Unit I - Partial Differential Equations

CONTENTS

Introduction to PDE

Partial differential equations arise in geometry, physics and in Engineering branches when the number of independent variables in the given problems discussion is two or more. In such cases any dependent variable is likely to be a function of more than one variables, so that it possesses not ordinary derivatives with respect to single variable but partial derivatives with respect to several variables.

For example, in the study of thermal effects in a solid body the temperature u may vary from point to point as well as from time to time and as a consequence, the derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ and $\frac{\partial u}{\partial t}$.

Definition of PDE:

A partial differential equation (PDE) is an equation which involves partial derivatives of a dependent variable with respect to more than one independent variable.

Example:

$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2 z}{\partial x \partial y} = 0$

Here z-dependent variable,

x, y-independent variables.

Notation If $z = f(x, y)$ is a function of two independent variables x and y then the usual notations are

$\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$ (First derivatives)

$\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial x \partial y} = s$ and $\frac{\partial^2 z}{\partial y^2} = t$ (Second derivatives)

Order: The order of a PDE is the order of the highest derivative.

Degree: The degree of the PDE is the power of its highest derivative occurring in it.

Example: (1). $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1.$

order-1, degree-1.

(2). $\left(\dfrac{\partial z}{\partial x}\right)^2 + 2\left(\dfrac{\partial^2 z}{\partial x \partial y}\right) + \left(\dfrac{\partial z}{\partial y}\right)^2 = 0.$
order-2, degree-1.

(3). $\left(\dfrac{\partial^2 z}{\partial x^2}\right)^{\frac{3}{2}} = \dfrac{\partial z}{\partial x} \Rightarrow \left(\dfrac{\partial^2 z}{\partial x^2}\right)^3 = \left(\dfrac{\partial z}{\partial x}\right)^2.$
order-2, degree-3.

Linear and Non-linear PDE

A PDE is said to be linear if the dependent variable and the partial derivatives occur in the first degree and there is no product of partial derivatives or product of derivative and dependent variable.

A PDE which is not linear is said to be non-linear.

Example:

$\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial t^2}$ →linear.

$x^2\dfrac{\partial^2 z}{\partial x^2} + x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} + z = e^{x+y}$ →linear.

$x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$ →linear.

$\dfrac{\partial^2 z}{\partial x^2} + z\left(\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}\right) = 0$ →non linear.

$\dfrac{\partial^2 z}{\partial x^2} + z^2 = \sin(x+y)$ →non linear.

$\dfrac{\partial^2 z}{\partial y^2} + \left(\dfrac{\partial z}{\partial x}\right)^3 = \cos(x+y)$ →non linear.

Formation of PDE: PDE can be formed in two ways

(i) By eliminating two or more arbitrary Constants

(ii) By eliminating one or more arbitrary functions

By elimination of arbitrary constants: Let us take the function

$$f(x, y, z, a, b) = 0 \tag{1}$$

where a and b are arbitrary constants.

Now we have to eliminate a and b

Differentiating the equation (1) partially with respect to x and y, we get

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial x} = 0$ and $\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial y} = 0$. Therefore

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0 \text{ and } \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0 \tag{2}$$

Now, eliminating the two arbitrary constants a and b from equation (1) and using (2), we get $g(x, y, z, p, q) = 0$

which is the required PDE of the first order of the form.

Remarks:

(1) If the number of constants to be eliminated is equal to the number of independent variables, the resulting PDE will be of the first order.

(2) If the number of constants to be eliminated is more than the number of independent variables, the resulting PDE will be of the second or orders.

Example 1. Form the PDE by eliminating the arbitrary constants a and b from $z = ax + by + a^2 + b^2$.

Sol.: Given

$$z = ax + by + a^2 + b^2 \tag{3}$$

Differentiating equation (3) partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = a \text{ and } \frac{\partial z}{\partial y} = b.$$

$$\Rightarrow p = a \text{ and } q = b \tag{4}$$

Substituting (4) in equation (3), we get

$z = px + qy + p^2 + q^2$ which is the required PDE.

Example 2. Form the PDE by eliminating the arbitrary constants a and b from $(x - a)^2 + (y - b)^2 + z^2 = c^2$.

Sol.: Given

$$(x - a)^2 + (y - b)^2 + z^2 = c^2 \tag{5}$$

Differentiating equation (5) partially with respect to x and y, we get

$$2(x - a) + 2z\frac{\partial z}{\partial x} = 0 \text{ and } 2(y - b) + 2z\frac{\partial z}{\partial y} = 0.$$

$$\Rightarrow (x - a) = -zp \text{ and } (y - b) = -zq. \tag{6}$$

Substituting (6) in (5), we get

$(p^2 + q^2 + 1)z^2 = c^2$ which is the required PDE.

Example 3. Form the PDE by eliminating the arbitrary constants a and b from $\log_e(az - 1) = x + ay + b$.

Sol.: Given

$$\log_e(az - 1) = x + ay + b \tag{7}$$

Differentiating equation (7) partially with respect to x and y, we get

$$\frac{1}{az-1}.a\frac{\partial z}{\partial x} = 1 \text{ and } \frac{1}{az-1}.a\frac{\partial z}{\partial y} = a$$

$$\Rightarrow \frac{ap}{az-1} = 1 \text{ and } \frac{q}{az-1} = 1$$

$$ap = az - 1 \tag{8}$$

and

$$q = az - 1 \tag{9}$$

From equations (8) and (9), we get $ap = q$

$$a = \frac{q}{p} \tag{10}$$

Substituting equation (10) in (8) or (9), we get $p = q(z - p)$.

Example 4. Find the PDE of the family of spheres having their centres on the line $x = y = z$.

Sol.:

Given centres of the spheres lie on the line $x = y = z$.

Therefore centre of the sphere is (a, a, a) and let r be the radius.

So, the equation of the family of sphere is

$$(x - a)^2 + (y - a)^2 + (z - a)^2 = r^2 \tag{11}$$

where a and r are arbitrary constants.

Differentiating equation (11) partially with respect to x and y, we get

$$2(x - a) + 2(z - a)\frac{\partial z}{\partial x} = 0 \tag{12}$$

and

$$2(y-a)+2(z-a)\frac{\partial z}{\partial y}=0 \tag{13}$$

Equation (12) $\Rightarrow (x-a)=-p(z-a)$

$\Rightarrow x+pz=a(p+1)$

Therefore

$$a=\frac{x+pz}{p+1} \tag{14}$$

Similarly equation (13) implies that

$$a=\frac{y+qz}{q+1} \tag{15}$$

From equations (14) and (15), we get

$(q+1)(x+pz)=(p+1)(y+qz)$

$\Rightarrow p(y-z)+q(z-x)=x-y.$

Example 5. Form a PDE by eliminating a,b,c from $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1.$

Sol.:

Given

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{16}$$

Differentiating equation (16) partially with respect to x and y, we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0 \text{ or } c^2x+a^2z\frac{\partial z}{\partial x}=0 \tag{17}$$

and

$$\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0 \text{ or } c^2y+b^2z\frac{\partial z}{\partial y}=0 \tag{18}$$

Again differentiating (17) w.r.t y, we get

$$0+a^2\left[z\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial x}.\frac{\partial z}{\partial y}\right]=0$$

$\Rightarrow zs+pq=0.$

Remark:

(i) If we differentiate (17) with respect to x, we get

$$c^2 + a^2\left[z\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}.\frac{\partial z}{\partial x}\right] = 0$$

$$c^2 + a^2(zr + p^2) = 0$$

$$\frac{c^2}{a^2} = -(zr + p^2) \tag{19}$$

From (17) implies that

$$\frac{c^2}{a^2} = -\frac{pz}{x} \tag{20}$$

Equating (19) and (20), we get

$x(zr + p^2) = pz.$

(ii) Instead of differentiating (17), if we differentiate (18) with respect to y, we get

$$c^2 + b^2\left[z\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}.\frac{\partial z}{\partial y}\right] = 0$$

$$c^2 + b^2(zt + q^2) = 0$$

$$\frac{c^2}{b^2} = -(zt + q^2) \tag{21}$$

From (18) implies that

$$\frac{c^2}{b^2} = -\frac{qz}{y} \tag{22}$$

Equating (21) and (22), we get

$y(zr + q^2) = qz.$

Note:

(i) Three different PDE's could be formed. So, the resulting PDE is not unique when the number of constants is more than the number of independent variables.

(ii) The order of the resulting PDE will be equal to the number of arbitrary functions to be eliminated.

Self Practice (i) Form the PDE by eliminating the arbitrary constants a and b from $z = (x^2 + a^2)(y^2 + b^2)$.

(ii) Find the differential equation of all spheres whose center lie on the z-axis.

(iii) Form a PDE by eliminating a, b, c from $z = ax^2 + bxy + cy^2$.

Formation of PDE - By Elimination of Arbitrary Functions

Model I: By eliminating single arbitrary function in $z = f(x, y)$

The resulting PDE will be of first order.

Model II: By eliminating two arbitrary functions in $z = f(x, y) + g(x, y)$

To form the PDE first we find the equations

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, t = \frac{\partial^2 z}{\partial y^2}, s = \frac{\partial^2 z}{\partial x \partial y}$$

From these six equations, we choose the suitable equations to eliminate f and g. The resulting PDE will be a second order equation.

Model III: By eliminating arbitrary function ϕ from $\phi(u, v) = 0$ where u and v are functions of x, y, z.

Method 1: $\phi(u,\ v) = 0$ can be written as $v = f(u)$ or $u = g(v)$ where u and v are functions of x, y, z. Then proceed as in eliminating single arbitrary function.

Method 2: Let $\phi(u,\ v) = 0$ be given function. Then we can construct the PDE as follows:

Step 1: Differentiate u and v w.r.t x, y and z

Step 2: Find

$$\begin{aligned} P &= \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} = \frac{\partial(u, v)}{\partial(y, z)} \\ Q &= \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z} = \frac{\partial(u, v)}{\partial(z, x)} \\ R &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial(u, v)}{\partial(x, y)} \end{aligned}$$

Step 3: Write the PDE $Pp + Qq = R$.

Method 3: Let $\phi(u,\ v) = 0$ be given function. Then we can construct the PDE as follows:

Step 1: Differentiate u and v w.r.t x and y

Step 2: Find $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0$

Step 3: Write the PDE $Pp + Qq = R$.

Remark

The order of the resulting PDE will be equal to the number of arbitrary functions to be eliminated.

Model I:

Example 1 Form the PDE by eliminating the arbitrary function from $z = f(x^2+y^2)$.

Sol.: Given

$$z = f(x^2 + y^2) \tag{1}$$

Differentiating equation (1) partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = f^{'}(x^2 + y^2).2x \Rightarrow f^{'}(x^2 + y^2) = \frac{p}{2x} \tag{2}$$

and

$$\frac{\partial z}{\partial y} = f^{'}(x^2 + y^2).2y \Rightarrow f^{'}(x^2 + y^2) = \frac{q}{2y} \tag{3}$$

From (2) and (3), we get

$$\frac{p}{x} = \frac{q}{y} \Rightarrow py = qx.$$

Example 2 Form the PDE by eliminating the arbitrary function from $z = x^2 + 2f\left(\frac{1}{y} + \log x\right)$.

Sol.: Differentiating partially w. r. t. x and y separately, we get

$$\frac{\partial z}{\partial x} = 2x + 2f^{'}\left(\frac{1}{y} + \log x\right)\left(\frac{1}{x}\right) \Rightarrow \frac{x(p-2x)}{2} = f^{'}\left(\frac{1}{y} + \log x\right) \tag{4}$$

and

$$\frac{\partial z}{\partial y} = 2f^{'}\left(\frac{1}{y} + \log x\right)\left(\frac{-1}{y^2}\right) \Rightarrow -\frac{y^2 q}{2} = f^{'}\left(\frac{1}{y} + \log x\right) \tag{5}$$

From (4) and (5), we get

$$\frac{x(p-2x)}{2} = -\frac{y^2 q}{2} \Rightarrow xp + y^2 q = 2x^2.$$

Example 3 Form the PDE by eliminating the arbitrary function from $xyz = f(x^2 + y^2 - z^2)$.

Sol.: Differentiating partially w. r. t. x and y separately, we get

$$y\left[1.z + x.\frac{\partial z}{\partial x}\right] = f^{'}(x^2 + y^2 + z^2)\left(2x - 2z\frac{\partial z}{\partial x}\right)$$

and

$$x\left[1.z + y.\frac{\partial z}{\partial y}\right] = f^{'}(x^2 + y^2 + z^2)\left(2y - 2z\frac{\partial z}{\partial y}\right)$$

Therefore

$$\frac{y(z+xp)}{2(x-zp)} = f^{'}(x^2+y^2-z^2) \tag{6}$$

and

$$\frac{x(z+yq)}{2(y-zq)} = f^{'}(x^2+y^2-z^2) \tag{7}$$

From (6) and (7), we get

$$\frac{y(z+xp)}{2(x-zp)} = \frac{x(z+yq)}{2(y-zq)}$$

$$\Rightarrow y(z+xp)(y-zq) = x(z+yq)(x-zp)$$

$$\Rightarrow x(y^2+z^2)p - y(z^2+x^2)q + z(y^2-x^2) = 0.$$

Example 4 Form the PDE by eliminating the arbitrary function from $z = f\left(\frac{xy}{z}\right)$.

Sol.: Differentiating partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = f^{'}\left(\frac{xy}{z}\right).y.\left[\frac{z.1 - x\frac{\partial z}{\partial x}}{z^2}\right] \Rightarrow f^{'}\left(\frac{xy}{z}\right) = \frac{pz^2}{y(z-xp)} \tag{8}$$

and

$$\frac{\partial z}{\partial y} = f^{'}\left(\frac{xy}{z}\right).x.\left[\frac{z.1 - y\frac{\partial z}{\partial y}}{z^2}\right] \Rightarrow f^{'}\left(\frac{xy}{z}\right) = \frac{qz^2}{x(z-yq)} \tag{9}$$

From (8) and (9), we get

$$\frac{pz^2}{y(z-xp)} = \frac{qz^2}{x(z-yq)} \Rightarrow px = qy.$$

Model II:

Example 5

Form the PDE by eliminating the arbitrary functions f and g in $z = f(x+ct) + g(x-ct)$.

Sol.: Differentiating partially with respect to x and t, we get

$$\frac{\partial z}{\partial x} = f^{'}(x+ct) + g^{'}(x-ct)$$
$$\frac{\partial z}{\partial t} = cf^{'}(x+ct) - cg^{'}(x-ct)$$

Again differentiating partially with respect to x and t, we get

$$\frac{\partial^2 z}{\partial x^2} = f^{''}(x+ct) + g^{''}(x-ct) \tag{10}$$

$$\frac{\partial^2 z}{\partial t^2} = c^2 f^{''}(x+ct) + c^2 g^{''}(x-ct) \tag{11}$$

Equation (11) implies that

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2} &= c^2 \left[f^{''}(x+ct) + g^{''}(x-ct) \right] \\ &= a^2 \left[\frac{\partial^2 z}{\partial x^2} \right]. \text{ by equation (10)} \end{aligned}$$

Example 6

Form the PDE by eliminating the arbitrary functions f and g in $z = x^2 f(y) + y^2 g(x)$.

Sol.: Given

$$z = x^2 f(y) + y^2 g(x) \tag{12}$$

Differentiating partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = 2x f(y) + y^2 g^{'}(x) \tag{13}$$

and

$$\frac{\partial z}{\partial y} = x^2 f^{'}(y) + 2y g(x) \tag{14}$$

Again differentiating partially with respect to x and y, we get

$$\frac{\partial^2 z}{\partial x^2} = 2f(y) + y^2 g^{''}(x)$$
$$\frac{\partial^2 z}{\partial y^2} = x^2 f^{''}(y) + 2g(x)$$

Again differentiating equation (13) partially with respect to y, we get

$$\frac{\partial^2 z}{\partial x \partial y} = 2x f^{'}(y) + 2y g^{'}(x) \tag{15}$$

$$\text{From (13)} \Rightarrow \frac{p - 2x f(y)}{y^2} = g^{'}(x) \tag{16}$$

$$\text{From (14)} \Rightarrow \frac{q - 2y g(x)}{x^2} = f^{'}(y) \tag{17}$$

Substituting (16) and (17) in (15), we get

$$\frac{\partial^2 z}{\partial x \partial y} = 2x\left(\frac{q-2yg(x)}{x^2}\right) + 2y\left(\frac{p-2xf(y)}{y^2}\right)$$
$$\Rightarrow s = \frac{2q}{x} - \frac{4yg(x)}{x} + \frac{2p}{y} - \frac{4xf(y)}{y}$$
$$sxy = 2qy - 4y^2g(x) + 2px - 4x^2f(y)$$
$$= 2(px+qy) - 4\left[y^2g(x) + x^2f(y)\right]$$
$$= 2(px+qy) - 4z \text{ by equation (12)}$$

Self Practice Problems:

Form the PDE by eliminating the arbitrary functions f and g in

(i) $z = f(x^3+2y) + g(x^3-2y)$

(ii) $z = xf\left(\frac{y}{x}\right) + yg(x)$

Model III:

Example 7 Form the PDE by eliminating the arbitrary function ϕ from $\phi(x+y+z,\ x^2+y^2-z^2) = 0$.

Sol.: Here $u = x+y+z$ and $v = x^2+y^2-z^2$

$$\Rightarrow u_x = 1 + \frac{\partial z}{\partial x}, u_y = 1 + \frac{\partial z}{\partial y}, v_x = 2x - 2z\frac{\partial z}{\partial x} \text{ and } v_y = 2y - 2z\frac{\partial z}{\partial y}$$

Therefore $u_x = 1+p, u_y = 1+q, v_x = 2x-2zp$ and $v_y = 2y-2zq$.

Hence $\begin{vmatrix} 1+p & 1+q \\ 2x-2zp & 2y-2zq \end{vmatrix} = 0$

$\Rightarrow (1+p)(2y-2zq) - (1+q)(2x-2zp) = 0$

$\Rightarrow (y+z)p - (z+x)q = x-y$

Example 9 Form the PDE by eliminating the arbitrary function ϕ from $\phi(z^2 - xy, \frac{x}{z}) = 0$.

Sol.: Here $u = z^2 - xy$ and $v = \frac{x}{z}$

$$\Rightarrow u_x = 2z\frac{\partial z}{\partial x} - y, u_y = 2z\frac{\partial z}{\partial y} - x, v_x = \frac{z.1 - x\frac{\partial z}{\partial x}}{z^2} \text{ and } v_y = x.\frac{-1}{z^2}.\frac{\partial z}{\partial y}$$

Therefore $u_x = 2zp - y, u_y = 2zq - x, v_x = \dfrac{z - xp}{z^2}$ and $v_y = \dfrac{-xq}{z^2}$.

Hence $\begin{vmatrix} 2zp - y & 2zq - x \\ \dfrac{z - xp}{z^2} & \dfrac{-xq}{z^2} \end{vmatrix} = 0 \Rightarrow (2zp - y)\dfrac{-xq}{z^2} - (2zq - x)\dfrac{z - xp}{z^2} = 0$

$\Rightarrow x^2p + (2z^2 - xy)q = xz.$

Solution of Non-Linear PDE's

A solution of a PDE is a relation between the independent variables and dependent variable which satisfies the given PDE.

A solution is also known as an integral of the PDE.

Note: Two types of solutions may occur as solutions of the same equation.

For example: Consider the equations

$$z = ax + by \tag{1}$$

and

$$z = xf\left(\frac{y}{x}\right) \tag{2}$$

If we eliminate the arbitrary constants a and b from the equation (1) and arbitrary function from the equation (2), we get the same differential equation $xp + yq = z$. Hence $z = ax + by$ and $z = xf\left(\frac{y}{x}\right)$ are the solutions of the equation $xp + yq = z$.

There are four types of solutions:

- **Complete Integral**

  A solution of a PDE which contains as many arbitrary constants as the number of independent variables is called the complete integral or complete solution.

  Example: If $f(x, y, z, p, q) = 0$ where x and y are independent variables is a given PDE, then the solution of the form $\phi(x, y, z, a, b) = 0$ is a complete integral.

- **Particular Integral**

  In complete integral by giving particular values to the arbitrary constants is called a particular integral.

  Example: If $\phi(x, y, z, a, b) = 0$ is a complete integral, then $\phi(x, y, z, 1, 2) = 0$, $\phi(x, y, z, 0, 1) = 0$, $\phi(x, y, z, 3, 9) = 0$ all are possible particular integral.

- **General Integral**

  A solution of a PDE which contains as many arbitrary functions as the order of the equation is called general integral.

  In geometrically, the envelope of the family of surfaces with arbitrary functions is called general integral.

- **Singular Integral**

  In geometrically, the envelope of the family of surface with two parameters a and b is called singular integral.

Procedure to find general integral and singular integral:

Let

$$F(x, y, z, p, q) = 0 \tag{3}$$

first order PDE.

Let

$$\phi(x, y, z, a, b) = 0 \tag{4}$$

be the complete integral, where a and b are arbitrary constants.

To find general integral or general solution:

Put $b = f(a)$ [or $a = g(b)$] in (4) where f (or g) is an arbitrary function.

$$\Rightarrow \phi(x, y, z, a, f(a)) = 0. \tag{5}$$

Differentiating (5) partially w.r.to a, we get

$$\frac{\partial}{\partial a}\left[\phi(x, y, z, a, f(a))\right] = 0 \tag{6}$$

The eliminate of a from (5) and (6), if it exists, is called the general integral.

To find singular integral or singular solution:

Differentiating (4) partially w.r.to a and b, we obtain

$$\frac{\partial \phi}{\partial a} = 0 \text{ and } \frac{\partial \phi}{\partial b} = 0. \tag{7}$$

The eliminate of a and b from equations (4) and (7), when it exists, is the singular integral.

In order to find the solutions of some standard types of first order non-linear PDEs, let us classify the given PDE as below:

- **Type I:** $F(p, q) = 0$
- **Type II:** $z = px + qy + f(p, q)$ (or) Clairaut's form
- **Type III:** $F(z, p, q) = 0$
- **Type IV:** $f(x, p) = g(y, q)$ (or) Separable equations

Type I: If $F(p,\ q) = 0$, i.e it contains only p and q then the solution can be obtained as below:

Let us consider the first order PDEs of the form $f(x,\ y,\ z,\ p,\ q) = 0$ where $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$

- **Step 1 :** Put $p = a$ and $q = b$ which gives $F(a,\ b) = 0$
- **Step 2 :** Suppose that $z = ax + by + c$ is a solution of the equation.
- **Step 3 :** From $F(a,\ b) = 0$, write $b = f(a)$ or $a = g(b)$
- **Step 4 (Complete Integral):** Hence the complete integral is

$$z = ax + f(a)y + c \tag{8}$$

- **Step 5 (Singular Integral):**

  Differentiating (8) with respect to a and c partially, we get

$$\frac{\partial z}{\partial a} = x + f^{'}(a)y \Rightarrow 0 = x + f^{'}(a)y$$

  and

$$\frac{\partial z}{\partial c} = 1 \Rightarrow 0 = 1 \text{ which is absurd.}$$

  Therefore there is no singular integral for $F(p, q) = 0$.

- **Step 6 (General Integral):** Put $c = g(a)$ in (8) where g is arbitrary function

$$z = ax + f(a)y + g(a) \tag{9}$$

Differentiating (9) with respect to a, we get

$$\frac{\partial z}{\partial a} = x + f^{'}(a)y + g^{'}(a) \Rightarrow 0 = x + f^{'}(a)y + g^{'}(a) \tag{10}$$

Eliminating a from (9) and (10), we get the general integral.

Example 1: Solve $\sqrt{p} + \sqrt{q} = 1$.

Solution: This is of the form $F(p, q) = 0$.

Assume the solution is $z = ax + by + c$ where $\sqrt{a} + \sqrt{b} = 1$.

$\Rightarrow \sqrt{b} = 1 - \sqrt{a} \ \Rightarrow b = (1 - \sqrt{a})^2$

Therefore, the complete integral is given by

$$z = ax + (1 - \sqrt{a})^2 y + c \tag{11}$$

where a and c are arbitrary constants.

To find the singular solution (or) singular integral

Differentiate (11) w.r.t c and equate with 0, we get

$$\frac{\partial z}{\partial c} = 1 \ \Rightarrow 0 = 1, \text{ which is absurd.}$$

Therefore, there is no singular solution.

To find the general solution (or) general integral

Put $c = f(a)$ in (11), we get

$$z = ax + (1 - \sqrt{a})^2 y + f(a). \tag{12}$$

Now differentiating (12) w.r.t a gives

$$\frac{\partial z}{\partial a} = x + y.2(1 - \sqrt{a}) \times \frac{-1}{2\sqrt{a}} + f^{'}(a)$$

$$\Rightarrow 0 = x - \frac{(1 - \sqrt{a})y}{\sqrt{a}} + f^{'}(a) \tag{13}$$

Eliminating a from (12) and (13), we get general solution.

Example 2: Solve $p^2 + q^2 = npq$.

Solution: This is of the form $F(p, q) = 0$.

Assume the solution is $z = ax + by + c$ where $a^2 + b^2 = nab$.

$\Rightarrow b^2 - nab + a^2 = 0$.This is quadratic in$b$.

$\Rightarrow b = \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right]$.

Hence the complete integral is

$$z = ax + \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right] y + c. \tag{14}$$

where a and c are arbitrary constants.

To find the singular integral:

Differentiating (14) w.r.t c, we get

$$\frac{\partial z}{\partial c} = 0 \;\Rightarrow 1 = 0, \text{ which is absurd.}$$

Therefore, there is no singular integral.

To find the general integral:

Put $c = f(a)$ in (14), we get

$$z = ax + \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right] y + f(a). \tag{15}$$

Now differentiation of (15) w.r.t a gives

$$\frac{\partial z}{\partial a} = x + \frac{1}{2} \times \left[n \pm \sqrt{n^2 - 4}\right] y + f^{'}(a)$$

$$0 = x + \frac{1}{2} \times \left[n \pm \sqrt{n^2 - 4}\right] y + f^{'}(a) \tag{16}$$

Eliminating a from (15) and (16), we get general integral.

Example 3: Solve $p + q = pq$.

Solution: This is of the form $F(p, q) = 0$.

Assume the solution is $z = ax + by + c$ where $a + b = ab$.

$\Rightarrow a + b - ab = 0 \Rightarrow b = \dfrac{a}{a-1}$.
Hence the complete integral is

$$z = ax + \frac{a}{a-1}y + c. \tag{17}$$

where a and c are arbitrary constants.

To find the singular integral:

Differentiating (17) w.r.t c, we get

$$\frac{\partial z}{\partial c} = 0 \ \Rightarrow 1 = 0, \text{ which is absurd.}$$

Therefore, there is no singular integral.

To find the general integral:

Put $c = f(a)$ in (17), we get

$$z = ax + \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right] y + f(a). \tag{18}$$

Now differentiation of (9) w.r.t a gives

$$\frac{\partial z}{\partial a} = x + \frac{1}{2}\left[n \pm \sqrt{n^2 - 4}\right] y + f^{'}(a)$$

$$0 = x + \frac{1}{2}\left[n \pm \sqrt{n^2 - 4}\right] y + f^{'}(a) \tag{19}$$

Eliminating a from (18) and (19), we get general integral.

Type II : Clairaut's form

Let the first order PDEs is of the form $z = px + qy + f(p, q)$ where $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$. then the solution can be obtained as below:

- **Complete integral:** Put $p = a$ and $q = b$ and write the complete integral

$$z = ax + by + f(a, b). \tag{20}$$

- **Singular integral:** Differentiating (20) w.r.to a and b, we get

$$x + \frac{\partial f}{\partial a} = 0 \text{ and} \tag{21}$$

$$y + \frac{\partial f}{\partial b} = 0. \tag{22}$$

Eliminatinating a and b from (20), (21) and (22), we get the singular integral.

- **General integral:** Putting $b = \psi(a)$ in (20), we get

$$z = ax + \psi(a)y + f(a, \psi(a)). \tag{23}$$

Differentiation of (23) w.r.t a gives

$$\frac{\partial z}{\partial a} = 0 = x + \psi^{'}(a)y + f^{'}(a) \tag{24}$$

Eliminating a from (23) and (24), one can get the general integral.

Example 1: Solve $z = px + qy + p^2 - q^2$.

Solution: This is of the form $z = px + qy + f(p, q)$ where $f(p, q) = p^2 - q^2$. Therefore the complete integral is

$$z = ax + by + a^2 - b^2. \tag{25}$$

To find singular integral: Differentiate (25) w.r.t a and b, we get

$$\frac{\partial z}{\partial a} = x + 2a \Rightarrow 0 = x + 2a \tag{26}$$

and

$$\frac{\partial z}{\partial b} = y - 2b \Rightarrow 0 = y - 2b \tag{27}$$

$$\text{From (26)} \Rightarrow a = -\frac{x}{2} \tag{28}$$

$$\text{From (27)} \Rightarrow b = \frac{y}{2} \tag{29}$$

Substituting (28) and (29) in (25), we get

$$4z = y^2 - x^2.$$

To find general integral: Put $b = \psi(a)$ in (25) becomes

$$z = ax + \psi(a)y + a^2 - (\psi(a))^2. \tag{30}$$

Differentiating (30) w.r.t a, we get

$$\frac{\partial z}{\partial a} = 0 = x + \psi'(a)y + 2a - 2\psi(a)\psi'(a). \tag{31}$$

Eliminating a from (30) and (31), we will get the general solution.

Example 2: Find the singular solution of $z = px + qy + \sqrt{1 + p^2 + q^2}$.

Solution: This is of the form $z = px + qy + f(p, q)$ where $f(p, q) = \sqrt{1 + p^2 + q^2}$. Therefore the complete integral is

$$z = ax + by + \sqrt{1 + a^2 + b^2}. \tag{32}$$

To find the singular integral: Differentiate (32) w.r.t a and b, we get

$$\frac{\partial z}{\partial a} = 0 = x + \frac{a}{\sqrt{1 + a^2 + b^2}} \Rightarrow x = -\frac{a}{\sqrt{1 + a^2 + b^2}} \text{ and} \tag{33}$$

$$\frac{\partial z}{\partial b} = 0 = y + \frac{b}{\sqrt{1 + a^2 + b^2}} \Rightarrow y = -\frac{b}{\sqrt{1 + a^2 + b^2}}. \tag{34}$$

Now

$$\begin{aligned} x^2 + y^2 &= \frac{a^2}{1 + a^2 + b^2} + \frac{b^2}{1 + a^2 + b^2} \\ \Rightarrow 1 - x^2 - y^2 &= \frac{1}{1 + a^2 + b^2} \\ \Rightarrow 1 + a^2 + b^2 &= \frac{1}{1 - x^2 - y^2}. \end{aligned}$$

$$\Rightarrow \sqrt{1 + a^2 + b^2} = \frac{1}{\sqrt{1 - x^2 - y^2}} \tag{35}$$

Substituting (35) in (33) and (34), we get

$$\Rightarrow a = -\frac{x}{\sqrt{1 - x^2 - y^2}} \text{ and } b = -\frac{y}{\sqrt{1 - x^2 - y^2}}.$$

Substituting a and b in (32), we get

$$\begin{aligned} z &= -\frac{x^2}{\sqrt{1 - x^2 - y^2}} - \frac{y^2}{\sqrt{1 - x^2 - y^2}} + \frac{1}{1 - x^2 - y^2} \\ z &= \frac{1 - x^2 - y^2}{\sqrt{1 - x^2 - y^2}} \\ z &= \sqrt{1 - x^2 - y^2} \end{aligned}$$

Therefore $z^2 = 1 - x^2 - y^2$ (or) $x^2 + y^2 + z^2 = 1$.

Example 3: Find the singular solution of $z = px + qy + p^2q^2$

Solution: This is of the form $z = px + qy + f(p, q)$ where $f(p, q) = p^2q^2$.

Therefore the complete integral is

$$z = ax + by + a^2b^2. \tag{36}$$

To find the singular integral: Differentiate (36) w.r.t a and b, we get

$$\begin{aligned} x + 2ab^2 &= 0 \Rightarrow x = -2ab^2 \\ &\text{and} \\ y + 2a^2b &= 0 \Rightarrow y = -2a^2b. \end{aligned} \tag{37}$$

From the above equations one can find

$$\frac{x}{b} = \frac{y}{a} = -2ab = \frac{1}{k} \Rightarrow a = ky \text{ and } b = kx. \tag{38}$$

Using (38) in (37) we get

$$\begin{aligned} x &= -2k^3x^2y \Rightarrow k^3 = -\frac{1}{2xy} \\ z &= kxy + kxy + k^4x^2y^2 = 2kxy + kx^2y^2\left(-\frac{1}{2xy}\right) = \frac{3}{2}kxy \\ \Rightarrow z^3 &= \frac{27}{8}k^3x^3y^3 = \frac{27}{8}k^3x^3y^3\left(-\frac{1}{2xy}\right) = -\frac{27}{16}x^2y^2 \end{aligned}$$

Therefore $16z^3 + 27x^2y^2 = 0$.

Example 4: Solve $z = px + qy + p^2 + pq + q^2$.

Solution: This is of the form $z = px + qy + f(p, q)$ where $f(p, q) = p^2 + pq + q^2$.

Therefore the complete integral is

$$z = ax + by + a^2 + ab + b^2. \tag{39}$$

To find the singular integral: Differentiate (39) w.r.t a and b, we get

$$\frac{\partial z}{\partial a} = x + 2a + b \Rightarrow 0 = x + 2a + b \tag{40}$$

and

$$\frac{\partial z}{\partial b} = y + a + 2b \Rightarrow 0 = y + a + 2b \tag{41}$$

Solving (40) and (41) by rule of cross multiplication, we get

$$\frac{a}{y-2x} = \frac{b}{x-2y} = \frac{1}{4}$$

$$\Rightarrow a = \frac{y-2x}{3} \text{ and } b = \frac{x-2y}{3} \tag{42}$$

Substituting (43) in (39), we get $3z = xy - x^2 - y^2$.

To find the general integral: Put $b = \psi(a)$ in (39) becomes

$$z = ax + \psi(a)y + a^2 + a\psi(a) + (\psi(a))^2.$$

Differentiating the above equation w.r.t a we find

$$0 = x + \psi^{'}(a)y + 2a + \psi(a) + a\psi^{'}(a) + 2\psi(a)\psi^{'}(a).$$

Eliminating a from the above equations, we will get the general solution.

Self Practice Problems:

(i) Solve $z = px + qy + \dfrac{p}{q} - p$.

(ii) Solve $z = px + qy + 2\sqrt{pq}$.

Type III: The PDEs of the form $F(z, p, q) = 0$ which does not contain x and y.

Assume that solution $z = f(x + ay)$ is a solution, where a is a constant.

$$\begin{aligned} \text{Put } u &= x + ay \text{ then } z = f(u) \\ \Rightarrow p &= \frac{\partial z}{\partial x} = \frac{dz}{du}.\frac{\partial u}{\partial x} = \frac{dz}{du} \text{ and } q = \frac{\partial z}{\partial y} = \frac{dz}{du}.\frac{\partial u}{\partial y} = a\frac{dz}{du}. \end{aligned}$$

Then find the solutions as follows:

- **Step I:** Putting $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$ with $u = x + by$, we will get the ODE of the form $\frac{dz}{du} = \phi(z, a)$.
- **Step II:** Solve the ODE $\frac{dz}{du} = \phi(z, a)$ as $\frac{dz}{\phi(z, a)} = du$.

  Integrating this we get $\int \frac{dz}{\phi(z, a)} = u + c = x + by + c$, where c is arbitrary constant.
- **Step III:** Singular and general integrals are found out as usual.

Example 1: Solve $p(1 + q) = qz$.

Put $u = x + ay$ then $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$ in the given equation.

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = az\frac{dz}{du} \Rightarrow 1 + a\frac{dz}{du} = az$$

$$\Rightarrow a\frac{dz}{du} = (az - 1) \Rightarrow \frac{adz}{(az - 1)} = du$$

$$\Rightarrow \int \frac{adz}{(az - 1)} = \int du + c \Rightarrow \log(az - 1) = u + c.$$

Therefore, the complete integral is

$$\log(az - 1) = x + ay + c \tag{43}$$

Singular integral: Differentiate (43) w.r.t. c, we get

$$\frac{\partial z}{\partial c} = 0 = 1, \text{ which is absurd.}$$

Therefore Singular solution does not exist.

General integral: Put $c = \psi(a)$ in (43), we get

$$\log(az - 1) = x + ay + \psi(a) \tag{44}$$

Differentiate w.r.t. a, we get

$$\frac{z}{(az - 1)} = y + \psi'(a) \tag{45}$$

eliminating a from (44) and (45), we will get the general integral.

Example 2: Solve $9(p^2z+q^2)=4$.

Solution: This is of the form $F(z,p,q)=0$.

Put $u=x+ay$ then $p=\dfrac{dz}{du}$ and $q=a\dfrac{dz}{du}$ in the given equation.

$$9\left[z\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2\right]=4 \;\Rightarrow 9\left(\frac{dz}{du}\right)^2(z+a^2)=4$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2=\frac{4}{9(z+a^2)} \;\Rightarrow \frac{dz}{du}=\frac{2}{3\sqrt{(z+a^2)}}$$

$$\Rightarrow 3\sqrt{(z+a^2)}dz=2du$$

Integrating both sides, we get $(z+a^2)^{3/2}=u+c$. Therefore the complete integral is given by

$$(z+a^2)^{3/2}=x+ay+c. \tag{46}$$

Singular integral: Differentiate (46) w.r.t. c, we get

$$\frac{\partial z}{\partial c}=0=1, \text{ which is absurd.}$$

Therefore Singular solution does not exist.

General integral: Put $c=\psi(a)$ in (46), we get

$$(z+a^2)^{3/2}=x+ay+\psi(a) \tag{47}$$

Differentiating (47) w.r.t. a partially, and eliminating a we get the general integral.

Example 3: Solve $z^2(p^2+q^2+1)=1$.

Solution: This is of the form $F(z,p,q)=0$.

Put $u=x+ay$ then $p=\dfrac{dz}{du}$ and $q=a\dfrac{dz}{du}$ in the given equation.

$$z^2\left[\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2+1\right]=1 \;\Rightarrow z^2\left[(1+a^2)\left(\frac{dz}{du}\right)^2+1\right]=1$$

$$\Rightarrow z^2(1+a^2)\left(\frac{dz}{du}\right)^2=1-z^2 \;\Rightarrow z\sqrt{1+a^2}\frac{dz}{du}=\sqrt{1-z^2}$$

$$\sqrt{1+a^2}\frac{z}{\sqrt{1-z^2}}\,dz = \,du$$

Integrating both sides, we get $-\sqrt{1+a^2}\sqrt{1-z^2} = u + c$.

Therefore the complete integral is given by

$$-\sqrt{1+a^2}\sqrt{1-z^2} = x + ay + c. \tag{48}$$

Singular integral: Differentiate (48) w.r.t. c, we get

$$\frac{\partial z}{\partial c} = 0 = 1, \text{ which is absurd.}$$

Therefore Singular solution does not exist.
General integral: Put $c = \psi(a)$ in (46), we get

$$-\sqrt{1+a^2}\sqrt{1-z^2} = x + ay + \psi(a) \tag{49}$$

Differentiating (49) w.r.t. a partially, and eliminating a we get the general integral.

Self Practice Problems:

(i) Solve $z = p^2 + q^2$.

(ii) Solve $p(1-q^2) = q(1-z)$.

(iii) Solve $q^2 = z^2p^2(1-p^2)$.

Type IV (Separable Equations):

The first order PDE is said to be separable equation it it can put in the form $f(x,p) = \phi(y,p)$.

For such PDE the solutions can be obtained as:

- **Step I:** Put $f(x,p) = g(y,p) = a$
- **Step II:** Write $p = f_1(x,a)$ and $q = g_1(y,a)$

- **Step III:** Putting in $dz = pdx + qdy$ and integrating, get the complete integral as
$$z = \int f_1(x,a)dx + \int g_1(y,a)dy + c.$$
- **Step IV:** Singular and general integrals are found out as usual.

Example 1: Solve $p^2y(1+x^2) = qx^2$.

Solution: The equation is separable. Therefore
$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a \text{ where } a \text{ is arbitrary constant.}$$
$$\Rightarrow p^2 = \frac{ax^2}{(1+x^2)} \text{ and } q = ay.$$
$$\Rightarrow p = \frac{\sqrt{a}x}{\sqrt{1+x^2}} \text{ and } q = ay.$$
We know that $dz = pdx + qdy$.
$$\text{Therefore } dz = \frac{\sqrt{a}x}{\sqrt{1+x^2}}dx + aydy.$$
Integrating the above equation we get the complete integral as
$$z = \sqrt{a}\sqrt{(1+x^2)} + \frac{ay^2}{2} + c \tag{50}$$
Singular integral: Differentiate (50) w.r.t. c, we get
$$\frac{\partial z}{\partial c} = 0 = 1, \text{ which is absurd.}$$
Therefore Singular solution does not exist.
General integral: The general integral is obtained by putting $c = \psi(a)$ in (50) differentiating w.r.t. a partially, and eliminating a.

Example 2: Solve $p^2 + q^2 = x + y$.

Solution: The equation is separable. Therefore
$$p^2 - x = y - q^2 = a \quad \Rightarrow \quad p^2 = x + a \Rightarrow p = \sqrt{x+a} \text{ and}$$
$$q^2 = y - a \Rightarrow q = \sqrt{y-a}.$$

Now

$$dz = pdx + qdy = \sqrt{x+a}dx + \sqrt{y-a}dy.$$

Integrating the above equation we get the complete integral as

$$z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + c \tag{51}$$

Singular integral: Differentiate (51) w.r.t. c, we get

$$0 = 1, \text{ which is absurd.}$$

Therefore Singular solution does not exist.

General integral: The general integral is obtained by putting $c = \psi(a)$ in (51) differentiating w.r.t. a partially, and eliminating a.

Example 3: Solve $p^2 + q^2 = x^2 + y^2$.

Solution: The equation is separable. Therefore

$$p^2 - x^2 = y^2 - q^2 = a^2 \quad \Rightarrow \quad p^2 = x^2 + a^2 \Rightarrow p = \sqrt{x^2+a^2} \text{ and}$$
$$q^2 = y^2 - a^2 \Rightarrow q = \sqrt{y^2-a^2}.$$

Now

$$dz = pdx + qdy = \sqrt{x^2+a^2}dx + \sqrt{y^2-a^2}dy.$$

Integrating the above equation we get the complete integral as

$$z = \frac{a^2}{2}\sin h^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{x^2+a^2} + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cos h^{-1}\frac{y}{a} + c \tag{52}$$

Singular integral: Differentiate (52) w.r.t. c, we get

$$0 = 1, \text{ which is absurd.}$$

Therefore Singular solution does not exist.

General integral: The general integral is obtained by putting $c = \psi(a)$ in (52) differentiating w.r.t. a partially, and eliminating a.

Self Practice Problems:

(i) Solve $yp = 2xy + \log q$.

(ii) Solve $p - x^2 = q + y^2$.

Lagrange's Linear Equation

A linear first order PDE of the form $Pp + Qq = R$, where $P,\ Q,\ R$ are are functions of $x,\ y,\ z$. This is called Lagrange's linear equation.

Working Rule:

- Form the subsidiary equations (or auxiliary equations)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

- Solving the subsidiary equations, find two independent solutions $u(x, y, z) = a$ and $v(x, y, z) = b$ where a, b are arbitrary constants.
- Then the required general solution is $\phi(u, v) = 0\ \ [u = f(v) \text{ or } v = g(u)]$.

Generally, the subsidiary equation can be solved in two ways.

(1) Method of grouping
(2) Method of multipliers

Method of grouping

In the subsidiary equation $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$ if the variables can be separated in any pair of equations, then we get a solution.

Example 1: Find the general solution of $px + qy = z$.

Solution: This is of the form $Pp + Qq = R$.

Here $P = x, Q = y$ and $R = z$.

Therefore the auxiliary equation is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Considering the first two ratios and integrating, we get

$$\frac{dx}{x} = \frac{dy}{y}$$
$$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y}$$
$$\Rightarrow \log x = \log y + \log a$$
$$\Rightarrow \log\left(\frac{x}{y}\right) = \log a \Rightarrow \frac{x}{y} = a$$

Considering the second and third ratios and integrating, we get

$$\frac{dy}{y} = \frac{dz}{z}$$
$$\Rightarrow \int \frac{dy}{y} = \int \frac{dz}{z}$$
$$\Rightarrow \log y = \log z + \log b$$
$$\Rightarrow \log\left(\frac{y}{z}\right) = \log b \Rightarrow \frac{y}{z} = b$$

Therefore the general solution is given by

$$\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0.$$

Example 2: Solve $\frac{y^2z}{x}p + xzq = y^2$.

Solution: This is of the form $Pp + Qq = R$.

Here $P = \frac{y^2z}{x}, Q = xz$ and $R = y^2$.

Therefore the auxiliary equation is

$$\frac{dx}{\frac{y^2z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$$
$$\Rightarrow \frac{xdx}{y^2z} = \frac{dy}{xz} = \frac{dz}{y^2}$$

Considering the first two ratios and integrating, we get

$$\frac{xdx}{y^2z} = \frac{dy}{xz}$$
$$\Rightarrow x^2dx = y^2dy$$
$$\Rightarrow \int x^2dx = \int y^2dy$$
$$\frac{x^3}{3} = \frac{y^3}{3} + a$$
$$\Rightarrow x^3 - y^3 = 3a = a_1$$

Considering the first and last ratios, we get

$$\frac{xdx}{y^2z} = \frac{dz}{y^2} \Rightarrow xdx = zdz$$
$$\Rightarrow \int xdx = \int zdz$$
$$\frac{x^2}{2} = \frac{z^2}{2} + b$$
$$\Rightarrow x^2 - z^2 = 2b = b_1$$

Therefore the general solution is given by

$$\phi(x^3 - y^3, x^2 - y^2) = 0.$$

Example 3: Solve $\tan x\ p + \tan y\ q = \tan z$.

Solution: This is of the form $Pp + Qq = R$.

Here $P = \tan x, Q = \tan y$ and $R = \tan z$.

Therefore the auxiliary equation is

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$
$$\Rightarrow \cot x\, dx = \cot y\, dy = \cot z\, dz$$

Taking $\cot x\, dx = \cot y\, dy$ and integrating one can get

$$\log \sin x = \log \sin y + \log a \ \Rightarrow \frac{\sin x}{\sin y} = a.$$

Similarly taking

$$\cot y\, dy = \cot z\, dz \text{ and integrating we get } \frac{\sin y}{\sin z} = b.$$

Therefore the general solution is given by

$$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0.$$

Example 4: Solve $px^2 + qy^2 = z^2$.

Solution: This is of the form $Pp + Qq = R$.

Here $P = x^2, Q = y^2$ and $R = z^2$.

Therefore the auxiliary equation is

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

Considering the first two ratios and integrating, we get

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$\Rightarrow \int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$
$$\Rightarrow -\frac{1}{x} = -\frac{1}{y} + a$$
$$\Rightarrow \frac{1}{y} - \frac{1}{x} = a$$

Considering the second and third ratios and integrating, we get

$$\frac{dy}{y^2} = \frac{dz}{z^2}$$
$$\Rightarrow \int \frac{dy}{y^2} = \int \frac{dz}{z^2}$$
$$\Rightarrow -\frac{1}{y} = -\frac{1}{z} + b$$
$$\Rightarrow \frac{1}{z} - \frac{1}{y} = b$$

Therefore the general solution is given by

$$\phi\left(\frac{1}{y}-\frac{1}{x},\frac{1}{z}-\frac{1}{y}\right)=0.$$

Method of Multipliers

Choose any 3 multiples l, m, n which may be constants or functions of x, y, z, we have

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{l\,dx+m\,dy+n\,dz}{lP+mQ+nR}$$

where l, m, n are called Lagrangian multipliers.

If it is possible to choose l, m, n such that $lP+mQ+nR=0$ then $l\,dx+m\,dy+n\,dz=0$.

If $l\,dx+m\,dy+n\,dz=0$ is an exact differential then on integration we get $lx+my+nz=a$.

Similarly we can find another set of independent multipliers $l^{'}, m^{'}, n^{'}$ to find another solution.

Remark: Since we have to find two independent solutions $u=a$ and $v=b$, to find one solution by grouping method and other by multiplier method or both by two independent set of multipliers.

Example 5: Solve $(mz-ny)p+(nx-lz)q=(ly-mx)$.

Solution: This is of the form $Pp+Qq=R$.

Here $P=mz-ny, Q=nx-lz$ and $R=ly-mx$.

Therefore the auxiliary equation is

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

Using $x,\ y,\ z$ as multipliers, we get

$$\begin{aligned}\text{each ratio} &= \frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}\\ &= \frac{xdx+ydy+zdz}{0}.\end{aligned}$$

Therefore $xdx+ydy+zdz=0$.

Integrating, $\int xdx+\int ydy+\int zdz=0 \Rightarrow \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=a$.

$\Rightarrow x^2+y^2+z^2=2a$.

Again using $l,\ m,\ n$ as multipliers, we get

$$\begin{aligned}\text{each ratio} &= \frac{ldx+mdy+ndz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}\\ &= \frac{ldx+mdy+ndz}{0}.\end{aligned}$$

Therefore $ldx+mdy+ndz=0$.

Integrating, $\int ldx+\int mdy+\int ndz=0 \Rightarrow lx+my+nz=b$.

Therefore the general solution is given by

$$\phi(x^2+y^2+z^2, lx+my+nz)=0.$$

Example 6: Solve $x(z^2-y^2)p+y(x^2-z^2)q=z(y^2-x^2)$.

Solution: This is of the form $Pp+Qq=R$.

Here $P=x(z^2-y^2), Q=y(x^2-z^2)$ and $R=z(y^2-x^2)$.

Therefore the auxiliary equation is

$$\frac{dx}{x(z^2-y^2)}=\frac{dy}{y(x^2-z^2)}=\frac{dz}{z(y^2-x^2)}$$

Using $x, \ y, \ z$ as multipliers, we get

$$\text{each ratio} = \frac{xdx + ydy + zdz}{x^2(z^2 - y^2) + y^2(x^2 - z^2) + z^2(y^2 - x^2)}$$
$$= \frac{xdx + ydy + zdz}{0}.$$

Therefore $xdx + ydy + zdz = 0$.

Integrating, $\int xdx + \int ydy + \int zdz = 0 \Rightarrow \dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{z^2}{2} = a.$

$\Rightarrow x^2 + y^2 + z^2 = 2a.$

Again using $\dfrac{1}{x}, \ \dfrac{1}{y}, \ \dfrac{1}{z}$ as multipliers, we get

$$\frac{\frac{dx}{x}}{y^2 - z^2} = \frac{\frac{dy}{y}}{x^2 - z^2} = \frac{\frac{dz}{z}}{y^2 - x^2}$$
$$\text{each ratio} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{z^2 - y^2 + x^2 - z^2 + y^2 - x^2}$$
$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}.$$

Therefore $\dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{z} = 0.$

Integrating, $\displaystyle\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$

$\Rightarrow \log x + \log y + \log z = \log b \Rightarrow xyz = b.$

Therefore the general solution is given by

$$\phi(x^2 + y^2 + z^2, xyz) = 0.$$

Example 7: Solve $x(y - z)p + y(z - x)q = z(x - y)$.

Solution: This is of the form $Pp + Qq = R$.

Here $P = x(y - z), Q = y(z - x)$ and $R = z(x - y)$.

Therefore the auxiliary equation is

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

Using 1, 1, 1 as multipliers, we get

$$\begin{aligned}\text{each ratio} &= \frac{dx + dy + dz}{x(y-z) + y(z-x) + z(x-y)} \\ &= \frac{dx + dy + dz}{0}.\end{aligned}$$

Therefore $dx + dy + dz = 0$.

Integrating, $\int dx + \int dy + \int dz = 0 \Rightarrow x + y + z = a$.

Again using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get

$$\frac{\frac{dx}{x}}{y-z} = \frac{\frac{dy}{y}}{z-x} = \frac{\frac{dz}{z}}{x-y}$$

$$\begin{aligned}\text{each ratio} &= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y - z + z - x + x - y} \\ &= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}.\end{aligned}$$

Therefore $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$.

Integrating, $\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$

$\Rightarrow \log x + \log y + \log z = \log b \Rightarrow xyz = b$.

Therefore the general solution is given by

$$\phi(x + y + z, xyz) = 0.$$

Example 8: Solve $(y + z)p + (z + x)q = (x + y)$.

Solution: This is of the form $Pp + Qq = R$.

Here $P = y + z, Q = z + x$ and $R = x + y$.

Therefore the auxiliary equation is

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

It can also be written as below:

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$$

Considering the first two ratios and integrating, we get

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{-(x-y)}$$

$$\log(x+y+z) = -2\log(x-y) + \log a$$

$$\log(x+y+z) + 2\log(x-y) = \log a$$

$$\log(x+y+z).(x-y)^2 = \log a$$

$$\Rightarrow a = (x+y+z)(x-y)^2$$

Considering the second and third ratios and integrating, we get

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} \;\Rightarrow \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

$$\log(x-y) = \log(y-z) + \log b \;\Rightarrow b = \frac{(x-y)}{(y-z)}$$

Therefore the general solution is given by

$$\phi\left((x+y+z)(x-y)^2, \frac{(x-y)}{(y-z)}\right) = 0.$$

Example 9: Solve $zp + yq = x$.

Solution: This is of the form $Pp + Qq = R$.

Here $P = z, Q = y$ and $R = x$.

Therefore the auxiliary equation is

$$\frac{dx}{z} = \frac{dy}{y} = \frac{dz}{x}$$

Taking the first and last ratios and integrating, we get

$$\frac{dx}{z} = \frac{dz}{x}$$
$$\int x\,dx = \int z\,dz$$
$$\frac{x^2}{2} = \frac{z^2}{2} + a$$
$$x^2 - z^2 = 2a$$

Similarly considering

$$\frac{dx + dy + dz}{x + y + z} = \frac{dy}{y} \ \Rightarrow \frac{d(x + y + z)}{x + y + z} = \frac{dy}{y}$$

and integrating we get

$$\log(x + y + z) = \log y + \log b \ \Rightarrow b = \frac{(x + y + z)}{y}$$

Therefore the general solution is given by

$$\phi\left(x^2 - z^2, \frac{(x + y + z)}{y}\right) = 0.$$

Example 10: Solve $(p - q)z = z^2 + x + y$.

Solution: This is of the form $Pp + Qq = R$.

Here $P = z, Q = -z$ and $R = z^2 + x + y$.

Therefore the auxiliary equation is

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + x + y}$$

Considering the first two ratios and integrating, we get

$$\frac{dx}{z} = \frac{dy}{-z}$$
$$\Rightarrow \int dx = -\int dy$$

$$\Rightarrow x = -y + a \Rightarrow x + y = a \tag{1}$$

Note: Here neither the grouping method nor the multiplier method can be used to find the second solution. So, we use the first solution to find the second solution. Therefore

$$\begin{aligned}
\frac{dx}{z} &= \frac{dz}{z^2+x+y} \\
\frac{dx}{z} &= \frac{dz}{z^2+a} \text{ by (1)} \\
\Rightarrow \int dx &= \int \frac{zdz}{z^2+a} \\
\Rightarrow x &= \frac{1}{2}\log(z^2+a)+b \\
\Rightarrow 2x &- \log(z^2+a) = 2b \\
\Rightarrow 2x &- \log(z^2+x+y) = c
\end{aligned}$$

Therefore the general solution is given by

$$\phi\left(x+y, 2x-\log(z^2+x+y)\right) = 0.$$

Example 11: Solve $p-q=\log(x+y)$.

Solution: This is of the form $Pp+Qq=R$.

Here $P=1, Q=-1$ and $R=\log(x+y)$.

Therefore the auxiliary equation is

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)}$$

Considering the first two ratios and integrating, we get

$$\begin{aligned}
dx &= -\,dy \\
\Rightarrow \int dx &= -\int dy
\end{aligned}$$

$$\Rightarrow x = -y+a \Rightarrow x+y=a \tag{2}$$

Considering the first and last ratios and integrating, we get

$$\begin{aligned}
dx &= \frac{dz}{\log(x+y)} \\
\Rightarrow \int dx &= \int \frac{dz}{\log a} \text{ by (2)} \\
\Rightarrow x &= \frac{1}{\log a} z + b \\
\Rightarrow x - \frac{z}{\log(x+y)} &= b
\end{aligned}$$

Therefore the general solution is given by

$$\phi\left(x+y, x - \frac{z}{\log(x+y)}\right) = 0.$$

Example 12: Solve $z(x-y) = px^2 - qy^2$.

Solution: This is of the form $Pp + Qq = R$.

Here $P = x^2, Q = -y^2$ and $R = z(x-y)$.

Therefore the auxiliary equation is

$$\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{z(x-y)}$$

Each equal to $\dfrac{dx+dy}{x^2-y^2}$

Taking the first two ratios and integrating, we get

$$\begin{aligned}
\frac{dx}{x^2} &= \frac{dy}{-y^2} \\
\int x^{-2}\, dx &= -\int y^{-2}\, dy \\
-\frac{1}{x} &= \frac{1}{y} + a \\
\frac{1}{x} + \frac{1}{y} &= a_1
\end{aligned}$$

Also

$$\frac{dz}{z(x-y)} = \frac{dx+dy}{x^2-y^2}$$
$$\Rightarrow \frac{dz}{z(x-y)} = \frac{d(x+y)}{(x+y)(x-y)}$$
$$\Rightarrow \frac{dz}{z} = \frac{d(x+y)}{(x+y)}$$

Integrating $\log z = \log(x+y) + \log b \;\Rightarrow \dfrac{z}{x+y} = b.$

Therefore the general solution is given by

$$\phi\left(\frac{1}{x}+\frac{1}{y}, \frac{z}{x+y}\right) = 0.$$

Example 13: Solve $(2z-y)p + (x+z)q + 2x + y = 0.$

Solution: This is of the form $Pp + Qq = R$.

Here $P = 2z - y, Q = x + z$ and $R = -(2x+y)$.

Therefore the auxiliary equation is

$$\frac{dx}{2z-y} = \frac{dy}{x+z} = \frac{dz}{-(2x+y)}$$

Using $-1,\ 2,\ 1$ as multipliers, we get

$$\text{each ratio} = \frac{-\,dx + 2\,dy + \,dz}{-2z+y+2x+2z-2x-y}$$
$$= \frac{-\,dx + 2\,dy + \,dz}{0}.$$

Therefore $-\,dx + 2\,dy + \,dz = 0.$

Integrating, $-\int dx + 2\int dy + \int dz = 0 \Rightarrow -x + 2y + z = a.$

Again using $x,\ y,\ z$ as multipliers, we get

$$\begin{aligned}\text{each ratio} &= \frac{x\,dx + y\,dy + z\,dz}{x(2z-y)+y(x+z)-z(2x+y)}\\ &= \frac{x\,dx + y\,dy + z\,dz}{2zx - xy + yx + yz - 2xz - zy}\\ &= \frac{x\,dx + y\,dy + z\,dz}{0}.\end{aligned}$$

Therefore $x\,dx + y\,dy + z\,dz = 0$.

Integrating, $\displaystyle\int x\,dx + \int y\,dy + \int z\,dz = 0$

$\Rightarrow \dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{z^2}{2} = b \Rightarrow x^2 + y^2 + z^2 = b_1.$

Therefore the general solution is given by

$$\phi(-x + 2y + z, x^2 + y^2 + z^2) = 0.$$

Example 14: Solve $(3z - 4y)p + (4x - 2z)q = 2y - 3x$.

Solution: This is of the form $Pp + Qq = R$.

Here $P = 3z - 4y, Q = 4x - 2z$ and $R = 2y - 3x$.

Therefore the auxiliary equation is

$$\frac{dx}{3z-4y} = \frac{dy}{4x-2z} = \frac{dz}{2y-3x}$$

Using $2,\ 3,\ 4$ as multipliers, we get

$$\begin{aligned}\text{each ratio} &= \frac{2\,dx + 3\,dy + 4\,dz}{6z - 8y + 12x - 6z + 8y - 12x}\\ &= \frac{2\,dx + 3\,dy + 4\,dz}{0}.\end{aligned}$$

Therefore $2\,dx + 3\,dy + 4\,dz = 0$.

Integrating, $2\int dx + 3\int dy + 4\int dz = 0 \Rightarrow 2x + 3y + 4z = a.$

Again using $x,\ y,\ z$ as multipliers, we get

$$\begin{aligned}\text{each ratio} &= \frac{x\,dx + y\,dy + z\,dz}{3xz - 4xy - +4xy - 2yz + 2yz - 3xz}\\ &= \frac{x\,dx + y\,dy + z\,dz}{0}.\end{aligned}$$

Therefore $x\,dx + y\,dy + z\,dz = 0$.

Integrating, $\displaystyle\int x\,dx + \int y\,dy + \int z\,dz = 0$

$\Rightarrow \dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{z^2}{2} = b \Rightarrow x^2 + y^2 + z^2 = b_1$.

Therefore the general solution is given by

$$\phi(2x + 3y + 4z, x^2 + y^2 + z^2) = 0.$$

Example 15: Solve $(y^2 + z^2 - x^2)p - 2xyq + 2xz = 0$.

Solution: This is of the form $Pp + Qq = R$.

Here $P = y^2 + z^2 - x^2, Q = -2xy$ and $R = -2xz$.

Therefore the auxiliary equation is

$$\frac{dx}{y^2 + z^2 - x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}$$

Taking the last two ratios and integrating, we get

$$\begin{aligned}\frac{dy}{-2xy} &= \frac{dz}{-2xz}\\ \Rightarrow \int \frac{dy}{y} &= \int \frac{dz}{z}\\ \Rightarrow \log y &= \log z + \log a\\ \Rightarrow \frac{y}{z} &= a\end{aligned}$$

using $x,\ y,\ z$ as multipliers, we get

$$\begin{aligned}\text{each ratio} &= \frac{x\,dx + y\,dy + z\,dz}{x(y^2+z^2-x^2) - 2xy^2 - 2xz^2}\\ &= \frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)}.\end{aligned}$$

Therefore $\dfrac{dy}{-2xy} = \dfrac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)}$

$\Rightarrow \dfrac{dy}{y} = \dfrac{2(x\,dx + y\,dy + z\,dz)}{x^2+y^2+z^2}.$

Integrating, $\displaystyle\int \frac{dy}{y} = \int \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}$

$\Rightarrow \log y = \log(x^2+y^2+z^2) + \log b \Rightarrow \dfrac{y}{x^2+y^2+z^2} = b.$

Therefore the general solution is given by

$$\phi\left(\frac{y}{z}, \frac{y}{x^2+y^2+z^2}\right) = 0.$$

Homogeneous Linear PDE of the Second and Higher Order with Constant Coefficients

A linear PDE in which all the partial derivatives are of same order is called a homogeneous linear PDE.

Example:

(1) $\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = x + y.$

(2) $\dfrac{\partial^3 z}{\partial x^3} - \dfrac{\partial^3 z}{\partial x^2 \partial y} + \dfrac{\partial^3 z}{\partial x \partial y^2} + 5\dfrac{\partial^3 z}{\partial y^3} = e^{x+y}.$

General form of a homogeneous Linear PDE of the n^{th} order with constant coefficients:

$$a_0\frac{\partial^n z}{\partial x^n} + a_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \dots\dots\dots + +a_n\frac{\partial^n z}{\partial y^n} = R(x,\ y)$$

where $a_0,\ a_1,\ \dots\dots\ a_n$ are the constants.

If we take $\dfrac{\partial}{\partial x} = D, \dfrac{\partial}{\partial y} = D', \dfrac{\partial^r}{\partial x^r} = D^r, \dfrac{\partial^s}{\partial y^s} = D'^s$ and $\dfrac{\partial^{r+s}}{\partial x^r \partial y^s} = D^r D'^s$, the above equation can be written as

$$\left(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots\dots\dots + a_n D'^n\right) z = R(x,\ y)$$

$$\Rightarrow F\left(D,\ D'\right) z = R(x,\ y). \tag{1}$$

The general solution of (1) is z =C.F.+P.I. where C.F.=complementary function and P.I.=particular integral.

To find complementary function of $F\left(D,\ D'\right) z = R(x,\ y)$

Consider $F\left(D,\ D'\right) z = 0$

The auxiliary equation is $F(m, 1) = 0$ where $D = m, D' = 1$.

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots\dots\dots + a_n = 0.$$

Solve the auxiliary above equation and find the roots. Let $m_1,\ m_2,\ \dots\dots m_n$ are the n roots of this equation, which are real or complex.

Case-I: If $m_1 \neq m_2 \neq \dots\dots \neq m_n$ i.e. all the roots are distinct, then

C.F.=$z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \phi_3(y + m_3 x) + \dots\dots\dots + \phi_n(y + m_n x)$.

Case II: If two roots are equal, say $m_1 = m_2 = m$ and others are different, then

C.F.=$z = \phi_1(y + mx) + x\phi_2(y + mx) + \phi_3(y + m_3 x) + \dots\dots + \phi_n(y + m_n x)$.

Note:

(1) If three roots are equal, say $m_1 = m_2 = m_3 = m$ and others are different, then

C.F.=$z = \phi_1(y+mx)+x\phi_2(y+mx)+x^2\phi_3(y+mx)+\phi_4(y+m_4 x)+\dots\dots+\phi_n(y+m_n x)$.

(2) In general, if $m_1 = m_2 = \dots\dots = m_n$ i.e. all roots are equal then

C.F.=$z = \phi_1(y + m_1 x) + x\phi_2(y + m_1 x) + x^2\phi_3(y + m_1 x) + \dots\dots + x^{n-1}\phi_n(y + m_1 x)$.

(iii) There is no separate rule for complex roots as in the case of ODE.

For Example: Solve $(D^4 - D^{'4})z = 0$.

The auxiliary equation is $m^4 - 1 = 0$.

$\Rightarrow (m^2 + 1)(m^2 - 1) = 0$.

$\Rightarrow m = \pm i, \pm 1$.

Therefore the general solution is

$z = \phi_1(y + ix) + x\phi_2(y - ix) + \phi_3(y + x) + +\phi_3(y - x)$.

To find particular integral of $F\,(D,\ D')\,z = R(x,\ y)$

Type 1: If $R(x,\ y) = e^{ax+by}$, then P.I. is given by

$$\text{P.I} = \frac{1}{F\,(D,\ D')} e^{ax+by} = \frac{1}{F\,(a,\ b)} e^{ax+by},\ \text{if } F\,(a,\ b) \neq 0.$$

Note: If $F\,(a,\ b) = 0$, then multiply the numerator by x and differentiate $F\,(D,\ D')$ in the denominator w.r.to D, and then replace D by a, D' by b. Even then if the denominator is 0, proceed as above again.

Type 2: If $R(x,\ y) = \sin(ax + by)$ or $\cos(ax + by)$, then P.I. is given by

$$\begin{aligned} \text{P.I} &= \frac{1}{F\left(D^2,\ DD',\ D'^2\right)} \sin(ax + by) \text{ or } \cos(ax + by) \\ &= \frac{1}{F\,(-a^2,\ -ab,\ -b^2)} \sin(ax + by) \text{ or } \cos(ax + by), \end{aligned}$$

provided $F\,(-a^2,\ -ab,\ -b^2) \neq 0$.

Note: If $F\,(-a^2,\ -ab,\ -b^2) = 0$, then multiply the numerator by x and differentiate $F\left(D^2,\ DD',\ D'^2\right)$ in the denominator w.r.to D, and then replace D^2 by $-a^2$, $DD' = -ab$, D'^2 by $-b^2$. Even then if the denominator is 0, proceed as above again.

Type 3: If $R(x,\ y) = x^m y^n$, then P.I. is given by

$$\text{P.I} = \frac{1}{F\,(D,\ D')} x^m y^n$$

(i) If $m \geq n$, rewrite $[F\,(D,\ D')]$ by taking out the highest power of D, as

$\left[1 \pm F\left(\frac{D'}{D}\right)\right]^{-1}$ and expand using binomial expansion in power of $\frac{D'}{D}$.

(ii) If $m < n$, rewrite $[F(D,\ D')]$ by taking out the highest power of D', as $\left[1 \pm F\left(\frac{D}{D'}\right)\right]^{-1}$ and expand using binomial expansion in power of $\frac{D}{D'}$.

Note: $\frac{1}{D}F(x,y) = \int F(x,y)\,dx, y$ constant

$\frac{1}{D'}F(x,y) = \int F(x,y)\,dy, x$ constant

Type 4: If $R(x,\ y) = e^{ax+by}f(x,y)$, then P.I. is given by

$$\text{P.I} \;=\; \frac{1}{F(D,\ D')}e^{ax+by}f(x,y) = \frac{e^{ax+by}}{F(D+a,\ D'+b)}f(x,y),$$

This can be evaluated by any of the above methods.

Type 5 (General Method): $R(x,y)$ may not always be of the above types. If $R(x,\ y)$ is any functions of x, y then the general method to find the P.I is given by

$$\text{P.I} = \frac{1}{(D-m_1D')(D-m_2D').......(D-m_nD')}R(x,\ y).$$

Then we shall use one of the following formulas

Formula 1:

$$\frac{1}{(D-mD')}R(x,\ y) = \int R(x,\ c-mx)dx,\ \text{where } c = y+mx.$$

Formula 2:

$$\frac{1}{(D+mD')}R(x,\ y) = \int R(x,\ c+mx)dx,\ \text{where } c = y-mx.$$

Problems based on Type I

Example 1: Solve $(D^2 - 2DD' + D'^2)z = 8e^{x+2y}$.

Solution: The auxiliary equation is $m^2 - 2m + 1 = 0$ where $D = m, D^{'} = 1$.

$\Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1.$

Therefore, the C.F.= $\phi_1(y+x)+x\phi_2(y+x)$.

$$\begin{aligned} P.I. &= \frac{1}{D^2-2DD'+D'^2}.8e^{x+2y}, \\ &= 8.\frac{1}{(1)^2-2(1)(2)+(2)^2}.e^{x+2y} \\ &= 8.e^{x+2y} \end{aligned}$$

Therefore the general solution is

$$z = C.F. + P.I. = \phi_1(y+x)+x\phi_2(y+x)+8e^{x+2y}$$

Example 2: Solve $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial y^3} = e^{x+2y}$.

Solution: The above equation can be written as

$$\left(D^3 - 3D^2D' + 4D'^3\right) z = e^{x+2y}$$

The auxiliary equation is $m^3 - 3m^2 + 4 = 0$ where $D = m, D' = 1$.

$$\begin{array}{r|rrrr} & 1 & -3 & 0 & 4 \\ -1 & & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$\Rightarrow (m+1)(m^2-4m+4) = 0 \Rightarrow (m+1)(m-2)^2 = 0 \Rightarrow m = -1,\ 2,\ 2.$

Therefore, the C.F.= $\phi_1(y-x)+\phi_2(y+2x)+x\phi_2(y+2x)$.

$$\begin{aligned} P.I. &= \frac{1}{D^3-3D^2D'+4D'^3}.e^{x+2y} \\ &= \frac{1}{(1)^3-3(1)^2(2)+4(2)^3}.e^{x+2y} \\ &= \frac{e^{x+2y}}{27}. \end{aligned}$$

Therefore the general solution is

$$z = C.F. + P.I. = \phi_1(y-x)+\phi_2(y+2x)+x\phi_3(y+2x)+\frac{e^{x+2y}}{27}.$$

Example 3: Solve $\left(D^3 - 7DD'^2 - 6D'^3\right) z = e^{2x+y}$.

Solution: The auxiliary equation is $m^3 - 7m - 6 = 0$.

$$\begin{array}{r|rrrr} & 1 & 0 & -7 & -6 \\ -1 & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$\Rightarrow (m+1)(m^2 - m - 6) = 0$

$\Rightarrow (m+1)(m-3)(m+2) = 0 \Rightarrow m = -1,\ 3,\ -2.$

Therefore the C.F.$= \phi_1(y - x) + \phi_2(y + 3x) + \phi_3(y - 2x)$.

$$\begin{aligned} P.I. &= \frac{1}{D^3 - 7DD'^2 - 6D'^3} e^{2x+y} \\ &= \frac{1}{(2)^3 - 7(2)(1)^2 - 6(1)^3} e^{2x+y} \\ &= -\frac{e^{2x+y}}{12}. \end{aligned}$$

Therefore the general solution is

$$z = C.F. + P.I. = \phi_1(y - x) + \phi_2(y + 3x) + \phi_3(y - 2x) - \frac{e^{2x+y}}{12}.$$

Example 4: Solve $\left(D^2 - 4DD' + 4D'^2\right) z = e^{2x+y}$. (or) $r - 4s + 4t = e^{2x+y}$.

Solution: The auxiliary equation is $m^2 - 4m + 4 = 0$.

$\Rightarrow (m-2)^2 \ \Rightarrow m = 2,\ 2.$

Therefore the C.F.$= \phi_1(y + 2x) + x\phi_2(y + 2x)$.

$$\begin{aligned} P.I. &= \frac{1}{D^2 - 4DD' + 4D'^2} . e^{2x+y} \\ &= \frac{1}{(2)^2 - 4(2)(1) + 4(1)^2} . e^{2x+y} \\ &= \frac{e^{2x+y}}{0}. \end{aligned}$$

Differentiate the denominator w.r.t D and multiply the numerator by x

$$P.I. = \frac{x}{2D - 4D'}.e^{2x+y} = x.\frac{e^{2x+y}}{0}$$

Again differentiating the denominator w.r.t. D and multiply the numerator by x, we get

$$P.I. = \frac{x^2}{2}e^{2x+y}.$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1(y + 2x) + x\phi_2(y + 2x) + \dfrac{x^2 e^{2x+y}}{2}.$

Example 5: Solve $\left(D^3 - 3DD'^2 + 2D'^3\right) z = e^{2x-y} + e^{x+y}.$

Solution: The auxiliary equation is $m^3 - 3m + 2 = 0.$

$$\begin{array}{c|cccc} & 1 & 0 & -3 & 2 \\ 1 & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

$\Rightarrow (m - 1)(m^2 + m - 2) = 0 \Rightarrow (m - 1)(m - 1)(m + 2) = 0.$

$\Rightarrow m = 1,\ 1,\ -2.$

Therefore the C.F.$= \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y - 2x).$

$P.I. = P.I._1 + P.I._2$ where $P.I._1 = \dfrac{1}{D^3 - 3DD'^2 + 2D'^3}.e^{2x-y}$ and

$P.I._2 = \dfrac{1}{D^3 - 3DD'^2 + 2D'^3}.e^{x+y}.$

$$\begin{aligned} P.I._1 &= \frac{1}{D^3 - 3DD'^2 + 2D'^3}.e^{2x-y} \\ &= \frac{1}{8 - 6 - 2}.e^{2x-y} = \frac{e^{2x-y}}{0} \\ &= \frac{x}{3D^2 - 3D'^2}.e^{2x-y} = \frac{xe^{2x-y}}{9}. \end{aligned}$$

$$\begin{aligned}
P.I._2 &= \frac{1}{D^3 - 3DD'^2 + 2D'^3}.e^{x+y} \\
&= \frac{1}{1-3+2}.e^{x+y} = \frac{e^{x+y}}{0}
\end{aligned}$$

$$\begin{aligned}
&= \frac{x}{3D^2 - 3D'^2}.e^{x+y} = x.\frac{e^{x+y}}{0} \\
&= x^2\frac{e^{x+y}}{6D} = \frac{x^2e^{x+y}}{6}
\end{aligned}$$

Therefore the general solution is given by

$z = C.F. + P.I. = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y-2x) + \dfrac{xe^{2x-y}}{9} + \dfrac{x^2e^{x+y}}{6}$.

Example 6: Solve $\left(D^2 - 6DD' + 5D'^2\right)z = e^x \sinh y$.

Solution:

$$\begin{aligned}
\left(D^2 - 6DD' + 5D'^2\right)z &= e^x \sinh y \\
&= e^x\left(\frac{e^y - e^{-y}}{2}\right) \\
&= \frac{1}{2}\left[e^{x+y} - e^{x-y}\right]
\end{aligned}$$

The auxiliary equation is $m^2 - 6m + 5 = 0$.

$\Rightarrow (m-1)(m-5) = 0 \Rightarrow m = 1, 5$.

Therefore the C.F.$= \phi_1(y+x) + \phi_2(y+5x)$.

$P.I. = P.I._1 + P.I._2$ where $P.I._1 = \dfrac{1}{D^2 - 6DD' + 5D'^2}.\dfrac{1}{2}e^{x+y}$ and

$P.I._2 = \dfrac{1}{D^2 - 6DD' + 5D'^2}.\dfrac{1}{2}e^{x-y}$.

$$\begin{aligned}
P.I._1 &= \frac{1}{D^2 - 6DD' + 5D'^2}.\frac{1}{2}e^{x+y} \\
&= \frac{1}{1-6+5}.\frac{1}{2}e^{x+y} = \frac{e^{x+y}}{0} \\
&= \frac{x}{2D - 6D'}.\frac{1}{2}e^{x+y} = -\frac{xe^{x+y}}{8}.
\end{aligned}$$

$$P.I._2 = \frac{1}{D^2 - 6DD' + 5D'^2} \cdot \frac{-1}{2} e^{x-y}$$
$$= -\frac{1}{2(1+6+5)} . e^{x-y} = -\frac{1}{24} e^{x-y}.$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1(y+x) + \phi_2(y+5x) - \dfrac{xe^{x+y}}{8} - \dfrac{1}{24} e^{x-y}.$

Example 7: Solve $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial y^3} = e^{x+2y}.$

Solution: The above equation can be written as $(D^3 - 3D^2D' + 4D'^3)z = e^{x+2y}$.

The auxiliary equation is $m^3 - 3m^2 + 4 = 0$.

$$\begin{array}{r|rrrr} & 1 & -3 & 0 & 4 \\ -1 & & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$\Rightarrow (m+1)(m^2 - 4m + 4) = 0 \Rightarrow (m+1)(m-2)^2 = 0.$

$\Rightarrow m = -1,\ 2,\ 2.$

Therefore the C.F.$= \phi_1(y-x) + \phi_2(y+2x) + x\phi_3(y+2x)$.

$$P.I. = \frac{1}{D^3 - 3D^2D' + 4D'^3} . e^{x+2y}$$
$$= \frac{1}{(1)^3 - 3(1)^2(2) + 4(2)^3} . e^{x+2y}$$
$$= \frac{e^{x+2y}}{27}.$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1(y-x) + \phi_2(y+2x) + x\phi_3(y+2x) + \dfrac{e^{x+2y}}{27}.$

Example 8: Solve $(D^2 - 2DD')\, z = e^{2x}$.

Solution: The auxiliary equation is $m^2 - 2m = 0$.

$\Rightarrow m(m-2) = 0\ \Rightarrow m = 0,\ 2.$

Therefore the C.F.$= \phi_1(y) + \phi_2(y+2x)$.

$$\begin{aligned} P.I. &= \frac{1}{D^2 - 2DD'} e^{2x} \\ &= \frac{1}{(2)^2 - 2(2)(0)} e^{2x} \\ &= \frac{e^{2x}}{4}. \end{aligned}$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1(y) + \phi_2(y+2x) + \dfrac{e^{2x}}{4}$.

Problems based on Type II

Example 9: Solve $(D^3 - 2D^2D')\, z = \sin(x+2y)$.

Solution: The auxiliary equation is

$m^3 - 2m^2 = 0 \;\Rightarrow m^2(m-2) = 0 \;\Rightarrow m = 0,\; 0,\; 2.$

Therefore the C.F.$= \phi_1(y) + x\phi_2(y) + \phi_2(y+2x)$.

$$\begin{aligned} P.I. &= \frac{1}{D^3 - 2D^2D'} \sin(x+2y) \\ &= \frac{1}{D(D^2 - 2DD')} \sin(x+2y) \\ &= \frac{1}{D(-(1)^2 - 2(-1.2))} \sin(x+2y) \\ &= \frac{1}{3D} \sin(x+2y) \\ &= -\frac{1}{3} \cos(x+2y). \text{ since } \frac{1}{D} \text{ is integration w.r.t } x \end{aligned}$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1(y) + x\phi_2(y) + \phi_2(y+2x) - \dfrac{1}{3}\cos(x+2y)$.

Example 10: Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = \cos(3x+2y)$.

Solution: The above equation can be written as

$(D^2 + DD' - 6D'^2)z = \cos(3x+2y)$.

The auxiliary equation is $m^2 + m - 6 = 0$.

$\Rightarrow (m+3)(m-2) = 0 \ \Rightarrow m = -3,\ 2.$

Therefore the C.F.= $\phi_1(y-3x) + \phi_2(y+2x)$.

$$P.I. = \frac{1}{D^2 + DD' - 6D'^2} \cos(3x+2y)$$
$$= \frac{1}{-(3)^2 + (-3.2) - 6.(-(2)^2)} \cos(3x+2y)$$
$$P.I. = \frac{1}{-9-6+24} \cos(3x+2y)$$
$$= \frac{1}{9} \cos(3x+2y).$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1(y-3x) + \phi_2(y+2x) + \frac{1}{9}\cos(3x+2y).$

Example 11: Solve $(D^3 - 4D^2D' + 4DD'^2)z = 6\sin(3x+6y)$.

Solution: The auxiliary equation is $m^3 - 4m^2 + 4m = 0$.

$\Rightarrow m(m^2 - 4m + 4) = 0 \Rightarrow m(m-2)^2 = 0.$

$\Rightarrow m = 0,\ 2,\ 2.$

Therefore the C.F. is $= \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x)$.

$$P.I. = \frac{1}{D^3 - 4D^2D' + 4DD'^2}.6\sin(3x+6y)$$
$$= \frac{1}{D[-(3)^2 - 4(-3.6) + 4.(-(6)^2)]}.6\sin(3x+6y)$$
$$= \frac{1}{-81D}.6\sin(3x+6y)$$
$$P.I. = \frac{6}{-81}.\frac{1}{D}\sin(3x+6y)$$
$$= -\frac{6}{81}\left(\frac{-\cos(3x+6y)}{3}\right) = \frac{2}{81}\cos(3x+6y).$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) + \frac{2}{81}\cos(3x+6y).$

Example 12: Solve $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x + 2y)$.

Solution: The auxiliary equation is $m^3 - 7m - 6 = 0$.

$$\begin{array}{r|rrrr} & 1 & 0 & -7 & -6 \\ -1 & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$\Rightarrow (m + 1)(m^2 - m - 6) = 0 \Rightarrow (m + 1)(m + 2)(m - 3) = 0.$

$\Rightarrow m = -1,\ -2,\ 3.$

Therefore the C.F.$= \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x)$.

$$\begin{aligned}
P.I. &= \frac{1}{D^3 - 7DD'^2 - 6D'^3} \sin(x + 2y) \\
&= \frac{1}{D.D^2 - 7D.D'^2 - 6D'.D'^2} \sin(x + 2y) \\
&= \frac{1}{D(-(1)^2) - 7D(-(2)^2) - 6.D'(-(2)^2)} \sin(x + 2y) \\
&= \frac{1}{27D + 24D'} \sin(x + 2y) = \frac{1}{3}.\frac{1}{9D + 8D'} \sin(x + 2y) \\
&= \frac{1}{3}.\frac{(9D - 8D')}{(9D + 8D')(9D - 8D')} \sin(x + 2y) \\
&= \frac{1}{3}.\frac{(9D - 8D')}{81D^2 - 64D'^2} \sin(x + 2y) \\
&= \frac{1}{3}.\frac{9D(\sin(x + 2y)) - 8D'(\sin(x + 2y))}{81(-1) - 64(-4)} \\
&= \frac{1}{3}.\frac{9\cos(x + 2y)) - 8.2\cos(x + 2y))}{175} \\
&= -\frac{7}{525} \cos(x + 2y) = -\frac{1}{75} \cos(x + 2y)
\end{aligned}$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x) - \frac{1}{75}.cos(x + 2y).$

Example 13: Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = \cos(x + y)$.

Solution: The auxiliary equation is $m^3 + m^2 - m - 1 = 0$.

$$\begin{array}{c|cccc} & 1 & 1 & -1 & -1 \\ 1 & & 1 & 2 & 1 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

$\Rightarrow (m-1)(m^2+2m+1) = 0 \Rightarrow (m-1)(m+1)^2 = 0.$

$\Rightarrow m = 1,\ -1,\ -1.$

Therefore the C.F.$= \phi_1(y+x) + \phi_2(y-x) + x\phi_3(y-x)$.

$$\begin{aligned}
P.I. &= \frac{1}{D^3 + D^2D' - DD'^2 - D'^3}\cos(x+y) \\
&= \frac{1}{D.D^2 + D^2D' - D.D'^2 - D'.D'^2}\cos(x+y) \\
&= \frac{1}{D(-1) + (-1)D' - D(-1) - D'(-1)}\cos(x+y) \\
&= \frac{1}{-D - D' + D + D'}\cos(x+y) \\
&= \frac{1}{0}\cos(x+y) \\
&= x.\frac{1}{3D^2 + 2DD' - D'^2}\cos(x+y) \\
&= x.\frac{1}{3(-1) + 2(-1) - (-1)}\cos(x+y) \\
&= -\frac{x}{4}.\cos(x+y)
\end{aligned}$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1(y+x) + \phi_2(y-x) + x\phi_3(y-x) - \frac{x}{4}.\cos(x+y).$

Example 14: Solve $(4D^2 - 4DD' + D'^2)z = \sin x$

The auxiliary equation is $4m^2 - 4m + 1 = 0$.

$\Rightarrow (2m-1)^2 = 0 \ \Rightarrow m = \frac{1}{2},\ \frac{1}{2}.$

Therefore the C.F.$= \phi_1\left(y + \frac{1}{2}x\right) + x\phi_2\left(y + \frac{1}{2}x\right).$

$$\begin{aligned}
P.I. &= \frac{1}{4D^2 - 4DD' + D'^2}\sin x \\
&= \frac{1}{4(-1) - 4(1.0) + 0}\sin x \\
&= -\frac{1}{4}\sin x
\end{aligned}$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1\left(y + \frac{1}{2}x\right) + x\phi_2\left(y + \frac{1}{2}x\right) - \frac{1}{4}\sin x.$

Example 15: Solve $(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x + y)$.

Solution: The auxiliary equation is $2m^2 - 5m + 2 = 0$.

$\Rightarrow (2m^2 - 4m - m + 2) = 0 \Rightarrow 2m(m - 2) - 1(m - 2) = 0.$

$\Rightarrow (2m - 1)(m - 2) = 0 \Rightarrow m = \frac{1}{2},\ 2.$

Therefore the C.F.$= \phi_1\left(y + \frac{1}{2}x\right) + \phi_2(y + 2x)$.

$$\begin{aligned}
P.I. &= \frac{1}{2D^2 - 5DD' + 2D'^2}.5\sin(2x + y) \\
&= \frac{1}{2(-4) - 5(-2.1) + 2(-1)}.5\sin(2x + y) \\
&= \frac{1}{0}.5\sin(2x + y) \\
&= 5x.\frac{1}{4D - 5D'}\sin(2x + y) \\
&= 5x.\frac{(4D + 5D')}{(4D - 5D')(4D + 5D')}\sin(2x + y) \\
&= 5x.\frac{(4D + 5D')}{16D^2 - 25D'^2}\sin(2x + y) \\
&= 5x.\frac{4D(\sin(2x + y)) + 5D'(\sin(2x + y))}{16D^2 - 25D'^2} \\
&= 5x.\frac{4.2\cos(2x + y)) + 5\cos(2x + y))}{16(-4) - 25(-1)} \\
&= 5x.\frac{13}{(-39)}\cos(2x + y) = -\frac{5x}{3}\cos(2x + y)
\end{aligned}$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1\left(y + \frac{1}{2}x\right) + \phi_2(y + 2x) - \frac{5x}{3}\cos(2x + y).$

Example 16: Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y.$

Solution: The above equation can be written as

$$(D^2 - 2DD')z = \sin x \cos 2y \tag{2}$$

We know that $2\sin A \cos B = \sin(A + B) + \sin(A - B)$.

Therefore $\sin x \cos 2y = \frac{1}{2}\left[\sin(x + 2y) + \sin(x - 2y)\right].$

Hence equation (2) implies that

$(D^2 - 2DD')z = \frac{1}{2}\left[\sin(x + 2y) + \sin(x - 2y)\right].$

The auxiliary equation is $m^2 - 2m = 0$.

$\Rightarrow m(m - 2) = 0 \Rightarrow m = 0,\ 2.$

Therefore the C.F.$= \phi_1(y) + \phi_2(y + 2x)$.

$P.I. = P.I._1 + P.I._2$ where $P.I._1 = \frac{1}{D^2 - 2DD'}.\frac{1}{2}\sin(x + 2y)$ and

$P.I._2 = \frac{1}{D^2 - 2DD'}.\frac{1}{2}\sin(x - 2y).$

$$\begin{aligned}
P.I._1 &= \frac{1}{D^2 - 2DD'}.\frac{1}{2}\sin(x + 2y) \\
&= \frac{1}{2}.\frac{1}{-1 - 2(-2)}.\sin(x + 2y) \\
&= \frac{1}{6}\sin(x + 2y) \\
P.I._2 &= \frac{1}{D^2 - 2DD'}.\frac{1}{2}\sin(x - 2y) \\
&= \frac{1}{2}.\frac{1}{-1 - 2(2)}.\sin(x - 2y) \\
&= -\frac{1}{10}\sin(x - 2y)
\end{aligned}$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1(y) + \phi_2(y+2x) + \frac{1}{6}\sin(x+2y) - \frac{1}{10}\sin(x-2y).$

Example 17: Solve $(D^2 - 2DD')z = \sin x \sin 2y$.

Solution: We know that $2\sin A \sin B = \cos(A-B) - \cos(A+B)$.

Therefore $\sin x \sin 2y = \frac{1}{2}\left[\cos(x-2y) - \cos(x+2y)\right]$.

Hence $(D^2 - 2DD')z = \frac{1}{2}\left[\cos(x-2y) - \cos(x+2y)\right]$.

The auxiliary equation is $m^2 - 2m = 0$.

$\Rightarrow m(m-2) = 0 \Rightarrow m = 0,\ 2.$

Therefore the C.F.$= \phi_1\,(y) + \phi_2(y+2x)$.

$P.I. = P.I._1 + P.I._2$ where $P.I._1 = \frac{1}{D^2 - 2DD'}.\frac{1}{2}\cos(x-2y)$ and

$P.I._2 = \frac{1}{D^2 - 2DD'}.\frac{1}{2}\cos(x+2y)$.

$$\begin{aligned}
P.I._1 &= \frac{1}{D^2 - 2DD'}.\frac{1}{2}\cos(x-2y) \\
&= \frac{1}{2}.\frac{1}{-1-2(2)}.\cos(x-2y) \\
&= -\frac{1}{10}\cos(x-2y)
\end{aligned}$$

$$\begin{aligned}
P.I._2 &= \frac{1}{D^2 - 2DD'}.\left[-\frac{1}{2}\cos(x+2y)\right] \\
&= -\frac{1}{2}.\frac{1}{-1-2(-2)}.\cos(x+2y) \\
&= \frac{1}{6}\cos(x+2y)
\end{aligned}$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1(y) + \phi_2(y+2x) - \frac{1}{10}\cos(x-2y) - \frac{1}{6}\cos(x+2y).$

Example 18: Solve $(D^2 - D'^2)z = \cos 2x \cos 3y$.

Solution: We know that $2\cos A \cos B = \cos(A+B) + \cos(A-B)$.

Therefore $\sin x \sin 2y = \frac{1}{2}\left[\cos(2x+3y)+\cos(2x-3y)\right].$

Hence $(D^2 - D'^2)z = \frac{1}{2}\left[\cos(2x+3y)+\cos(2x-3y)\right].$

The auxiliary equation is $m^2 - 1 = 0.$

$\Rightarrow m^2 = 1 \Rightarrow m = -1,\ 1.$

Therefore the C.F.$= \phi_1\,(y-x) + \phi_2(y+x).$

$P.I. = P.I._1 + P.I._2$ where $P.I._1 = \frac{1}{D^2 - D'^2}.\frac{1}{2}\cos(2x+3y)$ and

$P.I._2 = \frac{1}{D^2 - D'^2}.\frac{1}{2}\cos(2x-3y).$

$$\begin{aligned}
P.I._1 &= \frac{1}{D^2 - D'^2}.\frac{1}{2}\cos(2x+3y)\\
&= \frac{1}{2}.\frac{1}{-4-(-9)}.\cos(2x+3y)\\
&= \frac{1}{10}\cos(2x+3y)\\
P.I._2 &= \frac{1}{D^2 - D'^2}.\frac{1}{2}\cos(2x-3y)\\
&= \frac{1}{2}.\frac{1}{-4-(9)}.\cos(2x-3y)\\
&= -\frac{1}{26}\cos(2x-3y)
\end{aligned}$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1\,(y-x) + \phi_2(y+x) + \frac{1}{10}\cos(2x+3y) - \frac{1}{26}\cos(2x-3y).$

Example 19: Solve $\left(D^2 - 3DD' + 2D'^2\right)z = \sin x \cos y.$

We know that $2\sin A \cos B = \sin(A+B) + \sin(A-B).$

Therefore $\sin x \cos 2y = \frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right].$

Hence $(D^2 - 3DD' + 2D'^2)z = \frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right].$

The auxiliary equation is $m^2 - 3m + 2 = 0.$

$\Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1,\ 2.$

Therefore the C.F.$= \phi_1\,(y+x) + \phi_2(y+2x)$.

$P.I. = P.I._1 + P.I._2$ where $P.I._1 = \dfrac{1}{D^2 - 3DD' + 2D'^2}.\dfrac{1}{2}\sin(x+y)$ and

$P.I._2 = \dfrac{1}{D^2 - 3DD' + 2D'^2}.\dfrac{1}{2}\sin(x-y)$.

$$\begin{aligned}
P.I._1 &= \frac{1}{D^2 - 3DD' + 2D'^2}.\frac{1}{2}\sin(x+y) \\
&= \frac{1}{2}.\frac{1}{-1-3(-1.1)+2(-1)}.\sin(x+y) \\
&= \frac{1}{2}.\frac{1}{0}\sin(x+y) \\
&= \frac{1}{2}.x\frac{1}{2D-3D'}\sin(x+y) \\
&= \frac{x}{2}.\frac{(2D+3D')}{(2D-3D')(2D+3D')}\sin(x+y) \\
&= \frac{x}{2}.\frac{(2D+3D')}{4D^2-9D'^2}\sin(x+y) \\
&= \frac{x}{2}.\frac{2D(\sin(x+y)) + 3D'(\sin(x+y))}{4(-1)-9(-1)} \\
&= \frac{x}{10}.\left[2\cos(x+y) + 3\cos(x+y)\right] \\
&= \frac{x}{2}\cos(x+y)
\end{aligned}$$

$$\begin{aligned}
P.I._2 &= \frac{1}{D^2 - 3DD' + 2D'^2}.\frac{1}{2}\sin(x-y) \\
&= \frac{1}{2}.\frac{1}{-1-3(-1.(-1))+2(-1)}.\sin(x-y) \\
&= -\frac{1}{12}\sin(x-y)
\end{aligned}$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1\,(y-x) + \phi_2(y+x) + \dfrac{x}{2}\cos(x+y) - \dfrac{1}{12}\sin(x-y)$.

Problems based on Type III

Example 20: Solve $(D^3 - 2D^2D')\,z = 3x^2y$.

Solution: The auxiliary equation is

$m^3 - 2m^2 = 0 \;\Rightarrow m^2(m-2) = 0 \;\Rightarrow m = 0,\; 0,\; 2.$

Therefore the C.F.$= \phi_1(y) + x\phi_2(y) + \phi_3(y+2x)$.

$$\begin{aligned}
P.I. &= \frac{1}{D^3 - 2D^2D'}3x^2y = \frac{1}{D^3\left(1 - \dfrac{2D'}{D}\right)}3x^2y \\
&= \frac{1}{D^3}\left(1 - \frac{2D'}{D}\right)^{-1}.3x^2y \\
&= \frac{1}{D^3}\left[1 + \frac{2D'}{D} - \frac{4D'^2}{D^2} + \ldots\right]3x^2y. \\
&= \frac{1}{D^3}\left[1 + \frac{2D'}{D}\right]3x^2y = \frac{1}{D^3}\left[3x^2y + \frac{2.3x^2D'(y)}{D}\right] \\
&= \frac{1}{D^3}\left[3x^2y + \frac{6x^2}{D}\right] = \frac{1}{D^3}\left[3x^2y + 2x^3\right] \\
&= \frac{1}{D^2}\left[\frac{3yx^3}{3} + \frac{2x^4}{4}\right] = \frac{1}{D}\left[\frac{yx^4}{4} + \frac{x^5}{10}\right] \\
&= \frac{x^5y}{20} + \frac{x^6}{60}.
\end{aligned}$$

Therefore the general solution is given by:

$$z = C.F. + P.I. = \phi_1(y) + x\phi_2(y) + \phi_3(y+2x) + \frac{x^5y}{20} + \frac{x^6}{60}.$$

Example 21: Solve $(D^2 - 2DD')\,z = x^3y$.

Solution: The auxiliary equation is

$m^2 - 2m = 0 \;\Rightarrow m(m-2) = 0 \;\Rightarrow m = 0,\; 2.$

Therefore the C.F. is $= \phi_1(y) + \phi_2(y+2x)$.

$$
\begin{aligned}
P.I. &= \frac{1}{D^2 - 2DD'}.x^3y = \frac{1}{D^2\left(1 - \frac{2D'}{D}\right)}.x^3y \\
&= \frac{1}{D^2}\left(1 - \frac{2D'}{D}\right)^{-1}.x^3y \\
&= \frac{1}{D^2}\left[1 + \frac{2D'}{D} - \frac{4D'^2}{D^2} + \dots\right].x^3y \\
&= \frac{1}{D^2}\left[x^3y + \frac{2x^3}{D}\right] \\
&= \frac{1}{D^2}\left[x^3y + \frac{2x^4}{4}\right] \\
&= \frac{1}{D}\left[\frac{x^4y}{4} + \frac{x^5}{10}\right] \\
&= \frac{x^5y}{20} + \frac{x^6}{60}.
\end{aligned}
$$

Therefore the general solution is given by:

$$
z = C.F. + P.I. = \phi_1(y) + \phi_2(y+2x) + \frac{x^5y}{20} + \frac{x^6}{60}.
$$

Example 22: Solve $\left(D^2 + 2DD' + D'^2\right)z = x^2y.$

Solution: The auxiliary equation is $m^2 + 2m + 1 = 0.$

$\Rightarrow (m+1)^2 = 0 \ \Rightarrow m = -1,\ -1.$

Therefore the C.F.$= \phi_1(y-x) + x\phi_2(y-x).$

$$
\begin{aligned}
P.I. &= \frac{1}{D^2 + 2DD' + D'^2}.x^2y = \frac{1}{D^2\left(1 + \frac{2D'}{D} + \frac{D'^2}{D^2}\right)}.x^2y \\
&= \frac{1}{D^2}\left[1 + \left(\frac{2D'}{D} + \frac{D'^2}{D^2}\right)\right]^{-1}.x^2y \\
&= \frac{1}{D^2}\left[1 - \left(\frac{2D'}{D} + \frac{D'^2}{D^2}\right) + \left(\frac{2D'}{D} + \frac{D'^2}{D^2}\right)^2 + \dots\right].x^2y \\
&= \frac{1}{D^2}\left[1 - \frac{2D'}{D}\right]x^2y = \frac{1}{D^2}\left[x^2y - \frac{2x^2}{D}\right]
\end{aligned}
$$

$$
\begin{aligned}
P.I. &= \frac{1}{D^2}\left[x^2y - \frac{2x^3}{3}\right] \\
&= \frac{1}{D}\left[\frac{x^3y}{3} - \frac{2x^4}{3.4}\right] \\
&= \frac{x^4y}{12} - \frac{x^5}{30}.
\end{aligned}
$$

Therefore the general solution is given by:

$$z = C.F. + P.I. = \phi_1(y-x) + x\phi_2(y-x) + \frac{x^4y}{12} - \frac{x^5}{30}.$$

Example 23: Solve $\left(D^2 + 3DD' + 2D'^2\right)z = x + y$.

Solution: The auxiliary equation is $m^2 + 3m + 2 = 0$.

$\Rightarrow (m+1)(m+2) = 0 \ \Rightarrow m = -1,\ -2.$

Therefore the C.F.$= \phi_1(y-x) + \phi_2(y-2x)$.

$$
\begin{aligned}
P.I. &= \frac{1}{D^2 + 3DD' + 2D'^2}(x+y) = \frac{1}{D^2\left(1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right)}(x+y) \\
&= \frac{1}{D^2}\left[1 + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)\right]^{-1}(x+y) \\
&= \frac{1}{D^2}\left[1 - \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right) + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)^2 + \ldots\right](x+y) \\
&= \frac{1}{D^2}\left[(x+y) - \frac{3D'(x+y)}{D}\right] \\
&= \frac{1}{D^2}\left[(x+y) - \frac{3(0+1)}{D}\right] = \frac{1}{D^2}\left[(x+y) - 3x\right] \\
&= \frac{1}{D^2}(y-2x) = \frac{1}{D}[yx - \frac{2x^2}{2}] \\
&= \frac{yx^2}{2} - \frac{x^3}{3}
\end{aligned}
$$

Therefore the general solution is

$$z = C.F. + P.I. = \phi_1(y-x) + \phi_2(y-2x) + \frac{yx^2}{2} - \frac{x^3}{3}.$$

Example 24: Solve $\left(D^2 - 6DD' + 5D'^2\right) z = xy$.

Solution: The auxiliary equation is $m^2 - 6m + 5 = 0$.

$\Rightarrow (m-1)(m-5) = 0 \ \Rightarrow m = 1,\ 5.$

Therefore the C.F.$= \phi_1(y+x) + \phi_2(y+5x)$.

$$
\begin{aligned}
P.I. &= \frac{1}{D^2 - 6DD' + 5D'^2} xy = \frac{1}{D^2\left(1 - \frac{6D'}{D} + \frac{5D'^2}{D^2}\right)} xy \\
&= \frac{1}{D^2}\left[1 - \left(\frac{6D'}{D} - \frac{5D'^2}{D^2}\right)\right]^{-1} xy \\
&= \frac{1}{D^2}\left[1 + \left(\frac{6D'}{D} - \frac{5D'^2}{D^2}\right) + \left(\frac{6D'}{D} - \frac{5D'^2}{D^2}\right)^2 + \dots\right] xy \\
&= \frac{1}{D^2}\left[xy + \frac{6D'(xy)}{D}\right] \\
&= \frac{1}{D^2}\left[xy + \frac{6x}{D}\right] = \frac{1}{D^2}\left[xy + \frac{6x^2}{2}\right] \\
&= \frac{1}{D}\left[\frac{yx^2}{2} + \frac{3x^3}{3}\right] \\
&= \frac{yx^3}{6} + \frac{x^4}{4}
\end{aligned}
$$

Therefore the complete integral is

$z = C.F. + P.I. = \phi_1(y+x) + \phi_2(y+5x) + \dfrac{yx^3}{6} + \dfrac{x^4}{4}.$

Problems based on Type IV

Example 25: Solve $\left(D^2 - 3DD' + 2D'^2\right) z = (4x+2)e^{x+2y}$.

Solution: The auxiliary equation is $m^2 - 3m + 2 = 0$.

$\Rightarrow (m-1)(m-2) = 0 \ \Rightarrow m = 1,\ 2.$

Therefore the C.F. $= \phi_1(y+x) + \phi_2(y+2x)$.

$$
\begin{aligned}
P.I. &= \frac{1}{D^2 - 3DD' + 2D'^2}(4x+2)e^{x+2y} \\
&= \frac{1}{(D-D')(D-2D')}(4x+2)e^{x+2y} \\
&= e^{x+2y}.\frac{1}{[D+1-(D'+2)][D+1-2(D'+2)]}(4x+2) \\
&= e^{x+2y}.\frac{1}{[D-D'-1][D-2D'-3]}(4x+2) \\
&= e^{x+2y}.\frac{1}{3[1-(D-D')]\left[1-\left(\frac{D-2D'}{3}\right)\right]}(4x+2) \\
&= \frac{e^{x+2y}}{3}[1-(D-D')]^{-1}\left[1-\left(\frac{D-2D'}{3}\right)\right]^{-1}(4x+2) \\
&= \frac{e^{x+2y}}{3}\left[1+(D-D')+(D-D')^2+\ldots\right] \\
&\times \left[1+\left(\frac{D-2D'}{3}\right)+\left(\frac{D-2D'}{3}\right)^2+\ldots\right](4x+2) \\
&= \frac{e^{x+2y}}{3}(1+D)\left(1+\frac{D}{3}\right)(4x+2) \\
&= \frac{e^{x+2y}}{3}\left[1+\frac{4D}{3}\right](4x+2) \\
&= \frac{e^{x+2y}}{3}\left[4x+2+\frac{4D(4x+2)}{3}\right] \\
&= \frac{e^{x+2y}}{3}\left[4x+2+\frac{4}{3}.4\right] = \frac{e^{x+2y}}{3}\left(4x+\frac{22}{3}\right)
\end{aligned}
$$

Therefore $z = C.F. + P.I. = \phi_1(y+x) + \phi_2(y+2x) + \dfrac{e^{x+2y}}{3}\left(4x+\dfrac{22}{3}\right)$.

Example 26: Solve $\left(D^3 + D^2D' - DD'^2 - D'^3\right) z = e^x \cos 2y$.

Solution: The auxiliary equation is $m^3 + m^2 - m - 1 = 0$.

$$\begin{array}{c|cccc} & 1 & 1 & -1 & -1 \\ 1 & & 1 & 2 & 1 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

$\Rightarrow (m-1)(m^2+2m+1) = 0 \Rightarrow (m-1)(m+1)^2 = 0$.

$\Rightarrow m = 1,\ -1,\ -1$.

Therefore the C.F. $= \phi_1(y+x) + x\phi_2(y-x) + x\phi_3(y-x)$.

$$\begin{aligned}
P.I. &= \frac{1}{D^3 + D^2D' - DD'^2 - D'^3} e^x \cos 2y \\
&= \frac{1}{(D-D')(D+D')^2} e^x \cos 2y \\
&= e^x . \frac{1}{\left[D+1-(D'+0)\right]\left[D+1+D'+0\right]^2} \cos 2y \\
&= e^x . \frac{1}{\left[D-D'+1\right]\left[D+D'+1\right]^2} \cos 2y \\
&= e^x . \frac{1}{\left[D-D'+1\right]\left[D+D'+1\right]^2} \text{Real part of } e^{2iy} \\
&= e^x . \text{Real part of } \frac{1}{\left[0-2i+1\right]\left[0+2i+1\right]^2} e^{2iy} \text{ by Type I} \\
&= e^x . \text{Real part of } \frac{1}{5(1+2i)} e^{2iy} \\
&= \frac{e^x}{5} . \text{Real part of } \frac{(1-2i)}{(1+2i)(1-2i)} e^{2iy} \\
&= \frac{e^x}{5} . \text{Real part of } \frac{(1-2i)}{5} e^{2iy} \\
&= \frac{e^x}{25} . \text{Real part of } (1-2i)\left(\cos 2y + i \sin 2y\right) \\
&= \frac{e^x}{25} . \text{Real part of } \left[\cos 2y + i \sin 2y - 2i \cos 2y + 2 \sin 2y\right] \\
&= \frac{e^x}{25} . \text{Real part of } \left[\cos 2y + 2 \sin 2y + i(\sin 2y - 2 \cos 2y)\right]
\end{aligned}$$

$$P.I. = \frac{e^x}{25}(\cos 2y + 2\sin 2y)$$

Therefore $z = \phi_1(y+x) + x\phi_2(y-x) + x\phi_3(y-x) + \dfrac{e^x}{25}(\cos 2y + 2\sin 2y)$.

Example 26: Solve $(D^2 - 2DD' + D'^2)z = x^2y^2e^{x+y}$.

Solution: The auxiliary equation is $m^2 - 2m + 1 = 0$.

$\Rightarrow (m-1)^2 = 0 \ \Rightarrow m = 1,\ 1.$

Therefore the C.F. $= \phi_1(y+x) + x\phi_2(y+x)$.

$$\begin{aligned}
P.I. &= \frac{1}{D^2 - 2DD' + D'^2}x^2y^2e^{x+y} \\
&= \frac{1}{(D-D')^2}x^2y^2e^{x+y} \\
&= e^{x+y}.\frac{1}{[D+1-(D'+1)]^2}x^2y^2 \\
&= e^{x+y}.\frac{1}{(D-D')^2}x^2y^2
\end{aligned}$$

$$\begin{aligned}
P.I. &= e^{x+y}.\frac{1}{D^2\left[1-\dfrac{D'}{D}\right]^2}x^2y^2 \\
&= e^{x+y}.\frac{1}{D^2}\left[1-\frac{D'}{D}\right]^{-2}x^2y^2 \\
&= e^{x+y}.\frac{1}{D^2}\left[1+2\frac{D'}{D}+3\frac{D'^2}{D^2}+\dots\right]x^2y^2
\end{aligned}$$

Since $(1-x)^{-2} = 1 + 2x + 3x^2 + \dots$

$$\begin{aligned}
P.I. &= e^{x+y}.\frac{1}{D^2}\left[x^2y^2 + 2x^2.\frac{D'(y^2)}{D} + 3x^2.\frac{D'^2(y^2)}{D^2}\right] \\
&= e^{x+y}.\frac{1}{D^2}\left[x^2y^2 + \frac{2x^2}{D}.2y + \frac{3x^2}{D^2}.2\right] \\
&= e^{x+y}.\frac{1}{D^2}\left[x^2y^2 + \frac{4x^3y}{3} + \frac{x^4}{2}\right] \\
&= e^{x+y}.\frac{1}{D}\left[\frac{x^3y^2}{3} + \frac{x^4y}{3} + \frac{x^5}{10}\right] = e^{x+y}\left[\frac{x^4y^2}{12} + \frac{x^5y}{15} + \frac{x^6}{60}\right]
\end{aligned}$$

Therefore $z = C.F. + P.I. = \phi_1(y+x) + x\phi_2(y+x) + e^{x+y}\left[\frac{x^4y^2}{12} + \frac{x^5y}{15} + \frac{x^6}{60}\right]$.

Example 27: Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x-y}\sin(2x+3y)$.

Solution: The above equation can be written as $(D^2 - D'^2)z = e^{x-y}\sin(2x+3y)$.

The auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = -1,\ 1$.

Therefore the C.F. $= \phi_1(y-x) + \phi_2(y+x)$.

$$\begin{aligned}
P.I. &= \frac{1}{D^2 - D'^2} e^{x-y}\sin(2x+3y)\\
&= e^{x-y}.\frac{1}{(D+1)^2 - (D'-1)^2}\sin(2x+3y)\\
&= e^{x-y}.\frac{1}{D^2 + 2D + 1 - D'^2 + 2D' - 1}\sin(2x+3y)\\
&= e^{x-y}.\frac{1}{D^2 - D'^2 + 2(D+D')}\sin(2x+3y)
\end{aligned}$$

$$\begin{aligned}
P.I. &= e^{x-y}.\frac{1}{2(D+D')+5}\sin(2x+3y)\\
&= e^{x-y}.\frac{[2(D+D')-5]}{[2(D+D')+5][2(D+D')-5]}\sin(2x+3y)\\
&= e^{x-y}.\frac{[2(D+D')-5]}{4(D+D')^2 - 25}\sin(2x+3y)\\
&= e^{x-y}.\frac{[2(D+D')-5]}{4(D^2 + D'^2 + 2DD') - 25}\sin(2x+3y)\\
&= e^{x-y}.\frac{[2D(\sin(2x+3y)) + 2D'(\sin(2x+3y)) - 5\sin(2x+3y)]}{4(-4-9+2(-6)) - 25}\\
&= \frac{e^{x-y}}{-125}\left[4\cos(2x+3y) + 6\cos(2x+3y) - 5\sin(2x+3y)\right]\\
&= -\frac{e^{x-y}}{25}\left[2\cos(2x+3y) - \sin(2x+3y)\right]
\end{aligned}$$

Therefore $z = C.F. + P.I. = \phi_1(y-x) + \phi_2(y+x) - \frac{e^{x-y}}{25}\left[2\cos(2x+3y) - \sin(2x+3y)\right]$.

Problems based on General Rule:

Example 28: Solve $\left(D^2 + DD' - 6D'^2\right) z = y\cos x$.

Solution: The auxiliary equation is $m^2 + m - 6 = 0$.

$\Rightarrow (m-2)(m+3) = 0 \ \Rightarrow m = 2,\ -3.$

Therefore the C.F. $= \phi_1(y+2x) + \phi_2(y-3x)$.

$$\begin{aligned}
P.I. &= \frac{1}{D^2 + DD' - 6D'^2} y\cos x = \frac{1}{(D-2D')(D+3D')} y\cos x \\
&= \frac{1}{(D-2D')}\left[\frac{1}{(D+3D')} y\cos x\right] \\
& \frac{1}{(D-2D')}\left[\int (c+3x)\cos x dx\right] \text{ where } y = c+3x \\
&= \frac{1}{(D-2D')}\left[(c+3x)\sin x - 3\int \sin x dx\right] \\
&= \frac{1}{(D-2D')}\left[(c+3x)\sin x + 3\cos x\right] \\
&= \frac{1}{(D-2D')}\left[y\sin x + 3\cos x\right] \\
&= \int \left[(c-2x)\sin x + 3\cos x\right] dx \text{ where } y = c-2x \\
&= (c-2x)(-\cos x) - (-2)\int -\cos x dx + 3\sin x \\
&= -y\cos x - 2\sin x + 3\sin x = -y\cos x + \sin x.
\end{aligned}$$

Therefore the general solution is

$z = C.F. + P.I. = \phi_1(y+2x) + \phi_2(y-3x) - y\cos x + \sin x.$

Example 29: Solve $\left(D^2 - DD' - 2D'^2\right) z = (y-1)e^x$.

Solution: The auxiliary equation is $m^2 - m - 2 = 0$.

$\Rightarrow (m+1)(m-2) = 0 \ \Rightarrow m = -1,\ 2.$

Therefore the C.F.$= \phi_1(y-x) + \phi_2(y+2x)$.

$$
\begin{aligned}
P.I. &= \frac{1}{D^2 - DD' - 2D'^2}(y-1)e^x \\
&= \frac{1}{(D+D')(D-2D')}(y-1)e^x \\
&= \frac{1}{(D+D')}\left[\frac{1}{(D-2D')}(y-1)e^x\right] \\
&= \frac{1}{(D+D')}\left[\int (c-2x-1)e^x dx\right] \text{ where } c = y-2x \\
&= \frac{1}{(D+D')}\left[(c-2x-1)e^x - \int e^x(-2)dx\right] \\
&= \frac{1}{(D+D')}\left[(c-2x-1)e^x + 2e^x\right] \\
&= \frac{1}{(D+D')}\left[(c-2x+1)e^x\right] = \frac{1}{(D+D')}\left[(y+1)e^x\right] \\
&= \int (c+x+1)e^x dx \text{ where } y = c+x \\
&= (c+x+1)e^x - e^x \text{ integrating by part} \\
&= (c+x)e^x = ye^x
\end{aligned}
$$

Therefore the general solution is $z = C.F. + P.I. = \phi_1(y-x) + \phi_2(y+2x) + ye^x$.

Example 30: Solve $\left(D^2 + 2DD' + D'^2\right) z = 2\cos y - x \sin y$.

Solution: The auxiliary equation is $m^2 - m - 2 = 0$.

$\Rightarrow (m+1)(m+1) = 0 \;\Rightarrow m = -1,\ -1.$

Therefore the C.F.$= \phi_1(y-x) + x\phi_2(y-x)$.

$$
\begin{aligned}
P.I. &= \frac{1}{D^2 + 2DD' + D'^2}[2\cos y - x\sin y] \\
&= \frac{1}{(D+D')(D+D')}[2\cos y - x\sin y] \\
&= \frac{1}{(D+D')}\left[\frac{1}{(D+D')}(2\cos y - x\sin y)\right] \\
&= \frac{1}{(D+D')}\int [2\cos(a+x) - x\sin(a+x)]dx \text{ where } y = x+a
\end{aligned}
$$

$$\begin{aligned}
P.I. &= \frac{1}{(D+D')}\left[2\sin(a+x) + x\cos(a+x) - \sin(x+a)\right] \\
&= \frac{1}{(D+D')}\left[x\cos(a+x) + \sin(x+a)\right] \\
&= \frac{1}{(D+D')}\left[x\cos y + \sin y\right] \\
&= \int \left[x\cos(a+x) + \sin(a+x)\right] dx \text{ where } y = x+a \\
&= x\sin(a+x) + \cos(a+x) - cos(a+x) = x\sin y
\end{aligned}$$

Therefore $z = \phi_1(y-x) + x\phi_2(y-x) + x\sin y$.

Example 1: Solve $(D^2 - 3DD' + 2D'^2)z = 2\cos h(3x+4y)$.

Solution:

$$\begin{aligned}
\left(D^2 - 3DD' + 2D'^2\right) z &= 2\cos h(3x+4y) \\
&= 2\left[\frac{e^{3x+4y} + e^{-(3x+4y)}}{2}\right] \\
&= e^{3x+4y} + e^{-(3x+4y)}
\end{aligned}$$

The auxiliary equation is $m^2 - 3m + 2 = 0$.

$\Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1, 2$.

Therefore the C.F.$= \phi_1(y+x) + \phi_2(y+2x)$.

$P.I. = P.I._1 + P.I._2$ where $P.I._1 = \dfrac{1}{D^2 - 3DD' + 2D'^2}.e^{3x+4y}$ and

$P.I._2 = \dfrac{1}{D^2 - 3DD' + 2D'^2}.e^{-(3x+4y)}$.

$$\begin{aligned}
P.I._1 &= \frac{1}{D^2 - 3DD' + 2D'^2}.e^{3x+4y} \\
&= \frac{1}{9 - 3(3)(4) + 2(16)}.e^{3x+4y} \\
&= \frac{1}{5}e^{3x+4y}
\end{aligned}$$

$$\begin{aligned}
P.I._2 &= \frac{1}{D^2 - 3DD' + 2D'^2}.e^{-(3x+4y)} \\
&= \frac{1}{9 - 3(-3)(-4) + 2(16)}.e^{-(3x+4y)} \\
&= \frac{1}{5}e^{-(3x+4y)}
\end{aligned}$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1(y + x) + \phi_2(y + 2x) + \frac{1}{5}e^{3x+4y} + \frac{1}{5}e^{-(3x+4y)}$. (or)

$z = C.F. + P.I. = \phi_1(y + x) + \phi_2(y + 2x) + \frac{2}{5}\cos h(3x + 4y)$.

Example 2: Solve $(D^2 + D'^2)z = 2\sin^2(x + y)$.

Solution: We know that $\sin^2 x = \frac{1 - \cos 2x}{2}$.

Therefore $(D^2 + D'^2)z = 2\left[\frac{1 - \cos 2(x + y)}{2}\right]$.

$\Rightarrow (D^2 + D'^2)z = 1 - \cos(2x + 2y)$.

The auxiliary equation is $m^2 + 1 = 0$.

$\Rightarrow m^2 = -1 \Rightarrow m = -i, i$.

Therefore the C.F.$= \phi_1(y - ix) + \phi_2(y + ix)$.

$$\begin{aligned}
P.I. &= \frac{1}{D^2 + D'^2}\left[1 - \cos(2x + 2y)\right] \\
&= \frac{1}{D^2}\left[1 + \frac{D'^2}{D^2}\right]^{-1}(1) - \frac{1}{D^2 + D'^2}\cos(2x + 2y) \\
&= \frac{1}{D^2}(1) - \frac{1}{-4 - 4}\cos(2x + 2y) \\
&= \frac{x^2}{2} + \frac{1}{8}\cos(2x + 2y)
\end{aligned}$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1(y - ix) + \phi_2(y + ix) + \frac{x^2}{2} + \frac{1}{8}\cos(2x + 2y)$.

Example 3: Solve $(9D^2 + 6DD' + D'^2)z = (e^x + e^{-2y})^2$.

Solution: The auxiliary equation is $9m^2 + 6m + 1 = 0$.

$\Rightarrow (3m+1)^2 = 0 \Rightarrow m = -\dfrac{1}{3},\ -\dfrac{1}{3}$.

Therefore the C.F.$= \phi_1\left(y - \dfrac{1}{3}x\right) + x\phi_2\left(y - \dfrac{1}{3}x\right)$.

$$\begin{aligned}
P.I. &= \frac{1}{9D^2 + 6DD' + D'^2}(e^x + e^{-2y})^2 \\
&= \frac{1}{(3D + D')^2}\left[e^{2x} + e^{-4y} + 2e^{x-2y}\right] \\
&= \frac{1}{(3D + D')^2}e^{2x} + \frac{1}{(3D + D')^2}e^{-4y} + 2\frac{1}{(3D + D')^2}e^{x-2y} \\
&= \frac{1}{36}e^{2x} + \frac{1}{16}e^{-4y} + 2e^{x-2y}
\end{aligned}$$

Therefore the general solution is given by:

$z = C.F. + P.I. = \phi_1\left(y - \dfrac{1}{3}x\right) + x\phi_2\left(y - \dfrac{1}{3}x\right) + \dfrac{1}{36}e^{2x} + \dfrac{1}{16}e^{-4y} + 2e^{x-2y}$.