UNIT-III

06/09/23

MATHEMATICAL LOGICS

PROPOSITIONS (statement)

- A deklarative sentence which is either true or false is called proposition.

Eg: Chennai is the capital of India ⇒ Proposition.

What is time ⇒ Not a proposition.

CONNECTIVES :

1. Conjuction ($\wedge$)

When P and q are two propositions then p and q demoted by $p \wedge q$ is defined as a proposition that is true when both p and q are true and is false otherwise.

Truth Table.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

2. DISJUNCTION ($\vee$)

When p and q are two propositions, then p (or) q, denoted by $p \vee q$ is defined as a proposition that is false. When both p and q are false and true otherwise.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

CONDITIONAL PROPOSITION :

If p and q are two propositions, then the proposition if p then q that is denoted $p \rightarrow q$ is ealled conditional proposition which is false when p is true and q is false and true otherwise.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

BICONDITIONAL PROPOSITION

If p and q are two propositions, then the proposition p if and only if q that is denoted $p \rightleftarrows q$ is called a biconditional proposition which is true when both p and q are having same Truth values.

p	q	$p \rightleftarrows q$
T	T	T
T	F	F
F	T	F
F	F	T

PROBLEM

Construct a truth Table for the proposition. $(p \vee q) \rightarrow q$

p	q	$p \vee q = a$	$a \rightarrow q$
T	T	T	T
T	F	T	F
F	T	T	T
F	F	F	T

PROBLEM

Construct the truth table for $(p \wedge q) \rightarrow r$

p	q	r	$p \wedge q = a$	$a \rightarrow r$
T	T	T	T	T
T	F	T	F	T
T	T	F	T	F
F	T	T	F	T
T	F	F	F	T
F	F	T	F	T
F	T	F	F	T
F	F	F	F	T

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TAUTOLOGY

A compound proposition $P(p_1, p_2, \ldots p_n)$ is called a tautology if it is true for every truth values of $P_1, P_2 \ldots P_n$.

NOTE

Negation: It is denoted by $\neg$.

Truth Table for Negation

P	$\neg$P
T	F
F	T

PROBLEM

Prove that $P \vee \neg P$ is a tautology.

P	$\neg$P	$P \vee \neg P$
T	F	T
F	T	T

CONTRADICTION

A compound Proposition $P(P_1, P_2 \ldots P_n)$ is called a contradiction if it is False for every truth values of $P_1, P_2 \ldots P_n$.

PROBLEM

Prove that $P \wedge \neg P$ is a contradiction.

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

$P \wedge \neg P$ is a contradiction

PROBLEM

Check whether the following proposition $(\neg q \wedge (P \rightarrow q)) \rightarrow \neg P$ is a tautology on contradiction or not.

P	q	(a) $\neg q$	(b) $P \rightarrow q$	(c) (a) $\wedge$ (b)	(d) $\neg P$	(c) $\rightarrow$ (d)
T	T	F	T	F	F	T
T	F	T	F	F	F	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T

$\therefore$ It is a tautology.

PROBLEM

$((p \to q) \wedge (q \to r)) \to (p \to r)$ is a tautology.

p	q	r	(a) $p \to q$	(b) $q \to r$	(c) (a) $\wedge$ (b)	(d) $p \to r$	(c) $\to$ (d)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	T	T	T	T	T	T	T

Hence, it is a tautology.

PROBLEM

$\neg(q \to r) \wedge r \wedge (p \to q)$ is a contradiction.

p	q	r	(a) $q \to r$	(b) $\neg$(a)	(c) (b) $\wedge$ r	(d) $p \to q$	(c) $\wedge$ (d)
T	T	T	T	F	F	T	F
T	T	F	F	T	F	T	F
T	F	T	T	F	F	F	F
T	F	T	T	F	F	F	F
F	F	T	T	F	F	T	F
F	F	T	T	F	F	T	F
F	T	F	F	T	F	T	F
F	T	T	T	F	F	T	F

Hence, proved.

LAWS OF ALGEBRA OF PROPOSITIONS.

Idempotent law

(i) $P \vee P = P$
(ii) $P \wedge P = P$

Identity law

(i) $P \vee F = P$
(ii) $P \wedge T = P$

Dominant law

(i) $P \vee T = T$
(ii) $P \wedge F = F$

Complement law

(i) $P \vee \neg P = T$
(ii) $P \wedge \neg P = F$

Commutative law.

(i) $P \vee q = q \vee P$
(ii) $p \wedge q = q \wedge p$

Associative law

(i) $P \vee (q \vee r) = (P \vee q) \vee r$
(ii) $P \wedge (q \wedge r) = (P \wedge q) \wedge r$

Distributive law

(i) $P \vee (q \wedge r) = \cancel{P \wedge q}\ (P \vee q) \wedge (P \vee r)$
(ii) $P \wedge (q \vee r) = (P \wedge q) \vee (P \wedge r)$

Demorgan's law

(i) $\neg(P \vee q) = \neg P \wedge \neg q$

(ii) $\neg(P \wedge q) = \neg P \vee \neg q$.

EQUIVALENCES INVOLVING CONDITIONALS.

1. $P \rightarrow q = \neg P \vee q$
2. $P \rightarrow q = \neg q \rightarrow \neg P$
3. $(P \rightarrow q) \wedge (P \rightarrow r) = P \rightarrow (q \wedge r)$
4. $(P \rightarrow r) \wedge (P \rightarrow r) = (P \vee q) \rightarrow r$
5. $(P \rightarrow q) \vee (P \rightarrow r) = P \rightarrow (q \vee r)$
6. $(P \rightarrow r) \vee (q \rightarrow r) = (P \wedge q) \rightarrow r$.

NOTE:

$\neg\neg P = P$

PROBLEM.

Prove that $(\neg q \wedge (P \rightarrow q)) \rightarrow \neg P$ is a tautology.

$(\neg q \wedge (P \rightarrow q)) \rightarrow \neg P$

$(\neg q \wedge (\neg P \vee q)) \rightarrow \neg P$ [∵ formula ①]

$= \neg(\neg q \wedge (\neg P \vee q)) \vee \neg P$ [∵ for ①]

$= (\neg\neg q) \vee \neg(\neg P \vee q) \vee \neg P$

$= q \vee (P \wedge \neg q) \vee \neg P$

$= [(q \vee P) \wedge (q \vee \neg q)] \vee \neg P$ [Distributive]

$= [(q \vee P) \wedge T] \vee \neg P$ [Identity]

$= (q \vee P) \vee \neg P$

$= q \vee (P \vee \neg P)$

$= q \vee T$ [complement]

$= T$

∴ It is a tautology.

PROBLEM

Prove $\neg(q \to r) \land r \land (p \to q)$ is a contradiction

$$= \neg(q \to r) \land r \land (p \to q)$$
$$= \neg(\neg q \lor r) \land r \land (\neg p \to q)$$
$$= (q \land \neg r) \land r \land (\neg p \lor q)$$
$$= q \land (\neg r \land r) \land (\neg p \lor q)$$
$$= q \land F \land (\neg p \lor q)$$
$$= F$$

PROBLEM

Prove that $((p \to q) \land (q \to r)) \to (p \to r)$ is a Tautology $[\because (p \to q) \land (q \to r) \equiv p \to r]$

$$= ((p \to q) \land (q \to r)) \longrightarrow (p \to r)$$
$$= (p \to r) \longrightarrow (p \to r)$$
$$= \neg(p \to r) \lor (p \to r)$$
$$= T$$

$\because (p \to q) \land (p \to r) = p \to r$

$p \to q \equiv \neg p \lor q$

$\neg p \lor p \equiv T$

PROBLEM

Prove that $(\neg p \lor q) \land (p \land (p \land q)) \equiv p \land q$

$$(\neg p \lor q) \land (p \land (p \land q))$$
$$= (\neg p \lor q) \land (p \land q)$$
$$= (\neg p \land (p \land q)) \lor (q \land (p \land q))$$
$$= (\neg p \land p \land q) \lor (q \land (q \land p))$$
$$= (F \land q) \lor ((q \land q) \land p)$$
$$= F \lor (q \land p)$$

$\because (a \lor b) \land c =$
$(a \land c) \lor (b \land c)$

$= q \wedge p$
$= p \wedge q$

PROBLEM

Prove that $p \rightarrow (q \rightarrow p) \equiv \neg p \rightarrow (p \rightarrow q)$

$$
\begin{aligned}
LHS = p \rightarrow (q \rightarrow p) &\equiv \neg p \vee (q \rightarrow p) \\
&\equiv \neg p \vee (\neg q \vee p) \\
&\equiv \neg p \vee (p \vee \neg q) \\
&\equiv (\neg p \vee p) \vee \neg q \\
&= T \vee \neg q \\
&= T
\end{aligned}
$$

$$
\begin{aligned}
R.H.S &= \neg p \rightarrow (p \rightarrow q) \\
&= \neg(\neg p) \vee (p \rightarrow q) \\
&= p \vee (\neg p \vee q) \\
&= T \vee q \\
&= T
\end{aligned}
$$

$\boxed{p \rightarrow q = \neg p \vee q}$

$\therefore$ LHS = RHS

PROBLEM

Prove that $(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

$$
\begin{aligned}
LHS = (p \vee q) \rightarrow r &= \neg(p \vee q) \vee r \\
&= (\neg p \wedge \neg q) \vee r \\
&= (\neg p \vee r) \wedge (\neg q \vee r) \\
&= (p \rightarrow r) \wedge (q \rightarrow r)
\end{aligned}
$$

$\boxed{\because p \rightarrow q \equiv \neg p \vee q}$

Equivalence involving Bi conditionals

1. $P \rightleftharpoons q \equiv (P \to q) \wedge (q \to P)$
2. $P \rightleftharpoons q = \neg P \rightleftharpoons \neg q$
3. $P \rightleftharpoons q \equiv (P \wedge q) \vee (\neg P \wedge \neg q)$

PROBLEM

Prove that $\neg(P \rightleftharpoons q) \equiv (P \vee q) \wedge \neg(P \vee q)$

$$\neg(P \rightleftharpoons q) = \neg((P \to q) \wedge (q \to P))$$
$$= \neg(P \to q) \vee \neg(q \to P)$$
$$= \neg(\neg P \vee q) \vee \neg(\neg P \vee P)$$
$$= (P \wedge \neg q) \vee (q \wedge \neg P)$$
$$= (P \vee q) \wedge (\neg q \vee q) \wedge (P \vee \neg P) \wedge (\neg q \vee \neg P)$$
$\boxed{\because T \wedge a \equiv a}$
$$= (P \vee q) \wedge T \wedge T \wedge (\neg q \vee \neg P)$$
$$= (P \vee q) \wedge (\neg q \vee \neg P)$$
$$= (P \vee q) \wedge \neg(q \vee \neg P)$$
$$= (P \vee q) \wedge \neg(q \wedge P)$$
$$= (P \vee q) \wedge \neg(P \wedge q)$$

PROBLEM.

Prove that $(\neg P \to \neg q) \to (q \to P) \equiv T$

$$(\neg P \to \neg q) \to (q \to P)$$
$$= (q \to P) \to (q \to P)$$
$\boxed{\because P \to q \equiv \neg q \to \neg P}$
$$= \neg(q \to P) \vee (q \to P)$$
$$= T$$

TAUTOLOGICAL IMPLICATION

A proposition P is said to tautologically imply q if $P \to q$ is a tautology.

(or)

$p \Rightarrow q$ if $p \to q$ is

Tautological Implication.

PROBLEM.

Prove that $p \to (q \to r) \Rightarrow (p \to q) \to (p \to r)$ tautologically.

To show $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ is a tautology.

p	q	r	$q \to r$	(a) $p \to (q \to r)$	(b) $p \to q$	(c) $p \to r$	(d) (b) $\to$ (c)	(a) $\to$ (d)
T	T	T	T	T	T	T	T	T
T	F	T	T	T	F	T	T	T
T	F	F	~~F~~ T	T	F	F	T	T
T	T	F	F	F	T	F	F	T
F	F	F	~~F~~ T	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T

$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

$= (\neg p \vee (q \to r)) \to (\neg(p \to q) \vee (p \to r))$

$= (\neg p \vee \neg q \vee r) \to (\neg(\neg p \vee q) \vee (\neg p \vee r))$

$= (\neg p \vee \neg q \vee r) \to ((p \wedge \neg q) \vee (\neg p \vee r))$

$= \neg(\neg p \vee \neg q \vee r) \to ((p \wedge \neg q) \vee (\neg p \vee r))$

$= (p \wedge q \wedge \neg r) \vee ((p \wedge \neg q) \vee (\neg p \vee r))$

$= (p \wedge q \wedge \neg r) \vee (p \vee (\neg p \vee r)) \wedge (\neg q \vee \neg p \vee r)$

$= (p \wedge q \wedge \neg r) \vee (T \vee r) \wedge (\neg q \vee \neg p \vee r)$

$= T$

PROBLEM

Prove that $(P \vee q) \wedge (P \rightarrow r) \wedge (q \rightarrow r) \Rightarrow r$

To prove $(P \vee q) \wedge (P \rightarrow r) \wedge (q \rightarrow r) \rightarrow r$ is a tautology.

$$\left[\begin{array}{l} P \Rightarrow q \\ P \rightarrow q \end{array}\right]$$

$(P \vee q) \wedge (P \rightarrow r) \wedge (q \rightarrow r) \rightarrow r$

$= (P \vee q) \wedge (\neg P \vee r) \wedge (\neg q \vee r) \rightarrow r$

$= (P \vee q) \wedge ((\neg P \wedge \neg q) \vee r) \rightarrow r$

$= (P \vee q) \wedge (\neg (P \vee q) \vee r) \rightarrow r$

$= ((P \vee q) \wedge (\neg (P \vee q))) \vee ((P \vee q) \wedge r) \rightarrow r$

$=$

$= F \vee \{(P \vee q) \wedge r) \rightarrow r$

$= (P \vee q) \wedge r \rightarrow r$

$= \neg ((P \vee q) \wedge r) \vee r$

$= (\neg (P \vee q) \vee \neg r) \vee r$

$= \neg (P \vee q) \vee (\neg r \vee r)$

$= \neg (P \vee q) \vee T$

$= T$

PROBLEM

Prove that $((P \vee \neg P) \to q) \to ((P \vee \neg P) \to r) \Rightarrow q \to r$

$((P \vee \neg P) \to q) \to (P \vee \neg P) \to r) \to (q \to r)$

$= (T \to q) \to (T \to r) \to (q \to r)$

$= (\neg T \vee q) \to (\neg T \vee r) \to (\neg q \vee r)$

$= (F \vee q) \to (F \vee r) \to (\neg q \vee r)$

$= q \to r \to (\neg q \vee r)$

$= (\neg q \vee r) \to (\neg q \vee r)$

$= \neg(\neg q \vee r) \vee (\neg q \vee r)$

$= T$

DUALITY

The dual of a compound proposition is a compound proposition obtained by replacing $\vee$ by $\wedge$, $\wedge$ by $\vee$ T by F and F by T.

PROBLEM

Obtain the duality of

(i) $(P \vee q) \wedge (r)$

(ii) $(P \to q) \vee (q \to r)$

(i) The dual of (i) is $(P \wedge q) \vee (r)$

(ii) $(P \rightarrow q) \vee (q \rightarrow r)$

$= (\neg P \vee q) \vee (\neg q \vee r)$

dual $= (\neg P \wedge q) \wedge (\neg q \wedge r)$

[∵ Cannot do the duality of $(\rightarrow)$, so; change it $\wedge$ or $\vee$ form].

THEORY OF INFERENCE

A conclusion C is said to follow from a set of Propositions $P_1, P_2, \ldots P_n$ if $P_1 \wedge P_2 \wedge \cdots \wedge P_n \Rightarrow C$

(or)

C logically follows from a set of n propositions $P_1, P_2, \ldots P_n$ if $P_1 \wedge P_2 \wedge \cdots \wedge P_n \Rightarrow C$

RULES OF INFERENCES:

Rule P:

A proposition is introduced (or) assumed at any step in the derivation.

Rule T:

A formula S is introduced in the derivation of S is tautologically implied by one or more preceding formula in the derivation.

Rule CP :

If a formula s can be derived from another r and a set of propositions then the proposition $(r \to s)$ can be derived from the test of given propositions alone.

NOTE

1. P_1, P_2
$\Rightarrow P_1 \wedge P_2$

2. $P, P \to q \Rightarrow q$

PROOF
$$P \wedge (P \to q)$$
$$= P \wedge (\neg P \vee q)$$
$$= (P \wedge \neg P) \vee (P \wedge q)$$
$$= F \vee (P \wedge q)$$
$$= P \wedge q \qquad [P \wedge q \Rightarrow P \text{ or } q]$$
$$= q$$

3. (i) $P \wedge q \Rightarrow P$ or q
(ii) $P \Rightarrow P \vee q$
(iii) $q \Rightarrow P \vee q$
(iv) $P \to q, q \to r \Rightarrow P \to r$
(v) $P, P \to q \Rightarrow q$

PROBLEM

Show that $t \wedge s$ can be derived from $P \to q$, $q \to \neg r$, r, $P \vee (t \wedge s)$

(or)

Show that $t \wedge s$ logically derived from $P \to q$, $q \to \neg r$, r, $P \vee (t \wedge s)$

(or)

$P \to q,\ q \to \neg r,\ r,\ P \vee (t \wedge s) \Rightarrow t \wedge s$

1. $P \to q$ (Rule P)
2. $P \to \neg r$ (Rule P)
3. r (Rule P)
4. $P \vee (t \wedge s)$ (Rule P)
5. $P \to \neg r$ (Rule T, 1, 2)
6. $r \to \neg P$ (Rule T, 5)

$P,\ P \to q \Rightarrow q$

$P \to q$

$= \neg q \to \neg P$

$P \to \neg r$

$\neg(\neg r) \to \neg P$

7. $\neg P$ (Rule T, 6, 3)
8. $\neg(\neg P) \vee (t \wedge s)$ (Rule T, 4)
9. $\neg P \to (t \wedge s)$ (Rule T, 8)
10. $(t \wedge s)$ (Rule T, 7, 9)

PROBLEM

Show that $a \vee b$ follows logically from $p \vee q$ $(p \vee q) \to \neg r$, $\neg r \to (s \wedge \neg t)$ and $(s \wedge \neg t) \to (a \vee b)$

1. $(p \vee q)$ (Rule P)
2. $(p \vee q) \to \neg r$ (Rule P)
3. $\neg r \to (s \wedge \neg t)$ (Rule P)
4. $(s \wedge \neg t) \to (a \vee b)$ (Rule P)
5. $\neg r$ (Rule T, 1, 2)
6. $(s \wedge \neg t)$ (Rule T, 5, 3)
7. $(a \vee b)$ (Rule T, 6, 4)

PROBLEM.

Show that $(a \to b) \land (a \to c)$, $\neg(b \land c)$, $(d \lor a) \Rightarrow d$.

1.	$(a \to b) \land (a \to c)$	(Rule P)
2.	$\neg(b \land c)$	(Rule P)
3.	$(d \lor a)$	(Rule P)
4.	$a \to b$	(Rule T, 1)
	$[P \land q \Rightarrow P$ or $q]$	
5.	$a \to c$	(Rule T, 1)
6.	$\neg b \lor \neg c$	(Rule T, 2)
7.	$\neg b \to \neg a$	(Rule T, 4)
8.	$b \to \neg c$	(Rule T, 6)
9.	$a \to \neg c$	(Rule T, 8, 4)
10	$\neg c \to \neg a$	(Rule T, 5)
11.	$(\neg b \lor \neg c) \to \neg a$	(Rule T, 10, 6, 7)
12.	$\neg(b \land c) \to \neg a$	(Rule T, 11)
13.	$\neg a$	(Rule T, 12, 2)
14	$(d \lor a) \land \neg a$	(Rule T, 13, 3)
15	$(d \land \neg a) \lor (a \land \neg a)$	(Rule T, 14)
16	$(d \land \neg a) \lor F$	(Rule T, 15)
17	$d \land \neg a$	(Rule T, 16)
18	d	(Rule T, 17)