18MAB101T- Calculus and Linear Algebra

CLA 50%

CLAT1 $\longrightarrow$ 10% Module 1
CLAT2 $\longrightarrow$ 15% Module 2, 3
CLAT3 $\longrightarrow$ 15% Module 4, 5
CLAT4 $\longrightarrow$ 10% $\left\langle \begin{array}{l} \text{Assignment } 5\% \\ \text{surprise } 5\% \end{array} \right\} \longrightarrow 10$

Module - 1

Matrix

Square matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Determinant $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $|A| = 1 \times 1 - 2 \times 0$
$= 1 - 0 = 1$

Eigenvalue

Vector : It is an element of a vectorspace.

λ, X such that $X = (x_1, x_2, \ldots x_n)$
$$AX = \lambda X$$

Eigenvalue

Let A be a square matrix. A number λ is called eigenvalue of the matrix A if it satisfies
$$Ax = \lambda X \text{ for some vector } X.$$
Also the vect. x is called eigen vector.

$size of(A) = n \times n$, and $size of (x) = n \times 1$

$$Ax = \lambda X \qquad I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$A X = \lambda X$$
$$= \lambda I X$$

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 1 \end{bmatrix}$$

$$\Rightarrow AX - \lambda I X = 0 \qquad IX = X$$

$$\Rightarrow \underline{(A - \lambda I) X = 0} \qquad X = (0\,0,\,0\,0 \cdots 0)$$

Note: Eigenvector is a nonzero vector

$$A X = b \qquad b \text{ is some non zero vector.}$$

$$A^{-1}, \quad A^{-1} A X = A^{-1} b$$
$$\underline{\underline{X}} = A^{-1} b$$

b is zero vector.

$$A X = 0$$
$$A^{-1} A X = A^{-1} 0 = 0$$
$$X = 0$$

$$(A - \lambda I) X = 0$$

| $A - \lambda I$ is non singular then it has only zero solution.

In order to find the nonzero solution of $(A - \lambda I) X = 0$,
$|A - \lambda I| = 0$

$|A - \lambda I| = 0$ is called the characteristic equation

Note. 1) If A is a square matrix of size n, then the characteristic equation will be n^{th} degree polynomial.

2) The n^{th} degree polynomial has n roots, these roots are called eigenvalues

3) The vector corresponding to each eigenvalue is called the eigenvector.

Example

1)
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A-\lambda I = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 0\cdot 0$$
$$= (1-\lambda)(2-\lambda) = 0$$

$$(1-\lambda)(2-\lambda) = 0$$
$$(\lambda-1)(\lambda-2) = 0 \Rightarrow \lambda = 1, 2$$

2) If $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, then find the eigenvalues of A.

$$A-\lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix}$$

$$|A-\lambda I| = (1-\lambda)(3-\lambda) = 0$$

The characteristic equation is $|A-\lambda I| = 0$

$$(1-\lambda)(3-\lambda) = 0$$

The eigenvalues are $\lambda = 1, 3$

Note: If A is either diagonal, or triangular matrices then the eigen values are the diagonal entries.

3) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ Find the eigenvalues of A.

Soln.

$$A-\lambda I = \begin{bmatrix} 1-\lambda & 2 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{bmatrix}$$

$$|A-\lambda I| = (1-\lambda)(2-\lambda)(4-\lambda)$$

The characteristic equation is $|A-\lambda I| = 0$

$$(1-\lambda)(2-\lambda)(4-\lambda)=0$$

The eigenvalues are 1, 2, 4.

④ $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$, Find the Characteristic equation and hence find the eigenvalues

Soln

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (4-\lambda)(2-\lambda) - 3$$

The characteristic equation $|A - \lambda I| = 0$

$$(4-\lambda)(2-\lambda) - 3 = 0$$

$$(\lambda-4)(\lambda-2) - 3 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 4\lambda + 8 - 3 = 0 \qquad 5$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$\lambda = 1, \lambda = 5$

Note: If A is 3×3 matrix, then the characteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = sum of the diagonal elements = trace of A.

S_2 = sum of the minors of the main diagonal

S_3 = the determinant of A.

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 4 & 2 & 5 \\ 3 & 2 & \boxed{4} \end{bmatrix} \qquad \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} \qquad \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$$

⑤ Find the characteristic equation of

(i) $A = \begin{bmatrix} 6 & 8 \\ 4 & 5 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

Soln.

(i) The characteristic equation of A is

i.e. $|A - \lambda I| = 0$

ie $\lambda^2 - S_1 \lambda + S_2 = 0$

$S_1 = 11$, $S_2 = 30 - 32 = -2$

$\therefore$ The characteristic equation is $\lambda^2 - 11\lambda - 2 = 0$

(ii) $|A - \lambda I| = 0$

ie, $\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$

$S_1 = 4$,

$$S_2 = \begin{vmatrix} -2 & 4 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix}$$

$$= (-6+4) + (9-4) + (-6+4)$$

$$= 1$$

$$S_3 = \begin{vmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{vmatrix} = 3\begin{vmatrix} -2 & 4 \\ -1 & 3 \end{vmatrix} - (-4)\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix}$$

$$= -6$$

$\therefore$ The characteristic equation $\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$

Note Suppose the characteristic equation 3x3 matrix is

$\lambda^3 - S_1\lambda^2 + S_2\lambda = 0$ $(S_3 = 0)$ $S_3 = 0 \Rightarrow |A| = 0$

$\lambda(\lambda^2 - S_1\lambda + S_2) = 0$

$\Rightarrow \lambda = 0$ is one of the eigenvalues.

① Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

soln

The characteristic equation $|A - \lambda I| = 0$

(ie) $\lambda^2 - S_1\lambda + S_2 = 0$

(ie) $\lambda^2 - 6\lambda + 5 = 0$

$\lambda = 1, 5$

$\lambda_1 = 1$, $\lambda_2 = 5$

The eigenvector corresponding to $\lambda_1 = 1$ is

$A X = \lambda_1 X$

$$AX = \lambda_1 X$$

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} X = 1\, X , \qquad X = (x_1, x_2)^T$$

$$\Rightarrow \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left.\begin{aligned} 4x_1 + x_2 &= x_1 \\ 3x_1 + 2x_2 &= x_2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 3x_1 + x_2 &= 0 \\ 3x_1 + x_2 &= 0 \end{aligned}\right\} \checkmark$$

$$x_2 = -3x_1$$

$x_1 = 1 \Rightarrow x_2 = -3$

$$\therefore X_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad AX = \lambda X \checkmark$$

$$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$4 - 3 = 1$$
$$3 - 6 = -3 \quad \checkmark$$

The eigenvector corresponding to $\lambda_2 = 5$

$$AX = 5X , \qquad X = (x_3, x_4)$$

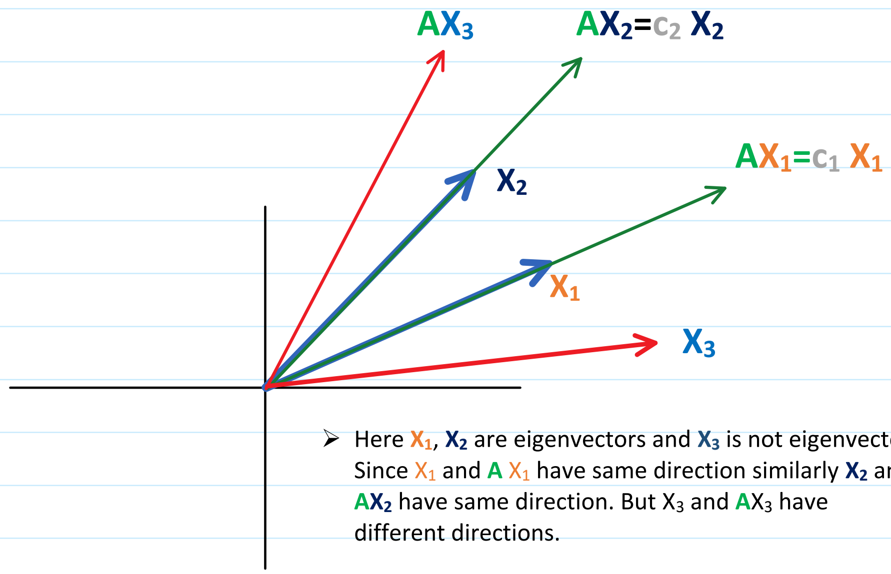
$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = 5 \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

$$\left.\begin{aligned} 4x_3 + x_4 &= 5x_3 \\ 3x_3 + 2x_4 &= 5x_4 \end{aligned}\right\} \Rightarrow \begin{aligned} -x_3 + x_4 &= 0 \\ 3x_3 - 3x_4 &= 0 \end{aligned}$$

$$-x_3 + x_4 = 0 \Rightarrow x_4 = x_3$$

$x_3 = 1$, then $x_4 = 1$ $\therefore$ $X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Geometrical Meaning



- Here X_1, X_2 are eigenvectors and X_3 is not eigenvector. Since X_1 and $A X_1$ have same direction similarly X_2 and AX_2 have same direction. But X_3 and AX_3 have different directions.
- C1 and C2 are the eigenvalues of the matrix A. It is also called characteristic values. Sometimes it is viewed as a stretching factor of eigenvector.

① Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

Soln.

The ch. equation equation $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

S_1 = trace of $A = 3 - 2 + 3 = 4$

S_2 = sum of the minors of the diagonal

$$= \begin{vmatrix} -2 & 4 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix}$$

$$= -6 + 4 + 9 - 4 + -6 + 4 = 1$$

$$S_3 = \begin{vmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{vmatrix} = 3\begin{vmatrix} -2 & 4 \\ -1 & 3 \end{vmatrix} - (-4)\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix}$$

$= -6$

$\therefore$ The ch. polynomial $1\lambda^3 - \underline{\underline{4}}\lambda^2 + 1\lambda + \underline{\underline{6}} = 0$

$-4 + 6 = 2$

$1 + 1 = 2$

$\lambda = -1 \Rightarrow (-1)^3 - 4(-1)^2 + (-1) + 6$

$\Rightarrow -1 - 4 - 1 + 6 = 0$

$\Rightarrow$ one of the eigenvalues is $\lambda = -1$

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 1 & 6 \\ & 0 & -1 & 5 & -6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = (\lambda+1)(\lambda^2 - 5\lambda + 6)$$

$$= (\lambda+1)(\lambda-2)(\lambda-3)$$

$\therefore$ The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 3$

The eigenvector corresponding to $\lambda_1 = -1$ is

$$AX = \lambda_1 X \qquad , \quad X = (x_1, x_2, x_3)^T$$

$$\begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$4x_1 - 4x_2 + 4x_3 = 0 \rightarrow x_1 - x_2 + x_3 = 0$$

$$x_1 - x_2 + 4x_3 = 0 \rightarrow x_1 - x_2 + 4x_3 = 0$$

$$x_1 - x_2 + 4x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & 4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-3} = \frac{x_2}{-3} = \frac{x_3}{0}$$

$x_1 = -3$, $x_2 = -3$, $x_3 = 0$

$x_1 = +1$, $x_2 = +1$, $x_3 = 0$

$$X_{\lambda_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

The eigenvector corresponding to $\lambda_2 = 2$

$$(A - \lambda_2 I)X = 0$$

$$(3-2)x_1 - 4x_2 + 4x_3 = 0 \Rightarrow x_1 - 4x_2 + 4x_3 = 0$$

$$x_1 - (4)x_2 + 4x_3 = 0 \Rightarrow x_1 - 4x_2 + 4x_3 = 0$$

$$x_1 - x_2 + x_3 = 0 \qquad x_1 - x_2 + x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -4 & 4 \\ -1 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -4 \\ 1 & -1 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{3} = \frac{x_3}{3} \qquad X_{\lambda_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow x_1 = 0$, $x_2 = 3$, $x_3 = 3$

$\Rightarrow x_1 = 0$, $x_2 = 1$, $x_3 = 1$

The eigenvector corresponding to $\lambda_3 = 3$

$$0x_1 - 4x_2 + 4x_3 = 0 \quad \Rightarrow$$
$$x_1 - 5x_2 + 4x_3 = 0$$
$$x_1 - x_2 + 0x_3 = 0$$

$x_1 = 1,\ x_2 = 1,\ x_3 = 1$

$$X_{\lambda_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(2) Find the eigenvalues and eigenvectors of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

<u>Soln.</u> $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

The ch. equation is $|A - \lambda I| = 0$

$$\Rightarrow \quad \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 7$$
$$S_2 = 4 + 3 + 4 = 11$$
$$S_3 = 8 - 2 + 1(-1) = 8 - 3 = 5$$

The characteristic polynomial is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

sum of all the co-efficients equal to zero $\Rightarrow$ $\lambda = 1$ is a root

$$\begin{array}{c|cccc} & 1 & -7 & 11 & -5 \\ 1 & 0 & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & \boxed{0} \end{array}$$

$$\therefore \lambda^3 - 7\lambda^2 + 11\lambda - 5 = (\lambda - 1)(\lambda^2 - 6\lambda + 5) = 0$$
$$= (\lambda - 1)(\lambda - 1)(\lambda - 5)$$

$\therefore$ The eigenvalues are $\lambda_1 = 1,\ \lambda_2 = 1,\ \lambda_3 = 5$.

The eigenvector corresponding to $\lambda_1 = 1$.

$(A - \lambda I)X = 0 \longleftarrow \quad AX = \lambda_1 X \quad , \quad X = (x_1, x_2, x_3)$

$$\left.\begin{aligned} (2-\lambda_1)x_1 + 2x_2 + x_3 &= 0 \\ x_1 + (3-\lambda_1)x_2 + x_3 &= 0 \\ x_1 + 2x_2 + (2-\lambda_1)x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0. \end{aligned}$$

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 = -2x_2 - x_3$$ Here x_2, x_3 are arbitrary

$x_3 = 0, \quad x_2 = 1 \Rightarrow \quad x_1 = -2 - 0 = -2$

$$X_{\lambda_1} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$x_2 = 0, \quad x_3 = 1 \quad \Rightarrow \quad x_1 = -1$

$$X_{\lambda_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The eigenvector corresponding to $\lambda_3 = 5$

$(A - 5I)X = 0$

$$-3x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 2x_2 + x_3 = 0$$
$$x_1 + 2x_2 - 3x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4} \quad \Rightarrow \quad x_1 = x_2 = x_3 = 1$$

$$X_{\lambda_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

③ Find the eigenvalues and eigenvectors of $\begin{bmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{bmatrix}$

Soln. The ch. equation is $|A - \lambda I| = 0$

ie $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = -3$

$$S_2 = \begin{vmatrix} -13 & 10 \\ -6 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 5 \\ 7 & 4 \end{vmatrix} + \begin{vmatrix} 6 & -6 \\ 14 & -13 \end{vmatrix}$$
$$= 3$$

$$S_3 = \begin{vmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{vmatrix} = -1$$

$(A+B)^3 = A^3 + 3AB^2 + 3A^2B + B^3$

$\therefore$ The ch. equation $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

$$\Rightarrow (\lambda + 1)^3 = 0$$

$$\lambda = -1, -1, -1.$$

The the eigenvectors corresponding to the eigenvalues $\lambda_1 = -1$ is

$$(A+I)X=0$$

$$7x_1-6x_2+5x_3=0$$

$x_1=0$, $\quad 6x_2=5x_3 \Rightarrow x_2=5, \quad x_3=6$

$$X_1=\begin{pmatrix}0\\5\\6\end{pmatrix}$$

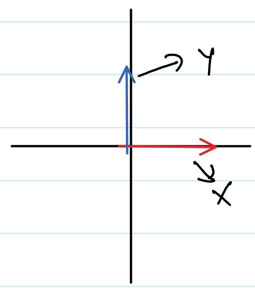
$x_2=0 \Rightarrow \quad 7x_1=-5x_3 \quad \Rightarrow x_1=-5, \quad x_3=7$

$$X_2=\begin{pmatrix}-5\\0\\7\end{pmatrix}$$

$x_3=0 \Rightarrow 7x_1-6x_2=0 \quad \Rightarrow x_1=6, \quad x_2=7$

$$X_3=\begin{pmatrix}6\\7\\0\end{pmatrix}$$

$X=(1,0)$, $\quad Y=(0,1)$



α, β

$\alpha X+\beta Y=(0,0)$

Then $\alpha=0, \quad \beta=0$

$X=(1,0)$, $\quad Y=(0,1)$

$\alpha X+\beta Y=(\alpha,0)+(0,\beta)=(\alpha,\beta)=(0,0)$

$\alpha=0, \quad \beta=0 \qquad$ X and Y are linearly independent

$X=(1,1)$, $\qquad Y=(2,2)$

$$\alpha X+\beta Y=(\alpha,\alpha)+(\beta,\beta)=(0,0)$$

$\Rightarrow \alpha+\beta=0$

$\Rightarrow \alpha=-\beta, \quad \beta\neq 0$

$\therefore$ X and Y are linearly dependent

<u>Ex.</u> α, β, γ, $\qquad \alpha X_1+\beta X_2+\gamma X_3=(0,0,0)$

$\alpha\neq 0, \ \beta\neq 0, \ \gamma\neq 0$

$(1,1)$, $\quad (2,2)$

$(2,2)=\underline{2}(1,1)$

$$(2,2) = \underline{\underline{2}}(1,1)$$

$$\alpha \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix} + \beta \begin{pmatrix} -5 \\ 0 \\ 7 \end{pmatrix} + \gamma \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad 0\alpha + (-5)\beta + 6\gamma = 0 \quad \Rightarrow 6\gamma = 5\beta \quad \Rightarrow \gamma = 5/6(\beta)$$

$$5\alpha + 0\beta + 7\gamma = 0 \quad \Rightarrow 5\alpha = -7\gamma$$

$$= -7(5/6)\beta$$

$$\alpha = \frac{-7}{6}\beta$$

$$6\alpha + 7\beta = 0$$

$$\beta \neq 0, \quad \beta = 1$$

$$\boxed{\alpha = -7/6, \quad \gamma = 5/6, \quad \beta = 1.}$$

① $x_1 = (1,0,0)$, $\quad x_2 = (1,1,0)$, $\quad x_3 = (1,1,1)$

$$\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} \alpha + \beta + \gamma &= 0 \\ \beta + \gamma &= 0 \\ \gamma &= 0 \end{aligned}\right\} \Rightarrow \alpha = 0, \ \beta = 0, \ \gamma = 0$$

Defn.

Let $x_1, x_2, \ldots x_n$ be the vectors. The vectors are said to be linearly independent if, there exists the scalars $\alpha_1, \alpha_2, \ldots \alpha_n$ such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0 \Rightarrow \alpha_i = 0, \text{ for all } i = 1,2,\ldots n$$

Suppose atleast one of $\alpha_i \neq 0$ then the vectors are called linearly dependent

Properties of eigenvalues

Property 1 A square matrix A and its transpose have the same eigenvalues.

Proof

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \end{pmatrix}$$

Same eigenvalues.

Proof

$$|A-\lambda I|=0$$

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn}-\lambda \end{vmatrix}$$

$$A^t = A' = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & \cdots & a_{n2} \\ \vdots & & & & \\ a_{1n} & a_{2n} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

$$|A^t-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22}-\lambda & \cdots & \cdots & a_{n2} \\ \vdots & & & & \\ a_{1n} & a_{2n} & \cdots & \cdots & a_{nn}-\lambda \end{vmatrix}$$

$$\therefore |A-\lambda I| = |A^t-\lambda I|$$

Property 2 The product of the eigenvalues of the matrix A is equal to $|A|$.

Proof The ch. equation is $|A-\lambda I|=0$

$$\lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} - S_3\lambda^{n-3} + \cdots + (-1)^n S_n = 0$$

From the theory equation, the product of all the roots is equal to $= (-1)^n \dfrac{\text{Constant term}}{\text{the coefficient } \lambda^n}$

$S_n =$ Product of all the roots.

$$\lambda_1\lambda_2 \cdots \lambda_n = (-1)^n(-1)^n S_n = |A|$$

$$\Rightarrow S_n = |A|$$

$$\lambda_1\lambda_2\lambda_3 \cdots \lambda_n = |A|$$

Note

$S_1 =$ trace (A)

$=$ sum of the root of the polynomial

$S_2 =$ sum of the product of the roots 2 at a time

$S_3 =$ sum of the product of the roots 3 at a time.

$\vdots$

$S_n =$ sum of the product of the roots n at time.

$= \lambda_1\lambda_2\lambda_3 \cdots \lambda_n$.

(i.e) $S_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \cdots + \lambda_1\lambda_{n-1} + \lambda_1\lambda_n$
$+ \lambda_2\lambda_3 + \lambda_2\lambda_4 + \cdots + \lambda_2\lambda_n + \lambda_3\lambda_4 + \lambda_3\lambda_5 + \cdots + \lambda_3\lambda_n$
$+ \cdots + \lambda_{n-1}\lambda_n$.

$S_3 = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \cdots + \lambda_1\lambda_2\lambda_n + \cdots$
$\cdots + \lambda_{n-2}\lambda_{n-1}\lambda_n$

Property 3 If $\lambda_1, \lambda_2 \ldots \lambda_n$ are non zero eigenvalues of the nonsingular matrix, then $1/\lambda_1, 1/\lambda_2, \ldots 1/\lambda_n$ are the eigenvalues of A^{-1}.

Proof Let λ be ~~the~~ an eigenvalue, Then.

$$AX = \lambda X$$

A^{-1} exists $\because$ A is nonsingular

$$A^{-1}AX = A^{-1}\lambda X$$
$$\Rightarrow IX = \lambda A^{-1}X$$
$$\Rightarrow \frac{1}{\lambda} IX = A^{-1}X.$$
$$\Rightarrow A^{-1}X = \frac{1}{\lambda}X$$

Property 4 If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of A, Then

(i) $c\lambda_1, c\lambda_2, \ldots c\lambda_n$ are the eigenvalues of cA, where c is a constant.

(ii) $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ are the eigen values of A^m for any positive integer m.

Proof

(i) Let λ be a eigen value of A, then

$$AX = \lambda X$$

Let $c \neq 0$ be a constant, then

$$cAX = c\lambda X$$
$$\Rightarrow (cA)X = (c\lambda)X$$

$\Rightarrow$ $c\lambda$ is the eigenvalue of the matrix cA

(ii) Let λ be a eigenvalue of the matrix A. Then.

$$AX = \lambda X$$
$$\Rightarrow A(AX) = A\lambda X$$
$$\Rightarrow A^2X = \lambda(AX)$$
$$= \lambda(\lambda X) = \lambda^2 X$$

$\Rightarrow A^2 x = \lambda^2 x \Rightarrow \lambda^2$ is the eigenvalue of A^2

In general for $A^m x = \lambda^m x \Rightarrow$ ~~λ_1^m, λ~~

λ^m is the eigenvalue of A^m

Property 5 If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of A, then

(i) $\lambda_1 - k, \lambda_2 - k, \ldots \lambda_n - k$ are the eigenvalues of $A - kI$.

(ii) $\alpha_0\lambda_1^2 + \alpha_1\lambda + \alpha_2, \alpha_0\lambda_2^2 + \alpha_1\lambda_2 + \alpha_2, \ldots$

$, \alpha_0\lambda_n^2 + \alpha_1\lambda_n + \alpha_2$ are the eigenvalues

the matrix $\alpha_0 A^2 + \alpha_1 A + \alpha_2 I$

Problems

If 2 and 3 are the eigenvalues $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ then

find the third eigenvalue and hence find the eigenvalues

A^{-1} and A^3

Soln.

~~λ_1~~

$\lambda_1 = 2, \lambda_2 = 3, \lambda = ?$

We know that $\lambda_1 + \lambda_2 + \lambda = S_1 = \text{trace of } (A)$

$= 3 - 3 + 7$

$= 7$

$\lambda = 7 - \lambda_1 - \lambda_2$

$= 7 - 2 - 3 = 2.$

ie the third eigenvalue is $\lambda_3 = 2$.

Note that all the eigenvalues are non zero eigenvalues

$\Rightarrow A^{-1}$ exists.

Then the eigenvalues of A^{-1} are $\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$.

We know that If λ is the eigenvalue of A, then λ^m is the eigenvalue of A^m.

$\therefore$ The eigenvalues of A^3 are $8, 8, 27$

(2) If $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$, then find the eigenvalues of A^{-1}, $A^2 - 2A + I$

Soln. The given matrix is upper triangular matrix. Therefore the eigenvalues are $2, 3, 5$

Note that all the eigenvalues are non zero $\Rightarrow A^{-1}$ exists

The eigenvalues of A^{-1} are $1/2, 1/3, 1/5$

$\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 5$

$1A^2 - 2A + I = \alpha_0 A^2 + \alpha_1 A + \alpha_2 I.$

$\alpha_0 = 1, \alpha_1 = -2, \alpha_2 = 1$

$\therefore$ The eigenvalues of $\alpha_0\lambda_1^2 + \alpha_1\lambda_1 + \alpha_2 = 1(4) - 2(2) + 1 = 1$

$\alpha_0\lambda_2^2 + \alpha_1\lambda_2 + \alpha_2 = 1(9) - 2(3) + 1 = 4$

$\alpha_0\lambda_3^2 + \alpha_1\lambda_3 + \alpha_2 = 25 - 10 + 1 = 16$

$\therefore$ The eigenvalues of $A^2 - 2A + I$ are 1, 4, 16

③ The product of two eigenvalues of $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16.

Find the third eigen value

Soln. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

Then $\lambda_1\lambda_2\lambda_3 = |A|$

$\lambda_1\lambda_2 = 16 \Rightarrow 16\lambda_3 = |A|$

$\lambda_3 = \frac{1}{16}|A|$

$= \frac{1}{16}(32) = 2$

(ie) $\lambda_3 = 2$

Ex: If one the eigenvalue $\lambda = 2$ of $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ Then find the other two eigenvalues

———— x ————

Cayley - Hamilton Theorem

Statement: Every square matrix satisfies its characteristic equation.

Suppose $\lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} + \dots + (-1)^n S_n = 0$ is the ch. equation of A. Then by Cayley-Hamilton Theorem.

$$A^n - S_1 A^{n-1} + S_2 A^{n-2} + \dots + (-1)^n S_n I = 0 \rightarrow \text{Zero matrix.}$$

① Verify that $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ satisfies its characteristic equation and hence find A^4.

Soln. The ch. equation $|A - \lambda I| = 0$

$$\Rightarrow \lambda^2 - S_1\lambda + S_2 = 0$$

$$S_1 = 0, \quad S_2 = |A| = -5$$

$$\Rightarrow \lambda^2 - 5 = 0$$

$A^2 - 5I = ?$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$-5I = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$$

$$A^2 - 5I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \checkmark$$ ie it satisfies the Cayley-Hamilton Theorem.

$$A^2 - 5I = O$$

$$\Rightarrow \boxed{A^2 = 5I} \longrightarrow (*)$$

$$\Rightarrow A^2(A^2) = A^2 \, 5I$$

$$\Rightarrow A^4 = 5A^2 I = 5A^2 = 5(5I) = 25I$$

$$\Rightarrow A^4 = 25I = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$$

(2) Verify the Cayley-Hamilton Theorem for $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and find its inverse. Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A.

Soln. $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, The ch. equation is $|A - \lambda I| = 0$

$$\text{ie } \lambda^2 - S_1\lambda + S_2 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + (-5) = 0$$

$$(\text{ie}) \; \lambda^2 - 4\lambda - 5 = 0.$$

To verify $A^2 - 4A - 5I$.

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$$-4A = \begin{bmatrix} -4 & -16 \\ -8 & -12 \end{bmatrix}$$

$$-5I = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^2 - 4A - 5I = O$$

$$\Rightarrow 5I = A^2 - 4A$$

$$I = \frac{1}{5}(A^2 - 4A)$$

$S_2 \neq 0 \Rightarrow A^{-1}$ exists, $\quad A^{-1} I = \frac{1}{5}\left(A^{-1}A^2 - 4A^{-1}A\right)$

$$A^{-1} = \tfrac{1}{5}(A - 4I)$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$.

$\rightarrow \lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 = \cancelto{0}{(\lambda^2 - 4\lambda - 5)}(q(\lambda)) + (r(\lambda))^{\neq 0}$

$$\begin{array}{r|l}
 & \lambda^3 - 2\lambda + 3 \\
\hline
\lambda^2 - 4\lambda - 5 & \lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 \\
 & \cancel{\lambda^5} \; \cancel{-4\lambda^4} - 5\lambda^3 \\
 & \quad (-) \quad (+) \quad (+) \\
\hline
 & -2\lambda^3 + 11\lambda^2 - \lambda - 10 \\
 & \cancel{-2\lambda^3} + 8\lambda^2 + 10\lambda \\
 & \quad (+) \quad (-) \quad (-) \\
\hline
 & 3\lambda^2 - 11\lambda - 10 \\
 & \cancel{3\lambda^2} - 12\lambda - 15 \\
 & \quad (-) \quad (+) \quad (+) \\
\hline
 & \lambda + 5
\end{array}$$

$\because$ it is ch. polynomial

$\therefore \quad \lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 = \cancelto{0}{(\lambda^2 - 4\lambda - 5)}(\lambda^3 - 2\lambda + 3) + (\lambda + 5)$

$$= 0 + (\lambda + 5)$$

$\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 = \lambda + 5$

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5I$

ie the required linear polynomial in A is $\quad A + 5I$

(3) Use the Cayley-Hamilton Theorem to find the matrix equation $A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$ if $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

Soln.

The ch. equation is $|A - \lambda I| = 0$

$$\Rightarrow \quad \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 5, \quad S_2 = 7, \quad S_3 = 3$$

$$\Rightarrow \quad \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

$\lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + 8\lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1 = (\lambda^3 - 5\lambda^2 + 7\lambda - 3)\, q(\lambda)$
$+ r(\lambda)$

$$\begin{array}{r|l}
 & \lambda^5 + 8\lambda + 35 \\
\hline
\lambda^3 - 5\lambda^2 + 7\lambda - 3 & \lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + 8\lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1 \\
 & \cancel{\lambda^8} \; \cancel{-5\lambda^7} + \cancel{7\lambda^6} \; \cancel{-3\lambda^5} \\
 & \quad (-) \quad (+) \quad (-) \quad (+) \\
\hline
 & 8\lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1
\end{array}$$

$$\cancel{\lambda^8} - 5\lambda^7 + 7\lambda^6 - 3\lambda^5$$
$$- \quad + \quad - \quad +$$

$$8\lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1$$
$$\cancel{8\lambda^4} - 40\lambda^3 + 56\lambda^2 - 24\lambda$$
$$- \quad + \quad - \quad +$$

$$35\lambda^3 - 48\lambda^2 + 22\lambda + 1$$
$$\cancel{35\lambda^3} - 175\lambda^2 + 245\lambda - 105$$
$$- \quad + \quad - \quad +$$

$$127\lambda^2 - 223\lambda + 106$$

$$q(\lambda) = \lambda^5 + 8\lambda + 35$$
$$r(\lambda) = 127\lambda^2 - 223\lambda + 106$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I = 0 + 127A^2 - 223A + 106I$$
$$= 127A^2 - 223A + 106I$$

$$A^2 = AA = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix}$$

$$\therefore 127A^2 - 223A + 106I = 127\begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} - 223\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + 106\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 295 & 285 & 285 \\ 0 & 10 & 0 \\ 285 & 285 & 295 \end{pmatrix}.$$

Definition Similar matrices.

Let A and B be two ~~nxn~~ square matrices. The matrices A and B are said to be similar if there exists a non singular matrix P such that

$$A = P^{-1}BP$$
$$B = PAP^{-1}$$
$$= (P^{-1})^{-1}A(P^{-1}) \qquad , \quad Q = P^{-1}$$
$$= Q^{-1}AQ$$

Result 1: Similar matrices have the same eigenvalues.

Proof:

Let A and B be two similar matrices.

Then $B = P^{-1}AP$

The ch. polynomial of B is $|B - \lambda I| = 0$

$$|B - \lambda I| = |P^{-1}AP - \lambda I|$$
$$= |P^{-1}AP - \lambda P^{-1}IP|$$

$$= |P^{-1}(A-\lambda I)P|$$

$\because$ $|MN| = |M||N|$ $\Rightarrow$

$$= |P^{-1}||A-\lambda I||P|.$$

$$= |A-\lambda I|$$

$$0 = |B-\lambda I| = |A-\lambda I| = 0$$

i.e the eigenvalues of A and B are one and the same

<u>Diagonal matrix</u> ~~A~~ A square matrix A is said to be diagonal if $a_{ij} = 0$, $i \neq j$

<u>Diagonalisation of a square matrix</u>

Let A be a square matrix. The matrix A is said to be diagonalizable if there exist a non singular matrix P such that

$P^{-1}AP = D$, where D is a diagonal matrix.

<u>Note</u> If A is diagonalizable, then A and D are similar matrices.

$$A = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \qquad A^2 = AA = \begin{pmatrix} d_1^2 & 0 \\ 0 & d_2^2 \end{pmatrix}, \quad A^3 = A^2A = \begin{pmatrix} d_1^2 & 0 \\ 0 & d_2^2 \end{pmatrix}\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

$$= \begin{pmatrix} d_1^3 & 0 \\ 0 & d_2^3 \end{pmatrix}$$

$$A^n = \begin{pmatrix} d_1^n & 0 \\ 0 & d_2^n \end{pmatrix}$$

<u>Computation of powers of a square matrix</u>

Let A be a diagonalizable matrix, i.e ~~st~~ there exists a nonsingular matrix M such that

$$D = M^{-1}AM.$$

$$D^2 = DD = (M^{-1}AM)(M^{-1}AM)$$

$$= M^{-1}AMM^{-1}AM = M^{-1}AIAM$$

$$= M^{-1}A^2M$$

$$D^3 = M^{-1}A^3M$$

$$D^n = M^{-1}A^nM$$

$$\Rightarrow A^n = MD^nM^{-1}$$

<u>Note</u> If A has n-distinct eigenvalues then there must be n linearly independent eigenvectors. $X_1, X_2, X_3, \ldots X_n$.

Note If A has n-distinct eigenvalues then there must be n linearly independent eigenvectors. $X_1, X_2, X_3, \dots X_n$.

Then $M = [X_1 \quad X_2 \quad X_3 \quad \dots \quad X_n]$

Orthogonal matrix

A real square matrix A is said to be an orthogonal matrix if $AA^T = A^TA = I$

$$AA^T = I$$
$$A^T = A^{-1}$$

Note: If A is orthogonal then $A^{-1} = A^T$

Symmetric matrix

A real square matrix A is said to be symmetric if $A^T = A$

Example $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Some Properties of orthogonal and symmetric matrices

① If A is orthogonal then A^T is also orthogonal

Proof

Given that A is orthogonal $\Rightarrow AA^T = I$

$$\Rightarrow (AA^T)^T = (A^T)^T A^T = AA^T = I.$$
$$(A^T)(A^T)^T = A^TA = I.$$

$\Rightarrow A^T$ is also orthogonal.

② If A is orthogonal then $|A| = -1$ or $+1$

Proof Given that A is orthogonal $\Rightarrow AA^T = I$

$$|A||A^T| = 1$$
$$\Rightarrow |A||A^T| = 1 \longrightarrow (*)$$

We know that $|A| = |A^T|$

$$(*) \Rightarrow |A| \cdot |A| = 1$$
$$|A|^2 = 1$$
$$\Rightarrow |A| = \pm 1$$

③ If λ is an eigenvalue of the orthogonal matrix, A then

$1/\lambda$ is also eigenvalue of A.

④ If A and B are orthogonal matrices then AB is also.

<u>Proof</u> $A^T = A^{-1}$, $B^T = B^{-1}$

$$(AB)^T = B^T A^T = B^{-1} A^{-1} = (AB)^{-1}$$

⑤ If A is orthogonal matrix then the magnitude of the eigenvalues of A is 1

⑥ Eigenvalues of a real symmetric matrix are real

<u>Proof</u> Let A be real symmetric matrix. and λ be an eigenvalue of A.

Then there exists a vector x such that

$$Ax = \lambda x$$

$$\overline{Ax} = \overline{\lambda x} \qquad \Big| \quad \bar{A} = A$$

$$\Rightarrow \bar{A}\bar{x} = \bar{\lambda}\bar{x}$$

$$\Rightarrow A\bar{x} = \bar{\lambda}\bar{x}$$

$$\Rightarrow (A\bar{x})^T = (\bar{\lambda}\bar{x})^T$$

$$\Rightarrow \bar{x}^T A^T = \bar{\lambda}\bar{x}^T$$

$$\Rightarrow \bar{x}^T A = \bar{\lambda}\bar{x}^T$$

$$(\bar{x}^T A)x = (\bar{\lambda}\bar{x})^T x \qquad \bar{x} \neq 0$$

$$\bar{x}^T(Ax) = \bar{\lambda}(\bar{x}^T x) \qquad \bar{x}^T x \neq 0$$

$$\bar{x}^T \lambda x = \bar{\lambda}(\bar{x}^T x)$$

$$\Rightarrow \lambda(\bar{x}^T x) = \bar{\lambda}(\bar{x}^T x)$$

$$\Rightarrow \lambda = \bar{\lambda}$$

$\Rightarrow \lambda$ is a real eigenvalue

<u>Property</u> Eigenvectors corresponds to distinct eigenvalues of a symmetric matrix are orthogonal vectors

$$\lambda_1 \neq \lambda_2,$$

$$x_1 \quad x_2 \quad , \quad \langle x_1, x_2 \rangle = 0$$

$$x_1 \perp r \; x_2$$

<u>Unit vector / Unit normalized vector.</u>

Let X be a non zero vector, then its normalized unit vector is $\hat{x}_1$ $\qquad X_1 = (a, b, c)$

Let X be a non zero vector, then its normalized unit vector is

$$\hat{X}_1 = \frac{X_1}{\|X_1\|}, \qquad X_1 = (a, b, c), \quad \|X_1\| = \sqrt{a^2+b^2+c^2}$$

$$\hat{X}_1 = \frac{X_1}{\sqrt{a^2+b^2+c^2}}$$

$X = (-1, 2, 6)$,

$$\hat{X} = \frac{(-1, 2, 6)}{\sqrt{41}} = \left(\frac{-1}{\sqrt{41}}, \frac{2}{\sqrt{41}}, \frac{6}{\sqrt{41}}\right)$$

$$\|\hat{X}\| = \sqrt{\left(\frac{-1}{\sqrt{41}}\right)^2 + \left(\frac{2}{\sqrt{41}}\right)^2 + \left(\frac{6}{\sqrt{41}}\right)^2} = 1$$

Diagonalization by orthogonal Transformation / orthogonal reduction

Step 1 Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Step 2 Find the linearly independent eigenvectors and they are pairwise orthogonal ie $X_j \perp X_i$, $i \neq j$

Step 3 Calculate $\hat{X}_1 = \frac{X_1}{\|X_1\|}$, $\hat{X}_2 = \frac{X_2}{\|X_2\|}$, ..., $\hat{X}_n = \frac{X_n}{\|X_n\|}$

Step 4 Form the modal matrix $N = [\hat{X}_1 \ \hat{X}_2 \ \ldots \ \hat{X}_n]$

Step 5 $N^T = N^{-1}$

Step 6 $N^{-1}AN = D \Rightarrow N^TAN = D$

Problem Diagonalize the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ by means of orthogonal transformation.

Soln. The Ch. equation ~~equation~~ $|A - \lambda I| = 0$

ie, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = \text{trace of } (A) = 9$

$S_2 = 8 + 8 + 8 = 24$

$S_3 = |A| = 16$

$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$

$1 - 9 + 24 - 16 = 0$

$\Rightarrow \lambda = 1$ is root.

$$\begin{array}{c|cccc} 1 & 1 & -9 & 24 & -16 \\ & 0 & 1 & -8 & 16 \end{array}$$

$$\begin{array}{c|cccc} & 1 & -9 & 24 & -16 \\ 1 & 0 & 1 & -8 & 16 \\ \hline & 1 & -8 & 16 & \boxed{0} \end{array}$$

$$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = (\lambda-1)(\lambda^2 - 8\lambda + 16)$$
$$= (\lambda-1)(\lambda-4)^2$$
$$\Rightarrow \lambda = 1, \lambda = 4, \lambda = 4.$$

To find the first eigenvector $\lambda = 1$

$$AX = \lambda X \quad , \quad X = (x_1, x_2, x_3)$$

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{array}{l} 2x_1 + x_2 + x_3 = 0 \\ x_1 + 2x_2 - x_3 = 0 \\ x_1 - x_2 + 2x_3 = 0 \end{array}$$

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}$$

$x_1 = -3, \; x_2 = 3, \; x_3 = 3$

$\Rightarrow x_1 = -1, \; x_2 = 1, \; x_3 = 1$

$$X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\hat{X}_1 = \frac{X_1}{\|X_1\|} \quad , \quad \|X_1\| = \sqrt{3}$$

$$\hat{X}_1 = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix},$$

To find the second eigenvector $\lambda = 4$.

$$AX = 4X \quad , \quad X = (x_1, x_2, x_3)$$

$$\Rightarrow \begin{array}{l} -x_1 + x_2 + x_3 = 0 \\ x_1 - x_2 - x_3 = 0 \\ x_1 - x_2 - x_3 = 0 \end{array}$$

$x_1 = x_2 + x_3$, x_2 and x_3 are free variable

$x_3 = 0, \; x_2 = 1,$ then $x_1 = 1$

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{X}_2 = \frac{X_2}{\|X_2\|} \quad , \quad \|X_2\| = \sqrt{2}.$$

$$\hat{X}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

To find the ~~&~~ third orthogonal vector $X_3 = (a, b, c)$

$\rightarrow$ inner product / dot product.

$X_1 \perp X_3 \Rightarrow X_1^T X_3 = 0$

$X_2 \perp_r x_3 \Rightarrow X_2^T x_3 = 0$

$$x_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$0 = X_1^T \cdot x_3 = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -a+b+c$$

$$-a+b+c=0 \longrightarrow (*).$$

$$0 = X_2^T \cdot X_3 = (1,1,0) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a+b+0$$

$$\therefore a+b=0 \longrightarrow (**)$$

$$b=-a$$

$$c=2a, \quad a=1 \qquad X_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$b=-1, \quad c=2$$

$$\hat{X}_3 = \frac{X_3}{\|X_3\|} = \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}, \qquad \|X_3\| = \sqrt{1^2+(-1)^2+2^2} = \sqrt{6}$$

$$M = \begin{bmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{3}} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$

To find the diagonal matrix

$$D = M^{-1}AM \qquad \text{but } M^{-1} = M^T$$

$$= \underbrace{\begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{pmatrix}}_{M^T} \underbrace{\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix}}_{M}$$

$$AM = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{4}{\sqrt{2}} & \frac{4}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{2}} & -\frac{4}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & 8/\sqrt{6} \end{pmatrix}$$

$$M^TAM = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 1/\sqrt{3} & 1/\sqrt{3} \\ \frac{1}{\sqrt{2}} & 1/\sqrt{2} & 0 \\ \end{pmatrix} \begin{pmatrix} -1/\sqrt{3} & 4/\sqrt{2} & 4/\sqrt{6} \\ 1/\sqrt{3} & 4/\sqrt{2} & -4/\sqrt{6} \\ \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ \end{pmatrix}$$

$$M^T A M = \begin{pmatrix} -\frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & \frac{2}{\sqrt6} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt3} & \frac{4}{\sqrt2} & \frac{4}{\sqrt6} \\ \frac{1}{\sqrt3} & \frac{4}{\sqrt2} & -\frac{4}{\sqrt6} \\ \frac{1}{\sqrt3} & 0 & \frac{8}{\sqrt6} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad 1, 4, 4.$$

$$M^T A M = M^{-1} A M = D$$

$$A = M D M^{-1} = M D M^T$$

$$A^{1000} = M D^{1000} M^T$$

$$= M \begin{pmatrix} 1^{1000} & 0 & 0 \\ 0 & 4^{1000} & 0 \\ 0 & 0 & 4^{1000} \end{pmatrix} M^T$$

(2) Diagonalise the matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ and hence find A^2.

Soln. The ch. equation of A is $|A - \lambda I| = 0$

ie $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

where $s_1 = \text{trace of } A = 7$

$s_2 = 6 + 3 + 6 = 15$

$s_3 = |A| = 9$

$\therefore$ The ch. equation is $\lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$

sum of the coefficients is $1 - 7 + 15 - 9 = 0 \Rightarrow \lambda = 1$ is a root

$$\begin{array}{c|cccc} 1 & 1 & -7 & 15 & -9 \\ & 0 & 1 & -6 & 9 \\ \hline & 1 & -6 & 9 & \boxed{0} \end{array}$$

Ch. polynomial $= (\lambda - 1)(\lambda^2 - 6\lambda + 9)$

$= (\lambda - 1)(\lambda - 3)^2 = 0$

$\Rightarrow \lambda = 1, \lambda = 3, \lambda = 3.$

To find the eigenvector, $AX = \lambda X$, $\lambda = 1$.

$AX = 1X$, $X = (x_1, x_2, x_3)^T$

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\Rightarrow 2x_1 + x_3 = x_1 \quad \Rightarrow x_1 + x_3 = 0$

$3x_2 = x_2 \quad \Rightarrow 2x_2 = 0 \Rightarrow x_2 = 0$

$x_1 + 2x_3 = x_3 \quad \Rightarrow x_1 + x_3 = 0$

$x_1 + x_3 = 0 \quad \Rightarrow x_3 = -x_1$

$$x_1 + 2x_3 = x_3 \Rightarrow x_1 + x_3 = 0$$

$$x_1 + x_3 = 0 \Rightarrow x_3 = -x_1$$

$$x_1 = 1, \quad x_3 = -1, \quad x_2 = 0 \quad \therefore \quad X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\|X_1\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$\hat{X}_1 = \frac{X_1}{\|X_1\|} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

To find the second eigenvector $AX = \lambda X$, $\lambda = 3$

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 3 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow 2x_1 + x_3 = 3x_1 \Rightarrow -x_1 + x_3 = 0$$

$\Rightarrow 3x_2 = 3x_2 \Rightarrow x_2$ is free variable

$$\Rightarrow x_1 + 2x_3 = 3x_3 \Rightarrow x_1 - x_3 = 0$$

$$x_1 = x_3, \quad x_1 = 1, \; x_3 = 1, \quad x_2 = 0$$

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \|X_2\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\hat{X}_2 = \frac{X_2}{\|X_2\|} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

To find the third linearly independent and pairwise orthogonal eigenvector, $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$X_1^T \cdot X_3 = 0 \Rightarrow \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a - c = 0$

and $X_2^T \cdot X_3 = 0$

$$X_2^T \cdot X_3 = 0 \quad \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow a + c = 0$$

$$\left.\begin{matrix} a - c = 0 \\ a + c = 0 \end{matrix}\right\} \; a = 0 \Rightarrow c = 0, \quad b \text{ is free variable} \Rightarrow b = 1.$$

Then $X_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\|X_3\| = \sqrt{0^2 + 1^2 + 0^2} = \sqrt{1} = 1$

$$\therefore \hat{X}_3 = \frac{X_3}{\|X_3\|} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Let N be the modal matrix $N = [\hat{X}_1 \;\; \hat{X}_2 \;\; \hat{X}_3]$

Let N be the modal matrix $N = [\hat{x}_1 \quad \hat{x}_2 \quad \hat{x}_3]$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$\boxed{\text{Ex Prove that } NN^T = I.}$

$$N^{-1} = N^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$$

$$D = N^{-1} A N = N^T A N$$

$$AN = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & 1/\sqrt{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \\ 0 & 0 & 3 \\ -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \end{pmatrix}$$

$$N^T A N = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -1/\sqrt{2} \\ \frac{1}{\sqrt{2}} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \\ 0 & 0 & 3 \\ -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

~~$d_1 = 1, d_2 = 3, d_3$~~ $= 3$

$\text{dia}(A) = (1, 3, 3)$

$A^3 = ?$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = N^{-1} A N.$$

$$\Rightarrow A = N D N^{-1}$$

$$A^3 = N D^3 N^{-1}$$

$$= N \begin{pmatrix} 1^3 & 0 & 0 \\ 0 & 3^3 & 0 \\ 0 & 0 & 3^3 \end{pmatrix} N^{-1}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{27}{\sqrt{2}} & 0 \\ 0 & 0 & 27 \\ -\frac{1}{\sqrt{2}} & \frac{27}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -1/\sqrt{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 14 & 0 & 13 \\ 0 & 27 & 0 \\ 13 & 0 & 14 \end{pmatrix}$$

$A = NDN^{-1}$

$A^2 = A \times A = (NDN^{-1})(NDN^{-1})$

$= ND(N^{-1}N)DN^{-1}$

$= NDIDN^{-1}$

$= ND^2N^{-1}$

$A^3 = A^2 A = (ND^2N^{-1})(NDN^{-1})$

$= N D^2 D N^{-1}$

$= N D^3 N^{-1}$

$A^n = N D^n N^{-1}$

Real Quadratic form | Reduction to Canonical form.

Defn. Quadratic form: A homogeneous polynomial of second degree in n-number of variables is called quadratic form.

Ex

1. $x^2 + 2xy + y^2 \rightarrow$ quadratic form
2. $x^2 + 2x + y^2 + 4y \rightarrow$ not a quadratic form.
3. $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx \rightarrow$ quadratic form.
4. $x_1^2 + x_1x_2 + x_2x_3 + x_3x_4$
5. $a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{44}x_4^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{14}x_1x_4$

Defn. The general form of the Quadratic form: Suppose there are n variables $x_1, x_2, x_3, \ldots x_n$ in the quadratic form, then the quadratic form can be written as $\boxed{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\, x_i x_j}$ where a_{ij} are real numbers such that $a_{ij} = a_{ji}$ for all $i = 1, 2, 3, \ldots n$, $j = 1, 2, 3, \ldots n$.

Example

$$\sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}\, x_i x_j = \sum_{i=1}^{n}\left(a_{i1} x_i x_1 + a_{i2} x_i x_2 \right)$$

$$= a_{11}x_1x_1 + a_{12}x_1x_2 + a_{21}x_2x_1 + a_{22}x_2x_2$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_1x_2 + a_{22}x_2^2$$

$$= a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 \quad \because a_{12} = a_{21}$$

$$\boxed{\begin{aligned} X &= (x_1 \;\; x_2)^T \\ X^T &= (x_1, x_2) \end{aligned}}$$

$$= (x_1 \;\; x_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 & a_{12}x_1 + a_{22}x_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= x_1(a_{11}x_1 + a_{12}x_2) + x_2(a_{12}x_1 + a_{22}x_2)$$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + a_{12}x_1x_2 + a_{22}x_2^2$$

$$= a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

$$Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j = X^T A X$$

Defn. Canonical form of Q.

A quadratic form Q which contains only the squared terms is called canonical form

1. $x^2 + xy + y^2 \rightarrow$ it is not a canonical form.
2. $x^2 - y^2 \rightarrow$ it is a canonical form.
3. $x^2 + y^2 - z^2 \rightarrow$ it is a canonical form.

(4) $x^2 + xy + y^2 - z^2 \rightarrow$ it is not Canonical form

Reduction of Q to Canonical form by orthogonal transformation

Step 1. Rewrite the given Q into the matrix equation

$$Q = X^T A X.$$

Step 2 Find the eigenvalues of A and hence find the normalized eigenvectors of A

Step 3 Form the modal matrix $N = [\hat{x}_1\ \hat{x}_2\ \dots\ \hat{x}_n]$

Step 4 Use the orthogonal transformation $\boxed{X = NY}$

Then
$$Q = X^T A X = (NY)^T A (NY)$$
$$= Y^T \boxed{N^T A N}\ Y$$
$$= Y^T (N^{-1} A N) Y = Y^T D Y$$

$Y^T = (y_1, y_2, y_3 \dots y_n)$

$$= (y_1, y_2, \dots y_n) \begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & & d_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$= d_1 y_1^2 + d_2 y_2^2 + \dots + d_n y_n^2$$

(1) Write down the matrix of the quadratic form

$$Q = 2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$$

Soln $Q = X^T A X$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$A = \begin{array}{c|c} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} x_1 \rightarrow \\ x_2 \rightarrow \\ x_3 \rightarrow \end{array} & \begin{pmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{pmatrix} \end{array}$$

(2) If the matrix $A = \begin{pmatrix} 2 & 4 & 5 \\ 4 & 3 & 1 \\ 5 & 1 & 1 \end{pmatrix}$ Then write the quadratic form

Soln. $Q = 2x^2 + 3y^2 + z^2 + 8xy + 10xz + 2zy$

Nature of the quadratic form.

Nature of the quadratic form.

Defn Rank of Q. If A is the matrix of the quadratic form Q in n-variables say, $x_1, x_2, \ldots x_n$, then the rank of the the Q is equal to the rank of ~~the~~ the matrix A.

Defn Singular form
If the rank of $A < n$, where n is the size of the matrix A (or) $|A| = 0$, then Q is called singular form.

Defn. Index:

Let $Q = X^T A X$ be a quadratic form with n-variables, say $x_1, x_2, x_3, \ldots x_n$.
The index of the quadratic form is the number of positive eigenvalues of A.

Defn. Signature
The signature of of the quadratic form is the difference between the number of positive and negative eigenvalues of A.

Defn. The rank of the quadratic form is the number of positive and negative eigenvalues of A.

Notation
The index is denoted as p
The signature is denoted by s
The rank is denoted by r

Defn Definite and Indefinite quadratic forms
Let $Q = X^T A X$ be a quadratic form. Then
(i) Q is said to be positive definite if all the eigenvalues of A are positive

(ii) Q is said to be negative definite if all the eigenvalues are negative

(iii) Q is said to be positive semidefinite if all the eigenvalues are nonnegative and atleast one of the eigen-value is zero

(iv) Q is said to be negative semidefinite if all the eigenvalues are nonpositive and atleast one of the eigenvalue is zero.

(v) Q is said to be indefinite if A has positive and negative eigenvalues (or) Q is not satisfies the above four definitions.

__Note:__ (i) If Q is positive definite then $r=n$ and $p=n$.
(ii) If Q is negative definite then $r=n$ and $p=0$
(iii) If Q is positive semidefinite then $r<n$ and $p=r$
(iv) If Q is negative semidefinite then $r<n$ and $p=0$

__Defn.__ Let $Q = X^T A X$ be a quadratic form in n-variables say $x_1, x_2, \ldots, x_n$ and A be the real symmetric matrix $X^T = (x_1, x_2\ x_3\ \ldots x_n)$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

→ is not modulus

Let $D_1 = |a_{11}| = a_{11}$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad D_4 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \ldots$$

$D_n = |A|$

These determinants are called the principal minors of A

(i) The Q is positive definite if all $D_i > 0$, $i = 1, 2, \ldots, n$.

(ii) ~~I~~. The Q is said to be negative definite if $(-1)^i D_i > 0$, $i = 1, 2, 3, \ldots n$.

~~$D_1 > 0$~~ $D_1 < 0$, $D_2 > 0$, $D_3 < 0$, $D_4 > 0$, $D_5 < 0$, $\ldots$ $(-1)^n D_n > 0$

(iii) The Q is said to be positive semi definite if all $D_i \geq 0$ and atleast one $D_i = 0$

(iv) The Q is said to be negative semidefinite if $(-1)^i D_i \geq 0$ and atleast one of $D_i = 0$

(v) The Q is indefinite for all other cases

——— x ———

① Discuss the nature of the quadratic form.

$$6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$$

Soln.

$$Q = 6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$$

$$= X^T A X, \quad X^T = (x \; y \; z)$$

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$D_1 = 6, \quad D_2 = \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} = 14, \quad D_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 32$$

$D_1 > 0$, $D_2 > 0$, $D_3 > 0 \Rightarrow$ so The Q is positive definite

② Find the nature of the $Q = 10x^2 + 2y^2 + 5z^2 + 6yz - 10xz - 4xy$

Soln.

$$Q = X^T A X, \quad X^T = (x \; y \; z), \quad A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$$

$D_1 = 10$, $D_2 = 16$, $D_3 = |A| = 0$

$D_1 > 0$, $D_2 > 0$, $D_3 = 0 \Rightarrow$ Q is positive semidefinite

Ex Find the nature of the Q

(i) $6x^2 + 3y^2 + 4z^2 + 4yz + 18xz + 4xy$

(ii) $xy + yz + zx$

——— x ———

Reduce the $Q = 6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$ into a canonical form by an orthogonal ~~trantransformation~~ transformation and then find the rank, index, and ~~nature~~ signature and nature

Soln.

$$Q = 6x^2+3y^2+3z^2-4xy-2yz+4xz$$
$$= X^TAX, \quad X^T=(x,y,z)$$
$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The ch. equation $|A-\lambda I|=0$

ie $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 12, \quad S_2 = 8+14+14 = 36$

$S_3 = |A| = 32$

Ch. equation $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$$\begin{array}{r|rrrr} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & 0 \end{array}$$

$$\lambda^3-12\lambda^2+36\lambda-32 = (\lambda-2)(\lambda^2-10\lambda+16)$$
$$= (\lambda-2)(\lambda-2)(\lambda-8)$$

$\Rightarrow \lambda = 2, 2, 8$

To find the eigenvectors,

$$(A-\lambda I)X = 0$$

If $\lambda = 8$, then $(A-8I)X=0$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \quad (6-8)x_1 - 2x_2 + 2x_3 = 0$$
$$-2x_1 + (3-8)x_2 - x_3 = 0$$
$$2x_1 - x_2 + (3-8)x_3 = 0$$

$$\Rightarrow x_1 + x_2 - x_3 = 0$$
$$2x_1 + 5x_2 + x_3 = 0$$
$$2x_1 - x_2 - 5x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} 1 & -1 \\ 5 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix}} \Rightarrow \frac{x_1}{6} = \frac{x_2}{-3} = \frac{x_3}{3}$$

$$\Rightarrow x_1 = 2, \; x_2 = -1, \; x_3 = 1$$

$X_1 = \begin{pmatrix} 2 \\ -1 \\ \ldots \end{pmatrix}$ $\quad \hat{X}_1 = \frac{X_1}{\ldots}$, $\|X_1\| = \sqrt{2^2+(-1)^2+1^2}$

$\Rightarrow x_1 = 2, \; x_2 = -1, \; x_3 = 1$

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \hat{X}_1 = \frac{X_1}{\|X_1\|}, \quad \|X_1\| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$\hat{X}_1 = \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

To find the second eigenvector for $\lambda = 2$

$$4x_1 - 2x_2 + 2x_3 = 0 \Rightarrow 2x_1 - x_2 + x_3 = 0$$
$$-2x_1 + x_2 - x_3 = 0 \Rightarrow 2x_1 - x_2 + x_3 = 0$$
$$2x_1 - x_2 + x_3 = 0$$

$$2x_1 = x_2 - x_3, \quad x_3 = 0$$

$$\Rightarrow x_2 = 2x_1, \quad x_1 = 1$$
$$\Rightarrow x_2 = 2$$

$$\therefore X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \hat{X}_2 = \frac{X_2}{\|X_2\|}, \quad \|X_2\| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$$

$$\therefore \hat{X}_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix}$$

To find the third eigenvector we hence use the orthogonal definition,

$$X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad X_2^T X_3 = 0, \quad X_1^T X_3 = 0$$

$$X_1^T X_3 = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2a - b + c = 0 \longrightarrow (1)$$

$$X_2^T X_3 = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a + 2b + 0 = 0 \longrightarrow (2)$$

From eqn (2) $a = -2b$

$$(1) \Rightarrow 2(-2b) - b + c = 0$$
$$c = 5b$$

$b = 1$, then $a = -2$, $c = 5$

$$X_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}, \quad \hat{X}_3 = \frac{X_3}{\|X_3\|}, \quad \|X_3\| = \sqrt{(-2)^2 + 1^2 + 5^2}$$

$$= \sqrt{30}$$

$$\hat{X}_3 = \begin{pmatrix} -\frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \\ 5/\sqrt{30} \end{pmatrix}$$

Define the modal matrix $N = \begin{bmatrix} \hat{X}_1 & \hat{X}_2 & \hat{X}_3 \end{bmatrix}$

$$= \begin{pmatrix} \frac{2}{\sqrt{6}} & 1/\sqrt{5} & -\frac{2}{\sqrt{30}} \\ -\frac{1}{\sqrt{6}} & 2/\sqrt{5} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & 5/\sqrt{30} \end{pmatrix}$$

$D = N^T A N.$

$$= \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & 2/\sqrt{5} & 0 \\ \frac{-2}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & 5/\sqrt{30} \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{Verify !!}$$

$$Q = X^T A X, \qquad X = NY, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= Y^T D Y$$

$$=$$

$$= 8y_1^2 + 2y_2^2 + 2y_3^2$$

It is the required canonical form.

$\lambda = 2, 2, 8$ all the eigenvalues are positives

Index $p = 3$

Signature $s = 3 - 0 = 3$

rank $r = 3$

All the eigenvalues are positive $\Rightarrow$ the given quadratic form is positive definite

(2) Reduce the $Q = 8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz$ to the canonical form by an orthogonal reduction. Then find one set of values of x, y, z which will make the quadratic form zero.