$A x = \lambda \cancel{x}$

A = $n \times n$ matrix

$\underline{\underline{A}}\, \underline{\underline{X}} = \underline{\underline{\lambda}}\, X$

Non-zero Column Vector $\underline{\underline{X}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

λ = Eigen value

① Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix} \quad 3 \times 3$$

$A X = \lambda X$

$A X - \lambda X = 0$

$(A - \lambda I) X = 0$

Char. eqn

$|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -4 & 4 \\ 1 & -2-\lambda & 4 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

Short-cut Method :-

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$$

S_1 = Sum of diagonal elements

$= 3 - 2 + 3 = 4$

S_2 = sum of minors of diagonal elements of A.

$$= \begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} + \begin{vmatrix} -2 & 4 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix}$$

$= (-6+4) + (-6+4) + (9-4)$

$= -2 - 2 + 5 = 1$

$S_3 = |A| = \begin{vmatrix} 3 & -4 & 4 \end{vmatrix}$ -6

$$S_3 = |A| = \begin{vmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{vmatrix} = -6$$

$$\boxed{\lambda^3 - 4\lambda^2 + \lambda + 6 = 0}$$

$1-4+1+6 \neq 0$

$$\begin{array}{r|rrrr} 1 & 1 & -4 & 1 & 6 \\ & 0 & 1 & -3 & -2 \\ \hline & 1 & -3 & -2 & \boxed{4} \end{array}$$

Add

$\times$ ly

Remainder $\neq 0$
1 is not a root

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 1 & 6 \\ & 0 & -1 & 5 & -6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

-1 is a root

$$(\lambda+1)(\lambda^2 - 5\lambda + 6) = 0$$

$$(\lambda+1)(\lambda-2)(\lambda-3) = 0$$

X distinct roots

$$\lambda = -1, 2, 3$$

$= 0$ l.d

$\neq 0$ l.i

To find eigen Vector :-

$\boxed{\lambda = -1}$

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 3-\lambda & -4 & 4 \\ 1 & -2-\lambda & 4 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

sub $\lambda = -1$

$$\begin{pmatrix} 4 & -4 & 4 \\ 1 & -1 & 4 \\ 1 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$4x_1 - 4x_2 + 4x_3 = 0$; $x_1 - x_2 + x_3 = 0$

$x_1 - x_2 + 4x_3 = 0$; $x_1 - x_2 + 4x_3 = 0$

$x_1 - x_2 + 4x_3 = 0$; $x_1 - x_2 + 4x_3 = 0$

getting soln of system of eqns. we need distinct eqns.

eqns. distinct eqns. $\quad x_1 - x_2 + 4x_3 = 0$

$$x_1 - x_2 + x_3 = 0$$
$$x_1 - x_2 + 4x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & 4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{-3} = \frac{-x_2}{3} = \frac{x_3}{0}$$

$$X_1 = \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix} \Rightarrow X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = 2$

$$\begin{pmatrix} 1 & -4 & 4 \\ 1 & -4 & 4 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} x_1 - 4x_2 + 4x_3 &= 0 \\ x_1 - 4x_2 + 4x_3 &= 0 \end{aligned}\right\} \times$$

$$\checkmark \left\{\begin{aligned} x_1 - 4x_2 + 4x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \end{aligned}\right.$$

$$\frac{x_1}{\begin{vmatrix} -4 & 4 \\ -1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -4 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{-4+4} = \frac{-x_2}{1-4} = \frac{x_3}{-1+4}$$

$$X_2 = \begin{pmatrix} 0 \\ +3 \\ 3 \end{pmatrix} \Rightarrow X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = 3$

$$\begin{pmatrix} 0 & -4 & 4 \\ 1 & -5 & 4 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0 \cdot x_1 - 4x_2 + 4x_3 = 0$$

$$\left.\begin{aligned} x_1 - 5x_2 + 4x_3 &= 0 \\ x_1 - x_2 + 0x_3 &= 0 \end{aligned}\right\}$$

$$\frac{x_1}{\begin{vmatrix} -5 & 4 \\ -1 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 4 \\ 1 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -5 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{-x_2}{-4} = \frac{x_3}{4}$$

$$X_3 = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} \Rightarrow X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda_1 = -1$ $\quad X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 2$ $\quad X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\lambda_3 = 3$ $\quad X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Column vectors in rows $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$

$= l.i$

Note:- Distinct eigen values will have linearly independent eigen vectors.

(2) Find the eigen values and eigen vectors of

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$S_1 = 2+3+2 = 7$$

$$S_2 = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= (6-2) + (6-2) + (4-1) = 4+4+3 = 11$$

$$S_3 = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 5$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$\boxed{\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0}$$

1	1	-7	11	-5
	0	1	-6	5
1	1	-6	5	0
	0	1	-5	
	1	-5	0	

$$(\lambda-1)(\lambda-1)(\lambda-5) = 0$$

Repeated roots $\quad \lambda = 1, 1, 5$

To find eigenvectors :- $\quad (A - \lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = 5$

$$\begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -3x_1 + 2x_2 + x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 + 2x_2 - 3x_3 &= 0 \end{aligned}\right\}$$

$$\frac{x_1}{\begin{vmatrix} -2 & 1 \\ 2 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix}}$$

$$\frac{x_1}{6-2} = \frac{-x_2}{-3-1} = \frac{x_3}{2+2}$$

$$\frac{x_1}{4} = \frac{+x_2}{+4} = \frac{x_3}{4}$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\underline{\underline{\lambda = 1}}$

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \end{aligned}\right\} \text{All eqns are same.}$$

$$\boxed{x_1 + 2x_2 + x_3 = 0}$$

No of unknowns $\underbrace{= \text{No of eqn}}_{\text{unique}}$

Any 2 variables as free variables

$x_1 = 1$; $x_2 = 0$; $x_3 = -1$

$x_1 = 0$; $x_2 = 1$; $x_3 = -2$

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$\lambda_1 = 5 \qquad x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda_2 = \lambda_3 = 1 \quad ; \quad x_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \; ; \; x_3 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

Note :- Repeated eigen Values and different eigen Vectors

(3) $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

$S_1 = 3 - 3 + 7 = 7$

$S_2 = \begin{vmatrix} 3 & 10 \\ -2 & -3 \end{vmatrix} + \begin{vmatrix} -3 & -4 \\ 5 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 3 & 7 \end{vmatrix}$

$= (-9+20) + (-21+20) + (21-15) = 11 - 1 + 6 = 16$

$S_3 = \begin{vmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{vmatrix} = 12$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\boxed{\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0}$$

$$\begin{array}{r|rrrr} & 1 & -7 & 16 & -12 \\ 2 & 0 & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & \boxed{0} \\ 2 & 0 & 2 & -6 & \\ \hline & 1 & -3 & \boxed{0} & \end{array}$$

$(\lambda - 2)(\lambda - 2)(\lambda - 3) = 0$

Repeated roots. $\lambda = 2, 2, 3$

To find eigen Vectors :- $(A - \lambda I)X = 0$

To find eigen Vectors :- $(A - \lambda I)x = 0$

$$\begin{pmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = 3$

$$0x_1 + 10x_2 + 5x_3 = 0$$
$$\left.\begin{aligned} -2x_1 - 6x_2 - 4x_3 &= 0 \\ 3x_1 + 5x_2 + 4x_3 &= 0 \end{aligned}\right\}$$

$$\frac{x_1}{\begin{vmatrix} -6 & -4 \\ 5 & 4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & -4 \\ 3 & 4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -6 \\ 3 & 5 \end{vmatrix}}$$

$$\frac{x_1}{-24+20} = \frac{-x_2}{-8+12} = \frac{x_3}{-10+18}$$

$$\frac{x_1}{-4} = \frac{-x_2}{4} = \frac{x_3}{8}$$

$$x_1 = \begin{pmatrix} -4 \\ -4 \\ 8 \end{pmatrix} \Rightarrow X_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$\underline{\underline{\lambda = 2}}$

$$\begin{pmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} x_1 + 10x_2 + 5x_3 &= 0 \\ -2x_1 - 5x_2 - 4x_3 &= 0 \\ 3x_1 + 5x_2 + 5x_3 &= 0 \end{aligned}\right\}$$

All diff eqns.

$$\frac{x_1}{\begin{vmatrix} -5 & -4 \\ 5 & 5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & -4 \\ 3 & 5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -5 \\ 3 & 5 \end{vmatrix}}$$

x_1 x_2 x_3

$$\frac{x_1}{-25+20} = \frac{-x_2}{-10+12} = \frac{x_3}{-10+15}$$

$$\frac{x_1}{-5} = \frac{-x_2}{2} = \frac{x_3}{5}$$

$$X_2 = \begin{pmatrix} -5 \\ -2 \\ 5 \end{pmatrix} \Rightarrow X_2 = \begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix} = X_3$$

$$\lambda_1 = 3 \qquad \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\lambda_2 = 2 = \lambda_3 \qquad X_2 = X_3 = \begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix}$$

Note:- Repeated roots and linearly dependent eigen vectors.