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Chapter 1

Operations Research (O.R.)

(Resource Management Techniques)

1.1 Introduction

Operations Research is the study of optimisation techniques. It is applied decision theory. The existence of optimisation techniques can be traced at least to the days of Newton and Lagrange. Rapid development and invention of new techniques occurred since the World War II essentially, because of the necessity to win the war with the limited resources available. Different teams had to do ***research*** on military ***operations*** in order to invent techniques to ***manage*** with available ***resources*** so as to obtain the desired objective. Hence the nomenclature ***Operations Research or Resource Management Techniques.***

Scope or Uses or Applications of O.R (Some O.R. Models)

O.R. is useful for solving

(1) Resource allocation problems

(2) Inventory control problems

(3) Maintenance and Replacement Problems

(4) Sequencing and Scheduling Problems

(5) Assignment of jobs to applicants to maximise total profit or minimize total cost.

(6) Transportation Problems

(7) Shortest route problems like travelling sales person problems

(8) Marketing Management problems

(9) Finance Management problems

(10) Production, Planning and control problems

(11) Design Problems

(12) Queuing problems, *etc.* to mention a few.

1.2 Role of operations research in Business and Management

1. ***Marketing Management Operations research techniques have definitely a role to play in***
   (a) Product selection
   (b) Competitive strategies
   (c) Advertising strategy *etc.*

2. Production Management

The O.R techniques are very useful in the following areas of production management.

(a) Production scheduling
(b) Project scheduling
(c) Allocation of resources
(d) Location of factories and their sizes
(e) Equipment replacement and Maintenance
(f) Inventory policy etc.

3. Finance Management

The techniques O.R are applied to Budgeting and Investment areas and especially to

(a) Cash flow analysis
(b) Capital requirement
(c) Credit policies
(d) Credit risks *etc.*

4. Personal Management

(a) Recruitment policies and
(b) Assignment of jobs are some of the areas of personnel management where O.R techniques are useful.

5. Purchasing and Procurement

(a) Rules for purchasing
(b) Determining the quantity
(c) Determining the time of purchaser are some of the areas where O.R. techniques can be applied

6. Distribution

In determining

(a) location of warehouses
(b) size of the warehouses
(c) rental outlets
(d) transportation strategies O.R. techniques are useful.

1.3 Role of O.R in Engineering

1. Optimal design of water resources systems
2. Optimal design of structures
3. Production, Planning, Scheduling and control
4. Optimal design of electrical networks
5. Inventory control
6. Planning of maintenance and replacement of equipment
7. Allocation of resources of services to maximise the benefit
8. Design of material handling
9. Optimal design of machines
10. Optimum design of control systems
11. Optimal selection of sites for an industry to mention a few.

1.4 Classification of Models

The first thing one has to do to use O.R. techniques after formulating a practical problem is to construct a suitable model to represent the practical problem. A model is a reasonably simplified representation of a real–world situation. It is an abstraction of reality. The models can broadly be classified as

Iconic (Physical) Models
Analogue Models
Mathematical Models

Also as

Static Models
Dynamic Models and, in addition, as
Deterministic Models
Stochastic Models

Models can further be subdivided as

Descriptive Models
Prescriptive Models
Predictive Models
Analytic Models
Simulation Models

Iconic Model :

This is a physical, or pictorial representation of various aspects of a system.

Example : Toy, Miniature model of a building, scaled up model of a cell in biology *etc.*

Analogue or Schematic model :

This uses one set of properties to represent another set of properties which a system under study has

Example : A network of water pipes to represent the flow of current in an electrical network or graphs, organisational charts etc.

Mathematical model or Symbolic model :

This uses a set of mathematical symbols (letters, numbers etc) to represent the decision variables of a system under consideration. These variables related by mathematical equations or inequalities which describes the properties of the system.

Example : A linear programming model, A system of equations representing an electrical network or differential equations representing dynamic systems etc.

Static Model :

This is a model which does not take time into account. It assumes that the values of the variables do not change with time during a certain period of time horizon.

Example : A linear programming problem, an assignment problem, transportation problem *etc.*

Dynamic model is a model which considers time as one of the important variables.

Example: A dynamic programming problem, A replacement problem

Deterministic model is a model which does not take uncertainty into account.

Example : A linear programming problem, an assignment problem *etc.*

Stochastic model is a model which considers uncertainty as an important aspect of the problem.

Example : Any stochastic programming problem, stochastic inventory models *etc.*

Descriptive model is one which just describes a situation or system.

Example : An opinion poll, any survey.

Predictive model is one which predicts something based on some data. Predicting election results before actually the counting is completed.

Prescriptive model is one which prescribes or suggests a course of action for a problem.

Example : Any programming (linear, nonlinear, dynamic, geometric *etc.*) problem.

Analytic model is a model in which exact solution is obtained by mathematical methods in closed form.

Simulation model is a representation of reality through the use of a model or device which will react in the same manner as reality under a given set of conditions. Once a simulation model is designed, it takes only a little time, in general, to run a simulation on a computer.

It is usually less mathematical and less time consuming and generally least expensive as well, in many situations.

Example : queuing problems, inventory problems.

1.5 Some characteristics of a good Model

(1) It should be reasonably simple.

(2) A good model should be capable of taking into account new changes in the situation affecting its frame significantly with ease *i.e.,* updating the models should be as simple and easy as possible,

(3) Assumptions made to simplify the model should be as small as possible.

(4) Number of variables used should be as small in number as possible.

(5) The model should be open to parametric treatment.

1.6 Principles of Modelling

(1) Do not build up a complicated model while a simple one will suffice.

(2) Beware of moulding the problems to fit a (favourite !) technique

(3) Deductions must be made carefully.

(4) Models should be validated prior to implementation.

(5) A model should neither be pressed to do nor criticised for failing to do that for which it was never intended.

(6) Beware of overselling the model in cases where assumption made for the construction of the model can be challenged.

(7) The solution of a model cannot be more accurate than the accuracy of the information that goes into the construction of the model.

(8) Models are only aids in decision making.

(9) Model should be as accurate as possible.

1.7 General Methods for solving O.R models

(1) *Analytic Procedure :* Solving models by classical Mathematical techniques like differential calculus, finite differences etc. to obtain analytic solutions.

(2) *Iterative Procedure :* Starts with a trial solution and a set of rules for improving it by repeating the procedure until further improvement is not possible.

(3) *Monte–carlo technique :* Taking sample observations, computing probability distributions for the variable using random numbers and constructing some functions to determine values of the decision variables.

1.8 Main Phases of O.R.

(1) *Formulation of the Problems :* Identifying the objective, the decision variables involved and the constraints that arise involving the decision variables

(2) *Construction of a Mathematical Model:* Expressing the measure of effectiveness which may be total profit, total cost, utility etc., to be optimised by a Mathematical function called ***objective function.*** Representing the constraints like budget constraints, raw materials constraints, ***resource constraints,*** quality constraints etc., by means of mathematical equations or inequalities.

(3) *Solving the Model constructed:* Determining the solution by analytic or iterative or Monto–carlo method depending upon the structure of the mathematical model.

(4) *Controlling and updating :* A solution which is optimum today may not be so tomorrow. The values of the variables may change, new variables may emerge. The structural relationship between the variables may also undergo a change. All these are determined in updating.

A solution from a model remains a solution only so long as the uncontrollable variables retain their values and the relationship between the variables does not change. The solution itself goes "out of control" if the values of one or more controlled variables vary or relationship between the variables undergoes a change. Therefore controls must be established to indicate the limits within which the model and its solution can be considered as reliable. This is called controlling.

(5) ***Testing the model*** and ***its solution i.e., validating the model*** : checking as far as possible either from the past available data or by expertise and experience whether the model gives a solution which can be used in practice.

(6) ***Implementation :*** Implement using the solution to achieve the desired goal.

1.9 Limitation

Mathematical models which are the essence of OR do not take into account qualitative or emotional or some human factors which are quite real and influence the decision making. All such influencing factors find no place in O.R. This is the main limitation of O.R. Hence O.R. is only an aid in decision making.

EXERCISE

1. **What is O.R ?** *[MU. BE. Oct 96]*
2. **What is the scope of O.R ?**
3. **What are the applications of O.R. ?**
4. **List the uses of O.R.**
5. **Write a short note on the importance of operations research in production management** *[MU. MBA April, 97]*
6. **Write a short note on the role of operations research in marketing management.** *[MU. MBA Nov.96, Nov. 97, April 98]*
7. **Write a short note on the role of operations research in production planning.** *[MU. MBA April, 96]*
8. **What are the different phases of O.R.?** *[MKU. BE. Nov 97]*
9. **What are the characteristics of an O.R. problem ?** *[MKU. BE. Nov 97]*
10. **What is a model ?**
11. **What is a Mathematical Model ?**
12. **What are the different types of Models ?**

13. What are the characteristics of a good model ?

14. What are the limitations of a mathematical model ?

15. What are the limitations of an O.R. Model ?

[MKU. BE. Nov 97]

16. Explain the general methods of solving O.R. Models.

17. Explain the principles of modelling.

18. State the different types of Models used in O.R.

[MU. BE. Oct 97]

19. What are the various phases in the study of operations research ? *[BRU. BE. Apr 97]*

20. Answer the following questions with examples wherever necessary.

    (a) Necessities of OR in industry,

    (b) Fields of application of OR in industry

    (c) Deterministic models,

    (d) Mention atleast eight mathematical models.

[MKU. BE. Nov 97]

21. What is an iconic model in the study of operations research ?

[MU. BE. Oct. '96]

Chapter 2

Linear Programming Formulation and Graphical Method

(Formulation and Graphical Solution)

2.1 Introduction

Linear programming problems deal with determining optimal allocations of limited resources to meet given objectives. The resources may be in the form of men, raw materials, market demand, money and machines etc. The objective is usually maximizing profit, minimizing total cost, maximizing utility etc. There are certain restrictions on the total amount of each resource available and on the quantity or quality of each product made.

Linear programming problem deals with the optimization (Maximization or Minimization) of a function of decision variables (The variables whose values determine the solution of a problem are called ***decision variables*** of the problem) known as ***objective function,*** subject to a set of simultaneous linear equations (or inequalities) known as ***constraints***. The term ***linear*** means that all the variables occurring in the objective function and the constraints are of the first degree in the problems under consideration and the term ***programming*** means the process of determining a particular course of action.

Linear programming techniques are used in many industrial and economic problems. They are applied in product mix, blending, diet, transportation and assignment problems. Oil refineries, airlines, railways, textile industries, chemical industries, steel industries, food processing industries and defence establishments are also the users of this technique.

2.2 Requirements for employing LPP Technique :

[BRU. BE. Nov 96]

1. There must be a well defined objective function.
2. There must be alternative courses of action to choose.

3. At least some of the resources must be in limited supply, which give rise to constraints.
4. Both the objective function and constraints must be linear equations or inequalities.

2.3 Mathematical Formulation of L.P.P

If x_j ($j = 1, 2, \ldots, n$) are the n decision variables of the problem and if the system is subject to m constraints, the general Mathematical model can be written in the form :

$$\text{Optimize } Z = f(x_1, x_2, \ldots x_n)$$

$$\text{subject to } g_i(x_1, x_2, \ldots x_n) \leq, =, \geq b_i, (i = 1, 2, \ldots, m)$$

(called structural constraints)

$$\text{and } x_1, x_2, \ldots x_n \geq 0,$$

(called the non–negativity restrictions or constraints)

Procedure for forming a LPP Model :

Step 1 : Identify the unknown decision variables to be determined and assign symbols to them.

Step 2 : Identify all the restrictions or constraints (or influencing factors) in the problem and express them as linear equations or inequalities of decision variables.

Step 3 : Identify the objective or aim and represent it also as a linear function of decision variables.

Step 4 : Express the complete formulation of LPP as a general mathematical model.

We consider only those situations where this will help the reader to put proper inequalities in the formulation.

1. Usage of manpower, time, raw materials etc are always less than or equal to the availability of manpower, time, raw materials etc.

2. Production is always greater than or equal to the requirement so as to meet the demand.

Example 1 : A firm manufactures two types of products A and B and sells them at a profit of Rs. 2 on type A and Rs. 3 on type B. Each product is processed on two machines M_1 and M_2. Type A requires 1 minute of processing time on M_1 and 2 minutes on M_2. Type B requires 1 minute on M_1 and 1 minute on M_2. Machine M_1 is available for not more than 6 hours 40 minutes while machine M_2 is

available for 10 hours during any working day. Formulate the problem as a LPP so as to maximize the profit.

Solution : Let the firm decide to produce x_1 units of product A and x_2 units of product B to maximize its profit.

To produce these units of type A and type B products, it requires

$x_1 + x_2$ processing minutes on M_1

$2x_1 + x_2$ processing minutes on M_2

Since machine M_1 is available for not more than 6 hours and 40 minutes and machine B is available for 10 hours doing any working day, the constraints are

$$x_1 + x_2 \leq 400$$
$$2x_1 + x_2 \leq 600$$
$$\text{and } x_1, x_2 \geq 0.$$

Since the profit from type A is Rs. 2 and profit from type B is Rs. 3, the total profit is $2x_1 + 3x_2$. As the objective is to maximize the profit, the objective function is maximize $Z = 2x_1 + 3x_2$.

$\therefore$ The complete formulation of the LPP is

$$\text{Maximize } Z = 2x_1 + 3x_2$$

subject to the constraints

$$x_1 + x_2 \leq 400$$
$$2x_1 + x_2 \leq 600$$
$$\text{and } x_1, x_2 \geq 0.$$

Example 2 : (Production Allocation Problem)

A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines are given in the table below :

Machine	Time per unit (minutes)			Machine capacity (Minutes/day)
	Product 1	Product 2	Product3	
M_1	2	3	2	440
M_2	4	–	3	470
M_3	2	5	–	430

It is required to determine the number of units to be manufactured for each product daily. The profit per unit for product 1,2 and 3 is Rs.4, Rs.3 and Rs.6 respectively. It is assumed that all the amounts produced are consumed in the market. Formulate the mathematical model for the problem. *[MU. B. Tech. Leather. Oct 96]*

Solution : Let x_1, x_2 and x_3 be the number units of products 1, 2 and 3 produced respectively.

To produce these amount of products 1, 2 and 3, it requires :

$$2x_1 + 3x_2 + 2x_3 \quad \text{minutes on } M_1$$
$$4x_1 + 3x_3 \quad \text{minutes on } M_2$$
$$2x_1 + 5x_2 \quad \text{minutes on } M_3.$$

But the capacity of the machines M_1, M_2 and M_3 are 440,470 and 430 (minutes/day).

∴ The constraints are

$$2x_1 + 3x_2 + 2x_3 \leq 440$$
$$4x_1 + 3x_3 \leq 470$$
$$2x_1 + 5x_2 \leq 430$$
$$\text{and } x_1, x_2, x_3 \geq 0$$

Since the profit per unit for product 1, 2, and 3 is Rs.4, Rs. 3 and Rs.6 respectively, the total profit is $4x_1 + 3x_2 + 6x_3$. As the objective is to maximize the profit, the objective function is maximize $Z = 4x_1 + 3x_2 + 6x_3$.

∴ The complete formulation of the LPP is

$$\text{Maximize } Z = 4x_1 + 3x_2 + 6x_3$$

subject to the constraints

$$2x_1 + 3x_2 + 2x_3 \leq 440$$
$$4x_1 + 3x_3 \leq 470$$
$$2x_1 + 5x_2 \leq 430$$
$$\text{and } x_1, x_2, x_3 \geq 0.$$

Example 3 : (Blending Problem)

A firm produces an alloy having the following specifications :

(i) **Specific gravity ≤ 0.98**

(ii) **Chromimum $\geq 8\%$**

(iii) **Melting point $\geq 450°C$**

Raw materials A, B and C having the properties shown in the table can be used to make the alloy.

Property	**Raw material**		
	A	**B**	**C**
Specific gravity	**0.92**	**0.97**	**1.04**
Chromium	**7%**	**13%**	**16%**
Melting point	**440°C**	**490°C**	**480°C**

Cost of the various raw materials per unit ton are : Rs. 90 for A, Rs. 280 for B and Rs. 40 for C. Find the proportions in which A, B and C be used to obtain an alloy of desired properties while the cost of raw materials is minimum.

Solution : Let x_1, x_2 and x_3 be the tons of raw materials A,B and C to be used for making the alloy.

From these raw materials, the firm requires :

$$0.92\,x_1 + 0.97x_2 + 1.04\,x_3 \quad \text{specific gravity.}$$
$$7x_1 + 13x_2 + 16x_3 \quad \text{chromium}$$
$$440\,x_1 + 490\,x_2 + 480\,x_3 \quad \text{melting point.}$$

∴ By the given specifications, the constraints are

$$0.92\,x_1 + 0.97\,x_2 + 1.04\,x_3 \leq 0.98$$
$$7\,x_1 + 13\,x_2 + 16\,x_3 \geq 8$$
$$440\,x_1 + 490\,x_2 + 480\,x_3 \geq 450$$
$$\text{and } x_1, x_2, x_3 \geq 0.$$

Since the cost of the various raw materials per unit ton are : Rs. 90 for A, Rs. 280 for B and Rs. 40 for C, the total cost is

$90x_1 + 280x_2 + 40x_3$. As the objective is to minimize the total cost, the objective function is

$$\text{Minimize } Z = 90\,x_1 + 280\,x_2 + 40\,x_3$$

∴ The complete formulation of the LPP is

$$\text{Minimize } Z = 90\,x_1 + 280\,x_2 + 40\,x_3$$

subject to

$$0.92\,x_1 + 0.97\,x_2 + 1.04\,x_3 \leq 0.98$$
$$7\,x_1 + 13\,x_2 + 16\,x_3 \geq 8$$
$$440\,x_1 + 490\,x_2 + 480\,x_3 \geq 450$$
$$\text{and } x_1, x_2, x_3 \geq 0.$$

$\boxed{\text{Example 4}}$ **: (Diet Problem)**

A person wants to decide the constituents of a diet which will fulfil his daily requirements of proteins, fats and carbohydrates at the minmum cost. The choice is to be made from four different types of foods. The yields per unit of these foods are given in the following table:

Food type	Yield/unit			cost/unit (Rs)
	Proteins	Fats	carbohydrates	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum requirement	800	200	700	

Formulate the L.P model for the problem.

Solution : Let x_1, x_2, x_3 and x_4 be the units of food of type 1, 2, 3 and 4 used respectively.

From these units of food of type 1, 2, 3 and 4 he requires

$3x_1 + 4x_2 + 8x_3 + 6x_4$ Proteins/day

$2x_1 + 2x_2 + 7x_3 + 5x_4$ Fats / day

$6x_1 + 4x_2 + 7x_3 + 4x_4$ Carbohydrates/day

Since the minimum requirement of these proteins, fats and carbohydrates are 800, 200 and 700 respectively, the constraints are

$$3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$$
$$2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$$
$$6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$$
$$\text{and } x_1, x_2, x_3, x_4 \geq 0.$$

Since, the costs of these food of type 1, 2, 3 and 4 are Rs. 45, Rs.40, Rs.85 and Rs. 65 per unit, the total cost is Rs.$45x_1 + 40x_2 + 85x_3 + 65x_4$. As the objective is to minimize the total cost, the objective function is

$$\text{Minimize } Z = 45x_1 + 40x_2 + 85x_3 + 65x_4$$

∴ The complete formulation of the L.P.P is

$$\text{Minimize } Z = 45x_1 + 40x_2 + 85x_3 + 65\,x_4$$

subject to

$$3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$$
$$2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$$
$$6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$$
$$\text{and } x_1, x_2, x_3, x_4 \geq 0.$$

$\boxed{\text{Example 5}}$**: A farmer has 100 acre farm. He can sell all tomatoes, lettuce, or radishes he can raise. The price he can obtain is Rs.1.00 per kg for tomatoes, Rs. 0.75 a head for lettuce and Rs. 2.00 per kg for radishes. The average yield per - acre is 2,000 kgs of tomatoes, 3000 heads of lettuce, and 1000 kgs of radishes. Fertilizer is available at Rs.0.50 per kg and the amount required per acre is 100 kgs each for tomatoes and lettuce, and 50 kgs for radishes. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes, and 6 man-days for lettuce. A total of 400 man-days of labour are available at Rs. 20.00 per man-day**

Formulate this problem as a L.P. model to maximise the farmer's total profit.

Solution : Let the farmer decide to allot x_1, x_2 and x_3 acre of his farm to grow tomatoes, lettuce, and radishes respectively to maximize his total profit.

Since the total area of the form is restricted to 100 acre and the total man-days labour is restricted to 400 man-days, the constraints are

$$x_1 + x_2 + x_3 \leq 100$$
$$5x_1 + 6x_2 + 5x_3 \leq 400$$
$$\text{and } x_1, x_2 \geq 0$$

The Farmer will produce 2000 x_1 kgs of tomatoes, 3000 x_2 heads of lettuce, and 1000 x_3 kgs of radishes.

∴ The total sale of farmer will be

$$= \text{Rs } [1 \infty 2000\,x_1 + 0.75 \infty 3000\,x_2 + 2 \infty 1000\,x_3]$$

Fertilizer expenditure will be

$$= \text{Rs } 0.50\,[100\,x_1 + 100\,x_2 + 50\,x_3]$$

Labour expenditure will be $= \text{Rs } 20\,[5x_1 + 6x_2 + 5x_3]$

$\therefore$ Farmer's net profit will be

$$= \text{Rs [Sale – total expenditure]}$$
$$= \text{Rs}\,[\,2000\,x_1 + 2250\,x_2 + 2000\,x_3 - 50\,x_1 - 50\,x_2]$$
$$- 25x_3 - 100\,x_1 - 120\,x_2 - 100\,x_3]$$
$$= \text{Rs}\,[\,1850\,x_1 + 2080\,x_2 + 1875\,x_3]$$

$\therefore$ The objective function is

$$\text{Maximize } Z = 1850\,x_1 + 2080\,x_2 + 1875\,x_3$$

$\therefore$ The complete formulation of the L.P.P is

$$\text{Maximize } Z = 1850\,x_1 + 2080\,x_2 + 1875\,x_3$$

subject to the constraints

$$x_1 + x_2 + x_3 \leq 100$$
$$5x_1 + 6x_2 + 5x_3 \leq 400$$
$$\text{and } x_1, x_2, x_3 \geq 0.$$

Example 6 : **Old hens can be bought at Rs. 2 each and young ones at Rs.5 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week, each egg being worth 30 paise. A hen costs Rs. 1 per week to feed. A person has only Rs.80 to spend for hens. How many of each kind should he buy to give a profit of more than Rs.6 per week, assuming that he cannot house more than 20 hens. Formulate this as a L.P.P.** *[MU. BE. 89]*

Solution : The person decides to buy x_1 old hens and x_2 young hens to maximize his profit.

Since he has only Rs. 80 to spend for hens and old hen costs Rs. 2 and young hen costs Rs. 5 each,

$$2x_1 + 5x_2 \leq 80$$

Also, since he can not house more than 20 hens,

$$x_1 + x_2 \leq 20$$

The total sale of eggs will be

$$= \text{Rs. } 0.3\,(3x_1 + 5x_2)$$

Expenditure on feeding will be

$$= \text{Rs.} 1\,(x_1 + x_2)$$

$\therefore$ The net profit is $= \text{Rs. } [0.3(3x_1 + 5x_2) - 1(x_1 + x_2)]$

$$= \text{Rs. } (0.5x_2 - 0.1x_1)$$

$\therefore$ $$0.5x_2 - 0.1\,x_1 \geq 6.$$

$\therefore$ The constraints are

$$2x_1 + 5x_2 \leq 80$$
$$x_1 + x_2 \leq 20$$
$$0.5x_2 - 0.1\,x_1 \geq 6.$$
$$\text{and } x_1, x_2 \geq 0.$$

$\therefore$ The complete formulation of the L.P.P is

$$\text{Maximze } Z = 0.5x_2 - 0.1x_1$$

subject to the constraints

$$2x_1 + 5x_2 \leq 80$$
$$x_1 + x_2 \leq 20$$
$$0.5x_2 - 0.1\,x_1 \geq 6.$$
$$\text{and } x_1, x_2 \geq 0.$$

Example 7 : **A production planner in a soft drink plant has two bottling machines A and B. A is designed for 8–ounce bottles and B for 16-ounce bottles. However, each can be used on both types with some loss of efficiency. The following data is available :**

Machine	8–ounce bottles	16–ounce bottles
A	100/minute	40/minute
B	60/minute	75/minute

The machine can be run 8–hours per day, 5 days per week. Profit on 8–ounce bottle is 15 paise and on 16–ounce bottle is 25 paise. Weekly production of the drink cannot exceed 3,00,000 ounces and the market can absorb 25,000 eight–ounce bottles and 7000 sixteen–ounce bottles per week. The planner wishes to maximize his profit. Formulate this as a L.P.P. *[BRU. BE. Nov 96]*

Solution : Let the production planner decide to produce x_1 number of units of 8–ounce bottles and x_2 number of units of 16–ounce bottles to maximize his profit.

To produce these x_1 and x_2 units of 8–ounce and 16–ounce bottles, he requires

$$\frac{x_1}{100} + \frac{x_2}{40} \quad \text{minutes on machine A}$$

$$\frac{x_1}{60} + \frac{x_2}{75} \quad \text{minutes on machine B}$$

Since the machines A and B are available for 8–hours per day, 5–days per week, we have

$$\frac{x_1}{100} + \frac{x_2}{40} \leq 2400 \Rightarrow 2x_1 + 5x_2 \leq 4,80,000$$

$$\frac{x_1}{60} + \frac{x_2}{75} \leq 2400 \Rightarrow 5x_1 + 4x_2 \leq 7,20,000$$

Since the weekly production of the drink should not exceed 3,00,000 ounces and the market can absorb only upto 25,000 eight-ounce bottles and 7,000 sixteen–ounce bottles per week, we have

$$8x_1 + 16x_2 \leq 3,00,000$$

$$x_1 \leq 25,000 \text{ and } x_2 \leq 7000$$

∴ The constraints are :

$$\begin{aligned} 2x_1 + 5x_2 &\leq 4,80,000 \\ 5x_1 + 4x_2 &\leq 7,20,000 \\ 8x_1 + 16x_2 &\leq 3,00,000 \\ x_1 &\leq 25000 \\ x_2 &\leq 7000 \\ \text{and } x_1, x_2 &\geq 0. \end{aligned}$$

Since the profit on 8–ounce bottle is Rs. 0.15 and on 16-ounce bottle is Rs.0.25, the total profit is Rs.$0.15x_1 + 0.25x_2$. As the objective is to maximize the profit, the objective function is

$$\text{Maximize } Z = 0.15x_1 + 0.25x_2$$

∴ The complete formulation of the L.P.P is

$$\text{Maximize } Z = 0.15x_1 + 0.25\,x_2$$

subject to the constraints

$$\begin{aligned} 2x_1 + 5x_2 &\leq 4,80,000 \\ 5x_1 + 4x_2 &\leq 7,20,000 \\ 8x_1 + 16x_2 &\leq 3,00,000 \\ x_1 &\leq 25000 \\ x_2 &\leq 7000 \\ \text{and } x_1, x_2 &\geq 0. \end{aligned}$$

EXERCISE

1. A company makes two types of leather products A and B. Product A is of high quality and product B is of lower quality. The respective profits are Rs.4 and Rs.3 per product. Each product A requires twice as much time as product B and if all products were of type B, the company could make 1000 per day. The supply of leather is sufficient for only 800 products per day (Both A and B combined). Product A requires a special spare part and only 400 per day are available. There are only 700 special spare parts a day available for product B. Formulate this as a L.P.P.

2. A firm engaged in producing two models A and B performs three operations–painting, Assembly and testing. The relevant data are as follows:

Model	Unit sale Price	Hours required for each unit		
		Assembly	Painting	Testing
A	Rs.50	1.0	0.2	0.0
B	Rs.80	1.5	0.2	0.1

Total number of hours available are : Assembly 600, painting 100, testing 30. Determine weekly production schedule to maximize the profit.

3. A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs Rs.100 for preparation, requires 7 man-days of work and yields a profit Rs. 30. An acre of wheat costs Rs.120 to prepare, requires 10 man-days of work and yields a profit Rs.40. An acre of soyabeans costs Rs.70 to prepare, requires 8 man-days of work and yields a profit Rs.20. The farmer has Rs.1,00,000 for preparation and 8000 man-days of work. Formulate this as a L.P.P

4. A television company operates two assembly sections, section A and section B. Each section is used to assemble the components of three types of televisions : Colour, Standard and Economy. The expected daily production on each section is as follows :

T.V. Model	Section A	Section B
Colour	3	1
Standard	1	1
Economy	2	6

The daily running costs for two sections average Rs. 6000 for section A and Rs.4000 for section B. It is given that the company must produce atleast 24 colours, 16 standard and 40 Economy TV sets for which an order is pending. Formulate this as a L.P.P so as to minimize the total cost.

5. A transistor Radio company manufactures four models A, B, C and D which have profit contributions of Rs.8, Rs.15 and Rs.25 on models A, B and C respectively and a loss of Rs.1 on model D. Each type of radio requires a certain amount of time for the manufacturing of components for assembling and for packing. Specially a dozen units of model A require 1 hour of manufacturing, 2 hours for assembling and 1 hour for packing. The corresponding figures for a dozen units of model B are 2, 1 and 2 and for a dozen units of model C are 3, 5 and 1, while a dozen units of model D require 1 hour of packing only. During the forthcoming week, the company will be able to make available 15 hours of manufacturing, 20 hours of assembling and 10 hours of packing time. Formulate this as a L.P.P.

6. A soft drinks firm has two bottling plants, one located at P and the other at Q. Each plant produces three different soft drinks A, B and C. The capacities of the two plants in number of bottles per day, are as follows :

Products	Plants	
	P	Q
A	3000	1000
B	1000	1000
C	2000	6000

A market survey indicates that during the month of April, there will be a demand for 24000 bottles of A, 16000 bottles of B and 48000 bottles of C. The operating costs per day of running plants P and Q respectively are Rs.6000 and Rs. 4000. How many days should the firm run each plant in April so that the production cost is minimized?

7. Consider two different types of food stuffs A and B. Assume that these food stuffs contain vitamins V_1, V_2 and V_3 respectively. Minimum daily requirements of these vitamins are 1 mg. of V_1, 50 mg. of V_2 and 10 mg. of V_3. Suppose that the food stuff A contains 1 mg. of V_1, 100 mg. of V_2 and 10 mg. of V_3, whereas foodstuff B contains 1 mg. of V_1, 10 mg. of V_2 and 100 mg. of V_3. Cost of one unit of food stuff A is Rs.1 and that of B is Rs.1.5. Find the minimum cost diet that would supply the body atleast the minimum requirements of each vitamin. ✗

8. A company produces refrigerators in Unit I and heaters in Unit II. The two products are produced and sold on a weekly basis. The weekly production cannot exceed 25 in Unit I and 36 in Unit II, due to constraints 60 workers are employed. A refrigerator requires 2 man – week of labour, while a heater requires 1 man–week of labour. The profit available is Rs. 600 per refrigerator and Rs.400 per heater. Formulate the LPP problem. *[BNU. BE. Nov 96]*

ANSWERS

1. Maximize $Z = 4x_1 + 3x_2$

subject to
$$2x_1 + x_2 \le 1000$$
$$x_1 + x_2 \le 800$$
$$x_1 \le 400$$
$$x_2 \le 700$$
and $x_1, x_2 \ge 0$.

2. Maximize $Z = 50x_1 + 80x_2$

subject to
$$x_1 + 1.5x_2 \le 600$$
$$0.2x_1 + 0.2x_2 \le 100$$
$$0.1x_2 \le 30$$
and $x_1, x_2 \ge 0$.

3. Maximize $Z = 30x_1 + 40x_2 + 20x_3$

subject to
$$10x_1 + 12x_2 + 7x_3 \le 10000$$
$$7x_1 + 10x_2 + 8x_3 \le 8000$$
$$x_1 + x_2 + x_3 \le 1000$$
and $x_1, x_2, x_3 \ge 0$.

4. Minimize $Z = 6000x_1 + 4000x_2$

subject to
$$3x_1 + x_2 \geq 24$$
$$x_1 + x_2 \geq 16$$
$$2x_1 + 6x_2 \geq 40$$
and $x_1, x_2 \geq 0$.

5. Maximize $Z = 8x_1 + 15x_2 + 25x_3 - x_4$

subject to
$$x_1 + 2x_2 + 3x_3 \leq 15$$
$$2x_1 + x_2 + 5x_3 \leq 20$$
$$x_1 + 2x_2 + x_3 + x_4 \leq 10$$
and $x_1, x_2, x_3, x_4 \geq 0$.

6. Minimize $Z = 6000x_1 + 4000x_2$

subject to
$$3x_1 + x_2 \geq 24$$
$$x_1 + x_2 \geq 16$$
$$x_1 + 3x_2 \geq 24$$
and $x_1, x_2 \geq 0$.

7. Minimize $Z = x_1 + 1.5x_2$

subject to
$$x_1 + x_2 \geq 1$$
$$100x_1 + 10x_2 \geq 50$$
$$10x_1 + 100x_2 \geq 10$$
and $x_1, x_2 \geq 0$.

8. Maximize $Z = 600x_1 + 400x_2$

subject to
$$2x_1 + x_2 \leq 60$$
$$x_1 \leq 25$$
$$x_2 \leq 36$$
and $x_1, x_2 \geq 0$.

2.4 Basic Assumptions

[MU. MCA. Nov 98]

The linear programming problems are formulated on the basis of the following assumptions.

1. Proportionality : The contribution of each variable in the objective function or its usage of the resources is directly proportional to the value of the variable. *i.e.*, if resource availability increases by some percentage, then the output shall also increase by the same percentage.

2. Additivity : Sum of the resources used by different activities must be equal to the total quantity of resources used by each activity for all the resources individually or collectively.

3. Divisibility : The variables are not restricted to integer values.

4. Certainty or Deterministic : Co–efficients in the objective function and constraints are completely known and do not change during the period under study in all the problems considered.

5. Finiteness : Variables and constraints are finite in number.

6. Optimality : In a linear programming problem we determine the decision variables so as to extremise (optimize) the objective function of the LPP.

7. The problem involves only one objective namely profit maximization or cost minimization.

2.5 Graphical method of the solution of a L.P.P

Linear Programming problems involving only two variables can be effectively solved by a graphical method which provides a pictorial representation of the problems and its solutions and which gives the basic concepts used in solving general L.P.P. which may involve any finite number of variables. This method is simple to understand and easy to use. The redundant constraints are automatically eliminated from the system. Multiple solutions, unbounded solutions and infeasible solutions get highlighted very clearly in graphical analysis. Sensitivity analysis can be illustrated easily by drawing the graph of the changes.

Graphical method is not a powerful tool of linear programming as most of the practical situations do involve more than two variables. But the method is really useful to explain the basic concepts of L.P.P to the persons who are not familiar with this. Though graphical method can deal with any number of constraints but since each constraint is shown as a line on a graph, a large number of lines make the graph difficult to read.

Working procedure for Graphical method :

Given a L.P.P, optimize $Z = f(x_j)$ subject to the constraints $g_j(x_i) \leq, =, \geq b_j$ and the non-negativity restrictions $x_i \geq 0$, $i = 1, 2$; $j = 1,2,3......m$

Step 1 : Consider the inequality constraints as equalities. Draw the straight lines in the XOY plane corresponding to each equality and non–negativity restrictions.

Step 2 : Find the permissible region (feasible region or solution space or convex region) for the values of the variables which is the region bounded by the lines drawn in step 1.

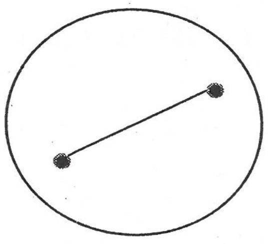
Step 3 : Find the points of intersection of the bounded lines (vertices of the permissible region) by solving the equations of the corresponding lines.

Step 4 : Find the values of Z at all vertices of the permissible region.

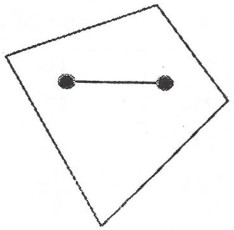
Step 5 : *(i)* For maximization problem, choose the vertex for which Z is maximum. *(ii)* For minimization problem, choose vertex for which Z is minimum.

Note : A ***region*** or a ***set*** of points is said to be ***convex*** if the line joining any two of its points lies completely within the region (or the set).

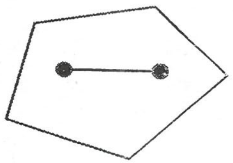
Example :



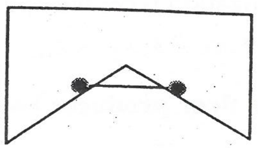
Convex



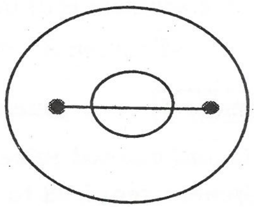
Convex



Convex



Not convex



Not convex

Example 1 : Solve the following L.P.P by the graphical method

$$\text{Max } Z = 3x_1 + 2x_2$$

subject to

$$-2x_1 + x_2 \leq 1$$
$$x_1 \leq 2$$
$$x_1 + x_2 \leq 3$$
$$\text{and } x_1, x_2 \geq 0$$

[MU. BE. Nov 94]

Solution : First consider the inequality constraints as equalities.

$$-2x_1 + x_2 = 1 \quad (1)$$
$$x_1 = 2 \quad (2)$$
$$x_1 + x_2 = 3 \quad (3)$$
$$\text{and } x_1 = 0 \quad (4)$$
$$x_2 = 0 \quad (5)$$

For the line, $-2x_1 + x_2 = 1$,

put $x_1 = 0 \Rightarrow x_2 = 1 \Rightarrow (0,1)$

put $x_2 = 0 \Rightarrow -2x_1 = 1 \Rightarrow x_1 = -0.5 \Rightarrow (-0.5, 0)$

So, the line (1) passes through the points (0,1) and (–0.5,0). The points on this line satisfy the equation $-2x_1 + x_2 = 1$ Now origin (0,0), on substitution, gives $-0 + 0 = 0 < 1$; hence it also satisfies the inequality $-2x_1 + x_2 \leq 1$. Thus all points on the origin side and on this line satisfy the inequality $-2x_1 + x_2 \leq 1$. Similarly interpreting the other constraints we get the feasible region OABCD. The feasible region is also known as solution space of the L.P.P. Every point within this area satisfies all the constraints.

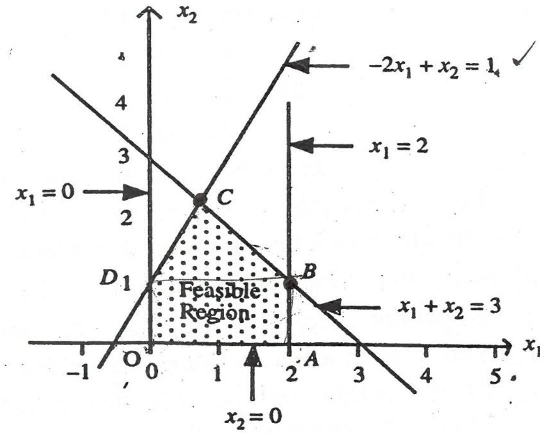


Fig. 2.1

Now our aim is to find the vertices of the solution space. B is the point of intersection of $x_1 = 2$ and $x_2 + x_2 = 3$. Solving these two equations, we have $x_1 = 2$, $x_2 = 1$. $\therefore$ We have the vertex B(2,1). Similarly, C is the intersectin of $-2x_1 + x_2 = 1$ and $x_1 + x_2 = 3$. Solving these we have C $\left(\frac{2}{3}, \frac{7}{3}\right)$.

$\therefore$ The vertices of the solution space are O (0,0), A (2,0), B (2,1), C $\left(\frac{2}{3}, \frac{7}{3}\right)$ and D (0,1)

The values of Z at these vertices are given by

Vertex	Value of Z
O (0, 0)	0
A(2,0)	6
B(2,1)	8
C $\left(\frac{2}{3}, \frac{7}{3}\right)$	$\frac{20}{3}$
D(0,1)	2

$(\because Z = 3x_1 + 2x_2)$

Since the problem is of maximization type, the optimum solution to the L.P.P is

$$\text{Maximum } Z = 8, x_1 = 2, x_2 = 1.$$

Example 2 : **Solve the following L.P.P by the graphical method**

$$\text{Minimize } Z = 3x_1 + 5x_2$$

subject to

$$-3x_1 + 4x_2 \leq 12$$
$$x_1 \leq 4$$
$$2x_1 - x_2 \geq -2$$
$$x_2 \geq 2$$
$$2x_1 + 3x_2 \geq 12 \text{ and } x_1, x_2 \geq 0.$$

Solution : Let us consider $-3x_1 + 4x_2 \leq 12$ as the equation $-3x_1 + 4x_2 = 12$, then it gives a line in the $x_1 0 x_2$ plane. All the points on the origin side and on this line satisfy the inequality $-3x_1 + 4x_2 \leq 12$. If we consider the constraint $2x_1 + 3x_2 \geq 12$ as the equation $2x_1 + 3x_2 = 12$, then it gives a line in the $x_1 0 x_2$ plane. All the points on the other side of the origin and on this line satisfy the inequality $2x_1 + 3x_2 \geq 12$. Similarly interpreting the other constraints, we get the feasible region ABCDE. Every point with in this area satisfies all the

constraints. The vertices of the solution space are A (3, 2), B (4, 2), C (4, 6) D $\left(\frac{4}{5}, \frac{18}{5}\right)$, and E $\left(\frac{3}{4}, \frac{7}{2}\right)$.

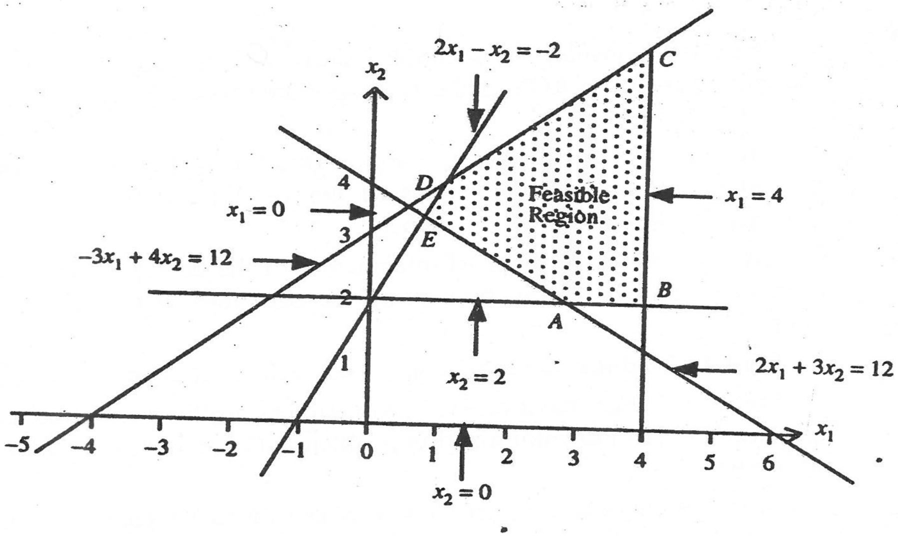


Fig 2.2

The values of Z at these vertices are given by

Vertex	Value of Z
A (3, 2)	19
B (4, 2)	22
C (4, 6)	42
D $\left(\frac{4}{5}, \frac{18}{5}\right)$	$\frac{102}{5}$
E $\left(\frac{3}{4}, \frac{7}{2}\right)$	$\frac{79}{4}$

$(\because Z = 3x_1 + 5x_2)$

Since the problem is of minimization type, the optimum solution is

$$\text{Minimum } Z = 19, x_1 = 3, x_2 = 2.$$

Example 3 : **A pineapple firm produces two products canned pineapple and canned juice. The specific amounts of material, labour and equipment required to produce each product and the availability of each of these resources are shown in the table given below :**

	Canned Juice	Canned Pineapple	Available resources
Labour (Man hours)	3	2.0	12.0
Equipment (M/c hours)	1	2.3	6.9
Material (Unit)	1	1.4	4.9

Assuming one unit of canned juice and canned Pineapple has profit margins Rs.2 and Rs.1 respectively. Formulate this as a L.P.P. and solve it graphically also. *[MU. BE. Nov 91]*

Solution : Let x_1 be the number of units of canned juice and x_2 be the number of units of canned pineapple to be produced.

The constraints or restrictions in this problem are the labour, equipment and material.

$$\begin{aligned} \text{For labour, } 3x_1 + 2x_2 &\leq 12 \\ \text{For Equipment, } x_1 + 2.3x_2 &\leq 6.9 \\ \text{For material, } x_1 + 1.4\,x_2 &\leq 4.9 \\ \text{and } x_1, x_2 &\geq 0. \end{aligned}$$

Here our objective is to maximize the profit.

$\therefore$ The objective function is Maximize $Z = 2x_1 + x_2$

$\therefore$ The complete formulation of the L.P.P.is Maximize $Z = 2x_1 + x_2$ subject to the constraints

$$\begin{aligned} 3x_1 + 2x_2 &\leq 12 \\ x_1 + 2.3x_2 &\leq 6.9 \\ x_1 + 1.4\,x_2 &\leq 4.9 \text{ and } x_1, x_2 \geq 0. \end{aligned}$$

The solution space is given below with the shaded area with vertices O (0, 0), A (0, 3). B (1.8, 2.2), C (3.2, 1.2) and D (4, 0)

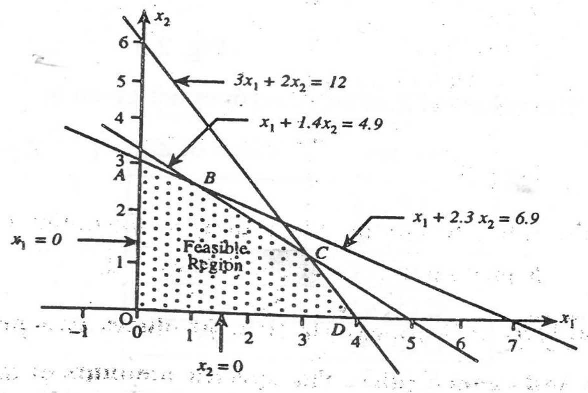


Fig. 2.3

The values of Z at these vertices are given by

Vertex	Value of Z
O (0, 0)	0
A (0, 3)	3
B (1.8, 2.2)	5.8
C (3.2, 1.2)	7.6
D (4,0)	8

($\because$ $Z = 2x_1 + x_2$)

Since the problem is of maximization type, the optimum solution is

$$\text{Maximum } Z = 8, x_1 = 4, x_2 = 0.$$

Example 4 : A Company manufactures 2 types of printed circuits. The requirements of transistors, resistors and capacitors for each type of printed circuits along with other data are given below :

	Circuit		Stock available
	A	B	
Transistor	15	10	180
Resistor	10	20	200
Capacitor	15	20	210
Profit	Rs.5	Rs.8	

How many circuits of each type should the company produce from the stock to earn maximum profit. *[MU. BE. Oct 95]*

Solution : Let x_1 be the number of type A circuits and x_2 be the number of type B circuits to be produced.

To produce these units of type A and type B circuits, the company requires

$$\begin{aligned} \text{Transistors} &= 15x_1 + 10x_2 \\ \text{Resistor} &= 10x_1 + 20x_2 \\ \text{Capacitors} &= 15x_1 + 20x_2 \end{aligned}$$

Since the availability of these transistors, resistors and capacitors are 180, 200 and 210 respectively, the constraints are

$$\begin{aligned} 15x_1 + 10x_2 &\leq 180 \\ 10x_1 + 20x_2 &\leq 200 \\ 15x_1 + 20\,x_2 &\leq 210 \\ \text{and } x_1 \geq 0, \; x_2 &\geq 0. \end{aligned}$$

Since the profit from type A is Rs.5 and from type B is Rs.8 per units, the total profit is $5x_1 + 8x_2$

$\therefore$ The complete formulation of the L.P.P.is

$$\begin{aligned} \text{Maximize } Z &= 5x_1 + 8x_2 \\ \text{subject to} \quad 15x_1 + 10x_2 &\leq 180 \\ 10x_1 + 20x_2 &\leq 200 \\ 15x_1 + 20x_2 &\leq 210 \\ \text{and } x_1, x_2 &\geq 0. \end{aligned}$$

By using graphical method, the solution space is given below with shaded area OABCD with vertices O (0, 0), A (12, 0), B (10, 3), C (2, 9) and D (0, 10)

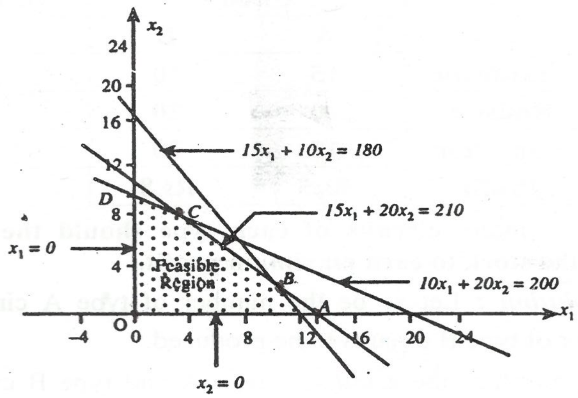


Fig. 2.4

The values of Z of these vertices are given by

Vertex	Value of Z
O (0, 0)	0
A (12, 0)	60
B (10, 3)	74
C (2, 9)	82
D (0,10)	80

($\because$ $Z = 5x_1 + 8x_2$)

Since the problem is of maximization type, the optimum solution is

Maximum $Z = 82$, $x_1 = 2$, $x_2 = 9$.

Example 5 : Apply graphical method to solve the LPP: Maximise $Z = x_1 - 2x_2$ subject to $-x_1 + x_2 \leq 1$, $6x_1 + 4x_2 \geq 24$, $0 \leq x_1 \leq 5$ and $2 \leq x_2 \leq 4$. *[MU.MBA.Nov.96]*

Solution : By using graphical method, the solution space is given below with shaded area ABCDE with vertices A $\left(\frac{8}{3}, 2\right)$, B (5, 2), C (5, 4), D (3, 4) and E (2, 3)

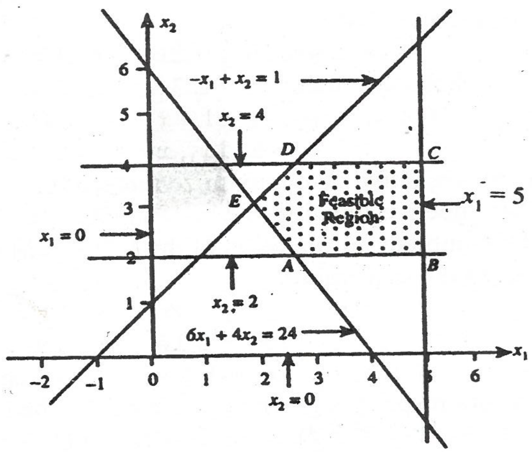


Fig. 2.5

The values of Z at these vertices are given by

Vertex	Value of Z
A $\left(\frac{8}{3}, 2\right)$	$-\frac{4}{3}$
B (5, 2)	1
C (5, 4)	– 3
D (3, 4)	– 5
E (2, 3)	– 4

($\because$ $Z = x_1 - 2x_2$)

Since the problem is of maximization type, the optimal solution

Maximum $Z = 1$, $x_1 = 5$, $x_2 = 2$.

Example 6 : **A Company manufactures two types of cloth, using three different colours of wool. One yard length of type A cloth require 4 OZ of red wool, 5 OZ of green wool and 3 OZ of yellow wool. One yard length of type B cloth requires 5 OZ of red wool, 2 OZ of green wool and 8 OZ of yellow wool. The wool available for manufacturer is 1000 OZ of red wool, 1000 OZ of green wool and 1200 OZ of yellow wool. The manufacturer can make a profit of Rs. 5 on one yard of type A cloth and Rs.3 on one yard of type B cloth. Find the best combination of the quantities of type A and type B cloth which gives him maximum profit by solving the L.P.P. graphically.**

[MU. BE. Apr 92]

Solution : Let the manufacturer decide to produce x_1 yards of type A cloth and x_2 yards of type B cloth.

To produce these yards of type A and type B cloth, he requires

$$\begin{aligned} \text{red wool} &= 4x_1 + 5x_2 \text{ OZ} \\ \text{green wool} &= 5x_1 + 2x_2 \text{ OZ} \\ \text{yellow wool} &= 3x_1 + 8x_2 \text{ OZ} \end{aligned}$$

Since the availability of these red wool, green wool and yellow wool are 1000 OZ, 1000 OZ and 1200 OZ respectively, the constraints are

$$\begin{aligned} 4x_1 + 5x_2 &\leq 1000 \\ 5x_1 + 2x_2 &\leq 1000 \\ 3x_1 + 8x_2 &\leq 1200 \\ \text{and } x_1 \geq 0, \ x_2 &\geq 0. \end{aligned}$$

Since the profit from one yard of type A cloth is Rs.5 and the profit from one yard of type B cloth is Rs. 3, the total profit is $5x_1 + 3x_2$

∴ The complete formulation of the L.P.P.is

$$\begin{aligned} \text{Maximize } Z &= 5x_1 + 3x_2 \\ \text{subject to} \quad 4x_1 + 5x_2 &\leq 1000 \\ 5x_1 + 2x_2 &\leq 1000 \\ 3x_1 + 8x_2 &\leq 1200 \\ \text{and } x_1, x_2 &\geq 0. \end{aligned}$$

By using graphical method, the feasible region is given below with shaded area OABCD with vertices O (0, 0), A (200, 0), B $\left(\frac{3000}{17}, \frac{1000}{17}\right)$, C $\left(\frac{2000}{17}, \frac{1800}{17}\right)$ and D (0, 150)

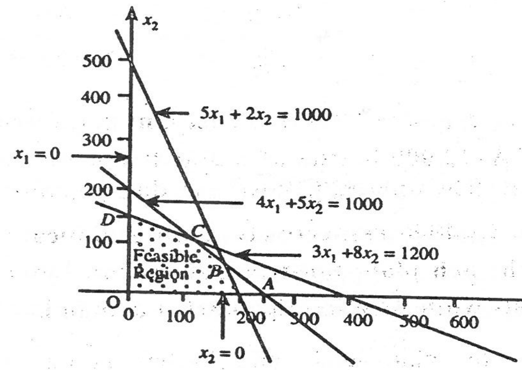


Fig 2.6

The values of Z at these vertices are given by

Vertex	Value of Z
O (0, 0)	0
A (200, 0)	1000
B $\left(\frac{3000}{17}, \frac{1000}{17}\right)$	1058.8
C $\left(\frac{2000}{17}, \frac{1800}{17}\right)$	905.8
D (0, 150)	450

($\because$ $Z = 5x_1 + 3x_2$)

Since the problem is of maximization type, the optimum solution is

$$\text{Maximum } Z = 1058.8, \ x_1 = \frac{3000}{17}, \ x_2 = \frac{1000}{17}.$$

Note : In a given L.P.P, if any constraint does not affect the feasible region (or solution space), then the constraint is said to be a ***redundant constraint.***

Example 7 : A Company making cold drinks has two bottling plants located at towns T_1 and T_2. Each plant produces three drinks A, B and C and their production capacity per day is given below :

Cold drinks	Plant at	
	T_1	T_2
A	6000	2000
B	1000	2500
C	3000	3000

The marketing department of the company forecasts a demand of 80,000 bottles of A, 22,000 bottles of B and 40,000 bottles of C during the month of June. The operating costs per day of plants at T_1 and T_2 are Rs. 6000 and Rs.4000 respectively. Find graphically, the number of days for which each plant must be run in June so as to minimize the operating costs while meeting the market demand.

Solution : Let the plant at T_1 and T_2 be run for x_1 and x_2 days respectively.

Since the plants at T_1 and T_2 run for x_1 and x_2 days, they will produce,

$6000\,x_1 + 2000\,x_2$ bottles of A

$1000\,x_1 + 2500\,x_2$ bottles of B

$3000\,x_1 + 3000\,x_2$ bottles of C

Since the demand for the cold drinks A, B and C are 80,000, 22,000 and 40,000 respectively and the production is always greater than or equal to the demand, the constraints are

$$6000\,x_1 + 2000\,x_2 \geq 80000 \Rightarrow 6x_1 + 2x_2 \geq 80$$

$$\Rightarrow 3x_1 + x_2 \geq 40$$

$$1000\,x_1 + 2500\,x_2 \geq 22000 \Rightarrow x_1 + 2.5x_2 \geq 22$$

$$3000\,x_1 + 3000\,x_2 \geq 40000 \Rightarrow 3x_1 + 3x_2 \geq 40$$

$$\text{and } x_1, x_2 \geq 0.$$

Since the operating costs per day at T_1 is Rs. 6000 and at T_2 is Rs. 4000 and T_1, T_2 run for x_1 and x_2 days, the total operating cost is Rs. $6000x_1 + 4000x_2$.

Here our objective is to minimize the total operating cost. Therefore the objective function is minimize $Z = 6000\,x_1 + 4000\,x_2$.

$\therefore$ The complete formulation of the L.P.P.is

$$\text{Minimize } Z = 6000x_1 + 4000x_2$$

subject to

$$3x_1 + x_2 \geq 40$$
$$x_1 + 2.5x_2 \geq 22$$
$$3x_1 + 3x_2 \geq 40$$
$$\text{and } x_1, x_2 \geq 0.$$

By using graphical method, the feasible region is given below with shaded area with vertices A (22, 0), B (12, 4), C (0, 40)

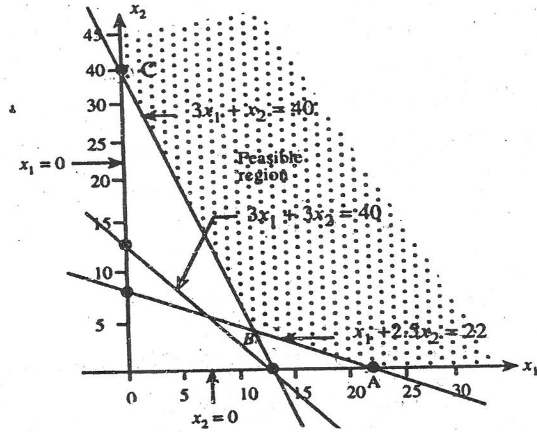


Fig 2.7

From the figure, we see that the constraint $3x_1 + 3x_2 \geq 40$ does not affect the solution space. So $3x_1 + 3x_2 \geq 40$ is a ***redundant constraint.*** Also from the direction of the arrows, we see that the solution space is unbounded above.

The values of Z at these vertices A (22,0), B(12,4) and C(0,40) are given by

Vertex	Value of Z
A (22, 0)	1,32,000
B (12, 4)	88,000
C (0, 40)	1,60,000

$(\because Z = 6000x_1 + 4000x_2)$

Since the problem is of minimization type, the optimum solution is

$$\text{Minimum } Z = \text{Rs. } 88{,}000,\ x_1 = 12 \text{ days},\ x_2 = 4 \text{ days.}$$

Note : From the above examples, for problems involving two variables and having a finite solution, we observed that the optimal solution existed at a vertex of the feasible region. That is, ***"if there exists an optimal solution of an L.P.P, it will be at one of the vertices of the feasible region".***

2.6 Some more cases

The constraints generally, give region of feasible solution which may be bounded or unbounded. We discussed seven linear programming problems and the optimal solution for either of them was unique. However, it may not be true for every problem. In general, a linear programming problem may have :

(i) a unique optimal solution

(ii) an infinite number of optimal solutions

(iii) an unbounded solution

(iv) no solution

We now give a few examples to illustrate the cases.

Example 8 : A firm manufactures two products A and B on which the profits earned per unit are Rs.3 and Rs.4 respectively. Each product is processed on two machines M_1 and M_2. Product A requires one minute of processing time on M_1 and two minutes on M_2 while B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 7 hours 30 minutes while machine M_2 is available for 10 hours during any working day. Find the number of units of products A and B to be manufactured to get maximum profit. Formulate the above as a L.P.P. and solve by graphical method. *[MU. BE. Apr 92, MSU. BE. Nov 96]*

Solution : Let the firm decide to manufacture x_1 units of product A and x_2 units of product B.

To produce these units of products A and B, it requires

$x_1 + x_2$ hours of processing times on M_1

$2x_1 + x_2$ hours of processing times on M_2

But the availability of these two machines M_1 and M_2 are 450 minutes and 600 minutes respectively, the constraints are

$$x_1 + x_2 \le 450$$
$$2x_1 + x_2 \le 600$$
$$\text{and } x_1, x_2 \ge 0$$

Since the profit from product A is Rs.3 per unit and from product B is Rs. 4 per unit, the total profit is Rs. $3x_1 + 4x_2$ and our objective is to maximize the profit.

$\therefore$ The complete formulation of the L.P.P is

$$\text{Maximize } Z = 3x_1 + 4x_2$$

subject to

$$x_1 + x_2 \le 450 \quad \ldots (i)$$
$$2x_1 + x_2 \le 600 \quad \ldots (ii)$$
$$\text{and } x_1, x_2 \ge 0 \quad \ldots (iii)$$

By graphical method, the solution space satisfying the constraints *(i)*, *(ii)* and meeting the non-negativity restriction *(iii)* is shown shaded in the following figure.

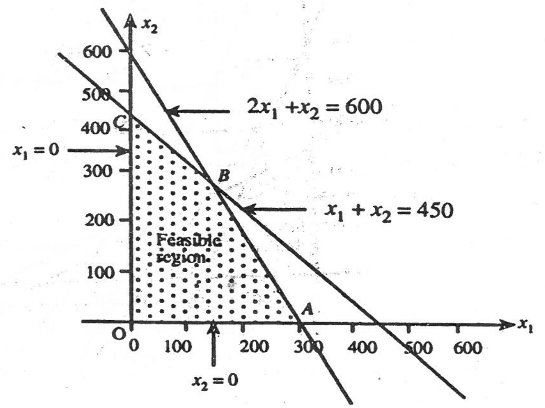


Fig. 2.8

The solution space is the region OABC. The vertices of this solution space are O (0, 0), A (300, 0), B (150, 300) and C (0, 450)

The values of Z at these vertices are given by

Vertex	Value of Z
O (0, 0)	0
A (300, 0)	900
B (150, 300)	1650
C (0, 450)	1800

$(\because Z = 3x_1 + 4x_2)$

Since the problem is of maximization type and the maximum value of Z is attained at a single vertex, this problem has a ***unique optimal solution.***

$\therefore$ The optimal solution is

$$\text{Maximum } Z = 1800, x_1 = 0, x_2 = 450.$$

Example 9 : Solve the following L.P.P. graphically.

$$\text{Maximize } Z = 100x_1 + 40x_2$$

subject to

$$5x_1 + 2x_2 \le 1000$$
$$3x_1 + 2x_2 \le 900$$
$$x_1 + 2x_2 \le 500$$
$$\text{and } x_1, x_2 \ge 0.$$

Solution : By using graphical method, the solution space OABC shown shaded in the following figure.

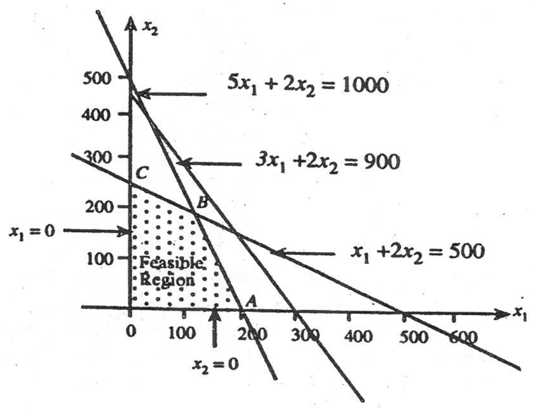


Fig. 2.9

The vertices of this convex region are O (0, 0), A (200, 0), B (125, 187.5) and C (0, 250)

The values of Z at these vertices are given by

Vertex	Value of Z
O (0, 0)	0
A (200, 0)	20,000
B (125, 187.5)	20,000
C (0, 250)	10,000

$(\because Z = 100x_1 + 40x_2)$

Here the maximum value of Z occurs at two vertices A and B.

Any point on the line joining A and B will also give the same maximum value of Z.

Since, there are infinite number of points between any ponits, there are infinite number of points on the line joining A and B gives the same maximum value of Z.

Thus, there are ***infinite number of optimal solutions*** for this L.P.P.

Note : An L.P.P having more than one optimal solution is said to have ***alternative*** or ***multiple optimal solutions.*** That is, the resources can be combined in more than one way to maximize the profit.

Example 10 : Using graphical method, solve the following L.P.P.

$$\textbf{Maximize } Z = 2x_1 + 3x_2$$

subject to

$$x_1 - x_2 \le 2$$
$$x_1 + x_2 \ge 4 \text{ and } x_1, x_2 \ge 0.$$

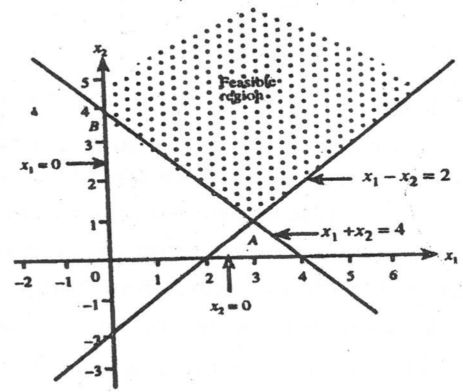


Fig. 2.10

Solution : By using graphical method, the solution space is shaded in the following figure.

Here the solution space is unbounded. The vertices of the feasible region (in the finite plane) are A (3,1) and B (0,4)

Valule of the objective function $Z = 2x_1 + 3x_2$ at these vertices are $Z(A) = 9$ and $Z(B) = 12$.

But there are points in this convex region for which Z will have much higher values. In fact, the maximum value of Z occurs at infinity. Hence this problem has an ***unbounded solution.***

Example 11 : Solve graphically the following L.P.P.:

$$\text{Maximize } Z = 4x_1 + 3x_2$$

subject to

$$x_1 - x_2 \le -1 \quad \ldots (i)$$

$$-x_1 + x_2 \le 0 \quad \ldots (ii)$$

$$\text{and } x_1, x_2 \ge 0 \quad \ldots (iii)$$

Solution : Any point satisfying the non–negativity restrictions *(iii)* lies in the first quadrant only. The two solution spaces, one satisfying *(i)*, and other satisfying *(ii)* are shown shaded in the following figure.

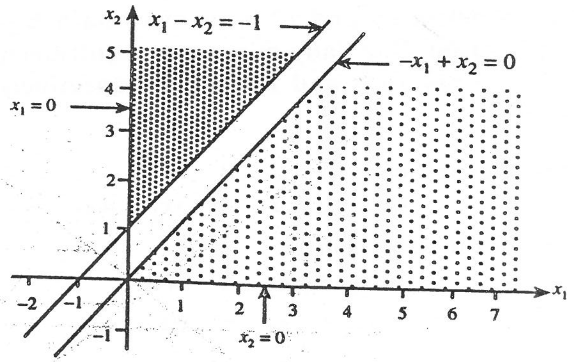


Fig. 2.11

There being no point (x_1, x_2) common to both the shaded regions. That is, we can not find a convex region for this problem. So the problem cannot be solved. Hence the problem have ***no feasible solution.***

2.7 Advantage of Linear Programming :

1. It provides an insight and perspective in to the problem environment. This generally results in clear picture of the true problem.
2. It makes a scientific and mathematical analysis of the problem situations.
3. It gives an opportunity to the decision maker to formulate his strategies consistent with the constraints and the objectives.
4. It deals with changing situations. Once a plan is arrived through the linear programming it can also be revaluated for changing conditions.
5. By using linear programming the decision maker makes sure that he is considering the best solution.

2.8 Limitations of Linear Programming : *[MU. MCA. Nov 98]*

1. The major limitation of linear programming is that it treats all relationships as linear. But it is not true in many real life situations.
2. The decision variables in some LPP would be meangingful only if they have integer values. But some times we get fractional values to the optimal solution, where only integer values are meaningful.
3. All the parameters in the linear programming model are assumed to be known constants. But in real life they may not be known completely or they may be probabilistic and they may be liable for changes from time to time.
4. The problems are complex if the number of variables and constraints are quite large.
5. Linear Programming deals with only a single objective problems, whereas in real life situations, there may be more than one objective.

EXERCISE

1. **Explain the essential characteristics and limitations of Linear Programming Problem.** ***[MU. MCA. Nov 98]***
2. **What is feasibility region in an LP problem ? Is it necessary that it should always be a convex set ?** ***[MU. MCA. Nov. 97]***
3. **What is a redundant constraint ? What does it imply ? Does it affect the optimal solution to an LPP ?** ***[MU. MCA. Nov. 97]***
4. **Explain the advantages of linear programming problems.**
5. **State the limitations of the graphical method of solving a LPP.** ***[BRU. BE. Apr. 97, Nov. 97]***

6. Solve the following by graphical method :

Maximize $x_1 - 3x_2$

subject to
$$\begin{aligned} x_1 + x_2 &\leq 300 \\ x_1 - 2x_2 &\leq 200 \\ 2x_1 + x_2 &\geq 100 \\ x_2 &\leq 200 \\ \text{and } x_1, x_2 &\geq 0. \end{aligned}$$
[MU. BE. Apr 94]

7. By graphical method solve the following problem :

Maximize $Z = 3x_1 + 4x_2$

subject to
$$\begin{aligned} 5x_1 + 4x_2 &\leq 200 \\ 3x_1 + 5x_2 &\leq 150 \\ 5x_1 + 4x_2 &\geq 100 \\ 8x_1 + 4x_2 &\geq 80 \\ \text{and } x_1, x_2 &\geq 0. \end{aligned}$$
[MU. BE. Nov 93]

8. Solve the following problem by graphical method :

Maximize $5x + 8y$

subject to the constraints
$$\begin{aligned} 3x + 2y &\leq 36 \\ x + 2y &\leq 20 \\ 3x + 4y &\leq 42 \\ \text{and } x &\geq 0 \\ y &\geq 0 \end{aligned}$$
[MU. BE. Apr 93]

9. Maximize $2x_1 + x_2$

subject to
$$\begin{aligned} 3x_1 + 2x_2 &\leq 12.0 \\ x_1 + 2.3x_2 &\leq 6.9 \\ x_1 + 1.4x_2 &\leq 4.9 \\ \text{and } x_1, x_2 &\geq 0 \end{aligned}$$
Using graphical method. *[MU. BE. Apr 91]*

10. Using graphical method to solve L.P.P :

Minimize $Z = 3x_1 + 2x_2$

subject to
$$\begin{aligned} 5x_1 + x_2 &\geq 10 \\ x_1 + x_2 &\geq 6 \\ x_1 + 4x_2 &\geq 12 \\ \text{and } x_1, x_2 &\geq 0 \end{aligned}$$
[MU. BE. Nov 93]

11. Solve the following problem graphically.

Maximize $Z = 2x_1 + x_2$

subject to
$$\begin{aligned} 3x_1 + 2x_2 &\leq 12.0 \\ x_1 + 2x_2 &\leq 7 \\ x_1 + x_2 &\leq 5 \\ \text{and } x_1, x_2 &\geq 0 \end{aligned}$$
[MU. BE. Apr 90, Apr 91]

12. Use graphical method to Maximize $Z = 6x_1 + 4x_2$

subject to
$$\begin{aligned} -2x_1 + x_2 &\leq 2 \\ x_1 - x_2 &\leq 2 \\ 3x_1 + 2x_2 &\leq 9 \\ x_1 \geq 0, \ x_2 &\geq 0 \end{aligned}$$
[MU. BE. Apr 90]

13. Minimize $Z = x - 3y$

subject to the constraints
$$\begin{aligned} x + y &\leq 300 \\ x - 2y &\leq 200 \\ 2x + y &\geq 100 \\ y &\geq 200 \\ \text{and } x, y &\geq 0 \end{aligned}$$
by graphical method. *[MU. BE. Apr 93]*

14. Egg contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and cost 12 paise per gram. Milk contains 8 units of vitamin A per gram and 12 units of vitamin B per gram, and costs 20 paise per gram. The daily minimum requirement of vitamin A and vitamin B are 100 units and 120 units respectively. Find the optimal product mix. *[MU. BE. Nov 93]*

15. Solve graphically the L.P.P

Maximize $Z = 3x_1 + 4x_2$

subject to the constraints
$$\begin{aligned} 2x_1 + 5x_2 &\leq 120 \\ 4x_1 + 2x_2 &\leq 80 \\ \text{and } x_1, x_2 &\geq 0 \end{aligned}$$
[MU. BE. Nov 92]

16. A company produces 2 types of hats. Each hat A require twice as much labour time as the second hat B. If all are of hat B only, the company can produce a total of 500 hats a day. The market limits daily sales of the hat A and hat B to 150 and 250 hats. The profits on hat A and B are Rs. 8 and Rs.5 respectively. Solve graphically to get the optimal solution . *[MU. BE. Apr 95]*

17. Solve graphically the following L.P.P :

$$\text{Maximize } Z = 5x_1 + 3x_2$$

subject to constraints

$$3x_1 + 5x_2 \leq 15$$
$$5x_1 + 2x_2 \leq 10$$
$$\text{and } x_1, x_2 \geq 0$$

[MU. BE. Apr 93]

18. Solve graphically the following L.P.P.

$$\text{Minimize } Z = 20x_1 + 10x_2$$

subject to

$$x_1 + 2x_2 \leq 40$$
$$3x_1 + x_2 \geq 30$$
$$4x_1 + 3x_2 \geq 60$$
$$\text{and } x_1, x_2 \geq 0$$

19. Solve graphically the following L.P.P.

$$\text{Max } Z = 3x + 2y$$

subject to

$$-2x + 3y \leq 9$$
$$x - 5y \geq -20$$
$$\text{and } x, y \geq 0$$

20. Solve graphically the following L.P.P :

$$\text{Minimize } Z = -6x_1 - 4x_2$$

subject to

$$2x_1 + 3x_2 \geq 30$$
$$3x_1 + 2x_2 \leq 24$$
$$x_1 + x_2 \geq 3$$
$$\text{and } x_1, x_2 \geq 0$$

21. Solve graphically the following L.P.P.

$$\text{Maximize } Z = 3x_1 - 2x_2$$

subject to

$$x_1 + x_2 \leq 1$$
$$2x_1 + 2x_2 \geq 4$$
$$\text{and } x_1, x_2 \geq 0$$

22. Solve graphically the following L.P.P.

$$\text{Maximize } Z = x_1 + x_2$$

subject to

$$x_1 - x_2 \geq 0$$
$$-3x_1 + x_2 \geq 3$$
$$\text{and } x_1, x_2 \geq 0$$

23. A manufacturer of furniture makes two products, chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machines B. A table requires 5 hours on machine A and no time on machine B. There are 16 hours of time per day available on machine A and 30 hours on machine B. Profit gained by the manufacturer from a chair and a table is Rs. 2 and Rs. 10 respectively. What should be the daily production of each of the products.

24. A person requires 10, 12 and 12 units of chemicals A, B and C respectively for his garden. A liquid product contains 5, 2 and 1 units of A, B and C respectively per Jar. A dry product contains 1,2 and 4 units of A, B and C per carton. How many of each should he purchase in order to minimize the cost and meet the requirement.

25. Solve the following problem graphically,

$$\text{Maximize } Z = 40x_1 + 100x_2$$

subject to

$$12x_1 + 6x_2 \leq 3000$$
$$4x_1 + 10x_2 \leq 2000$$
$$2x_1 + 3x_2 \leq 900$$
$$\text{and } x_1, x_2 \geq 0$$

[MU. B. Tech. Leather Oct 96]

26. A garment manufacturing company can make two products, Prima and Seconda. Each of the products requires time on a cutting machine and a finishing Machine. Relevant data are

	Product	
	Prima	seconda
Cutting hours (per unit)	2	1
Finishing hours (per unit)	3	3
Unit cost Rs.	128	120
Selling price Rs.	134	129
Maximum sales (units per week)	200	200

The number of cutting hours available per week is 390 and the number of finishing hours available per week is 810. How much should each product be produced in order to maximize the profit ?

[BRU. BE. Apr 94]

27. A firm manufactures refrigerators and air coolers. Production takes place in two separate departments. Refrigerators are produced in department I and air coolers are produced in department II. The firm's two products are produced and sold on a weekly basis. The weekly pro-duction cannot exceed 25 refrigerators in department I and 35 air coolers in department II, because of the limited available facilities in these two departments. The firm regularly employs a total

of 60 workers in two departments. A refrigerator requires 2 man–weeks of labour, while an air coolers requires 1 man–week of labour. The firm receives a profit margin of Rs.300 and Rs.200 per refrigerator and cooler respectively. Determine the product mix in order to maximize profit. *[BRU. BE. Nov 94]*

28. The production and planning department of a soft drink plant faces the following problem. The bottling plant has two bottling machines A and B. A is designed for 80cc bottle and B for 160 cc bottles. However each can be used on both types with loss of efficiency. The following data is available.

Machine	80 cc bottle	160 cc bottle
A	100 per min	40 per min
B	60 per min	75 per min

The machine can run 8 hour per day 5 days per week. Profit on 80 cc bottle is 15 paise and 160 cc bottle is 25 paise. Weekly production on the drink cannot exceed 3 million cc and the market can absorb 250000 of 80 cc bottles and 70000 of 160 cc bottles per week. The department wishes to maximize the profit subject to production and marketing restrictions. Solve the problem by graphical method or by simplex method. *[BRU. BE. Nov 96]*

29. A company manufactures two products A and B. Each unit of B takes twice as long to produce as one unit of A and if the company were to produce only A it would have time to produce 2000 units per day. The availability of the raw material is sufficient to produce 1500 units per day of both A and B combined. Product B requiring a special ingredient only 600 units can be made per day. If A fetches a profit of Rs. 2 per unit and B a profit of Rs. 4 per unit find the optimum product mix by graphical method. *[BRU. BE. Apr 97]*

30. A company produces two different products A and B. The company makes a profit of Rs. 40 and Rs. 30 per unit on A and B respectively. The production process has a capacity of 30,000 man hours. It takes 3 hours to produce one unit of A and one hour to produce one unit of B. The market survey indicates that the maximum number of units A that can be sold is 8,000 and those of B is 12,000 units. Formulate the problem and solve it by graphical method to get maximum profit. *[MU. BE. Apr 97]*

31. Apply graphical method to find non–negative values of x_1 and x_2 which minimise $z = 10x_1 + 25x_2$ subject to $x_1 + x_1 \geq 50$, $x_1 \geq 20$, and $x_2 \leq 40$. *[M.U. M.B.A. Apr '97]*

ANSWERS

6. Max Z $= 200, x_1 = 200, x_2 = 0.$
7. Max Z $= \frac{1800}{13}, x_1 = \frac{400}{13}, x_2 = \frac{150}{13}$
8. Max Z $= 82, x = 2, y = 9.$
9. Max Z $= 8, x_1 = 4, x_2 = 0.$
10. Min Z $= 13, x_1 = 1, x_2 = 5.$ 11. Max Z $= 8, x_1 = 4, x_2 = 0.$
12. An infinite number of solutions with Max Z = 18,

    (i) $x_1 = \frac{13}{5}, x_2 = \frac{3}{5}$ (ii) $x_1 = \frac{5}{7}, x_2 = \frac{24}{7}$ etc.
13. Min Z $= -600, x = 0, y = 200.$
14. Min Z $= 12x_1 + 20x_2$

    subject to
    $$6x_1 + 8x_2 \geq 100$$
    $$7x_1 + 12x_2 \geq 120$$
    $$\text{and } x_1, x_2 \geq 0.$$
    Also Min Z $= 205, x_1 = 15, x_2 = 1.25$
15. Max Z $= 110, x_1 = 10, x_2 = 20$
16. Max Z $= 8x_1 + 5x_2$

    subject to
    $$2x_1 + x_2 \leq 500$$
    $$x_1 \leq 150$$
    $$x_2 \leq 250$$
    $$\text{and } x_1, x_2 \geq 0.$$
    Also Max Z $= 2250, x_1 = 125, x_2 = 250$
17. Max Z $= \frac{235}{19}, x_1 = \frac{20}{19}, x_2 = \frac{45}{19}$
18. Min Z $= 240, x_1 = 6, x_2 = 12$
19. Unbounded solution
20. An infinite number of optimal solutions with Min Z = –48

    (i) $x_1 = 8, x_2 = 0$

    (ii) $x_1 = \frac{12}{5}, x_2 = \frac{42}{5}$, *etc ......*
21. No feasible solution

22. No feasible solution

23. Max $Z = 2x_1 + 10x_2$

subject to
$$2x_1 + 5x_2 \le 16$$
$$6x_1 \le 30$$
$$\text{and } x_1, x_2 \ge 0.$$

Also Max $Z = 32, x_1 = 0, x_2 = 3.2$.

24. Min $Z = 3x_1 + 2x_2$

subject to
$$5x_1 + x_2 \ge 10$$
$$2x_1 + 2x_2 \ge 12$$
$$x_1 + 4x_2 \ge 12$$
$$\text{and } x_1, x_2 \ge 0.$$

Also Min $Z = 13, x_1 = 1, x_2 = 5$.

25. An infinite number of optimal solutions with Max $Z = 20{,}000$

(i) $x_1 = 0, x_2 = 200$

(ii) $x_1 = \frac{375}{2}, x_2 = 125$, *etc.*

26. Max $Z = 6x_1 + 9x_2$

subject to
$$2x_1 + x_2 \le 390$$
$$3x_1 + 3x_2 \le 810$$
$$x_1 \le 200$$
$$x_2 \le 200 \text{ and } x_1, x_2 \ge 0.$$

Also Max $Z = 2220, x_1 = 70, x_2 = 200$.

27. Max $Z = 300x_1 + 200x_2$

subject to
$$2x_1 + x_2 \le 60$$
$$x_1 \le 25$$
$$x_2 \le 35 \quad \text{and } x_1, x_2 \ge 0.$$

Also Max $Z = \text{Rs.}10750, x_1 = \frac{25}{2}, x_2 = 35$.

28. Max $Z = 0.15x_1 + 0.25x_2$

subject to
$$2x_1 + 5x_2 \le 4{,}80{,}000$$
$$5x_1 + 4x_2 \le 7{,}20{,}000$$

$$80x_1 + 160x_2 \le 3{,}00{,}000$$
$$x_1 \le 2{,}50{,}000$$
$$x_2 \le 70{,}000 \quad \text{and } x_1, x_2 \ge 0.$$

Also Max $Z = 562.50, x_1 = 3750, x_2 = 0$.

29. Max $Z = 2x_1 + 4x_2$

subject to
$$x_1 + 2x_2 \le 2000$$
$$x_1 + x_2 \le 1500$$
$$x_2 \le 600 \quad \text{and } x_1, x_2 \ge 0.$$

Also Max $Z = 4000$ with

(i) $x_1 = 800, x_2 = 600$, *(ii)* $x_1 = 1000, x_2 = 500$ etc.

30. Max $Z = 40x_1 + 30x_2$

subject to
$$3x_1 + x_2 \le 30{,}000$$
$$x_1 \le 8000$$
$$x_2 \le 12000 \quad \text{and } x_1, x_2 \ge 0.$$

Also Max $Z = 6{,}00{,}000, x_1 = 6000, x_2 = 12000$

31. Min $Z = 500, x_1 = 50, x_2 = 0$.

Chapter 3

General Linear Programming Problems – Simplex Methods

3.1.1 General Linear Programming Problem

The linear programming involving more than two variables may be expressed as follows :

Maximize (or) Minimize $Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots c_nx_n$

subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n \leq \text{ or } = \text{ or } \geq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n \leq \text{ or } = \text{ or } \geq b_2$$
$$a_{31}x_1 + a_{32}x_2 + \ldots\ldots\ldots\ldots + a_{3n}x_n \leq \text{ or } = \text{ or } \geq b_3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n \leq \text{ or } = \text{ or } \geq b_m$$

and the non-negativity restrictions

$$x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_n \geq 0.$$

Note : Some of the constraints may be equalities, some others may be inequalities of ($\leq$) type and remaining ones inequalities of ($\geq$) type or all of them are of same type.

Definition (1) : A set of values $x_1, x_2 \ldots\ldots x_n$ which satisfies the constraints of the LPP is called its ***solution.***

Definition (2) : Any solution to a LPP which satisfies the non-negativity restrictions of the LPP is called its ***feasible solution.***

Definition (3) : Any feasible solution which optimizes (maximizes or minimizes) the objective function of the LPP is called its ***optimum solution*** or ***optimal solution.***

Definition (4) : If the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij}\, x_j \leq b_i \qquad (i = 1, 2, 3, \ldots k) \qquad \ldots(1)$$

then the non-negative variables s_i which are introduced to convert the inequalities (1) to the equalities

$$\sum_{j=1}^{n} a_{ij}\, x_j + s_i = b_i \qquad (i = 1, 2, 3, \ldots k)$$

are called ***slack variables.*** The value of these variables can be interpreted as the amount of unused resource.

Definition (5) : If the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij}\, x_j \geq b_i \qquad (i = k, k+1, \ldots) \qquad \ldots(1)$$

then the non-negative variables s_i which are introduced to convert the inequalities (1) to the equalities

$$\sum_{j=1}^{n} a_{ij}\, x_j - s_i = b_i \qquad (i = k, k+1, \ldots)$$

are called ***surplus variables.*** The value of these variables can be interpreted as the amount over and above the required level.

3.1.2 Canonical and Standard forms of LPP :

After the formulation of LPP, the next step is to obtain its solution. But before any method is used to find its solution, the problem must be presented in a suitable from. Two forms are dealt with here, the canonical form and the standard form.

The canonical form : The general linear programming problem can always be expressed in the following form :

Maximize $Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots c_nx_n$

subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n \leq b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n \leq b_m$$

and the non-negativity restrictions

$$x_1, x_2, \ldots\ldots\ldots\ldots, x_n \geq 0.$$

This form of LPP is called the ***canonical form*** of the LPP.

In matrix notation the canonical form of LPP can be expressed as :

$$\begin{aligned} \text{Maximize } Z &= CX \text{ (objective function)} \\ \text{subject to } AX &\leq b \text{ (constraints)} \\ \text{and } X &\geq 0 \text{ (non-negativity restrictions)} \\ \text{where } C &= (c_1\ c_2 \ldots\ldots c_n), \end{aligned}$$

Chapter 7

Transportation Model

7.1 Introduction

Transportation deals with the transportation of a commodity (single product) from 'm' sources (origins or supply or *capacity* centres) to 'n' destinations (sinks or demand or requirement centres). It is assumed that

i) Level of supply at each source and the amount of demand at each destination and

ii) The unit transportation cost of commodity from each source to each destination are known [given].

It is also assumed that the cost of transportation is linear.

The objective is to determine the amount to be shifted from each source to each destination such that the total transportation cost is minimum.

Note : The transportation model also can be modified to account for multiple commodities.

I. Mathematical Formulation of a Transportation Problem :

Let us assume that there are m sources and n destinations.

Let a_i be the supply (capacity) at source i, b_j be the demand at destination j, c_{ij} be the unit transportation cost from source i to destination j and x_{ij} be the number of units shifted from source i to destination j.

Then the transportation problem can be expressed mathematically as

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, 2, 3 \ldots m.$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, 3 \ldots n.$$

$$\text{and } \; x_{ij} \geq 0, \; \text{for all } i \text{ and } j.$$

Note 1 : The two sets of constraints will be *consistent* if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

(total supply) *(total demand)*

which is the necessary and sufficient condition for a transportation problem to have a feasible solution. Problems satisfying this condition are called ***balanced transportation problems.***

Note 2 : If $\sum a_i \neq \sum b_j$, then the transportation problem is said to be ***unbalanced.***

Note 3 : For any transportation problem, the coefficients of all x_{ij} in the constraints are unity.

Note 4 : The objective function and the constraints being all linear, the transportation problem is a special class of linear programming problem. Therefore it can be solved by simplex method. But the number of variables being large, there will be too many calculations. So we can look for some other technique which would be simpler than the usual simplex method.

Standard transportation table :

Transportation problem is explicitly represented by the following transportation table.

Destination

		D_1	D_2	D_3	...	D_j	...	D_n	Supply
	S_1	c_{11}	c_{12}	c_{13}		c_{1j}		c_{1n}	a_1
Source	S_2	c_{21}	c_{22}	c_{23}		c_{2j}		c_{2n}	a_2
									$\vdots$
	S_i	c_{i1}	c_{i2}			c_{ij}		c_{in}	$\vdots$
	S_m	c_{m1}	c_{m2}			c_{mj}		c_{mn}	a_m
	Demand	b_1	b_2	b_3	...	...	...	b_n	$\boxed{\Sigma a_i = \Sigma b_j}$

The mn squares are called ***cells***. The unit transportation cost c_{ij} from the i^{th} source to the j^{th} destination is displayed in the ***upper left side of the (i, j)th cell.*** Any feasible solution is shown in the table by entering the value of x_{ij} ***in the centre of the*** $(i,j)^{th}$ cell. The various a's and b's are called ***rim requirements***. The feasibility of a solution can be verified by summing the values of x_{ij} along the rows and down the columns.

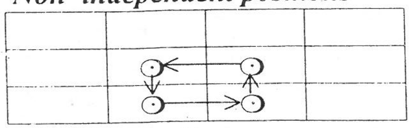
Definition 1: A set of non-negative values x_{ij}, $i = 1,2, \ldots m$; $j = 1,2\ldots n$. that satisfies the constraints (rim conditions and also the non-negativity restrictions) is called a ***feasible solution*** to the transportation problem.

Note : A balanced transportation problem will always have a feasible solution.

Definition 2: A feasible solution to a $(m \times n)$ transportation problem that contains no more than $m + n - 1$ non-negative allocations is called a ***basic feasible solution*** (BFS) to the transportation problem.

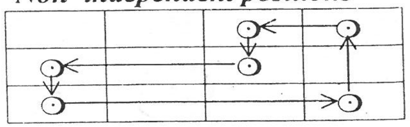
The allocations are said to be in ***independent positions*** if it is impossible to increase or decrease any allocation without either changing the position of the allocation or violating the rim requirements. A simple rule for allocations to be in independent positions is that it is impossible to travel from any allocation, back to itself by a series of horizontal and vertical jumps from one occupied cell to another, without a direct reversal of the route. Example

Non–independent positions



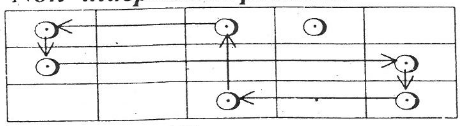
(i)

Non–independent positions



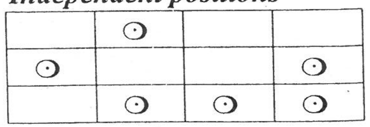
(ii)

Non–independent positions



(i)

Independent positions



(ii)

Definition 3 : A basic feasible solution to a $(m \times n)$ transportation problem is said to be a ***non-degenerate basic feasible solution*** if it contains exactly $m + n - 1$ non-negative allocations in independent positions.

Definition 4: A basic feasible solution that contains less than $m + n - 1$ non-negative allocations is said to be a ***degenerate basic*** feasible solution.

Definition 5: A feasible solution (not necessarily basic) is said to be an ***optimal solution*** if it minimizes the total transportation cost.

Note : The number of basic variables in an $m \times n$ balanced transportation problem is atmost $m + n - 1$.

Note : The number of non–basic variables in an $m \times n$ balanced transportation problem is atleast $mn - (m + n - 1)$

II. Methods for finding initial basic feasible solution

The transportation problem has a solution if and only if the problem is balanced. Therefore before starting to find the initial basic feasible solution, check whether the given transportation problem is balanced. If not one has to balance the transportation problem first. The way of doing this is discussed in section 7.4 page 7.40 In this section all the given transportation problems are balanced.

Method 1 : North west Corner Rule :

Step 1 : The first assignment is made in the cell occupying the upper left–hand (north–west) corner of the transportation table. The maximum possible amount is allocated there. That is $x_{11} = \min\{a_1, b_1\}$.

Case (i) : If min $\{a_1, b_1\} = a_1$, then put $x_{11} = a_1$, decrease b_1 by a_1 and move vertically to the 2nd row (*i.e.,*) to the cell (2,1) cross out the first row.

Case (ii) : If min $\{a_1, b_1\} = b_1$, then put $x_{11} = b_1$, and decrease a_1 by b_1 and move horizontally right (*i.e.,*) to the cell (1,2) cross out the first column

Case (iii) : If min $\{a_1, b_1\} = a_1 = b_1$ then put $x_{11} = a_1 = b_1$ and move diagonally to the cell (2,2) cross out the first row and the first column.

Step 2: Repeat the procedure until all the rim requirements are satisfied.

Method 2 : Least Cost method (or) Matrix minima method (or) Lowest cost entry method :

Step 1 : Identify the cell with smallest cost and allocate x_{ij} =Min $\{a_i, b_j\}$

Case (i) : If min $\{a_i, b_j\} = a_i$, then put $x_{ij} = a_i$, cross out the i^{th} row and decrease b_j by a_i, Go to step (2).

Case (ii) : If min $\{a_i, b_j\} = b_j$ then put $x_{ij} = b_j$ cross out the j^{th} column and decrease a_i by b_j Go to step (2).

Case (iii) : If min $\{a_i, b_j\} = a_i = b_j$, then put $x_{ij} = a_i = b_j$, cross out either i^{th} row or j^{th} column but not both, Go to step (2).

Step 2 : Repeat step (1) for the resulting reduced transportation table until all the rim requirements are satisfied.

Method 3: Vogel's approximation method (VAM) (or) Unit cost penalty method : *[MU. MBA. Nov 96, Apr 95, Apr 97]*

Step 1 : Find the difference (penalty) between the smallest and next smallest costs in each row (column) and write them in brackets against the corresponding row (column).

Step 2 : Identify the row (or) column with largest penalty. If a tie occurs, break the tie arbitrarily. Choose the cell with smallest cost in that selected row or column and allocate as much as possible to this cell and cross out the satisfied row or column and go to step (3).

Step 3 : Again compute the column and row penalties for the reduced transportation table and then go to step (2). Repeat the procedure until all the rim requirements are satisfied.

Example 1 : Determine basic feasible solution to the following transportation problem using North West Corner Rule :

		Sink					
		A	B	C	D	E	Supply
Origin	P	2	11	10	3	7	4
	Q	1	4	7	2	1	8
	R	3	9	4	8	12	9
Demand		3	3	4	5	6	

[MU. BE. Apr 94]

Solution: Since $\sum a_i = \sum b_j = 21$, the given problem is balanced.

$\therefore$ There exists a feasible solution to the transportation problem.

2 (3)	11	10	3	7	4
1	4	7	2	1	8
3	9	4	8	12	9
3	3	4	5	6	

Following North West Corner rule, the first allocation is made in the cell (1,1).

Here $x_{11} = \min\{a_1, b_1\} = \min\{4, 3\} = 3$

$\therefore$ Allocate 3 to the cell (1,1) and decrease 4 by 3 *ie.*, $4 - 3 = 1$

As the first column is satisfied, we cross out the first column and the resulting reduced Transportation table is

11 (1)	10	3	7	1
4	7	2	1	8
9	4	8	12	9
3	4	5	6	

Here the North West Corner cell is (1,2).

So Allocate $x_{12} = \min\{1,3\} = 1$ to the cell (1,2) and move vertically to cell (2,2). The resulting reduced transportation table is

4 (2)	7	2	1	8
9	4	8	12	9
2	4	5	6	

Allocate $x_{22} = \min\{8, 2\} = 2$ to the cell (2, 2) and move horizontally to the cell (2,3). The resulting transportation table is

7 (4)	2	1	6
4	8	12	9
4	5	6	

Allocate x_{23} = min {6,4} = 4 and move horizontally to the cell (2,4).

The resulting reduced transportation table is

2 2	1	2
8	12	9
5	6	

Allocate x_{24} = {2,5} = 2 and move vertically to the cell (3,4). The resulting reduced transportation table is

8 3	12	9
3	6	

Allocate x_{34} = min {9, 3} = 3 and move horizontally to the cell (3, 5), which is

12 6	6
6	

Allocate x_{35} = min {6,6} = 6

Finally the initial basic feasible solution is as shown in the following table.

2 3	11 1	10	3	7
1	4 2	7 4	2 2	1
3	9	4	8 3	12 6

From this table we see that the number of positive independent allocations is equal to $m + n - 1 = 3 + 5 - 1 = 7$. This ensures that the solution is non degenerate basic feasible.

$\therefore$ The initial transportation cost $= \text{Rs. } 2 \times 3 + 11 \times 1 + 4 \times 2 + 7 \times 4 + 2 \times 2 + 8 \times 3 + 12 \times 6$

$=$ Rs.153/-

Example 2 : Find the initial basic feasible solution for the following transportation problem by Least Cost Method.

	To				Supply
From	1	2	1	4	30
	3	3	2	1	50
	4	2	5	9	20
Demand	20	40	30	10	

[MU. BE. Apr 95, MSU. BE. Nov 96]

Solution : Since $\sum a_i = \sum b_j = 100$, the given TPP is balanced.

$\therefore$ There exists a feasible solution to the transportation problem.

1 20	2	1	4	30
3	3	2	1	50
4	2	5	9	20
20	40	30	10	

By least cost method, min $c_{ij} = c_{11} = c_{13} = c_{24} = 1$

Since more than one cell having the same minimum c_{ij}, break the tie.

Let us choose the cell (1,1) and allocate x_{11} = min $\{a_1, b_1\}$ = min {30, 20} = 20 and cross out the satisfied column and decrease 30 by 20.

The resulting reduced transportation table is

2	1 10	4	10
3	2	1	50
2	5	9	20
40	30	10	

Here min $c_{ij} = c_{13} = c_{24} = 1$

Choose the cell (1,3) and allocate $x_{13} = \min \{a_1, b_3\} = \min \{10,30\} =$ 10 and cross out the satisfied row.

The resulting reduced transportation table is

3	2	1 **10**	50
2	5	9	20
40	20	10	

Here min $c_{ij} = c_{24} = 1$,

$\therefore$ Allocate $x_{24} = \min \{a_2, b_4\} = \min (50,10) = 10$ and cross out the satisfied column.

The resulting transportation table is

3	2 **20**	40
2	5	20
40	20	

Here min $c_{ij} = c_{23} = c_{32} = 2$. Choose the cell (2,3) and allocate $x_{23} = \min \{a_2, b_3\} = \min (40,20) = 20$ and cross out the satisfied column.

The resulting reduced transportation table is

3	20
2 20	20
40	

Here min $c_{ij} = c_{32} = 2$. Choose the cell (3, 2) and allocate $x_{32} = \min \{a_3, b_2\} = \min (20,40) = 20$ and cross out the satisfied row.

The resulting reduced transportation table is

3 **20**	20
20	

Finally the initial basic feasible solution is as shown in the following table.

1 **20**	2	1 **10**	4
3	3 **20**	2 **20**	1 **10**
4	2 **20**	5	9

From this table we see that the number of positive independent allocations is equal to $m + n - 1 = 3 + 4 - 1 = 6$. This ensures that the solution is non degenerate basic feasible.

$\therefore$ The initial transportation cost $= \text{Rs. } 1 \times 20 + 1 \times 10 + 3 \times 20 + 2 \times 20 + 1 \times 10 + 2 \times 20$

$= 20 + 10 + 60 + 40 + 10 + 40$

$= \text{Rs.}180/-$

Example 3 : **Find the initial basic feasible solution for the following transportation problem by VAM.**

Distribution Centres

		D_1	D_2	D_3	D_4	Availability
	S_1	11	13	17	14	250
Origin	S_2	16	18	14	10	300
	S_3	21	24	13	10	400
Requirements		200	225	275	250	

Solution : Since $\Sigma a_i = \Sigma b_j = 950$, the given problem is balanced.

$\therefore$ There exists a feasible solution to this problem.

11 **200**	13	17	14	250	(2)
16	18	14	10	300	(4)
21	24	13	10	400	(3)
200	225	275	250		
(5)	(5)	(1)	(0)		

First let us find the difference (penalty) between the smallest and next smallest costs in each row and colum and write them in brackets against the respective rows and columns.

The largest of these differences is (5) and is associated with the first two columns of the transportation table. We choose the first column arbitrarily.

In this selected column, the cell (1,1) has the minimum unit transportation cost $c_{11} = 11$.

$\therefore$ Allocate x_{11} = min {250, 200} = 200 to this cell (1,1) and decrease 250 by 200 and cross out the satisfied column.

The resulting reduced transportation table is

13 50	17	14	50	(1)
18	14	10	300	(4)
24	13	10	400	(3)
225	275	250		
(5)	(1)	(0)		

The row and column differences are now computed for this reduced transportation table. The largest of these is (5) which is associated with the second column. Since $c_{12} = 13$ is the minimum cost, we allocate x_{12} = min {50,225} = 50 to the cell (1,2) and decrease 225 by 50 and cross out the satisfied row.

Continuing in this manner, the subsequent reduced transportation tables and the differences for the surviving rows and columns are shown below :

18 175	14	10	300	(4)
24	13	10	400	(3)
175	275	250		
(6)	(1)	(0)		

(i)

14	10 125	125	**(4)**
13	10	400	(3)
	250		
(1)	(0)		

(ii)

13	10 125	400	(3)
275	125		

(iii)

13 275	275
275	

(iv)

Finally the initial basic feasible solution is as shown in the following table.

11 **200**	13 **50**	17	14
16	18 **175**	14	10 **125**
21	24	13 **275**	10 **125**

From this table we see that the number of positive independent allocations is equal to $m + n - 1 = 3 + 4 - 1 = 6$. This ensures that the solution is non degenerate basic feasible.

$\therefore$ The initial transportation cost $=$ Rs. $11 \times 200 + 13 \times 50 + 18 \times 175 + 10 \times 125 + 13 \times 275 + 10 \times 125$

$=$ Rs. 12075/-

Example 4 : Find the starting solution of the following transportation model

1	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

using *(i)* ***North West Corner rule***

(ii) ***Least Cost method***

(iii) ***Vogel's approximation method.***

Solution : Since $\Sigma a_i = \Sigma b_j = 30$, the given Transportation problem is balanced. Hence there exists a basic feasible solution to this problem.

(i) North West Corner rule : Using this method, the allocations are shown in the tables below :

1 **7**	**2**	**6**	7
0	4	2	12
3	1	5	11
10	10	10	

(i)

0 **3**	4	2	12
3	1	5	11
3	10	10	

(ii)

4 **8**	**2**	9
1	5	11
10	10	

(iii)

1 **1**	5	11
1	10	

(iv)

5 10	10
10	

(v)

The starting solution is as shown in the following table

1 **7**	2	6
0 **3**	4 **9**	2
3	1 **1**	5 **10**

$\therefore$ The initial transportation cost $=$ Rs. $1 \times 7 + 0 \times 3 + 4 \times 9 + 1 \times 1 + 5 \times 10$

$=$ Rs. 94/-

(ii) Least Cost Method : Using this method, the allocations are as shown in the table below :

1	2	6 7	7
0 10	4	2	12
3	1	5	11
10	10	10	

(i)

2	6	7
4	2	2
1 10	5	11
10	10	

(ii)

6	7
2 2	2
5	1
10	

(iii)

6	7
5 **1**	1.
8	

(iv)

6 7	7
7	

(v)

The starting solution is as shown in the following table :

1	2	6 **7**
0 **10**	4	2 **2**
3	1 **10**	5 **1**

$\therefore$ The initial transportation cost $=$ Rs. $6 \times 7 + 0 \times 10 + 2 \times 2 + 1 \times 10 + 5 \times 1$

$=$ Rs. 61/-

(iii) Vogel's Approximation Method: Using this method, the allocations are shown in the tables below :

1	2	6	7	(1)
0	4	2 10	12	(2)
3	1	5	11	(2)
10	10	10		
(1)	(1)	(3)		

(i)

1	2	7	(1)
0 2	4	2	**(4)**
3	1	11	(2)
10	10		
(1)	(1)		

(ii)

1	2	7	(1)
3	1 10	11	**(2)**
8	10		
(2)	(1)		

(iii)

1 7	7
3	1
8	

(iv)

3 1	1
1	

(v)

The starting solution is as shown in the following table :

1 **7**	2	6
0 **2**	4	2 **10**
3 **1**	1 **10**	5

$\therefore$ The initial transportation cost $=$ Rs. $1 \times 7 + 0 \times 2 + 2 \times 10 + 3 \times 1 + 1 \times 10$

$=$ Rs. 40/-

Note : For the above problem, the number of positive allocations in independent positions is $(m + n - 1)$ (*i.e.*, $m + n - 1 = 3 + 3 - 1 = 5$). This ensures that the given problem has a non-degenerate basic feasible solution by using all the three methods. This need not be the case in all the problems.

7.2 Transportation Algorithm (or) MODI Method (modified distribution method) (Test for optimal solution).

[MU. MBA. Apr 96, Apr 97]

Step 1 : Find the initial basic feasible solution of the given problem by Northwest Corner rule (or) Least Cost method or VAM.

Step 2 : Check the number of occupied cells. If these are less than $m + n - 1$, there exists degeneracy and we introduce a very small positive assignment of $\in$ (≈ 0) in suitable independent positions, so that the number of occupied cells is exactly equal to $m + n - 1$.

Step 3 : Find the set of values u_i, v_j ($i = 1,2,3, \ldots m$; $j = 1, 2, 3 \ldots n$) from the relation $c_{ij} = u_i + v_j$ for each occupied cell (i, j), by starting initially with $u_i = 0$ or $v_j = 0$ preferably for which the corresponding row or column has maximum number of individual allocations.

Step 4 : Find $u_i + v_j$ for each unoccupied cell (i, j) and enter at the upper right corner of the corresponding cell (i, j).

Step 5 : Find the cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ (d_{ij} = upper left – upper right) for each unoccupied cell (i, j) and enter at the lower right corner of the corresponding cell (i, j).

Step 6 : Examine the cell evaluations d_{ij} for all unoccupied cells (i, j) and conclude that

(i) if all $d_{ij} > 0$, then the solution under the test is optimal and unique.

(ii) if all $d_{ij} > 0$, with atleast one $d_{ij} = 0$, then the solution under the test is optimal and an alternative optimal solution exists.

(iii) if atleast one $d_{ij} < 0$, then the solution is not optimal. Go to the next step.

Step 7 : Form a new B.F.S by giving maximum allocation to the cell for which d_{ij} is most negative by making an occupied cell empty. For that draw a closed path consisting of horizontal and vertical lines beginning and ending at the cell for which d_{ij} is most negative and having its ***other corners at some allocated cells.*** Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners. Choose minimum of the allocations from the cells having $-\theta$. Add this minimum allocation to the cells with $+\theta$ and subtract this minimum allocation from the allocation to the cells with $-\theta$.

Step 8: Repeat steps (2) to (6) to test the optimality of this new basic feasible solution.

Step 9: Continue the above procedure till an optimum solution is attained.

Note: The Vogels approximation method (VAM) takes into account not only the least cost c_{ij} but also the costs that just exceed the least cost c_{ij} and therefore yields better initial solution than obtained from other methods in general. This can be justified by the above example (4). So to find the initial solution, give preference to VAM unless otherwise specified.

Example 1 **: Solve the transportation problem :**

	1	2	3	4	Supply
I	21	16	25	13	11
II	17	18	14	23	13
III	32	27	18	41	19
Demand	6	10	12	15	

[MU. BE. Apr 91, Apr 92, Apr 93, Apr 97, MSU. BE. Apr 97]

Solution : Since $\Sigma a_i = \Sigma b_j = 43$, the given transportation problem is balanced. $\therefore$ There exists a basic feasible solution to this problem.

By Vogel's approximation method, the initial solution is as shown in the following table:

21	16	25	13 **11**	(3) – – –
17 **6**	18 **3**	14	23 **4**	(3) (3) (3) (4)
32	27 **7**	18 **12**	41	(9) (9) (9) (9)

(4)	(2)	(4)	**(10)**
(15)	(9)	(4)	**(18)**
(15)	(9)	(4)	–
–	**(9)**	(4)	–

That is

21	16	25	13 **11**
17 **6**	18 **3**	14	23 **4**
32	27 **7**	18 **12**	41

From this table, we see that the number of non-negative independent allocations is $(m + n - 1) = (3 + 4 - 1) = 6$. Hence the solution is non-degnerate basic feasible.

$\therefore$ The initial transportation cost

$= \text{Rs. } 13 \times 11 + 17 \times 6 + 18 \times 3 + 23 \times 4 + 27 \times 7 + 18 \times 12$

$=$ Rs. 796/-

To find the optimal solution

Consider the above transportation table. Since $m + n - 1 = 6$, we apply MODI method,

Now we determine a set of values u_i and v_j for each occupied cell (i, j) by using the relation $c_{ij} = u_i + v_j$. As the 2nd row contains maximum number of allocations, we choose $u_2 = 0$.

Therefore

$$c_{21} = u_2 + v_1 \Rightarrow 17 = 0 + v_1 \Rightarrow v_1 = 17$$
$$c_{22} = u_2 + v_2 \Rightarrow 18 = 0 + v_2 \Rightarrow v_2 = 18$$

$$c_{24} = u_2 + v_4 \Rightarrow 23 = 0 + v_4 \Rightarrow v_4 = 23$$
$$c_{14} = u_1 + v_4 \Rightarrow 13 = u_1 + 23 \Rightarrow u_1 = -10$$
$$c_{32} = u_3 + v_2 \Rightarrow 27 = u_3 + 18 \Rightarrow u_3 = 9$$
$$c_{33} = u_3 + v_3 \Rightarrow 18 = 9 + v_3 \Rightarrow v_3 = 9$$

Thus we have the following transportation table :

21	16	25	13 **11**	$u_1 = -10$
17 **6**	18 **3**	14	23 **4**	$u_2 = 0$
32	27 **7**	18 **12**	41	$u_3 = 9$
$v_1 = 17$	$v_2 = 18$	$v_3 = 9$	$v_4 = 23$	

Now we find $u_i + v_j$ for each unoccupied cell (i, j) and enter at the upper right corner of the corresponding unoccupied cell (i, j).

Then we find the cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ (*ie.*, upper left corner – upper right corner) for each unoccupied cell (i, j) and enter at the lower right corner of the corresponding unoccupied cell (i, j)

Thus we get the following table :

21 7 / 14	16 8 / 8	25 −1 / 26	13 **11**	$u_1 = -10$
17 **6**	18 **3**	14 9 / 5	23 **4**	$u_2 = 0$
32 26 / 6	27 **7**	18 **12**	41 32 / 9	$u_3 = 9$
$v_1 = 17$	$v_2 = 18$	$v_3 = 9$	$v_4 = 23$	

Since all $d_{ij} > 0$, the solution under the test is optimal and unique.

$\therefore$ The optimum allocation schedule is given by $x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$, $x_{32} = 7$, $x_{33} = 12$ and the optimum (minimum) transportation cost

$= \text{Rs. } 13 \times 11 + 17 \times 6 + 18 \times 3 + 23 \times 4 + 27 \times 7 + 18 \times 12$

$=$ Rs. 796/-

Example 2 : **Obtain on optimum basic feasible solution to the following transportation problem :**

	To			Available
From	7	3	2	2
	2	1	3	3
	3	4	6	5
Demand	4	1	5	10

[MU. MCA. May 93]

Solution : Since $\sum a_i = \sum b_j = 10$, the given transportation problem is balanced. ∴ There exists a basic feasible solution to this problem.

By Vogel's approximation method, the initial solution is as shown in the following table :

7	3	2 **2**	(1)	**(5)**	–
2	1 **1**	3 2	(1)	(1)	(1)
3 **4**	4	6 **1**	(1)	(3)	**(3)**
(1) (1) (1)	**(2)** – –	(1) (1) (3)			

That is

7	3	2 **2**
2	1 **1**	3 **2**
3 **4**	4	6 **1**

From this table we see that the number of non-negative allocations is $m + n - 1 = (3 + 3 - 1) = 5$.

Hence the solution is non-degenerate basic feasible

∴ The initial transportation cost

$$= \text{Rs. } 2 \times 2 + 1 \times 1 + 3 \times 2 + 3 \times 4 + 6 \times 1$$
$$= \text{Rs. } 29/-$$

For optimality : Since the number of non-negative independent allocations is $m + n - 1$, we apply MODI method.

Since the third column contains maximum number of allocations, we choose $v_3 = 0$.

Now we determine a set of values u_i and v_j by using the occupied cells and the relation $c_{ij} = u_i + v_j$.

That is,

7 –1	3 0	2 **2**	$u_1 = 2$
2	1 **1**	3 **2**	$u_2 = 3$
3 **4**	4	6 **1**	$u_3 = 6$
$v_1 = -3$	$v_2 = -2$	$v_3 = 0$	

Now we find $u_i + v_j$ for each unoccupied cell (i,j) and enter at the upper right corner of the corresponding unoccupied cell (i,j).

Then we find the cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ for each unoccupied cell (i,j) and enter at the lower right corner of the corresponding unoccupied cell (i,j).

Thus we get the following table

7 –1 8	3 0 3	2 **2**	$u_1 = 2$
2 0 2	1 **1**	3 **2**	$u_2 = 3$
3 **4**	4 4 0	6 **1**	$u_3 = 6$
$v_1 = -3$	$v_2 = -2$	$v_3 = 0$	

Since all $d_{ij} > 0$, with $d_{32} = 0$, the current solution is optimal and there exists an alternative optimal solution.

$\therefore$ The optimum allocation schedule is given by $x_{13} = 2$, $x_{22} = 1$, $x_{23} = 2$, $x_{31} = 4$, $x_{33} = 1$ and the optimum (minimum) transportation cost

$= \text{Rs. } 2 \times 2 + 1 \times 1 + 3 \times 2 + 3 \times 4 + 6 \times 1 = \text{Rs. } 29/-$

$\boxed{\textbf{Example 3}}$: **Find the optimal transportation cost of the following matrix using least cost method for finding the critical solution.**

Market

	A	B	C	D	E	**Available**
P	4	1	2	6	9	100
Factory Q	6	4	3	5	7	120
R	5	2	6	4	8	120
Demand	40	50	70	90	90	

[MU. MCA. Apr 93]

Solution :

Since $\Sigma a_i = \Sigma b_j = 340$, the given transportation problem is balanced.

$\therefore$ There exists a basic feasible solution to this problem.

By using Least cost method, the initial solution is as shown in the following table :

4	1 / **50**	2 / **50**	6	9
6 / **10**	4	3 / **20**	5	7 / **90**
5 / **30**	2	6	4 / **90**	8

$\therefore$ The initial transportation cost

$= \text{Rs. } 1 \times 50 + 2 \times 50 + 6 \times 10 + 3 \times 20 + 7 \times 90 + 5 \times 30 + 4 \times 90$

$= \text{Rs. } 1410/-$

For optimality : Since the number of non-negative independent allocations is $(m + n - 1)$, we apply MODI method :

That is

4 / 5 / −1	1 / **50**	2 / **50**	6 / 4 / 2	9 / 6 / 3	$u_1 = -1$
6 / **10**	4 / 2 / 2	3 / **20**	5 / 5 / 0	7 / **90**	$u_2 = 0$
5 / **30**	2 / 1 / 1	6 / 2 / 4	4 / **90**	8 / 6 / 2	$u_3 = -1$
$v_1 = 6$	$v_2 = 2$	$v_3 = 3$	$v_4 = 5$	$v_5 = 7$	

Since $d_{11} = -1 < 0$, the current solution is not optimal.

Now let us form a new basic feasible solution by giving maximum allocation to the cell (i,j) for which d_{ij} is most negative by making an occupied cell empty. Here the cell (1,1) having the negative value $d_{11} = -1$. We draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (1,1) and having its other corners at some occupied cells. Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners. we have

4 / $+\theta$	1 / **50**	2 / **50** $-\theta$	6	9
6 / **10** $-\theta$	4	3 / **20** $+\theta$	5	7 / **90**
5 / **30**	2	6	4 / **90**	8

From the two cells (1,3), (2,1) having $-\theta$, we find that the minimum of the allocations 50,10 is 10. Add this 10 to the cells with $+\theta$ and subtract this 10 to the cells with $-\theta$.

Hence the new basic feasible solution is displayed in the following table :

4 / **10**	1 / **50**	2 / **40**	6	9
6	4	3 / **30**	5	7 / **90**
5 / **30**	2	6	4 / **90**	8

We see that the above table satisfies the rim conditions with $(m+n-1)$ non-negative allocations at independent positions. So we apply MODI method.

4 / **10**	1 / **50**	2 / **40**	6 / 3 / 3	9 / 6 / 3	$u_1 = 0$
6 / 5 / 1	4 / 2 / 2	3 / **30**	5 / 4 / 1	7 / **90**	$u_2 = 1$
5 / **30**	2 / 2 / 0	6 / 3 / 3	4 / **90**	8 / 7 / 1	$u_3 = 1$
$v_1 = 4$	$v_2 = 1$	$v_3 = 2$	$v_4 = 3$	$v_5 = 6$	

Since all $d_{ij} > 0$, with $d_{32} = 0$, the current solution is optimal and there exists an alternative optimal solution.

$\therefore$ The optimum allocation schedule is given by $x_{11} = 10$, $x_{12} = 50$, $x_{13} = 40$, $x_{23} = 30$, $x_{25} = 90$, $x_{31} = 30$, $x_{34} = 90$ and the optimum (minimum) transportation cost

$= \text{Rs. } 4 \times 10 + 1 \times 50 + 2 \times 40 + 3 \times 30 + 7 \times 90 + 5 \times 30 + 4 \times 90$.

$=$ Rs. 1400/-

7.3 Degeneracy in Transportation Problems

In a transportation problem, whenever the number of non-negative independent allocations is less than $m + n - 1$, the transportation problem is said to be a ***degenerate*** one. Degeneracy may occur either at the initial stage or at an intermediate stage at some subsequent iteration.

To resolve degeneracy, we allocate an extremely small amount (close to zero) to one or more empty cells of the transportation table (generally minimum cost cells if possible), so that the total number of occupied cells becomes $(m + n - 1)$ at independent positions. We denote this small amount by $\in$ (epsilon) satisfying the following conditions :

(i) $0 < \in < x_{ij}$, for all $x_{ij} > 0$

(ii) $x_{ij} \pm \in = x_{ij}$, for all $x_{ij} > 0$

The cells containing $\in$ are then treated like other occupied cells and the problem is solved in the usual way. The $\in$'s are kept till the optimum solution is attained. Then we let each $\in \to 0$.

Example 1 : Find the non-degenerate basic feasible solution for the following transportation problem using

(i) North west corner rule

(ii) Least cost method

(iii) Vogel's approximation method.

	To				Supply
From	10	20	5	7	10
	13	9	12	8	20
	4	5	7	9	30
	14	7	1	0	40
	3	12	5	19	50
Demand	60	60	20	10	

[MU. MCA. Apr 93]

Solution : Since $\sum a_i = \sum b_j = 150$, the given transportation problem is balanced.

$\therefore$ There exists a basic feasible solution to this problem.

***(i)* The starting solution by NWC rule is as shown in the following table.**

10 **10**	20	5	7
13 **20**	9	12	8
4 **30**	5	7	9
14	7 **40**	1	0
3	12 **20**	5 **20**	19 **10**

Since the number of non-negative allocations at independent positions is 7 which is less than $(m + n - 1) = (5 + 4 - 1) = 8$, this basic feasible solution is a degenerate one.

To resolve this degeneracy, we allocate a very small quantity $\in$ to the unoccupied cell (5,1) so that the number of occupied cells becomes $(m + n - 1)$. Hence the non-degenerate basic feasible solution is as shown in the following table.

10 **10**	20	5	7
13 **20**	9	12	8
4 **30**	5	7	9
14	7 **40**	1	0
3 $\in$.	12 **20**	5 **20**	19 **10**

The initial transportation cost $= \text{Rs. } 10 \times 10 + 13 \times 20 + 4 \times 30 + 7 \times 40 + 3 \times \in + 12 \times 20 + 5 \times 20 + 19 \times 10$

$= \text{Rs.}(1290 + 3\in)$

$= \text{Rs. } 1290/-$, as $\in \to 0$.

***(ii)* Least cost method :** Using this method the starting solution is as shown in the following table :

10	20 **10**	5	7
13	9 **20**	12	8
4 **10**	5 **20**	7	9
14	7 **10**	1 **20**	0 **10**
3 **50**	12	5	19

m+n-1

5+4-1 = 8

8 balanced

No.of allocations

Since the number of non-negative allocations at independent positions is $(m + n - 1) = 8$, the solution is non-degenerate basic feasible.

The initial transportation cost $= \text{Rs. } 20 \times 10 + 9 \times 20 + 4 \times 10 + 5 \times 20 + 7 \times 10 + 1 \times 20 + 0 \times 10 + 3 \times 50$

$= \text{Rs. } 760/-$

***(iii)* Vogel's approximation Method :** The starting solution by this method is as shown in the following table :

10 **10**	20	5	7
13	9 **20**	12	8
4	5 **30**	7	9
14	7 **10**	1 **20**	0 **10**
3 **50**	12	5	19

Since the number of non-negative allocations is 7 which is less than $(m + n - 1) = (5 + 4 - 1) = 8$, this basic solution is a degenerate one.

To resolve this degeneracy, we allocate a very small quantity $\in$ to the unoccupied cell (5, 2) so that the number of occupied cells becomes $(m + n - 1)$. Hence the non-degenerate basic feasible solution is as shown in the following table.

10 **10**	20	5	7
13	9 **20**	12	8
4	5 **30**	7	9
14	7 **10**	1 **20**	0 **10**
3 **50**	12 $\in$	5	19

$\therefore$ The initial transportation cost

$= \text{Rs. } 10 \times 10 + 9 \times 20 + 5 \times 30 + 7 \times 10 + 1 \times 20 + 0 \times 10 + 3 \times 50 + 12 \times \in$

$= \text{Rs. } (670 + 12 \in)$

$= \text{Rs. } 670/- = \text{as } \in \to 0.$ *[Please refer note in page 7.17]*

Example 2 : **Solve the following transportation problem using Vogel's method.**

		Warehouse					
	A	**B**	**C**	**D**	**E**	**F**	**Available**
Factory 1	9	12	9	6	9	10	5
2	7	3	7	7	5	5	6
3	6	5	9	11	3	11	2
4	6	8	11	2	2	10	9
Requirement	4	4	6	2	4	2	

[MU. MCA. Apr 92]

Solution : Since $\Sigma a_i = \Sigma b_j = 22$, the given transportation problem is balanced. $\therefore$ There exists a basic feasible solution to this problem. By Vogel's approximation method, the initial solution is as shown in the following table :

9	12	9 **5**	6	9	10
7	3 **4**	7	7	5	5 **2**
6 **1**	5	9 **1**	11	3	11
6 **3**	8	11	2 **2**	2 **4**	10

Since the number of non-negative allocations is 8 which is less than $(m + n - 1) = (4 + 6 - 1) = 9$, this basic solution is degenerate one.

To resolve degeneracy, we allocate a very small quantity $\in$ to the cell (3,2), so that the number of occupied cells becomes $(m + n - 1)$. Hence the non-degenerate basic feasible solution is as shown in the following table.

9	12	9 **5**	6	9	10	u_1
7	3 **4**	7	7	5	5 **2**	u_2
6 **1**	5 $\in$	9 **1**	11	3	11	$u_3 = 0$
6 **3**	8	11	2 **2**	2 **4**	10	$u_4 = 0$
v_1	v_2	v_3	v_4	v_5	v_6	

The initial transportation cost $= \text{Rs. } 9 \times 5 + 3 \times 4 + 5 \times 2 + 6 \times 1 + 5 \times \in + 9 \times 1 + 6 \times 3 + 2 \times 2 + 2 \times 4$

$= \text{Rs. } (112 + 5 \in) = \text{Rs. } 112/-, \in \to 0.$

To find the optimal solution

Now the number of non-negative allocations at independent positions is $(m + n - 1)$. We apply the MODI method.

9 6 3	12 5 7	9 **5**	6 2 4	9 2 7	10 7 3	$u_1 = 0$
7 4 3	3 **4**	7 7 0	7 0 7	5 0 5	5 **2**	$u_2 = -2$
6 **1**	5 $\in$	9 **1**	11 2 9	3 2 1	11 7 4	$u_3 = 0$
6 **3**	8 5 3	11 9 2	2 **2**	2 **4**	10 7 3	$u_4 = 0$
$v_1 = 6$	$v_2 = 5$	$v_3 = 9$	$v_4 = 2$	$v_5 = 2$	$v_6 = 7$	

Since all $d_{ij} > 0$ with $d_{23} = 0$, the solution under the test is optimal and an alternative optimal solution is also exists.

$\therefore$ The optimum allocation schedule is given by $x_{13} = 5$, $x_{22} = 4$, $x_{26} = 2$, $x_{31} = 1$, $x_{32} = \in$, $x_{33} = 1$, $x_{41} = 3$, $x_{44} = 2$, $x_{45} = 4$ and the optimum (minimum) transportation cost is

$= \text{Rs. } 9 \times 5 + 3 \times 4 + 5 \times 2 + 6 \times 1 + 5 \times \in + 9 \times 1 + 6 \times 3 + 2 \times 2 + 2 \times 4$

$= \text{Rs. } (112 + 5 \in)$

$= \text{Rs. } 112$ as $\in \to 0$. very ny

$\boxed{\text{Example 3}}$ **: Solve the transportation problem:**

To

	1	2	3	4	**Supply**
From	1	2	3	4	6
	4	3	2	0	8
	0	2	2	1	10
Demand	4	6	8	6	

[MU. MCA. Apr 87]

Solution : Since $\Sigma a_i = \Sigma b_j = 24$, the given transportation problem is balanced. $\therefore$ There exists a basic feasible solution.

By using Vogel's approximation method, the initial solution is as shown in the following table :

1	2 **6**	3	4
4	3	2 **2**	0 **6**
0 **4**	2	2 **6**	1
	4		

Since the number of non-negative allocations at independent positions is 5, which is less than $(m + n - 1) = (3 + 4 - 1) = 6$, this basic feasible solution is degenerate.

To resolve degeneracy, we allocate a very small quantity $\in$ to the cell (3,2) so that the number of occupied cells becomes $(m + n - 1)$. Hence the non-degenerate initial basic feasible solution is given by

1	2 **6**	3	4
4	3	2 **2**	0 **6**
0 **4**	2 $\in$	2 **6**	1

The initial transportation cost

$= \text{Rs. } 2 \times 6 + 2 \times 2 + 0 \times 6 + 0 \times 4 + 2 \times \in + 2 \times 6$

$= \text{Rs. } (28 + 2 \in)$

$= \text{Rs. } 28/-$, as $\in \to 0$.

To find the optimal solution

Now the number non-negative allocations at independent positions is $(m + n - 1)$. We apply the MODI method.

1 0 1	2 **6**	3 2 1	4 0 4	$u_1 = 0$
4 0 4	3 2 1	2 **2**	0 **6**	$u_2 = 0$
0 **4**	2 $\in$	2 **6**	1 0 1	$u_3 = 0$
$v_1 = 0$	$v_2 = 2$	$v_3 = 2$	$v_4 = 0$	

Since all $d_{ij} > 0$ the solution under the test is optimal and unique.

$\therefore$ The optimum allocation schedule is given by $x_{12} = 6$, $x_{23} = 2$, $x_{24} = 6$, $x_{31} = 4$, $x_{32} = \in$, $x_{33} = 6$ and the optimum (minimum) transportation cost.

$= \text{Rs. } 2 \times 6 + 2 \times 2 + 0 \times 6 + 0 \times 4 + 2 \times \in + 2 \times 6$

$= \text{Rs. } (28 + 2 \in) = \text{Rs. } 28, \text{ as } \in \to 0.$

Example 4 : **Find the optimal solution of the following problem**

Destination

	X	Y	Z	Supply
P	1	2	0	30
Origin Q	2	3	4	35
R	1	5	6	35
Demand	30	40	30	

[MU. BE. Apr 95]

Solution : Since $\Sigma a_i = \Sigma b_j = 100$, the given transportation problem is balanced.

By using the Vogel's approximation method, the basic feasible solution is displayed in the following table.

1	2	0 **30**
2	3 **35**	4
1 **30**	5 **5**	6

Since the number of non-negative allocations at independent positions is 4 which is less than $(m + n - 1) = 3 + 3 - 1 = 5$, this initial solution is degenerate.

To resolve degeneracy we allocate a very small quantity $\in$ to the cell (3, 3), so that the number of occupied cells becomes $(m + n - 1)$. Hence the non-degenerate basic feasible solution is given by

1	2	0 **30**
2	3 **35**	4
1 **30**	5 **5**	6 $\in$

Now the number of non-negative allocations at independent positions is $(m + n - 1) = 5$. We apply MODI method.

1 −5 6	2 −1 3	0 **30**	$u_1 = -6$
2 −1 3	3 **35**	4 4 0	$u_2 = -2$
1 **30**	5 **5**	6 $\in$	$u_3 = 0$
$v_1 = 1$	$v_2 = 5$	$v_3 = 6$	

Since all $d_{ij} > 0$ with $d_{23} = 0$, the solution under the test is optimal and there exists an alternative optimal solution.

$\therefore$ The optimal allocation schedule is given by $x_{13} = 30$, $x_{22} = 35$, $x_{31} = 30$, $x_{32} = 5$, $x_{33} = \in$ and the optimum (minimum) transportation cost.

$= \text{Rs. } 0 \times 30 + 3 \times 35 + 1 \times 30 + 5 \times 5 + 6 \times \in$

$= \text{Rs. } (160 + 6\in)$

$= \text{Rs. } 160/- \text{ as } \in \to 0.$

Example 5 : Solve the following transportation problem to minimize the total cost of transportation.

		Destination				
		1	2	3	4	Supply
	1	14	56	48	27	70
Origin	2	82	35	21	81	47
	3	99	31	71	63	93
Demand		70	35	45	60	210

[BNU. BE. Nov 96]

Solution: Since $\sum a_i = \sum b_j = 210$, the given transportation problem is balanced. $\therefore$ There exists a basic feasible solution to this problem.

By using Vogel's approximation method, the initial solution is as shown in the following table :

14 **70**	56	48	27
82	35	21 **45**	81 **2**
99	31 **35**	71	63 **58**

Since the number of non-negative allocations is 5, which is less than $(m+n-1) = (3+4-1) = 6$, this basic feasible solution is degenerate.

To resolve degeneracy, we allocate a very small quantity $\in$ to the cell (1,4), so that the number of occupied cells becomes $(m+n-1)$. Hence the non-degenerate basic feasible solution is given in the following table

14 **70**	56	48	27 $\in$
82	35	21 **45**	81 **2**
99	31 **35**	71	63 **58**

$u_3+v_4=63$, $u_3=63$; $u_1+v_2=3$..., $v_2=-32$; $u_2+v_4=81$, $u_2=81$; $u_2+v_3=21$, $v_3=-60$; $u_1+v_4=27$, $u_1=27$

To find the optimal solution :

Now the number of non-negative allocations at independent positions is $(m+n-1) = 6$. We apply MODI method.

14 **70**	56 −5 61	48 −33 81	27 $\in$	$u_1 = 27$
82 68 14	35 49 −14	21 **45**	81 **2**	$u_2 = 81$
99 50 49	31 **35**	71 3 68	63 **58**	$u_3 = 63$
$v_1 = -13$	$v_2 = -32$	$v_3 = -60$	$v_4 = 0$	

Since $d_{22} = -14 < 0$, the solution under the test is not optimal.

Now let us form a new basic feasible solution by giving maximum allocation to the cell (2,2) by making an occupied cell empty. We draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (2,2) and having its other corners at some occupied cells. Along this closed loop, indicate $+\theta$ and $-\theta$ alternatively at the corners.

14 **70**	56	48	27 $\in$
82	35 $+\theta$	21 **45**	81 **2** $-\theta$
99	31 **35** $-\theta$	71	63 $+\theta$ **58**

From the two cells (2,4), (3,2) having $-\theta$ we find that the minimum of the allocations 2,35 is 2. Add this 2 to the cells with $+\theta$ and subtract this 2 to the cells with $-\theta$. Hence the new basic feasible solution is given by

14 **70**	56	48	27 $\in$
82	35 **2**	21 **45**	81
99	31 **33**	71	63 **60**

We see that the above table satisfies the rim conditions with $(m + n - 1)$ non-negative allocations at independent positions. We apply MODI method for optimality.

14 **70**	56 −5 61	48 −19 67	27 $\in$	$u_1 = -40$
82 54 28	35 **2**	21 **45**	81 67 14	$u_2 = 0$
99 50 49	31 **33**	71 17 54	63 **60**	$u_3 = -4$
$v_1 = 54$	$v_2 = 35$	$v_3 = 21$	$v_4 = 67$	

$u_2+v_2=35$
$v_2=35$
$u_3+v_2=31$
$u_3=-4$
$u_3+v_4=63$
$v_4=67$
$u_1+v_4=27$
$u_1=-40$
$u_1+v_1=14$
$v_1=54$

Since all $d_{ij} > 0$, the solution under the test is optimal.

$\therefore$ The optimal allocation schedule is given by $x_{11} = 70$, $x_{14} = \in$ $x_{22} = 2$, $x_{23} = 45$, $x_{32} = 33$, $x_{34} = 60$ and the optimum (minimum) transportation cost

$= \text{Rs. } 14 \times 70 + 27 \times \in + 35 \times 2 + 21 \times 45 + 31 \times 33 + 63 \times 60$

$= \text{Rs. } 6798/-$ as $\in \to 0$.

$\boxed{\text{Example 6}}$: Solve the following transportation problem, in which a_i is the availability at origin O_i and b_j is the requirement at the destination D_j and cell entries are unit costs of transportation from any origin to any destination :

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	4	7	3	8	2	4
O_2	1	4	7	3	8	7
O_3	7	2	4	7	7	9
O_4	4	8	2	4	7	2
b_j	8	3	7	2	2.	

[BRU. BE. Nov 96]

Solution : Since $\Sigma a_i = \Sigma b_j = 22$, the given problem is balanced. $\therefore$ There exists a basic feasible solution to this problem.

By using Vogel's approximation method, the initial solution is as shown in the following table :

4 **1**	7	3 **1**	8	2 **2**
1 **7**	4	7	3	8
7	2 **3**	4 **6**	7	7
4	8	2 $\in$	4 **2**	7

Since the number of non-negative allocations is 7, which is less than $(m + n - 1) = (4 + 5 - 1) = 8$, this basic feasible solution is degenerate.

To resolve this degeneracy, we allocate a very small quantity $\in$ to the cell (4,3), so that the number of occupied cells becomes $(m + n - 1)$. Hence the non-degenerate basic feasible solution is given in the following table

4 **1**	7	3 **1**	8	2 **2**
1 **7**	4	7	3	8
7	2 **3**	4 **6**	7	7
4	8	2 $\in$	4 **2**	7

optimal solution : Now the number of non-negative independent positions is $(m + n - 1) = 8$. We apply MODI

	7 1 6	3 **1**	8 5 3	2 **2**	$u_1 = 0$
1 **7**	4 −2 6	7 0 7	3 2 1	8 −1 9	$u_2 = -3$
7 5 2	2 **3**	4 **6**	7 6 1	7 3 4	$u_3 = 1$
4 3 1	8 0 8	2 $\in$	4 **2**	7 1 6	$u_4 = -1$
$v_1 = 4$	$v_2 = 1$	$v_3 = 3$	$v_4 = 5$	$v_5 = 2$	

Since all $d_{ij} > 0$, the solution under the test is optimal.

$\therefore$ The optimal allocation schedule is given by $x_{11} = 1$, $x_{13} = 1$, $x_{15} = 2$, $x_{21} = 7$, $x_{32} = 3$, $x_{33} = 6$, $x_{43} = \in$, $x_{44} = 2$ and the optimum (minimum) transportation cost

$= \text{Rs. } 4 \times 1 + 3 \times 1 + 2 \times 2 + 1 \times 7 + 2 \times 3 + 4 \times 6 + 2 \times \in + 4 \times 2$

$= \text{Rs. } (56 + 2\in)$

$= \text{Rs. } 56/\text{-} \text{ as } \in \to 0.$

7.4 Unbalanced Transportation Problems

If the given transportation problem is unbalanced one, *i.e.*, if $\Sigma a_i \neq \Sigma b_j$, then convert this into a balanced one by introducing a dummy source or dummy destination with zero cost vectors (zero unit transportation costs) as the case may be and then solve by usual method.

When the total supply is greater than the total demand, a dummy destination is included in the matrix with zero cost vectors. The excess supply is entered as a rim requirement for the dummy destination.

When the total demand is greater than the total supply, a dummy source is included in the matrix with zero cost vectors. The excess demand is entered as a rim requirement for the dummy source.

Example 1 : **Solve the transportation problem**

Destination

	A	B	C	D	Supply
Source 1	11	20	7	8	50
Source 2	21	16	20	12	40
Source 3	8	12	18	9	70
Demand :	30	25	35	40	160

Solution : Since the total supply $(\Sigma a_i = 160)$ is greater than the total demand $(\Sigma b_j = 130)$, the given problem is an unbalanced transportation problem. To convert this into a balanced one, we introduce a dummy destination E with zero unit transportation costs and having demand equal to $160 - 130 = 30$ units.

$\therefore$ The given problem becomes

Destination

	A	B	C	D	E	Supply
Source 1	11	20	7	8	0	50
Source 2	21	16	20	12	0	40
Source 3	8	12	18	9	0	70
	30	25	35	40	30	160

By using VAM the initial solution is as shown in the following table

11	20	7 **35**	8 **15**	0
21	16	20	12 **10**	0 **30**
8 **30**	12 **25**	18	9 **15**	0

$\therefore$ The initial transportation cost

$= \text{Rs. } 7 \times 35 + 8 \times 15 + 12 \times 10 + 0 \times 30 + 8 \times 30 + 12 \times 25 + 9 \times 15$

$= \text{Rs. } 1160/\text{-}$

For optimality: Since the number non-negative allocations at independent positions is $(m + n - 1)$, we apply the MODI method.

11 7 4	20 11 9	7 **35**	8 **15**	0 −4 4	$u_1 = 8$
21 11 10	16 15 1	20 11 9	12 **10**	0 **30**	$u_2 = 12$
8 **30**	12 **25**	18 8 10	9 **15**	0 −3 3	$u_3 = 9$
$v_1 = -1$	$v_2 = 3$	$v_3 = -1$	$v_4 = 0$	$v_5 = -12$	

Since all $d_{ij} > 0$, the solution under the test is optimal and unique.

$\therefore$ The optimum allocation schedule is

$x_{13} = 35, x_{14} = 15, x_{24} = 10, x_{25} = 30, x_{31} = 30, x_{32} = 25, x_{34} = 15$

It can be noted that $x_{25} = 30$ means that 30 units are despatched from source 2 to the dummy destination E. In other words, 30 units are left undespatched from source 2.

The optimum (minimum) transportation cost

= Rs. $7 \times 35 + 8 \times 15 + 12 \times 10 + 0 \times 30 + 8 \times 30 + 12 \times 25 + 9 \times 15$

= Rs. 1160/-

Example 2 : **Solve the transportation problem with unit transportation costs, demands and supplies as given below :**

		Destination				
		D_1	D_2	D_3	D_4	Supply
	S_1	6	1	9	3	70
Source	S_2	11	5	2	8	55 195
	S_3	10	12	4	7	70
Demand		85	35	50	45	

[MU MBA Apr 95]

Solution : Since the total demand ($\Sigma b_i = 215$) is greater than the total supply ($\Sigma a_j = 195$), the given problem is unbalanced transportation problem. To convert this into a balanced one, we introduce a dummy source S_4 with zero unit transportation costs and having supply equal to $215 - 195 = 20$ units. $\therefore$ The given problem becomes

		Destination				
		D_1	D_2	D_3	D_4	Supply
	S_1	6	1	9	3	70
Source	S_2	11	5	2	8	55
	S_3	10	12	4	7	70
	S_4	0	0	0	0	20
		85	35	50	45	215

As this problem is balanced, there exists a basic feasible solution to this problem. By using Vogel's approximation method, the initial solution is as shown in the following table.

6 **65**	1 **5**	9	3
11	5 **30**	2 **25**	8
10	12	4 **25**	7 **45**
0 **20**	0	0	0

$\therefore$ The initial transportation cost

= Rs. $6 \times 65 + 1 \times 5 + 5 \times 30 + 2 \times 25 + 4 \times 25 + 7 \times 45 + 0 \times 20$

= Rs. 1010/-

For optimality : Since number of non-negative allocations at independent positions is $(m + n - 1)$, we apply the MODI method :

6 **65**	1 **5**	9 −2 11	3 1 2	$u_1 = 6$
11 10 1	5 **30**	2 **25**	8 5 3	$u_2 = 10$
10 12 −2	12 7 5	4 **25**	7 **45**	$u_3 = 12$
0 **20**	0 −5 5	0 −8 8	0 −5 5	$u_4 = 0$
$v_1 = 0$	$v_2 = -5$	$v_3 = -8$	$v_4 = -5$	

Since $d_{31} = -2 < 0$, the solution under the test is not optimal.

Now let us form a new basic feasible solution by giving maximum allocation to the cell (3,1) (since d_{31} is $-ve$) by making an occupied cell empty. For this, we draw a closed path consisting of horizontal and vertical lines beginning and ending at this cell (3,1) and having its other corners at some occupied cells. Along this closed loop, indicate $+\theta$ and $-\theta$ alternatively at the corners.

We have

6 **65** $-\theta$	1 **5** $+\theta$	9	3
11	5 **30** $-\theta$	2 **25** $+\theta$	8
10 $+\theta$	12	4 **25** $-\theta$	7 **45**
0 **20**	0	0	0

From the three cells (1,1), (2,2), (3,3) having $-\theta$, we find that the minimum of the allocations 65, 30, 25 is 25. Add this 25 to the cells with $+\theta$ and subtract this 25 to the cells with $-\theta$. Finally, the new basic feasible solution is displayed in the following table.

6 **40**	1 **30**	9	3
11	5 **5**	2 **50**	8
10 **25**	12	4	7 **45**
0 **20**	0	0	0

We see that the above table satisfies the rim conditions with $(m+n-1)$ non-negative allocations at independent positions. Now we check for optimality

6 **40**	1 **30**	9 −2 11	3 3 0	$u_1 = 6$
11 10 1	5 **5**	2 **50**	8 7 1	$u_2 = 10$
10 **25**	12 5 7	4 2 2	7 **45**	$u_3 = 10$
0 **20**	0 −5 5	0 −8 8	0 −3 3	$u_4 = 0$
$v_1 = 0$	$v_2 = -5$	$v_3 = -8$	$v_4 = -3$	

Since all $d_{ij} > 0$ with $d_{14} = 0$, the solution under the test is optimal and an alternative optimal solution exists.

$\therefore$ The optimum allocation schedule is given by

$x_{11} = 40, x_{12} = 30, x_{22} = 5, x_{23} = 50, x_{31} = 25, x_{34} = 45, x_{41} = 20.$

It can be noted that $x_{41} = 20$ means that 20 units are despatched from the dummy source S_4 to the destination D_1. In other words, 20 units are not fulfilled for the destination D_1.

The optimum (minimum) transportation cost

$= \text{Rs. } 6 \times 40 + 1 \times 30 + 5 \times 5 + 2 \times 50 + 10 \times 25 + 7 \times 45 + 0 \times 20$

$= \text{Rs. } 960/-$

Example 3 : **Solve the transportation problem with unit transportation costs in rupees, demands and supplies as given below :**

		Destination D_1	D_2	D_3	Supply (units)
Origin	A	5	6	9	100
	B	3	5	10	75
	C	6	7	6	50
	D	6	4	10	75
Demand (units)		70	80	120	

[MU MBA Apr 98]

Solution : Since the total supply ($\Sigma a_i = 300$) is greater than the total demand ($\Sigma b_j = 270$), the given transportation problem is unbalanced.

To convert this into a balanced one, we introduce a dummy source D_4 with zero unit transportation costs and having demand equal to $300 - 270 = 30$ units. $\therefore$ The given problem becomes

		Destination				
		D_1	D_2	D_3	D_4	Supply
Origin	A	5	6	9	0	100
	B	3	5	10	0	75
	C	6	7	6	0	50
	D	6	4	10	0	75
Demand		70	80	120	30	300

By using VAM the initial solution is given by

5	6	9 **100**	0
3 **70**	5 **5**	10	0
6	7	6 **20**	0 **30**
6	4 **75**	10	0

Since the number of non-negative allocations is 6, which is less than $(m + n - 1) = 4 + 4 - 1 = 7$, this basic feasible solution is degenerate.

To resolve this degeneracy, we allocate a very small quantity $\in$ to the cell (2, 4), so that the number of occupied cells becomes $(m + n - 1)$. Hence the non-degenerate basic feasible solution is given in the following table

5	6	9 **100**	0
3 **70**	5 **5**	10	0 $\in$
6	7	6 **20**	0 **30**
6	4 **75**	10	0

Now the number of non-negative allocations at independent positions is $(m + n - 1)$. We apply MODI method.

5 6 −1	6 8 −2	9 **100**	0 3 −3	$u_1 = 3$
3 **70**	5 **5**	10 6 4	0 $\in$	$u_2 = 0$
6 3 3	7 5 2	6 **20**	0 **30**	$u_3 = 0$
6 2 4	4 **75**	10 5 5	0 −1 1	$u_4 = -1$
$v_1 = 3$	$v_2 = 5$	$v_3 = 6$	$v_4 = 0$	

Since there are some $d_{ij} < 0$, the current solution is not optimal.

Since $d_{14} = -3$ is the most negative, let us form a new basic feasible solution by giving maximum allocation to the corresponding cell (1,4) by making an occupied cell empty. We draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (1,4) and having its other corners at some occupied cells. Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners.

We have

5	6	9 $-\theta$ **100**	0 $+\theta$
3 **70**	5 **5**	10	0 $\in$
6	7	6 **20** $+\theta$	0 **30** $-\theta$
6	4 **75**	10	0

From the two cells (1,3), (3,4) having $-\theta$, we find that the minimum of the allocations 100,30 is 30. Add this 30 to the cells with $+\theta$ and subtract this 30 to the cells with $-\theta$. Hence the new basic feasible solution is given in the following table.

5	6	9 70	0 30
3 70	5 5	10	0 $\in$
6	7	6 50	0
6	4 75	10	0

We see that the above table satisfies the rim conditions with $(m + n - 1)$ non-negative allocations at independent positions. So we apply MODI method.

5 3 2	6 5 1	9 70	0 30	$u_1 = 0$
3 70	5 5	10 9 1	0 $\in$	$u_2 = 0$
6 0 6	7 2 5	6 50	0 −3 3	$u_3 = -3$
6 2 4	4 75	10 8 2	0 −1 1	$u_4 = -1$
$v_1 = 3$	$v_2 = 5$	$v_3 = 9$	$v_4 = 0$	

Since all $d_{ij} > 0$, the current solution is optimal and unique.

The optimum allocation schedule is given by

$x_{13} = 70, x_{14} = 30, x_{21} = 70, x_{22} = 5, x_{24} = \in, x_{33} = 50, x_{42} = 75$ and the optimum (minimum) transportation cost

$= \text{Rs. } 9 \times 70 + 0 \times 30 + 3 \times 70 + 5 \times 5 + 0 \times \in + 6 \times 50 + 4 \times 75$

$= \text{Rs. } 1465/-$

7.5 Maximization case in Transportation Problems

So far we have discussed the transportation problems in which the objective has been to minimize the total transportation cost and algorithms have been designed accordingly.

If we have a transportation problem where the objective is to maximize the total profit, first we have to convert the maximization problem into a minimization problem by multiplying all the entries by −1 (or) by subtracting all the entries from the highest entry in the given transportation table. The modified minimization problem can be solved in the usual manner.

Example 1 : **Solve the following transportation problem to maximize profit**

Profits (Rs)/Unit

Destination

	A	B	C	D	Supply
Source 1	40	25	22	33	100
Source 2	44	35	30	30	30
Source 3	38	38	28	30	70
Demand	40	20	60	30	

Solution : Since the given problem is of maximization type, first convert this into a minimization problem by subtracting the cost elements (entries or c_{ij}) from the highest cost element ($c_{ij} = 44$) in the given transportation problem. Then the given problem becomes.

Destination

	A	B	C	D	Supply
Source 1	4	19	22	11	100
Source 2	0	9	14	14	30
Source 3	6	6	16	14	70
Demand	40	20	60	30	

This modified minimization problem is unbalanced ($\Sigma a_i = 200$, $\Sigma b_j = 150$ and $\Sigma a_i \neq \Sigma b_j$). To make it balanced, we introduce a dummy destination E with demand $(200 - 150) = 50$ units with zero costs c_{ij}. Hence the balanced minimization transportation problem becomes

Destination

	A	B	C	D	E	Supply
Source 1	4	19	22	11	0	100
Source 2	0	9	14	14	0	30
Source 3	6	6	16	14	0	70
Demand	40	20	60	30	50	200

Since $\Sigma a_i = \Sigma b_j = 200$, there exists a basic feasible solution to this problem and is displayed in the following table by using VAM. [Try least cost method]

4 **10**	19	22 **60**	11 **30**	0
0 **30**	9	14	14	0
6	6 **20**	16	14	0 **50**

Since the number of non-negative allocations at independent position is 6, which is less than $(m + n - 1) = (3 + 5 - 1) = 7$, this initial solution is degenerate.

To resolve degeneracy, we allocate a very small quantity $\in$ to the cell (3, 3), so that the number of occupied cells becomes $(m + n - 1)$. Hence the initial solution is given by

4 **10**	19	22 **60**	11 **30**	0
0 **30**	9	14	14	0
6	6 **20**	16 $\in$	14	0 **50**

Now the number of non-negative allocations at independent positions is $(m + n - 1)$. We apply MODI method for optimal solution.

4 **10**	19 12 7	22 **60**	11 **30**	0 6 –6	$u_1 = 0$
0 **30**	9 8 1	14 18 –4	14 7 7	0 2 –2	$u_2 = -4$
6 –2 8	6 **20**	16 $\in$	14 5 9	0 **50**	$u_3 = -6$
$v_1 = 4$	$v_2 = 12$	$v_3 = 22$	$v_4 = 11$	$v_5 = 6$	

Since d_{15}, d_{23}, d_{25} are less than zero, the current solution under the test is not optimal. Here $d_{15} = -6$ is the most negative value of d_{ij}.

Let us form a new basic feasible solution by giving maximum allocation to the cell (1,5) by making an occupied cell empty. For this, we draw a closed path consisting of horizontal and vertical lines beginning and ending at this cell (1,5) and having its other corners at some occupied cells. Along this closed loop, indicate $+\theta$ and $-\theta$ alternatively at the corners.

4 **10**	19	22 **60** $-\theta$	11 **30**	0 $+\theta$
0 **30**	9	14	14	0
6	6 **20**	16 $+\theta$ $\in$	14	0 $-\theta$ **50**

From the two cells (1,3), (3,5) having $-\theta$, we find that the minimum of 60, 50 is 50. Add this 50 to the cells with $+\theta$ and subtract this 50 to the cells with $-\theta$. Hence the new basic feasible solution is displayed in the following table.

4 **10**	19	22 **10**	11 **30**	0 **50**
0 **30**	9	14	14	0
6	6 **20**	16 **50**	14	0

We see that the above table satisfies the rim conditions with $(m + n - 1)$ non-negative allocations at independent positions.

Now we apply the MODI method for optimality.

4 **10**	19 12 7	22 **10**	11 **30**	0 **50**	$u_1 = 0$
0 **30**	9 8 1	14 18 –4	14 7 7	0 –4 4	$u_2 = -4$
6 –2 8	6 **20**	16 **50**	14 5 9	0 –6 6	$u_3 = -6$
$v_1 = 4$	$v_2 = 12$	$v_3 = 22$	$v_4 = 11$	$v_5 = 0$	

Since $d_{23} = -4 < 0$ the current solution is not optimal.

Let us form a new basic feasible solution by giving maximum allocation to the cell (2,3) by making an occupied cell empty. For this, we draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (2,3) and having its other corner at some occupied cells. Along this closed loop, indicate $+\theta$ and $-\theta$ alternatively at the corners.

10

4 **20** $+\theta$	19	22 **10** $-\theta$	11 **30**	0 **50**
0 **30** $-\theta$	9	14 $+\theta$	14	0
6	6 **20**	16 **50**	14	0

From the two cells (1,3), (2,1) having $-\theta$, we find that the minimum of the allocations 10, 30 is 10. Add this 10 to the cells wih $+\theta$ and subtract this 10 to the cells with $-\theta$. Hence the new basic feasible solution is displayed in the following table.

4 **20**	19	22	11 **30**	0 **50**
0 **20**	9	14 **10**	14	0
6	6 **20**	16 **50**	14	0

Now the number non-negative allocations at independent positions is $(m + n - 1)$. We apply MODI method for the optimality

4 **20**	19 8 11	22 18 4	11 **30**	0 **50**	$u_1 = 0$
0 **20**	9 4 5	14 **10**	14 7 7	0 –4 4	$u_2 = -4$
6 2 4	6 **20**	16 **50**	14 9 5	0 –2 2	$u_3 = -2$
$v_1 = 4$	$v_2 = 8$	$v_3 = 18$	$v_4 = 11$	$v_5 = 0$	

Since all $d_{ij} > 0$, the current solution is optimal and unique.

$\therefore$ The optimum allocation schedule is given by

$x_{11} = 20, x_{14} = 30, x_{15} = 50, x_{21} = 20, x_{23} = 10, x_{32} = 20, x_{33} = 50.$

The optimum profit

$= \text{Rs. } 40 \times 20 + 33 \times 30 + 0 \times 50 + 44 \times 20 + 30 \times 10 + 38 \times 20 + 28 \times 50$

$= \text{Rs. } 5130/-.$

Example 2: **Solve the following transportation problem to maximize profit.**

Destination

		A	**B**	**C**	**D**	**Supply**
	1	**15**	**51**	**42**	**33**	**23**
Source	**2**	**80**	**42**	**26**	**81**	**44**
	3	**90**	**40**	**66**	**60**	**33**
Demand		**23**	**31**	**16**	**30**	**100**

Solution : Since the given problem is of maximization type, we convert this in to minimization problem by multiplying the profit costs c_{ij} by -1.

Destination

	A	B	C	D	Supply
Source 1	−15	−51	−42	−33	23
Source 2	−80	−42	−26	−81	44
Source 3	−90	−40	−66	−60	33
Demand	23	31	16	30	100

Since $\sum a_i = \sum b_j = 100$, there exists a basic feasible solution to this problem and is displayed in the following table by using VAM.

−15	−51 **23**	−42	−33
−80 **6**	−42 **8**	−26	−81 **30**
−90 **17**	−40	−66 **16**	−60

Since the number of non-negative allocations at independent positions is $(m + n - 1) = 6$, we apply MODI method for optimal solution.

−15 −89 74	−51 **23**	−42 −65 23	−33 −90 57	$u_1 = -9$
−80 **6**	−42 **8**	−26 −56 30	−81 **30**	$u_2 = 0$
−90 **17**	−40 −52 12	−66 **16**	−60 −91 31	$u_3 = -10$
$v_1 = -80$	$v_2 = -42$	$v_3 = -56$	$v_4 = -81$	

Since all $d_{ij} > 0$, the current solution is optimal and unique.

$\therefore$ The optimum allocations are given by $x_{12} = 23$, $x_{21} = 6$, $x_{22} = 8$, $x_{24} = 30$, $x_{31} = 17$, $x_{33} = 16$

$\therefore$ The optimum profit

$= \text{Rs. } 51 \times 23 + 80 \times 6 + 42 \times 8 + 81 \times 30 + 90 \times 17 + 66 \times 16$

$= \text{Rs. } 7005$ /-

EXERCISE

1. **What do you mean by transportation model ?**
2. **Define : Feasible solution, Basic feasible solution, Degenerate basic feasible solution, Non-degenerate basic feasible solution and optimal solution of a transportation problem.**
3. **Explain, in brief, with examples**
   (i) **North West Corner rule.** ***(ii)*** **Lowest Cost entry method.** ***(iii)*** **Vogel's approximation method.**
   [MU. MBA. Nov. 96, Apr 95]
4. **What do you mean by balanced and unbalanced transportation problems? Explain how would you convert the unbalanced problem into a balanced one?**
5. **State all the constraints in a transportation problem and how they are different from linear programming problem.**
   [MU. BE. Apr 94]
6. **Write down the dual of a transportation problem. Explain how this helps us in identifying whether the current solution is optimal or not.** ***[MU. MCA. Nov 96]***
7. **Explain an algorithm for solving a transportation problem.**
   [BRU. BE. Apr 95]
8. **Describe the method of solving unbalanced transportation problem.** ***[MU. MCA. Apr 95]***
9. **State the classical transportation problem and write down its mathematical model.**
10. **Give mathematical formulation of a transportation problem.**
    [MU. BE. Nov 93, Nov 89]
11. **How the problem of degeneracy arises in a transportation problem ? Explain how does one over come it.**
    [MU. BE. Apr 92, BRU. BE. Apr 96]
12. **Describe a transportation problem and give a method of finding an initial feasible solution .** ***[MU. BE. Nov 91]***
13. **Obtain the initial (starting) solution for the following transportation problem.**

Destination

	A	B	C	Supply
Source 1	2	7	4	5
Source 2	3	3	1	8
Source 3	5	4	7	7
Source 4	1	6	2	14
Demand	7	9	18	34

(i) **North West Corner rule** ***(ii)*** **Least Cost method**
(iii) **Vogel's approximation method**

14. Solve the following transportation problem

	To A	B	C	Availability
From I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
Requirement	4	2	2	

[MU. BE. Nov 89]

15. Obtain an optimum basic feasible solution to the transportation problem :

	Warehouse W_1	W_2	W_3	W_4	Capacity
Factory F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	

[MU. BE. Apr 90, B. Tech. Leather. Oct 96]

16. Obtain an optimum basic feasible solution to the following transportation problem :

	To			
From	7	3	4	2
	2	1	3	3 Available
	3	4	6	5
Demand	4	1	5	10

[MU. BE. Nov 91]

17. Solve the transportation problem :

	Destinations 1	2	3	
Source 1	2	2	3	10
2	4	1	2	15 Capacities
3	1	3	1	40
Demands	20	15	30	

[MU. BE. Nov 92]

18. Solve the following transportation problem where the cell entries denote the unit transportation costs.

	Destination A	B	C	D	Available
Origin P	5	4	2	6	20
Q	8	3	5	7	30
R	5	9	4	6	50
Required	10	40	20	30	

[MU. BE. Nov 91]

19. A company has 4 warehouses and 6 stores, the cost of shipping one unit from warehouse i to store j is c_{ij}

$$C_{ij} = \begin{matrix} 7 & 10 & 7 & 4 & 7 & 8 \\ 5 & 1 & 5 & 5 & 3 & 3 \\ 4 & 3 & 7 & 9 & 1 & 9 \\ 4 & 6 & 9 & 0 & 0 & 8 \end{matrix}$$

and the requirements of six stores are 4, 4, 6, 2, 4, 2 and quantities at warehouses are 5, 6, 2, 9, find the minimum cost solution.

[MU. BE. Nov 93]

20. A company has four warehouses *a, b, c, d*. It is required to deliver a product from these warehouses to three customers A, B and C. The warehouses have the following amounts in stock :

Ware house	:	*a*	*b*	*c*	*d*
No.of Units	:	15	16	12	13

and the customer's requirements are :

Customer	:	*A*	*B*	*C*
No. of Units	:	18	20	18

The table below shows the costs of transporting one unit from warehouse to the customer :

	Warehouse *a*	*b*	*c*	*d*
Customer A	8	9	6	3
B	6	11	5	10
C	3	8	7	9

Find the optimal transportation routes. *[MU. BE. Nov 93]*

21. Explain the concept of degeneracy in transportation problems:
Obtain an optimum basic feasible solution to the following transportation problem

	To			Supply
From	7	3	4	2
	2	1	3	3
	3	4	6	3
Demand	4	1	3	8

[MU. MCA. May 92]

22. Solve the transportation problem with unit transportation costs, demands and supplies as given below :

Destination Centre

		D_1	D_2	D_3	D_4	Supply
	F_1	3	3	4	1	100
Factory	F_2	4	2	4	2	125
	F_3	1	5	3	2	75
Demand		120	80	75	25	

[MU. MBA Nov 95]

23. Find the minimum cost of transportation, given

Warehouses

		D_1	D_2	D_3	D_4	Supply
Factory	F_1	19	30	50	10	7
	F_2	70	30	40	60	9
	F_3	40	8	70	20	18
Demand		5	8	7	14	

[MU. BE. Apr 90, Nov 94]

24. Compute the initial feasible solution to the following transportation problem, given the cost of transportation in rupees.

To

		P	Q	R	Supply
	A	5	1	7	10
From	B	6	4	6	80
	C	3	2	5	15
Demand		75	20	50	

[MU. BE. Oct 95]

25. Solve the transportation problem where the cell entries are transportation costs.

	D_1	D_2	D_3	D_4	Capacity
O_1	1	2	4	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Required	4	6	8	6	

[MU. BE. Apr 88]

26. **Solve the transportation problem**

Demand point

		1	2	3	4	Supply
	1	2	3	11	7	6
Source	2	1	0	6	1	1
	3	5	8	15	9	10
Demand		7	5	3	2	

[MU. BE. Apr 89]

27. **An automobile dealer is faced with the problem of determining the minimum cost policy for supplying dealers with the desired number of automobiles. The relevant data are given below. Obtain the minimum total cost of transportation**

Dealers

		1	2	3	4	5	Supply
	A	1.2	1.7	1.6	1.8	2.4	300
Plant	B	1.8	1.5	2.2	1.2	1.6	400
	C	1.5	1.4	1.2	1.5	1.0	100
Requirement		100	50	300	150	200	

The cost unit is in 100 rupees.

[MU. BE. Apr 89]

28. **Food packets have to be air lifted by three aircrafts from an airport and air–dropped to five villages. The quantities that can be carried in one trip by these aircrafts to the village are given below. The total number of trips per day an aircraft can make to the villages are also given. Find the number of trips each aircraft should make to each village so that the total quantity of food transported is maximum.**

	V_1	V_2	V_3	V_4	V_5	Trips/day by air crafts
A_1	10	8	6	9	12	50
A_2	5	3	8	4	10	90
A_3	7	9	6	10	4	60
Trips/day to village	100	80	70	40	20	

19

29. A company produces a small component for an industrial products and distributes it to five whole salers at a fixed delivered price of Rs. 2.50 per unit. Sales forecasts indicate that monthly deliveries will be 3000, 3000, 10,000, 5000 and 4000 units to wholesalers 1,2,3,4, and 5 respectively. The monthly production capacities are 5000, 10000 and 12500 at plants 1,2 and 3 respectively. The direct costs of production of each unit are Rs.1.00, Rs.0.90 and Rs.0.80 at plants 1,2 and 3 respectively. The transportation cost of shipping a unit from a plant to a wholesaler are given below :

		wholesaler				
		1	2	3	4	5
	1	0.05	0.07	0.10	0.15	0.15
Plant	2	0.08	0.06	0.09	0.12	0.14
	3	0.10	0.09	0.08	0.10	0.15

Find how many components each plant supplies to each wholesaler in order to maximize its profit.

30. A company has three plants A, B, C and three warehouses X, Y, Z. The number of units available at the plants is 60,70, 80 and the demand at X, Y, Z are 50, 80, 80 respectively. The unit costs of transportation is given by the following table.

	X	Y	Z
A	8	7	3
B	3	8	9
C	11	3	5

Find the allocation so that the total transportation cost is minimum. *[MU. BE. Oct 96]*

31. An oil corporation has got three refineries R_1, R_2, R_3 and it has to send petrol to four different depots D_1, D_2, D_3 and D_4. The cost of supplying of one unit of petrol from each refinery to each depot is given below. The requirements of the depot and the available petrol at the refineries are also given. Find the minimum cost of shipping after obtaining the initial solution by Vogel's Approximation method.

		Depot				
		D_1	D_2	D_3	D_4	Available
	R_1	10	12	15	8	130
Refinery	R_2	14	11	9	10	150
	R_3	20	5	7	18	170
Required :		90	100	140	120	

[MU. BE. Oct 96]

32. Find the optimum solution to the following transportation problem in which cells contain the transportation cost in rupees.

Table

	W_1	W_2	W_3	W_4	W_5	Available
F_1	7	6	4	5	9	40
F_2	8	5	6	7	8	30
F_3	6	8	9	6	5	20
F_4	5	7	7	8	6	10
Required :	30	30	15	20	5	100 Total

[MU. B. Tech. Leather. Oct 96]

33. The following is a transportation problem relating three warehouses (A, B and C) and four customers (1,2,3 and 4). The capacities at the warehouses and the demands from the customers are shown around the perimeter. Per unit transportation costs are shown in the cells. Cost minimization is the objective.

	1	2	3	4	Cap
A	7	8	11	10	30
B	10	12	5	4	45
C	6	11	10	9	35
Dem	20	28	19	33	–

Find the optimal solution and the total cost of transportation.

[BRU. BE. Apr 94]

34. A fan manufacturing company has its plants at calcutta and Delhi. The company is having warehouses in Nagpur, Patna and Baroda. The following table indicates the maximum capacities of the plants and the demands of the warehouses. The cost in rupees of shipping one unit from the particular plant to the given warehouses are shown on the corner of the cells. Find the least cost shipping assignment.

Warehouses

Plants	Nagpur	Patna	Baroda	Capacity
Delhi	3	4	2	600
Calcutta	3	2	5	700
Demand	400	500	400	–

[BRU. BE. Nov 95]

35. The projects X, Y, Z require truck loads of 45,50 and 20 respectively per week. The availabilities in plants A,B,C are 40, 40 and 40 of truck loads respectively per week. The cost of transport per unit of truck load from plant to project is given below :

Project

		X	Y	Z
	A	5	20	5
Plant	B	10	30	8
	C	10	20	12

(i) Determine an initial feasible solution by VAM.

(ii) Obtain an optimal solution by MODI method. The objective is to minimize the total cost of transportation.

[BRU. BE. Nov 94, MSU. BE. Apr 97]

36. Production shops P, Q, R can produce 5 new products A, B, C, D, E with their excess production capacity. The unit costs are given below with sale potential and availability of the capacity. It is given that the production shop R cannot produce the fifth product E. Find the optimal production schedule. Start the solution by VAM.

New Products

		A	B	C	D	E	Availability
	P	20	19	14	21	16	40
Production shop	Q	15	20	13	19	16	60
	R	18	15	18	20	–	90
Sales potential		30	40	70	40	60	

Find the allocation so that the total transportation cost is minimum. *[BRU. BE. Nov 96]*

37. Solve the transportation problem with unit transportation costs in rupees, requirements and availability as given below :

Distribution centre

		D_1	D_2	D_3	D_4	Availability
	F_1	10	15	12	12	200
Factory	F_2	8	10	11	9	150
	F_3	11	12	13	10	120
Requirement		140	120	80	220	

[MU. MBA. Apr. 97]

38. Solve the transportation problem with unit transportation costs in rupees and units of demand and supply as given below:

Destination

		D_1	D_2	D_3	D_4	Availability
	S_1	10	13	11	8	65
Source	S_2	9	12	12	10	44
	S_3	13	9	11	9	41
Demand		60	40	55	45	

[MU. MBA. Nov. 97]

39. Solve the transportation problem with unit transportation costs, demands and supplies as given below:

Destination

		D_1	D_2	D_3	Supply
	S_1	4	1	7	80
Source	S_2	3	2	2	20
	S_3	5	3	4	50
Demand		60	40	35	

[MU. MBA. Apr. 96]

ANSWERS

13. (i) $x_{11} = 5, x_{21} = 2, x_{22} = 6, x_{32} = 3, x_{33} = 4, x_{41} = 14$. and the transportation cost = Rs. 102/-
(ii) $x_{11} = 5, x_{21} = 2, x_{22} = 6, x_{32} = 3, x_{33} = 4, x_{41} = 14$. and the transportation cost = Rs. 102/-
(iii) $x_{11} = 5, x_{21} = 2, x_{22} = 6, x_{32} = 3, x_{33} = 4, x_{41} = 14$. and the transportation cost = Rs. 102/-

14. $x_{11} = 1, x_{21} = 3, x_{31} = \epsilon, x_{32} = 2$, and $x_{33} = 2$ and minimum T.P. cost = Rs. 743.

15. $x_{12} = 5, x_{14} = 2, x_{22} = 2, x_{23} = 7, x_{32} = 6$ and $x_{34} = 12$. and minimum T.P. cost = Rs. 743.

16. $x_{13} = 2, x_{22} = 1, x_{23} = 2, x_{31} = 4, x_{33} = 1$ and the optimum transportation cost is Rs. 33.

17. $x_{11} = 10, x_{22} = 15, x_{31} = 10, x_{33} = 30$, and the optimum transportation cost is Rs. 75/-.

18. $x_{12} = 10, x_{13} = 10, x_{22} = 30, x_{31} = 10, x_{33} = 10, x_{34} = 30$ and the optimum transportation cost is Rs. 420/-.

19. $x_{13} = 5, x_{22} = 4, x_{26} = 2, x_{31} = 1, x_{32} = \epsilon, x_{33} = 1, x_{41} = 3, x_{44} = 2, x_{45} = 4$, and the optimal transportation cost = Rs. $(68 + 3\epsilon)$ = Rs. 68/- , as $\epsilon \to 0$.

20. $x_{12} = 5, x_{14} = 13, x_{22} = 8, x_{23} = 12, x_{31} = 15, x_{32} = 3$, and the optimum transportation cost = Rs. 301/-.

21. $x_{13} = 2, x_{21} = 1, x_{22} = 1, x_{23} = 1, x_{31} = 3$ and the optimum transportation cost = Rs.23/-.

22. $x_{11} = 45, x_{13} = 30, x_{14} = 25, x_{22} = 80, x_{23} = 45, x_{31} = 75$, and the optimum transportation cost = Rs.695/-.

23. $x_{11} = 5, x_{14} = 2, x_{22} = 2, x_{23} = 7, x_{32} = 6, x_{34} = 12$, and the optimum transportation cost = Rs.743/-.

24. $x_{12} = 10, x_{21} = 60, x_{22} = 10, x_{23} = 10, x_{31} = 15, x_{43} = 40$, and the optimum transportation cost = Rs. 515/-

25. $x_{12} = 6, x_{23} = 2, x_{24} = 6, x_{31} = 4, x_{33} = 6$ and the optimum transportation cost = Rs. 28/-

26. $x_{12} = 5, x_{23} = 1, x_{33} = 1, x_{13} = 1, x_{31} = 7, x_{34} = 2$, and the optimum transportation cost = Rs. 100/-

27. $x_{11} = 100, x_{13} = 200, x_{22} = 50, x_{24} = 150, x_{25} = 200, x_{33} = 100$, and the optimum transportation cost = Rs. 1,13,500/-

28. $x_{11} = 50, x_{23} = 70, x_{25} = 20, x_{32} = 20, x_{34} = 40, x_{41} = 50, x_{42} = 60$ and the maximum quantity of transportation is 1840 units.

29. $x_{11} = 2500, x_{16} = 2500, x_{21} = 500, x_{22} = 3000, x_{23} = 2500, x_{25} = 4000, x_{33} = 7500, x_{34} =$ **5000** units respectively. The total transportation cost = Rs. 23,730/-. Total sale = Rs. 62,500/-. Total production cost = Rs. 23,600/-.
Therefore, the net maximum profit
= Total sale – (Total transportation cost + Total production cost)
= 62,500 – (23,730 + 23,600)
= Rs. 15,170/-

30. $x_{13} = 60, x_{21} = 50, x_{23} = 20, x_{32} = 80$ and the optimum transportation cost is Rs. 750/-

31. $x_{11} = 90, x_{14} = 40, x_{23} = 70, x_{24} = 80, x_{32} = 100, x_{33} = 70$ and the minimum (optimum) shipping cost is Rs. 3640/-

32. $x_{11} = 5, x_{13} = 15, x_{14} = 20, x_{21} = \epsilon, x_{22} = 30, x_{31} = 15, x_{35} = 5, x_{41} = 10$, and the optimum solution is Rs.510/- as $\epsilon \to 0$.

33. $x_{12} = 28, x_{15} = 2, x_{23} = 19, x_{24} = 26, x_{31} = 20, x_{34} = 7, x_{35} = 8$, and the optimum (minimum) transportation cost is Rs. 606/-.

34. $x_{11} = 200, x_{13} = 400, x_{21} = 200, x_{22} = 500$, and the least shipping cost is Rs. 3000/-.

35. (i) $x_{11} = 40, x_{21} = 10, x_{22} = 15, x_{23} = 20, x_{32} = 35, x_{34} = 5$, and the initial transportation cost is Rs.1560/-.
(ii) $x_{11} = 30, x_{12} = 10, x_{21} = 15, x_{23} = 20, x_{24} = 5, x_{32} = 40$, and the optimum (minimum) transportation cost is Rs.1460/-

36. $x_{13} = 10, x_{15} = 30, x_{23} = 60, x_{31} = 30, x_{32} = 40, x_{34} = 20, x_{44} = 20, x_{45} = 30$ and the optimum production cost is Rs. 2940/-. Note that an alternative optimum production schedule also exists.

37. $x_{11} = 140, x_{13} = 60, x_{22} = 50, x_{24} = 100, x_{34} = 120, x_{42} = 70, x_{43} = 20$. The minimum transportation cost is Rs. 4720/-.

38. $x_{11} = 16, x_{13} = 4, x_{14} = 45, x_{21} = 44, x_{32} = 40, x_{33} = 1, x_{43} = 50$. The minimum transportation cost = Rs. 1331/-

39. $x_{11} = 40, x_{12} = 40, x_{23} = 20, x_{31} = 20, x_{33} = 15, x_{34} = 15$. The Minimum transportation cost = Rs. 400/-.

Chapter 8

Assignment Problem

8.1 Introduction

The assignment problem is a particular case of the transportation problem in which the objective is to assign a number of tasks (Jobs or origins or sources) to an equal number of facilities (machines or persons or destinations) at a minimum cost (or maximum profit).

Suppose that we have 'n' jobs to be performed on 'm' machines (one Job to one machine) and our objective is to assign the jobs to the machines at the minimum cost (or maximum profit) under the assumption that each machine can perform each job but with varying degree of efficiencies.

The assignment problem can be stated in the form of $m \times n$ matrix (c_{ij}) called a ***Cost matrix*** (or) ***Effectiveness matrix*** where c_{ij} is the cost of assigning i^{th} machine to the j^{th} job.

		Jobs 1	2	3	....	n
	1	c_{11}	c_{12}	c_{13}	....	c_{1n}
	2	c_{21}	c_{22}	c_{23}	....	c_{2n}
Machines	3	c_{31}	c_{32}	c_{33}	....	c_{3n}
	:	...	...	...	....	...
	:	...	...	...	....	...
	:	...	...	...	....	...
	m	c_{m1}	c_{m2}	c_{m3}	....	c_{mn}

8.2 Mathematical formulation of an assignment problem.

[MU. BE. Apr 93, Oct 96, MU. MBA. Nov 96]

Consider an assignment problem of assigning n jobs to n machines (one job to one machine). Let c_{ij} be the unit cost of assigning i^{th} machine to the j^{th} job and

$$\text{let } x_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ job is assigned to } i^{th} \text{ machine} \\ 0, & \text{if } j^{th} \text{ job is not assigned to } i^{th} \text{ machine} \end{cases}$$

The assignment model is then given by the following LPP

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

subject to the constraints

$$\sum_{i=1}^{n} x_{ij} = 1, \; j = 1, 2, \ldots n$$

$$\sum_{j=1}^{n} x_{ij} = 1, \; i = 1, 2, \ldots n$$

$$\text{and } x_{ij} = 0 \text{ (or) } 1.$$

8.3 Comparison with Transportation Model

[MU. MBA Nov 95, Apr. 97]

The assignment problem may be considered as a special case of the transportation problem. Consider a transportation problem with 'n' sources and 'n' destinations.

		Destination 1	2	3	....	n	Supply (a_i)
	1	c_{11}	c_{12}	c_{13}	....	c_{1n}	a_1
	2	c_{21}	c_{22}	c_{23}	....	c_{2n}	a_2
Source	3	c_{31}	c_{32}	c_{33}	....	c_{3n}	a_3
	:	...	...	...	....	...	...
	:	...	...	...	....	...	...
	:	...	...	...	....	...	...
	n	c_{n1}	c_{n2}	c_{n3}	....	c_{nn}	a_n
Demand (b_j)		b_1	b_2	b_3	....	b_n	

We have to find x_{ij} $(i, j = 1, 2, 3 \ldots n)$ for which the total transportation cost

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

is minimized.

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, 2, \ldots n$$

$$\sum_{i=1}^{n} x_{ij} = b_j, \qquad j = 1, 2, \ldots n$$

$$\Sigma a_i = \Sigma b_j, \qquad i, j = 1, 2, \ldots n$$

$$\text{and } x_{ij} \geq 0, \qquad i, j = 1, 2, 3, \ldots n$$

Here the 'sources' represent 'facilities' or 'machines' and 'destinations' represent 'jobs'.

Suppose that the supply available at each source is 1 *i.e.*, $a_i = 1$ and the demand required at each destination is 1 *i.e.*, $b_j = 1$.

Let c_{ij} be the unit transportation cost from the i^{th} source to the j^{th} destination. Here it means the cost of assigning the i^{th} machine to the j^{th} job.

Let x_{ij} be the amount to be shipped from i^{th} source to the j^{th} destination. Here it means the assignment of the i^{th} machine to the j^{th} job. We can restrict the value of x_{ij} to be either 0 (or) 1. $x_{ij} = 0$ means that the i^{th} machine does not get the j^{th} job and $x_{ij} = 1$ means that the i^{th} machine gets the j^{th} job.

Since each machine should be assigned to only one job and each job requires only one machine, the total assignment value of the i^{th} machine is 1, (*i.e.*,) $\Sigma x_{ij} = 1$ and the total assignment value of the j^{th} job is 1, (*i.e.*,) $\Sigma x_{ij} = 1$.

Hence the assignment problem can be expressed as

$$\text{Minimize Z} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

where c_{ij} is the cost of assigning i^{th} machine to the j^{th} job subject to the constraints.

$$x_{ij} = \begin{cases} 1, & \text{if } i^{th} \text{ machine is assigned to the } j^{th} \text{ job} \\ 0, & \text{if } i^{th} \text{ machine is not assigned to the } j^{th} \text{ job} \end{cases}$$

$$i.e., x_{ij} = 0 \text{ or } 1 \Rightarrow x_{ij}(x_{ij} - 1) = 0 \Rightarrow x_{ij}^2 = x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = 1, \; i = 1, 2, \ldots n \text{ and}$$

$$\sum_{i=1}^{n} x_{ij} = 1, \; j = 1, 2, \ldots n.$$

From this we see that assignment problem represents a transportation problem with all demands and supplies equal to 1.

The units available at each source and units demanded at each destination are equal to 1. It means exactly that there is only one occupied cell in each row and each column of the transportation table. *i.e.*, only 'n' occupied cells in place of the required $n + n - 1 = 2n - 1$ occupied cells. Hence ***an assignment problem is always a degenerate form of a transportation problem.***

But the transportation technique (or) simplex method can not be used to solve the assignment problem because of degeneracy. In fact a very convenient iterative procedure is available for solving an assignment problem.

The technique used for solving assignment problem makes use of the following two theorems.

Theorem 1 : The optimum assignment schedule remains unaltered if we add or subtract a constant from all the elements of the row or column of the assignment cost matrix.

Theorem 2 : If for an assignment problem all $c_{ij} > 0$, then an assignment schedule (x_{ij}) which satisfies $\Sigma\, c_{ij}\, x_{ij} = 0$, must be optimal.

8.4 Difference between the transportation problem and the assignment problem.

[MU. BE. Nov 94]

	Transportation Problem	*Assignment Problem*
(a)	Supply at any source may be any positive quantity a_i	Supply at any source (machine) will be 1 *i.e.*, $a_i = 1$.
(b)	Demand at any destination may be any positive quantity b_j	Demand at any destination (job) will be 1. *i.e.*, $b_j = 1$
(c)	One or more source to any number of destinations	One source (machine) to only one destination (job).

8.5 Assignment Algorithm (or) Hungarian Method.

[MU. MBA. Apr 95, BRU. BE. Nov 96, MU. BE. Apr 95, MU. MCA. Nov 95, Nov 98]

First check whether the number of rows is equal to the number of columns, If it is so, the assignment problem is said to be **balanced**. Then proceed to step 1. If it is not balanced, then it should be balanced before applying the algorithm. The method of balancing is discussed in sec 5.6 page 5.15.

Step 1 : Subtract the smallest cost element of each row from all the elements in the row of the given cost matrix. See that each row contains atleast one zero.

Step 2 : Subtract the smallest cost element of each column from all the elements in the column of the resulting cost matrix obtained by step 1.

Step 3 : (Assigning the zeros)

(a) Examine the rows successively until a row with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it. Cross all other zeros in the column of this encircled zero, as these will not be considered for any future assignment. Continue in this way until all the rows have been examined.

(b) Examine the columns successively until a column with exactly one unmarked zero is found. Make an assignment to this single one unmarked zero by encircling it and cross any other zero in its unmarked zero by encircling it and cross any other zero in its row. Continue until all the columns have been examined.

Step 4 : (Apply optimal Test)

(a) If each row and each column contain exactly one encircled zero, then the current assignment is optimal.

(b) If atleast one row/column is without an assignment (i.e., if there is atleast one row/column is without one encircled zero), then the current assignment is not optimal. Go to step 5.

Step 5 : Cover all the zeros by drawing a minimum number of straight lines as follows.

(a) Mark (✓) the rows that do not have assignment.

(b) Mark (✓) the columns (not already marked) that have zeros in marked rows.

(c) Mark (✓) the rows (not already marked) that have assignments in marked columns.

(d) Repeat *(b)* and *(c)* until no more marking is required.

(e) Draw lines through all unmarked rows and marked columns. If the number of these lines is equal to the order of the matrix then it is an optimum solution otherwise not.

Step 6 : Determine the smallest cost element not covered by the straight lines. Subtract this smallest cost element from all the uncovered elements and add this to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

Step 7 : Repeat steps (1) to (6), until an optimum assignment is attained.

Note 1 : In case some rows or columns contain more than one zero, encircle any unmarked zero arbitrarily and cross all other zeros in its column or row. Proceed in this way until no zero is left unmarked or encircled.

Note 2 : The above assignment algorithm is only for minimization problems.

Note 3 : If the given assignment problem is of maximization type, convert it to a minimization assignment problem by max Z = – min (–Z) and multiply all the given cost elements by –1 in the cost matrix and then solve by assignment algorithm.

Note 4 : Some times, a final cost matrix contains more than required number of zeros at independent positions. This implies that there is more than one optimal solution (multiple optimal solutions) with the same optimum assignment cost.

$\boxed{\textbf{Example 1}}$ **: Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows :**

$$\begin{array}{cc} & \textbf{Job} \\ & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \textbf{Person} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{pmatrix} 8 & 4 & 2 & 6 & 1 \\ 0 & 9 & 5 & 5 & 4 \\ 3 & 8 & 9 & 2 & 6 \\ 4 & 3 & 1 & 0 & 3 \\ 9 & 5 & 8 & 9 & 5 \end{pmatrix} \end{array}$$

Determine the optimum assignment schedule.

[MU. BE. Apr 90, Apr 91]

Solution : The cost matrix of the given assignment problem is

$$\begin{pmatrix} 8 & 4 & 2 & 6 & 1 \\ 0 & 9 & 5 & 5 & 4 \\ 3 & 8 & 9 & 2 & 6 \\ 4 & 3 & 1 & 0 & 3 \\ 9 & 5 & 8 & 9 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 13 & 49 & 2 & 10 \\ 0 & 35 & 29 & 7 & 20 \\ 13 & 0 & 63 & 9 & 17 \\ 47 & 15 & 0 & 22 & 12 \\ 25 & 0 & 46 & 11 & 14 \end{pmatrix}$$

Step 2 : Select the smallest cost element in each column and subtract this from all the elements of the corresponding column, we get the following reduced matrix

$$\begin{pmatrix} 0 & 13 & 49 & 0 & 0 \\ 0 & 35 & 29 & 5 & 10 \\ 13 & 0 & 63 & 7 & 7 \\ 47 & 15 & 0 & 20 & 2 \\ 25 & 0 & 46 & 9 & 4 \end{pmatrix}$$

Step 3 : Now we shall examine the rows successively. Second row contains a single unmarked zero, encircle this zero and cross all other zeros in its column. Third row contains a single unmarked zero, encircle this zero and cross all other zeros in its column. Fourth row contains a single unmarked zero, encircle this zero and cross all other zeros in its column. After this no row is with exactly one unmarked zero. So go for columns.

Examine the columns successively. Fourth column contains exactly one unmarked zero, encircle this zero and cross all other zeros in its row. After examining all the rows and columns, we get

$$\begin{pmatrix} \cancel{0} & 13 & 49 & (0) & \cancel{0} \\ (0) & 35 & 29 & 5 & 10 \\ 13 & (0) & 63 & 7 & 7 \\ 47 & 15 & (0) & 20 & 2 \\ 25 & \cancel{0} & 46 & 9 & 4 \end{pmatrix}$$

2 3 4

Step 4 : Since the 5th row and 5th column do not have any assignment, the current assignment is not optimal.

Step 5 : Cover all the zeros by drawing a minimum number of straight lines as follows :

(a) Mark (✓) the rows that do not have assignment. The row 5 is marked.

(b) Mark (✓) the columns (not already marked) that have zeros in marked rows. Thus column 2 is marked.

(c) Mark (✓) the rows (not already marked) that have assignments in marked columns. Thus row 3 is marked.

(d) Repeat *(b)* and *(c)* until no more marking is required. In the present case this repetition is not necessary.

(e) Draw lines through all unmarked rows (rows 1, 2 and 4). and marked columns (column 2). We get

$$\begin{array}{c} \begin{pmatrix} 0 & 13 & 49 & 0 & 0 \\ 0 & 35 & 29 & 5 & 10 \\ 13 & (0) & 63 & 7 & 7 \\ 47 & 15 & 0 & 20 & 2 \\ 25 & 0 & 46 & 9 & 4 \end{pmatrix} \begin{array}{c} \\ \\ ✓ \\ \\ ✓ \end{array} \\ \quad ✓ \end{array}$$

Min ✓
Sub ✓
Add intersc

Step 6 : Here 4 is the smallest element not covered by these straight lines. Subtract this 4 from all the uncovered elements and add this 4 to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines, we get the following matrix.

$$\begin{pmatrix} 0 & 17 & 49 & 0 & 0 \\ 0 & 39 & 29 & 5 & 10 \\ 9 & 0 & 59 & 3 & 3 \\ 47 & 19 & 0 & 20 & 2 \\ 21 & 0 & 42 & 5 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we examine the rows and columns successively, *i.e.*, repeat step 3 above, we get

$$\begin{pmatrix} \cancel{0} & 17 & 49 & (0) & \cancel{0} \\ (0) & 39 & 29 & 5 & 10 \\ 9 & (0) & 59 & 3 & 3 \\ 47 & 19 & (0) & 20 & 2 \\ 21 & \cancel{0} & 42 & 5 & (0) \end{pmatrix}$$

Since the number of rows is equal to the number of columns in the cost matrix, the given assignment problem is balanced.

Step 1: Select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix

$$\begin{pmatrix} 7 & 3 & 1 & 5 & 0 \\ 0 & 9 & 5 & 5 & 4 \\ 1 & 6 & 7 & 0 & 4 \\ 4 & 3 & 1 & 0 & 3 \\ 4 & 0 & 3 & 4 & 0 \end{pmatrix}$$

Step 2 : Select the smallest cost element in each column and subtract this from all the elements of the corresponding column, we get the reduced matrix.

$$\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{pmatrix} 7 & 3 & \cancel{0} & 5 & 0 \\ 0 & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & 0 & 4 \\ 4 & 3 & 0 & \cancel{0} & 3 \\ 4 & 0 & 2 & 4 & \cancel{0} \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make assignments in the reduced matrix.

Step 3 : Examine the rows successively until a row with exactly one unmarked zero is found. Since the 2nd row contains a single zero, encircle this zero and cross all other zeros of its column. The 3rd row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. The 4th row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. The 1st row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. Finally the last row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. Like wise examine the columns successively. The assignments in rows and columns in the reduced matrix is given by

$$\begin{pmatrix} 7 & 3 & \cancel{0} & 5 & (0) \\ (0) & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & (0) & 4 \\ 4 & 3 & (0) & \cancel{0} & 3 \\ 4 & (0) & 2 & 4 & \cancel{0} \end{pmatrix}$$

Step 4 : Since each row and each column contains exactly one assignment (*i.e.,* exactly one encircled zero) the current assignment is optimal.

$\therefore$ The optinum assignment schedule is given by A $\to$ 5, B $\to$ 1, C $\to$ 4, D $\to$ 3, E $\to$ 2.

The optimum (minimum) assignment cost = (1 + 0 + 2 + 1 + 5) cost units = 9 units of cost.

$\boxed{\text{Example 2}}$: The processing time in hours for the jobs when allocated to the different machines are indicated below. Assign the machines for the jobs so that the total processing time is minimum

		Machines				
		M_1	M_2	M_3	M_4	M_5
Jobs	J_1	9	22	58	11	19
	J_2	43	78	72	50	63
	J_3	41	28	91	37	45
	J_4	74	42	27	49	39
	J_5	36	11	57	22	25

[MU. MBA. Nov 95]

Solution : The cost matrix of the given problem is

9	22	58	11	19
43	78	72	50	63
41	28	91	37	45
74	42	27	49	39
36	11	57	22	25

Since the number of rows is equal to the number of columns in the cost matrix, the given assignment problem is balanced.

Step 1 : Select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix.

In the above matrix, each row and each column contains exactly one assignment (*i.e.,* exactly one encircled zero), therefore the current assignment is optimal.

$\therefore$ The optimum assignment schedule is $J_1 \rightarrow M_4$, $J_2 \rightarrow M_1$, $J_3 \rightarrow M_2$, $J_4 \rightarrow M_3$, $J_5 \rightarrow M_5$ and the optimum (minimum) processing time

$= 11 + 43 + 28 + 27 + 25$ hours

$=$ 134 hours.

Example 3 : Four different jobs can be done on four different machines. The set up and take down time costs are assumed to be prohibitively high for change overs. The matrix below gives the cost in rupees of processing job *i* on machine *j*.

		Machines			
		M_1	M_2	M_3	M_4
	J_1	5	7	11	6
Jobs	J_2	8	5	9	6
	J_3	4	7	10	7
	J_4	10	4	8	3

How should the jobs be assigned to the various machines so that the total cost is minimized? *[MU. BE. Nov 92]*

Solution : The assignment problem is given by the cost matrix

$$\begin{pmatrix} 5 & 7 & 11 & 6 \\ 8 & 5 & 9 & 6 \\ 4 & 7 & 10 & 7 \\ 10 & 4 & 8 & 3 \end{pmatrix}$$

Since the number of rows is equal to the number of columns in the cost matrix, the given assignment problem is balanced.

Select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix

$$\begin{pmatrix} 0 & 2 & 6 & 1 \\ 3 & 0 & 4 & 1 \\ 0 & 3 & 6 & 3 \\ 7 & 1 & 5 & 0 \end{pmatrix}$$

Select the smallest cost element in each column and subtract this from all the elements of the corresponding column, we get the following reduced matrix

$$\begin{pmatrix} 0 & 2 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 2 & 3 \\ 7 & 1 & 1 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns of this reduced matrix.

Examine the rows successively until a row with exactly one unmarked zero is found. The first row contains exactly one unmarked zero, encircle this zero and cross all other zeros of its column. The fourth row contains exactly one unmarked zero, encircle this zero and cross all other zeros in its column. The 2nd column contains exactly one unmarked zero, encircle this zero and cross all other zeros in its row. We get

$$\begin{pmatrix} (0) & 2 & 2 & 1 \\ 3 & (0) & \cancel{0} & 1 \\ \cancel{0} & 3 & 2 & 3 \\ 7 & 1 & 1 & (0) \end{pmatrix}$$

Since there are some rows and columns without assignment (*i.e.,* without encircled zero), the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

We get

$$\begin{pmatrix} 0 & 2 & 2 & (1) \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 2 & 3 \\ 7 & 1 & 1 & 0 \end{pmatrix}$$

Here 1 is the smallest cost element not covered by these straight lines. Add this 1 to those elements which lie in the intersection of these straight lines, subtract this 1 from all the uncovered elements and do not change the remaining elements which lie on the straight lines we get

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 4 & 0 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 8 & 1 & 1 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make assignment in the rows and columns of this reduced matrix. We get

$$\begin{pmatrix} \cancel{0} & 1 & 1 & \cancel{0} \\ 4 & (0) & \cancel{0} & 1 \\ (0) & 2 & 1 & 2 \\ 8 & 1 & 1 & (0) \end{pmatrix}$$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

We get

$$\begin{array}{cc} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 4 & 0 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 8 & (1) & 1 & 0 \end{pmatrix} & \begin{matrix} \checkmark \\ \\ \checkmark \\ \checkmark \end{matrix} \\ \begin{matrix} \checkmark & \quad & \quad & \checkmark \end{matrix} & \end{array}$$

Here 1 is the smallest cost element not covered by these straight lines. Subtract this 1 from all the uncovered elements, add this 1 to those elements which lie in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines. We get

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 8 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in the rows and columns of this reduced matrix. We get

$$\begin{pmatrix} (0) & \cancel{0} & \cancel{0} & \cancel{0} \\ 5 & (0) & \cancel{0} & 2 \\ \cancel{0} & 1 & (0) & 2 \\ 8 & \cancel{0} & \cancel{0} & (0) \end{pmatrix}$$

Since each row and each column contains exactly one assignment (*i.e.*, exactly one encircled zero) the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by $J_1 \to M_1$, $J_2 \to M_2$, $J_3 \to M_3$, $J_4 \to M_4$ and the optimum (minimum) assignment cost

$= \text{Rs. } (5 + 5 + 10 + 3) = \text{Rs. } 23$

Example 4 : The assignment cost of assigning any one operator to any one machine is given in the following table

		Operators			
		I	**II**	**III**	**IV**
Machine	**A**	**10**	$\boxed{5}$	**13**	**15**
	B	$\boxed{3}$	**9**	**18**	**3**
	C	**10**	**7**	$\boxed{3}$	**2**
	D	**5**	**11**	**9**	$\boxed{7}$

Find the optimal assignment by Hungarian method.

[BNU. BE. Nov 96]

Solution : The cost matrix of the given assignment problem is

$$\begin{pmatrix} 10 & 5 & 13 & 15 \\ 3 & 9 & 18 & 3 \\ 10 & 7 & 3 & 2 \\ 5 & 11 & 9 & 7 \end{pmatrix}$$

Since the number of rows is equal to the number of columns in the cost matrix, the given assignment problem is balanced.

Select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix.

$$\begin{pmatrix} 5 & 0 & 8 & 10 \\ 0 & 6 & 15 & 0 \\ 8 & 5 & 1 & 0 \\ 0 & 6 & 4 & 2 \end{pmatrix}$$

Select the smallest cost element in each column and subtract this from all the elements of the corresponding column, we get the reduced matrix

$$\begin{pmatrix} 5 & 0 & 7 & 10 \\ 0 & 6 & 14 & 0 \\ 8 & 5 & 0 & 0 \\ 0 & 6 & 3 & 2 \end{pmatrix}$$

Since each row and each column contain atleast one zero, we shall make the assignment in rows and columns of this reduced cost matrix.

$$\begin{pmatrix} 5 & (0) & 7 & 10 \\ \cancel{0} & 6 & 14 & (0) \\ 8 & 5 & (0) & \cancel{0} \\ (0) & 6 & 3 & 2 \end{pmatrix}$$

4 = 4//

Since each row and each column contain exactly one assignment (i.e., exactly one encircled zero), the current assignment is optimal.

The optimum assignment schedule is

A → II, B → IV, C → III, D → I

and the optimum (minimum) assignment cost

= Rs. (5 + 3 + 3 + 5) = Rs. 16/-

A – II D – A

B – IV

C – III

8.6 Unbalanced Assignment Models

If the number of rows is not equal to the number columns in the cost matrix of the given assignment problem, then the given assignment problem is said to be unbalanced.

First convert the unbalanced assignment problem in to a balanced one by adding dummy rows or dummy columns with zero cost elements in the cost matrix depending upon whether $m < n$ or $m > n$ and then solve by the usual method.

Example 1 : A company has four machines to do three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table.

Machines

		1	2	3	4
	A	18	24	28	32
Jobs	B	8	13	17	19
	C	10	15	19	22

What are job assignments which will minimize the cost?

[MU. BE. 1977]

Solution : The cost matrix of the given assignment problem is

$$\begin{pmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \end{pmatrix}$$

Since the number of rows is less than the number of columns in the cost matrix, the given assignment problem is unbalanced.

To make it a balanced one, add a dummy job D (row) with zero cost elements. The balanced cost matrix is given by

$$\begin{pmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now select the smallest cost element in each row (column) and subtract this from all the elements of the corresponding row (column), we get the reduced matrix

$$\begin{pmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In this reduced matrix, we shall make the assignment in rows and columns having single zero. We have

$$\begin{pmatrix} (0) & 6 & 10 & 14 \\ \cancel{0} & 5 & 9 & 11 \\ \cancel{0} & 5 & 9 & 12 \\ \cancel{0} & (0) & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover the all zeros by drawing a minimum number of straight lines. Choose the smallest cost element not covered by these straight lines.

$$\begin{array}{c} \begin{pmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & (5) & 9 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \checkmark \\ \checkmark \\ \checkmark \\ \\ \end{matrix} \\ \checkmark \qquad\qquad\qquad\qquad \end{array}$$

Here 5 is the smallest cost element not covered by these straight lines. Subtract this 5 from all the uncovered elements, add this 5 to those elements which lie in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines. We get

$$\begin{pmatrix} 0 & 1 & 5 & 9 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 4 & 7 \\ 5 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make assignment in the rows and columns having single zero. We get

$$\begin{pmatrix} (0) & 1 & 5 & 9 \\ \cancel{0} & (0) & 4 & 6 \\ \cancel{0} & \cancel{0} & 4 & 7 \\ 5 & \cancel{0} & (0) & \cancel{0} \end{pmatrix}$$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

$$\begin{array}{cc} \begin{pmatrix} 0 & 1 & 5 & 9 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & (4) & 7 \\ 5 & 0 & 0 & 0 \end{pmatrix} & \begin{matrix} \checkmark \\ \checkmark \\ \checkmark \\ \end{matrix} \\ \begin{matrix} \checkmark & \checkmark & & \end{matrix} & \end{array}$$

Choose the smallest cost element not covered by these straight lines, subtract this from all the uncovered elements, add this to those elements which are in the intersection of the lines and do not change the remaining elements which lie on these straight lines. Thus we get

$$\begin{pmatrix} 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 9 & 4 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in the rows and columns having single zero. We get

$$\begin{pmatrix} (0) & 1 & 1 & 5 \\ \cancel{0} & (0) & \cancel{0} & 2 \\ \cancel{0} & \cancel{0} & (0) & 3 \\ 9 & 4 & \cancel{0} & (0) \end{pmatrix}$$

Since each row and each column contains exactly one assignment (*i.e.*, exactly one encircled zero) the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by $A \to 1$, $B \to 2$, $C \to 3$, $D \to 4$ and the optimum (minimum) assignment cost

$= (18 + 13 + 19 + 0)$ cost units $=$ 50/- units of cost

Note 1 : For this problem, the alternative optimum schedule is $A \to 1$, $B \to 3$, $C \to 2$, $D \to 4$, with the same optimum assignment cost = Rs. $(18 + 17 + 15 + 0)$ = 50/- units of cost.

Note 2 : Here the assignment $D \to 4$ means that the dummy Job D is assigned to the 4th Machine. It means that machine 4 is left without any assignment.

Example 2 : Assign four trucks 1,2,3 and 4 to vacant spaces A, B, C, D, E and F so that the distance travelled is minimized. The matrix below shows the distance.

	1	2	3	4
A	4	7	3	7
B	8	2	5	5
C	4	9	6	9
D	7	5	4	8
E	6	3	5	4
F	6	8	7	3

Solution : The matrix of the assignment problem is

$$\begin{pmatrix} 4 & 7 & 3 & 7 \\ 8 & 2 & 5 & 5 \\ 4 & 9 & 6 & 9 \\ 7 & 5 & 4 & 8 \\ 6 & 3 & 5 & 4 \\ 6 & 8 & 7 & 3 \end{pmatrix}$$

Since the number of rows is more than the number of columns, the given assignment problem is unbalanced. To make it balanced, let us introduce two dummy trucks (columns) with zero costs. We get

$$\begin{pmatrix} 4 & 7 & 3 & 7 & 0 & 0 \\ 8 & 2 & 5 & 5 & 0 & 0 \\ 4 & 9 & 6 & 9 & 0 & 0 \\ 7 & 5 & 4 & 8 & 0 & 0 \\ 6 & 3 & 5 & 4 & 0 & 0 \\ 6 & 8 & 7 & 3 & 0 & 0 \end{pmatrix}$$

Select the smallest cost element in each row (column) and subtract this from all the elements of the corresponding row (column). We get

$$\begin{pmatrix} 0 & 5 & 0 & 4 & 0 & 0 \\ 4 & 0 & 2 & 2 & 0 & 0 \\ 0 & 7 & 3 & 6 & 0 & 0 \\ 3 & 3 & 1 & 5 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 0 \\ 2 & 6 & 4 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} \cancel{0} & 5 & (0) & 4 & \cancel{0} & \cancel{0} \\ 4 & (0) & 2 & 2 & \cancel{0} & \cancel{0} \\ (0) & 7 & 3 & 6 & \cancel{0} & \cancel{0} \\ 3 & 3 & 1 & 5 & (0) & \cancel{0} \\ 2 & 1 & 2 & 1 & \cancel{0} & (0) \\ 2 & 6 & 4 & (0) & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since each row and each column contains exactly one assignment (*i.e.*, exactly one encircled zero), the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by $A \to 3$, $B \to 2$, $C \to 1$, $D \to 5$, $E \to 6$, $F \to 4$, and the optimum (minimum) distance.

$$= (3 + 2 + 4 + 0 + 0 + 3)$$

units of distance $=$ 12/- units of distance.

$\boxed{\text{Example 3}}$: A batch of 4 jobs can be assigned to 5 different machines. The set up time (in hours) for each job on various machines is given below :

		Machine				
		1	**2**	**3**	**4**	**5**
	1	**10**	**11**	**4**	**2**	**8**
	2	**7**	**11**	**10**	**14**	**12**
Job	**3**	**5**	**6**	**9**	**12**	**14**
	4	**13**	**15**	**11**	**10**	**7**

Find an optimal assignment of jobs to machines which will minimize the total set up time.

[BRU. BE. Nov 96, BNU. BE. Nov 96, MSU. BE. Nov 97]

Solution : The matrix of the given assignment problem is

$$\begin{pmatrix} 10 & 11 & 4 & 2 & 8 \\ 7 & 11 & 10 & 14 & 12 \\ 5 & 6 & 9 & 12 & 14 \\ 13 & 15 & 11 & 10 & 7 \end{pmatrix}$$

Since the number of rows is less than the number of columns in the cost matrix, the given assignment problem is unbalanced.

To make it a balanced one, add a dummy job 5 (row) with zero cost elements. The balanced cost matrix is given by

$$\begin{pmatrix} 10 & 11 & 4 & 2 & 8 \\ 7 & 11 & 10 & 14 & 12 \\ 5 & 6 & 9 & 12 & 14 \\ 13 & 15 & 11 & 10 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now select the smallest cost element in each row (column) and subtract this from all the elements of the corresponding row (column), we get the reduced cost matrix.

$$\begin{pmatrix} 8 & 9 & 2 & 0 & 6 \\ 0 & 4 & 3 & 7 & 5 \\ 0 & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each columns contains atleast one zero, we shall make the assignment in rows and columns of this reduced cost matrix

$$\begin{pmatrix} 8 & 9 & 2 & (0) & 6 \\ (0) & 4 & 3 & 7 & 5 \\ \cancel{0} & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & (0) \\ \cancel{0} & (0) & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since there are some rows and columns with out assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

$$\begin{array}{c c} \begin{pmatrix} 8 & 9 & 2 & 0 & 6 \\ 0 & 4 & 3 & 7 & 5 \\ 0 & (1) & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} \\ \checkmark \\ \checkmark \\ \\ \\ \end{array} \\ \begin{array}{ccccc} \checkmark & & & & \end{array} & \end{array}$$

Here 1 is the smallest cost element not covered by these straight lines. Subtract this 1 from all the uncovered elements, add this 1 to those elements which lie in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines. We get

$$\begin{pmatrix} 9 & 9 & 2 & 0 & 6 \\ 0 & 3 & 2 & 6 & 4 \\ 0 & 0 & 3 & 6 & 8 \\ 7 & 8 & 4 & 3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns of this reduced cost matrix

$$\begin{pmatrix} 9 & 9 & 2 & (0) & 6 \\ (0) & 3 & 2 & 6 & 4 \\ \cancel{0} & (0) & 3 & 6 & 8 \\ 7 & 8 & 4 & 3 & (0) \\ 1 & \cancel{0} & (0) & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since each row and each column contains exactly one assignment (i.e., exactly one encircled zero), the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by Job 1 $\rightarrow$ M/c 4, Job 2 $\rightarrow$ M/c 1, Job 3 $\rightarrow$ M/c 2, Job 4 $\rightarrow$ M/c 5. M/c 3 is left without any assignment.

The optimum (minimum) total set up time

$= 2 + 7 + 6 + 7$ hours

$= 22$ hours.

8.7 Maximization case in Assignment Problems

In an assignment problem, we may have to deal with maximization of an objective function. For example, we may have to assign persons to jobs in such a way that the total profit is maximized. The maximization problem has to be converted into an equivalent minimization problem and then solved by the usual Hungarian Method.

The conversion of the maximization problem into an equivalent minimization problem can be done by any one of the following methods :

(i) Since max Z = – min (–Z), multiply all the cost elements c_{ij} of the cost matrix by –1.

(ii) Subtract all the cost elements c_{ij} of the cost matrix from the highest cost element in that cost matrix.

Example 1 : A company has a team of four salesmen and there are four districts where the company wants to start its business. After taking into account the capabilities of salesman and the nature of districts, the company estimates that the profit per day in rupees for each salesman in each district is as below :

		Districts			
		1	2	3	4
Salesmen	A	16	10	14	11
	B	14	11	15	15
	C	15	15	13	12
	D	13	12	14	15

Find the assignment of salesmen to various districts which will yield maximum profit. *[MU. BE. Nov 93]*

Solution : The cost matrix of the given assignment problem is

$$\begin{pmatrix} (16) & 10 & 14 & 11 \\ 14 & 11 & 15 & 15 \\ 15 & 15 & 13 & 12 \\ 13 & 12 & 14 & 15 \end{pmatrix}$$

Since this is a maximization problem, it can be converted into an equivalent minimization problem by subtracting all the cost elements in the cost matrix from the highest cost element 16 of this cost matrix. Thus the cost matrix of the equivalent minimization problem is

$$\begin{pmatrix} 0 & 6 & 2 & 5 \\ 2 & 5 & 1 & 1 \\ 1 & 1 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

Select the smallest cost element in each row (column) and subtract this from all the cost elements of the corresponding row (column). We get the reduced cost matrix

$$\begin{pmatrix} 0 & 6 & 2 & 5 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 2 & 3 & 1 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} (0) & 6 & 2 & 5 \\ 1 & 4 & (0) & \cancel{0} \\ \cancel{0} & (0) & 2 & 3 \\ 2 & 3 & 1 & (0) \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by A $\rightarrow$ 1, B $\rightarrow$ 3, C $\rightarrow$ 2, D $\rightarrow$ 4 and the optimum (maximum) profit

$$= \text{Rs. } (16 + 15 + 15 + 15)$$
$$= \text{Rs. } 61/-.$$

$\boxed{\textbf{Example 2}}$ **: Solve the assignment problem for maximization given the profit matrix (profit in rupees).**

		Machines			
		P	**Q**	**R**	**S**
Job	**A**	**51**	**53**	**54**	**50**
	B	**47**	**50**	**48**	**50**
	C	**49**	**50**	**60**	**61**
	D	**63**	**64**	**60**	**60**

[MU. BE. Apr 95]

Solution : The profit matrix of the given assignment problem is

$$\begin{pmatrix} 51 & 53 & 54 & 50 \\ 47 & 50 & 48 & 50 \\ 49 & 50 & 60 & 61 \\ 63 & (64) & 60 & 60 \end{pmatrix}$$

Since this is a maximization problem, it can be converted into an equivalent minimization problem by subtracting all the profit elements in the profit matrix from the highest profit element 64 of this profit matrix. Thus the cost matrix of the equivalent minimization problem is

$$\begin{pmatrix} 13 & 11 & 10 & 14 \\ 17 & 14 & 16 & 14 \\ 15 & 14 & 4 & 3 \\ 1 & 0 & 4 & 4 \end{pmatrix}$$

Select the smallest cost in each row and subtract this from all the cost elements of the corresponding row. We get

$$\begin{pmatrix} 3 & 1 & 0 & 4 \\ 3 & 0 & 2 & 0 \\ 12 & 11 & 1 & 0 \\ 1 & 0 & 4 & 4 \end{pmatrix}$$

Select the smallest cost element in each column and subtract this from all the cost elements of the corresponding column. We get

$$\begin{pmatrix} 2 & 1 & 0 & 4 \\ 2 & 0 & 2 & 0 \\ 11 & 11 & 1 & 0 \\ 0 & 0 & 4 & 4 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} 2 & 1 & (0) & 4 \\ 2 & (0) & 2 & \cancel{0} \\ 11 & 11 & 1 & (0) \\ (0) & \cancel{0} & 4 & 4 \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by A $\rightarrow$ R, B $\rightarrow$ Q, C $\rightarrow$ S, D $\rightarrow$ P and the optimum (maximum) profit

$$= \text{Rs. } (54 + 50 + 61 + 63)$$
$$= \text{Rs. } 228/-$$

Example 3 : A company is faced with the problem of assigning four different salesman to four territories for promoting its sales. Territories are not equally rich in their sales potential and the salesman also differ in their ability to promote sales. The following table gives the expected annual sales (in thousands of Rs) for each salesman if assigned to various territories. Find the assignment of salesman so as to maximize the annual sales.

		Territories			
		1	**2**	**3**	**4**
	1	**60**	**50**	**40**	**30**
Salesmen	**2**	**40**	**30**	**20**	**15**
	3	**40**	**20**	**35**	**10**
	4	**30**	**30**	**25**	**20**

[BRU. BE. Apr 95]

Solution : The cost matrix of the given assignment problem is

$$\begin{pmatrix} (60) & 50 & 40 & 30 \\ 40 & 30 & 20 & 15 \\ 40 & 20 & 35 & 10 \\ 30 & 30 & 25 & 20 \end{pmatrix}$$

Since this is a maximization problem, it can be converted into an equivalent minimization problem by subtracting all the cost elements in the cost matrix from the highest cost element 60 of this cost matrix. Thus the cost matrix of the equivalent minimization problem is

$$\begin{pmatrix} 0 & 10 & 20 & 30 \\ 20 & 30 & 40 & 45 \\ 20 & 40 & 25 & 50 \\ 30 & 30 & 35 & 40 \end{pmatrix}$$

Select the smallest cost element in each row (column) and subtract this from all the cost elements of the corresponding row (column). We get the reduced cost matrix

$$\begin{pmatrix} 0 & 10 & 15 & 20 \\ 0 & 10 & 15 & 15 \\ 0 & 20 & 0 & 20 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we make the assignment in rows and columns of this reduced cost matrix

$$\begin{pmatrix} (0) & 10 & 15 & 20 \\ \cancel{0} & 10 & 15 & 15 \\ \cancel{0} & 20 & (0) & 20 \\ \cancel{0} & (0) & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines

$$\begin{pmatrix} 0 & 10 & 15 & 20 \\ 0 & (10) & 15 & 15 \\ 0 & 20 & 0 & 20 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Here 10 is the smallest cost element not covered by these straight lines. Subtract this 10 from all the uncovered elements, add this 10 to those elements which lie in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines. We get

$$\begin{pmatrix} 0 & 0 & 5 & 10 \\ 0 & 0 & 5 & 5 \\ 10 & 20 & 0 & 20 \\ 10 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we make the assignment in rows and columns of this reduced cost matrix.

$$\begin{pmatrix} (0) & \cancel{0} & 5 & 10 \\ \cancel{0} & (0) & 5 & 5 \\ 10 & 20 & (0) & 20 \\ 10 & \cancel{0} & \cancel{0} & (0) \end{pmatrix}$$

Since each row and each column contains exactly one assignment (*i.e.*, exactly one encircled zero), the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by Salesman 1 $\rightarrow$ Territory 1, Salesman 2 $\rightarrow$ Territory 2, Salesman 3 $\rightarrow$ Territory 3, Salesman 4 $\rightarrow$ Territory 4.

The optimum (maximum) annual sales

$= 60 + 30 + 35 + 20$ (in thousand of rupees)

$= 145$ (in thousand of rupees)

$=$ Rs. 1,45,000/-.

Note : For this problem, there exists alternative optimal assignment schedule with the same maximum sales Rs. 1,45,000/-.

8.8 Restrictions in Assignments

The assignment technique assumes that the problem is free from practical restrictions and any task could be assigned to any facility. But in some cases, it may not be possible to assign a particular task to a particular facility due to space, size of the task, process capability of the facility, technical difficulties or other restrictions. This can be overcome by assigning a very high processing time or cost element (it can be ∞) to the corresponding cell. This cell will be automatically excluded in the assignment because of the unused high time cost associated with it.

Example 1 : A machine shop purchased a drilling machine and two lathes of different capacities. The positioning of the machines among 4 possible locations on the shop floor is important from the standard of materials handling. Given the cost estimate per unit time of materials below, determine the optimal location of the machines

	Location			
	1	**2**	**3**	**4**
Lathe 1	**12**	**9**	**12**	**9**
Drill	**15**	**not suitable**	**13**	**20**
Lathe 2	**4**	**8**	**10**	**6**

[MU. BE. Apr 93]

Solution : Since the drilling machine is not suitable for location 2, the corresponding cost element should be taken as ∞. Thus the cost matrix of the given assignment problem is

$$\begin{pmatrix} 12 & 9 & 12 & 9 \\ 15 & \infty & 13 & 20 \\ 4 & 8 & 10 & 6 \end{pmatrix}$$

Since the number of rows is less than the number of columns, we add a dummy row (a dummy drilling machine or a dummy lathe 3) with zero cost elements. The cost matrix for the balanced assignment problem is

$$\begin{pmatrix} 12 & 9 & 12 & 9 \\ 15 & \infty & 13 & 20 \\ 4 & 8 & 10 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Select the smallest cost in each row (column) and subtract this from all the cost elements of the corresponding row (column). We get the reduced matrix

$$\begin{pmatrix} 3 & 0 & 3 & 0 \\ 2 & \infty & 0 & 7 \\ 0 & 4 & 6 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} 3 & (0) & 3 & \cancel{0} \\ 2 & \infty & (0) & 7 \\ (0) & 4 & 6 & 2 \\ \cancel{0} & \cancel{0} & \cancel{0} & (0) \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by

Lathe 1 $\rightarrow$ Location 2, Drill $\rightarrow$ Location 3,
Lathe 2 $\rightarrow$ Location 1, Dummy drill $\rightarrow$ Location 4

and the optimum (minimum) assignment cost

$$= (9 + 13 + 4 + 0) \text{ unit of cost}$$
$$= 26/- \text{ units of cost.}$$

Note : For this the alternate optimum assignment is

Lathe 1 $\rightarrow$ Location 4, Drill $\rightarrow$ Location 3,
Lathe 2 $\rightarrow$ Location 1, Dummy drill $\rightarrow$ Location 2.

with the same optimum (minimum) assignment cost

$$= (9 + 13 + 4 + 0) \text{ units of cost}$$
$$= 26/- \text{ units of cost.}$$

Example 2 : Five workers are available to work with the machines and the respective costs (in rupees) associated with each worker - machine assignment is given below. A sixth machine is available to replace one of the existing machines and the associated costs are also given below :

Machines

		M_1	M_2	M_3	M_4	M_5	M_6
	W_1	12	3	6	–	5	8
	W_2	4	11	–	5	–	3
Workers	W_3	8	2	10	9	7	5
	W_4	–	7	8	6	12	10
	W_5	5	8	9	4	6	–

(i) **Determine whether the new machine can be accepted ?**

(ii) **Determine also optimal assignment and the associated saving in cost.**

Solution : The cost matrix of the given assignment problem is

$$\begin{pmatrix} 12 & 3 & 6 & \infty & 5 & 8 \\ 4 & 11 & \infty & 5 & \infty & 3 \\ 8 & 2 & 10 & 9 & 7 & 5 \\ \infty & 7 & 8 & 6 & 12 & 10 \\ 5 & 8 & 9 & 4 & 6 & \infty \end{pmatrix}$$

Since the number of rows is less than the number of columns, the given assignment problem is unbalanced. Add a dummy worker W_6 (dummy row) with zero cost elements.

Thus the cost matrix of the balanced assignment problem is

$$\begin{pmatrix} 12 & 3 & 6 & \infty & 5 & 8 \\ 4 & 11 & \infty & 5 & \infty & 3 \\ 8 & 2 & 10 & 9 & 7 & 5 \\ \infty & 7 & 8 & 6 & 12 & 10 \\ 5 & 8 & 9 & 4 & 6 & \infty \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Select the smallest cost in each row and column and subtract this from all the cost elements of the corresponding row and column of the cost matrix. We get

$$\begin{pmatrix} 9 & 0 & 3 & \infty & 2 & 5 \\ 1 & 8 & \infty & 2 & \infty & 0 \\ 6 & 0 & 8 & 7 & 5 & 3 \\ \infty & 1 & 2 & 0 & 6 & 4 \\ 1 & 4 & 5 & 0 & 2 & \infty \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} 9 & (0) & 3 & \infty & 2 & 5 \\ 1 & 8 & \infty & 2 & \infty & (0) \\ 6 & \cancel{0} & 8 & 7 & 5 & 3 \\ \infty & 1 & 2 & (0) & 6 & 4 \\ 1 & 4 & 5 & \cancel{0} & 2 & \infty \\ (0) & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since some rows and columns are without assignment, the current assignment is not optimal.

Cover all the zeros by drawing minimum number of straight lines.

$$\begin{array}{c c} \begin{pmatrix} 9 & 0 & 3 & \infty & 2 & 5 \\ 1 & 8 & \infty & 2 & \infty & 0 \\ 6 & 0 & 8 & 7 & 5 & 3 \\ \infty & 1 & 2 & 0 & 6 & 4 \\ (1) & 4 & 5 & 0 & 2 & \infty \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} \checkmark \\ \\ \checkmark \\ \checkmark \\ \checkmark \\ \\ \end{array} \\ \begin{array}{cccccc} \quad & \checkmark & \quad & \checkmark & \quad & \quad \end{array} & \end{array}$$

Choose the smallest cost element not covered by these straight lines. Here 1 is such element. Subtract this 1 from all the uncovered elements, add this 1 to those elements which are in the intersection of the straight lines and do not change the remaining elements which lie on these straight lines. We get

$$\begin{pmatrix} 8 & 0 & 2 & \infty & 1 & 4 \\ 1 & 9 & \infty & 3 & \infty & 0 \\ 5 & 0 & 7 & 7 & 4 & 2 \\ \infty & 1 & 1 & 0 & 5 & 3 \\ 0 & 4 & 4 & 0 & 1 & \infty \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Now we shall make the assignment in rows and columns having single zero.

$$\begin{pmatrix} 8 & (0) & 2 & \infty & 1 & 4 \\ 1 & 9 & \infty & 3 & \infty & (0) \\ 5 & \cancel{0} & 7 & 7 & 4 & 2 \\ \infty & 1 & 1 & (0) & 5 & 3 \\ (0) & 4 & 4 & \cancel{0} & 1 & \infty \\ 0 & 1 & (0) & 1 & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since some rows and columns are without assignment, the current assignment is not optimal.

Cover all the zeros by drawing minimum number of straight lines.

$$\begin{array}{c c} \begin{pmatrix} 8 & 0 & 2 & \infty & (1) & 4 \\ 1 & 9 & \infty & 3 & \infty & 0 \\ 5 & 0 & 7 & 7 & 4 & 2 \\ \infty & 1 & 1 & 0 & 5 & 3 \\ 0 & 4 & 4 & 0 & 1 & \infty \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} & \begin{array}{c} \checkmark \\ \\ \checkmark \\ \\ \\ \\ \end{array} \\ \begin{array}{cccccc} \quad & \checkmark & \quad & \quad & \quad & \quad \end{array} & \end{array}$$

Subtract the smallest uncovered element 1 from all the uncovered elements, add this 1 to those elements which are in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines. We get

$$\begin{pmatrix} 7 & 0 & 1 & \infty & 0 & 3 \\ 1 & 10 & \infty & 3 & \infty & 0 \\ 4 & 0 & 6 & 6 & 3 & 1 \\ \infty & 2 & 1 & 0 & 5 & 3 \\ 0 & 5 & 4 & 0 & 1 & \infty \\ 0 & 2 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Now we shall make the assignment in rows and columns having single zeros.

$$\begin{pmatrix} 7 & \cancel{0} & 1 & \infty & (0) & 3 \\ 1 & 10 & \infty & 3 & \infty & (0) \\ 4 & (0) & 6 & 6 & 3 & 1 \\ \infty & 2 & 1 & (0) & 5 & 3 \\ (0) & 5 & 4 & \cancel{0} & 1 & \infty \\ \cancel{0} & 2 & (0) & 1 & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimum assignment schedule is given by

$W_1 \rightarrow M_5$, $W_2 \rightarrow M_6$, $W_3 \rightarrow M_2$, $W_4 \rightarrow M_4$, $W_5 \rightarrow M_1$, $W_6 \rightarrow M_3$, and the optimum (minimum) assignment cost according to this schedule is

$$= \text{Rs. } (5 + 3 + 2 + 6 + 5 + 0)$$
$$= \text{Rs. } 21/-$$

Now, if the sixth machine M_6 is not assigned to any of the workers, the given problem reduces to balanced one (deleting the sixth column). Applying the assignment algorithm to this balanced problem (reduced problem), the optimal assignment schedule is given by

$W_1 \rightarrow M_5$, $W_2 \rightarrow M_1$, $W_3 \rightarrow M_2$, $W_4 \rightarrow M_3$, $W_5 \rightarrow M_4$, and the optimum (minimum) assignment cost according to this schedule is

$$= \text{Rs. } (5 + 4 + 2 + 8 + 4)$$
$$= \text{Rs. } 23/-$$

It is clear from the above that the minimum cost is more when there are only five machines. Hence, the sixth machine should be accepted. By accepting this sixth machine the associated saving cost will be Rs. (23 – 21) = Rs. 2.

8.9 Travelling Salesman Problem

A salesman normally must visit a number of cities starting from his head quarters. The distance (or time or cost) between every pair of cities are assumed to be known. The problem of finding the shortest distance (or minimum time or minimum cost) if the salesman starts from his headquarters and passes through each city under his Jurisdiction exactly once and returns to the headquarters is called the ***Travelling salesman problem or A Travelling salesperson problem.***

A travelling salesman problem is very similar to the assignment problem with the additional constraints.

(a) The salesman should go through every city exactly once except the starting city (headquarters).

(b) The salesman starts from one city (head quarters) and comes back to that city (headquarters).

(c) Obviously going from any city to the same city directly is not allowed (*i.e.*, no assignments should be made along the diagonal line).

Note 1 : Conditions *(a)* and *(b)* are usually called ***route conditions.***

Note 2 : If a salesman has to visit n cities, then he will have a total of $(n-1)!$ possible round trips.

Therefore, the necessary basic steps to solve a travelling salesman problem are :

(i) Assigning an infinitely large element (∞) in each of the squares along the diagonal line in the cost matrix.

(ii) Solving the problem as a routine assignment problem.

(iii) Scrutinizing the solution obtained under *(ii)* to see if the "route" conditions are satisfied.

(iv) If not, making adjustments in assignments to satisfy the condition with minimum increase in total cost (*i.e.*, to satisfy route condition "next best solution" may require to be considered).

Example 1 : Solve the following travelling salesman problem

		To			
		A	B	C	D
	A	–	46	16	40
From	B	41	–	50	40
	C	82	32	–	60
	D	40	40	36	–

[MU. BE. Apr 93]

Solution : The cost matrix of the given travelling salesman problem is

$$\begin{pmatrix} \infty & 46 & 16 & 40 \\ 41 & \infty & 50 & 40 \\ 82 & 32 & \infty & 60 \\ 40 & 40 & 36 & \infty \end{pmatrix}$$

Solve this as a routine assignment problem

Subtract the smallest cost element in each row from all the elements of the corresponding row. We get.

$$\begin{pmatrix} \infty & 30 & 0 & 24 \\ 1 & \infty & 10 & 0 \\ 50 & 0 & \infty & 28 \\ 4 & 4 & 0 & \infty \end{pmatrix}$$

Subtract the smallest cost element in each column from all the elements of the corresponding column. We get

$$\begin{pmatrix} \infty & 30 & 0 & 24 \\ 0 & \infty & 10 & 0 \\ 49 & 0 & \infty & 28 \\ 3 & 4 & 0 & \infty \end{pmatrix}$$

Now we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} \infty & 30 & (0) & 24 \\ (0) & \infty & 10 & \cancel{0} \\ 49 & (0) & \infty & 28 \\ 3 & 4 & \cancel{0} & \infty \end{pmatrix}$$

Since some rows and columns are with out assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

$$\begin{pmatrix} \infty & 30 & 0 & 24 \\ 0 & \infty & 10 & 0 \\ 49 & 0 & \infty & 28 \\ (3) & 4 & \cancel{0} & \infty \end{pmatrix} \begin{matrix} \checkmark \\ \\ \\ \checkmark \end{matrix}$$

Subtract the smallest uncovered cost element 3 from all uncovered elements, add this 3 to those elements which are in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines. We have

$$\begin{pmatrix} \infty & 27 & 0 & 21 \\ 0 & \infty & 13 & 0 \\ 49 & 0 & \infty & 28 \\ 0 & 1 & 0 & \infty \end{pmatrix}$$

Now we shall make the assignment in rows and columns having single zeros. We get

$$\begin{pmatrix} \infty & 27 & (0) & 21 \\ \cancel{0} & \infty & 13 & (0) \\ 49 & (0) & \infty & 28 \\ (0) & 1 & \cancel{0} & \infty \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal for the assignment problem.

$\therefore$ The optimum assignment schedule is given by

$A \rightarrow C, \quad B \rightarrow D, \quad C \rightarrow B, \quad D \rightarrow A,$

i.e., $A \rightarrow C, \quad C \rightarrow B, \quad B \rightarrow D, \quad D \rightarrow A,$

i.e., $A \rightarrow C \rightarrow B \rightarrow D \rightarrow A$

Check whether the route conditions are satisfied.

$A \rightarrow C \rightarrow B \rightarrow D \rightarrow A$ satisfies the route condition.

$\therefore$ The required minimum costs.

$= (16 + 32 + 40 + 40)$ units of cost.

$=$ 128/- units of cost.

Example 2 : **Solve the following travelling salesman problem so as to minimize the cost per cycle.**

		To				
		A	**B**	**C**	**D**	**E**
	A	–	3	6	2	3
From	**B**	3	–	5	2	3
	C	6	5	–	6	4
	D	2	2	6	–	6
	E	3	3	4	6	–

[MU. BE. 85, Nov 93]

Solution : The cost matrix of the given travelling salesman problem is

$$\begin{pmatrix} \infty & 3 & 6 & 2 & 3 \\ 3 & \infty & 5 & 2 & 3 \\ 6 & 5 & \infty & 6 & 4 \\ 2 & 2 & 6 & \infty & 6 \\ 3 & 3 & 4 & 6 & \infty \end{pmatrix}$$

Subtract the smallest cost element in each row from all the elements of the corresponding row. We get

$$\begin{pmatrix} \infty & 1 & 4 & 0 & 1 \\ 1 & \infty & 3 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 4 & \infty & 4 \\ 0 & 0 & 1 & 3 & \infty \end{pmatrix}$$

Subtract the smallest cost element in each column from all the elements of the corresponding column. We get

$$\begin{pmatrix} \infty & 1 & 3 & 0 & 1 \\ 1 & \infty & 2 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 3 & \infty & 4 \\ 0 & 0 & 0 & 3 & \infty \end{pmatrix}$$

Now we shall make the assignment in rows and columns having single zeros. We get

$$\begin{pmatrix} \infty & 1 & 3 & (0) & 1 \\ 1 & \infty & 2 & \cancel{0} & 1 \\ 2 & 1 & \infty & 2 & (0) \\ \cancel{0} & (0) & 3 & \infty & 4 \\ \cancel{0} & \cancel{0} & (0) & 3 & \infty \end{pmatrix}$$

Since some rows and columns are without assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

$$\begin{array}{c c} \begin{pmatrix} \infty & (1) & 3 & 0 & 1 \\ 1 & \infty & 2 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 3 & \infty & 4 \\ 0 & 0 & 0 & 3 & \infty \end{pmatrix} & \begin{matrix} \checkmark \\ \checkmark \\ \\ \\ \\ \end{matrix} \\ \begin{matrix} \quad & \quad & \quad & \quad\quad\quad \checkmark & \quad \end{matrix} & \end{array}$$

Subtract the smallest uncovered cost element 1 from all uncovered elements, add this 1 to those elements which are in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines. We have

$$\begin{pmatrix} \infty & 0 & 2 & 0 & 0 \\ 0 & \infty & 1 & 0 & 0 \\ 2 & 1 & \infty & 3 & 0 \\ 0 & 0 & 3 & \infty & 4 \\ 0 & 0 & 0 & 4 & \infty \end{pmatrix}$$

Now we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} \infty & \cancel{0} & 2 & (0) & \cancel{0} \\ (0) & \infty & 1 & \cancel{0} & \cancel{0} \\ 2 & 1 & \infty & 3 & (0) \\ \cancel{0} & (0) & 3 & \infty & 4 \\ \cancel{0} & \cancel{0} & (0) & 4 & \infty \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

$\therefore$ The optimal assignment schedule is given by

$A \to D,\quad B \to A,\quad C \to E,\quad D \to B,\quad E \to C$

i.e., $A \to D \to B \to A,\quad C \to E \to C$

and the corresponding optimum (minimum) assignment cost

$= (2 + 3 + 4 + 2 + 4)$ units of cost

$=$ 15/- units of cost.

But this assignment schedule does not provide the solution of this travelling salesman problem, because it does not satisfy the 'route' condition.

We try to find the next best solution which satisfies the route condition also. The next minimum (non-zero) cost element in the cost matrix is 1. So we try to bring 1 in to the solution. But the 1 occurs at two places. We shall consider all the cases separately until the acceptable solution is reached.

We start with making an assignment at (2, 3) instead of zero assignment at (2, 1). The resulting feasible solution will then be

$$\begin{pmatrix} \infty & \cancel{0} & 2 & (0) & \cancel{0} \\ \cancel{0} & \infty & (1) & \cancel{0} & \cancel{0} \\ 2 & 1 & \infty & 3 & (0) \\ \cancel{0} & (0) & 3 & \infty & 4 \\ (0) & \cancel{0} & \cancel{0} & 4 & \infty \end{pmatrix}$$

$\therefore$ The optimum assignment is given by

$A \to D,\quad B \to C,\quad C \to E,\quad D \to B,\quad E \to A,$

i.e., $A \to D \to B \to C \to E \to A$

Also, when an assignment is made at (3, 2) instead of zero assignment at (3, 5), the resulting feasible solution will be

$$\begin{pmatrix} \infty & \cancel{0} & 2 & \cancel{0} & (0) \\ \cancel{0} & \infty & 1 & (0) & \cancel{0} \\ 2 & (1) & \infty & 3 & \cancel{0} \\ (0) & \cancel{0} & 3 & \infty & 4 \\ \cancel{0} & \cancel{0} & (0) & 4 & \infty \end{pmatrix}$$

$\therefore$ The optimum assignment is given by

$A \to E, \quad B \to D, \quad C \to B, \quad D \to A, \quad E \to C,$

i.e., $A \to E \to C \to B \to D \to A$

$\therefore$ For the given travelling salesman problem, the optimum assignment schedule is given by

$A \to D \to B \to C \to E \to A$, (or)

$A \to E \to C \to B \to D \to A$

In both cases, the optimum (minimum) assignment cost is 16/- units of cost.

EXERCISE

1. **What are assignment problems? Describe Mathematical formulation of an assignment problem?**

   [MU. BE. Apr 93, Oct 96, BRU. BE. Nov 96, MU. MBA. Nov 96,]

2. **Distinguish between transportation model and assignment model.** ***[MU. BE. Nov 94]***

3. **Explain how the assignment problem can be treated as a particular case of transportation problem? Why this method is not preferred ?** ***[MU. MBA Nov. 95, Apr.97]***

4. **Explain the steps in the Hungarian Method used for solving assignment problems.**

   [MU. MBA. Apr 95, BRU. BE. Nov 96, MU. MCA. Nov 95]

5. **Define an unbalanced assignment problem and describe the steps involved in solving it.**

6. **Explain how maximization problems are solved using assignment model technique ?**

7. **What do you understand by restricted assignments? Explain how should one overcome it ?**

8. **Enumerate the steps to solve an unbalanced profit maximization problem containing one or more restricted assignments ?**

9. **Is it possible to have more than one optimal solution to an assignment problem ? How is the presence of an alternate solution established ?**

10. **What is the difference between assignment problem and travelling salesman problem?**

11. **What is the objective of the travelling salesman problem ?** ***[MU. BE. Oct 96]***

12. **What is travelling salesman problem ?** ***[BRU. BE. Nov 96]***

13. **Solve the assignment problem**

	A	B	C	D
I	1	4	6	3
II	9	7	10	9
III	4	5	11	7
IV	8	7	8	5

[MU. BE. Apr 91]

14. **Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows :**

Jobs

Person		1	2	3	4	5
	A	8	5	2	6	1
	B	0	9	5	5	4
	C	3	8	9	2	6
	D	4	3	1	0	3
	E	9	5	8	9	5

Determine the optimum assignment schedule.

[MU. BE. Apr 90, Apr 91]

15. **A department has four subordinates and four tasks are to be performed. The subordinates differ in efficiency and tasks differ in their intrinsic difficulties. The estimate of time (in hours) each man would take to perform each task is given by**

		Tasks		
	I	II	III	IV
1	8	26	17	11
Subordinate 2	13	28	4	26
3	38	19	18	15
4	19	26	24	10

Find out how the tasks be allotted to each subordinate so as to optimize the total man-hours. ***[MU. BE. Nov 91]***

16. **Solve the assignment problem**

	Job			
	P	Q	R	S
A	18	26	17	11
Machine B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

[MU. BE. Oct 95]

17. **A department head has four tasks to be performed by three subordinates, the subordinates differing in efficiency. The estimates of the time, each subordinate would take to perform, is given below in the matrix. How should he allocate the tasks one to each man, so as to minimize the total man-hour ?**

	Men		
	1	2	3
I	9	26	15
Tasks II	13	27	6
III	35	20	15
IV	18	30	20

[MU. BE. Nov 92]

18. **A company has four machines on which to do three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table :**

	Machine			
	P	Q	R	S
A	18	24	28	32
Job B	8	13	17	19
C	10	15	19	22

What are the job assignments which will minimize the cost ? ***[MU. BE. Apr 87, MBA. Apr 95]***

19. **Solve the following unbalanced problem of assigning four jobs to three different men (only one job to each man). The time to perform the job by different men is given in the following table :**

	Job			
	J_1	J_2	J_3	J_4
Men M_1	7	5	8	4
M_2	5	6	7	4
M_3	8	7	9	8

[MU. BE. Nov 92]

20. **Solve the following assignment problem to find the maximum total expected sale.**

	Area			
	I	II	III	IV
A	42	35	28	21
Salesman B	30	25	20	15
C	30	25	20	15
D	24	20	16	12

[MU. BE. Nov 90]

21. **A sales manager has to assign salesmen to four territories. He has 4 candidates of varying experience and capabilities and assesses the possible profit for each salesman in each territory as given below. Find the assignment which maximises the profit.**

	Territory			
	A	B	C	D
1	35	27	28	37
Salesman 2	28	34	29	40
3	35	24	32	28
4	24	32	25	28

[MU. BE. Nov 92]

22. A company has 5 jobs to be done. The following data shows the return (in rupees) by assigning the ith machine to the jth job. Using Hungarian method, assign the 5 jobs to the 5 machines so as to maximize the total expected profit.

Job

Machine	1	2	3	4	5
A	62	78	50	101	82
B	71	84	61	73	59
C	87	92	111	70	81
D	45	64	87	77	80
E	60	70	98	66	83

[MU. BE. Apr 91, Apr 98, BRU. ME. 81, MKU. BE. Nov 97]

23. A company is faced with the problem of assigning 4 machines to 6 different jobs (one machine to one job only). The profits are estimated as follows :

Machine

Job	A	B	C	D
1	3	6	2	6
2	7	1	4	4
3	3	8	5	8
4	6	4	3	7
5	5	2	4	3
6	5	7	6	4

Solve the problem to maximize the total profit.

[MU. BE. Apr 90, Nov 94]

24. Five operators have to be assigned to five machines. The assignment costs are given in the table below :

Machine

Operator	I	II	III	IV	V
A	5	5	–	2	6
B	7	4	2	3	4
C	9	3	5	–	3
D	7	2	6	7	2
E	6	5	7	9	1

Operator A cannot operate machine III and operator C cannot operate machine IV. Find the optimal assignment schedule.

25. Solve the following assignment problem

Task

Machine	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	Not suitable	5	4
M_3	Not suitable	6	9	6	2
M_4	9	3	7	2	3

26. Solve the following assignment problem

Machine

Task	1	2	3	4	5
A	7	7	∞	4	8
B	9	6	4	5	6
C	11	5	7	∞	5
D	9	4	8	9	4
E	8	7	9	11	3

27. Given the following matrix of setup costs show how to sequence production so as to minimize setup cost per cycle.

To

From	A	B	C	D	E
A	–	2	5	7	1
B	6	–	3	8	2
C	8	7	–	4	7
D	12	4	6	–	5
E	1	3	2	8	–

28. Solve the following travelling salesman problem so as to minimize cost per cycle :

To city

From city	1	2	3	4	5
1	∞	10	25	25	10
2	1	∞	10	15	2
3	8	9	∞	20	10
4	14	10	24	∞	15
5	10	8	25	27	∞

29. A salesman has to visit five cities A, B, C, D, and E. The distances (in hundred km) between the five cities are as follows :

To

From	A	B	C	D	E
A	–	7	6	8	4
B	7	–	8	5	6
C	6	8	–	9	7
D	8	5	9	–	8
E	4	6	7	8	–

If the salesman starts from city A and has to come back to city A, which route he should select so that the total distance travelled by him is minimized.

30. Solve the following travelling salesman problem :

	A	B	C	D	E	F
A	∞	5	12	6	4	8
B	6	∞	10	5	4	3
C	8	7	∞	6	3	11
D	5	4	11	∞	5	8
E	5	2	7	8	∞	4
F	6	3	11	5	4	∞

31. Solve the travelling salesman problem given by the following data.

$c_{12} = 20,\quad c_{13} = 4,\quad c_{14} = 10,\quad c_{23} = 5,\quad c_{34} = 6,$

$c_{25} = 10,\quad c_{35} = 6,\quad c_{45} = 20,\quad$ where $c_{ij} = c_{ji}$

and there is no route between cities i and j if a value for c_{ij} is not shown above.

32. A company has five jobs to be done one five machines ; any job can be done on any machine. The costs of doing the jobs in different machines are given below. Assign the jobs for different machines so as to minimize the total cost.

Machines

Jobs	A	B	C	D	E
I	13	8	16	18	19
II	9	15	24	9	12
III	12	9	4	4	4
IV	6	12	10	8	13
V	15	17	18	12	20

[MU. BE. Oct 96]

33. The owner of a small machine shop has four machinists available to assign to jobs for the day. Five jobs are offered with expected profit for each machinist on each job as follows:

Jobs

Machinist	A	B	C	D	E
M_1	12	28	0	51	32
M_2	12	34	11	23	9
M_3	37	42	61	21	31
M_4	0	14	37	27	30

Assign machinists to jobs which results in overall maximum profit. Which job should be declined ?

[MU. MBA. Nov 96]

34. A machine tool company decides to make four sub assemblies through four contractors. Each contractor is to receive only one subassembly. The cost of each subassembly is determined by the bids submitted by each contractor and is shown in table given below in hundreds of rupees. Assign the different subassemblies to contractors so as to minimize the total cost.

Table

		Contractors			
		1	2	3	4
	1	15	13	14	17
Subassemblies	2	11	12	15	13
	3	13	12	10	11
	4	15	17	14	16

Assign machinists to jobs which results in overall maximum profit. Which job should be declined ?

[MU. B. Tech. Leather. Oct 96]

35. The R and D company has recently requested the skill testing agency to test four applicants for the three jobs that are available at this time. Each job has a primary skill and R and D's objective is to pick the three applicants whose aptitude test scores will maximize R and D's total performance. Only one applicant can be assigned to each job. Their aptitude test scores is listed below :

		Job		
		A	B	C
	1	95	110	103
Applicant	2	89	95	100
	3	120	132	118
	4	107	119	112

Determine the three best applicants for the three jobs. What are their total aptitude test scores ? *[BRU. BE. Apr 96]*

ANSWERS

13. $1 \longrightarrow A, 2 \longrightarrow C, 3 \longrightarrow B, 4 \longrightarrow D,$
minimum cost Rs. 21.

14. $A \longrightarrow 5, B \longrightarrow 1, C \longrightarrow 4, D \longrightarrow 3, E \longrightarrow 2,$
minimum cost Rs. 9.

15. $1 \longrightarrow I, 2 \longrightarrow III, 3 \longrightarrow II, 4 \longrightarrow IV,$
minimum time 41 hours.

16. $A \longrightarrow R, B \longrightarrow P, C \longrightarrow Q, D \longrightarrow S,$
minimum cost Rs. 59.

17. $I \longrightarrow 1, II \longrightarrow 3, III \longrightarrow 2, IV \longrightarrow 4$
minimum time 35 hours.

18. $A \longrightarrow P, B \longrightarrow Q, C \longrightarrow R, D \longrightarrow S,$
minimum cost Rs. 50.

19. $M_1 \longrightarrow J_4, M_2 \longrightarrow J_1, M_3 \longrightarrow J_2, M_4 \longrightarrow J_3,$
minimum cost Rs. 16.

20. $A \longrightarrow I, B \longrightarrow II, C \longrightarrow III, D \longrightarrow IV,$ (or)
$A \longrightarrow I, B \longrightarrow III, C \longrightarrow II, D \longrightarrow IV,$
maximum total expected sale = Rs. 99000.

21. $1 \longrightarrow A, 2 \longrightarrow D, 3 \longrightarrow C, 4 \longrightarrow B,$
maximum profit Rs. 139.

22. $A \longrightarrow 4, B \longrightarrow 2, C \longrightarrow 1, D \longrightarrow 5, E \longrightarrow 3,$
maximum profit is Rs. 450.

23. $2 \longrightarrow A, 3 \longrightarrow B, 4 \longrightarrow D, 6 \longrightarrow C$
maximum profit is Rs. 28.

24. $A \longrightarrow IV, B \longrightarrow III, C \longrightarrow II, D \longrightarrow I, E \longrightarrow V,$ (or)
$A \longrightarrow IV, B \longrightarrow III, C \longrightarrow V, D \longrightarrow II, E \longrightarrow I,$
minimum cost Rs. 15.

25. $M_1 \longrightarrow A$, $M_2 \longrightarrow B$, $M_3 \longrightarrow E$, $M_4 \longrightarrow D$, $M_5 \longrightarrow C$

minimum assignment cost is Rs. 12.

26. $A \longrightarrow 4$, $B \longrightarrow 3$, $C \longrightarrow 2$, $D \longrightarrow 1$, $E \longrightarrow 5$

minimum assignment cost is Rs. 25.

27. $A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow A$

minimum cost is Rs. 15.

28. $1 \longrightarrow 5 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 1$

minimum cost is Rs. 62.

29. $A \longrightarrow E \longrightarrow B \longrightarrow D \longrightarrow C \longrightarrow A$

minimum distance is Rs. 30 (in hundred km).

30. $A \longrightarrow B \longrightarrow F \longrightarrow E \longrightarrow C \longrightarrow D \longrightarrow A$

minimum cost is Rs. 30.

31. $A \longrightarrow C \longrightarrow B \longrightarrow D \longrightarrow A$

minimum cost is Rs. 19.

32. $I \longrightarrow B$, $II \longrightarrow E$, $III \longrightarrow C$, $IV \longrightarrow A$, $V \longrightarrow D$

minimum cost is Rs. 42/-.

33. $M_1 \longrightarrow D$, $M_2 \longrightarrow B$, $M_3 \longrightarrow C$, $M_4 \longrightarrow E$, $M_5 \longrightarrow A$

The maximum profit is Rs. 176/-. Job A should be declined.

34. $1 \longrightarrow 2$, $2 \longrightarrow 1$, $3 \longrightarrow 4$, $4 \longrightarrow 3$, and the minimum assignment cost is Rs. 49/-.

35. Applicant 1 $\rightarrow$ Job 3, Applicant 3 $\rightarrow$ Job 1, Applicant 4 $\rightarrow$ Job 2, and their total aptitude test scores = 103 + 120 + 119 = 342.

Chapter 9

Integer Programming

9.1 Introduction

A linear programming problem in which some or all of the variables in the optimal solution are restricted to assume non-negative integer values is called an ***integer programming problem*** [or ***I.P.P*** or ***integer linear programming***].

In a linear programming problem, if all the variables in the optimal solution are restricted to assume non-negative integer values, then it is called the ***Pure (all) integer programming problem [Pure I.P.P].***

In a linear programming problem, if only some of the variables in the optimal solution are restricted to assume non-negative integer values, while the remaining variables are free to take any non-negative values, then it is called a ***Mixed integer programming problem [Mixed I.P.P]***

Further, if all the variables in the optimum solution are allowed to take values either 0 or 1 as in 'do' or 'not to do' type decisions, then the problem is called the ***Zero–one programming problem (or) standard discrete programming problem.***

The general integer programming problem is given by

$$\text{Maximize } Z = CX$$

subject to the constraints

$$AX \leq b,$$

$$X \geq 0 \text{ and some or all variables are integers.}$$

Importance of integer Programming :

In linear programming problem, all the decision variables were allowed to take any non-negative real (continuous or fractional) values, as it is quite possible and appropriate to have fractional values in many situations. For example, it is quite possible to use 6.38 kg of raw material, or 5.62 machine hours etc. However in many situations, especially in business and industry, these decision variables make sense only if they have integer values in the optimal solution. For example, it is meaningless to produce 8.13 chairs or 6.85 tables, or to open 3.83 branches of a bank