Chapter 4

ORDINARY DIFFERENTIAL EQUATIONS

4.1. INTRODUCTION

A **differential equation** is a mathematical equation involving an unknown function and its derivatives.

For example,

(i) $$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^{-2x}$$

(ii) $$\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2+2y=\sin 3x$$

(iii) $$\left(\frac{d^2y}{dx^2}\right)^2+2\frac{dy}{dx}+y=5x$$

(iv) $$\frac{dy}{dx}+3y=5x$$

Differential equations arise in many areas of science and technology; whenever a deterministic relationship involving some continuously changing quantities (modeled by functions) and their rates of change (expressed as derivatives) is known or postulated. This is well illustrated by classical mechanics, where the motion of a body is described by its position and velocity as the time varies. Newton's Laws allow one to relate the position, velocity, acceleration and various forces acting on the body and state this relation as a differential equation for the unknown position of the body as a function of time.

A simple example is Newton's second law of motion, which leads to the differential equation $F = m\frac{dv}{dt}$, where F is the force vector, m is the mass of the body, v is the velocity vector and t is time.

The **order** of a differential equation is the order of the highest derivative of the unknown function involved in the equation. The order of the differential equations (i), (ii) and (iii) is two where as the order of the differential equation (iv) is one.

The **degree** of a differential equation is the degree of the highest derivative of the unknown function involved in the equation, after it is expressed free from radicals. The degree of the differential equations (i), (ii) and (iv) is one where as the order of the differential equation (iii) is two and degree is also two.

The relation between the dependent and independent variables not involving derivatives is called the **solution** (integral) of the differential equation.

For example, consider the differential equation $\frac{dy}{dx} = 2x$. Its solution is $y = x^2 + c$, where c is some constant. Also consider the differential equation $\frac{d^2y}{dx^2} + y = 0$. Its solution is $y = A\cos x + B\sin x$, where A and B are constants.

In general, the number of arbitrary constants in the solution of a differential equation is equal to the order of that differential equation. Such a solution is called **general (complete) solution** of the differential equation. The above two solutions are general solutions.

4. 2. LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form

$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + a_n y = F(x)$, where $a_0, a_1, a_2, a_3, \ldots\ldots, a_n$ are constants, is called a **Linear differential equation of degree n with `constant coefficients.**

Let $\frac{d}{dx} = D, \frac{d^2}{dx^2} = D^2$, *etc.* Then the above equation can be written as $(a_0 D^n + a_1 D^{n-1} + + a_2 D^{n-2} + \dots\dots + a_n)y = F(x)$.

i.e. $\phi(D)y = F(x)$ (1)

The general or complete solution of (1) consists of two parts namely (i) Complementary Function (CF) and the (ii) Particular Integral (PI).

That is $y = CF+PI$ (2)

To find the Complementary function

Form the auxiliary equation (AE) by putting $D = m$ in $\phi(D) = 0$. Therefore the auxiliary equation of (1) is $\phi(m) = 0$ (3)

which will be a polynomial equation of degree n. By solving this equation we get n roots say $m_1, m_2, m_3, \dots\dots, m_n$.

Case (i) If all the roots are real and unequal, i.e. if $m_1 \neq m_2 \neq m_3 \neq \dots \dots \neq m_n$, then $CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$.

Case (ii) If $m_1 = m_2 = m$ and the remaining be real and unequal, then

$CF = (c_1 + c_2 x)e^{mx} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$.

Case (iii) If $m_1 = m_2 = m_3 = m$ and the remaining be real and unequal, then $CF = (c_1 + c_2 x + c_3 x^2)e^{mx} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$.

Case (iv) If roots are imaginary, i.e. if $m = \alpha \pm i\beta$, then

$CF = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$.

To find the Particular integral

Let the given differential equation be $\phi(D)y = F(x)$. If the RHS is zero, i. e. if $F(x) = 0$, then there is no particular integral. In this case the complementary function alone constitute the complete solution of the given differential equation. On the other hand if $F(x) \neq 0$, then we have PI also. The PI is given by $PI = \frac{1}{\phi(D)} F(x)$.

$\boxed{TYPE\ 1:}$ If $F(x) = e^{ax}$, then

$$PI = \frac{1}{\phi(D)} F(x) = \frac{1}{\phi(D)} e^{ax}$$

$$= \frac{1}{\phi(a)} e^{ax}, \quad provided\ \phi(a) \neq 0.$$

$$If\ \phi(a)=0,\ then\ PI = \frac{1}{\phi(D)} e^{ax} = x.\frac{1}{\phi'(D)} e^{ax}$$

$$= x.\frac{1}{\phi'(a)} e^{ax}, provided\ \phi'(a) \neq 0$$

Here $\phi'(D)$ means derivative of $\phi(D)$ with respect to D.

$$\text{If } \phi'(a)=0,\ then\ PI = x^2 \frac{1}{\phi''(D)} e^{ax}$$

$$= x^2 \frac{1}{\phi''(a)} e^{ax},\ provided\ \phi''(a) \neq 0$$

and so on.

Example 1

Solve: $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$

Solution: Given $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$

i. e. $(D^2 - 7D + 12)y = 0$

i. e. $\phi(D)y = 0$

To find CF:

Put $D = m$ in $\phi(D) = 0$, the auxiliary equation is $m^2 - 7m + 12 = 0$

$\Rightarrow (m - 3)(m - 4) = 0 \Rightarrow m = 3, 4 \Rightarrow m_1 = 3, m_2 = 4$ are real

and $m_1 \neq m_2$

$$\therefore CF = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{3x} + C_2 e^{4x}$$

Since $F(x) = 0$, there is no PI. The complete solution is

$$y = C_1e^{3x} + C_2e^{4x}.$$

Example 2 ....

Solve: $(D^3 + 3D^2 + 3D + 2)y = 0$. **SRM Nov 2005**

Solution: Given $(D^3 + 3D^2 + 3D + 2)y = 0$. i. e. $\phi(D)y = 0$

To find *CF*: Put $D = m$ in $\phi(D) = 0$, the auxiliary equation is

$m^3 + 3m^2 + 3m + 2 = 0$. The roots of this equation are

$$m = -2,\ m = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\Rightarrow m_1 = -2,\ m = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} = \alpha \pm i\beta \Rightarrow \alpha = -\frac{1}{2},\ \beta = \frac{\sqrt{3}}{2}.$$

Hence $CF = C_1e^{m_1x} + e^{\alpha x}(C_2\cos\beta x + C_3\sin\beta x)$

$$= C_1e^{-2x} + e^{-\frac{1}{2}x}\left(C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right).$$

Since $F(x) = 0$, there is no *PI*. The complete solution

$$y = C_1e^{-2x} + e^{-\frac{1}{2}x}\left(C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right).$$

Example 3 ....

Solve: $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{-2x}$.

Solution: Given $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{-2x}$

i. e. $(D^2+3D+2)\,y = e^{-2x}$

i. e. $\phi(D)\,y = F(x)$

To find CF: Put $D = m$ in $\phi(D) = 0$, the auxiliary equation is

$m^2+3m+2 = 0$, $(m+1)(m+2) = 0 \Rightarrow m = -1, -2$

$\Rightarrow m_1 = -1,\ m_2 = -2$ are real and $m_1 \neq m_2$.

$$\therefore\ CF = C_1e^{m_1x} + C_2e^{m_2x} = C_1e^{-x} + C_2e^{-2x}$$

To find *PI*: Let $PI = \dfrac{1}{\phi(D)}F(x) = \dfrac{1}{D^2+3D+2}e^{-2x}$, put $D=-2$

$$= \frac{1}{4-6+2}e^{-2x}, \quad here\ \phi(2)=0$$

$$= x.\frac{1}{2D+3}e^{-2x} = x.\frac{1}{-4+3}e^{-2x} = -xe^{-2x}$$

The complete solution is $y = CF+PI$

$$y = C_1e^{-x} + C_2e^{-2x} - xe^{-2x}.$$

Example 4 ....

Solve: $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 3e^{4x}$

Solution: Given $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 3e^{4x}$

i. e. $(D^2+6D+9)y = 3e^{4x}$

i. e. $\phi(D)y = F(x)$

To find *CF*: Put $D = m$ in $\phi(D)=0$, the auxiliary equation is $m^2+6m+9 = 0$

$\Rightarrow (m+3)^2 = 0 \Rightarrow m = -3, -3 \Rightarrow m_1 = -3,\ m_2 = -3$ and $m_1 = m_2$

The roots are real and equal.

$$\therefore CF = (c_1 + c_2x)e^{-3x}$$

To find *PI*:

$$\text{Let } PI = \frac{1}{\phi(D)}F(x) = \frac{1}{D^2+6D+9}3e^{4x} = 3.\frac{1}{(D+3)^2}e^{4x}$$

$$= 3.\frac{1}{(4+3)^2}e^{4x} = 3.\frac{1}{49}e^{4x} = \frac{3}{49}e^{4x}$$

The complete solution $y = CF+PI$

$$y=(c_1+c_2x)e^{-3x}+\frac{3}{49}e^{4x}.$$

Example 5

Solve: $(D^2+9)y=e^{-2x}$

Solution: Given $(D^2+9)y=e^{-2x}$ i. e. $\phi(D)y=F(x)$

To find CF:

Put $D = m$ in $\phi(D)=0$, the auxiliary equation is $m^2+9 = 0$
$\Rightarrow m^2 = -9,\ m = \pm 3i = 0 \pm 3i = \alpha \pm i\beta \Rightarrow \alpha = 0,\ \beta = 3.$

The roots are imaginary.

$$\therefore\ CF=e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x) = C_1\cos 3x + C_2\sin 3x$$

To find PI: $PI=\dfrac{1}{\phi(D)}F(x)=\dfrac{1}{D^2+9}e^{-2x}=\dfrac{1}{4+9}e^{-2x}=\dfrac{e^{-2x}}{13}$

The complete solution is $y = CF+PI = C_1\cos 3x + C_2\sin 3x + \dfrac{e^{-2x}}{13}$.

Example 6

Solve: $(D^2+2D+1)y = e^{-x}+3$. **SRM Dec 2005, Nov 2007.**

Solution: Given $(D^2+2D+1)y = e^{-x}+3$ i. e. $\phi(D)y=F(x)$.

The auxiliary equation is $m^2+2m+1 = 0$

$\Rightarrow (m+1)(m+1) = 0$ $\Rightarrow m = -1,-1.$

$\therefore\ CF = (c_1+c_2x)e^{-x}$.

$$\text{Now } PI=\frac{1}{\phi(D)}F(x)=\frac{1}{D^2+2D+1}e^{-x}+3$$

$$=\frac{1}{D^2+2D+1}e^{-x}+3\frac{1}{D^2+2D+1}e^{0x}$$

$$=\frac{1}{1=-2+1}e^{-x}+3\frac{1}{0+0+1}e^{0x}$$

$$=x\ \frac{1}{2(D+1)}e^{-x}+3\,e^{0x}$$

$$= x \frac{1}{2(-1+1)} e^{-x} + 3 = x^2 \frac{1}{2} e^{-x} + 3$$

$$= \frac{x^2}{2} e^{-x} + 3 \cdot$$

The complete solution is $y = CF+PI.$

$$i.e. \quad y = (c_1 + c_2 x) e^{-x} + \frac{x^2}{2} e^{-x} + 3 .$$

TYPE 2: If $F(x) = sinax \text{ or } cosax$, then

$$\text{Let} \quad PI = \frac{1}{\phi(D)} F(x) = \frac{1}{\phi(D)} \sin ax \text{ or } \cos ax$$

$$= \frac{1}{\phi(-a^2)} \sin ax \text{ or } \cos ax, \text{ provided } \phi(-a^2) \neq 0$$

[*i.e. on* $\phi(D)$, *replace* D^2 *by* $-a^2$, *provided* $\phi(D) \neq 0$]

If $\phi(D) = 0$, *when* $D^2 = -a^2$, *then*

$$PI = x.\frac{1}{\phi'(D)} \sin ax \text{ or } \cos ax$$

$$= x.\frac{1}{\phi'(-a^2)} \sin ax \text{ or } \cos ax, \text{ provided } \phi'(-a^2) \neq 0$$

[*i.e. Again put* $D = -a^2$ *in* $\phi'(D)$, *provided* $\phi'(D) \neq 0$]

If $\phi'(D) = 0$, *when* $D^2 = -a^2$, *then*

$$PI = x^2.\frac{1}{\phi''(D)} \sin ax \text{ or } \cos ax$$

$$= x^2.\frac{1}{\phi''(-a^2)} \sin ax \text{ or } \cos ax, \text{ provided } \phi''(-a^2) \neq 0 .$$

This process may be repeated till the denominator becoming n[on] zero when replacing D^2 by $-a^2$.

Example 7

Solve: $(D^2+3D+2)\, y = sinx$

Solution: Given $(D^2+3D+2)\, y = sinx$

i. e. $\phi(D)y=F(x)$

The auxiliary equation is $m^2+3m+2 = 0 \Rightarrow (m+1)\,(m+2) = 0$

$\Rightarrow m = -1, -2 \Rightarrow\ m_1 = -1,\ m_2 = -2.$

$\therefore CF = C_1 e^{-x} + C_2 e^{-2x}$

$$Now\ PI = \frac{1}{\phi(D)}F(x) = \frac{1}{D^2+3D+2}\sin x, \quad put\ D^2=-1$$

$$= \frac{1}{-1+3D+2}\sin x$$

$$= \frac{1}{3D+1}\sin x$$

$$= \frac{(3D-1)}{(3D-1)\,(3D+1)}\sin x$$

$$= \frac{(3D-1)}{(9D^2-1)}\sin x$$

$$= \frac{1}{(-9-1)}(3D\sin x - \sin x)$$

$$= -\frac{1}{10}(3\cos x - \sin x)$$

The complete solution: $y = CF+PI$

$$y = C_1 e^{-x} + C_2 e^{-2x} - \frac{1}{10}(3\cos x - \sin x).$$

Example 8

Solve: $(D^2+6D+8)\, y = e^{-2x}+cos^2x$

Solution: Given $(D^2+6D+8)\, y = e^{-2x}+cos^2x$

i. e, $\phi(D)y=F(x)$

The auxiliary equation is $m^2+6m+8 = 0 \Rightarrow (m+2)\,(m+4) = 0$

$\Rightarrow m = -2, -4 \Rightarrow m_1 = -2, m_2 = -4.$

$\therefore CF = C_1 e^{-2x} + C_2 e^{4x}$

$$\text{Now } PI = \frac{1}{\phi(D)} F(x) = \frac{1}{D^2+6D+8}(e^{-2x} + \cos^2 x)$$

$$= \frac{1}{D^2+6D+8} e^{-2x} + \frac{1}{D^2+6D+8} \cos^2 x$$

$$= \frac{1}{4-12+8} e^{-2x} + \frac{1}{D^2+6D+8}\left(\frac{1+\cos 2x}{2}\right)$$

$$= x.\frac{1}{2D+6} e^{-2x} + \frac{1}{2}.\frac{1}{D^2+6D+8} e^{0x} + \frac{1}{2}.\frac{1}{D^2+6D+8} \cos 2x$$

$$= x.\frac{1}{-4+6} e^{-2x} + \frac{1}{2}.\frac{1}{0+0+8} e^{0x} + \frac{1}{2}.\frac{1}{-4+6D+8} \cos 2x$$

$$= \frac{x}{2} e^{-2x} + \frac{1}{16} + \frac{1}{2}.\frac{1}{6D+4} \cos 2x$$

$$= \frac{x}{2} e^{-2x} + \frac{1}{16} + \frac{1}{2}.\frac{(6D-4)}{(6D+4)(6D-4)} \cos 2x$$

$$= \frac{x}{2} e^{-2x} + \frac{1}{16} + \frac{1}{2}.\frac{(6D-4)}{(36D^2-16)} \cos 2x$$

$$= \frac{x}{2} e^{-2x} + \frac{1}{16} + \frac{1}{2}.\frac{(6D-4)}{[36(-4)-16]} \cos 2x$$

$$= \frac{x}{2} e^{-2x} + \frac{1}{16} + \frac{1}{2}.\frac{1}{-160}(6D\cos 2x - 4\cos 2x)$$

$$= \frac{x}{2} e^{-2x} + \frac{1}{16} - \frac{1}{320}(-12\sin 2x - 4\cos 2x)$$

$$= \frac{x}{2} e^{-2x} + \frac{1}{16} + \frac{1}{80}(3\sin 2x + \cos 2x)$$

The complete solution $y = CF+PI$

$$y = C_1 e^{-2x} + C_2 e^{-4x} + \frac{x}{2} e^{-2x} + \frac{1}{16} + \frac{1}{80}(3\sin 2x + \cos 2x).$$

Example 9 ....

Solve: $(D^2-4D+3)\, y = \sin 3x \cos 2x$

Solution: Given $(D^2-4D+3)y = \sin 3x \cos 2x$

i.e $\phi(D)y = F(x)$

The auxiliary equation is $m^2-4m+3=0 \Rightarrow (m-1)(m-3)=0$

$\Rightarrow m = 1, 3 \Rightarrow m_1 = 1, m_2 = 3$

$\therefore CF = C_1e^{x} + C_2e^{3x}$

Now $PI = \dfrac{1}{\phi(D)} = \dfrac{1}{D^2-4D+3}\sin 3x\cos 2x$

$$= \frac{1}{2}\cdot\frac{1}{D^2-4D+3}(\sin 5x + \sin x)$$

$$\left[\because \sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]\right]$$

$$= \frac{1}{2}\cdot\frac{1}{D^2-4D+3}\sin 5x + \frac{1}{2}\cdot\frac{1}{D^2-4D+3}\sin x$$

$$= \frac{1}{2}\cdot\frac{1}{-25-4D+3}\sin 5x + \frac{1}{2}\cdot\frac{1}{-1-4D+3}\sin x$$

$$= -\frac{1}{2}\cdot\frac{1}{22+4D}\sin 5x + \frac{1}{2}\cdot\frac{1}{2-4D}\sin x$$

$$= -\frac{1}{4}\cdot\frac{1}{(11+2D)}\sin 5x + \frac{1}{4}\cdot\frac{1}{(1-2D)}\sin x$$

$$= -\frac{1}{4}\frac{(11-2D)}{(121-4D^2)}\sin 5x + \frac{1}{4}\frac{(1+2D)}{(1-4D^2)}\sin x$$

$$= -\frac{1}{4}\frac{(11-2D)}{(121+100)}\sin 5x + \frac{1}{4}\frac{(1+2D)}{(1+4)}\sin x$$

$$= -\frac{1}{4(221)}(11\sin 5x - 2D\sin 5x) + \frac{1}{20}(\sin x + 2D\sin x)$$

$$= -\frac{1}{884}(11\sin 5x - 10\cos 5x) + \frac{1}{20}(\sin x + 2\cos x)$$

The complete solution $y = CF + PI$

$$y = C_1e^x + C_2e^{3x} - \frac{1}{884}(11\sin 5x - 10\cos 5x) + \frac{1}{20}(\sin x + 2\cos x)$$

Example 10

Solve: $(D^2 - 3D + 2)\,y = \cos 3x \cos 2x$. **SRM Dec 2005**

Solution: Given $(D^2 - 3D + 2)\,y = \cos 3x \cos 2x$

The auxiliary equation is $m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1, 2$

$\Rightarrow m_1 = 1,\ m_2 = 2$ *and* $m_1 \neq m_2$.

$\therefore CF = C_1e^x + C_2e^{2x}$

$$\text{Now } PI = \frac{1}{\phi(D)}F(x) = \frac{1}{D^2 - 3D + 2}\cos 3x \cos 2x$$

$$\left\{\because \cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]\right\}$$

$$= \frac{1}{2}\cdot\frac{1}{(D^2 - 3D + 2)}(\cos 5x + \cos x)$$

$$= \frac{1}{2}\cdot\frac{1}{(D^2 - 3D + 2)}\cos 5x + \frac{1}{2}\cdot\frac{1}{(D^2 - 3D + 2)}\cos x$$

$$= \frac{1}{2}\cdot\frac{1}{(-25 - 3D + 2)}\cos 5x + \frac{1}{2}\cdot\frac{1}{(-1 - 3D + 2)}\cos x$$

$$= \frac{1}{2}\cdot\frac{1}{(-23 - 3D)}\cos 5x + \frac{1}{2}\cdot\frac{1}{(1 - 3D)}\cos x$$

$$= -\frac{1}{2}\cdot\frac{1}{(23 + 3D)}\cos 5x + \frac{1}{2}\cdot\frac{1}{(1 - 3D)}\cos x$$

$$= -\frac{1}{2}\cdot\frac{(23 - 3D)}{(23 + 3D)(23 - 3D)}\cos 5x + \frac{1}{2}\cdot\frac{(1 + 3D)}{(1 - 3D)(1 + 3D)}\cos x$$

$$= -\frac{1}{2}\frac{(23 - D)}{(23^2 - D^2)}\cos 5x + \frac{1}{2}\frac{(1 + 3D)}{(1 - 9D^2)}\cos x$$

$$= -\frac{1}{2}\frac{(23 - D)}{(23^2 + 25)}\cos 5x + \frac{1}{2}\frac{(1 + 3D)}{(1 + 9)}\cos x$$

$$=-\frac{1}{2}\frac{1}{(529+25)}(23\cos 5x-D\cos 5x)+\frac{1}{20}(\cos x+3D\cos x)$$

$$=-\frac{1}{1108}(23\cos 5x+5\sin 5x)+\frac{1}{20}(\cos x-3\sin x)$$

The complete solution $y = CF+PI$

$$y=C_1e^{x}+C_2e^{2x}-\frac{1}{1108}(23\cos 5x+5\sin 5x)+\frac{1}{20}(\cos x-3\sin x).$$

Example 11 ....

Solve: $(D^3+2D^2+D)\,y=e^{2x}+\sin 2x$.

Solution: Given $(D^3+2D^2+D)\,y=e^{2x}+\sin 2x$. The auxiliary equation is $m^3+2m^2+m=0 \Rightarrow m(m^2+2m+1)=0$

$$\Rightarrow m=0 \text{ or } (m+1)^2=0 \Rightarrow m=0,\ m=-1,\ m=-1$$

$$\Rightarrow m_1=0\neq m_2=-1=m_3=m.$$

$$\therefore CF=c_1+(c_2+c_3x)e^{-x}.$$

Now $PI=\dfrac{1}{\phi(D)}F(x)=\dfrac{1}{(D^3+2D^2+D)}e^{2x}+\sin 2x$

$$=\frac{1}{(D^3+2D^2+D)}e^{2x}+\frac{1}{(D^3+2D^2+D)}\sin 2x$$

$$=\frac{1}{(8+8+2)}e^{2x}+\frac{1}{(-4D-8+D)}\sin 2x$$

$$=\frac{1}{18}e^{2x}+\frac{1}{(-3D-8)}\sin 2x$$

$$=\frac{1}{18}e^{2x}-\frac{(3D-8)}{(3D+8)(3D-8)}\sin 2x$$

$$= \frac{x}{2}\frac{D}{D^2}(\sin 2x) = \frac{x}{2}\frac{1}{-4}D(\sin 2x) = \frac{x}{2}\frac{1}{-4}(2\cos 2x)$$

$$= -\frac{x}{4}\cos 2x$$

The complete solution is $y = CF + PI$

$$y = c_1\cos 2x + c_2\sin 2x - \frac{x}{4}\cos 2x\ .$$

TYPE 3: If $F(x) = x^n$, where n is a constant (+ ve integer), then

$$PI = \frac{1}{\phi(D)}F(x) = \frac{1}{\phi(D)}x^n = \frac{1}{[1 \pm f(D)]}x^n = [1 \pm f(D)]^{-1}x^n$$

(Express $\phi(D)$ as $1 \pm f(D)$, bring it to the Nr and expand $(1 \pm f(D))^{-1}$ as a Binomial series. Operate x^n on each term of this expansion)

Note: Binomial expansion

(*i*) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots.$

(*ii*) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

(*iii*) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots.$

(*iv*) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

Example 13 ....

Solve: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2 + 3x - 1$

Solution: Given $(D^2 - 5D + 6)y = x^2 + 3x - 1$

i. e. $\phi(D)y = F(x)$

The auxiliary equation is $m^2\text{-}6m+5=0 \Rightarrow (m\text{-}2)\ (m\text{-}3) = 0 \Rightarrow m = 2, 3$

$\Rightarrow m_1 = 2, m_2 = 3$ and $m_1 \neq m_2$.

$\therefore\ CF = c_1e^{2x} + c_2e^{3x}$.

Now $PI = \dfrac{1}{\phi(D)}F(x) = \dfrac{1}{(D^2 - 5D + 6)}(x^2 + 3x - 1)$

$$= \frac{1}{6\left[1 + \left(\dfrac{D^2 - 5D}{6}\right)\right]}(x^2 + 3x - 1)$$

$$= \frac{1}{6}\left[1 + \left(\frac{D^2 - 5D}{6}\right)\right]^{-1}(x^2 + 3x - 1)$$

$$= \frac{1}{6}\left[1 - \left(\frac{D^2 - 5D}{6}\right) + \left(\frac{D^2 - 5D}{6}\right)^2 - \ldots\right](x^2 + 3x - 1)$$

$$= \frac{1}{6}\left[1 - \frac{D^2}{6} + \frac{5D}{6} + \frac{D^4}{36} - \frac{10D^3}{36} + \frac{25D^2}{36} \ldots\right](x^2 + 3x - 1)$$

$$= \frac{1}{6}\left[1 - \frac{D^2}{6} + \frac{5D}{6} + \frac{25D^2}{36}\right](x^2 + 3x - 1)$$

$$+ \frac{25}{36}D^2(x^2 + 3x - 1)\Big]$$

$$= \frac{1}{6}\left[(x^2 + 3x - 1) - \frac{1}{6}D^2(x^2 + 3x - 1) + \frac{5}{6}D(x^2 + 3x - 1)\right.$$

$$= \frac{1}{6}\left[x^2 + 3x - 1 + \frac{1}{6}(2) + \frac{5}{6}(2x + 3) + \frac{25}{36}(2)\right]$$

$$= \frac{1}{6}\left[x^2 + 3x - 1 + \frac{1}{3} + \frac{5}{6}(2x) + \frac{5}{6}.3 + \frac{25}{36}(2)\right]$$

$$= \frac{1}{6}\left[x^2 + \left(3 + \frac{5}{3}\right)x + \left(\frac{1}{3} - 1 + \frac{5}{2} + \frac{25}{18}\right)\right]$$

$$= \frac{1}{6}\left[x^2 + \frac{14}{3}x + \frac{58}{18}\right] = \frac{1}{6}\left[x^2 + \frac{14}{3}x + \frac{26}{9}\right]$$

The complete solution is $y = CF+PI$

$$y=C_1e^{2x}+C_2e^{3x}+\frac{1}{6}\left[x^2+\frac{14}{3}x+\frac{26}{9}\right].$$

Example 14 ....

Solve: $(D^2+5D+6)\,y = x^2+4e^{3x}$

Solution: Given $(D^2+5D+6)\,y = x^2+4e^{3x}$ i. e. $\phi(D)y=F(x)$

The auxiliary equation is $m^2+5m+6 = 0 \Rightarrow (m+2)\,(m+3) = 0$

$\Rightarrow m = -2,\ -3 \Rightarrow m_1 = -2,\ m_2 = -3$ and $m_1 \neq m_2$.

$\therefore CF = C_1e^{-2x}+C_2e^{-3x}$

$$\text{Now } PI=\frac{1}{\phi(D)}F(x)=\frac{1}{D^2+5D+6}(x^2+4e^{3x})$$

$$=\frac{1}{D^2+5D+6}x^2+4.\frac{1}{D^2+5D+6}e^{3x}$$

$$=\frac{1}{6\left[1+\left(\dfrac{D^2+5D}{6}\right)\right]}x^2+4.\frac{1}{9+15+6}e^{3x}$$

$$=\frac{1}{6}\left[1+\left(\frac{D^2+5D}{6}\right)\right]^{-1}x^2+\frac{4}{30}e^{3x}$$

$$=\frac{1}{6}\left[1-\frac{D^2}{6}-\frac{5D}{6}+\frac{D^4}{36}+\frac{10D^3}{36}+\frac{25D^2}{36}\right]x^2+\frac{2}{15}e^{3x}$$

$$=\frac{1}{6}\left[x^2-\frac{1}{6}D^2(x^2)-\frac{5D}{6}(x^2)+0+0+\frac{25D^2}{36}(x^2)\right]+\frac{2}{15}e^{3x}$$

$$=\frac{1}{6}\left[x^2-\frac{1}{6}(2)-\frac{5}{6}(2x)+\frac{25}{36}(2)\right]+\frac{2}{15}e^{3x}$$

$$=\frac{1}{6}\left(x^2-\frac{5}{3}x+\frac{19}{18}\right)+\frac{2}{15}e^{3x}$$

$$= \frac{1}{-6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25x}{18}\right]$$

The complete solution is $y = CF + PI$

$$y = c_1 + c_2 e^{-2x} + c_3 e^{3x} - \frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25x}{18}\right].$$

TYPE 4: If $F(x) = e^{ax} f(x)$, where $f(x) = x^n$ or $\sin ax$ or $\cos ax$, etc., then $PI = \frac{1}{\phi(D)} F(x) = \frac{1}{\phi(D)} e^{ax} f(x) = e^{ax} \frac{1}{\phi(D+a)} f(x)$

(i. e) replace D by $(D+a)$.

Note that $\frac{1}{\phi(D+a)} f(x)$ will be in any one of the previous known forms.

Example 17 ....

Solve: $(D^2 + D + 1)y = x^2 e^{-x}$

Solution: Given $(D^2 + D + 1)y = x^2 e^{-x}$. i. e. $\phi(D)y = F(x)$

The auxiliary equation is $m^2 + m + 1 = 0$

$$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = \frac{-1}{2} \pm \frac{i\sqrt{3}}{2} = \alpha \pm i\beta$$

$$\Rightarrow \alpha = \frac{-1}{2},\ \beta = \frac{\sqrt{3}}{2}$$

Roots are imaginary.

$$\therefore CF = e^{\frac{-1}{2}x}\left(C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right)$$

Now $PI = \frac{1}{\phi(D)} F(x) = \frac{1}{(D^2 + D + 1)} e^{-x} x^2$

$$= e^{-x} \cdot \frac{1}{[(D-1)^2 + (D-1) + 1]} x^2$$

$$= e^{-x} \cdot \frac{1}{(D^2 - 2D + 1 + D - 1 + 1)} x^2$$

$$= e^{-x} \cdot \frac{1}{(D^2 - D + 1)} x^2$$

$$= e^{-x} \cdot \frac{1}{[1 + (D^2 - D)]} x^2 = e^{-x} \left[1 + (D^2 - D)\right]^{-1} x^2$$

$$= e^{-x} \left[1 - (D^2 - D) + (D^2 - D)^2\right] x^2$$

$$= e^{-x} \left[1 - D^2 + D + D^4 - 2D^3 + D^2\right] x^2$$

$$= e^{-x} (x^2 + 2x)$$

The complete solution is $y = CF + PI$

$$y = e^{\frac{-1}{2}x} \left(C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right) + e^{-x} (x^2 + 2x).$$

Example 18 ....

Solve: $(D^2 + 9)\, y = (x^2 + 1)\, e^{3x}$

Solution: Given $(D^2 + 9)\, y = (x^2 + 1)\, e^{3x}$. i. e. $\phi(D) y = F(x)$

The auxiliary equation is $m^2 + 9 = 0 \Rightarrow m^2 = -9 = 9i^2 \Rightarrow m = \pm i3$

$\Rightarrow m = 0 \pm i3 = \alpha \pm i\beta \Rightarrow \alpha = 0,\ \beta = 3$. Roots are imaginary.

$\therefore CF = C_1 \cos 3x + C_2 \sin 3x$

$$\text{Now } PI = \frac{1}{\phi(D)} F(x) = \frac{1}{D^2 + 9} e^{3x} (x^2 + 1)$$

$$= e^{3x} \frac{1}{(D+3)^2 + 9} (x^2 + 1)$$

$$= e^{3x} \frac{1}{D^2 + 6D + 9 + 9} (x^2 + 1)$$

$$= e^{3x}\frac{1}{D^2+6D+18}(x^2+1)$$

$$= \frac{e^{3x}}{18}\frac{1}{\left[1+\left(\frac{D^2+6D}{18}\right)\right]}(x^2+1)$$

$$= \frac{e^{3x}}{18}\left[1+\left(\frac{D^2+6D}{18}\right)\right]^{-1}(x^2+1)$$

$$= \frac{e^{3x}}{18}\left[1-\left(\frac{D^2+6D}{18}\right)+\left(\frac{D^2+6D}{18}\right)^2\right](x^2+1)$$

$$= \frac{e^{3x}}{18}\left[(x^2+1)-\frac{D^2}{18}(x^2+1)-\frac{6D}{18}(x^2+1)+0+0+\frac{36D^2}{324}(x^2+1)\right]$$

$$= \frac{e^{3x}}{18}\left[x^2+1-\frac{2}{18}-\frac{6}{18}(2x)+\frac{36}{324}(2)\right]$$

$$= \frac{e^{3x}}{18}\left(x^2-\frac{2}{3}x+\frac{10}{9}\right)$$

The complete solution is $y = CF+PI$

$$y = C_1\cos 3x + C_2\sin 3x + \frac{e^{3x}}{18}\left(x^2-\frac{2}{3}x+\frac{10}{9}\right).$$

Example 19 ....

Solve: $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y = e^{-2x}+e^{3x}\sin x$

Solution: Given $(D^2+4D+4)y = e^{-2x}+e^{3x}\sin x$, *i. e.* $\phi(D)y = F(x)$

The auxiliary equation is $m^2+4m+4 = 0 \Rightarrow (m+2)^2 = 0 \Rightarrow m = -2, -2$
Roots are real and equal.

$$\therefore\ CF = (C_1+C_2x)e^{-2x}$$

4. 3. LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

We will now study two types of linear differential equations with variables coefficients which can be reduced to linear differential equations with constant coefficients by suitable substitution.

(a) Cauchy's homogeneous linear equation (Euler type):

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_n y = F(x) \qquad (1)$$

where $a_1, a_2, \ldots a_n$ are constants and $F(x)$ is a function of x is called Cauchy's (Euler's) homogeneous linear differential equation.

Equation (1) can be transformed to a linear differential equation with constant coefficients by the transformation

$$x = e^z \text{ or } z = \log x \text{ and } \frac{dz}{dx} = \frac{1}{x}$$

Now $\dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx} = \dfrac{dy}{dz} \cdot \dfrac{1}{x} \Rightarrow x\dfrac{dy}{dx} = \dfrac{dy}{dz}$. Hence $xDy = D'y$, where

$$D = \frac{d}{dx}, \quad D' = \frac{d}{dz} \qquad (2)$$

Also $$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz} = \frac{1}{x}\frac{d^2 y}{dz^2}\frac{1}{x} - \frac{1}{x^2}\frac{dy}{dz}$$

$$\frac{d^2 y}{dx^2} = \frac{1}{x^2}\frac{d^2 y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz} = \frac{1}{x^2}\left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right)$$

$$\Rightarrow x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz}$$

That is $x^2 D^2 y = D'^2 y - D' y = (D'^2 - D')y$

$$x^2 D^2 y = D'(D' - 1)y \qquad (3)$$

Similarly, $$x^3 D^3 y = D'(D' - 1)(D' - 2)y \qquad (4)$$

Substituting (2), (3), (4) and so on in (1) we get a linear differential equation with constant coefficients and can be solved by any one of the known method.

Example 1 ....

Solve: $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=\frac{12\log x}{x^2}$ **SRM Dec 2005, June 2006, Nov 2007**

Solution: Given $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=\frac{12\log x}{x^2}$

Multiplying throughout by x^2, we have $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=12\log x$

i.e $(x^2D^2+xD)y=12\log x$ (1)

Let $x = e^z$ *or* $z = \log x$ so that $xD = D'$, $x^2D^2 = D'(D'-1)$, where $D = d/dx$ and $D' = d/dz$.

Now equation (1) becomes $(D'(D'-1)+D')y=12z$

$$(D'^2-D'+D')y=12z \Rightarrow D'^2y=12z$$

$$\frac{d^2y}{dz^2}=12z$$

Integrating w. r. to z, we have

$$\frac{dy}{dz}=12\frac{z^2}{2}+C_1 \quad and \quad y=6\frac{z^3}{3}+C_1z+C_2$$

$y=2z^2+C_1z+C_2$. *But* $z=\log x$, *so that*

$y=2(\log x)^3+C_1z+C_2$ *is the required solution.*

Example 2 ....

Solve: $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=4\sin(\log x)$

Solution: Given $(x^2D^2+xD+1)y=4\sin(\log x)$ (1)

Let $x = e^z$ *or* $z = \log x$ so that $xD = D'$, $x^2D^2 = D'(D'-1)$, where $D = d/dx$ and $D' = d/dz$.

Now equation (1) becomes

$$\therefore \quad (D'(D'-1)+D'+1)y = 4\sin z$$

$$\Rightarrow (D'^2 - D' + D' + 1)y = 4\sin z$$

$$\Rightarrow (D'^2+1)y = 4\sin z \qquad (2)$$

$$i.e.\ \phi(D')y = F(z)$$

We have to solve equation (2). The auxiliary equation is $m^2+1=0$
$\Rightarrow m^2 = -1 \Rightarrow m = \pm i \Rightarrow m = 0 \pm i = \alpha \pm i\beta \Rightarrow \alpha = 0,\ \beta = 1.$

Roots are imaginary.

$\therefore\ CF = C_1\cos z + C_2\sin z.$

$$\text{Now}\ \ PI = \frac{1}{\phi(D')}F(z) = \frac{1}{D'^2+1}4\sin z$$

$$= z.\frac{1}{2D'}4\sin z$$

$$= 2z.\frac{1}{D'}\sin z$$

$$= 2z.\frac{D'}{D'^2}\sin z$$

$$= 2z.\frac{D'}{-1}\sin z \quad (\because\ D'^2 = -1,\ D'^2+1 = 0.\ i.e.\ Dr = 0)$$

$$= -2z\cos z$$

The complete solution of (2) is $y = C_1\cos z + C_2\sin z - 2z\cos z$.

Then the required solution is

$$y = C_1\cos(\log x) + C_2\sin(\log x) - 2(\log x)\cos(\log x).$$

Example 3

Solve: $(x^2D^2+4xD+2)\,y = x\log x$

Solution: Given $(x^2D^2+4xD+2)y = x\log x \qquad (1)$

Let $x = e^z$ or $z = \log x$ so that $xD = D'$, $x^2D^2 = D'(D'-1)$, where $D = d/dx$ and $D' = d/dz$.

Now equation (1) becomes

$$(D'(D'-1)+4D'+2)y = e^z z$$

$$(D'^2 - D' + 4D' + 2)y = e^z z$$

$$(D'^2 + 3D' + 2)y = e^z z$$

$$\text{i. e. } \phi(D')y = F(z) \qquad (2)$$

The auxiliary equation is $m^2+3m+2 = 0 \Rightarrow (m+1)(m+2) = 0$

$\Rightarrow m = -1, -2.$

$\therefore CF = C_1e^{-z} + C_2e^{-2z}.$

$$\text{Now } PI = \frac{1}{\phi(D')}F(z) = \frac{1}{D'^2+3D'+2}e^z z$$

$$= e^z \frac{1}{(D'+1)^2 + 3(D'+1)+2} z$$

$$= e^z \frac{1}{D'^2+5D'+6} z \quad = e^z \frac{1}{6\left[1+\left(\frac{D'^2+5D'}{6}\right)\right]} z$$

$$= \frac{e^z}{6}\left[1+\left(\frac{D'^2+5D'}{6}\right)\right]^{-1} z$$

$$= \frac{e^z}{6}\left(1 - \frac{D'^2}{6} - \frac{5D'}{6} + \frac{D'^4}{36} + \frac{10D'^3}{36} + \frac{25D'^2}{36}\right) z$$

$$= \frac{e^z}{6}\left(z - \frac{5}{6}\right)$$

Hence the complete solution of (2) is

$$y = C_1e^{-z} + C_2e^{-2z} + \frac{e^z}{6}\left(z - \frac{5}{6}\right).$$

The required solution (1) is $y = \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{1}{6}x\left(\log x - \frac{5}{6}\right).$

Substituting these in (1), we get a linear differential equation with constant coefficients which can be solved by any one of the known methods.

Example 7 ....

Solution: Given $(2x+5)^2\dfrac{d^2y}{dx^2}-6(2x+5)\dfrac{dy}{dx}+8y=0$

i.e. $\left[(2x+5)^2D^2-6(2x+5)D+8\right]y=0$ (1)

Let $(2x+5)=e^z$ or $z=\log(2x+5)$ so that $(2x+5)D=2D'$,

$(2x+5)^2D^2=2^2D'(D'-1)$, where $D=\dfrac{d}{dx}$ and $D'=\dfrac{d}{dz}$.

Now equation (1) becomes

$$(4D'(D'-1)-6.2D'+8)y=0$$

$$(4D'^2-4D'-12D'+8)y=0$$

i.e. $(4D'^2-16D'+8)y=0$ (2)

The auxiliary equation of (2) is $4m^2-16m+8=0\Rightarrow m^2-4m+2=0$

$$\Rightarrow m=\frac{4\pm\sqrt{16-8}}{2}=\frac{4\pm\sqrt{8}}{2}=\frac{4\pm2\sqrt{2}}{2}=2\pm\sqrt{2}$$

$$\Rightarrow m=2+\sqrt{2},2-\sqrt{2}.$$

$$\therefore CF=C_1e^{(2+\sqrt{2})z}+C_2e^{(2-\sqrt{2})z} \text{ and } PI=0$$

The complete solution of (2) is $y=CF+PI$.

$$y=C_1e^{(2+\sqrt{2})z}+C_2e^{(2-\sqrt{2})z}$$

$\therefore$ The complete solution of (1) is

$$y=C_1(2x+5)^{(2+\sqrt{2})}+C_2(2x+5)^{(2-\sqrt{2})}.$$

Example 8

Solve: $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos[\log(1+x)]$

Solution: Given $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos[\log(1+x)]$

i. e. $\left((1+x)^2 D^2 + (1+x)D + 1\right) y = 4\cos[\log(1+x)]$ (1)

Let $1+x = e^z$ or $z = \log(1+x)$ so that $(1+x)D = D'$,

$(1+x)^2 D^2 = D'(D'-1)$, where $D = \frac{d}{dx}$ and $D' = \frac{d}{dz}$.

Now equation (1) becomes $\left(D'(D'-1) + D' + 1\right) y = 4\cos z$

i. e. $(D'^2+1)y = 4\cos z$ (2)

$\phi(D')y = F(z)$

The auxiliary equation of (2) is $m^2+1 = 0$

$\Rightarrow m = \pm i = 0 \pm i = \alpha \pm i\beta \;\; \Rightarrow \alpha = 0,\; \beta = 1$

$\therefore \; CF = C_1 \cos z + C_2 \sin z.$

Now $PI = \frac{1}{\phi(D')} F(z) = \frac{1}{D'^2+1} 4\cos z$

$= z.\frac{1}{2D'} 4\cos z$

$= 2z.\frac{1}{D'} \cos z$

$= 2z \int \cos z dz$

$= 2z \sin z$

The complete solution of (2) is $y = C_1 \cos z + C_2 \sin z + 2z \sin z$.

Hence the complete solution of (1) is

$$y = C_1 \cos[\log(1+x)] + C_2 \sin[\log(1+x)] + 2[\log(1+x)]\sin[\log(1+x)]$$