Properties of eigen values and eigen vectors

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(1) A square matrix A and its transpose A^T have the same eigen values.

$\boxed{\text{Eigen value of } A = \text{Eigen value of } A^T}$ X.

(2) Sum of eigen values of a square matrix A is equal to the sum of the elements on its main diagonal. (Trace)

$\boxed{\text{Sum of eigen values} = \text{Trace of } A}$ X.

(3) Product of eigen values of a square matrix A is equal to $|A|$

$\boxed{\text{Product of eigenvalues} = |A|}$ X.

(4) If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are non-zero eigen values of square matrix A of order n, Then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_n}$ are eigen values of A^{-1}.

λ_i is eigen value of A

then $\frac{1}{\lambda_i}$ " " " A^{-1} $(\lambda_i \neq 0)$

X.

(5) If $\lambda_1, \lambda_2, \cdots \lambda_n$ are eigen values of A, then

$\lambda_1^m, \lambda_2^m, \cdots \lambda_n^m$ " " " " A^n

(6) If $\lambda_1, \lambda_2, \cdots \lambda_n$ are eigen values of A, then

$c\lambda_1, c\lambda_2, \cdots c\lambda_n$ " " " " cA. $c \neq 0$

Problems:-

(1) Find the sum and product of the eigen values of the

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{bmatrix}$$

Soln

Sum of eigen values = Trace

$$= 1 + 0 - 3 = -2$$

Product of eigen values $= |A| = -1$

(2) If 2 and 3 are eigen values of $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

find the third eigen value. And hence find eigen values of A^{-1} and A^3.

Soln W.K.T sum of eigen values = TRACE of A

$$2 + 3 + \lambda_3 = 3 - 3 + 7$$

$$5 + \lambda_3 = 7$$

$$\lambda_3 = 2$$

Eigen values of A are $2, 2, 3$

" " A^{-1} " $1/2, 1/2, 1/3$

" " A^3 " $8, 8, 27$

(3) The product of two eigen values of the matrix

$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the third eigen value.

$$\underbrace{\lambda_1 \cdot \lambda_2}\cdot \lambda_3 = |A|$$

$$16 \cdot \lambda_3 = 32$$

$$\lambda_3 = \frac{32}{16} = 2$$

(4) Find the sum of the squares of eigen values of

$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

$S_1 = 13$

$S_2 = \begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 0 & 6 \end{vmatrix}$

$= 12 + 24 + 18$

$$\begin{bmatrix} & & \\ 0 & 0 & 6 \end{bmatrix}$$

$= 12 + 24 + 18$

$= 54$

Eigen values of $A = 3, 4, 6$

Sum of squares $= 3^2 + 4^2 + 6^2$

$= 9 + 16 + 36 = 61$

$S_3 = 3(24) = 72$

$\lambda^3 - 13\lambda^2 + 54\lambda - 72 = 0$

Cayley-Hamilton Theorem:-

Every square matrix satisfies its own characteristic eqn.

(1) $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

$S_1 = 1 - 1 = 0$

$S_2 = |A| = -1 - 4 = -5$

$\lambda^2 - 0\lambda - 5 = 0$

$\lambda^2 - 5 = 0$

Don't find eigen values

By C-H Thm, $A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = 0$

C-H Thm has two important applications

1) to find inverse of a non-singular matrix

2) " " higher powers of A.

(1) Verify C-H Thm & hence find A^{-1}

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$S_1 = 2 + 2 + 2 = 6$

$S_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$

$= (4-1) + (4-1) + (4-1) = 9$

$S_3 = |A| = 4$

The Char eqn is $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

Consider

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$$\boxed{A^3-6A^2+9A-4I} = $$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3-6A^2+9A-4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ C-H Thm is verified

$$A^3-6A^2+9A-4I = 0$$

$$A^{-1}(A^3-6A^2+9A-4I) = 0$$

$$A^2-6A+9I-4A^{-1} = 0$$

$$4A^{-1} = A^2-6A+9I$$

$$A^{-1} = \frac{1}{4}\left[A^2-6A+9I\right]$$

$$= \frac{1}{4}\left[\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$