Unit-III: Application of PDE

Content:

(1) Classification of Second order PDE's

(2) Method of Separation of Variables

(3) One dimensional Wave equation and its Possible Solution:

- Method I: Initial displacement with zero initial velocity
  - ☞ Type 1: Algebraic Function
  - ☞ Type 2: Trigonometric Function
  - ☞ Type 3: Midpoint of the string is displaced
- Method II: Initial displacement with non-zero initial velocity
  - ☞ Type I: Algebraic Function
  - ☞ Type II: Trigonometric Function
  - ☞ Type III: Split function

(4) One dimensional Heat equation and its Possible Solution:

- Steady state conditions with zero velocity
- Steady state conditions and Non-zero boundary conditions

Classification of PDE:

The general linear PDE of second order can be written as

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0 \tag{1}$$

$$\text{(OR) } Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$$

where A, B, C, D, E, F are in general functions of x and y.

The above equation of second order (1) is said to

(1) elliptic if $B^2 - 4AC < 0$

(2) hyperbolic if $B^2 - 4AC > 0$

(3) parabolic if $B^2 - 4AC = 0$

Example 1: Classify the following equations

(a) $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

(b) $xf_{xx} + yf_{yy} = 0, x > 0, y > 0.$

(c) $xu_{xx} + u_{yy} = 0.$

(d) $(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0.$

(e) $\dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial x \partial y} + 4\dfrac{\partial^2 u}{\partial y^2} - 12\dfrac{\partial u}{\partial y} + 7u = x^2 + y^2.$

(f) $u_{xx} + 4u_{xy} + (x^2 + 4y^2)u_{yy} = \sin(x+y).$

Solution:

(a) Here $A = 1, B = 2, C = 1$

$\Rightarrow B^2 - 4AC = 4 - 4 = 0.$

Hence, the equation is parabolic.

(b) Here $A = x, B = 0, C = y$

$\Rightarrow B^2 - 4AC = 0 - 4xy = -ve$. Since $x > 0, y > 0$.

Hence the equation is elliptic for all $x > 0, y > 0$.

(c) Here $A = x, B = 0, c = 1$

$$\begin{aligned} B^2 - 4AC &= 0 - 4x \\ &= -4x = \begin{cases} \text{elliptic if } x > 0 \\ \text{hyperbolic if } x < 0 \\ \text{parabolic if } x = 0 \end{cases} \end{aligned}$$

(d) Here $A = x + 1, B = -2(x+2), C = x + 3$

$$\begin{aligned} B^2 - 4AC &= 4(x+2)^2 - 4(x+1)(x+3) \\ &= 4(x^2 + 4x + 4) - 4(x^2 + 4x + 3) = 4. \end{aligned}$$

Hence the equation is hyperbolic.

(e) Here $A = 1, B = 4, C = 4$

$\Rightarrow B^2 - 4AC = 16 - 4(1)(4) = 0.$

Hence the equation is parabolic.

(f) Here $A = 1, B = 4, C = x^2 + 4y^2$

$\Rightarrow B^2 - 4AC = 16 - 4(x^2 + 4y^2) = 4(4 - x^2 - 4y^2)$

(i) The equation is elliptic if $4 - x^2 - 4y^2 < 0$

That is $x^2 + 4y^2 > 4 \Rightarrow \dfrac{x^2}{4} + \dfrac{y^2}{1} > 1$.

Therefore it is elliptic in the region outside the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$.

(ii) The equation is hyperbolic if $4 - x^2 - 4y^2 > 0$

That is $4 > x^2 + 4y^2 \Rightarrow \dfrac{x^2}{4} + \dfrac{y^2}{1} < 1$.

Therefore it is hyperbolic in the region inside the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$.

(iii) It is parabolic on the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$.

Example 2: Classify the equation: $x^2 f_{xx} + (1 - y^2) f_{yy} = 0$.

Solution:

Here $A = x^2, B = 0, C = 1 - y^2$

$\Rightarrow B^2 - 4AC = -4x^2(1 - y^2) = 4x^2(y^2 - 1)$

For all x except $x = 0$, x^2 is positive.

Case (i): For $x = 0$ for all y or for all x, $y = \pm 1$, the equation is parabolic.

Case (ii): For $x \neq 0$ $(-\infty < x < \infty)$, $-1 < y < 1$, the equation is elliptic.

Case (iii): For $x \neq 0$ $(-\infty < x < \infty)$, $y < 1$ or $y > 1$, the equation is hyperbolic.

Example 3: Classify the following equations

(a) The Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

(b) The Poisson equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y)$.

(c) One dimensional heat equation $\alpha^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$.

(e) One dimensional wave equation $\alpha^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$.

Solution:

(a) and (b)$\Rightarrow$ Here $A = 1, B = 0, C = 1$

Therefore $B^2 - 4AC = -4 < 0$. Hence the equation is elliptic.

(c) Here $A = \alpha^2, B = 0, C = 0$.

Therefore $B^2 - 4AC = 0$. Hence the equation is parabolic.

(d) Here $A = \alpha^2, B = 0, C = -1$.

Therefore $B^2 - 4AC = 4\alpha^2 > 0$. Hence the equation is hyperbolic.

Method of Separation of Variables

Example 1: Solve $2x\dfrac{\partial z}{\partial x} - 3y\dfrac{\partial z}{\partial y} = 0$ by method of separation of variables.

Solution:

Assume a solution of the form

$$z = X(x)Y(y) \tag{1}$$

where X is a function of x alone and y is a function of y only.

Differentiating (1) partially w.r.to x and y, we get

$$\frac{\partial z}{\partial x} = X^{'}Y \text{ and } \frac{\partial z}{\partial y} = XY^{'}.$$

where $X^{'} = \dfrac{dX}{dx}$ and $Y^{'} = \dfrac{dY}{dy}$

Hence the given equation becomes $2xX^{'}Y - 3yXY^{'} = 0$.

Separating the variables,

$$\frac{2xX^{'}}{X} = \frac{3yY^{'}}{Y}$$

The L.H.S. is a function x only and the R.H.S. is a function of y only. Also, x and y are independent variables. When y varies, keeping x fixed, the L.H.S. is constant and hence the R.H.S. must also be the same constant.

$$\frac{2xX^{'}}{X} = \frac{3yY^{'}}{Y} = k, \text{say where } k \text{ is a constant}$$

$$\Rightarrow \frac{X^{'}}{X} = \frac{k}{2x}$$

$$\text{Integrating w.r.t } x, \text{ we get } \log X = \frac{k}{2}\log x + \log c_1$$

$$\Rightarrow \log X = \log(c_1 x^{k/2}) \Rightarrow X = c_1 x^{k/2}$$

$$\text{Similarly } \frac{Y^{'}}{Y} = \frac{k}{3y}$$

$$\text{Integrating w.r.t } y, \text{ we get } \log Y = \frac{k}{3}\log y + \log c_2 \Rightarrow \log Y = \log(c_2 x^{k/3})$$

$$\Rightarrow Y = c_2 y^{k/3}$$

Hence $z = XY = c_1x^{k/2}.c_2y^{k/3} = cx^{k/2}y^{k/3}$, where c and k are arbitrary constants.

Example 2: Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ by method of separation of variables.

Solution:

Assume a solution of the form

$$z = X(x)Y(y) \tag{2}$$

where X is a function of x alone and y is a function of y only.

Differentiating (2) partially w.r.to x and y, we get

$$\frac{\partial z}{\partial x} = X'Y, \text{ and } \frac{\partial z}{\partial y} = XY'.$$

where $X' = \frac{dX}{dx}$ and $Y' = \frac{dY}{dy}$

Again diff. w.r. to x, we get

$$\frac{\partial^2 z}{\partial x^2} = X''Y$$

where $X'' = \frac{d^2X}{dx^2}$

Hence the given equation becomes $X''Y - 2X'Y + XY' = 0$.

Separating the variables,

$$\frac{X' - 2X}{X} = -\frac{Y'}{Y} \tag{3}$$

The L.H.S. is a function x only and the R.H.S. is a function of y only. Also, x and y are independent variables. When y varies, keeping x fixed, the L.H.S. is constant and hence the R.H.S. must also be the same constant.

$$\frac{X' - 2X}{X} = -\frac{Y'}{Y} = k, \text{say where } k \text{ is a constant}$$

$$\Rightarrow X'' - 2X' - kX = 0 \text{ and } Y' + kyY = 0$$

Now $\Rightarrow (D^2 - 2D - k)X = 0$ where $D = \frac{d}{dx}$

The auxiliary equation is $m^2 - 2m - k = 0$

$\Rightarrow m = \frac{2 \pm \sqrt{4 + 4k}}{2} = 1 \pm \sqrt{1 + k}$

Therefore $X = Ae^{(1+\sqrt{1+k})x} + Be^{(1-\sqrt{1+k})x}$

Solving $Y' + kyY = 0 \Rightarrow \frac{Y'}{Y} = -k$

Integrating w.r.t y, we get $\log Y = -ky + \log c_2$

$\Rightarrow Y = e^{-ky+c_2} = De^{-ky}$

Hence $z = XY = [Ae^{(1+\sqrt{1+k})x} + Be^{(1-\sqrt{1+k})x}]De^{-ky}$.

$\Rightarrow z = e^{-ky}\left[ae^{(1+\sqrt{1+k})x} + be^{(1-\sqrt{1+k})x}\right]$ where $a = AD, b = BD$.

Example: 3 By the method of separation of variables, solve $4\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 3z$ subject to $z = e^{-5y}$ when $x = 0$.

Solution:

Let $z = X(x)Y(y) = XY$

The given equation reduces to $4X^{'}Y + XY^{'} = 3XY$

$$\Rightarrow \frac{4X^{'}}{X} + \frac{Y^{'}}{Y} = 3$$

$$\Rightarrow \frac{4X^{'}}{X} = 3 - \frac{Y^{'}}{Y} = k \text{ constant}$$

$$\Rightarrow \frac{4X^{'}}{X} = k \text{ and } 3 - \frac{Y^{'}}{Y} = k$$

$$\Rightarrow \frac{X^{'}}{X} = \frac{k}{4} \text{ and } \frac{Y^{'}}{Y} = 3 - k$$

Integrating

$$\log X = \frac{k}{4}x + A \text{ and } \log Y = (3-k)y + B$$

Therefore $X = e^{A+\frac{kx}{4}}$ and $y = e^{B+(3-k)y}$

$\Rightarrow X = ae^{\frac{kx}{4}}$ and $Y = be^{(3-k)y}$ where $a = e^A, b = e^B$

Hence $z = XY = ce^{\frac{kx}{4}}e^{(3-k)y}$ where $c = ab$

$\Rightarrow z = Ce^{\frac{kx}{4}+(3-k)y}$

$\Rightarrow e^{-5y} = ce^{(3-k)y}$ Since $x = 0, z = e^{-5y}$

Therefore $c = 1, 3 - k = -5$

Hence $z = e^{2x-5y}$.

Transverse vibrations of a stretched elastic string

Let us consider small transverse vibrations of an elastic string of length l, which is stretched and then fixed at its two ends. Now we will study the transverse vibration of the string when no external forces act on it. Take an end end of the string as the origin and the string in the equilibrium position as the x-axis and the line through the origin and perpendicular to the x-axis as the y-axis.

Following are the assumptions made to derive PDE

(1) The motion takes place entirely in one plane. This plane is chosen as the xy plane.

(2) In this plane, each particle of the string moves in a perpendicular to the equilibrium position of the string.

(3) The tension T is very large compared with the weight of the string and hence the gravitational force may be neglected.

(4) The tension T caused by stretching the string before fixing it an end points is constant at all times at all points of the deflected string.

(5) The effect of friction is negligible.

(6) The string is perfectly flexible. It can transmit only tension but not bending or sharing forces.

(7) The slope of the deflection curve is small at all points and at all times.

One dimensional Wave equation

Equation governing the transverse vibration of an elastic string is known as One dimensional Wave equation.

One dimensional Wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

where $a^2 = \dfrac{T}{m}$, T is tension and m is mass per unit length.

Solution of one dimensional wave equation by the method of separation of variables

Let $y = X(x).T(t)$ be a solution of (1) where $X(x)$ is a function of x only and $T(t)$ is a function of t only.

Therefore $\frac{\partial y}{\partial t} = XT'$ and $\frac{\partial y}{\partial x} = X'T$

$\Rightarrow \frac{\partial^2 y}{\partial t^2} = XT''$ and $\frac{\partial^2 y}{\partial x^2} = X''T$

Therefore the equation (1) becomes $XT'' = a^2X''T$

$$\Rightarrow \frac{X''}{X} = \frac{T''}{a^2T} \tag{2}$$

The L.H.S. of (2) is a function of x only whereas the R.H.S. is a function of time t only. But x and t are independent variables. Hence (2) is true only if each is equal to a constant.

Therefore $\frac{X''}{X} = \frac{T''}{a^2T} = k$ (say) where k is any constant. Hence

$$X'' - kX = 0 \text{ and } T'' - ka^2T = 0 \tag{3}$$

Thus we get two second order ordinary linear differential equation with constant coefficients. The solutions of (3) depend upon the nature of k. that is $k > 0$ or <0 or 0.

Case 1: If $k > 0$, let $k = \lambda^2$, a positive value.

Now the equation (3) are $X'' - \lambda^2X = 0$ and $T'' - \lambda^2a^2T = 0$.

Therefore Auxiliary equation is $m^2 - \lambda^2 = 0$ and $m^2 - \lambda^2a^2 = 0$

$\Rightarrow m = \pm\lambda$ and $m = \pm\lambda a$

Therefore $X = Ae^{\lambda x} + Be^{-\lambda x}$ and $T = Ce^{\lambda at} + De^{-\lambda at}$

Hence the solution is $y(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})(Ce^{\lambda at} + De^{-\lambda at})$.

Case 2: If $k < 0$, let $k = \lambda^2$, a negative value.

Now the equation (3) are $X'' + \lambda^2X = 0$ and $T'' + \lambda^2a^2T = 0$.

Therefore Auxiliary equation is $m^2 + \lambda^2 = 0$ and $m^2 + \lambda^2a^2 = 0$

$\Rightarrow m = \pm i\lambda$ and $m = \pm i\lambda a$

Therefore $X = A\cos\lambda x + B\sin\lambda x$ and $T = C\cos\lambda at + D\sin\lambda at$

Hence the solution is $y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)$.

Case 3: Let $k = 0$.

Now the equation (3) are $X^{''} = 0$ and $T^{''} = 0$.

$\Rightarrow X^{'} = A$ and $T^{'} = C$

$\Rightarrow X = Ax + B$ and $T^{'} = Ct + D$

Hence the solution is $y(x,t) = (Ax + B)(Ct + D)$.

Thus there are three possible solutions of the wave equation and they are

$$y(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})(Ce^{\lambda at} + De^{-\lambda at}) \tag{4}$$

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \tag{5}$$

$$y(x,t) = (Ax + B)(Ct + D) \tag{6}$$

Proper Choice of the solution

Out of these solutions, we have to choose the solution which is consistent with the physical nature of the problem and the given boundary conditions. In the case of vibrations of the elastic string, the displacement $y(x,t)$ of the string at any point x and at any time $t > 0$ must be a periodic function of x and t.

Hence the solution (5) consisting of trigonometric functions, which are periodic functions, is the suitable solution to the one-dimensional wave equation.

Note:

- The constants A, B, C and D are determined by using the boundary value conditions of the given problem. Hence four conditions are required to solve the one dimensional wave equation.
- The conditions to be satisfied by the solution $y(x,t)$ of the one dimensional wave equation are (i) $y(0,t) = 0$ (ii)$y(l,t) = 0$ for all $t \geq 0$. Since the string is fixed at the end points, there is no displacement at the end points.
- If the string if pulled up into a curve $y = f(x)$ and then released the conditions are (iii) $y(x,0) = f(x)$ and (iv) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$ for all $0 \leq x \leq 1$.
- The conditions (i) and (ii) are the boundary conditions and the conditions (iii) and (iv) are the initial conditions.
- If the string is initially release from rest then $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$.

- If the string is initially in the equilibrium position then $y(x,0) = 0$.

Method I: Initial displacement with zero initial velocity

Type 1: Algebraic function

Example 1: A tightly string of length l has its end fastened at $x = 0,\ x = l$. At $t = 0$, the string is in the form $f(x) = k(lx - x^2)$ and then released. Find the displacement at any point on the string at a distance x from one end and at any time $t > 0$.

Solution: The displacement of the string $y(x,t)$ is governed by $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$.

The boundary conditions are

(i) $y(0,t) = 0, t \geq 0$

(ii) $y(l,t) = 0, t \geq 0$.

The initial conditions are

(iii) $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0, 0 \leq x \leq l$

(iv) $y(x,0) = k(lx - x^2), 0 \leq x \leq l$.

The proper solution is

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \tag{7}$$

Using boundary condition (i) in (7), we get

$$y(x,0) = A(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow 0 = A(C\cos\lambda at + D\sin\lambda at)$$

$\Rightarrow A = 0$ and $C\cos\lambda at + D\sin\lambda at \neq 0$.

$A = 0$ in (7), we get

$$y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at) \tag{8}$$

Applying the boundary condition (ii) in (8), we get

$$y(l,t) = B\sin\lambda l(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow 0 = B\sin\lambda l(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow B \neq 0 \text{and} \sin nl = 0 \Rightarrow \sin nl = \sin n\pi$$

$$\Rightarrow \lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{l}.$$

$\lambda = \dfrac{n\pi}{l}$ in (8), we get

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \tag{9}$$

Differentiating (9) partially w.r. to t and using the initial condition (iii), we get
$\dfrac{\partial y}{\partial t} = B\sin\dfrac{n\pi x}{l}\left[-C\sin\dfrac{n\pi at}{l}.\dfrac{n\pi a}{l} + D\cos\dfrac{n\pi at}{l}.\dfrac{n\pi a}{l}\right]$
$\Rightarrow \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = B\sin\dfrac{n\pi x}{l}\left[0 + D.\dfrac{n\pi a}{l}\right]$
$\Rightarrow 0 = B\sin\dfrac{n\pi x}{l}.D\dfrac{n\pi a}{l}$
$\Rightarrow B \neq 0, D = 0$
$D = 0$ in equation (9), we get $y(x,t) = B\sin\dfrac{n\pi x}{l}C\cos\dfrac{n\pi at}{l}$
$\Rightarrow y(x,t) = B_n\sin\dfrac{n\pi x}{l}\cos\dfrac{n\pi at}{l}$, since $BC = B_n$
Therefore the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \tag{10}$$

Using initial condition (iv) in (10), we get $y(x,0) = \sum\limits_{n=1}^{\infty} B_n\sin\dfrac{n\pi x}{l}$
$\Rightarrow K(lx - x^2) = \sum\limits_{n=1}^{\infty} B_n\sin\dfrac{n\pi x}{l}$
This is half-range Fourier sine series. Therefore

$$B_n = \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2k}{l}\left[(lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$
\begin{aligned}
B_n &= \frac{2k}{l}\left[-2\cos n\pi.\frac{l^3}{n^3\pi^3}+2\frac{l^3}{n^3\pi^3}\right], \text{ since } \sin 0=\sin n\pi=0\\
&= \frac{2k}{l}.\frac{2l^3}{n^3\pi^3}[-(-1)^n+1]\\
&= \frac{4kl^2}{n^3\pi^3}[1-(-1)^n]=\begin{cases}\frac{8kl^2}{n^3\pi^3} \text{ if } n \text{ is odd}\\ 0 \text{ if } n \text{ is even}\end{cases}
\end{aligned}
$$

Substituting the value of B_n in (10), we get

$$y(x,t)=\sum_{n=odd}\frac{8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}.$$

Example 2: A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y=3(lx-x^2)$ from which it is released at time $t=0$. Find the displacement of any point on the string at a distance of x from one end at any time t.

Solution: In the above problem, k replaced by 3.

Type 2: Trigonometric function

Example 3: A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y=a\sin\frac{\pi x}{l}$ from which it is released at time $t=0$. Find the displacement $y(x,t)$.

Sol: The displacement of the string $y(x,t)$ is governed by $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$.

The boundary conditions are

(i) $y(0,t)=0, t\geq 0$

(ii) $y(l,t)=0, t\geq 0$

The initial conditions are

(iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0}=0, 0\leq x\leq l$

(iv) $y(x,0)=a\sin\frac{\pi x}{l}, 0\leq x\leq l.$

The proper solution is

$$y(x,t)=(A\cos\lambda x+B\sin\lambda x)(C\cos\lambda at+D\sin\lambda at) \tag{11}$$

Using boundary condition (i) in (11), we get

$$y(0,t)=A(C\cos\lambda at+D\sin\lambda at)$$

$$\Rightarrow 0 = A(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow A = 0$$

$A = 0$ in (11), we get

$$y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at) \tag{12}$$

Applying the boundary condition (ii) in (12), we get

$$y(l,t) = B\sin\lambda l(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow 0 = B\sin\lambda l(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow B \neq 0 \text{ and } \sin nl = 0 \Rightarrow \sin nl = \sin n\pi$$

$$\Rightarrow \lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{l}.$$

$\lambda = \dfrac{n\pi}{l}$ in (12), we get

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \tag{13}$$

Differentiating (13) partially w.r. to t and using the initial condition (iii), we get

$\dfrac{\partial y}{\partial t} = B\sin\dfrac{n\pi x}{l}\left[-C\sin\dfrac{n\pi at}{l}.\dfrac{n\pi a}{l} + D\cos\dfrac{n\pi at}{l}.\dfrac{n\pi a}{l}\right]$

$\Rightarrow \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = B\sin\dfrac{n\pi x}{l}\left[0 + D.\dfrac{n\pi a}{l}\right]$

$\Rightarrow 0 = B\sin\dfrac{n\pi x}{l}.D\dfrac{n\pi a}{l}$

$\Rightarrow B \neq 0, D = 0$

$D = 0$ in equation (13), we get $y(x,t) = B\sin\dfrac{n\pi x}{l}C\cos\dfrac{n\pi at}{l}$

$\Rightarrow y(x,t) = B_n\sin\dfrac{n\pi x}{l}\cos\dfrac{n\pi at}{l}$, since $BC = B_n$

Therefore the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \tag{14}$$

Using initial condition (iv) in (14), we get $y(x,0) = \sum\limits_{n=1}^{\infty} B_n\sin\dfrac{n\pi x}{l}$

$\Rightarrow a\sin\dfrac{\pi x}{l} = \sum\limits_{n=1}^{\infty} B_n\sin\dfrac{n\pi x}{l}.$

$\Rightarrow a\sin\frac{\pi x}{l} = B_1\sin\frac{\pi x}{l} + B_2\sin\frac{2\pi x}{l} + B_3\sin\frac{3\pi x}{l} + \cdots$

Equating the like coefficients, we get $B_1 = a, B_n = 0, n \neq 1$.

Substituting the value of $B_1 = a, B_n = 0, n \neq 1$ in (14), we get

$$y(x,t) = a\sin\frac{\pi x}{l}\cos\frac{\pi a t}{l}$$

Example 4: A tightly stretched string with fixed end point $x = 0$ and $x = l$ is initially in a position given by $y(x,0) = y_0\sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest in this position, find the displacement $y(x,t)$ of the string at any point.

Sol: The displacement of the string $y(x,t)$ is governed by $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$.

The boundary conditions are

(i) $y(0,t) = 0, t \geq 0$

(ii) $y(l,t) = 0, t \geq 0$

The initial conditions are

(iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0, 0 \leq x \leq l$

(iv) $y(x,0) = y_0\sin^3\left(\frac{\pi x}{l}\right), 0 \leq x \leq l$.

The proper solution is

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \tag{15}$$

Using boundary condition (i) in (15), we get

$$y(0,t) = A(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow 0 = A(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow A = 0$$

$A = 0$ in (15), we get

$$y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at) \tag{16}$$

Applying the boundary condition (ii) in (16), we get

$$y(l,t) = B\sin\lambda l(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow 0 = B\sin\lambda l(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow B \neq 0 \text{ and } \sin nl = 0 \Rightarrow \sin nl = \sin n\pi$$

$$\Rightarrow \lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{l}.$$

$\lambda = \frac{n\pi}{l}$ in (16), we get

$$y(x,t) = B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi at}{l} + D \sin \frac{n\pi at}{l} \right) \tag{17}$$

Differentiating (17) partially w.r. to t and using the initial condition (iii), we get
$\frac{\partial y}{\partial t} = B \sin \frac{n\pi x}{l} \left[-C \sin \frac{n\pi at}{l} . \frac{n\pi a}{l} + D \cos \frac{n\pi at}{l} . \frac{n\pi a}{l} \right]$
$\Rightarrow \left(\frac{\partial y}{\partial t} \right)_{t=0} = B \sin \frac{n\pi x}{l} \left[0 + D . \frac{n\pi a}{l} \right]$
$\Rightarrow 0 = B \sin \frac{n\pi x}{l} . D \frac{n\pi a}{l}$
$\Rightarrow B \neq 0, D = 0$
$D = 0$ in equation (17), we get $y(x,t) = B \sin \frac{n\pi x}{l} C \cos \frac{n\pi at}{l}$
$\Rightarrow y(x,t) = B_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$, since $BC = B_n$
Therefore the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \tag{18}$$

Using initial condition (iv) in (18), we get $y(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$

$$\Rightarrow y_0 \sin^3 \left(\frac{\pi x}{l} \right) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \tag{19}$$

We know that $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$
$\Rightarrow \sin^3 \theta = \frac{1}{4} [3 \sin \theta - \sin 3\theta]$
Therefore $\sin^3 \left(\frac{\pi x}{l} \right) = \frac{1}{4} \left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right]$.
Therefore equation (19) becomes

$$\frac{y_0}{4} \left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right] = B_1 \sin \frac{\pi x}{l} + B_2 \sin \frac{2\pi x}{l} + B_3 \sin \frac{3\pi x}{l} + \cdots$$

Equating like coefficients, we get

$$B_1 = \frac{3y_0}{4}, B_2 = 0, B_3 = \frac{-y_0}{4}, B_4 = B_5 = \cdots = 0.$$

Substituting these value in (18), we get

$$y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l}.$$

Type 3: Midpoint of the string is displaced

Example 5: The taut string of length $2l$ is fastened at both ends. The midpoint of the string is taken to a height b and then releases from the rest in that position. Find the displacement of the string.

Solution:

The displacement of the string $y(x,t)$ is governed by $\dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2}$.

The proper solution is

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \tag{20}$$

The boundary conditions are

(i) $y(0,t) = 0, t \geq 0$

(ii) $y(2l,t) = 0, t \geq 0$

The initial conditions are

(iii) $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0, 0 \leq x \leq 2l$

(iv) $y(x,0) = \begin{cases} \dfrac{bx}{l}, & 0 \leq x \leq l \\ \dfrac{-b}{l}(x-2l), & l \leq x \leq 2l \end{cases}$

$\Bigg[$Since, equation of line joining OA is $\dfrac{x-0}{l-0} = \dfrac{y-0}{b-0} \Rightarrow y = \dfrac{bx}{l}$

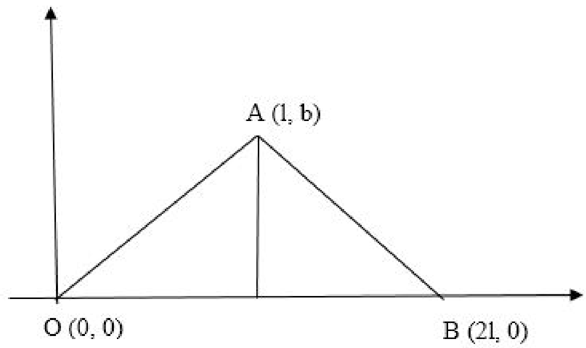
and equation of line joining AB is $\dfrac{x-l}{2l-l} = \dfrac{y-b}{0-b} \Rightarrow y = -\dfrac{b}{l}(x-2l)$ $\Bigg]$

Therefore the new boundary conditions and initial conditions are

(I) $y(0,t) = 0, t \geq 0$

(II) $y(L,t) = 0, t \geq 0$ where $L = 2l$

(III) $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0, 0 \leq x \leq L$



(IV) $y(x,0) = \begin{cases} \dfrac{2bx}{L}, & 0 \le x \le \dfrac{L}{2} \\ \dfrac{-2b}{L}(x-L), & \dfrac{L}{2} \le x \le L \end{cases}$

Using boundary condition (I) in (20), we get

$$y(0,t) = A(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow 0 = A(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow A = 0$$

$A = 0$ in (20), we get

$$y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at) \tag{21}$$

Applying the boundary condition (II) in (21), we get

$$y(L,t) = B\sin\lambda L(C\cos\lambda at + D\sin\lambda at)$$

$$0 = B\sin\lambda L(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow B \neq 0 \text{ and } \sin nL = 0 \Rightarrow \sin nL = \sin n\pi$$

$$\Rightarrow \lambda L = n\pi \Rightarrow \lambda = \frac{n\pi}{L}.$$

$\lambda = \dfrac{n\pi}{L}$ in (21), we get

$$y(x,t) = B\sin\frac{n\pi x}{L}\left(C\cos\frac{n\pi at}{L} + D\sin\frac{n\pi at}{L}\right) \tag{22}$$

Differentiating (22) partially w.r. to t and using the initial condition (III), we get

$\dfrac{\partial y}{\partial t} = B\sin\dfrac{n\pi x}{L}\left[-C\sin\dfrac{n\pi at}{L}.\dfrac{n\pi a}{L} + D\cos\dfrac{n\pi at}{l}.\dfrac{n\pi a}{L}\right]$

$\Rightarrow \left(\frac{\partial y}{\partial t}\right)_{t=0} = B \sin \frac{n\pi x}{L} \left[0 + D.\frac{n\pi a}{L}\right]$

$\Rightarrow 0 = B \sin \frac{n\pi x}{L}.D\frac{n\pi a}{L}$

$\Rightarrow B \neq 0, D = 0$

$D = 0$ in equation (22), we get $y(x,t) = B \sin \frac{n\pi x}{L} C \cos \frac{n\pi at}{L}$

$\Rightarrow y(x,t) = B_n \sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L}$, since $BC = B_n$

Therefore the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L} \tag{23}$$

Using initial condition (IV) in (23), we get $y(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$

This is half range sine series.

$$\begin{aligned}
B_n &= \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \\
&= \frac{2}{L}\left\{\int_0^{L/2} \frac{2bx}{L} \sin \frac{n\pi x}{L}\, dx + \int_{L/2}^{L} \frac{-2b}{L}(x-L) \sin \frac{n\pi x}{L}\, dx\right\} \\
&= \frac{4b}{L^2}\left\{\int_0^{L/2} x \sin \frac{n\pi x}{L}\, dx + \int_{L/2}^{L} (L-x) \sin \frac{n\pi x}{L}\, dx\right\} \\
&= \frac{4b}{L^2}\left\{ \left[x.\left(-\cos\frac{n\pi x}{L}\right).\frac{L}{n\pi} - 1.\left(-\sin\frac{n\pi x}{L}\right).\frac{L^2}{n^2\pi^2}\right]_0^{L/2}\right. \\
&\left. + \left[(L-x).\left(-\cos\frac{n\pi x}{L}\right).\frac{L}{n\pi} - (-1).\left(-\sin\frac{n\pi x}{L}\right).\frac{L^2}{n^2\pi^2}\right]_{L/2}^{L}\right\} \\
&= \frac{4b}{L^2}\left[-\frac{L}{2}\cos\frac{n\pi}{2}.\frac{L}{n\pi} + \sin\frac{n\pi}{2}.\frac{L^2}{n^2\pi^2} - 0 - \sin 0.\frac{L^2}{n^2\pi^2} + 0\right. \\
&\left. - \sin n\pi.\frac{L^2}{n^2\pi^2} + \frac{L}{2}\cos\frac{n\pi}{2}.\frac{L}{n\pi} + \sin\frac{n\pi}{2}.\frac{L^2}{n^2\pi^2}\right]
\end{aligned}$$

$$\begin{aligned}
B_n &= \frac{4b}{L^2}.2\sin\frac{n\pi}{2}.\frac{L^2}{n^2\pi^2} \text{ Since } \sin 0 = 0, \sin n\pi = 0 \\
&= \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2} \\
&= \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{8b}{n^2\pi^2}\sin\dfrac{n\pi}{2}, & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

Substituting B_n values in (23), we get

$$y(x,t) = \sum_{\text{odd}} \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}$$

where $L = 2l$.

Method II: Initial displacement with non-zero initial velocity

Type 1: Algebraic function

Example 1: A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $3x(l - x)$, find the displacement.

Solution: The displacement of the string $y(x,t)$ is governed by $\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}$.

The boundary conditions are

(i) $y(0,t) = 0, t \geq 0$

(ii) $y(l,t) = 0, t \geq 0$.

The initial conditions are

(iii) $y(x,0) = 0, 0 \leq x \leq l$

(iv) $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 3x(l - x), 0 \leq x \leq l$

The proper solution is

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \tag{1}$$

Using boundary condition (i) in (1), we get $A(C\cos\lambda at + D\sin\lambda at) = 0$

$\Rightarrow A = 0$ and $C\cos\lambda at + D\sin\lambda at \neq 0$.

$A = 0$ in (1), we get

$$y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at) \tag{2}$$

Applying the boundary condition (ii) in (2), we get

$$y(x,l) = B\sin\lambda l(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow 0 = B\sin\lambda l(C\cos\lambda at + D\sin\lambda at)$$

$\Rightarrow B \neq 0$ and $\sin nl = 0 \Rightarrow \sin nl = \sin n\pi$

$\Rightarrow \lambda l = n\pi \Rightarrow \lambda = \dfrac{n\pi}{l}$.

$\lambda = \dfrac{n\pi}{l}$ in (2), we get

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \tag{3}$$

Again applying the initial condition (iii) in (3), we get

$$y(x,0) = B\sin\frac{n\pi x}{l}(C+0) \Rightarrow 0 = B\sin\frac{n\pi x}{l}.C$$

$$\Rightarrow C = 0, B \neq 0.$$

$C = 0$ in (3), we get $y(x,t) = B\sin\dfrac{n\pi x}{l} D\sin\dfrac{n\pi at}{l}$

$\Rightarrow y(x,t) = B_n \sin\dfrac{n\pi x}{l}\cos\dfrac{n\pi at}{l}$, since $BD = B_n$

Therefore the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \tag{4}$$

Differentiating (4), partially w.r. to t and using initial condition (iv), we get

$$\begin{aligned}
\frac{\partial y}{\partial t} &= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}.\frac{n\pi a}{l} \\
\left(\frac{\partial y}{\partial t}\right)_{t=0} &= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}.\frac{n\pi a}{l} \\
3x(l-x) &= \sum_{n=1}^{\infty} B_n.\frac{n\pi a}{l}\sin\frac{n\pi x}{l}
\end{aligned}$$

This is half-range Fourier sine series. Therefore

$$\begin{aligned}
b_n &= \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx \\
B_n.\frac{n\pi a}{l} &= \frac{2}{l}\int_0^l 3(lx-x^2)\sin\frac{n\pi x}{l}dx \\
B_n &= \frac{6}{l}.\frac{l}{n\pi a}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right. \\
&\quad \left. + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l
\end{aligned}$$

$$B_n = \frac{6}{n\pi a}\left[-2\cos n\pi . \frac{l^3}{n^3\pi^3} + 2\frac{l^3}{n^3\pi^3}\right], \text{ since } \sin 0 = \sin n\pi = 0$$
$$= \frac{6}{n\pi a}.\frac{2l^3}{n^3\pi^3}[-(-1)^n + 1]$$
$$= \frac{12l^3}{n^4\pi^4 a}[1-(-1)^n] = \begin{cases} \dfrac{24l^3}{n^4\pi^4 a} \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even} \end{cases}$$

Substituting the value of B_n in (4), we get

$$y(x,t) = \sum_{n=odd} \frac{24l^3}{n^4\pi^4 a} sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}.$$

Example 2: A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in equilibrium position. If it is set vibrating giving each point a velocity $\lambda x(l - x)$, find the displacement.

Solution: In the above problem, 3 replaced by λ.

Type 2: Trigonometric function

Example 3: If a string of length l is initially at rest equilibrium position and each point of it is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3\left(\frac{\pi x}{l}\right), 0 < x < l$, determine the transverse displacement $y(x,t)$.

Sol: The displacement of the string $y(x,t)$ is governed by $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$.

The boundary conditions are

(i) $y(0,t) = 0, t \geq 0$

(ii) $y(l,t) = 0, t \geq 0$

The initial conditions are

(iii) $y(x,0) = 0, 0 < x < l$

(iv) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3\left(\frac{\pi x}{l}\right), 0 < x < l.$

The proper solution is

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \tag{5}$$

Using boundary condition (i) in (5), we get $y(x,0) = A(C\cos\lambda at + D\sin\lambda at)$

$$\Rightarrow A(C\cos\lambda at + D\sin\lambda at) = 0 \Rightarrow A = 0$$

$A = 0$ in (5), we get

$$y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at) \tag{6}$$

Again using boundary condition (ii) in (6), we get

$$y(x,l) = B\sin\lambda l(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow 0 = B\sin\lambda l(C\cos\lambda at + D\sin\lambda at)$$

$\Rightarrow B \neq 0$ and $\sin nl = 0 \Rightarrow \sin nl = \sin n\pi$
$\Rightarrow \lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{l}$.
$\lambda = \frac{n\pi}{l}$ in (6), we get

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \tag{7}$$

Again applying the initial condition (iii) in (7), we get

$$y(x,0) = B\sin\frac{n\pi x}{l}(C+0) \Rightarrow 0 = B\sin\frac{n\pi x}{l}.C$$

$$\Rightarrow C = 0, B \neq 0.$$

$C = 0$ in (7), we get $y(x,t) = B\sin\frac{n\pi x}{l}D\sin\frac{n\pi at}{l}$

$\Rightarrow y(x,t) = B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$, since $BD = B_n$
Therefore the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \tag{8}$$

Differentiating (8), partially w.r. to t and using initial condition (iv), we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}.\frac{n\pi a}{l}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}.\frac{n\pi a}{l}$$

$$v_0\sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} B_n.\frac{n\pi a}{l}\sin\frac{n\pi x}{l} \tag{9}$$

We know that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
$\Rightarrow \sin^3\theta = \frac{1}{4}[3\sin\theta - \sin 3\theta]$

Therefore $\sin^3\left(\frac{\pi x}{l}\right) = \frac{1}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$.
Therefore equation (9) becomes

$$\frac{v_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = B_1.\frac{\pi a}{l}.\sin\frac{\pi x}{l} + B_2.\frac{2\pi a}{l}.\sin\frac{2\pi x}{l} + B_3.\frac{3\pi a}{l}.\sin\frac{3\pi x}{l} + \cdots$$

Equating like coefficients, we get

$$\frac{3v_0}{4} = B_1.\frac{\pi a}{l}, B_2 = 0, \frac{-v_0}{4} = B_3.\frac{3\pi a}{l}, B_4 = B_5 = \cdots = 0$$

$$\Rightarrow B_1 = \frac{3lv_0}{4\pi a}, B_2 = 0, B_3 = -\frac{v_0 l}{12\pi a}, B_4 = B_5 = \cdots = 0.$$

Substituting these value in (8), we get

$$y(x,t) = \frac{3lv_0}{4\pi a}\sin\frac{\pi x}{l}\sin\frac{\pi at}{l} - \frac{v_0 l}{12\pi a}\sin\frac{3\pi x}{l}\sin\frac{3\pi at}{l}.$$

Type 3: Split function

Example 4: A string is stretched between two fixed points at a distance $2l$ apart and the points of string are given initial velocities v where $v = \begin{cases} \frac{cx}{l}, & 0 < x < l \\ \frac{c}{l}(2l - x), & l < x < 2l \end{cases}$, x being the distance from an end point. Find the displacement of the string at any subsequent time.

Solution:

The displacement of the string $y(x,t)$ is governed by $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$.

The boundary conditions are

(i) $y(0,t) = 0, t \geq 0$

(ii) $y(L,t) = 0, t \geq 0$ where $L = 2l$

The initial conditions are

(iii) $y(x,0) = 0, 0 \leq x \leq L$

(iv) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \begin{cases} \frac{2cx}{L}, & 0 < x < \frac{L}{2} \\ \frac{2c}{L}(L - x), & \frac{L}{2} < x < L \end{cases}$

The proper solution is

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \tag{10}$$

Using boundary condition (i) in (10), we get

$$y(0,t) = A(C\cos\lambda at + D\sin\lambda at)$$

$$0 = A(C\cos\lambda at + D\sin\lambda at)$$

$$\Rightarrow A = 0$$

$A = 0$ in (10), we get

$$y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at) \tag{11}$$

Applying the boundary condition (ii) in (11), we get

$$B\sin\lambda L(C\cos\lambda at + D\sin\lambda at) = 0$$

$\Rightarrow B \neq 0$ and $\sin nL = 0 \Rightarrow \sin nL = \sin n\pi$
$\Rightarrow \lambda L = n\pi \Rightarrow \lambda = \dfrac{n\pi}{L}$.
$\lambda = \dfrac{n\pi}{L}$ in (11), we get

$$y(x,t) = B\sin\frac{n\pi x}{L}\left(C\cos\frac{n\pi at}{L} + D\sin\frac{n\pi at}{L}\right) \tag{12}$$

Again applying the initial condition (iii) in (12), we get

$$y(x,0) = B\sin\frac{n\pi x}{L}(C+0)$$

$$\Rightarrow 0 = B\sin\frac{n\pi x}{L}.C$$

$$\Rightarrow C = 0, B \neq 0.$$

$C = 0$ in (12), we get $y(x,t) = B\sin\dfrac{n\pi x}{L}D\sin\dfrac{n\pi at}{L}$
$\Rightarrow y(x,t) = B_n\sin\dfrac{n\pi x}{L}\cos\dfrac{n\pi at}{L}$, since $BD = B_n$
Therefore the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{L}\sin\frac{n\pi at}{L} \tag{13}$$

Differentiating (13), partially w.r. to t and using initial condition (iv), we get

$$\begin{aligned}
\frac{\partial y}{\partial t} &= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}.\frac{n\pi a}{L} \\
\left(\frac{\partial y}{\partial t}\right)_{t=0} &= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}.\frac{n\pi a}{L} \\
&= \sum_{n=1}^{\infty} B_n.\frac{n\pi a}{L}\sin\frac{n\pi x}{L}
\end{aligned}$$

This is half-range Fourier sine series. Therefore

$$\begin{aligned}
b_n &= \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx \\
B_n.\frac{n\pi a}{L} &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \\
&= \frac{2}{L}\left\{\int_0^{L/2}\frac{2cx}{L}\sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L}\frac{-2c}{L}(x-L)\sin\frac{n\pi x}{L}\,dx\right\} \\
&= \frac{4c}{L^2}\left\{\int_0^{L/2} x\sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}\,dx\right\} \\
B_n &= \frac{4c}{n\pi a L}\Bigg\{\left[x.\left(-\cos\frac{n\pi x}{L}\right).\frac{L}{n\pi} - 1.\left(-\sin\frac{n\pi x}{L}\right).\frac{L^2}{n^2\pi^2}\right]_0^{L/2} \\
&+ \left[(L-x).\left(-\cos\frac{n\pi x}{L}\right).\frac{L}{n\pi} - (-1).\left(-\sin\frac{n\pi x}{L}\right).\frac{L^2}{n^2\pi^2}\right]_{L/2}^{L}\Bigg\} \\
&= \frac{4c}{n\pi a L}\Bigg[-\frac{L}{2}\cos\frac{n\pi}{2}.\frac{L}{n\pi} + \sin\frac{n\pi}{2}.\frac{L^2}{n^2\pi^2} - 0 - \sin 0.\frac{L^2}{n^2\pi^2} + 0 \\
&- \sin n\pi.\frac{L^2}{n^2\pi^2} + \frac{L}{2}\cos\frac{n\pi}{2}.\frac{L}{n\pi} + \sin\frac{n\pi}{2}.\frac{L^2}{n^2\pi^2}\Bigg] \\
&= \frac{4c}{n\pi a L}.2\sin\frac{n\pi}{2}.\frac{L^2}{n^2\pi^2}\ \text{Since } \sin 0 = 0, \sin n\pi = 0
\end{aligned}$$

$$B_n = \frac{8cL}{n^3\pi^3 a}\sin\frac{n\pi}{2}$$
$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{8cL}{n^3\pi^3 a}\sin\dfrac{n\pi}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Substituting B_n values in (13), we get

$$y(x,t) = \sum_{\text{odd}} \frac{8cL}{n^3\pi^3 a}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

where $L = 2l$.

One Dimensional Heat Flow (In a Rod)

Consider the flow of heat and the consequent variation of temperature with position and time in conducting solids.

In the derivation of the one-dimensional heat equation, we use the following empirical laws.

(1) Heat flows from a higher to lower temperature.

(2) The amount of heat required to produce a given temperature change in a body is proportional to the mass of the body and to the temperature change. This constant of proportionality is known a the specific heat (c) of the conducting material.

(3) The rate at which heat flows through an area is proportional to the area and to the temperature gradient normal to the area. this constant of proportionality is known as the thermal conductivity (k) of the material.

The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where $\alpha^2 = \dfrac{k}{\rho c}$ is called the diffusivity (cm^2/sec.) of the substance, $\rho =$ material of the density.

Note:

The one dimensional heat equation is also known as one dimensional diffusion equation.

Solution of heat equation by the method of separation of variables

Let $u(x,t) = X(x).T(t)$ be a solution of (1).

Then $\dfrac{\partial u}{\partial t} = XT'$ and $\dfrac{\partial^2 u}{\partial x^2} = X''T$

Therefore the equation (1) becomes $XT' = \alpha^2 X''T$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{\alpha^2 T} \tag{2}$$

The L.H.S. of (2) is a function of x only whereas the R.H.S. is a function of time t only. But x and t are independent variables. Hence (2) is true only if each is equal to a constant.

Therefore $\dfrac{X''}{X} = \dfrac{T'}{\alpha^2 T} = k$ (say) where k is any constant. Hence

$$X'' - kX = 0 \text{ and } T' - k\alpha^2 T = 0 \tag{3}$$

Thus we get two order ordinary linear differential equation with constant coefficients. The solutions of (3) depend upon the nature of k. that is $k > 0$ or <0 or 0.

Case 1: If $k > 0$, let $k = \lambda^2$, a positive value.

Now the equation (3) are $X'' - \lambda^2 X = 0$ and $T' - \lambda^2\alpha^2 T = 0$.

Therefore Auxiliary equation is $m^2 - \lambda^2 = 0$

$\Rightarrow m = \pm\lambda$

Therefore $X = A_1 e^{\lambda x} + B_1 e^{-\lambda x}$

and $\dfrac{T'}{T} = \lambda^2\alpha^2$

Therefore $\displaystyle\int \frac{T'}{T}\,dt = \lambda^2\alpha^2 \int dt$

$\Rightarrow \log T = \lambda^2\alpha^2 t + \log C_1$

$\Rightarrow \log \dfrac{T}{C_1} = \lambda^2\alpha^2 t \Rightarrow \dfrac{T}{C_1} = e^{\lambda^2\alpha^2 t}$

$\Rightarrow T = C_1 e^{\lambda^2\alpha^2 t}$

Hence the solution is $u(x,t) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x}) C_1 e^{\lambda^2\alpha^2 t}$.

Case 2: If $k < 0$, let $k = \lambda^2$, a negative value.

Now the equation (3) are $X'' + \lambda^2 X = 0$ and $T' + \lambda^2\alpha^2 T = 0$.

Therefore Auxiliary equation is $m^2 + \lambda^2 = 0$

$\Rightarrow m = \pm i\lambda$

Therefore $X = A_2 \cos\lambda x + B_2 \sin\lambda x$

and $\dfrac{T'}{T} = -\lambda^2\alpha^2$

Therefore $\displaystyle\int \frac{T'}{T}\,dt = -\lambda^2\alpha^2 \int dt$

$\Rightarrow \log T = -\lambda^2\alpha^2 t + \log C_2$

$\Rightarrow \log \dfrac{T}{C_2} = -\lambda^2\alpha^2 t \Rightarrow \dfrac{T}{C_2} = e^{-\lambda^2\alpha^2 t}$

$\Rightarrow T = C_2 e^{-\lambda^2\alpha^2 t}$

Hence the solution is $u(x,t) = (A_2 \cos\lambda x + B_2 \sin\lambda x)C_2 e^{-\lambda^2\alpha^2 t}$.

Case 3: Let $k = 0$.

Now the equation (3) are $X'' = 0$ and $T' = 0$.

$\Rightarrow X' = A_3$ and $= C_3$

$\Rightarrow X = A_3 x + B_3$

Hence the solution is $y(x,t) = (Ax + B)C$.

Thus there are three possible solutions of the heat equation and they are

$$u(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})e^{\alpha^2\lambda^2 t} \tag{4}$$

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t} \tag{5}$$

$$u(x,t) = Ax + B \tag{6}$$

Proper Choice of the solution

- Of the three possible solutions, we choose the solution which is consistent with the physical nature of the problem and the given boundary value conditions.
- Since $u(x,t)$ represent the temperature at any time t and at a distance x from one end of the rod, the temperature cannot be increasing as t is increasing. So as t increases, u must decrease hence the suitable solution for unsteady state conditions (or transient) is (5).
- A, B, λ are independent constants to be determined. Hence three conditions are required to solve the one-dimensional heat in transient state.
- In steady state conditions, the temperature, at any point is independent of time (i.e it does not change with time). Hence the suitable solution for steady state state heat flow is (6).

Types:

(1) Problems with zero boundary values. i.e. the temperature at the ends of the rod are kept at zero.

The boundary value conditions are

(i) $u(0,t) = 0$ and (ii) $u(l,t) = 0$ for all $t \geq 0$

The initial condition is

(iii) $u(x,0) = f(x)$ for all $x \in (0,l)$ where $f(x)$ is algebraic form.

(2) Non zero temperature at the end points of the bar in steady state and zero temperature in unsteady state.

In steady state the temperature $u(x,t)$ is a function of x alone, as it is independent of t.

Therefore $u(x,t) = u(x)$

Hence $\dfrac{\partial u}{\partial t} = 0 \Rightarrow \dfrac{d^2u}{dx^2} = 0$

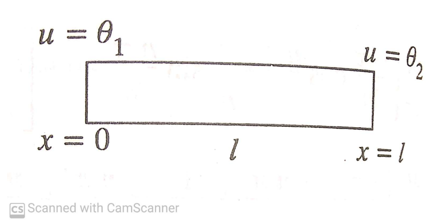
Integrating, we get $\dfrac{du}{dx} = A$

Again integrating, we get

$$u = Ax + B \tag{7}$$

When the state changes from steady to unsteady, the temperature at the ends are reduced to zero.

Example: The bar of length l cm has its ends kept at $\theta_1^\circ C$ and $\theta_2^\circ C$ until steady state condition prevails. Find the steady state temperature of the rod.



The boundary conditions are

(i) $u(0) = \theta_1$ (ii) $u(l) = \theta_2$

Using (i) in (7), we get

$$u(0) = A.0 + B \Rightarrow \theta_1 = B$$

$B = \theta_1$ in (7), we get

$$u = Ax + \theta_1 \tag{8}$$

Again using (ii) in (8), we get

$$u(l) = A.l + \theta_1$$

$$\Rightarrow A = \frac{\theta_2 - \theta_1}{l}$$

Substituting A value in (8), we get

$$u(x) = \left(\frac{\theta_2 - \theta_1}{l}\right) x + \theta_1$$

(3) Non-zero temperature at the ends of the bar, both in steady state and unsteady state

In type 1 and type 2, the temperatures at the ends in unsteady (or transient) state are kept at $0°C$. So, the constants in the suitable solution could be obtained easily.

In this type the temperatures at the ends are non-zero in the unsteady state and so the computations of constants cannot be done as before.

We split the required solution $u(x,t)$ into two parts as $u(x,t) = u_1(x) + u_2(x,t)$ where $u_1(x)$ is the steady state solution and $u_2(x,t)$ is the unsteady steady solution, in order to get zero boundary conditions.

Type 1:

Example 1: Solve $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$
subject to the conditions:

(i) u is not infinite as $t \to \infty$

(ii) $u = 0$ for $x = 0$ and $x = \pi$ for all t

(iii) $u = \pi x - x^2$ for $t = 0$ in $(0, \pi)$

Solution:

The solution of one dimensional heat equation is

$$u(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})e^{\alpha^2\lambda^2 t} \tag{9}$$

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t} \tag{10}$$

$$u(x,t) = (Ax + B) \tag{11}$$

As $t \to \infty$, in solution (9).

Hence we reject solution as u is not infinite as $t \to \infty$.

Solution (11) is independent of t.

Hence the proper solution is (10).

By (ii) $[u(0,t) = 0]$ in (10), we get

$$u(0,t) = A.e^{-\alpha^2\lambda^2 t} \Rightarrow 0 = A.e^{-\alpha^2\lambda^2 t}$$

$$\Rightarrow A = 0$$

$A = 0$ in (10), we have

$$u(x,t) = B\sin\lambda x e^{-\alpha^2\lambda^2 t} \tag{12}$$

By (ii) $[u(\pi,t) = 0]$ in (12), we get

$$u(\pi,t) = B\sin\lambda\pi e^{-\alpha^2\lambda^2 t} \Rightarrow 0 = B\sin\lambda\pi e^{-\alpha^2\lambda^2 t}$$

$$\Rightarrow B \neq 0, \sin\lambda\pi = 0$$

$$\Rightarrow \sin\lambda\pi = \sin n\pi \Rightarrow \lambda = n$$

$\lambda = n$ in (12), we get $u(x,t) = B\sin nx e^{-\alpha^2\lambda^2 t}$, where n is any constant.

Hence the most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin nx e^{-\alpha^2 n^2 t} \tag{13}$$

By (iii) $[u(x,0) = \pi x - x^2]$ in (13), we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx$$

$$\Rightarrow \pi x - x^2 = \sum_{n=1}^{\infty} B_n \sin nx$$

This is half range Fourier sine series.

$$\begin{aligned}
B_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx\,dx \\
&= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) + (\pi - 2x)\left(\frac{\sin nx}{n^2}\right) - 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[0 + 0 - \frac{2\cos n\pi}{n^3} + 0 - 0 + \frac{2\cos 0}{n^3}\right] \\
&= \frac{4}{\pi n^3}\left[-(-)^n + 1\right] \\
&= \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{8}{\pi n^3}, & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

Substituting B_n value in (13), we get

$$u(x,t) = \sum_{\text{odd}} \frac{8}{\pi n^3} \sin nx e^{-\alpha^2 n^2 t}.$$

Example 2: Solve $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$
subject to the conditions:

(i) $u(0,t) = 0$ for $t \geq 0$

(ii) $u(l,t) = 0$ for $t \geq 0$

(iii) $u(x,0) = \begin{cases} x, & 0 \leq x \leq \dfrac{l}{2} \\ l - x & \dfrac{l}{2} \leq x \leq l \end{cases}$

Solution:

The proper solution is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t} \tag{14}$$

By (i) in (14), we get

$$u(0,t) = A.e^{-\alpha^2\lambda^2 t} \Rightarrow 0 = A.e^{-\alpha^2\lambda^2 t}$$

$$\Rightarrow A = 0$$

$A = 0$ in (14), we have

$$u(x,t) = B\sin\lambda x e^{-\alpha^2\lambda^2 t} \tag{15}$$

By (ii) in (15), we get

$$u(l,t) = B\sin\lambda l e^{-\alpha^2\lambda^2 t} \Rightarrow 0 = B\sin\lambda l e^{-\alpha^2\lambda^2 t}$$

$$\Rightarrow B \neq 0, \sin\lambda l = 0$$

$$\Rightarrow \sin\lambda l = \sin n\pi \Rightarrow \lambda = \frac{n\pi}{l}$$

$\lambda = \dfrac{n\pi}{l}$ in (15), we get

$$u(x,t) = B\sin\frac{n\pi x}{l}e^{-\alpha^2 n^2\pi^2 t/l^2},$$

where n is any constant.

Hence the most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}e^{-\alpha^2 n^2\pi^2 t/l^2} \tag{16}$$

By (iii) in (16), we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$\Rightarrow \begin{cases} x, & 0 \leq x \leq \dfrac{l}{2} \\ l - x & \dfrac{l}{2} \leq x \leq l \end{cases} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

This is half range Fourier sine series.

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left\{\int_0^{\frac{l}{2}} x\sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{2}}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right\}$$

$$= \frac{2}{l}\left\{\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1.\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{\frac{l}{2}}\right.$$

$$\left.+\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1).\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{\frac{l}{2}}^{l}\right\}$$

$$= \frac{2}{l}\left[-\frac{l}{2}.\frac{l}{n\pi}.\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + 0 - 0 + 0\right.$$

$$\left.-\sin n\pi.\frac{l^2}{n^2\pi^2} + \frac{l}{2}.\frac{l}{n\pi}.\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}.\sin\frac{n\pi}{2}\right]$$

$$= \frac{4}{l}.\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Substituting B_n value in (16), we get

$$u(x,t) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}e^{-\alpha^2 n^2\pi^2 t/l^2}.$$

Example 3: Solve $\dfrac{\partial u}{\partial t} = \alpha^2\dfrac{\partial^2 u}{\partial x^2}$,

Subject to the conditions:

(i) $u(0,t) = 0$ for $t \geq 0$

(ii) $u(l,t) = 0$ for $t \geq 0$

(iii) $u(x,0) = x, 0 \leq x \leq l$

Solution:

The proper solution is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t} \tag{17}$$

By (i) in (17), we get

$$u(0,t) = A.e^{-\alpha^2\lambda^2 t} \Rightarrow 0 = A.e^{-\alpha^2\lambda^2 t}$$

$$\Rightarrow A = 0$$

$A = 0$ in (17), we have

$$u(x,t) = B\sin\lambda x e^{-\alpha^2\lambda^2 t} \tag{18}$$

By (ii) in (18), we get

$$u(l,t) = B\sin\lambda l e^{-\alpha^2\lambda^2 t} \Rightarrow 0 = B\sin\lambda l e^{-\alpha^2\lambda^2 t}$$

$$\Rightarrow B \neq 0, \sin\lambda l = 0$$

$$\Rightarrow \sin\lambda l = \sin n\pi \Rightarrow \lambda = \frac{n\pi}{l}$$

$\lambda = \dfrac{n\pi}{l}$ in (18), we get

$$u(x,t) = B\sin\frac{n\pi x}{l} e^{-\alpha^2 n^2\pi^2 t/l^2}$$

Hence the most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\alpha^2 n^2\pi^2 t/l^2} \tag{19}$$

By (iii) in (19), we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$\Rightarrow x = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

This is half range Fourier sine series.

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^{\frac{l}{2}} x\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[x\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1.\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$B_n = \frac{2}{l}\left[-l.\frac{l}{n\pi}\cos n\pi + \frac{l^2}{n^2\pi^2}\sin n\pi - 0 - 0\right]$$
$$= \frac{-2l(-1)^n}{n\pi}$$

Substituting B_n value in (19), we get

$$u(x,t) = \frac{-2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l}e^{-\alpha^2 n^2\pi^2 t/l^2}.$$

Example 4: A rod l cm. with insulated lateral surface is initially at temperature $f(x)$ at an inner point distance x cm. from one end. If both the ends are kept at zero temperature, find the temperature at anu point of the rod at any subsequent time.

Solution: The temperature distribution in the bar is given by the one dimensional heat equation $\dfrac{\partial u}{\partial t} = \alpha^2\dfrac{\partial^2 u}{\partial t^2}$.

The boundary conditions are

(i) $u(0,t) = 0$ for all $t \geq 0$

(ii) $u(l,t) = 0$ for all $t \geq 0$

The initial condition is

(iii) $u(x,0) = f(x)$ for $0 < x < l$

The proper solution is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t} \tag{20}$$

Substituting the boundary condition (i) in (20), we get

$$u(0,t) = Ae^{-\alpha^2\lambda^2 t} \Rightarrow 0 = Ae^{-\alpha^2\lambda^2 t}$$

$$\Rightarrow A = 0$$

$A = 0$ in (20), we get

$$u(x,t) = B\sin\lambda x e^{-\alpha^2\lambda^2 t} \tag{21}$$

Using (ii) in (21), we get

$$u(l,t) = B\sin\lambda l e^{-\alpha^2\lambda^2 t} \Rightarrow 0 = B\sin\lambda l e^{-\alpha^2\lambda^2 t}$$

$$\Rightarrow B \neq 0, \sin\lambda l = 0$$

$$\Rightarrow \sin \lambda l = \sin n\pi \Rightarrow \lambda = \frac{n\pi}{l}$$

$\lambda = \dfrac{n\pi}{l}$ in (21), we get $u(x,t) = B \sin \dfrac{n\pi x}{l} e^{-\dfrac{\alpha^2 n^2 \pi^2 t}{l^2}}$ where n is any constant.
Hence the most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{-\dfrac{\alpha^2 n^2 \pi^2 t}{l^2}} \tag{22}$$

Using (iii) in (22), we get $u(x,0) = \sum\limits_{n=1}^{\infty} B_n \sin \dfrac{n\pi x}{l}$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

This is half range sine series.

$$B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

Substituting B_n in (22), we get

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx \right) \sin \frac{n\pi x}{l} e^{-\dfrac{\alpha^2 n^2 \pi^2 t}{l^2}}.$$

Example 5: Heat flows through a uniform bar of length l which has its sides insulated and the temperature at the ends kept at zero. If the initial temperature at the interior points of the bar is given by $k(lx - x^2), 0 < x < l$, find the temperature distribution in the bar at time t.

Solution:

The temperature distribution in the bar is given by the one dimensional heat equation $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial t^2}$.
The boundary conditions are

(i) $u(0,t) = 0$ for all $t \geq 0$

(ii) $u(l,t) = 0$ for all $t \geq 0$

The initial condition is

(iii) $u(x, 0) = k(lx - x^2)$ for $0 < x < l$

The proper solution is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t} \tag{23}$$

Substituting the boundary condition (i) in (23), we get

$$u(0,t) = Ae^{-\alpha^2\lambda^2 t} \Rightarrow 0 = Ae^{-\alpha^2\lambda^2 t}$$

$$\Rightarrow A = 0$$

$A = 0$ in (23), we get

$$u(x,t) = B\sin\lambda x e^{-\alpha^2\lambda^2 t} \tag{24}$$

Using (ii) in (24), we get

$$u(l,t) = B\sin\lambda l e^{-\alpha^2\lambda^2 t} \Rightarrow 0 = B\sin\lambda l e^{-\alpha^2\lambda^2 t}$$

$$\Rightarrow B \neq 0, \sin\lambda l = 0$$

$$\Rightarrow \sin\lambda l = \sin n\pi \Rightarrow \lambda = \frac{n\pi}{l}$$

$\lambda = \dfrac{n\pi}{l}$ in (24), we get $u(x,t) = B\sin\dfrac{n\pi x}{l}e^{-\dfrac{\alpha^2 n^2\pi^2 t}{l^2}}$ where n is any constant.
Hence the most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}e^{-\dfrac{\alpha^2 n^2\pi^2 t}{l^2}} \tag{25}$$

Using (iii) in (25), we get $u(x,0) = \sum_{n=1}^{\infty} B_n \sin\dfrac{n\pi x}{l}$

$$\Rightarrow k(lx - x^2) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

This is half range sine series.

$$
\begin{aligned}
B_n &= \frac{2}{l}\int\limits_0^l k(lx-x^2)\sin\frac{n\pi x}{l}\,dx \\
&= \frac{2k}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)+(-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l \\
&= \frac{2k}{l}\left[-\frac{l}{n\pi}(lx-x^2)\cos\frac{n\pi x}{l}+\frac{l^2}{n^2\pi^2}(l-2x)\sin\frac{n\pi x}{l}-\frac{2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l \\
&= \frac{2k}{l}\left[0+0-\frac{2l^3}{n^3\pi^3}\cos n\pi-\left(0+0-\frac{2l^3}{n^3\pi^3}\right)\right] \\
&= \frac{2k}{l}.\frac{2l^3}{n^3\pi^3}\left[-(-1)^n+1\right]=\frac{4kl^2}{n^3\pi^3}\left[-(-1)^n+1\right] \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{8kl^2}{n^3\pi^3} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

Substituting B_n in (25), we get

$$u(x,t)=\sum_{\text{odd}}\frac{8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}e^{-\dfrac{\alpha^2n^2\pi^2t}{l^2}}.$$

Type 2:

Example 6: A rod, 30 cm, long has its ends A and B kept at $20°C$ and $80°C$ respectively, until steady state conditions prevail. The temperature at each end is then suddenly reduced to $0°C$ and kept so. Find the resulting temperature function $u(x,t)$, taking $x=0$ at A.

Solution:

The one dimensional heat equation is

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2} \tag{26}$$

Initially steady state conditions prevails with $u=20$ at $x=0$ and $u=80$ at $x=30$. i.e. $u(0)=20$ and $u(30)=80$.

In steady state, u is independent of time:

$$\begin{aligned} u &= \frac{\theta_2 - \theta_1}{l}x + \theta_1 \\ &= \frac{80 - 20}{30}x + 20 \\ &= 2x + 20 \end{aligned}$$

When the temperature at the ends are changed to $0°C$, the heat flow or the temperature distribution in the bar will not be in steady state and so will depend on time also. So, the temperature distribution $u(x,t)$ is given by (26).
The proper solution is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t} \tag{27}$$

The boundary conditions of the unsteady state are
(i) $u(0,t) = 0$ and (ii) $u(30,t) = 0$ for all $t \geq 0$.
The initial distribution of temperature is given by
(iii) $u(x,0) = 2x + 20$, for $0 < x < 30$
Using condition (i) in (27), we get

$$0 = A.e^{-\alpha^2\lambda^2 t} \Rightarrow A = 0$$

$A = 0$ in (27), we get

$$u(x,t) = B\sin\lambda x e^{-\alpha^2\lambda^2 t} \tag{28}$$

Using condition (ii) in (28), we get

$$B\sin 30\lambda e^{-\alpha^2\lambda^2 t} = 0$$

$$\Rightarrow \sin 30\lambda = 0 = \sin n\pi$$

$$\Rightarrow \lambda = \frac{n\pi}{30}$$

$\lambda = \frac{n\pi}{30}$ in (28), we get

$$u(x,t) = B\sin\frac{n\pi x}{30}e^{-\frac{\alpha^2 n^2\pi^2 t}{30^2}}$$

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30} e^{-\dfrac{\alpha^2 n^2 \pi^2 t}{30^2}} \tag{29}$$

Using condition (iii) in (29), we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30}$$

$$\Rightarrow 2x + 20 = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30}$$

This is half range sine series.

$$\begin{aligned}
B_n &= \frac{2}{30} \int_0^{30} (2x+20) \sin \frac{n\pi x}{30}\, dx \\
&= \frac{1}{15} \left[(2x+20) \left(\frac{-\cos \dfrac{n\pi x}{30}}{\dfrac{n\pi}{30}} \right) - 2 \left(\frac{-\sin \dfrac{n\pi x}{30}}{\dfrac{n^2\pi^2}{30^2}} \right) \right]_0^{30} \\
&= \frac{1}{15} \left[-\frac{30}{n\pi}(2x+20) \cos \frac{n\pi x}{30} + \frac{1800}{n^2\pi^2} \sin \frac{n\pi x}{30} \right]_0^{30} \\
&= \frac{1}{15} \left[-\frac{30}{n\pi} \times 80. \cos n\pi + 0 + \frac{600}{n\pi} - 0 \right] \\
&= \frac{600}{15} \left[\frac{1 - 4(-1)^n}{n\pi} \right] = \frac{40}{n\pi}[1 - 4(-1)^n]
\end{aligned}$$

Substituting B_n in (29), we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}[1 - 4(-1)^n] \sin \frac{n\pi x}{30} e^{-\dfrac{\alpha^2 n^2 \pi^2 t}{900}}.$$

Example 7: A rod of length l has its ends A and B kept at $0°C$ and $100°C$ respectively, until steady state conditions prevail. If the temperature at B is suddenly to $0°C$ and maintained at $0°C$, find the temperature at a distance x from a and at any time t.

Solution:

The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{30}$$

Initially steady state conditions prevails with $u = 0$ at $x = 0$ and $u = 100$ at $x = l$. i.e. $u(0) = 0$ and $u(l) = 100$.

In steady state, u is independent of time:

$$\begin{aligned} u &= \frac{\theta_2 - \theta_1}{l} x + \theta_1 \\ &= \frac{100 - 0}{l} x + 0 \\ &= \frac{100}{l} x \end{aligned}$$

If the temperature at B is reduced to $0°C$, then the temperature distribution changes from steady state to unsteady state. So, the temperature distribution $u(x, t)$ is given by (30).

The proper solution is

$$u(x, t) = (A \cos \lambda x + B \sin \lambda x) e^{-\alpha^2 \lambda^2 t} \tag{31}$$

The boundary conditions of the unsteady state are

(i) $u(0, t) = 0$ and (ii) $u(l, t) = 0$ for all $t \geq 0$.

The initial distribution of temperature is given by

(iii) $u(x, 0) = \frac{100}{l} x$, for $0 < x < l$

Using condition (i) in (31), we get

$$0 = A.e^{-\alpha^2 \lambda^2 t} \Rightarrow A = 0$$

$A = 0$ in (31), we get

$$u(x, t) = B \sin \lambda x e^{-\alpha^2 \lambda^2 t} \tag{32}$$

Using condition (ii) in (32), we get

$$B \sin \lambda l e^{-\alpha^2 \lambda^2 t} = 0$$

$$\Rightarrow \sin \lambda l = 0 = \sin n\pi$$

$$\Rightarrow \lambda = \frac{n\pi}{l}$$

$\lambda = \dfrac{n\pi}{l}$ in (32), we get

$$u(x,t) = B\sin\frac{n\pi x}{l}e^{-\dfrac{\alpha^2 n^2\pi^2 t}{l^2}}$$

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}e^{-\dfrac{\alpha^2 n^2\pi^2 t}{l^2}} \tag{33}$$

Using condition (iii) in (33), we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$\Rightarrow \frac{100}{l}x = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

This is half range sine series.

$$\begin{aligned}
B_n &= \frac{2}{l}\int_0^l \frac{100}{l}x\sin\frac{n\pi x}{l}\,dx \\
&= \frac{200}{l^2}\left[x.\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1.\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l \\
&= \frac{200}{l^2}\left[-\frac{l}{n\pi}.x.\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^l \\
&= \frac{200}{l^2}\left[-\frac{l^2}{n\pi}.\cos n\pi + 0 + 0 - 0\right] \\
&= \frac{-200(-1)^n}{n\pi}
\end{aligned}$$

Substituting B_n in (33), we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{-200(-1)^n}{n\pi}\sin\frac{n\pi x}{l}e^{-\dfrac{\alpha^2 n^2\pi^2 t}{l^2}}.$$

Type 3:

Example 8: A bar, 10 cm long, with insulted sides, has its ends A and B kept $20°C$ and $40°C$ respectively until steady state conditions prevail. The temperature A is then suddenly raised to $50°C$ and at the same instant at B is lowered to $10°C$. Find the subsequent temperature at any point of the bar at any time.

Solution:

The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{34}$$

Initially steady state conditions prevails with $u = 0$ at $x = 0$ and $u = 100$ at $x = l$.
i.e. $u(0) = 20$ and $u(10) = 40$.

In steady state, u is independent of time:

$$\begin{aligned} u &= \frac{\theta_2 - \theta_1}{l} x + \theta_1 \\ &= \frac{40 - 20}{l0} x + 20 \\ &= 2x + 20 \end{aligned}$$

Suddenly the temperature at A is increased to $50°C$ and that at B is decreased to $10°C$. So the temperature distribution in the bar is changed from steady state to unsteady state.

Then the temperature $u(x,t)$ satisfies (34).

The boundary conditions of the unsteady state are

(i) $u(0,t) = 50$ and (ii) $u(10,t) = 10$ for all $t \geq 0$.

The initial distribution of temperature is given by

(iii) $u(x,0) = 2x + 20$, for $0 < x < 10$

As the temperatures at the ends are not equal to $0°C$, we split the solution $u(x,t)$ of (34) into two parts in order to get zero boundary conditions.

$$u(x,t) = u_1(x) + u_2(x,t) \tag{35}$$

where $u_1(x)$ is steady state solution of (34

Therefore $u_1 = \dfrac{10-50}{10}x + 50$

$$\Rightarrow u_1(x) = -4x + 50,\ 0 \le x \le 10 \tag{36}$$

and $u_2(x,t)$ is the transient solution of (34).

Therefore the proper solution is

$$u_2(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t} \tag{37}$$

But $u_2(x,t) = u(x,t) - u_1(x)$

Therefore the boundary conditions of $u_2(x,t)$ are

$$\begin{aligned}
\text{(iv) } u_2(x,t) &= u(0,t) - u_1(0)\\
&= 50 - 50 = 0 \text{ Using (i)}\\
\text{(v) } u_2(x,t) &= u(10,t) - u_1(10)\\
&= 10 - 10 = 0 \text{ Using (ii)}\\
\text{and (iv) } u_2(x,0) &= u(x,0) - u_1(x)\\
&= 2x + 20 - (-4x + 50)\\
&= 6x - 30
\end{aligned}$$

Using condition (iv) in (37), we get

$$A.e^{-\alpha^2\lambda^2 t} = 0 \Rightarrow A = 0$$

Substituting $A = 0$ in (37), we get

$$u_2(x,t) = B\sin\lambda x e^{-\alpha^2\lambda^2 t} \tag{38}$$

Using condition (v) in (38), we get

$$B\sin 10\lambda.e^{-\alpha^2\lambda^2 t} = 0$$

$$B \ne 0, \sin 10\lambda = 0$$

$$\Rightarrow \lambda = \frac{n\pi}{10}$$

Substituting $\lambda = \dfrac{n\pi}{10}$ in (38), we get

$$u_2(x,t) = B \sin \frac{n\pi x}{10} e^{-\dfrac{\alpha^2 n^2 \pi^2 t}{100}}$$

The most general solution is

$$u_2(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{10} e^{-\dfrac{\alpha^2 n^2 \pi^2 t}{100}} \tag{39}$$

Using condition (vi) in (39), we get

$$u_2(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{10}$$

$$\Rightarrow 6x - 30 = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{10}$$

This is half range sine series.

$$\begin{aligned}
B_n &= \frac{2}{10} \int_0^{10} (6x-30) \sin \frac{n\pi x}{10}\, dx \\
&= \frac{1}{5} \left[(6x-30) \left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}} \right) - 6. \left(\frac{-\sin \frac{n\pi x}{10}}{\frac{n^2\pi^2}{10^2}} \right) \right]_0^{10} \\
&= \frac{1}{5} \left[-\frac{10}{n\pi}(6x-30) \cos \frac{n\pi x}{10} + \frac{600}{n^2\pi^2} \sin \frac{n\pi x}{10} \right]_0^{10} \\
&= \frac{1}{5} \left[-\frac{10}{n\pi}.30. \cos n\pi + \frac{600}{n^2\pi^2} \sin n\pi - \left(-\frac{10}{n\pi}(-30) + 0 \right) \right] \\
&= \frac{1}{5} \left[-\frac{300}{n\pi}(-1)^n - \frac{300}{n\pi} \right] \\
&= -\frac{60}{n\pi}[1 + (-1)^n] \\
&= \begin{cases} 0; & \text{if } n \text{ is odd} \\ -\dfrac{120}{n\pi}; & \text{if } n \text{ is even} \end{cases}
\end{aligned}$$

Substituting B_n in (39), we get

$$u_2(x,t) = \sum_{\text{even}} -\frac{120}{n\pi} \sin \frac{n\pi x}{10} e^{-\dfrac{\alpha^2 n^2 \pi^2 t}{100}} \tag{40}$$

Substituting (36) and (40) in (35) , we get

$$u(x,t) = -4x + 50 - \frac{120}{\pi}\sum_{\text{even}} \frac{1}{n}\sin\frac{n\pi x}{10} e^{-\dfrac{\alpha^2 n^2 \pi^2 t}{100}}.$$

Example 9: The ends A and B of a rod l cm long have their temperature kept at $30°C$ and $80°C$, until steady state conditions prevail. The temperature of the end B is suddenly reduced to $60°C$ and that of A increases to $40°C$. Find the temperature distribution of the rod after time t.

Solution:

The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{41}$$

Initially steady state conditions prevails with $u = 0$ at $x = 0$ and $u = 100$ at $x = l$. i.e. $u(0) = 30$ and $u(l) = 80$.

In steady state, u is independent of time:

$$\begin{aligned} u &= \frac{\theta_2 - \theta_1}{l}x + \theta_1 \\ &= \frac{80-30}{l}x + 30 \\ &= \frac{50}{l}x + 30 \end{aligned}$$

Suddenly the temperature at A and B are changed to $40°C$ and $60°C$. So, the temperature distribution in the bar is changed from steady state to unsteady state. Then the temperature $u(x,t)$ satisfies (41).

The boundary conditions of the unsteady state are

(i) $u(0,t) = 40$ and (ii) $u(l,t) = 60$ for all $t \geq 0$.

The initial distribution of temperature is given by

(iii) $u(x,0) = \dfrac{50}{l}x + 30$, for $0 < x < l$

As the temperatures at the ends are not equal to $0°C$, we split the solution $u(x,t)$ of (41) into two parts in order to get zero boundary conditions.

$$u(x,t) = u_1(x) + u_2(x,t) \tag{42}$$

where $u_1(x)$ is steady state solution of (41
Therefore $u_1 = \dfrac{60-40}{l}x + 40$

$$\Rightarrow u_1(x) = \frac{20}{l}x + 40,\ 0 \leq x \leq l \tag{43}$$

and $u_2(x,t)$ is the transient solution of (41).
Therefore the proper solution is

$$u_2(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t} \tag{44}$$

But $u_2(x,t) = u(x,t) - u_1(x)$
Therefore the boundary conditions of $u_2(x,t)$ are

$$\begin{aligned}
\text{(iv) } u_2(x,t) &= u(0,t) - u_1(0) \\
&= 40 - 40 = 0 \text{ Using (i)} \\
\text{(v) } u_2(x,t) &= u(10,t) - u_1(10) \\
&= 60 - 60 = 0 \text{ Using (ii)} \\
\text{and (iv) } u_2(x,0) &= u(x,0) - u_1(x) \\
&= \frac{50}{l}x + 30 - \left(\frac{20}{l}x + 40\right) \\
&= \frac{30}{l}x - 10
\end{aligned}$$

Using condition (iv) in (44), we get

$$A.e^{-\alpha^2\lambda^2 t} = 0 \Rightarrow A = 0$$

Substituting $A = 0$ in (44), we get

$$u_2(x,t) = B\sin\lambda x e^{-\alpha^2\lambda^2 t} \tag{45}$$

Using condition (v) in (45), we get

$$B\sin 10\lambda.e^{-\alpha^2\lambda^2 t} = 0$$

$$B \neq 0, \sin\lambda.l = 0$$

$$\Rightarrow \lambda = \frac{n\pi}{l}$$

Substituting $\lambda = \dfrac{n\pi}{l}$ in (45), we get

$$u_2(x,t) = B \sin \frac{n\pi x}{l} e^{-\dfrac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

The most general solution is

$$u_2(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{-\dfrac{\alpha^2 n^2 \pi^2 t}{l^2}} \tag{46}$$

Using condition (vi) in (46), we get

$$u_2(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

$$\Rightarrow \frac{30}{l} x - 10 = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

This is half range sine series.

$$\begin{aligned}
B_n &= \frac{2}{l} \int_0^l \left(\frac{30}{l} x - 10 \right) \sin \frac{n\pi x}{l}\, dx \\
&= \frac{2}{l} \left[\left(\frac{30}{l} x - 10 \right) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - \frac{30}{l} . \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right]_0^l \\
&= \frac{2}{l} \left[-\frac{l}{n\pi} \left(\frac{30}{l} x - 10 \right) \cos \frac{n\pi x}{l} + \frac{30l}{n^2\pi^2} \sin \frac{n\pi x}{l} \right]_0^l \\
&= \frac{2}{l} \left[-\frac{l}{n\pi} .20. \cos n\pi + \frac{30l}{n^2\pi^2} \sin n\pi - \left(-\frac{l}{n\pi} (-10) + 0 \right) \right] \\
&= \frac{2}{l} \left[-\frac{20l}{n\pi} (-1)^n - \frac{10l}{n\pi} \right] \\
&= -\frac{20}{n\pi} [1 + 2(-1)^n]
\end{aligned}$$

Substituting B_n in (46), we get

$$u_2(x,t) = \sum_{n=1}^{\infty} -\frac{20}{n\pi} [1 + 2(-1)^n] \sin \frac{n\pi x}{l} e^{-\dfrac{\alpha^2 n^2 \pi^2 t}{l^2}} \tag{47}$$

Substituting (43) and (47) in (42), we get

$$u(x,t) = \frac{20}{l}x + 40 - \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}.$$

Example 10: (Type I) Find the temperature $u(x,t)$ in a silver bar (of length 10 cm, constant cross-section of 1 cm^2 area, density 10.6 gm/cm^3, thermal conducivity 1.04 cal/cm deg.sec; specific heat 0.056 cal/gm.deg.) which is perfectly insulated laterally, if the ends are kept at $0°C$. and if, intially, the temperature is $5°C$. at the centre of the bar and falls uniformly to zero at its ends.

Solution: One dimensional heat equation $\dfrac{\partial u}{\partial t} = \alpha^2\dfrac{\partial^2 u}{\partial t^2}$

where $\alpha^2 = \dfrac{k}{\rho c} = \dfrac{1.04}{(10.6)(0.056)} = 1.75$

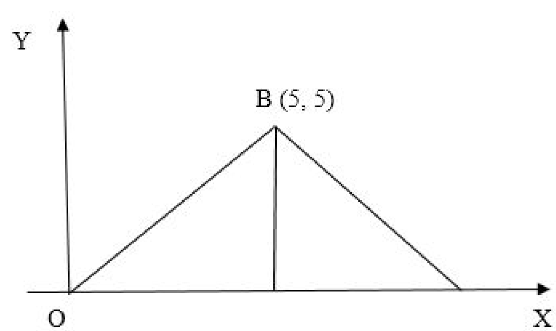
The boundary conditions are

(i) $u(0,t) = 0$ for all $t \geq 0$

(ii) $u(10,t) = 0$ for all $t \geq 0$

The initial condition is

(iii) $u(x,0) = \begin{cases} x, & 0 < x < 5 \\ 10 - x, & 5 < x < 10 \end{cases}$



The proper solution is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t} \tag{48}$$

Substituting the boundary condition (i) in (48), we get

$$u(0,t) = Ae^{-\alpha^2\lambda^2 t} \Rightarrow 0 = Ae^{-\alpha^2\lambda^2 t}$$

$$\Rightarrow A = 0$$

$A = 0$ in (48), we get

$$u(x,t) = B\sin\lambda x e^{-\alpha^2\lambda^2 t} \tag{49}$$

Using (ii) in (49), we get

$$u(10,t) = B\sin\lambda l e^{-\alpha^2\lambda^2 t} \Rightarrow 0 = B\sin\lambda 10 e^{-\alpha^2\lambda^2 t}$$

$$\Rightarrow B \neq 0, \sin\lambda 10 = 0$$

$$\Rightarrow \sin\lambda 10 = \sin n\pi \Rightarrow \lambda = \frac{n\pi}{10}$$

$\lambda = \dfrac{n\pi}{10}$ in (49), we get $u(x,t) = B\sin\dfrac{n\pi x}{10} e^{-\dfrac{\alpha^2 n^2\pi^2 t}{10^2}}$ where n is any constant.
Hence the most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} e^{-\dfrac{\alpha^2 n^2\pi^2 t}{10^2}} \tag{50}$$

Using (iii) in (50), we get $u(x,0) = \sum\limits_{n=1}^{\infty} B_n \sin\dfrac{n\pi x}{10}$
This is half range sine series.

$$\begin{aligned}
B_n &= \frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi x}{10}\,dx \\
&= \frac{1}{5}\left\{\int_0^5 x\sin\frac{n\pi x}{10}\,dx + \int_5^{10}(10-x)\sin\frac{n\pi x}{10}\,dx\right\} \\
&= \frac{1}{5}\Bigg\{\left[x\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - 1.\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{10^2}}\right)\right]_0^5 \\
&+ \left[(10-x)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - (-1).\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{10^2}}\right)\right]_5^{10}\Bigg\}
\end{aligned}$$

$$B_n = \frac{1}{5}\bigg[-\frac{50x}{n\pi}.\cos\frac{n\pi}{2} + \frac{10^2}{n^2\pi^2}\sin\frac{n\pi}{2} + 0 - 0 + 0$$

$$-\sin n\pi.\frac{10^2}{n^2\pi^2} + \frac{50}{n\pi}.\cos\frac{n\pi}{2} + \frac{10^2}{n^2\pi^2}.\sin\frac{n\pi}{2}\bigg]$$

$$= \frac{1}{5}.\frac{200}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{40}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Substituting B_n in (50), we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}e^{-\frac{\alpha^2 n^2\pi^2 t}{10^2}}.$$