

# SEQUENCES AND SERIES

1

CHAPTER

## Sequences

**Definition.** A set of numbers,  $a_1, a_2, a_3, \dots, a_n, \dots$  such that to each positive integer  $n$ , there corresponds a number  $a_n$  of the set, is called a *sequence* and it is denoted by  $\{a_n\}$ .

In a *sequence*, the elements are arranged in a definite order while in a *set* order of the elements may be in any order. Thus, a sequence and the set of natural numbers  $N$  have one to one correspondence.

**Examples.**

1. If  $a_n = \frac{1}{n}$ , the sequence is  $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$
2. If  $a_n = n^3$ , the sequence is  $1^3, 2^3, 3^3, \dots, n^3, \dots$
3. If  $a_n = k$ , the sequence is  $k, k, k, \dots$
4. If  $a_n = (-1)^n$ , then the sequence is  $-1, 1, -1, 1, \dots, (-1)^n, \dots$
5. If  $a_n = 1$  for  $n$  even  
and  $= -1$  for  $n$  odd, then the sequence is  
 $-1, 1, -1, 1, \dots$  which is oscillating.

**Limit of a sequence or Convergence of a sequence.** Let  $\{a_n\}$  be a sequence and  $l$ , a real number. We say that  $\{a_n\}$  tends to the limit  $l$  or converges to  $l$  if given any positive number,  $\varepsilon$ , however small it may be, there exists a positive integer  $N$  (depending upon  $\varepsilon$ ) such that

$$|a_n - l| < \varepsilon \text{ for all } n \geq N$$

We write  $\lim_{n \rightarrow \infty} a_n = l$

Here  $l$  is the *limit* of the sequence  $\{a_n\}$ .

**Note.**

$$|a_n - l| < \varepsilon \text{ implies } l - \varepsilon < a_n < l + \varepsilon$$

**Bounded Sequence.** A sequence  $\{a_n\}$  is said to be *bounded above* if there exists a real number  $M$  such that  $a_n \leq M$  for all  $n$ .

If such a number  $M$  does not exist, the  $\{a_n\}$  is *unbounded above*.

A sequence  $\{a_n\}$  is said to be *bounded below* if there exists a real number  $m$  such that  $a_n \geq m$  for all  $n$ . If such a 'm' does not exist, then the sequence  $\{a_n\}$  is *unbounded below*.

**Definition.** A sequence  $\{a_n\}$  is said to be *bounded* if it is bounded above and bounded below. In other words, there exists two real numbers  $m$  and  $M$  such that  $m \leq a_n \leq M$  for all  $n (\in N)$ .

**Divergent Sequence.** A sequence  $\{a_n\}$  is said to *diverge to  $+\infty$*  if given any positive number  $M$ , however large it may be, there exists a positive integer  $N$  such that

$$a_n > M \text{ for all } n \geq N$$

We write  $\lim_{n \rightarrow \infty} a_n = +\infty$

A sequence  $\{a_n\}$  is said to *diverge to  $-\infty$*  if given any positive number  $M$ , however large it may be, there exists a positive integer  $N$  such that

$$a_n < -M \text{ for all } n \geq N$$

We write  $\lim_{n \rightarrow \infty} a_n = -\infty$

**Monotonic Sequences:** A sequence  $\{a_n\}$  is said to be *monotonically increasing* if  $a_n \leq a_{n+1}$  for all  $n$ .

That is if  $a_1 \leq a_2 \leq a_3 \leq a_4 \dots \leq a_n \leq a_{n+1} \leq \dots$

A sequence  $\{a_n\}$  is said to be *monotonically decreasing* if  $a_n \geq a_{n+1}$  for all  $n$ .

That is, if  $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq a_{n+1} \geq \dots$  A sequence which is either monotonically increasing or decreasing is called a *monotonic sequence*.

**Oscillatory Sequences :** A sequence  $\{a_n\}$  is said to be *oscillatory* (or oscillating) if it does not converge and does not diverge to  $+\infty$  or  $-\infty$ .

A sequence is said to oscillate *finitely* if it is bounded and is an *oscillatory sequence*.

A sequence is said to oscillate *infinitely* if it is not bounded and is an oscillatory sequence.

**Theorems.** The following theorems can be assumed (without proof).

1.  $\lim (a_n \pm b_n) = \lim a_n \pm \lim b_n$

2.  $\lim (a_n \cdot b_n) = (\lim a_n) (\lim b_n)$

3.  $\lim \left( \frac{1}{a_n} \right) = \frac{1}{\lim a_n}$

where  $\{a_n\}, \{b_n\}$  are convergent sequences.

4. If  $\{a_n\}$  and  $\{b_n\}$  are convergent sequences, then  $\{a_n \pm b_n\}$  is also a convergent sequence.

5. A sequence  $\{a_n\}$  cannot converge to two distinct limits.

6. If  $\{a_n\}$  converges to 'a' and  $\{b_n\}$  converges to 'b' then  $\{a_n b_n\}$  converges to  $ab$ .

7. If  $\{a_n\}$  converges to 'a' and  $\{b_n\}$  converges to 'b' ( $\neq a$ ), then  $\left\{ \frac{a_n}{b_n} \right\}$  converges to  $\frac{a}{b}$ .

8. A monotonic increasing sequence which is bounded above converges.

9. A monotonic increasing sequence which is not bounded above diverges to  $+\infty$ .

10. A monotonic decreasing sequence which is bounded below converges.

11. A monotonic decreasing sequence which is unbounded below diverges to  $-\infty$ .

12. If a sequence  $\{a_n\}$  of positive terms converges to 'l' then  $l \geq 0$ .

13. If  $|x| < 1$  then  $\lim_{n \rightarrow \infty} x^n = 0$

14. If  $x > 1$  then  $\lim_{n \rightarrow \infty} x^n = \infty$

15. A monotonic sequence always tends to a limit, finite or infinite.

**Example 1.** Show that the sequence (i)  $\left\{ \frac{1}{n} \right\}$  (ii)  $\left\{ \frac{n+1}{2n+7} \right\}$

(iii)  $\{a_n\}$  where  $a_n = 3 + \frac{(-1)^n}{n}$  are convergent.

**Solution.** (i)  $a_n = \frac{1}{n}$  and  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

We will apply the definition of convergence and find the existence of  $N$ .

For a given positive number  $\varepsilon$ ,

$$\left| \frac{1}{n} - 0 \right| < \varepsilon \text{ for all } n \geq N.$$

i.e.,  $\frac{1}{n} < \varepsilon$  for all  $n \geq N$

i.e., for  $n > \frac{1}{\varepsilon}$ , this is possible.

Therefore select  $N$ , a positive integer, such that  $N > \frac{1}{\varepsilon}$ .

Then for all  $n \geq N$ ,  $\left| \frac{1}{n} - 0 \right| < \varepsilon$ .

$\therefore \left\{ \frac{1}{n} \right\}$  converges to zero.

$$(ii) \text{ Let } a_n = \frac{n+1}{2n+7} = \frac{1 + \frac{1}{n}}{2 + \frac{7}{n}}$$

This tends to  $\frac{1}{2}$  as  $n \rightarrow \infty$

Let us find the existence of  $N$ .

$$\begin{aligned} |a_n - l| &= \left| \frac{n+1}{2n+7} - \frac{1}{2} \right| = \left| \frac{-5}{2(2n+7)} \right| \\ &\left| a_n - \frac{1}{2} \right| < \varepsilon \text{ implies } \frac{5}{2(2n+7)} < \varepsilon \end{aligned}$$

$$\text{i.e., } 2n+7 > \frac{5}{2\varepsilon}$$

$$\text{i.e., } n > \frac{5}{4\varepsilon} - \frac{7}{2}$$

$\therefore$  For all  $n \geq N$  where  $N > \frac{5}{4\varepsilon} - \frac{7}{2}$ ,  $\left| a_n - \frac{1}{2} \right| < \varepsilon$

Hence  $\{a_n\} \rightarrow \frac{1}{2}$  and it is convergent.

$$(iii) |a_n - 3| = \left| 3 + \frac{(-1)^n}{n} - 3 \right| = \left| \frac{1}{n} \right| \text{ since, here } l = 3$$

$$|a_n - 3| < \varepsilon \text{ implies } \frac{1}{n} < \varepsilon \text{ i.e., } n > \frac{1}{\varepsilon}$$

We find a positive integer  $N > \frac{1}{\varepsilon}$  such that for all  $n \geq N$ , the sequence is convergent.

**Example 2.** The sequences  $\{n^2\}$ ,  $\{-n^2\}$ ,  $\{3n+2\}$ ,  $\{n\}$  are examples of divergent sequences since  $N$  cannot be found out such that  $|a_n - l| < \varepsilon$  for  $n \geq N$ .  $\{(-1)^n\}$ ,  $\{4(-1)^n\}$  are examples of oscillating sequences.

**Example 3.** The sequence  $\{n\}$ ,  $\{n^2\}$ ,  $\{2n+7\}$  are examples of monotonically increasing sequences which are unbounded above.

The sequences  $\left\{\frac{1}{n}\right\}$ ,  $\left\{\frac{1}{2n+3}\right\}$  are examples of monotonically decreasing sequences which are bounded below.

**Solution.** Let  $a_n = n$ ; then  $a_{n+1} - a_n = (n+1) - n = 1$

i.e.,  $a_{n+1} - a_n = 1 > 0$

$a_{n+1} > a_n$  for all  $n$ .

Hence  $\{a_n\}$  is monotonically increasing.

Since no  $M$  can be found out such that  $n < M$  the sequence is not bounded above.  
Similarly all the other problems can be worked out.

**Example 4.** Prove  $\left\{\frac{n+1}{2n+3}\right\}$  is monotonically increasing.

**Solution.**  $a_{n+1} - a_n = \frac{(n+1)+1}{2(n+1)+3} - \frac{n+1}{2n+3} = \frac{1}{(2n+5)(2n+3)} > 0$  for all  $n$   
 $\therefore a_{n+1} > a_n$  for all  $n$ .

$\therefore \{a_n\}$  is monotonically increasing (bounded above by  $\frac{1}{2}$ )

### EXERCISE 1

1. Show that the following sequences  $\{a_n\}$  is convergent where  $a_n$  is given by

$$\begin{array}{llll} ((i)) \frac{n+1}{n} & ((ii)) x^n \text{ where } x > 0 & ((iii)) \frac{2n+3}{3n+7} & ((iv)) \frac{1}{n(n+1)} \\ ((v)) n \tan\left(\frac{1}{n}\right) & ((vi)) \left(1 + \frac{1}{n}\right)^n & ((vii)) \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} & \\ ((viii)) \frac{1}{n^2+1} & ((ix)) \frac{an^2+bn+c}{\alpha n^2+\beta n+\gamma} & & \end{array}$$

2. Show that the following sequences  $\{a_n\}$  where  $a_n$  is given below are divergent or oscillating

$$\begin{array}{llll} ((i)) \frac{1+n^2}{n} & ((ii)) n + \frac{(-1)^n}{n^2} & ((iii)) 3 + (-1)^n & ((iv)) n(-1)^n \\ ((v)) (-1)^n & ((vi)) 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} & ((vii)) n^3 + n & \end{array}$$

3. Prove that the sequence  $\{a_n\}$  where  $a_n = \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$  is convergent.

4. Prove that the sequence  $\{a_n\}$  where  $a_n = \frac{a+nb}{c+nd}$  where  $a, b, c, d$  are positive is descending or ascending according as  $ad > bc$ .

5. Prove  $\left\{\frac{n}{n+1}\right\}$  is monotonic increasing and bounded above. Hence what is your conclusion?

6. If  $a_n = 3 + \frac{1}{2^n}$ , find the least positive integer  $m$  such that  $|a_n - 3| < 10^{-6}$  for all  $n > m$ . Find the limit of  $\{a_n\}$ .

### Cauchy's General Principle of Convergence

(Ans.  $m = 20$ )

A necessary and sufficient condition for the existence of a limit to the sequence  $\{a_n\}$  is that, given any positive value  $\epsilon$ , however small it may be, there exists a positive integer  $m$ , such that  $|a_{n+p} - a_n| < \epsilon$  for all  $n > m$  and for all integral values of  $p \geq 0$ .

### INFINITE SERIES

If  $\{a_n\}$  is a sequence, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots \text{ to } \infty \text{ is called an infinite series.}$$

**Convergence or divergence of a series.** Let

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots \text{ to } \infty \text{ be a given series.}$$

Define  $s_n = a_1 + a_2 + \dots + a_n$ .

That is  $s_n$  is the sum of the first  $n$  terms of the series  $\sum a_n$ .

The convergence or divergence of the series  $\sum a_n$  is defined in terms of the convergence or divergence of the sequence  $\{s_n\}$ .

That is,  $\sum a_n$  is said to converge if sequence  $\{s_n\}$  converges.

$\sum a_n$  converges or diverges according as  $\{s_n\}$  converges or diverges.

If  $\{s_n\}$  diverges to  $\infty$ , then  $\sum a_n$  diverges to  $\infty$ .

If  $\{s_n\}$  diverges to  $-\infty$ , then  $\sum a_n$  diverges to  $-\infty$ .

If  $\{s_n\}$  oscillates finitely or infinitely, then we say  $\sum a_n$  oscillates finitely or infinitely.

If  $\{s_n\}$  converges to  $l$ , then we say the series  $\sum a_n$  converges to  $l$ . That is,  $\sum a_n \rightarrow l$ .

**Geometric series.** The geometric series

$$1 + r + r^2 + \dots + r^n + \dots \text{ to } \infty = \sum_{n=0}^{\infty} r^n$$

(i) converges to  $\frac{1}{1-r}$  if  $|r| < 1$       (ii) diverges if  $r \geq 1$

(iii) oscillates finitely if  $r = -1$       (iv) oscillates infinitely if  $r < -1$

**Theorems 1.** If  $\sum u_n$  is convergent then  $\sum ku_n$  is convergent where  $k$  is a constant.

1. If  $\sum u_n$  and  $\sum v_n$  are two convergent series then  $\sum (u_n + v_n)$  is convergent and its sum is the sum of the two series.

### SERIES OF POSITIVE TERMS

A series whose terms are all positive is called a series of positive terms.

**Theorem 1.** A series of positive terms either converges or diverges. It cannot oscillate.

2. If  $\sum u_n$  is convergent then  $\lim_{n \rightarrow \infty} u_n = 0$ . This is a necessary condition but not a sufficient condition.

That is, if  $\lim_{n \rightarrow \infty} u_n = 0$ , then  $\sum u_n$  need not converge.

(Converse is not true)

For example, in the series

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots, u_n = \frac{1}{n} \text{ and } \lim_{n \rightarrow \infty} \frac{1}{n} = 0;$$

but still  $\sum u_n = \sum \frac{1}{n}$  is not convergent; it diverges although  $\frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$ .

**Note.** If the  $n$ th term of a positive termed series tends to zero as  $n \rightarrow \infty$ , we cannot say anything; whereas if the  $n$ th term of the series does not tend to zero as  $n \rightarrow \infty$  we can say that the series is not convergent (therefore, divergent).

TESTS. (proofs of tests are assumed)

**Auxiliary series test or Harmonic series test.** The series

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots \text{to } \infty$$

(i) convergent if  $p > 1$  and

(ii) divergent if  $p \leq 1$

### COMPARISON TEST, (different forms)

1. If  $\sum u_n$  and  $\sum v_n$  are two series of positive terms and the second series  $\sum v_n$  is convergent and if  $u_n \leq k v_n$  for all  $n$  ( $k$  constant) then  $\sum u_n$  is also convergent and it converges to the same limit as  $\sum v_n$ . It is less or equal to  $k$  times the convergent value of  $\sum v_n$ .

#### Useful forms

2. If  $\sum u_n$  and  $\sum v_n$  are two positive termed series and if  $\sum v_n$  is convergent and  $u_n \leq c v_n$  for all  $n$  where  $c$  is a constant, then  $\sum u_n$  is also convergent to a limit other than zero as  $n$  tends to infinity then  $\sum u_n$  is also convergent.

3. If  $\sum u_n$  and  $\sum v_n$  are two series of positive terms and if  $\sum v_n$  is divergent and  $u_n \geq c v_n$  for all  $n$  where  $c$  is a constant, then  $\sum u_n$  is also divergent to a limit other than zero as  $n$  tends to infinity, then  $\sum u_n$  is also divergent.

4. If  $\sum u_n$  and  $\sum v_n$  are two positive termed series and if  $u_n \leq k v_n$  for all  $n$  and if  $\sum v_n$  is divergent then  $\sum u_n$  is divergent.

Note.  $\sum \frac{1}{n}$  is divergent while  $\sum \frac{1}{n^2}$  is convergent.

### 5. Cauchy's general principle of convergence of an infinite series.

A necessary and sufficient condition for the convergence of the series  $\sum u_n$  is that for any positive number  $\epsilon$ , however small it may be, there exists a positive integer  $m$  such that

$$|u_{n+1} + u_{n+2} + \dots + u_{n+p}| < \epsilon \text{ for all } n \geq m \text{ and for all integers } p \geq 0.$$

**Example 1.** Test the convergence of the series :

- (i)  $\sum \frac{1}{\sqrt{n^2 + 1}}$       (ii)  $\sum \frac{1}{\sqrt{n + 1}}$       (iii)  $\sum \frac{1}{(n+1)(2n+1)}$   
 (iv)  $\sum (\sqrt{n^2 + 1} - n)$       (v)  $\sum (\sqrt{n^4 + 1} - \sqrt{n^4 - 1})$

**Solution.** (i) Let  $u_n = \frac{1}{\sqrt{n^2 + 1}}$  Take  $v_n = \frac{1}{n}$ .

$$\text{Now } \frac{u_n}{v_n} = \frac{\frac{1}{\sqrt{n^2 + 1}}}{\frac{1}{n}} = \frac{n}{\sqrt{n^2 + 1}} = \frac{1}{\sqrt{1 + \frac{1}{n^2}}}$$

$\frac{u_n}{v_n} \rightarrow 1$  as  $n \rightarrow \infty$ . Further  $\sum v_n = \sum \frac{1}{n}$  is divergent. Hence,  $\sum u_n$  is divergent by comparison test.

$$(ii) \quad u_n = \frac{1}{\sqrt{n+1}}; \text{ take } v_n = \frac{1}{n^{1/2}}$$

$$\frac{u_n}{v_n} = \frac{\sqrt{n}}{\sqrt{n+1}} = \sqrt{\frac{1}{1 + \frac{1}{n}}}$$

$\frac{u_n}{v_n} \rightarrow 1$  as  $n \rightarrow \infty$  and  $\sum v_n = \sum \frac{1}{n^{1/2}}$  is divergent.

$\therefore \sum u_n$  is divergent.

$$(iii) \quad u_n = \frac{1}{(n+1)(2n+1)} \left( = \frac{1}{\text{second degree in } n} \right)$$

So, take  $v_n = \frac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{n^2}{(n+1)(2n+1)} = \frac{1}{\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)} = \frac{1}{\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)} \frac{u_n}{v_n} \rightarrow \frac{1}{2} \text{ as } n \rightarrow \infty.$$

Also  $\sum v_n = \sum \frac{1}{n^2}$  is convergent.

$\therefore \sum \frac{1}{(n+1)(2n+1)}$  is convergent.

$$(iv) \text{ Let } u_n = \sqrt{n^2 + 1 - n}$$

$$= \frac{(\sqrt{n^2 + 1} - n)(\sqrt{n^2 + 1} + n)}{(\sqrt{n^2 + 1} + n)} = \frac{1}{\sqrt{n^2 + 1 + n}}$$

Therefore, take  $v_n = \frac{1}{n}$

$$\frac{u_n}{v_n} = \frac{n}{\sqrt{n^2 + 1 + n}} = \frac{1}{\sqrt{1 + \frac{1}{n^2} + 1}} \quad \frac{u_n}{v_n} \rightarrow \frac{1}{2} \neq 0 \text{ as } n \rightarrow \infty$$

Also  $\sum v_n$  is divergent. By comparison test,  $\sum u_n$  is also divergent.

$$(v) \text{ Let } u_n = \sqrt{n^4 + 1} - \sqrt{n^4 - 1}$$

$$= \frac{(\sqrt{n^4 + 1} - \sqrt{n^4 - 1})(\sqrt{n^4 + 1} + \sqrt{n^4 - 1})}{(\sqrt{n^4 + 1} + \sqrt{n^4 - 1})}$$

$$= \frac{(n^4 + 1) - (n^4 - 1)}{\sqrt{n^4 + 1} + \sqrt{n^4 - 1}} = \frac{2}{\sqrt{n^4 + 1} + \sqrt{n^4 - 1}}$$

Take

$$v_n = \frac{1}{n^2}$$

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$$\frac{u_n}{v_n} = \frac{2n^2}{\sqrt{n^4 + 1} + \sqrt{n^4 - 1}} = \frac{2}{\sqrt{1 + \frac{1}{n^4}} + \sqrt{1 - \frac{1}{n^4}}}$$

$$\frac{u_n}{v_n} \rightarrow 1 \neq 0 \text{ as } n \rightarrow \infty$$

Since  $\sum v_n = \sum \frac{1}{n^2}$  is convergent,  $\sum u_n$  is also convergent.

**Example 2.** Discuss the convergence of  $\sum \frac{1}{(n+a)^p (n+b)^q}$

**Solution.** Take  $v_n = \frac{1}{n^{p+q}}$  and use comparison test.

$$\frac{u_n}{v_n} = \frac{n^{p+q}}{(n+a)^p (n+b)^q} = \frac{1}{\left(1 + \frac{a}{n}\right)^p \left(1 + \frac{b}{n}\right)^q}$$

$$\frac{u_n}{v_n} \rightarrow 1 \text{ as } n \rightarrow \infty$$

$\therefore \sum u_n$  and  $\sum v_n$  converge or diverge together.

$\therefore \sum u_n$  is convergent if  $p+q > 1$  and divergent if  $p+q \leq 1$

**Example 3.** Discuss the convergence of the series

$$(i) \sum \frac{1}{n} \sin\left(\frac{1}{n}\right) \text{ and } (ii) \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right)$$

**Solution.** (i) Let  $u_n = \frac{1}{n} \sin\left(\frac{1}{n}\right)$ ; take  $v_n = \frac{1}{n^2}$ .

$$\frac{u_n}{v_n} = \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{\frac{1}{n} \rightarrow 0} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = 1 \neq 0$$

$\therefore \sum u_n$  is convergent since  $\sum v_n = \sum \frac{1}{n^2}$  is convergent.

(ii) Let

$$u_n = \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right); v_n = \frac{1}{n^{3/2}}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{\frac{1}{n} \rightarrow 0} \frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1 \neq 0$$

$\therefore \sum u_n$  is convergent since  $\sum v_n$  is convergent.

**Example 4.** Show that the series  $\frac{1}{1 \cdot 2 \cdot 3} + \frac{3}{2 \cdot 3 \cdot 4} + \frac{5}{3 \cdot 4 \cdot 5} + \dots$  to  $\infty$  is convergent.

**Solution.**  $u_n$  =  $n$ th term of the series

$$= \frac{2n-1}{n(n+1)(n+2)}$$

Since the right hand side is of the form  $\frac{\text{first degree}}{\text{third degree}}$

Take

$$v_n = \frac{1}{\text{second degree}} = \frac{1}{n^2}$$

Now

$$\frac{u_n}{v_n} = \frac{(2n-1) \cdot n^2}{n(n+1)(n+2)} = \frac{(2n-1)n}{(n+1)(n+2)} = \frac{\left(2 - \frac{1}{n}\right)}{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 2 \neq 0$$

$\therefore \sum u_n$  is convergent since  $\sum v_n = \sum \frac{1}{n^2}$  is convergent.

**Example 5.** If  $\sum u_n$  and  $\sum v_n$  are two convergent series of positive terms, then prove

$$(i) \sum (u_n + v_n) \quad (ii) \sum \sqrt{u_n v_n} \quad (iii) \sum u_n v_n$$

(iv)  $\sum u_n^2, \sum u_n^3, \dots$  are also convergent.

**Solution.** Let  $\sum u_n$  and  $\sum v_n$  converge to  $a$  and  $b$  i.e., if  $U_n = u_1 + u_2 + \dots + u_n$  and  $V_n = v_1 + v_2 + \dots + v_n$

$$\lim_{n \rightarrow \infty} U_n = a \text{ and } \lim_{n \rightarrow \infty} V_n = b.$$

$$(i) \text{ Let } s_n = (u_1 + v_1) + (u_2 + v_2) + \dots + (u_n + v_n)$$

$$= \sum_{n=1}^{\infty} (u_n + v_n)$$

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{n=1}^{\infty} (u_n + v_n)$$

$$= \lim_{n \rightarrow \infty} \sum_{n=1}^{\infty} u_n + \lim_{n \rightarrow \infty} \sum_{n=1}^{\infty} v_n = a + b.$$

$\therefore \sum_{n=1}^{\infty} (u_n + v_n)$  converges.

(ii) Since A.M. > G.M.

$$\frac{u_n + v_n}{2} > \sqrt{u_n v_n}$$

Since  $\sum (u_n + v_n)$  is convergent,  $\sum \left( \frac{u_n + v_n}{2} \right)$  is also convergent.

$\therefore \sum \sqrt{u_n v_n}$  is convergent.

(iii)  $u_n v_n < a v_n$  since  $u_n < a$   
 Since  $\sum a v_n = a \sum v_n$  is convergent

$\sum u_n v_n$  is also convergent.

(iv) Taking  $v_n = u_n$ ,  $\sum u_n^2$  is convergent

Similarly  $\sum u_n^3, \sum u_n^4, \dots$  are all convergent.

**Example 6.** Discuss the convergence of the series  $\sum \frac{1}{x^n + x^{-n}}$  for  $x > 0$ .

**Solution. Case 1.** Let  $x > 1$ ; hence  $\frac{1}{x}$  is positive and less than 1

$\therefore \sum \left(\frac{1}{x}\right)^n$  is convergent since it is a geometric series.

$$x^n + x^{-n} > x^n \quad (\because x^{-n} > 0)$$

$$\frac{1}{x^n + x^{-n}} < \frac{1}{x^n} \text{ and } \sum \frac{1}{x^n} \text{ is convergent.}$$

$\therefore \sum \frac{1}{x^n + x^{-n}}$  is also convergent for  $x > 1$ .

**Case 2.** Let  $x = 1$ ;  $\frac{1}{x^n + x^{-n}} = \frac{1}{2}$  and  $\sum \left(\frac{1}{2}\right)$  is divergent

$\therefore \sum \frac{1}{x^n + x^{-n}}$  is divergent for  $x = 1$

**Case 3.** Let  $0 < x < 1$

$x^n + x^{-n} > x^{-n}$  and  $\frac{1}{x^n + x^{-n}} < \frac{1}{x^{-n}} = x^n$  and  $\sum x^n$  is convergent

$\therefore \sum \frac{1}{x^n + x^{-n}}$  is convergent for  $0 < x < 1$

**Example 7.** If  $a_n > 0$  and (i) if  $\sum a_n^2$  is convergent, can you say that  $\sum a_n$  is convergent?  
 If  $\sum a_n$  is divergent can you say  $\sum a_n^2$  is divergent?

**Solution.** (i) No. As a counter example,  $\sum \frac{1}{n^2}$  is convergent while  $\sum \frac{1}{n}$  is not convergent  
 (it is divergent)

(ii) No.  $\sum \frac{1}{n}$  is divergent while  $\sum \frac{1}{n^2}$  is not divergent.

**Example 8.** If  $a_n > 0$  and  $\sum a_n$  is convergent, prove  $\sum \frac{a_n}{1+a_n}$  and  $\sum \frac{a_n}{1+n^2 a_n}$  are convergent.

**Solution.** If  $a_n > 0$ ,  $1 + a_n > 1$  and  $\frac{1}{1 + a_n} < 1$

$\therefore \frac{a_n}{1 + a_n} < a_n$  and  $\sum a_n$  is convergent.

$\sum \frac{a_n}{1+a_n}$  is also convergent.

Similarly  $\frac{a_n}{1+n^2 a_n} < a_n$  and  $\sum a_n$  is convergent.

Hence  $\sum \frac{a_n}{1+n^2 a_n}$  is also convergent.

**Example 9.** Discuss the convergence the series

$$(i) \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n^p} \quad (ii) \sum_{n=1}^{\infty} \frac{n^p}{\sqrt{n+1} + \sqrt{n}} \quad (iii) \sum_{n=1}^{\infty} \frac{n^p}{(n+1)^{n+1}} \quad (iv) \sum_{n=10}^{\infty} \frac{1}{n+5}$$

**Solution.** (i) Here,  $u_n = \frac{\sqrt{n+1} - \sqrt{n}}{n^p} = \frac{\sqrt{(n+1) - n} (\sqrt{n+1} + \sqrt{n})}{n^p (\sqrt{n+1} + \sqrt{n})} = \frac{1}{n^p (\sqrt{n+1} + \sqrt{n})}$

Take

$$v_n = \frac{1}{n^{p+\frac{1}{2}}}$$

$$\text{Now } \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{n^{p+\frac{1}{2}}}{n^p (\sqrt{n+1} + \sqrt{n})} = \lim_{n \rightarrow \infty} \frac{n^{1/2}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} = 1/2 \neq 0$$

$\sum u_n$  is convergent for  $p + \frac{1}{2} > 1$  and divergent for  $p + \frac{1}{2} \leq 1$ .

Hence, the given series is convergent for  $p > \frac{1}{2}$  and divergent for  $p + \frac{1}{2} \leq 1$ .

(ii) Let  $u_n = \frac{n^p}{\sqrt{n+1} + \sqrt{n}}$ ; Take  $v_n = \frac{1}{n^{1/2-p}}$

$$\text{Now, } \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{n^p \cdot n^{1/2-p}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} = \frac{1}{2}$$

$\sum v_n = \sum \frac{1}{n^{1/2-p}}$  is convergent if  $\frac{1}{2} - p > 1$  and divergent if  $\frac{1}{2} - p \leq 1$

$\therefore$  The given series is convergent if  $p < -\frac{1}{2}$  and divergent if  $p \geq -\frac{1}{2}$

(iii) Let  $u_n = \frac{n^n}{(n+1)^{n+1}}$ ; Take  $v_n = \frac{1}{n}$

$$\text{Now, } \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{n^{n+1}}{(n+1)^{n+1}} = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^{n+1}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\left( 1 + \frac{1}{n} \right)^n} \cdot \frac{1}{1 + \frac{1}{n}} = \frac{1}{e} \neq 0$$

is convergent?

is not convergent

$\frac{a_n}{1+n^2 a_n}$  are also

Since  $\sum v_n = \sum \frac{1}{n}$  is divergent, the given series  $\sum u_n$  is also divergent by comparison test.

(iv) **Method 1.**

The series is  $\frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \dots$  to  $\infty$ .

This is nothing but the series  $\sum_{n=1}^{\infty} \frac{1}{n}$  from which the first few terms are removed. Since  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent, the given series is also divergent.

**Method 2.** Take  $v_n = \frac{1}{n}$ ;

Then,

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{n}{n+5} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{5}{n}} = 1 \neq 0$$

$\therefore \sum_{n=1}^{\infty} u_n$  is divergent since  $\sum_{n=1}^{\infty} v_n$  is divergent by comparison test.

$\therefore \sum_{n=10}^{\infty} u_n$  is also divergent.

**Example 10.** Show that the series  $\sum_{n=1}^{\infty} e^{-n^2}$  converges.

**Solution.** Let  $u_n = e^{-n^2} = \frac{1}{e^{n^2}}$

we know, for  $x > 0$ ,  $e^x > x$

$$e^{n^2} > n^2$$

$$\frac{1}{e^{n^2}} < \frac{1}{n^2} \text{ and } \sum \frac{1}{n^2} \text{ is convergent.}$$

$\therefore \sum u_n = \sum \frac{1}{e^{n^2}}$  is convergent by comparison test.

**Example 11.** Find the sum to  $n$  terms of the series  $\sum \frac{1}{4n^2 - 1}$ . Hence discuss the convergence of the series.

**Solution.** Let  $u_n = \frac{1}{4n^2 - 1} = \frac{1}{(2n-1)(2n+1)} = \frac{1}{2} \left[ \frac{1}{2n-1} - \frac{1}{2n+1} \right]$

Sum to  $n$  terms of the series =  $u_1 + u_2 + \dots + u_n$

$$= \frac{1}{2} \left[ \left( \frac{1}{1} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \left( \frac{1}{5} - \frac{1}{7} \right) + \dots + \left( \frac{1}{2n-1} - \frac{1}{2n+1} \right) \right] = \frac{1}{2} \left[ 1 - \frac{1}{2n+1} \right]$$

$$\text{Since } \sum_{n=1}^{\infty} u_n = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{1}{2} \left[ 1 - \frac{1}{2n+1} \right] = \frac{1}{2}$$

The series converges and it converges to the value  $1/2$ .

**Note.** Here, we are using the basic idea of the convergence of a series, by finding  $s_n$  and its limit as  $n \rightarrow \infty$ .

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Example 1

Solution 1

Here

$\sum v_n$  is co

$\therefore$  The giv

Test for convergence

$$1. \frac{1}{3 \cdot 4 \cdot 5} + \dots$$

$$3. \frac{1}{x} + \frac{1}{x+1} + \dots$$

$$5. 1 + \frac{1}{3} + \frac{1}{5} + \dots$$

$$7. 1 + \frac{1+2}{1+2^2} + \dots$$

$$9. \frac{1}{1^2 - x} + \dots$$

Test for convergence

$$11. \frac{\sqrt{n+1}}{n+2}$$

$$15. \frac{n^p}{\sqrt{n+1} + \dots}$$

$$19. \frac{n+1}{n^p}$$

Answers. C-Cor

$$1. (C)$$

$$11. (D) \quad 1$$

$$18. (D), \quad 1$$

If  $\sum u_n$

- (i) converges
- (ii) diverges

## Sequences and Series

**Example 12.** Discuss the convergence of  $\frac{1^3}{1^p + 2^p} + \frac{2^3}{2^p + 3^p} + \frac{3^3}{3^p + 4^p} + \dots$  to  $\infty$

**Solution.** Let  $u_n = \frac{n^3}{n^p + (n+1)^p}$  Take  $v_n = \frac{1}{n^{p-3}}$

$$\begin{aligned} \text{Here } \lim_{n \rightarrow \infty} \frac{u_n}{v_n} &= \lim_{n \rightarrow \infty} \frac{n^{p-3} \cdot n^3}{n^p + (n+1)^p} = \lim_{n \rightarrow \infty} \frac{n^p}{n^p + (n+1)^p} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 + \left(1 + \frac{1}{n}\right)^p} = \frac{1}{2} \neq 0 \end{aligned}$$

$\sum v_n$  is convergent for  $p - 3 > 1$  and divergent for  $p - 3 \leq 1$

$\therefore$  The given series is convergent for  $p > 4$  and divergent for  $p \leq 4$ , by comparison test.

**EXERCISE 2****Test for convergence of the series**

1.  $\frac{1}{3 \cdot 4 \cdot 5} + \frac{2}{4 \cdot 5 \cdot 6} + \frac{3}{5 \cdot 6 \cdot 7} + \dots$  to  $\infty$

2.  $\frac{1 \cdot 2}{3 \cdot 4 \cdot 5} + \frac{2 \cdot 3}{4 \cdot 5 \cdot 6} + \frac{3 \cdot 4}{5 \cdot 6 \cdot 7} + \dots$  to  $\infty$

3.  $\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} + \dots$  to  $\infty$ , if  $x > 0$

4.  $\frac{1}{x} + \frac{1}{x+y} + \frac{1}{x+2y} + \frac{1}{x+3y} + \dots$ , if  $x, y > 0$

5.  $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$  to  $\infty$

6.  $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \dots$  to  $\infty$

7.  $1 + \frac{1+2}{1+2^2} + \frac{1+3}{1+3^2} + \dots + \frac{1+n}{1+n^2} + \dots$

8.  $\sum \frac{(n+a)(n+b)}{n(n+1)(n+2)}$

9.  $\frac{1}{1^2 - x} + \frac{1}{2^2 - x} + \frac{1}{3^2 - x} + \dots$

10.  $\frac{1}{\sqrt{1^2 - x^2}} + \frac{1}{\sqrt{2^2 - x^2}} + \frac{1}{\sqrt{3^2 - x^2}} + \dots$

**Test for convergence of the series whose  $n$ th term is**

11.  $\frac{\sqrt{n+1}}{n+2}$

12.  $\frac{1}{1+3n^4}$

13.  $\sin^2 \left( \frac{1}{n} \right)$

14.  $\sqrt[3]{n^3+1} - n$

15.  $\frac{n^p}{\sqrt{n+1} - \sqrt{n}}$

16.  $\sqrt{n^3+1} - \sqrt{n^3}$

17.  $\frac{1}{\sqrt{n+1} - \sqrt{n}}$

18.  $\frac{1}{an+b}$

19.  $\frac{n+1}{n^p}$

20.  $\frac{1}{\sqrt{n(n+1)}}$

21.  $\frac{n+1}{n^2}$

22.  $n^2$

23.  $\sin \left( \frac{1}{n} \right)$

**Answers.** C-Convergent; D-divergent.

- |          |  |         |  |          |          |          |         |
|----------|--|---------|--|----------|----------|----------|---------|
| 1. (C)   | 2. to                                    | 5. (D)  | 6. (C)   | 7. (D)   | 8. (D)   | 9. (C)   | 10. (D) |
| 11. (D)  | 12. to                                   | 14. (C) | 15. (C) if $p < -3/2$ and (D) if $p \geq -3/2$ | 16. (C), | 17. (D), |          |         |
| 18. (D), | 19. (C) if $p > 2$ and (D) if $p \leq 2$ |         | 20. (D),                                       | 21. (D)  | 22. (D)  | 23. (D). |         |

**D'ALEMBERT'S RATIO TEST**

If  $\sum u_n$  is a series of positive terms and  $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = l$ , then the series  $\sum u_n$  is

- (i) convergent if  $l < 1$ , and
- (ii) divergent if  $l > 1$ .

**Example 1.** Test the convergence of the series

$$\frac{3}{4} + \frac{3 \cdot 4}{4 \cdot 6} + \frac{3 \cdot 4 \cdot 5}{4 \cdot 6 \cdot 8} + \dots \text{to } \infty$$

**Solution.** Here

$$u_n = \frac{3 \cdot 4 \cdot 5 \dots (n+2)}{4 \cdot 6 \cdot 8 \dots (2n+2)}$$

$$u_{n+1} = \frac{3 \cdot 4 \cdot 5 \dots (n+2)(n+3)}{4 \cdot 6 \cdot 8 \dots (2n+2)(2n+4)}$$

$$\frac{u_{n+1}}{u_n} = \frac{n+3}{2n+4} = \frac{1 + \frac{3}{n}}{2 + \frac{4}{n}}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1$$

∴ By ratio test, the given series is convergent.

**Example 2.** Test the convergence or divergence of

$$1 + \frac{a+1}{b+1} + \frac{(a+1)(2a+1)}{(b+1)(2b+1)} + \frac{(a+1)(2a+1)(3a+1)}{(b+1)(2b+1)(3b+1)} + \dots \text{to } \infty$$

**Solution.** Here  $u_n = \frac{(a+1)(2a+1)(3a+1) \dots [(n-1)a+1]}{(b+1)(2b+1)(3b+1) \dots [(n-1)b+1]}$

$$\frac{u_{n+1}}{u_n} = \frac{na+1}{nb+1} = \frac{a + \frac{1}{n}}{b + \frac{1}{n}} \quad \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \frac{a}{b}$$

If  $a, b$  are positive, the given series become positive termed series.  
∴ by Ratio rest,

If  $\frac{a}{b} < 1$ , i.e.,  $a < b$  and  $a, b > 0$ , the given series is convergent.

If  $\frac{a}{b} > 1$ , i.e.,  $a > b > 0$ , the series is divergent.

If  $a = b$ , then  $u_n = 1$ . Hence the series is  $1 + 1 + 1 + \dots$  which is divergent.  
∴ If  $a = b$ , the given series diverges.

**Example 3.** Test for convergency the series  $\sum_{n=1}^{\infty} \frac{n^3 + 1}{2^n + 1}$

**Solution.** Here

$$u_n = \frac{n^3 + 1}{2^n + 1}; u_{n+1} = \frac{(n+1)^3 + 1}{2^{n+1} + 1}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^3 + 1}{n^3 + 1} \cdot \frac{2^n + 1}{2^{n+1} + 1} = \frac{\left(1 + \frac{1}{n}\right)^3 + \frac{1}{n^3}}{1 + \frac{1}{n^3}} \cdot \frac{1 + \frac{1}{2^n}}{2 + \frac{1}{2^n}}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 1 \times \frac{1}{2} = \frac{1}{2} < 1$$

∴ The series is convergent.

**Example 4.** Test for convergency the series  $\sum \frac{n!}{n^n}$ .

**Solution.** Here  $u_n = \frac{n!}{n^n}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n+1}{(n+1)^n \cdot (n+1)} \cdot n^n = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} \rightarrow \frac{1}{e} < 1$$

$\therefore$  The series is convergent.

**Example 5.** Test for convergency the series  $\sum \frac{x^n}{1+x^n}$ ,  $x > 0$ .

**Solution.** Here,  $\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{1+x^{n+1}} \cdot \frac{1+x^n}{x^n} = \frac{1+x^n}{1+x^{n+1}} \cdot x$

If  $x < 1$  and  $x > 0$ ,  $x^n \rightarrow 0$  as  $n \rightarrow \infty$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = x \text{ since } x^n, x^{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$\therefore \sum u_n$  is convergent if  $0 < x < 1$ .

If  $x > 1$ , the series is divergent.

When  $x = 1$ , the series is  $\sum u_n = \sum \frac{1^n}{1+1^n} = \sum \frac{1}{2}$

$\therefore \sum u_n$  is divergent when  $x = 1$ .

**Example 6.** Test for convergence of (i)  $\sum \frac{x^n}{n!}$

(ii)  $\sum \sqrt{\frac{n}{n+1}} \cdot x^n$ ,  $x > 0$

**Solution.** (i) Let  $u_n = \frac{x^n}{n!}$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 0 < 1.$$

$\therefore \sum u_n$  is convergent for all real  $x$ .

(ii) Let  $u_n = \sqrt{\frac{n}{n+1}} x^n$ ;  $u_{n+1} = \sqrt{\frac{n+1}{n+2}} x^{n+1}$

$$\therefore \frac{u_{n+1}}{u_n} = \sqrt{\frac{n+1}{n+2} \cdot \frac{n+1}{n}} \cdot x = \sqrt{\frac{\left(1 + \frac{1}{n}\right)^2}{\left(1 + \frac{2}{n}\right)}} \cdot x$$

$$\lim_{n \rightarrow \infty} \frac{u_n + 1}{u_n} = x$$

If  $0 < x < 1$ , the series is convergent.

If  $x > 1$  the series is divergent.

If  $x = 1$ , D'Alembert's test fails.

when

$$x = 1, u_n = \sqrt{\frac{n}{n+1}} = \sqrt{\frac{1}{\left(1 + \frac{1}{n}\right)}}$$

$\lim_{n \rightarrow \infty} u_n = 1$  which does not tend to zero.

∴ The series is divergent.

∴ The series is convergent if  $0 < x < 1$  and divergent if  $x \geq 1$ .

**Example 7.** Examine the convergence of the series

$$(i) \frac{x}{1} + \frac{1}{2} \cdot \frac{x^2}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^3}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^4}{7} + \dots \text{to } \infty, x > 0$$

$$(ii) \frac{1}{2} + \frac{4}{9} x + \frac{9}{28} x^2 + \dots + \frac{n^2}{1+n^3} x^{n-1} \dots \text{to } \infty, x > 0$$

**Solution.** Let

$$u_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n-2)} \cdot \frac{x^n}{2n-1}$$

$$u_{n+1} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)(2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n-2)(2n)} \cdot \frac{x^{n+1}}{2n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{2n-1}{2n} \cdot \frac{2n-1}{2n+1} \cdot x = \frac{\left(1 - \frac{1}{2n}\right)^2}{\left(1 + \frac{1}{2n}\right)} \cdot x$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = x$$

∴ The series is convergent if  $0 < x < 1$

The series diverges if  $x > 1$

If  $x = 1$ ,  $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 1$  and hence the ratio test fails, In this case we can use Raabe's test.

(ii) Let

$$u_n = \frac{n^2}{1+n^3} x^{n-1}; u_{n+1} = \frac{(n+1)^2}{1+(n+1)^3} x^n$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{n^2} \cdot \frac{1+n^3}{1+(n+1)^3} \cdot x$$

$$= \left(1 + \frac{1}{n}\right)^2 \frac{\left(1 + \frac{1}{n^3}\right)}{\frac{1}{n^3} + \left(1 + \frac{1}{n}\right)^3} \cdot x \rightarrow x \text{ as } n \rightarrow \infty$$

$\therefore$  If  $0 < x < 1$ , then  $\sum u_n$  converges

and if  $x > 1$  the series diverges.

$$\text{When } x = 1, u_n = \frac{n^2}{1+n^3}$$

We will use comparison test. Take  $v_n = \frac{1}{n}$

$$\frac{u_n}{v_n} = \frac{n^3}{1+n^3} = \frac{1}{1+\frac{1}{n^3}} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

$\therefore \sum u_n$  and  $\sum v_n$  converge or diverge together.

Since  $\sum v_n = \sum \frac{1}{n}$  is divergent,  $\sum u_n$  also diverges.

### RAABE'S TEST

The positive termed series  $\sum u_n$  is convergent or divergent according as

$$\lim_{n \rightarrow \infty} \left\{ n \left( \frac{u_n}{u_{n+1}} - 1 \right) \right\} > 1 \text{ or } < 1.$$

Note : If D'Alembert's test fails, use Raabe's test.

**Example 1.** Test for convergence of the series.

$$(i) \frac{2}{3 \cdot 4} + \frac{2 \cdot 4}{3 \cdot 5 \cdot 6} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7 \cdot 8} + \frac{2 \cdot 4 \cdot 6 \cdot 8}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 10} + \dots \text{to } \infty$$

$$(ii) \frac{3}{4} \cdot \frac{x}{5} + \frac{3 \cdot 6}{4 \cdot 7} \cdot \frac{x^2}{8} + \frac{3 \cdot 6 \cdot 9}{4 \cdot 7 \cdot 10} \cdot \frac{x^3}{11} + \dots \text{to } \infty, x > 0$$

**Solution.** (i)

$$u_n = \frac{2 \cdot 4 \cdot 6 \cdot 8 \dots (2n)}{3 \cdot 5 \cdot 7 \dots (2n+1)} \cdot \frac{1}{2n+2}$$

$$u_{n+1} = \frac{2 \cdot 4 \cdot 6 \cdot 8 \dots (2n)(2n+2)}{3 \cdot 5 \cdot 7 \dots (2n+3)} \cdot \frac{1}{2n+4}$$

$$\frac{u_{n+1}}{u_n} = \frac{2n+2}{2n+3} \cdot \frac{2n+2}{2n+4} = \frac{\left(1 + \frac{1}{n}\right)^2}{\left(1 + \frac{3}{2n}\right)\left(1 + \frac{4}{2n}\right)}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 1$$

Hence ratio test fails.

$$n \left( \frac{u_n}{u_{n+1}} - 1 \right) = n \left[ \frac{(2n+3)(2n+4)}{(2n+2)^2} - 1 \right] = \frac{n(6n+8)}{(2n+2)^2} \rightarrow \frac{3}{2} \text{ as } n \rightarrow \infty$$

$\therefore$  By Raabe's test,  $\sum u_n$  is convergent.

(ii) Let

$$u_n = \frac{3 \cdot 6 \cdot 9 \dots (3n)}{4 \cdot 7 \cdot 10 \dots (3n+1)} \cdot x^n$$

$$\frac{u_{n+1}}{u_n} = \frac{(3n+3)}{(3n+4)} \cdot \frac{3n+2}{3n+5} \cdot x \rightarrow x \text{ as } n \rightarrow \infty$$

$\therefore \sum u_n$  is convergent if  $0 < x < 1$  and divergent if  $x > 1$

If  $x = 1$ , the test fails.

$\therefore$  When  $x = 1$ ,

$$\begin{aligned} n \left[ \frac{u_n}{u_{n+1}} - 1 \right] &= n \left[ \frac{(3n+4)(3n+5)}{(3n+3)(3n+2)} - 1 \right] = n \left[ \frac{12n+14}{(3n+3)(3n+2)} \right] \\ &= \frac{\left( 12 + \frac{14}{n} \right)}{\left( 3 + \frac{3}{n} \right) \left( 3 + \frac{2}{n} \right)} \rightarrow \frac{12}{9} \text{ as } n \rightarrow \infty \\ \lim_{n \rightarrow \infty} n \left( \frac{u_n}{u_{n+1}} - 1 \right) &= \frac{12}{9} > 1 \end{aligned}$$

$\therefore \sum u_n$  is convergent when  $x = 1$ .

**Example 2.** Test for convergence of the series

$$(i) \sum \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)} \cdot \frac{1}{n}$$

$$(ii) 1 + \frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \dots \text{ to } \infty, \text{ if } a, b > 0$$

$$(iii) \sum \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)} x^n.$$

**Solution.** (i) Let

$$u_n = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)} \cdot \frac{1}{n}$$

$$u_{n+1} = \frac{1 \cdot 3 \cdot 5 \dots (2n+1)}{2 \cdot 4 \cdot 6 \dots (2n)(2n+2)} \cdot \frac{1}{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{2n+1}{2n+2} \cdot \frac{n}{n+1} = \frac{\left( 1 + \frac{1}{2n} \right)}{\left( 1 + \frac{2}{2n} \right)} \cdot \frac{1}{\left( 1 + \frac{1}{n} \right)}$$

$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 1$ . Therefore, D'Alembert's ratio test fails.

We will try Raabe's test,

$$n \left( \frac{u_n}{u_{n+1}} - 1 \right) = n \left[ \frac{2n+2}{2n+1} \cdot \frac{n+1}{n} - 1 \right] = \frac{3n+2}{2n+1} = \frac{3 + \frac{2}{n}}{2 + \frac{1}{n}} \rightarrow \frac{3}{2} (> 1) \text{ as } n \rightarrow \infty$$

$\therefore$  By Raabe's test, the series is convergent.

(ii)

$\therefore$  Ratio test fails.

We will apply Raabe

$$n \left( \frac{u_{n+1}}{u_n} - 1 \right)$$

By Raabe's test,  $\sum$   
If  $b - a = 1$ , Raabe's

In this case

Taking

If  $a \neq 0$ ,  $\sum u_n$  is c

(iii)

$\therefore \sum u_n$  is conve  
If  $x = 1$ , D'Alemb

$$n \left( \frac{u_{n+1}}{u_n} - 1 \right)$$

$\therefore$  when  $x = 1$ ,  $\sum$

**Example 3.** Prov

$$1 + \frac{1}{2} \cdot \frac{a}{b} + \frac{1 \cdot 3}{2 \cdot 4}$$

$$b > a + \frac{1}{2}$$

**Solution.** Let

$$(ii) \quad u_n = \frac{a(a+1)(a+2)\dots(a+n-1)}{b(b+1)(b+2)\dots(b+n-1)}$$

$$u_{n+1} = \frac{a(a+1)\dots(a+n)}{b(b+1)\dots(b+n)}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{a+n}{b+n} = \frac{1 + \frac{a}{n}}{1 + \frac{b}{n}} \rightarrow 1 \text{ as } n \rightarrow \infty$$

$\therefore$  Ratio test fails.

We will apply Raabe's test

$$n \left( \frac{u_n}{u_{n+1}} - 1 \right) = n \left( \frac{n+b}{n+a} - 1 \right) = \frac{n(b-a)}{(n+a)} \rightarrow b-a \text{ as } n \rightarrow \infty$$

By Raabe's test,  $\sum u_n$  is convergent if  $b-a > 1$  and divergent if  $b-a < 1$

If  $b-a = 1$ , Raabe's test also fails.

$$\text{In this case } u_n = \frac{a(a+1)(a+2)\dots(a+n-1)}{(a+1)(a+2)\dots(a+n)} = \frac{a}{a+n}$$

$$\text{Taking } v_n = \frac{1}{n}, \frac{u_n}{v_n} = \frac{an}{n+a} = \frac{a}{1+\frac{a}{n}} \rightarrow a \text{ if } n \rightarrow \infty$$

If  $a \neq 0$ ,  $\sum u_n$  is divergent since  $\sum v_n$  is divergent by comparison test.

$$(iii) \quad u_n = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)} x^n$$

$$\frac{u_{n+1}}{u_n} = \frac{2n+1}{(2n+2)} \cdot x = \frac{1 + \frac{1}{2n}}{1 + \frac{2}{2n}} \cdot x \rightarrow x \text{ as } n \rightarrow \infty$$

$\therefore \sum u_n$  is convergent if  $0 < x < 1$  and divergent if  $x > 1$ .

If  $x = 1$ , D'Alembert's ratio test fails. Apply Raabe's test

$$n \left( \frac{u_n}{u_{n+1}} - 1 \right) = \frac{n}{2n+1} \rightarrow \frac{1}{2} \text{ as } n \rightarrow \infty$$

$\therefore$  when  $x = 1$ ,  $\sum u_n$  is divergent.

**Example 3.** Prove that the series

$$1 + \frac{1}{2} \cdot \frac{a}{b} + \frac{1 \cdot 3}{2 \cdot 4} \frac{a(a+1)}{b(b+1)} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \dots \text{ to } \infty \text{ is convergent if } a > 0, b > 0 \text{ and}$$

$$b > a + \frac{1}{2}$$

$$\text{Solution. Let } u_n = \frac{1 \cdot 3 \cdot 5 \dots (2n-3)}{2 \cdot 4 \cdot 6 \dots (2n-2)} \cdot \frac{a(a+1)\dots(a+n-2)}{b(b+1)\dots(b+n-2)}$$

01 - 37 - 14.8

02 - 37 - 14.8

03 - 27 - 10.8

04 - 29 - 11.6

05 - 34 - 13.6

06 - 40 - 16

07 - 30 - 12

08 - 25 - 10

09 - 30 - 12

10 - 32 - 12.8

11 - 25 - 10

12 - 35 - 14

J M A 20 A 01 T

13 - 30 - 12

14 - 30 - 12

15 - 32 - 12.8

$$\frac{u_{n+1}}{u_n} = \frac{(2n-1)(a+n-2)}{2n(b+n-1)} = \left(1 - \frac{1}{2n}\right) \frac{\left(1 + \frac{a-1}{n}\right)}{\left(1 + \frac{b-1}{n}\right)} \rightarrow 1 \text{ as } n \rightarrow \infty$$

Since  $a, b > 0$ , it is a positive termed series.

Ratio test fails.

Using Raabe's test,

$$\begin{aligned} n \left( \frac{u_n}{u_{n+1}} - 1 \right) &= n \left[ \frac{2n(b+n-1)}{(2n-1)(a+n-1)} - 1 \right] = n \left[ \frac{2n(b-a) + a+n-1}{(2n-1)(a+n-1)} \right] \\ &= \frac{(2b-2a+1) + \frac{a-1}{n}}{\left(2 - \frac{1}{n}\right)\left(1 + \frac{a-1}{n}\right)} \rightarrow b-a+\frac{1}{2} \text{ as } n \rightarrow \infty \end{aligned}$$

By Raabe's test,  $\sum u_n$  is convergent if  $b-a+\frac{1}{2} > 1$  i.e., if  $b > a + \frac{1}{2}$

**Example 4.** Discuss the convergence of  $\sum \frac{(n!)^2}{(2n)!} x^n$ .

**Solution.** Let

$$\begin{aligned} u_n &= \frac{(n!)^2}{(2n)!} x^n; u_{n+1} = \frac{[(n+1)!]^2}{(2n+2)!} x^{n+1} \\ \frac{u_{n+1}}{u_n} &= \frac{(n+1)^2}{(2n+1)(2n+2)^2} \cdot x = \frac{n+1}{2(2n+1)} x \end{aligned}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{2\left(2 + \frac{1}{n}\right)} \cdot x = \frac{x}{4}$$

Hence the series is convergent if  $0 \leq x < 4$  and divergent if  $x > 4$ .  
If  $x = 4$ , D'Alembert's ratio test fails.

$$\lim_{n \rightarrow \infty} n \left( \frac{u_n}{u_{n+1}} - 1 \right) = \lim_{n \rightarrow \infty} n \left[ \frac{2(2n+1)}{4(n+1)} - 1 \right] = \lim_{n \rightarrow \infty} \frac{-n}{2(n+1)} = -1/2 < 1$$

∴ when  $x = 4$ , the series is divergent.

**Example 5.** Discuss the convergency or otherwise of the series whose  $n$ th term is ( $x > 0$ )

$$(i) \frac{1}{1+nx^n}$$

$$(ii) \frac{x^n}{1+x^n}$$

$$(iii) \frac{x^n}{1+n^2x^{2n}}$$

$$(iv) \frac{x^n}{1+x^{2n}}$$

**Solution.** (i) Let

$$u_n = \frac{1}{1+nx^n}; \therefore u_{n+1} = \frac{1}{1+(n+1)x^{n+1}}$$

$$\frac{u_{n+1}}{u_n} = \frac{1+nx^n}{1+(n+1)x^{n+1}} = \frac{\frac{1}{nx^n} + 1}{\frac{1}{nx^n} + \left(1 + \frac{1}{n}\right)x}$$

If  $0 < x <$

∴  $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 1$

when  $x >$

∴  $\sum u_n$

If

∴ The gi

If  $x > 1$ ,

The series

For  $0 < x <$

Since  $\frac{1}{2} +$   
when  $0 < x < 1$ .

(ii) Let  $u_n$

If  $0 < x <$

∴  $\lim_{n \rightarrow \infty} \frac{u_n}{n} =$

If  $x > 1$ ,

∴ The nt

∴  $\sum u_n$

when

(iii) Let

If  $0 < x <$

∴  $\lim_{n \rightarrow \infty} \frac{u_n}{n} =$

when  $x >$

∴  $\sum u_n$

If

∴ The gi

## Sequences and Series

$$\text{If } x > 1, \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \frac{1}{x} < 1$$

$\rightarrow 1$  as  $n \rightarrow \infty$ .

The series is convergent for  $x > 1$ . When  $x = 1$ , the series  $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$  to  $\infty$  which is divergent.

For  $0 < x < 1$ ,

$$\frac{1}{1+x} + \frac{1}{1+2x^2} + \frac{1}{1+3x^3} + \dots > \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Since  $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$  is divergent, the given series is also divergent

when  $0 < x < 1$ .

$$(ii) \text{ Let } u_n = \frac{x^n}{1+x^n}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{1+x^n}{1+x^{n+1}} \cdot x$$

If  $0 < x < 1, x^n \rightarrow 0$  as  $n \rightarrow \infty$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = x < 1 \quad \therefore \sum u_n \text{ is convergent.}$$

$$\text{If } x > 1, \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{1+\frac{1}{x^n}} = 1 \neq 0$$

$\therefore$  The  $n$ th term does not tend to zero as  $n \rightarrow \infty$

$\therefore \sum u_n$  is divergent.

$$\text{when } x = 1, u_n = \frac{1}{2} \quad \therefore \sum u_n = \sum \frac{1}{2} \text{ is divergent.}$$

$$(iii) \text{ Let } u_n = \frac{x^n}{1+n^2 x^{2n}}; u_{n+1} = \frac{x^{n+1}}{1+(n+1)^2 x^{2n+2}}$$

$$\frac{u_{n+1}}{u_n} = \frac{1+n^2 x^{2n}}{1+(n+1)^2 x^{2n+2}} \cdot x$$

If  $0 < x < 1, n^2 x^{2n}, (n+1)^2 x^{2n+2}$  all tend to zero as  $n \rightarrow \infty$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = x < 1 \quad \therefore \sum u_n \text{ is convergent if } 0 < x < 1$$

$$\text{when } x > 1, \frac{u_{n+1}}{u_n} = \frac{\frac{1}{n^2 x^{2n}} + 1}{\frac{1}{n^2 x^{2n}} + \left(1 + \frac{1}{n}\right)^2 x^2} \cdot x \rightarrow \frac{1}{x^2} \cdot x = \frac{1}{x} < 1$$

$\therefore \sum u_n$  is convergent if  $x > 1$ .

$$\text{If } x = 1, u_n = \frac{1}{1+n^2} \text{ and } \sum u_n \text{ is convergent.}$$

$\therefore$  The given series is convergent for all real  $x$ .

(iv) Let

$$u_n = \frac{x^n}{1+x^{2n}}, u_{n+1} = \frac{x^{n+1}}{1+x^{2n+2}}$$

$$\frac{u_{n+1}}{u_n} = \frac{1+x^{2n}}{1+x^{2n+2}} \cdot x$$

If  $0 < x < 1$ ,  $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = x < 1 \therefore \sum u_n$  is convergent.

If  $x > 1$ ,  $\frac{u_{n+1}}{u_n} = \frac{\frac{1}{x^{2n}} + 1}{\frac{1}{x^{2n}} + x^2} \cdot x \rightarrow \frac{1}{x} < 1$  since  $\frac{1}{x^{2n}} \rightarrow 0$  as  $n \rightarrow \infty$

$\therefore \sum u_n$  is convergent if  $x > 1$

When  $x = 1$ ,  $u_n = \frac{1}{2}$ ; The series is  $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$  which is divergent.

$\therefore \sum u_n$  is convergent if  $x < 1$  and it is divergent if  $x = 1$

**Example 6.** Test the convergence of (i)  $\sum_{n=1}^{\infty} \frac{n^p}{n!}$  ( $p > 0$ )

$$(i) \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$(ii) \sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$$

$$(iii) \sum \frac{n!}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$$

**Solution.** (i) Let  $u_n = \frac{n^p}{n!}$

Then

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^p}{(n+1)!} \cdot \frac{n!}{n^p} = \left(\frac{n+1}{n}\right)^p \cdot \frac{1}{n+1} = \left(1 + \frac{1}{n}\right)^p \cdot \frac{1}{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 0 < 1$$

$\therefore$  The given series is convergent, by Ratio test.

(ii) Let

$$u_n = \frac{x^n}{n}$$

Then

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} = \lim_{n \rightarrow \infty} \frac{x}{\left(1 + \frac{1}{n}\right)} = x$$

$\therefore$  The series is convergent if  $0 < x < 1$  and diverges for  $x > 1$ . If  $x = 1$ , then  $u_n = \frac{1}{n}$  and hence  $\sum u_n$  is divergent.

$$(iii) \text{ Let } u_n = \frac{3^n n!}{n^n}$$

$$\frac{u_{n+1}}{u_n} = \frac{3^{n+1} (n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n n!} = \frac{3(n+1)}{\left(\frac{n+1}{n}\right)^n} \cdot \frac{1}{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}}$$

Hence,

(iii) Let

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{3}{\left(1 + \frac{1}{n}\right)^n} = \frac{3}{e} > 1$$

$\therefore \sum u_n$  is divergent by Ratio test.

(iv) Let

$$u_n = \frac{n!}{3 \cdot 5 \cdot 7 \dots (2n+1)}$$

Then

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{3 \cdot 5 \cdot 7 \dots (2n+3)} \cdot \frac{3 \cdot 5 \cdot 7 \dots (2n+1)}{n!} = \frac{n+1}{2n+3}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1 \therefore \sum u_n \text{ is convergent.}$$

**Example 7.** Test for the convergence of (i)  $\sum_{n=1}^{\infty} \frac{3n^2 + 1}{4^n}$

$$(ii) \sum_{n=1}^{\infty} \left( \frac{1}{2^n} + \frac{1}{3^n} \right)$$

$$(iii) \sum_{n=1}^{\infty} \frac{1}{n!}$$

$$(iv) \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \dots \text{to } \infty$$

**Solution.** (i) Let  $u_n = \frac{3n^2 + 1}{4^n}$

$$\frac{u_{n+1}}{u_n} = \frac{3(n+1)^2 + 1}{4^{n+1}} \cdot \frac{4^n}{3n^2 + 1} = \frac{3n^2 + 6n + 4}{3n^2 + 1} \cdot \frac{1}{4}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{3 + \frac{6}{n} + \frac{4}{n^2}}{3 + \frac{1}{n^2}} \cdot \frac{1}{4} = \frac{1}{4} < 1$$

$\therefore \sum u_n$  is convergent by Ratio test.

(ii) Let

$$u_n = \frac{1}{2^n} + \frac{1}{3^n} = \frac{3^n + 2^n}{6^n}$$

$$\frac{u_{n+1}}{u_n} = \frac{3^{n+1} + 2^{n+1}}{6^{n+1}} \cdot \frac{6^n}{3^n + 2^n} = \frac{1}{6} \cdot \frac{3^{n+1} + 2^{n+1}}{3^n + 2^n}$$

$$= \frac{1}{6} \cdot \frac{3^{n+1}}{3^n} \left( \frac{1 + \left(\frac{2}{3}\right)^{n+1}}{1 + \left(\frac{2}{3}\right)^n} \right)$$

then  $u_n =$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1 \text{ since } \left(\frac{2}{3}\right)^n \rightarrow 0 \text{ as } n \rightarrow \infty$$

Hence, by Ratio test, the given series converges.

(iii) Let  $u_n = \frac{1}{n!}$ ; Here  $u_n > 0$

$$\frac{u_{n+1}}{u_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

For  $x = 1, \infty$ 

10. Convergent

15. Convergent

17, 18, 19. Convergent

22. Divergent

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 0 < 1$$

$\therefore$  The given series is convergent.

$$(iv) \text{ Here, } u_n = \frac{n!}{3.5.7...(2n+1)}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)}{(2n+3)} = \frac{\left(1 + \frac{1}{n}\right)}{2 + \frac{3}{n}}$$

$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1$ ; By Ratio test, the series converges.

### EXERCISE 3

Test for convergency and divergency of the following series.  
(Take  $x > 0$ )

1.  $\sum \frac{1}{1+x^n}$

2.  $\sum n^{100}x^n$

3.  $\sum (\sqrt{n^2+1} - n)x^n$

4.  $\sum \frac{\sqrt{n}}{n^2+3} \cdot x^n$

5.  $\sum \frac{x^n}{\sqrt{n}}$

6.  $\sum \frac{a^n x^n}{n^2+1}, a > 0$

7.  $\sum n!x^n$

8.  $\sum \frac{x^n}{n(n+1)}$

9.  $\sum_{n=1}^{\infty} \frac{x^{n-1}}{(2n-1)^p}$

10.  $\frac{1 \cdot x}{2 \cdot 4} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} x^2 + \dots + \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n+2)} x^n + \dots$

11.  $1 + \frac{2^2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + \dots \text{to } \infty$

12.  $\sum \sqrt{\frac{2^n-1}{3^n-1}}$

13.  $\sum \frac{n^2}{n!}$

14.  $\sum \frac{n!}{n^n}$

15.  $\sum \frac{x^n}{1+n^2}$

16.  $\sum \sqrt{\frac{n}{n^2+1}} \cdot x^n$

17.  $\sum \frac{n^2}{3^n}$

18.  $\sum \frac{n}{1+2^n}$

19.  $\sum \frac{n^3+k}{2^n+k}$

20.  $\sum \frac{n^2-1}{n^2+1} x^n$

21.  $3x + 5x^2 + 7x^3 + \dots \text{to } \infty$

22.  $\frac{1}{1+\sqrt{2}} + \frac{2}{1+2\sqrt{3}} + \frac{3}{1+3\sqrt{4}} + \dots \text{to } \infty$

23.  $\sum \frac{10^n}{n!}$

24.  $\sum \frac{3^n-1}{4^n+1}$

25.  $\sum \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{4 \cdot 6 \cdot 8 \dots (2n+2)}$

26.  $\sum \frac{x^n}{n^2+1}$

27.  $\frac{1}{2} + \frac{1.6}{2.6} + \frac{1.6 \cdot 11}{2.6 \cdot 10} + \dots \text{to } \infty$

### ANSWERS

1. Convergent if  $x > 1$  and divergent if  $0 < x \leq 1$ 2, 3. Convergent if  $0 < x < 1$  and divergent if  $x \geq 1$ 4. Convergent if  $0 < x \leq 1$  and divergent if  $x > 1$ 5. Convergent if  $0 < x < 1$  and divergent if  $x \geq 1$ 6. Convergent if  $0 < x \leq \frac{1}{a}$  and divergent if  $x > \frac{1}{a}$ 7. Divergent for all real  $x (x \neq 0)$ , and convergent for  $x = 0$ 8. Convergent for  $0 < x \leq 1$  and divergent for  $x > 1$ 9. Convergent for  $x > 1$ , for all  $p$  divergent for  $x > 1$  for all  $p$ Taking log<sub>e</sub> $\lim_{n \rightarrow \infty} u_n^{1/n}$ 

By Cauchy

(ii) Let

Let

## Sequences and Series

For  $x = 1$ , convergent if  $p > 1$ , divergent if  $p \leq 1$

10. Convergent if  $0 < x \leq 1$ , divergent if  $x > 1$       11. Divergent. 12, 13, 14 Convergent  
 15. Convergent for  $x \leq 1$ , divergent for  $x > 1$       16. Convergent for  $x < 1$ , divergent if  $x \geq 1$   
 17, 18, 19. Convergent. 20, 21. Convergent if  $x < 1$ , divergent if  $x \geq 1$   
 22. Divergent. 23, 24, 25. Convergent      26. Convergent if  $0 < x \leq 1$  27. Convergent

## CAUCHY'S ROOT TEST

If  $\sum_{n=1}^{\infty} u_n$  is a series of positive terms, then the series is convergent or divergent according as

$\lim_{n \rightarrow \infty} u_n^{1/n}$  is less than 1 or greater than 1.

**Example 1.** Test the convergence of the series

$$(i) \sum \frac{1}{(\log n)^n}$$

$$(ii) \sum \frac{(n+1)(n+2)\dots(n+n)}{n^n}$$

$$(iii) \sum \frac{1}{\left(1 + \frac{1}{n}\right)^{n^2}}$$

$$(iv) \sum \frac{x^n}{(n+1)^n}, x > 0$$

$$(v) \sum \frac{r(n+1)^n}{n^{n+1}} \text{ where } r \text{ is a constant.}$$

**Solution.** (i) Let  $u_n = \frac{1}{(\log n)^n}$ ; since the power contains  $n$  we will use Root test.

$$u_n^{1/n} = \left\{ \frac{1}{(\log n)^n} \right\}^{1/n} = \frac{1}{\log n}$$

$\lim_{n \rightarrow \infty} u_n^{1/n} \rightarrow 0$  which is less than 1 as  $n \rightarrow \infty$

By Cauchy's Root test, the series converges.

$$(ii) \text{ Let } u_n = \frac{(n+1)(n+2)\dots(n+n)}{n^n}$$

$$\begin{aligned} k &= \lim_{n \rightarrow \infty} u_n^{1/n} = \lim_{n \rightarrow \infty} \left\{ \frac{n+1}{n} \cdot \frac{n+2}{n} \cdots \frac{n+n}{n} \right\}^{1/n} \\ &= \lim \left\{ \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \cdots \left(1 + \frac{n}{n}\right) \right\}^{1/n} \end{aligned}$$

Taking logarithm

$$\begin{aligned} \log k &= \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \sum_{r=1}^n \log \left(1 + \frac{r}{n}\right) \right\} \\ &= \int_0^1 \log(1+x) dx \text{ using integration as a summation} \\ &= [x \log(1+x)]_0^1 - \int_0^1 \frac{x}{x+1} dx = 2 \log 2 - 1 = \log \frac{4}{e} \\ k &= \frac{4}{e} > 1 \text{ since } 2 < e < 3 \end{aligned}$$

$\therefore \sum u_n$  is divergent.

(iii) Let  $u_n = \frac{1}{\left(1 + \frac{1}{n}\right)^n}$  for divergent series and use root test.

$$u_n^{1/n} = \left\{ \frac{1}{\left(1 + \frac{1}{n}\right)^n} \right\}^{1/n} = \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

$$\lim_{n \rightarrow \infty} u_n^{1/n} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

$\therefore$  The given series is convergent.

(iv) Let

$$u_n = \frac{x^n}{(n+1)^n}; u_n^{1/n} = \frac{x}{n+1}$$

$\lim_{n \rightarrow \infty} u_n^{1/n} = 0$  for all real  $x$ .

Since this limit is  $< 1$ , the series is convergent.

(v) Let  $u_n = \frac{\{r(n+1)\}^n}{n^{n+1}}$

$$u_n^{1/n} = \frac{r(n+1)}{n^{1/n}} = \frac{r \cdot \left(1 + \frac{1}{n}\right)}{n^{1/n}}$$

$\lim_{n \rightarrow \infty} u_n^{1/n} = r$  since  $\lim_{n \rightarrow \infty} n^{1/n} = 1$

$\therefore \sum u_n$  is convergent if  $r < 1$  and divergent if  $r > 1$  when  $r = 1$ , Root test fails. Therefore go to basic idea.

When  $r = 1$ ,  $u_n = \left(1 + \frac{1}{n}\right)^n \cdot \frac{1}{n}$

Take  $v_n = \frac{1}{n}$  and use comparison test.

$$\frac{u_n}{v_n} = \left(1 + \frac{1}{n}\right)^n \rightarrow e \text{ as } n \rightarrow \infty$$

$\therefore \sum u_n$  and  $\sum v_n$  converge or diverge together. But  $\sum v_n$  is divergent.

$\therefore \sum u_n$  is divergent when  $r = 1$ .

**Example 2.** Test the convergence of (i)  $\sum \frac{n^3}{3^n}$  (ii)  $\sum_{n=1}^{\infty} e^{\sqrt{n}} x^n$  (iii)  $\sum_{n=1}^{\infty} \left(\frac{n+a}{n+b}\right)^n x^n$

**Solution.** (i) Let

$$u_n = \frac{n^3}{3^n}; u_n^{1/n} = \left(\frac{n^3}{3^n}\right)^{1/n} = \frac{(n^{1/n})^3}{3}$$

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$$\lim_{n \rightarrow \infty} u_n^{\frac{1}{n}} = \frac{1}{3} < 1 \text{ since } \lim_{n \rightarrow \infty} n^{1/n} = 1$$

By Root test,  $\sum u_n$  converges.

(ii) Let  $u_n = e^{\sqrt{n}} x^n$

$$u_n^{\frac{1}{n}} = (e^{\sqrt{n}} x^n)^{1/n} = e^{1/\sqrt{n}} \cdot x$$

$$\lim_{n \rightarrow \infty} u_n^{\frac{1}{n}} = x$$

∴ By Root test,  $\sum u_n$  converges if  $0 < x < 1$  and diverges if  $x > 1$

If  $x = 1$ ,  $u_n = e^{\sqrt{n}}$   $\lim_{n \rightarrow \infty} u_n = \infty \therefore \sum u_n$  diverges if  $x = 1$

(iii) Let  $u_n = \left( \frac{n+a}{n+b} \right)^n \cdot x^n$

$$u_n^{\frac{1}{n}} = \frac{n+a}{n+b} \cdot x = \frac{\left(1 + \frac{a}{n}\right)}{1 + \frac{b}{n}} \cdot x \therefore \lim_{n \rightarrow \infty} u_n^{\frac{1}{n}} = x$$

∴  $\sum u_n$  converges if  $0 < x < 1$  and diverges if  $x > 1$  by Root test.

If  $x = 1$ ,  $u_n = \left( \frac{n+a}{n+b} \right)^n = \frac{\left(1 + \frac{a}{n}\right)^n}{\left(a + \frac{b}{n}\right)^n}$

$$\lim_{n \rightarrow \infty} u_n = \frac{\lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n}{\lim_{n \rightarrow \infty} \left(1 + \frac{b}{n}\right)^n} = \frac{e^a}{e^b} = e^{a-b}$$

$\lim_{n \rightarrow \infty} u_n$  does not tend to zero.

∴  $\sum u_n$  diverges if  $x = 1$

**Example 3.** Test the convergence of  $\sum_{n=1}^{\infty} \frac{n^k}{a^n}$  and hence discuss the convergence of  $\sum_{n=1}^{\infty} \frac{n^k + b}{a^n + c}$

**Solution.** Let  $u_n = \frac{n^k}{a^n}; u_n^{\frac{1}{n}} = \left( \frac{n^k}{a^n} \right)^{1/n} = \frac{(n^{1/n})^k}{a}$

$$\lim_{n \rightarrow \infty} u_n^{\frac{1}{n}} = \frac{1}{a} \lim_{n \rightarrow \infty} (n^{1/n})^k = \frac{1}{a} \cdot 1^k = \frac{1}{a}$$

∴ If  $\frac{1}{a} < 1$ ,  $\sum \frac{n^k}{a^n}$  converges.

That is,  $\sum \frac{n^k}{a^n}$  converges if  $a > 1$  and diverges if  $a < 1$

If  $a = 1$ ,  $\sum u_n = \sum n^k = \sum \frac{1}{n^{-k}}$   
which converges if  $-k > 1$  and diverges if  $-k \leq 1$

Taking

$$u_n = \frac{n^k}{a^n} \text{ and } v_n = \frac{n^k + b}{a^n + c}$$

$$\frac{v_n}{u_n} = \frac{n^k + b}{a^n + c} \cdot \frac{a^n}{n^k} = \left( \frac{n^k + b}{n^k} \right) \left( \frac{a^n}{a^n + c} \right) = \left( 1 + \frac{b}{n^k} \right) \left( \frac{1}{1 + \frac{c}{a^n}} \right)$$

$$\lim_{n \rightarrow \infty} \frac{v_n}{u_n} = 1 \text{ since } n^k \rightarrow \infty \text{ if } k > 0 \text{ and } a^n \rightarrow \infty \text{ if } a > 1$$

$\sum v_n$  and  $\sum u_n$  converge or diverge together.

$\therefore \sum v_n$  converges if  $a > 1$  and  $k > 0$

If  $0 < a < 1$ ,  $a^n \rightarrow 0$  and then  $\lim_{n \rightarrow \infty} \frac{v_n}{u_n} = 0$

Hence comparison test fails.

**Example 4.** Test for convergence of  $\sum \frac{n!}{n^n}$

**Solution.** Let  $u_n = \frac{n!}{n^n}$

$$k = \lim_{n \rightarrow \infty} u_n^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left\{ \frac{n!}{n^n} \right\}^{1/n} = \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} \cdot \frac{2}{n} \cdots \frac{n}{n} \right\}^{1/n}$$

$$\log k = \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \sum_{r=1}^n \log \frac{x}{r} \right\} = \int_0^1 \log x \, dx$$

by using the concept that integration is the summation

$$= (x \log x)_0^1 - \int_0^1 dx = -1$$

$$k = e^{-1} = \frac{1}{e} < 1$$

Hence  $\sum u_n$  is convergent.

#### EXERCISE 4

Test for convergence of the series below:

$$1. \sum \frac{x^n}{n^n}, x > 0$$

$$2. \sum \left( 1 + \frac{1}{\sqrt{n}} \right)^{n^{-3/2}}$$

$$3. (\sqrt[n]{n} - 1)^n$$

$$4. a + b + a^2 + b^2 + a^3 + b^3 + \dots \text{to } \infty$$

$\therefore f(n+1)$

That is,  $f(n)$

$$5. \sum \left( \frac{n}{2n+1} \right)^n$$

$$6. \sum \left( \frac{n}{n^2+1} \right)^n$$

$$7. \sum \frac{2^n n!}{n^n}$$

$$8. \sum \frac{3^n n!}{n^n}$$

$$9. \sum e^{-\sqrt{n}} x^n$$

$$10. \sum_{n=1}^{\infty} \frac{n^3 + 100}{2^n + 10}$$

1. 2, 3, Con  
5. 6, 7, Con  
10. Converge

If  $\sum f(n)$   
increases, then

$$\int_1^{\infty} f(x) \, dx$$

**Example**  
and diverges for

**Solution.**

and  $\rightarrow \infty$

Here  $\frac{1}{p} -$

Hence,  $\sum$

**Example**

**Solution.**

Let

## ANSWERS

1. 2, 3, Convergent      4. Convergent if  $0 < a < 1$  and  $0 < b < 1$  and divergent if  $a \geq 1$  or  $b \geq 1$   
 5. 6, 7, Convergent      8. Divergent      9. Converges if  $0 < x < 1$  and diverges if  $x > 1$   
 10. Convergent

## INTEGRAL TEST

If  $\sum f(n) = f(1) + f(2) + \dots + f(n) + \dots$  is a series of positive terms and  $f(n)$  decreases as  $n$  increases, then  $\sum f(n)$  converges or diverges according as the integral

$$\int_1^{\infty} f(x) dx \text{ is finite or infinite}$$

**Example 1.** By using integral test, show that the harmonic series  $\sum \frac{1}{n^p}$  converges for  $p \geq 1$

and diverges for  $p \leq 1$

**Solution.** Here,  $f(n) = \frac{1}{n^p}$  decrease as  $n$  increases. For  $p \neq 1$ ,  $\int_1^{\infty} \frac{1}{x^p} dx = \lim_{N \rightarrow \infty} \int_1^N \frac{dx}{x^p}$

$$= \lim_{N \rightarrow \infty} \left( \frac{x^{1-p}}{1-p} \right)_1^N = \lim_{N \rightarrow \infty} \frac{N^{1-p} - 1}{1-p} = \frac{1}{p-1} \text{ if } p > 1$$

and  $\rightarrow \infty$  if  $p < 1$

Here  $\frac{1}{p-1}$  is finite. If  $p = 1$ ,  $\int_1^{\infty} \frac{1}{x^p} dx = \int_1^{\infty} \frac{1}{x} dx = (\log x)_1^{\infty} \rightarrow \infty$

Hence,  $\sum \frac{1}{n^p} = \sum f(n)$  converges if  $p > 1$  and diverges if  $p \leq 1$ .

**Example 2.** Using integral test, show that the series  $\frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + \dots$ , to  $\infty$ , diverges

**Solution.**  $n$ th term  $= \frac{n}{(2n-1)(2n+1)}$

Let  $f(n) = \frac{n}{(2n-1)(2n+1)}$

$$\begin{aligned} f(n) - f(n+1) &= \frac{n}{(2n-1)(2n+1)} - \frac{n+1}{(2n+1)(2n+3)} \\ &= \frac{1}{2n+1} \left[ \frac{n}{2n-1} - \frac{n+1}{2n+3} \right] = \frac{1}{(2n+1)} \cdot \frac{2n+1}{(2n-1)(2n+3)} \\ &= \frac{1}{(2n-1)(2n+3)} = +\text{ve} \end{aligned}$$

$\therefore f(n+1) < f(n)$

That is,  $f(n)$  decreases as  $n$  increases.

$$\begin{aligned} \int_1^{\infty} f(x) dx &= \int_1^{\infty} \frac{x}{(2x-1)(2x+1)} dx = \int_1^{\infty} \frac{1}{4} \left[ \frac{1}{2x-1} + \frac{1}{2x+1} \right] dx \\ &= \lim_{N \rightarrow \infty} \int_1^N \frac{1}{4} \left[ \frac{1}{2x-1} + \frac{1}{2x+1} \right] dx \end{aligned}$$

$$\begin{aligned}
 &= \lim_{N \rightarrow \infty} \frac{1}{8} [\log(2x-1) + \log(2x+1)]_1^N \\
 &= \lim_{N \rightarrow \infty} \frac{1}{8} [\log(4x^2-1)]_1^N \\
 &= \lim_{N \rightarrow \infty} \frac{1}{8} [\log(4N^2-1) - \log 3] \rightarrow \infty
 \end{aligned}$$

$\therefore$  Given series diverges by integral test.

Note : This could be done easily by comparison test

### EXERCISE 5

Using integral test, test for convergence or divergence of the following series. (Given under Exercise 2) whose  $n$ th term is

1.  $\frac{n}{(n+2)(n+3)(n+4)}$
2.  $\frac{n(n+1)}{(n+2)(n+3)(n+4)}$
3.  $\frac{1}{x+n}$  if  $x > 0$
4.  $\frac{1}{x+ny}$  if  $x, y > 0$
5.  $\frac{1}{2n-1}$
6.  $\frac{1}{(2n-1)(2n)}$

### ANSWERS

1. 6, Convergent; 2 to 5 divergent.

Note : All these problems can be easily done by comparison list.

### ALTERNATING SERIES

Hitherto, we were dealing with infinite series whose terms are all positive. In practice, this need not be the case. There are series whose terms are positive and negative.

**Alternating series.** A series in which the terms are alternately positive and negative is called an alternating series.

**Absolute convergence of series.** The series  $\sum u_n$  containing positive and negative terms is said to be *absolutely convergent*, if the series formed by the numerical values of the terms of  $\sum u_n$  is convergent. That is,  $\sum u_n$  is absolutely convergent if  $\sum |u_n|$  is convergent.

Hence the  
Example  
Solution

Here

**Conditionally convergent.** The series  $\sum u_n$  containing positive and negative terms is said to be conditionally convergent or semi convergent, if

(i)  $\sum u_n$  is convergent while

(ii)  $\sum |u_n|$  is not convergent. (That is, divergent)

Example  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \dots$  is absolutely convergent since,

$\sum \left| \frac{(-1)^{n-1}}{n^2} \right| = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$  is convergent.

While,

$\sum u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$  is convergent whereas

$\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \dots$  is divergent and hence

$\therefore$  The se

If  $|x| > 1$

Example  
Solution.

A function  $f(x, y)$  of two variables is continuous at a point  $(a, b)$  of the domain of  $f(x, y)$ , if for any small given positive number  $\epsilon$ , there exists a positive number  $\delta$  such that

$$|f(x, y) - f(a, b)| < \epsilon \text{ for}$$

all points  $(x, y)$  of the neighbourhood

$$\{(x, y); a - \delta \leq x \leq a + \delta; b - \delta \leq y \leq b + \delta\} \text{ of } (a, b)$$

In other words, for continuity at  $(a, b)$ , there exists a region bounded by the square,  $x = a - \delta$ ,  $x = a + \delta$ ,  $y = b - \delta$ ,  $y = b + \delta$ , such that for every  $(x, y)$  of the region of the square  $f(x, y)$  lies between

$$f(a, b) - \epsilon \text{ and } f(a, b) + \epsilon$$

**Definition 2.** The function  $f(x, y)$  is said to be **continuous** at the point  $(a, b)$  if

$$(i) \quad \underset{\substack{x \rightarrow a \\ y \rightarrow b}}{\text{Lt}} \quad f(x, y) \text{ exists and equals to } l \text{ and}$$

$$(ii) \quad f(a, b) = l.$$

**Continuity of a function :** A function  $f(x, y)$  is said to be continuous in the domain, if it is continuous at every point of its domain.

**Note :** Since one variable can be kept constant within the square, a **continuous function of two variables is always continuous function of each variable**.

### Limit of a continuous function

If  $f(x, y)$  is a continuous function in some domain

$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = f(a, b)$$

That is, the limit of a continuous function is equal to the value of the function at the point

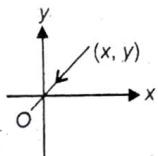
**Example 1.** Evaluate the limit of the following functions at the given points.

$$(i) \quad \underset{\substack{x \rightarrow 0 \\ y \rightarrow 0}}{\text{Lt}} \quad \frac{2y + x}{2x^2 + y^2 - 3} \quad (ii) \quad \underset{\substack{x \rightarrow 1 \\ y \rightarrow 2}}{\text{Lt}} \quad \frac{xy}{x^2 + y^2 + 2} \quad (iii) \quad \underset{\substack{x \rightarrow 1 \\ y \rightarrow 1}}{\text{Lt}} \quad \frac{2x + 1}{3y + 2} \quad (iv) \quad \underset{\substack{x \rightarrow 1 \\ y \rightarrow 0}}{\text{Lt}} \quad \frac{x^2 + y}{x + 1}$$

**Solution.**  $f(x, y)$  is continuous in all the problems and  $f(a, b)$  exists.

$$(i) \quad \underset{\substack{x \rightarrow 0 \\ y \rightarrow 0}}{\text{Lt}} \quad \frac{2y + x}{2x^2 + y^2 - 3} = f(0, 0) = \frac{0}{-3} = 0 \quad (ii) \quad \underset{\substack{x \rightarrow 1 \\ y \rightarrow 2}}{\text{Lt}} \quad \frac{xy}{x^2 + y^2 + 2} = f(1, 2) = \frac{2}{7}$$

$$(iii) \quad \underset{\substack{x \rightarrow 1 \\ y \rightarrow 1}}{\text{Lt}} \quad \frac{2x + 1}{3y + 2} = f(1, 1) = \frac{3}{5} \quad (iv) \quad \underset{\substack{x \rightarrow 1 \\ y \rightarrow 0}}{\text{Lt}} \quad \frac{x^2 + y}{x + 1} = \frac{1}{2}$$



**Example 2.** Show that the following limits do not exist :

$$(i) \quad \underset{\substack{x \rightarrow 0 \\ y \rightarrow 0}}{\text{Lt}} \quad \frac{2x^2 + 3y^2}{4xy}$$

$$(ii) \quad \underset{\substack{x \rightarrow 0 \\ y \rightarrow 0}}{\text{Lt}} \quad \frac{x^2}{2x^2 + 3y^2}$$

**Solution.**  $y = mx$  passes through the origin. Take a neighbouring point  $(x, y)$  on this line and let  $x \rightarrow 0$  and  $y \rightarrow 0$

$$(i) \quad \text{Let } f(x, y) = \frac{2x^2 + 3y^2}{4xy},$$

$\underset{\substack{x \rightarrow 0 \\ y \rightarrow 0}}{\text{Lt}} f(x, y)$  is of the form  $\frac{0}{0}$ .

$$\text{On } y = mx, f(x, y) = \frac{2x^2 + 3m^2x^2}{4mx^2} = \frac{2 + 3m^2}{4m}, x \neq 0 \quad \underset{\substack{x \rightarrow 0 \\ y \rightarrow 0}}{\text{Lt}} f(x, y) = \frac{2 + 3m^2}{4m}.$$

As  $x \rightarrow 0, y \rightarrow 0$  on different paths  $y = mx$  ( $m$  can take different values such as 1, 2) the limit values differ (as  $m$  differs). Therefore the limit is not unique. Hence, the limit does not exist.

(ii) Let

$$f(x, y) = \frac{x^2}{2x^2 + 3y^2}.$$

Taking the path  $y = mx$  as in the previous problem, on the path,  $f(x, y) = \frac{x^2}{2x^2 + 3m^2x^2} = \frac{1}{2 + 3m^2}$

Therefore

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y) = \frac{1}{2 + 3m^2}$$

which is different for different values of  $m$ . Hence the limit is not unique. Therefore, the limit does not exist.

**Example 3.** Discuss the continuity of the following functions.

$$(i) \frac{x+y}{2x+y}$$

$$(ii) \frac{x-y}{x+y}$$

$$(iii) \frac{3x^2y^2}{2x^4+y^4}$$

$$(iv) \frac{x^4+y^4}{2x^2y^2+y^4}$$

**Solution.** On

$$y = mx, (i) f(x, y) = \frac{x+y}{2x+y} \text{ becomes } \frac{1+m}{2+m}$$

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y) = \frac{1+m}{2+m} \text{ which varies as } m \text{ varies.}$$

Hence the limit does not exist. Hence  $f(x, y)$  is not continuous at  $(0, 0)$ . Therefore  $f(x, y)$  is not continuous in the  $xy$  plane. (ii), (iii), (iv) : In the other three problems also, taking the path as  $y = mx$ , the limits do not exist at the origin. Hence, the function is not continuous at the origin and therefore not continuous in the  $xy$  plane.

**Example 4.** Prove that the functions given below are continuous at the origin

$$(i) \frac{y}{2x^2 + y^2 + 7}$$

$$(ii) f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0 & \text{when } (x, y) = (0, 0) \end{cases}$$

$$\text{Solution. (i)} f(x, y) = \frac{y}{2x^2 + y^2 + 7}$$

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y) = 0 \text{ and } f(0, 0) = 0$$

Hence it is continuous at  $(0, 0)$ .

$$(ii) f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}, (x, y) \neq (0, 0)$$

$$\text{on path } y = mx, f(x, y) = \frac{mx^2}{x\sqrt{1+m^2}} = \frac{mx}{\sqrt{1+m^2}}$$

$$\lim_{x \rightarrow 0} f(x, y) = 0 \text{ for all values of } m.$$

Sit  
problem.  
He

Evaluate

$$1. \quad \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y)$$

$$3. \quad \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y)$$

$$5. \quad \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y)$$

Discuss th

$$7. \quad \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y)$$

11. Is the

12. Its th

13. If a f

14. If  $f(x,$

15. If  $f(x,$

cont

$$1. \quad \frac{2}{3} \quad 2$$

10. not c

$1 - \frac{1}{2} + \frac{1}{3} - \dots$  is conditionally convergent.

**Theorems.** (without proof)

1. An absolutely convergent series is convergent.
2. If the terms of an absolutely convergent series are rearranged, the series remains convergent and its sum is unaltered.
3. In a conditionally convergent series, a rearrangement of terms may alter the sum of the series.

4. If  $\sum u_n$  is an alternating series and if  $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = l < 1$ , the series  $\sum u_n$  is absolutely convergent and hence it is convergent while if  $l > 1$ , the series  $\sum u_n$  is not convergent.

**Example 1.** Prove that the exponential series  $1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$  to  $\infty$  is absolutely convergent and hence convergent for all values of  $x$ .

**Solution.** Let  $u_n = \frac{x^{n-1}}{(n-1)!}$ ;  $u_{n+1} = \frac{x^n}{n!}$

$$\frac{u_{n+1}}{u_n} = \frac{x^n}{n!} \times \frac{(n-1)!}{x^{n-1}} = \frac{x}{n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{n} = 0 < 1 \text{ for all } x.$$

Hence the series is absolutely convergent and hence convergent for all  $x$ . (real)

**Example 2.** Discuss the convergence of the Binomial series.

**Solution.** The Binomial series is

$$1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots \text{ to } \infty$$

$$\text{Here } u_r = \frac{n(n-1)(n-2)\dots(n-r+2)}{(r-1)!} x^{r-1}$$

$$u_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} x^r$$

$$\frac{u_{r+1}}{u_r} = \frac{n-r+1}{r} \cdot x = \left( \frac{n+1}{r} - 1 \right) x$$

$$\left| \frac{u_{r+1}}{u_r} \right| = \left| \frac{n+1}{r} - 1 \right| |x|$$

$$\lim_{r \rightarrow \infty} \left| \frac{u_{r+1}}{u_r} \right| = |x|. \lim_{r \rightarrow \infty} \left| \frac{n+1}{r} - 1 \right| = |x|$$

$\therefore$  The series is absolutely convergent if  $|x| < 1$  and hence convergent if  $|x| < 1$

If  $|x| > 1$ , the series  $\sum |u_n|$  is divergent.

**Example 3.** Discuss the convergence of the logarithmic series.

**Solution.** The logarithmic series is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + \dots \text{to } \infty$$

$$u_n = (-1)^{n-1} \frac{x^n}{n}$$

$$\frac{u_{n+1}}{u_n} = -\frac{n}{n+1} x = \frac{-x}{\left(1 + \frac{1}{n}\right)}$$

$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = |x|$ .  $\therefore$  If  $|x| < 1$ ,  $\sum u_n$  is absolutely convergent and hence converges. If  $|x| > 1$ ,  $u_n$  does not tend to zero as  $n \rightarrow \infty$  hence the series is not convergent.

If  $x = 1$ , the series is  $\sum \frac{(-1)^{n-1}}{n}$  which is convergent by alternating series test since

$$\frac{1}{n} > \frac{1}{n+1} \text{ and } \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

If  $x = -1$ , the series becomes  $\sum \left(-\frac{1}{n}\right)$  which is divergent. Hence the logarithmic series converges if  $-1 < x \leq 1$  and diverges if  $x \leq -1$  and  $x > 1$ .

### LEIBNITZ'S TEST

The alternating series  $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$  converges if

- (i)  $\{u_n\}$  is a monotonic decreasing sequence
- and (ii)  $\lim_{n \rightarrow \infty} u_n = 0$ .

In other words,

The alternating series  $u_1 - u_2 + u_3 - u_4 + \dots \text{to } \infty$  converges if

(i) Each term is numerically less than its preceding term and

(ii)  $\lim_{n \rightarrow \infty} u_n = 0$ .

**Note.** In the above theorem, if  $\lim_{n \rightarrow \infty} u_n \neq 0$ , then the alternating series  $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$  oscillates.

**Example 1.** Prove  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\log(n+1)}$  converges.

**Solution.** Let  $u_n = \frac{1}{\log(n+1)}$

The given series is an alternating series.

$\log(n+2) > \log(n+1)$  for  $n > 0$

$$\frac{1}{\log(n+2)} < \frac{1}{\log(n+1)} \quad u_{n+1} < u_n \text{ for } n > 0$$

$\{u_n\}$  is monotonic decreasing.

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{\log(n+1)} = 0$$

Since decreasing

$\therefore$  By

(ii) L

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i.e.,

Hence by Leibnitz's test, the given series converges.

**Example 2.** Test for convergence of the series

$$(1+1) - \left(1 + \frac{1}{2}\right) + \left(1 + \frac{1}{3}\right) - \left(1 + \frac{1}{4}\right) + \dots \text{to } \infty$$

**Solution.** Here,  $a_n = 1 + \frac{1}{n}$  where the given series is  $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$

Since  $\frac{1}{n+1} < \frac{1}{n}$ ,  $a_{n+1} < a_n$  and

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1 \neq 0; \text{ Hence the series given is not convergent.}$$

**Example 3.** Bring out the fallacy in the following proof:

$$\text{"We know, } 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \text{to } \infty = \log_e 2.$$

$$\begin{aligned} \text{Also, } 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \text{to } \infty &= \left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \text{to } \infty\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots \text{to } \infty\right) \\ &= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \text{to } \infty\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots \text{to } \infty\right) = 0'' \end{aligned}$$

**Solution.** In the above argument, we have done rearrangement of terms which is not permissible. Also, to find the sum of an infinite series, we should get  $s_n = \text{sum to } n \text{ terms}$  and then get limit  $s_n$  as  $n \rightarrow \infty$ . This is not done here in the above proof. Hence the fallacy.

**Example 4.** Discuss the convergence of

$$(i) \sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n}) \quad (ii) \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{n}{3n-2}$$

**Solution.** (i) The series is evidently alternating series

$$u_n = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$u_{n+1} = \frac{1}{\sqrt{n+2} + \sqrt{n+1}}$$

Since  $(\sqrt{n+2} + \sqrt{n+1}) > (\sqrt{n+1} + \sqrt{n})$ , evidently  $u_{n+1} < u_n$  for all  $n$  i.e.,  $\{u_n\}$  is a monotonically decreasing sequence

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0.$$

∴ By Leibnitz's test, the given series converges.

(ii) Let  $u_n = \frac{n}{3n-2}$ . The given series is an alternating series.

We have to prove  $u_n > u_{n+1}$  for all  $n$ .

$$\text{i.e., } \frac{n}{3n-2} > \frac{n+1}{3(n+1)-2}$$

$$\text{i.e., } \frac{3n-2}{n} < \frac{3(n+1)-2}{n+1}$$

i.e.,

$$3 - \frac{2}{n} < 3 - \frac{2}{n+1}$$

i.e.,

$$-\frac{2}{n} < -\frac{2}{n+1}$$

i.e.,

$$\frac{2}{n} > \frac{2}{n+1}$$

i.e.,

$$n < n+1 \text{ which is true.}$$

Hence  $u_n > u_{n+1}$  for all

$$\text{Further } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{n}{3n-2} = \frac{1}{3} \neq 0$$

 $\therefore$  The given series oscillates.

**Note.** By testing  $\lim_{n \rightarrow \infty} u_n = \frac{1}{3} \neq 0$ , we can immediately say that the given series does not converge; the first part need not be proved.

**Example 5.** Discuss the convergence of

$$(i) \sum (-1)^n \left(1 + \frac{1}{n}\right) \quad (ii) \sum (-1)^n \sin\left(\frac{1}{n}\right) \quad (iii) \frac{1}{1.2} - \frac{1}{3.4} + \frac{1}{5.6} - \dots \text{to } \infty$$

**Solution.** (i) Let  $u_n = 1 + \frac{1}{n}$

$$u_{n+1} = 1 + \frac{1}{n+1}$$

 $\therefore u_{n+1} < u_n \text{ for all } n$ 

and

$$\lim_{n \rightarrow \infty} 1 + \frac{1}{n} = 1 \neq 0$$

Hence the given series does not converge. In fact, it oscillates.

$$(ii) \text{ Here } u_n = \sin\left(\frac{1}{n}\right); u_{n+1} = \sin\left(\frac{1}{n+1}\right)$$

$$\frac{1}{n+1} < \frac{1}{n} \text{ and } \sin\left(\frac{1}{n+1}\right) < \sin\left(\frac{1}{n}\right)$$

i.e.,  $u_{n+1} < u_n$  for all  $n$ .

$$\text{Further } \lim_{n \rightarrow \infty} \sin\left(\frac{1}{n}\right) = 0$$

Hence, by alternating series test, it converges.

$$(iii) \text{ Here } u_n = \frac{1}{(2n-1)(2n)}$$

and

$$u_{n+1} = \frac{1}{(2n+1)(2n+2)}$$

$$\text{Since } (2n+1)(2n+2) > (2n-1)(2n)$$

$$u_{n+1} = u_{n+1} < u_n$$

and

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{(2n-1)(2n)} = 0$$

Hence, the given series is convergent.

**Example 6.** Discuss the convergence of the series**Solu**

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$$(i) 1 - \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{3} - \frac{\sqrt{4}}{4} + \dots \infty$$

$$(ii) 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \infty$$

$$(iii) \sum \frac{(-1)^{n+1}}{n^p} \text{ where } p > 0$$

$$(iv) \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{2n-1}$$

**Solution.** Consider  $1 - \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{3} - \frac{\sqrt{4}}{4} + \dots \text{to } \infty$ .

(i) In alternating series, we try to use Leibnitz's test.

### Points to check

**P1.** The terms are alternately positive and negative

$$\mathbf{P2.} \quad u_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$$

$$u_{n+1} = \frac{1}{\sqrt{n+1}}$$

Evidently  $u_{n+1} < u_n$  since  $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$  for all  $n$

$$\mathbf{P3.} \text{ Further } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

All the conditions of the Leibnitz's test are satisfied.

Hence the given series is convergent.

However, it is not absolutely convergent (why?)

$$(ii) \text{ Consider } 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \text{ to } \infty$$

$$\text{Here } u_n = \frac{1}{n} \text{ and } u_{n+1} = \frac{1}{n+1}$$

**P1.** The series is alternately positive and negative.

$$\mathbf{P2.} \quad u_{n+1} < u_n \text{ since } \frac{1}{n+1} < \frac{1}{n} \text{ for all } n.$$

$$\mathbf{P3.} \quad \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Hence, by Leibnitz's test, the given series is convergent.

**Note.** This series is semi-convergent or conditionally convergent but not absolutely convergent.

(why?)

$$(iii) \quad u_n = \frac{1}{n^p} \text{ and } u_{n+1} = \frac{1}{(n+1)^p}$$

$$\frac{u_{n+1}}{u_n} = \frac{n^p}{(n+1)^p} = \frac{1}{\left(1 + \frac{1}{n}\right)^p} < 1$$

$$\text{since } p > 0, \left(1 + \frac{1}{n}\right)^p > 1$$

$\therefore u_{n+1} < u_n$   
and the series is alternating.

$$\text{Also } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{n^p} = 0 \text{ if } p > 0$$

Hence the series converges if  $p > 0$

(iv) Here

$$u_n = \frac{n}{2n-1}$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{2 - \frac{1}{n}} = \frac{1}{2} \neq 0$$

∴ The series is not convergent.

In fact, it is oscillatory.

**Example 7.** Discuss the convergence of the series  $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \dots \text{to } \infty$ , if  $0 < x < 1$ .

**Solution.** Since  $x$  is positive, the series is alternately positive and negative.

Here

$$u_n = \frac{x^n}{1+x^n} \text{ and } u_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$$

$$\begin{aligned} u_n - u_{n+1} &= \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}} = x^n \left( \frac{1}{1+x^n} - \frac{x}{1+x^{n+1}} \right) \\ &= \frac{x^n(1-x)}{(1+x^n)(1+x^{n+1})} = \text{a positive value since } 0 < x < 1 \end{aligned}$$

∴  $u_{n+1} < u_n$  for all  $n$ .

$$\text{Also, } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{x^n}{1+x^n} = 0$$

since  $x^n \rightarrow 0$  as  $n \rightarrow \infty$ 

∴ The series is convergent.

## EXERCISE 6

Discuss the convergence of the series :

1.  $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \dots \text{to } \infty$

2.  $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$

3.  $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \dots$

4.  $\sum \frac{(-1)^n}{n+7}$

5.  $\sum \frac{(-1)^{n-1}}{x+n}$

6.  $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots \text{to } \infty$

7.  $\sum (-1)^n (\sqrt{n+1} - \sqrt{n})$

8.  $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \dots \text{to } \infty$

9.  $1 - \frac{x}{1+a} + \frac{x^2}{1+2a} - \dots + \frac{(-1)^{n-1} x^{n-1}}{1+(n-1)a} + \dots$

10.  $\sum \frac{(-1)^n n}{2n+7}$

11.  $\sum \frac{(-1)^n x^n}{\sqrt{n}}$

12.  $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \text{to } \infty$

13.  $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots \text{to } \infty$

14.  $\sum (-1)^n \frac{1+n^2}{1+n^3}$

## ANSWERS

1 to 7. Convergent

8., 9. Convergent if  $|x| < 1$ 

10. Not convergent

11. Convergent if  $|x| < 1$  12. abs. convergent

13, 14. Convergent

## SHORT ANSWER QUESTIONS

1. Define convergence and divergence of a sequence.

2. Define oscillating sequence, bounded sequence.

- 3. Defi
- 4. State
- 5. Wha
- 6. Give
- 7. Stat
- 8. Def
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- 16. Un
- 17. If
- 18.  $\frac{1}{n}$
- 19.  $\frac{1}{n^2}$
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- 28. .
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- 30. .