2 VECTOR CALCULUS

INTRODUCTION

A physical quantity, which is a function of the position of a point in space, is called a **scalar point function** or a **vector point function** according as the quantity is a scalar or vector. Temperature, velocity are respectively scalar and vector point function. When a point function is defined at every point of certain region of space, then that region is called field.

Vector Differential Operator ∇(del)

The operator ∇ is defined as $\nabla = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$

2.1 Gradient of a scalar point function

Let $f(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space. Then the gradient of f, written as ∇f or *gradf* is defined as

$$\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) f = \left(\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}\right)$$

If f is a scalar point function, then ∇f is a vector point function.

Level Surface

Let $f(x, y, z)$ be a scalar field over a region R. The points satisfying an equation of the type $f(x, y, z) = c$(arbitrary constant) constitute a family of surfaces in three dimensional space. The surfaces of this family are called level surfaces.

2.2 Directional Derivatives of a Scalar Point Function

Definition: Let $f(x, y, z)$ define a scalar field in a region R and let P be any point in this region. Suppose Q is a point in this region in the neighborhood of P in the direction of a given unit vector $\hat{a}$. Then $\lim_{Q\to P}\dfrac{f(Q) - f(P)}{PQ}$, if it exits is called the directional derivative of $f(x, y, z)$ at P in the direction of $\hat{a}$.

Result 1: The directional derivative of a scalar field f at a point $P(x, y, z)$ in the direction of a unit vector $\hat{a}$ is given by $\nabla f.\hat{a}$. The directional derivative is maximum in the direction of *gradf* and its maximum value is $|gradf|$.

Result 2: Unit normal to a surface $f = c$ is given by $\dfrac{\nabla f}{|\nabla f|}$

Result 3: Angle between two surfaces $\phi_1 = c, \phi_2 = c$ is given by $\cos\theta = \dfrac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

2.3 Divergence of a Vector Point Function

Definition: Let $\overline{V}$ be any given differentiable vector point function. Then the divergence of $\overline{V}$ written as $\nabla.\overline{V}$ or $div\overline{V}$ is defined as $div\overline{V} = \nabla\overline{V} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\overline{V} = \sum \vec{i}.\frac{\partial \overline{V}}{\partial x}$

If $\overline{V} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ then $div\overline{V} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$.

Solenoidal Vector

A vector $\overline{V}$ is said to be solenoidal if $div\overline{V} = 0$.

Curl of a vector Point Function

Definition: Let $\overline{F}$ be a vector point function. Then $curl\overline{F}$ written as $\nabla \times \overline{F}$ is defined as $curl\overline{F} = \sum \vec{i} \times \frac{\partial \overline{F}}{\partial x}$

If $\overline{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ then $curl\overline{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

Irrotational Vector

A vector $\overline{F}$ is said to be irrotational if $\nabla \times \overline{F} = 0$. In this case we can represent $\overline{F} = \nabla\phi$, where ϕ is called scalar potential.

Note: 1 Physical meaning of $\nabla.\overline{V}$.

If $\overline{V} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is a vector point function representing the instantaneous velocity of a moving fluid at the point (x, y, z), then $\nabla.\overline{V}$ represents rate of loss of fluid per unit volume at that point.

Note: 2 Physical meaning of $curl\overline{F}$.

If $\overline{F}$ represents the linear velocity of the point (x, y, z) of a rigid body that rotates about a fixed axis with constant angular velocity $\overline{w}$, then $curl\overline{F}$ at that point represents $2\overline{w}$.

Example 2.1. *If $\phi(x, y, z) = x^2y + y^2x + z^2$, find $\nabla\phi$ at the point* $(1, 1, 1)$.

Solution: Let $\nabla\phi = \sum \vec{i}\frac{\partial \phi}{\partial x} = \sum \vec{i}\frac{\partial}{\partial x}(x^2y + y^2x + z^2)$

$= \vec{i}(2xy + y^2) + \vec{j}(x^2 + 2yx) + \vec{k}(2z)$

$\therefore \nabla\phi_{(1,1,1)} = \vec{i}(2+1) + \vec{j}(1+2) + \vec{k}(2)$

$= 3\vec{i} + 3\vec{j} + 2\vec{k}$

Example 2.2. *Find a unit normal vector to the level surface $x^2y + 2xz = 4$ at the point* $(2, -2, 3)$.

Solution: The unit normal vector $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$, where $\phi = x^2y + 2xz - 4$.

Now $\nabla\phi = \sum \vec{i}\frac{\partial \phi}{\partial x} = \sum \vec{i}\frac{\partial}{\partial x}(x^2y + 2xz - 4)$

$= \vec{i}(2xy + 2z) + \vec{j}(x^2) + \vec{k}(2x)$

$\nabla\phi_{(2,-2,1)} = \vec{i}(-8+6) + \vec{j}(4) + \vec{k}(4)$

$= -2\vec{i} + 4\vec{j} + 4\vec{k}$

$\Rightarrow |\nabla\phi| = \sqrt{4+16+16} = 6$

$\hat{n} = -\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$

Example 2.3. *Find the directional derivative of $f(x,y,z) = x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of the vector $2\vec{i} - \vec{j} - 2\vec{k}$.*

Solution: The required directional derivative is $\nabla f.\hat{a}$.
Here $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$.

So $|\vec{a}| = \sqrt{4+1+4} = 3$ and $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{2\vec{i} - \vec{j} - 2\vec{k}}{3}$.

Now *grad*$f = \nabla f = \sum \vec{i}\dfrac{\partial f}{\partial x} = \sum \vec{i}\dfrac{\partial}{\partial x}(x^2yz + 4xz^2)$

$= \vec{i}(2xyz + 4z^2) + \vec{j}(x^2z) + \vec{k}(x^2y + 8xz)$

Hence $(\nabla f)_{(1,-2,-1)} = \vec{i}(8) + \vec{j}(-1) + \vec{k}(-10)$

$\therefore$ The directional derivative $= (8\vec{i} - \vec{j} - 10\vec{k}).\dfrac{2\vec{i} - \vec{j} - 2\vec{k}}{3}$

$= \dfrac{16}{3} + \dfrac{1}{3} + \dfrac{20}{3} = \dfrac{37}{3}$ Units.

Example 2.4. *Find the directional derivative of $f(x,y,z) = x^2 - 2y^2 + 4z^2$ at the point $(1,1,-1)$ in the direction of $2\vec{i} + \vec{j} - \vec{k}$.*

Solution: The required directional derivative is $\nabla f.\vec{a}$.

Here $\hat{a} = \dfrac{2\vec{i} + \vec{j} - \vec{k}}{\sqrt{4+1+1}} = \dfrac{1}{\sqrt{6}}(2\vec{i} + \vec{j} - \vec{k})$

Now $\nabla f = \vec{i}.\dfrac{\partial}{\partial x}(x^2 - 2y^2 + 4z^2) + \vec{j}.\dfrac{\partial}{\partial y}(x^2 - 2y^2 + 4z^2) + \vec{k}.\dfrac{\partial}{\partial z}(x^2 - 2y^2 + 4z^2) = \vec{i}(2x) + \vec{j}(-4y) + \vec{k}(8z)$

and $(\nabla f)_{(1,1,-1)} = 2\vec{i} - 4\vec{j} - 8\vec{k}$

Directional derivative $= (2\vec{i} - 4\vec{j} - 8\vec{k}).\dfrac{1}{\sqrt{6}}(2\vec{i} + \vec{j} - \vec{k})$

$= \dfrac{4-4+8}{\sqrt{6}} = \dfrac{8}{\sqrt{6}}$

Example 2.5. *Find the directional derivative of $f(x,y,z) = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ, where Q is the point* $(5,0,4)$.

Solution: Given $f(x,y,z) = x^2 - y^2 + 2z^2$

Hence *grad*$f = \vec{i}\dfrac{\partial f}{\partial x} + \vec{j}\dfrac{\partial f}{\partial y} + \vec{k}\dfrac{\partial f}{\partial z}$

$= 2x\vec{i} - 2y\vec{j} + 4z\vec{k} = 2\vec{i} - 4\vec{j} + 12\vec{k}$ at the point $(1,2,3)$.

Also $\overline{PQ}$ = Position of Q - Position vector of P.

$= (5\vec{i} + 0\vec{j} + 4\vec{k}) - (\vec{i} + 2\vec{j} + 3\vec{k}) = 4\vec{i} - 2\vec{j} + \vec{k}$

If $\hat{a}$ be the unit vector in the direction of $\overrightarrow{PQ}$ then

$$\hat{a} = \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{16+4+1}} = \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{21}}$$

$\therefore$ The required directional derivative

$$= (gradf).\hat{a} = (2\vec{i} - 4\vec{j} + 12\vec{k}).\frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{21}} = \frac{28}{\sqrt{21}} =$$

$$\frac{28}{21}\sqrt{21} = \frac{4}{3}\sqrt{21}$$

Example 2.6. *In what direction from the point $(1,1,-1)$ is the directional derivative of $f(x,y,z) = x^2 - 2y^2 + 4z^2$ a maximum? Also find the value of this maximum directional derivative.*

Solution: We have $gradf = 2x\vec{i} - 4y\vec{j} + 8z\vec{k} = 2\vec{i} - 4\vec{j} - 8\vec{k}$ at the point $(1,1,-1)$.

The directional derivative of f is a maximum in the direction of $gradf = 2\vec{i} - 4\vec{j} - 8\vec{k}$

The maximum value of this directional derivative

$$= |gradf| = |2\vec{i} - 4\vec{j} - 8\vec{k}| = \sqrt{4+16+64} = 2\sqrt{21}$$

Example 2.7. *Show that the vector field $\overline{F}$ given by $\overline{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ is irrotational. Find a scalar ϕ such that $\vec{F} = \nabla\phi$.*

Solution: We have

$$curl\overline{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z^2 - xy) - \frac{\partial}{\partial z}(y^2 - zx)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(z^2 - xy) - \frac{\partial}{\partial z}(x^2 - yz)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(y^2 - zx) - \frac{\partial}{\partial y}(x^2 - yz)\right]$$

$$= (-x + x)\vec{i} - (-y + y)\vec{j} + (-z + z)\vec{k} = 0$$

$\therefore$ The vector field $\overline{F}$ is irrotational.

Let $F = \nabla\phi$ (or)

$$(x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k} = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

Then $\frac{\partial\phi}{\partial x} = x^2 - yz$ whence $\phi = \frac{x^3}{3} - xyz + f_1(y,z)$

$$\therefore f_1(y,z) = \frac{y^3}{3} + \frac{z^3}{3}$$

$\frac{\partial\phi}{\partial y} = y^2 - zx$ whence $\phi = \frac{y^3}{3} - xyz + f_2(x,z)$

$\therefore f_2(x,z) = \dfrac{x^3}{3} + \dfrac{y^3}{3}$

$\dfrac{\partial\phi}{\partial z} = z^2 - xy$ whence $\phi = \dfrac{z^3}{3} - xyz + f_3(x,y)$

$\therefore f_3(x,y) = \dfrac{y^3}{3} + \dfrac{z^3}{3}$

We choose ϕ satisfying all the above requirements of f_1, f_2, f_3.

$\Rightarrow \phi = \dfrac{x^3+y^3+z^3}{3} - xyz + C.$

Example 2.8. *Prove that the vector* $(y^2\cos x+z^3)\vec{i}+(2y\sin x-4)\vec{j}+(3xz^2+2)\vec{k}$ *is irrotational and find its scalar potential.*

Solution: Let $\overline{F} = (y^2\cos x+z^3)\vec{i}+(2y\sin x-4)\vec{j}+(3xz^2+2)\vec{k}$.

Then $curl\overline{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2\cos x+z^3 & 2y\sin x-4 & 3xz^2+2 \end{vmatrix}$

$= \vec{i}\left[\dfrac{\partial}{\partial y}(3xz^2+2) - \dfrac{\partial}{\partial z}(2y\sin x-4)\right]$

$-\vec{j}\left[\dfrac{\partial}{\partial x}(3xz^2+2) - \dfrac{\partial}{\partial z}(y^2\cos x+z^3)\right]$

$+\vec{k}\left[\dfrac{\partial}{\partial x}(2y\sin x-4) - \dfrac{\partial}{\partial y}(y^2\cos x+z^3)\right]$

$= \vec{i}(0-0) - \vec{j}(3z^2-3z^2) + \vec{k}(2y\cos x - 2y\cos x) = 0$

$\Rightarrow \overline{F}$ is irrotational.

If ϕ is scalar potential let $\overline{F} = \nabla\phi$

$\therefore \overline{F} = \sum \vec{i}\dfrac{\partial\phi}{\partial x}$

$\Rightarrow (y^2\cos x+z^3)\vec{i}+(2y\sin x-4)\vec{j}+(3xz^2+2)\vec{k} = \dfrac{\partial\phi}{\partial x}\vec{i} + \dfrac{\partial\phi}{\partial y}\vec{j} + \dfrac{\partial\phi}{\partial z}\vec{k}$

$\Rightarrow \dfrac{\partial\phi}{\partial x} = y^2\cos x + z^3;\ \dfrac{\partial\phi}{\partial y} = 2y\sin x - 4;\ \dfrac{\partial\phi}{\partial z} = 3xz^2+2$

Integrating partially, we get

$\phi = y^2\sin x + z^3x + f_1(y,z)$

$\phi = y^2\sin x - 4y + f_2(x,z)$

$\phi = xz^3 + 2z + f_3(x,y)$

$\therefore \phi = y^2\sin x + z^3x - 4y + 2z + C$

Example 2.9. *Find the angle between the surfaces* $x^2+yz=2$ *and* $x+2y-z=2$ *at* $(1,1,1)$.

Solution: The first surface is $\phi_1 = x^2+yz-2 = 0$

Then $\nabla\phi_1 = \left(\vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z}\right)(x^2+yz-2)$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+yz-2) + \vec{j}\frac{\partial}{\partial y}(x^2+yz-2) + \vec{k}\frac{\partial}{\partial z}(x^2+yz-2)$$

$$= \vec{i}(2x) + \vec{j}(z) + \vec{k}(y) = 2\vec{i} + \vec{j} + \vec{k} \text{ at } (1,1,1).$$

The second surface $\phi_2 = x + 2y - z - 2 = 0$.

Then $\nabla\phi_2 = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x+2y-z-2)$

$$= \vec{i}\frac{\partial}{\partial x}(x+2y-z-2) + \vec{j}\frac{\partial}{\partial y}(x+2y-z-2) + \vec{k}\frac{\partial}{\partial z}(x+2y-z-2)$$

$$= \vec{i}(1) + \vec{j}(2) + \vec{k}(-1) = \vec{i} + 2\vec{j} - \vec{k} \text{ at } (1,1,1).$$

Let 'θ' be the angle between the surfaces ϕ_1 and ϕ_2.

$$\therefore \cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{(2\vec{i} + \vec{j} + \vec{k})(\vec{i} + 2\vec{j} - \vec{k})}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\therefore \theta = \frac{\pi}{3}$$

Example 2.10. *Find a and b so that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1,-1,2)$.*

Solution:

$$\text{Here } \phi_1 = ax^2 - byz - (a+2)x = 0 \tag{2.1}$$

$$\text{and } \phi_2 = 4x^2y + z^3 - 4 = 0 \tag{2.2}$$

Now $\nabla\phi_1 = \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi_1}{\partial z}$

$$= (2ax - a - 2)\vec{i} + (-bz)\vec{j} + (-by)\vec{k}$$

$$\Rightarrow (\nabla\phi_1) \text{ at } (1,-1,2) = (2a - a - 2)\vec{i} - 2b\vec{j} + b\vec{k}$$

and $\nabla\phi_2 = \vec{i}\frac{\partial\phi_2}{\partial x} + \vec{j}\frac{\partial\phi_2}{\partial y} + \vec{k}\frac{\partial\phi_2}{\partial z} = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$

$$\Rightarrow (\nabla\phi_2) \text{ at } (1,-1,2) = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

Since $\nabla\phi_1.\nabla\phi_2 = 0$, [$\because$ surfaces intersect orthogonally]

we get $[(a-2)\vec{i} - 2b\vec{j} + b\vec{k}].(-8\vec{i} + 4\vec{j} + 12\vec{k}) = 0$

(i.e.) $-8(a-2) - 8b + 12b = 0$ (or)

$$-2a + 4 + b = 0 \tag{2.3}$$

Also the point $(1,-1,2)$ lies on $\phi_1 = 0$

Putting $x = 1, y = -1, z = 2$ in 2.1, we get

$$a + 2b - (a+2) = 0 \Rightarrow b = 1 \tag{2.4}$$

Putting $b = 1$ in eqn(2.3), we get $-2a + 5 = 0 \Rightarrow a = \frac{5}{2}$

Example 2.11. *Find the angle between the surfaces $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at the point $(2, -1, 2)$.*

Solution: Let $\nabla\phi_1 = x^2 + y^2 - z - 3$ and $\nabla\phi_2 = x^2 + y^2 + z^2 - 9$.

Now, $\nabla\phi_1 = \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi_1}{\partial z} = \vec{i}(2x) + \vec{j}(2y) - \vec{k}$

$\Rightarrow (\nabla\phi_1)$ at $(2, -1, 2) = 4\vec{i} - 2\vec{j} - \vec{k}$

and $\nabla\phi_2 = \vec{i}\frac{\partial\phi_2}{\partial x} + \vec{j}\frac{\partial\phi_2}{\partial y} + \vec{k}\frac{\partial\phi_2}{\partial z} = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

$\Rightarrow (\nabla\phi_2)_{(2,-1,2)} = 4\vec{i} - 2\vec{j} + 4\vec{k}$

If 'θ' be the angle between the surfaces ϕ_1 and ϕ_2 then

$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$\cos\theta = \frac{(4\vec{i} - 2\vec{j} - \vec{k}).(4\vec{i} - 2\vec{j} + 4\vec{k})}{\sqrt{16+4+1}\sqrt{16+4+16}}$$

$$= \frac{16+4-4}{\sqrt{21}\sqrt{36}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Example 2.12. *Find the equation of the tangent plane to the surface $xz^2 + x^2y = z - 1$ at the point $(1, -3, 2)$.*

Solution: Let $\phi = xz^2 + x^2y - z + 1 = 0$

$$\therefore \nabla\phi = \sum \vec{i}\frac{\partial\phi}{\partial x} = \vec{i}(z^2 + 2xy) + \vec{j}(x^2) + \vec{k}(2xz - 1)$$

$$(\nabla\phi)_{(1,-3,2)} = -2\vec{i} + \vec{j} + 3\vec{k}$$

Direction ratios of the normal to the plane is $(-2, 1, 3)$.

Required tangent plane passes through $(1, -3, 2)$. [Plane equation through (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where a, b, c are dR's of normal to the plane]

$\Rightarrow$ Its equation is $-2(x-1) + (y+3) + 3(z-2) = 0$.

(or) $-2x + y + 3z + 2 + 3 - 6 = 0$

(or) $2x - y - 3z + 1 = 0$

Example 2.13. *Find the constants a, b, c so that $\overline{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ may be irrotational. For these values of a, b, c, find its scalar potential.*

Solution: We know that $\overline{F}$ is irrotational if $curl\overline{F} = 0$.

$$\nabla \times \overline{F} = 0 \Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = 0$$

$$\Rightarrow \vec{i}\left[\frac{\partial}{\partial y}(4x + cy + 2z) - \frac{\partial}{\partial z}(bx - 3y - z)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(4x + cy + 2z) - \frac{\partial}{\partial z}(x + 2y + az)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(bx-3y-z)-\frac{\partial}{\partial y}(x+2y+az)\right]=0$$

$$\Rightarrow \vec{i}(c+1)-\vec{j}(4-a)+\vec{k}(b-2)=0$$

(i.e.) $c+1=0, 4-a=0, b-2=0$

(i.e.) $a=4, b=2$ and $c=-1$

Using these values in $\overline{F}$, we get

$$\overline{F}=(x+2y+4z)\vec{i}+(2x-3y-z)\vec{j}+(4x-y+2z)\vec{k}$$

Let ϕ be the scalar potential. We consider $\overline{F}=\nabla\phi$.

$$\therefore (x+2y+4z)\vec{i}+(2x-3y-z)\vec{j}+(4x-y+2z)\vec{k}=\sum\vec{i}\frac{\partial\phi}{\partial x}$$

$$\Rightarrow \frac{\partial\phi}{\partial x}=x+2y+4z;\ \frac{\partial\phi}{\partial y}=2x-3y-z;\ \frac{\partial\phi}{\partial z}=4x-y+2z$$

Integrating, $\phi=\frac{x^2}{2}+2yx+4zx+f_1(y,z), \phi=2xy-3\frac{y^2}{2}-yz+f_2(x,z)$ and $\phi=4xz-yz+z^2+f_3(x,y)$

$$f_1(y,z)=\frac{-3y^2}{2}+y^2+z^2, f_2(x,z)=4zx+z^2+\frac{x^2}{2},$$

$$f_3(x,y)=2xy-\frac{3y^2}{2}+\frac{x^2}{2}$$

We must choose ϕ, satisfying the above condition.

$$\therefore \phi=\frac{x^2}{2}-3\frac{y^2}{2}+z^2+2xy+4zx-yz+C$$

Example 2.14. *If* $\overline{F}=3x^2\vec{i}+5xy^2\vec{j}+xyz^3\vec{k}$, *find* $\nabla.\overline{F}$, $\nabla(\nabla.\overline{F})$, $\nabla\times\overline{F}$, $\nabla.(\nabla\times\overline{F})$ *and* $\nabla\times(\nabla\times\overline{F})$ *at the point* $(1,2,3)$.

Solution: Let $\nabla.\overline{F}=\frac{\partial}{\partial x}(F_1)+\frac{\partial}{\partial y}(F_2)+\frac{\partial}{\partial z}(F_3)$

Here $F_1=3x^2,\ F_2=5xy^2,\ F_3=xyz^3$

$$\therefore \nabla.\overline{F}=\frac{\partial}{\partial x}(3x^2)+\frac{\partial}{\partial y}(5xy^2)+\frac{\partial}{\partial z}(xyz^3)$$

$$=6x+10xy+3xyz^2$$

$$\Rightarrow (\nabla.\overline{F})_{(1,2,3)}=6+20+54=80.$$

Also $\nabla(\nabla.\overline{F})=\sum\vec{i}\frac{\partial}{\partial x}(\nabla.\overline{F})$

$$=\sum\vec{i}\frac{\partial}{\partial x}(6x+10xy+3xyz^2)$$

$$=\vec{i}(6+10y+3yz^2)+\vec{j}(10x+3xz^2)+\vec{k}(6xyz)$$

$$\Rightarrow \nabla(\nabla.\overline{F})_{(1,2,3)}=\vec{i}(6+20+54)+\vec{j}(10+27)+\vec{k}(36)$$

$$=80\vec{i}+37\vec{j}+36\vec{k}$$

$$\text{Now } \nabla \times \overline{F} = curl\overline{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & 5xy^2 & xyz^3 \end{vmatrix}$$

$$= \vec{i}(xz^3) - \vec{j}(yz^3) + \vec{k}(5y^2)$$

$$(\nabla \times \overline{F})_{(1,2,3)} = 27\vec{i} - 54\vec{j} + 20\vec{k}$$

$$\text{Also } \nabla(\nabla \times \overline{F}) = div\, curl\overline{F} = \frac{\partial}{\partial x}(xz^3) + \frac{\partial}{\partial y}(-yz^3) + \frac{\partial}{\partial z}(5y^2)$$

$= z^3 - z^3 = 0$ [Note: $\div curl\overline{F} = 0$ is a vector identity]

$$\text{and } \nabla \times (\nabla \times \overline{F}) = curl\, curl\overline{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -yz^3 & 5y^2 \end{vmatrix}$$

$$= \vec{i}(10y + 3z^2y) - \vec{j}(-3xz^2) + \vec{k}(0)$$

$$\Rightarrow (\nabla \times (\nabla \times \overline{F}))_{(1,2,3)} = (20 + 54)\vec{i} - (-27)\vec{j} = 74\vec{i} + 27\vec{j}$$

Example 2.15. *If $\vec{r}$ is the position vector of the point (x, y, z) with respect to the origin, prove that $div\,\vec{r} = 3$, $curl\,\vec{r} = 0$, $\nabla(r^n) = nr^{n-2}\vec{r}$*

Solution: Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\text{Then } div\,\vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

$$\text{Also } curl\,\vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] - \vec{j}\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right] = 0$$

$$\text{and } \nabla(r^n) = \sum \vec{i}\frac{\partial}{\partial x}(r^n) = \sum \vec{i}\, nr^{n-1}\frac{\partial r}{\partial x}$$

Here $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow r = \sqrt{x^2 + y^2 + z^2} = |\vec{r}|$

(i.e.) $r^2 = x^2 + y^2 + z^2$.

Differentiating partially w.r.to x, we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.$$

$$\text{Similarly, } \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{Now } grad(r^n) = \nabla(r^n) = \sum \vec{i}\, nr^{n-1}\frac{x}{r}$$

$$= \frac{nr^{n-1}}{r}(x\vec{i} + y\vec{j} + z\vec{k}) = nr^{n-2}\vec{r}$$

Example 2.16. *Prove that i) $curl\, grad\phi = 0$ ii) $div\, curl\overline{F} = \nabla(\nabla \times \overline{F}) = 0$ iii) $div(\phi\overline{u}) = \phi div\overline{u} + \nabla\phi.\overline{u}$*

Solution: (i) We know that $grad\phi = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

$$\therefore curl\, grad\phi = \nabla\times(\nabla\phi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right] - \vec{j}\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right]$$

$$+\vec{k}\left[\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right] = 0$$

(ii) Let $\overline{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\therefore curl\,\overline{F} = (\nabla\times\overline{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - \vec{j}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + \vec{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

Now $div\, curl\,\overline{F} = \nabla.(\nabla\times\overline{F})$

$$= \frac{\partial}{\partial x}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial F_1}{\partial x} - \frac{\partial F_3}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$$= \frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} + \frac{\partial^2 F_1}{\partial y\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} + \frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_1}{\partial z\partial y} = 0$$

(iii) We know that $div\overline{F} = \sum\vec{i}.\frac{\partial\overline{F}}{\partial x}$

$$\Rightarrow div(\phi\overline{u}) = \sum\vec{i}.\frac{\partial}{\partial x}(\phi\overline{u})$$

$$= \sum\vec{i}.\left[\overline{u}\frac{\partial\phi}{\partial x} + \phi\frac{\partial\overline{u}}{\partial x}\right]$$

$$= \sum\left[\vec{i}.\left(\frac{\partial\phi}{\partial x}.\overline{u}\right)\right] + \sum\left[\vec{i}.\left(\phi.\frac{\partial\overline{u}}{\partial x}\right)\right]$$

$$= \left[\sum\vec{i}\frac{\partial\phi}{\partial x}\right].\overline{u} + \phi\left[\sum\vec{i}.\frac{\partial\overline{u}}{\partial x}\right] = \nabla\phi.\overline{u} + \phi\, div\,\overline{u}$$

Example 2.17. *Find the value of n, if $r^n\overline{r}$ is both solenoidal and irrotational, when $\overline{r} = x\vec{i} + y\vec{j} + z\vec{k}$*

Solution:

$$\text{Consider } div(r^n\overline{r}) = \nabla r^n.\overline{r} + r^n div\overline{r} \qquad (2.5)$$

$[\because div(\phi\overline{u}) = \phi div\overline{u} + \nabla\phi.\overline{u}]$

We know that $div\overline{r} = 3, \nabla r^n = nr^{n-2}\overline{r}$

From (2.5), $div\, r^n\overline{r} = (nr^{n-2}\overline{r}.\overline{r}) + r^n(3)$

$= nr^{n-2}r^2 + 3r^n = nr^n + 3r^n = (n+3)r^n$

$\Rightarrow r^n\overline{r}$ is solenoidal means $div\, r^n\overline{r} = 0$, which is possible only when $n = -3$.

We know that $curl(\phi\overline{u}) = \nabla\phi \times \overline{u} + \phi curl\overline{u}$

$\therefore curl(r^n\overline{r}) = \nabla r^n \times \overline{r} + r^n curl\overline{r}$

$= nr^{n-2}(\overline{r}\times\overline{r}) + r^n.0 = 0$ (as $\overline{r}\times\overline{r} = 0, curl\overline{r} = 0$)

$\Rightarrow$ $r^n\overline{r}$ is irrotational if $curl(r^n\overline{r}) = 0$, (i.e.) whatever be the value of n.

Hence $r^n\overline{r}$ is solenoidal as well as irrotational when $n = -3$.

Example 2.18. *Prove that $f(r)\overline{r}$ is an irrotational vector.*

Solution:

Consider $curl(f(r)\overline{r}) = \nabla f(r) \times \overline{r} + f(r) curl\overline{r}$ (2.6)

$$\therefore \nabla f(r) = \sum \vec{i}\frac{\partial}{\partial x} f(r) = \sum \vec{i} f'(r)\frac{\partial r}{\partial x}$$

$$= \sum \vec{i} f'(r)\frac{x}{r} = \frac{f'(r)}{r}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{f'(r)}{r}\overline{r}$$

Further $curl\overline{r} = 0$.

From (2.6), $curl(f(r)\overline{r}) = \dfrac{f'(r)}{r}(\overline{r}\times\overline{r}) + f(r).0 = 0 + 0 = 0.$

$\Rightarrow f(r)\overline{r}$ is an irrotational vector.

Example 2.19. *Prove that $\nabla^2 r^n = n(n+1)r^{n-2}$ where $\overline{r} = x\vec{i} + y\vec{j} + z\vec{k}$.*

Solution: We know that $\nabla^2\phi = \nabla.(\nabla\phi)$.

Hence $\nabla^2 r^n = \nabla(\nabla r^n) = \nabla(nr^{n-2}\overline{r})$.

Since $\nabla.(\phi\overline{u}) = \nabla\phi.\overline{u} + \phi\, div\,\overline{u}$

We have $\nabla.(nr^{n-2}\overline{r}) = \nabla nr^{n-2}.\overline{r} + nr^{n-2}\, div\,\overline{r}$ (2.7)

$$\text{Now } \nabla(nr^{n-2}) = \sum \vec{i}\frac{\partial}{\partial x} nr^{n-2} = \sum \vec{i}\, n(n-2)r^{n-3}\frac{\partial r}{\partial x}$$

$$= \sum \vec{i}\, n(n-2)r^{n-3}\frac{x}{r} = \sum n(n-2)r^{n-4}x\vec{i} \qquad (2.8)$$

Now put (2.8) in (2.7), we have

$\nabla(nr^{n-2}) = n(n-2)r^{n-4}\overline{r}.\overline{r} + nr^{n-2}.3$

$= nr^{n-2}(n-2+3) = n(n+1)r^{n-2}$

Example 2.20. *Find the angle between the normals to the surface $x^2 = yz$ at the points $(1,1,1)$ and $(2,4,1)$.*

Solution: Let $\overline{n_1}$ = unit normal to $x^2 - yz = 0$ at $(2,4,1)$.

(i.e.) $\overline{n_1} = \dfrac{\nabla\phi}{|\nabla\phi|}$.

$\nabla\phi = \sum \vec{i}\dfrac{\partial}{\partial x}(x^2 - yz)$ as $\phi = x^2 - yz$

$= \vec{i}(2x) + \vec{j}(-z) + \vec{k}(-y)$

at $(2,4,1) \Rightarrow \overline{n_1} = 4\vec{i} - \vec{j} - 4\vec{k}$

Similarly $\overline{n_2} = \dfrac{\sum \vec{i}\frac{\partial}{\partial x}(x^2 - yz)}{|\nabla\phi|}$ at $(1,1,1)$.

$\therefore \overline{n_2} = 2\vec{i} - \vec{j} - \vec{k}$

If 'θ' is the required angle then

$$\cos\theta = \frac{\overline{n_1}.\overline{n_2}}{|\overline{n_1}||\overline{n_2}|} = \frac{(4\vec{i} - \vec{j} - 4\vec{k})}{\sqrt{16+1+16}}.\frac{(2\vec{i} - \vec{j} - \vec{k})}{\sqrt{4+1+1}}$$

$$= \frac{8+1+4}{\sqrt{33}\sqrt{6}} = \frac{13}{\sqrt{198}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{13}{3\sqrt{22}}\right)$$

Example 2.21. *Prove that* $\vec{A} = (2x + yz)\vec{i} + (4y + zx)\vec{j} - (6z - xy)\vec{k}$ *is solenoidal as well as irrotational. Also find the scalar potential of* $\vec{A}$.

Solution: We know that

$$div\,\vec{A} = \frac{\partial}{\partial x}(2x + yz) + \frac{\partial}{\partial y}(4y + zx) + \frac{\partial}{\partial z}(-6z + xy)$$

$$= 2 + 4 - 6 = 0$$

$\therefore \vec{A}$ is solenoidal.

$$curl\,\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + yz & 4y + zx & -6z + xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(-6z + xy) - \frac{\partial}{\partial z}(4y + zx)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(-6z + xy) - \frac{\partial}{\partial z}(2x + yz)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(4y + zx) - \frac{\partial}{\partial y}(2x + yz)\right]$$

$$= \vec{i}(x - x) - \vec{j}(y - y) + \vec{k}(z - z) = 0$$

$\therefore \vec{A}$ is irrotational.

Let $\phi(x, y, z)$ be the scalar potential of $\vec{A}$.

Then $\vec{A} = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

Hence $\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (2x + yz)\vec{i} + (4y + zx)\vec{j} - (6z - xy)\vec{k}$

Equating the components by $\vec{i}, \vec{j}, \vec{k}$

$$\frac{\partial\phi}{\partial x} = 2x + yz \tag{2.9}$$

$$\frac{\partial\phi}{\partial y} = 4y + zx \tag{2.10}$$

$$\frac{\partial\phi}{\partial z} = -6z + xy \tag{2.11}$$

Partially integrating (2.9), (2.10) and (2.11) w.r.to x, y, z respectively, we get

$\phi = x^2 + xyz +$ a constant independent of x

$\therefore \overline{n_2} = 2\vec{i} - \vec{j} - \vec{k}$

If 'θ' is the required angle then

$$\cos\theta = \frac{\overline{n_1}.\overline{n_2}}{|\overline{n_1}||\overline{n_2}|} = \frac{(4\vec{i} - \vec{j} - 4\vec{k})}{\sqrt{16+1+16}} . \frac{(2\vec{i} - \vec{j} - \vec{k})}{\sqrt{4+1+1}}$$

$$= \frac{8+1+4}{\sqrt{33}\sqrt{6}} = \frac{13}{\sqrt{198}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{13}{3\sqrt{22}}\right)$$

Example 2.21. *Prove that* $\vec{A} = (2x+yz)\vec{i} + (4y+zx)\vec{j} - (6z-xy)\vec{k}$ *is solenoidal as well as irrotational. Also find the scalar potential of* $\vec{A}$.

Solution: We know that

$$div\,\vec{A} = \frac{\partial}{\partial x}(2x+yz) + \frac{\partial}{\partial y}(4y+zx) + \frac{\partial}{\partial z}(-6z+xy)$$

$$= 2+4-6 = 0$$

$\therefore \vec{A}$ is solenoidal.

$$curl\,\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+yz & 4y+zx & -6z+xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(-6z+xy) - \frac{\partial}{\partial z}(4y+zx)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(-6z+xy) - \frac{\partial}{\partial z}(2x+yz)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(4y+zx) - \frac{\partial}{\partial y}(2x+yz)\right]$$

$$= \vec{i}(x-x) - \vec{j}(y-y) + \vec{k}(z-z) = 0$$

$\therefore \vec{A}$ is irrotational.

Let $\phi(x,y,z)$ be the scalar potential of $\vec{A}$.

Then $\vec{A} = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

Hence $\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (2x+yz)\vec{i} + (4y+zx)\vec{j} - (6z-xy)\vec{k}$

Equating the components by $\vec{i}, \vec{j}, \vec{k}$

$$\frac{\partial\phi}{\partial x} = 2x+yz \tag{2.9}$$

$$\frac{\partial\phi}{\partial y} = 4y+zx \tag{2.10}$$

$$\frac{\partial\phi}{\partial z} = -6z+xy \tag{2.11}$$

Partially integrating (2.9), (2.10) and (2.11) w.r.to x, y, z respectively, we get

$\phi = x^2 + xyz + a$ constant independent of x

$\phi = 2y^2 + xyz + a$ constant independent of y

$\phi = -3z^2 + xyz + a$ constant independent of z

Hence a possible form of ϕ is $\phi = x^2 + 2y^2 - 3z^2 + xyz + a$.

Example 2.22. *Prove that the directional derivative of* $\phi = x^3y^2z$ *at* $(1, 2, 3)$ *is maximum along the direction* $9\vec{i} + 3\vec{j} + \vec{k}$. *Also find the maximum directional derivative.*

Solution: Given $\phi = x^3y^2z$

Hence $\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^3y^2z)$

$= \vec{i}\frac{\partial}{\partial x}(x^3y^2z) + \vec{j}\frac{\partial}{\partial y}(x^3y^2z) + \vec{k}\frac{\partial}{\partial z}(x^3y^2z)$

$= (3x^2y^2z)\vec{i} + (2x^3yz)\vec{j} + (x^3y^2)\vec{k}$

$= 36\vec{i} + 12\vec{j} + 4\vec{k} = 4(9\vec{i} + 3\vec{j} + \vec{k})$

We know that the directional derivative of ϕ is maximum along the direction $\nabla\phi$.

Hence it is maximum along the direction $4(9\vec{i} + 3\vec{j} + \vec{k})$. (i.e.) along $(9\vec{i} + 3\vec{j} + \vec{k})$.

Magnitude of this vector $= 4\sqrt{9^2 + 3^2 + 1^2} = 4\sqrt{91}$ and this is the maximum directional derivative.

EXERCISE

1. Find the unit normal to the surface $x^2 + 2y^2 + z^2 = 7$ at the point (1, -1, 2). $\left[\text{Ans:} \frac{\vec{i} - 2\vec{j} + 3\vec{k}}{3}\right]$

2. If $\nabla\phi = yz\vec{i} + zx\vec{j} + xy\vec{k}$, find ϕ. [*Ans* : $\phi = xyz + c$]

3. If $\phi(x, y, z) = x^2y - 2y^2z^3$, find $\nabla\phi$ at the point (1, -1, 2). [*Ans* : $-2\vec{i} + 33\vec{j} - 24\vec{k}$]

4. Show that the vector $\vec{v} = (x + 3y)\vec{i} + (y - 3z)\vec{j} + (x - 2z)\vec{k}$ is solenoidal.

5. Show that the vector $\vec{A} = 3x^2y\vec{i} + (x^3 - 2yz^2)\vec{j} + (3z^2 - 2y^2z)\vec{k}$ is irrotational.

6. If $\bar{a}$ is a constant, find $div(\bar{r} \times \bar{a})$. [Ans: 0]

7. Determine the constant $'a'$ so that the vector $\vec{v} = (x + 3y)\vec{i} + (y - 2z)\vec{j} + (x + az)\vec{k}$ is solenoidal. [Ans: a = -2]

8. Evaluate $div\vec{f}$ where $\vec{f} = 2x^2z\vec{i} - xy^2z\vec{j} + 3y^2x\vec{k}$

   [*Ans* : $2xz(2 - y)$]

9. If $u = x^2 - y^2 + 4z$, show that $\nabla^2 u = 0$.

10. If $\vec{F} = x^2y\,\vec{i} + xz\,\vec{j} + 2yz\,\vec{k}$, verify that $div\, curl\, \vec{F} = 0$.

11. Find the equation of the tangent plane to the surface $z = x^2 + y^2$ at the point (2, -1, 5). [*Ans* : $4x - 2y - z = 5$]

12. If $\vec{F} = (x + y + 1)\,\vec{i} + \vec{j} - (x + y)\,\vec{k}$, show that $\vec{F}$ is perpendicular to $curl\,\vec{F}$

13. Find the maximum directional derivative of $\phi = x^3y^2z$ at the point (1, 1, 1) [*Ans* : $\sqrt{14}$]

14. Find the value of a, if $\vec{F} = (axy - z^3)\,\vec{i} + (a - 2)x^2\,\vec{j} + (1 - a)xz^2\,\vec{k}$ is irrotational.

15. If $\overline{u}$ and $\overline{v}$ are irrotational, prove that $(\overline{u} \times \overline{v})$ is solenoidal.

16. Show that $\vec{F} = (\sin y + z)\,\vec{i} + (x\cos y - z)\,\vec{j} + (x - y)\,\vec{k}$ is irrotational.

17. When is a vector said to be (i) solenoidal, (ii) irrotational?

18. Find the unit normal to the surface $x^4 - 3xyz + z^2 + 1 = 0$ at the point (1, 1, 1). $\left[\text{Ans:}\frac{1}{\sqrt{11}}(\vec{i} - 3\,\vec{j} - \vec{k})\right]$

19. If $r = |\vec{r}|$, where $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$, prove that $\nabla \log|\vec{r}| = \frac{\vec{r}}{r^2}$

20. Find $grad\, f$, where f is given by $f = x^3 - y^3 + xz^2$ at the point (1, -1, 2). [*Ans* : $7\,\vec{i} - 3\,\vec{j} + 4\,\vec{k}$]

21. What is the greatest rate of increase of $u = xyz^2$ at the point (1, 0, 3)? [Ans: 9]

22. Prove that $grad\, f(u) = f'(u)\, grad u$

23. Find the directional derivative of the function $\phi = xy^2 + yz^3$ at the point (2, -1, 1) in the direction of the normal to the surface $x\log z - y^2 + 4$ at the point (-1, 2, 1). $\left[\text{Ans:}\frac{15}{\sqrt{17}}\text{units}\right]$

24. Find the constants a and b, so that the surfaces $5x^2 - 2yz - 9x = 0$ and $ax^2y + bz^3 = 4$ may cut orthogonally at the point (1, -1, 2). [Ans: a=4, b=1]

25. Find the angle between the surfaces $x^2 - y^2 - z^2 = 11$ and $xy + yz - zx = 18$ at the point (6, 4, 3).

$$\left[\text{Ans: } \theta = \cos^{-1}\left(\frac{-24}{\sqrt{5246}}\right)\right]$$

26. Show that $\vec{u} = (2x^2 + 8xy^2z)\,\vec{i} + (3x^3y - 3xy)\,\vec{j} - (4y^2z^2 + 2x^3z)\,\vec{k}$ is not solenoidal, but $\vec{v} = xyz^2\,\vec{u}$ is solenoidal.

27. Find the directional derivative of $\phi = 2xy + z^2$ at the point (1, -1, 3) in the direction of $\vec{i} + 2\,\vec{j} + 2\,\vec{k}$ $\left[\text{Ans:}\frac{14}{3}\right]$

28. If $\nabla\phi = 2xyz^3\,\vec{i} + x^2z^3\,\vec{j} + 3x^2yz^2\,\vec{k}$, find $\phi(x, y, z)$ given that $\phi(1, -2, 2) = 4$. [*Ans* : $\phi = x^2yz^3 + 20$]

29. If $r = |\vec{r}|$, where $\vec{r}$ is the position vector of the point

(x, y, z) with respect to the origin, prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

30. Find the directional derivative of $\phi = xy + yz + zx$ in the direction of vector $\vec{i} + 2\vec{j} + 2\vec{k}$ at (1, 2, 0). $\left[\text{Ans:} \frac{10}{3}\right]$

31. Find the angle of intersection at (4, -3, 2) of spheres $x^2 + y^2 + z^2 = 29$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z - 47 = 0$. $[Ans : \cos^{-1} \sqrt{19/29}]$

32. Find $\nabla\phi$ and $|\nabla\phi|$ when $\phi = (x^2 + y^2 + z^2)e^{-(x^2+y^2+z^2)^{1/2}}$ $[Ans : (2-r)e^{-r}\vec{r}, (2-r)e^{-r}r]$

33. If $u = 3x^2y$ and $v = xz^2 - 2y$, then find $grad[(grad u).(grad v)]$ $[Ans : (6yz^2 - 4x)\vec{i} + 6xz^2\vec{j} + 12xyz\vec{k}]$

34. Find $div\vec{f}$ and $curl\vec{f}$ where $\vec{f} = grad(x^3 + y^3 + z^3 - 3xyz)$ $[Ans : div\vec{f} = 6(x + y + z).curl\vec{f} = 0]$

35. Find the directional derivative of $\phi = xy^2 + yz^3$ at the point $P(2, -1, 1)$ in the direction of PQ where Q is the point (3, 1, 3). $\left[\text{Ans:} -\frac{11}{3}\right]$

36. Find the values of λ and μ if the surfaces $\lambda x^2 - \mu yz = (\lambda + 2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at the point (1, -1, 2). $\left[\text{Ans:} \lambda = \frac{5}{2}, \mu = 1\right]$

37. Prove that $yz^2\vec{i} + (xz^2 - 1)\vec{j} + 2(xyz - 1)\vec{k}$ is irrotational

and find its scalar potential. $[Ans : xy^2 - y - z + c]$

38. Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2y = 2 - z$ at the point (1, 1, 1). $\left[\text{Ans: } \cos^{-1}\left(\frac{1}{\sqrt{30}}\right)\right]$

39. Find the directional derivative of $\phi = 2x^2 + 3y^2 + z^2$ at (2, 1, 3) in the direction of the vector $\vec{i} - 2\vec{k}$. $\left[\text{Ans:} -\frac{4}{\sqrt{5}}\right]$

40. Prove that $e^x[(2y + 3z)\vec{i} + 2\vec{j} + 3\vec{k}]$ is irrotational and find its scalar potential. $[Ans : e^x(2y + 3z) + c]$

41. Given the vector field $\vec{v} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$. Find $curl\vec{v}$. Show that the vectors given by $curl\vec{v}$ at $P_0(1, 2, -3)$ and $P_1(2, 3, 12)$ are orthogonal.

42. A fluid motion is given by $\vec{v} = (y+z)\vec{i} + (z+x)\vec{j} + (x+y)\vec{k}$. Is this motion irrotational? If so, find the velocity potential. [Ans: yes, $\phi = xy + yz + zx$]

43. In what direction from (3, 1, -2) is the directional derivative $\phi = x^2y^2z^4$ is maximum? Find the magnitude of this maximum. $[Ans : 96\sqrt{19}]$

44. Establish the relation: $curl\, curl\vec{f} = grad\, div\vec{f} - \nabla^2\vec{f}$.