Let $f(x,y) = c$ be a function or a curve.

$$f(x,y) = c \iff y = g(x)$$

The radius of curvature of the curve $y(x)$ is denoted as ρ,

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\text{(or)} \quad \rho = \frac{\left[1 + y_1^2\right]^{3/2}}{y_2}, \quad y_1 = \frac{dy}{dx}, \quad y_2 = \frac{d^2y}{dx^2}$$

① Find the radius of curvature of the curve $x^4 + y^4 = 2$ at the point $(1,1)$

Soln.

$$\rho = \frac{\left[1 + y_1^2\right]^{3/2}}{y_2}, \quad y_1 = \frac{dy}{dx}, \quad y_2 = \frac{d^2y}{dx^2}$$

$$f(x,y) = x^4 + y^4 - 2 \qquad f(x,y) = 0$$

$$\Rightarrow x^4 + y^4 = 2 \longrightarrow ①$$

diff ① with respect to x

$$4x^3 + 4y^3\frac{dy}{dx} = 0$$

$$4y^3\frac{dy}{dx} = -4x^3$$

$$\frac{dy}{dx} = \frac{-4x^3}{4y^3} = -\frac{x^3}{y^3}$$

$$\left.\frac{dy}{dx}\right|_{(1,1)} = -1.$$

$$\frac{dy}{dx} = -\frac{x^3}{y^3} \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right)$$

$$\frac{dy}{dx} = -\frac{x^3}{y^3} \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d}{dx}\left(\frac{-x^3}{y^3}\right)$$

$$= -\frac{y^3 3x^2 - x^3 3y^2 \frac{dy}{dx}}{y^6}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(1,1)} = -\frac{1\times 3\times 1 - 1\times 3\times(-1)}{1}$$

$$= -6$$

$$\left.\frac{dy}{dx}\right|_{(1,1)} = -1, \quad \left.\frac{d^2y}{dx^2}\right|_{(1,1)} = -6$$

$\therefore$ The radius of curvature at $(1,1)$ is

$$\rho(1,1) = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+(-1)^2)^{3/2}}{-6} = \frac{(2)^{3/2}}{|-6|}$$

$$= \frac{\sqrt{2}}{3}$$

(2) Find the radius of curvature of the curve $xy = 1$ at the point $(1,1)$.

Soln. Given that $xy = 1 \rightarrow (1)$

Diff (1) with respect to $x \Rightarrow \frac{d}{dx}(xy) = \frac{d}{dx}(0)$

$$\Rightarrow x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{dy}{dx} = -y/x$$

$$y_1 = \left.\frac{dy}{dx}\right|_{(1,1)} = -1.$$

$$\frac{dy}{dx} = -y/x$$

$$\frac{x\frac{dy}{dx} - y}{x^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(-y/x) = x \quad \frac{2y}{x^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left[-y/x\right] = x \frac{-y}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{2y}{x^2}$$

$$y_2 = \frac{d^2y}{dx}\Big|_{(1,1)} = 2$$

$$\rho(1,1) = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

(3) Find the radius of curvature of $y = e^x$ at the point $(0,1)$

Soln

$$\frac{dy}{dx} = e^x \qquad y_1 = \frac{dy}{dx}\Big|_{(0,1)} = 1$$

$$\frac{d^2y}{dx^2} = e^x \qquad y_2 = \frac{d^2y}{dx^2}\Big|_{(0,1)} = 1$$

$$\rho(0,1) = \frac{(1+y_1^2)^{3/2}}{y_2} = 2\sqrt{2}$$

———— x ————

(4) Find the radius of curvature of $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point $(a/4, a/4)$

Soln. $\sqrt{x} + \sqrt{y} = \sqrt{a} \longrightarrow (1)$

Diff (1) with respect to x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\sqrt{y/x}$$

$$y_1 = \frac{dy}{dx}\Big|_{(a/4,\, a/4)} = -1$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{\sqrt{y}}{\sqrt{x}}\right) = -\frac{\sqrt{x}\frac{1}{2\sqrt{y}}\frac{dy}{dx} - \sqrt{y}\times\frac{1}{2\sqrt{x}}}{x}$$

$$y_2 = \frac{d^2y}{dx^2}\Big|_{(a/4,\, a/4)} = \frac{4}{a}$$

$$(1+y_1^2)^{3/2}$$

$$\frac{d^2y}{dx^2}\Big|_{(a/4,\,a/4)} = \frac{4}{a}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{4/a} = \frac{a}{\sqrt{2}}$$

<u>Note</u> If either $\frac{dy}{dx} = \infty$ or $\frac{d^2y}{dx^2} = \infty$ at a particular point then the radius of curvature at the point is

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

(3) Find the radius of curvature of $xy^2 = a^3 - x^3$ at the point $(a,0)$

<u>Soln.</u> $xy^2 = a^3 - x^3 \longrightarrow (1)$ $\Rightarrow y^2 = \frac{a^3-x^3}{x} = \frac{a^3}{x} - x^2$

Diff. (1) with respect to x

$$\frac{dy}{dx} = -\frac{(a^3+2x^3)}{2x^2y}$$

$$2y\frac{dy}{dx} = -\frac{a^3}{x^2} - 2x$$

$$\frac{dy}{dx} = -\frac{(a^3+2x^3)}{2x^2y}$$

$$\frac{dx}{dy} = -\frac{2x^2y}{(a^3+2x^3)}$$

at the point $(a,0)$, $\frac{dy}{dx} = \infty$

$$\therefore \quad \rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

$$\frac{dx}{dy}\Big|_{(a,0)} = 0$$

$$\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{-2x^2y}{(a^3+2x^3)}\right)$$

$$= -\frac{(a^3+2x^3)\frac{d}{dy}(2x^2y) - 2x^2y\frac{d}{dy}(a^3+2x^3)}{(a^3+2x^3)^2}$$

$$= -\frac{2(a^3+2x^3)\left(x^2+y\,2x\frac{dx}{dy}\right) - 2x^2y\,6x^2\frac{dx}{dy}}{(a^3+2x^3)^2}$$

$$\frac{d^2x}{dy^2}\Big|_{(a,0)} = \frac{-2}{3a}$$

$$\left.\frac{}{dy^2}\right|_{(a,0)} = 3a$$

$$\therefore \rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} = \frac{3a}{2}.$$

Find the radius of curvature of $\frac{\sqrt{x}}{\sqrt{a}} + \frac{\sqrt{y}}{\sqrt{b}} = 1$ at any point.

Soln.

$$\frac{dy}{dx} = -\sqrt{\frac{by}{ax}} = -\sqrt{\frac{b}{a}}\frac{\sqrt{y}}{\sqrt{x}}$$

$$\frac{d^2y}{dx^2} = -\sqrt{\frac{b}{a}}\frac{d}{dx}\left(\frac{\sqrt{y}}{\sqrt{x}}\right)$$

$$= -\sqrt{\frac{b}{a}}\frac{\sqrt{x}\frac{1}{2\sqrt{y}}\frac{dy}{dx} - \sqrt{y}\frac{1}{2\sqrt{x}}}{x}$$

$$= -\sqrt{\frac{b}{a}}\frac{\frac{\cancel{\sqrt{x}}}{2\cancel{\sqrt{y}}}\frac{\cancel{\sqrt{y}}}{\cancel{\sqrt{x}}}\left(-\sqrt{\frac{b}{a}}\right) - \frac{\sqrt{y}}{2\sqrt{x}}}{x}$$

$$= -\sqrt{\frac{b}{a}}\left[-\frac{1}{2}\sqrt{\frac{b}{a}} \quad \frac{?}{} \right]$$

$$\frac{\sqrt{x}}{\sqrt{a}} + \frac{\sqrt{y}}{\sqrt{b}} = 1$$

$$\sqrt{x} + \frac{\sqrt{a}}{\sqrt{b}}\sqrt{y} = \sqrt{a}$$

$$1 + \frac{\sqrt{a}}{\sqrt{b}}\frac{\sqrt{y}}{\sqrt{x}} = \frac{\sqrt{a}}{\sqrt{x}}$$

$$\frac{\sqrt{a}}{\sqrt{b}}\frac{\sqrt{y}}{\sqrt{x}} = \frac{\sqrt{a}}{\sqrt{x}} - 1 = \frac{\sqrt{a}-\sqrt{x}}{\sqrt{x}}$$

$$\Rightarrow \frac{\sqrt{y}}{\sqrt{x}} = \frac{\sqrt{b}}{\sqrt{a}}\left(\frac{\sqrt{a}-\sqrt{x}}{\sqrt{x}}\right)$$

$$= \frac{b}{2\sqrt{a}\,x^{3/2}}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1+\frac{by}{ax}\right)^{3/2}}{\frac{b}{2\sqrt{a}\,x\sqrt{x}}}$$

$$= \frac{2(ax+by)^{3/2}}{ab}$$

Ex

Find the radius of curvature of $y = 4\sin x - \sin 2x$ at the point $x = \pi/2$ (or) $(\pi/2, 4)$

if $x = \pi/2 \Rightarrow y = 4(1) - 0 = 4$.

Soln.

if $x = \pi/2 \Rightarrow y = 4(1) - 0 = 4.$

Soln.

$$\rho = \frac{5\sqrt{5}}{4}.$$

Find radius of curvature of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

at $(0, b)$

Soln.

$$\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left(\frac{a^2y^2 + b^2x^2}{a^2y^3}\right)$$

$$\left.\frac{dy}{dx}\right|_{(0,b)} = 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{(0,b)} = -\frac{b}{a^2}$$

$$\rho = \frac{(1+0)^{3/2}}{\left|-\frac{b}{a^2}\right|} = \frac{a^2}{b}$$

Ex

Find the radius of curvature of the function $y^2 = 8x$ at $(9/8, 3)$ and $(9/8, -3)$

— x —

Find the radius of curvature of $x^3 + y^3 = 3axy$ at $(3a/2, 3a/2)$

Soln.

Given that $x^3 + y^3 = 3axy \rightarrow (1)$

Diff (1) with respect to x

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(y + x\frac{dy}{dx}\right) \rightarrow (2)$$

$$\frac{dy}{dx}(y^2 - ax) = ay - x^2$$

$$\frac{dy}{dx} = \frac{ay - x^2}{}$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} \rightarrow ③$$

$$y_1 = \left.\frac{dy}{dx}\right|_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = -1$$

$$③ \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{ay - x^2}{y^2 - ax}\right)$$

$$= \frac{(y^2 - ax)\left(a\frac{dy}{dx} - 2x\right) - (ay - x^2)\left(2y\frac{dy}{dx} - a\right)}{(y^2 - ax)^2}$$

$\frac{dy}{dx} = -1$, $x = \frac{3a}{2}$, $y = 3a/2$

$$y_2 = \left.\frac{d^2y}{dx^2}\right|_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \frac{-32}{3a}$$

$$\rho = \frac{\left[1 + y_1^2\right]^{3/2}}{y_2}$$

$$= \frac{3\sqrt{2}\, a}{16} = \frac{3a}{8\sqrt{2}}$$

—x—

Radius of curvature in parametric form

Let $x = f(t)$, $y = g(t)$, then the radius of curvature $\rho = \dfrac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$

—x—

Show that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ is $4a\cos(\theta/2)$.

Soln.

$x = x(\theta)$, $y = y(\theta)$

$$x' = \frac{dx}{d\theta}, \quad y' = \frac{dy}{d\theta}, \quad x'' = \frac{d^2x}{d\theta^2}, \quad y'' = \frac{d^2y}{d\theta^2}$$

$$x' = \frac{dx}{d\theta} = a(1 + \cos\theta)$$

$$y' = \frac{dy}{d\theta} = a\sin\theta$$

$$x' = \frac{dx}{d\theta} = a(1+\cos\theta)$$

$$y' = \frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{a\sin(2\frac{\theta}{2})}{a(1+\cos(2\theta/2))}$$

$$= \frac{\cancel{2}\sin\theta/2\cos\theta/2}{\cancel{2}\cos^2\theta/2} = \frac{\sin\theta/2}{\cos\theta/2} = \tan\theta/2$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}(\tan\theta/2)\frac{1}{a(1+\cos\theta)}$$

$$= \frac{1}{2}\sec^2(\theta/2)\frac{1}{2a\cos^2\theta/2} = \frac{1}{4a}\sec^4(\theta/2)$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+\tan^2\theta/2)^{3/2}}{\frac{1}{4a}\sec^4(\theta/2)}$$

$$= \frac{4a(1+\tan^2\theta/2)^{3/2}}{\sec^4(\theta/2)} = \frac{4a}{\sec\theta/2}$$

$$= 4a\cos\theta/2$$

(d) Find the radius of curvature at any point on the curve $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$

Soln.

$$x' = \frac{dx}{dt} = a(-\sin t + t\cos t + \sin t) = at\cos t$$

$$y' = \frac{dy}{dt} = a(\cos t + t\sin t - \cos t) = at\sin t$$

$$x'' = \frac{d^2x}{dt^2} = \frac{d}{dt}(at\cos t) = a(\cos t - t\sin t)$$

$$y'' = \frac{d^2y}{dt^2} = \frac{d}{dt}(at\sin t) = a(\sin t + t\cos t)$$

$$x'y'' - y'x'' = at\cos t(a(\sin t + t\cos t))$$
$$- at\sin t(a(\cos t - t\sin t))$$

$$x'y'' - y'x'' = at\cos t\,(a(\sin t + t\cos t))$$
$$- at\sin t\,(a(\cos t - t\sin t))$$
$$= a^2 t^2$$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = at$$

—— x ——

Show that the radius of curvature at an end of the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to the semi-latus rectum.

Soln. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x b^2}{y a^2}$$

one of the major axis can be taken as $(a, 0)$

$$\left.\frac{dy}{dx}\right|_{(a,0)} = \infty$$

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

$$\frac{dx}{dy} = 0, \qquad \frac{dx}{dy} = \frac{-a^2 y}{x b^2}$$

$$\frac{d^2x}{dy^2} = \frac{-a^2}{b^2}\left[\frac{x - y\frac{dx}{dy}}{x^2}\right] \Rightarrow \left.\frac{d^2x}{dy^2}\right|_{(a,0)} = -\frac{a^2}{b^2}\cdot\frac{a}{a^2} = -\frac{a}{b^2}$$

$$\rho = \frac{(1+0)^{3/2}}{-a/b^2} = \frac{1}{a/b^2} = \frac{b^2}{a}$$

Total length of the latus rectum $= 2\frac{b^2}{a}$.

—— x ——

Find ρ at any point $(at^2, 2at)$ on the parabola $y^2 = 4ax$. Hence prove that, if s is the focus, then ρ^2 varies as $\cancel{SP^3}$ D^3

Soln

$$y^2 = 4ax. \qquad x = at^2,\; y = 2at$$

$$x' = 2at, \quad y' = 2a$$

$y^2 = 4ax$. $\quad x = at^2, \; y = 2at$

$$x' = 2at, \quad y' = 2a$$
$$x'' = 2a, \quad y'' = 0$$

$$\frac{dy}{dx} = \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

$$= \frac{d}{dt}\left(1/t\right) \frac{1}{2at} = \frac{-1}{2at^3}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= 2a(t^2+1)^{3/2}$$

$$\rho^2 = 4a^2(t^2+1)^3$$

For the parabola $y^2 = 4ax$, the focus is $(a,0)$ at any point on the parabola $(at^2, 2at)$.

D = The distance $= \sqrt{(at^2-a)^2 + (2at-0)^2}$

$$= a\sqrt{t^4 + 1 + 2t^2}$$

$$= a\left[(t^2+1)^2\right]^{1/2} = a(t^2+1)$$

$$D = a(t^2+1)$$

$$D^3 = a^3(t^2+1)^3$$

$\dfrac{\rho^2}{D^3} = \dfrac{4}{a} =$ constant (i.e) ρ^2 varies as D^3

<u>Note</u>

① The curvature of the curve is $k = \frac{1}{\rho}$, where ρ is the radius of curvature.

② What is the curvature of the straight line?

<u>Soln.</u> $\quad y = ax + b$

Soln.

$$y = ax + b$$

$$\frac{dy}{dx} = a \quad , \frac{d^2y}{dx} = 0 \quad \Rightarrow \rho = \infty$$

$$\Rightarrow \quad k = 0$$

—— x ——

Find the radius of curvature at the point $(a\cos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$

Soln.

$$x = a\cos^3\theta, \quad y = a\sin^3\theta$$

$$x' = \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

$$y' = \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$x'' = \frac{d^2x}{d\theta^2} = \frac{d}{d\theta}(-3a\cos^2\theta\sin\theta)$$

$$= -3a(\cos^3\theta - 2\cos\theta\sin^2\theta)$$

$$y'' = \frac{d^2y}{d\theta^2} = \frac{d}{d\theta}(3a\sin^2\theta\cos\theta)$$

$$= 3a\left[2\sin\theta\cos^2\theta - \sin^3\theta\right]$$

$$\rho = \frac{\left[x'^2 + y'^2\right]^{3/2}}{x'y'' - y'x''} = \frac{\left[9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta\right]^{3/2}}{9a^2\left[-\cos^2\theta\sin\theta(2\sin\theta\cos^2\theta - \sin^3\theta)\right] + 9a^2\left[\sin^2\theta\cos\theta(\cos^3\theta - 2\cos\theta\sin^2\theta)\right]}$$

$$= \frac{\left[9a^2\cos^2\theta\sin^2\theta\right]^{3/2}\left[\cos^2\theta + \sin^2\theta\right]^{3/2}}{9a^2\left[-\cos^2\theta\sin^2\theta(2\cos^2\theta - \sin^2\theta)\right] + 9a^2\left[\sin^2\theta\cos^2\theta(\cos^2\theta - 2\sin^2\theta)\right]}$$

$$= \frac{3^3a^3\cos^3\theta\sin^3\theta\,(1)}{9a^2\cos^2\theta\sin^2\theta\left[\cos^2\theta - 2\sin^2\theta - 2\cos^2\theta + \sin^2\theta\right]}$$

$$= \frac{3a\cos\theta\sin\theta}{(-1)}$$

$$= \frac{3a\cos\theta\sin\theta}{(-1)}$$

$$\rho = 3a\cos\theta\sin\theta$$

— x —

Find the radius of curvature for the curve $y = c\cosh\left(\frac{x}{c}\right)$ at the point where the curve cross the y-axis

Soln.

$$y = c\cosh\left(\frac{x}{c}\right)$$

$$\frac{dy}{dx} = c\sinh\left(\frac{x}{c}\right)\left(\frac{1}{c}\right)$$

$$= \sinh\left(\frac{x}{c}\right)$$

$$\cosh x = \frac{e^{x}+e^{-x}}{2}$$

$$\sinh x = \frac{e^{x}-e^{-x}}{2}$$

$\Rightarrow$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d^2y}{dx^2} = \cosh\left(\frac{x}{c}\right)\cdot\frac{1}{c}$$

$$= \frac{1}{c}\cosh(x/c)$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

$$= \frac{\left[1+\left(\sinh\left(\frac{x}{c}\right)\right)^2\right]^{3/2}}{\frac{1}{c}\cosh(x/c)} = \underline{\underline{c}}\;\frac{\left(1+\sinh^2\left(\frac{x}{c}\right)\right)^{3/2}}{\cosh\left(\frac{x}{c}\right)}$$

$$y = c\cosh\left(\frac{x}{c}\right)$$

$x=0,\ y=c,\quad (0, c)$

i.e the curve cross the y-axis at the point $(0, c)$

$$\rho\big|_{(0,c)} = c\,\frac{1}{1} = c$$

— x —

Find the radius of curvature of $y = x^2(x-3)$ at the points where the tangent is parallel to the x-axis.

Soln.

$$y = x^3 - 3x^2$$

$$\frac{dy}{dx} = 3x^2 - 6x, \quad \frac{d^2y}{dx^2} = 6x - 6$$

$y = x^3 - 3x^2$

$$\frac{dy}{dx} = 3x^2 - 6x, \quad \frac{d^2y}{dx^2} = 6x - 6$$

$$\frac{dy}{dx} = 0 \Rightarrow x = 0 \text{ and } x = 2$$

$x = 0,\ y = 0,\quad (0,0)$

$x = 2,\ y = -4,\quad (2,-4)$

$$\rho = \left.\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}\right|_{(0,0)} = -\frac{1}{6}$$

$$\rho = \left.\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}\right|_{(2,-4)} = \frac{1}{6}$$

so The radius of curvature at $(0,0)$ and $(2,-4)$ are $1/6$.

——— x ———

Find the radius of curvature at any point (r,θ) on the equiangular spiral $r = ae^{\theta\cot\alpha}$

Soln.

$$\rho = \frac{\left[r^2 + r_1^2\right]^{3/2}}{r^2 + 2r_1^2 - rr_2}, \quad r_1 = \frac{dr}{d\theta}, \quad r_2 = \frac{d^2r}{d\theta^2}$$

$$r = ae^{\theta\cot\alpha}$$

$$\Rightarrow \log r = \log a + \theta\cot\alpha$$

$$\frac{1}{r}\frac{dr}{d\theta} = \cot\alpha$$

$$\Rightarrow r_1 = r\cot\alpha$$

$$r_2 = \frac{d^2r}{d\theta^2} = \frac{d(r\cot\alpha)}{d\theta} = \cot\alpha\frac{dr}{d\theta}$$

$$\Rightarrow r_1 = r \cot \alpha$$

$$r_2 = \frac{d^2r}{d\theta^2} = \frac{d}{d\theta}(r\cot\alpha) = \cot\alpha \frac{dr}{d\theta}$$

$$= \cot\alpha\, r_1$$

$$= r\cot^2\alpha$$

$$\rho = \frac{[r^2+r_1^2]^{3/2}}{r^2+2r_1^2-rr_2} = \frac{(r^2+r^2\cot^2\alpha)^{3/2}}{r^2+2r^2\cot^2\alpha - r^2\cot^2\alpha}$$

$$= \frac{(r^2(1+\cot^2\alpha))^{3/2}}{r^2+r^2\cot^2\alpha} = \frac{r^3(1+\cot^2\alpha)^{3/2}}{r^2(1+\cot^2\alpha)}$$

$$= r(1+\cot^2\alpha)^{3/2-1} = r(1+\cot^2\alpha)^{1/2}$$

$$= r\left[\frac{\sin^2\alpha+\cos^2\alpha}{\sin^2\alpha}\right]^{1/2} \qquad \cot^2\alpha = \frac{\cos^2\alpha}{\sin^2\alpha}$$

$$= r[\operatorname{cosec}^2\alpha]^{1/2} = r\operatorname{cosec}\alpha$$

Find the radius of curvature of the curve $r = a\cos\theta$

Soln.

$$r_1 = -a\sin\theta$$

$$r_2 = -a\cos\theta$$

$$\rho = \frac{a}{2}$$

Show that the radius of curvature of $r^n = a^n\cos n\theta$ is

$$\frac{a^n r^{-n+1}}{n+1}.$$

Soln. $r^n = a^n\cos n\theta$

$$\Rightarrow n\log r = n\log a + \log(\cos n\theta)$$

$$\Rightarrow \cancel{n}\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta}\cdot -n\sin n\theta = -\frac{\cancel{n}\sin n\theta}{\cos n\theta}$$

$$\Rightarrow \cancel{n}\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta} \cdots \qquad = \frac{-\cancel{y}\cdots}{\cos n\theta}$$

$$r_1 = \frac{dr}{d\theta} = -r\tan n\theta$$

$$r_1 = -r\tan n\theta$$

$$r_2 = \frac{d^2r}{d\theta^2} = \frac{d}{d\theta}(-r\tan n\theta)$$

$$= -\left[nr\sec^2 n\theta + \tan n\theta\, r_1\right]$$

$$= -nr\sec^2 n\theta + r\tan^2 n\theta$$

$$= r\tan^2 n\theta - nr\sec^2 n\theta$$

$$\rho = \frac{[r^2 + r_1^2]^{3/2}}{r^2 + 2r_1^2 - rr_2} = \frac{[r^2 + r^2\tan^2 n\theta]^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r^2\tan^2 n\theta + nr^2\sec^2 n\theta}$$

$$= \frac{r^3\sec^3 n\theta}{r^2\sec^2 n\theta\,(n+1)} = \frac{r\sec n\theta}{n+1} = \frac{r}{n+1\,\cos n\theta}$$

$$r^n = a^n\cos n\theta \Rightarrow \cos n\theta = \frac{r^n}{a^n} \qquad =$$

$$\rho = \frac{r}{(n+1)\left(\frac{r^n}{a^n}\right)} = \frac{a^n}{n+1}r^{1-n} = \frac{a^n}{n+1}r^{-n+1}$$

Centre of curvature

Friday, November 26, 2021
3:06 PM

The centre ~~of~~ curvature of a curve is given by

$$\boxed{\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}}, \text{ where } y_1 = \frac{dy}{dx},\ y_2 = \frac{d^2y}{dx^2}$$

$$\boxed{\bar{y} = y + \frac{1}{y_2}(1+y_1^2)}$$

$$\boxed{y_1 = y'} \qquad \boxed{y_2 = y''}$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{\ }$$

$$\bar{y} = y + \frac{1}{\ }(1+y_1^2)$$

$$\bar{y} = y + \frac{1}{y_2}(1+y_1^2)$$

Find the centre of Curvature of $y^2 = 2x$ at $(1,1)$

Soln.

$$2y\frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = 2/2y = 1/y$$

$$y_1 = \left.\frac{dy}{dx}\right|_{(1,1)} = \frac{1}{1} = 1$$

$$y_2 = \left.\frac{d^2y}{dx^2}\right|_{(1,1)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(1/y\right) = -\frac{1}{y^2}\,dy/dx = \frac{-1}{y^3}$$

$$y_2 = -1$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= 1 - \frac{1}{(-1)}(1+1) = 1+2 = 3$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= 1 + \frac{1+1}{-1} = 1-2 = -1$$

$\therefore$ The centre of curvature is $(\bar{x}, \bar{y}) = (3, -1)$

— x —

Find the coordinates of Centre of Curvature of the curve $y = x^3 - 6x^2 + 3x + 1$ at $(1,-1)$

Soln.

$$\frac{dy}{dx} = 3x^2 - 12x + 3$$

$$\frac{d^2y}{dx^2} = 6x - 12$$

$$y_1 = \left.\frac{dy}{dx}\right|_{(1,-1)} = -6$$

$$y_2 = -6$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

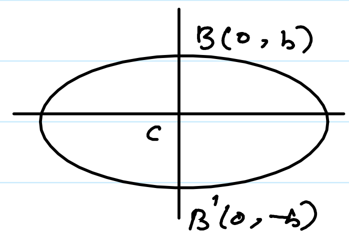
$$\bar{x} = -36,$$

$$\bar{y} = y + \frac{1}{y_2}(1+y_1^2) = -\frac{43}{6}$$

$$(\bar{x}, \bar{y}) = \left(-6, -\frac{43}{6}\right)$$

---x---

Prove that, if the centre of curvature of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at one end of the minor axis lies at the other hand, then the eccentricity of the ellipse is $1/\sqrt{2}$



Soln-

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \longrightarrow (1)$$

Here BB' is the minor axis.

B is $(0, b)$, B' is $(0, -b)$

Diff (1) with respect to x

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

$$\frac{d^2y}{dx^2} = \frac{-b^2}{a^2}\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$= -\frac{b^2}{a^2}\left[\frac{y - x\,dy/dx}{y^2}\right]$$

At $B(0,b)$, $y_1 = \left.\frac{dy}{dx}\right|_{(0,b)} = 0$

$$y_2 = \left.\frac{d^2y}{dx^2}\right|_{(0,b)} = -\frac{b}{a^2}$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2) = 0,$$

$$\bar{y} = b + \frac{1}{\;} = b - \frac{a^2}{\;} = b^2 - a^2$$

$$x = x - \frac{y_1}{y_2}(1+y_1^2) = 0,$$

$$\bar{y} = y + \frac{1}{y_2}(1+y_1^2) = b + \frac{1}{-b/a^2} = b - \frac{a^2}{b} = \frac{b^2-a^2}{b}$$

$\therefore$ The centre of curvature is $\left(0, \frac{b^2-a^2}{b}\right)$ and this given to the point $(0, -b)$.

Hence $b - \frac{a^2}{b} = -b \Rightarrow \underline{2b^2 = a^2}$

$$e^2 = \frac{a^2-b^2}{a^2} = \frac{a^2 - \frac{a^2}{2}}{a^2} = \frac{a^2}{2a^2} = \frac{1}{2}$$

$$\Rightarrow e = \frac{1}{\sqrt{2}}.$$

—— x ——

Circle of curvature

The circle of curvature of any curve is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$, where ρ is the radius of curvature.

—— x ——

Find the equation of circle of curvature of the parabola $y^2 = 12x$ at $(3,6)$.

Soln.

$$y^2 = 12x$$

$$\Rightarrow 2y\frac{dy}{dx} = 12 \Rightarrow \frac{dy}{dx} = \frac{6}{y}$$

$$\frac{d^2y}{dx^2} = -6\frac{dy/dx}{y^2} = \frac{-6y'}{y^2}$$

$$y_1 = \frac{dy}{dx}\Big|_{(3,6)} = 1, \quad y_2 = \frac{-1}{6}$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2) = 3 - \frac{1}{(-1/6)}(1+1) = 3+6(2) = 15$$

$$\bar{y} = y + \frac{1}{y_2}(1+y_1^2) = 6 - 6(2) = -6$$

$$\bar{y} = y + \frac{1}{y_2}(1+y_1^2) = 6 - 6(2) = -6$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{(-1/6)} = 6 \times 2\sqrt{2} = 12\sqrt{2}$$

Hence the circle of curvature is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$(x-15)^2 + (y+6)^2 = 2 \times 144 = 288$$

—— x ——

Find the circle of curvature of $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$

Soln.

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

$$\frac{d^2y}{dx} = \frac{d}{dx}\left(-\frac{\sqrt{y}}{\sqrt{x}}\right)$$

$$= -\frac{\sqrt{x}\frac{1}{2\sqrt{y}}\frac{dy}{dx} - \sqrt{y}\frac{1}{2\sqrt{x}}}{x}$$

$$y_1 = \frac{dy}{dx}\Big|_{(a/4,\, a/4)} = -1$$

$$y_2 = \frac{d^2y}{dx^2}\Big|_{(a/4,\, a/4)} = \frac{4}{a}$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2) = \frac{3a}{4}$$

$$\bar{y} = y + \frac{1}{y_2}(1+y_1^2) = \frac{3a}{4}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{a}{\sqrt{2}}$$

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{a^2}{2} \longrightarrow \text{Circle of Curvature.}$$

—— x ——

Find the circle of curvature of $xy = c^2$ at (c, c)

Soln..

$$xy = c^2$$

Soln..

$$xy = c^2$$

$$\Rightarrow \quad y + x\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

$$y_1 = \frac{dy}{dx}\Big|_{(c,c)} = -1$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$= + \frac{y - x\frac{dy}{dx}}{x^2}$$

$$y_2 = \frac{d^2y}{dx^2}\Big|_{(c,c)} = \frac{2}{c}$$

$$\rho = \frac{\left[1+y_1^2\right]^{3/2}}{y_2}$$

$$= \sqrt{2}c$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= 2c$$

$$\bar{y} = y + \frac{1}{y_2}(1+y_1^2)$$

$$= 2c$$

$\therefore$ The circle of curvature is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$(x-2c)^2 + (y-2c)^2 = 2c^2$$

——— x ———

Evolutes, Involutes and Envelopes

Let P be any point on the curve $y = f(x)$. Let C be the centre of curvature corresponding to the point P of the curve. As P moves on the curve then the centre of curvature of the curve C will also trace out a locus which is called the evolute of the given curve (σ). If Σ is the evolute on the given curve σ, then the σ is called involute.

the evolute on the given curve σ, then the σ is called involute.

Note

To find the evolute of the given curve, elliminate the parameter from the centre of curvature, which gives the evolute of the curve.

① Obtain the equation of the evolute of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$

Soln..

$$x = a(\cos\theta + \theta\sin\theta)$$

$$\frac{dx}{d\theta} = a\theta\cos\theta$$

$$y = a(\sin\theta - \theta\cos\theta)$$

$$\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta)$$

$$= a\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

$$y_1 = \tan\theta$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right).\left(\frac{d\theta}{dx}\right)$$

$$= \frac{d}{d\theta}(\tan\theta)\frac{1}{a\theta\cos\theta}$$

$$= \frac{1}{a\theta}\sec^3\theta$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}, \quad \bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$\bar{x} = a\cos\theta + a\theta\sin\theta - \frac{\tan\theta}{\frac{1}{a\theta}\sec^3\theta}(1+\tan^2\theta)$$

$$= a\cos\theta + a\theta\sin\theta - \frac{a\theta\tan\theta}{\sec^3\theta}\sec^2\theta$$

$$= a\cos\theta + \cancel{a\theta\sin\theta} - \cancel{a\theta\frac{\sin\theta}{\cancel{\cos\theta}}} \times \cancel{\cos\theta}$$

$$= a\cos\theta + \cancel{a\theta\sin\theta} - \cancel{a\theta}\frac{\sin\theta}{\cancel{\cos\theta}} \times \cancel{\cos\theta}$$

$$\bar{x} = a\cos\theta$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= a\sin\theta - a\theta\cos\theta + \frac{(1+\tan^2\theta)}{\frac{1}{a\theta}\sec^3\theta}$$

$$= a\sin\theta - a\theta\cos\theta + a\theta\frac{\sec^2\theta}{\sec^3\theta}$$

$$= a\sin\theta - \cancel{a\theta\cos\theta} + \cancel{a\theta\cos\theta}$$

$$\bar{y} = a\sin\theta$$

$$\bar{x}^2 + \bar{y}^2 = a^2(\cos^2\theta + \sin^2\theta) = a^2$$

$$\bar{x}^2 + \bar{y}^2 = a^2$$

———x———

Find the evolute of the parabola $y^2 = 4ax$

Soln $x = at^2$, $y = 2at$

$$\frac{dx}{dt} = 2at, \quad \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{t} = y_1$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{1}{t}\right)\frac{1}{2at} = \frac{-1}{2at^3}$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}, \quad \bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$\bar{x} = at^2 - \frac{\frac{1}{t}\left(1+\frac{1}{t^2}\right)}{\left(-\frac{1}{2at^3}\right)}$$

$$\bar{x} = 3at^2 + 2a$$

$$\bar{x} - 2a = 3at^2$$

(or) $$\boxed{\frac{\bar{x} - 2a}{3a} = t^2} \longrightarrow (1)$$

$$\bar{y} = y + \frac{(1+y_1^2)}{}$$

$3a$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= 2at + \frac{(1+1/t^2)}{-1/2at^3}$$

$$= 2at - 2at(1+t^2)$$

$$= -2at^3$$

$$\boxed{\frac{\bar{y}}{-2a} = t^3} \longrightarrow (2)$$

(1) $\Rightarrow t^6 = \left(\frac{\bar{x}-2a}{3a}\right)^3$

(2) $\Rightarrow t^6 = \left(\frac{\bar{y}}{-2a}\right)^2$

$\Rightarrow \left(\frac{\bar{x}-2a}{3a}\right)^3 = \left(\frac{\bar{y}}{2a}\right)^2$

---x---

Find the evolute the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Soln. $x = a\sec\theta, \quad y = b\tan\theta.$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta, \quad \frac{dy}{d\theta} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{b\sec\theta}{a\tan\theta} = \frac{b}{a}\operatorname{cosec}\theta = \frac{b}{a\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx} = \frac{d}{d\theta}\left(\frac{b}{a\sin\theta}\right)\frac{1}{a\sec\theta\tan\theta}$$

$$y_2 = \frac{-b}{a^2}\cot^3\theta$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a\sec\theta + \frac{\cancel{b}}{\cancel{a}\sin\theta}\,\frac{\cancel{a^2}\sin^{\cancel{3}}\theta}{\cancel{b}\cos^3\theta}\left(1+\frac{b^2}{a^2\sin^2\theta}\right)$$

$$= \frac{a}{\cos\theta} + \frac{\cancel{a\sin^2\theta}}{b\cos^3\theta}\left(\frac{a^2\sin^2\theta+b^2}{\cancel{a^2\sin^2\theta}}\right)$$

$$= \frac{a^2\cos^2\theta + a^2\sin^2\theta + b^2}{a\cos^3\theta} = \frac{a^2+b^2}{a\cos^3\theta} = \frac{a^2+b^2}{a}\sec^3\theta$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= b\tan\theta - \frac{a^2\sin^3\theta}{b\cos^3\theta}\left(1+\frac{b^2}{a^2\sin^2\theta}\right)$$

$$= b\sin\theta - \cancel{a^2}\sin^{\cancel{3}}\theta \left(a^2\sin^2\theta + b^2 \right.$$

$$b\cos^3\theta \qquad a\sin^2\theta))$$

$$= \frac{b\sin\theta}{\cos\theta} - \frac{\cancel{a^2}\sin^{\cancel{3}}\theta}{b\cos^3\theta}\left(\frac{a^2\sin^2\theta + b^2}{\cancel{a^2\sin^2\theta}}\right)$$

$$= \frac{b\sin\theta}{\cos\theta} - \frac{\sin\theta}{b\cos^3\theta}(a^2\sin^2\theta + b^2)$$

$$= \frac{\sin\theta}{b\cos^3\theta}\left(b^2\cos^2\theta - a^2\sin^2\theta - b^2\right)$$

$$= \frac{\sin\theta}{b\cos^3\theta}\left(-a^2\sin^2\theta - b^2(1-\cos^2\theta)\right)$$

$$= \frac{\sin\theta}{b\cos^3\theta}\left(-a^2\sin^2\theta - b^2\sin^2\theta\right)$$

$$\bar{y} = -\frac{(a^2+b^2)\sin^3\theta}{b\cos^3\theta} = -\frac{(a^2+b^2)}{b}\tan^3\theta$$

$$\left.\begin{aligned} \tan^3\theta &= \frac{-\bar{y}\,b}{(a^2+b^2)} \\ \sec^3\theta &= \frac{\bar{x}\,a}{a^2+b^2} \end{aligned}\right\} \qquad \begin{aligned} \tan^2\theta &= \left(-\frac{\bar{y}\,b}{a^2+b^2}\right)^{2/3} \\ \sec^2\theta &= \left(\frac{\bar{x}\,a}{a^2+b^2}\right)^{2/3} \end{aligned}$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$\Rightarrow \left(\frac{\bar{x}a}{a^2+b^2}\right)^{2/3} - \left(\frac{-b\bar{y}}{a^2+b^2}\right)^{2/3} = 1$$

$$\Rightarrow (\bar{x}a)^{2/3} - (-b\bar{y})^{2/3} = (a^2+b^2)^{2/3}$$

———x———

Find the equation of the evolute of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

<u>Soln.</u>

$$x = a\cos\theta, \quad y = b\sin\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta, \quad \frac{dy}{d\theta} = b\cos\theta$$

$$y_1 = \frac{dy}{dx} = -\frac{b}{a}\frac{\cos\theta}{\sin\theta} = -\frac{b}{a}\cot\theta$$

$$y_2 = \frac{d^2y}{\quad} = \frac{d}{\quad}\left(\frac{dy}{\quad}\right)\cdot\frac{d\theta}{\quad}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx}$$

$$= \frac{-b}{a^2} \operatorname{cosec}^3\theta$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a\cos\theta - \frac{\cancel{b}}{\cancel{a}}\cot\theta \times \frac{a^{\cancel{2}}}{\cancel{b}} \frac{1}{\operatorname{cosec}^3\theta}\left(1 + \frac{b^2\cot^2\theta}{a^2}\right)$$

$$= a\cos\theta - \frac{\cos\theta}{\sin\theta} \times a\sin^3\theta \left(\frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2\sin^2\theta}\right)$$

$$= a\cos\theta - \frac{\cos\theta}{a}(a^2\sin^2\theta + b^2\cos^2\theta)$$

$$= a\cos\theta - \frac{\cos\theta}{a}\left(a^2(1-\cos^2\theta) + b^2\cos^2\theta\right)$$

$$= a\cos\theta - \frac{\cos\theta}{a}\left(a^2 - a^2\cos^2\theta + b^2\cos^2\theta\right)$$

$$= a\cos\theta - a\cos\theta + \frac{(a^2-b^2)\cos^3\theta}{a}$$

$$\bar{x} = \frac{(a^2-b^2)\cos^3\theta}{a}$$

$$\boxed{\frac{\bar{x}a}{a^2-b^2} = \cos^3\theta} \longrightarrow (1)$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= b\sin\theta + \frac{\left(1 + \frac{b^2\cot^2\theta}{a^2}\right)}{(-b/a^2)\operatorname{cosec}^3\theta}$$

$$= b\sin\theta - \frac{\cancel{a^2}}{b}\sin^3\theta\left(\frac{a^2\sin^2\theta + b^2\cos^2\theta}{\cancel{a^2}\cancel{\sin^2\theta}}\right)$$

$$= b\sin\theta - \frac{\sin\theta}{b}(a^2\sin^2\theta + b^2\cos^2\theta)$$

$$= b\sin\theta - \frac{\sin\theta}{b}\left(a^2\sin^2\theta + b^2(1-\sin^2\theta)\right)$$

$$= b\sin\theta - \frac{\sin\theta}{b}\left(a^2\sin^2\theta + b^2 - b^2\sin^2\theta\right)$$

$$= b\sin\theta - b\sin\theta - \frac{\sin^3\theta}{b}(a^2-b^2)$$

$$= b\sin\theta - b\sin\theta - \frac{\sin^3\theta}{b}\left(a^2-b^2\right)$$

$$= \frac{b^2-a^2}{b}\sin^3\theta$$

$$\boxed{\frac{\bar{y}\,b}{b^2-a^2} = \sin^3\theta} \longrightarrow (2)$$

$$(1) \Rightarrow \quad \cos\theta = \left(\frac{a\bar{x}}{a^2-b^2}\right)^{1/3}$$

$$(2) \Rightarrow \quad \sin\theta = \left(-\frac{b\bar{y}}{a^2-b^2}\right)^{1/3}$$

$$\cos^2\theta + \sin^2\theta = \left(\frac{a\bar{x}}{a^2-b^2}\right)^{2/3} + \left(-\frac{b\bar{y}}{a^2-b^2}\right)^{2/3} = 1$$

$$\Rightarrow \left(\left(\frac{a\bar{x}}{a^2-b^2}\right)^{1/3}\right)^2 + \left(-\left(\frac{b\bar{y}}{a^2-b^2}\right)^{1/3}\right)^2 = 1$$

$$\Rightarrow \frac{(a\bar{x})^{2/3}}{(a^2-b^2)^{2/3}} + \frac{(b\bar{y})^{2/3}}{(a^2-b^2)^{2/3}} = 1$$

$$\text{(OR)} \quad (a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2-b^2)^{2/3}$$

Envolopes

If $f(x,y,\alpha)=0$ is a family of curves. Then the locus of their points of intersection is called the envolope of that family.

Rules to find the envolope of the family of curves $f(x,y,\alpha)=0$.

Eliminate the parameter α from $f(x,y,\alpha)=0$ and $\frac{\partial f}{\partial \alpha}(x,y,\alpha)=0$

and $\frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0$

Remarks

1) Let the equation of the family of curves be $A\alpha^2 + B\alpha + C = 0 \rightarrow (1)$
Which is the quadratic equation in α. Then the equation of ~~envlt~~ envelope of the family is $B^2 - 4AC = 0$

2) Evolute of a curve is the envelope of the normals of the curve

— x —

Find the envelope of the family of straight lines
$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

Soln.

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$
$$y - mx = \pm\sqrt{a^2m^2 + b^2}$$
$$\Rightarrow (y - mx)^2 = a^2m^2 + b^2$$
$$y^2 + m^2x^2 - 2mxy = a^2m^2 + b^2$$
$$m^2(x^2 - a^2) - 2mxy + y^2 - b^2 = 0$$

$Am^2 + Bm + C = 0$, Where m is the Parameter

$\therefore$ The equation of envelope is $B^2 - 4AC = 0$

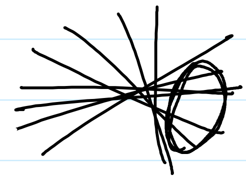
Here $A = x^2 - a^2$, $B = -2xy$, $C = y^2 - b^2$

$$B^2 - 4AC = 4x^2y^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$$
$$\Rightarrow x^2y^2 - (x^2 - a^2)(y^2 - b^2) = 0$$
$$\Rightarrow x^2y^2 - x^2y^2 + x^2b^2 + a^2y^2 - a^2b^2 = 0$$

$$\Rightarrow x^2y^2 - x^2y^2 + x^2b^2 + a^2y^2 - a^2b^2 = 0$$

$$\Rightarrow x^2b^2 + a^2y^2 = a^2b^2$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



---x---

Find the envelope of straight lines

$x\cos\alpha + y\sin\alpha = a\sin\alpha\cos\alpha$, where α is the parameter.

Soln.

$$x\cos\alpha + y\sin\alpha = a\sin\alpha\cos\alpha.$$

$$\Rightarrow \frac{x}{\sin\alpha} + \frac{y}{\cos\alpha} = a$$

$$\Rightarrow x\,\mathrm{cosec}\,\alpha + y\sec\alpha = a$$

$$\frac{d}{d\alpha}(\mathrm{cosec}\,\alpha) = -\cot\alpha\,\mathrm{cosec}\,\alpha$$

$$\frac{d}{d\alpha}(\sec\alpha) = \sec\alpha\tan\alpha.$$

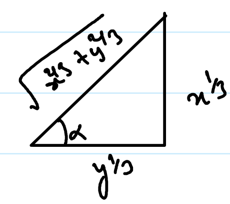
$$\Rightarrow \frac{d}{d\alpha} \Rightarrow$$

$$-x\,\mathrm{cosec}\,\alpha\cot\alpha + y\sec\alpha\tan\alpha = 0$$

$$\Rightarrow -\frac{x\cos\alpha}{\sin^2\alpha} + y\frac{\sin\alpha}{\cos^2\alpha} = 0$$

$$\Rightarrow y\sin^3\alpha = x\cos^3\alpha$$

$$\therefore \tan^3\alpha = \frac{x}{y}$$



$$\tan\alpha = \frac{x^{1/3}}{y^{1/3}}$$

$$\mathrm{cosec}\,\alpha = \frac{\sqrt{x^{2/3}+y^{2/3}}}{x^{1/3}}, \quad \sec\alpha = \frac{\sqrt{x^{2/3}+y^{2/3}}}{y^{1/3}}$$

$$\therefore x\,\mathrm{cosec}\,\alpha + y\sec\alpha = a$$

$$x\frac{\sqrt{x^{2/3}+y^{2/3}}}{x^{1/3}} + y\frac{\sqrt{x^{2/3}+y^{2/3}}}{y^{1/3}} = a$$

It is a required envolope.

it is a required envelope.

Find the envelope of the straight lines $\frac{x}{a}+\frac{y}{b}=1$ where a and b are connected by the relation $a+b=c$, where c ~~is~~ a constant

Soln.

$a+b=c \Rightarrow b=c-a$

$\therefore \frac{x}{a}+y$

$\Rightarrow \frac{x}{a}+\frac{y}{c-a}=1$

$x(c-a)+ya=a(c-a)$

$xc-xa+ya=ca-a^2$

$\Rightarrow a^2+a(y-x-c)+cx=0$

$\Rightarrow \therefore$ The equation of envelope is

$B^2-4AC=0$, where $B=y-x-c$

$A=1, \quad C=cx$

$(y-x-c)^2-4(1)cx=0$

Find the envelope of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where a and b connected by the relation $a^2+b^2=c^2$, c - is a constant.

Soln.

$a^2+b^2=c^2 \longrightarrow (1)$

$b^2=c^2-a^2$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \longrightarrow (2)$

$\therefore$ equation (2) $\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{c^2-a^2}=1$

$\Rightarrow (c^2-a^2)x^2+a^2y^2=a^2(c^2-a^2)$

$$\Rightarrow \quad x^2c^2 - a^2x^2 + a^2y^2 = a^2c^2 - a^4$$

$$\Rightarrow \quad a^4 + a^2(y^2 - x^2 - c^2) + x^2c^2 = 0$$

Let $\lambda = a^2$, then

$$\lambda^2 + \lambda(y^2 - x^2 - c^2) + x^2c^2 = 0$$

$\therefore$ The equation of envelope is

$$B^2 - 4AC = 0 \qquad \text{where}$$

$$B = y^2 - x^2 - c^2, \quad A = 1, \quad C = x^2c^2$$

$$B^2 - 4AC = (y^2 - x^2 - c^2)^2 - 4x^2c^2 = 0$$

$$\Rightarrow \quad (y^2 - x^2 - c^2)^2 - (2xc)^2 = 0$$

$$X^2 - Y^2 = (X - Y)(X + Y)$$

$$(y^2 - x^2 - c^2 - 2cx)(y^2 - x^2 - c^2 + 2cx) = 0$$

$$\Rightarrow \quad y^2 - x^2 - c^2 = 2cx$$

another one is $y^2 - x^2 - c^2 = -2cx$

—— x ——

Find the envelope of the straight lines represented by $x\cos\alpha + y\sin\alpha = a\sec\alpha$, α is parameter.

Soln.

The given equation is $x\cos\alpha + y\sin\alpha = a\sec\alpha$.

$$x + y\tan\alpha = a\frac{\sec\alpha}{\cos\alpha}$$

$$= a\sec^2\alpha$$

$$= a(1 + \tan^2\alpha)$$

$$\Rightarrow \quad a\tan^2\alpha - y\tan\alpha + (a - x) = 0$$

Let $\lambda = \tan\alpha$, then

$$a\lambda^2 - y\lambda + (a - x) = 0$$

$\therefore$ The equation of envelope is $B^2 - 4AC = 0$

where $B = -y$, $A = a$, $C = a - x$

$\therefore$ the equation of envelope is $y^2 - 4a(a - x) = 0$

—— x ——

Find the envelope of the family of lines

$y = mx - am^3$, m is parameter.

$$y = mx - am^3, \quad m \text{ is parameter.}$$

—— x ——

~~soln.~~

Find the envelope of $\frac{x}{a} + \frac{y}{b} = 1$ where a and b are parameters having the relation $ab = c^2$, c is constant.

Soln.

Given that $\frac{x}{a} + \frac{y}{b} = 1 \longrightarrow (1)$

and $ab = c^2 \longrightarrow (2)$

$$\Rightarrow \frac{x}{a} + \frac{y}{c^2/a} = 1$$

$$\Rightarrow \frac{x}{a} + \frac{ay}{c^2} = 1$$

$$\Rightarrow xc^2 + a^2 y = ac^2$$

$$\Rightarrow a^2 y - ac^2 + xc^2 = 0$$

$\therefore$ The equation of envelope is $B^2 - 4AC = 0$

where $B = -c^2$, $A = y$, $C = xc^2$

$$\therefore B^2 - 4AC = c^4 - 4xyc^2 = 0$$

$$\Rightarrow 4xy = c^2$$

$$xy = \left(c/2\right)^2$$

—— x ——

Find the envelope of family of lines $x\cos^3\alpha + y\sin^3\alpha = a$ where α is a parameter.

Soln.

$$x\cos^3\alpha + y\sin^3\alpha = a \longrightarrow (1)$$

diff. (1) $\Rightarrow 3x\cos^2\alpha(-\sin\alpha) + 3y\sin^2\alpha(\cos\alpha) = 0$

$$\Rightarrow x\cos\alpha - y\sin\alpha = 0$$

$$\Rightarrow \tan\alpha = \frac{x}{y}$$

$$\sin\alpha = \frac{x}{\sqrt{x^2+y^2}}, \quad \cos\alpha = \frac{y}{\sqrt{x^2+y^2}}$$

$$x \frac{y^3}{(x^2+y^2)^{3/2}} + y \times \frac{x^3}{(x^2+y^2)^{3/2}} = a$$

$$(x^2+y^2)^{3/2}$$

$$\Rightarrow \frac{xy^3+yx^3}{(x^2+y^2)^{3/2}} = a$$

$$\Rightarrow xy^3 + yx^3 = a(x^2+y^2)^{3/2}$$

$$\Rightarrow xy(y^2+x^2) = a(x^2+y^2)^{3/2}$$

$$xy = a(x^2+y^2)^{3/2-1}$$

$$= a\sqrt{x^2+y^2}$$

$$\boxed{x^2y^2 = a^2(x^2+y^2)}$$

—— x ——

Find the envelope of system of Concentric and coaxial ellipses of Constant area.

Soln.

The area of ellipse is $\pi ab = k$, Where k is constant

$$ab = \frac{k}{\pi} = c$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{(c/a)^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2a^2}{c^2} = 1$$

$$\Rightarrow x^2c^2 + y^2a^4 = a^2c^2$$

$$\Rightarrow (a^2)^2y^2 - a^2c^2 + x^2c^2 = 0$$

Let $\lambda = a^2$, then

$$\lambda^2y^2 - \lambda c^2 + x^2c^2 = 0$$

The equations of envelope $B^2 - 4AC = 0$

$$B = -c^2,\quad A = y^2,\quad C = x^2c^2$$

$$\therefore B^2 - 4AC = c^4 - 4y^2x^2c^2 = 0$$

$$c^2 - 4x^2y^2 = 0$$

$$\Rightarrow x^2y^2 = \frac{c^2}{4}$$

$\Rightarrow xy = \pm c/2$