18MAB101T-Calculus and Linear Algebra

UNIT-I : MATRICES

Definition

Matrix A rectangular array of numbers that represents a multidimensional object. Matrices are used to solve linear systems, as well as in vector operations.

4 x 3 Matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

Application of Matrices:

- **Encryption**

  In encryption, we use it to scramble data for security purposes to encode and to decode this data we need matrices. There is a key that helps encode and decode data which is generated by matrices.

- **Games especially 3D**

  They use it to alter the object, in 3d space. They use the 3d matrix to 2d matrix to convert it into the different objects as per requirement.

- **Economics and business**

  To study the trends of a business, shares, and more. To create business models etc.

- **In Geology**, matrices are used for making seismic surveys. They are used for plotting graphs, statistics and also to do scientific studies and research in almost different fields.

- **In robotics and automation**, matrices are the basic components for robot movements. The inputs for controlling robots are obtained based on the calculations from matrices and these are very accurate movements

CHARACTERISTIC EQUATION:

Consider the linear transformation Y = AX

In general, this transformation transforms a column vector$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ into another column vector $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

By means of the square matrix A where A= $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

If a vector X is transformed into a scalar multiple of the same vector. i.e., X is transformed into λX, then Y = λX = AX

i.e., AX = λX = $\lambda I X$, where I is the unit matrix of order n.

Now (A - λI)X =0 implies

$$a_{11} - \lambda)x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0$$
$$a_{21}x_1 + (a_{22} - \lambda)x_2 + \ldots + a_{2n}x_n = 0$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + ((a_{nn} - \lambda)x_n = 0$$

This system of equations will have a n solution if

|(A - λI)| = 0

$|(A - \lambda I)| = 0$ implies

$$\boxed{\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0}$$

The equation $|(A - \lambda I)| = 0$ is said to be the characteristic equation of the transformation or the characteristic equation of the matrix A.

Characteristic roots or Eigenvalues or Latent values of the Matrix A

Solving $|(A - \lambda I)| = 0$, we get n roots for λ, these roots are called Eigenvalues of A

Characteristic Vectors or Eigenvectors or Latent vectors of the Matrix A

Solving $|(A - \lambda I)| = 0$, we get n roots for λ, corresponding to each value of λ, the equation $AX = \lambda X$ has a non-zero solution vector. If X_r be the non-zero vector satisfying $AX = \lambda X$ for each λ_r, then X_r is said to be the Eigenvectors.

characteristic polynomial

Expand $|(A - \lambda I)|$

Working Rule to find Eigenvalues and Eigenvectors

Step-1. Find the Characteristic equation $|(A - \lambda I)| = 0$

Step-2. Solve characteristic equation and find eigenvalues

Step-3. Find n distinct eigenvectors corresponding to n distinct eigen values

Note:

1. If two or more eigenvalues are equal, it may or may not be possible to get linearly independent Eigenvectors corresponding to the repeated Eigenvalues.
2. If X_i is a solution of for a Eigenvalue λ_i then it follows that C X_i is also solution, where C is an arbitrary constant. Thus Eigenvector corresponding to a Eigenvalue is not unique but may be any one of the vectors C X_i

3. Non-repeated eigenvalues of a non-symmetric matrix implies linearly independent sets of Eigen vectors

4. Repeated eigenvalues of a non-symmetric matrix implies linearly independent sets of Eigen vectors may or may not be possible

5. Diagonalisation through similarity transformation is possible for linearly independent sets of eigenvectors

6. In a symmetric matrix the eigenvalues are non-repeated then we get a linearly independent and pairwise orthogonal set of eigenvectors

7. In a symmetric matrix the Eigen values are repeated then we may or may not be possible to get linearly independent and pairwise orthogonal sets of eigenvectors. If we form a linearly independent and pairwise orthogonal sets of eigenvectors then diagonalisation is possible through orthogonal transformation.

Properties of Eigen values

Property 1

Every square matrix and its transpose have the same Eigen values.

Property 2

If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the Eigen values of matrix A, then $1/\lambda_1, 1/\lambda_2, \ldots 1/\lambda_n$ are the Eigen values of A^{-1}.

Property 3

If $\lambda_1, \lambda_2, \ldots\lambda_n$ are Eigen values of the matrix A, then $\lambda_1^2, \lambda_2^2, \ldots\lambda_n^2$ are the Eigen values of A^2.

Property 4

If $\lambda_1, \lambda_2, \ldots\lambda_n$ are the Eigen values of the matrix A, then $k\lambda_1, k\lambda_2, \ldots k\lambda_n$ are the Eigen values of kA.

Property 5

The Eigen value of a real symmetric matrix are all real.

Property 6

The Eigen values of a triangular matrix are the diagonal elements of the matrix.

Property 7

Zero is an Eigen value of a matrix A if and only if A is singular.

Property 8

The sum of the Eigen values of a matrix A is equal to the sum of the principal diagonal elements of A.(The sum of the principal diagonal elements is called the Trace of the matrix)

Property 9

The product of the Eigen values of a matrix A is equal to the determinant of A.

Property 10

The Eigen vectors corresponding to distinct Eigen values of a real symmetric matrix are orthogonal.

Problems Using Properties

1. Find the sum and product of the Eigen values of the matrix

$$A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$$

Solution

Sum of the Eigen values of the matrix = Sum of the leading diagonal elements of the matrix = -1

Product of Eigen values of the matrix = $|A| = \begin{vmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{vmatrix}$

= -2 (0 – 12) -2 (0 – 6) -3 (-4 + 1)

= 45

2. If two of the Eigen values of A are 3 and 15. Find the third Eigen value of $A = \begin{bmatrix} 8 & -6 & 2 \\ 6 & 7 & -4 \\ -2 & -4 & 3 \end{bmatrix}$.

Solution

Let λ_1, λ_2, λ_3 are the Eigen values of A. Sum of all the Eigen value is

$$\lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3 = 18$$

$$\lambda_3 + 3 + 15 = 18$$

$\lambda_3 = 0$

3. If two of the Eigen values of the matrix

$$A=\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & 1 \\ 1 & -1 & 3 \end{bmatrix}$$ are 3 and 6. Find the Eigen values of A^{-1}.

Solution

Let $\lambda_1, \lambda_2, \lambda_3$ are the Eigen values of A. Sum of all the Eigen value is

$$\lambda_1 + \lambda_2 + \lambda_3 = 3 + 5 + 3 = 11$$

$$\lambda_3 + 3 + 6 = 11$$

$$\lambda_3 = 2.$$

By Property 2, the Eigen values of A^{-1} are 1/3,1/6,1/2.

4. Find the Eigen values of A^3 if $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$

Solution

By property 6, the Eigen values of A = 1, 2, 3.

By property 3, the Eigen values of $A^3 = 1^3, 2^3, 3^3$.

5. Find the constants a and b such that $\begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$ matrix has 3 and -2 as its Eigen values.

Solution

Given $\lambda_1 = 3$ and $\lambda_2 = -2$

Sum of the Eigen values of A = Trace of A

$3+(-2) = a+b$

$a+b = 1$ implies $b=1-a$ (1)

Product of the Eigen values = $|A|$

$-6 = ab-4$ implies $ab=-2$ (2)

Solving (1) and (2),

$a = 2, -1$ and $b = -1, 2$ respectively.

6. Find the Eigen values of the inverse of the matrix $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$.

Solution

Characteristic equation is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix}=0$$

$$\lambda^2-5\lambda-6=0$$

$$\lambda= -1, 6$$

By Property 2, the Eigen values are -1,1/6.

7. If 2 is an Eigen value of A= $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$, find the other two Eigen values.

Solution

Let $\lambda_1, \lambda_2, \lambda_3$ are the Eigen values of A.

Sum of all the Eigen value is 2, $\lambda_1 + \lambda_2 + \lambda_3 = 2$

Product of the Eigen values = $|A| = -8$, $\lambda_1 \lambda_2 \lambda_3 = -8$.

Given $\lambda_1 = 2$,

$\lambda_2 + \lambda_3 = 0$ and $\lambda_2 \lambda_3 = -4$

Solving , $\lambda_2 = 2$, $\lambda_3 = -2$ or $\lambda_2 = -2$, $\lambda_3 = 2$.

Therefore, 2 and -2 are the two Eigen values.

Problem:

Find eigen Values and the corresponding eigen vectors of the matrix $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$.

Solution:

Eigen Values are given by $|A-\lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^3 - (\text{sum of diagonal elements})\lambda^2 +$
$(\text{sum of cofactor of diagonal elements})\lambda - |A| = 0$

$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

$\lambda =$ 1, -2 , 3 are required eigen values

Eigen vectors are given by $(A-\lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} &(2-\lambda)x-2y+3z=0 \\ &x+(1-\lambda)y+z=0 \\ &x+3y+(1-\lambda)z=0 \end{aligned}\right\}\cdots(1)$$

Case: 1

Let $\lambda = 1$ in (1), we get

$$x-2y+3z=0$$

$$x+0y+z=0$$

$$x+3y-2z=0$$

Case: 1

Let $\lambda = 1$ in (1), we get

$$x - 2y + 3z = 0$$

$$x + 0y + z = 0$$

$$x + 3y - 2z = 0$$

Consider first two equations

$$\begin{array}{ccc} x & y & z \\ 1 & -2 & 3 \\ 1 & 0 & 1 \end{array}$$

$$\frac{x}{-2-0} = \frac{-y}{1-3} = \frac{z}{0+2}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$$

Eigen Vector is $X_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Case: 2

$$4x - 2y + 3z = 0$$

Let $\lambda =$ -2 in (1), we get, $1x + 3y + z = 0$

$$x + 3y + z = 0$$

Consider first two equations,

$$\begin{array}{ccc} x & y & z \\ 4 & -2 & 3 \\ 1 & 3 & 1 \end{array}$$

$$\frac{x}{-2-9} = \frac{-y}{4-3} = \frac{z}{12+2}$$

$$\frac{x}{11} = \frac{y}{1} = \frac{z}{-14}$$

Eigen Vector is $X_2 = \begin{pmatrix} 11 \\ 1 \\ -14 \end{pmatrix}$

Case: 3

Let $\lambda = -2$ in (1), we get,

$$-x - 2y + 3z = 0$$
$$x - 2y + z = 0$$
$$x + 3y - 4z = 0$$

Consider last two equations,

$$\begin{array}{ccc} x & y & z \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{array}$$

$$\frac{x}{8-3} = \frac{-y}{-4-1} = \frac{z}{3+2}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

Eigen Vector is $X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Problem:

Find eigen Values and the corresponding eigen vectors of the matrix

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution:

Eigen Values are given by

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (\text{sum of diagonal elements})\lambda^2 +$$

$$(\text{sum of cofactor of diagonal elements})\lambda - |A| = 0$$

$$\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$$

$\lambda = 0$, 3, 15 are required eigen values.

Eigen vectors are given by $(A-\lambda I)X=0$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (8-\lambda)x-6y+2z&=0 \\ -6x+(7-\lambda)y-4z&=0 \\ 2x-4y+(3-\lambda)z&=0 \end{aligned}\right\}\cdots(1)$$

Case: 1

Let $\lambda=0$ in (1), we get

$$8x-6y+2z=0$$

$$-6x+7y-4z=0$$

$$2x-4y+3z=0$$

Consider last two equations

$$\begin{array}{ccc} x & y & z \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{array}$$

$$\frac{x}{21-16} = \frac{-y}{-18+8} = \frac{z}{24-14}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

Eigen Vector is $X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$

Case: 2

Let $\lambda = 3$ in (1), we get

$$5x - 6y + 2z = 0$$

$$-6x + 4y - 4z = 0$$

$$2x - 4y + 0z = 0$$

Consider last two equations

$$\begin{array}{ccc} x & y & z \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{array}$$

$$\frac{x}{0-16} = \frac{-y}{0+8} = \frac{z}{24-8}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

Eigen Vector is $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

Case: 3

Let $\lambda = 15$ in (1), we get

$$7x - 6y + 2z = 0$$

$$-6x - 8y - 4z = 0$$

$$2x - 4y - 12z = 0$$

Consider last two equations

$$\begin{array}{ccc} x & y & z \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{array}$$

$$\frac{x}{96-16} = \frac{-y}{72+8} = \frac{z}{24+16}$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$$

Eigen Vector is $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Problem:

Find eigen Values and the corresponding eigen vectors of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

Solution:

Eigen Values are given by

$$|A-\lambda I|=0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{vmatrix}=0$$

$$\lambda^3-(\text{sum of diagonal elements})\lambda^2+$$
$$(\text{sum of cofactor of diagonal elements})\lambda-|A|=0$$

$$\lambda^3-9\lambda^2+15\lambda-7=0$$

$\lambda=1,\ 1,\ 7$ are required eigen values

(Repeated roots with non – symmetric matrix)

Eigen vectors are given by $(A-\lambda I)X=0$

$$\begin{pmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x+y+z&=0 \\ 2x+(3-\lambda)y+2z&=0 \\ 3x+3y+(4-\lambda)z&=0 \end{aligned}\right\}\cdots(1)$$

Case: 1

Let $\lambda=1$ in (1), we get

$$x+y+z=0$$

$$2x+2y+2z=0$$

$$3x+3y+3z=0$$

But all the above three equations are same

i.e., $x + y + z = 0$

let x = 0 and y =1 we get z = -1

Eigen Vector is $X_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Case: 2

let x = 1 and y =0 we get z = -1

Eigen Vector is $X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Case: 3

Let $\lambda = 7$ in (1), we get

$$-5x + y + z = 0$$

$$2x - 4y + 2z = 0$$

$$3x + 3y - 3z = 0$$

Consider last two equations

$$\begin{array}{ccc} x & y & z \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{array}$$

$$\frac{x}{12-6} = \frac{-y}{-6-6} = \frac{z}{6+12}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$

Eigen Vector is $X_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Problem:

Find eigen Values and the corresponding eigen vectors

of the matrix $\begin{bmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{bmatrix}$

Solution:

Eigen Values are given by

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (\text{sum of diagonal elements})\lambda^2 +$$

$$(\text{sum of cofactor of diagonal elements})\lambda - |A| = 0$$

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

λ =-1, -1, -1 are required eigen values

(Repeated roots with non – symmetric matrix)

Eigen vectors are given by $(A-\lambda I)X = 0$

$$\begin{pmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (6-\lambda)x-6y+5z &= 0 \\ 14x+(-13-\lambda)y+10z &= 0 \\ 7x-6y+(4-\lambda)z &= 0 \end{aligned}\right\}\cdots(1)$$

Case: 1

Let $\lambda =$-1 in (1), we get

$$7x-6y+5z=0$$

$$14x-12y+10z=0$$

$$7x-6y+5z=0$$

But all the above three equations are same
i.e., $7x - 6y + 5z = 0$

let x = 0 and y =1 we get z = $\frac{6}{5}$

Eigen Vector is $X_1 = \begin{pmatrix} 0 \\ 1 \\ 6/5 \end{pmatrix}$

Case: 2

let x = 1 and y = 0 we get z = $\frac{7}{5}$

Eigen Vector is $X_2 = \begin{pmatrix} 0 \\ 1 \\ -7/5 \end{pmatrix}$

Case: 3

let x = 1 and z = 0 we get y = $\frac{7}{6}$

Eigen Vector is $X_3 = \begin{pmatrix} 0 \\ 7/6 \\ 0 \end{pmatrix}$

Note:

If X_1, X_2 and X_3 are eigen vectors of a symmetric matrix X_1, X_2 and X_3 are orthogonal

Then,

$$X_1 X_2^T = 0,$$

$$X_2\, X_3^T = 0 \text{ and}$$

$$X_1\, X_3^T = 0$$

Problem:

Find eigen Values and the corresponding eigen vectors of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution:

Eigen Values are given by

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\lambda^3 - (\text{sum of diagonal elements})\lambda^2 +$
$(\text{sum of cofactor of diagonal elements})\lambda - |A| = 0$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$\lambda = 8, 2, 2$ are required eigen values (Repeated roots with symmetric matrix

Eigen vectors are given by $(A-\lambda I)X=0$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} (6-\lambda)x-2y+2z=0 \\ -2x+(3-\lambda)y-z=0 \\ 2x-y+(3-\lambda)z=0 \end{array}\right\}\cdots(1)$$

Case: 1

Let $\lambda=8$ in (1), we get

$$-2x-2y+2z=0$$

$$-2x-5y-z=0$$

$$2x-y-5z=0$$

Consider first two equations

$$\begin{array}{ccc} x & y & z \\ -2 & -2 & 2 \\ -2 & -5 & -1 \end{array}$$

$$\frac{x}{2+10} = \frac{-y}{2+4} = \frac{z}{10-4}$$

$$\frac{x}{12} = \frac{y}{-6} = \frac{z}{6}$$

Eigen Vector is $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Case: 2

Let $\lambda = 2$ in (1), we get

$$4x - 2y + 2z = 0$$

$$-2x + y - z = 0$$

$$2x - y + z = 0$$

But all the above three equations are same

i.e., $2x - y + z = 0$

let x = 0 and y =1 we get z = 1

Eigen Vector is $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Case: 3

Since the given matrix A is symmetric

($X_1 X_2^T = 0$, $X_2 X_3^T = 0$ and $X_1 X_3^T = 0$)

$X_2 X_3^T = 0$ and $X_1 X_3^T = 0$, we get

$$2x - y + z = 0$$
$$0x + y + z = 0$$

x	y	z
2	-1	1
0	1	1

$$\frac{x}{-1-1} = \frac{-y}{2+0} = \frac{z}{2-0}$$
$$\frac{x}{-2} = \frac{y}{-2} = \frac{z}{2}$$

Eigen Vector is $X_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

Problem:

Find Eigen Values and the corresponding eigen vectors of the matrix $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Solution:

Eigen Values are given by

$|A-\lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\lambda^3 - (\text{sum of diagonal elements})\lambda^2 +$

$(\text{sum of cofactor of diagonal elements})\lambda - |A| = 0$

$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

$\lambda = 4, 1, 1$ are required eigen values (Repeated roots with symmetric matrix)

Eigen vectors are given by $(A-\lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} &(2-\lambda)x - y + z = 0 \\ &-x + (2-\lambda)y - z = 0 \\ &x - y + (2-\lambda)z = 0 \end{aligned}\right\}\cdots(1)$$

Case: 1

Let $\lambda = 4$ in (1), we get

$$-2x - y + z = 0$$

$$-x - 2y - z = 0$$

$$x - y - 2z = 0$$

Consider first two equations

$$\begin{array}{ccc} x & y & z \\ -2 & -1 & 1 \\ -1 & -2 & -1 \end{array},$$

$$\frac{x}{1+2} = \frac{-y}{2+1} = \frac{z}{4-1}, \quad \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

Eigen Vector is $X_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Case: 2

Let $\lambda = 1$ in (1), we get

$$x - y + z = 0$$

$$-x + y - z = 0$$

$$x - y + z = 0$$

But all the above three equations are same

i.e., $x - y + z = 0$

let x = 0 and y =1 we get z = 1

Eigen Vector is $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Case: 3

Since the given matrix A is symmetric

($X_1 X_2^T = 0$, $X_2 X_3^T = 0$ and $X_1 X_3^T = 0$)

$X_2 X_3^T = 0$ and $X_1 X_3^T = 0$, we get

$$x - y + z = 0$$

$$0x + y + z = 0$$

$$\begin{array}{ccc} x & y & z \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{array}$$

$$\frac{x}{-1-1} = \frac{-y}{1+0} = \frac{z}{1-0}$$

$$\frac{x}{-2} = \frac{y}{-1} = \frac{z}{1}$$

$$\text{Eigen Vector is } X_3 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

Thank you