Unit-II: Fourier Series

Content:

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Fourier Series-related problems in $(0, 2l)$ and $(-l, l)$

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(6) Half range Sine series-related problems in $(0, \pi)$ and $(0, l)$

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- related problems in Fourier series

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-for finding harmonic in $(0, 2\pi)$

-for finding harmonic in $(0, 2l)$

-for finding harmonic in $(0, T)$

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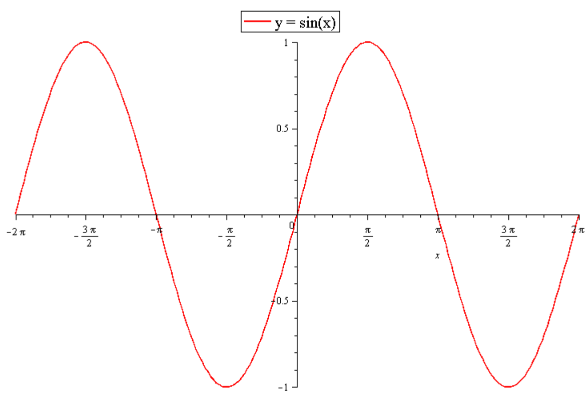
Periodic Function:

A function $f(x)$ is said to be periodic if and only if $f(x+p) = f(x)$ is true for some value of p and every value of x. The smallest value of p is called the period of the function.

Examples: (i) $\sin x$ is a periodic function with period 2π.

Now $\sin(2\pi + x) = \sin x, \sin(4\pi + x) = \sin x, \sin(6\pi + x) = \sin x$ and so on.

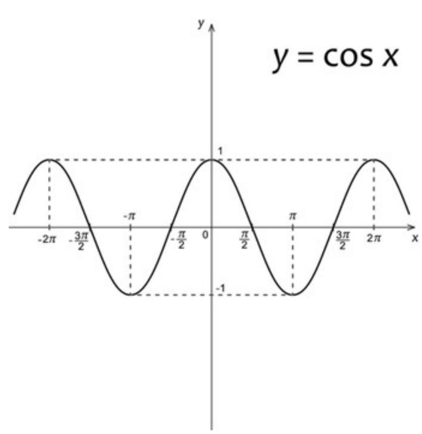
2π is least value. Therefore 2π is the period of $\sin x$.



(ii) $\cos x$ is a periodic function with period 2π.

$\cos(2\pi + x) = \cos x, \cos(4\pi + x) = \cos x, \cos(6\pi + x) = \cos x$ and so on.

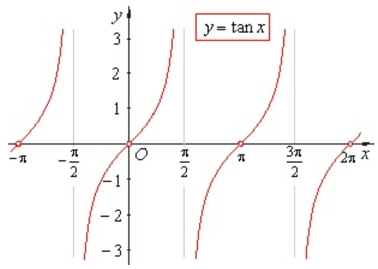
2π is least value. Therefore 2π is the period of $\cos x$.



(iii) $\tan x$ is a periodic function with period π.

$\tan(\pi + x) = \tan x, \tan(2\pi + x) = \tan x$ and so on.

π is least value. Therefore π is the period of $\tan x$.



Continuity of a function:

Let f be a real function on a subset of the real numbers and a be a point in the domain of f. Then f is continuous at a if $\lim_{x \to a} f(x) = f(a)$.

In other words, if the left-hand limit, right-hand limit and the value of the function at $x = a$ exist and are equal to each other, i.e., $\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$ then f is said to be continuous at $x = a$.

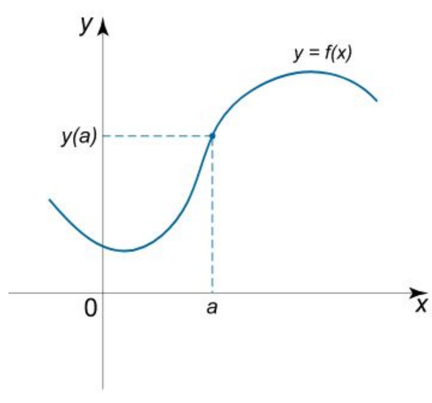


Fig 1. Continuous function.

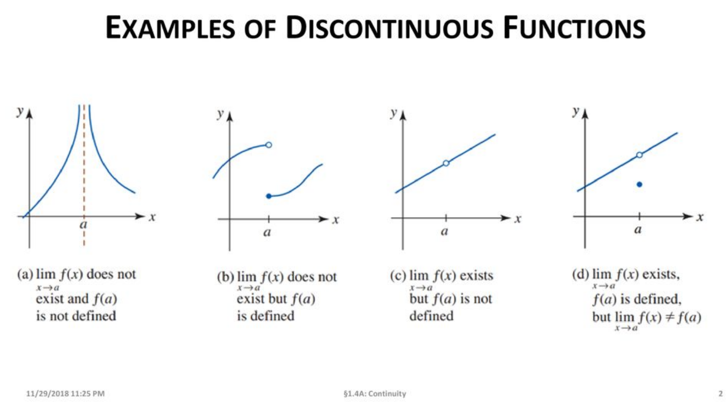
Note: A function is said to be continuous in a given interval if there is no break in the graph of the function in the entire interval range.

Discontinuity of a function:

The function f will be discontinuous at $x = a$ in any of the following cases:

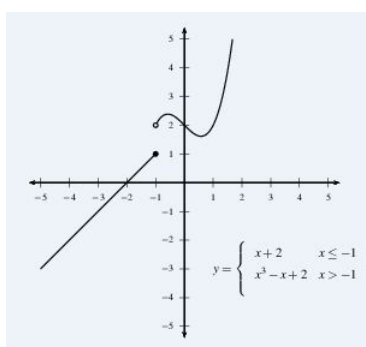
(i) $f(a)$ is not defined. (ii) $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ exist but are not equal.

(iii) if either $\lim\limits_{x\to a^-} f(x)$ or $\lim\limits_{x\to a^+} f(x)$ does not exist.



Jump Discontinuity (or) Piecewise Discontinuity (or) Sectionally Continuous

A function is continuous everywhere except for a finite number of jumps in a given interval is called a piecewise continuous in that interval.



Fourier Series: A development of periodic function into a series of sines and cosines was effected by French Physicist and Mathematician Joseph Fourier in the year (1768-1830).

Definition:

If $f(x)$ is a periodic function with period 2π and if $f(x)$ can be represented by a trigonometric series in the interval $c < x < c + 2\pi$, then

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ where the coefficients a_n and b_n are given by

$$a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\cos nx\,dx \; n \geq 0 \text{ and } b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\sin nx\,dx, n \geq 1.$$

Here a_n and b_n are sometimes called Euler formulas.

Note:

(i) The Fourier series for $f(x)$ in $(0, 2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ where}$$

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\,dx,\; a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx\,dx \text{ and } b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin nx\,dx, n \geq 1.$$

(ii) The Fourier series for $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ where}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx,\; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, n \geq 1.$$

Dirichlet Condition (or) Sufficient condition for $f(x)$:

For a function $f(x)$ can be expanded as a Fourier series. The following conditions are satisfied:

(1) $f(x)$ is a single valued and finite in $(c, c + 2\pi)$.

(2) $f(x)$ is continuous or piece-wise continuous with finite number of finite discontinuous in $(c, c + 2\pi)$.

(3) $f(x)$ has a finite number of maxima or minima in $(c, c + 2\pi)$.

Convergence of Fourier Series:

Let $f(x)$ be defined in $(c, c+2\pi)$ and satisfy Dirichlet's conditions, then $f(x)$ can be expanded as a Fourier series and $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$.

Let α be any point in the interval $(c, c+2\pi)$. Then α is either a point of continuity or discontinuity of $f(x)$.

- If α is a point of continuity, then the sum of the Fourier series when $x = \alpha$ is $f(\alpha)$.
- If α is an interior point of discontinuity of $f(x)$, then the sum of the Fourier series when $x = \alpha$ is $=\frac{1}{2}\left[f(\alpha^-) + f(\alpha^+)\right]$.
- At an end point of the interval $x = c$ or $x = c + 2\pi$ the sum of the series is $=\frac{1}{2}\left[f(c^+) + f(c+2\pi^-)\right]$.

Note: Formulae

(1) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

(2) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$

(3) Bernoulli's generalized formula $\int uv \, dx = uv_1 - u^{'}v_2 + u^{''}v_3 - u^{'''}v_4 + \ldots$

(4) $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

Problems based on $(0, 2\pi)$

Example 1: Find the Fourier series for $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $0 < x < 2\pi$. Hence prove that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution: The Fourier series for $f(x)$ is

$f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx + \sum\limits_{n=1}^{\infty} b_n \sin nx$ where

$a_0 = \frac{1}{\pi}\int\limits_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int\limits_0^{2\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int\limits_0^{2\pi} f(x)\sin nx\,dx$.

To find a_0, a_n and b_n

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int\limits_0^{2\pi} f(x)\,dx \\
&= \frac{1}{\pi}\int\limits_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 dx = \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} \\
&= -\frac{1}{12\pi}[-\pi^3 - \pi^3] = \frac{\pi^2}{6}.
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int\limits_0^{2\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int\limits_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 \cos nx\,dx \\
&= \frac{1}{4\pi}\left[(\pi - x)^2.\frac{\sin nx}{n} - 2(\pi - x)(-1)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi} \\
&= \frac{1}{4\pi}\left[(\pi - x)^2.\frac{\sin nx}{n} - 2(\pi - x)\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{2\pi} \\
&= \frac{1}{4\pi}\Bigg[(\pi - 2\pi)^2.\frac{\sin 2\pi}{n} - 2(\pi - 2\pi)\frac{\cos 2\pi}{n^2} - 2\frac{\sin 2\pi}{n^3} \\
&\quad - \left((\pi - 0)^2.\frac{\sin 0}{n} - 2(\pi - 0)\frac{\cos 0}{n^2} - 2\frac{\sin 0}{n^3}\right)\Bigg]
\end{aligned}
$$

$$a_n = \frac{1}{4\pi}\left[0 + \frac{2\pi}{n^2} - 0 - \left(0 - \frac{2\pi}{n^2} - 0\right)\right] \text{ Since } \cos 2\pi = 1, \sin 2\pi = 0$$
$$= \frac{1}{4\pi}.\frac{4\pi}{n^2} = \frac{1}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$
$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \sin nx\,dx$$
$$= \frac{1}{4\pi}\left[(\pi - x)^2.\left(\frac{-\cos nx}{n}\right) - 2(\pi - x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$
$$= \frac{1}{4\pi}\left[-(\pi - x)^2.\frac{\cos nx}{n} + 2(\pi - x)\frac{\sin nx}{n^2} - 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$
$$= \frac{1}{4\pi}\left[-(\pi - 2\pi)^2.\frac{\cos 2\pi}{n} + 2(\pi - 2\pi)\frac{\sin 2\pi}{n^2} - 2\frac{\cos 2\pi}{n^3}\right.$$
$$\left. - \left(-(\pi - 0)^2.\frac{\cos 0}{n} + 2(\pi - 0)\frac{\sin 0}{n^2} - 2\frac{\cos 0}{n^3}\right)\right]$$
$$= \frac{1}{4\pi}\left[-\frac{\pi^2}{n} + 0 - \frac{2}{n^3} - \left(-\frac{\pi^2}{n} + 0 - \frac{2}{n^3}\right)\right] \text{ Since } \cos 2\pi = 1, \sin 2\pi = 0$$
$$= 0.$$

Therefore $\left(\dfrac{\pi - x}{2}\right)^2 = \dfrac{\pi^2}{12} + \sum\limits_{n=1}^{\infty}\left(\dfrac{\cos nx}{n^2}\right).$

$$\Rightarrow f(x) = \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots$$

To deduce $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2} = \dfrac{\pi^2}{6}$

Put $x = 0$ in the Fourier series. But $x = 0$ is an end point of the interval $(0, 2\pi)$ for the function $f(x) = \left(\dfrac{\pi - x}{2}\right)^2$.

Therefore the sum of the Fourier series when $x = 0$ is

$$f(0) = \frac{1}{2}\left[f(0+) + f(2\pi-)\right]$$
$$= \frac{1}{2}\left[\left(\frac{\pi - 0}{2}\right)^2 + \left(\frac{\pi - 2\pi}{2}\right)^2\right] = \frac{\pi^2}{4}$$

Therefore $\dfrac{\pi^2}{4} = \dfrac{\pi^2}{12} + \dfrac{1}{1^2} + + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots.$

$\Rightarrow \sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{4} - \dfrac{\pi^2}{12} = \dfrac{\pi^2}{6}.$

Example 2: Expand $f(x) = x \sin x$, $0 < x < 2\pi$ in a Fourier series. Hence deduce the result $\dfrac{1}{1.3} - \dfrac{1}{3.5} + \dfrac{1}{5.7} - \ldots = \dfrac{\pi - 2}{4}$.

Solution: Given $f(x) = x \sin x$

The Fourier series for $f(x)$ is

$f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ where

$a_0 = \dfrac{1}{\pi} \int_0^{2\pi} f(x)\, dx$, $a_n = \dfrac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$ and $b_n = \dfrac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx.$

To find a_0, a_n **and** b_n

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx \\
&= \frac{1}{\pi} \int_0^{2\pi} x \sin x\, dx \\
&= \frac{1}{\pi} \left[x(-\cos x) - 1.(-\sin x)\right]_0^{2\pi} \\
&= \frac{1}{\pi}[-x \cos x + \sin x]_0^{2\pi} \\
&= \frac{1}{\pi}[-2\pi \cos 2\pi + \sin 2\pi - (0 + 0)]_0^{2\pi} \\
&= \frac{1}{\pi}(-2\pi) = -2.
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx \\
&= \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos nx\, dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} x \left[\sin(x + nx) + \sin(x - nx)\right]\, dx
\end{aligned}$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]\,dx \text{ Since } \sin(-\theta) = -\sin\theta$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\sin(n+1)x\,dx - \int_0^{2\pi} x\sin(n-1)x\,dx\right]$$

$$= \frac{1}{2\pi}\Bigg\{\left[x.\left(\frac{-\cos(n+1)x}{n+1}\right) - 1.\left(\frac{-\sin(n+1)x}{(n+1)^2}\right)\right]_0^{2\pi}$$

$$-\left[x.\left(\frac{-\cos(n-1)x}{n-1}\right) - 1.\left(\frac{-\sin(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}\Bigg\}$$

$$= \frac{1}{2\pi}\Bigg\{\left[\frac{-x\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{2\pi}$$

$$-\left[\frac{-x\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}\Bigg\}$$

$$= \frac{1}{2\pi}\Bigg\{\frac{-2\pi\cos(n+1)2\pi}{n+1} + \frac{\sin(n+1)2\pi}{(n+1)^2} - 0$$

$$+\frac{2\pi\cos(n-1)2\pi}{n-1} - \frac{\sin(n-1)2\pi}{(n-1)^2} - 0\Bigg\}$$

$$= \frac{1}{2\pi}\left\{\frac{-2\pi}{n+1} + \frac{2\pi}{n-1}\right\} = \frac{2}{(n-1)(n+1)}\ \ n \neq 1$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx \text{ Since } \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1.\left(\frac{-\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$
\begin{aligned}
a_1 &= \frac{1}{2\pi}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{2\pi} \\
&= \frac{1}{2\pi}\left[\frac{-2\pi\cos 4\pi}{2} + \frac{\sin 4\pi}{4} - (0+0)\right] \\
&= -\frac{1}{2}.
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int\limits_0^{2\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int\limits_0^{2\pi} x\sin x\sin nx\,dx \\
&= \frac{1}{2\pi}\int\limits_0^{2\pi} x\left[\cos(x-nx) - \cos(x+nx)\right]\,dx \\
&= \frac{1}{2\pi}\int\limits_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x\right]\,dx \text{ Since } \cos(-\theta) = \cos\theta \\
&= \frac{1}{2\pi}\left[\int\limits_0^{2\pi} x\cos(n-1)x\,dx - \int\limits_0^{2\pi} x\cos(n+1)x\,dx\right] \\
&= \frac{1}{2\pi}\Bigg\{\left[x.\left(\frac{\sin(n-1)x}{n-1}\right) - 1.\left(\frac{-\cos(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi} \\
&- \left[x.\left(\frac{\sin(n+1)x}{n+1}\right) - 1.\left(\frac{-\cos(n+1)x}{(n+1)^2}\right)\right]_0^{2\pi}\Bigg\} \\
&= \frac{1}{2\pi}\Bigg\{\left[\frac{x\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2}\right]_0^{2\pi} \\
&- \left[\frac{x\sin(n+1)x}{n+1} + \frac{\cos(n+1)x}{(n+1)^2}\right]_0^{2\pi}\Bigg\} \\
&= \frac{1}{2\pi}\Bigg\{\left[\frac{2\pi\sin(n-1)2\pi}{n-1} + \frac{\cos(n-1)2\pi}{(n-1)^2} - \left(0 + \frac{\cos 0}{(n-1)^2}\right)\right] \\
&- \left[\frac{2\pi\sin(n+1)2\pi}{n+1} + \frac{\cos(n+1)2\pi}{(n+1)^2} - \left(0 + \frac{\cos 0}{(n-1)^2}\right)\right]\Bigg\} \\
&= \frac{1}{2\pi}\left\{0 + \frac{1}{(n-1)^2} - \frac{1}{(n-1)^2} - \left[0 + \frac{1}{(n+1)^2} - \frac{1}{(n+1)^2}\right]\right\} = 0
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1-\cos 2x}{2}\right)\,dx \\
&= \frac{1}{2\pi}\left\{\int_0^{2\pi} x\,dx - \int_0^{2\pi}\cos 2x\,dx\right\} \\
&= \frac{1}{2\pi}\left\{\left[\frac{x^2}{2}\right]_0^{2\pi} - \left[x.\frac{\sin 2x}{2} - 1.\left(\frac{-\cos 2x}{4}\right)\right]_0^{2\pi}\right\} \\
&= \frac{1}{2\pi}\left\{\frac{4\pi^2}{2} - \left[\frac{2\pi\sin 4\pi}{2} + \frac{\cos 4\pi}{4} - \left(0 + \frac{\cos 0}{4}\right)\right]\right\} \\
&= \frac{1}{2\pi}\left\{2\pi^2 - \left[0 + \frac{1}{4} - \frac{1}{4}\right]\right\} = \pi.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f(x) &= -1 - \frac{\cos x}{2} + \sum_{n=2}^{\infty}\frac{2\cos nx}{(n-1)(n+1)} + \pi\sin x \\
&= -1 - \frac{\cos x}{2} + \pi\sin x + 2\left[\frac{1}{1.3}\cos 2x + \frac{1}{2.4}\cos 3x\right. \\
&\quad \left. + \frac{1}{3.5}\cos 4x + \frac{1}{4.6}\cos 5x + \dots\right]
\end{aligned}
$$

To deduce that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots = \frac{\pi - 2}{4}$

Put $x = \frac{\pi}{2}$ in the Fourier series and $x = \frac{\pi}{2}$ is a point of continuity for the function $f(x) = x\sin x$.

Hence the sum of the Fourier series when $x = \frac{\pi}{2}$ is

$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}. \sin\frac{\pi}{2} = \frac{\pi}{2}.$

Therefore

$$\frac{\pi}{2} = -1 - \frac{1}{2}.\cos\frac{\pi}{2} + \pi\sin\frac{\pi}{2} + 2\left[\frac{1}{1.3}\cos\pi + 0 + \frac{1}{3.5}\cos 2\pi\right.$$
$$\left. + 0 + \frac{1}{5.7}\cos 3\pi + \dots\right]$$
$$= -1 + \pi + 2\left[-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \dots\right]$$
$$\Rightarrow -\frac{\pi}{2} + 1 = -2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\right]$$
$$\Rightarrow \frac{\pi}{2} - 1 = 2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\right]$$
$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots = \frac{\pi - 2}{4}.$$

Example 3: Find the Fourier series to represent the function $f(x)$ given by $f(x) = \begin{cases} x & \text{for } 0 < x \leq \pi \\ 2\pi - x & \text{for } \pi \leq x < 2\pi \end{cases}$. Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$.

Solution: The Fourier series for $f(x)$ is

$f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx + \sum\limits_{n=1}^{\infty} b_n \sin nx$ where

$a_0 = \frac{1}{\pi}\int\limits_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int\limits_0^{2\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int\limits_0^{2\pi} f(x)\sin nx\,dx$.

To find a_0, a_n and b_n

$$a_0 = \frac{1}{\pi}\int\limits_0^{2\pi} f(x)\,dx$$
$$= \frac{1}{\pi}\left[\int\limits_0^{\pi} f(x)\,dx + \int\limits_{\pi}^{2\pi} f(x)\,dx\right]$$

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} (2\pi - x)\,dx\right] \\
&= \frac{1}{\pi}\left\{\left[\frac{x^2}{2}\right]_0^{\pi} + \left[\frac{(2\pi - x)^2}{-2}\right]_{\pi}^{2\pi}\right\} \\
&= \frac{1}{\pi}\left\{\frac{\pi^2}{2} - 0 - \frac{1}{2}\left[0 - \pi^2\right]\right\} \\
&= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right] = \pi. \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos nx\,dx + \int_{\pi}^{2\pi} f(x)\cos nx\,dx\right] \\
&= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi} (2\pi - x)\cos nx\,dx\right] \\
&= \frac{1}{\pi}\Bigg\{\left[x.\frac{\sin nx}{n} - 1.\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi} \\
&+ \left[(2\pi - x)\frac{\sin nx}{n} - (-1).\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\Bigg\} \\
&= \frac{1}{\pi}\Bigg\{\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} \\
&+ \left[(2\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{\pi}^{2\pi}\Bigg\} \\
&= \frac{1}{\pi}\Bigg\{\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - \left(0 + \frac{\cos 0}{n^2}\right) \\
&+ \left[0 - \frac{\cos 2n\pi}{n^2} - \left(0 - \frac{\cos n\pi}{n^2}\right)\right]\Bigg\} \\
&= \frac{1}{\pi}\left[\frac{2\cos n\pi}{n^2} - \frac{1}{n^2} - \frac{1}{n^2}\right]
\end{aligned}
$$

$$a_n = \frac{1}{\pi}\left[\frac{2(-1)^n}{n^2} - \frac{2}{n^2}\right] = \frac{2}{\pi n^2}\left\{(-1)^n - 1\right\}$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-4}{\pi n^2} & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int\limits_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int\limits_0^{\pi} f(x)\sin nx\,dx + \int\limits_{\pi}^{2\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int\limits_0^{\pi} x\sin nx\,dx + \int\limits_{\pi}^{2\pi} (2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[x.\left(\frac{-\cos nx}{n}\right) - 1.\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}\right.$$

$$\left. + \left[(2\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1).\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}\right.$$

$$\left. + \left[-(2\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2} - 0\right.$$

$$\left. + \left[0 - \frac{\sin 2n\pi}{n^2} - \left(\frac{-(2\pi - \pi)\cos n\pi}{n} - 0\right)\right]\right\}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n}\cos n\pi + \frac{\pi}{n}\cos n\pi\right] = 0.$$

Therefore the Fourier series is

$$f(x) = \frac{\pi}{2} + \sum_{odd} \frac{-4}{\pi n^2}\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right]$$

To deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$.

Put $x = 0$ in the Fourier series. But $x = 0$ is an end point of the interval $(0, 2\pi)$ for the function $f(x)$.

Therefore the sum of the Fourier series when $x = 0$ is

$$\begin{aligned} f(0) &= \frac{1}{2}\left[f(0+) + f(2\pi-)\right] \\ &= \frac{1}{2}\left[0 + 0\right] = 0. \end{aligned}$$

Therefore

$$\begin{aligned} 0 &= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2}\cos 0 + \frac{1}{3^2}\cos 0 + \frac{1}{5^2}\cos 0 + \ldots\right] \\ \Rightarrow -\frac{\pi}{2} &= -\frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right] \\ \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots &= \frac{\pi^2}{8}. \end{aligned}$$

Example 4: Expand $x(2\pi - x)$ as Fourier series in $(0, 2\pi)$. Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$.

Solution: Given $f(x) = x(2\pi - x)$

The Fourier series for $f(x)$ is

$f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx + \sum\limits_{n=1}^{\infty} b_n \sin nx$ where

$a_0 = \frac{1}{\pi}\int\limits_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int\limits_0^{2\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int\limits_0^{2\pi} f(x)\sin nx\,dx$.

To find a_0, a_n **and** b_n

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int\limits_0^{2\pi} f(x)\,dx \\ &= \frac{1}{\pi}\int\limits_0^{2\pi} x(2\pi - x)\,dx \end{aligned}$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi}(2\pi x - x^2)\,dx \\
&= \frac{1}{\pi}\left[\pi x^2 - \frac{x^3}{3}\right]_0^{2\pi} \\
&= \frac{1}{\pi}\left[4\pi^3 - \frac{8\pi^3}{3}\right] = \frac{4\pi^2}{3}. \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi}(2\pi x - x^2)\cos nx\,dx \\
&= \frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (2\pi - 2x)\left(\frac{-\cos nx}{n^2}\right)\right. \\
&\left. + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi} \\
&= \frac{1}{\pi}\left[(2\pi x - x^2)\frac{\sin nx}{n} + (2\pi - 2x)\frac{\cos nx}{n^2} + 2\frac{\sin nx}{n^3}\right]_0^{2\pi} \\
&= \frac{1}{\pi}\left[(4\pi^2 - 4\pi^2)\frac{\sin 2n\pi}{n} + (2\pi - 4\pi)\frac{\cos 2n\pi}{n^2} + 2\frac{\sin 2n\pi}{n^3}\right. \\
&\left. - \left(0 + \frac{2\pi}{n^2}\cos 0 + 0\right)\right] \\
&= \frac{1}{\pi}\left[-\frac{2\pi}{n^2} - \frac{2\pi}{n^2}\right] = -\frac{4}{n^2}. \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi}(2\pi x - x^2)\sin nx\,dx
\end{aligned}$$

$$b_n = \frac{1}{\pi}\Bigg[(2\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (2\pi - 2x)\left(\frac{-\sin nx}{n^2}\right)$$

$$+ (-2)\left(\frac{\cos nx}{n^3}\right)\Bigg]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-(2\pi x - x^2)\frac{\cos nx}{n} + (2\pi - 2x)\frac{\sin nx}{n^2} - 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\Bigg[-(4\pi^2 - 4\pi^2)\frac{\cos 2n\pi}{n} + (2\pi - 4\pi)\frac{\sin 2n\pi}{n^2} - 2\frac{\cos 2n\pi}{n^3}$$

$$- \left(0 + 0 - 2\frac{\cos 0}{n^3}\right)\Bigg]$$

$$= \frac{1}{\pi}\left[-\frac{2}{n^3} + \frac{2}{n^3}\right] = 0.$$

Therefore

$$f(x) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4}{n}\cos nx$$

$$= \frac{2\pi^2}{3} - 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots\right]$$

To deduce the value of $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots.$

Put $x = 0$ in the Fourier series.

But $x = 0$ is end point of the interval $(0, 2\pi)$ for the function $f(x) = x(2\pi - x)$.

Therefore the sum of the Fourier series when $x = 0$ is

$$\frac{1}{2}\left[f(0+) + f(2\pi-)\right] = \frac{1}{2}(0 + 0) = 0.$$

$$\text{Therefore } 0 = \frac{2\pi^2}{3} - 4\left[\frac{\cos 0}{1^2} + \frac{\cos 0}{2^2} + \frac{\cos 0}{3^2} + \dots\right]$$

$$\Rightarrow -\frac{2\pi^2}{3} = -4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}.$$

Example 5: Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.

Solution: The Fourier series for $f(x)$ is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ where

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$.

To find a_0, a_n and b_n

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx \\
&= \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi} \\
&= -\frac{1}{\pi}\left(e^{-2\pi} - e^{-0}\right) = \frac{1}{\pi}(1 - e^{-2\pi}).
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx \\
&= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-\cos nx + n\sin nx\right)\right]_0^{2\pi} \\
&= \frac{1}{\pi(1+n^2)}\Bigg[e^{-2\pi}\left(\cos 2n\pi + n\sin 2n\pi\right) - e^{0}\left(-\cos 0 + n\sin 0\right)\Bigg] \\
&= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-1+0) - (-1+0)\right] \\
&= \frac{1}{\pi(1+n^2)}(1 - e^{-2\pi})
\end{aligned}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$
$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$
$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-\sin nx - n\cos nx\right)\right]_0^{2\pi}$$
$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}\left(-\sin 2n\pi - n\cos 2n\pi\right) - e^0\left(-\sin 0 - n\cos 0\right)\right]$$
$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(0-n) - (0-n)\right]$$
$$= \frac{n}{\pi(1+n^2)}(1-e^{-2\pi})$$

Therefore the Fourier series is

$$e^{-x} = \frac{1}{2\pi}(1-e^{-2\pi}) + \sum_{n=1}^{\infty}\frac{1}{\pi(1+n^2)}(1-e^{-2\pi})\cos nx$$
$$+\sum_{n=1}^{\infty}\frac{n}{\pi(1+n^2)}(1-e^{-2\pi})\sin nx$$
$$= \frac{1}{\pi}(1-e^{-2\pi})\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos nx}{1+n^2} + \sum_{n=1}^{\infty}\frac{\sin nx}{1+n^2}\right]$$

Example 6: If $f(x) = \begin{cases} \sin x & \text{in } 0 \le x \le \pi \\ 0 & \text{in } \pi \le x < 2\pi \end{cases}$. find a Fourier series of periodicity 2π and hence evaluate $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots \text{to}\infty$.

Solution:

The Fourier series for $f(x)$ is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ where

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$.

To find a_0, a_n and b_n

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right\}\\
&= \frac{1}{\pi}\left\{\int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} 0.\,dx\right\}\\
&= \frac{1}{\pi}\left[-\cos x\right]_0^{\pi}\\
&= \frac{1}{\pi}\left[-\cos\pi + \cos 0\right] = \frac{2}{\pi}.\\
a_n &= \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\cos nx\,dx + \int_{\pi}^{2\pi} f(x)\cos nx\,dx\right\}\\
&= \frac{1}{\pi}\left\{\int_0^{\pi} \sin x\cos nx\,dx + \int_{\pi}^{2\pi} 0.\cos nx\,dx\right\}\\
&= \frac{1}{2\pi}\int_0^{\pi}\left[\sin(x+nx)+\sin(x-nx)\right]\,dx\\
&= \frac{1}{2\pi}\int_0^{\pi}\left[\sin(1+n)x+\sin(1-n)x\right]\,dx\\
&= \frac{1}{2\pi}\left[-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}\\
&= \frac{1}{2\pi}\left[-\frac{\cos(1+n)\pi}{1+n} - \frac{\cos(1-n)\pi}{1-n} - \left(-\frac{\cos 0}{1+n} - \frac{\cos 0}{1-n}\right)\right]\\
&= \frac{1}{2\pi}\left[-\frac{\cos(\pi+n\pi)}{1+n} - \frac{\cos(\pi-n\pi)}{1-n} + \frac{\cos 0}{1+n} + \frac{\cos 0}{1-n}\right]\\
&= \frac{1}{2\pi}\left[\frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right]
\end{aligned}
$$

Since $\cos(\pi+n\pi) = -\cos n\pi$, $\cos(\pi-n\pi) = -\cos n\pi$

$$
\begin{aligned}
&= \frac{1}{2\pi}\left[\frac{1}{1+n}\left[(-1)^n+1\right] + \frac{1}{1-n}\left[(-1)^n+1\right]\right]\\
&= \frac{\left[(-1)^n+1\right]}{2\pi}\left[\frac{1}{1+n}+\frac{1}{1-n}\right]
\end{aligned}
$$

$$a_n = \frac{-\left[(-1)^n + 1\right]}{\pi(n-1)(n+1)}$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd, } n \neq 1 \\ \dfrac{-2}{\pi(n-1)(n+1)} & \text{if } n \text{ is even} \end{cases}$$

$$a_1 = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\cos x\,dx + \int_{\pi}^{2\pi} f(x)\cos x\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} \sin x\cos x\,dx + 0\right\}$$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin 2x\,dx \text{ Since } \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{1}{2\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = \frac{1}{4\pi}\left[-\cos 2\pi + \cos 0\right] = 0.$$

$$b_n = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\sin nx\,dx + \int_{\pi}^{2\pi} f(x)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} \sin x\sin nx\,dx + \int_{\pi}^{2\pi} 0.\cos nx\,dx\right\}$$

$$= \frac{1}{2\pi}\int_0^{\pi} x\left[\cos(x-nx) - \cos(x+nx)\right]\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \left[\cos(1-n)x - \cos(1+n)x\right]\,dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\sin(1-n)\pi}{1-n} - \frac{\sin(1+n)\pi}{1+n} - \left(\frac{\sin 0}{1-n} - \frac{\sin 0}{1+n}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{\sin n\pi}{1-n} + \frac{\sin n\pi}{1+n} - 0 + 0\right] = 0.$$

Since $\sin(\pi - n\pi) = \sin n\pi$, $\sin(\pi + n\pi) = -\sin n\pi$

$$
\begin{aligned}
b_1 &= \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\sin x\,dx + \int_{\pi}^{2\pi} f(x)\sin x\,dx\right\} \\
&= \frac{1}{\pi}\int_0^{\pi} \sin x \sin x\,dx = \frac{1}{\pi}\int_0^{\pi} \sin^2 x\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right)\,dx \\
&= \frac{1}{2\pi}\left\{\int_0^{\pi} dx - \int_0^{\pi}\cos 2x\,dx\right\} \\
&= \frac{1}{2\pi}\left\{[x]_0^{\pi} - \left[\frac{\sin 2x}{2}\right]_0^{\pi}\right\} \\
&= \frac{1}{2\pi}\left\{(\pi-0) - \frac{1}{2}(\sin 2\pi - \sin 0)\right\} = \frac{1}{2}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f(x) &= \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=\text{even}} \frac{1}{(n-1)(n+1)}\cos nx + \frac{1}{2}\sin x \\
&= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\left[\frac{1}{1.3}\cos 2x + \frac{1}{3.5}\cos 4x\right. \\
&\left. + \frac{1}{5.7}\cos 6x + \frac{1}{7.9}\cos 8x + \dots\right]
\end{aligned}
$$

To deduce $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots$

Put $x = 0$ in the Fourier series and $x = 0$ is an end point the of the interval $(0, 2\pi)$. Therefore the sum of the Fourier series when $x = 0$ is

$f(0) = \frac{1}{2}\left[f(0+) + f(2\pi-)\right] = \frac{1}{2}\left[\sin 0 + 0\right] = 0.$ Therefore

$$
\begin{aligned}
0 &= \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{1}{1.3}\cos 0 + \frac{1}{3.5}\cos 0 + \frac{1}{5.7}\cos 0 + \frac{1}{7.9}\cos 0 + \dots\right] \\
-\frac{1}{\pi} &= -\frac{2}{\pi}\left[\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots\right] \\
&\Rightarrow \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots = \frac{1}{2}.
\end{aligned}
$$

Problems based on $(-\pi, \pi)$

Example 1: Find the Fourier series for $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$.

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$.

Solution: The Fourier series for $f(x)$ is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ where

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$.

To find a_0, a_n **and** b_n

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right\} \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx\right\} \\
&= \frac{1}{\pi}\left\{-\pi\,[x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}\right\} \\
&= \frac{1}{\pi}\left\{-\pi(0+\pi) + \frac{\pi^2}{2} - 0\right\} = -\frac{\pi}{2}.
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right\} \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right\}
\end{aligned}$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\left\{-\pi\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[x.\frac{\sin nx}{n} - 1.\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}\right\} \\
&= \frac{1}{\pi}\left\{\frac{-\pi}{n}\left(\sin 0 - \sin(-n\pi)\right) + \left[\frac{\pi \sin n\pi}{n} + \frac{\cos nx}{n^2} - \left(0 + \frac{\cos 0}{n^2}\right)\right]\right\} \\
&= \frac{1}{\pi}\left\{0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right\} = \frac{1}{\pi n^2}\left[(-1)^n - 1\right] \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-2}{\pi n^2} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right\} \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right\} \\
&= \frac{1}{\pi}\left\{-\pi\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \left[x.\frac{-\cos nx}{n} - 1.\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}\right\} \\
&= \frac{1}{\pi}\left\{\frac{\pi}{n}\left(\cos 0 - \cos(-n\pi)\right) + \left[\frac{-\pi\cos n\pi}{n} + \frac{\sin nx}{n^2} - 0\right]\right\} \\
&= \frac{1}{\pi}\left\{\frac{\pi}{n}\left[1 - (-1)^n\right] - \frac{\pi}{n}(-1)^n\right\} = \frac{1}{n}\left[1 - 2(-1)^n\right]
\end{aligned}
$$

Therefore the Fourier series is

$$
\begin{aligned}
f(x) &= -\frac{\pi}{4} + \sum_{n \text{ is odd}} \frac{-2}{\pi n^2}\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}\left[1 - 2(-1)^n\right]\sin nx \\
&= -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right] \\
&\quad + \left[3\sin x - \frac{1}{2}\sin 2x + \sin 3x - \frac{1}{4}\sin 4x + \dots\right]
\end{aligned}
$$

To deduce the series $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$.

Put $x=0$ in the Fourier series. But $x=0$ is not defined and so $f(x)$ is discontinuous at$x=0$.

Hence the sum of the Fourier series when $x=0$ is

$$\frac{1}{2}\left[f(0-)+f(0+)\right]=\frac{1}{2}[-\pi+0]=-\frac{\pi}{2}.$$

Therefore

$$\begin{aligned}-\frac{\pi}{2}&=-\frac{\pi}{4}-\frac{2}{\pi}\left[\frac{\cos 0}{1^2}+\frac{\cos 0}{3^2}+\frac{\cos 0}{5^2}+\ldots\right]\\ \Rightarrow-\frac{\pi}{4}&=-\frac{2}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right]\\ \Rightarrow&\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}.\end{aligned}$$

Example 2: Find the Fourier series for $f(x)$ defined by $f(x)=\begin{cases}0, & -\pi\le x\le 0\\ \sin x, & 0\le x\le\pi\end{cases}$.

Hence deduce the values of (i) $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots$

(ii) $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\ldots$

Solution: The Fourier series for $f(x)$ is

$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$ where

$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$, $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$ and $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$.

To find a_0, a_n and b_n

$$\begin{aligned}a_0&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx\\ &=\frac{1}{\pi}\left\{\int_{-\pi}^{0}f(x)\,dx+\int_{0}^{\pi}f(x)\,dx\right\}\end{aligned}$$

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} \sin x\,dx\right\} \\
&= \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} \\
&= -\frac{1}{\pi}\left\{\cos\pi - \cos 0\right\} = \frac{2}{\pi}. \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right\} \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0.\cos nx\,dx + \int_{0}^{\pi} \sin x\cos nx\,dx\right\} \\
&= \frac{1}{\pi}\left\{\int_{0}^{\pi} \left[\sin(x+nx) + \sin(x-nx)\right]\,dx\right\} \\
&= \frac{1}{2\pi}\left\{\int_{0}^{\pi} \left[\sin(n+1)x - \sin(n-1)x\right]\,dx\right\} \\
&= \frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \\
&= \frac{1}{2\pi}\left[\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \left(\frac{-\cos 0}{n+1} + \frac{\cos 0}{n-1}\right)\right] \\
&= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \\
&= \frac{1}{2\pi}\left[(-1)^n\left(\frac{1}{n+1} - \frac{1}{n-1}\right) + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right] \\
&= \frac{1}{2\pi}\left[(-1)^n + 1\right]\left(\frac{1}{n+1} - \frac{1}{n-1}\right) \\
&= \frac{1}{2\pi}\left[(-1)^n + 1\right]\left(\frac{n-1-n-1}{(n+1)(n-1)}\right) \\
&= \frac{-\left[(-1)^n + 1\right]}{\pi(n+1)(n-1)} \\
&= \begin{cases} 0 & \text{if } n = 3, 5, 7, \ldots \\ \dfrac{-2}{\pi(n-1)(n+1)} & \text{if } n = 2, 4, 6, \ldots \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
a_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right\} \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0.\cos nx\,dx + \int_{0}^{\pi} \sin x\cos x\,dx\right\} \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin 2x\,dx \text{ Since } \sin 2\theta = 2\sin\theta\cos\theta \\
&= \frac{1}{2\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} \\
&= \frac{1}{2\pi}\left[\frac{-\cos 2\pi}{2} + \frac{\cos 0}{2}\right] \\
&= \frac{1}{2\pi}\left[-\frac{1}{2} + \frac{1}{2}\right] = 0.
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right\} \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0.\sin nx\,dx + \int_{0}^{\pi} \sin x\sin nx\,dx\right\} \\
&= \frac{1}{\pi}\left\{\int_{0}^{\pi} \left[\cos(x-nx) - \cos(x+nx)\right]\,dx\right\} \\
&= \frac{1}{2\pi}\left\{\int_{0}^{\pi} \left[\cos(n-1)x - \cos(n+1)x\right]\,dx\right\} \\
&= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi}
\end{aligned}
$$

$$b_n = \frac{1}{2\pi}\left[\frac{\sin(n-1)\pi}{n-1} - \frac{\sin(n+1)\pi}{n+1} - \left(\frac{\sin 0}{n-1} - \frac{\sin 0}{n+1}\right)\right] = 0.$$

$$\begin{aligned} b_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx \\ &= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin x\,dx + \int_{0}^{\pi} f(x)\sin x\,dx\right\} \\ &= \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0.\sin x\,dx + \int_{0}^{\pi} \sin x\sin x\,dx\right\} \\ &= \frac{1}{2\pi}\int_{0}^{\pi}(1-\cos 2x)\,dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}. \end{aligned}$$

Therefore the Fourier series is

$$\begin{aligned} f(x) &= \frac{1}{\pi} + \sum_{n \text{ is even}} \frac{-2}{\pi(n-1)(n+1)}\cos nx + \frac{1}{2}\sin x \\ &= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\left[\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} + \dots\right] \end{aligned}$$

To deduce the series (i) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots$

Put $x = 0$ in the series and it is a point of continuity for $f(x)$.

Therefore the sum of the Fourier series when $x = 0$ is $f(0) = 0$

Therefore

$$\begin{aligned} 0 =& \frac{1}{\pi} + \frac{1}{2}\sin 0 - \frac{2}{\pi}\left[\frac{\cos 0}{1.3} + \frac{\cos 0}{3.5} + \frac{\cos 0}{5.7} + \dots\right] \\ -\frac{1}{\pi} =& -\frac{2}{\pi}\left[\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots\right] \\ \Rightarrow& \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots = \frac{1}{2}. \end{aligned}$$

To deduce the series (ii) $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots$

Put $x = \frac{\pi}{2}$ in the series and it is a point of continuity for $f(x)$.

Therefore the sum of the Fourier series when $x = \frac{\pi}{2}$ is $f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

Therefore

$$\begin{aligned}
1 =& \frac{1}{\pi} + \frac{1}{2}\sin\frac{\pi}{2} - \frac{2}{\pi}\left[\frac{\cos\pi}{1.3} + \frac{\cos 2\pi}{3.5} + \frac{\cos 3\pi}{5.7} + \dots\right] \\
1 =& \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\left[-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \dots\right] \\
\frac{1}{2} =& \frac{1}{\pi} + \frac{2}{\pi}\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\right] \\
\frac{\pi - 2}{2\pi} =& \frac{2}{\pi}\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\right] \\
\Rightarrow& \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots = \frac{\pi - 2}{4}.
\end{aligned}$$

Example 3: If $f(x) = \begin{cases} -k, & \text{when } -\pi < x < 0 \\ k, & \text{when } 0 < x < \pi \end{cases}$.

and $f(x + 2\pi) = f(x)$ for all x, derive the Fourier series for $f(x)$. Deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$.

Solution: The Fourier series for $f(x)$ is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ where

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$.

To find a_0, a_n and b_n

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right\} \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\,dx + \int_{0}^{\pi} k\,dx\right\}
\end{aligned}$$

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\left\{-k\,[x]_{-\pi}^{0} + k\,[x]_{0}^{\pi}\right\} \\
&= \frac{1}{\pi}\left\{-k(0+\pi) + k(\pi - 0)\right\} = 0. \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right\} \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right\} \\
&= \frac{1}{\pi}\left\{-k\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + k\left[\frac{\sin nx}{n}\right]_{0}^{\pi}\right\} \\
&= \frac{1}{\pi}\left\{\frac{-k}{n}\left(\sin 0 - \sin(-n\pi)\right) + \frac{k}{n}\left(\sin n\pi - \sin 0\right)\right\} \\
&= 0. \text{ Since } \sin(-n\pi) = \sin n\pi = 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right\} \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right\} \\
&= \frac{1}{\pi}\left\{-k\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + k\left[\frac{-\cos nx}{n}\right]_{0}^{\pi}\right\} \\
&= \frac{1}{\pi}\left\{\frac{k}{n}\left(\cos 0 - \cos(-n\pi)\right) - \frac{k}{n}\left(\cos n\pi - \cos 0\right)\right\} \\
&= \frac{1}{\pi}\left\{\frac{k}{n}\left[1 - (-1)^n\right] - \frac{k}{n}\left[(-1)^n - 1\right]\right\}, \text{ Since } \cos(-n\pi) = \cos n\pi
\end{aligned}
$$

$$b_n = \frac{k}{n\pi}\left[1-(-1)^n-(-1)^n+1\right] = \frac{2k}{n\pi}\left[1-(-1)^n\right]$$
$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{4k}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

Therefore the Fourier series is

$$f(x) = \sum_{n=\text{odd}} \frac{4k}{n\pi}\sin nx$$
$$= \frac{4k}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\dots\right]$$

To deduce the series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{\pi}{4}$.

Put $x=\frac{\pi}{2}$ in the Fourier series and $x=\frac{\pi}{2}$ is a point of continuity for the function.

Hence the sum of the Fourier series when $x=\frac{\pi}{2}$ is $f\left(\frac{\pi}{2}\right)=k$. Therefore

$$k = \frac{4k}{\pi}\left[\frac{\sin\left(\frac{\pi}{2}\right)}{1}+\frac{\sin 3\left(\frac{\pi}{2}\right)}{3}+\frac{\sin 5\left(\frac{\pi}{2}\right)}{5}+\dots\right]$$
$$\Rightarrow 1 = \frac{4}{\pi}\left[\frac{1}{1^2}-\frac{1}{3}+\frac{1}{5}-\dots\right]$$
$$\Rightarrow \frac{1}{1}+\frac{1}{3}+\frac{1}{5}-\ldots=\frac{\pi}{4}.$$

Example 4: Find the Fourier series of $f(x)=e^x$ in the interval $(-\pi,\pi)$ of periodicity 2π.

Solution: The Fourier series for $f(x)$ is

$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$ where

$a_0=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}f(x)\,dx$, $a_n=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}f(x)\cos nx\,dx$ and $b_n=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}f(x)\sin nx\,dx$.

To find a_0, a_n and b_n

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx$$
$$= \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right)$$
$$= \frac{2}{\pi}\sinh\pi.\ \text{Since } \sinh x = \frac{e^x - e^{-x}}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$
$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left(\cos nx + n\sin nx\right)\right]_{-\pi}^{\pi}$$
$$= \frac{1}{\pi(1+n^2)}\left[e^{\pi}\left(\cos n\pi + n\sin n\pi\right) - e^{-\pi}\left(\cos n\pi - n\sin n\pi\right)\right]$$
$$= \frac{1}{\pi(1+n^2)}\left[e^{\pi}((-1)^n + 0) - e^{-\pi}((-1)^n + 0)\right]$$
$$= \frac{(-1)^n}{\pi(1+n^2)}(e^{\pi} - e^{-\pi}) = \frac{2(-1)^n}{\pi(1+n^2)}\sinh\pi.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\,dx$$
$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left(\sin nx - n\cos nx\right)\right]_{-\pi}^{\pi}$$
$$= \frac{1}{\pi(1+n^2)}\left[e^{\pi}\left(\sin n\pi - n\cos n\pi\right) - e^{-\pi}\left(-\sin n\pi - n\cos n\pi\right)\right]$$
$$= \frac{1}{\pi(1+n^2)}\left[e^{\pi}(0 - n(-1)^n) - e^{-\pi}(0 - n(-1)^n)\right]$$
$$= \frac{-(-1)^n n}{\pi(1+n^2)}(e^{\pi} - e^{-\pi}) = \frac{-2(-1)^n n}{\pi(1+n^2)}\sinh\pi.$$

Therefore the Fourier series is

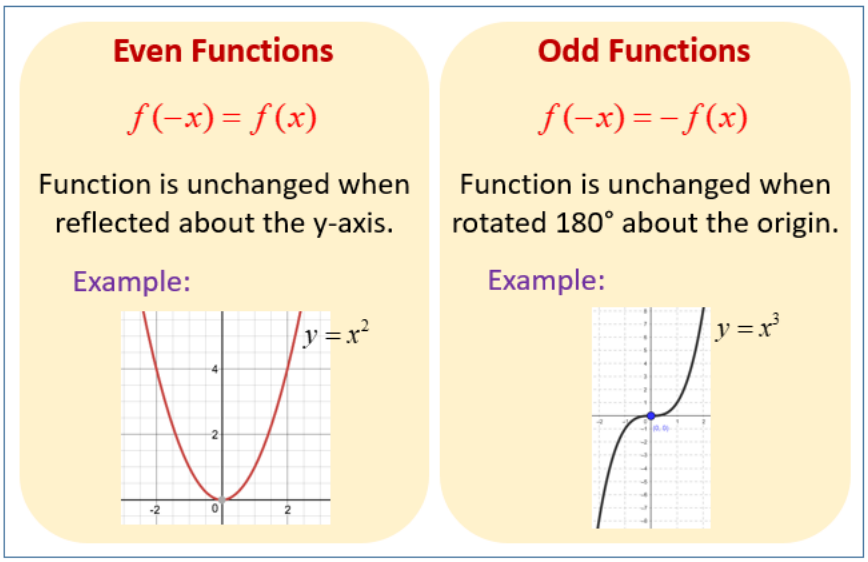
$$e^x = \frac{1}{\pi}\sinh\pi + \sum_{n=1}^{\infty}\frac{2(-1)^n}{\pi(1+n^2)}\sinh\pi\cos nx + \sum_{n=1}^{\infty}\frac{-2(-1)^n n}{\pi(1+n^2)}\sinh\pi\sin nx$$
$$= \frac{\sinh\pi}{\pi}\left[1 + \sum_{n=1}^{\infty}\frac{2(-1)^n}{\pi(1+n^2)}(\cos nx - n\sin nx)\right]$$

Even and Odd Functions

A function $f(x)$ defined in the interval $(-a, a)$ is said be **even if** $f(-x) = f(x)$ and **odd if** $f(-x) = -f(x)$ for all $x \in (-a, a)$.

$$\text{Therefore } \int_{-a}^{a} f(x)\,dx = \begin{cases} 0; & \text{if } f(x) \text{ is odd function of } x \\ 2\int_{0}^{a} f(x)\,dx; & \text{if } f(x) \text{ is even function of } x \end{cases}$$

Example:



Note:

(1) If $f(x)$ is an odd function of x, then the Fourier Coefficients $a_0 = 0, a_n = 0$ and $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$.

Hence the resulting Fourier series is a sine series and $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$.

(2) If $f(x)$ is an even function of x, then the Fourier Coefficients $b_n = 0$,

$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$ and $a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$.

Hence the resulting Fourier series is a cosine series and

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

Example 5:

Expand $f(x) = x^2$, when $-\pi < x < \pi$, in a Fourier series of periodicity 2π. Hence deduce that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \text{to } \infty = \frac{\pi^2}{6}$.

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots \text{to } \infty = \frac{\pi^2}{12}$.

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \text{to } \infty = \frac{\pi^2}{8}$.

Solution: Given the function is an even function. Therefore $b_n = 0$.

The Fourier Cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$.

To find a_0, a_n :

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi} x^2 dx \\
&= \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} \\
&= \frac{2\pi^2}{3} \\
a_n &= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx \\
&= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n} - \frac{2\sin nx}{n^3}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n}\right], \text{ since } \sin 0 = \sin n\pi = 0 \\
&= \frac{4(-1)^n}{n^2}
\end{aligned}$$

Therefore the Fourier Cosine series is

$$
\begin{aligned}
x^2 &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos nx \\
&= \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} - \dots\right] \\
&= \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \dots\right]
\end{aligned}
$$

(i) To deduce $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ **to** $\infty = \frac{\pi^2}{6}$.

Put $x = \pi$ in the Fourier series and $x = \pi$ is an end point of the $(-\pi, \pi)$ for the function $f(x)$.

Therefore the sum of the Fourier series when $x = \pi$ is

$$
\frac{1}{2}\left[f(-\pi+) + f(\pi-)\right] = \frac{1}{2}\left[(-\pi^2)^2 + \pi^2\right] = \pi^2.
$$

Hence

$$
\begin{aligned}
\pi^2 &= \frac{\pi^2}{3} - 4\left[\frac{\cos \pi}{1^2} - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} - \frac{\cos 4\pi}{4^2} + \dots\right] \\
\pi^2 - \frac{\pi^2}{3} &= -4\left[-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \dots\right] \\
\frac{2\pi^2}{3} &= 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right]
\end{aligned}
$$

$$
\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}. \tag{1}
$$

(ii) To deduce $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots$ **to** $\infty = \frac{\pi^2}{12}$.

Put $x = 0$ in the Fourier series and $x = 0$ is an interior point of continuity for $f(x)$.

Therefore the sum of the Fourier series when $x = 0$ is $f(0) = 0$. Hence

$$
\begin{aligned}
0 &= \frac{\pi^2}{3} - 4\left[\frac{\cos 0}{1^2} - \frac{\cos 0}{2^2} + \frac{\cos 0}{3^2} - \frac{\cos 0}{4^2} + \dots\right] \\
-\frac{\pi^2}{3} &= -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right]
\end{aligned}
$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}. \tag{2}$$

(iii) To deduce $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ **to** $\infty = \frac{\pi^2}{8}$.

Adding (1) and (2), we get

$$\begin{aligned}
2.\frac{1}{1^2} + 2.\frac{1}{3^2} + 2.\frac{1}{5^2} + \ldots &= \frac{\pi^2}{6} + \frac{\pi^2}{12} \\
2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right] &= \frac{\pi^2}{4} \\
\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \text{to } \infty &= \frac{\pi^2}{8}.
\end{aligned}$$

Example 6:

Find the Fourier series of $f(x) = x + x^2$ in $(-\pi, \pi)$ of periodicity 2π. Hence deduce that $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution: Given the function $f(x)$ is neither even nor odd.
Let $f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx + \sum\limits_{n=1}^{\infty} b_n \sin nx$

where $a_0 = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cos nx dx$ and $b_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\sin nx dx$.

To find a_0, a_n, b_n :

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int\limits_{-\pi}^{\pi}(x + x^2)dx = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} xdx + \frac{1}{\pi}\int\limits_{-\pi}^{\pi} x^2 dx \\
&= 0 + \frac{2}{\pi}\int\limits_{0}^{\pi} x^2 dx, \text{ since } x \text{ is odd and } x^2 \text{ is even.} \\
&= \frac{2\pi^2}{3}.
\end{aligned}$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx dx \\
&= 0 + \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx dx, \text{ since } x\cos nx \text{ is odd} \\
&= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n} - \frac{2\sin nx}{n^3}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n}\right], \text{ since } \sin 0 = \sin n\pi = 0 \\
&= \frac{4(-1)^n}{n^2} \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx dx + 0, \text{ since } x^2\sin nx \text{ is odd} \\
&= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n}\right)\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right], \text{ since } \sin 0 = \sin n\pi = 0 \\
&= \frac{-2(-1)^n}{n}
\end{aligned}
$$

Therefore

$$\begin{aligned}
x + x^2 &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2}\cos nx - \frac{2(-1)^n}{n}\sin nx\right] \\
&= \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} - \dots\right] \\
&- 2\left[-\frac{\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \frac{\sin 4x}{4} - \dots\right]
\end{aligned}$$

To deduce that $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.

Put $x = \pi$ in the Fourier series and $x = \pi$ is an end point of the $(-\pi, \pi)$ for the function $f(x)$. Therefore the sum of the Fourier series when $x = \pi$ is

$$\begin{aligned}
&= \frac{1}{2}\left[f(-\pi+) + f(\pi-)\right] \\
&= \frac{1}{2}\left[(-\pi + \pi^2) + (\pi + \pi^2)\right] = \pi^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\pi^2 &= \frac{\pi^2}{3} + 4\left[-\frac{\cos \pi}{1^2} + \frac{\cos 2\pi}{2^2} - \frac{\cos 3\pi}{3^2} + \frac{\cos 4\pi}{4^2} - \dots\right] \\
&- 2\left[-\frac{\sin \pi}{1} + \frac{\sin 2\pi}{2} - \frac{\sin 3\pi}{3} + \frac{\sin 4\pi}{4} - \dots\right] \\
\pi^2 - \frac{\pi^2}{3} &= 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right] \text{ Since } \sin \pi = \sin 2\pi = \dots = 0 \\
\frac{2\pi^2}{3} &= 4\sum \frac{1}{n^2} \\
\Rightarrow \sum \frac{1}{n^2} &= \frac{\pi^2}{6}.
\end{aligned}$$

Example 7: Find the Fourier series to represent $(x - x^2)$ in the interval $(-\pi, \pi)$. Deduce the value of $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$.

Solution: Given the function $f(x)$ is neither even nor odd.

Let $f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx + \sum\limits_{n=1}^{\infty} b_n \sin nx$

where $a_0 = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cos nx dx$ and $b_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\sin nx dx$.

To find a_0, a_n, b_n :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} xdx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2dx$$

$$= 0 - \frac{2}{\pi}\int_{0}^{\pi} x^2dx, \text{ since } x \text{ is odd and } x^2 \text{ is even}$$

$$= \frac{-2\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nxdx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nxdx$$

$$= 0 - \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nxdx, \text{ since } x\cos nx \text{ is odd}$$

$$= -\frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\left[\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n} - \frac{2\sin nx}{n^3}\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n}\right], \text{ since } \sin 0 = \sin n\pi = 0$$

$$= \frac{-4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nxdx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nxdx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\sin nxdx - 0, \text{ since } x^2\sin nx \text{ is odd}$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{-\sin nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right]$$

$$= \frac{-2(-1)^n}{n}$$

Therefore

$$\begin{aligned}
x - x^2 &= -\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{-4(-1)^n}{n^2}\cos nx - \frac{2(-1)^n}{n}\sin nx\right] \\
&= \frac{\pi^2}{3} - 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} - \dots\right] \\
&- 2\left[-\frac{\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \frac{\sin 4x}{4} - \dots\right]
\end{aligned}$$

To deduce $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$.

Put $x = 0$ in the Fourier series and $x = 0$ is an interior point of continuity for $f(x)$.

Therefore the sum of the Fourier series when $x = 0$ is $f(0) = 0$. Hence

$$\begin{aligned}
0 &= -\frac{\pi^2}{3} - 4\left[-\frac{\cos 0}{1^2} + \frac{\cos 0}{2^2} - \frac{\cos 0}{3^2} + \frac{\cos 0}{4^2} - \dots\right] \\
&- 2\left[-\frac{\sin 0}{1} + \frac{\sin 0}{2} - \frac{\sin 0}{3} + \frac{\sin 0}{4} - \dots\right] \\
\frac{\pi^2}{3} &= 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right] \\
&\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}.
\end{aligned}$$

Example 8: Determine the Fourier series for the function

$$f(x) = \begin{cases} -1 + x, & -\pi < x < 0 \\ 1 + x, & 0 < x < \pi \end{cases}.$$

Solution: The function $f(x)$ is given in the symmetrical interval $(-\pi, \pi)$.

$$f(-x) = \begin{cases} -1 - x, & -\pi < -x < 0 \\ 1 - x, & 0 < -x < \pi \end{cases}$$

$$
\begin{aligned}
f(-x) &= \begin{cases} -(1+x), & \pi > x > 0 \\ -(-1+x), & 0 > x > -\pi \end{cases} \\
&= \begin{cases} -(-1+x), & -\pi < x < 0 \\ -(1+x), & 0 < x < \pi \end{cases} \\
&= -f(x)
\end{aligned}
$$

Therefore $f(x)$ is an odd function in $(-\pi, \pi)$ and Hence $a_0 = 0, a_n = 0$.

Therefore the Fourier sine series is $f(x) = \sum\limits_{n=1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{\pi} \int\limits_0^{\pi} f(x) \sin nx \, dx$.

$$
\begin{aligned}
b_n &= \frac{2}{\pi} \int\limits_0^{\pi} (1+x) \sin nx \, dx \\
&= \frac{2}{\pi} \left[(1+x) \left(\frac{-\cos nx}{n} \right) - 1. \left(\frac{-\sin nx}{n^2} \right) \right]_0^{\pi} \\
&= \frac{2}{\pi} \left[\frac{-(1+x)\cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi} \\
&= \frac{2}{\pi} \left[\frac{-(1+\pi)\cos n\pi}{n} + \frac{\sin n\pi}{n^2} - \left(\frac{-\cos 0}{n} + 0 \right) \right] \\
&= \frac{2}{\pi} \left[\frac{-(1+\pi)}{n} (-1)^n + \frac{1}{n} \right] \\
&= \frac{2}{n\pi} \left[1 - (1+\pi)(-1)^n \right]
\end{aligned}
$$

Therefore the Fourier sine series is $f(x) = \sum\limits_{n=1}^{\infty} \frac{2}{n\pi} \left[1 - (1+\pi)(-1)^n \right]$.

Example 9: If

$$
f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & \text{if } -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & \text{if } 0 \le x \le \pi \end{cases}.
$$

Show that $f(x) = \frac{8}{\pi^2} \left[\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right]$.

Hence show that $\sum\limits_{n=1}^{\infty}(2n-1)^{-2} = \frac{\pi^2}{8}$.

Solution: The function $f(x)$ is given in the symmetrical interval $(-\pi, \pi)$.

$$\begin{aligned}
f(-x) &= \begin{cases} 1-\frac{2x}{\pi}, & -\pi \le -x \le 0 \\ 1+\frac{2x}{\pi}, & 0 \le -x \le \pi \end{cases} \\
&= \begin{cases} 1-\frac{2x}{\pi}, & \pi \ge x \ge 0 \\ 1+\frac{2x}{\pi}, & 0 \ge x \ge -\pi \end{cases} \\
&= \begin{cases} 1+\frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi}, & 0 \le x \le \pi \end{cases} \\
&= f(x)
\end{aligned}$$

Therefore $f(x)$ is an even function in $(-\pi, \pi)$ and Hence $b_n = 0$.

Therefore the Fourier cosine series is $f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi}\int\limits_0^{\pi} f(x)\,dx$ and $a_n = \frac{2}{\pi}\int\limits_0^{\pi} f(x)\cos nx\,dx$.

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int\limits_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx \\
&= \frac{2}{\pi}\left[x-\frac{x^2}{\pi}\right]_0^{2\pi} = 0. \\
a_n &= \frac{2}{\pi}\int\limits_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx \\
&= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)-\left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n}-\frac{2}{\pi}\frac{\cos nx}{n^2}\right]_0^{\pi}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\left[\left(1-\frac{2\pi}{\pi}\right)\frac{\sin n\pi}{n} - \frac{2}{\pi}\frac{\cos n\pi}{n^2} - \left(0 - \frac{2\cos 0}{n^2\pi}\right)\right] \\
&= \frac{2}{\pi}\left[\frac{-2}{\pi n^2}(-1)^n + \frac{2}{\pi n^2}\right] \\
&= \frac{4}{\pi n^2}\left[1-(-1)^n\right] \\
&= \begin{cases} 0; & \text{if } n \text{ is even} \\ \dfrac{8}{\pi^2 n^2}; & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

Therefore the Fourier Cosine series is

$$\begin{aligned}
f(x) &= \sum_{n=\text{odd}} \frac{8}{\pi^2 n^2}\cos nx \\
&= \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right]
\end{aligned}$$

To deduce $\sum_{n=1}^{\infty}(2n-1)^{-2} = \frac{\pi^2}{8}$.

Put $x = 0$ in the Fourier series, since $x = 0$ ia a point of continuity, as $f(0-) = 1 = f(0+)$ and $f(0) = 1$, the sum of the Fourier series when $x = 0$ is $f(0) = 1$. Therefore

$$\begin{aligned}
1 &= \frac{8}{\pi^2}\left[\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + \dots\right] \\
&= \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right] \\
\Rightarrow & \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.
\end{aligned}$$

Example 10: In $-\pi < x < \pi$, express $\sinh ax$ and $\cosh ax$ in Fourier series of periodicity 2π.

Solution: We know that $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

Therefore $f(x) = \sinh ax = \dfrac{e^{ax} - e^{-ax}}{2}$

$f(-x) = \sinh a(-x) = \dfrac{e^{a(-x)} - e^{-a(-x)}}{2}$

$$\Rightarrow f(-x) = \frac{e^{-ax} - e^{ax}}{2} = \frac{-(e^{ax} - e^{-ax})}{2} = -f(x).$$

Hence $\sinh ax$ is an odd function. Since $f(-x) = -f(x)$.

Similarly $\cosh ax$ is an even function. Since $f(-x) = f(x)$.

(i) The Fourier sine series for $f(x) = \sinh ax$ is

$f(x) = \sum\limits_{n=1}^{\infty} b_n \sin nx$ where $b_n = \frac{2}{\pi}\int\limits_0^{\pi} f(x)\sin nx\, dx$.

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int\limits_0^{\pi} \sinh ax \sin nx\, dx \\
&= \frac{2}{\pi}\int\limits_0^{\pi} \left(\frac{e^{ax} - e^{-ax}}{2}\right) \sin nx\, dx \\
&= \frac{1}{\pi}\left\{\int\limits_0^{\pi} e^{ax}\sin nx\, dx - \int\limits_0^{\pi} e^{-ax}\sin nx\, dx\right\} \\
&= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}\left(a\sin nx - n\cos nx\right) - \frac{e^{-ax}}{a^2+n^2}\left(-a\sin nx - n\cos nx\right)\right]_0^{\pi} \\
&= \frac{1}{\pi(a^2+n^2)}\Bigg[e^{a\pi}(a\sin n\pi - n\cos n\pi) - e^{-a\pi}(-a\sin n\pi - n\cos n\pi) \\
&\quad - \left(e^0\left(a\sin 0 - n\cos 0\right) - e^{-0}\left(-a\sin 0 - n\cos 0\right)\right)\Bigg] \\
&= \frac{1}{\pi(a^2+n^2)}\left[e^{a\pi}(0 - n(-1)^n) - e^{-a\pi}(0 - n(-1)^n) - (0 - n - (0 - n))\right] \\
&= \frac{1}{\pi(a^2+n^2)}\left[-n(-1)^n e^{a\pi} + n(-1)^n e^{-a\pi}\right] \\
&= \frac{-n(-1)^n}{\pi(a^2+n^2)}\left(e^{a\pi} - e^{-a\pi}\right) = \frac{-2n(-1)^n}{\pi(a^2+n^2)}\sinh a\pi.
\end{aligned}$$

Hence $\sinh ax = \sum\limits_{n=1}^{\infty} \frac{-2n(-1)^n}{\pi(a^2+n^2)} \sinh a\pi \sin ax$.

(ii) The Fourier cosine series for $f(x) = \cosh ax$ is

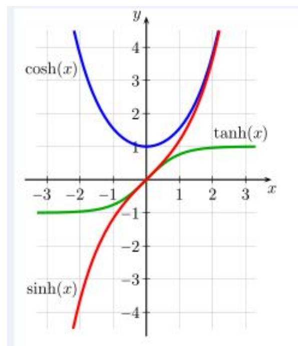
$f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$ where

$a_0 = \frac{2}{\pi} \int\limits_0^{\pi} f(x)\, dx,\ a_n = \frac{2}{\pi} \int\limits_0^{\pi} f(x) \cos nx\, dx.$

$$
\begin{aligned}
a_0 &= \frac{2}{\pi} \int\limits_0^{\pi} \cosh ax\, dx \\
&= \frac{2}{\pi} \left(\frac{\sinh ax}{a} \right)_0^{\pi} \\
&= \frac{2}{\pi a} \sinh a\pi. \text{ Since } \sinh 0 = 0 \\
a_n &= \frac{2}{\pi} \int\limits_0^{\pi} \cosh ax \cos nx\, dx \\
&= \frac{2}{\pi} \int\limits_0^{\pi} \left(\frac{e^{ax} + e^{-ax}}{2} \right) \cos nx\, dx \\
&= \frac{1}{\pi} \left\{ \int\limits_0^{\pi} e^{ax} \cos nx\, dx + \int\limits_0^{\pi} e^{-ax} \cos nx\, dx \right\} \\
&= \frac{1}{\pi} \left[\frac{e^{ax}}{a^2 + n^2} (a \cos nx + n \sin nx) + \frac{e^{-ax}}{a^2 + n^2} (-a \cos nx + n \sin nx) \right]_0^{\pi} \\
&= \frac{1}{\pi(a^2 + n^2)} \Big[e^{a\pi} (a \cos n\pi + n \sin n\pi) + e^{-a\pi} (-a \cos n\pi + n \sin n\pi) \\
&\quad - \left(e^0 (a \cos 0 + n \sin 0) + e^{-0} (-a \cos 0 + n \sin 0) \right) \Big] \\
&= \frac{1}{\pi(a^2 + n^2)} \left[e^{a\pi} (a(-1)^n + 0) + e^{-a\pi} (-a(-1)^n + 0) - (a + 0 - a + 0) \right] \\
&= \frac{1}{\pi(a^2 + n^2)} \left[a e^{a\pi} (-1)^n - a e^{-a\pi} (-1)^n \right] \\
&= \frac{a(-1)^n}{\pi(a^2 + n^2)} \left(e^{a\pi} - e^{-a\pi} \right) = \frac{2a(-1)^n}{\pi(a^2 + n^2)} \sinh a\pi.
\end{aligned}
$$

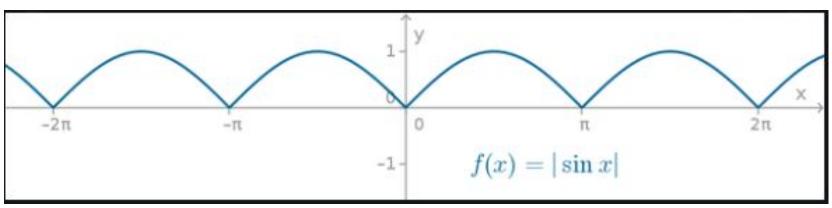
Hence $\cosh ax = \dfrac{1}{\pi a} + \sum_{n=1}^{\infty} \dfrac{2a(-1)^n}{\pi(a^2+n^2)} \sinh a\pi \sin nx$.

Note:



Example 11: Find the Fourier series for $f(x) = |\sin x|$ in $(-\pi, \pi)$ of periodicity 2π.

Solution:



$f(x) = |\sin x|$ is an even function. Therefore $b_n = 0$.

Hence the Fourier cosine series is $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ where

$a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\,dx$, $a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\cos nx\,dx$.

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} |\sin x|\; dx \\ &= \frac{2}{\pi}\int_0^{\pi} \sin x\,dx \\ &= \frac{2}{\pi}(-\cos x)_0^{\pi} = \frac{4}{\pi}. \end{aligned}$$

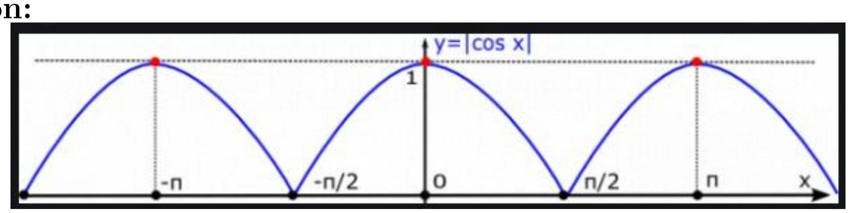
$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx \\
&= \frac{1}{\pi}\int_0^{\pi} [\sin(1+n)x + \sin(1-n)x] \, dx \\
&= \frac{1}{\pi}\left[-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left[-\frac{\cos(1+n)\pi}{1+n} - \frac{\cos(1-n)\pi}{1-n} + \frac{\cos 0}{1+n} + \frac{\cos 0}{1-n}\right] \\
&= \frac{1}{\pi}\left[\frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right] \\
&\text{Since } \cos(1+n)\pi = \cos(\pi + n\pi) = -\cos n\pi, \cos(1-n)\pi = -\cos n\pi \\
&= \frac{1}{\pi}\left[\frac{(-1)^n}{1+n} + \frac{(-1)^n}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right] \\
&= \frac{1}{\pi}\left[\frac{1}{1+n}[(-1)^n + 1] + \frac{1}{1-n}[(-1)^n + 1]\right] \\
&= \frac{[(-1)^n + 1]}{\pi}\left(\frac{1}{1+n} + \frac{1}{1-n}\right) \\
&= \frac{[(-1)^n + 1]}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) = \frac{-2[(-1)^n + 1]}{\pi(n^2 - 1)} \\
&= \begin{cases} 0 & \text{if } n = 3, 5, 7, \ldots \; n \neq 1 \\ -\dfrac{4}{\pi(n^2 - 1)} & \text{if } n = 2, 4, 6, \ldots \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
a_1 &= \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx \\
&= \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx \text{ Since } \sin 2\theta = 2\sin\theta\cos\theta \\
&= \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} \\
&= \frac{1}{2\pi}[-\cos 2\pi + \cos 0] \\
&= \frac{1}{2\pi}[-1 + 1] = 0.
\end{aligned}
$$

Hence the Fourier Cosine Series is $f(x) = \frac{2}{\pi} + \sum_{n=\text{even}} \frac{-4}{\pi(n^2-1)} \cos nx$.

Example 12: Find the Fourier series for $f(x) = |\cos x|$ in $(-\pi, \pi)$ of periodicity 2π.

Solution:



$f(x) = |\cos x|$ is an even function. Therefore $b_n = 0$.

Hence the Fourier cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ where

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$.

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi} |\cos x|\ dx \\
&= \frac{2}{\pi}\left\{\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx\right\} \\
&= \frac{2}{\pi}\left[(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}\right] = \frac{4}{\pi}. \\
a_n &= \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx\,dx \\
&= \frac{2}{\pi}\left\{\int_0^{\pi/2} \cos x \cos nx\,dx + \int_{\pi/2}^{\pi} -\cos x \cos nx\,dx\right\} \\
&= \frac{1}{\pi}\left\{\int_0^{\pi/2} [\cos(1+n)x + \cos(1-n)x]\,dx\right. \\
&\quad \left. - \int_{\pi/2}^{\pi} [\cos(1+n)x + \cos(1-n)x]\,dx\right\}
\end{aligned}$$

$$a_n = \frac{1}{\pi}\left\{ \left[\frac{\sin(1+n)x}{1+n} + \frac{\sin(1-n)x}{1-n}\right]_0^{\pi/2} - \left[\frac{\sin(1+n)x}{1+n} + \frac{\sin(1-n)x}{1-n}\right]_{\pi/2}^{\pi} \right\}$$

$$= \frac{1}{\pi}\left\{ \left[\frac{\sin(1+n)\pi/2}{1+n} + \frac{\sin(1-n)\pi/2}{1-n} - \frac{\sin 0}{1+n} - \frac{\sin 0}{1-n}\right]\right.$$

$$\left. - \left[\frac{\sin(1+n)\pi}{1+n} + \frac{\sin(1-n)\pi}{1-n} - \frac{\sin(1+n)\pi/2}{1+n} - \frac{\sin(1-n)\pi/2}{1-n}\right] \right\}$$

$$= \frac{1}{\pi}\left\{ \frac{\cos\frac{n\pi}{2}}{1+n} + \frac{\cos\frac{n\pi}{2}}{1-n} - 0 - 0 - \left[\frac{-\sin n\pi}{1+n} + \frac{\sin n\pi}{1-n} - \frac{\cos\frac{n\pi}{2}}{1+n} - \frac{\cos\frac{n\pi}{2}}{1-n}\right] \right\}$$

$$= \frac{1}{\pi}\left\{ \frac{\cos\frac{n\pi}{2}}{1+n} + \frac{\cos\frac{n\pi}{2}}{1-n} + \frac{\cos\frac{n\pi}{2}}{1+n} + \frac{\cos\frac{n\pi}{2}}{1-n} \right\}$$

$$= \frac{2}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]\cos\frac{n\pi}{2}$$

$$= -\frac{4}{\pi(n^2-1)}\cos\frac{n\pi}{2}, \;\; n \neq 1$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos x \, dx$$

$$= \frac{2}{\pi}\left\{ \int_0^{\pi/2} \cos x.\cos x \, dx + \int_{\pi/2}^{\pi} -\cos x.\cos x \, dx \right\}$$

$$= \frac{2}{\pi}\left\{ \int_0^{\pi/2} \cos^2 x \, dx - \int_{\pi/2}^{\pi} \cos^2 x \, dx \right\}$$

$$= \frac{2}{\pi}\left\{ \int_0^{\pi/2} \left(\frac{1+\cos 2x}{2}\right) dx - \int_{\pi/2}^{\pi} \left(\frac{1+\cos 2x}{2}\right) dx \right\}$$

$$= \frac{1}{\pi}\left\{ \left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} - \left[x + \frac{\sin 2x}{2}\right]_{\pi/2}^{\pi} \right\}$$

$$= \frac{1}{\pi}\left\{ \left[\frac{\pi}{2} + \frac{\sin\pi}{2} - 0 - \frac{\sin 0}{2}\right] - \left[\pi + \frac{\sin 2\pi}{2} - \frac{\pi}{2} - \frac{\sin\pi}{2}\right] \right\}$$

$$a_n = \frac{1}{\pi}\left\{\frac{\pi}{2} + 0 - 0 - 0 - \pi - 0 + \frac{\pi}{2} + 0\right\} = 0.$$

Hence the Fourier Cosine Series is $f(x) = \dfrac{2}{\pi} + \sum\limits_{n=\text{even}} \dfrac{-4}{\pi(n^2-1)}\cos\dfrac{n\pi}{2}\cos nx.$

Half range Fourier Series

In several engineering and physical applications it is required to obtain the Fourier series expansion of a function in an interval $[0, \pi]$ where π is the half the period. Such an expression is called half range Fourier series.

Half range Cosine Series:

Suppose $f(x)$ is defined in the interval $[0, \pi]$.

We define $F(x) = \begin{cases} f(x) & \text{if } 0 \le x \le \pi \\ f(-x) & \text{if } -\pi \le x \le 0 \end{cases}$

Clearly $F(x)$ is an even function defined in the interval $[-\pi, \pi]$. Hence the Fourier series of $F(x)$ contains only cosine terms. In the interval $[0, \pi]$, $F(x) = f(x)$ and hence the cosine series of $F(x)$ gives the cosine series of $f(x)$ in $[0, \pi]$. Thus

$f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi} \int\limits_0^{\pi} f(x) dx$ and $a_n = \frac{2}{\pi} \int\limits_0^{\pi} f(x) \cos nx dx$.

Half range Sine Series:

Suppose $f(x)$ is defined in the interval $[0, \pi]$.

We define $F(x) = \begin{cases} f(x) & \text{if } 0 \le x \le \pi \\ -f(-x) & \text{if } -\pi \le x \le 0 \end{cases}$

Clearly $F(x)$ is an odd function defined in the interval $[-\pi, \pi]$. Hence the Fourier series of $F(x)$ contains only sine terms. In the interval $[0, \pi]$, $F(x) = f(x)$ and hence the sine series of $F(x)$ gives the sine series of $f(x)$ in $[0, \pi]$. Thus

$f(x) = \sum\limits_{n=1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{\pi} \int\limits_0^{\pi} f(x) \sin nx dx$.

Example 1:

Expand the function $f(x) = \sin x$, $0 < x < \pi$, in a Fourier Cosine series.

Solution: The Fourier Cosine series is $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \dfrac{2}{\pi} \int\limits_0^{\pi} f(x)dx$ and $a_n = \dfrac{2}{\pi} \int\limits_0^{\pi} f(x) \cos nx dx$.

To find a_0, a_n :

$$
\begin{aligned}
a_0 &= \frac{2}{\pi} \int\limits_0^{\pi} \sin x \, dx \\
&= \frac{2}{\pi} \left(-\cos x\right)_0^{\pi} \\
&= \frac{2}{\pi} \left[-\cos \pi + \ cos 0\right] \\
&= \frac{2}{\pi}(1+1) = \frac{4}{\pi}. \\
a_n &= \frac{2}{\pi} \int\limits_0^{\pi} \sin x \cos nx \, dx \\
&= \frac{1}{\pi} \int\limits_0^{\pi} \left[\sin(x+nx) + \sin(x-nx)\right] \, dx \\
&= \frac{1}{\pi} \left[\frac{-\cos(n+1)x}{(n+1)} - \frac{\cos(1-n)x}{(1-n)}\right]_0^{\pi} \\
&= \frac{1}{\pi} \left[\frac{-\cos(n+1)\pi}{(n+1)} - \frac{\cos(1-n)\pi}{(1-n)} - \left(\frac{-\cos 0}{(1+n)} - \frac{\cos 0}{(1-n)}\right)\right] \\
&= \frac{1}{\pi} \left[\frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right] \\
&= \frac{1}{\pi} \left[\frac{1+\cos n\pi}{1+n} + \frac{1+\cos n\pi}{1-n}\right] \\
&= \frac{(1+\cos n\pi)}{\pi} \left[\frac{1}{1+n} + \frac{1}{1-n}\right] = \frac{-2\left[1+(-1)^n\right]}{\pi(n-1)(n+1)} \\
&= \begin{cases} 0; \text{if } n = 3, 5, 7, \ldots \\ \dfrac{-4}{\pi(n-1)(n+1)}; \text{if } n = 2, 4, 6, \ldots \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
a_1 &= \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx \\
&= \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx \text{ Since } \sin 2\theta = 2\sin\theta\cos\theta \\
&= \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left[\frac{-\cos 2\pi}{2} + \frac{\cos 0}{2}\right] \\
&= \frac{1}{\pi}\left[-\frac{1}{2} + \frac{1}{2}\right] = 0.
\end{aligned}
$$

Therefore the Fourier Cosine series is

$$
\begin{aligned}
\sin x &= \frac{2}{\pi} - \sum_{n=\text{even}} \frac{4}{\pi(n-1)(n+1)} \cos nx \\
&= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} + \dots\right]
\end{aligned}
$$

Example 2:

Find half-range Fourier cosine series and sine series for $f(x) = x$ in $0 < x < \pi$.

Solution: (i) **Half range cosine series:**

The Fourier Cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$.

To find a_0, a_n :

$$
\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi} x \, dx \\
&= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right] \\
&= \pi.
\end{aligned}
$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx \\
&= \frac{2}{\pi}\left[x.\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - \left(0 + \frac{\cos 0}{n^2}\right)\right] \\
&= \frac{2}{\pi}\left[0 + \frac{(-1)^n}{n^2} - 0 - \frac{1}{n^2}\right] \\
&= \frac{2}{\pi}\left[\frac{[(-1)^n - 1]}{n^2}\right] = \begin{cases} 0; \text{if } n \text{ is even} \\ \dfrac{-4}{\pi n^2}; \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

Therefore the Fourier Cosine series is $x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=\text{odd}} \frac{1}{n^2}\cos nx$.

(ii) **Half range sine series:** The Fourier sine series is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$.

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx \\
&= \frac{2}{\pi}\left[x.\left(\frac{-\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2} - \left(-0 + \frac{\sin 0}{n^2}\right)\right] \\
&= \frac{2}{\pi}\left[\frac{-\pi(-1)^n}{n}\right] = \frac{2}{n}(-1)^{n-1}
\end{aligned}$$

Therefore the Fourier sine series is $x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\sin nx$.

Example 3:

Express $f(x) = x(\pi - x)$, in $0 < x < \pi$ as a Fourier series of periodicity 2π containing (i) Sine terms only (ii) Cosine terms only. Hence deduce,

$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots = \frac{\pi^3}{32}$ and $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$.

Solution: (i) **Half range sine series:**

The Fourier sine series is $f(x) = \sum\limits_{n=1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{\pi} \int\limits_0^{\pi} f(x) \sin nx dx$.

$$\begin{aligned}
b_n &= \frac{2}{\pi} \int\limits_0^{\pi} x(\pi - x) \sin nx \, dx \\
&= \frac{2}{\pi} \left[(\pi x - x^2). \left(\frac{-\cos nx}{n} \right) - (\pi - 2x) \left(-\frac{\sin nx}{n^2} \right) + (-2) \left(\frac{\cos nx}{n^3} \right) \right]_0^{\pi} \\
&= \frac{2}{\pi} \left[-(\pi x - x^2) \frac{\cos nx}{n} + \frac{(\pi - 2x) \sin nx}{n^2} - \left(\frac{2 \cos nx}{n^3} \right) \right]_0^{\pi} \\
&= \frac{2}{\pi} \Bigg[-(\pi^2 - \pi^2) \frac{\cos n\pi}{n} + \frac{(\pi - 2\pi) \sin n\pi}{n^2} - \left(\frac{2 \cos n\pi}{n^3} \right) \\
&\quad - \left(-0 + \frac{(\pi - 0) \sin 0}{n^2} - \left(\frac{2 \cos 0}{n^3} \right) \right) \Bigg] \\
&= \frac{2}{\pi} \left[-\frac{2(-1)^n}{n^3} + \frac{2}{n^3} \right] = \frac{4}{\pi n^3} \left[1 - (-1)^n \right] \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8}{\pi n^3} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

Therefore the Fourier sine series is

$$\begin{aligned}
x(\pi - x) &= 8 \sum_{n=\text{odd}} \frac{1}{\pi n^3} \sin nx \\
&= \frac{8}{\pi} \left[\frac{1}{1^3} \sin x + \frac{1}{3^3} \sin 3x + \frac{1}{5^3} \sin 5x + \ldots \right]
\end{aligned}$$

Put $x = \frac{\pi}{2}$ in the series and it is a point of continuity for $f(x)$.

Therefore the sum of the Fourier series when $x = \frac{\pi}{2}$ is

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{\pi^2}{4}.$$

Therefore

$$\begin{aligned}\frac{\pi^2}{4} &= \frac{8}{\pi}\left[\frac{1}{1^3}\sin\frac{\pi}{2} + \frac{1}{3^3}\sin 3\frac{\pi}{2} + \frac{1}{5^3}\sin 5\frac{\pi}{2} + \dots\right] \\ &= \frac{8}{\pi}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots\right] \\ \Rightarrow & \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots = \frac{\pi^3}{8}.\end{aligned}$$

(ii) **Half range cosine series:**

The Fourier Cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$.

To find a_0, a_n :

$$\begin{aligned}a_0 &= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,dx \\ &= \frac{2}{\pi}\left(\frac{\pi x^2}{2} - \frac{x^3}{3}\right)_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] \\ &= \frac{\pi^2}{3}.\end{aligned}$$

$$\begin{aligned}a_n &= \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\cos nx\,dx \\ &= \frac{2}{\pi}\left[(\pi x - x^2).\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[(\pi x - x^2).\frac{\sin nx}{n} + (\pi - 2x).\frac{\cos nx}{n^2} + \frac{2\sin nx}{n^3}\right]_0^{\pi}\end{aligned}$$

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\Bigg[(\pi^2-\pi^2).\frac{\sin n\pi}{n}+(\pi-2\pi).\frac{\cos n\pi}{n^2}+\frac{2\sin n\pi}{n^3}\\
&\quad -\left(0+\frac{\pi\cos 0}{n^2}+\frac{2\sin 0}{n^3}\right)\Bigg]\\
&= \frac{2}{\pi}\left[0+\frac{(-\pi)(-1)^n}{n^2}+0-\left(0+\frac{\pi}{n^2}+0\right)\right]\\
&= -\frac{2}{n^2}\left[(-1)^n+1\right] = \begin{cases} 0; \text{if } n \text{ is odd}\\ \dfrac{-4}{n^2}; \text{if } n \text{ is even}\end{cases}
\end{aligned}
$$

Therefore the Fourier cosine series is

$$
\begin{aligned}
x(\pi-x) &= \frac{\pi^2}{6}-4\sum_{n=\text{even}}\frac{1}{n^2}\cos nx\\
&= \frac{\pi^2}{6}-4\left[\frac{1}{2^2}\cos 2x+\frac{1}{4^2}\cos 4x+\frac{1}{6^2}\cos 6x+\dots\right]
\end{aligned}
$$

Put $x=\frac{\pi}{2}$ in the series and it is a point of continuity for $f(x)$.

Therefore the sum of the Fourier series when $x=\frac{\pi}{2}$ is

$$
f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}\left(\pi-\frac{\pi}{2}\right)=\frac{\pi^2}{4}.
$$

Therefore

$$
\begin{aligned}
\frac{\pi^2}{4} &= \frac{\pi^2}{6}-4\left[\frac{1}{2^2}\cos 2x+\frac{1}{4^2}\cos 4x+\frac{1}{6^2}\cos 6x+\dots\right]\\
&= \frac{\pi^2}{6}-4\left[\frac{1}{2^2}\cos\frac{2\pi}{2}+\frac{1}{4^2}\cos\frac{4\pi}{2}+\frac{1}{6^2}\cos\frac{6\pi}{2}+\dots\right]\\
&= \frac{\pi^2}{6}-4\left[-\frac{1}{2^2}+\frac{1}{4^2}-\frac{1}{6^2}+\dots\right]\\
\frac{\pi^2}{4}-\frac{\pi^2}{6} &= \frac{4}{2^2}\left[1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots\right]\\
&\Rightarrow 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots=\frac{\pi^2}{12}.
\end{aligned}
$$

Example 4:

Expand the function $f(x) = x\sin x$ as a cosine series in $0 < x < \pi$ and show that $1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \ldots = \frac{\pi}{2}$.

Solution: The Fourier Cosine series is $f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi}\int\limits_0^{\pi} f(x)dx$ and $a_n = \frac{2}{\pi}\int\limits_0^{\pi} f(x)\cos nx dx$.

To find a_0, a_n :

$$
\begin{aligned}
a_0 &= \frac{2}{\pi}\int\limits_0^{\pi} x\sin x\, dx \\
&= \frac{2}{\pi}\left[x(-\cos x) - 1(-\sin x)\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[-x\cos x + \sin x\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[-\pi\cos\pi + \sin\pi - (0 + \sin 0)\right] \\
&= \frac{2}{\pi}(\pi) = 2.
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int\limits_0^{\pi} x\sin x\cos nx\, dx \\
&= \frac{1}{\pi}\int\limits_0^{\pi} x\left[\sin(x+nx) + \sin(x-nx)\right]\, dx \\
&= \frac{1}{\pi}\left\{\int\limits_0^{\pi} x\sin(1+n)x\, dx + \int\limits_0^{\pi} x\sin(1-n)x\, dx\right\} \\
&= \frac{1}{\pi}\left\{x\left(\frac{-\cos(1+n)x}{(1+n)}\right) - 1\left(-\frac{\sin(1+n)x}{(1+n)^2}\right)\right. \\
&\left. + \left[x\left(\frac{-\cos(1-n)x}{(1-n)}\right) - 1\left(\frac{-\sin(1-n)x}{(1-n)^2}\right)\right]\right\}_0^{\pi} \\
&= \frac{1}{\pi}\left[\frac{-x\cos(1+n)x}{(1+n)} + \frac{\sin(1+n)x}{(1+n)^2} + \frac{-x\cos(1-n)x}{(1-n)} + \frac{\sin(1-n)x}{(1-n)^2}\right]_0^{\pi}
\end{aligned}
$$

$$a_n = \frac{1}{\pi}\left\{\frac{-\pi\cos(1+n)\pi}{(1+n)} + \frac{\sin(1+n)\pi}{(1+n)^2} + \frac{-\pi\cos(1-n)\pi}{(1-n)} + \frac{\sin(1-n)\pi}{(1-n)^2}\right.$$
$$\left. - \left[0 + \frac{\sin 0}{(1+n)^2} + 0 + \frac{\sin 0}{(1-n)^2}\right]\right\}$$
$$= \frac{1}{\pi}\left[\frac{\pi\cos n\pi}{(1+n)} - \frac{\sin n\pi}{(1+n)^2} + \frac{\pi\cos n\pi}{(1-n)} + \frac{\sin n\pi}{(1-n)^2}\right]$$

Since $\cos(\pi \pm \theta) = -\cos\theta,\ \sin(\pi-\theta) = \sin\theta,\ \sin(\pi+\theta) = -\sin\theta$

$$= \frac{1}{\pi}\left[\frac{\pi(-1)^n}{1+n} + \frac{\pi(-1)^n}{1-n}\right]$$
$$= (-1)^n\left[\frac{1-n+1+n}{(1+n)(1-n)}\right] = -\frac{2(-1)^n}{(n+1)(n-1)} \text{ if } n \neq 1$$

$$a_1 = \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx$$
$$= \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx \text{ Since } \sin 2\theta = 2\sin\theta\cos\theta$$
$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]_0^\pi$$
$$= \frac{1}{\pi}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^\pi$$
$$= \frac{1}{\pi}\left[\frac{-\pi\cos 2\pi}{2} + \frac{\sin 2\pi}{4} - \left(0 + \frac{\sin 0}{4}\right)\right]$$
$$= -\frac{1}{2}$$

Therefore the Fourier Cosine series is

$$x\sin x = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{(n-1)(n+1)}\cos nx$$
$$= 1 - \frac{1}{2}\cos x - 2\left[\frac{\cos 2x}{1.3} - \frac{\cos 3x}{2.4} + \frac{\cos 4x}{3.5} - \frac{\cos 5x}{4.6} + \dots\right]$$

Put $x = \frac{\pi}{2}$ in the series and it is a point of continuity for $f(x)$.

Therefore the sum of the Fourier series when $x = \frac{\pi}{2}$ is

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

Therefore

$$\frac{\pi}{2} = 1 - \frac{1}{2}\cos\frac{\pi}{2} - 2\left[\frac{\cos 2\left(\frac{\pi}{2}\right)}{1.3} - \frac{\cos 3\left(\frac{\pi}{2}\right)}{2.4} + \frac{\cos 4\left(\frac{\pi}{2}\right)}{3.5} - \frac{\cos 5\left(\frac{\pi}{2}\right)}{4.6} + \dots\right]$$
$$= 1 - 2\left[-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \dots\right]$$
$$\Rightarrow 1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \dots = \frac{\pi}{2}.$$

Example 5: Find sine half-range series for the function $f(x) = x$ in $0 \leq x \leq \frac{\pi}{2}$ and $f(x) = \pi - x$ in $\frac{\pi}{2} < x < \pi$.

Solution: The Fourier sine series is $f(x) = \sum\limits_{n=1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{\pi}\int\limits_0^{\pi} f(x)\sin nx dx.$

$$b_n = \frac{2}{\pi}\left\{\int\limits_0^{\pi/2} x\sin nx\,dx + \int\limits_{\pi}^{\pi/2} (\pi - x)\sin nx\,dx\right\}$$
$$= \frac{2}{\pi}\left\{\left[x.\left(\frac{-\cos nx}{n}\right) - 1.\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi/2}\right.$$
$$\left. + \left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1).\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$
$$= \frac{2}{\pi}\left\{\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi/2} + \left[\frac{-(\pi - x)\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{\pi/2}^{\pi}\right\}$$
$$= \frac{2}{\pi}\left\{\frac{-\frac{\pi}{2}\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2} + 0 - \frac{\sin 0}{n^2} + \left[0 - \frac{\sin n\pi}{n^2}\right.\right.$$
$$\left.\left. - \left(\frac{-\left(\pi - \frac{\pi}{2}\right)\cos\frac{n\pi}{2}}{n} - \frac{\sin\frac{n\pi}{2}}{n^2}\right)\right]\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-\frac{\pi}{2}\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2} + 0 - 0 + 0 - 0 + \frac{\frac{\pi}{2}\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right]\right\}$$

$$= \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{4}{\pi n^2}\sin\dfrac{n\pi}{2} & \text{if } n \text{ is odd} \end{cases}$$

Therefore the Fourier sine series is $f(x) = \sum\limits_{n=\text{even}} \dfrac{4}{\pi n^2}\sin\dfrac{n\pi}{2}\sin nx.$

Example 6: If $f(x) = \begin{cases} \dfrac{\pi x}{4}, & 0 < x < \dfrac{\pi}{2} \\ \dfrac{\pi}{4}(\pi - x), & \dfrac{\pi}{2} < x < \pi \end{cases}$ Express $f(x)$ in a series of cosines only.

Solution: The Fourier Cosine series is $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)dx$ and $a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\cos nx dx.$

To find a_0, a_n :

$$a_0 = \frac{2}{\pi}\left\{\int_0^{\pi/2} \frac{\pi x}{4}\,dx + \int_{\pi/2}^{\pi} \frac{\pi}{4}(\pi - x)\,dx\right\} = \frac{1}{2}\left\{\left[\frac{x^2}{2}\right]_0^{\pi/2} + \left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{2}\left[\frac{\pi^2}{8} - 0 + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right] = \frac{\pi^2}{8}.$$

$$a_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} \frac{\pi x}{4}\cos nx\,dx + \int_{\pi/2}^{\pi} \frac{\pi}{4}(\pi - x)\cos nx\,dx\right\}$$

$$= \frac{1}{2}\left\{\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right\}$$

$$= \frac{1}{2}\left\{\left[x.\left(\frac{\sin nx}{n}\right) - 1.\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi/2}\right.$$

$$\left. + \left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\frac{\cos nx}{n^2}\right]_{\pi/2}^{\pi}\right\}$$

$$a_n = \frac{1}{2}\left\{ \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi/2} + \left[\frac{(\pi - x)\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{\pi/2}^{\pi} \right\}$$

$$= \frac{1}{2}\left\{ \frac{\frac{\pi}{2}\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2} - 0 - \frac{\cos 0}{n^2} + 0 + \frac{\cos n\pi}{n^2} \right.$$

$$\left. - \frac{\left(\pi - \frac{\pi}{2}\right)\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2} \right\}$$

$$= \frac{1}{2}\left[\frac{\cos\frac{n\pi}{2}}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

Therefore $f(x) = \frac{\pi^2}{16} + \sum_{n=1}^{\infty} \frac{1}{2}\left[\frac{\cos\frac{n\pi}{2}}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right]\cos nx.$

Change of Interval

Suppose we have to obtain the Fourier series of a function $f(x)$ defined in the interval $c < x < c + 2l$.

Introduce the new variable z to vary in an interval of length 2π as x varies in an interval of length $2l$.

From concept of proportion, we have

$$\frac{z}{2\pi} = \frac{x}{2l} \Rightarrow z = \frac{\pi x}{l}.$$

When $x = c \Rightarrow z = \frac{\pi c}{l} = d$ (say).

When $x = c + 2l \Rightarrow z = \frac{\pi}{l}(c + 2l) = \dfrac{\pi c}{l} + 2\pi = d + 2\pi.$

Therefore z lies in the interval $(d, d+2\pi)$ of length 2π and $f(x) = f\left(\dfrac{lz}{\pi}\right) = g(z)$, $d < z < d + 2\pi$.

If $f(x)$ satisfies Dirichlet's conditions, then $g(z)$ also satisfies them. Therefore we can expand $g(z)$ as a Fourier series of the standard form.

$$g(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \text{ where}$$

$$a_n = \frac{1}{\pi} \int_d^{d+2\pi} g(z) \cos nz \, dz \;\; n \geq 0 \text{ and } b_n = \frac{1}{\pi} \int_d^{d+2\pi} g(z) \sin nz \, dz, n \geq 1.$$

Since $g(z) = f(x)$ and $z = \dfrac{\pi x}{l}$, we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \text{ where}$$

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x) \, dx, \; a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l} \, dx \text{ and } b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l} \, dx.$$

Note:

(i) If the interval is $(0, 2l)$, put $c = 0$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \text{ where}$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx,\ a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx \text{ and } b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

(ii) If the interval is $(-l, l)$, put $c = -l$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{ where}$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx,\ a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \text{ and } b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

(iii) Even and odd functions and half-range series will be similar to in the previous sections.

- If $f(x)$ is an even function in $(-l, l)$ then $b_n = 0$ and
  $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$
  where $a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx,\ a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx.$
- If $f(x)$ is an odd function in $(-l, l)$ then $a_0 = a_n = 0$ and
  $$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$
  where $b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$

(iv) The half range cosine series for $f(x)$ in $(0, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx,\ a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx.$

(v) The half range sine series for $f(x)$ in $(0, l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$

Problems based on $(0, 2l)$

Example 1: Find the Fourier series expansion of period $2l$ for the function $f(x) = (l-x)^2$ in $0 < x < 2l$. Deduce the sum of the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$.

Solution: The Fourier series for $f(x)$ is

$f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos\dfrac{n\pi x}{l} + \sum\limits_{n=1}^{\infty} b_n \sin\dfrac{n\pi x}{l}$ where

$a_0 = \dfrac{1}{l}\int\limits_0^{2l} f(x)\,dx$, $a_n = \dfrac{1}{l}\int\limits_0^{2l} f(x)\cos\dfrac{n\pi x}{l}\,dx$ and $b_n = \dfrac{1}{l}\int\limits_0^{2l} f(x)\sin\dfrac{n\pi x}{l}\,dx$.

To find a_0, a_n **and** b_n

$$a_0 = \frac{1}{l}\int\limits_0^{2l} (l-x)^2\,dx = \frac{1}{l}\left[\frac{(l-x)^3}{-3}\right]_0^{2l}$$

$$= -\frac{1}{3l}[-l^3 - l^3] = \frac{2l^2}{3}.$$

$$a_n = \frac{1}{l}\int\limits_0^{2l} (l-x)^2 \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(l-x)^2.\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - 2(l-x)(-1)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[(l-x)^2.\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - 2(\pi - l)\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} - 2\frac{\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^{2l}$$

$$= \frac{1}{l}\Bigg[(l-2l)^2.\frac{l}{n\pi}.\sin 2n\pi - 2(l-2l).\frac{l^2}{n^2\pi^2}.\cos 2n\pi - 2\frac{l^3}{n^3\pi^3}.\sin 2n\pi$$

$$-\left((l-0)^2.\frac{l}{n\pi}.\sin 0 - 2(l-0)\frac{l^2}{n^2\pi^2}.\cos 0 - 2\frac{l^3}{n^3\pi^3}.\sin 0\right)\Bigg]$$

$$= \frac{1}{l}\left[0 + \frac{2l^3}{n^2\pi^2} - 0 - \left(0 - \frac{2l^2}{n^2\pi^2} - 0\right)\right] \text{ Since } \cos 2n\pi = 1, \sin 2n\pi = 0$$

$$= \frac{1}{l}.\frac{4l^3}{n^2\pi^2} = \frac{4l^2}{n^2\pi^2}.$$

$$b_n = \frac{1}{l}\int_0^{2l} (l-x)^2 \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{1}{l}\left[(l-x)^2.\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2(l-x)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[-(l-x)^2.\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} + 2(l-x)\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} - 2\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^{2l}$$

$$= \frac{1}{l}\Bigg[-(l-2l)^2.\frac{l}{n\pi}.\cos 2n\pi + 2(l-2l)\frac{l^2}{n^2\pi^2}.\sin 2n\pi - 2\frac{l^3}{n^3\pi^3}.\cos 2n\pi$$

$$-\left(-(l-0)^2.\frac{l}{n\pi}.\cos 0 + 2(l-0)\frac{l^2}{n^2\pi^2}.\sin 0 - 2\frac{l^3}{n^3\pi^3}.\cos 0\right)\Bigg]$$

$$= \frac{1}{l}\left[-\frac{l^3}{n\pi} + 0 - \frac{2l^3}{n^3\pi^3} - \left(-\frac{l^3}{n\pi} + 0 - \frac{2l^3}{n^3\pi^3}\right)\right] \text{ Since } \cos 2n\pi = 1, \sin 2n\pi = 0$$

$$= 0.$$

Therefore $(l-x)^2 = \frac{l^2}{3} + \sum\limits_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}$.

$$\Rightarrow (l-x)^2 = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \dots\right]$$

To deduce $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$.

Put $x = 0$ in the Fourier series. But $x = 0$ is an end point of the interval $(0, 2l)$ for $f(x)$.

Therefore the sum of the Fourier series when $x = 0$ is

$$f(0) = \frac{1}{2}\left[f(0+) + f(2l-)\right]$$
$$= \frac{1}{2}\left[(l-0)^2 + (l-2l)^2\right] = l^2.$$

Therefore $l^2 = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos 0 + \frac{1}{2^2}\cos 0 + \frac{1}{3^2}\cos 0 + \dots\right]$.

$$\Rightarrow \frac{2l^2}{3} = \frac{4l^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right] \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Example 2: Expand $x(2l - x)$ in $(0, 2l)$ as a Fourier series of period $2l$. Deduce the sum $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots$.

Solution:

The Fourier series for $f(x)$ is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ where

$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$, $a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$ and $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$.

To find a_0, a_n and b_n

$$
\begin{aligned}
a_0 &= \frac{1}{l}\int_0^{2l} x(2l - x)\,dx \\
&= \frac{1}{l}\int_0^{2l} (2lx - x^2)\,dx \\
&= \frac{1}{l}\left[lx^2 - \frac{l^3}{3}\right]_0^{2l} \\
&= \frac{1}{l}\left[4l^3 - \frac{8l^3}{3}\right] = \frac{4l^2}{3}.
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{l}\int_0^{2l} (2lx - x^2)\cos\frac{n\pi x}{l}\,dx \\
&= \frac{1}{l}\left[(2lx - x^2)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (2l - 2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l} \\
&= \frac{1}{l}\left[(2lx - x^2)\frac{l}{n\pi}.\sin\frac{n\pi x}{l} + (2l - 2x)\frac{l^2}{n^2\pi^2}.\cos\frac{n\pi x}{l} + 2\frac{l^3}{n^3\pi^3}.\sin\frac{n\pi x}{l}\right]_0^{2l} \\
&= \frac{1}{l}\Bigg[(4l^2 - 4l^2)\frac{l}{n\pi}.\sin 2n\pi + (2l - 4l)\frac{l^2}{n^2\pi^2}.\cos 2n\pi + 2\frac{l^3}{n^3\pi^3}.\sin 2n\pi \\
&\quad - \left(0 + \frac{2l^3}{n^2\pi^2}.\cos 0 + 0\right)\Bigg]
\end{aligned}
$$

$$a_n = \frac{1}{l}\left[-\frac{2l^3}{n^2\pi^2} - \frac{2l^3}{n^2\pi^2}\right] = -\frac{4l^2}{n^2\pi^2}.$$

$$b_n = \frac{1}{\pi}\int_0^{2l} (2lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(2lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (2l - 2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right.$$

$$\left. + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[-(2lx - x^2)\frac{l}{n\pi}.\cos\frac{n\pi x}{l} + (2l - 2x)\frac{l^2}{n^2\pi^2}.\sin\frac{n\pi x}{l} - 2\frac{l^3}{n^3\pi^3}.\cos\frac{n\pi x}{l}\right]_0^{2\pi}$$

$$= \frac{1}{l}\left[-(4l^2 - 4l^2)\frac{l}{n\pi}.\cos 2n\pi + (2l - 4l)\frac{l^2}{n^2\pi^2}.\sin 2n\pi - 2\frac{l^3}{n^3\pi^3}.\cos 2n\pi\right.$$

$$\left. - \left(0 + 0 - 2\frac{l^3}{n^3\pi^3}\cos 0\right)\right]$$

$$= \frac{1}{l}\left[-\frac{2l^3}{n^3\pi^3} + \frac{2l^3}{n^3\pi^3}\right] = 0.$$

Therefore

$$f(x) = \frac{2l^2}{3} + \sum_{n=1}^{\infty}\frac{-4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$= \frac{2l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \ldots\right]$$

To deduce the value of $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots.$

Put $x = l$ in the Fourier series.

But $x = l$ is an interior point of continuity $(0, 2l)$ for the function $f(x)$.

Therefore the sum of the Fourier series when $x = l$ is $f(l) = l(2l - l) = l^2$.

$$\text{Therefore } l^2 = \frac{2l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi + \dots\right]$$
$$\Rightarrow \frac{l^2}{3} = -\frac{4l^2}{\pi^2}\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots\right]$$
$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}.$$

Example 3: If $f(x) = \begin{cases} \dfrac{x}{l} & \text{for } 0 < x \le l \\ \dfrac{2l-x}{l} & \text{for } l < x < 2l \end{cases}$ express $f(x)$ as a Fourier series of periodicity $2l$.

Solution: The Fourier series for $f(x)$ is

$f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos\dfrac{n\pi x}{l} + \sum\limits_{n=1}^{\infty} b_n \sin\dfrac{n\pi x}{l}$ where

$a_0 = \dfrac{1}{l}\int\limits_0^{2l} f(x)\,dx$, $a_n = \dfrac{1}{l}\int\limits_0^{2l} f(x)\cos\dfrac{n\pi x}{l}\,dx$ and $b_n = \dfrac{1}{l}\int\limits_0^{2l} f(x)\sin\dfrac{n\pi x}{l}\,dx$.

To find a_0, a_n and b_n

$$a_0 = \frac{1}{l}\left[\int\limits_0^{l} f(x)\,dx + \int\limits_l^{2l} f(x)\,dx\right]$$
$$= \frac{1}{l}\left[\int\limits_0^{l} \frac{x}{l}\,dx + \int\limits_l^{2l} \frac{2l-x}{l}\,dx\right]$$
$$= \frac{1}{l^2}\left\{\left[\frac{x^2}{2}\right]_0^l + \left[2lx - \frac{x^2}{2}\right]_l^{2l}\right\}$$
$$= \frac{1}{l^2}\left\{\left(\frac{l^2}{2} - 0\right) + \left(4l^2 - \frac{4l^2}{2}\right) - \left(2l^2 - \frac{l^2}{2}\right)\right\}$$
$$= \frac{1}{l^2}\left[\frac{l^2}{2} + 2l^3 + \frac{3l^2}{2}\right] = 1.$$

$$
\begin{aligned}
a_n &= \frac{1}{l}\left[\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx + \int_l^{2l} f(x)\cos\frac{n\pi x}{l}\,dx\right]\\
&= \frac{1}{l}\left[\int_0^l \frac{x}{l}\cos\frac{n\pi x}{l}\,dx + \int_l^{2l} \frac{2l-x}{l}\cos\frac{n\pi x}{l}\,dx\right]\\
&= \frac{1}{l^2}\left\{\left[x.\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - 1.\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{\pi}\right.\\
&\left.+\left[(2l-x)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - (-1).\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_l^{2l}\right\}\\
&= \frac{1}{l^2}\left\{\left[x.\frac{l}{n\pi}\sin\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}.\cos\frac{n\pi x}{l}\right]_0^l\right.\\
&\left.+\left[(2l-x).\frac{l}{n\pi}\sin\frac{n\pi x}{l} - \frac{l^2}{n^2\pi^2}.\cos\frac{n\pi x}{l}\right]_l^{2l}\right\}\\
&= \frac{1}{l^2}\left\{\frac{l^2}{n\pi}\sin n\pi + \frac{l^2}{n^2\pi^2}\cos n\pi - \left(0+\frac{l^2}{n^2\pi^2}\cos 0\right)\right.\\
&\left.+\left[0-\frac{l^2}{n^2\pi^2}\cos 2n\pi - \left(0-\frac{l^2}{n^2\pi^2}\cos n\pi\right)\right]\right\}\\
&= \frac{1}{l^2}.\frac{l^2}{n^2\pi^2}\left[\cos n\pi - 1 - 1 + \cos n\pi\right]\\
&= \frac{2}{n^2\pi^2}\left[2(-1)^n - 2\right] = \frac{2}{\pi^2 n^2}\left\{(-1)^n - 1\right\}\\
&= \begin{cases} 0 & \text{if } n \text{ is even}\\ \dfrac{-4}{\pi n^2} & \text{if } n \text{ is odd}\end{cases}\\
b_n &= \frac{1}{l}\left[\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx + \int_l^{2l} f(x)\sin\frac{n\pi x}{l}\,dx\right]\\
&= \frac{1}{l}\left[\int_0^l \frac{x}{l}\sin\frac{n\pi x}{l}\,dx + \int_l^{2l} \frac{2l-x}{l}\sin\frac{n\pi x}{l}\,dx\right]
\end{aligned}
$$

$$b_n = \frac{1}{l^2}\left\{ \left[x.\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1.\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l \right.$$

$$\left. + \left[(2l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1).\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_l^{2l} \right\}$$

$$= \frac{1}{l^2}\left\{ \left[\frac{-xl\cos\frac{n\pi x}{l}}{n\pi} + \frac{l^2\sin\frac{n\pi x}{l}}{n^2\pi^2}\right]_0^l \right.$$

$$\left. + \left[-(2l-x)\frac{l\cos\frac{n\pi x}{l}}{n\pi} - \frac{l^2\sin\frac{n\pi x}{l}}{n^2\pi^2}\right]_l^{2l} \right\}$$

$$= \frac{1}{l^2}\left\{ \frac{-l^2\cos n\pi}{n\pi} + \frac{l^2\sin n\pi}{n^2\pi^2} + 0 - 0 \right.$$

$$\left. + \left[0 - \frac{l^2\sin 2n\pi}{n^2\pi^2} - \left(\frac{-l^2\cos n\pi}{n\pi} - 0\right)\right] \right\}$$

$$= \frac{1}{l^2}.\frac{l^2}{n\pi}\left[-\cos n\pi + \cos n\pi\right] = 0. \text{ Since } \sin n\pi = 0$$

Therefore the Fourier series is $f(x) = \frac{1}{2} + \sum_{odd} \frac{-4}{\pi n^2}\cos\frac{n\pi x}{l}$.

Example 4: Find the Fourier series of periodicity 3 for $f(x) = 2x - x^2$, in $0 < x < 3$.

Solution: Here $2l = 3 \Rightarrow l = \frac{3}{2}$.

Therefore the Fourier series for $f(x)$ is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{3}$ where

$a_0 = \frac{2}{3}\int_0^3 f(x)\,dx$, $a_n = \frac{2}{3}\int_0^3 f(x)\cos\frac{2n\pi x}{3}\,dx$ and $b_n = \frac{2}{3}\int_0^3 f(x)\sin\frac{2n\pi x}{3}\,dx$.

Now $a_0 = \frac{2}{3}\int_0^3 \left(2x - x^2\right)\,dx = \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3 = 0.$

$$a_n = \frac{2}{3}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x-x^2).\frac{\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}} - (2-2x)\left(-\frac{\cos\frac{2n\pi x}{3}}{\frac{2^2n^2\pi^2}{3^2}}\right) + (-2)\left(\frac{-\sin\frac{2n\pi x}{3}}{\frac{2^3n^3\pi^3}{l^3}}\right)\right]_0^3$$

$$= \frac{2}{3}\left[(2x-x^2).\frac{3}{2n\pi}\sin\frac{2n\pi x}{3} + (2-2x).\frac{3^2}{2^2n^2\pi^2}\cos\frac{2n\pi x}{3} + 2\frac{3^3}{2^3n^3\pi^3}\sin\frac{2n\pi x}{3}\right]_0^3$$

$$= \frac{2}{3}\left[(6-9).\frac{3}{2n\pi}.\sin 2n\pi + (2-6).\frac{3^2}{2^2n^2\pi^2}.\cos 2n\pi + 2\frac{3^3}{2^3n^3\pi^3}.\sin 2n\pi\right.$$

$$\left.-\left(0 + (2-0)\frac{3^2}{2^2n^2\pi^2}.\cos 0 + 2\frac{3^3}{2^3n^3\pi^3}.\sin 0\right)\right]$$

$$= \frac{2}{3}\left[0 - \frac{9}{n^2\pi^2} - 0 - \left(0 + \frac{9}{2n^2\pi^2} + 0\right)\right] \text{ Since } \cos 2n\pi = 1, \sin 2n\pi = 0$$

$$= -\frac{2}{3}.\frac{9}{n^2\pi^2}\left(1 + \frac{1}{2}\right) = -\frac{9}{n^2\pi^2}.$$

$$b_n = \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x-x^2).\left(\frac{-\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right) - (2-2x)\left(-\frac{\sin\frac{2n\pi x}{3}}{\frac{2^2n^2\pi^2}{3^2}}\right) + (-2)\left(\frac{\cos\frac{2n\pi x}{3}}{\frac{2^3n^3\pi^3}{l^3}}\right)\right]_0^3$$

$$= \frac{2}{3}\left[-(2x-x^2).\frac{3}{2n\pi}\cos\frac{2n\pi x}{3} + (2-2x).\frac{3^2}{2^2n^2\pi^2}\sin\frac{2n\pi x}{3} - 2\frac{3^3}{2^3n^3\pi^3}\cos\frac{2n\pi x}{3}\right]_0^3$$

$$= \frac{2}{3}\left[-(6-9).\frac{3}{2n\pi}.\cos 2n\pi + (2-6).\frac{3^2}{2^2n^2\pi^2}.\sin 2n\pi - 2\frac{3^3}{2^3n^3\pi^3}.\cos 2n\pi\right.$$

$$\left.-\left(0 + (2-0)\frac{3^2}{2^2n^2\pi^2}.\sin 0 - 2\frac{3^3}{2^3n^3\pi^3}.\cos 0\right)\right]$$

$$= \frac{2}{3}\left[\frac{9}{2n\pi} - 0 - \frac{27}{4n^3\pi^3} - \left(0 - 0 + -\frac{27}{4n^3\pi^3}\right)\right] \text{ Since } \cos 2n\pi = 1, \sin 2n\pi = 0$$

$$= \frac{2}{3}.\frac{9}{2n\pi} = \frac{3}{n\pi}.$$

Therefore $2x - x^2 = \sum\limits_{n=1}^{\infty} \frac{-9}{n^2\pi^2} \cos\frac{n\pi x}{l} + \sum\limits_{n=1}^{\infty} \frac{3}{n\pi} \sin\frac{n\pi x}{l}$.

Example 5: Determine the Fourier series for the function

$$f(x) = \begin{cases} \pi x & \text{for } 0 < x < 1 \\ \pi(2-x) & \text{for } 1 < x < 2 \end{cases}.$$

Solution: Here $2l = 2 \Rightarrow l = 1$. The Fourier series for $f(x)$ is

$f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos n\pi x + \sum\limits_{n=1}^{\infty} b_n \sin n\pi x$ where

$a_0 = \int\limits_0^2 f(x)\,dx$, $a_n = \int\limits_0^2 f(x)\cos n\pi x\,dx$ and $b_n = \int\limits_0^2 f(x)\sin n\pi x\,dx$.

To find a_0, a_n and b_n

$$\begin{aligned}
a_0 &= \int\limits_0^1 f(x)\,dx + \int\limits_1^2 f(x)\,dx \\
&= \int\limits_0^1 \pi x\,dx + \int\limits_1^2 \pi(2-x)\,dx \\
&= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[\frac{(2-x)^2}{-2}\right]_1^2 \\
&= \frac{\pi}{2} - 0 - \frac{\pi}{2}[0-1] = \pi.
\end{aligned}$$

$$\begin{aligned}
a_n &= \int\limits_0^1 f(x)\cos n\pi x\,dx + \int\limits_1^2 f(x)\cos n\pi x\,dx \\
&= \int\limits_0^1 \pi x\cos n\pi x\,dx + \int\limits_1^2 \pi(2-x)\cos n\pi x\,dx \\
&= \pi\left[x.\frac{\sin n\pi x}{n\pi} - 1.\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 \\
&+ \pi\left[(2-x)\frac{\sin n\pi x}{n\pi} - (-1).\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_1^2 \\
&= \pi\left[\frac{x\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right]_0^1 + \pi\left[(2-x)\frac{\sin n\pi x}{n\pi} - \frac{\cos n\pi x}{n^2\pi^2}\right]_1^2
\end{aligned}$$

$$a_n = \pi\left[\frac{\sin n\pi}{n\pi} + \frac{\cos n\pi}{n^2\pi^2} - \left(0 + \frac{\cos 0}{n^2\pi^2}\right)\right]$$

$$+\,\pi\left[0 - \frac{\cos 2n\pi}{n^2\pi^2} - \left(\frac{\sin n\pi}{n\pi} - \frac{\cos n\pi}{n^2\pi^2}\right)\right]$$

$$= \pi\left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} + \frac{(-1)^n}{n^2\pi^2}\right]$$

$$= \frac{2}{\pi n^2}\left\{(-1)^n - 1\right\}$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-4}{\pi n^2} & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \int_0^1 f(x)\sin n\pi x\,dx + \int_1^2 f(x)\sin n\pi x\,dx$$

$$= \int_0^1 \pi x \sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \pi\left[x.\left(\frac{-\cos n\pi x}{n\pi}\right) - 1.\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$+\,\pi\left[(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-1).\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \pi\left[\frac{-x\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 + \pi\left[-(2-x)\frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right]_1^2$$

$$= \pi\left[\frac{-\cos n\pi}{n\pi} + \frac{\sin n\pi}{n^2\pi^2} - (0+0)\right]$$

$$+\,\pi\left[0 - \frac{\sin 2n\pi}{n^2\pi^2} - \left(\frac{-\cos n\pi}{n\pi} - \frac{\sin n\pi}{n^2\pi^2}\right)\right]$$

$$= \pi\left[\frac{-(-1)^n}{n} + \frac{(-1)^n}{n}\right] = 0.$$

Therefore the Fourier series is $f(x) = \dfrac{\pi}{2} + \sum_{odd} \dfrac{-4}{\pi n^2}\cos n\pi x$.

Example 6: Find the Fourier series expansion of $f(x)$ given by

$$f(x) = \begin{cases} x & \text{for } 0 < x < 2 \\ 0 & \text{for } 2 < x < 4 \end{cases}.$$

Solution: Here $2l = 4 \Rightarrow l = 2$.

Therefore the Fourier series for $f(x)$ is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$ where

$a_0 = \frac{1}{2}\int_0^4 f(x)\,dx$, $a_n = \frac{1}{2}\int_0^4 f(x)\cos\frac{n\pi x}{2}\,dx$ and $b_n = \frac{1}{2}\int_0^4 f(x)\sin\frac{n\pi x}{2}\,dx$.

$$\begin{aligned}
a_0 &= \frac{1}{2}\left\{\int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx\right\}\\
&= \frac{1}{2}\left\{\int_0^2 x\,dx + 0\right\} = \frac{1}{2}\left(\frac{x^2}{2}\right)_0^2 = 1.\\
a_n &= \frac{1}{2}\left\{\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx + \int_2^4 f(x)\cos\frac{n\pi x}{2}\,dx\right\}\\
&= \frac{1}{2}\left\{\int_0^2 x\cos\frac{n\pi x}{2}\,dx + 0\right\}\\
&= \frac{1}{2}\left[x.\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - 1.\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{2^2}}\right)\right]_0^2\\
&= \frac{1}{2}\left[\frac{2x}{n\pi}.\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}.\cos\frac{n\pi x}{2}\right]_0^2\\
&= \frac{1}{2}\left[0 + \frac{4(-1)^n}{n^2\pi^2} - \left(0 + \frac{4}{n^2\pi^2}\right)\right]\\
&= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} 0 & \text{if } n \text{ is even}\\ \frac{-4}{\pi^2 n^2} & \text{if } n \text{ is odd}\end{cases}\\
b_n &= \frac{1}{2}\left\{\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx + \int_2^4 f(x)\sin\frac{n\pi x}{2}\,dx\right\}\\
&= \frac{1}{2}\left\{\int_0^2 x\sin\frac{n\pi x}{2}\,dx + 0\right\}
\end{aligned}$$

$$b_n = \frac{1}{2}\left[x.\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1.\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{2^2}}\right)\right]_0^2$$
$$= \frac{1}{2}\left[-\frac{2x}{n\pi}.\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}.\sin\frac{n\pi x}{2}\right]_0^2$$
$$= \frac{1}{2}\left[-\frac{4}{n\pi}.\cos n\pi + \frac{4}{n^2\pi^2}.\sin n\pi + 0 - 0\right]$$
$$= \frac{-2(-1)^n}{n\pi}$$

Therefore $f(x) = \frac{1}{2} + \sum\limits_{n=\text{odd}} \frac{-4}{\pi^2 n^2}\cos\frac{n\pi x}{2} + \sum\limits_{n=1}^{\infty} \frac{-2(-1)^n}{n\pi}\sin\frac{n\pi x}{2}$.

Problems based on $(-l, l)$

Example 7: If $f(x)$ is defined in (-2, 2) as follows. Express $f(x)$ in a Fourier series of periodicity 4.

$$f(x) = \begin{cases} 0; & -2 < x < -1 \\ 1 + x; & -1 < x < 0 \\ 1 - x; & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}.$$

Solution: Here $2l = 4 \Rightarrow l = 2$.

Therefore the Fourier series for $f(x)$ is

$f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum\limits_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$ where

$a_0 = \frac{1}{2}\int\limits_{-2}^{2} f(x)\,dx$, $a_n = \frac{1}{2}\int\limits_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$ and $b_n = \frac{1}{2}\int\limits_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$.

$$a_0 = \frac{1}{2}\left\{\int\limits_{-2}^{-1} f(x)\,dx + \int\limits_{-1}^{0} f(x)\,dx + \int\limits_{0}^{1} f(x)\,dx + \int\limits_{1}^{2} f(x)\,dx\right\}$$

$$
\begin{aligned}
a_0 &= \frac{1}{2}\left\{\int\limits_{-1}^{0}(1+x)\,dx + \int\limits_{0}^{1}(1-x)\,dx\right\} \\
&= \frac{1}{2}\left[\left(\frac{(1+x)^2}{2}\right)_{-1}^{0} + \left(\frac{(1-x)^2}{-2}\right)_{0}^{1}\right] \\
&= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \\
a_n &= \frac{1}{2}\Bigg\{\int\limits_{-2}^{-1} f(x)\cos\frac{n\pi x}{2}\,dx + \int\limits_{-1}^{0} f(x)\cos\frac{n\pi x}{2}\,dx \\
&+ \int\limits_{0}^{1} f(x)\cos\frac{n\pi x}{2}\,dx + \int\limits_{1}^{2} f(x)\cos\frac{n\pi x}{2}\,dx\Bigg\} \\
&= \frac{1}{2}\left\{\int\limits_{-1}^{0}(1+x)\cos\frac{n\pi x}{2}\,dx + \int\limits_{0}^{1}(1-x)\cos\frac{n\pi x}{2}\,dx\right\} \\
&= \frac{1}{2}\left[(1+x).\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - 1.\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{2^2}}\right)\right]_{-1}^{0} \\
&+ \frac{1}{2}\left[(1-x).\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - (-1).\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{2^2}}\right)\right]_{0}^{1} \\
&= \frac{1}{2}\left[\frac{2(1+x)}{n\pi}.\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}.\cos\frac{n\pi x}{2}\right]_{-1}^{0} \\
&+ \frac{1}{2}\left[\frac{2(1-x)}{n\pi}.\sin\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}.\cos\frac{n\pi x}{2}\right]_{0}^{1} \\
&= \frac{1}{2}\left[0 + \frac{4}{n^2\pi^2} - 0 - \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} + 0 - \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - 0 + \frac{4}{n^2\pi^2}\right] \\
&= \frac{4}{n^2\pi^2}\left[1 - \cos\frac{n\pi}{2}\right]
\end{aligned}
$$

$$b_n = \frac{1}{2}\Bigg\{ \int\limits_{-2}^{-1} f(x)\sin\frac{n\pi x}{2}\,dx + \int\limits_{-1}^{0} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$+ \int\limits_{0}^{1} f(x)\sin\frac{n\pi x}{2}\,dx + \int\limits_{1}^{2} f(x)\sin\frac{n\pi x}{2}\,dx \Bigg\}$$

$$= \frac{1}{2}\left\{ \int\limits_{-1}^{0} (1+x)\sin\frac{n\pi x}{2}\,dx + \int\limits_{0}^{1} (1-x)\sin\frac{n\pi x}{2}\,dx \right\}$$

$$= \frac{1}{2}\left[(1+x).\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1.\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{2^2}}\right)\right]_{-1}^{0}$$

$$+ \frac{1}{2}\left[(1-x).\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1).\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{2^2}}\right)\right]_{0}^{1}$$

$$= \frac{1}{2}\left[-\frac{2(1+x)}{n\pi}.\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}.\sin\frac{n\pi x}{2}\right]_{-1}^{0}$$

$$+ \frac{1}{2}\left[-\frac{2(1-x)}{n\pi}.\cos\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}.\sin\frac{n\pi x}{2}\right]_{0}^{1}$$

$$= \frac{1}{2}\left[-\frac{2}{n\pi} + 0 + 0 + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} - 0 - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2}{n\pi}\right] = 0.$$

Therefore $f(x) = \frac{1}{4} + \sum\limits_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left[1 - \cos\frac{n\pi}{2}\right]\cos\frac{n\pi x}{2}$.

Example 8: Find Fourier series of periodicity 2 for $f(x)$ given

$$f(x) = \begin{cases} 0; & \text{in } -1 < x < 0 \\ 1; & \text{in } 0 < x < 1 \end{cases}.$$

Solution: The Fourier series for $f(x)$ is

$f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos n\pi x + \sum\limits_{n=1}^{\infty} b_n \sin n\pi x$ where

$a_0 = \int\limits_{-1}^{1} f(x)\,dx$, $a_n = \int\limits_{-1}^{1} f(x)\cos n\pi x\,dx$ and $b_n = \int\limits_{-1}^{1} f(x)\sin n\pi x\,dx$.

To find a_0, a_n and b_n

$$a_0 = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$$

$$= 0 + \int_{0}^{1} dx = [x]_0^1 = 1.$$

$$a_n = \int_{-1}^{0} f(x)\cos n\pi x\,dx + \int_{0}^{1} f(x)\cos n\pi x\,dx$$

$$= 0 + \int_{0}^{1} \cos n\pi x\,dx = \left[\frac{\sin n\pi x}{n\pi}\right]_0^1 = 0.$$

$$b_n = \int_{-1}^{0} f(x)\sin n\pi x\,dx + \int_{0}^{1} f(x)\sin n\pi x\,dx$$

$$= 0 + \int_{0}^{1} \sin n\pi x\,dx = \left[\frac{-\cos n\pi x}{n\pi}\right]_0^1$$

$$= -\frac{1}{n\pi}\left[\cos n\pi - cos 0\right]$$

$$= -\frac{1}{n\pi}\left[(-1)^n - 1\right] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

Therefore $f(x) = \dfrac{1}{2} + \sum\limits_{n=\text{odd}} \dfrac{2}{n\pi}\sin n\pi x.$

Example 9: Find Fourier series of periodicity 2 for

$f(x) = \begin{cases} x; & \text{in } -1 < x < 0 \\ x+2; & \text{in } 0 < x < 1 \end{cases}$ and deduce the sum of $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots$ to ∞.

Solution: The Fourier series for $f(x)$ is

$f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos n\pi x + \sum\limits_{n=1}^{\infty} b_n \sin n\pi x$ where

$a_0 = \int\limits_{-1}^{1} f(x)\,dx,\ a_n = \int\limits_{-1}^{1} f(x)\cos n\pi x\,dx$ and $b_n = \int\limits_{-1}^{1} f(x)\sin n\pi x\,dx.$

To find a_0, a_n and b_n

$$a_0 = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$$

$$= \int_{-1}^{0} x\,dx + \int_{0}^{1} (x+2)\,dx$$

$$= \left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{(x+2)^2}{2}\right]_{0}^{1}$$

$$= 0 - \frac{1}{2} + \frac{1}{2}\,[9-4] = 2.$$

$$a_n = \int_{-1}^{0} f(x)\cos n\pi x\,dx + \int_{0}^{1} f(x)\cos n\pi x\,dx$$

$$= \int_{-1}^{0} x\cos n\pi x\,dx + \int_{0}^{1} (x+2)\cos n\pi x\,dx$$

$$= \left[\frac{x\sin n\pi x}{n\pi} - 1.\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_{-1}^{0}$$

$$+ \left[(x+2)\frac{\sin n\pi x}{n\pi} - \left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_{0}^{1}$$

$$= \left[\frac{x\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right]_{-1}^{0} + \left[(x+2)\frac{\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right]_{0}^{1}$$

$$= 0 + \frac{\cos 0}{n^2\pi^2} - \left(0 + \frac{\cos n\pi}{n^2\pi^2}\right) + 0 + \frac{\cos n\pi}{n^2\pi^2} - \left(0 + \frac{\cos 0}{n^2\pi^2}\right)$$

$$= \frac{1}{n^2\pi^2} - \frac{cos n\pi}{n^2\pi^2} + \frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2} = 0.$$

$$b_n = \int_{-1}^{0} f(x)\sin n\pi x\,dx + \int_{0}^{1} f(x)\sin n\pi x\,dx$$

$$= \int_{-1}^{0} x\sin n\pi x\,dx + \int_{0}^{1} (x+2)\sin n\pi x\,dx$$

$$= \left[x.\left(\frac{-\cos n\pi x}{n\pi}\right) - 1.\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_{-1}^{0}$$

$$+ \left[(x+2)\left(\frac{-\cos n\pi x}{n\pi}\right) - 1.\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_{0}^{1}$$

$$
\begin{aligned}
b_n &= \left[\frac{-x\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_{-1}^{0} + \left[-(x+2)\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_{0}^{1} \\
&= 0 + 0 - \left(\frac{\cos n\pi}{n\pi} - \frac{\sin n\pi}{n^2\pi^2}\right) - \frac{3\cos n\pi}{n\pi} + \frac{\sin n\pi}{n^2\pi^2} - \left(\frac{-2\cos 0}{n\pi} + 0\right) \\
&= -\frac{4\cos n\pi}{n\pi} + \frac{2}{n\pi} = \frac{2}{n\pi}\left[1 - 2\cos n\pi\right] \\
&= \frac{2}{n\pi}\left[1 - 2(-1)^n\right].
\end{aligned}
$$

Therefore the Fourier series is

$$
\begin{aligned}
f(x) &= 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1 - 2(-1)^n\right]\sin n\pi x \\
&= 1 + \frac{2}{\pi}\left[3\sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}.3\sin 3\pi x - \frac{1}{4}\sin 4\pi x\right. \\
&\quad \left. + \frac{1}{5}.3\sin 5\pi x - \frac{1}{6}\sin 6\pi x + \ldots\right]
\end{aligned}
$$

To deduce the value of $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ **to** ∞.

Put $x = \frac{1}{2}$ in the Fourier series. But $x = \frac{1}{2}$ is a point of continuity of $f(x)$.

Therefore the sum of the Fourier series when $x = \frac{1}{2}$ is $f\left(\frac{1}{2}\right) = \frac{1}{2} + 2 = \frac{5}{2}$.

$$
\begin{aligned}
\frac{5}{2} &= 1 + \frac{2}{\pi}\left[3\sin\frac{\pi}{2} - \frac{1}{2}\sin\frac{2\pi}{2} + \frac{1}{3}.3\sin\frac{3\pi}{2} - \frac{1}{4}\sin\frac{4\pi}{2}\right. \\
&\quad \left. + \frac{1}{5}.3\sin\frac{5\pi}{2} - \frac{1}{6}\sin\frac{6\pi}{2} + \ldots\right] \\
\frac{5}{2} - 1 &= \frac{2}{\pi}\left[3 - 0 - 1 - 0 + \frac{3}{5} + \ldots\right] \\
\frac{3}{2} &= \frac{6}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right] \\
&\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}.
\end{aligned}
$$

Example 10: Find the Fourier series of represent $f(x) = x^2 - 2$ in the interval $-2 < x < 2$.

Solution: $f(x) = x^2 - 2$ is an even function. Therefore $b_n = 0$.

Here $l = 2$, so the Fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$

where $a_0 = \int_0^2 f(x)\,dx$, $a_n = \int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$.

$$a_0 = \int_0^2 (x^2 - 2)\,dx = \left[\frac{x^3}{3} - 2x\right]_0^2 = -\frac{4}{3}.$$

$$\begin{aligned}
a_n &= \int_0^2 (x^2 - 2)\cos\frac{n\pi x}{2}\,dx \\
&= \left[(x^2-2).\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - (2x).\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{2^2}}\right) + 2.\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^3\pi^3}{2^3}}\right)\right]_0^2 \\
&= \left[\frac{2}{n\pi}(x^2-2)\sin\frac{n\pi x}{2} + \frac{8x}{n^2\pi^2}\cos\frac{n\pi x}{2} - \frac{16}{n^3\pi^3}\sin\frac{n\pi x}{2}\right]_0^2 \\
&= \frac{4}{n\pi}\sin n\pi + \frac{16}{n^2\pi^2}\cos n\pi - \frac{16}{n^3\pi^3}\sin n\pi - (0+0-0) \\
&= \frac{16(-1)^n}{n^2\pi^2}.
\end{aligned}$$

Therefore the Fourier series is $x^2 - 2 = -\frac{2}{3} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{n^2\pi^2}\cos\frac{n\pi x}{2}$

Problems based on Half Range series

Example 11: Express $f(x) = x$ as a half range sine series in $0 < x < 2$.

Solution: Here $l = 2$.

Therefore the half range sine series is $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$

where $b_n = \int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$.

$$b_n = \int_0^2 x \sin\frac{n\pi x}{2}\,dx$$
$$= \left[x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1.\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{2^2}}\right)\right]_0^2$$
$$= \left[-\frac{2x}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2$$
$$= -\frac{4}{n\pi}\cos n\pi + \frac{4}{n^2\pi^2}\sin n\pi - (0+0)$$
$$= -\frac{4(-1)^2}{n\pi}.$$

Therefore the half-range sine series is $x = \sum_{n=1}^{\infty} -\frac{4(-1)^2}{n\pi}\sin\frac{n\pi x}{2}$.

Example 12: Obtain the half-range cosine series for $f(x) = (x-1)^2$ in the interval $0 < x < 1$ and hence deduce the value of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$.

Solution: Here $l = 1$.

Therefore the half range cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$

where $a_0 = 2\int_0^1 f(x)\,dx$, $a_n = 2\int_0^1 f(x)\cos n\pi x\,dx$.

$$a_0 = \int_0^1 (x-1)^2\,dx = 2\left[\frac{(x-1)^3}{3}\right] = \frac{2}{3}.$$

$$a_n = \int_0^1 (x-1)^2\cos n\pi x\,dx$$
$$= 2\left[(x-1)^2.\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1).\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$
$$= 2\left[\frac{(x-1)^2\sin n\pi x}{n\pi} + \frac{2(x-1)\cos n\pi x}{n^2\pi^2} - \frac{2\sin n\pi x}{n^3\pi^3}\right]_0^1$$
$$= 2\left[0 + 0 - \frac{2\sin n\pi}{n^3\pi^3} - \left(0 - \frac{2\cos 0}{n^2\pi^2} - 0\right)\right] = \frac{4}{n^2\pi^2}.$$

Therefore the half-range cosine series is

$$\begin{aligned}(x-1)^2 &= \frac{1}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\cos n\pi x\\ &= \frac{1}{3}+\frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\pi x+\frac{1}{2^2}\cos 2\pi x+\frac{1}{3^2}\cos 3\pi x+\ldots\right]\end{aligned}$$

To deduce the value of $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots.$

Put $x=0$ in the Fourier series. Since half range cosine series $F(x)$ in the interval $(-1,1)$ is an even function. Hence $F(0-)=F(0+)=f(0+)=1$.

Therefore $\frac{F(0-)+F(0+)}{2}=\frac{1+1}{2}=1.$

Hence

$$\begin{aligned}1 &= \frac{1}{3}+\frac{4}{\pi^2}\left[\frac{1}{1^2}\cos 0+\frac{1}{2^2}\cos 0+\frac{1}{3^2}\cos 0+\ldots\right]\\ 1-\frac{1}{3} &= \frac{4}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right]\\ &\Rightarrow\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}.\end{aligned}$$

Example 13: Find Fourier cosine series of $f(x)=\begin{cases}x^2, & 0<x<1\\ 2-x, & 1<x<2\end{cases}.$

Solution: Here $l=2$.

Therefore the Fourier cosine series is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{2}$

where $a_0=\int_0^2 f(x)\,dx$, $a_n=\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx.$

$$\begin{aligned}a_0 &= \int_0^1 x^2\,dx+\int_1^2(2-x)\,dx\\ &= \left[\frac{x^3}{3}\right]_0^1+\left[\frac{(2-x)^2}{-2}\right]_0^2\\ &= \frac{1}{3}+\frac{1}{2}=\frac{5}{6}.\end{aligned}$$

$$a_n = \int_0^1 x^2 \cos\frac{n\pi x}{2}\,dx + \int_1^2 (2-x)\cos\frac{n\pi x}{2}\,dx$$

$$= \left[x^2.\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - (2x).\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{2^2}}\right) + 2.\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^3\pi^3}{2^3}}\right)\right]_0^1$$

$$+ \left[(2-x).\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - (-1).\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{2^2}}\right)\right]_1^2$$

$$= \left[\frac{2x^2}{n\pi}\sin\frac{n\pi x}{2} + \frac{8x}{n^2\pi^2}\cos\frac{n\pi x}{2} - \frac{16}{n^3\pi^3}\sin\frac{n\pi x}{2}\right]_0^1$$

$$+ \left[\frac{2(2-x)}{n\pi}\sin\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_1^2$$

$$= \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{16}{n^3\pi^3}\sin\frac{n\pi}{2} - (0+0-0)$$

$$+ 0 - \frac{4}{n^2\pi^2}\cos n\pi - \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}$$

$$= \frac{12}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{16}{n^3\pi^3}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\cos n\pi.$$

Therefore the Fourier cosine series is

$$f(x) = \frac{5}{12} + \sum_{n=1}^{\infty}\left(\frac{12}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{16}{n^3\pi^3}\sin\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\cos n\pi\right)\cos\frac{n\pi x}{2}.$$

Example 14: Expand $f(x) = (x-1)^2$, $0 < x < 1$ in a Fourier series of sine only.

Solution: Here $l = 1$.

Therefore the half range sine series is $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$

where $b_n = 2\int_0^1 f(x)\sin n\pi x\,dx$.

$$b_n = 2\int_0^1 (x-1)^2 \sin n\pi x\,dx$$

$$= 2\left[(x-1)^2.\left(\frac{-\cos n\pi x}{n\pi}\right) - 2(x-1).\left(\frac{-\sin n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$b_n = 2\left[\frac{-(x-1)^2\cos n\pi x}{n\pi} + \frac{2(x-1)\sin n\pi x}{n^2\pi^2} + \frac{2\cos n\pi x}{n^3\pi^3}\right]_0^1$$
$$= 2\left[0+0+\frac{2\cos n\pi}{n^3\pi^3} - \left(\frac{-1}{n\pi}+0+\frac{2\cos 0}{n^3\pi^3}\right)\right]$$
$$= \frac{4}{n^3\pi^3}\left[(-1)^n - 1\right] + \frac{2}{n\pi}.$$

Therefore the half-range sine series is

$$(x-1)^2 = \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3}\left[(-1)^n - 1\right] + \frac{2}{n\pi}\sin n\pi x.$$

Example 15: If If $f(x) = \begin{cases} \sin x, & \text{for } 0 \le x \le \frac{\pi}{4} \\ \cos x, & \text{for } \frac{\pi}{4} \le x \le \frac{\pi}{2} \end{cases}$ express $f(x)$ in a series of sines.

Solution: The Fourier sine series is $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{\pi x}{l}dx.$

Here $l = \frac{\pi}{2}$

$$b_n = \frac{2}{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} f(x)\sin 2nx\,dx$$
$$= \frac{2}{\frac{\pi}{2}}\left\{\int_0^{\pi/2} f(x)\sin 2nx\,dx + \int_{\pi/4}^{\pi/2} f(x)\sin 2nx\,dx\right\}$$
$$= \frac{4}{\pi}\left\{\int_0^{\pi/2} \sin x\sin 2nx\,dx + \int_{\pi/4}^{\pi/2} \cos x\sin 2nx\,dx\right\}$$
$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} \left[\cos(1-2n)x - \cos(1+2n)x\right]\,dx\right.$$
$$\left. + \int_{\pi/4}^{\pi/2} \left[\sin(1+2n)x - \sin(1-2n)x\right]\,dx\right\}$$

$$b_n = \frac{2}{\pi}\left\{ \int_0^{\pi/2} [\cos(2n-1)x - \cos(2n+1)x]\, dx \right.$$

$$\left. + \int_{\pi/4}^{\pi/2} [\sin(2n+1)x + \sin(2n-1)x]\, dx \right\}$$

Since $\cos(-\theta) = \cos\theta, \sin(-\theta) = -\sin\theta$

$$= \frac{2}{\pi}\left\{ \left[\frac{\sin(2n-1)x}{2n-1} - \frac{\sin(2n+1)x}{2n+1}\right]_0^{\pi/4} \right.$$

$$\left. + \left[\frac{-\cos(2n+1)x}{2n+1} - \frac{\cos(2n-1)x}{2n-1}\right]_{\pi/4}^{\pi/2} \right\}$$

$$= \frac{2}{\pi}\left\{ \frac{\sin(2n-1)\frac{\pi}{4}}{2n-1} - \frac{\sin(2n+1)\frac{\pi}{4}}{2n+1} - \frac{\sin 0}{2n-1} + \frac{\sin 0}{2n+1} \right.$$

$$\left. - \left[\frac{\cos(2n+1)\frac{\pi}{2}}{2n+1} + \frac{\cos(2n-1)\frac{\pi}{2}}{2n-1} - \left(\frac{\cos(2n+1)\frac{\pi}{4}}{2n+1} + \frac{\cos(2n-1)\frac{\pi}{4}}{2n-1}\right)\right]\right\}$$

$$= \frac{2}{\pi}\left\{ \frac{\sin\left(\frac{n\pi}{2} - \frac{\pi}{4}\right)}{2n-1} - \frac{\sin\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)}{2n+1} - 0 + 0 - \frac{\cos\left(n\pi + \frac{\pi}{2}\right)}{2n+1} \right.$$

$$\left. - \frac{\cos\left(n\pi + \frac{\pi}{2}\right)}{2n-1} + \frac{\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)}{2n+1} + \frac{\cos\left(\frac{n\pi}{2} - \frac{\pi}{4}\right)}{2n-1} \right\}$$

$$= \frac{2}{\pi}\left\{ \frac{1}{2n-1}\left[\sin\frac{n\pi}{2}\cos\frac{\pi}{4} - \cos\frac{n\pi}{2}\sin\frac{\pi}{4}\right] - \frac{1}{2n+1}\left[\sin\frac{n\pi}{2}\cos\frac{\pi}{4}\right.\right.$$

$$\left. + \cos\frac{n\pi}{2}\sin\frac{\pi}{4}\right] + \frac{\sin n\pi}{2n+1} - \frac{\sin n\pi}{2n-1} + \frac{1}{2n+1}\left[\cos\frac{n\pi}{2}\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\sin\frac{\pi}{4}\right]$$

$$\left. + \frac{1}{2n-1}\left[\cos\frac{n\pi}{2}\cos\frac{\pi}{4} + \sin\frac{n\pi}{2}\sin\frac{\pi}{4}\right] \right\}$$

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\left\{\frac{1}{2n-1}\left[\sin\frac{n\pi}{2}\cos\frac{\pi}{4}-\cos\frac{n\pi}{2}\sin\frac{\pi}{4}+\cos\frac{n\pi}{2}\cos\frac{\pi}{4}+\sin\frac{n\pi}{2}\sin\frac{\pi}{4}\right]\right.\\
&\left.-\frac{1}{2n+1}\left[\sin\frac{n\pi}{2}\cos\frac{\pi}{4}+\cos\frac{n\pi}{2}\sin\frac{\pi}{4}-\left(\cos\frac{n\pi}{2}\cos\frac{\pi}{4}-\sin\frac{n\pi}{2}\sin\frac{\pi}{4}\right)\right]\right\}\\
&= \frac{2}{\pi}\left\{\frac{1}{2n-1}\left[\frac{1}{\sqrt{2}}\sin\frac{n\pi}{2}-\frac{1}{\sqrt{2}}\cos\frac{n\pi}{2}+\frac{1}{\sqrt{2}}\cos\frac{n\pi}{2}+\frac{1}{\sqrt{2}}\sin\frac{n\pi}{2}\right]\right.\\
&\left.-\frac{1}{2n+1}\left[\frac{1}{\sqrt{2}}\sin\frac{n\pi}{2}+\frac{1}{\sqrt{2}}\cos\frac{n\pi}{2}-\frac{1}{\sqrt{2}}\cos\frac{n\pi}{2}+\frac{\pi}{\sqrt{2}}\sin\frac{n\pi}{2}\right]\right\}\\
&= \frac{2}{\pi}\left\{\frac{1}{2n-1}.\frac{2}{\sqrt{2}}\sin\frac{n\pi}{2}-\frac{1}{2n+1}.\frac{2}{\sqrt{2}}\sin\frac{n\pi}{2}\right\}\\
&= \frac{4}{\pi\sqrt{2}}\sin\frac{n\pi}{2}\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]=\frac{8}{\pi\sqrt{2}(2n-1)(2n+1)}\sin\frac{n\pi}{2}
\end{aligned}
$$

Therefore $f(x)=\sum\limits_{n=1}^{\infty}\frac{8}{\pi\sqrt{2}(2n-1)(2n+1)}\sin\frac{n\pi}{2}\sin 2nx.$

Root Mean Square Value (RMS Value):

Let $f(x)$ be a function defined in an interval (a,b). Then $\sqrt{\dfrac{\int_a^b (f(x))^2\,dx}{b-a}}$ is called the root mean square or effective value of $f(x)$ and is denoted by $\bar{y}$.

The RMS value is also known as the effective value of the function. The RMS value is used in the theory of mechanical vibrations and in electric circuit theory.

Parseval's Theorem:

If the Fourier series for $f(x)$ in the interval $(c, c+2l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l},$$

then $\displaystyle \frac{1}{2l}\int_c^{c+2l} (f(x))^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2).$

Corollary

(1) The half range cosine series for $f(x)$ in $(0,l)$ is $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\dfrac{n\pi x}{l}$ then $\displaystyle \frac{1}{l}\int_0^{l} (f(x))^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2.$

(2) The half range sine series for $f(x)$ in $(0,l)$ is $f(x) = \sum_{n=1}^{\infty} b_n \sin\dfrac{n\pi x}{l}$ then $\displaystyle \frac{1}{l}\int_0^{l} (f(x))^2\,dx = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2.$

(3) The half range cosine series for $f(x)$ in $(0,\pi)$ is $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ then $\displaystyle \frac{1}{\pi}\int_0^{\pi} (f(x))^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2.$

(4) The half range sine series for $f(x)$ in $(0,\pi)$ is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ then $\displaystyle \frac{1}{\pi}\int_0^{\pi} (f(x))^2\,dx = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2.$

Example 1: Find the Fourier series of periodicity 2π for $f(x) = x^2$, in $-\pi < x < \pi$. Hence show that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots + \text{to } \infty = \frac{\pi^4}{90}$.

Solution: Given the function is an even function. Therefore $b_n = 0$.

The Fourier Cosine series is $f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi}\int\limits_0^{\pi} f(x)dx$ and $a_n = \frac{2}{\pi}\int\limits_0^{\pi} f(x)\cos nx dx$.

To find a_0, a_n :

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int\limits_0^{\pi} x^2 dx \\ &= \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2\pi^2}{3} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int\limits_0^{\pi} x^2 \cos nx dx \\ &= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n} - \frac{2\sin nx}{n^3}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n}\right], \text{ since } \sin 0 = \sin n\pi = 0 \\ &= \frac{4(-1)^n}{n^2} \end{aligned}$$

Therefore the Fourier Cosine series is $x^2 = \frac{\pi^2}{3} + \sum\limits_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos nx$.

Deduction: Use Parseval's Theorem

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}(f(x))^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

$$\Rightarrow \frac{1}{4}.\frac{4\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4} = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} x^4\,dx$$

$$\frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{2}{2\pi}\int_0^{\pi} x^4\,dx$$
$$= \frac{1}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi}$$
$$8\sum_{n=1}^{\infty}\frac{16}{n^4} = \frac{\pi^4}{5} - \frac{\pi^4}{9} = \frac{4\pi^4}{45}$$
$$\sum_{n=1}^{\infty}\frac{16}{n^4} = \frac{\pi^4}{90}.$$

Example 2: Prove that in $0 < x < l$, $x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \dots\right)$ and deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$.

Solution:

The Fourier Cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ and $a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$.

To find a_0, a_n :

$$a_0 = \frac{2}{l}\int_0^{l} x\,dx = \frac{2}{l}\left(\frac{x^2}{2}\right)_0^{l}$$
$$= \frac{2}{l}\left[\frac{l^2}{2} - 0\right] = l.$$
$$a_n = \frac{2}{l}\int_0^{l} x\cos\frac{n\pi x}{l}\,dx$$
$$= \frac{2}{l}\left[x.\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l}$$
$$= \frac{2}{l}\left[\frac{xl\sin\frac{n\pi x}{l}}{n\pi} + \frac{l^2\cos\frac{n\pi x}{l}}{n^2\pi^2}\right]_0^{l}$$

$$
\begin{aligned}
a_n &= \frac{2}{l}\left[\frac{l^2 \sin n\pi}{n\pi} + \frac{l^2 \cos n\pi}{n^2\pi^2} - \left(0 + \frac{l^2 \cos 0}{n^2\pi^2}\right)\right] \\
&= \frac{2}{l}\left[0 + \frac{l^2(-1)^n}{n^2\pi^2} - 0 - \frac{l^2}{n^2\pi^2}\right] \\
&= \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} 0; \text{if } n \text{ is even} \\ \dfrac{-4l}{\pi^2 n^2}; \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

Therefore the Fourier Cosine series is

$$
\begin{aligned}
x &= \frac{l}{2} - \frac{4l}{\pi^2} \sum_{n=\text{odd}} \frac{1}{n^2} \cos \frac{n\pi x}{l} \\
&= \frac{l}{2} - \frac{4l}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \dots\right]
\end{aligned}
$$

Deduction: By Parseval's theorem $\frac{1}{l}\int\limits_0^l (f(x))^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum\limits_{n=1}^{\infty} a_n^2.$

$$
\begin{aligned}
\frac{1}{l}\int\limits_0^l x^2\,dx &= \frac{l^2}{4} + \frac{1}{2}\sum_{n=\text{odd}} \frac{16l^2}{n^4\pi^4} \\
\frac{1}{l}\left[\frac{x^3}{3}\right]_0^l &= \frac{l^2}{4} + \frac{8l^2}{\pi^4}\sum_{n=\text{odd}} \frac{1}{n^4} \\
\frac{l^2}{3} &= \frac{l^2}{4} + \frac{8l^2}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right] \\
&\Rightarrow \frac{8l^2}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right] = \frac{l^2}{3} - \frac{l^2}{4} \\
&\Rightarrow \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{l^2}{12}.\frac{\pi^4}{8l^2} \\
&\Rightarrow \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}.
\end{aligned}
$$

Example 3: Find the half-range sine series for $f(x) = x$ in $0 < x < l$, then show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$.

Solution:

The Fourier sine series is $f(x) = \sum\limits_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

where $b_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$.

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^l x\sin\frac{n\pi x}{l}\,dx\\
&= \frac{2}{l}\left[x.\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^\pi\\
&= \frac{2}{l}\left[-\frac{xl\cos\frac{n\pi x}{l}}{n\pi}+\frac{l^2\sin\frac{n\pi x}{l}}{n^2\pi^2}\right]_0^l\\
&= \frac{2}{l}\left[-\frac{l^2\cos n\pi}{n\pi}+\frac{l^2\sin n\pi}{n^2\pi^2}-\left(-0+\frac{l^2\sin 0}{n^2\pi^2}\right)\right]\\
&= \frac{2}{l}\left[\frac{-l^2(-1)^n}{n\pi}\right]=\frac{2l}{n\pi}(-1)^{n+1}
\end{aligned}$$

Therefore the Fourier sine series is $x = \dfrac{2l}{\pi}\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n}\sin\dfrac{n\pi x}{l}$.

Deduction: By Parseval's theorem

$$\frac{1}{l}\int_0^l (f(x))^2\,dx = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2.$$

$$\begin{aligned}
\frac{1}{l}\int_0^l x^2\,dx &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}\\
\frac{1}{l}\left[\frac{x^3}{3}\right]_0^l &= \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\\
\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{\pi^2}{6}\\
\Rightarrow \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots &= \frac{\pi^2}{6}.
\end{aligned}$$

Example 4: Find the half-range sine series for $f(x) = x$ in $0 < x < \pi$, then show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$.

Solution:

The Fourier sine series is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$.

$$
\begin{aligned}
b_n &= \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx \\
&= \frac{2}{\pi} \left[x. \left(\frac{-\cos nx}{n} \right) - (1) \left(-\frac{\sin nx}{n^2} \right) \right]_0^{\pi} \\
&= \frac{2}{\pi} \left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi} \\
&= \frac{2}{\pi} \left[-\frac{\pi \cos n\pi}{n} + \frac{\sin n\pi}{n^2} - \left(-0 + \frac{\sin 0}{n^2} \right) \right] \\
&= \frac{2}{\pi} \left[\frac{-\pi(-1)^n}{n} \right] = \frac{2}{n}(-1)^{n+1}
\end{aligned}
$$

Therefore the Fourier sine series is $x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$.

Deduction: By Parseval's theorem

$$\frac{1}{\pi} \int_0^{\pi} (f(x))^2 \, dx = \frac{1}{2} \sum_{n=1}^{\infty} b_n^2.$$

$$
\begin{aligned}
\frac{1}{\pi} \int_0^{\pi} x^2 \, dx &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{4}{n^2} \\
\frac{1}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} &= 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \\
&\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}.
\end{aligned}
$$

Example 5: Find the half range cosine series for $f(x) = x(\pi - x)$, in $0 < x < \pi$. Deduce that $\dfrac{1}{1^4} + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \ldots = \dfrac{\pi^4}{90}$.

Solution:

The Fourier Cosine series is $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \dfrac{2}{\pi}\int\limits_0^{\pi} f(x)dx$ and $a_n = \dfrac{2}{\pi}\int\limits_0^{\pi} f(x)\cos nx dx$.

To find a_0, a_n :

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int\limits_0^{\pi} x(\pi - x)\, dx \\
&= \frac{2}{\pi}\left(\frac{\pi x^2}{2} - \frac{x^3}{3}\right)_0^{\pi} \\
&= \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{\pi^2}{3}.
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int\limits_0^{\pi} (\pi x - x^2)\cos nx\, dx \\
&= \frac{2}{\pi}\left[(\pi x - x^2).\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[(\pi x - x^2).\frac{\sin nx}{n} + (\pi - 2x).\frac{\cos nx}{n^2} + \frac{2\sin nx}{n^3}\right]_0^{\pi} \\
&= \frac{2}{\pi}\Bigg[(\pi^2 - \pi^2).\frac{\sin n\pi}{n} + (\pi - 2\pi).\frac{\cos n\pi}{n^2} + \frac{2\sin n\pi}{n^3} \\
&- \left(0 + \frac{\pi\cos 0}{n^2} + \frac{2\sin 0}{n^3}\right)\Bigg] \\
&= \frac{2}{\pi}\left[0 + \frac{(-\pi)(-1)^n}{n^2} + 0 - \left(0 + \frac{\pi}{n^2} + 0\right)\right] \\
&= -\frac{2}{n^2}\left[(-1)^n + 1\right] = \begin{cases} 0; \text{if } n \text{ is odd} \\ \dfrac{-4}{n^2}; \text{if } n \text{ is even} \end{cases}
\end{aligned}$$

Therefore the Fourier cosine series is $x(\pi - x) = \dfrac{\pi^2}{6} - 4\sum\limits_{n=\text{even}} \dfrac{1}{n^2}\cos nx$.

Deduction: By Parseval's theorem

$$\frac{1}{\pi}\int_0^{\pi}(f(x))^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2.$$

$$\frac{1}{\pi}\int_0^{\pi}(\pi x - x^2)^2\,dx = \frac{\pi^4}{36} + \frac{1}{2}\sum_{n=\text{even}}\frac{16}{n^4}$$

$$\frac{1}{\pi}\int_0^{\pi}\left[\pi^2x^2 - 2\pi x^3 + x^4\right]\,dx = \frac{\pi^4}{36} + 8\sum_{n=\text{even}}\frac{1}{n^4}$$

$$\frac{1}{\pi}\left[\frac{\pi^2x^3}{3} - \frac{2\pi x^4}{4} + \frac{x^5}{5}\right]_0^{\pi} = \frac{\pi^4}{36} + 8\left[\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \dots\right]$$

$$\frac{\pi^4}{30}[10 - 15 + 6] = \frac{\pi^4}{36} + \frac{8}{2^4}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right]$$

$$\frac{1}{2}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right] = \frac{\pi^4}{30} - \frac{\pi^4}{36}$$

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}.$$

Example 6: Find the half-range cosine series for $f(x) = (x-1)^2$ in $0 < x < 1$. Hence find the sum of the series $\sum\limits_{n=1}^{\infty}\frac{1}{n^4}$.

Solution: Here $l = 1$.

Therefore the half range cosine series is $f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos n\pi x$

where $a_0 = 2\int\limits_0^1 f(x)\,dx$, $a_n = 2\int\limits_0^1 f(x)\cos n\pi x\,dx$.

$$a_0 = \int_0^1 (x-1)^2\,dx = 2\left[\frac{(x-1)^3}{3}\right] = \frac{2}{3}.$$

$$a_n = \int_0^1 (x-1)^2 \cos n\pi x\,dx$$

$$= 2\left[(x-1)^2.\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1).\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$a_n = 2\left[\frac{(x-1)^2 \sin n\pi x}{n\pi} + \frac{2(x-1)\cos n\pi x}{n^2\pi^2} - \frac{2\sin n\pi x}{n^3\pi^3}\right]_0^1$$
$$= 2\left[0 + 0 - \frac{2\sin n\pi}{n^3\pi^3} - \left(0 - \frac{2\cos 0}{n^2\pi^2} - 0\right)\right] = \frac{4}{n^2\pi^2}.$$

Therefore the half-range cosine series is $(x-1)^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos n\pi x$.

Deduction: By Parseval's theorem

$$\frac{1}{l}\int_0^l (f(x))^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2.$$

$$\int_0^1 (x-1)^4\,dx = \frac{1}{4}.\frac{4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}$$
$$\left[\frac{(x-1)^5}{5}\right]_0^1 = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$
$$\frac{1}{5} = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$
$$\frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}$$
$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Example 7: Find the RMS value of $f(x) = 1 - x$ in $0 < x < 1$.

Solution:

$$\text{RMS value of } f(x) = \sqrt{\frac{\int_a^b (f(x))^2 dx}{b-a}}$$
$$= \sqrt{\int_0^1 (1-x)^2 dx}$$
$$= \sqrt{\left(\frac{(1-x)^3}{-3}\right)_0^1}$$
$$= \sqrt{\frac{1}{3}}.$$

Example 8: Find the RMS value of $f(x) = x - x^2$, in $-1 < x < 1$.

Solution:

$$\begin{aligned}
\text{RMS value of } f(x) &= \sqrt{\frac{\int_{-1}^{1} (x - x^2)^2 dx}{2}} \\
&= \sqrt{\frac{1}{2} \int_{-1}^{1} (x^2 + x^4 - 2x^3) dx} \\
&= \sqrt{\frac{1}{2} \times 2 \int_{0}^{1} (x^2 + x^4) dx} \text{ since } x^3 \text{ is an odd function} \\
&= \sqrt{\frac{8}{15}}.
\end{aligned}$$

Example 9: Find the RMS value of $f(x) = x$ in $-1 \leq x \leq 1$

Solution:

$$\begin{aligned}
\text{RMS value of } f(x) &= \sqrt{\frac{\int_{-1}^{1} x^2 dx}{2}} \\
&= \sqrt{\frac{2\int_{0}^{1} x^2 dx}{2}} \text{ since } x^2 \text{ is an even function} \\
&= \frac{1}{\sqrt{3}}.
\end{aligned}$$

Harmonic Analysis

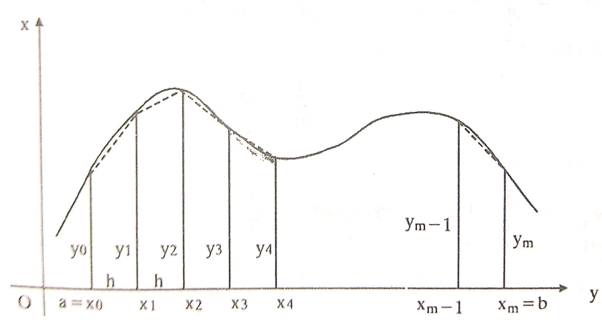
In many practical problems, the function $f(x)$ is represented in a tabular form obtained from an experiment. In such cases approximate values of the Fourier coefficients are obtained by numerical methods of integration.

Definition: The process of finding the Fourier series for a function given by numerical value is known as harmonic analysis.

Trapezoidal rule: If $f(x)$ is defined in (a, b) then

$$\int_a^b f(x)\,dx = h\left[\frac{1}{2}(y_0 + y_m) + (y_1 + y_2 + y_3 + \ldots + y_{m-1})\right]$$

where $h = \dfrac{b-a}{m}$ is the width of each sub-interval, m is the number of sub-intervals and values at each point of division is $y_i = f(x_i)$, $i = 0, 1, 2, \ldots, m$.



Because of periodicity, if we assume $y_0 = y_m$ then

$$\int_a^b f(x)\,dx = h\left[y_0 + y_1 + y_2 + y_3 + \ldots + y_{m-1}\right]$$

(or)

$$\int_a^b f(x)\,dx = h\left[y_1 + y_2 + y_3 + \ldots + y_{m-1} + y_m\right]$$

Suppose $f(x)$ be considered in an interval of length 2π, $0 < x < 2\pi$.

Then $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx + \sum\limits_{n=1}^{\infty} b_n \sin nx$

Divide the interval into m equal parts with $m+1$ points $x_0, x_1, \ldots, x_m$ with $h = \dfrac{2\pi}{m}$.

Let $y_i = f(x_i),\ i = 0, 1, 2, \ldots, m$ then

$$a_0 = \frac{1}{2}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}.\frac{2\pi}{m}\,[y_0 + y_1 + \ldots + y_{m-1}]$$

$$= \frac{2}{m}\sum_{i=0}^{m-1} y_i \text{ or } = 2[\text{mean value of } f(x)]$$

$$a_n = \frac{1}{2}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}.\frac{2\pi}{m}\sum_{i=0}^{m-1} y_i\cos nx_i$$

$$= \frac{2}{m}\sum_{i=0}^{m-1} y_i\cos nx_i \text{ or } = 2[\text{mean value of } f(x)\cos nx]$$

$$b_n = \frac{1}{2}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}.\frac{2\pi}{m}\sum_{i=0}^{m-1} y_i\sin nx_i$$

$$= \frac{2}{m}\sum_{i=0}^{m-1} y_i\sin nx_i \text{ or } = 2[\text{mean value of } f(x)\sin nx]; n \geq 1.$$

Note:

(1) Usually take $m = 6$ or 12.

(2) In a Fourier expansion, the term $a_1\cos x + b_1\sin x$ is called the first harmonic or fundamental harmonic of Fourier expansion of $f(x)$, the term $a_2\cos 2x + b_2\sin 2x$ is called the second harmonic and so on.

Example 1: Determine the first two harmonics of the Fourier series for the following data:

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$
y	1.98	1.30	1.05	1.30	-0.88	-0.25

Solution: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{2} a_n \cos nx + \sum_{n=1}^{2} b_n \sin nx$ where $a_0 = \frac{2}{m}\sum y$, $a_n = \frac{2}{m}\sum y\cos nx$ and $b_n = \frac{2}{m}\sum y \sin nx,\ n = 1, 2.$

x	y	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$
0	1.98	1	0	1	0	1.98	0	1.98	0
$\frac{\pi}{3}$	1.3	0.5	0.866	-0.5	0.866	0.65	1.126	-0.65	1.126
$\frac{2\pi}{3}$	1.05	-0.5	0.866	-0.5	-0.866	-0.525	0.909	-0.525	-0.909
π	1.3	-1	0	1	0	-1.3	0	1.3	0
$\frac{4\pi}{3}$	-0.88	-0.5	-0.866	0.5	0.866	0.44	0.762	0.44	-0.762
$\frac{5\pi}{3}$	-0.25	0.5	-0.866	0.5	-0.866	-0.125	0.217	0.125	0.217
	4.5					1.12	3.014	-2067	-0.328

Now $a_0 = \frac{2}{6}(4.5) = 1.5$

$a_1 = \frac{2}{6}\sum y\cos x = \frac{1}{3}(1.12) = 0.3733$

$a_2 = \frac{2}{6}\sum y\cos 2x = \frac{1}{3}(2.67) = 0.89$

$b_1 = \frac{2}{6}\sum y\sin x = \frac{1}{3}(3.014) = 1.0048$

$b_2 = \frac{2}{6}\sum y\sin 2x = \frac{1}{3}(-.328) = -0.1093$

Hence $f(x) = 0.75 + 0.3733\cos x + 1.0048\sin x + 0.89\cos 2x - 0.1093\sin 2x.$

Example 2: Find the Fourier series upto third harmonic from the following data:

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y	1	1.4	1.9	1.7	1.5	1.2	1

Solution: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{3} a_n \cos nx + \sum_{n=1}^{3} b_n \sin nx$ where $a_0 = \frac{2}{m}\sum y$, $a_n = \frac{2}{m}\sum y \cos nx$ and $b_n = \frac{2}{m}\sum y \sin nx,\ n = 1, 2, 3.$

x	y	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$	$y\cos 3x$	$y\sin 3x$
0	1	1	0	1	0	1	0
$\frac{\pi}{3}$	1.4	0.7	1.212	-0.7	1.212	-1.4	0
$\frac{2\pi}{3}$	1.9	-0.95	1.65	-0.95	-1.65	1.9	0
π	1.7	-1.7	0	1.7	0	-1.7	0
$\frac{4\pi}{3}$	1.5	-0.75	-1.299	0.75	1.299	1.5	0
$\frac{5\pi}{3}$	1.2	0.6	-1.039	-0.6	-1.039	-1.2	0
	8.7	-1.1	0.519	-0.3	-0.173	0.1	0

Now $a_0 = \frac{2}{6}(8.7) = 2.9$

$a_1 = \frac{2}{6}\sum y \cos x = \frac{1}{3}(-1.1) = -0.3667$

$a_2 = \frac{2}{6}\sum y \cos 2x = \frac{1}{3}(-0.3) = -0.1$

$a_3 = \frac{2}{6}\sum y \cos 3x = \frac{1}{3}(0.1) = 0.033$

$b_1 = \frac{2}{6}\sum y \sin x = \frac{1}{3}(0.519) = 0.173$

$b_2 = \frac{2}{6}\sum y \sin 2x = \frac{1}{3}(-0.173) = -0.0577$

$b_3 = 0$

Hence $f(x) = 1.45 - 0.3667\cos x + 0.173\sin x - 0.1\cos 2x - 0.0577\sin 2x + 0.033\cos 3x.$

Example 3: Find the Fourier sine series upto third harmonic for the function $y = f(x)$ in $(0, \pi)$ from the table:

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π
y	2.34	2.2	1.6	0.83	0.51	0.88	2.34

Solution: Let $f(x) = \sum_{n=1}^{3} b_n \sin nx$ where $b_n = \frac{2}{m} \sum f(x) \sin nx,\ n = 1, 2, 3.$

x	y	$\sin x$	$\sin 2x$	$\sin 3x$	$y \sin x$	$y \sin 2x$	$y \sin 3x$
0	2.34	0	0	0	0	0	0
$\frac{\pi}{6}$	2.2	0.5	0.866	1	1.1	1.905	2.2
$\frac{2\pi}{6}$	1.6	0.866	0.866	0	1.386	1.386	0
$\frac{3\pi}{6}$	0.83	1	0	-1	0.83	0	-0.83
$\frac{4\pi}{6}$	0.51	0.866	-0.866	0	0.442	-0.442	0
$\frac{5\pi}{6}$	0.88	0.5	-0.866	1	0.44	-0.762	0.88
					4.198	2.087	2.25

Now $b_1 = \frac{2}{6} \sum y \sin x = \frac{1}{3}(4.198) = 1.399$

$b_2 = \frac{2}{6} \sum y \sin 2x = \frac{1}{3}(2.087) = 0.696$

$b_3 = \frac{2}{6} \sum y \sin 2x = \frac{1}{3}(2.25) = 0.75$

Hence $f(x) = 1.399 \sin x + 0.696 \sin 2x + 0.75 \sin 3x$.

Example 4: Find the Fourier cosine series of $y = f(x)$ in $(0, \pi)$ upto the third harmonic from the following table:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$
y	0	9.2	14.4	17.8	17.3	11.7

Solution: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{3} a_n \cos nx$

where $a_0 = \frac{2}{m} \sum y, a_n = \frac{2}{m} \sum y \cos nx,\ n = 1, 2, 3.$

x	y	$\cos x$	$\cos 2x$	$\cos 3x$	$y\cos x$	$y\cos 2x$	$y\cos 3x$
0	0	1	1	1	0	0	0
$\frac{\pi}{6}$	9.2	0.866	0.5	0	7.967	4.6	0
$\frac{\pi}{3}$	14.4	0.5	-0.5	-1	7.2	-7.2	-14.4
$\frac{\pi}{2}$	17.8	0	-1	0	0	-17.8	0
$\frac{2\pi}{3}$	17.3	-0.5	-0.5	1	-8.65	-8.65	17.3
$\frac{5\pi}{6}$	11.7	-0.866	0.5	0	-10.132	5.85	0
	70.4				-3.615	-23.2	2.9

Now $a_0 = \frac{2}{6}(70.4) = 23.467$

$a_1 = \frac{2}{6}\sum y\cos x = \frac{1}{3}(-3.615) = -1.205$

$a_2 = \frac{2}{6}\sum y\cos 2x = \frac{1}{3}(-23.2) = -7.733$

$a_2 = \frac{2}{6}\sum y\cos 3x = \frac{1}{3}(2.9) = 0.9667$

Hence $f(x) = 11.734 - 1.205\cos x - 7.733\cos 2x + 0.9667\cos 3x$.

Example 5: Find the Fourier series as far as the second harmonic to represent the function given below:

x	0	1	2	3	4	5
$f(x)$	9	18	24	28	26	20

Solution:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{2} a_n \cos n\theta + \sum_{n=1}^{2} b_n \sin n\theta \quad$ where $\theta = \frac{\pi x}{l}$,

$a_0 = \frac{2}{m}\sum f(x), a_n = \frac{2}{m}\sum f(x)\cos n\theta, b_n = \frac{2}{m}\sum f(x)\sin n\theta,\ n = 1, 2.$

Here $2l = 6 \Rightarrow l = 3$.

Therefore $\theta = \frac{\pi x}{3}$.

x	$\theta = \frac{\pi x}{3}$	$f(x)$	$f(x)\cos\theta$	$f(x)\cos 2\theta$	$f(x)\sin\theta$	$f(x)\sin 2\theta$
0	0	9	9	9	0	0
1	$\frac{\pi}{3}$	18	9	-9	15.588	15.588
2	$\frac{2\pi}{3}$	24	-12	-12	20.784	-20.784
3	π	28	-28	28	0	0
4	$\frac{4\pi}{3}$	26	-13	-13	-22.516	22.516
5	$\frac{5\pi}{3}$	20	10	-10	-17.32	-17.32
		125	-25	-7	-3.464	0

Now $a_0 = \frac{2}{6}(125) = 41.667$

$a_1 = \frac{2}{6}\sum f(x)\cos x = \frac{1}{3}(-25) = -8.333$

$a_2 = \frac{2}{6}\sum f(x)\cos 2x = \frac{1}{3}(-7) = -2.333$

$b_1 = \frac{2}{6}\sum f(x)\sin x = \frac{1}{3}(-3.464) = -1.155$

$b_2 = \frac{2}{6}\sum f(x)\sin 2x = 0$

Hence $f(x) = 20.834 - 8.333\cos\theta - 1.155\sin\theta - 2.333\cos 2\theta$.

(or) $f(x) = 20.834 - 8.333\cos\frac{\pi x}{3} - 1.155\sin\frac{\pi x}{3} - 2.333\cos\frac{2\pi x}{3}$.

Example 6: Find the first three harmonics in the Fourier Cosine series of $y = f(x)$ in (0,6) using the following table

x	0	1	2	3	4	5
$f(x)$	4	8	15	7	6	2

Solution:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{3} a_n \cos n\theta \quad$ where $\theta = \frac{\pi x}{l}$,

$a_0 = \frac{2}{m}\sum f(x), a_n = \frac{2}{m}\sum f(x)\cos n\theta,\ n = 1, 2, 3.$

Here $l = 6$.

x	$\theta = \frac{\pi x}{6}$	$f(x)$	$f(x)\cos\theta$	$f(x)\cos 2\theta$	$f(x)\cos 3\theta$
0	0	4	4	4	4
1	$\frac{\pi}{6}$	8	6.928	4	0
2	$\frac{\pi}{3}$	15	7.5	-7.5	-15
3	$\frac{\pi}{2}$	17	0	-7	0
4	$\frac{2\pi}{3}$	6	-3	-3	6
5	$\frac{5\pi}{6}$	2	-1.732	-1	0
		42	13.696	-8.5	-5

Now $a_0 = \frac{2}{6}(42) = 14$, $a_1 = \frac{2}{6}(13.696) = 4.5653$, $a_2 = \frac{2}{6}(-8.5) = -2.8333$ $a_3 = \frac{2}{6}(-5) = -1.6667$

Therefore $f(x) = 7 + 4.5653\cos\theta - 2.8333\cos 2\theta - 1.6667\cos 3\theta$ where $\theta = \frac{\pi x}{6}$.

Example 7: The values of x and the corresponding values of $f(x)$ over a period T are given below. Show that $f(x) = 0.75 + 0.37\cos\theta + 1.004\sin\theta$ where $\theta = \frac{2\pi x}{T}$.

x	0	$\frac{T}{6}$	$\frac{T}{3}$	$\frac{T}{2}$	$\frac{2T}{3}$	$\frac{5T}{6}$	T
$f(x)$	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Solution:

x	$\theta = \frac{2\pi x}{T}$	$f(x)$	$f(x)\cos\theta$	$f(x)\sin\theta$
0	0	1.98	1.98	0
$\frac{T}{6}$	$\frac{\pi}{3}$	1.30	0.65	1.1258
$\frac{T}{3}$	$\frac{2\pi}{3}$	1.05	-0.525	0.9093
$\frac{T}{2}$	π	1.30	-1.3	0
$\frac{2T}{3}$	$\frac{4\pi}{3}$	-0.88	0.44	0.762
$\frac{5T}{6}$	$\frac{5\pi}{6}$	-0.25	-0.125	0.2165
		4.6	1.12	3.013

Let $f(x) = \dfrac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta$

$a_0 = \dfrac{2}{m}\sum f(x), a_1 = \dfrac{2}{m}\sum f(x)\cos\theta, b_1 = \dfrac{2}{m}\sum f(x)\sin\theta.$

Now $a_0 = \dfrac{2}{6}(4.6) = 1.5,\ a_1 = \dfrac{2}{6}(1.12) = 0.37,\ b_1 = \dfrac{2}{6}(3.013) = 1.005.$

Therefore $f(x) = 0.75 + 0.37\cos\theta + 1.005\sin\theta$ where $\theta = \dfrac{2\pi x}{T}$.

Example 8: Compute the first three harmonics of the Fourier series for $f(x)$ from the following data:

x	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$f(x)$	2.34	3.01	3.68	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

Solution: Let $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{3} a_n \cos nx + \sum_{n=1}^{3} b_n \sin nx$ where $a_0 = \dfrac{2}{m}\sum f(x)$, $a_n = \dfrac{2}{m}\sum f(x)\cos nx$ and $b_n = \dfrac{2}{m}\sum f(x)\sin nx,\ n = 1, 2, 3.$

x	$f(x)$	$f(x)\cos x$	$f(x)\sin x$	$f(x)\cos 2x$	$f(x)\sin 2x$	$f(x)\cos 3x$	$f(x)\sin 3x$
30°	2.34	2.026	1.17	1.17	2.026	0	2.34
60°	3.01	1.505	2.607	-1.505	2.607	-3.01	0
90°	3.68	0	3.68	-3.68	0	0	-3.68
120°	4.15	-2.075	3.594	-2.075	-3.594	4.15	0
150°	3.69	-3.196	1.845	1.845	-3.196	0	3.69
180°	2.20	-2.20	0	2.20	0	-2.20	0
210°	0.83	-0.719	-0.415	0.415	0.719	0	-0.83
240°	0.51	-0.255	-0.442	-0.255	0.442	0.51	0
270°	0.88	0	-0.88	-0.88	0	0	0.88
300°	1.09	0.545	-0.944	-0.545	-0.944	-1.09	0
330°	1.19	1.031	-0.595	0.595	-1.031	0	-1.19
360°	1.64	1.64	0	1.64	0	1.64	0
	25.21	-1.698	9.620	-1.075	-2.971	0	1.21

Now $a_0 = \dfrac{2}{12}(25.21) = 4.202,\ a_1 = \dfrac{2}{12}(-1.698) = -0.283,$

$a_2 = \dfrac{2}{12}(-1.075) = -0.179,\ a_3 = 0,\ b_1 = \dfrac{2}{12}(9.620) = 1.603,$

$b_2 = \frac{2}{12}(-2.971) = -0.495$, $b_3 = \frac{2}{12}(1.21) = 0.202$. Therefore

$f(x) = 2.101 - 0.283\cos x + 1.603\sin x - 0.179\cos 2x - 0.495\sin 2x + 0.202\sin 3x.$

Example 9: Compute the first three harmonics of the Fourier series of $f(x)$ given by the following table:

x	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$
$f(x)$	6.824	7.976	8.026	7.204	5.676	3.674	1.764	0.552	0.262	0.904	2.492	4.736

Solution: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{3} a_n \cos nx + \sum_{n=1}^{3} b_n \sin nx$ where $a_0 = \frac{2}{m}\sum f(x)$, $a_n = \frac{2}{m}\sum f(x)\cos nx$ and $b_n = \frac{2}{m}\sum f(x)\sin nx$, $n = 1, 2, 3$.

x	$f(x)$	$f(x)\cos x$	$f(x)\sin x$	$f(x)\cos 2x$	$f(x)\sin 2x$	$f(x)\cos 3x$	$f(x)\sin 3x$
$0°$	6.824	6.824	0	6.824	0	6.824	0
$30°$	7.976	6.907	3.988	3.988	6.907	0	7.976
$60°$	8.026	4.013	6.950	-4.013	6.950	-8.026	0
$90°$	7.204	0	7.204	-7.204	0	0	-7.24
$120°$	5.676	-2.838	4.916	-2.838	-4.918	5.676	0
$150°$	3.674	-3.182	1.187	1.837	-3.182	0	3.674
$180°$	1.764	-1.764	0	1.764	0	-1.764	0
$210°$	0.552	-0.478	-0.276	0.276	0.478	0	-0.552
$240°$	0.262	0.131	-0.227	-0.131	0.227	0.262	0
$270°$	0.904	0	-0.904	-0.904	0	0	0.904
$300°$	2.492	1.246	-2.158	-1.246	-2.158	-2.492	0
$330°$	4.736	4.102	-2.368	2.368	-4.102	0	-4.736
	50.09	14.699	18.962	0.721	0.204	18.962	0.062

Now $a_0 = \frac{2}{12}(50.09) = 8.348$, $a_1 = \frac{2}{12}(14.699) = 2.450$,

$a_2 = \frac{2}{12}(0.721) = 0.12$, $a_3 = \frac{2}{12}(0.48) = 0.08$, $b_1 = \frac{2}{12}(18.962) = 3.16$,

$b_2 = \frac{2}{12}(0.204) = 0.034$, $b_3 = \frac{2}{12}(0.062) = 0.010$. Therefore

$f(x) = 4.174 + 2.45\cos x + 3.16\sin x + 0.12\cos 2x + 0.034\sin 2x + 0.08\cos 3x + 0.01\sin 3x$.