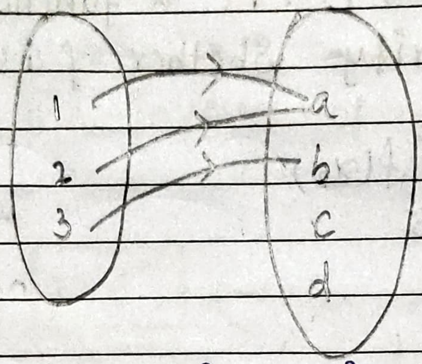
Range of $f = \{ y / y = f(x) \in Y \}$



Domain $= \{1, 2, 3\}$
Range $= \{a, b\}$

PROBLEM.
Let $f: Z^+ \longrightarrow Z^+$ be a function defined by $f(x) = 3x$ verify whether it is bijection.

sol: Let $f(x_1) = f(x_2)$

$3x_1 = 3x_2$

$3(x_1 - x_2) = 0$

$x_1 - x_2 = 0$

$x_1 = x_2$

f is 1-1.

Let $y \in Z^+$

Then $y = 3x$

$x = y/3$

For $y \in Z^+$ we can not find $x \in Z^+$ such that $y = 3x$.

[If $y = 2$, $x = \frac{2}{3} \notin Z^+$]

f is not onto.

$\boxed{\text{It is not a bijection.}}$

PROBLEM:

If $f: \mathbb{Z} \to \mathbb{Z}$ is a function given by $f(x) = x + 5$. verify whether f is bijection.

Let $f(x_1) = f(x_2)$

$x_1 + 5 = x_2 + 5$

$\boxed{x_1 = x_2}$

f is 1-1

Let $y \in \mathbb{Z}$.

Then $y = x + 5$

$x = y - 5 \in \mathbb{Z}$.

$\boxed{f \text{ is onto}}$

f is a bijection.

IDENTITY FUNCTION

Let $f: A \to A$ be a function Then $f(a) = a \; \forall a$ is called the identity function.

Usually it is denoted by I_A.

$\boxed{I_A(a) = a}$

COMPOSITION OF TWO FUNCTIONS.

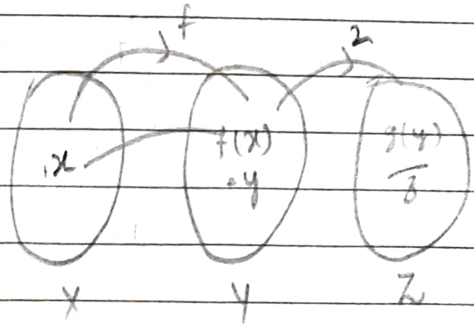
Let $f: X \to Y$ be a function.

and $g: Y \to Z$ be another function.

Then $(g \circ f)(x)$ is called the composition of f and g

$(g \circ f)(x) = g(f(x))$

Ex: $\sin(\log x)$



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INVERSE OF A FUNCTION

If $f: A \longrightarrow B$ and $g: B \longrightarrow A$ then g is called the inverse of f if $g \circ f = I_A$
$f \circ g = I_B$

NOTE : $\boxed{g = f^{-1}}$

PROPERTIES OF FUNCTION

① Composition is associative

(or)

Let $f: A \rightarrow B$ and $g: B \rightarrow C$, $h: C \rightarrow D$

Then $h \circ (g \circ f) = (h \circ g) \circ f$.

Sol: Given. $f: A \longrightarrow B$

$\exists b$ in B Such that $b = f(a)$

$g: B \rightarrow C$

$\exists c$ in C such that $c = g(b)$

$h: C \rightarrow D$

$\exists d$ in D such that $h(c) = d$.

$(g \circ f)(a) = g(f(a))$

$= g(b) = c$

$(h \circ (g \circ f))(a)$

$= h((g \circ f)(a))$

$= h(c)$

$= d$.

$(h \circ (g \circ f))(a) = d$ —— ①

NOW

$((h \circ g) \circ f)(a)$

$= ((h \circ g)(f(a)))$

$= h(g(h))$

$= h(c)$

$(ho(gof))(a) = d$ —— ②

PROPERTY 2

If $f: A \rightarrow B$ and $g: B \rightarrow C$ are bijective then $gof: A \rightarrow C$ is also bijective

Sol

As $f: A \rightarrow B$ is 1-1

$f(a_1) = f(a_2)$

$\Rightarrow a_1 = a_2$

As $g: B \rightarrow C$ is 1-1

$g(b_1) = g(b_2)$

$\Rightarrow b_1 = b_2$.

To Prove $gof: A \rightarrow C$ is 1-1

Let $(gof)(a_1) = (gof)(a_2)$

$g(f(a_1)) = g(f(a_2))$.

$\Rightarrow f(a_1) = f(a_2)$ ($\because g$ is 1-1)

$\Rightarrow \boxed{a_1 = a_2}$ ($\because f$ is 1-1

gof is 1-1

As $g: B \rightarrow C$ is onto. then $c \in C$ such that $\boxed{g(b) = c}$

As $f: A \rightarrow B$ is onto,

$\exists$ $b \in B$ such that $b = f(a)$

Now For $c \in C$,

$g(b) = c$

$g(f(a)) = c$

$(gof)(a) = c$

$\Rightarrow$ gof is onto

gof is bijective.

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PROPERTY 3 :

The necessary and sufficient condition for the function $f: A \to B$ to be invertible is that f is bijective

PROOF

Assume that

$f: A \to B$ is invertible

By definition of invertible, function

∴ there existing $g: B \to A$ such that

$g o f = I_A$ and $f o g = I_B$

Let $a_1, a_2 \in A$ such that

$f(a_1) = f(a_2)$

As $g: B \to A$ is a function.

$g(f(a_1)) = g(f(a_2))$

$(g o f)(a_1) = (g o f)(a_2)$

$I_A(a_1) = I_A(a_2)$

$a_1 = a_2 \Rightarrow \boxed{f \text{ is } 1\text{-}1}$

Let $b \in B$ then $g(b) \in A$

Now $b = I_B(b)$

$b = (f o g)(b) = f(g(b)) \Rightarrow \boxed{f \text{ is onto}}$

∴ f is a bijection.

THEOREM

If $f: A \to B$ and $g: B \to C$ are invertible functions, then $g o f: A \to C$ is also invertible and $(g o f)^{-1} = f^{-1} o g^{-1}$

Sol:- As $f: A \to B$ is invertible.

f is both 1-1 and onto

As $g: B \to C$ is invertible

g is both 1-1 and onto

$\Rightarrow g \circ f$ is both 1-1 and onto
$\Rightarrow g \circ f$ is invertible
$g \circ f : A \to C$ is invertible

Now

$g^{-1} : C \to B$, $f^{-1} : B \to A$
$f^{-1} \circ g^{-1} : C \to A$.
$(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are functions from $C \to A$.
for any $a \in A$, $b = f(a)$
for any $c \in C$, $c = g(b)$

$$(g \circ f)(a) = g(f(a))$$
$$= g(b)$$
$$= c.$$

$(g \circ f)(a) = c$
$(g \circ f)^{-1}(c) = a$ — ①

$$(f^{-1} \circ g^{-1})(c) = f^{-1}(g^{-1}(c))$$
$$= f^{-1}(b)$$
$$= a$$

$(f^{-1} \circ g^{-1})(c) = a$ — ②
from ① and ②

$$\boxed{(g \circ f)^{-1} = f^{-1} \circ g^{-1}}$$

PROBLEM.

Determine whether each of the following function is an injection and/or surjection or both

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x^3 + x$
(b) $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by $f(x) = x^2 + 2$.

Sol: (a) Let $f(x_1) = f(x_2)$

$3x_1^3 + x_1 = 3x_2^3 + x_2$

$3(x_1^3 - x_2^3) + (x_1 - x_2) = 0$

$(x_1 - x_2)[3(x_1^2 + x_1x_2 + x_2^2) + 1] = 0$

$\Rightarrow 3(x_1^2 + x_1x_2 + x_2^2) + 1 \neq 0$

$x_1 - x_2 = 0$

$\boxed{x_1 = x_2}$

f is injective

Let $y \in R$

$y = 3x^3 + x$

$\Rightarrow$ one of the root of the above equation is a real $x \in R$

For $y \in R$, $\exists x \in R$ such

$y = 3x^3 + x$

f is subjective.

(b) Let $f(x_1) = f(x_2)$

$x_1^2 + 2 = x_2^2 + 2$

$x_1^2 \pm x_2^2$

$(x_1^2 - x_2^2) = 0$

$(x_1 - x_2)(x_1 + x_2) = 0$

$x_1 - x_2 = 0$

$x_1 = x_2$

f is injective

Let $y \in z^+$

$y = x^2 + 2$

$x^2 = y - 2$

$x = \sqrt{y-2}$

For $y = 5 \in z^+$

$x = \sqrt{5-2} = \sqrt{3} \notin z^+$

For $y \in z^+$, $x \in z^+$ such that

$\boxed{y = x^2 + 2}$

$f(x)$ is not subjective

TO COMPUTE NO. OF FUNCTIONS.

Let $f: A \rightarrow B$ be a function such that $|A| = m$ and $|B| = n$

The total number of functions $= n^m$

NO OF 1-1 FUNCTIONS

If $m > n$, no. 1-1 function is possible

If $m \leq n$ no. of 1-1 functions

$= n(n-1)(n-2)(n-3)\cdots(n-(n-1))$

NO OF onto FUNCTIONS.

If $m < n$, no. onto function is possible.

If $m \geq n$, no onto functions

$$= \sum_{r=0}^{n-1} (-1)^r \, {}^nC_{n-r} (n-r)^m$$

m = cardinality of domain

n = Cardinality of co-domain

PROBLEM

If $f: A \rightarrow B$ be a function where $|A| = 4$ and $|B| = 3$

(1) Find no. of functions

(2) Find no. of onto functions

Sol. $m = 4$ $n = 3$

No. of functions $= 3^4 = 81$

No. of Onto functions.

$$= \sum_{r=0}^{2} (-1)^r \, {}^3C_{3-r} (3-r)^4$$

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$= {}^3C_3 \times 3^4 + (-1)\,{}^3C_2\, 2^4 + (-1)^2\, {}^3C_1 (1)^4$

$= 81 - 3\times16 + 3$

$= 36.$

PROBLEM

If $f: Z \longrightarrow N$ is defined by

$$f(x) = \begin{cases} 2x-1 & \text{if } x>0 \\ -2x & \text{if } x\leq 0. \end{cases}$$

(i) prove that f is a bijection

(ii) find f^{-1}

Sol: Let $x_1, x_2 \in Z$ and let $f(x_1) = f(x_2)$

case (i)

$f(x_1)$ and $f(x_2)$ are odd

$f(x_1) = f(x_2)$

$2x_1 - 1 = 2x_2 - 1$

f is 1-1

$x_1 = x_2$

case ii

$f(x_1)$ and $f(x_2)$ are even

$f(x_1) = f(x_2)$

$-2x_1 = -2x_2$

$x_1 = x_2$

f is 1-1

To prove f is onto

Let $y \in N$ and y is odd

$2x-1 = y$

$x = \frac{y+1}{2} \in Z$

Let $y \in N$ and y is even

$$y = -2x$$
$$x = -\frac{y}{2} \in \mathbb{Z}$$

For $y \in \mathbb{N}$ $\exists$ $x \in \mathbb{Z}$

$$y = \begin{cases} 2x-1 & \text{if } x > 0 \\ -2x & \text{if } x = 0 \end{cases}$$

f is onto.

(iii) $$y = \begin{cases} 2x-1 & \text{if } x > 0 \\ -2x & \text{if } x \leq 0 \end{cases}$$

$2x - 1 = y$ for $x > 0$ | $-2x = y$ for $x \leq 0$

$2x = y + 1$ | $\boxed{x = -\frac{y}{2}}$

$x = \frac{y+1}{2}$