Chapter 3

FUNCTIONS OF SEVERAL VARIABLES

3.1 INTRODUCTION

This chapter is devoted to a study of functions depending on more than one independent variable. A real function $z = f(x, y)$ of two independent variables x and y, can be thought to represent a surface in the three-dimensional space referred to a set of co-ordinate axes X, Y, Z.

A simple example of a function of two independent variables x and y is $z = xy$, which represents the area of a rectangle whose sides are x and y.

CONTINUITY OF A FUNCTION OF TWO VARIABLES:

Definition: A function $z = f(x, y)$ is said to be continuous at the point (x_0, y_0) provided that a small change in the values of x and y produces a corresponding (small) change in the value of z. More precisely, if the value of $z = f(x, y)$ at (x_0, y_0) is z_0, then the continuity of the function at the point (x_0, y_0) means that

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x,y) = f(x_0, y_0) = z_0$$

If a function is continuous at all points of some region R in the XY plane, then it is said to be continuous in the region R.

The definition of continuity of a function of more than two independent variables is similar.

PARTIAL DERIVATIVES

The analytical definition of the derivatives of a function $y = f(x)$ of a single variable x is

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Let Δz denote the increment in the function $z = f(x, y)$ where y is kept fixed and x is changed by an amount Δx,

(i.e) $\Delta z = f(x+\Delta x, y) - f(x, y)$. Then,

$\lim\limits_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$ is called the partial derivative of z with respect to x and is denoted by $\frac{\partial z}{\partial x}$ or z_x.

Similarly, the partial derivative of z with respect y is defined and denoted as $\frac{\partial z}{\partial y}$ or z_y.

In general, if $z = f(x_1, x_2, \ldots x_n)$ is a function of independent variables $x_1, x_2, x_3, \ldots\ldots x_n$ then $\frac{\partial z}{\partial x_i}$ denote the partial derivatives of z with respect to $x_i (i = 1, 2, \ldots., n)$, when the remaining variables are treated as constants.

Example 1

If $z = x^2 + y^2 + 3xy$, then $\frac{\partial z}{\partial x} = 2x + 3y$, $\frac{\partial z}{\partial y} = 2y + 3x$

Example 2

If $u = e^x \sin y \cos z$, then $\frac{\partial u}{\partial x} = e^x \sin y \cos z$ (when y and z are held constants)

$\frac{\partial u}{\partial y} = e^x \cos y \cos z$ (Here x and z are treated as constants) and

$\frac{\partial u}{\partial z} = -e^x \sin y \sin z$ (both x and y are held constants).

Total Differential: If $z = f(x, y)$ is a function of two independent variables x and y, then the total differential of z is denoted as dz and defined as

In general, if $u = f(x_1, x_2, \ldots x_n)$, then the total differential of u is given by $du = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots\ldots + \frac{\partial f}{\partial x_n} dx_n$

It may thus be noted that the total differential is equal to the sum of the partial differentials.

Example 3

Example 1

If $f(x,y)=x^2+y^2$, where $x=r\cos\theta$ and $y=r\sin\theta$, find (i) $\frac{df}{dr}$ and $\frac{df}{d\theta}$ and also (iii) df.

Solution: Given $f(x,y)=x^2+y^2$, where $x=r\cos\theta$ and $y=r\sin\theta$

(i) Now, $\frac{df}{dr}=\frac{\partial f}{\partial x}\frac{dx}{dr}+\frac{\partial f}{\partial y}\frac{dy}{dr}=2x\cos\theta+2y\sin\theta$

$$=2r(\cos^2\theta+\sin^2\theta)=2r$$

(ii) $\frac{df}{d\theta}=\frac{\partial f}{\partial x}\frac{dx}{d\theta}+\frac{\partial f}{\partial y}\frac{dy}{d\theta}=2x(-r\sin\theta)+2y(r\cos\theta)$

$$=-2r^2\cos\theta\sin\theta+2r^2\cos\theta\sin\theta=0$$

(iii) $df=2rdr \quad \left(\because \text{from (i) } \frac{df}{dr}=2r\right)$

$$=2xdx+2ydy \quad (\because r^2=x^2+y^2)$$

Homogeneous Function: A function $f(x, y)$ is said to be a homogeneous function of degree 'n' if it can be expressed of the form $x^n\phi\left(\frac{y}{x}\right)$.

Euler's theorem: If u is a homogeneous function of degree 'n', then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

and $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$

Example 2

If $u=\sin^{-1}\left(\frac{x^2+y^2}{x-y}\right)$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$.

Solution: Given $u=\sin^{-1}\left(\frac{x^2+y^2}{x-y}\right)$

(i.e.) $\sin u = \dfrac{x^2+y^2}{x-y} = x^1\phi(y/x)$. '$\sin u$' is a homogeneous function of degree 1. $\therefore$ Using Euler's theorem for the homogeneous function $\sin u$, we have

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = 1.\sin u$$

$$\cos u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \sin u$$

$$(or)\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u} = \tan u.$$

Chain Rule:

If $z = f(x, y)$, where x and y are functions of given two variables u and v, then $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}$ and $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$

Example 3

Find $\dfrac{dz}{dt}$, when $z = xy^2 + x^2y$, where $x = at^2$, $y = 2at$

Solution: We know that $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$

$$= (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$$

$$= 2at(2at)^2 + 4at(at^2)(2at) + 4a(at^2)(2at) + 2a(at^2)^2$$

$$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^3t^4$$

$$= 16a^3t^3 + 10a^3t^4$$

$$= 2a^3t^3(8 + 5t)$$

Example 4

If $u = \sin(x/y)$, $x = e^t$, $y = t^2$, find $\dfrac{du}{dt}$.

Solution: Let $u = \sin(x/y)$, $x = e^t$, $y = t^2$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= \cos(x/y)\left(\frac{1}{y}\right)(e^t) + \cos(x/y)\left(\frac{-x}{y^2}\right)(2t)$$

$$= \cos\left(\frac{e^t}{t^2}\right)\left(\frac{e^t}{t^2}\right) - \cos\left(\frac{e^t}{t^2}\right)2.\frac{e^t}{t^3}$$

$$= \frac{e^t}{t^2}\left[\cos\left(\frac{e^t}{t^2}\right) - \frac{2}{t}\right]$$

Example 5

If $u = x^2 - y^2,\ v = 2xy,\ f(x,y) = \varphi(u,v)$ show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 \varphi}{\partial u^2} + \frac{\partial^2 \varphi}{\partial v^2}\right)$$

Solution: Let $\frac{\partial f}{\partial x} = \frac{\partial \varphi}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial v}.\frac{\partial v}{\partial x} = \frac{\partial \varphi}{\partial u}(2x) + \frac{\partial \varphi}{\partial v}(2y)$

$$\therefore \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 \varphi}{\partial u^2}(4x^2) + 2\frac{\partial \varphi}{\partial u}(1) + \frac{\partial^2 \varphi}{\partial u^2}(4y^2) \qquad (1)$$

$$\frac{\partial f}{\partial y} = \frac{\partial \varphi}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial \varphi}{\partial v}.\frac{\partial v}{\partial y} = \frac{\partial \varphi}{\partial u}(-2y) + \frac{\partial \varphi}{\partial v}(2x)$$

$$\therefore \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 \varphi}{\partial u^2}(4y^2) + \frac{\partial \varphi}{\partial u}(-2) + \frac{\partial^2 \varphi}{\partial v^2}(4x^2) \qquad (2)$$

Equation (1) + (2) gives, $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial^2 \varphi}{\partial u^2} + \frac{\partial^2 \varphi}{\partial v^2}\right)(4x^2 + 4y^2)$

$$= 4(x^2 + y^2)\left(\frac{\partial^2 \varphi}{\partial u^2} + \frac{\partial^2 \varphi}{\partial v^2}\right)$$

Example 6

If $u = f(r),\ x = r\cos\theta,\ y = r\sin\theta$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$.

Solution: Given $u = f(r)$ where $r = \sqrt{x^2 + y^2}$

Now, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = f'(r).\frac{2x}{2\sqrt{x^2 + y^2}} = f'(r)\frac{x}{r}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{x}{r}f''(r).\frac{\partial r}{\partial x} + f'(r)\frac{1}{r} + xf'(r)\frac{(-1)}{r^2}\frac{\partial r}{\partial x}$$

$$= \frac{x^2}{r^2}f''(r) + f'(r)\frac{1}{r} - \frac{x^2}{r^3}f'(r) \quad \because \frac{\partial r}{\partial x} = \frac{x}{r} \qquad (1)$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{r^2} f''(r) + f'(r)\frac{1}{r} - \frac{y^2}{r^3} f'(r)$ (2)

Equations (1) + (2) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$.

Differentiation of Implicit Functions

Consider the implicit function $f(x, y) = 0$. Then, $\frac{dy}{dx} = -\frac{(\partial f\ \partial x)}{(\partial f\ \partial y)}$.

Example 7

Find $\frac{dy}{dx}$ if $xe^{-y} - 2ye^{x} = 1$.

Solution: Given $f(x,y) = xe^{-y} - 2ye^{x} - 1 = 0$

$$\frac{\partial f}{\partial x} = e^{-y} - 2ye^{x} \ \&\ \frac{\partial f}{\partial y} = -xe^{-y} - 2e^{x}$$

$$\frac{dy}{dx} = -\frac{(\partial f\ \partial x)}{(\partial f\ \partial y)} = -\frac{(e^{-y} - 2ye^{x})}{(-xe^{-y} - 2e^{x})} = \frac{e^{-y} - 2ye^{x}}{xe^{-y} + 2e^{x}}$$

Example 8

If $u = x\log(xy)$, where $x^3 + y^3 + 3xy = 1$, find du/dx.

Solution: $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$ (1)

Now $f(x, y) = 0$ is $x^3 + y^3 + 3xy - 1 = 0$

$\frac{dy}{dx} = -\frac{(\partial f\ \partial x)}{(\partial f\ \partial y)} = -\frac{(3x^2 + 3y)}{(3y^2 + 3x)} = -\left(\frac{x^2 + y}{y^2 + x}\right)$ (2)

Equation (1) becomes,

$$\frac{du}{dx} = x.\frac{1 \times y}{xy} + \log(xy) + x.\frac{1 \times x}{xy}\left(-\frac{x^2 + y}{y^2 + x}\right) \quad = 1 + \log(xy) - \frac{x}{y}\left(\frac{x^2 + y}{y^2 + x}\right)$$

Example 9

Find dy/dx, if $(\cos x)^{y} = (\sin y)^{x}$

Solution: Given $(\cos x)^{y} = (\sin y)^{x}$

Taking *log* on both sides, we get, $y\ \log(\cos x) = x\ \log(\sin y)$

(i. e) $f(x, y) = y\log(\cos x) - x\log(\sin y) = 0$

Example1

Expand $x^2y+3y-2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's theorem upto terms of third degree.

Solution: The Taylor series expansion of $f(x, y)$ in powers of $(x-a)$ and $(y-b)$ is given by

$$f(x,y)=f(a,b)+\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]$$

$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]+\ldots$$

Here $f(x,y)=x^2y+3y-2$, $a = 1$, $b = -2 \Rightarrow f(1,-2) = -10$

$f_x(x,y)=2xy,\quad f_x(1,-2)=-4$

$f_y(x,y)=x^2+3\quad f_{xy}(x,y)=2x$

$f_y(1,-2)=4\quad f_{xy}(1,-2)=2$

$f_{xx}(x,y)=2y\quad f_{yy}(x,y)=0$

$f_{xx}(1,-2)=-4\quad f_{yy}(1,-2)=0$

$f_{xxx}(x,y)=0\quad f_{yyy}(x,y)=0$

$f_{xxy}(x,y)=2\quad f_{xyy}(x,y)=0$

From (1),

$$x^2y+3y-2=-10+\frac{1}{1!}\left[(x-1)(-4)+(y+2)(4)\right]$$

$$+\frac{1}{2!}\left[(x-1)^2(-4)+2(x-1)(y+2)(2)\right]$$

$$+\frac{1}{3!}\left[3(x-1)^2(y+2)(2)\right]+\ldots$$

(or) $$x^2y+3y-2=-10-4(x-1)+4(y+2)-2(x-1)^2$$

$$+2(x-1)(y+2)+(x-1)^2(y+2)+\ldots.$$

Example 2

Expand $e^x\cos y$ in powers of x and y as far as the terms of the third degree.

Solution: Let $f(x,y)=e^x\cos y \Rightarrow f_x(x,y)=e^x\cos y\quad f_x(0,0)=1$

$f_y(x,y)=-e^x\sin y$

$f_y(0,0)=0\quad f_{xx}(x,y)=e^x\cos y$

$f_{xx}(0,0)=1$

$f_{xx}(x,y) = e^x \cos y$ $\quad$ $f_{xxy}(x,y) = -e^x \sin y$

$f_{xx}(0,0) = 1$ $\quad$ $f_{xxy}(0,0) = 0$

$f_{xy}(x,y) = -e^x \sin y$ $\quad$ $f_{xyy}(x,y) = -e^x \cos y$

$f_{xy}(0,0) = 0$ $\quad$ $f_{xyy}(0,0) = -1$

$f_{yy}(x,y) = -e^x \cos y$ $\quad$ $f_{yyy}(x,y) = e^x \sin y$

$f_{yy}(0,0) = -1$ $\quad$ $f_{yyy}(0,0) = 0$

Taylor's series of $f(x, y)$ in powers of x and y is

$$f(x,y) = f(0,0) + \left[xf_x(0,0) + yf_y(0,0)\right]$$

$$+\frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \ldots$$

$$\therefore e^x \cos y = 1 + \frac{1}{1!}[x.1 + y.0] + \frac{1}{2!}[x^2.1 + 2xy.0 + y^2(-1)]$$

$$+\frac{1}{3!}[x^3.1 + 3x^2 y.0 + 3xy^2(-1) + y^3.0] + \ldots$$

$$= 1 + \frac{x}{1!} + \frac{x^2 - y^2}{2!} + \frac{x^3 - 3xy^2}{3!} + \ldots$$

Example 3

Expand $\tan^{-1}\left(\frac{y}{x}\right)$ using Taylor's series near (1,1) upto quadratic terms.

Solution: The Taylor series expansion for $f(x, y)$ in powers of $(x-a)$ and $(y-b)$ is given by

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$

$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \ldots \rightarrow (1)$$

Here $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$, $a = 1 = b \Rightarrow f(a,b) = f(1,1) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$$f_x(x,y) = \frac{1}{1+(y^2/x^2)}\left(\frac{-y}{x^2}\right) = \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} \Rightarrow f_x(1,1) = \frac{-1}{2}$$

$$f_y(x,y)=\frac{1}{1+\frac{y^2}{x^2}}\times\frac{1}{x}=\frac{x}{x^2+y^2}\Rightarrow f_y(1,1)=\frac{1}{2}$$

$$f_{xx}(x,y)=(-y)(x^2+y^2)^{-2}2x=\frac{-2xy}{(x^2+y^2)^2}\quad\therefore f_{xx}(1,1)=\frac{-2}{4}=\frac{-1}{2}$$

$$f_{xy}(x,y)=\frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}\Rightarrow f_{xy}(1,1)=0$$

$$f_{yy}(x,y)=(x)(x^2+y^2)^{-2}2y=\frac{2xy}{(x^2+y^2)^2}\Rightarrow f_{yy}(1,1)=\frac{1}{2}.$$

Using these in (1) we get,.

$$\tan^{-1}\left(\frac{y}{x}\right)=\frac{\pi}{4}+\frac{1}{1!}\left[(x-1)\left(-\frac{1}{2}\right)+(y-1)\frac{1}{2}\right]+\frac{1}{2!}\left[(x-1)^2\left(-\frac{1}{2}\right)+(x-1)(y-1)\times 0\right.$$
$$\left.+(y-1)^2\left(\frac{1}{2}\right)+\dots\right]$$

(or) $$\tan^{-1}\left(\frac{y}{x}\right)=\frac{\pi}{4}-\left(\frac{x-1}{2}\right)+\left(\frac{y-1}{2}\right)-\frac{(x-1)^2}{4}+\frac{(y-1)^2}{4}+\dots$$

Example 4

Using Taylor's series expand $e^x\log(1+y)$ up to terms of the third degree in the neighborhood of origin.

Solution: The Taylor's series expansion of $f(x, y)$ near $(0, 0)$ is

$$f(x,y)=f(0,0)+\frac{1}{1!}\left[(x-0)f_x(0,0)+(y-0)f_y(0,0)\right]$$
$$+\frac{1}{2!}\left[(x-0)^2f_{xx}(0,0)+2(x-0)(y-0)f_{xy}(0,0)+(y-0)^2f_{yy}(0,0)\right]$$
$$+\frac{1}{3!}\left[\begin{array}{l}(x-0)^3f_{xxx}(0,0)+3(x-0)^2(y-0)f_{xxy}(0,0)+3(x-0)(y-0)^2\\ f_{xyy}(0,0)+(y-0)^3f_{yyy}(0,0)+\dots\end{array}\right]\rightarrow(1)$$

Now $f(x,y)=e^x\log(1+y)\Rightarrow f(0,0)=0$ as $\log 1=0$

$f_x(x,y) = e^x \log(1+y)$ $\qquad f_{xxx}(x,y) = e^x \log(1+y)$

$f_y(x,y) = \dfrac{e^x}{1+y}$ $\qquad f_{xxy}(x,y) = \dfrac{e^x}{1+y}$

$f_{xx}(x,y) = e^x \log(1+y)$ $\qquad f_{xyy}(x,y) = -e^x(1+y)^{-2}$

$f_{xy}(x,y) = \dfrac{e^x}{1+y}$ $\qquad f_{yyy}(x,y) = 2e^x(1+y)^{-3}$

$f_{yy}(x,y) = -e^x(1+y)^{-2}$

$f_x(0,0) = 0;$ $\quad f_{xx}(0,0) = 0;$ $\quad f_{xxx}(0,0) = 0$

$f_y(0,0) = 0;$ $\quad f_{xy}(0,0) = 1;$ $\quad f_{xxy}(0,0) = 1$

$f_{yy}(0,0) = -1;$ $\quad f_{xyy}(0,0) = -1$

$f_{yyy}(0,0) = 2$

Using in (1),

$$e^x \log(1+y) = 0 + \frac{1}{1!}\left[(x-0)\times 0 + (y-0)(1)\right]$$

$$+\frac{1}{2!}\left[(x-0)^2\times 0 + 2(x-0)(y-0)(1) + (y-0)^2(-1)\right]$$

$$+\frac{1}{3!}\left[(x-0)^3\times 0 + 3x^2y(1) + 3xy^2(-1) + (y-0)^3(2)\right] + \ldots$$

$$\text{(or) } e^x \log(1+y) = y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \frac{y^3}{3} + \ldots$$

Example 5

Expand $xy^2 + 2x - 3y$ in powers of $(x+2)$ and $(y-1)$ up to third degree terms.

Solution: The Taylor series expansion of $f(x, y)$ in powers of $(x-a)$ and $(y-b)$ is given by

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$

$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \ldots$$

Here $f(x, y) = xy^2 + 2x - 3y$; $a = -2$, $b = 1$ $\therefore f(-2,1) = (-2)(1) + 2(-2) - 3(1) = -9$

$f_x(x,y) = y^2 + 2$	$f_{xx}(x,y) = 0$
$f_x(-2,1) = 3$	$f_{xy}(x,y) = 2y$
$f_y(x,y) = 2xy - 3$	$f_{xy}(-2,1) = 2$
$f_y(-2,1) = -7$	$f_{yy}(x,y) = 2x$
$f_{xxx}(x,y) = 0$	$f_{yy}(-2,1) = -4$
$f_{xyy}(x,y) = 2$	$f_{yyy}(x,y) = 0$
$f_{xxy}(x,y) = 0$	

Substituting in (1)

$$xy^2 + 2x - 3y = -9 + 3(x+2) - 7(y-1) + 2(x+2)(y-1) - 2(y-1)^2$$
$$+(x+2)(y-1)^2 + \ldots$$

Example 6

Find the Taylor series expansion of e^{xy} at (1, 1) up to third degree terms.

Solution: Taylor series expansion of $f(x, y)$ at (1, 1) is given by

$$f(x,y) = f(1,1) + \left[(x-1)f_x(1,1) + (y-1)f_y(1,1)\right] + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1)\right.$$
$$\left. + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right] + \ldots \quad \rightarrow (1)$$

Here $f(x,y) = e^{xy}$; $a = 1 = b$

$$\therefore f(1,1) = e^{1\cdot 1} = e$$

$$f_x(x,y) = e^{xy}.y \Rightarrow f_x(1,1) = e^{1\cdot 1} \times 1 = e$$
$$f_y(x,y) = e^{xy}.x \Rightarrow f_y(1,1) = e^{1\cdot 1} \times 1 = e$$

$$f_{xx}(x,y) = y.e^{xy}.y \Rightarrow f_{xx}(1,1) = 1^2 \times e^{1\cdot 1} = e$$
$$f_{xy}(x,y) = e^{xy}.1 + xe^{xy}.y \Rightarrow f_{xy}(1,1) = 2e$$

Note: The point (a, b) is called a stationary point if $f_x(a,b) = 0$, $f_y(a,b) = 0$. The value $f(a, b)$ is called a stationary value. Thus every extreme value is a stationary value but the converse may not be true.

Rule to find the extreme value of a function $z = f(x, y)$

i) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

ii) Solve $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$, similarly. Let (a, b); (c, d),.... be the solutions of these equations.

iii) For each solution in step (ii) find $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

iv) a) If $rt - s^2 > 0$ and $r < 0$ for a particular solution (a, b) of step (ii) then z has a maximum value at (a, b).

b) If $rt - s^2 > 0$ and $r > 0$ for a particular solution (a, b) of step (ii) then z has a minimum value at (a, b).

c) If $rt - s^2 < 0$ for a particular solution (a, b) of step (ii) then z has no extreme value at (a, b).

d) If $rt - s^2 = 0$, this case is doubtful and requires further investigation.

Example 1

Examine for extreme values of $x^2 + y^2 + 6x + 12$.

Solution: Let $f(x,y) = x^2 + y^2 + 6x + 12$ and $f_x = 2x + 6$; $f_y = 2y$.

For maximum or minimum, $f_x = 0, f_y = 0 \Rightarrow 2x+6 = 0;\ 2y = 0$

Solving we get $x = -3$, $y = 0$. Stationary point is (-3, 0)

$r = f_{xx} = 2$ at (-3,0), $rt - s^2$ gives 4 (i.e) $rt - s^2 > 0$, $t = f_{xy} = 0$,

$s = f_{yy} = 2$. As r is 0, $rt - s^2 > 0 \Rightarrow$ (-3, 0) is a minimum point.

And $f(-3,0) = 9-18+12 = 3$.

$\therefore$ Minimum value = 3.

Example 2

Examine for extreme values of $xy + \frac{a^3}{x} + \frac{a^3}{y}$.

Solution: Let $z = f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$

$$f_x = y - \frac{a^3}{x^2};\ f_y = x - \frac{a^3}{y^2}$$

Equation f_x and f_y to zero

$$y - \frac{a^3}{x^2} = 0 \qquad (1)$$

$$x - \frac{a^3}{y^2} = 0 \qquad (2)$$

From (1), $y = \frac{a^3}{x^2}$. Put this value in (2), we get $x - \frac{a^3}{\left(\frac{a^6}{x^4}\right)} = 0$

$$(i.e)\ x - \frac{a^3x^4}{a^6} = 0 \quad (i.e)\ x\left(1 - \frac{a^3x^3}{a^6}\right) = 0 \quad (i.e)\ x\left(1 - \frac{x^3}{a^3}\right) = 0$$

(i. e) $x = 0, a$.

When $x = 0 \Rightarrow y = \infty$, when $x = a \Rightarrow y = a$.

Omit $(0,\infty)$; $\Rightarrow$ stationary point is (a, a)

$$r = 2\frac{a^3}{x^3};\ t = 2\frac{a^3}{y^3};\ s = 1 \text{ at } (a, a)$$

$$\Rightarrow r = \frac{2a^3}{a^3} = 2;\ t = 2;\ s = 1.\ \therefore rt - s^2 \text{ gives } 4 - 1 = 3 > 0$$

As $rt - s^2 > 0$, $r > 0 \Rightarrow$ the point (a, a) is a minimum point.

$$f(a,a) = a^2 + \frac{a^3}{a} + \frac{a^3}{a}$$

(i. e) $3a^2$ is the minimum value.

Example 3

Examine the function $x^3 + y^3 - 12x - 3y + 20$ for extreme values.

Solution: Let $f(x,y) = x^3 + y^3 - 12x - 3y + 20$

$f_x = 3x^2 - 12;\ f_y = 3y^2 - 3\ \ f_{xx} = 6x;\ f_{yy} = 6y;\ f_{xy} = 0$

The stationary points are given by $f_x = 0, f_y = 0$

$\Rightarrow 3x^2 - 12 = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$

Also $3y^2 - 3 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

The critical points are (2, 1), (2, -1), (-2, 1), (-2, -1).

Now at (2, 1); $r = f_{xx} = 12, t = f_{yy} = 6, s = 0$

$\therefore \Delta = rt - s^2 \Rightarrow (12 \times 6) - 0^2 = 72 > 0$

(i. e) (2,1) is a minimum point as $r > 0,\ \Delta > 0$

$\therefore f(2,1) = 8+1-24-3+20 = 2$

$\therefore$ Minimum value = 2

At $(2,-1)$: $r = 12,\ t = -6,\ s = 0$

$\Rightarrow \Delta < 0 \Rightarrow$ (2,-1) is a saddle point.

Similarly (-2, 1) is also a saddle point.

At $(-2,-1)$: $r = -12,\ t = -6,\ s = 0$

$\Rightarrow \Delta = rt - s^2 = 72 - 0$

$\Rightarrow \Delta > 0,\ r < 0$ at (-2,-1)

$\therefore$ (-2,-1) is a maximum point.

$\therefore f(-2,-1) = -8-1+24+3+20 = 38$

$\therefore$ At (-2,-1); the maximum value = 38.

Example 4 ....

Identify the saddle point and extreme points of $f(x,y) = x^4 - y^4 - 2x^2 + 2y^2$.

Solution: Let $f(x,y) = x^4 - y^4 - 2x^2 + 2y^2$.

$f_x = 4x^3 - 4x$, $f_y = -4y^3 + 4y$, $f_{xx} = 12x^2 - 4$; $f_{xy} = 0$; $f_{yy} = -12y^2 + 4$

The stationary points are given by $f_x = 0, f_y = 0$

$\therefore 4x^3 - 4x = 0; \quad 4y - 4y^3 = 0$

$x^3 - x = 0; \quad y - y^3 = 0$

$x(x^2 - 1) = 0; \quad y(1 - y^2) = 0$

$x = 0, \pm 1, \quad y = 0, \pm 1$

The stationary points are (0, 0), $(0, \pm 1)$, $(\pm 1, 0)$, $(\pm 1, \pm 1)$.

At (0, 0): $r = -4,\ t = 4,\ s = 0$

$\therefore \Delta = rt - s^2 = -16 < 0$

(0, 0) is a saddle point.

At *(0, 1)*: $r = -4,\ s = 0,\ t = -8$

$\therefore \Delta = rt - s^2 > 0$

As $rt - s^2 > 0$, $r < 0$, *(0, 1)* is a maximum point.

$f(0,1) = -1 + 2 = 1$

Similarly (0,-1) is a maximum point.

At $(\pm 1, 0)$: $r = 12 - 4 = 8,\ s = 0,\ t = 4$

$\therefore \Delta = rt - s^2 = 32 > 0$ as $\Delta > 0, r > 0 \Rightarrow (\pm 1, 0)$ is a minimum point.

$f(\pm 1, 0) = 1 - 2 = -1$

The points $(\pm 1, \pm 1)$ are saddle points.

Example 5

Examine $f(x,y) = x^3 + y^3 - 3axy$ for maxima and minima.

Solution: Given $f(x,y) = x^3 + y^3 - 3axy$

Now, $f_x = 3x^2 - 3ay$; $\quad f_y = 3y^2 - 3ax$

$f_{xx} = 6x$; $\quad f_{yy} = 6y$

$f_{xy} = -3a$

The stationary points are obtained from $f_x = \dfrac{\partial f}{\partial x} = 0$ & $f_y = \dfrac{\partial f}{\partial y} = 0$

where $3x^2 - 3ay = 0$ and $3y^2 - 3ax = 0$. (i. e) $x^2 - ay = 0$ and $y^2 - ax = 0$

Using these two equations, we get the stationary points as $(0, 0)$ and (a, a).

Now, at $(0, 0)$: $r = f_{xx} = 0$, $t = f_{yy} = 0$ and $s = f_{xy} = -3a$

$\therefore\ rt - s^2 = 0 - 9a^2 < 0.$

The point $(0, 0)$ is neither a maximum nor a minimum point.

At (a, a): $r = f_{xx} = 6a$, $t = f_{yy} = 6a$ and $s = f_{xy} = -3a$ such that

$rt - s^2 = 36a^2 - 9a^2 > 0.$

Also, $r = f_{xx}$ at $(a, a) = 6a$

$\Rightarrow$ is positive when a is positive and r is negative when a is negative.

(i. e) The point (a, a) is a minimum if $a > 0$ and (a, a) is a maximum if $a<0$.

Example 6

Find the maximum or minimum value of $\sin x + \sin y + \sin(x+y)$.

Solution: Given, $f(x,y) = \sin x + \sin y + \sin(x+y)$

$f_x = \cos x + \cos(x+y)$; $f_y = \cos y + \cos(x+y)$

$f_{xx} = -\sin x - \sin(x+y)$, $f_{yy} = -\sin y - \sin(x+y)$ $f_{xy} = -\sin(x+y)$

The stationary points are obtained by equating f_x to 0 and f_y to 0.

(i.e) $f_x = 0 \Rightarrow \cos x + \cos(x+y) = 0 \qquad (1)$

$f_y = 0 \Rightarrow \cos y + \cos(x+y) = 0 \qquad (2)$

From (1), $\cos x = -\cos(x+y) = \cos(\pi-(x+y)) \Rightarrow x = \pi-(x+y)$

(i.e) $2x+y=\pi$ (3)

Solving (3) and (4), we get, $x=\frac{\pi}{3}, y=\frac{\pi}{3}$.

Then the stationary point is $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$.

At $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$: $r=f_{xx}=-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}=-\sqrt{3}$, $s=f_{xy}=-\frac{\sqrt{3}}{2}$, $t=f_{yy}=-\sqrt{3}$

$\therefore rt-s^2=3-\frac{3}{4}=\frac{9}{4}>0.$

Also, $r<0$ at $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$.

$\therefore$ The point $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$ is a maximum point.

Hence the maximum value of the given function is

$$f\left(\frac{\pi}{3},\frac{\pi}{3}\right)=\sin\frac{\pi}{3}+\sin\frac{\pi}{3}+\sin\frac{2\pi}{3}$$

$$=\sin\frac{\pi}{3}+\sin\frac{\pi}{3}+\sin\left(\pi-\frac{\pi}{3}\right)$$

$$=\sin\frac{\pi}{3}+\sin\frac{\pi}{3}+\sin\frac{\pi}{3}$$

$$=3\sin\frac{\pi}{3}=3\frac{\sqrt{3}}{2}$$

Example 7

Determine the maxima or minima of the function $x^3y^2(12-x-y)$ where $x>0, y>0$. Also find the extreme value.

Solution: Given $f(x,y)=x^3y^2(12-x-y)=12x^3y^2-x^4y^2-x^3y^3$

Now, $f_x=36x^2y^2-4x^3y^2-3x^2y^3$ *and* $f_y=24x^3y-2x^4y-3x^3y^2$

$f_x=0$ & $f_y=0 \Rightarrow 36x^2y^2-4x^3y^2-3x^2y^3=0 \Rightarrow x^2y^2(36-4x-3y)=0$

$\Rightarrow 36-4x-3y=0$ (since $x>0, y>0$)

(i.e) $4x+3y=36$ (1)

and $24x^3y - 2x^4y - 3x^3y^2 = 0$

$x^3y(24 - 2x - 3y) = 0$

$\Rightarrow 24 - 2x - 3y = 0$ (since $x > 0, y > 0$)

(i. e) $2x + 3y = 24$ (2)

Solving (1) and (2), we get, $y = 4$ and $x = 6$.

Now, $f_{xx} = 72xy^2 - 12x^2y^2 - 6xy^3$, $f_{xy} = 72x^2y - 8x^3y - 9x^2y^2$

$f_{yy} = 24x^3 - 2x^4 - 6x^3y$

Then at $(6, 4)$, $r = f_{xx} = -2204$, $s = -1728$, $t = -2592$

$\therefore rt - s^2 > 0$. Also $r < 0$.

Then point (6, 4) is a maximum point.

The maximum value of the given function is 6912.

Lagrange's method of undetermined multipliers

This method is to find the maximum or minimum value of a function of three or more variables, given the constraints.

Let $f(x, y, z)$ be a function of three variables which is to be tested for maximum or minimum value. Let the variables x, y, z be connected by a relation

$$\varphi(x, y, z) = 0 \tag{1}$$

The conditions for $f(x, y, z)$ to have a maximum point, or a minimum point are

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0$$

By total differentials, we have

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \tag{2}$$

Similarly from (1), we have that

$$\frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz = 0 \tag{3}$$

Equation (2) + λ.equation (3), ultimately gives the following:

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \varphi}{\partial x}=0,\ \frac{\partial f}{\partial y}+\lambda\frac{\partial \varphi}{\partial y}=0\ \text{and}\ \frac{\partial f}{\partial z}+\lambda\frac{\partial \varphi}{\partial z}=0$$ (Here 'λ' is called the Language multiplier).

Solving the above equations along with the given relation, we will get the values of x, y, z and the Lagrange's multiplier λ.

These values give finally the required maximum or minimum value of the function $f(x, y, z)$.

Example 1

A rectangular box open at the top is to have volume of 32 cubit ft. Find the dimensions in order that the total surface area is minimum.

Solution: Given volume, $\varphi(x,y,z) = xyz - 32 = 0$ (1)

The required function is the total surface area

$S = f(x, y, z) = xy + 2xz + 2yz$ (2)

At the critical points, we have

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \varphi}{\partial x}=0 \Rightarrow y+2z+\lambda yz=0 \quad (3)$$

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial \varphi}{\partial y}=0 \Rightarrow x+2z+\lambda xy=0 \quad (4)$$

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial \varphi}{\partial z}=0 \Rightarrow 2x+2y+\lambda xy=0 \quad (5)$$

(3) $\times x$ – (4) $\times y$ gives, $2(zx - zy) = 0,\ z \neq 0 \Rightarrow x - y = 0$

$x = y$ (6)

(4)) $\times y$ – (5)) $\times z$ gives, $xy - 2xz = 0$

$y^2 - 2yz = 0$ (using (6)) $\Rightarrow y(y-2z) = 0 \Rightarrow y - 2z = 0\ (y \neq 0)$

$z = y/2$ (7)

Using (6) and (7) in (1), we get,

$x.x.x/2 = 32 \Rightarrow x^3 = 64 \Rightarrow x = 4,\ \therefore y = 4$ and $z = 2$.

The dimensions are 4 cm, 4 cm and 2 cm.

Example 2

7

Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution: The given ellipsoid is $\varphi(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ (1)

The required function is the volume of the parallelopiped; given by

$V=8xyz=f(x,y,z)$

Where the dimensions are $2x$, $2y$ and $2z$. (2)

At the critical points we have

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial\varphi}{\partial x}=0\Rightarrow 8yz+\lambda\left(\frac{2x}{a^2}\right)=0 \quad (3)$$

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial\varphi}{\partial y}=0\Rightarrow 8xz+\lambda\left(\frac{2y}{b^2}\right)=0 \quad (4)$$

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial\varphi}{\partial z}=0\Rightarrow 8xy+\lambda\left(\frac{2z}{c^2}\right)=0 \quad (5)$$

Equation $(3)\times x+(4)\times y+(5)\times z$ gives

$$24xyz+2\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=0,\ \text{(i. e)}\ 2\lambda=-24xyz\ \ (\text{using}\ (1))$$

$\therefore\ \lambda=-12xyz$ (6)

Using (6) in (5), we get

$$8xy+(-12xyz)\left(\frac{2z}{c^2}\right)=0\Rightarrow 8xy\left(1-\frac{3z^2}{c^2}\right)=0$$

$$\Rightarrow\frac{3z^2}{c^2}=1\ (or)\ z=\frac{c}{\sqrt{3}}\quad (\because x\neq 0, y\neq 0)$$

Similarly by using (6) in (4) and (3), we get

$$y=\frac{b}{\sqrt{3}}\quad and\quad x=\frac{a}{\sqrt{3}}$$

The maximum volume of rectangular parallelepiped is

$$V=8xyz=\frac{8abc}{3\sqrt{3}}\ \text{cu.units}$$

Example 3

Find the dimensions of the rectangular box, open at the top of maximum capacity whose surface is 432 sq.cm.

Solution: Let x, y, z be the dimensions of the rectangular box, open at the top.

Given its surface area

$$\phi(x,y,z) = xy + 2yz + 2zx - 432 = 0 \qquad (1)$$

The required function is its volume $V = xyz = f(x, y, z)$ (2)

At the critical point we get

$$yz + \lambda(y + 2z) = 0 \qquad (3)$$

$$xz + \lambda(x + 2z) = 0 \qquad (4)$$

$$xy + \lambda(2y + 2x) = 0 \qquad (5)$$

Equation (3)×x - (4) ×y gives, $2\lambda z(x-y)=0 \Rightarrow x=y \; (z\neq 0, \lambda\neq 0)$ (6)

Equation (3)×x - (5)×z gives,

$$\lambda y(x-2z) = 0 \Rightarrow z = \frac{x}{2} \; (y \neq 0, \lambda \neq 0) \qquad (7)$$

Using (6) and (7) in (1), we get,

$$x^2 + x^2 + x^2 = 432 \Rightarrow 3x^2 = 432 \Rightarrow x^2 = 144 \therefore x = 12$$

Hence $y = 12$ and $z = 6$.

Thus, the dimensions of the rectangular box open at the top of maximum capacity are 12 cm, 12 cm and 6 cm.

Example 4

Find the maximum and minimum distance of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 1$.

Solution: Given $\phi(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$ (1)

Example 5

If $\frac{3}{x}+\frac{4}{y}+\frac{5}{z}=6$, find the values of x, y, z which make $x + y + z$ is minimum.

Solution: Given $\varphi(x,y,z)\equiv\frac{3}{x}+\frac{4}{y}+\frac{5}{z}-6=0$ (1)

The required function is $f(x, y, z) = x + y + z$ (2)

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \varphi}{\partial x}=0\Rightarrow\left(1-\frac{3\lambda}{x^2}\right)=0 \qquad (3)$$

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial \varphi}{\partial y}=0\Rightarrow\left(1-\frac{4\lambda}{y^2}\right)=0 \qquad (4)$$

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial \varphi}{\partial z}=0\Rightarrow\left(1-\frac{5\lambda}{z^2}\right)=0 \qquad (5)$$

From (3), (4) and (5), we get

$$3y^2=4x^2 \qquad (6)$$

$$4z^2=5y^2 \qquad (7)$$

(i.e) $y=\pm\frac{2}{\sqrt{3}}x$ (8)

and $y=\pm\frac{2}{\sqrt{5}}z$ (9)

Using (8) and (9) in (1), we get,

$$\frac{3}{x}+\frac{2\sqrt{3}}{x}+\frac{5}{\sqrt{5}\left(\frac{x}{\sqrt{3}}\right)}=6\Rightarrow\frac{3}{x}+\frac{2\sqrt{3}}{x}+\frac{\sqrt{15}}{x}=6$$

$$\sqrt{3}(\sqrt{3}+\sqrt{5}+2)=6x\Rightarrow 3+2\sqrt{3}+\sqrt{15}=6x$$

(or) $x=\frac{\sqrt{3}}{6}(\sqrt{3}+\sqrt{5}+2)$ $\therefore y=\frac{1}{3}(\sqrt{3}+\sqrt{5}+2)$ & $z=\frac{\sqrt{5}}{6}(\sqrt{3}+\sqrt{5}+2)$