48. Prove that Laplace transform of the triangular wave of period 2π defined by $f(t) = \begin{cases} t, & 0 \le t \le \pi \\ 2\pi - t, & \pi \le t \le 2\pi \end{cases}$

is $\frac{1}{s^2} \tanh\left(\frac{\pi s}{2}\right)$

49. Find the Laplace transform of $f(t) = \begin{cases} t, & 0 < t \le 2 \\ 4 - t, & 2 \le t < 4 \end{cases}$ and satisfy $f(t+4) = f(t)$.

50. Given that $L^{-1}\left[\frac{s^2+4}{(s^2-4)^2}\right] = t\cosh^2 t$, find $L^{-1}\left[\frac{s^2+1}{(s^2-1)^2}\right]$

[Ans: $t \cosh t$]

4 ANALYTIC FUNCTIONS

Introduction

Let $z = x + iy$ be a complex variable where x and y are real variables. If for every z, there exists one or more values of w, then w can be represented as a function of z.

(i.e.) $w = f(z) = u(x, y) + iv(x, y)$ is a function of the complex variable $z = x + iy$.

Example:

Let $w = z^2$, here for every z there exist a value of w.

Now $z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy = u + iv$

$f(z) = u + iv$ and $u = x^2 - y^2$ and $v = 2xy$

Limit of a function

Let $f(z)$ be a function defined in a set D and z_0 be a limit point of D. Then A is said to be limit of $f(z)$ at z_0, if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - A| < \epsilon$ for all z in D other than z_0 with $|z - z_0| < \delta$. It is denoted by

$\lim_{\triangle z \to 0} f(z) = A, |z - z_0| < \delta$.

Continuity of a function

Let $f(z)$ be a function defined in a set D and z_0 be a limit point of D. If the limit of $f(z)$ at z_0 exists and if it is finite and is equal to $f(z_0)$. (i.e.) if $\lim_{\triangle z \to 0} f(z) = f(z_0)$, then $f(z)$ is said to be continuous at z_0.

Derivative as a Complex Function

A function $f(z)$ is said to be differentiable at a point $z = z_0$ if $\lim_{\triangle z \to 0} \dfrac{f(z_0 + \triangle z) - f(z)}{\triangle z}$ exists and is the same in what ever way $\triangle z$ approaches zero. It is denoted by $f'(z_0)$.

$$\therefore f'(z_0) = \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z)}{\triangle z}$$

4.1 Analytic Function or Regular Function or Holomorphic Function

A function defined at a point z_0 is said to be analytic at z_0, if it has a derivative at z_0 and at every point in the neighbourhood of z_0. If it is analytic at every point in a region R, then it is said to be analytic in the region 'R'.

The necessary condition for the function f(z) to be analytic

The necessary condition for the function $f(z) = u + iv$ to be

analytic are the Cauchy - Riemann (C.R.) equations.

$$\text{(i.e.)} \ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof: Let $f(z) = u(x, y) + iv(x, y)$ be an analytic functions. Hence its derivative $f'(z)$ exists and is given by

$$f'(z) = \lim_{\triangle z \to 0} \frac{f(z + \triangle z) - f(z)}{\triangle z} \tag{4.1}$$

Let $z = x + iy$ so that $\triangle z = \triangle x + i\triangle y$. Also

$$f(z + \triangle z) = u(x + \triangle x, y + \triangle y) + iv(x + \triangle x, y + \triangle y)$$

Equation (4.1) becomes,

$$f'(z) = \lim_{\triangle z \to 0} \frac{f(z + \triangle z) - f(z)}{\triangle x + i\triangle y}$$

$$= \lim_{\triangle z \to 0} \frac{[u(x + \triangle x, y + \triangle y) + iv(x + \triangle x, y + \triangle y)] - [u(x, y) + iv(x, y)]}{\triangle x + i\triangle y}$$

$$= \lim_{\triangle z \to 0} \frac{[u(x + \triangle x, y + \triangle y) - u(x, y)] + i[v(x + \triangle x, y + \triangle y) - v(x, y)]}{\triangle x + i\triangle y}$$

$$= \lim_{\triangle z \to 0} \frac{u(x + \triangle x, y + \triangle y) - u(x, y)}{\triangle x + i\triangle y} + i \lim_{\triangle z \to 0} \frac{v(x + \triangle x, y + \triangle y) - v(x, y)}{\triangle x + i\triangle y} \tag{4.2}$$

Case (i): Let us consider $\triangle z$ is purely real, then $\triangle z = \triangle x$ and $\triangle y = 0$. Now equation (4.2) becomes,

$$f'(z) = \lim_{\triangle x \to 0} \frac{u(x + \triangle x, y) - u(x, y)}{\triangle x}$$

$$+i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{4.3}$$

Case(ii): Now let us consider Δz to be imaginary, then $\Delta z = \Delta y$ and $\Delta x = 0$. Then equation (4.2) becomes,

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y}$$

$$+i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

$$= \frac{1}{i} \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$$

$$+\frac{i}{i} \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{4.4}$$

From equation (4.3) and (4.4) gives

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Equating real and imaginary parts, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ (or) } u_x = v_y \text{ and } u_y = -v_x$$

These equations are called Cauchy-Riemann equations (or) C-R equations.

Sufficient condition for f(z) to be analytic

Statement: Sufficient condition for the function $f(z) = u + iv$ to be analytic in D is that (i) u and v are differentiable in D and $u_x = v_y$ and $u_y = -v_x$ (ii) the partial derivatives u_x, u_y, v_x and v_y are all continuous in D.

Proof: Let us consider the Taylor series for function of two variables is

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x,y)$$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y) + \cdots$$

$$\therefore f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$$

$$= u(x,y) + \left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right) + \frac{1}{2!}\left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right)^2 + \cdots$$

$$\cdots + i\left[v(x,y) + \left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right) + \frac{1}{2!}\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right)^2 + \cdots\right]$$

$$= u(x,y) + iv(x,y) + \left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right) + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right) +$$

other term.

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y$$

$\therefore f(z+\Delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y$ using

C - R equations

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\triangle x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\triangle y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\triangle x + \left(i^2\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\triangle y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\triangle x + i\left(i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right)\triangle y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\triangle x + i\triangle y)$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\triangle z$$

Now $\lim\limits_{\triangle z\to 0}\dfrac{f(z+\triangle z)-f(z)}{\triangle z} = \lim\limits_{\triangle z\to 0}\left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}\right) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

This implies that four partial derivatives u_x, u_y, v_x and v_y exist and are continuous.

Polar form of C-R equations

Let $f(z) = w = u + iv$ where $z = re^{i\theta}$

$$f(re^{i\theta}) = u + iv \tag{4.5}$$

Differentiate (4.5) partially w.r.to r

$$f'(re^{i\theta})e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \tag{4.6}$$

Differentiate (4.5) partially w.r.to θ

$$f'(re^{i\theta})rie^{i\theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$\Rightarrow f'(re^{i\theta})e^{i\theta} = \frac{1}{ir}\frac{\partial u}{\partial \theta} + \frac{i}{ir}\frac{\partial v}{\partial \theta} = -\frac{i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{4.7}$$

From equation (4.6) and (4.7), we have

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = -\frac{i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Example 4.1. *Test whether $w = \overline{z}$ is analytic.*

Solution: Let $f(z) = \overline{z} \Rightarrow u + iv = x - iy \Rightarrow u = x$ and $v = -y$

$$\Rightarrow \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = -1.$$

Here $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$. Hence the given function is not analytic.

Example 4.2. *Examine the analyticity of the function $f(z) = z^2$.*

Solution: Let $f(z) = z^2 \Rightarrow u + iv = (x+iy)^2 = x^2 - y^2 + 2ixy$.

$$\Rightarrow u = x^2 - y^2, v = 2xy \Rightarrow \frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y,$$

$$\frac{\partial v}{\partial x} = 2y, \frac{\partial v}{\partial y} = 2x.$$

Here $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ and all the partial derivatives are continuous, as u and v are algebraic functions.

$\therefore f(z) = z^2$ is analytic.

Example 4.3. *Show that $f(z) = \bar{z}$ is nowhere differentiable.*

Solution: Let $f(z) = \bar{z} \Rightarrow u + iv = x - iy \Rightarrow u = x$ and $v = -y$

$$\Rightarrow \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = -1.$$

Here $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$. Here C - R equations are not satisfied. $\therefore f(z) = \bar{z}$ is nowhere differentiable.

Example 4.4. *Show that $f(z)$ is discontinuous at the origin, given that* $f(z) = \begin{cases} \dfrac{xy(x-2y)}{x^3+y^3}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

Solution:

$$\lim_{z\to 0} |f(z)| = \lim_{\substack{y = mx \\ x \to 0}} |f(z)| = \lim_{x\to 0} \left| \frac{m(1-2m)x^3}{(1+m^3)x^3} \right|$$

$$= \frac{m(1-2m)}{(1+m^3)}$$

Thus the $\lim_{z\to 0} |f(z)|$ depends on the value of $'m'$ and hence does not have a unique value. So $\lim_{z\to 0} |f(z)|$ does not exist.

$\therefore f(z)$ is discontinuous at the origin.

Example 4.5. *Show that $f(z)$ is discontinuous at $z = 0$, given that $f(z) = \dfrac{2xy^2}{x^2+3y^4}$, when $z \neq 0$ and $f(0) = 0$.*

Solution:

$$\text{Let } \lim_{z\to 0} f(z) = \lim_{\substack{y = mx \\ x \to 0}} f(z) = \lim_{x\to 0} \left[\frac{2xm^2}{1+3m^4x^2} \right] = 0$$

Now let us take the limit by approaching zero along the curve $x = y^2$ then

$$\lim_{z\to 0}[f(z)] = \lim_{\substack{x = y^2 \\ y \to 0}} [f(z)] = \lim_{y\to 0} \left[\frac{2y^4}{y^4+3y^4} \right]$$

$$= \lim_{y\to 0} \left[\frac{2}{4} \right] = \frac{1}{2} \neq 0.$$

$\therefore \lim_{z\to 0}[f(z)]$ does not exist and hence $f(z)$ is not continuous.

Example 4.6. *Show that $f(z) = \dfrac{2xy}{x^2+y^2}$ is discontinuous at $z = 0$, given that $f(z) = 0$.*

Solution: Given $f(z) = 0$ at $z = 0$. Let us find the limit of $f(z)$ as $z \to 0$.

Since $z = x + iy$ as $z \to 0$ we have $x \to 0$ and $y \to 0$.

Let $z \to 0$ such that $y \to 0$ first and then $x \to 0$.

$$\lim_{z\to 0} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{2xy}{x^2+y^2} = 0$$

Let $z \to 0$ such that $x \to 0$ first and then $y \to 0$.

$$\lim_{z\to 0} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{2xy}{x^2+y^2} = 0.$$

Let $z \to 0$ such that x and y simultaneously tend to zero, along the path $y = mx$. Then

$$\lim_{z\to 0} f(z) = \lim_{\substack{y = mx \\ x \to 0}} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{2mx^2}{x^2(1+m^2)}$$

$$= \lim_{x\to 0} \frac{2m}{1+m^2} = \frac{2m}{1+m^2}$$

This limit changes its value, for different values of m.

When $m = 1, \frac{2m}{1+m^2} = 1$ and $m = 2, \frac{2m}{1+m^2} = \frac{4}{5}$ and so on.

Hence $\lim_{z\to 0} f(z)$ is not equal to zero, the actual value $f(0)$ of $f(z)$ at $z = 0$ is 0. So $f(z)$ is not continuous at the origin.

Example 4.7. *Show that $f(z) = \sin z$ is an analytic function.*

Solution: Given $f(z) = \sin z \Rightarrow u + iv = \sin(x + iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh y$

Equating real and imaginary parts, we have $u = \sin x \cosh y$ and $v = \cos x \sinh y$

$\Rightarrow \frac{\partial u}{\partial x} = \cos x \cosh y, \frac{\partial u}{\partial y} = \sin x \sinh y, \frac{\partial v}{\partial x} = -\sin x \sinh y$ and $\frac{\partial v}{\partial y} = \cos x \cosh y$. Here $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and all the partial derivatives are continuous.

$\therefore f(z) = \sin z$ is an analytic function.

Example 4.8. *Prove that $f(z) = z^n$ is analytic, where n is a positive integer.*

Solution: Let $f(z) = z^n$

$\Rightarrow u + iv = (re^{i\theta})^n = r^n e^{in\theta} = (r^n \cos n\theta + ir^n \sin n\theta)$

$\Rightarrow u = r^n \cos n\theta, v = r^n \sin n\theta$ so that

$\frac{\partial u}{\partial r} = nr^{n-1} \cos n\theta, \frac{\partial v}{\partial r} = nr^{n-1} \sin n\theta, \frac{\partial u}{\partial \theta} = -nr^n \sin n\theta$ and

$\frac{\partial v}{\partial \theta} = nr^n \cos n\theta.$

Here $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

The partial derivatives exist and are continuous. $\therefore$ The given function is analytic as trigonometric functions are continuous.

Example 4.9. *Test for analyticity of the function $f(z) = e^x(\cos y + i\sin y)$.*

Solution: Let $f(z) = e^x(\cos y + i\sin y)$

$\Rightarrow u + iv = e^x \cos y + ie^x \sin y$

$\Rightarrow u = e^x \cos y, v = e^x \sin y$ so that $\frac{\partial u}{\partial x} = e^x \cos y$, $\frac{\partial v}{\partial x} = e^x \sin y$, $\frac{\partial u}{\partial y} = -e^x \sin y$ and $\frac{\partial v}{\partial y} = e^x \cos y$.

Here $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

The partial derivatives and are continuous. $\therefore f(z)$ is an analytic.

Example 4.10. *Show that the function $|z|^2$ is differentiable at $z = 0$ but it is not analytic at any point.*

Solution: Let $z = x + iy \Rightarrow \overline{z} = x - iy$.

Now $f(z) = |z|^2 = z\overline{z} = x^2 + y^2$

$\Rightarrow u + iv = x^2 + y^2 + i0 \Rightarrow u = x^2 + y^2$ and $v = 0$, $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$. Here C - R equations are not

satisfied except at $z = 0$.

$\therefore f(z)$ is differentiable only at $z = 0$ only.

Now $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$ are continuous everywhere and in particular at $(0, 0)$. Hence the sufficient condition for differentiability are satisfied by $f(z)$ at $z = 0$.

$\therefore f(z)$ is differentiable only at $z = 0$.

$$\lim_{\triangle z \to 0}\left[\frac{f(z+\triangle z) - f(z)}{\triangle z}\right] = \lim_{\triangle z \to 0}\left[\frac{|z+\triangle z|^2 - |z|^2}{\triangle z}\right]$$

$$= \lim_{\triangle z \to 0}\left[\frac{(z+\triangle z)(\overline{z}+\triangle\overline{z}) - z\overline{z}}{\triangle z}\right]$$

$$= \lim_{\triangle z \to 0}\left[\frac{z\triangle\overline{z}}{\triangle z} + \overline{z} + \triangle\overline{z}\right]$$

$$= \lim_{\triangle z \to 0}\left[(x+iy)\frac{(\triangle x - i\triangle y)}{\triangle x + i\triangle y} + (x - iy) + (\triangle x - i\triangle y)\right]$$

Let us find the value of this limit by taking $\triangle z \to 0$ in the two different ways.

$$P_1 = \lim_{\substack{\triangle x \to 0 \\ \triangle y \to 0}}\left[(x+iy)\frac{(\triangle x - i\triangle y)}{\triangle x + i\triangle y} + (x - iy) + (\triangle x - i\triangle y)\right]$$

$$= \lim_{\triangle x \to 0}\left[(x+iy)\frac{\triangle x}{\triangle x} + (x - iy) + \triangle x\right]$$

$$= x + iy + x - iy + 0 = 2x$$

$$P_2 = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[(x+iy)\frac{(\Delta x - i\Delta y)}{\Delta x + i\Delta y} + (x-iy) + (\Delta x - i\Delta y)\right]$$

$$= \lim_{\Delta y \to 0}\left[(x+iy)\frac{-i\Delta y}{i\Delta y} + (x-iy) + i\Delta y\right]$$

$$= -x - iy + x - iy + 0 = -2iy$$

$P_1 \neq P_2$ for all values of x and y.

$\therefore f(z)$ is not differentiable at any point $z \neq 0$.

$\therefore f(z)$ is not analytic at any point $z \neq 0$.

Even though $f(z)$ is differentiable at $z = 0$, it is not differentiable at any point in the neighbourhood of $z = 0$.

$\therefore f(z)$ is not analytic even at $(0, 0)$.

$\therefore f(z) = |z|^2$ is not analytic at any point.

Example 4.11. *If $w = f(z)$ is analytic then it is independent of $\overline{z}$.*

$u \begin{cases} x \begin{cases} z \\ \bar z \end{cases} \\ y \begin{cases} z \\ \bar z \end{cases} \end{cases}$

Solution:

Let $z = x + iy \Rightarrow \overline{z} = x - iy \Rightarrow x = \dfrac{z + \overline{z}}{2}, y = \dfrac{z - \overline{z}}{2i}$

$$\therefore \frac{\partial x}{\partial \overline{z}} = \frac{1}{2} \text{ and } \frac{\partial y}{\partial \overline{z}} = \frac{1}{2i}.$$

To prove $w = f(z) = u + iv$ is independent of $\overline{z}$, we have to prove

that $\dfrac{\partial w}{\partial \overline{z}} = 0$.

Consider $\dfrac{\partial w}{\partial \overline{z}} = \dfrac{\partial u}{\partial \overline{z}} + i\dfrac{\partial v}{\partial \overline{z}}$.

Now u and v are functions of x and y and x and y are functions of z and $\overline{z}$.

$$\therefore \frac{\partial w}{\partial \overline{z}} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial \overline{z}} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial \overline{z}} + i\left(\frac{\partial v}{\partial x}.\frac{\partial x}{\partial \overline{z}} + \frac{\partial v}{\partial y}.\frac{\partial y}{\partial \overline{z}}\right)$$

$$= \frac{\partial u}{\partial x}.\frac{1}{2} - \frac{\partial u}{\partial y}.\frac{1}{2i} + i\left(\frac{\partial v}{\partial x}.\frac{1}{2} - \frac{\partial v}{\partial y}.\frac{1}{2i}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \text{ using C - R equations}$$

$$= \frac{1}{2}(0) + \frac{i}{2}(0) = 0 \Rightarrow \frac{\partial w}{\partial \overline{z}} = 0$$

$\therefore w = f(z)$ is independent of $\overline{z}$.

$\overline{f(z)}$

Example 4.12. *If $f(z)$ and $f(\overline{z})$ are analytic function of z, prove that $f(z)$ is constant.*

Solution: Let $f(z) = u + iv \Rightarrow \overline{f(\overline{z})} = u - iv$.

Given $f(z)$ is analytic

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{4.8}$$

Also $f(\bar{z})$ is analytic

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{4.9}$$

From (4.8) and (4.9), $2\frac{\partial v}{\partial y} = 0$ and $2\frac{\partial v}{\partial x} = 0$

$\frac{\partial v}{\partial y} = 0$ and $\frac{\partial v}{\partial x} = 0 \Rightarrow v = C_1$ (constant).

Again $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0 \Rightarrow u = C_2$ (constant).

$\therefore f(z) = C_2 + iC_1 =$ constant. (i.e.) $f(z)$ is constant.

Example 4.13. *Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin although Cauchy - Riemann equation are satisfied at that point.*

Solution: Let $f(z) = u + iv = \sqrt{|xy|} \Rightarrow u = \sqrt{|xy|}$ and $v = 0$.

Since $\frac{\partial u}{\partial x} = \lim_{\triangle x \to 0} \frac{u(x+\triangle x, y) - u(x,y)}{\triangle x}$, we have

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{\triangle x \to 0} \left(\frac{\sqrt{|(x+\triangle x)y|} - \sqrt{|xy|}}{\triangle x}\right)_{(0,0)} = \lim_{\triangle x \to 0} \frac{0-0}{\triangle x} = 0$$

Similarly $\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = 0, \left(\frac{\partial v}{\partial x}\right)_{(0,0)} = 0$ and $\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = 0$

Here $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at the origin.

$\therefore$ C - R equations are satisfied at the origin.

Now $f'(0) = \lim_{\triangle z \to 0} \left[\frac{f(0+\triangle z) - f(0)}{\triangle z}\right]$

$= \lim_{\triangle z \to 0} \left[\frac{\sqrt{|\triangle x \triangle y|} - 0}{\triangle x + i\triangle y}\right]$

$= \lim_{\substack{\triangle y = m\triangle x \\ \triangle x \to 0}} \left[\frac{\sqrt{|m|\triangle x^2}}{\triangle x(1+im)}\right]$

$= \lim_{\triangle x \to 0} \left[\frac{\sqrt{|m|}}{(1+im)}\right] = \frac{\sqrt{|m|}}{(1+im)}$

Since the limit depends upon m, the limit is not unique.

$\therefore$ $f'(0)$ does not exist and hence $f(z)$ is not analytic at the origin.

Example 4.14. *Show that $u + iv = \frac{x - iy}{x - iy + a}$ $(a \neq 0)$ is not an analytic function of z but $u - iv$ is an analytic function at all points where $z \neq -a$.*

Solution: Let $f(z) = u + iv = \frac{\bar{z}}{\bar{z}+a}$ is a function of $\bar{z}$. Since a function of $\bar{z}$ cannot be analytic, $(u+iv)$ is not an analytic function of z.

Now $u - iv =$ conjugate of $u + iv = \frac{z}{z+a}$.

Let $f(z) = \frac{z}{z+a}$. $f(z)$ is a function of z alone and $f'(z) =$

$\dfrac{a}{(z+a)^2}$ that exists everywhere except at $z=-a$.

$\therefore f(z)$ is analytic except at $z=-a$.

Example 4.15. *Find the values of C_1 and C_2 such that the function $f(z)=C_1xy+i(C_2x^2+y^2)$ is analytic.*

Solution: Let $f(z)=C_1xy+i(C_2x^2+y^2)=u+iv$ then $u=C_1xy$ and $v=C_2x^2+y^2$

$$\Rightarrow \frac{\partial u}{\partial x}=C_1y;\ \frac{\partial u}{\partial y}=C_1x;\ \frac{\partial v}{\partial x}=2C_2x;\ \frac{\partial v}{\partial y}=2y$$

If the function is analytic, the C - R equations must be satisfied.

(i.e.) $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}\Rightarrow C_1y=2y\Rightarrow C_1=2$.

Also $C_1x=2C_2x\Rightarrow 2x=-2C_2x\Rightarrow C_2=-1$.

Example 4.16. *Show that $f(z)=\dfrac{1}{z^2+1}$ is analytic everywhere except at $z=\pm i$.*

Solution: Let $f(z)=\dfrac{1}{z^2+1}\Rightarrow f'(z)=\dfrac{2z}{(z^2+1)^2}$.

Now $f'(z)$ becomes infinite when $z^2+1=0$, (i.e.) $z^2=-1\Rightarrow z=\pm i$.

$\therefore$ The function $f(z)$ is analytic everywhere except at $z=\pm i$.

Example 4.17. *Show that an analytic function with (i) constant real part is constant and (ii) constant modulus is constant.*

Solution: Let $f(z)=u+iv$ can be analytic function.

(i) Given $u=c$. Then $\dfrac{\partial u}{\partial x}=0$ and $\dfrac{\partial u}{\partial y}=0$.

By C - R equations $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$. Since the partial derivatives of v with respect to x and y are zero, v is a constant (say c').

$\therefore f(z)=c+c'=$ constant.

$$(ii)\ \text{Given}\ |f(z)|=\sqrt{u^2+v^2}=c,\ (i.e.)\ u^2+v^2=k \tag{4.10}$$

Differentiate (4.10) with respect to x and y

$2u\dfrac{\partial u}{\partial x}+2v\dfrac{\partial v}{\partial x}=0\Rightarrow u\dfrac{\partial u}{\partial x}+v\dfrac{\partial v}{\partial x}=0$ and

$2u\dfrac{\partial u}{\partial y}+2v\dfrac{\partial v}{\partial y}=0\Rightarrow u\dfrac{\partial u}{\partial y}+v\dfrac{\partial v}{\partial y}=0$

Using C - R equations $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$

$$u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}=0 \tag{4.11}$$

$$u\frac{\partial u}{\partial y}+v\frac{\partial u}{\partial x}=0 \tag{4.12}$$

Solving (4.11) and (4.12), $\dfrac{\partial u}{\partial x}=0,\ \dfrac{\partial u}{\partial y}=0,\ \dfrac{\partial v}{\partial x}=0$ and $\dfrac{\partial v}{\partial y}=0$

$\therefore f'(z)=\dfrac{\partial u}{\partial x}+i\dfrac{\partial v}{\partial x}=0+i0=0\Rightarrow f(z)$ is constant.

Harmonic function

Any function which has continuous second order partial derivatives and which satisfies the Laplace equation is called harmonic function. For example if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, then u is said to be a harmonic function.

4.1.1 Properties of analytic function:

(i) Both real and imaginary parts of an analytic function satisfies the Laplace equation (or) the real and imaginary parts of an analytic function are harmonic functions.

Proof: Let $f(z) = u + iv$ be an analytic function. Then by C - R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{4.13}$$

$$\text{and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{4.14}$$

Differentiate equation (4.13) w.r.to x and (4.14) w.r.to y and adding, we get $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0 \Rightarrow \nabla^2 u = 0$

$\therefore$ u is harmonic.

Now differentiate (4.13) w.r.to y and (4.14) w.r.to x and subtracting $\frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0 \Rightarrow \nabla^2 v = 0$. $\therefore$ v is harmonic.

$\therefore$ u and v are harmonic functions.

Note: The converse of the above result need not be true.

(ii) If $f(z) = u + iv$ be an analytic function, then the family of curves $u(x,y) = C_1$ and $v(x,y) = C_2$ (where C_1 and C_2 are constants) cut each other orthogonally (or) the real and imaginary parts of an analytic function form an orthogonal system.

Proof: Let $u(x,y) = C_1$. Then $du = 0 \Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1.$$

Also $v(x,y) = C_2 \Rightarrow \frac{dy}{dx} = \frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} = m_2$

$$\text{Now } m_1 m_2 = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} . \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} = -1.$$

The curves cut each other orthogonally.

Example 4.18. *Verify that the families of curves $u = C_1, v = C_2$ cut each other orthogonally when $w = z^3$.*

Solution:

Let $f(z) = z^3 \Rightarrow u + iv = (x+iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3$

$\Rightarrow u = x^3 - 3xy^2 = C_1, v = 3x^2y - y^3 = C_2$

Differentiate with respect to x, we have

$$3x^2 - 3\left(y^2 + 2xy\frac{dy}{dx}\right) = 0 \Rightarrow \frac{dy}{dx} = \frac{3(y^2-x^2)}{6xy} = m_1 \text{ and}$$

$$3\left(2xy + x^2\frac{dy}{dx}\right) - 3y^2\frac{dy}{dx} = 0 \Rightarrow 6xy + 3x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-6xy}{3(y^2-x^2)} = m_2.$$

Now $m_1.m_2 = \dfrac{y^2-x^2}{2xy} \times \dfrac{-2xy}{y^2-x^2} = -1.$

Hence the curves $u = C_1, v = C_2$ cut each other orthogonally.

Example 4.19. *Find the constants a, b, c if $f(z) = x + ay + i(bx + cy)$ is analytic.*

Solution:

Let $u + iv = x + ay + i(bx + cy) \Rightarrow u = x + ay, v = bx + cy$

Hence $\dfrac{\partial u}{\partial x} = 1, \dfrac{\partial u}{\partial y} = a, \dfrac{\partial v}{\partial x} = b$ and $\dfrac{\partial v}{\partial y} = c.$

Using C - R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow c = 1 \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow a = -b.$$

Example 4.20. *Examine whether the function xy^2 can be the real part of an analytic function.*

Solution:

Let $u = xy^2 \Rightarrow \dfrac{\partial u}{\partial x} = y^2, \dfrac{\partial^2 u}{\partial x^2} = 0, \dfrac{\partial u}{\partial y} = 2xy, \dfrac{\partial^2 u}{\partial y^2} = 2x$

$\Rightarrow \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 + 2x = 2x \neq 0.$ Does not satisfy Laplace equation and hence it cannot be the real part of an analytic function.

Example 4.21. *What is the relation between 'a' and 'b' if $ax^2 + by^2$ can be the real part of an analytic function.*

Solution: Let $u = ax^2 + by^2$

$$\Rightarrow \frac{\partial u}{\partial x} = 2ax, \frac{\partial^2 u}{\partial x^2} = 2a, \frac{\partial u}{\partial y} = 2by, \frac{\partial^2 u}{\partial y^2} = 2b$$

If u is the real part of an analytic function, it must satisfy Laplace equation. So $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \Rightarrow 2a + 2b = 0 \Rightarrow a + b = 0$ which is the required relation.

Example 4.22. *Find the harmonic conjugate of $u = e^x \cos y$*

Solution:

Let $u = e^x \cos y \Rightarrow \frac{\partial u}{\partial x} = e^x \cos y, \frac{\partial u}{\partial y} = -e^x \sin y$

$\frac{\partial^2 u}{\partial x^2} = e^x \cos y, \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$ and

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$

$\therefore$ u is harmonic.

The C - R equations are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x \cos y \tag{4.15}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^x \sin y \tag{4.16}$$

Integrating (4.15) w.r.to y, we get

$v = e^x \int \cos y dy +$ constant independent of y

$$v = e^x \sin y + f(x) \tag{4.17}$$

$$\text{For (4.17), } \frac{\partial v}{\partial x} = e^x \sin y + f'(x) \tag{4.18}$$

Equating (4.16) and (4.18), we get

$e^x \sin y = e^x \sin y + f'(x)$

$\therefore f'(x) = 0 \Rightarrow f(x) = C$

Hence from (4.17), $v = e^x \sin y + C$

Example 4.23. *If u and v are harmonic functions of x and y and $s = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$ and $t = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, prove that $s + it$ is an analytic function of $z = x + iy$.*

Solution:

$$\text{Let } s = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \tag{4.19}$$

$$\text{and } t = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{4.20}$$

So $\frac{\partial s}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2}, \frac{\partial t}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}$

$\frac{\partial s}{\partial x} - \frac{\partial t}{\partial y} = -\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0$

$\Rightarrow \frac{\partial s}{\partial x} = \frac{\partial t}{\partial y}$. Similarly, we can prove that $\frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x}$. Hence $f(z) = s + it$ is an analytic function.

Example 4.24. *If $f(z) = u + iv$ is an analytic function of z, show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2$*

Solution: We know that $|z|^2 = z\overline{z}$. $\therefore |f(z)|^2 = f(z)f(\overline{z})$ and

$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \overline{z}}$.

Now $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4\frac{\partial^2}{\partial z \partial \overline{z}} f(z) f(\overline{z})$

$= 4f'(z)f'(\overline{z}) = 4|f'(z)|^2$

Aliter:

Solution: Let $f(z) = u + iv$ so that $|f(z)| = \sqrt{u^2 + v^2}$

$|f(z)|^2 = u^2 + v^2 = \varphi(x, y)$ (say)

Now $\dfrac{\partial \varphi}{\partial x} = 2u\dfrac{\partial u}{\partial x} + 2v\dfrac{\partial v}{\partial x}$,

$$\frac{\partial^2 \varphi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly, $\dfrac{\partial^2 \varphi}{\partial y^2} = 2\left[u\dfrac{\partial^2 u}{\partial y^2} + \left(\dfrac{\partial u}{\partial y}\right)^2 + v\dfrac{\partial^2 v}{\partial y^2} + \left(\dfrac{\partial v}{\partial y}\right)^2\right]$

Adding, we get

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 2\Big[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

$$+v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\Big] \qquad \text{(I)}$$

Since $f(z) = u + iv$ is an analytic function of z.

$u_x = v_y, u_y = -v_x$ and $\nabla^2 u = 0, \nabla^2 v = 0$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 2\Big[0 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + 0 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\Big]$$

$$= 4\Big[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\Big] \qquad \text{(II)}$$

Now $f(z) = u + iv \Rightarrow f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$ and

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

From (II), $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)\varphi = 4|f'(z)|^2$ (or)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Example 4.25. *If $f(z) = u + iv$ is an analytic function of z, prove that*

(i) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2|f(z)|^{p-2}|f'(z)|^2$

(ii) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)\log|f(z)| = 0$

Solution:

(i) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^p = 4\dfrac{\partial^2}{\partial z \partial \overline{z}}[f(z)f(\overline{z})]^{p/2}$

$= 4\dfrac{\partial}{\partial z}[f(z)]^{p/2}\dfrac{\partial}{\partial \overline{z}}[f(\overline{z})]^{p/2}$

$= 4\dfrac{p}{2}[f(z)]^{p/2-1}f'(z)\dfrac{p}{2}[f(\overline{z})]^{p/2-1}f'(\overline{z})$

$= p^2|f(z)|^{p-2}|f'(z)|^2$

(ii) Let $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)\log|f(z)| = 4\dfrac{\partial^2}{\partial z\partial \bar{z}}\dfrac{1}{2}\log|f(z)|^2$

$$= 2\frac{\partial^2}{\partial z\partial \bar{z}}[\log f(z) + \log f(\bar{z})] = 0$$

Aliter:

If $f(z) = u + iv$ is a regular function of z, prove that

$\nabla^2[\log|f(z)|] = 0$

Solution: Let $f(z) = u + iv$ is analytic function.

$u_x = v_y, u_y = -v_x$ (By C - R equations)

$u_{xx} + v_{yy} = 0$ and $v_{xx} + u_{yy} = 0$

$$\log|f(z)| = \frac{1}{2}\log(u^2+v^2)$$

$$\therefore \frac{\partial}{\partial x}\log|f(z)| = \frac{1}{2}\left(\frac{2uu_x + 2vv_x}{u^2+v^2}\right) = \frac{uu_x+vv_x}{u^2+v^2}$$

$$\therefore \frac{\partial^2}{\partial x^2}\log|f(z)|$$

$$= \frac{(u^2+v^2)(uu_{xx}+u_x^2+vv_{xx}+v_x^2) - (uu_x+vv_x)(2uu_x+2vv_x)}{(u^2+v^2)^2}$$

$$= \frac{1}{u^2+v^2}(uu_{xx}+u_x^2+vv_{xx}+v_x^2) - \frac{2}{(u^2+v^2)^2}(uu_x+vv_x)^2$$

Similarly, $\dfrac{\partial^2}{\partial y^2}\log|f(z)|$

$$= \frac{1}{u^2+v^2}(uu_{yy}+u_y^2+vv_{yy}+v_y^2) - \frac{2}{(u^2+v^2)^2}(uu_y+vv_y)^2$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f(z)|^2$$

$$= \frac{1}{u^2+v^2}[u(u_{xx}+u_{yy}) + v(v_{xx}+v_{yy}) + u_x^2+v_x^2+u_y^2+v_y^2]$$

$$- \frac{2}{(u^2+v^2)^2}[(uu_x+vv_x)^2 + (uu_y+vv_y)^2]$$

$$= \frac{1}{u^2+v^2}\left[2(u_x^2+v_x^2) - \frac{2}{(u^2+v^2)^2}(uu_x+vv_x)^2 + (-uv_x+vu_x)\right]$$

$$= \frac{2(u_x^2+v_x^2)}{u^2+v^2} - \frac{2}{(u^2+v^2)^2}[u^2(u_x^2+v_x^2) + v^2(u_x^2+v_x^2)]$$

$$= \frac{2(u_x^2+v_x^2)}{u^2+v^2} - \frac{2(u^2+v^2)(u_x^2+v_x^2)}{(u^2+v^2)^2} = 0$$

Example 4.26. *If* $u = x^2 - y^2$ *and* $v = -\dfrac{y}{x^2+y^2}$, *prove that both* u *and* v *are satisfy Laplace equations, but* $(u+iv)$ *is not an analytic function of* z.

Solution:

Given $u = x^2 - y^2 \Rightarrow \dfrac{\partial u}{\partial x} = u_x = 2x, \dfrac{\partial u}{\partial y} = u_y = -2y$

$$\frac{\partial^2 u}{\partial x^2} = u_{xx} = 2, \frac{\partial^2 u}{\partial y^2} = u_{yy} = -2 \text{ and } u_{xx} + u_{yy} = 0$$

$\therefore u$ satisfies Laplace equation.

Again $v = -\dfrac{y}{x^2+y^2} \Rightarrow v_x = \dfrac{2xy}{(x^2+y^2)^2}$ and

$$v_{xx} = \frac{2y(x^2+y^2)^2 - 2xy(x^2+y^2)(4x)}{(x^2+y^2)^4} = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$v_y = -\left[\frac{(x^2+y^2).1 - 2y^2}{(x^2+y^2)^2}\right] = \frac{y^2-x^2}{(x^2+y^2)^2} \text{ and}$$

$$v_{yy} = \frac{(x^2+y^2)^2.2y - (y^2-x^2)2(x^2+y^2).2y}{(x^2+y^2)^4} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

$\therefore v_{xx} + v_{yy} = 0$

$\therefore v$ satisfies Laplace equation.

Here $u_x \neq v_y$ and $u_y \neq -v_x$.

(i.e.) C - R equations are not satisfied by u and v.

Hence $u + iv$ is not an analytic function of z.

CONSTRUCTION OF ANALYTIC FUNCTION

4.2 Milne - Thomson Method

To find f(z) when u is given (real part)

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \text{ (using C-R equations)} \quad (4.21)$$

Assume that

$$\frac{\partial}{\partial x}u(x,y) = \varphi_1(z,0) \quad (4.22)$$

$$\text{and } \frac{\partial}{\partial y}u(x,y) = \varphi_2(z,0) \quad (4.23)$$

Using (4.22) and (4.23) in (4.21), we get

$f'(z) = \varphi_1(z,0) - i\varphi_2(z,0)$

Integrating we get

$f(z) = \int \varphi_1(z,0)dz - i\int \varphi_2(z,0)dz + C$

To find f(z) when v is given (imaginary part)

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \quad (4.24)$$

Assume $\dfrac{\partial}{\partial y}v(x,y) = \varphi_1(z,0)$, $\dfrac{\partial}{\partial x}v(x,y) = \varphi_2(z,0)$.

So that $f'(z) = \varphi_1(z,0) - i\varphi_2(z,0)$

Integrating we get

$f(z) = \int \varphi_1(z,0)dz - i\int \varphi_2(z,0)dz + C$

Method to find the harmonic conjugate

Let $f(z) = u + iv$ be an analytic function.

Case(i): If the real part u is given, to find v.

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

Integrating v is obtained.

Case(ii): If the imaginary part v is given, to find u.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy$$

Integrating u is obtained.

Example 4.27. *Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2$ is harmonic and determine its harmonic conjugate. Also find $f(z)$.*

Solution:

Given $u = x^3 - 3xy^2 + 3x^2 - 3y^2 \Rightarrow \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x,$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 6, \frac{\partial u}{\partial y} = -6xy - 6y, \frac{\partial^2 u}{\partial y^2} = -6x - 6$$

and $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

$\therefore u$ is a harmonic function. Since the real part u is given.

By Milne - Thomson method

$$f'(z) = \frac{\partial u}{\partial x}(z,0) - i\frac{\partial u}{\partial y}(z,0) = 3z^2 + 6z \text{ (using } x = z, y = 0)$$

$f(z) = \int(3z^2 + 6z)dz = z^3 + 3z^2 + C$

To find the harmonic conjugate: $z = x + iy$

$f(z) = (x+iy)^3 + 3(x+iy)^2 + (C_1 + iC_2)$

$$u + iv = (x^3 - iy^3 + 3yix^2 - 3xy^2) + 3(x^2 - y^2 + 2ixy) + C_1 + iC_2$$

Comparing the imaginary parts

$v = -y^3 + 3x^2y + 6xy + C_2$

Example 4.28. *Find the analytic function $w = x + iy$ if $w = e^x(x\sin y + y\cos x)$* u+iv

Solution: Given $w = e^x(x\sin y + y\cos y)$

$$\frac{\partial u}{\partial x} = \sin y(e^x + xe^x) + e^x(y\cos y)$$

$$\frac{\partial u}{\partial y} = xe^x\cos y + (\cos y - y\sin y)e^x$$

Since the real part u is given.

By Milne - Thomson method

$f'(z) = \frac{\partial u}{\partial x}(z,0) - i\frac{\partial u}{\partial y}(z,0) = -ie^z(z+1)$

$f(z) = -i\int e^z(z+1)dz = -i(ze^z - e^z + e^z) + C$

$\therefore f(z) = -i(ze^z) + C$

Example 4.29. *Verify whether the function* $\frac{1}{2}\log(x^2+y^2)$ *is harmonic. Find the harmonic conjugate. Also find* $f(z)$.

Solution: Let $u = \frac{1}{2}\log(x^2+y^2)$

$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}, \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2},$

$\frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$ so that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

$\therefore u$ is a harmonic function.

To find v:

$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial y}dy$

$= -\left(\frac{y}{x^2+y^2}\right)dx + \left(\frac{x}{x^2+y^2}\right)dy = \frac{xdy - ydx}{x^2+y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

Integrating $v = \tan^{-1}\frac{y}{x} + f_1(y)$

$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) + f_1'(y)$

Using C - R equations $\frac{\partial u}{\partial x} = \frac{x^2}{x^2+y^2}\frac{1}{x} + f_1'(y)$

(i.e.) $\frac{x}{x^2+y^2} = \frac{x}{x^2+y^2} + f_1'(y) \Rightarrow f_1'(y) = 0$

$\therefore v = \tan^{-1}\frac{y}{x}$

To find f(z): $f(z) = u + iv$

$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x} = \log(x+iy)$

$\therefore f(z) = \log z$

Example 4.30. *If* $u + v = (x-y)(x^2+4xy+y^2)$ *and* $f(z) = u + iv$. *Find* $f(z)$ *interms of* z.

Solution: Let $f(z) = u + iv \Rightarrow if(z) = iu - v$

$\Rightarrow (1+i)f(z) = (u-v) + i(u+v) = U + iV$

Given $V = (x-y)(x^2+4xy+y^2)$

$\frac{\partial V}{\partial x} = (x-y)(2x+4y) + x^2 + 4xy + y^2$

$\frac{\partial V}{\partial x} = (4x+2y)(x-y) + (x^2+4xy+y^2)(-1)$

$F'(z) = V_y(z, 0) + iV_x(z, 0)$

By Milne - Thomson method

$F'(z) = 3z^2(1 + i) + C$

Integrating, we get $F(z) = (1 + i)f(z) = 3\frac{z^3}{3}(1 + i) + C$

$\therefore f(z) = z^3 + k$

Example 4.31. *Find the analytic function* $f(z) = u + iv$ *if* $u - v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

Solution:

$$\text{Let } u + iv = f(z) \tag{4.25}$$

$$\text{and } iu - v = if(z) \tag{4.26}$$

Equation (4.25) + (4.26) $\Rightarrow (1 + i)f(z) = (u - v) + i(u + v)$

$F(z) = U + iV$

Given $U = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\frac{\partial U}{\partial x} = \frac{(\cosh 2y - \cos 2x)2\cos 2x - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial U}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$U_x(z, 0) = \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} = \frac{-2}{1 - \cos 2z} = \frac{-2}{2\sin^2 z} = -\csc^2 z$$

$U_y(z, 0) = 0$

By Milne - Thomson method

$F'(z) = U_x(z, 0) - iU_y(z, 0) = -\csc^2 z$

Integrating, we get $F(z) = \int(-\csc^2 z)dz + C = \cot z + C$

$(1 + i)f(z) = \cot z + C \quad \therefore f(z) = \dfrac{\cot z}{1 + i} + C$

Example 4.32. *Find the analytic function* $f(z) = u + iv$ *if* $u + v = \dfrac{x}{x^2 + y^2}$ *and* $f(1) = 1$.

Solution:

$$\text{Let } u + iv = f(z) \tag{4.27}$$

$$\text{and } iu - v = if(z) \tag{4.28}$$

Equation (4.27) + (4.28) $\Rightarrow (1 + i)f(z) = (u - v) + i(u + v)$

$F(z) = U + iV$

Here $V = u + v = \dfrac{x}{x^2 + y^2}$

$$\varphi_1(x, y) = V_x = \frac{(x^2 + y^2)(1) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\varphi_1(z, 0) = -\frac{z^2}{(z^2)^2} = -\frac{1}{z^2}$$

$$\varphi_2(x, y) = V_y = \frac{(x^2 + y^2)(0) - x(2y)}{(x^2 + y^2)^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

$\varphi_2(z,0) = 0$

By Milne - Thomson method

$F(z) = \int \varphi_1(z,0)dz - i\int \varphi_2(z,0)dz = 0 + i\int \frac{(-1)}{z^2}dz$

$\therefore F(z) = -i\int z^{-2}dz$

$(1+i)f(z) = \frac{i}{z} + C$

$f(z) = \frac{i}{z(1+i)} + C = \frac{i+1}{2z} + C$

Given $f(1) = 1 \Rightarrow 1 = \frac{i+1}{2} + C \Rightarrow C = \frac{1-i}{2}$

$\therefore f(z) = \frac{1+i}{2z} + \frac{1-i}{2}$

Example 4.33. *Find the analytic function $f(z) = u + iv$ if $u - v = e^x(\cos y - \sin y)$.*

Solution:

$$\text{Let } u + iv = f(z) \tag{4.29}$$

$$\text{and } iu - v = if(z) \tag{4.30}$$

Equation $(4.29) + (4.30) \Rightarrow (1+i)f(z) = (u-v) + i(u+v)$

$F(z) = U + iV$

Take $F(z) = (1+i)f(z)$, F(z) will be analytic as $(1+i)f(z)$ is analytic.

Here $U = u - v = e^x(\cos y - \sin y)$

$\varphi_1(x,y) = U_x = e^x(\cos y - \sin y) \Rightarrow \varphi_1(z,0) = e^z$

$\varphi_2(x,y) = U_y = e^x(-\sin y - \cos y) \Rightarrow \varphi_2(z,0) = e^z(-1) = -e^z$

By Milne - Thomson method

$F(z) = \int \varphi_1(z,0)dz - i\int \varphi_2(z,0)dz$

$= e^z dz - i\int(-e^z)dz = (1+i)e^z$

$(1+i)f(z) = (1+i)e^z + C_1 \Rightarrow f(z) = e^z + C$

Example 4.34. *Find the analytic function $f(z) = u + iv$ given that $2u + v = e^x(\cos y - \sin y)$.*

Solution:

$$\text{Let } u + iv = f(z) \Rightarrow 2f(z) = 2u + i2v \tag{4.31}$$

$$\text{and } if(z) = iu - v \Rightarrow -if(z) = v - iu \tag{4.32}$$

Equation $(4.31) + (4.32)$ gives $(2u+v) + i(2v-u) = (2-i)f(z)$

$F(z) = U + iV$

Here $U = 2u + v = e^x(\cos y - \sin y)$.

$\varphi_1(x,y) = U_x = e^x(\cos y - \sin y) \Rightarrow \varphi_1(z,0) = e^z$

$\varphi_2(x,y) = U_y = e^x(-\sin y - \cos y) \Rightarrow \varphi_2(z,0) = e^z(-1) = -e^z$

By Milne - Thomson method

$F(z)=\int\varphi_1(z,0)dz-i\int\varphi_2(z,0)dz=\int e^z dz-i\int(-e^z)dz$

$(2-i)f(z)=(1+i)e^z+C_1\Rightarrow f(z)=\dfrac{1+3i}{5}e^z+C$

Example 4.35. *Find the analytic function* $f(z)=u+iv$ *if* $-2v=e^x(\cos y-\sin y)$.

Solution:

$$\text{Let } f(z)=u+iv \tag{4.33}$$

$$if(z)=-iu+v \tag{4.34}$$

Equations (4.34) $\times(-2)\Rightarrow 2if(z)=2iu-2v=-(2v)+i(2u)$

$F(z)=U+iV$

Here $F(z)=2if(z), U=-2v$ and $V=2u$.

$\varphi_1(x,y)=U_x=e^x(\cos y-\sin y)\Rightarrow\varphi_1(z,0)=e^z$

$\varphi_2(x,y)=U_y=e^x(-\sin y-\cos y)\Rightarrow\varphi_2(z,0)=e^z(-1)=-e^z$

By Milne - Thomson method

$F(z)=\int\varphi_1(z,0)dz-i\int\varphi_2(z,0)dz=\int e^z dz-i\int(-e^z)dz$

$2if(z)=(1+i)e^z+C_1\Rightarrow f(z)=\dfrac{1+i}{2i}e^z+C=\dfrac{1-i}{2}e^z+C$

Example 4.36. *Find the analytic function* $f(z)=u+iv$ *if* $u-2v=e^x(\cos y-\sin y)$.

Solution:

$$\text{Let } f(z)=u+iv \tag{4.35}$$

$$-if(z)=-iu+v \tag{4.36}$$

Equations (4.35) + (−2)(4.36) gives,

$\Rightarrow f(z)+2if(z)=u+iv+2iu-2v$

$\Rightarrow(1+2i)f(z)=u-2v+i(v+2u)$

$F(z)=U+iV$

Here $U=u-2v=e^x(\cos y-\sin y)$

$\varphi_1(x,y)=U_x=e^x(\cos y-\sin y)\Rightarrow\varphi_1(z,0)=e^z$

$\varphi_2(x,y)=U_y=e^x(-\sin y-\cos y)\Rightarrow\varphi_2(z,0)=e^z(-1)=-e^z$

By Milne - Thomson method

$F(z)=\int\varphi_1(z,0)dz-i\int\varphi_2(z,0)dz=\int e^z dz-i\int(-e^z)dz$

$(1+2i)f(z)=(1+i)e^z+C_1\Rightarrow f(z)=\dfrac{1+i}{1+2i}e^z+C=\dfrac{3-i}{5}e^z+C$

EXERCISE

1. Find the real part of $f(z)=e^{2z}$ [Ans: $e^x\cos y$]
2. Find the real and imaginary part of $w=\log z$

   [Ans: $u=\log r, v=0$]
3. Test for analyticity of the functions

(i) $f(z) = e^x(\cos y + i \sin y)$ (ii) $f(z) = \frac{1}{z}$ and

(iii) $f(z) = z^3$ [Ans: (i) Yes (ii) Yes and (iii) Yes]

4. Verify whether the function $e^y \cosh x$ is harmonic.

   [Ans: No.]

5. Verify whether $f(z) = z^3$ is harmonic. [Ans: Yes]

6. if $f(z)$ is analytic where $f(z) = r^2 \cos 2\theta + ir^2 \sin p\theta$, find p. [Ans: P=2]

7. Find the analytic function if the imaginary part is $r = e^{2x}(y \cos 2y + x \sin 2y)$ [Ans: $ze^{2z} + C$]

8. Find $f(z)$ if $u - v = (x - y)(x^2 + 4xy + y^2)$ [Ans: $iz^3 + C$]

9. Find $f(z)$ if $u - v = e^x(\cos y - \sin y)$ [Ans: $e^z + C$]

10. Find the analytic function $w = u + iv$ if $v = e^{2x}(x \cos 2y - y \sin 2y)$, also find the harmonic conjugate.

    [Ans: $f(z) = ize^{2z} + C, u = e^{2x}(-y \cos 2y - x \sin 2y) + C$]

11. Find the analytic function $f(z) = u + iv$ where

$$u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^y - e^{-y}} \left[\text{Ans:} f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right)\right]$$

12. Find the analytic function $f(z) = u + iv$ given that

$$u + v = \frac{2x}{x^2 + y^2} \left[\text{Ans:} f(z) = \frac{1+i}{z} - i\right]$$

13. Prove that the analytic function $f(z) = u + iv$ is independent of $\overline{z}$. [(i.e) if f(z) is analytic it is a function of z only].

14. Find the points at which the function $f(z) = \frac{1}{z^2 + 1}$

    [Ans: $\pm i$], fails to be analytic

    Hint: $f'(z) \to \infty$

15. If u and v are harmonic, can we say that $u+iv$ is an analytic function? [Ans: No]

    Hint: Take $u = x^2 - y^2, v = \frac{-y}{x^2 + y^2}$

16. Test the analyticity of the function $w = \sin z$

    [Ans: Analytic]

17. If ϕ and ψ are functions of x and y satisfying (i) Laplace equation, namely $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$, show that $u+iv$ is analytic where $u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}$, (ii) If $w = f(z)$ is analytic prove that $\frac{\partial w}{\partial x} = \frac{dw}{dz} = -i\frac{\partial w}{\partial y}$

18. Prove that the function $f(z) = \frac{x - iy}{x^2 + y^2}$ is not analytic.

4.3 Conformal Mapping

To each point (x, y) in the z-plane the function $w = f(z)$ determines a point (u, v) in the w-plane if $f(z)$ in a single valued function. If the point z moves along some curve C in the z-plane, the correspondng point w will move along a curve C in the w plane. The correspondence thus designed is called a mapping or transformation of z-plane into w-plane.

The function $w = f(z)$ is called the mapping or transformation function.

Definition: A transformation that preserve angles between every pair of curves through a point, both in magnitude and sense is said to be conformal at that point.

A transformation under which angles between every pair of curves through a point are preserved in magnitude, but altered in sense is said to be isogonal at that point.

Magnification: The transformation $w = az$ where $'a'$ is a real constant, represents magnification. The transformation equation is given by

$$u + iv = a(x + iy) \Rightarrow u = ax, v = ay$$

The image of the point (x, y) is the point (ax, ay). Hence the size of any figure in the z-plane is magnified $'a'$ times, but there will be no change in the shape and orientation. Here circles are transformed into circles.

4.3.1 Magnification and Rotation

A transformation $w = az$ where $'a'$ is a complex constant, represent both magnification and rotation.

Let $z = re^{i\theta}, w = Re^{i\phi}, c = \rho e^{i\alpha}$ then $Re^{i\phi} = (\rho e^{i\alpha})(re^{i\theta}) = \rho r e^{i(\theta+\alpha)}$

The transformation equations are $R = \rho r$ and $\varphi = \theta + \alpha$.

Thus the point (r, θ) in the z-plane is mapped into the point $(\rho r, \theta + \alpha)$. This means that the magnitude of the vector representing z is magnified by $\rho = |a|$ and its direction is rotated through an angle $\alpha = amp(a)$. hence the transformation consists of a magnification and a rotation clearly circles in the z-plane are mapped into circles by this transformation.

4.3.2 Inversion and Reflection

The transformation $w = \frac{1}{z}$ represents inversion with respect to the unit circle $|z| = 1$ followed by reflection in the real axis.

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2} = \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2}$$

$$\Rightarrow x = \frac{u}{u^2 + v^2}, y = \frac{-v}{u^2 + v^2}.$$

General equation of the circle in the z plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{4.37}$$

$$\Rightarrow c(u^2 + v^2) + 2gu - 2fv + 1 = 0 \tag{4.38}$$

which is the equation of the circle in the w plane. $\therefore$ Under the transformation $w = \dfrac{1}{z}$, a circle in the z plane transforms another circle in the w plane. When the circle passes through $(0, 0)$, we have $c = 0$ in equation (4.37) and when $c = 0$ in equation (4.38) we get a straight line.

Example 4.37. *Find the image of the rectangular region in the z-plane bounded by the lines $x = 0, y = 0, x = 2$ and $y = 1$ under the transformation $w = 2z$.*

Solution:

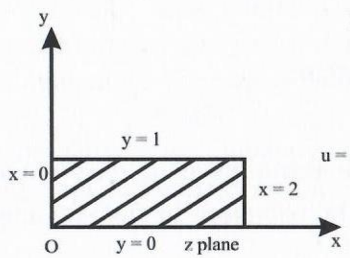

Fig. 1

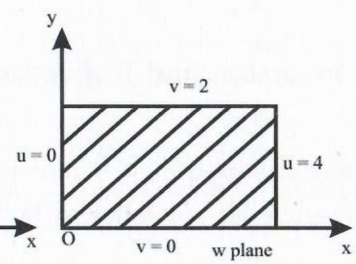

Fig. 2

Let $w = 2z \Rightarrow u + iv = 2(x + iy) \Rightarrow u = 2x, v = 2y$

when $x = y = 0, u = v = 0$, when $x = 2, u = 4$,

when $y = 1, v = 2$

In this transformation rectangle in the z-plane is mapped into w-plane but it is magnified twice.

Example 4.38. *Show that the transformation $w = \sin z$ transform the semi-infinite strip $0 \le x \le \pi/2, y \ge 0$ onto the upper w-plane.*

Solution: Let $w = \sin z \Rightarrow u + iv = \sin(x + iy)$

$\Rightarrow u = \sin x \cosh y, v = \cos x \sinh y; 0 \le x \le \pi/2, y \ge 0$

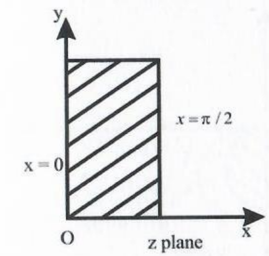

Fig. 3

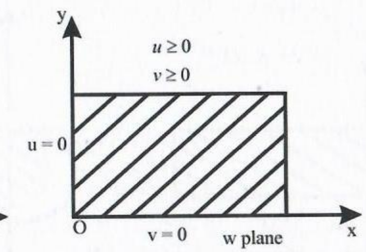

Fig. 4

when $x = 0, y > 0$ we get $u = 0$ and $v \ge 0$, when $x = \pi/2, y \ge 0$ we get $u \ge 0$ and $v \ge 0$.

Example 4.39. *Find the images of the infinite strips (i) $1/4 < y < 1/2$ (ii) $0 < y < 1/2$ under the transformation $w = 1/z$.*

Solution: Let $w = \dfrac{1}{z} \Rightarrow z = \dfrac{1}{w} \Rightarrow x + iy = \dfrac{u - iv}{(u + iv)(u - iv)}$

$$\Rightarrow x = \frac{u}{u^2 + v^2}, y = \frac{-v}{u^2 + v^2}$$

(i) when $y = \frac{1}{4} \Rightarrow \frac{1}{4} = \frac{-v}{u^2+v^2} \Rightarrow u^2 + (v+2)^2 = 4$ which is the equation of a circle with centre $(0, -2)$ and radius 2 units.

(ii) when $y = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{-v}{u^2+v^2} \Rightarrow u^2 + (v+1)^2 = 1$ which is the equation of a circle with centre $(0, -1)$ and radius 1 unit.

(iii) when $y = 0 \Rightarrow \frac{-v}{u^2+v^2} = 0 \Rightarrow v = 0 \Rightarrow u^2 + v^2 \geq -4v$

$\Rightarrow u^2 + (v+2)^2 \leq 2^2$. The interior of the circle $u^2 + (v+2)^2 = 2^2$

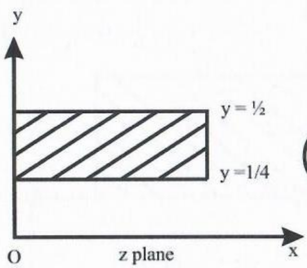

Fig. 5

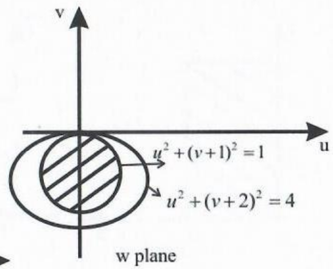

Fig. 6

The image of $y \leq 1/2$ is given by

$$\frac{-v}{u^2+v^2} \leq \frac{1}{2} \Rightarrow u^2 + v^2 \geq -2v \Rightarrow u^2 + (v+1)^2 \geq 1$$

The exterior of the circle $u^2 + (v+1)^2 = 1$.

Example 4.40. *Determine the region D of the w plane into which the triangular region D enclosed by the lines $x = 0, y =$*

$0, x + y = 1$ is transformed under the transform $w = 2z$.

Solution: Let $w = 2z \Rightarrow u + iv = 2(x + iy) \Rightarrow u = 2x, v = 2y$

$x = 0 \Rightarrow u = 0, y = 0 \Rightarrow v = 0, x + y = 1 \Rightarrow u + v = 2$

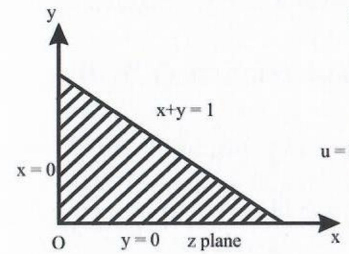

Fig. 7

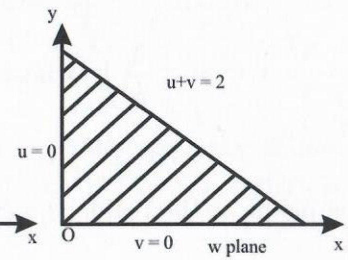

Fig. 8

Example 4.41. *Find the image of the following regions under the transformation $w = \frac{1}{z}$*
(i) the half plane $x > c$, when $c > 0$ (ii) the half plane $y > c$, when $c < 0$ (iii) the infinite strip $\frac{1}{4} \leq y \leq \frac{1}{2}$

Solution: Let $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

The transformation equations are $\quad x = \dfrac{u}{u^2+v^2} \qquad (4.39)$

$$y = \frac{-v}{u^2+v^2} \qquad (4.40)$$

(i) The image of the origin $x > c$ is given by using (4.39)

$\Rightarrow \dfrac{u}{u^2+v^2} > c$

$$c(u^2+v^2) < u \Rightarrow u^2+v^2 < \frac{u}{c} \Rightarrow u^2+v^2-\frac{u}{c} < 0$$

$$\left(u-\frac{1}{2c}\right)^2+v^2 < \left(\frac{1}{2c}\right)^2 \tag{4.41}$$

the equation (4.41) represents the interior if the circle

$\left(u-\frac{1}{2c}\right)^2+v^2 = \left(\frac{1}{2c}\right)^2$ whose centre is $(1/2c, 0)$ and radius is $1/2c$.

(ii) The image of the region $y > c$ is given by $\frac{-v}{u^2+v^2} > c$

$$\Rightarrow c(u^2+v^2) < -v \Rightarrow u^2+v^2 > \frac{-v}{c} \Rightarrow u^2+v^2+\frac{v}{c} > 0$$

$$\Rightarrow u^2+\left(v+\frac{1}{2c}\right)^2 > \left(\frac{1}{2c}\right)^2 \tag{4.42}$$

the equation (4.42) represents the exterior of the circle

$u^2+\left(v+\frac{1}{2c}\right)^2 = \left(\frac{1}{2c}\right)^2$ whose centre is $(0, -1/2c)$ and radius is $\frac{1}{2|c|}$.

(iii) The image of $y \geq \frac{1}{4}$ is given by $\frac{-v}{u^2+v^2} \geq \frac{1}{4}$

Example 4.42. *Find the image of the circle $|z| = 2$ under the transformation $w = \sqrt{2}e^{i\pi/4}z$.*

Solution:

Let $w = \sqrt{2}e^{i\pi/4}z = \sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)z = (1+i)(x+iy)$

Here $u = x-y, v = x+y \quad \therefore x = \frac{u+v}{2}, y = \frac{v-u}{2}$

Given $|z| = 2 \Rightarrow x^2+y^2 = 4 \Rightarrow \frac{(u+v)^2}{4}+\frac{(v-u)^2}{4} = 4$

$\Rightarrow u^2+v^2 = 8$

Example 4.43. *Find the image of the rectangular in the z-plane bounded by the lines $x = 0, y = 0, x = 2, y = 1$ under the transformation $w = (1+2i)z+(1+i)$.*

Solution: Let $w = u+iv = (1+2i)z+(1+i)$

The image of $(0,0)$ is given by $u+iv = (1+2i)(0+0i)+1+i = 1+i$

(i.e.) The point is $(1,1)$.

The image of $(2,0)$ is given by $u+iv = (1+2i)(2+0i)+1+i = 3+5i$

(i.e.) The point is $(3,5)$.

The image of $(2,1)$ is given by $u+iv = (1+2i)(2+i)+1+i = 1+6i$

(i.e.) The point is $(1,6)$.

The image of $(0,1)$ is given by $u+iv = (1+2i)(i)+1+i = -1+2i$

(i.e.) The point is $(-1,2)$.

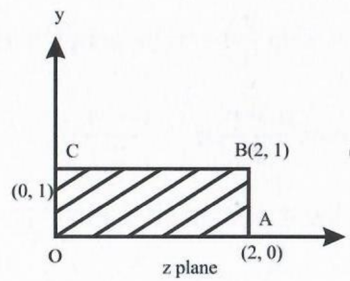

Fig. 9

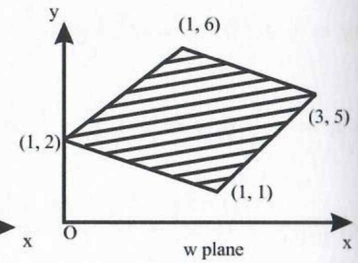

Fig. 10

Example 4.44. *Show that the transformation $w = \frac{1}{z}$ transforms circles and straight lines in the z-plane into circles or straight lines in the w-plane.*

Solution: Let $w = \frac{1}{z} \Rightarrow z = \frac{1}{w} \Rightarrow x + iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$

$\Rightarrow x^2 + y^2 = \frac{1}{u^2+v^2}$

$$\text{Consider } a(x^2+y^2) + bx + cy + d = 0 \tag{4.43}$$

If $a \neq 0$, equation (4.43) represents a circle and if $a = 0$, equation (4.43) a straight line.

Substitute the values of x and y in (4.43),

$$\frac{a}{u^2+v^2} + \frac{bu}{u^2+v^2} - \frac{cv}{u^2+v^2} + d = 0 \Rightarrow d(u^2+v^2) + bu - cv + a = 0 \tag{4.44}$$

If $d \neq 0$, equation (4.44) represents a circle and if $d = 0$, equation (4.44) represents a straight line.

Example 4.45. *What is the region of the w-plane into which the rectangular region in the z-plane bounded by the lines $x = 0, y = 0, x = 1$ and $y = 2$ is mapped under the transformation $w = z + 2 - i$.*

Solution:

Let $w = z+2-i \Rightarrow u+iv = x+iy+(2-i) \Rightarrow u = x+2, v = y-1$

when $x = 0, u = 0+2 = 2$, $x = 1, u = 1+2 = 3$, $y = 0, v = 0-1 = -1$, $y = 2, v = 2-1 = 1$

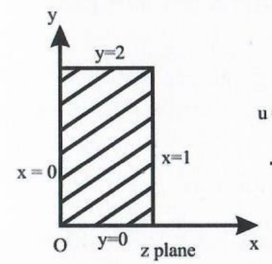

Fig. 11

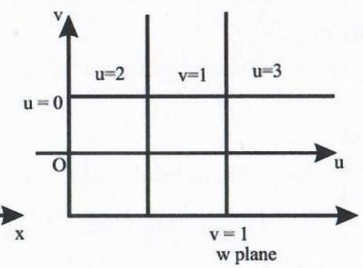

Fig. 12

Example 4.46. *Find the image of the circle $|z| = 1$ by the transformation $w = z + 2 + 4i$.*

Solution: Let $w = z + 2 + 4i \Rightarrow u + iv = (x+iy) + 2 + 4i$

Here $u = x + 2, v = y + 4 \Rightarrow x = u - 2, y = v - 4$

Given $|z| = 1 \Rightarrow x^2 + y^2 = 1 \Rightarrow (u-2)^2 + (v-4)^2 = 1$.

$\therefore$ The circle $x^2 + y^2 = 1$ is mapped into $(u-2)^2 + (v-4)^2 = 1$.

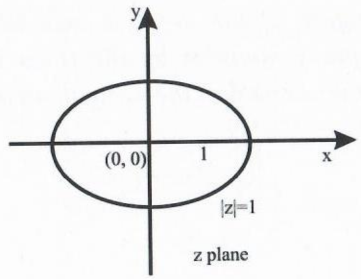

Fig. 13

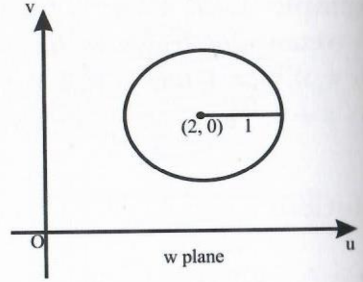

Fig. 14

Example 4.47. *Find the image of $|z-2i|=2$ under the transformation $w=\dfrac{1}{z}$.*

Solution:

Let $w=\dfrac{1}{z}\Rightarrow z=\dfrac{1}{w}\Rightarrow x+iy=\dfrac{1}{u+iv}=\dfrac{u-iv}{u^2+v^2}$

$$x=\frac{u}{u^2+v^2} \tag{4.45}$$

$$y=\frac{-v}{u^2+v^2} \tag{4.46}$$

Given $|z-2i|=2\Rightarrow |x+iy-2i|=2\Rightarrow x^2+(y-2)^2=4$

$$\Rightarrow x^2+y^2-4y=0 \tag{4.47}$$

Substitute (4.45) and (4.46) in (4.47),

$$\frac{u^2}{(u^2+v^2)^2}+\left(\frac{v}{u^2+v^2}\right)^2-4\left(\frac{-v}{u^2+v^2}\right)=0$$

$$\Rightarrow \frac{(u^2+v^2)(1+4v)}{(u^2+v^2)^2}=0$$

$\Rightarrow 1+4v=0$ which is a straight line in the w-plane.

4.4 Bilinear Transformation

The transformation $w=\dfrac{az+b}{cz+d}$ where a, b, c, d are complex constants such that $ad-bc\neq 0$ is called a bilinear transformation. It is also called linear fractional or Mobius transformation.

If $ad-bc=0$, every point of the z plane becomes a critical point of the bilinear transformation.

The term $(ad-bc)$ is called the determinant of the bilinear transformation. The inverse of the transformation $w=\dfrac{az+b}{cz+d}$ is $z=\dfrac{-dw+b}{cw-a}$ which is also a bilinear transformation.

Definition of cross-ratio of four points

If z_1, z_2, z_3 and z_4 are four points in the z-plane, then

$\dfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$ is called the cross ratio of these points.

Cross ration property of bilinear transformation

If cross-ratio of four points is invariant under bilinear transformation. If w_1, w_2, w_3 and w_4 are images of z_1, z_2, z_3 and z_4 respectively under a bilinear transformation, then

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

Proof: Let the bilinear transformation be $w = \dfrac{az+b}{cz+d}$

Then $w_1 - w_2 = \dfrac{az_1+b}{cz_1+d} - \dfrac{az_2+b}{cz_2+d} = \dfrac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)}$

$$(w_1-w_2)(w_3-w_4) = \frac{(ad-bc)^2(z_1-z_2)(z_3-z_4)}{(cz_1+d)(cz_2+d)(cz_3+d)(cz_4+d)} \text{ and}$$

$$(w_1-w_4)(w_3-w_2) = \frac{(ad-bc)^2(z_1-z_4)(z_3-z_2)}{(cz_1+d)(cz_2+d)(cz_3+d)(cz_4+d)}$$

$$\Rightarrow \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

Fixed point of the transformation

The image of a point z under a transformation $w = f(z)$ is itself, then the point is called fixed point or an invariant point of the transformation.

Example 4.48. *Find the invariant points of the transformation* $w = \dfrac{2z+6}{z+7}$

Solution: Put $w = z \Rightarrow z = \dfrac{2z+6}{z+7}$

$\Rightarrow z^2 + 5z - 6 = 0 \Rightarrow (z+6)(z-1) = 0$

The invariant points are $-6, 1$.

Example 4.49. *Find the invariant points of the transformation* $w = \dfrac{2z+4}{1+iz}$

Solution:

Put $w = z \Rightarrow w = \dfrac{2z+4i}{1+iz} \Rightarrow z^2 - 3iz + 4 = 0$

$\Rightarrow (z-4i)(z+i) = 0$. The invariant points are $4i, -i$.

Example 4.50. *Find the bilinear transformations which maps the points $z_1 = 1, z_2 = i, z_3 = -1$ into the points $w_1 = i, w_2 = 0, w_3 = -i$ and hence find the image $|z| < 1$.*

Solution: Let the bilinear transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(z-1)(i+1)}{(1-i)(1-z)} = \frac{(w-i)((0+i)}{(i-0)(-i-w)}$$

Using C/D rule

$$\frac{(z-1)(i+1)+(1-i)(z+1)}{(z-1)(i+1)-(1-i)(-1-z)} = \frac{(w-i)+((w+i)}{(w-i)-(i+w)}$$

$\Rightarrow \dfrac{2z-2i}{2iz-2} = -\dfrac{w}{i} \Rightarrow w = \dfrac{1+iz}{1-iz}$ which is a bilinear transformation.

Now inverse mapping $z = i\left(\frac{1-w}{1+w}\right)$

Given $|z| < 1 \Rightarrow \left|\frac{1-w}{1+w}\right| < 1 \Rightarrow \left|\frac{1-(u+iv)}{1+(u+iv)}\right| < 1$

$\Rightarrow |(1-u) - iv| < |(1+u) + iv|$

$\Rightarrow (1-u)^2 + v^2 < (1+u)^2 + v^2 \Rightarrow u > 0.$

Hence the interior of the circle $x^2 + y^2 = 1$ in the z plane is mapped onto the entire half of the w plane to the right of the imaginary axis.

Aliter:

$$\text{Let the bilinear transformation be } w = \frac{az+b}{cz+d} \tag{4.48}$$

Substituting the values of w and z in (4.48), we get

$$i = \frac{a+b}{c+d} \tag{4.49}$$

$$0 = \frac{ai+b}{ci+d} \tag{4.50}$$

$$-i = \frac{-a+b}{-c+d} \tag{4.51}$$

Equations (4.49), (4.50) and (4.51)

$$\Rightarrow (a+b) - i(c-d) = 0 \tag{4.52}$$

$$b + ia = 0 \tag{4.53}$$

$$(-a+b) + i(-c+d) = 0 \tag{4.54}$$

Substituting (4.52), (4.53) and (4.54), we get

$c = \frac{b}{i} = -a$ and $d = \frac{a}{i} = -ia$

$\therefore$ Equation (4.48) $\Rightarrow w = \frac{az - ia}{-az - ia} = \frac{i-z}{i+z}$ which is the bilinear transformation.

Example 4.51. *Find the bilinear transformations which maps the points $z_1 = 0, z_2 = -i, z_3 = -1$ onto the points $w_1 = i, w_2 = 1, w_3 = 0$ respectively.*

Solution: Let the bilinear transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(w-i)(1-0)}{(i-1)(-w)} = \frac{(z-0)(-i+1)}{(0+i)(-1-z)}$$

Using c and d rule $\frac{2w-i}{-i} = \frac{3z+1}{z-1} \Rightarrow w = \frac{(z+1)i}{1-z}$ which is a bilinear transformation.

Example 4.52. *Find the bilinear transformations which maps the points $z_1 = \infty, z_2 = i, z_3 = 0$ onto the points $w_1 = 0, w_2 = i, w_3 = \infty$ respectively.*

Solution: Let the bilinear transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)(i-w_3)}{(0-i)(w_3-w)} = \frac{(z-z_1)(i-0)}{(z_1-i)(0-z)}$$

$$\frac{w\left(\frac{i}{w_3}-1\right)}{(-i)\left(1-\frac{w}{w_3}\right)} = \frac{\left(\frac{z}{z_1}-1\right)(i)}{\left(1-\frac{i}{z_1}(-z)\right)} \Rightarrow -\frac{w}{i} = \frac{i}{3} \Rightarrow w = -\frac{1}{z}$$

which is the bilinear transformation.

$\left(\text{As } w_3 \to \infty, \frac{w}{w_3} \to 0, \frac{i}{w_3} \to 0 \text{ as } z_1 \to \infty, \frac{z}{z_1} \to 0, \frac{i}{z_1} \to 0\right)$.

Example 4.53. *Find the bilinear transformation that maps the points $-1, 0, 1$ in the z-plane into the points $0, i, 3i$ in the w-plane.*

Solution: Let $z_1 = -1, z_2 = 0, z_3 = 1, w_1 = 0, w_2 = i, w_3 = 3i$.

Let the bilinear transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)(i-3i)}{(0-i)(3i-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)}$$

$$\Rightarrow \frac{w}{-i}.\frac{(-2i)}{(3i-w)} = \frac{z+1}{-1}.\frac{(-1)}{1-z}$$

$$\Rightarrow \frac{2w}{3i-w} = \frac{z+1}{1-z} \Rightarrow \frac{2w}{w-3i} = \frac{z+1}{z-1}$$

$$2w(z-1) = (z+1)(w-3i) = w(z+1)(1+z)$$

$$\Rightarrow 2w(z-1) - w(z+1) = -3i(1+z)$$

$$\Rightarrow w(2z-2-z-1) = -3i(1+z) \Rightarrow w(z-3) = -3i(1+z)$$

$w = -3i\left(\frac{1+z}{z-3}\right)$ which is the required transformation.

Example 4.54. *Find the bilinear transformation that maps $z = 0, 1, \infty$ onto $w = i, -1, -i$. Also find the invariant point of the transformation.*

Solution: Let $z_1 = 0, z_2 = 1, z_3 = \infty, w_1 = i, w_2 = -1, w_3 = -i$.

Let the bilinear transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-i)(-1+i)}{(i+1)(-1-w)} = \frac{(z-0)(1-\infty)}{(0-1)(\infty-z)}$$

$$\Rightarrow \frac{(w-i)(-1+i)}{(i+1)(-i-w)} = \frac{(z-0)}{(0-1)}(-1)$$

$$\Rightarrow \frac{(w-i)(-1+i)}{-(i+1)(i+w)} = z \Rightarrow \frac{w-i}{w+i} = \frac{z}{-i}$$

By components and dividendo

$$\frac{(w-i)+(w+i)}{(w-i)-(w+i)} = \frac{z-i}{z+i} \Rightarrow \frac{2w}{-2i} = \frac{z-i}{z+i} \Rightarrow w = (-i)\left(\frac{z-i}{z+i}\right)$$

EXERCISE

1. Find the fixed point of (i) $w = \dfrac{1}{z-2i}$ (ii) $w = \dfrac{z-2}{z+3}$ (iii) $w = \dfrac{5z+4}{z+5}$ [Ans: (i) $z = i$, (ii) $z = -1 \pm i$, (iii) $z = \pm 2$]

2. Find the bilinear transformation which maps the points

   (i) $z_1 = 2, z_2 = i, z_3 = -2$ onto the points $w_1 = 1, w_2 = i, w_3 = -1$

   (ii) $z_1 = -i, z_2 = 0, z_3 = i$ onto the points $w_1 = -1, w_2 = i, w_3 = 1$

   (iii) $z_1 = 0, z_2 = 1, z_3 = \infty$ onto the points $w_1 = i, w_2 = -1, w_3 = -i$

   $\left[\text{Ans:}i)w = \dfrac{3z+2i}{iz+6}\ ii)w = -\left(\dfrac{z-1}{z+1}\right)\ iii)w = -i\left(\dfrac{z-i}{z+i}\right)\right]$

3. Find the critical points of the transformation $w = z^2$.

   [Ans: $z = 0$]

4. Find the invariant points of the transformation $w = \dfrac{2z+6}{z+7}$
   [Ans: $z = -6, 1$]

5. Define a bilinear or mobius transformation and its determinant.

6. Find the image of the infinite strip $0 \le x \le 2$ under the transformation $w = iz$. [Ans: u=-y, v=x]

7. If a and b are two fixed points of a bilinear transformation, show that it can be written in the form, $\dfrac{w-a}{w-b} = k\left(\dfrac{z-a}{z-b}\right)$, k is a constant, $a \neq b$.

8. Define Isogonal transformation.

9. Find the points at which the transformation $w = \sin z$ is not conformal. $\left[\text{Ans:}z = (2n+1)\dfrac{\pi}{2}\right]$

10. State the conditions for which the transformation $w = f(z)$ is conformal.

11. Find the image of the rectangular region in the z-plane bounded by the lines $x = 0, y = 0, x = 2$ and $y = 1$ under the transformation $w = 2z$. [Ans: Rectangle is magnified twice]

12. Find the image of $|z+1| = 1$ under the mapping $w = 1/z$.

13. Discuss the transformation $w = \sin z$.

14. If $f(z)$ and $f(\overline{z})$ are analytic in a region R, show that $f(z)$ is a constant in that region.