# Mathematical Induction :-

**Introduction:-** One of the most basic methods of proof is mathematical Induction, which is a method to establish the truth of a statement about all the natural numbers.

It will often help us to provide a general mathematical statement involving positive integers when certain instances of that statement suggest a general pattern.

**Statement of the principle of Mathematical Induction:**

Let $S(n)$ denotes the given statement, that involves one (or) more occurances of the variable 'n', which represents a positive integer

(a) If, $S(1)$ is true.

(b) If $S(k)$ is true for some particular, but arbitrarily chosen $k \in Z^+$, $S(k+1)$ is also true then $S(n)$ is true for all $n \in Z^+$

---

**Note:-** Step (a) is called the basic step.

Step (b) is known as inductive step.

1\. Prove by mathematical induction 2.

$$1^2+3^2+5^2+\cdots(2n-1)^2=\frac{1}{3}n(2n-1)(2n+1).$$

**Proof:-** Let $S_n = 1^2+3^2+5^2+\cdots(2n-1)^2=\frac{1}{3}n(2n-1)(2n+1)$

**Case ①:-** when $n=1$, $S_1 = 1^2+3^2+5^2+\cdots(2-1)^2=\frac{1}{3}(1).1(2+1)$

$$1^2 = \frac{\cancel{3}}{\cancel{3}}1$$

$$1=1$$

$\therefore S_1$ (or) $S(1)$ is true.

**Case ②:-** when $n=k$, $S(k)=1^2+3^2+5^2+\cdots(2k-1)^2=\frac{1}{3}k(2k-1)(2k+1).$

Now, $1^2+3^2+5^2+\cdots(2k-1)^2+(2k+1)^2$

$$=\frac{1}{3}k(2k-1)(2k+1)+(2k+1)^2$$

$$=(2k+1)\left[\frac{k(2k-1)}{3}+(2k+1)\right]$$

$$=(2k+1)\left[\frac{2k^2-k+6k+3}{3}\right]$$

$$=(2k+1)\left[\frac{2k^2+5k+3}{3}\right]$$

$$=\frac{(2k+1)[(k+1).(2k+3)]}{3}$$

$\Rightarrow S(k+1)$ is valid.

6
3 2

$2k^2+3k+2k+3$

$2k(k+1)+3(k+1)$

$(k+1)(2k+3)$

2. Prove by mathematical induction that,

$$\left.\begin{aligned}&1.2.3+2.3.4+3.4.5+\dots n(n+1)(n+2)\\&=\frac{1}{4}n(n+1)(n+2)(n+3).\end{aligned}\right\}\to ①$$

<u>Proof:-</u> $S(n)$ be the given series.

$\therefore \quad S(n) = 1.2.3 + \dots\dots n(n+1)(n+2)$

$= \frac{1}{4} n(n+1)(n+2)(n+3)$

$S(1) = 1.2.3 + \dots\dots 1(1+1)(1+2) = \frac{1}{4}[2.3.4]$

$2 \times 3 = 2 \times 3$

$6 = 6.$

$\therefore$ $S(1)$ is true.

To prove $S(k)$ be true.

$$\left.\begin{aligned}S(k) &= 1.2.3 + \dots\dots k(k+1)(k+2)\\&= \frac{1}{4}\{k.(k+1)(k+2)(k+3)\}\end{aligned}\right\}.$$

$$\left.\begin{aligned}S(k+1) &= 1.2.3 + \dots\dots [k(k+1)(k+2)] +\\&\{(k+1).(k+2)(k+3)\} = \frac{1}{4}.k(k+1)(k+2)(k+3)\\&\qquad + (k+1).(k+2)(k+3)\end{aligned}\right]$$

$= \frac{1}{4}(k+1)(k+2)(k+3)[k+4].$

$= \frac{1}{4}[(k+1).(k+2).(k+3)((k+4)]$

$\therefore S_{k+1}$ is true if $S_k$ is true.

3. Prove by mathematical induction that, (4).

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

<u>Proof</u>:- Let $S(n)=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\frac{1}{n(n+1)}=\frac{n}{n+1}.$ $\longrightarrow$ (1)

To prove $n=1$ is true.

$$S(1)=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\frac{1}{1(1+1)}=\frac{1}{1+1}$$

$$\frac{1}{2}=\frac{1}{2}.$$

To prove $n=k$ is true.

$$S(k)=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\frac{1}{k(k+1)}=\frac{k}{k+1}.$$ $\longrightarrow$ (2).

To prove $k=k+1$ is true.

$$S(k+1)=\frac{1}{1\cdot 2}+\cdots\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{1}{k+1}\left[\frac{k}{1}+\frac{1}{k+2}\right]$$

$$=\frac{1}{k+1}\left[\frac{k(k+2)+1}{k+2}\right].$$

$$=\frac{1}{k+1}\left[\frac{k^2+2k+1}{k+2}\right].=\frac{(k+1)^{\cancel{2}}}{(\cancel{k+1})(k+2)}$$

$\therefore S(k+1)=\frac{k+1}{k+2}$ is also true.

8



4. Use mathematical induction to show that,

$n! \geq 2^{n-1}$, $\{n = 1, 2, 3 \ldots\}$.

<u>Proof</u>:- $S(n) = n! \geq 2^{n-1}$. $\longrightarrow$ (1)

$S(1) = 1! \geq 2^{1-1} = 2^{0} = 1$

$\therefore 1 = 1 \Rightarrow S(1)$ is true.

To prove $n = k$ is true:-

$S(k) = k! \geq 2^{k-1}$. $\longrightarrow$ (2)

$\left\{ \begin{array}{l} \text{we've } n! = n(n-1)! \\ \therefore (n+1)! = (n+1)(n)! \end{array} \right\}$

$\therefore (k+1)! = (k+1)\, k!$

$\geq (k+1) \cdot 2^{k-1}$ $\{\because$ from (1)$\}$.

$\geq 2 \cdot 2^{k-1}$

$\geq 2 \cdot 2^{k} \cdot 2^{-1}$

$\geq 2^{k}$

$k! \geq 2^{k-1}$

$(k+1)! \geq 2^{k+1-1}$

$\geq 2^{k}$.

$\therefore (k+1)! \geq 2^{k}$ $\longrightarrow$ (3).

$\therefore$ (3) means that, $S_{k+1}$ is also true.

$\therefore$ By induction method, $S_n$ is true for $n = 1, 2, 3 \ldots$

6. Use by mathematical induction method, to prove that $n^3+2n$ is divisible by 3 for $n \geq 1$.

Soln:- $S_n = n^3 + 2n$ is divisible by 3. $\longrightarrow$ (1)

$S_1 = 1^3 + 2(1) = 1+2 = 3$ is divisible by 3.

To prove $n=k$ is true.

$S(k) = k^3 + 2k \longrightarrow$ (2)

ie., $k^3+2k$ is divisible by 3.

Now, $k=k+1$ is true. $\longrightarrow$ (3).

from (2), $(k+1)^3 + 2(k+1)$ {∴ $k = k+1$}.

$= k^3 + 3k^2 + 3.k + 1^3 + 2(k+1)$

$= k^3 + 3k^2 + 3k\cancel{+} 1 + 2k + 2.$

$= (k^3+2k) + (3k^2 + 3k + 3)$

$= k^3+2k$ is divisible by 3 {from 2}

also, $3k^2 + 3k + 3 = 3(k^2 + k + 1)$ is divisible by 3.

∴ $(k+1)^3 + 2(k+1)$ is divisible by 3.

∴ $S_{k+1}$ is also true.

6

7.

7. Prove by mathematical induction method,

to prove $n^3 + (n+1)^3 + (n+2)^3 + \dots$ is divisible by 9.

$n \geq 1$.

$S(n) = n^3 + (n+1)^3 + (n+2)^3$ is divi by 9. $\longrightarrow$ ①

Case ①:- $S(1) = 1^3 + (1+1)^3 + (1+2)^3 = 1 + 8 + 27 = 36$

$= 36$ is divisible by 9.

Case ②:- $S(k) = k^3 + (k+1)^3 + (k+2)^3$ is divisible by 9. $\longrightarrow$ ②

Case ③:- $S(k+1) = (k+1)^3 + (k+1+1)^3 + (k+1+2)^3$

$= (k+1)^3 + (k+2)^3 + (k+3)^3 \longrightarrow$ ③.

$= (k^3 + 1^3 + 3k^2 + 3k) + (k^3 + 2^3 + 3.2.k^2 + 3.k.2^2)$

$+ (k^3 + 3^3 + 3.3.k^2 + 3k.3^2)$

$= (k+1)^3 + (k+2)^3 + [k^3 + 3^3 + 9k^2 + 27k]$.

{from ③}

$S(k+1) = \{k^3 + (k+1)^3 + (k+2)^3\} + 9[k^2 + 3k + 3]$

from ②, $[k^3 + (k+1)^3 + (k+2)^3]$ is divisible by 9.

also $9[k^2 + 3k + 3]$ is divisible by 9.

$\therefore S(k+1)$ is true. and $S_n$ is true.

Hence, Induction method is verified.