5 COMPLEX INTEGRATION

Introduction

Let $f(z)$ be a continuous function of the complex variable $z = x + iy$ defined at every point of a curve $'C'$ whose end points are A and B. Let us divide the curve C into $'n'$ parts by points $A = p_0(z_0), p_1(z_1), p_2(z_2), \cdots p_i(z_i) = B$. Let $\delta z_i = z_{i-1} - z_i$ and let ε_i be a point on the arc $p_{i-1} - p_i$. Then the limit of the sum $\sum_{i=1}^{n} f(\varepsilon_i)\delta z_i$ and $n \to \infty$ in such a way that each $\delta z_i \to 0$, if it exits is called the line integral of $f(z)$ along C and is denoted by $\int_C f(z)dz$.

If C is a closed curve, i.e., if p_0 and p_n coincide the integral is called the contour integral and is denoted by $\oint_C f(z)dz$

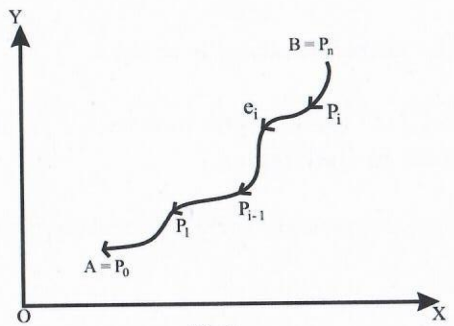

Fig. 1

Complex Integration: The integral of a complex function is defined as the limit of a certain sum in the same manner as integral of a real function along the x-axis is defined.

Multiple Point: If a curves intersects itself at a point then the point is said to be a multiple point of the curve.

Simple curve: A continuous curve which does not have a point of self intersection is called a simple curve. Simple curves are called Jordan curves.

Positively oriented simple closed curve: A simple closed curve C encloses a region, if the region lies to the left of a person when he travels along $'C'$, then curve $'C'$ is called positively oriented simple closed curve.

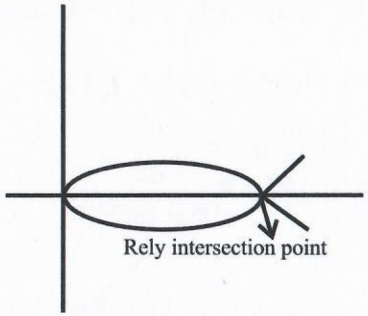

Fig. 2

Contour Integral: An integral along a simple closed curve is called a contour integral..

Simply - Connected Region: A region D is said to be simply - connected if, for every closed curve $'C'$ in D, C_i is wholly contained in D. Simply - connectedness of a region is equivalent to the absence of holes in it or to the situation in which every closed curve in D can shrink to a point, all the while being in D itself.

Multiply - Connected Region: A region which is not simply - connected is called multiply - connected region.

5.1 Evaluation of line integral

The evaluation of a line integral is reduced to the evaluation of two real line integrals as follows:

Since $z = x + iy, dz = dx + idy, f(z) = u + iv$

$\int_C f(z)dz = \int_C (u + iv)(dx + idy) = \int_C (udx - vdy) + i\int_C (vdx + idu)$

Example 5.1. *Evaluate* $\int_C \frac{dz}{(z - z_0)^{n+1}}$

Solution: Let $z - z_0 = re^{i\theta}$, so that θ varies from 0 to 2π as z describes the circle C.

$$I = \int_0^{2\pi} \frac{re^{i\theta}i}{(re^{i\theta})^{n+1}}d\theta = \frac{i}{r^n}\int_0^{2\pi} e^{-in\theta}d\theta$$

Case(i): when $n = 0, I = i\int_0^{2\pi} d\theta = 2\pi i$

Case(i): when $n \neq 0$,

$$I = \frac{1}{r^n}\int_0^{2\pi}(\cos n\theta - i\sin n\theta)d\theta = \frac{1}{nr^n}(\sin n\theta + i\cos n\theta)_0^{2\pi} = 0$$

Example 5.2. *Evaluate* $\int_0^{1+i}(x - y + ix^2)dz$
(i) along the straight line between the limits $(0,0)$ *and* $(1,1)$
(ii) over the path along the lines $y = 0, x = 1$
(iii) over the path along the lines $x = 0$ *and* $y = 1$
(iv) along the parabola $y^2 = x$

Solution:

(i) Along the path OP, $y = x, dy = dx$

$$\therefore I = \int_0^1 (x - y + ix^2)(dx + idy) = \int_0^1 (x - x + ix^2)(1 + i)dx$$

$$= (i - 1)\int_0^1 x^2dx = \frac{1}{3}(i - 1)$$

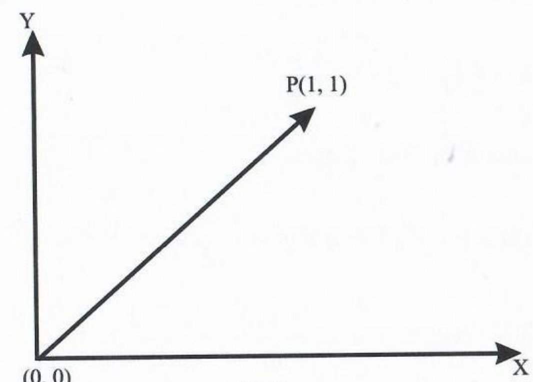

Fig. 3

(ii) Along the path OAP,

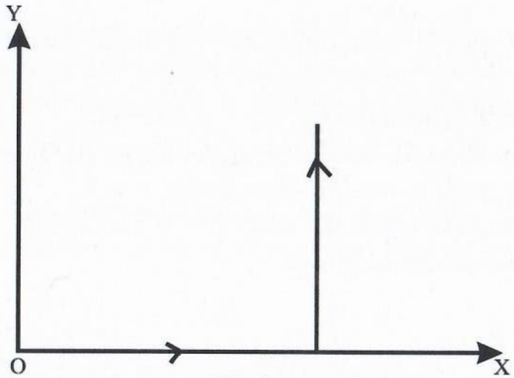

Fig. 4

$$I = \int_{OA} (x - y + ix^2)dz + \int_{AP} (x - y + ix^2)dz = I_1 + I_2 \quad (5.1)$$

Along the line OA, $y = 0, dy = 0$

$I_1 = \int_0^1 x dx + i \int_0^1 x^2 dx = \dfrac{3+2i}{6}$

Along the line AP, $x = 1, dx = 0$

$I_2 = \int_0^1 (-y)dy + i \int_0^1 (1-y)dy = \dfrac{i-2}{2}$

Equation (5.1) becomes $I = \dfrac{3+2i}{6} + \dfrac{i-2}{2} = \dfrac{5i-3}{6}$

(iii) Along the path OBP,

$$I = \text{along the line OB} + \text{along the line BP} = I_1 + I_2 \quad (5.2)$$

Along the line OB, $x = 1, dx = 0$

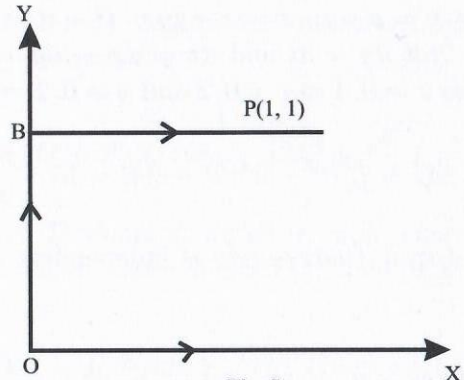

Fig. 5

$I_1 = i \int_0^1 (-y)dy = -\dfrac{i}{2}$

Along the line BP, $y = 1, dy = 0$

$I_2 = \int_0^1 (x-1)dx + i \int_0^1 x^2 dx = \dfrac{-1}{2} + \dfrac{i}{3}$

Equation (5.2) becomes $I = -\dfrac{i}{2} + \left(\dfrac{-1}{2} + \dfrac{i}{3}\right) = \dfrac{-3-i}{6}$

(iv) Along the parabola $y^2 = x \Rightarrow 2ydy = dx$

$I = \int_0^1 (x - y + ix^2)dz = \int_0^1 (y^2 - y + iy^4)(2y + i)dy$

$= \left[\dfrac{y^4}{2} - \dfrac{2y^3}{3} + i\dfrac{y^6}{3} + i\dfrac{y^3}{3} - i\dfrac{y^2}{2} - \dfrac{y^5}{5}\right]_0^1 = \dfrac{-11}{30} + \dfrac{i}{6}$

Example 5.3. *Evaluate $\int_C \overline{z} dz$ from $A(0,0)$ to $B(4,2)$ along the curve C and $z = t^2 + it$.*

Solution: Let $\overline{z} = x - iy, z = x + iy = t^2 + it \Rightarrow x = t^2, y = t$ so that $dx = 2tdt, dy = dt$ and $dz = dx + idy = 2tdt + idt = (2t+i)dt$. Also $x = 0, 4 \Rightarrow t = 0, 2$ and $y = 0, 2 \Rightarrow t = 0, 2$.

Hence $I = \int_C \overline{z}dz = \int_0^1 (t^2 - it)(2t+i)dt = 10 - \dfrac{8}{3}i$

Note: The integral $\int udx + vdy$ is independent of the path if $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$.

Example 5.4. *Find the value of the integral* $\int_C (x+y)dx + x^2ydy$, *(i) along* $y = x^2$ *having (0, 0), (3, 9) as end points, (ii) along* $y = 3x$ *between the same points. Do the values depend upon the path?*

Solution:

$$\text{Let } I = \int (x+y)dx + x^2ydy = \int udx + vdy \tag{5.3}$$

Here $u = x + y$ and $v = x^2y$ and we have $\dfrac{\partial u}{\partial x} = 1, \dfrac{\partial v}{\partial y} = x^2$ so that $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$. hence the integral is dependent on path.

(i) Along $y = x^2, dy = 2xdx$ and x varies from 0 to 3. Now equation (5.3) becomes

$I = \int_0^3 (x + x^2)dx + x^2.x^2.2xdx = \int_0^3 (x + x^2 + 2x^5)dx$

$= \left(\dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^6}{3}\right)_0^3 = \dfrac{9}{2} + \dfrac{27}{3} + \dfrac{729}{3} = 256.5$

(ii) Along $y = 3x, dy = 3dx$ and x varies from 0 to 3. Now equation (5.3) becomes

$I = \int_0^3 (x + 3x)dx + x^2(3x)3dx = \left[2x^2 + 9\dfrac{x^4}{4}\right]_0^3 = 200.25$

Example 5.5. *Evaluate* $\int_C 3y^2dx + 2ydy$ *where* C *is a circle* $x^2 + y^2 = 1$ *counter clockwise from* $(1, 0)$ *to* $(0, 1)$.

Solution: Let $I = \int_C 3y^2dx + 2ydy$. Given $x^2 + y^2 = 1$.

Differentiate w.r.to x and y, we get

$2xdx + 2ydy = 0 \Rightarrow ydy + xdx = 0 \Rightarrow ydy = -xdx$

$\therefore I = \int_1^0 3(1 - x^2)dx - 2xdx = [3x - x^3 - x^2]_1^0 = 1$

5.2 Cauchy's Integral Theorem

Statement: If $f(z)$ is analytic and $f'(z)$ is continuous at all points inside and on a simple closed curve C, then $\oint_C f(z)dz = 0$.

Proof: Let $f(z) = u + iv, z = x + iy$ and $dz = dx + idy$.

Then $\oint_C f(z)dz = \oint_C (u+iv)(dx+idy) = \oint_C (udx+iudy+ivdx-vdy)$

$$\oint_C f(z)dz = \oint_C (udx - vdy) + i(udy + vdx) \tag{5.4}$$

We know by Green's theorem in a plane,

$\oint_C (udx + vdy) = \int\int \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dxdy$

Equation (5.4) becomes

$\oint_C f(z)dz = \int\int \left[\left(-\frac{\partial u}{\partial y}dxdy - \frac{\partial v}{\partial x}dxdy\right) + i\left(\frac{\partial u}{\partial x}dxdy - \frac{\partial v}{\partial y}dxdy\right)\right]$

$= \int\int \left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) dxdy + i \int\int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$

$\int\int \left(-\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\right) dxdy + i \int\int \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dxdy = 0$, by C - R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

That is $\oint_C f(z)dz = 0$.

5.2.1 Cauchy's Theorem for Multiply Connected Region

Statement: If $f(z)$ is analytic and $f'(z)$ is continuous at all points in the region bounded by simple closed curves C_1 and C_2, then $\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$.

Corollary: If $f(z)$ is analytic and $f'(z)$ is continuous at all points in the region between the curves $C_1, C_2, C_3, \cdots C_n$ which lies entirely within a simple closed curve C, then

$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots \oint_{C_n} f(z)dz$

Cauchy's Fundamental Formula (OR) Cauchy's Integral Formula

Statement: If $f(z)$ is analytic and $f'(z)$ is continuous and if $'a'$ is any point inside C then $f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$.

Proof: The function $F(z) = \frac{f(z)}{z-a}$ is analytic at all points except $z = a$. Since $\frac{f(z)}{z-a}$ is analytic with $'a'$ as centre and radius $'r'$ draw a small circle C_1 which lies entirely within C.

$$\oint_C \frac{f(z)}{z-a} dz = \oint_{C_1} \frac{f(z)}{z-a} dz \tag{5.5}$$

Let $z - a = re^{i\theta} \Rightarrow z = a + re^{i\theta} \Rightarrow dz = 0 + re^{i\theta} d\theta = rie^{i\theta} d\theta$.

Equation (5.5) becomes

$$\oint_C \frac{f(z)}{z-a} dz = \oint_{C_1} \frac{f(a + re^{i\theta}) rie^{i\theta}}{re^{i\theta}} d\theta \tag{5.6}$$

As $r \to 0$, the circle C_1 shrinks to the point $'a'$, but $\theta = 0$ to 2π.

Equation (5.6) becomes

$\oint_C \frac{f(z)}{z-a} dz = i \int_0^{2\pi} f(a) d\theta = if(a)[\theta]_0^{2\pi} = if(a)2\pi = 2\pi i f(a)$

$$\Rightarrow f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz \tag{5.7}$$

Corollary: Differentiate (5.7) w.r.to a, we get

$$f'(a) = \frac{1}{2\pi i}\oint_C \left(\frac{(-1)}{(z-a)^2}(-1)\right) f(z)dz$$

$$\Rightarrow f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}dz$$

Differentiate again w.r.to a, we get

$$f''(a) = \frac{1}{2\pi i}\oint_C \left(\frac{(-2)}{(z-a)^3}(-1)\right) f(z)dz$$

$$f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}dz$$

Differentiate again w.r.to a, we get

$$f'''(a) = \frac{2!}{2\pi i}\oint_C \left(\frac{(-3)}{(z-a)^4}(-1)\right) f(z)dz = \frac{2!\times 3}{2\pi i}\oint_C \frac{f(z)}{(z-a)^4}dz.$$

That is $f'''(a) = \dfrac{3!}{2\pi i}\oint_C \dfrac{f(z)}{(z-a)^4}dz$ etc.

In general, $f^n(a) = \dfrac{n!}{2\pi i}\oint_C \dfrac{f(z)}{(z-a)^{n+1}}dz$

Note: We recall that a point z_0 at which a function $f(z)$ is not analytic is known as singular point or a singularity of $f(z)$. To find the singularity of $f(z)$, equate the denominator of $f(z)$ to zero and solve it for z. For example the singularity of $f(z) = \dfrac{z+3}{(z-1)(z-2)}$ is obtained as $(z-1)(z-2) = 0$

$z-1=0, z-2=0 \Rightarrow z=1, z=2$.

Example 5.6. *Evaluate* $\oint_C \dfrac{e^{-z}}{z+1}dz$ *where* C *is a circle* *(i)* $|z|=2$, *(ii)* $|z| = \dfrac{1}{2}$

Solution: Singular points are obtained by putting $z+1=0 \Rightarrow z=-1$.

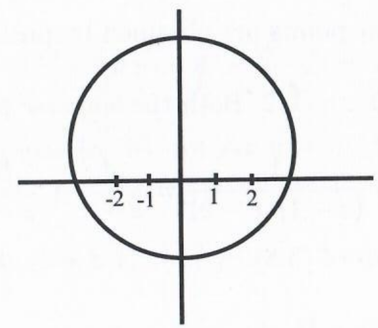

Fig. 6

(i) The singular point $z=-1$ lies within the circle $|z|=2$.

Then by Cauchy's integral formula $f(a) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(z)}{z-a}dz \Rightarrow \oint_C \dfrac{f(z)}{z-a}dz = 2\pi i f(a)$, we have

$\oint_C \dfrac{e^{-z}}{z+1}dz = \oint_C \dfrac{e^{-z}}{[z-(-1)]}dz = 2\pi i f(-1)$. Here $f(z) = e^{-z}, a = -1$ and $f(-1) = e$. Hence $\oint_C \dfrac{e^{-z}}{z+1}dz = 2\pi i e$.

(ii) The singular point $z = -1$ lies outside $|z| = \frac{1}{2}$.

Then $\oint_C \frac{f(z)}{z+1} dz = 0$ by Cauchy's theorem.

Example 5.7. *Evaluate* $\oint_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$ *where C is a circle* $|z| = 3$.

Solution: Singular points are obtained by putting

$(z-1)(z-2) = 0, z = 1, 2$. Both the singular points lies within $|z| = 3$.

$$\text{Now } \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \tag{5.8}$$

Multiply both sides of (5.8) by $(z-1)(z-2)$, we get

$1 = A(z-2) + B(z-1)$.

When $z = 2, 1 = 0 + B(2-1) \Rightarrow B = 1$.

When $z = 1, 1 = A(1-2) = -A \Rightarrow A = -1$.

Hence equation (5.8) becomes

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\text{Now } \oint_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = \oint_C \left(\frac{1}{(z-1)(z-2)} \right) \cos \pi z^2 dz$$

$$= -\oint_C \frac{\cos \pi z^2}{(z-1)} dz + \oint_C \frac{\cos \pi z^2}{(z-2)} dz$$

$$= \oint_C \left(\frac{-1}{z-1} + \frac{1}{z-2} \right) \cos \pi z^2 dz = -I_1 + I_2$$

$$I_1 = \oint_C \frac{\cos \pi z^2}{(z-1)} dz \tag{5.9}$$

By Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz \Rightarrow \oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a) \tag{5.10}$$

Comparing (5.9) and (5.10), we get $a = 1$, $f(z) = \cos \pi z^2 \Rightarrow f(1) = \cos \pi = -1$.

Equation (5.9) implies that

$$I_1 = \oint_C \frac{\cos \pi z^2}{(z-1)} dz = 2\pi i f(1) = 2\pi i(-1) = -2\pi i$$

$$\text{Also} \quad I_2 = \oint_C \frac{\cos \pi z^2}{(z-2)} dz = 2\pi i f(2) \tag{5.11}$$

We have $f(z) = \cos \pi z^2 \Rightarrow f(2) = \cos 4\pi = 1$.

Equation (5.11) becomes

$$I_2 = \oint_C \frac{\cos \pi z^2}{(z-2)} dz = 2\pi i f(2) = 2\pi i(1) = 2\pi i.$$

$$\therefore \oint_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = -(-2\pi i) + 2\pi i = 4\pi i$$

Example 5.8. *Using Cauchy's integral formula evaluate* $\oint_C \frac{e^{2z}}{(z+1)^4}dz$ *where C is a circle* $|z| = 2$.

Solution: Singular points are obtained by putting

$(z+1)^4 = 0 \Rightarrow z = -1$ which lies inside the circle C. By Cauchy's integral formula

$$f'''(a) = \frac{3!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^4}dz \Rightarrow \oint_C \frac{f(z)}{(z-a)^4}dz = \frac{2\pi i}{3!}f'''(a).$$

$$\text{Hence } \oint_C \frac{e^{2z}}{(z+1)^4}dz = \oint_C \frac{f(z)}{[z-(-1)]^4}dz = \frac{2\pi i}{3!}f'''(-1) \quad (5.12)$$

where $f(z) = e^{2z}, f'(z) = 2e^{2z}, f''(z) = 4e^{2z}, f'''(z) = 8e^{2z}$ and $f'''(-1) = 8e^{-2} = \frac{8}{e^2}$.

Now equation (5.12) becomes

$$\oint_C \frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{3!}f'''(-1) = \frac{2\pi i}{3!}\left(\frac{8}{e^2}\right) = \frac{16\pi i e^{-2}}{6} = \frac{8\pi i e^{-2}}{3}$$

Example 5.9. *Evaluate* $\oint_C \frac{3z^2+z}{z^2-1}dz$ *where C is a circle* $|z-1| = 1$.

Solution: Singular points are obtained by putting $z^2 - 1 = 0 \Rightarrow z^2 = 1$.

$\Rightarrow z = -1, 1$. The singular point $z = 1$ lies within the circle $|z-1| = 1$ but $z = -1$ lies outside the circle $|z-1| = 1$.

$$\text{Now } \frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1} \quad (5.13)$$

Multiply both sides of (5.13) by $(z+1)(z-1)$, we get

$1 = A(z-1) + B(z+1)$

When $z = 1, 1 = 0 + 2B \Rightarrow B = 1/2$

When $z = -1, 1 = A(-2) \Rightarrow A = -1/2$. hence equation (5.13) becomes

$$\frac{1}{z^2-1} = -\frac{1}{2(z+1)} + \frac{1}{2(z-1)}$$

$$\oint_C \frac{3z^2+z}{z^2-1}dz = \oint_C \frac{1}{2(z+1)}(3z^2+z)dz + \oint_C \frac{1}{2(z-1)}(3z^2+z)dz$$

$$= -\frac{1}{2}\oint_C \frac{3z^2+z}{(z+1)}dz + \frac{1}{2}\oint_C \frac{3z^2+z}{(z-1)}dz$$

$$= -\frac{1}{2}(2\pi i)f(-1) + \frac{1}{2}(2\pi i)f(1), \text{ where } f(z) = 3z^2 + z$$

$$= -\pi \times 0 + \pi i \times 4 = 4\pi i \text{ where } f(1) = 4.$$

Hence $\oint_C \frac{3z^2+z}{z^2-1}dz = 4\pi i$.

Example 5.10. *Evaluate* $\oint_C \frac{ze^{2z}}{(z-1)^3}dz$ *where C is a circle* $|z+i| = 2$.

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Solution: Singular points are obtained by putting $(z-1)^3 = 0 \Rightarrow z = 1$ is a singular point lies inside the circle $|z+i| = 2$.

By Cauchy's integral formula $f''(a) = \dfrac{2!}{2\pi i} \oint_C \dfrac{f(z)}{(z-a)^3} dz$.

$$\oint_C \frac{ze^{2z}}{(z-1)^3} dz = \oint_C \frac{f(z)}{(z-a)^3} dz = \frac{2\pi i}{2!} f''(1) \text{ where } f(z) = ze^{2z} \tag{5.14}$$

Let $f(z) = ze^{2z}$, $f'(z) = 2ze^{2z} + e^{2z}$, $f''(z) = 2[2ze^{2z} + e^{2z}] + 2e^{2z}$ and $f''(-1) = 2(2e^2 + e^2) = 8e^2$.

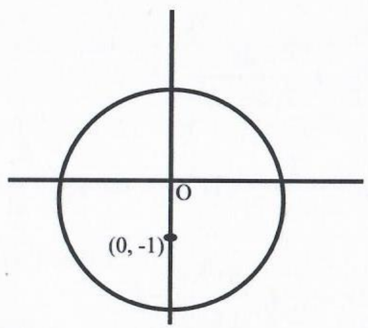

Fig. 7

Equation (5.14) becomes

$$\oint_C \frac{ze^{2z}}{(z-1)^3} dz = \frac{2\pi i}{2!} f''(1) = \frac{2\pi i}{2}(8e^2) = 8\pi i e^2$$

Example 5.11. *Evaluate* $\oint_C \dfrac{dz}{z^2 - 2z}$ *over the circle* $|z-2| = 1$.

Solution: Singular points are obtained by putting $z^2 - 2z = 0$

$\Rightarrow z(z-2) = 0 \Rightarrow z = 0, 2$ and $|z-2| = 1$ is a circle with centre $(2, 0)$ and radius 1. Only the singular point $z = 2$ lies within the circle $|z-2| = 1$. Hence by Cauchy's integral formula

$$I = \oint_C \frac{dz}{z^2 - 2z} = \oint_C \frac{dz}{z(z-2)} = \oint_C \frac{z^{-1} dz}{z-2} = \oint_C \frac{f(z) dz}{z-2}$$

$$= 2\pi i f(2), \text{ where } f(z) = \frac{1}{z}, f(2) = \frac{1}{2}$$

$$= 2\pi i \left(\frac{1}{2}\right) = \pi i$$

Example 5.12. *Evaluate the integral* $\oint \dfrac{\cos z dz}{z}$ *where* C *is an ellipse* $9x^2 + 4y^2 = 1$.

Solution: Here $z = 0$ is a singular point which lies within an ellipse $\dfrac{x^2}{1/9} + \dfrac{y^2}{1/4} = 1$. Hence by Cauchy's integral formula

$I = \oint \dfrac{\cos z dz}{z} = \oint \dfrac{f(z)}{z} dz = 2\pi i f(0) = 2\pi i \times 1 = 2\pi i$, where $f(z) = \cos z$

Example 5.13. *Evaluate* $\oint_C \dfrac{z^2+1}{z^2-1} dz$ *where* C *is a circle with unit radius and centre 1.*

Solution: Singular points are $z^2 - 1 = 0 \Rightarrow z = \pm 1$, hence only $z = 1$ lies within the circle. Hence by Cauchy's integral formula

$$I = \oint_C \frac{z^2+1}{z^2-1}dz = \oint_C \frac{z^2+1}{(z-1)(z+1)}dz = \oint_C \frac{\frac{z^2+1}{z+1}}{z-1}dz$$

$$= \oint_C \frac{f(z)}{z-1}dz = 2\pi i f(z) = 2\pi i, \text{ where } f(z) = \frac{z^2+1}{z+1}$$

and $f(1) = 1$.

Example 5.14. *Evaluate* $\oint \frac{dz}{z^3(z+4)}$ *where C is a circle* $|z| = 2$.

Solution: Singular points are obtained by putting $z^3(z+4) = 0 \Rightarrow z = 0$ and $z = -4$ of which only $z = 0$ lies inside the given circle.

By Cauchy's integral formula

$$I = \oint \frac{dz}{z^3(z+4)} = \oint \frac{\frac{1}{z+4}}{z^3}dz = \oint \frac{f(z)}{(z-0)^3}dz,\ f(z) = \frac{1}{z+4}$$

$$= \frac{2\pi i}{2!}f''(0), \text{ where } f'(z) = \frac{-1}{(z+4)^2}, f''(z) = \frac{2}{(z+4)^3}$$

$$= \pi i\left(\frac{1}{32}\right), \text{ where } f''(0) = \frac{2}{(0+4)^3} = \frac{1}{32}$$

EXERCISE

1. Evaluate $\oint \frac{\sin 3z dz}{z + (\pi/2)}$ if C is a circle $|z| = 5$.

2. Evaluate $\oint \frac{dz}{2z-3}$ where C is a circle $|z| = 1$.

3. Evaluate $\oint \frac{dz}{ze^z}$ where C is a circle $|z| = 1$.

4. Evaluate $\oint \frac{dz}{(z-3)^2}$ where C is a circle $|z| = 1$.

5. Evaluate $\oint \frac{\tan(z/2)dz}{(z-a)^2}, (-2 < a < 2)$ where C is the boundary of a square whose sides lie along $x = \pm 2$ and $y = \pm 2$ described in the positive sense.

6. Show that $\oint \frac{ze^z}{(z-a)^3}dz = \pi i e^a(a+2)$ where $z = a$ lies inside the closed curve C, using Cauchy's integral formula.

7. Show that $\oint \frac{z-2}{z(z-2)}dz = 2\pi i$ where C is a circle $|z| = 3$.

8. Evaluate $\oint \frac{dz}{z^2+4}$ where C is $|z| = 3$.

9. Evaluate $\oint \frac{(7z-1)dz}{z^2-3z-4}$ where C is an ellipse $x^2 + 4y^2 = 4$.

10. Without using residue theorem, evaluate $\oint \frac{(z+1)dz}{z^3-2z^2}$ where C is a unit circle $|z| = 1$.

11. Evaluate $\oint \frac{e^z dz}{z(1-z)^3}$ if (i) 0 lies inside C and 1 lies outside C, (ii) 1 lies inside C and 0 lies outside C and (iii) 0 lies

inside C.

12. If C is a circle $|z| = 3$ and if $g(z_0) = \oint_C \frac{2z^2 - z - 2}{z - z_0} dz$ then find $g(2)$. Hint: $g(z_0) = 2\pi i f(z_0) = 2\pi i f(2) = 8\pi i = g(2)$.

13. Evaluate $\oint \frac{z^2+1}{z^2-1} dz$ where C is the circle $|z+1| = 4$.

14. Find the value of $\oint \frac{dz}{(z^2+4)^2}$ if C is $|z - i| = 2$.

5.3 Taylor's Series and Laurent's Series

Now, we develop Taylor's series and Laurent's series to expand an analytic function $f(z)$ in powers in z.

Taylor's Theorem(Taylor's Series)

A function $f(z)$ is analytic at all points inside a circle C, with its centre at the point $'a'$ and radius R, we can expand

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a)$$

$$+ \cdots + \frac{(z-a)^n}{n!} f^n(a) + \cdots \infty.$$

$= \sum_{n=1}^{\infty} \frac{(z-a)^n}{n!} f^n(a)$, where $f^n(a)$ denotes the nth derivative.

Corollary 1. Putting $a = 0$, we get

$$f(z) = f(a) + \frac{z}{1!} f'(a) + \frac{z^2}{2!} f''(a) + \frac{z^3}{3!} f'''(a)$$

$$+ \cdots + \frac{z^n}{n!} f^n(a) + \cdots \infty.$$

Corollary 2. Putting $z - a = h \Rightarrow z = a + h$, we get

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a)$$

$$+ \cdots + \frac{h^n}{n!} f^n(a) + \cdots \infty.$$

Laurent's Theorem(Laurent's Series)

If $f(z)$ is analytic on two concentric circles C_1 and C_2 of radii r_1 and $r_2 (r_1 > r_2)$ with centre at $'a'$ and also on the annular region R bounded by C_1 and C_2 then for all z in R.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} = I_1 + I_2$$

I_1 is called the regular part (contain positive powers) and I_2 is called the principal part (contain negative powers) of f(z) and

$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz, b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{(z-a)^{1-n}} dz$ both the integrals being taken anticlockwise direction.

Note: Note the two formulae:

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

Example 5.15. *Obtain the expansion of $f(z) = \dfrac{z-1}{z^2}$ in Taylor's series in powers of $(z-1)$ and give the region of validity and Laurent's series for $|z-1| > 1$*

Solution: (i) Let $f(z) = \dfrac{z-1}{z^2} = \dfrac{1}{z} - \dfrac{1}{z^2}$. Now we want f(z) as a series in powers of $(z-1)$. The function is not analytic at $z = 0$.

Let $u = z - 1 \Rightarrow z = u + 1 \quad |u| < 1$.

$$f(z) = \frac{z-1}{z^2} = \frac{u}{(u+1)^2} = u(1+u)^{-2}$$

$$= u(1 - 2u + 3u^2 - 4u^3 + \cdots) = u - 2u^2 + 3u^3 - \cdots$$

$= [(z-1) - 2(z-1)^2 + 3(z-1)^3 - \cdots$ is a Taylor's series valid for $|z-1| > 1$.

(ii) Let $u = z - 1 \Rightarrow z = u + 1 \quad |u| > 1$. (i.e.) $1 < |u|, \dfrac{1}{|u|} < 1$.

$$\therefore f(z) = \frac{z-1}{z^2} = \frac{u}{(u+1)^2} = \frac{u}{u^2\left[1+\dfrac{1}{u}\right]^2} = \frac{1}{u}\left[1+\frac{1}{u}\right]^{-2}$$

$$= \frac{1}{u} - \frac{2}{u^2} + \frac{3}{u^3} - \cdots$$

$= \dfrac{1}{z-1} - \dfrac{2}{(z-1)^2} + \dfrac{3}{(z-1)^3} - \cdots$ valid expansion for $|z-1| < 1$.

Example 5.16. *Expand ze^{-z^2} in Laurent's series about $z = 0$.*

Solution: Let $ze^{-z^2} = z\left(1 - \dfrac{z^2}{1!} + \dfrac{z^4}{2!} - \dfrac{z^6}{3!} + \cdots\right)$

$= z - \dfrac{z^3}{1!} + \dfrac{z^5}{2!} - \dfrac{z^7}{3!} + \cdots$. Here this Laurent's series does not contain negative powers of z.

Example 5.17. *Expand $\dfrac{1-\cos z}{z}$ in Laurent's series about $z = 0$.*

Solution:

Let $\dfrac{1-\cos z}{z} = \dfrac{1}{z}\left[1 - \left(1 - \dfrac{z^2}{1!} + \dfrac{z^4}{2!} - \cdots\right)\right]$

$= \dfrac{z}{2!} - \dfrac{z^3}{4!} + \dfrac{z^5}{6!} - \cdots$ through $z = 0$ appears to be a singularity if $(1-\cos z)/z$, Laurent's series of $(1-\cos z)/z$ at $z = 0$ does not contain negative powers of z.

Example 5.18. *Find the residue at $z = 0$ of (i) $f(z) = e^{1/z}$*

(ii) $f(z) = \sin z / z^4$

Solution: (i) Here $f(z) = 1 + \dfrac{1}{z} + \dfrac{1}{z^2/2!} + \cdots$

Residue of $f(z)$ at $z = 0$ = coefficient of $1/z$ in the Laurent's expansion $= 1$.

(ii) Let $f(z) = \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right] = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \cdots$

Residue of $f(z)$ at $z = 0$ = coefficient of $1/z = -1/6$

Example 5.19. *Expand* $\frac{\sin z}{z-\pi}$ *about* $z = \pi$.

Solution: Put $z - \pi = t$.

Then $\frac{\sin z}{z-\pi} = \frac{\sin(\pi + t)}{t} = -\frac{\sin t}{t}$

$$= -\frac{1}{t}\left(t - \frac{t^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

$$= -1 + \frac{(z-\pi)^2}{3!} - \frac{(z-\pi)^4}{5!} + \cdots$$

Example 5.20. *Expand* $f(z) = \frac{z}{(z+1)(z+2)}$ *about* $z = -2$.

Solution: Put $z + 2 = t \Rightarrow z = t - 2$.

Then $f(z) = \frac{t-2}{t(t-1)} = \frac{2-t}{t}(1-t)^{-1}$

$$= \frac{1}{t}(2 + t + t^2 + \cdots) = \frac{2}{t} + 1 + t + t^2 + \cdots$$

$$= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \cdots$$

Example 5.21. *Expand* $\log\left(\frac{1+z}{1-z}\right)$ *at* $z = 0$ *using Taylor's series.*

Solution:

Let $f(z) = \log\left(\frac{1+z}{1-z}\right) = \log(1+z) - \log(1-z) \Rightarrow f(0) = 0$

$f'(z) = \frac{1}{1+z} + \frac{1}{1-z} \Rightarrow f'(0) = 2$

Then $f''(z) = \frac{-1}{(1+z)^2} + \frac{1}{(1-z)^2} \Rightarrow f''(0) = 0$ etc.

The Taylor's series of $f(z)$ about the point $z = a$ is

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots$$

The Taylor's series of $f(z)$ about the point $z = 0$ is

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \cdots = 2z + \frac{4}{3!}z^3 + \cdots$$

Example 5.22. *Expand* $f(z) = \frac{1}{z(z-1)}$ *as Laurent's series in powers valid in* $|z| < 1$ *and* $|z| > 1$.

Solution: Let $\frac{1}{z(z-1)} = -\frac{1}{z} - \frac{1}{1-z} = -\frac{1}{z} - (1-z)^{-1}$

(i) Let $f(z) = \left(-\frac{1}{z}\right)^n - \sum_0^\infty z^n$ is a valid expansion for $|z| < 1$.

(ii) Let $|z| > 1 \Rightarrow \frac{1}{|z|} < 1$.

Then $\dfrac{1}{z(z-1)} = -\dfrac{1}{z} + \dfrac{1}{z\left(1-\dfrac{1}{z}\right)}$

$= \left(-\dfrac{1}{z}\right) + \dfrac{1}{z}\left(1-\dfrac{1}{z}\right)^{-1} = \left(-\dfrac{1}{z}\right) + \dfrac{1}{z}\sum_0^\infty \left(\dfrac{1}{z}\right)^n$ is a valid expansion for $|z| > 1$.

Example 5.23. *If* $f(z) = \dfrac{1}{(z-1)(z-2)}$ *about* $z = 0$ *valid in (i)* $|z| < 1$, *(ii)* $1 < |z| < 2$ *and (iii)* $|z| > 2$.

Solution: Let $f(z) = \dfrac{1}{(z-1)(z-2)} = -\dfrac{1}{z-1} + \dfrac{1}{z-2}$

(i) In $|z| < 1$, $f(z) = \dfrac{1}{1-z} - \dfrac{1}{2\left(1-\dfrac{z}{2}\right)}$

$= (1+z+z^2+\cdots) - \dfrac{1}{2}\sum_0^\infty \left(\dfrac{z}{2}\right)^n$ is a valid expansion for $|z| < 1$.

(ii) Let $1 < |z| < 2 \Rightarrow \dfrac{1}{|z|} < 1, \dfrac{|z|}{2} < 1$.

Then $f(z) = -\dfrac{1}{z-1} + \dfrac{1}{z-2} = \dfrac{1}{z\left(1-\dfrac{1}{z}\right)} + \dfrac{1}{2\left(\dfrac{z}{2}-1\right)}$

$= -\dfrac{1}{z}\left(1-\dfrac{1}{z}\right)^{-1} - \dfrac{1}{2}\left(1-\dfrac{z}{2}\right)^{-1}$

$= -\dfrac{1}{z}\sum_0^\infty \left(\dfrac{1}{z}\right)^n - \dfrac{1}{2}\sum_0^\infty \left(\dfrac{z}{2}\right)^n$ is a valid expansion for

$1 < |z| < 2$

(iii) Let $|z| > 2 \Rightarrow \dfrac{2}{|z|} < 1$.

Then $f(z) = \dfrac{1}{z-1} + \dfrac{1}{z-2} = \dfrac{1}{z\left(1-\dfrac{1}{z}\right)} + \dfrac{1}{z\left(1-\dfrac{2}{z}\right)}$

$= -\dfrac{1}{z}\left(1-\dfrac{1}{z}\right)^{-1} + \dfrac{1}{z}\left(1-\dfrac{2}{z}\right)^{-1}$

$= -\dfrac{1}{z}\sum_{n=0}^\infty \left(\dfrac{1}{z}\right)^n + \dfrac{1}{z}\sum_{n=0}^\infty \left(\dfrac{2}{z}\right)^n$ is a valid expansion for

$|z| < 2$.

Example 5.24. *Expand* $f(z) = \dfrac{1}{(z-1)(z-2)}$ *in the region* $0 < |z-1| < 1$.

Solution: Let $f(z) = \dfrac{1}{(z-1)(z-2)}$

$= \dfrac{1}{[(z-1)-1]} - \dfrac{1}{z-1} = -[1-(z-1)]^{-1} - (z-1)^{-1}$

$= -(z-1)^{-1} + [1 + (z-1) + (z-1)^2 + \cdots]$

Example 5.25. *Find Laurent's series expansion of* $\dfrac{1}{(z+1)(z+3)}$ *in powers of* $(z+1)$ *for range* $0 < |z+1| < 2$.

Solution: Put $z + 1 = u$, then $0 < |z+1| < 2 \Rightarrow 0 < |u| < 2$.

Now $\dfrac{1}{(z+1)(z+3)} = \dfrac{1}{u(u+2)} = \dfrac{1}{2u\left(1+\frac{u}{2}\right)}$

$= \dfrac{1}{2u}\left(1+\frac{u}{2}\right)^{-1} = \dfrac{1}{2u}\left[1-\frac{u}{2}+\left(\frac{u}{2}\right)^2-\cdots\right]$

$= \dfrac{1}{2u}-\dfrac{1}{4}+\dfrac{u}{8}-\dfrac{u^2}{16}+\cdots$

Replace $u = z+1$, we have

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{4} + \frac{(z+1)}{8} - \frac{(z+1)^2}{16} + \cdots$$

Example 5.26. *Find the Taylor's series and Laurent's series expansion to represent the function* $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ *where (i)* $|z| < 2$, *(ii)* $2 < |z| < 3$ *and (iii)* $|z| > 3$.

Solution: Let $\dfrac{z^2-1}{(z+2)(z+3)} = A + \dfrac{B}{z+2} + \dfrac{C}{z+3}$

$$\Rightarrow z^2-1 = A(z+2)(z+3) + B(z+3) + C(z+2) \qquad (5.15)$$

Put $z=-3$ in (5.15), we get $9-1=-C \Rightarrow C=-8$.

Put $z=-2$ in (5.15), we get $4-1=B \Rightarrow B=3$ and put $z=0$ in (5.15), we get $-1 = 6A+9-16 \Rightarrow A=1$.

Then $\dfrac{z^2-1}{(z+2)(z+3)} = 1 + \dfrac{3}{z+2} - \dfrac{8}{z+3}$

(i) Let $|z| < 2 \Rightarrow \left|\dfrac{z}{2}\right| < 1$

$\dfrac{z^2-1}{(z+2)(z+3)} = 1 + \dfrac{3}{2\left(1+\frac{z}{2}\right)} - \dfrac{8}{3\left(1+\frac{z}{3}\right)}$

$= 1 + \dfrac{3}{2}\left(1+\dfrac{z}{2}\right)^{-1} - \dfrac{8}{3}\left(1+\dfrac{z}{3}\right)^{-1}$

$= 1 + \dfrac{3}{2}\sum(-1)^n\left(\dfrac{z}{2}\right)^n - \dfrac{8}{3}\sum(-1)^n\left(\dfrac{z}{3}\right)^n$ which is the expansion of Taylor's series valid for $|z| < 2$.

(ii) Let $2 < |z| < 3 \Rightarrow |z| > 2$ and $|z| < 3 \Rightarrow \dfrac{2}{|z|} < 1$ and $\dfrac{|z|}{3} < 1$.

$\therefore \dfrac{z^2-1}{(z+2)(z+3)} = 1 + \dfrac{3}{z+2} - \dfrac{8}{z+3}$

$= 1 + \dfrac{3}{z\left(\frac{2}{z}+1\right)} - \dfrac{8}{3\left(\frac{z}{3}+1\right)} = 1 + \dfrac{3}{z}\left(\dfrac{2}{z}+1\right)^{-1} - \dfrac{8}{3}\left(1+\dfrac{z}{3}\right)^{-1}$

$= 1 + \dfrac{3}{z}\sum(-1)^n\left(\dfrac{2}{z}\right)^n - \dfrac{8}{3}\sum(-1)^n\left(\dfrac{z}{3}\right)^n$ is a suitable expansion for $2 < |z| < 3$.

(iii) Let $|z| > 3 \Rightarrow \dfrac{|z|}{3} > 1 \Rightarrow \dfrac{3}{|z|} < 1$.

Now $\dfrac{z^2-1}{(z+2)(z+3)} = 1 + \dfrac{3}{z+2} - \dfrac{8}{z+3}$

$$= 1 + \frac{3}{z\left(\frac{2}{z}+1\right)} - \frac{8}{z\left(\frac{3}{z}+1\right)} = 1 + \frac{3}{z}\left(\frac{2}{z}+1\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$= 1 + \frac{3}{z}\sum(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum(-1)^n\left(\frac{3}{z}\right)^n$ is a suitable expansion for $|z| > 3$.

Example 5.27. *Represent the function* $\dfrac{4z+3}{z(z-3)(z+2)}$ *in Laurent's series, (i) within* $|z| = 2$, *(ii) in the annular region between* $|z| = 2$ *and* $|z| = 3$ *and (iii) exterior to* $|z| = 3$.

Solution: Let $\dfrac{4z+3}{z(z-3)(z+2)} = \dfrac{A}{z} + \dfrac{B}{z-3} + \dfrac{C}{z+2}$

$$\Rightarrow 4z + 3 = A(z-3)(z+2) + Bz(z+2) + Cz(z-3) \quad (5.16)$$

Put $z = 0$ in (5.16), we get $3 = A(6) \Rightarrow A = 1/2$.

Put $z = -2$ in (5.16), we get $-5 = C(-2)(-5) \Rightarrow C = -1/2$ and put $z = 3$ in (5.16), we get $15 = B(3) \Rightarrow B = 5$.

Hence $\dfrac{4z+3}{z(z-3)(z+2)} = \dfrac{1}{2z} + \dfrac{5}{z-3} - \dfrac{1}{2(z+2)}$

(i) Within $|z| = 2 \Rightarrow |z| < 2 \Rightarrow \dfrac{|z|}{2} < 1$ and $\dfrac{|z|}{3} < 1$

Now $\dfrac{4z+3}{z(z-3)(z+2)} = \dfrac{1}{2z} + \dfrac{5}{z-3} - \dfrac{1}{2(z+2)}$

$$= \frac{z^{-1}}{2} + \frac{5}{-3\left(1-\frac{z}{3}\right)} - \frac{1}{4\left(1+\frac{z}{2}\right)}$$

$$= \frac{z^{-1}}{2} - \frac{5}{3}\left(1-\frac{z}{3}\right)^{-1} - \frac{1}{4}\left(1+\frac{z}{2}\right)^{-1}$$

$$= \frac{z^{-1}}{2} - \frac{5}{3}\sum\left(\frac{z}{3}\right)^n - \frac{1}{4}\sum\left(\frac{z}{2}\right)^n$$

(ii) In the annular region between $|z| = 2$ and $|z| = 3$

$2 < |z| < 3 \Rightarrow 2 < |z|$ and $|z| < 3 \Rightarrow \dfrac{2}{|z|} < 1$ and $\dfrac{|z|}{3} < 1$

Now $\dfrac{4z+3}{z(z-3)(z+2)} = \dfrac{1}{2z} + \dfrac{5}{z\left(1-\frac{3}{z}\right)} - \dfrac{1}{4\left(1+\frac{z}{2}\right)}$

$$= \frac{z^{-1}}{2} + \frac{5}{z}\left(1-\frac{3}{z}\right)^{-1} - \frac{1}{4}\left(1+\frac{z}{2}\right)^{-1}$$

$$= \frac{z^{-1}}{2} + \frac{5}{z}\sum\left(\frac{3}{z}\right)^n - \frac{1}{4}\sum(-1)^n\left(\frac{z}{2}\right)^n$$

(iii) Exterior to $|z| = 3 \Rightarrow |z| > 3 \Rightarrow \dfrac{3}{|z|} < 1$

Now $\dfrac{4z+3}{z(z-3)(z+2)} = \dfrac{1}{2z} + \dfrac{5}{z-3} - \dfrac{1}{2(z+2)}$

$$= \frac{z^{-1}}{2} - \frac{5}{z\left(1-\frac{3}{z}\right)} - \frac{1}{2z\left(1+\frac{2}{z}\right)}$$

$$= \frac{z^{-1}}{2} - \frac{5}{z}\left(1-\frac{3}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= \frac{z^{-1}}{2} - \frac{5}{z}\Sigma\left(\frac{3}{z}\right)^n - \frac{1}{2z}\Sigma(-1)^n\left(\frac{2}{z}\right)^n$$

EXERCISE

1. Expand $\cos z$ as a Taylor's series about the point $z = \frac{\pi}{4}$.

   $\left[\text{Ans:} f(z) = \sum_{n=0}^{\infty} \frac{f^n\left(\frac{\pi}{4}\right)}{n!}\left(z - \frac{\pi}{4}\right)^n\right.$
   the region of convergence is $\left.\left|z - \frac{\pi}{4}\right| < \infty\right]$

2. Find the Taylor's series for $\frac{1}{z}$ about $z = 2, z = i$.

   $\left[\text{Ans:}(i) f(z) = \sum_0^{\infty} \frac{(-1)^n}{1^{n+1}(z-1)^n}\ (ii)\ f(z) = \sum_0^{\infty} \frac{(-1)^n n!}{i^{n+1}(z-1)^n}\right]$

3. Find the Taylor's series expansion of $\sec z$ in powers of $\left(z - \frac{\pi}{4}\right)$.

   $\left[\text{Ans:} f(z) = \sqrt{2} + \sqrt{2}\left(z - \frac{\pi}{4}\right) + \frac{3}{\sqrt{2}}\left(z - \frac{\pi}{4}\right)^2 + \cdots\right]$

4. Find the Taylor's series expansion about $z = 0$ of

   $f(z) = \frac{z}{(z+1)(z-3)}$.

   $\left[\text{Ans:} f(z) = \frac{1}{4}\left((1+z)^{-1} - \left(1 - \frac{z}{3}\right)^{-1}\right)\right]$

5. Show that $\frac{1}{z^2} = \frac{1}{4}\sum_0^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n$ when $|z-2| < 2$

6. Expand $\log\left(\frac{1+z}{1-z}\right)$ at $z = 0$ using Taylor's series.

   $\left[\text{Ans:} f(z) = 2\left[z + \frac{z^3}{3} + \frac{z^5}{3} + \cdots\right]\right.$ which converges for $\left.|z| < 1\right]$

7. Show that $\frac{1}{z^2} = \sum_0^{\infty}(n+1)(z+1)^n$ when $|z+1| < 1$.

8. Find the Laurent's series for $f(z) = \frac{1}{z^2+3z+2}$ in the region $1 < |z| < 2$.

   $\left[\text{Ans:} f(z) = \frac{1}{2}\left(1 + \frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1 + \frac{z}{2}\right)^{-1}\right]$

9. Expand $f(z) = \frac{z}{(z-1)(z-3)}$ as Laurent's series valid for (i) $1 < |z| < 3$ (ii) $0 < |z-1| < 2$.

   $\left[\text{Ans: (i)} -\frac{1}{2z}\left(1 - \frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1 - \frac{z}{3}\right)^{-1},\right.$
   $\left.\text{(ii)} \frac{-1}{2(z-1)} - \frac{3}{4}\left[1 + \left(\frac{z-1}{2}\right) + \left(\frac{z-1}{2}\right)^2 + \cdots\right]\right]$

10. Find the Laurent's expansion of of $f(z) = \dfrac{7z-2}{z(z-2)(z+1)}$ valid in $1 < |z+1| < 3$.

$$\left[\text{Ans: } -\frac{2}{z+1} + \sum_2^{\infty} \frac{1}{(z+1)^n} - \frac{2}{3}\sum_0^{\infty} \frac{(z+1)^n}{z^n}\right]$$

11. Find the Laurent's series of $f(z) = \dfrac{1}{z(1-z)}$ valid in the region (i) $|z+1| < 1$, (ii) $1 < |z+1| < 2$ and (iii) $|z+1| > 2$.

Ans: (i) $f(z) = \sum_0^{\infty} \left(-1 + \dfrac{1}{2^{n+1}}\right)(z+1)^n$

(ii) $f(z) = \sum_0^{\infty} \dfrac{1}{(z+1)^{n+1}} + \sum_0^{\infty} \dfrac{1}{2^{n+1}}(z+1)^n$

(iii) $f(z) = \sum_0^{\infty}(1-z^n)\dfrac{1}{(z+1)^{n+1}}$

12. Find the Laurent's series expansion of $f(z) = \dfrac{z}{(z-1)(z-2)}$ valid in the region (i) $|z+2| < 3$, (ii) $3 < |z+2| < 4$ and (iii) $|z+2| > 4$

Ans: (i) $\sum \left(\dfrac{1}{2.4^n} + \dfrac{1}{3^{n+1}}\right)(z+2)^n$

(ii) $-\dfrac{1}{2}\sum \dfrac{(z+2)^n}{4^n} - \sum \dfrac{3^n}{(z+2)^n}$

(iii) $\sum(2.4^n - 3^n)\dfrac{1}{(z+2)z^{n+1}}$

5.4 Poles and Residues

Zeros of an Analytic Function

A zero of an analytic function $f(z)$ is a value of z for which $f(z) = 0$. If $f(a) = 0$ and $f'(a) \neq 0$, then $z = a$ is called a simple zero or a zero of first order for $f(z)$.

If $f(a) = f'(a) = f''(a) = \cdots f^{m-1}(a) = 0$ and $f^m(a) \neq 0$, then $z = a$ is called a zero of order m for $f(z)$.

Singular Points

We know that a point z_0 at which a function $f(z)$ is not analytic is known as a singular point or a singularity of $f(z)$.

Types of Singular Points

(i) Isolated Singular Point

A singular point $z = z_0$ is said to be an isolated singular point of $f(z)$, if there is no other singular point in the neighbourhood of z_0.

Example: The function $f(z) = \dfrac{1}{z(z-2)}$ is analytic except at $z = 0$ and $z = 2$. There are no other singularities of $f(z)$ in the neighbourhood of $z = 0$ and $z = 2$.

(i.e.) $z = 0$ and $z = 2$ are isolated singular points of the function.

(ii) Removable Singular Point

A singular point $z = z_0$ is said to be a removable singular point of $f(z)$, if $\lim_{z\to z_0} f(z)$ exists and is finite.

Example: The function $f(z) = \dfrac{\sin z}{z}$ is analytic except at $z = 0$. So $z = 0$ is a singularity of $f(z)$.

Also $\lim_{z\to 0} f(z) = \lim_{z\to 0} \dfrac{\sin z}{z} = 1$. Hence $z = 0$ is a removable singular point of $f(z)$.

Example: The function $f(z) = \dfrac{\tan z}{z}$ is analytic except at $z = 0$. So $z = 0$ is a singularity of $f(z)$.

Also $\lim_{z\to 0} f(z) = \lim_{z\to 0} \dfrac{\tan z}{z} = 1$. Hence $z = 0$ is a removable singular point of $f(z)$.

Example: The function $f(z) = \dfrac{z^2 - a^2}{z + a}$ is analytic except

$z = -a$. So $z = -a$ is a singularity of $f(z)$. Also $\lim_{z\to -a} f(z) = \lim_{z\to -a} \dfrac{z^2 - a^2}{z + a} = \lim_{z\to -a} \dfrac{(z-a)(z+a)}{z+a} = -2a$.

Hence $z = -a$ is a removable singular point of $f(z)$.

Essential Singular Point

A singular point $z = z_0$ is said to be an essential singular point of $f(z)$, if it is neither an isolated singularity nor a removable

singularity.

Example: The function $f(z) = e^{\frac{1}{z+1}}$ is analytic except at

$z = -1$. So $z = -1$ is a singularity of $f(z)$. Also $z = -1$ is neither an isolated singularity nor a removable singularity. Hence $z = -1$ is an essential singular point of $f(z)$.

Example: The function $f(z) = e^{\frac{1}{z}}$ is analytic except at

$z = 0$. So $z = 0$ is a singularity of $f(z)$. Also $z = 0$ is neither an isolated singularity nor a removable singularity. Hence $z = 0$ is an essential singularity of $f(z)$.

Poles

Poles are nothing but isolated singular points. Pole of order one is called a simple pole. For a positive integer n, if

$\lim_z^a (z-a)^n f(z) \neq 0$, then $z = a$ is called a pole of order n for $f(z)$.

Residue at a Pole

The Laurent's series expansion of $f(z)$ about $z = a$ is given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \cdots$$

$$+\frac{b_1}{(z-a)}+\frac{b_2}{(z-a)^2}+\frac{b_3}{(z-a)^3}+\cdots$$

where $a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{(z-a)^{n+1}}dz, b_n = \frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{(z-a)^{1-n}}dz$ both the integrals being taken anticlockwise direction.

The co-efficient of $\frac{1}{z-a}$, i.e., b_1 is called the residue of $f(z)$ at $z=a$. That is $b_1 = \frac{1}{2\pi i}\oint_C f(z)dz$ is called the residue of $f(z)$ at $z=a$.

Evaluation of Residues of f(z)

(i) Residue of $f(z)$ at its simple pole $z=z_0$ is given by

$$R = Re(z=z_0) = \lim_{z\to z_0}(z-z_0)f(z).$$

(ii) If $f(z)=\frac{\phi(z)}{\psi(z)}$, where $\psi(a)=0$ but $\phi(a)\neq 0$, then

$$R = Re(z=z_0) = \lim_{z\to z_0}(z-z_0)f(z) = \frac{\phi(z)}{\psi'(z)}.$$

(iii) Residue of $f(z)$ at its pole $z=z_0$ of order n is given by

$$R = Re(z=z_0) = \lim_{z\to z_0}\left[\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z)\right]$$

5.5 Cauchy's Residue Theorem

Statement: If $f(z)$ is analytic at all points inside and on a simple closed curve C except for a finite number of isolated singular points (poles) $z_1, z_2, z_3, \cdots z_n$ within C, then $\oint_C f(z)dz = 2\pi i$ (sum of the Residues of f(z) at $z_1, z_2, z_3, \cdots z_n$)

Proof: Let us surround each of the poles $z_1, z_2, z_3, \cdots z_n$ by small circles $C_1, C_2, \cdots C_n$ all within C then the circles together with $'C'$ form a multiply connected region in which the function is analytic.

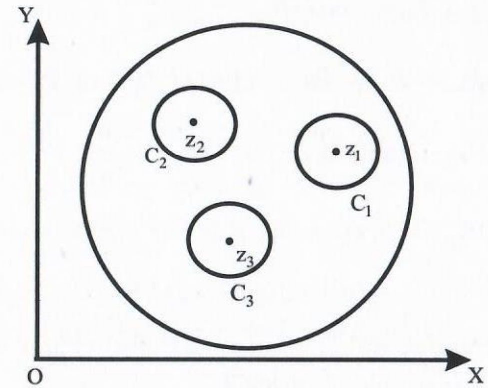

Fig. 8

Then by Cauchy's theorem,

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots + \oint_{C_n} f(z)dz \quad (5.17)$$

We know that $\frac{1}{2\pi i}\oint_C f(z)dz =$ Residue of f(z) at z.

$\Rightarrow \oint_C f(z)dz = 2\pi i$ (Residue of f(z) at z) $= 2\pi i R$. Hence

$\oint_{C_1} f(z)dz = 2\pi i$ (Residue of f(z) at $z = z_1$) $= 2\pi i R_1$

$\oint_{C_2} f(z)dz = 2\pi i$ (Residue of f(z) at $z = z_2$) $= 2\pi i R_2$

$\oint_{C_3} f(z)dz = 2\pi i$ (Residue of f(z) at $z = z_3$) $= 2\pi i R_3$ and so on.

Now equation (5.17) becomes

$\oint_C f(z)dz = 2\pi i R_1 + 2\pi i R_2 + 2\pi i R_3 + \cdots + 2\pi i R_n$

$= 2\pi i(R_1 + R_2 + R_3 + \cdots + R_n)$

$= 2\pi i \sum R_i$, where R_i is the residue of $f(z)$ at $z = z_i$.

$= 2\pi i$ (sum of the residues of f(z) at $z_1, z_2, z_3, \cdots z_n$)

Example 5.28. *Determine the poles of* $f(z) = \dfrac{z^2}{(z-1)(z-2)^2}$

Solution: The poles are $(z-1)(z-2)^2 = 0 \Rightarrow z = 1$ is a simple pole and $z = 2$ is a pole of order 2.

Example 5.29. *Find the poles of* $\cot z$.

Solution: Let $f(z) = \cot z = \dfrac{\cos z}{\sin z}$.

Poles are given by $\sin z = 0 \Rightarrow z = n\pi$

$\Rightarrow z = 0, \pm\pi, \pm 2\pi, \cdots$

Example 5.30. *Find the poles of (i)* $\tan z$ *(ii)* $\csc z$

Solution:

(i) Let $f(z) = \tan z = \dfrac{\sin z}{\cos z}$

The poles are given by $\cos z = 0$

$\Rightarrow z = (2n+1)\dfrac{\pi}{2} = n\pi + \dfrac{\pi}{2}, n = 0, \pm 1, \pm 2, \cdots$

(ii) Let $f(z) = \csc z = \dfrac{1}{\sin z}$.

Poles are given by $\sin z = 0 \Rightarrow z = n\pi \Rightarrow z = 0, \pm\pi, \pm 2\pi, \cdots$

Example 5.31. *Determine the poles of the function* $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ *and also find the residue at each pole.*

Solution: Poles are obtained by putting $(z-1)^2(z+2) = 0 \Rightarrow z = -2, 1, 1$. hence $f(z)$ has a simple pole at $z = -2$ and pole of order 2 at $z = 1$.

(i) Residue of $f(z)$ at the simple pole $z = -2$ is

$R_1 = \lim_{z\to -2}(z+2)f(z) = \lim_{z\to -2}\dfrac{(z+2)z^2}{(z-1)^2(z+2)}$

$= \lim_{z\to -2}\dfrac{z^2}{(z-1)^2} = \dfrac{4}{9}$

(ii) Residue of $f(z)$ at the pole $z = 1$ of order 2 is

$$R_2 = \lim_{z\to 1}\left[\frac{d}{dz}(z-1)^2 f(z)\right] = \lim_{z\to 1}\left[\frac{d}{dz}\frac{(z-1)^2 z^2}{(z-1)^2(z+2)}\right]$$

$$= \lim_{z\to 1}\left[\frac{d}{dz}\left(\frac{z^2}{z+2}\right)\right] = \lim_{z\to 1}\left[\frac{(z+2)2z - z^2}{(z+2)^2}\right] = \frac{5}{9}$$

Example 5.32. *Find the residues of* $f(z) = \dfrac{e^{2z}}{(z+1)^2}$.

Solution: The poles are $(z+1)^2 = 0 \Rightarrow z = -1$ of order 2.

Residue of $f(z)$ at the pole $z = -1$ is a pole of order 2 is

$$R = \lim_{z\to -1}\frac{1}{(2-1)!}\left[\frac{d}{dz}(z+1)^2 f(z)\right]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\frac{e^{2z}}{(z+1)^2}\right] = \lim_{z\to -1}\frac{d}{dz}e^{2z} = 2e^{-2}$$

Example 5.33. *Find the residues at their poles of* $f(z) = \dfrac{z}{(z-1)^2}$.

Solution: The poles are given by $(z-1)^2 = 0$. So $z = 1$ is a pole of order 2.

Residue of $f(z)$ at the pole $z = 1$ is a pole of order 2 is

$$R = \lim_{z\to 1}\frac{1}{(2-1)!}\left[\frac{d}{dz}(z-1)^2 f(z)\right]$$

$$= \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{z}{(z-1)^2}\right] = \lim_{z\to 1}\frac{d}{dz}(z) = 1$$

Example 5.34. *Evaluate* $\oint_C \dfrac{\sin\pi z + \cos\pi z^2}{z+z^2}dz$ *where* C *is a circle* $|z| = 2$.

Solution: The poles are given by $z(z+1) = 0 \Rightarrow z = 0, -1$ are simple poles and both lie within the circle $|z| = 2$.

$R_1 =$ Residue of $f(z)$ at the simple pole $z = 0$ is

$$\lim_{z\to 0}(z-0)f(z) = \lim_{z\to 0}\frac{\sin\pi z + \cos\pi z^2}{z(z+1)} = 1$$

$R_2 =$ Residue of $f(z)$ at the simple pole $z = -1$ is

$$Re(z=-1) = \lim_{z\to -1}(z+1)f(z)$$

$$= \lim_{z\to 0}(z+1)\frac{\sin\pi z + \cos\pi z^2}{z(z+1)} = 1$$

Now by Cauchy's residue theorem

$$\oint_C \frac{\sin\pi z + \cos\pi z^2}{z+z^2}dz = \oint_C f(z)dz = 2\pi i \text{ (sum of all residues)}$$

$$= 2\pi i(R_1 + R_2) = 2\pi i(1+1) = 4\pi i$$

Example 5.35. *Evaluate* $\oint_C \dfrac{\cos\pi z^2}{(z-1)(z-2)}dz$ *where* C *is a circle* $|z| = 3$.

Solution: Poles are given by $(z-1)(z-2) = 0 \Rightarrow z = 1, 2$ are simple poles and both lie within $|z| = 3$.

$R_1 =$ Residue of $f(z)$ at the simple pole $z = 1$ is

$$Re(z=1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1)\frac{\cos\pi z^2}{(z-1)(z-2)} = 1$$

$R_2 =$ Residue of $f(z)$ at the simple pole $z = 2$ is

$$Re(z=2) = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}(z-2)\frac{\cos\pi z^2}{(z-1)(z-2)} = 1$$

Now by Cauchy's residue theorem

$\oint_C \frac{\cos\pi z^2}{(z-1)(z-2)}dz = \oint_C f(z)dz = 2\pi i(R_1+R_2) = 2\pi i(1+1) =$ $4\pi i$

Example 5.36. *Find the residue at the poles of the function* $\frac{z}{z^2+1}$.

Solution: Poles are given by $z = \pm i \Rightarrow z = i, -i$ are simple poles.

$R_1 =$ Residue of $f(z)$ at the simple pole $z = i$ is

$$Re(z=i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}(z-i)\frac{z}{(z+i)(z-i)} = \frac{1}{2}$$

$R_2 =$ Residue of $f(z)$ at the simple pole $z = -i$ is

$$Re(z=-i) = \lim_{z\to -i}(z+i)f(z) = \lim_{z\to -i}(z+i)\frac{z}{(z+i)(z-i)} = -\frac{1}{2}$$

Example 5.37. *Evaluate* $\oint_C \frac{\cos\pi z}{z-1}$ *where C is a circle* $|z| = 1.5$

Solution: The simple pole $z = 1$ is lies within the circle $|z| = 1.5$.

R = Residue of f(z) at the pole z = 1 is

$= Re(z=1) = \lim_{z\to 1}(z-1)\frac{\cos\pi z}{z-1} = -1.$

By Cauchy's residue theorem

$\oint_C \frac{\cos\pi z}{z-1} = \oint_C f(z)dz = 2\pi i(R) = 2\pi i(-1) = -2\pi i$

Example 5.38. *Evaluate* $\oint_C \frac{z^2}{(z-1)^2(z+1)}$ *where C is* *(i)* $|z| = \frac{1}{2}$, *(ii)* $|z| = 2$.

Solution: Let $(z-1)^2(z+1) = 0$ implies that $f(z)$ has simple pole at $z = -1$ and a pole of order 2 at $z = 1$.

(i) If C is $|z| = \frac{1}{2}$, both poles lie outside C and hence by Cauchy's integral theorem

$$\oint_C f(z)dz = \oint_C \frac{z^2}{(z-1)^2(z+1)}dz = 0.$$

(ii) If C is $|z| = 2$, both poles lie ~~outside~~ inside C.

$$R_1 = \lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1}(z+1)\frac{z^2}{(z-1)^2(z+1)} = \frac{1}{4}$$

$$R_2 = \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{z^2}{(z-1)^2(z+1)}\right]$$

$$= \lim_{z\to 1}\left[\frac{(z+1)2z-z^2}{(z+1)^2}\right] = \frac{3}{2}$$

By Cauchy's residue theorem

$$\oint_C \frac{z^2}{(z-1)^2(z+1)} = \oint_C f(z)dz = 2\pi i(R_1+R_2)$$

$$= 2\pi i\left(\frac{1}{4}+\frac{3}{2}\right) = \frac{7\pi i}{2}$$

Example 5.39. *Evaluate* $\oint_C \frac{e^{2z}}{\cos \pi z}dz$ *where* C *is a circle* $|z|=1$.

Solution: Let f(z) has simple pole when $\cos \pi z = 0 \Rightarrow z = \pm\frac{1}{2}, \pm\frac{3}{2}, \cdots$ of which $z = \pm\frac{1}{2}$ lie inside the circle $|z|=1$.

$$R_1 = Re\left(z=\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{e^{2z}}{\cos \pi z} = \frac{0}{0} \text{ form.}$$

Using L'hospital rule, we have

$$R_1 = \lim_{z\to\frac{1}{2}} \frac{\left(z-\frac{1}{2}\right)2e^{2z}+e^{2z}}{-\pi \sin \pi z} = -\frac{e}{\pi}$$

Similarly, $R_2 = Re\left(z=-\frac{1}{2}\right) = \frac{e^{-1}}{\pi}$

By Cauchy's residue theorem,

$$\oint_C \frac{e^{2z}}{\cos \pi z}dz = \oint_C f(z)dz = 2\pi i(R_1+R_2) = 2\pi i\left(-\frac{e}{\pi}+\frac{e^{-1}}{\pi}\right)$$

$= -4i \sinh 1$

Example 5.40. *Evaluate* $\oint_C \tan z dz$ *where* C *is a circle* $|z|=2$.

Solution: The poles are given by $\cos z = 0 \Rightarrow z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots$ of which $z = \pm\frac{\pi}{2}$ lie within $|z|=2$.

$$R_1 = Re\left(z=\frac{\pi}{2}\right) = \lim_{z\to\frac{\pi}{2}} \frac{\left(z-\frac{\pi}{2}\right)\sin z}{\cos z}, \left(\frac{0}{0}\text{ form}\right)$$

$$\lim_{z\to\frac{\pi}{2}} \frac{\left(z-\frac{\pi}{2}\right)\cos z+\sin z}{-\sin z} = -1 \text{ (Using L'Hospital rule)}$$

$$R_2 = Re\left(z=-\frac{\pi}{2}\right) = \lim_{z\to-\frac{\pi}{2}} \frac{\left(z+\frac{\pi}{2}\right)\sin z}{\cos z}$$

$$= \lim_{z\to-\frac{\pi}{2}} \frac{\left(z+\frac{\pi}{2}\right)\cos z+\sin z}{-\sin z} = -1$$

By Cauchy's residue theorem,

$$\oint_C \tan z dz = \oint_C f(z)dz = 2\pi i(R_1+R_2) = 2\pi i(-1-1) = -4\pi i$$

Example 5.41. *Evaluate* $\oint_C \frac{z-1}{z^2+2z+5}$ *where* C *is the circle* *(i)* $|z|=1$, *(ii)* $|z+1+i|=2$ *and (iii)* $|z+1-i|=2$

Solution: The poles are given by $z^2 + 2z + 5 = 0$

$\Rightarrow (z+1+2i)(z+1-2i) = 0 \Rightarrow z = -1-2i, z = -1+2i$ are the simple poles.

(i) Since both poles lie outside the circle $|z| = 1$, by cauchy's integral theorem, we have

$$\oint_C \frac{z-1}{z^2+2z+5} = \oint_C f(z)dz = 0$$

(ii) The centre of the circle $|z+1+i| = 2$ is $(-1,-1)$ and radius is 2. The simple pole $-1-2i$ lies outside the circle.

$$R = Re(z=-1-2i) = \lim_{z\to -1-2i}(z+1+2i)\frac{z-1}{(z+1+2i)(z+1-2i)}$$

$$= \frac{1+i}{2i}$$

By Cauchy's residue theorem,

$$\oint_C \frac{z-1}{z^2+2z+5} = \oint_C f(z)dz = 2\pi i(R) = 2\pi i\left(\frac{1+i}{2i}\right) = \pi(1+i)$$

(iii) The centre of the circle $|z+1-i| = 2$ is $(-1,-1)$ and radius is 2. The simple pole $-1+2i$ lies within the circle $|z+1-i| = 2$.

$$R = Re(z=-1+2i) = \lim_{z\to -1+2i}(z+1-2i)\frac{z-1}{(z+1+2i)(z+1-2i)}$$

$$= \frac{-1+i}{2i}$$

By Cauchy's residue theorem,

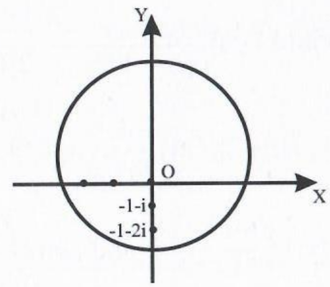

Fig. 9

$$\oint_C \frac{z-1}{z^2+2z+5} = \oint_C f(z)dz = 2\pi i(R)$$

$$= 2\pi i\left(\frac{-1+i}{2i}\right) = \pi(-1+i)$$

EXERCISE

1. Determine the poles of (i) $\dfrac{z(z-1)}{z(z-2)}$ (ii) $\dfrac{z+3}{z(z-1)(z+2)}$ (iii) $\dfrac{z}{\cos z}$ (iv) $\dfrac{1}{(z^2+a^2)^2}$ and (v) $\dfrac{1-e^{2z}}{z^4}$.

   [Ans: (i) 0, 2 (ii) 0, 1, -2 (iii) $(2n+1)\frac{\pi}{2}$ (iv) ai, -ia and (v) 0 of order 4]

2. Find the residues of (i) $\dfrac{e^z}{z^2+a^2}$ at $z = ai$ (ii) $\dfrac{e^z}{(z-1)^3}$

   (iii) $\dfrac{1}{(z^2+a^2)^2}$ at $z = ai$ (iv) $\dfrac{z^2}{(z-1)^2(z+2)}$ (v) $\dfrac{z+2}{(z+1)^2(z-2)}$

(vi) $\dfrac{1}{(z^2+1)^2}$ and (vii) $\dfrac{z^3}{(z+1)^2(z-2)(z+2)}$

$\Big[$Ans: (i)$2aie^{ai}$, (ii)$e/2$, (iii)$\dfrac{-i}{4a^3}$, (iv)$5/9, 4/9$,

(v)$4/9, -4/9$, (vi) $\left(\dfrac{-i}{4}, \dfrac{-i}{4}\right)$ and (vii) $\left(\dfrac{9}{4}, -8, \dfrac{27}{4}\right)\Big]$

3. Find the singular point of $f(z) = \dfrac{1}{z(e^z - 1}$

[Ans: $z = 0, \pm 2\pi, \pm 4\pi, \cdots$]

4. Find the residues of $f(z) = \dfrac{1-e^{2z}}{z^3}$ [Ans: -2]

5. Find the residue at $z = 0$ of $\dfrac{1+e^z}{z\cos z + \sin z}$ [Ans: 1]

6. Find the residue at $z = 0$ of the functions (i) $f(z) = e^{1/z}$ (ii) $\dfrac{\sin z}{z^4}$ and (iii) $z \cos \dfrac{1}{z}$

$\left[\text{Ans: (i)}1 \text{ (ii)} -\dfrac{1}{6} \text{ (iii)}\dfrac{-1}{2}\right]$

7. Find the residues of $f(z) = \dfrac{\sin z}{z \cos z}$ or $\dfrac{\tan z}{z}$ at each of its poles inside the circle $|z| = 2$

$\left[\text{Ans:}\dfrac{2}{11}\right]$

8. Evaluate $\oint_C \dfrac{z \sec z}{1-z^2}$ where C is the ellipse $4x^2 + 9y^2 = 9$

[Ans: $-2\pi i \sec 1$]

9. Using Cauchy's residue theorem evaluate the following:

(i) $\oint_C \dfrac{z}{(z-1)^2(z+1)}$ where C is a circle $|z| = \dfrac{3}{2}$ [Ans: 0]

(ii) $\oint_C \dfrac{z}{(z-1)(z-3)}$ over a circle $|z| = 3$ and $|z| = \dfrac{3}{2}$ [Ans: $2\pi i, -\pi i$]

(iii) $\oint_C \dfrac{e^{-2z}}{(z+1)^3}$ where C is a circle $|z| = 2$ [Ans: $4\pi i$]

(iv) $\oint_C \dfrac{z-2}{z(z-1)}$ where C is a circle $|z| = 3$ [Ans: $2\pi i$]

(v) $\oint_C \dfrac{7z-1}{z^2-3z-4}$ where C is an ellipse $x^2 + 4y^2 = 4$ $\left[\text{Ans:}\dfrac{16}{3}\pi i\right]$

(vi) $\oint_C \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$ where is a circle $|z| = 3$

[Ans: $4\pi i, 2\pi i$]

(vii) $\oint_C \dfrac{e^{-z^2}}{\cos \pi z}$ where C is a circle $|z| = 1$

[Ans: $2i(-e^{1/4} + e^{-1/4})$]

Application of Residues to evaluate real integrals

Integration around the unit circle of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$ where $f(\cos\theta, \sin\theta)d\theta$ is a rational function $\sin\theta$ and $\cos\theta$.

Here $z = e^{i\theta}$ [recall $z = re^{i\theta}$ where $r = 1$ for unit circle] so that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} \text{ and } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$$

Example 5.42. *Evaluate* $\int_0^{2\pi} \frac{d\theta}{5 + 3\cos\theta}$

Solution: Let $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$ and

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} = \frac{z^2 + 1}{2z}$$

$$\therefore I = \int_0^{2\pi} \frac{d\theta}{5 + 3\cos\theta} = \oint_C \frac{1}{5 + \frac{3}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$

$= \frac{2}{i} \oint_C \frac{dz}{3z^2 + 10z + 3} = \frac{2}{i} \oint_C f(z)dz = \frac{2}{i}$ [$2\pi i$ (sum of the residues of $f(z)$)]

$= 4\pi(R_1 + R_2 + \cdots)$ where $f(z) = \frac{1}{3z^2 + 10z + 3}$ and C is a unit circle $|z| = 1$.

Poles of $f(z)$ are $z = -1/3, 3$ out of which $z = -1/3$ lies inside C.

Hence $R = Re\left(z = -\frac{1}{3}\right) = \lim_{z \to -\frac{1}{3}} \left(z + \frac{1}{3}\right) \frac{1}{(3z + 1)(z + 3)}$

$$= \lim_{z \to -\frac{1}{3}} \frac{1}{3(z + 3)} = \frac{1}{8}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5 + 3\cos\theta} = 4\pi\left(\frac{1}{8}\right) = \frac{\pi}{2}$$

Example 5.43. *Show that* $\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1 - a^2}}, (a^2 < 1)$

Solution: Let $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$ and

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} = \frac{z^2 + 1}{2z}$$

$$\therefore I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \oint_C \frac{1}{1 + \frac{a}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$

$= \frac{2}{i} \oint_C \frac{dz}{az^2 + 2z + a} = \frac{2}{i} \oint_C f(z)dz$, where $f(z) = \frac{1}{az^2 + 2z + a}$.

Poles of $f(z)$ are given by $az^2 + 2z + a = 0 \Rightarrow z = \frac{-1 \pm \sqrt{1 - a^2}}{a}$

Let $\alpha = \frac{-1 + \sqrt{1 - a^2}}{a}, \beta = \frac{-1 + \sqrt{1 - a^2}}{a}$ it is clear that

$$\beta = -\frac{1}{a} - \sqrt{\frac{1}{a^2} - 1} > 1$$

Since $a < 1 \Rightarrow 1/a > 1$ and hence $|\beta| > 1$. Now $\alpha\beta = 1$

$\therefore |\alpha| < 1$.

$z = \alpha$ is a simple pole inside C.

$R = Re(z = \alpha) = \lim_{z\to\alpha}(z-\alpha)\dfrac{1}{a(z-\alpha)(z-\beta)}$

$= \dfrac{1}{\alpha(\alpha-\beta)} = \dfrac{1}{2\sqrt{1-a^2}}$

Hence $I = \int_0^{2\pi} \dfrac{d\theta}{1 + a\cos\theta} = \dfrac{2}{i}\oint_C f(z)dz$

$= \dfrac{2}{i}$ [$2\pi i$ (sum of the residues of f(z)]

$= 4\pi(R_1 + R_2 + \cdots) = 4\pi\left(\dfrac{1}{2\sqrt{1-a^2}}\right) = \dfrac{2\pi}{\sqrt{1-a^2}}$

Example 5.44. *Evaluate* $\int_0^{2\pi} \dfrac{d\theta}{13 + 5\sin\theta}$

Solution: Let $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \Rightarrow d\theta = \dfrac{dz}{iz}$ and

$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2} = \dfrac{z - z^{-1}}{2i} = \dfrac{z^2-1}{2iz}$

$\therefore I = \int_0^{2\pi} \dfrac{d\theta}{13 + 5\sin\theta} = \oint_C \dfrac{1}{13 + 5\left(\dfrac{z^2-1}{2iz}\right)}\dfrac{dz}{iz}$

$= \dfrac{2}{5}\oint_C \dfrac{dz}{z^2 + \dfrac{26}{5}iz - 1} = \dfrac{2}{5}\oint_C f(z)dz$, where $f(z) = \dfrac{1}{z^2 + \dfrac{26}{5}iz - 1}$

Poles of $f(z)$ are given by $z^2 + \dfrac{26}{5}iz - 1 = 0 \Rightarrow z = -\dfrac{i}{5}, -5i$ out of this $z = -\dfrac{i}{5}$ lies within $|z| = 1$.

$R = Re\left(z = -\dfrac{i}{5}\right) = \lim_{z\to -\frac{i}{5}}\left(z + \dfrac{i}{5}\right)\dfrac{1}{(z+5i)\left(z + \dfrac{i}{5}\right)} = -\dfrac{5i}{24}$

Hence $I = \int_0^{2\pi} \dfrac{d\theta}{13 + 5\sin\theta} = \dfrac{2}{5}\oint_C f(z)dz$

$= \dfrac{2}{5}$ [$2\pi i$ (sum of the residues of f(z)]

$= \dfrac{4\pi i}{5}(R_1 + R_2 + \cdots) = \left(\dfrac{4\pi i}{5}\right)\left(-\dfrac{5i}{24}\right) = \dfrac{\pi}{6}$

Example 5.45. *Evaluate* $\int_0^{2\pi} \dfrac{d\theta}{2 + \cos\theta}$

Solution: Let $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \Rightarrow d\theta = \dfrac{dz}{iz}$ and

$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{z + z^{-1}}{2} = \dfrac{z^2+1}{2z}$

$\therefore I = \int_0^{2\pi} \dfrac{d\theta}{2 + \cos\theta} = \oint_C \dfrac{dz}{iz\left(2 + \dfrac{z^2+1}{2z}\right)}$

$$= -2i \oint_C \frac{dz}{z^2+4z+1} = -2i \oint_C f(z)dz, \text{ where } f(z) = \frac{1}{z^2+4z+1}$$

Poles of $f(z)$ are given by $z^2+4z+1 = 0 \Rightarrow z = -2+\sqrt{3}, -2-\sqrt{3}$ of which $z = -2+\sqrt{3}$ lies within $|z| = 1$.

$$R = Re(z = -2+\sqrt{3}) = \lim_{z \to -2+\sqrt{3}} \frac{z+2-\sqrt{3}}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$$

$$= \frac{1}{2\sqrt{3}}$$

Hence $I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = -2i \oint_C f(z)dz$

$= 2i$ [$2\pi i$ (sum of the residues of f(z))]

$$= 4\pi(R_1 + R_2 + \cdots) = 4\pi\left(\frac{1}{2\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}$$

Example 5.46. *Evaluate* $\int_0^{2\pi} \frac{\sin^2\theta d\theta}{5+4\cos\theta}$

Solution: Let $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$ and

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2} = \frac{z^2+1}{2z}$$

$$\therefore I = \int_0^{2\pi} \frac{\sin^2\theta d\theta}{5+4\cos\theta} = \int_0^{2\pi} \frac{(1-\cos 2\theta)d\theta}{2(5+4\cos\theta)}$$

$$= R.P \frac{1}{2} \oint_C \frac{1-z^2}{5+\frac{4}{2}\left(\frac{z^2+1}{z}\right)} dz$$

$$I = \text{R.P} \frac{1}{2i} \int_C \frac{1-z^2}{2z^2+5z+2} dz \tag{5.18}$$

Poles are given by $2z^2+5z+2 = 0 \Rightarrow z = -2, -1/2$, out of these $z = -1/2$ lies within $|z| = 1$.

$$R = Re(z = -1/2) = \lim_{z \to 1/2} \left(z + \frac{1}{2}\right) \frac{1-z^2}{2\left(z^2 + \frac{5}{2}z + 1\right)}$$

$$= \lim_{z \to -1/2} \left(z + \frac{1}{2}\right) \frac{1-z^2}{\frac{1}{2}\left(z+\frac{1}{2}\right)(z+2)} = \frac{1}{4}$$

Hence equation (5.18) becomes

$$I = \int_0^{2\pi} \frac{\sin^2\theta d\theta}{5+4\cos\theta} = \text{R.P} \frac{1}{2i} \oint_C \frac{1-z^2}{2z^2+5z+2} dz$$

$= \text{R.P} \frac{1}{2i}$ [$2\pi i$ (sum of the residues of f(z))]

$$= \text{R.P}\, \pi(R_1 + R_2 + \cdots) = \text{R.P}\, \pi\left(\frac{1}{4}\right) = \frac{\pi}{4}$$

Example 5.47. *Evaluate* $\int_0^{2\pi} \frac{\cos 2\theta d\theta}{5-4\cos\theta}$

Solution: Let $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$ and

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z} \text{ and}$$

$z^2 = (e^{i\theta})^2 = \cos 2\theta + i\sin 2\theta$, R.P of $e^{2i\theta} = \cos 2\theta$.

Now $I = \int_0^{2\pi} \frac{\cos 2\theta d\theta}{5-4\cos\theta} = \text{R.P} \oint_C \frac{z^2}{5-4\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$

$$I = \text{R.P} \frac{-1}{i} \int_C \frac{z^2}{2z^2-5z+2} dz \qquad (5.19)$$

Poles are given by $2z^2-5z+2=0 \Rightarrow z=2, 1/2$ but only $z=1/2$ lies within $|z|=1$.

$R = Re(z=1/2) = \lim_{z\to 1/2}\left(z-\frac{1}{2}\right)\frac{z^2}{(z-2)(2z-1)} = -\frac{1}{12}$

Hence equation (5.19) becomes

$I = \int_0^{2\pi} \frac{\cos 2\theta d\theta}{5-4\cos\theta} = \text{R.P} \frac{-1}{i} \oint_C \frac{z^2}{2z^2-5z+2} dz$

$= \text{R.P}\left(\frac{-1}{i}\right)(2\pi i)(R_1+R_2+\cdots)$

$= \text{R.P}\left(\frac{-1}{i}\right)(2\pi i)\left(\frac{-1}{12}\right) = \frac{\pi}{6}$

Example 5.48. *Evaluate* $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}, a^2<1$

Solution: Let $z=e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$ and

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2} = \frac{z^2+1}{2z}$$

Now $I = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2} = \oint_C \frac{(dz/iz)}{1-2a\left(\frac{z^2+1}{2z}\right)+a^2}$

$= \oint_C \frac{(dz/iz)2z}{2z-2a(z^2+1)+a^2.2z} = \oint_C \frac{\frac{2dz}{i}}{2(z-az^2-2a+a^2z)}$

$= \oint_C \frac{-idz}{-az^2+a^2z+z-a} = \oint_C \frac{-idz}{az(-z+a)+(z-a)}$

$$I = -i\oint_C \frac{dz}{(z-a)(1-az)} \qquad (5.20)$$

Poles are given by $(z-a)(1-az)=0 \Rightarrow z=a, 1/a$. Given $a^2<1 \Rightarrow a<1$ and hence the pole $z=1/a$ lies outside the unit circle C and only $z=a$ lis within the unit circle.

$R = Re(z=a) = \lim_{z\to a}(z-a)\frac{1}{(z-a)(1-az)} = \frac{1}{1-a^2}$

Hence equation (5.20) becomes

$I = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2} = -i\oint_C \frac{dz}{(z-a)(1-az)}$

$= -i(2\pi i)\left(\frac{1}{1-a^2}\right) = \frac{2\pi}{1-a^2}$

Example 5.49. *Evaluate* $\int_0^{\pi} \frac{ad\theta}{a^2+\cos^2\theta}$

Solution: Put $2\theta = \varphi$ so that $2d\theta = d\varphi$. Also when $\theta = 0, \varphi = 0$ and when $\theta = \pi, \varphi = 2\pi$.

Now $I = \int_0^\pi \frac{ad\theta}{a^2 + \cos^2\theta} = \int_0^\pi \frac{ad\theta}{a^2 + \left(\frac{1+\cos 2\theta}{2}\right)}$

$= \int_0^\pi \frac{2ad\theta}{2a^2 + 1 + \cos 2\theta} = \int_0^{2\pi} \frac{2a\frac{d\varphi}{2}}{2a^2 + 1 + \cos\phi}$

$$I = a\int_0^{2\pi} \frac{d\varphi}{2a^2 + 1 + \cos\phi} \tag{5.21}$$

Let $z = e^{i\theta} dz = ie^{i\theta} d\varphi \Rightarrow d\varphi = \frac{dz}{iz}$ and

$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2 + 1}{2z}$

Now equation (5.21) becomes

$I = \int_0^\pi \frac{ad\theta}{a^2 + \cos^2\theta} = a\int_0^{2\pi} \frac{d\varphi}{2a^2 + 1 + \cos\phi}$

$= a\oint_C \frac{(dz/iz)}{2a^2 + 1 + \left(\frac{z^2+1}{2z}\right)} = a\oint_C \frac{(dz/iz)2z}{4a^2z + 2z + z^2 + 1}$

$$I = -2ai\oint_C \frac{dz}{z^2 + 2z(2a^2+1) + 1} \tag{5.22}$$

Poles are given by $z = \frac{-2(2a^2+1) \pm \sqrt{4(2a^2+1)^2 - 4}}{2}$

$= \frac{-2(2a^2+1) \pm 2\sqrt{4a^4 + 4a^2 + 1 - 1}}{2} = -(2a^2+1) \pm 2a\sqrt{a^2+1}$

Let the roots be $\alpha = -(2a^2+1) + 2a\sqrt{a^2+1}$ and

$\beta = -(2a^2+1) - 2a\sqrt{a^2+1}$

Since $|\beta| > 1$ and $|\alpha||\beta| = 1 \quad \therefore |\alpha| < 1$.

Then $R = Re(z = \alpha) = \lim_{z\to\alpha}(z-\alpha)\frac{1}{(z-\alpha)(z-\beta)} = \frac{1}{\alpha - \beta}$

Now equation (5.22) becomes

$I = \int_0^\pi \frac{ad\theta}{a^2 + \cos^2\theta} = -2ai\oint_C \frac{dz}{z^2 + 2z(2a^2+1) + 1}$

$= -2ai(2\pi i)(R) = -2ai(2\pi i)\left(\frac{1}{\alpha - \beta}\right) = \frac{\pi}{\sqrt{a^2+1}}$

EXERCISE

1. Evaluate $\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta}$ $\left[\text{Ans:}\frac{\pi}{6}\right]$

2. Evaluate $\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$ $\left[\text{Ans:}\frac{2\pi}{3}\right]$

3. Evaluate $\int_0^{2\pi} \frac{\cos 2\theta d\theta}{5 - 4\cos\theta}$ $\left[\text{Ans:}\frac{\pi}{6}\right]$

4. Evaluate $\int_0^{2\pi} \frac{\sin^2\theta d\theta}{5 + 4\cos\theta}$ $\left[\text{Ans:}\frac{\pi}{4}\right]$

5. Evaluate $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta}, a > |b|$ $\left[\text{Ans:} \frac{\pi}{\sqrt{a^2-b^2}}\right]$

6. Evaluate $\int_0^{\pi} \frac{d\theta}{a^2+\sin^2\theta}, a > 0$ $\left[\text{Ans:} \frac{\pi}{a\sqrt{a^2-1}}\right]$

7. Show that $\int_{-\pi}^{\pi} \frac{a\cos\theta}{a+\cos\theta} d\theta = 2a\pi\left[1-\frac{a}{\sqrt{a^2-1}}\right]$

8. Show that $\int_0^{\pi} \frac{(1+2\cos\theta)d\theta}{5+4\cos\theta} = 0$

9. Evaluate $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}, a > |b| > 0,$ $\left[\text{Ans:} \frac{\pi}{(a^2-b^2)^{3/2}}\right]$

Type II: Improper integrals of rational functions

Evaluation of $\int_{-\infty}^{\infty} f(x)dx$

If the function $f(z)$ is such that it has no poles on the real axis and may have some poles in the upper half of the z - plane and if $xf(x) \to 0$ as $x \to \infty$ then we can evaluate $\int_{-\infty}^{\infty} f(x)dx$ by using Cauchy's residue formula.

Method: Consider $I = \int_C f(z)dz$

where C is a curve, consisting of the upper half C_1 of the circle $|z| = R$ and part of the real axis from -R to R.

If there are no poles of $f(z)$ on the real axis, then the circle $|z| = R$ which is arbitrary can be taken such that there is no singularity of $f(z)$ on the circumference C_1 of the upper half of

the plane, but there may be poles of $f(z)$ inside the contour C.

Now, using Cauchy's theorem of residues, we have,

$\int_C f(z)dz = 2\pi i\times$ (sum of residues of f(z) at the poles within C).

(i.e.) $\int_{-R}^{R} f(x)dx + \int_{C_1} f(z)dz = 2\pi i$ (sum of residues)

(Note that along -R to R, f(z) is real and valued function f(x)).

$\therefore \int_{-R}^{R} f(x)dx = -\int_{C_1} f(z)dz + 2\pi i$ (sum of residues within C).

Taking limit as $R \to \infty$ on both sides, we get,

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx = -\lim_{R\to\infty} \int_{C_1} f(z)dz + 2\pi i \,(\text{sum of residues}) \tag{5.23}$$

Let $z = re^{i\theta}$

Consider $\lim_{R\to\infty} \int_{C_1} f(z)dz = \lim_{R\to\infty} \int_0^{2\pi} f(re^{i\theta}) rie^{i\theta} d\theta$

$= \lim_{R\to\infty} \int_{C_1} f(re^{i\theta})e^{i\theta} d\theta \quad (R \to \infty, r \to \infty)$

$= 0 \quad (\because xf(x) \to 0 \text{ as } x \to 0)$

Hence from (5.23)

$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx = 2\pi i$ (sum of residues of f(z) within C)

(i.e.) $\int_{-\infty}^{\infty} f(x)dx = 2\pi i$ (sum of residues of f(z) within C)

$\therefore \int_0^\infty f(x)dx = \frac{1}{2}\int_{-\infty}^\infty f(x)dx$

Cauchy's Lemma:

If f(z) is continuous function such that $|zf(z)| \to 0$ as $|z| \to \infty$ on the upper semi-circle C_1. $|z| = R$ then $\int_{C_1} f(z)dz \to 0$ as $R \to \infty$

Type I:

Integrals of the form $\int_{-\infty}^\infty \frac{P(x)}{Q(x)}dx$ where P(x) and Q(x) are polynomial in x such that the degree of Q(x) is at least two more than the degree of P(x) and Q(x) does not vanish for any real x.

Example 5.50. *Evaluate* $\int_{-\infty}^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$, $a > 0, b > 0$

Solution: The integrand is of the form $\frac{P(x)}{Q(x)}$ where Q(x) is two more than degree of P(x) and Q(x) does not vanish. Let $f(x) = \frac{x^2}{(x^2+a^2)(x^2+b^2)}$ (f(x) has no poles on real axis).

Hence $f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$

The poles of f(z) are $z = \pm ia, z = \pm ib$. Out of these $z = ia$ and $z = ib$ lie in the upper half of z - plane.

Let us find residues of f(z) at $z = ia$ and $z = ib$.

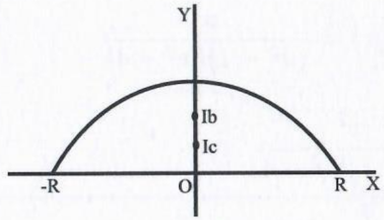

Fig. 10

Residue of f(z) at $z = ia$ is

$\lim_{z\to ia}(z-ia)f(z) = \lim_{z\to ia}\frac{(z-ia)z^2}{(z^2+a^2)(z^2+b^2)}$

$= \frac{-a^2}{(2ia)(b^2-a^2)} = \frac{a}{2i(b^2-a^2)}$ and residue of f(z) at $z = ib$ is

$= \lim_{z\to ib}(z-ib)\frac{z^2}{(z^2+a^2)(z^2+b^2)} = \lim_{z\to ib}\frac{z^2(z-ib)}{(z^2+a^2)(z+ib)}$

$= \frac{-b^2}{(2ib)(a^2-b^2)} = \frac{-b}{2i(a^2-b^2)}$

$\therefore \int_{-x}^{x} f(x)dx = 2\pi i$ (sum of residues of f(z) in the upper half of z - plane)

$= 2\pi i\left[\frac{a}{2i(b^2-a^2)} - \frac{b}{2i(a^2-b^2)}\right] = \pi\left[\frac{a-b}{a^2-b^2}\right] = \frac{\pi}{a+b}$

Example 5.51. *Evaluate* $\int_{-\infty}^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx$ *using contour integration.*

Solution: Let $f(x) = \dfrac{x^2}{(x^2+1)(x^2+4)}$

Let $f(z) = \dfrac{z^2}{(z^2+1)(z^2+4)}$

The poles of f(z) are given by $(z^2+1)(z^2+4) = 0$

$\Rightarrow z = \pm i, \pm 2i$ which are simple poles.

But only $z = i, 2i$ lie in the upper half of the z - plane.

Let us find the residue of f(z) at $z = i$ $z = 2i$.

$$Re(z = i) = \lim_{z\to i}(z-i)\frac{z^2}{(z+i)(z-i)(z^2+4)} = -\frac{1}{6i}$$

$$Re(z = 2i) = \lim_{z\to 2i}(z-2i)\frac{z^2}{(z+2i)(z-2i)(z^2+4)} = \frac{1}{3i}$$

$\therefore \int_{-x}^{x} f(x)dx = 2\pi i$ (sum of residues of f(z) in the upper half of z - plane)

$$= 2\pi i\left[-\frac{1}{6i} + \frac{1}{3i}\right] = \frac{\pi}{3}$$

Example 5.52. *Evaluate* $\int_0^\infty \dfrac{dx}{x^4+1}$

Solution: Let $f(x) = \dfrac{1}{x^4+1}$ (Note that here limits are 0 to ∞)

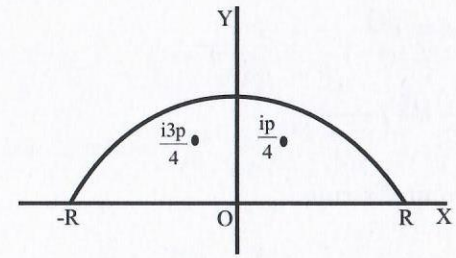

Fig. 11

Let $f(z) = \dfrac{1}{z^4+1}$. We find poles of f(z).

(i.e.) $z^4 = -1 = e^{i\pi} = e^{i(2n\pi+\pi)}$

$\therefore z = e^{i(2n+1)\frac{\pi}{4}}$ where $n = 0, 1, 2, 3.$

When (i) $n = 0, z = e^{i\frac{\pi}{4}} = \dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}$

(ii) $n = 1, z = e^{i\frac{3\pi}{4}} = \dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}$

(iii) $n = 2, z = e^{i\frac{5\pi}{4}} = \cos\dfrac{5\pi}{4} + i\sin\dfrac{5\pi}{4} = -\dfrac{1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}$

(iv) $n = 3, z = e^{i\frac{7\pi}{4}} = \dfrac{1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}$

Of these only $z = e^{i\frac{\pi}{4}}$ and $z = e^{i\frac{3\pi}{4}}$ lie above the upper half of z - plane.

We find residues of f(z) at $z = e^{i\frac{\pi}{4}}$ and $z = e^{i\frac{3\pi}{4}}$

(i) Residue at $z = e^{i\frac{\pi}{4}}$

$$= \lim_{z \to e^{i\frac{\pi}{4}}} (z - e^{i\frac{\pi}{4}}) \frac{1}{z^4+1}$$

Applying L'Hospital's rule

$$= \lim_{z \to e^{i\frac{\pi}{4}}} (z - e^{i\frac{\pi}{4}}) \frac{1}{4z^3} = \frac{1}{4e^{i\frac{3\pi}{4}}} = \frac{1}{4} e^{-i\frac{3\pi}{4}}$$

(ii) Residue of $z = e^{i\frac{3\pi}{4}}$

$$= \lim_{z \to e^{i\frac{3\pi}{4}}} (z - e^{i\frac{3\pi}{4}}) \frac{1}{z^4+1}$$

Applying L'Hospital's rule

$$= \lim_{z \to e^{i\frac{3\pi}{4}}} (z - e^{i\frac{3\pi}{4}}) \frac{1}{4z^3} = \frac{1}{4e^{i\frac{9\pi}{4}}} = \frac{1}{4} e^{-i\frac{9\pi}{4}}$$

Now $\int_0^\infty \frac{dx}{x^4+1} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{x^4+1}$ (Recall $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$, if f(x) is even)

$$= \frac{1}{2} \times 2\pi i \text{ (sum of residues)}$$

$$= \pi i \left[\frac{1}{4} e^{-i\frac{3\pi}{4}} + \frac{1}{4} e^{-i\frac{9\pi}{4}} \right]$$

$$= \frac{\pi i}{4} \left[\cos\frac{3\pi i}{4} - i \sin\frac{3\pi i}{4} + \cos\frac{9\pi i}{4} - i \sin\frac{9\pi i}{4} \right]$$

$$= \frac{\pi i}{4} \left[-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} \right] = \frac{\pi i}{4}(-i\sqrt{2}) = \frac{\pi\sqrt{2}}{4}$$

Example 5.53. *Evaluate* $\int_0^\infty \frac{dx}{(x^2+1)^3}$

Solution: Let $I = \int_0^\infty \frac{dx}{(x^2+1)^3}$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$$

$$f(x) = \frac{dx}{(x^2+1)^3}$$

Let $f(z) = \frac{dz}{(z^2+1)^3}$ and consider $\int_C \frac{dz}{(z^2+1)^3}$

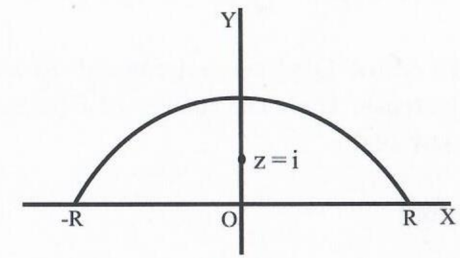

Fig. 12

Here f(z) has poles at $z = \pm i$ of order 3 each. Out of these $z = i$, a multiple pole of order 3 lies above x-axis.

We find residue at $z = i$

$$= \frac{1}{2!} \lim_{z\to i} \frac{d^2}{dz^2}(z-i)^3 \frac{1}{(z^2+1)^3} = \frac{1}{2} \lim_{z\to i} \frac{d^2}{dz^2}\left[\frac{1}{(z+i)^3}\right]$$

$$= \frac{1}{2} \lim_{z\to i} \frac{d}{dz}[-3(z+i)^{-4}] = \frac{1}{2} \lim_{z\to i}[12(z+i)^{-5}]$$

$$= 6(2i)^{-5} = \frac{6}{2^5}.\frac{1}{i^5} = \frac{3}{16}.\frac{1}{i} = \frac{-3i}{16}$$

$$\therefore \int_{-\infty}^{\infty} f(x)dx = 2\pi i \left(\frac{-3i}{16}\right) = \frac{3\pi}{8}$$

$$\therefore \int_0^{\infty} f(x)dx = \int_0^{\infty} \frac{dx}{(x^2+1)^3} = \frac{1}{2}\left(\frac{3\pi}{8}\right) = \frac{3\pi}{16}$$

Type II:

Integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)\sin mx}{Q(x)}dx$ and $\int_{-\infty}^{\infty} \frac{P(x)\cos mx}{Q(x)}dx$,

$m > 0$, where P(x) and Q(x) are polynomials in x such that the degree of Q(x) greater than the degree of P(x) and Q(x) does not vanish for any real x.

Jacobi's Lemma:

If f(x) is a continuous function such that $|f(z)| \to 0$ uniformly as $|z| \to 0$ then $\int_{C_1} f(z)e^{imz}dz \to 0$ as $R \to \infty$ where C_1 is the upper semi-circular $|z| = R$.

Example 5.54. *Evaluate* $\int_0^{\infty} \frac{\cos ax}{(x^2+1)}dx$

Solution: Let $I = \int_0^{\infty} \frac{\cos ax}{x^2+1}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}dx$

Consider $\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}dx = \text{R.P.} \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}dx$

Let $f(x) = \frac{e^{iax}}{x^2+1}$. We write $f(z) = \frac{e^{iaz}}{z^2+1}$

This has poles at $z = \pm i$, out of these $z = i$ lies in the upper half of z - plane.

Residue of f(z) at $z = i$ is

$$\lim_{z\to i}(z-i)\frac{e^{iaz}}{z^2+1} = \lim_{z\to i}(z-i)\frac{e^{iaz}}{(z-i)(z+i)} = \frac{e^{-a}}{2i}$$

$$\therefore \int_C \frac{e^{iaz}}{z^2+1}dz = \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}dx = 2\pi i\left(\frac{e^{-a}}{2i}\right) = \pi e^{-a}$$

$$\therefore \int_{-\infty}^{\infty} \frac{(\cos ax + i\sin ax)}{x^2+1}dx = \pi e^{-a}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}dx + \int_{-\infty}^{\infty} \frac{\sin ax}{=}x^2+1dx = \pi e^{-a}$$

equating real part, we get $\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}dx = \pi e^{-a}$

$$\int_0^{\infty} \frac{\cos ax}{(x^2+1)}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}dx = \frac{\pi}{2}e^{-a}$$

Example 5.55. *Evaluate* $\int_0^{\infty} \frac{\cos 2x dx}{(x^2+9)^2(x^2+16)}$

Solution: Let $I = \int_0^\infty \frac{\cos 2x dx}{(x^2+9)^2(x^2+16)}$

$= \frac{1}{2}\int_{-\infty}^\infty \frac{\cos 2x dx}{(x^2+9)^2(x^2+16)}$

$f(x) = \frac{\cos 2x}{(x^2+9)^2(x^2+16)}$

Consider $\int_C f(z)dz$ where $f(z) = \frac{\cos 2z}{(z^2+9)^2(z^2+16)}$.

f(z) has double poles at $z = \pm 3i$ and simple poles at $z = \pm 4i$. Out of these $z = 3i$ and $z = 4i$ lie above real axis.

To find residues

(i) Residue at double pole $z = 3i$.

$= \frac{1}{1!}\frac{d}{dz}[(z-3i)^2 f(z)]_{z=3i}$

$= \frac{d}{dz}\left[(z-3i)^2 \frac{\cos 2z}{(z-3i)^2(z+3i)^2(z^2+16)}\right]_{z=3i}$

$= \frac{d}{dz}\left[\frac{\cos 2z}{(z+3i)^2(z^2+16)}\right]_{z=3i} = -\frac{31ie^{-6}}{108\times 49}$

(ii) Residue at a simple pole $z = 4i$ is given by

$= \lim_{z\to 4i}(z-4i)f(z) = \lim_{z\to 4i}\frac{(z-4i)\cos 2z}{(z^2+9)^2(z^2+16)}$

$= \lim_{z\to 4i}\frac{\cos 2z}{(z^2+9)^2(z+4i)} = -\frac{ie^{-8}}{392}$

By Residue theorem,

$\int_{-\infty}^\infty \frac{\cos 2x dx}{(x^2+9)^2(x^2+16)} = 2\pi i\left[-\frac{31ie^{-6}}{108\times 49} - \frac{ie^{-8}}{392}\right]$

$= \frac{2\pi}{196}\left[\frac{31}{27}e^{-6} + \frac{e^{-8}}{2}\right]$

$\therefore \int_0^\infty \frac{\cos 2x dx}{(x^2+9)^2(x^2+16)} = \frac{\pi}{196}\left[\frac{31}{27}e^{-6} + \frac{e^{-8}}{2}\right]$

EXERCISE

Evaluate the following integrals:

1. $\int_{-\infty}^\infty \frac{dx}{x^6+1}$ $\left[\text{Ans:}\frac{\pi}{3}\right]$
2. $\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ $\left[\text{Ans:}\frac{\pi}{ab(a+b)}\right]$
3. $\int_{-\infty}^\infty \frac{(x^2-x+2)dx}{x^4+10x^2+9}$ $\left[\text{Ans:}\frac{5\pi}{12}\right]$
4. $\int_0^\infty \frac{x\sin x dx}{x^2+a^2}, a>0$ $\left[\text{Ans:}\frac{\pi e^{-a}}{2}\right]$
5. $\int_0^\infty \frac{x^2 dx}{(x^2+9)(x^2+4)^2}$ $\left[\text{Ans:}\frac{\pi}{200}\right]$

6. $\int_{-\infty}^{\infty} \dfrac{\cos x dx}{(x^2+a^2)(x^2+b^2)}, a > b > 0$

$\left[\text{Ans:} \dfrac{\pi}{a^2-b^2}\left(\dfrac{e^{-b}}{b} - \dfrac{e^{-a}}{a}\right)\right]$

7. $\int_{-\infty}^{\infty} \dfrac{\cos 3x dx}{(x^2+1)((x^2+4)}$ $\left[\text{Ans:} \dfrac{\pi}{3}\left(\dfrac{e^{-3}}{1} - \dfrac{e^{-6}}{2}\right)\right]$

8. $\int_{0}^{\infty} \dfrac{x^2 dx}{(x^2+1)^2}$ $\left[\text{Ans:} \dfrac{\pi}{4}\right]$

9. $\int_{0}^{\infty} \dfrac{x \sin mx dx}{x^2+a^2}, m > 0, a > b > 0$ $\left[\text{Ans:} \dfrac{\pi e^{-ma}}{2}\right]$

10. $\int_{0}^{\infty} \dfrac{x \cos mx dx}{(x^2+a^2)(x^2+b^2)}$ $\left[\text{Ans:} \dfrac{\pi(be^{-mb} - ae^{-ma})}{2(b^2-a^2)}\right]$

11. $\int_{0}^{\infty} \dfrac{\cos mx dx}{(1+x^2)^2}, m > 0$ $\left[\text{Ans:} \dfrac{\pi(m+1)e^{-m}}{4}\right]$

12. $\int_{0}^{\infty} \dfrac{\sin x dx}{x^2+4x+5}$ $\left[\text{Ans:} \dfrac{-\pi \sin 2}{e}\right]$

Reg. No. | | | | | | | | | |

B.Tech. Degree EXAMINATION, MAY 2014

(Second Semester)

MA 1002 - Advanced Calculus and Complex Analysis

(For the candidates admitted during the academic year 2013-14)

Time : Three Hours Maximum : 100

Part - A (20 × 1 = 20 Marsk)

Answer **ALL** Questions

1. If $I = \int_0^1 \int_1^X dxdy$ is

   (a) 1 (b) -1 (c) $\dfrac{1}{2}$ (d) $\dfrac{-1}{2}$

2. If R is the region bounded by $x = 0, y = 0, x + y = 1$ then $\int\int_R dxdy$

   (a) 1 (b) $\dfrac{1}{2}$ (c) $\dfrac{1}{3}$ (d) $\dfrac{2}{3}$

3. Area of double integral in Cartesian co-ordinate is equal to

   (a) $\int\int_R dydx$ (b) $\int\int_R drd\theta$
   (c) $\int\int_R xdxdy$ (d) $\int\int_R x^2 dxdy$

4. $\int_0^2 \int_1^2 \int_1^2 xy^2 z dzdydx$ is

   (a) 24 (b) 28 (c) 20 (d) 26

5. The unit normal vector to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$ is

   (a) $\dfrac{-\vec{i}}{3} + \dfrac{2}{3}\vec{j} + \dfrac{2}{3}\vec{k}$ (b) $\dfrac{\vec{i}}{3} + \dfrac{2}{3}\vec{j} + \dfrac{2}{3}\vec{k}$

   (c) $\dfrac{-\vec{i}}{3} - \dfrac{2}{3}\vec{j} + \dfrac{2}{3}\vec{k}$ (d) $\dfrac{\vec{i}}{3} - \dfrac{2}{3}\vec{j} - \dfrac{2}{3}\vec{k}$

6. If $\vec{r}$ is the position vector of the point (x, y, z) with respect to the origin, then $\nabla.\vec{r}$ is

   (a) 2 (b) 3 (c) 0 (d) 1