b_1 x + b_2}{a_0 D^2 + a_1 D + a_2} = \frac{1}{1 \pm f(D)}$$

$$\begin{cases} \dfrac{1}{1-x} = 1 + x + x^2 + x^3 - - - - \\ \dfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - - - - \end{cases}$$

Type-4

$$f(D)y = e^{ax} \cdot V$$

$$P.I = \frac{e^{ax} \cdot V}{f(D)} = e^{ax}\left[\frac{V}{f(D+a)}\right] \quad D \to D+a$$

Type-5

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = f(x)$$

$$x = e^z \quad , z = \log x$$

$$xD = D' \qquad D' = \frac{d}{dz}$$

$$x^2D^2 = D'(D'-1)$$

Lengendre Type Equation

$$(ax+b)^2 \frac{d^2y}{dx^2} + (ax+b)\frac{dy}{dx} + a_1 y = f(x)$$

Let $ax+b = e^z$, $z = \log(ax+b)$

$$(ax+b)D = aD'$$

$$(ax+b)^2 = a^2 D'(D'-1)$$

Method variation of Parameters

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = f(x)$$

Step 1: Find C.F

$C.F = C_1 f_1(x) + C_2 f_2(x)$

Step 2: Find ω $\quad \omega = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$

Step 3: $P = -\int \frac{f_1}{\omega} \cdot f(x)\,dx$

$Q = \int \frac{f_2}{\omega} \cdot f(x)\,dx$

$P.I = P.f_1 + Q f_2$.

Step 4: $y = C.F + P.f_1 + Q f_2$

UNIT-4

Radiuss of Curvature

Forms of the curve

1) corlesian form

$y = f(x)$

$$\rho = \frac{(1+(y')^2)^{3/2}}{y''} \qquad y' = \frac{dy}{dx} \qquad y'' = \frac{dy}{dx}$$

2) Parameter form

$x = x(t) \qquad y = y(t)$

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} \qquad \dot{x} = \frac{dx}{dt} \quad \dot{y} = \frac{dy}{dt}$$

$$\ddot{x} = \frac{d^2x}{dt^2} \quad \ddot{y} = \frac{d^2y}{dt^{c}}$$

3) Polar form

$r = f(\theta)$

$$\rho = \frac{(r^2+\dot{r}^2)^{3/2}}{r^2+2\dot{r}^2 - r\ddot{r}} \qquad \dot{r} = \frac{dr}{d\theta}, \quad \ddot{r} = \frac{d^2r}{d\theta^2}$$

Centre of the Curve

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} \qquad y_1 = \frac{dy}{dx}$$

$$y_2 = \frac{d^2y}{dx^2}$$

$$\bar{y} = y - \frac{(1+y_1^2)}{y_2}$$

Evolute of a curve

The locus of centre of curvature of a curve is called Evolute

Step 1: Find $(\bar{x}, \bar{y})$

Step 2: Eliminate the parameter present in $\bar{x}$ & $\bar{y}$

Equation of circle of a curvature is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

Parameteric form

Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$x = a\cos\theta \qquad y = b\sin\theta$

Parabola

$$y^2 = 4ax$$

$x = at^2 \qquad y = 2at$

Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$x = a\sec\theta \qquad y = b\tan\theta$

Circle

$$x^2 + y^2 = a^2$$

$x = a\cos\theta \qquad y = a\sin\theta$

Rectangular hyperbola

$xy = c^2 \qquad x = ct \quad y = \frac{c}{t}$

Envelope

1 Parameter

Suppose $f(x, y, \alpha) = 0$ be family of curves and α is a parameter

Step 1 :- find $\frac{\partial f}{\partial \alpha}$

$$\frac{\partial f}{\partial \alpha} = 0 \quad - (1) \quad \& \quad f(x, y, \alpha) = 0 \quad - (2)$$

Step 2 : Eliminating α from (1) & (2) we get envelope

Suppose the family of curve is expressed as a quadratic Equation is a parameter

ie $A\alpha^2 + B\alpha + C = 0$

Then Eqⁿ of Envelope as

$$B^2 - 4AC = 0$$

To Envelope when two parameters are given

Suppose $f(x, y, a, b) = 0$ — (1)

$\phi(a, b) = 0$ — (2)

Step 1:- Differentiate Partially (1) w.r.t 'a' by treating b as function of a and then call the eqn as (3)

Step 2: Diff partially (2) w.r.t 'a' and call the eqn as (4)

Step 3: Find $\frac{db}{da}$ from (4) and subsitute in (3)

Step 4: Eliminate a and b

Gamma functions & Beta functions

Gamma function

$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$ is called Gamma function

$\Gamma(n) = (n-1)!$ if n is integer

$\Gamma(n+1) = n\,\Gamma(n)$

$\int_0^\infty x^{n-1} e^{-ax}\, dx = \frac{\Gamma(n)}{a^n}$

$\Gamma(1/2) = \sqrt{\pi}$

Beta function

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

$\beta(m,n) = \beta(n,m)$

$\beta(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

$\beta(m,n) = \dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$

UNIT- 5

A set of Numbers, $a_1, a_2, a_3, \ldots a_n$ such that to each positive integer n, the corresponds a_n of the set is called Sequence and it denoted by $\{a_n\}$

Limit of a Sequence

A sequence converges to a limit (l) if given very positive number ε, there exist a positive small integer N such that

$$|a_n - l| < \varepsilon \quad \forall \; n \geq N$$

Bonded Sequence

A sequence a_n is said to be bounded if $a_n \leq M \quad \forall n$

Monotonically increasing Sequence

A Sequence a_n is said to be monotonically increasing if $a_n \leq a_{n+1} \quad \forall n$

Monotonically decreasing Sequence
A sequence a_n is said to be monotonically decreasing
if $a_n \geq a_{n+1}$ $\forall n$

Converging Series.
A sequence a_n is said to be converge to l
if $|a_n - l| < \varepsilon$ $\forall n \geq N$ where $\lim_{n \to \infty} a_n = l$

Comparison test
If ΣU_n and ΣV_n are two series and the Second series ΣV_n is convergent, $U_n \leq K V_n$
then ΣU_n is convergent

Limit form
- If ΣU_n and ΣV_n are two series and ΣV_n is convergent and $\frac{U_n}{V_n}$ tends to a limit other than zero as $n \to \infty$ then ΣU_n is convergent.

- If ΣV_n is divergent & $\frac{U_n}{V_n}$ tends to a limit other than zero as $n \to \infty$ then ΣU_n is divergent.

If two positive terms ΣU_n and ΣV_n be such that $\lim_{n\to\infty} \frac{U_n}{V_n}$ = finite number & non zero

then both series converge or diverge together

$\Sigma V_n = \frac{1}{n^p}$ $\quad p > 1 \rightarrow$ converges

$p \leq 1 \rightarrow$ diverges

D'Alembert's Ratio test

If ΣU_n is a positive term series such that

$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = K$ then

① series is convergent if $K < 1$

② series is divergent if $K > 1$

Raabe's test

If ΣU_n is a positive term series

$$\lim_{n\to\infty} \left\{ n \left(\frac{U_n}{U_{n+1}} - 1 \right) \right\} = l$$

(1) Series is convergent if $l > 1$

(2) Series is divergent if $l < 1$

Cauchy's Root test

If $\sum U_n$ is a positive term series such that

$$\lim_{n \to \infty} (U_n)^{1/n} = l$$

(1) series is convergent if $l < 1$

(2) series is divergent if $l > 1$