

## Unit V: $Z$ Transforms

- Introduction of  $Z$ -Transforms
- Properties of  $Z$ -transforms
- $Z$ -Transforms-problems
- Inverse  $Z$ -Transforms
  - Long division method
  - Partial fraction method
  - Residue theorem method
- Convolution Theorem
- Application of  $Z$ -transform  
(Solutions of difference equations with constant coefficients using  $Z$ -transform)

## Introduction

$Z$ -transforms play an important role in Communication Engineering and Control Systems. It is a discrete analogue of Laplace transforms. The two basic types of signals in communication Engineering are continuous time signals and discrete time signals. Continuous time signals are defined for a continuum of values of the independent variable (that is) time denoted by  $[f(t)]$ . Discrete time signals are defined only at a discrete set of values of the independent variable and are denoted by a sequence  $\{f(n)\}$ .

Laplace and Fourier transforms play an important role in the study of continuous time signals, while  $Z$ -transforms plays an important role in the study of discrete time signals.

$Z$ -transforms can be applied to solve difference equations which occur in systems with digital filters, found in Digital Signal Processing.

### Definition: Two sided $Z$ -transform

If  $\{f(n)\}$  is a sequence defined for  $n = 0, \pm 1, \pm 2, \pm 3, \dots$  then the two sided  $Z$ -transform of  $\{f(n)\}$  is defined as

$$Z \{f(n)\} = F(z) = \sum_{n=-\infty}^{\infty} f(n)z^{-n} \quad (1)$$

where  $z$  is a complex variable in general.

### Definition: One sided $Z$ -transform

If  $\{f(n)\}$  is a causal sequence (that is) defined only for  $n = 0, 1, 2, \dots$  and  $f(n) = 0$  for  $n < 0$  then the one sided  $Z$ -transform is defined as

$$Z \{f(n)\} = F(z) = \sum_{n=0}^{\infty} f(n)z^{-n} \quad (2)$$

### Note:

- The infinite series on the right hand side of (1) and (2) will be converge only for certain values of  $z$  depending on the sequence  $f(n)$ .
- This unit deals with one sided  $Z$ -transforms which shall be referred hereafter as  $Z$ -transform.
- The inverse  $Z$ -transform of  $Z \{f(n)\} = F(z)$  is defined as  $Z^{-1} [F(z)] = \{f(n)\}$ .

### Definition: Conversion of a continuous signal to a discrete signal

If  $f(t)$  is a continuous function defines for discrete values of  $t$  where  $t = nT, n = 0, 1, 2, \dots$ ,

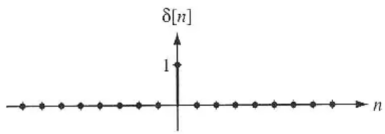
$T$  being the sampling period, then  $Z$ -transform of  $f(t)$  is defined as

$$Z[f(t)] = \sum_{n=0}^{\infty} f(t)z^{-n} = \sum_{n=0}^{\infty} f(nT)z^{-n}$$

### Unit impulse function (or) Unit sample sequence

The unit sample sequence  $\delta(n)$  is defined as

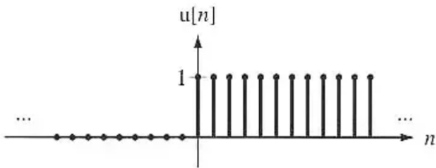
$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$



### Unit step sequence

The unit step sequence  $u(n)$  is defined as

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$



### Relation between $\delta(n)$ and $u(n)$

$$u(n) = \sum_{n=-\infty}^{\infty} \delta(n) \text{ and } \delta(n) = u(n) - u(n-1)$$

We have

$$\delta(n-k) = \begin{cases} 1 & \text{for } k = n \\ 0 & \text{for } k \neq n \end{cases}$$

and

$$u(n-k) = \begin{cases} 1 & \text{for } (n-k) \geq 0 \text{ or } n \geq k \\ 0 & \text{for } (n-k) < 0 \text{ or } n < k \end{cases}$$

Also  $f(n) = \sum_{k=-\infty}^{\infty} f(k)\delta(n-k)$

### Properties of Z-transform

**Property 1:** Linearity: Z-transform is linear

(i)  $Z[a\{f(n)\} + b\{g(n)\}] = aZ\{f(n)\} + bZ\{g(n)\}$

(ii)  $Z[af(t) + bg(t)] = aZ[f(t)] + bZ[g(t)]$

**Proof:**

$$\begin{aligned} (i) \quad Z[a\{f(n)\} + b\{g(n)\}] &= \sum_{n=0}^{\infty} [af(n) + bg(n)] z^{-n} \\ &= a \sum_{n=0}^{\infty} f(n)z^{-n} + b \sum_{n=0}^{\infty} g(n)z^{-n} \\ &= aZ\{f(n)\} + bZ\{g(n)\} \end{aligned}$$

$$\begin{aligned} (ii) \quad Z[af(t) + bg(t)] &= \sum_{n=0}^{\infty} [af(nT) + bg(nT)] z^{-n} \\ &= a \sum_{n=0}^{\infty} f(nT)z^{-n} + b \sum_{n=0}^{\infty} g(nT)z^{-n} \\ &= aZ[f(t)] + bZ[g(t)] \end{aligned}$$

**Property 2:** Frequency Shifting: (Damping rule or scaling property)

(i) If  $Z\{f(n)\} = F(z)$  then (a)  $Z\{a^n f(n)\} = F\left(\frac{z}{a}\right)$  (b)  $Z\{a^{-n} f(n)\} = F(az)$

(ii) If  $Z[f(t)] = F(z)$  then (a)  $Z[a^n f(t)] = F\left(\frac{z}{a}\right)$  and (b)  $Z[a^{-n} f(t)] = F(az)$

**Proof:**

$$\begin{aligned} (i) \quad (a) \quad Z\{a^n f(n)\} &= \sum_{n=0}^{\infty} a^n f(n)z^{-n} \\ &= \sum_{n=0}^{\infty} f(n)(a^{-1}z)^{-n} \\ &= F(a^{-1}z) = F\left(\frac{z}{a}\right) \end{aligned}$$

$$\begin{aligned} (b) \quad Z\{a^{-n} f(n)\} &= \sum_{n=0}^{\infty} a^{-n} f(n)z^{-n} \\ &= \sum_{n=0}^{\infty} f(n)(az)^{-n} \\ &= F(az) \end{aligned}$$

$$\begin{aligned}
 (ii) \quad (a) \quad Z[a^n f(t)] &= \sum_{n=0}^{\infty} a^n f(nT) z^{-n} \\
 &= a \sum_{n=0}^{\infty} f(nT) (a^{-1} z)^{-n} \\
 &= F(a^{-1} z) = F\left(\frac{z}{a}\right) \\
 (b) \quad Z[a^{-n} f(t)] &= \sum_{n=0}^{\infty} a^{-n} f(nT) z^{-n} \\
 &= \sum_{n=0}^{\infty} f(nT) (az)^{-n} \\
 &= F(az)
 \end{aligned}$$

**Property 3:** Differentiation in  $z$  domain or Multiplication by  $n$

(i) If  $Z\{f(n)\} = F(z)$  then  $Z\{nf(n)\} = -z \frac{d}{dz} F(z)$

(ii) If  $Z[f(t)] = F(z)$  then  $Z[nf(t)] = -z \frac{d}{dz} F(z)$

**Proof:**

(i) W.K.T.  $F(z) = Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$

Differentiating both sides with respect to  $z$ ,

$$\begin{aligned}
 \frac{d}{dz} F(z) &= \frac{d}{dz} \left[ \sum_{n=0}^{\infty} f(n) z^{-n} \right] \\
 &= \sum_{n=0}^{\infty} f(n) [-n z^{-n-1}] \\
 &= -z^{-1} \sum_{n=0}^{\infty} n f(n) z^{-n} \\
 &= -z^{-1} Z\{nf(n)\}
 \end{aligned}$$

Therefore  $Z\{nf(n)\} = -z \frac{d}{dz} F(z)$

(ii) W.K.T.  $F(z) = Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n}$

Differentiating both sides with respect to  $z$ ,

$$\begin{aligned}
 \frac{d}{dz}F(z) &= \frac{d}{dz} \left[ \sum_{n=0}^{\infty} f(nT)z^{-n} \right] \\
 &= \sum_{n=0}^{\infty} f(nT) [-nz^{-n-1}] \\
 &= -z^{-1} \sum_{n=0}^{\infty} nf(nT)z^{-n} \\
 &= -z^{-1}Z[nf(nT)]
 \end{aligned}$$

Therefore  $Z[nf(t)] = -z \frac{d}{dz}F(z)$

**Property 4:** First shifting theorem

If  $Z[f(t)] = F(z)$  then (i)  $Z[e^{-at}f(t)] = F(ze^{aT})$  or  $Z[e^{-at}f(t)] = [F(z)]_{z \rightarrow ze^{aT}}$   
(ii)  $Z[e^{at}f(t)] = F(ze^{-aT})$  or  $Z[e^{at}f(t)] = [F(z)]_{z \rightarrow ze^{-aT}}$

**Proof:**

$$\begin{aligned}
 (i) \quad Z[e^{-at}f(t)] &= \sum_{n=0}^{\infty} e^{-anT} f(nT)z^{-n} \\
 &= \sum_{n=0}^{\infty} f(nT)(e^{aT}z)^{-n} \\
 &= F[ze^{aT}] \text{ or } [F(z)]_{z \rightarrow ze^{aT}} \\
 (ii) \quad Z[e^{at}f(t)] &= \sum_{n=0}^{\infty} e^{anT} f(nT)z^{-n} \\
 &= \sum_{n=0}^{\infty} f(nT)(e^{-aT}z)^{-n} \\
 &= F[ze^{-aT}] \text{ or } [F(z)]_{z \rightarrow ze^{-aT}}
 \end{aligned}$$

**Property 5:** Time shifting theorem

(i) Shifting to the right: If  $Z\{f(n)\} = F(z)$  then  $Z\{f(n-k)\} = z^{-k}F(z)$  for  $k > 0$ .

(ii) Shifting to the left:

$$\text{If } Z\{f(n)\} = F(z) \text{ then } Z\{f(n+k)\} = z^k \left[ F(z) - f(0) - \frac{f(1)}{z} - \frac{f(2)}{z^2} - \dots - \frac{f(k-1)}{z^{k-1}} \right]$$

for  $k > 0$ .

(iii) If  $Z[f(t)] = F(z)$  then

$$Z[f(t+kT)] = z^k \left[ F(z) - f(0.T) - \frac{f(1.T)}{z} - \frac{f(2.T)}{z^2} - \dots - \frac{f(k-1).T}{z^{k-1}} \right]$$

**Proof:**

$$(i) \quad Z \{f(n-k)\} = \sum_{n=0}^{\infty} f(n-k)z^{-n}$$

Put  $m = n - k$

When  $n = 0 \Rightarrow m = -k$  and  $n = \infty \Rightarrow m = \infty$

$$\begin{aligned} Z \{f(n-k)\} &= \sum_{m=-k}^{\infty} f(m)z^{-(m+k)} \\ &= z^{-k} \sum_{m=-k}^{\infty} f(m)z^{-m} \\ &= z^{-k} \sum_{m=0}^{\infty} f(m)z^{-m} \text{ since } f(m) = 0 \text{ for } m < 0 \\ &= z^{-k} F(z) \text{ if } k > 0 \end{aligned}$$

$$(ii) \quad Z \{f(n+k)\} = \sum_{n=0}^{\infty} f(n+k)z^{-n}$$

Put  $m = n + k$

When  $n = 0 \Rightarrow m = k$  and  $n = \infty \Rightarrow m = \infty$

$$\begin{aligned} Z \{f(n+k)\} &= \sum_{m=k}^{\infty} f(m)z^{-(m-k)} \\ &= z^k \sum_{m=k}^{\infty} f(m)z^{-m} \\ &= z^k \left[ \sum_{m=0}^{\infty} f(m)z^{-m} - \sum_{m=0}^{k-1} f(m)z^{-m} \right] \\ &= z^k \left[ F(z) - f(0) - \frac{f(1)}{z} - \frac{f(2)}{z^2} - \dots - \frac{f(k-1)}{z^{k-1}} \right] \end{aligned}$$

**Corollary:**

- $Z \{f(n+1)\} = z [F(z) - f(0)]$
- $Z \{f(n+2)\} = z^2 \left[ F(z) - f(0) - \frac{f(1)}{z} \right]$
- $Z \{f(n+3)\} = z^3 \left[ F(z) - f(0) - \frac{f(1)}{z} - \frac{f(2)}{z^2} \right]$

$$(iii) \quad Z [f(t+kT)] = Z \{f[(n+k)T]\} = \sum_{n=0}^{\infty} f[(n+k)T] z^{-n}$$

Put  $m = n + k$

When  $n = 0 \Rightarrow m = k$  and  $n = \infty \Rightarrow m = \infty$

$$\begin{aligned}
 Z \{f[(n+k)T]\} &= \sum_{m=k}^{\infty} f(mT)z^{-(m-k)} \\
 &= z^k \sum_{m=k}^{\infty} f(mT)z^{-m} \\
 &= z^k \left[ \sum_{m=0}^{\infty} f(mT)z^{-m} - \sum_{m=0}^{k-1} f(mT)z^{-m} \right] \\
 &= z^k \left[ F(z) - f(0.T) - \frac{f(1.T)}{z} - \frac{f(2.T)}{z^2} - \dots - \frac{f[(k-1).T]}{z^{k-1}} \right]
 \end{aligned}$$

**Corollary:**

- $Z[f(t+T)] = z[F(z) - f(0)]$
- $Z[f(t+2T)] = z^2 \left[ F(z) - f(0) - \frac{f(1.T)}{z} \right]$
- $Z[f(t+3T)] = z^3 \left[ F(z) - f(0) - \frac{f(1.T)}{z} - \frac{f(2.T)}{z^2} \right]$

**Note:**

If  $Z\{f[(n+k)T]\}$  is denoted by  $f_{n+k}$  then

$$Z[f(t+kT)] = Z\{f_{n+k}\} = z^k \left[ F(z) - f(0) - \frac{f_1}{z} - \frac{f_2}{z^2} - \dots - \frac{f_{k-1}}{z^{k-1}} \right]$$

**Property 6:** Second Shifting theorem

If  $Z[f(t)] = F(z)$  then  $Z[f(t+T)] = z[F(z) - f(0)]$

**Proof:**

$$\begin{aligned}
 Z[f(t+T)] &= \sum_{n=0}^{\infty} f(nT+T)z^{-n} \\
 &= \sum_{n=0}^{\infty} f[(n+1)T]z^{-n} \\
 &= z \sum_{m=1}^{\infty} f(mT)z^{-m} \text{ put } m = n+1 \\
 &= z \left[ \sum_{m=0}^{\infty} f(mT)z^{-m} - f(0) \right] \\
 &= z[F(z) - f(0)]
 \end{aligned}$$



**Property 7:** Initial value theorem

- (i) If  $Z\{f(n)\} = F(z)$  then  $f(0) = \lim_{z \rightarrow \infty} F(z)$   
(ii) If  $Z[f(t)] = F(z)$  then  $f(0) = \lim_{z \rightarrow \infty} F(z)$

**Proof:**

$$\begin{aligned}
 (i) \quad F(z) &= Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)z^{-n} \\
 &= f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \frac{f(3)}{z^3} + \dots \\
 \lim_{z \rightarrow \infty} F(z) &= f(0) \text{ since } \frac{1}{z^n} \rightarrow 0 \text{ as } z \rightarrow \infty \text{ for any integer } n \\
 (ii) \quad F(z) &= Z[f(t)] = \sum_{n=0}^{\infty} f(nT)z^{-n} \\
 &= f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \frac{f(3T)}{z^3} + \dots \\
 \lim_{z \rightarrow \infty} F(z) &= f(0) \text{ since } \frac{1}{z^n} \rightarrow 0 \text{ as } z \rightarrow \infty \text{ for any integer } n
 \end{aligned}$$

**Corollary:**

- $f(1) = \lim_{z \rightarrow \infty} z[F(z) - f(0)]$
- $f(2) = \lim_{z \rightarrow \infty} z^2 \left[ F(z) - f(0) - \frac{f(1)}{z} \right]$

**Property 8:** Final value theorem

- (i) If  $Z\{f(n)\} = F(z)$  then  $\lim_{n \rightarrow \infty} f(n) = \lim_{z \rightarrow 1} (z-1)F(z)$   
(ii) If  $Z[f(t)] = F(z)$  then  $\lim_{t \rightarrow \infty} f(t) = \lim_{z \rightarrow 1} (z-1)F(z)$

**Proof:**

(i) W.K.T.  $Z\{f(n+1)\} = z[F(z) - f(0)]$

$\Rightarrow z[F(z) - f(0)] = Z\{f(n+1)\}$

Subtracting  $Z\{f(n)\}$  on both sides

$$\begin{aligned}
 zF(z) - zf(0) - Z\{f(n)\} &= Z\{f(n+1)\} - Z\{f(n)\} \\
 (z-1)F(z) - zf(0) &= Z\{f(n+1) - f(n)\} \\
 &= \sum_{n=0}^{\infty} [f(n+1) - f(n)] z^{-n}
 \end{aligned}$$

$$\begin{aligned}
 \lim_{z \rightarrow 1} [(z-1)F(z)] - f(0) &= \sum_{n=0}^{\infty} [f(n+1) - f(n)] \\
 &= f(1) - f(0) + f(2) - f(1) + f(3) - f(2) + \dots \\
 &\quad + f(n+1) - f(n) + \dots + f(\infty) \\
 &= f(\infty) - f(0)
 \end{aligned}$$

$$\lim_{z \rightarrow 1} [(z-1)F(z)] = \lim_{n \rightarrow \infty} f(n)$$

(ii) W.K.T.  $Z[f(t+T)] = z[F(z) - f(0)]$

$$\Rightarrow z[F(z) - f(0)] = Z[f(t+T)]$$

Subtracting  $Z[f(t)]$  on both sides

$$\begin{aligned}
 zF(z) - zf(0) - Z[f(t)] &= Z[f(t+T)] - Z[f(t)] \\
 (z-1)F(z) - zf(0) &= Z[f(t+T) - f(t)] \\
 &= \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] z^{-n} \\
 \lim_{z \rightarrow 1} [(z-1)F(z)] - f(0) &= \sum_{n=0}^{\infty} [f[(n+1)T] - f(nT)] \\
 &= f(T) - f(0) + f(2T) - f(T) + f(3T) - f(2T) + \dots \\
 &\quad + f[(n+1)T] - f(nT) + \dots + f(\infty) \\
 &= f(\infty) - f(0) \\
 \lim_{z \rightarrow 1} [(z-1)F(z)] &= \lim_{t \rightarrow \infty} f(t)
 \end{aligned}$$

## Problems

1.  $Z\{k\}$  where  $k$  is a constant.

**Sol:** By definition  $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)z^{-n}$

$$\begin{aligned}
 Z\{k\} &= \sum_{n=0}^{\infty} kz^{-n} \\
 &= k \left[ 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots \right] \\
 &= k \left( 1 - \frac{1}{z} \right)^{-1} \quad \text{if } \left| \frac{1}{z} \right| < 1 \\
 &= k \left( \frac{z-1}{z} \right)^{-1} = \frac{kz}{z-1}
 \end{aligned}$$

**Corollary:** When  $k = 1 \Rightarrow Z\{1\} = \frac{z}{z-1}, |z| > 1$

$$2. Z\{a^n\} = \frac{z}{z-a} \text{ if } |z| > |a| \text{ or } Z\{a^n u(n)\} = \frac{z}{z-a} \text{ if } |z| > |a|$$

**Proof:** By definition  $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)z^{-n}$

$$\begin{aligned} Z\{a^n\} &= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n \\ &= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots \\ &= \left(1 - \frac{a}{z}\right)^{-1} \text{ if } \left|\frac{a}{z}\right| < 1 \\ &= \left(\frac{z-a}{z}\right)^{-1} = \frac{z}{z-a} \end{aligned}$$

**Corollary:**  $Z\{1\} = \frac{z}{z-1}$  if  $a = 1$ ,  $Z\{(-1)^n\} = \frac{z}{z+1}$  if  $a = -1$ .

$$3. Z\{a^{n-1}\} = \frac{1}{z-a} \text{ if } |z| > |a|$$

**Proof:** We know that  $Z\{f(n-k)\} = z^{-k}Z\{f(n)\}$  by property 5

$$\begin{aligned} Z\{a^{n-1}\} &= z^{-1} \cdot Z\{a^n\} \\ &= z^{-1} \cdot \left(\frac{z}{z-a}\right) \text{ by problem 2} \\ &= \frac{1}{z-a} \end{aligned}$$

$$4. Z\{n\} = \frac{z}{(z-1)^2} \text{ if } |z| > |1| \text{ or } Z\{nu(n)\} = \frac{z}{(z-1)^2} \text{ if } |z| > |1|$$

$$\begin{aligned} Z\{n\} &= \sum_{n=0}^{\infty} n z^{-n} \\ &= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \dots \\ &= \frac{1}{z} \left(1 + \frac{2}{z} + \frac{3}{z^2} + \dots\right) \\ &= \frac{1}{z} \left(1 - \frac{1}{z}\right)^{-2} \text{ if } \left|\frac{1}{z}\right| < 1 \\ &\text{since } (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots \\ &= \frac{z}{(z-1)^2}; |z| > 1 \end{aligned}$$

$$5. \quad Z \left\{ \frac{1}{n} \right\} = \log \left( \frac{z}{z-1} \right) \text{ if } |z| > 1$$

$$\begin{aligned} Z \left\{ \frac{1}{n} \right\} &= \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} \\ &= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \dots \\ &= -\log \left( 1 - \frac{1}{z} \right) \text{ if } \left| \frac{1}{z} \right| < 1 \\ \text{since } -\log(1-x) &= x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \\ &= \log \left( \frac{z}{z-1} \right); |z| > 1 \end{aligned}$$

$$6. \quad Z \left\{ \frac{1}{n!} \right\} = e^{1/z}$$

$$\begin{aligned} Z \left\{ \frac{1}{n!} \right\} &= \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \\ &= 1 + \frac{1}{1!.z} + \frac{1}{2!.z^2} + \frac{1}{3!.z^3} + \dots \\ &= e^{1/z}; \text{ since } e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \end{aligned}$$

$$7. \quad Z \left\{ \frac{1}{(n+1)!} \right\} = z(e^{1/z} - 1)$$

$$\begin{aligned} Z \left\{ \frac{1}{(n+1)!} \right\} &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n} \\ &= \frac{1}{1!} + \frac{1}{2!.z} + \frac{1}{3!.z^2} + \dots \\ &= z \left[ \frac{1}{1!.z} + \frac{1}{2!.z^2} + \frac{1}{3!.z^3} + \dots \right] \\ &= z(e^{1/z} - 1); \text{ since } e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \end{aligned}$$

$$8. \quad Z \{e^{an}\} = \frac{z}{z - e^a}$$

$$\begin{aligned} Z \{e^{an}\} &= \sum_{n=0}^{\infty} e^{an} z^{-n} = \sum_{n=0}^{\infty} (e^a z^{-1})^n \\ &= 1 + (e^a z^{-1}) + (e^a z^{-1})^2 + (e^a z^{-1})^3 + \dots \\ &= (1 - e^a z^{-1})^{-1} \\ &= \frac{z}{z - e^a} \end{aligned}$$

$$9. \quad Z \left\{ \frac{a^n}{n!} \right\} = e^{a/z}$$

$$\begin{aligned} Z \left\{ \frac{a^n}{n!} \right\} &= \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n} \\ &= 1 + \frac{a}{1!z} + \frac{a^2}{2!z^2} + \frac{a^3}{3!z^3} + \dots \\ &= e^{a/z}; \text{ since } e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \end{aligned}$$

$$10. \quad Z \left\{ \frac{1}{n+1} \right\} = z \log \left( \frac{z}{z-1} \right)$$

$$\begin{aligned} Z \left\{ \frac{1}{n+1} \right\} &= \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n} \\ &= 1 + \frac{1}{2z} + \frac{1}{3z^2} + \frac{1}{4z^3} + \dots \\ &= z \left[ \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \dots \right] \\ &= z \left[ -\log \left( 1 - \frac{1}{z} \right) \right] = z \log \left( \frac{z}{z-1} \right) \end{aligned}$$

$$11. \quad Z \left\{ \frac{1}{n(n+1)} \right\}, n \geq 1$$

$$\text{Let } \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$A(n+1) + Bn = 1$$

$$\text{put } n = 0 \Rightarrow A = 1 \text{ and } n = -1 \Rightarrow B = -1$$

Therefore  $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$\begin{aligned}
 Z \left\{ \frac{1}{n(n+1)} \right\} &= Z \left\{ \frac{1}{n} - \frac{1}{n+1} \right\} \\
 &= Z \left\{ \frac{1}{n} \right\} - Z \left\{ \frac{1}{n+1} \right\} \\
 &= \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} - \sum_{n=1}^{\infty} \frac{1}{n+1} z^{-n} \\
 &= \left[ \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \dots \right] - \left[ \frac{1}{2z} + \frac{1}{3z^2} + \frac{1}{4z^3} + \dots \right] \\
 &= -\log \left( 1 - \frac{1}{z} \right) - z \left[ \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \dots \right] \\
 &= \log \left( \frac{z}{z-1} \right) - z \left[ \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \dots - \frac{1}{z} \right] \\
 &= \log \left( \frac{z}{z-1} \right) - z \left[ -\log \left( 1 - \frac{1}{z} \right) - \frac{1}{z} \right] \\
 &= (1-z) \log \left( \frac{z}{z-1} \right) + 1
 \end{aligned}$$

12.  $Z \{na^n\}$

We know that  $Z \{nf(n)\} = -z \frac{d}{dz} f(z)$  by property 3

$$\begin{aligned}
 Z \{na^n\} &= -z \frac{d}{dz} [Z \{a^n\}] \\
 &= -z \frac{d}{dz} \left( \frac{z}{z-a} \right) \\
 &= \frac{az}{(z-a)^2}
 \end{aligned}$$

13.  $Z \{n^2\}$

$$\begin{aligned}
 Z \{n.n\} &= -z \frac{d}{dz} Z \{n\} \\
 &= -z \frac{d}{dz} \left( \frac{z}{(z-1)^2} \right) \text{ by problem 4} \\
 &= \frac{z(z+1)}{(z-1)^3}
 \end{aligned}$$

14.  $Z \{n(n-1)\}$

$$\begin{aligned} Z \{n(n-1)\} &= Z \{n^2\} - Z \{n\} \\ &= \frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2} \text{ by problem 13 and 4} \\ &= \frac{2z}{(z-1)^3} \end{aligned}$$

15.  $Z \{n^2 + a^{n+3}\}$

$$\begin{aligned} Z \{n^2 + a^{n+3}\} &= Z \{n^2\} + Z \{a^{n+3}\} \\ &= \frac{z(z+1)}{(z-1)^3} + a^3 Z \{a^n\} \text{ by problem 13} \\ &= \frac{z(z+1)}{(z-1)^3} + \frac{a^3 z}{z-a} \text{ by problem 2} \end{aligned}$$

16.  $Z \{(n+1)(n+2)\}$

$$\begin{aligned} Z \{(n+1)(n+2)\} &= Z \{n^2 + 3n + 2\} \\ &= Z \{n^2\} + 3Z \{n\} + Z \{2\} \\ &= \frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{z-1} \text{ by problem 12, 4, 1} \\ &= \frac{2z^3}{(z-1)^3} \end{aligned}$$

17.  $Z \left\{ \frac{2n+3}{(n+1)(n+2)} \right\}$

Let  $\frac{2n+3}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$

$A(n+2) + B(n+1) = 2n+3$

put  $n = -1 \Rightarrow A = 1$  and  $n = -2 \Rightarrow B = 1$

Therefore  $\frac{2n+3}{(n+1)(n+2)} = \frac{1}{n+1} + \frac{1}{n+2}$

$$\begin{aligned} Z \left\{ \frac{2n+3}{(n+1)(n+2)} \right\} &= Z \left\{ \frac{1}{n+1} + \frac{1}{n+2} \right\} \\ &= Z \left\{ \frac{1}{n+1} \right\} + Z \left\{ \frac{1}{n+2} \right\} \\ &= z \log \left( \frac{z}{z-1} \right) + Z \left\{ \frac{1}{n+2} \right\} \text{ by problem 10} \end{aligned}$$

Now

$$\begin{aligned}
 Z \left\{ \frac{1}{n+2} \right\} &= \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n} \\
 &= 1 + \frac{1}{2} + \frac{1}{3z} + \frac{1}{4z^2} + \dots \\
 &= z^2 \left[ \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \dots \right] \\
 &= z^2 \left[ \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \dots - \frac{1}{z} \right] \\
 &= z^2 \left[ -\log \left( 1 - \frac{1}{z} \right) - \frac{1}{z} \right] = z^2 \log \left( \frac{z}{z-1} \right) - z
 \end{aligned}$$

Therefore  $Z \left\{ \frac{2n+3}{(n+1)(n+2)} \right\} = (z^2 + z) \log \left( \frac{z}{z-1} \right) - z$

18.  $Z \left\{ \frac{1}{n(n-1)} \right\}$

Let  $\frac{1}{n(n-1)} = \frac{A}{n} + \frac{B}{n-1}$

$A(n-1) + Bn = 1$

put  $n = 0 \Rightarrow A = -1$  and  $n = 1 \Rightarrow B = 1$

Therefore  $\frac{1}{n(n-1)} = -\frac{1}{n} + \frac{1}{n-1}$

$$\begin{aligned}
 Z \left\{ \frac{1}{n(n-1)} \right\} &= Z \left\{ -\frac{1}{n} + \frac{1}{n-1} \right\} \\
 &= Z \left\{ \frac{1}{n-1} \right\} - Z \left\{ \frac{1}{n} \right\} \\
 &= Z \left\{ \frac{1}{n-1} \right\} - \log \left( \frac{z}{z-1} \right) \text{ by problem 5}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } Z \left\{ \frac{1}{n-1} \right\} &= \sum_{n=2}^{\infty} \frac{1}{n-1} z^{-n} \\
 &= \frac{1}{z^2} + \frac{1}{2z^3} + \frac{1}{3z^4} + \frac{1}{4z^5} + \dots \\
 &= \frac{1}{z} \left[ \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \dots \right] \\
 &= \frac{1}{z} \left[ -\log \left( 1 - \frac{1}{z} \right) \right] \\
 &= \frac{1}{z} \log \left( \frac{z}{z-1} \right)
 \end{aligned}$$



Therefore  $Z \left\{ \frac{1}{n(n-1)} \right\} = \left( \frac{1-z}{z} \right) \log \left( \frac{z}{z-1} \right)$

19.  $Z \{r^n \cos n\theta\}$  and  $Z \{r^n \sin n\theta\}$

We know that  $Z \{a^n\} = \frac{z}{z-a}$

$$Z \{(re^{i\theta})^n\} = \frac{z}{z - re^{i\theta}}, |z| > |r|$$

$$\begin{aligned} Z \{r^n(\cos n\theta + i \sin n\theta)\} &= \frac{z}{z - r(\cos \theta + i \sin \theta)} \\ &= \frac{z}{(z - r \cos \theta) - ir \sin \theta} \times \frac{[(z - r \cos \theta) + ir \sin \theta]}{[(z - r \cos \theta) + ir \sin \theta]} \\ &= \frac{z[(z - r \cos \theta) + ir \sin \theta]}{[(z - r \cos \theta)^2 + r^2 \sin^2 \theta]} \end{aligned}$$

Equating real and imaginary part, we get

$$Z \{r^n \cos n\theta\} = \frac{z(z - r \cos \theta)}{z^2 - 2z \cos \theta + 1} \quad \text{and} \quad Z \{r^n \sin n\theta\} = \frac{zr \sin \theta}{z^2 - 2z \cos \theta + 1}$$

### Corollary

(1) Put  $r = 1$  in above problem, we get

$$Z \{\cos n\theta\} = \frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1} \quad \text{and} \quad Z \{\sin n\theta\} = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$$

(2) Put  $\theta = \frac{\pi}{2}$  in above corollary

$$Z \left\{ \cos \frac{n\pi}{2} \right\} = \frac{z^2}{z^2 + 1} \quad \text{and} \quad Z \left\{ \sin \frac{n\pi}{2} \right\} = \frac{z}{z^2 + 1}$$

20. Find the  $Z$ -transform of  $\sin^3 \frac{n\pi}{4}$

**Sol.:**  $Z \left\{ \sin^3 \frac{n\pi}{4} \right\} = \frac{3}{4} Z \left\{ \sin \frac{n\pi}{4} \right\} - \frac{1}{4} Z \left\{ \sin \frac{3n\pi}{4} \right\}$

We know that  $Z \{\sin n\theta\} = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$

$$\begin{aligned} Z \{\sin n\theta\} &= \frac{3}{4} \cdot \frac{z \sin \frac{\pi}{4}}{z^2 - 2z \cos \frac{\pi}{4} + 1} - \frac{1}{4} \cdot \frac{z \sin \frac{3\pi}{4}}{z^2 - 2z \cos \frac{3\pi}{4} + 1} \\ &= \frac{3}{4} \cdot \frac{\frac{z}{\sqrt{2}}}{4\sqrt{2}(z^2 - \sqrt{2}z + 1)} - \frac{1}{4} \cdot \frac{\frac{z}{\sqrt{2}}}{4\sqrt{2}(z^2 + \sqrt{2}z + 1)} \\ &= \frac{3z}{4\sqrt{2}(z^2 - \sqrt{2}z + 1)} - \frac{z}{4\sqrt{2}(z^2 + \sqrt{2}z + 1)} \end{aligned}$$

$$\begin{aligned}
 21. \quad Z \left\{ \cos \left( \frac{n\pi}{2} + \frac{\pi}{4} \right) \right\} \\
 Z \left\{ \cos \left( \frac{n\pi}{2} + \frac{\pi}{4} \right) \right\} &= Z \left\{ \cos \frac{n\pi}{2} \cos \frac{\pi}{2} - \sin \frac{n\pi}{2} \sin \frac{\pi}{2} \right\} \\
 &= Z \left\{ \cos \frac{n\pi}{2} \cdot \frac{1}{\sqrt{2}} - \sin \frac{n\pi}{2} \cdot \frac{1}{\sqrt{2}} \right\} \\
 &= \frac{1}{\sqrt{2}} \left[ Z \left\{ \cos \frac{n\pi}{2} \right\} - Z \left\{ \sin \frac{n\pi}{2} \right\} \right] \\
 &= \frac{1}{\sqrt{2}} \left[ \frac{z^2}{z^2 + 1} - \frac{z}{z^2 + 1} \right] = \frac{1}{\sqrt{2}} \frac{z(z-1)}{z^2 + 1}
 \end{aligned}$$

$$22. \quad Z \{u(n)\}$$

$$\begin{aligned}
 Z \{u(n)\} &= \sum_{n=0}^{\infty} u(n)z^{-n} \\
 &= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \\
 &= \left(1 - \frac{1}{z}\right)^{-1} = \frac{z}{z-1} \text{ if } |z| > 1
 \end{aligned}$$

$$23. \quad Z \{\delta(n)\}$$

$$Z \{\delta(n)\} = \sum_{n=0}^{\infty} \delta(n)z^{-n} = 1 \cdot z^0 = 1$$

$$24. \quad Z \{3^n \delta(n-1)\} = \sum_{n=1}^{\infty} 3^n \delta(n-1)z^{-n} = \frac{3}{z}$$

$$25. \quad Z \{u(n-1)\} = \sum_{n=1}^{\infty} u(n-1)z^{-n} = \frac{1}{z-1}$$

$$26. \quad Z \{e^{at}\} = Z \{e^{anT}\} = Z \{(e^{aT})^n\} = \frac{z}{z - e^{aT}}$$

$$27. \quad Z \{e^{-at}\} = Z \{e^{-anT}\} = Z \{(e^{-aT})^n\} = \frac{z}{z - e^{-aT}}$$

$$28. \quad Z \{t\}$$

$$\begin{aligned}
 Z \{t\} &= Z \{nT\} = \sum_{n=0}^{\infty} (nT)z^{-n} \\
 &= T \sum_{n=0}^{\infty} (n)z^{-n} = T \left[ -z \frac{d}{dz} Z \{1\} \right] \\
 &= -Tz \frac{d}{dz} \left( \frac{z}{z-1} \right) = \frac{Tz}{(z-1)^2}
 \end{aligned}$$

29.  $Z \{\sin \omega t\}$  and  $Z \{\cos \omega t\}$

$$Z \{\sin \omega t\} = Z \{\sin n(\omega T)\}$$

$$\text{W.K.T. } Z \{\sin n\theta\} = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$$

$$\text{Therefore } Z \{\sin n(\omega T)\} = \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1} \text{ if } |z| > 1$$

$$Z \{\cos \omega t\} = Z \{\cos n(\omega T)\}$$

$$\text{W.K.T. } Z \{\cos n\theta\} = \frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1}$$

$$\text{Therefore } Z \{\cos n(\omega T)\} = \frac{z(z - \cos \omega T)}{z^2 - 2z \cos \omega T + 1} \text{ if } |z| > 1$$

30.  $Z \{\cos^3 t\}$

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$$

$$Z \{\cos^3 t\} = Z \{\cos^3 nT\}$$

$$= Z \left\{ \frac{1}{4} (\cos 3nT + 3 \cos nT) \right\}$$

$$= \frac{1}{4} Z \{\cos 3nT\} + \frac{3}{4} Z \{\cos nT\}$$

$$= \frac{1}{4} \cdot \frac{z(z - \cos 3T)}{(z^2 - 2z \cos 3T + 1)} + \frac{3}{4} \cdot \frac{z(z - \cos T)}{(z^2 - 2z \cos T + 1)}$$

31.  $Z [e^{-at} \cos bt]$

$$Z [e^{-at} \cos bt] = Z [\cos bt]_{z \rightarrow ze^{aT}} \text{ by property 4}$$

$$= \left( \frac{z(z - \cos bT)}{z^2 - 2z \cos bT + 1} \right)_{z \rightarrow ze^{aT}}$$

$$= \frac{ze^{aT} [ze^{aT} - \cos bT]}{z^2 e^{2aT} - 2ze^{aT} \cos bT + 1}$$

32.  $Z [e^{-at} \sin bt]$

$$Z [e^{-at} \sin bt] = Z [\sin bt]_{z \rightarrow ze^{aT}} \text{ by property 4}$$

$$= \left( \frac{z \sin bT}{z^2 - 2z \cos bT + 1} \right)_{z \rightarrow ze^{aT}}$$

$$= \frac{ze^{aT} \sin bT}{z^2 e^{2aT} - 2ze^{aT} \cos bT + 1}$$

33. Find  $Z[te^{at}]$

$$\begin{aligned} Z[te^{at}] &= (Z[t])_{z \rightarrow ze^{-aT}} \\ &= \left( \frac{Tz}{(z-1)^2} \right)_{z \rightarrow ze^{-aT}} \\ &= \frac{Tze^{-aT}}{(ze^{-aT} - 1)^2} \end{aligned}$$

34. Find  $Z[t^2e^t]$

$$\begin{aligned} Z[t^2e^t] &= (Z[t^2])_{z \rightarrow ze^T} \\ \text{Now } Z[t^2] &= Z[(nT)^2] = T^2 Z\{n^2\} \\ Z[t^2e^t] &= \left( \frac{T^2 z(z+1)}{(z-1)^3} \right)_{z \rightarrow ze^T} \\ &= \frac{Tze^T(ze^T + 1)}{(ze^T - 1)^3} \end{aligned}$$

35.  $Z[e^t \sin 2t]$

$$\begin{aligned} Z[e^t \sin 2t] &= Z[\sin 2t]_{z \rightarrow ze^{-T}} \\ &= \left( \frac{z \sin 2T}{z^2 - 2z \cos 2T + 1} \right)_{z \rightarrow ze^{-T}} \\ &= \frac{ze^{-T} \sin 2T}{z^2 e^{-2T} - 2ze^{-T} \cos 2T + 1} \end{aligned}$$

36.  $Z[e^{3t} \cos 3t]$

$$\begin{aligned} Z[e^{3t} \cos 3t] &= Z[\cos t]_{z \rightarrow ze^{-3T}} \\ &= \left( \frac{z(z - \cos T)}{z^2 - 2z \cos T + 1} \right)_{z \rightarrow ze^{-3T}} \\ &= \frac{ze^{-3T}[ze^{-3T} - \cos T]}{z^2 e^{-6T} - 2ze^{-3T} \cos T + 1} \end{aligned}$$

37.  $Z[e^{3t+7}]$

$$Z[e^{3t+7}] = e^7 Z(e^{3t}) = \frac{ze^7}{z - e^{3T}}$$

38.  $Z[e^{-2t}t^3]$

$$Z[e^{-2t}t^3] = [Z(t^3)]_{z \rightarrow ze^{2T}}$$

$$Z(t^3) = Z\{n^3T^3\} = T^3Z\{n^3\}$$

$$= T^3Z\{n.n^2\}$$

$$= T^3 \left[ -z \frac{z(z+1)}{(z-1)^3} \right]$$

$$= T^3 \left[ \frac{z(z^2 + 4z + 1)}{()^4} \right]$$

$$[Z(t^3)]_{z \rightarrow ze^{2T}} = \frac{T^3ze^{2T}[z^2e^{4T} + 4ze^{2T} + 1]}{(ze^{2T} - 1)^4}$$

39. Find the  $Z$ -transform of (i)  $e^{2(t+T)}$  (ii)  $\sin(t+T)$  and (iii)  $(t+T)e^{-(t+T)}$

(i)  $Z[e^{2(t+T)}] = Z[f(t+T)]$  where  $f(t) = e^{2t}$

$$= z[F(z) - f(0)]$$

$$= z \left[ \frac{z}{z - e^{2T}} - 1 \right]$$

$$= \frac{ze^{2T}}{z - e^{2T}}$$

(ii)  $Z[\sin(t+T)] = Z[f(t+T)]$  where  $f(t) = \sin t$

$$= z[F(z) - f(0)]$$

$$= z \left[ \frac{z \sin T}{z^2 - 2z \cos T + 1} - 0 \right]$$

$$= \frac{z^2 \sin T}{z^2 - 2z \cos T + 1}$$

(iii)  $Z[(t+T)e^{-(t+T)}] = Z[f(t+T)]$  where  $f(t) = te^{-t}$

$$= z[F(z) - f(0)]$$

$$= z \left[ \frac{Tze^T}{(ze^T - 1)^2} - 0 \right]$$

$$= \frac{Tz^2e^T}{(ze^T - 1)^2}$$

40. Use final value theorem to find  $f(\infty)$  where  $F(z) = \frac{Tze^{aT}}{(ze^{aT} - 1)^2}$

$$f(\infty) = \lim_{z \rightarrow 1} (z-1)F(z) \text{ by final value theorem}$$

$$= \lim_{z \rightarrow 1} \left[ (z-1) \cdot \frac{Tze^{aT}}{(ze^{aT} - 1)^2} \right] = 0$$

41. Use initial value theorem to find  $f(0)$  when  $F(z) = \frac{ze^{aT}(ze^{aT} - \cos bT)}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}$

$$\begin{aligned} f(0) &= \lim_{z \rightarrow \infty} F(z) \text{ by initial value theorem} \\ &= \lim_{z \rightarrow \infty} \left[ \frac{z^2 e^{aT} \left( e^{aT} - \frac{1}{z} \cos bT \right)}{z^2 \left( e^{2aT} - \frac{2}{z} e^{aT} \cos bT + \frac{1}{z^2} \right)} \right] \\ &= \frac{e^{aT} \cdot e^{aT}}{e^{2aT}} = 1 \end{aligned}$$

42. Verify initial value theorem for  $f(n) = \frac{2^{n+1}}{n!}$

$$\text{Initial value theorem: } f(0) = \lim_{z \rightarrow \infty} F(z)$$

$$\text{L.H.S } f(0) = \frac{2^{0+1}}{0!} = 2$$

$$\begin{aligned} \text{R.H.S } F(z) &= Z \{f(n)\} = Z \left\{ \frac{2^{n+1}}{n!} \right\} \\ &= 2 \cdot Z \left\{ \frac{2^n}{n!} \right\} = 2e^{2/z} \text{ since } Z \left\{ \frac{a^n}{n!} \right\} = e^{a/z} \end{aligned}$$

$$\lim_{z \rightarrow \infty} F(z) = 2$$

Therefore L.H.S=R.H.S.

Hence the initial value theorem is verified.

43. Verify initial value theorem for  $f(t) = t^2$

$$\text{Initial value theorem: } f(0) = \lim_{z \rightarrow \infty} Z[f(t)]$$

$$\text{Given } f(t) = t^2$$

$$\text{L.H.S } f(0) = 0$$

$$\text{R.H.S } Z[f(t)] = Z[t^2] = \frac{T^2 z(z+1)}{(z-1)^3}$$

$$\lim_{z \rightarrow \infty} Z[f(t)] = \lim_{z \rightarrow \infty} \left[ \frac{T^2 z^2 \left( 1 + \frac{1}{z} \right)}{z^3 \left( 1 - \frac{1}{z} \right)^3} \right] = 0$$

Therefore L.H.S=R.H.S.

Hence the initial value theorem is verified.

44. Verify final value theorem for  $f(t) = e^{-at} \cos bt$

By final value theorem:  $f(\infty) = \lim_{z \rightarrow 1} (z - 1)Z[f(t)]$

Given  $f(t) = e^{-at} \cos bt$

$$\text{L.H.S } f(\infty) = 0$$

$$\text{R.H.S } Z[f(t)] = Z[e^{-at} \cos bt]$$

$$= \frac{ze^{aT}(ze^{aT} - \cos bT)}{z^2e^{2aT} - 2ze^{aT} \cos bT + 1}$$

$$\lim_{z \rightarrow 1} (z - 1)Z[f(t)] = \lim_{z \rightarrow 1} \left[ \frac{(z - 1)ze^{aT}(ze^{aT} - \cos bT)}{z^2e^{2aT} - 2ze^{aT} \cos bT + 1} \right] = 0$$

Therefore L.H.S=R.H.S.

Hence the final value theorem is verified.

### Inverse $Z$ -transform

As  $Z\{f(n)\} = F(z)$ , the inverse  $Z$ -transform of  $F(z)$  is defined as

$$Z^{-1}[F(z)] = \{f(n)\}$$

**Examples:** (i)  $Z\{a^n\} = \frac{z}{z-a} \Rightarrow Z^{-1}\left[\frac{z}{z-a}\right] = a^n$

(ii)  $Z\{n\} = \frac{z}{(z-1)^2} \Rightarrow Z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$

**Methods to find  $\{f(n)\}$  given  $F(z)$**

1. Long division method
2. Partial fraction method
3. Residue method or Inverse integral method
4. Convolution method

#### 1. Long division method

Since  $Z$ -transform is defined by the series  $F(z) = \sum_{n=0}^{\infty} f(n)z^{-n}$ , to find the inverse  $Z$ -transform of  $F(z)$ , expand  $F(z)$  in the proper power series and collect the coefficient of  $z^{-n}$  to get  $f(n)$ .

1. Find the inverse  $Z$ -transform of  $\frac{1+2z^{-1}}{1-z^{-1}}$  by long division method

Let  $F(z) = \frac{1+2z^{-1}}{1-z^{-1}}$

By actual division,

$$\begin{array}{r}
 1 - z^{-1} \overline{) \begin{array}{r} 1 + 3z^{-1} + 3z^{-2} + 3z^{-2} + \dots \\ 1 + 2z^{-1} \\ \hline 3z^{-1} \\ 3z^{-1} - 3z^{-2} \\ \hline 3z^{-2} \\ 3z^{-2} - 3z^{-3} \\ \hline 3z^{-3} \end{array}} \\
 \hline
 \end{array}$$

Therefore  $F(z) = 1 + 3z^{-1} + 3z^{-2} + 3z^{-2} + \dots$

$$\Rightarrow \sum_{n=0}^{\infty} f(n)z^{-n} = 1 + 3z^{-1} + 3z^{-2} + 3z^{-2} + \dots$$

$$\Rightarrow f(0) + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \dots = 1 + 3z^{-1} + 3z^{-2} + 3z^{-2} + \dots$$

Equating the like terms, we get

$$f(0) = 1, f(1) = 3, f(2) = 3, f(3) = 3, \dots$$

$$\text{Hence } f(n) = \begin{cases} 1, & n = 0 \\ 3, & n \geq 1 \end{cases}$$

2. Find  $Z^{-1} \left\{ \frac{z^2 + z}{(z-1)^3} \right\}$  by long division

$$\begin{aligned}
 \text{Let } F(z) &= \frac{z^2 + z}{(z-1)^3} \\
 &= \frac{z^2 + z}{z^3 - 3z^2 + 3z - 1} \\
 &= \frac{z^{-1} + z^{-2}}{1 - 3z^{-1} + 3z^{-2} - z^{-3}}
 \end{aligned}$$

By actual division,



$$\begin{array}{r}
 1 - 3z^{-1} + 3z^{-2} - z^{-3} \sqrt{\begin{array}{r}
 \begin{array}{rrrrr}
 z^{-1} & +4z^{-2} & +9z^{-3} & +16z^{-4} & +\dots \\
 \hline
 z^{-1} & +z^{-2} & & & \\
 \hline
 z^{-1} & -3z^{-2} & +3z^{-3} & -z^{-4} & \\
 \hline
 & 4z^{-2} & -3z^{-3} & +z^{-4} & \\
 \hline
 & 4z^{-2} & -12z^{-3} & +12z^{-4} & -4z^{-5} \\
 \hline
 & & 9z^{-3} & -11z^{-4} & +4z^{-5} \\
 \hline
 & & 9z^{-3} & -27z^{-4} & +27z^{-5} & -9z^{-6} \\
 \hline
 & & & 16z^{-4} & -23z^{-5} & +9z^{-6} \\
 \hline
 & & & 16z^{-4} & -48z^{-5} & +48z^{-6} & -16z^{-7} \\
 \hline
 & & & & 25z^{-5} & -39z^{-6} & +16z^{-7}
 \end{array}
 \end{array}}
 \end{array}$$

Therefore  $F(z) = z^{-1} + 4z^{-2} + 9z^{-3} + 16z^{-4} + \dots$

$$\Rightarrow \sum_{n=0}^{\infty} f(n)z^{-n} = z^{-1} + 4z^{-2} + 9z^{-3} + 16z^{-4} + \dots$$

$$\Rightarrow f(0) + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \dots = z^{-1} + 4z^{-2} + 9z^{-3} + 16z^{-4} + \dots$$

Equating the like terms, we get

$$f(0) = 0, f(1) = 1, f(2) = 4, f(3) = 9, f(4) = 16, \dots$$

$$\text{Hence } f(n) = \begin{cases} 0, & n = 0 \\ n^2, & n \geq 1 \end{cases}$$

3. Find  $Z^{-1} \left\{ \frac{1}{1 + 4z^{-2}} \right\}$  by the long division method

Let  $F(z) = \frac{1}{1 + 4z^{-2}}$  By actual division,

$$\begin{array}{r}
 1 + 4z^{-2} \sqrt{\begin{array}{r}
 \begin{array}{rrrrr}
 1 & -4z^{-2} & +16z^{-4} & -64z^{-6} & +256z^{-8} & +\dots \\
 \hline
 1 & & & & & \\
 \hline
 & -4z^{-2} & & & & \\
 \hline
 & -4z^{-2} & -16z^{-4} & & & \\
 \hline
 & & 16z^{-4} & & & \\
 \hline
 & & 16z^{-4} & +64z^{-6} & & \\
 \hline
 & & & -64z^{-6} & & \\
 \hline
 & & & -64z^{-6} & -256z^{-8} & \\
 \hline
 & & & & 256z^{-8} & 
 \end{array}
 \end{array}}
 \end{array}$$

Therefore  $F(z) = 1 - 4z^{-2} + 16z^{-4} - 64z^{-6} + \dots$

$$\Rightarrow \sum_{n=0}^{\infty} f(n)z^{-n} = 1 - 4z^{-2} + 16z^{-4} - 64z^{-6} + \dots$$

$$\Rightarrow f(0) + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \dots = 1 - 4z^{-2} + 16z^{-4} - 64z^{-6} + \dots$$

Equating the like terms, we get

$$f(0) = 1, f(1) = 0, f(2) = -4, f(3) = 0, f(4) = 16, f(5) = 0, f(6) = -6, \dots$$

$$\text{Hence } f(n) = 2^n \cos \frac{n\pi}{2}$$

4. Find the inverse  $Z$ -transform of  $\frac{10z}{(z-1)(z-2)}$  by long division method

$$\text{Let } F(z) = \frac{10z}{(z-1)(z-2)} = \frac{10z^{-1}}{1-3z^{-1}+2z^{-2}}$$

By actual division

$$\begin{array}{r} 1-3z^{-1}+2z^{-2} \overline{) \begin{array}{r} 10z^{-1} \quad +30z^{-2} \quad +70z^{-3} \quad +150z^{-4} \quad +\dots \\ 10z^{-1} \quad -30z^{-2} \quad +20z^{-3} \\ \hline 30z^{-2} \quad -20z^{-3} \\ 30z^{-2} \quad -90z^{-3} \quad +60z^{-4} \\ \hline 70z^{-3} \quad -60z^{-4} \\ 70z^{-3} \quad -210z^{-4} \quad +40z^{-5} \\ \hline 150z^{-4} \quad -140z^{-5} \end{array}} \end{array}$$

$$\text{Therefore } F(z) = 10z^{-1} + 30z^{-2} + 70z^{-3} + 150z^{-4} + \dots$$

$$\Rightarrow \sum_{n=0}^{\infty} f(n)z^{-n} = 10z^{-1} + 30z^{-2} + 70z^{-3} + 150z^{-4} + \dots$$

$$\Rightarrow f(0) + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \dots = 10z^{-1} + 30z^{-2} + 70z^{-3} + \dots$$

Equating the like terms, we get

$$f(0) = 0, f(1) = 10, f(2) = 30, f(3) = 70, \dots$$

$$\text{Hence } f(n) = 10(2^n - 1), n \geq 0$$

5. Find  $Z^{-1} \left\{ \frac{z^2 + 2z}{z^2 + 2z + 4} \right\}$  by long division

$$\begin{aligned} \text{Let } F(z) &= \frac{z^2 + 2z}{z^2 + 2z + 4} \\ &= \frac{1 + \frac{2}{z}}{1 + \frac{2}{z} + \frac{4}{z^2}} \\ &= \frac{1 + 2z^{-1}}{1 + 2z^{-1} + 4z^{-2}} \end{aligned}$$

By actual division,

$$\begin{array}{r}
 1 + 2z^{-1} + 4z^{-2} \sqrt{\begin{array}{r} \frac{1}{1} \frac{-4z^{-2}}{+2z^{-1}} \frac{+8z^{-3}}{+4z^{-2}} \frac{-32z^{-5}}{-4z^{-2}} \frac{+ \dots}{-4z^{-2}} \frac{-8z^{-3}}{8z^{-3}} \frac{-16z^{-4}}{+16z^{-4}} \frac{+32z^{-5}}{+32z^{-5}} \frac{-32z^{-5}}{-32z^{-5}} \end{array}}
 \end{array}$$

Therefore  $F(z) = 1 - 4z^{-2} + 8z^{-3} - 32z^{-5} + \dots$

$$\Rightarrow \sum_{n=0}^{\infty} f(n)z^{-n} = 1 - 4z^{-2} + 8z^{-3} - 32z^{-5} + \dots$$

$$\Rightarrow f(0) + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \dots = 1 - 4z^{-2} + 8z^{-3} - 32z^{-5} + \dots$$

Equating the like terms, we get

$$f(0) = 1, f(1) = 0, f(2) = -4, f(3) = 8, f(4) = 0, f(5) = -32, \dots$$

Therefore the sequence is 1, 0, -4, 8, 0, -32, ...

## 2. Partial fraction method

Step 1: When  $F(z)$  is a rational function in which the denominator can be factorized, resolve  $F(z)$  into partial fractions.

Step 2:  $Z^{-1}[F(z)]$  is the sum of the inverse  $Z$ -transforms of the partial fractions declare the result.

**Note:** (i) The degree of  $z$  in the numerator should be atleast one less than the degree of  $z$  in the denominator of  $F(z)$

(ii) Wherever possible rewrite the given functions as  $\frac{F(z)}{z}$  and apply the above steps.

1. Find the inverse  $Z$ -transform of (i)  $\frac{z}{z^2 + 7z + 10}$  (ii)  $\frac{z^2 + z}{(z - 1)(z^2 + 1)}$  and

(iii)  $\frac{z}{(z - 1)^2(z + 1)}$

(i) Given  $F(z) = \frac{z}{z^2 + 7z + 10}$

$$\begin{aligned}
 \Rightarrow \frac{F(z)}{z} &= \frac{1}{z^2 + 7z + 10} \\
 &= \frac{1}{(z + 2)(z + 5)}
 \end{aligned}$$

$$\text{Now } \frac{1}{(z + 2)(z + 5)} = \frac{A}{z + 2} + \frac{B}{z + 5}$$

$$1 = A(z + 5) + B(z + 2)$$

$$\text{put } z = -2 \Rightarrow A = \frac{1}{3} \text{ and } z = -5 \Rightarrow B = \frac{-1}{3}$$

$$\begin{aligned} \text{Therefore } \frac{F(z)}{z} &= \frac{1/3}{z+2} + \frac{-1/3}{z+5} \\ F(z) &= \frac{1}{3} \left[ \frac{z}{z+2} - \frac{z}{z+5} \right] \end{aligned}$$

Taking inverse on both sides, we have

$$\begin{aligned} Z^{-1}[F(z)] &= f(n) = \frac{1}{3} \left[ Z^{-1} \left( \frac{z}{z+2} \right) - Z^{-1} \left( \frac{z}{z+5} \right) \right] \\ &= \frac{1}{3} [(-2)^n - (-5)^n], \quad n = 0, 1, 2, \dots \end{aligned}$$

$$(ii) \text{ Given } F(z) = \frac{z^2 + z}{(z-1)(z^2+1)}$$

$$\begin{aligned} \Rightarrow \frac{F(z)}{z} &= \frac{z+1}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{(Bz+C)}{z^2+1} \\ \Rightarrow z+1 &= A(z^2+1) + (Bz+C)(z-1) \end{aligned}$$

$$\text{Put } z = 1 \Rightarrow A = 1$$

Equating the coefficients of  $z^2$  and constant term, we have

$$0 = A + B \tag{3}$$

$$1 = A - C \tag{4}$$

$A = 1$  in equations (3) and (4) we get  $B = -1$ ,  $C = 0$

$$\begin{aligned} \text{Therefore } \frac{F(z)}{z} &= \frac{1}{z-1} - \frac{z}{z^2+1} \\ \Rightarrow F(z) &= \frac{z}{z-1} - \frac{z^2}{z^2+1} \end{aligned}$$

Taking inverse on both sides, we have

$$\begin{aligned} Z^{-1}[F(z)] &= f(n) = Z^{-1} \left( \frac{z}{z-1} \right) - Z^{-1} \left( \frac{z^2}{z^2+1} \right) \\ &= 1^n - \cos \frac{n\pi}{2} \end{aligned}$$

$$(iii) \text{ Let } F(z) = \frac{z}{(z-1)^2(z+1)}$$

$$\begin{aligned} \Rightarrow \frac{F(z)}{z} &= \frac{1}{(z-1)^2(z+1)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+1} \\ \Rightarrow 1 &= A(z-1)(z+1) + B(z+1) + C(z-1)^2 \end{aligned}$$

Put  $z = 1 \Rightarrow B = \frac{1}{2}$ ,  $z = -1 \Rightarrow C = \frac{1}{4}$  and  $z = 0 \Rightarrow A = -\frac{1}{4}$

Therefore  $\frac{F(z)}{z} = \frac{-1/4}{z-1} + \frac{1/2}{(z-1)^2} + \frac{1/4}{z+1}$   
 $\Rightarrow F(z) = -\frac{1}{4} \cdot \frac{z}{z-1} + \frac{1}{2} \cdot \frac{z}{(z-1)^2} + \frac{1}{4} \cdot \frac{z}{z+1}$

Taking inverse on both sides, we get

$$f(n) = -\frac{1}{4} \cdot (1)^n + \frac{1}{2} \cdot n + \frac{1}{4} \cdot (-1)^n$$

2. Find  $Z^{-1} \left\{ \frac{3z^2 - 18z + 26}{(z-2)(z-3)(z-4)} \right\}$  by the partial fraction method

Consider  $\frac{3z^2 - 18z + 26}{(z-2)(z-3)(z-4)} = \frac{A}{(z-2)} + \frac{B}{(z-3)} + \frac{C}{(z-4)}$

$$\Rightarrow 3z^2 - 18z + 26 = A(z-3)(z-4) + B(z-2)(z-4) + C(z-2)(z-3)$$

Put  $z = 2 \Rightarrow A = 1$ ,  $z = 3 \Rightarrow B = 1$  and  $z = 4 \Rightarrow C = 1$

$$\frac{3z^2 - 18z + 26}{(z-2)(z-3)(z-4)} = \frac{1}{(z-2)} + \frac{1}{(z-3)} + \frac{1}{(z-4)}$$

Taking inverse on both sides, we have

$$\begin{aligned} Z^{-1} \left\{ \frac{3z^2 - 18z + 26}{(z-2)(z-3)(z-4)} \right\} &= Z^{-1} \left( \frac{1}{z-2} \right) + Z^{-1} \left( \frac{1}{z-3} \right) + Z^{-1} \left( \frac{1}{z-4} \right) \\ &= 2^{n-1} + 3^{n-1} + 4^{n-1} \end{aligned}$$

3. Find  $Z^{-1} \left\{ \frac{4z^3}{(2z-1)^2(z-1)} \right\}$  by the method of partial fraction

Let  $F(z) = \frac{4z^3}{(2z-1)^2(z-1)}$

$$\begin{aligned} \Rightarrow \frac{F(z)}{z} &= \frac{4z^2}{(2z-1)^2(z-1)} = \frac{A}{(2z-1)} + \frac{B}{(2z-1)^2} + \frac{C}{z-1} \\ \Rightarrow 4z^2 &= A(2z-1)(z-1) + B(z-1) + C(2z-1)^2 \end{aligned}$$

Put  $z = 1 \Rightarrow C = 4$ ,  $z = \frac{1}{2} \Rightarrow B = -2$  and  $z = 0 \Rightarrow A = -6$

Therefore  $\frac{F(z)}{z} = \frac{-6}{2z-1} + \frac{(-2)}{(2z-1)^2} + \frac{4}{z-1}$   
 $\Rightarrow F(z) = \frac{-6}{2} \cdot \frac{z}{z-\frac{1}{2}} - \frac{-2}{4} \cdot \frac{z}{\left(z-\frac{1}{2}\right)^2} + 4 \cdot \frac{z}{z-1}$

Taking inverse on both sides, we have

$$Z^{-1}[F(z)] = -3Z^{-1} \left( \frac{z}{z - \frac{1}{2}} \right) - \frac{-1}{2} Z^{-1} \left( \frac{z}{\left(z - \frac{1}{2}\right)^2} \right) + 4Z^{-1} \left( \frac{z}{z - 1} \right)$$

$$f(n) = -3 \left( \frac{1}{2} \right)^n - n \left( \frac{1}{2} \right)^n + 4(1)$$

$$= 4 - (n + 3) \left( \frac{1}{2} \right)^n$$

4. Find  $Z^{-1} \left\{ \frac{z^2}{(z+2)(z^2+4)} \right\}$  by method of partial fraction.

$$\text{Let } F(z) = \frac{z^2}{(z+2)(z^2+4)}$$

$$\Rightarrow \frac{F(z)}{z} = \frac{z}{(z+2)(z^2+4)} = \frac{A}{z+2} + \frac{(3z+C)}{(z^2+4)}$$

$$\Rightarrow z = A(z^2+4) + (Bz+4) + (Bz+C)(z+2)$$

$$\text{Put } z = -2 \Rightarrow A = -\frac{1}{4}$$

Equating the coefficient of  $z^2$ , constant term, we get

$$0 = A + B \quad (5)$$

$$0 = 4A + 2C \quad (6)$$

Sub.  $A = \frac{1}{4}$  in equations (5) and (6) we have  $B = \frac{1}{4}$ ,  $C = \frac{1}{2}$

$$\text{Therefore } \frac{F(z)}{z} = \frac{-1/4}{z+2} + \frac{(1/4z+1/2)}{z^2+4}$$

$$\Rightarrow F(z) = -\frac{1}{4} \frac{z}{z+2} + \frac{1}{4} \frac{z^2}{z^2+1} + \frac{1}{4} \frac{2z}{z^2+4}$$

$$Z^{-1}[F(z)] = -\frac{1}{4} Z^{-1} \left( \frac{z}{z+2} \right) + \frac{1}{4} Z^{-1} \left( \frac{z^2}{z^2+1} \right) + \frac{1}{4} Z^{-1} \left( \frac{2z}{z^2+4} \right)$$

$$f(n) = -\frac{1}{4}(-2)^n + \frac{1}{4} \cdot 2^n \cos \frac{n\pi}{2} + \frac{1}{2} \cdot 2^n \sin \frac{n\pi}{2}$$

5. Find  $Z^{-1} \left\{ \frac{4 - 8z^{-1} + 6z^{-2}}{(1 + z^{-1})(1 - 2z^{-1})^2} \right\}$  by method of partial fraction.

$$\begin{aligned}
 \text{Let } F(z) &= \frac{4 - 8z^{-1} + 6z^{-2}}{(1 + z^{-1})(1 - 2z^{-1})^2} \\
 &= \frac{4 - \frac{8}{z} + \frac{6}{z^2}}{\left(1 + \frac{1}{z}\right) \left(1 - \frac{2}{z}\right)^2} \\
 &= \frac{\left(\frac{1}{z^2}\right) (4z^2 - 8z + 6)}{\left(\frac{1}{z^3}\right) (z + 1)(z - 2)^2} \\
 &= \frac{4z^3 - 8z^2 + 6z}{(z + 1)(z - 2)^2} \\
 \Rightarrow \frac{F(z)}{z} &= \frac{4z^2 - 8z + 6}{(z + 1)(z - 2)^2} = \frac{A}{z + 1} + \frac{B}{z - 2} + \frac{C}{(z - 2)^2} \\
 \Rightarrow 4z^2 - 8z + 6 &= A(z - 2)^2 + B(z - 2)(z + 1) + C(z + 1)
 \end{aligned}$$

Put  $z = 2 \Rightarrow C = 2$ ,  $z = -1 \Rightarrow A = 2$

Equating the Coefficient of  $z^2$ , we have

$$4 = A + B \Rightarrow B = 2$$

$$\begin{aligned}
 \text{Therefore } \frac{F(z)}{z} &= \frac{2}{z + 1} + \frac{2}{z - 2} + \frac{2}{(z - 2)^2} \\
 \Rightarrow F(z) &= 2 \left[ \frac{z}{z + 1} + \frac{z}{z - 2} + \frac{z}{(z - 2)^2} \right] \\
 Z^{-1}[F(z)] &= 2Z^{-1} \left( \frac{z}{(z + 1)} \right) + 2Z^{-1} \left( \frac{z}{(z - 2)} \right) + 2Z^{-1} \left( \frac{z}{(z - 2)^2} \right) \\
 &= 2(-1)^n + 2.(2)^n + n.2^n
 \end{aligned}$$

### 3. Residue method or Inverse integral method

By using the relation between the  $Z$ -transform and Fourier transform of a sequence, it can be proved that  $f(n) = \frac{1}{2\pi i} \int_C F(z) z^{n-1} dz$  where  $C$  is a circle whose centre is the origin

and radius is sufficiently large to include all the isolated singularities of  $F(z)$ ,  $C$  may also be a closed contour including the origin and the isolated singularities of  $F(z)$ .

By Cauchy's residue theorem

$$\int_C F(z) z^{n-1} dz = 2\pi i \times \text{sum of the residues of } F(z) z^{n-1} \text{ at the isolated singularities.}$$

Therefore  $f(n)$  = Sum of the residues of  $F(z)z^{n-1}$  at the isolated singularities.

**Calculation of residue:**

(i) When  $z = a$  is a simple pole or a pole of order one, then the residue is given by

$$\text{Res}[F(z), z = a] = \lim_{z \rightarrow a} (z - a)F(z)$$

(ii) When  $z = a$  is a pole of order  $m$ , then the residue is given by

$$\text{Res}[F(z), z = a] = \frac{1}{(m-1)!} \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} [(z - a)F(z)]$$

1. Find  $Z^{-1} \left\{ \frac{z(z^2 - z + 2)}{(z+1)(z-1)^2} \right\}$  by using Residue theorem

$$\text{Let } F(z) = \frac{z(z^2 - z + 2)}{(z+1)(z-1)^2}$$

$$\Rightarrow F(z)z^{n-1} = \frac{z^n(z^2 - z + 2)}{(z+1)(z-1)^2}$$

The poles are given by  $z = -1, z = 1$

$z = -1$  is a simple pole and  $z = 1$  is a pole of order 2

$R_1$  = Residue at  $z = -1$

$$\begin{aligned} &= \lim_{z \rightarrow -1} (z+1) \cdot \frac{z^n(z^2 - z + 2)}{(z+1)(z-1)^2} \\ &= \lim_{z \rightarrow -1} \frac{z^n(z^2 - z + 2)}{(z-1)^2} = (-1)^n \end{aligned}$$

$R_2$  = Residue at  $z = 1$

$$\begin{aligned} &= \frac{1}{(2-1)!} \lim_{z \rightarrow 1} \frac{d}{dz} \left[ (z-1)^2 \cdot \frac{(z+1)z^n(z^2 - z + 2)}{(z+1)(z-1)^2} \right] \\ &= \lim_{z \rightarrow 1} \frac{d}{dz} \left[ \frac{z^n(z^2 - z + 2)}{(z+1)} \right] \\ &= \lim_{z \rightarrow 1} \left\{ \frac{(z+1)[nz^{n-1}(z^2 - z + 2) + z^n(2z - 1)] - z^n(z^2 - z + 2) \cdot 1}{(z+1)^2} \right\} \\ &= \frac{2[n \cdot 1^{n-1}(1 - 1 + 2) + 1^n(2 - 1)] - 1^n(1 - 1 + 2)}{4} \\ &= \frac{2(2n + 2) - 2}{4} = n \end{aligned}$$

Therefore  $f(n)$  = sum of the residues of  $F(z)z^{n-1}$  at poles

$$\Rightarrow f(n) = R_1 + R_2 = (-1)^n + n$$



2. Find  $Z^{-1} \left\{ \frac{2z^2 + 4z}{(z-2)^3} \right\}$  by using Residue theorem

$$\text{Let } F(z) = \frac{2z^2 + 4z}{(z-2)^3}$$

$$\Rightarrow F(z)z^{n-1} = \frac{2z^{n+1} + 4z^n}{(z-2)^3}$$

The poles are  $z = 2$  is pole of order 3

$$\begin{aligned} R &= \text{Residue at } z = 2 \\ &= \frac{1}{(3-1)!} \lim_{z \rightarrow 2} \frac{d^2}{dz^2} \left[ (z-2)^3 \cdot \frac{2z^{n+1} + 4z^n}{(z-2)^3} \right] \\ &= \frac{1}{2!} \lim_{z \rightarrow 2} \frac{d^2}{dz^2} [2z^{n+1} + 4z^n] \\ &= \frac{1}{2} \lim_{z \rightarrow 2} \frac{d}{dz} [2(n+1)z^n + 4nz^{n-1}] \\ &= \frac{1}{2} \lim_{z \rightarrow 2} [2(n+1)nz^{n-1} + 4n(n-1)z^{n-2}] \\ &= \frac{1}{2} [2(n+1).n2^{n-1} + 4n(n-1)2^{n-2}] \\ &= \frac{1}{2} [(n+1).n.2^n + n(n-1)2^n] \\ &= \frac{1}{2}.n2^n[n+1+n-1] \\ &= n2^n. \end{aligned}$$

Hence  $f(n) = \text{sum of the Residue of } F(z)z^{n-1} \text{ at poles inside } C = n2^n$

3. Find the inverse  $z$ -transform of  $\frac{z}{(z-1)(z-2)}$

$$\text{Let } F(z) = \frac{z}{(z-1)(z-2)}$$

$$\Rightarrow F(z)z^{n-1} = \frac{z^n}{(z-1)(z-2)}$$

The poles are  $z = 1, 2$ , each simple pole

$$\begin{aligned} R_1 &= \text{Residue at } z = 1 \\ &= \lim_{z \rightarrow 1} (z-1) \cdot \frac{z^n}{(z-1)(z-2)} \\ &= \lim_{z \rightarrow 1} \frac{z^n}{(z-2)} = -1 \end{aligned}$$

$$R_2 = \text{Residue at } z = 2$$

$$= \lim_{z \rightarrow 2} (z - 2) \cdot \frac{z^n}{(z - 1)(z - 2)}$$

$$= \lim_{z \rightarrow 2} \frac{z^n}{(z - 1)} = 2^n$$

$$\text{Hence } f(n) = R_1 + R_2 = 2^n - 1$$

4. Find  $Z^{-1} \left\{ \frac{z^2}{(z + 2)(z^2 + 4)} \right\}$  by the method of residues.

$$\text{Let } F(z) = \frac{z^2}{(z + 2)(z^2 + 4)}$$

$$\begin{aligned} \Rightarrow z^{n-1} F(z) &= \frac{z^{n+1}}{(z + 2)(z^2 + 4)} \\ &= \frac{z^{n+1}}{(z + 2)(z + 2i)(z - 2i)} \end{aligned}$$

The poles are given by  $z = -1, -2i, 2i$ , each of simple poles.

$$R_1 = \text{Residue at } z = -2$$

$$\begin{aligned} &= \lim_{z \rightarrow -2} (z + 2) \frac{z^{n+1}}{(z + 2)(z + 2i)(z - 2i)} \\ &= \lim_{z \rightarrow -2} \frac{z^{n+1}}{(z + 2i)(z - 2i)} \\ &= \frac{(-2)^{n+1}}{(z + 2i)(z - 2i)} \\ &= \frac{(-2)^{n+1}}{8} \end{aligned}$$

$$R_2 = \text{Residue at } z = -2i$$

$$\begin{aligned} &= \lim_{z \rightarrow -2i} (z + 2i) \frac{z^{n+1}}{(z + 2)(z + 2i)(z - 2i)} \\ &= \lim_{z \rightarrow -2i} \frac{z^{n+1}}{(z + 2)(z - 2i)} \\ &= \frac{(-2i)^{n+1}}{(-2i + 2)(-2i - 2i)} \\ &= \frac{(-2)^n (-2) (-i) (-i)^n}{2(1 - i)(-4i)} \\ &= \frac{(-2)^n (-1) (-i)^n}{4(1 - i)} \times \frac{(1 + i)}{(1 + i)} = \frac{(2)^n}{8} (-i)^n (1 + i) \end{aligned}$$

$R_3 = \text{Residue at } z = 2i$

$$\begin{aligned}
 &= \lim_{z \rightarrow 2i} (z - 2i) \frac{z^{n+1}}{(z+2)(z+2i)(z-2i)} \\
 &= \lim_{z \rightarrow 2i} \frac{z^{n+1}}{(z+2)(z+2i)} \\
 &= \frac{(2i)^{n+1}}{(2i+2)(4i)} \\
 &= \frac{(2)^n (i)^n}{4(1+i)} \times \frac{(1-i)}{(1-i)} \\
 &= \frac{(2)^n}{8} (i)^n (1-i)
 \end{aligned}$$

Therefore  $f(n) = \sum R = R_1 + R_2 + R_3$

$$\begin{aligned}
 f(n) &= \frac{(-2)^{n+1}}{8} + \frac{(2)^n}{8} (-i)^n (1+i) + \frac{(-2)^n}{8} (i)^n (1-i) \\
 &= \frac{(-2)^{n+1}}{8} + \frac{(2)^n (1+i)}{8} \left( \cos \frac{n\pi}{2} - \sin \frac{n\pi}{2} \right) \\
 &\quad + \frac{(2)^n (1-i)}{8} \left( \cos \frac{n\pi}{2} + \sin \frac{n\pi}{2} \right) \\
 &= \frac{(-2)^{n+1}}{8} + \frac{2^n}{8} \left\{ \cos \frac{n\pi}{2} - \sin \frac{n\pi}{2} + i \cos \frac{n\pi}{2} + \sin \frac{n\pi}{2} \right. \\
 &\quad \left. + \cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2} - i \cos \frac{n\pi}{2} + \sin \frac{n\pi}{2} \right\} \\
 &= \frac{(-2)^{n+1}}{8} + \frac{2^n}{4} \left( \cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2} \right)
 \end{aligned}$$

5. Find  $Z^{-1} \left\{ \frac{z}{z^2 + 2z + 2} \right\}$  by the method of residues

Let  $F(z) = \frac{z}{z^2 + 2z + 2}$

$\Rightarrow z^{n-1} F(z) = \frac{z^n}{z^2 + 2z + 2}$

The poles are given by  $z^2 + 2z + 2 = 0$

$$\begin{aligned}
 \Rightarrow z &= \frac{-2 \pm \sqrt{4-8}}{2} \\
 &= \frac{-1 \pm 2i}{2} \\
 &= -1 \pm i \text{ which are simple}
 \end{aligned}$$

$R_1$  = Residue at  $z = -1 + i$

$$\begin{aligned}
 &= \lim_{z \rightarrow (-1+i)} [z - (-1 + i)] \cdot \frac{z^n}{z^2 + 2z + 2} \\
 &= \lim_{z \rightarrow (-1+i)} [z - (-1 + i)] \cdot \frac{z^n}{[z - (-1 + i)][z - (-1 + i)]} \\
 &= \lim_{z \rightarrow (-1+i)} \frac{z^n}{z - (-1 - i)} \\
 &= \frac{(-1 + i)^n}{-1 + i + 1 + i} = \frac{(-1 + i)^n}{2i}
 \end{aligned}$$

$R_2$  = Residue at  $z = -1 - i$

$$\begin{aligned}
 &= \lim_{z \rightarrow (-1-i)} [z - (-1 - i)] \cdot \frac{z^n}{[z - (-1 + i)][z - (-1 - i)]} \\
 &= \lim_{z \rightarrow (-1-i)} \frac{z^n}{z - (-1 + i)} \\
 &= \frac{(-1 - i)^n}{-1 - i + 1 - i} = \frac{-(-1 - i)^n}{2i}
 \end{aligned}$$

Therefore  $f(n) = \sum R = R_1 + R_2$

$$f(n) = \frac{(-1 + i)^n}{2i} - \frac{(-1 - i)^n}{2i}$$

Let  $-1 + i = r(\cos \theta + i \sin \theta)$

Equating real and imaginary parts, we have

$$r \cos \theta = -1, \quad r \sin \theta = 1$$

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = 1 + 1$$

$$r^2 = 2 \implies r = \sqrt{2}$$

$$\text{Therefore } \cos \theta = -\frac{1}{\sqrt{2}} \text{ and } \sin \theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4}$$

$$\text{Therefore } -1 + i = \sqrt{2} \left[ \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right] \text{ and } -1 - i = \sqrt{2} \left[ \cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4} \right]$$

$$\begin{aligned}
 \text{Hence } f(n) &= \frac{1}{2i} \left\{ (\sqrt{2})^n \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)^n - (\sqrt{2})^n \left( \cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4} \right)^n \right\} \\
 &= \frac{(\sqrt{2})^n}{2i} \left\{ \cos \frac{3n\pi}{4} + i \sin \frac{3n\pi}{4} - \cos \frac{3n\pi}{4} + i \sin \frac{3n\pi}{4} \right\} \\
 &= \frac{(\sqrt{2})^n}{2i} \cdot 2i \sin \frac{3n\pi}{4} = (\sqrt{2})^n \sin \frac{3n\pi}{4}, n \geq 0
 \end{aligned}$$

#### 4. Convolution method

##### Convolution of sequence:

The Convolution of two sequence  $\{f(n)\}$  and  $\{g(n)\}$  is defined as

$$\{f(n) * g(n)\} = \sum_{r=0}^n f(r)g(n-r)$$

**Convolution theorem:**

If  $Z\{f(n)\} = F(z)$  and  $Z\{g(n)\} = G(z)$  then

$$\begin{aligned} Z\{f(n) * g(n)\} &= Z\{f(n)\} \cdot Z\{g(n)\} \\ &= F(z) \cdot G(z) \end{aligned}$$

That is  $Z$ -transform of Convolution of two sequence is equal to the product of the  $Z$ -transform.

**Note:**

- If  $Z\{f(n)\} = F(z)$  and  $Z\{g(n)\} = G(z)$  then  $Z^{-1}[F(z)] = f(n)$ ,  
 $Z^{-1}[G(z)] = g(n)$  and  $Z^{-1}[F(z)G(z)] = f(n) * g(n) = \sum_{r=0}^n f(r)g(n-r)$
- If  $Z[f(t)] = F(z)$  and  $Z[g(t)] = G(z)$  then  
 $Z^{-1}[F(z)G(z)] = f(t) * g(t) = \sum_{k=0}^n f(kT)g[(n-k)T]$

1. Find  $Z^{-1}\left[\frac{z^2}{(z+a)^2}\right]$  using convolution theorem

$$\begin{aligned} Z^{-1}\left[\frac{z^2}{(z+a)^2}\right] &= Z^{-1}\left[\frac{z}{(z+a)} \cdot \frac{z}{(z+a)}\right] \\ &= Z^{-1}\left(\frac{z}{z+a}\right) * Z^{-1}\left(\frac{z}{z+a}\right) \\ &= (-a)^n * (-a)^n \\ &= \sum_{r=0}^n (-a)^r (-a)^{n-r} \\ &= (-a)^n \sum_{r=0}^n 1 = (n+1)(-a)^n \end{aligned}$$

2. Find  $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$  using convolution theorem

$$\begin{aligned} Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] &= Z^{-1}\left[\frac{z}{z-a} \cdot \frac{z}{z-b}\right] \\ &= Z^{-1}\left(\frac{z}{z-a}\right) * Z^{-1}\left(\frac{z}{z-b}\right) \end{aligned}$$

$$\begin{aligned}
 Z^{-1} \left[ \frac{z^2}{(z-a)(z-b)} \right] &= a^n * b^n = \sum_{r=0}^n a^r b^{n-r} \\
 &= b^n \sum_{r=0}^n \left( \frac{a}{b} \right)^r \\
 &= b^n \left[ 1 + \left( \frac{a}{b} \right) + \left( \frac{a}{b} \right)^2 + \dots + \left( \frac{a}{b} \right)^n \right] \\
 &= b^n \left[ \frac{\left( \frac{a}{b} \right)^{n+1} - 1}{\frac{a}{b} - 1} \right] \\
 &= \frac{a^{n+1} - b^{n+1}}{a - b}, n \geq 0
 \end{aligned}$$

3. Find  $Z^{-1} \left[ \frac{8z^2}{(2z-1)(4z+1)} \right]$

$$\begin{aligned}
 Z^{-1} \left[ \frac{8z^2}{(2z-1)(4z+1)} \right] &= Z^{-1} \left[ \frac{8z^2}{2 \left( z - \frac{1}{2} \right) \cdot 4 \left( z + \frac{1}{4} \right)} \right] \\
 &= Z^{-1} \left[ \frac{z^2}{\left( z - \frac{1}{2} \right) \left( z + \frac{1}{4} \right)} \right] \\
 &= Z^{-1} \left( \frac{z}{z - \frac{1}{2}} \right) * Z^{-1} \left( \frac{z}{z + \frac{1}{4}} \right) \\
 &= \left( \frac{1}{2} \right)^n * \left( -\frac{1}{4} \right)^n \\
 &= \sum_{r=0}^n \left( \frac{1}{2} \right)^r * \left( -\frac{1}{4} \right)^{n-r} \\
 &= \left( -\frac{1}{4} \right)^n \sum_{r=0}^n (-2)^r \\
 &= \left( \frac{1}{4} \right)^n [1 + (-2) + (-2)^2 + \dots + (-2)^n] \\
 &= \left( \frac{1}{4} \right)^n \left[ \frac{(-2)^{n+1} - 1}{(-2) - 1} \right] \\
 &= \left( \frac{1}{4} \right)^n \left( -\frac{1}{3} \right) [(-2)^{n+1} - 1]
 \end{aligned}$$

$$4. \quad Z^{-1} \left[ \frac{z^2}{(z-1)(z-3)} \right]$$

$$\begin{aligned} Z^{-1} \left[ \frac{z^2}{(z-1)(z-3)} \right] &= Z^{-1} \left[ \frac{z}{z-1} \cdot \frac{z}{z-3} \right] \\ &= Z^{-1} \left( \frac{z}{z-1} \right) * Z^{-1} \left( \frac{z}{z-3} \right) \\ &= 1^n * 3^n \\ &= \sum_{r=0}^n 1^r \cdot 3^{n-r} \\ &= \frac{1}{2} (3^{n+1} - 1) \end{aligned}$$

### Application of $Z$ -transform to solve linear difference equation

**We know that:**

$$\begin{aligned} F(z) &= Z \{y_n\}, \quad Z \{y_{n+1}\} = z[F(z) - y_0], \quad Z \{y_{n+2}\} = z^2 \left[ F(z) - y_0 - \frac{y_1}{z} \right] \text{ and} \\ Z \{y_{n+3}\} &= z^3 \left[ F(z) - y_0 - \frac{y_1}{z} - \frac{y_2}{z^2} \right] \end{aligned}$$

1. Solve  $y_{n+1} - 2y_n = 0$  given  $y_0 = 3$ . (OR)  $y(n+1) - 2y(n) = 0$  given  $y_0 = 3$ .

**Solution:**

$$\text{Given } y_{n+1} - 2y_n = 0$$

Taking  $Z$ -transform on both sides, we have

$$Z \{y_{n+1}\} - 2Z \{y_n\} = 0$$

$$z[F(z) - y_0] - 2F(z) = 0$$

$$(z-2)F(z) - z.y_0 = 0$$

$$(z-2)F(z) - 3z = 0$$

$$\Rightarrow F(z) = \frac{3z}{z-2}$$

Taking inverse  $Z$ -transform on both sides, we have

$$Z^{-1} [F(z)] = 3Z^{-1} \left( \frac{z}{z-2} \right)$$

$$\Rightarrow f(n) = 3.2^n$$

2. Solve  $y_{n+2} - 7y_{n+1} + 12y_n = 2^n$ , given  $y_0 = y_1 = 0$ .

**Solution:**

Given  $y_{n+2} - 7y_{n+1} + 12y_n = 2^n$

Taking inverse  $Z$ -transform on both sides, we have

$$\begin{aligned} Z\{y_{n+2}\} - 7Z\{y_{n+1}\} + 12Z\{y_n\} &= Z\{2^n\} \\ z^2 \left[ F(z) - y_0 - \frac{y_1}{z} \right] - 7z[F(z) - y_0] + 12F(z) &= \frac{z}{z-2} \\ (z^2 - 7z + 12)F(z) &= \frac{z}{z-2} \\ F(z) &= \frac{z}{(z^2 - 7z + 12)(z-2)} \\ Z^{-1}[F(z)] &= Z^{-1} \left[ \frac{z}{(z-3)(z-4)(z-2)} \right] \\ f(n) &= Z^{-1} \left[ \frac{z}{(z-3)(z-4)(z-2)} \right] \\ &= Z^{-1}[\phi(z)] \end{aligned}$$

$$\text{where } \phi(z) = \frac{z}{(z-3)(z-4)(z-2)}$$

$$\Rightarrow z^{n-1}\phi(z) = \frac{z^n}{(z-3)(z-4)(z-2)}$$

The poles are given by  $z = 2, 3, 4$ ; each are simple pole

$$\begin{aligned} R_1 &= \text{Residue at } z = 2 \\ &= \lim_{z \rightarrow 2} (z-2) \cdot \frac{z^n}{(z-3)(z-4)(z-2)} \\ &= \lim_{z \rightarrow 2} \frac{z^n}{(z-3)(z-4)} = \frac{2^n}{(-1)(-2)} = \frac{2^n}{2} \end{aligned}$$

$$\begin{aligned} R_2 &= \text{Residue at } z = 3 \\ &= \lim_{z \rightarrow 3} (z-3) \cdot \frac{z^n}{(z-3)(z-4)(z-2)} \\ &= \lim_{z \rightarrow 3} \frac{z^n}{(z-2)(z-4)} = \frac{3^n}{(-1)(1)} = -3^n \end{aligned}$$

$$\begin{aligned} R_3 &= \text{Residue at } z = 4 \\ &= \lim_{z \rightarrow 4} (z-4) \cdot \frac{z^n}{(z-3)(z-4)(z-2)} \\ &= \lim_{z \rightarrow 4} \frac{z^n}{(z-2)(z-3)} = \frac{4^n}{2} \end{aligned}$$

$$\text{Therefore } f(n) = R_1 + R_2 + R_3 = \frac{2^n}{2} - 3^n + \frac{4^n}{2}$$



3. Solve  $y(n+2) + 4y(n+1) + 4y(n) = n$ , given that  $y(0) = 0$  and  $y(1) = 1$

**Solution:**

Given  $y(n+2) + 4y(n+1) + 4y(n) = n$

Taking  $Z$ -transform on both sides, we get

$$\begin{aligned} Z\{y(n+2)\} + 4Z\{y(n+1)\} + 4Z\{y_n\} &= Z\{n\} \\ z^2 \left[ F(z) - y(0) - \frac{y(1)}{z} \right] + 4z[F(z) - y(0)] + 4F(z) &= \frac{z}{(z-1)^2} \\ (z^2 + 4z + 4)F(z) - z &= \frac{z}{(z-1)^2} \\ (z^2 + 4z + 4)F(z) &= \frac{(z^3 - 2z^2 + 2z)}{(z-1)^2} \\ F(z) &= \frac{(z^3 - 2z^2 + 2z)}{(z-1)^2(z^2 + 4z + 4)} \end{aligned}$$

Taking inverse on both sides, we have

$$\begin{aligned} Z^{-1}[F(z)] &= Z^{-1} \left[ \frac{(z^3 - 2z^2 + 2z)}{(z-1)^2(z^2 + 4z + 4)} \right] \\ f(n) &= Z^{-1} \left[ \frac{(z^3 - 2z^2 + 2z)}{(z-1)^2(z+2)^2} \right] \\ &= Z^{-1}[\phi(z)] \end{aligned}$$

$$\begin{aligned} \text{where } \phi(z) &= \frac{(z^3 - 2z^2 + 2z)}{(z-1)^2(z+2)^2} = \frac{z(z^2 - 2z + 2)}{(z-1)^2(z+2)^2} \\ \Rightarrow \frac{\phi(z)}{z} &= \frac{(z^3 - 2z^2 + 2z)}{(z-1)^2(z+2)^2} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z+2)} + \frac{D}{(z+2)^2} \\ (z^3 - 2z^2 + 2z) &= A(z-1)(z+2)^2 + B(z+2)^2 + C(z+2)(z-1)^2 + D(z-1)^2 \\ \text{Put } z = 1 &\Rightarrow B = \frac{1}{9}, \quad z = -2 \Rightarrow D = \frac{10}{9} \end{aligned}$$

Equating the coefficient of  $z^3$  and constant term, we have

$$A + C = 0 \quad \text{and} \quad -4A + 4B + 2C + D = 2$$

$$\text{Solving above equations, we get } C = \frac{2}{27} \quad \text{and} \quad A = \frac{-2}{27}$$

$$\begin{aligned}
 \text{Therefore } \frac{\phi(z)}{z} &= \frac{-2}{z-1} + \frac{1}{(z-1)^2} + \frac{2}{(z+2)} + \frac{10}{(z+2)^2} \\
 \phi(z) &= \frac{-2}{27} \cdot \frac{z}{z-1} + \frac{1}{9} \cdot \frac{z}{(z-1)^2} + \frac{2}{27} \cdot \frac{z}{z+2} + \frac{10}{9} \cdot \frac{z}{(z+2)^2} \\
 Z^{-1}[\phi(z)] &= \frac{-2}{27} Z^{-1}\left(\frac{z}{z-1}\right) + \frac{1}{9} Z^{-1}\left(\frac{z}{(z-1)^2}\right) + \frac{2}{27} Z^{-1}\left(\frac{z}{(z+2)}\right) \\
 &\quad + \frac{10}{9} Z^{-1}\left(\frac{z}{(z+2)^2}\right) \\
 f(n) &= \frac{-2}{27} + \frac{1}{9} \cdot n + \frac{2}{27}(-2)^n + \frac{10}{9} \left[ \frac{-1}{2} \cdot n(-2)^n \right] \\
 &= \frac{-2}{27} + \frac{1}{9} \cdot n + \frac{2}{27}(-2)^n - \frac{5}{9} n(-2)^n
 \end{aligned}$$

4. Solve  $x(n+1) - 2x(n) = 1$ , given  $x(0) = 0$

**Solution:**

Given  $x(n+1) - 2x(n) = 1$

Taking  $Z$ -transform on both sides, we have

$$\begin{aligned}
 Z\{x(n+1)\} - 2Z\{x(n)\} &= Z\{1\} \\
 z[X(z) - x(0)] - 2X(z) &= \frac{z}{z-1} \\
 (z-2)X(z) &= \frac{z}{z-1} \\
 \Rightarrow X(z) &= \frac{z}{(z-1)(z-2)} \\
 \Rightarrow Z^{-1}\{X(z)\} &= Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right] \\
 \Rightarrow x(n) &= Z^{-1}[\phi(z)] \\
 \text{where } \phi(z) &= \frac{z}{(z-1)(z-2)} \\
 \Rightarrow z^{n-1}\phi(z) &= \frac{z^n}{(z-1)(z-2)}
 \end{aligned}$$

The poles are given by  $z = 1, 2$ ; each of simple poles.

$$\begin{aligned}
 R_1 &= \text{Residue at } z = 1 \\
 &= \lim_{z \rightarrow 1} (z-1) \cdot \frac{z^n}{(z-1)(z-2)} \\
 &= \lim_{z \rightarrow 1} \frac{z^n}{(z-2)} = -1
 \end{aligned}$$

$$\begin{aligned}
 R_2 &= \text{Residue at } z = 2 \\
 &= \lim_{z \rightarrow 2} (z - 2) \cdot \frac{z^n}{(z - 1)(z - 2)} \\
 &= \lim_{z \rightarrow 2} \frac{z^n}{(z - 1)} = 2^n
 \end{aligned}$$

Therefore  $x(n) = R_1 + R_2 = -1 + 2^n$

5. Solve  $y_{n+2} + y_n = 2$  given  $y_0 = y_1 = 0$

**Solution:**

Given  $y_{n+2} + y_n = 2$

Taking  $Z$ -transform on both sides, we have  $Z[y_{n+2}] + Z[y_n] = Z(2)$

$$z^2 \left[ Y(z) - y_0 - \frac{y_1}{z} \right] + Y(z) = \frac{2z}{(z - 1)}$$

$$z^2[Y(z) - 0 - 0] + Y(z) = \frac{2z}{(z - 1)}$$

$$(z^2 + 1)Y(z) = \frac{2z}{(z - 1)}$$

$$\Rightarrow Y(z) = \frac{2z}{(z - 1)(z^2 + 1)}$$

$$\frac{Y(z)}{z} = \frac{2}{(z - 1)(z^2 + 1)} = \frac{A}{z - 1} + \frac{Bz + C}{z^2 + 1}$$

$$\text{Now } 2 = A(z^2 + 1) + (Bz + C)(z - 1)$$

$$\text{Put } z = 1 \Rightarrow A = 1$$

Equating Co-efficient of  $z^2$  and constant term.

$$A + B = 0 \tag{7}$$

$$A - C = 2 \tag{8}$$

$A = 1$  in equations (7) and (8), we have  $B = -1$  and  $C = -1$

$$\text{Therefore } \frac{Y(z)}{z} = \frac{1}{z - 1} - \frac{z + 1}{z^2 + 1}$$

$$Y(z) = \frac{z}{z - 1} - \frac{z^2}{z^2 + 1} - \frac{z}{z^2 + 1}$$

Taking inverse on both sides, we get

$$y(n) = 1 - \cos \frac{n\pi}{2} - \sin \frac{n\pi}{2}$$

6. Solve  $y(n) - ay(n - 1) = u(n)$

**Solution**

Given  $y(n) - ay(n - 1) = u(n)$

Taking  $Z$ -transform on both sides

$$Z[y(n)] - aZ[y(n - 1)] = Z[u(n)]$$

$$Y(z) - a.z^{-1}Y(z) = \frac{z}{z-1} \text{ since } Z[x(n-m)] = z^{-m}X(z)$$

$$(1 - az^{-1})Y(z) = \frac{z}{z-1}$$

$$\Rightarrow \frac{z-a}{z}Y(z) = \frac{z}{z-1}$$

$$\text{Therefore } Y(z) = \frac{z^2}{(z-a)(z-1)}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{z}{(z-a)(z-1)} = \frac{A}{z-a} + \frac{B}{z-1}$$

$$\text{Therefore } A(z-1) + B(z-a) = z$$

$$\text{put } z=1 \Rightarrow B = \frac{1}{1-a} \text{ and } z=a \Rightarrow A = \frac{-a}{1-a}$$

$$\text{Therefore } \frac{y(z)}{z} = \frac{-a}{1-a} \cdot \frac{1}{z-a} + \frac{1}{1-a} \cdot \frac{1}{z-1}$$

$$y(z) = \frac{1}{1-a} \left[ \frac{-az}{z-a} + \frac{z}{z-1} \right]$$

Taking inverse  $Z$ -transform, we have

$$\begin{aligned} y(n) &= \frac{1}{1-a} \left[ -Z^{-1} \left( \frac{az}{z-a} \right) + Z^{-1} \left( \frac{z}{z-1} \right) \right] \\ &= \frac{1}{1-a} \left[ Z^{-1} \left( \frac{z}{z-1} \right) - aZ^{-1} \left( \frac{z}{z-a} \right) \right] \\ &= \frac{1}{1-a} [1 - a.a^n] \end{aligned}$$

7. Solve  $y(n) = y(n-1) = u(n) + u(n-1)$

**Solution:**

$$\text{Given } y(n) = y(n-1) = u(n) + u(n-1)$$

Taking  $Z$ -Transform on both sides,

$$Z[y(n)] - Z[y(n-1)] = Z[u(n)] + Z[u(n-1)]$$

$$Y(z) - z^{-1}Y(z) = \frac{z}{z-1} + z^{-1} \cdot \frac{z}{z-1} \text{ since } Z[x(n-m)] = z^{-m}X(z)$$

$$(1 - z^{-1})Y(z) = \frac{z+1}{z-1}$$

$$\Rightarrow Y(z) = \frac{z(z+1)}{(z-1)^2}$$

$$\frac{Y(z)}{z} = \frac{(z+1)}{(z-1)^2} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2}$$

$$\text{Now } z+1 = A(z-1) + B$$

$$\text{Put } z=1 \Rightarrow B=2 \text{ and equating co. eff. of } z, \text{ we get } A=1$$

$$\text{Therefore } \frac{Y(z)}{z} = \frac{1}{(z-1)} + \frac{2}{(z-1)^2}$$

$$\Rightarrow Y(z) = \frac{z}{(z-1)} + \frac{2z}{(z-1)^2}$$

Taking inverse  $Z$ -transform on both sides, we get

$$y(n) = Z^{-1}[Y(z)] = Z^{-1}\left(\frac{z}{z-1}\right) + 2Z^{-1}\left(\frac{z}{(z-1)^2}\right) = 1 + 2n$$

8. Solve  $x(k+2) - 3x(k+1) + 2x(k) = u(k)$  given  $x(k) = 0$  for  $k \leq 0$  and  $u(0) = 1, u(k) = 0$  for  $k < 0$  and  $k > 0$ .

**Solution:**

$$\text{Given } x(k+2) - 3x(k+1) + 2x(k) = u(k)$$

Taking  $Z$ -transform on both sides, we have

$$Z[x(k+2)] - 3Z[x(k+1)] + 2Z[x(k)] = Z[u(k)]$$

$$z^2 \left[ X(z) - x(0) - \frac{x(1)}{z} \right] - 3z[X(z) - x(0)] + 2X(z) = 1 \quad (9)$$

putting  $k = -1$  in given equation, we have

$$x(1) - 3x(0) + 2x(-1) = u(-1)$$

$$x(1) - 3 \cdot 0 + 2 \cdot 0 = 0 \text{ since } x(0) = 0, x(-1) = 0, u(-1) = 0$$

Therefore  $x(1) = 0$

From equation (9), becomes

$$z^2 X(z) - 3zX(z) + 2X(z) = 1$$

$$(z^2 - 3z + 2)X(z) = 1$$

$$\Rightarrow X(z) = \frac{1}{z^2 - 3z + 2}$$

$$X(z) = \frac{1}{(z-1)(z-2)}$$

Taking inverse  $Z$ -transform, we have

$$x(k) = Z^{-1}[X(z)] = Z^{-1}\left[\frac{1}{(z-1)(z-2)}\right]$$

$$\text{Let } \phi(z) = \frac{1}{(z-1)(z-2)}$$

$$\Rightarrow \phi(z)z^{k-1} = \frac{z^{k-1}}{(z-1)(z-2)}$$

The poles are given by  $z = 1, 2$ ; each are simple pole

$$R_1 = \text{Residue at } z = 1$$

$$= \lim_{z \rightarrow 1} (z-1) \cdot \frac{z^{k-1}}{(z-1)(z-2)} = -1$$

$$R_2 = \text{Residue at } z = 2$$

$$= \lim_{z \rightarrow 2} (z-2) \cdot \frac{z^{k-1}}{(z-1)(z-2)} = 2^{k-1}$$

$$\text{Hence } x(k) = R_1 + R_2 = -1 + 2^{k+1}$$

9. Solve  $y_{n+2} - 4y_n = 0$  using  $z$ -transform.

**Solution:**

Here, the conditions  $y_0$  and  $y_1$  are not given.

Take  $y_0 = A$ ,  $y_1 = B$

Given  $y_{n+2} - 4y_n = 0$

Taking  $Z$ -Transform on both sides, we get

$$\Rightarrow Z[y_{n+2}] - 4Z[y_n] = 0$$

$$\Rightarrow z^2 \left[ Y(z) - y_0 - \frac{y_1}{z} \right] - 4Y(z) = 0$$

$$\Rightarrow (z^2 - 4)Y(z) - Az^2 - Bz = 0$$

$$\Rightarrow Y(z) = \frac{Az^2 + Bz}{z^2 - 4}$$

$$\frac{Y(z)}{z} = \frac{Az}{z^2 - 4} + \frac{B}{z^2 - 4} \quad (10)$$

$$\text{Now } \frac{z}{z^2 - 4} = \frac{z}{(z+2)(z-2)} = \frac{A}{z+2} + \frac{B}{z-2}$$

$$z = A(z-2) + B(z+2)$$

$$\text{Put } z = 2 \Rightarrow B = \frac{1}{2} \text{ and } z = -2 \Rightarrow A = \frac{1}{2}$$

$$\text{Therefore } \frac{z}{z^2 - 4} = \frac{1}{2} \cdot \frac{1}{z+2} + \frac{1}{2} \cdot \frac{1}{z-2} \quad (11)$$

$$\text{and } \frac{1}{z^2 - 4} = \frac{1}{(z+2)(z-2)} = \frac{A}{z+2} + \frac{B}{z-2}$$

$$\Rightarrow 1 = A(z-2) + B(z+2)$$

$$\text{Put } z = 2 \Rightarrow B = \frac{1}{4} \text{ and } z = -2 \Rightarrow A = -\frac{1}{4}$$

$$\text{Therefore } \frac{1}{z^2 - 4} = \frac{-1}{4} \cdot \frac{1}{z+2} + \frac{1}{4} \cdot \frac{1}{z-2} \quad (12)$$

Substituting (11) and (12) in (10), we get

$$\begin{aligned} \frac{Y(z)}{z} &= \frac{A}{2} \left[ \frac{1}{z+2} + \frac{1}{z-2} \right] + \frac{B}{4} \left[ \frac{1}{z-2} - \frac{1}{z+2} \right] \\ \Rightarrow Y(z) &= \frac{A}{2} \left[ \frac{z}{z+2} + \frac{z}{z-2} \right] + \frac{B}{4} \left[ \frac{z}{z-2} - \frac{z}{z+2} \right] \end{aligned}$$

Taking inverse  $z$ -transform on both sides, we get

$$\begin{aligned} y(n) &= \frac{A}{2} [(-2)^n + 2^n] + \frac{B}{4} [2^n - (-2)^n] \\ &= \left( \frac{A}{2} + \frac{B}{4} \right) 2^n + \left( \frac{A}{2} - \frac{B}{4} \right) (-2)^n \\ &= C \cdot 2^n + D(-2)^n \text{ where } C = \frac{A}{2} + \frac{B}{4} \text{ and } D = \frac{A}{2} - \frac{B}{4} \end{aligned}$$

### SHIFTING PROPERTY

We know that  $Z\{x(n-m)\} = z^{-m}x(z)$

**Corollary:**  $x(n-m) = Z^{-1}[z^{-m}X(z)] = (Z^{-1}[X(z)])_{n \rightarrow n-m}$

1. Find  $Z^{-1}\left[\frac{1}{z - \frac{1}{2}}\right]$

**Solution:**

$$\begin{aligned} Z^{-1}\left[\frac{1}{z - \frac{1}{2}}\right] &= Z^{-1}\left[z^{-1}\left(z - \frac{1}{2}\right)\right] \\ &= Z^{-1}\left(\frac{z}{z - 1/2}\right)_{n \rightarrow n-1} \\ &= \left(\frac{1}{2}\right)^{n-1} \text{ or } \left(\frac{1}{2}\right)^{n-1} u(n-1) \end{aligned}$$

2. Evaluate  $Z^{-1}\left(\frac{1}{z+1}\right)$  given  $Z^{-1}\left(\frac{z}{z+1}\right) = (-1)^n$

**Solution:**

$$\begin{aligned} Z^{-1}\left(\frac{1}{z+1}\right) &= Z^{-1}\left[z^{-1}\left(\frac{z}{z+1}\right)\right] \\ &= Z^{-1}\left(\frac{z}{z+1}\right)_{n \rightarrow n-1} \\ &= (-1)^{n-1} \end{aligned}$$

3. Find  $Z^{-1}(X(z))$  where  $X(z) = \frac{4z^2 - 2z}{z^3 - 5z^2 + 8z - 4}$

**Solution:**

$$\begin{aligned} \text{Given } X(z) &= \frac{4z^2 - 2z}{z^3 - 5z^2 + 8z - 4} \\ \Rightarrow X(z)z^{n-1} &= \frac{z^{n-1} \cdot 2z(2z - 1)}{z^3 - 5z^2 + 8z - 4} \\ &= \frac{2z^n(2z - 1)}{z^3 - 5z^2 + 8z - 4} \end{aligned}$$

$$\begin{aligned} \text{The poles are given by } z^3 - 5z^2 + 8z - 4 &= 0 \\ \Rightarrow (z - 1)(z^2 - 4z + 4) &= 0 \end{aligned}$$

$$\Rightarrow (z-1)(z-2)^2 = 0$$

$\Rightarrow z = 1$  is a simple pole and  $z = 2$  is a pole of order 2

$R_1 = \text{Residue at } z = 1$

$$= \lim_{z \rightarrow 1} (z-1) \cdot \frac{2z^n(2z-1)}{(z-1)(z-2)^2} = 2$$

$R_2 = \text{Residue at } z = 2$

$$\begin{aligned} &= \frac{1}{(2-1)!} \lim_{z \rightarrow 2} \frac{d}{dz} \left[ (z-2)^2 \cdot \frac{2z^n(2z-1)}{(z-1)(z-2)^2} \right] \\ &= \lim_{z \rightarrow 2} \frac{d}{dz} \left( \frac{2z^2(2z-1)}{z-1} \right) \\ &= \lim_{z \rightarrow 2} 2 \left[ \frac{(z-1)[nz^{n-1}(2z-1) + z^n \cdot 2] - z^n(2z-1) \cdot 1}{(z-1)^2} \right] \\ &= 2[n \cdot 2^{n-1} \cdot 3 + 2 \cdot 2^n - 3 \cdot 2^n] = 2 \cdot 2^n \left[ \frac{3}{2}n - 1 \right] \\ &= 2^n(3n-2) \end{aligned}$$

Hence  $x(n) = R_1 + R_2 = 2 + 2^n(3n-2)$

4. Find the inverse  $Z$ -transform of  $\frac{z(z+1)}{(z-1)^3}$

**Solution:**

$$\text{Let } F(z) = \frac{z(z+1)}{(z-1)^3}$$

$$\Rightarrow F(z) \cdot z^{n-1} = \frac{z^n(z+1)}{(z-1)^3}$$

$\Rightarrow z = 1$  is a pole of order 3

$R_1 = \text{Residue at } z = 1$

$$\begin{aligned} &= \frac{1}{(3-1)!} \lim_{z \rightarrow 1} \left[ \frac{d^2}{dz^2} (z-1)^3 \cdot \frac{z^n(z+1)}{(z-1)^3} \right] \\ &= \frac{1}{2!} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} (z^n(z+1)) \\ &= \frac{1}{2} \lim_{z \rightarrow 1} \frac{d}{dz} [nz^{n-1}(z+1) + z^n \cdot 1] \\ &= \frac{1}{2} \lim_{z \rightarrow 1} \frac{d}{dz} [n(n-1)z^{n-2}(z+1) + nz^{n-1} + nz^{n-1}] \\ &= \frac{1}{2} [n(n-1) \cdot 2 + n + n] = n^2 \end{aligned}$$

Hence  $f(n) = n^2$



5. If  $X(z) = (z - 1) \log \left(1 - \frac{1}{z}\right) + 1$ , find  $x(n)$

**Solution:**

$$\begin{aligned}
 x(n) &= Z^{-1} \left[ (z - 1) \log \left(1 - \frac{1}{z}\right) + 1 \right] \\
 &= Z^{-1} \left[ (z - 1) \left( -\frac{1}{z} - \frac{1}{2z^2} - \frac{1}{3z^3} - \dots \right) + 1 \right] \\
 &= Z^{-1} \left[ z \left( -\frac{1}{z} - \frac{1}{2z^2} - \frac{1}{3z^3} - \dots \right) - 1 \left( -\frac{1}{z} - \frac{1}{2z^2} - \frac{1}{3z^3} - \dots \right) + 1 \right] \\
 &= Z^{-1} \left[ \left( -1 - \frac{1}{2z^2} - \frac{1}{3z^3} - \dots \right) + \left( \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} - \dots \right) + 1 \right] \\
 &= Z^{-1} \left[ \left( 1 - \frac{1}{2} \right) \frac{1}{z} + \left( \frac{1}{2} - \frac{1}{3} \right) \frac{1}{z^2} + \dots \right] \\
 &= Z^{-1} \left[ \frac{1}{1 \cdot 2} \frac{1}{z} + \frac{1}{2 \cdot 3} \frac{1}{z^2} + \dots \right] \\
 &= Z^{-1} \left[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \cdot \frac{1}{z^n} \right] \\
 &= \begin{cases} \frac{1}{n(n+1)}, & \text{if } n = 0 \\ 0, & \text{if } n \geq 1 \end{cases}
 \end{aligned}$$

6. If  $X(z) = (1 - az^{-1})^{-2}$ , find  $x(n)$

**Solution:**

$$\begin{aligned}
 X(z) &= (1 - az^{-1})^{-2} = \frac{1}{(1 - az^{-1})^2} \\
 &= \frac{z^2}{(z - a)^2} \\
 \Rightarrow z^{n-1} X(z) &= \frac{z^{n+1}}{(z - a)^2}
 \end{aligned}$$

$z = a$  is a pole of order 2

$R$  = Residue at  $z = a$

$$\begin{aligned}
 &= \lim_{z \rightarrow a} \frac{d}{dz} \left( (z - a)^2 \frac{z^{n+1}}{(z - a)^2} \right) \\
 &= \lim_{z \rightarrow a} \frac{d}{dz} z^{n+1} = (n + 1) a^n
 \end{aligned}$$

Hence  $f(n) = (n + 1) a^n$