Transforms and Partial Differential Equations

Third Edition

[For Semester III]

About the Author

T Veerarajan is Dean (Retd), Department of Mathematics, Velammal College of Engineering and Technology, Viraganoor, Madurai, Tamil Nadu. A Gold Medalist from Madras University, he has had a brilliant academic career all through. He has 53 years of teaching experience at undergraduate and postgraduate levels in various established engineering colleges in Tamil Nadu including Anna University, Chennai.

Transforms and Partial Differential Equations

Third Edition

[For Semester III]

T Veerarajan

(Retd) Dean
Department of Mathematics
Velammal College of Engineering and Technology
Viraganoor, Madurai
Tamil Nadu

McGraw Hill Education (India) Private Limited

NEW DELHI

McGraw Hill Education Offices

New Delhi New York St Louis San Francisco Auckland Bogotá Caracas
Kuala Lumpur Lisbon London Madrid Mexico City Milan Montreal
San Juan Santiago Singapore Sydney Tokyo Toronto

McGraw Hill Education (India) Private Limited

Published by McGraw Hill Education (India) Private Limited
P-24, Green Park Extension, New Delhi 110 016

Transforms and Partial Differential Equations (Third Edition)

Print Edition
ISBN 13: 978-93-5260-167-7
ISBN 10: 93-5260-167-X

E-Book Edition
ISBN 13: 978-93-5260-190-5
ISBN 10: 93-5260-190-4

Managing Director: *Kaushik Bellani*

Director—Products (Higher Education & Professional): *Vibha Mahajan*

Manager—Product Development: *Koyel Ghosh*
Specialist—Product Development: *Sachin Kumar*

Head—Production (Higher Education & Professional): *Satinder S Baveja*
Assistant Manager—Production: *Anuj K Shriwastava*
Senior Production Executive: *Jagriti Kundu*

Assistant General Manager—Product Management (Higher Education & Professional): *Shalini Jha*
Product Manager—Product Management: *Ritwick Dutta*

General Manager—Production: *Rajender P Ghansela*
Manager—Production: *Reji Kumar*

Typeset at Text-o-Graphics, B-1/56, Aravali Apartment, Sector-34, Noida 201 301, and printed at

Cover Printer:

Visit us at: www.mheducation.co.in

Contents

Preface

Transforms and Partial Differential Equations, 3e, has been designed specifically to cater to the needs of third semester B Tech students of Anna University. The current edition aims at preparing the students for examination alongside strengthening the fundamental concepts related to Transforms and Partial Differential Equations. Lucidity of the text, ample worked examples and notes highlighted within the text help students navigate through complex topics seamlessly. Stepwise explanation, use of multiple methods of problem solving, and additional information presented by the means of appendices are few other notable features of the content. In addition, solved question papers of 2009–2015 and plentiful practice exercises are the highlighting feature of this edition.

Salient Features

- Strict adherence to the latest AU syllabus
- In-depth coverage of topics like Fourier Series, Fourier Transform, Complex Analysis and Solution of PDE
- Addition of topics such as classification of PDE of the second order and inclusion of solved and unsolved examples
- Stepwise solutions of solved problems which will enable students to score marks
- Solved university questions papers from 2009 to 2015 (R-08, R-13)
- Rich pedagogy:
  - ✓ 284 examples within the chapter
  - ✓ 611 short and long answer type questions

Chapter Organization

The book is organised into 5 chapters. *Chapter 1* deals with Partial Differential Equations. *Chapter 2* explains in detail about Fourier Series. *Chapter 3* discusses the Applications of Partial Differential Equations and has been divided into three major parts—Vibrations of Strings; One Dimensional Heat Flow; and Steady State Heat Flow in Two Dimension. *Chapter 4* focuses on Fourier Transforms while *Chapter 5* elaborates on Z-Transforms and Difference Equations.

Acknowledgements

I am deeply grateful to all the students and teachers of India who have enthusiastically responded to the previous editions of *Transforms and Partial Differential Equations* (III Semester BE/B Tech courses). I hope that both the faculty and the students will receive the present edition as willingly as the earlier editions and my other books.

A number of reviewers took pains to provide valuable feedback for the book. We are grateful to all of them and their names are mentioned as follows:

A. Manoharan
Ganadipathy Tulsi's Jain Engineering College, Vellore, Tamil Nadu

K. Thanalakshmi
Kamaraj College of Engineering and Technology, Chennai, Tamil Nadu

R. Ranganathan
Dhirajlal Gandhi College of Technology, Salem, Tamil Nadu

K. Basari Kodi
Ramco Institute of Technology, Virudhunagar, Tamil Nadu

T Veerarajan

Publisher's Note

Critical evaluation and suggestions for the improvement of the book will be highly appreciated and acknowledged. Hence, we look forward to receiving your feedback, comments and ideas to enhance the quality of this book. You can reach us at *info.india@mheducation.com*. Kindly mention the title and the author's name as the subject. In case you spot piracy of this book, please do let us know.

Roadmap to the Syllabus

Transforms and Partial Differential Equations
Regulation – 2013

Module I: PARTIAL DIFFERENTIAL EQUATIONS

Formation of partial differential equations—Singular integrals—Solutions of standard types of first order partial differential equations—Lagrange's linear equation—Linear partial differential equations of second and higher order with constant coefficients of both homogeneous and non-homogeneous types.

Go to **CHAPTER 1. PARTIAL DIFFERENTIAL EQUATIONS**

Module II: FOURIER SERIES

Dirichlet's conditions—General Fourier series—Odd and even functions—Half range sine series—Half range cosine series—Complex form of Fourier series—Parseval's identity—Harmonic analysis.

Go to **CHAPTER 2. FOURIER SERIES**

Module III: APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

Classification of PDE—Method of separation of variables—Solutions of one dimensional wave equation—One dimensional equation of heat conduction—Steady state solution of two dimensional equation of heat conduction (excluding insulated edges).

Go to

CHAPTER 3. APPLICATION OF PARTIAL DIFFERENTIAL EQUATIONS

CHAPTER 3A. VIBRATIONS OF STRINGS

CHAPTER 3B. ONE DIMENSIONAL HEAT FLOW

CHAPTER 3C. STEADY STATE HEAT FLOW IN TWO DIMENSION

Module IV: FOURIER TRANSFORMS

Statement of Fourier integral theorem—Fourier transform pair—Fourier sine and cosine transforms—Properties—Transforms of simple function—Convolution theorem—Parseval's identity.

Go to CHAPTER 4. FOURIER TRANSFORMS

Module V: Z-TRANSFORMS AND DIFFERENCE EQUATIONS

Z-transforms—Elementary properties—Inverse Z-transform (using partial fraction and residues)—Convolution theorem—Formation of difference equations—Solution of difference equations using Z-transform.

Go to CHAPTER 5. Z-TRANSFORMS AND DIFFERENCE EQUATIONS

Chapter 1

Partial Differential Equations

1.1 INTRODUCTION

Partial differential equations are found in problems involving wave phenomena, heat conduction in homogeneous solids and potential theory. As an equation containing ordinary differential coefficients is called an ordinary differential equation, an equation containing partial differential coefficients is called a partial differential equation. Partial derivatives come into being only when there is a dependent variable which is a function of two or more independent variables. Hence in a partial differential equation, there will be one dependent variable and two or more independent variables. However we will mostly deal with partial differential equations containing only two independent variables. In what follows, z will be taken as the dependent variable and x and y the independent variables so that $z = f(x, y)$. We will use the following standard notations to denote the partial derivatives:

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s \text{ and } \frac{\partial^2 z}{\partial y^2} = t$$

The *order* of a partial differential equation is that of the highest order derivative occurring in it.

1.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

Though our main interest is to solve partial differential equations, it will be advantageous if we know how partial differential equations are formed. Knowledge of the formation of partial differential equations will help us to distinguish between two kinds of solutions of the equation. Partial differential equations can be formed by eliminating either arbitrary constants or arbitrary functions from functional relations satisfied by the dependent and independent variables. When we form partial differential equations the following points may be considered for proper procedure and checking.

1. If the number of arbitrary constants to be eliminated is equal to the number of independent variables, the process of elimination results in a partial differential equation of the first order.

Note

In the formation of ordinary differential equations, the order of the equation is equal to the number of constants eliminated.

2. If the number of arbitrary constants to be eliminated is more than the number of independent variables, the process of elimination will lead to a partial differential equation of second or higher orders.
3. If the partial differential equation is formed by eliminating arbitrary functions, the order of the equation will be, in general, equal to the number of arbitrary functions eliminated.

1.3 ELIMINATION OF ARBITRARY CONSTANTS

By way of verifying point 3 of Section 1.2, let us consider the functional relation among

$$x, y, z, \text{ i.e. } f(x, y, z, a, b) = 0 \tag{1}$$

where a and b are arbitrary constants to be eliminated.

Differentiating (1) partially with respect to x and y, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 0, \text{ i.e. } \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot p = 0 \tag{2}$$

and

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y} = 0, \text{ i.e. } \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot q = 0 \tag{3}$$

Equations (2) and (3) will contain a and b.

If we eliminate a and b from equations (1), (2) and (3), we get partial differential equation (involving p and q) of the first order. This justifies point 1 of Section 1.2.

1.4 ELIMINATION OF ARBITRARY FUNCTIONS

By way of verifying point 3 of Section 1.2 above, let us consider the relation

$$f(u, v) = 0 \tag{1}$$

where u and v are functions of x, y, z and f is an arbitrary function to be eliminated. Differentiating (1) partially with respect to x,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0 \tag{2}$$

[since u and v are functions of x, y, z and z is in turn, a function of x, y]

Differentiating (2) partially with respect to y,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0 \tag{3}$$

Instead of eliminating f, let us eliminate $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (2) and (3).

From (2) and (3), we get

$$\frac{u_x + u_z p}{u_y + u_z q} = \frac{v_x + v_z p}{v_y + v_z q}, \text{ where } u_x = \frac{\partial u}{\partial x}, \text{ etc.}$$

i.e. $$u_x v_y + u_x v_z q + u_z v_y p = u_y v_x + u_y v_z p + u_z v_x q$$

i.e. $$(u_y v_z - u_z v_y)p + (u_z v_x - u_x v_z)q = (u_x v_y - u_y v_x) \quad (4)$$

i.e. $Pp + Qq = R$, say, where P, Q and R are functions of x, y, z.

Now equation (4) is a partial differential equation of order 1.

This justifies point 3 of Section 1.2.

Note ✍

1. *To verify point 3 of Section 1.2, we could have taken a functional relation containing a function of one argument, but we have shown that the order of the partial differential equation formed depends only on the number of arbitrary functions eliminated and not on the number of arguments of the function.*
2. *The equation (4) is called Lagrange's linear equation, whose solution will be discussed later.*

Worked Examples 1(a)

Example 1

Form the partial differential equation by eliminating the arbitrary constants a and b from the following.

(i) $\log z = a \log x + \sqrt{1-a^2} \log y + b$

(ii) $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$

(i) $$\log z = a \log x + \sqrt{1-a^2} \log y + b \quad (1)$$

Differentiating (1) partially with respect to x and then with respect to y, we get

$$\frac{1}{z} p = a/x \quad (2)$$

and $$\frac{1}{z} q = \frac{\sqrt{1-a^2}}{y} \quad (3)$$

If we ignore (1), b is eliminated.

From (2), $a = \dfrac{px}{z}$ and using this in (3), we get

$$\frac{1}{z^2}q^2 = \frac{1}{y^2}\left\{1 - \frac{p^2x^2}{z^2}\right\}$$

i.e.
$$\frac{p^2x^2}{z^2} + \frac{q^2y^2}{z^2} = 1$$

or
$$p^2x^2 + q^2y^2 = z^2$$

(ii)
$$(x-a)^2 + (y-b)^2 = z^2\cot^2\alpha \qquad (1)$$

Differentiating (1) partially with respect to x and then with respect to y, we get

$$2(x-a) = 2zp\cot^2\alpha \qquad (2)$$

and
$$2(y-b) = 2zq\cot^2\alpha \qquad (3)$$

Using (2) and (3) in (1), we have

$$z^2(p^2+q^2)\cot^4\alpha = z^2\cot^2\alpha$$

i.e.
$$p^2+q^2 = \tan^2\alpha$$

Example 2

Form the partial differential equation by eliminating the arbitrary constants a and b from the following.

(i) $\sqrt{1+a^2}\log\left(z+\sqrt{z^2-1}\right) = x + ay + b$

(ii) $z = \dfrac{1}{2}x\sqrt{x^2+a^2} + \dfrac{1}{2}y\sqrt{y^2-a^2} + \dfrac{a^2}{2}\log\left\{\dfrac{x+\sqrt{x^2+a^2}}{y+\sqrt{y^2-a^2}}\right\} + b$

(i)
$$\sqrt{1+a^2}\log\left(z+\sqrt{z^2-1}\right) = x + ay + b \qquad (1)$$

Differentiating (1) partially with respect to x and then with respect to y, we get

$$\sqrt{1+a^2}\cdot\frac{1}{z+\sqrt{z^2-1}}\cdot\left\{1+\frac{z}{\sqrt{z^2-1}}\right\}p = 1$$

i.e.
$$\sqrt{1+a^2}\cdot p/\sqrt{z^2-1} = 1 \qquad (2)$$

and
$$\sqrt{1+a^2}\cdot\frac{1}{z+\sqrt{z^2-1}}\cdot\left\{1+\frac{z}{\sqrt{z^2-1}}\right\}q = a$$

i.e.
$$\sqrt{1+a^2}\cdot\frac{q}{\sqrt{z^2-1}} = a \qquad (3)$$

From (2) and (3), we get $$\frac{p}{q} = \frac{1}{a} \tag{4}$$

Using (4) in (2), we get

$$\sqrt{1 + \frac{q^2}{p^2}}.p = \sqrt{z^2 - 1}$$

i.e. $$\sqrt{p^2 + q^2} = \sqrt{z^2 - 1} \quad \text{or} \quad p^2 + q^2 + 1 = z^2$$

(*ii*) $$z = \frac{1}{2}x\sqrt{x^2 + a^2} + \frac{1}{2}y\sqrt{y^2 - a^2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2 + a^2}\right)$$

$$-\frac{a^2}{2}\log\left(y + \sqrt{y^2 - a^2}\right) + b \tag{1}$$

Differentiating (1) partially with respect to x,

$$p = \frac{1}{2}\left\{x.\frac{x}{\sqrt{x^2 + a^2}} + \sqrt{x^2 + a^2}\right\} + \frac{a^2}{2}.\frac{1}{x + \sqrt{x^2 + a^2}}\left\{1 + \frac{x}{\sqrt{x^2 + a^2}}\right\}$$

$$= \frac{1}{2}\left[\frac{2x^2 + a^2}{\sqrt{x^2 + a^2}} + \frac{a^2}{\sqrt{x^2 + a^2}}\right] = \sqrt{x^2 + a^2} \tag{2}$$

Similarly, differentiating (1) partially with respect to y, we get

$$q = \sqrt{y^2 - a^2} \tag{3}$$

From (2) and (3),

$$p^2 - x^2 = y^2 - q^2$$

i.e. $$p^2 + q^2 = x^2 + y^2$$

Example 3

Form a partial differential equation by eliminating the arbitrary constants a, b, c from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

We note that the number of constants is more than the number of independent variables. Hence the order of the resulting equation will be more than 1.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

Differentiating (1) partially with respect to x and then with respect to y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}p = 0 \tag{2}$$

and $$\frac{2y}{b^2} + \frac{2z}{c^2}q = 0 \tag{3}$$

Differentiating (2) partially with respect to x,

$$\frac{1}{a^2} + \frac{1}{c^2}(zr + p^2) \tag{4}$$

where $$r = \frac{\partial^2 z}{\partial x^2}$$

From (2), $$-\frac{c^2}{a^2} = \frac{zp}{x} \tag{5}$$

From (4), $$\frac{-c^2}{a^2} = zr + p^2 \tag{6}$$

From (5) and (6), we get

$xz\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 = z\frac{\partial z}{\partial x}$ which is the required partial differential equation. This is not the only way of eliminating a, b and c. Had we differentiated (2) partially with respect to y, we would have got

$$\frac{2}{c^2}\{zs + pq\} = 0, \text{ where } s = \frac{\partial^2 z}{\partial x \partial y}$$

i.e. $$z\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x}.\frac{\partial z}{\partial y} = 0$$

which is also a partial differential equation corresponding to (1).

If we differentiate (3) partially with respect to y and eliminate b and c, we will get yet another partial differential equation, namely

$$yz\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0$$

Example 4

Find the partial differential equation of the family of planes, the sum of whose x, y, z intercepts is unity.

The equation of a plane which cuts off intercepts a, b, c on the coordinate axes is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \tag{1}$$

If sum of the intercepts is unity, $a + b + c = 1$ or

$$c = 1 - a - b \tag{2}$$

Using (2) in (1), we get the equation of a plane, the sum of whose x, y, z - intercepts is unity as

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{1 - a - b} = 1$$

or $$b(1 - a - b)x + a(1 - a - b)y + abz = ab(1 - a - b) \tag{3}$$

If a and b are treated as arbitrary constants, (3) represents the family of planes having the given property. Differentiating (3) partially with respect to x and then with respect to y, we have

$$b(1-a-b)+abp=0 \text{ or } 1-a-b=-ap \tag{4}$$

and
$$a(1-a-b)+abq=0 \text{ or } 1-a-b=-bq \tag{5}$$

From (4) and (5), we get

$$ap=bq \text{ or } \frac{a}{q}=\frac{b}{p}=k \tag{6}$$

Using (6) in (4), $1-k(p+q)=-kpq$

i.e.
$$k=\frac{1}{p+q-pq}$$

$\therefore$
$$a=\frac{q}{p+q-pq},\ b=\frac{p}{p+q-pq} \text{ and } 1-a-b=\frac{-pq}{p+q-pq}$$

Using these values in (3), we have

$$-k^2p^2qx-k^2pq^2y+k^2pqz=-k^3p^2q^2$$

i.e.
$$-px-qy+z=-kpq$$

or $z=px+qy-\dfrac{pq}{p+q-pq}$, which is the required partial differential equation.

Example 5

Find the differential equation of all planes which are at a constant distance k from the origin.

The equation of a plane which is at a distance k from the origin is

$$x\cos\alpha+y\cos\beta+z\cos\nu=k$$

where $\cos\alpha$, $\cos\beta$, $\cos\nu$ are the direction cosines of a normal to the plane.

Taking $\cos\alpha=a$, $\cos\beta=b$ and $\cos\nu=c$ and noting that $a^2+b^2+c^2=1$, the equation of the plane can be assumed as

$$ax+by+\sqrt{1-a^2-b^2}z=k \tag{1}$$

If a and b are treated as arbitrary constants, equation (1) represents all planes having the given property.

Differentiating (1) partially with respect to x and then with respect to y, we have

$$a+\sqrt{1-a^2-b^2}p=0 \tag{2}$$

and
$$b+\sqrt{1-a^2-b^2}\,q=0 \tag{3}$$

From (2) and (3), $$\frac{a}{p} = \frac{b}{q} = -\sqrt{1 - a^2 - b^2} = \lambda, \text{ say}$$

$\therefore$ $$a = \lambda p, b = \lambda q \text{ and } \sqrt{1 - \lambda^2(p^2 + q^2)} = -\lambda$$

i.e. $$1 - \lambda^2(p^2 + q^2) = \lambda^2$$

$\therefore$ $$\lambda^2 = \frac{1}{1 + p^2 + q^2} \text{ or } \lambda = -\frac{1}{\sqrt{1 + p^2 + q^2}}$$

$$(\because \lambda \text{ is negative, as } \lambda = -\sqrt{1 - a^2 - b^2})$$

Using these values in (1), we get

$$\lambda px + \lambda qy - \lambda z = k$$

i.e. $$z = px + qy - \frac{k}{\lambda} \text{ or}$$

$z = px + qy + k\sqrt{1 + p^2 + q^2}$, which is the required partial differential equation.

Example 6

Find the differential equation of all spheres of the same radius c having their centres on the yoz-plane.

The equation of a sphere having its centre at $(0, a, b)$, that lies on the yoz-plane and having its radius equal to c is

$$x^2 + (y - a)^2 + (z - b)^2 = c^2 \tag{1}$$

If a and b are treated as arbitrary constants, (1) represents the family of spheres having the given property.

Differentiating (1) partially with respect to x and then with respect to y, we have

$$2x + 2(z - b)p = 0 \tag{2}$$

and $$2(y - a) + 2(z - b)q = 0 \tag{3}$$

From (2), $$z - b = -\frac{x}{p} \tag{4}$$

Using (4) in (3), $$y - a = \frac{qx}{p} \tag{5}$$

Using (4) and (5) in (1), we get

$$x^2 + \frac{q^2x^2}{p^2} + \frac{x^2}{p^2} = c^2$$

i.e. $(1 + p^2 + q^2)x^2 = c^2p^2$, which is the required partial differential equation.

Example 7

Find the differential equation of all spheres whose centres lie on the x-axis.
The equation of any sphere whose centre is $(a, 0, 0)$ (that lies on the x-axis) and whose radius is b is

$$(x-a)^2 + y^2 + z^2 = b^2 \tag{1}$$

If a and b are treated as arbitrary constants, (1) represents the family of spheres having the given property.

Differentiating (1) partially with respect to x and then with respect to y, we have

$$2(x-a) + 2zp = 0 \tag{2}$$

$$2y + 2zq = 0 \tag{3}$$

The required equation is provided by (3).

i.e. $$\text{it is } z\frac{\partial z}{\partial y} + y = 0$$

Example 8

Find the differential equation of all spheres whose radii are the same.
The equation of all spheres with equal radius can be taken as

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2 \tag{1}$$

where a, b, c are arbitrary constants and R is a given constant.

Differentiating (1) partially with respect to x and then with respect to y, we have

$$(x-a) + (z-c)p = 0 \tag{2}$$

and $$(y-b) + (z-c)q = 0 \tag{3}$$

Differentiating (2) and (3) with respect to x and y respectively, we get

$$1 + (z-c)r + p^2 = 0 \tag{4}$$

and $$1 + (z-c)t + q^2 = 0 \tag{5}$$

Eliminating $(z-c)$ from (4) and (5), we have

$$\frac{r}{t} = \frac{1+p^2}{1+q^2}$$

i.e. $$r(1+q^2) = t(1+p^2), \text{ where } r = \frac{\partial^2 z}{\partial x^2} \text{ and } t = \frac{\partial^2 z}{\partial y^2}.$$

Note ✍

The answer is not unique. We can get different partial differential equations.

Example 9

Form the partial differential equation by eliminating the arbitrary function 'f' from

(*i*) $z = e^{ay} f(x + by)$; and

(*ii*) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

(i) $$z = e^{ay}.f(x + by)$$

i.e. $$e^{-ay} z = f(x + by) \tag{1}$$

Differentiating (1) partially with respect to x and then with respect to y, we get

$$e^{-ay} p = f'(u) \cdot 1 \tag{2}$$

$$e^{-ay} q - ae^{-ay} z = f'(u)b \tag{3}$$

where $u = x + by$

Eliminating $f'(u)$ from (2) and (3), we get

$$\frac{q - az}{p} = b$$

i.e. $$q = az + bp$$

(*ii*) $$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$

i.e. $$z - y^2 = 2f\left(\frac{1}{x} + \log y\right) \tag{1}$$

Differentiating (1) partially with respect to x and then with respect to y, we get

$$p = 2f'(u).\left(\frac{-1}{x^2}\right) \tag{2}$$

and $$q - 2y = 2f'(u).\left(\frac{1}{y}\right) \tag{3}$$

where $u = \frac{1}{x} + \log y$

Dividing (2) by (3), we have

$$\frac{p}{q - 2y} = \frac{-y}{x^2}$$

i.e. $$px^2 + qy = 2y^2$$

which is the required partial differential equation.

Example 10

Form the partial differential equation by eliminating the arbitrary function 'f' from

(*i*) $xy + yz + zx = f\left(\dfrac{z}{x+y}\right)$ and

(*ii*) $f(z - xy, x^2 + y^2) = 0$

(*i*)
$$xy + yz + zx = f\left(\frac{z}{x+y}\right) \tag{1}$$

Differentiating (1) partially with respect to x and then with respect to y, we have

$$y + yp + xp + z = f'(u)\left\{\frac{(x+y)p - z}{(x+y)^2}\right\} \tag{2}$$

and
$$x + yq + z + xq = f'(u)\left\{\frac{(x+y)q - z}{(x+y)^2}\right\} \tag{3}$$

Dividing (2) by (3), we have

$$\frac{(y+z) + (x+y)p}{(z+x) + (x+y)q} = \frac{(x+y)p - z}{(x+y)q - z}$$

i.e. $\quad (x+y)(z+x)p - z(z+x) - z(x+y)q$
$$= (x+y)(y+z)q - z(y+z) - z(x+y)p$$

i.e.
$$(x+y)(x+2z)p - (x+y)(y+2z)q = z(x-y)$$

which is a Lagrange linear equation.

(*ii*)
$$f(z - xy, x^2 + y^2) = 0 \tag{1}$$

i.e.
$$f(u.v) = 0$$

If we assume that u can be expressed as a single-valued function of v, (1) can be rewritten as

$$z - xy = \phi(x^2 + y^2) \tag{2}$$

where ϕ is an arbitrary function.

Differentiating (2) partially with respect to x and then with respect to y, we have

$$p - y = \phi'(u).2x \tag{3}$$

and
$$q - x = \phi'(u).2y \tag{4}$$

Eliminating $\phi'(u)$ from (3) and (4), we get

$$\frac{p-y}{q-x} = \frac{x}{y} \text{ or } yp - xq = y^2 - x^2$$

Note ✍

Without assuming that $u = \phi(v)$, we can eliminate 'f' and form the equation alternatively as given in the following example.

Example 11

Form the partial differential equation by eliminating 'f' from

(*i*) $f(z - xy, x^2 + y^2) = 0$ and
(*ii*) $f(x^2 + y^2 + z^2, ax + by + cz) = 0$

(i) $$f(z - xy, x^2 + y^2) = 0 \quad (1)$$

By putting $z - xy = u$ and $x^2 + y^2 = v$, (1) becomes

$$f(u, v) = 0 \quad (2)$$

Differentiating (2) partially with respect to x and then with respect to y, we have

$$\frac{\partial f}{\partial u}.(p - y) + \frac{\partial f}{\partial v}(2x) = 0 \quad (3)$$

and $$\frac{\partial f}{\partial u}(q - x) + \frac{\partial f}{\partial v}(2y) = 0 \quad (4)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (3) and (4), we get

$$\begin{vmatrix} p - y & 2x \\ q - x & 2y \end{vmatrix} = 0$$

i.e. $$2y(p - y) - 2x(q - x) = 0$$

or $$yp - xq = y^2 - x^2$$

(*ii*) $$f(x^2 + y^2 + z^2, ax + by + cz) = 0 \quad (1)$$

Putting $u = x^2 + y^2 + z^2$ and $v = ax + by + cz$, (1) becomes

$$f(u, v) = 0 \quad (2)$$

Differentiating (2) partially with respect to x and then with respect to y, we have

$$\frac{\partial f}{\partial u}(2x + 2zp) + \frac{\partial f}{\partial v}(a + cp) = 0 \quad (3)$$

and $$\frac{\partial f}{\partial u}(2y + 2zq) + \frac{\partial f}{\partial v}(b + cq) = 0 \quad (4)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (3) and (4), we get

$$\begin{vmatrix} x + zp & a + cp \\ y + zq & b + cq \end{vmatrix} = 0$$

i.e. $$(x + zp)(b + cq) = (y + zq)(a + cp)$$

i.e. $$(cy - bz)p + (az - cx)q = bx - ay$$

Example 12

Form the partial differential equation by eliminating the arbitrary functions f and g from $z = f(2x + y) + g(3x - y)$

$$z = f(2x + y) + g(3x - y) \qquad (1)$$

Differentiating (1) partially with respect to x,

$$p = f'(u).2 + g'(v).3 \qquad (2)$$

where $u = 2x + y$ and $v = 3x - y$
Differentiating (1) partially with respect to y,

$$q = f'(u).1 + g'(v)(-1) \qquad (3)$$

Differentiating (2) partially with respect to x and then with respect to y,

$$r = f''(u).4 + g''(v).9 \qquad (4)$$

and

$$s = f''(u).2 + g''(v).(-3) \qquad (5)$$

Differentiating (3) partially with respect to y,

$$t = f''(u).1 + g''(v).1 \qquad (6)$$

Eliminating $f''(u)$ and $g''(v)$ from (4), (5) and (6) using determinants, we have

$$\begin{vmatrix} 4 & 9 & r \\ 2 & -3 & s \\ 1 & 1 & t \end{vmatrix} = 0$$

i.e.

$$5r + 5s - 30t = 0$$

or

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$$

Example 13

Form the differential equation by eliminating the arbitrary functions f and ϕ from $z = f(ax + by) + \phi(cx + dy)$.

$$z = f(u) + \phi(v) \qquad (1)$$

where $u = ax + by$ and $v = cx + dy$

Differentiating partially with respect to x and y,

$$p = f'(u) \cdot a + \phi'(v) \cdot c \tag{2}$$

$$q = f'(u) \cdot b + \phi'(v) d \tag{3}$$

$$r = f''(u) \cdot a^2 + \phi''(v) \cdot c^2 \tag{4}$$

$$s = f''(u) \cdot ab + \phi''(v) \cdot cd \tag{5}$$

$$t = f''(u) \cdot b^2 + \phi''(v) \cdot d^2 \tag{6}$$

Eliminating $f''(u)$ and $\phi''(v)$ from (4),(5),(6), we have

$$\begin{vmatrix} r & a^2 & c^2 \\ s & ab & cd \\ t & b^2 & d^2 \end{vmatrix} = 0$$

i.e. $$(abd^2 - b^2cd)r - (a^2d^2 - b^2c^2)s + (a^2cd - abc^2)t = 0$$

i.e. $$bd(ad - bc)r - (ad + bc)(ad - bc)s + ac(ad - bc)t = 0$$

i.e. $$bd\frac{\partial^2 z}{\partial x^2} - (ad + bc)\frac{\partial^2 z}{\partial x \partial y} + ac\frac{\partial^2 z}{\partial y^2} = 0.$$

Example 14

Form the differential equation by eliminating f and g from $z = xf(ax + by) + g(ax + by)$.

$$z = x \cdot f(u) + g(u) \tag{1}$$

where $u = ax + by$.
Differentiating partially with respect to x and y,

$$p = xf'(u) \cdot a + f(u) + g'(u) \cdot a \tag{2}$$

$$q = xf'(u) \cdot b + g'(u) \cdot b \tag{3}$$

$$r = x \cdot f''(u)a^2 + f'(u) \cdot 2a + g''(u) \cdot a^2 \tag{4}$$

$$s = xf''(u)ab + f'(u)b + g''(u)ab \tag{5}$$

$$t = xf''(u)b^2 + g''(u) \cdot b^2 \tag{6}$$

$[(4) \times b - (5) \times 2a]$ gives

$$br - 2as = -a^2b[xf''(u) + g''(u)] \tag{7}$$

$$= -a^2b \times \frac{1}{b^2}t, \text{ from (6)}$$

i.e. $$b^2\frac{\partial^2 z}{\partial x^2} - 2ab\frac{\partial^2 z}{\partial x \partial y} + a^2\frac{\partial^2 z}{\partial y^2} = 0$$

Example 15

Form the differential equation by eliminating the arbitrary functions f and g from

$$z = f(x + iy) + (x + iy)g(x - iy), \quad \text{where} \quad i = \sqrt{-1} \quad \text{and} \quad x + iy \neq z$$

$$z = f(u) + (x + iy)g(v) \tag{1}$$

where $u = x + iy$ and $v = x - iy$.

Differentiating partially with respect to x and y,

$$p = f'(u) \cdot 1 + (x + iy)g'(v) \cdot 1 + g(v) \tag{2}$$
$$q = f'(u) \cdot i + (x + iy)g'(v)(-i) + g(v) \cdot i \tag{3}$$
$$r = f''(u) \cdot 1 + (x + iy)g''(v) \cdot 1 + 2g'(v) \cdot 1 \tag{4}$$
$$s = f''(u) \cdot i + (x + iy)g''(v)(-i) \tag{5}$$
$$t = f''(u)(-1) + (x + iy)g''(v) \cdot (-1) + 2g'(v) \tag{6}$$

Adding (4) and (6), we get

$$r + t = 4g'(v) \tag{7}$$

From (2) and (3), we get

$$p + iq = 2(x + iy)g'(v) \tag{8}$$

Eliminating $g'(v)$ from (7) and (8), we get

$$r + t = 2\frac{(p + iq)}{x + iy}$$

i.e $$(x + iy)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = 2\left(\frac{\partial z}{\partial x} + i\frac{\partial z}{\partial y}\right)$$

Note ✍

Equation (5), giving the value of s, is not at all used.

Example 16

If $u = f(x^2 + y) + \phi(x^2 - y)$, show that $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{x}\dfrac{\partial u}{\partial x} - 4x^2\dfrac{\partial^2 u}{\partial y^2} = 0.$

$$u = f(v) + \phi(w) \tag{1}$$

where $v = x^2 + y$ and $w = x^2 - y$.

Differentiating partially with respect to x and y,

$$\frac{\partial u}{\partial x} = f'(v) \cdot 2x + \phi'(w) \cdot 2x \tag{2}$$

$$\frac{\partial u}{\partial y} = f'(v) \cdot 1 + \phi'(w) \cdot (-1) \tag{3}$$

$$\frac{\partial^2 u}{dx^2} = f'(v) \cdot 2 + f''(v) \cdot 4x^2 + \phi'(w) \cdot 2 + \phi''(w) \cdot 4x^2 \tag{4}$$

$$\frac{\partial^2 u}{\partial x \partial y} = f''(v) \cdot 2x + \phi''(w) \cdot (-2x) \tag{5}$$

$$\frac{\partial^2 u}{\partial y^2} = f''(v) \cdot 1 + \phi''(w) \cdot 1 \tag{6}$$

Eq. (4) can be rewritten as

$$\frac{\partial^2 u}{\partial x^2} = 2\{f'(v) + \phi'(w)\} + 4x^2\{f''(v) + \phi''(w)\}$$

$$= 2 \times \frac{1}{2x}\frac{\partial u}{\partial x} + 4x^2 \cdot \frac{\partial^2 u}{\partial y^2}, \qquad \text{from (2) and (6)}$$

i.e.

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{x}\frac{\partial u}{\partial x} - 4x^2\frac{\partial^2 u}{\partial y^2} = 0$$

Example 17

Form the differential equation by eliminating f and ϕ from $z = f(x+y) \cdot \phi(x-y)$.

$$z = f(u) \cdot \phi(v) \tag{1}$$

where $u = x + y$ and $v = x - y$.

Differentiating partially with respect to x and y, we get

$$p = f(u) \cdot \phi'(v) + f'(u) \cdot \phi(v) \tag{2}$$

$$q = f(u)\phi'(v)(-1) + f'(u)\phi(v) \tag{3}$$

$$r = f(u)\phi''(v) + 2f'(u)\phi'(v) + f''(u) \cdot \phi(v) \tag{4}$$

$$s = f(u)\phi''(v)(-1) + f''(u) \cdot \phi(v) \tag{5}$$

$$t = f(u) \cdot \phi''(v) - 2f'(u)\phi'(v) + f''(u)\phi(v) \tag{6}$$

Subtracting (5) from (3), we get

$$r - t = 4f'(u) \cdot \phi'(v) \tag{7}$$

From (1) and (2), we get

$$p^2 - q^2 = 4f(u)\cdot\phi(u)\cdot f'(u)\cdot\phi'(v)$$
$$= z(r-t) \text{ from (1) and (7)}$$

i.e. $$z\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$

Example 18

Form the differential equation by eliminating f and ϕ from $z = xf(y/x) + y\phi(x)$.

$$z = xf(u) + y\phi(x) \tag{1}$$

where $u = \frac{y}{x}$.
Differentiating partially with respect to x and y, we get

$$p = xf'(u)\cdot\left(-\frac{y}{x^2}\right) + f(u) + y\phi'(x)$$

i.e. $$p = -\frac{y}{x}\cdot f'(u) + f(u) + y\phi'(x) \tag{2}$$

$$q = x\cdot f'(u)\cdot\frac{1}{x} + \phi(x)$$

i.e. $$q = f'(u) + \phi(x) \tag{3}$$

$$r = -\frac{y}{x}\cdot f''(u)\left(-\frac{y}{x^2}\right) + y\phi''(x)$$

i.e. $$r = \frac{y^2}{x^3}f''(u) + y\phi''(x) \tag{4}$$

$$s = -\frac{y}{x^2}f''(u) + \phi'(x) \tag{5}$$

$$t = \frac{1}{x}f''(u) \tag{6}$$

Eliminating $f''(u)$ from (5) and (6), we get

$$s + \frac{y}{x}t = \phi'(x) \tag{7}$$

From (2) and (3), we get

$$px + qy = \{xf(u) + y\phi(x)\} + xy\phi'(x)$$

i.e. $$px + qy = z + xy\phi'(x) \tag{8}$$

Eliminating $\phi'(x)$ from (7) and (8), we get

$$xys + y^2t = px + qy - z$$

i.e. $$xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z$$

Example 19

Form the differential equation by eliminating f and ϕ from $z = f(y) + \phi(x + y + z)$

$$z = f(y) + \phi(u) \tag{1}$$

where $u = x + y + z$.
Differentiating partially with respect to x and y, we get

$$p = \phi'(u)(1 + p) \tag{2}$$

$$q = f'(y) + \phi'(u)(1 + q) \tag{3}$$

$$r = \phi'(u) \cdot r + \phi''(u) \cdot (1 + p)^2 \tag{4}$$

$$s = \phi'(u) \cdot s + \phi''(u)(1 + p)(1 + q) \tag{5}$$

$$t = f''(y) + \phi'(u)t + \phi'(u)(1 + q)^2 \tag{6}$$

From (4), $$r\{1 - \phi'(u)\} = (1 + p)^2\phi''(u) \tag{7}$$

From (5), $$s\{1 - \phi'(u)\} = (1 + p)(1 + q)\phi''(u) \tag{8}$$

Dividing (7) by (8), we get

$$\frac{r}{s} = \frac{1+p}{1+q}$$

i.e. $$\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial x}\right)\frac{\partial^2 z}{\partial x^2}$$

Example 20

Form the differential equation by eliminating the arbitrary function ϕ from

$$z = \frac{1}{x}\phi(y - x) + \phi'(y - x).$$

Note ✍

Though ϕ' is the derivative of ϕ, we should not assume that only one function is to be eliminated. We have to eliminate two functions ϕ and ϕ' and hence the resulting partial differential equation will be of order 2.

$$z = \frac{1}{x}\,\phi(u) + \phi'(u) \tag{1}$$

where $u = y - x$
Differentiating partially with respect to x and y, we get

$$p = \frac{1}{x}\phi'(u) \cdot (-1) - \frac{1}{x^2}\phi(u) + \phi''(u)(-1) \tag{2}$$

$$q = \frac{1}{x}\phi'(u) \cdot 1 + \phi''(u) \cdot 1 \tag{3}$$

$$r = \frac{1}{x}\phi''(u) \cdot 1 + \frac{2}{x^2}\phi'(u) + \frac{2}{x^3}\phi(u) + \phi'''(u) \cdot 1 \tag{4}$$

$$s = -\frac{1}{x^2}\phi'(u) - \frac{1}{x}\phi''(u) + \phi'''(u)(-1) \quad (5)$$

$$t = \frac{1}{x}\phi''(u) \cdot 1 + \phi'''(u) \cdot 1 \quad (6)$$

From (4) and (6), we get

$$\begin{aligned} r - t &= \frac{2}{x^2}\phi'(u) + \frac{2}{x^3}\phi(u) \\ &= \frac{2}{x^2}\left\{\frac{1}{x}\phi(u) + \phi'(u)\right\} \\ &= \frac{2}{x^2}z \end{aligned}$$

i.e.
$$x^2\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) = 2z$$

Exercise 1(a)

Part A (Short-Answer Questions)

1. Write down the form of the P.D.E. (partial differential equation), obtained by eliminating 'f' from $f(u, v) = 0$.

Form the P.D.E.s by eliminating the arbitrary constants a and b from the following relations:

2. $z = (x + a)(y + b)$
3. $z = (x^2 + a^2)(y^2 + b^2)$
4. $z = ax + by + ab$
5. $z = ax + by + a^2 + b^2$
6. $z = ax^3 + by^3$
7. $z = a(x + y) + b$
8. $ax^2 + by^2 + z^2 = 1$
9. $(x - a)^2 + (y - b)^2 = z^2$.

Form the P.D.E.s by eliminating the arbitrary functions from the following relations.

10. $z = f(x^2 + y^2)$
11. $z = \phi(x^3 - y^3)$
12. $z = f(bx - ay)$

13. $z = \phi(xy)$

14. $z = f\left(\dfrac{y}{x}\right)$

15. $z = f(x) + \phi(y)$

16. $z = f(x) + \phi(y) + axy$

17. $z = f(y) + x\phi(y)$

18. $z = yf(x) + \phi(x)$

19. $z = xf(y) + \phi(y) - \sin x$

20. $z = yf(x) + \phi(x) - \cos y$

Part B

21. Form the P.D.E. by eliminating a and b from $z = xy + y\sqrt{x^2 - a^2} + b$.

22. Form the P.D.E. by eliminating a and b from $z = ax - \dfrac{a}{a+1}y + b$.

23. Form the P.D.E. by eliminating a and b from $4z(1 + a^2) = (x + ay + b)^2$.

24. Form the P.D.E. by eliminating a and b from $z^2 + \left\{z\sqrt{z^2 - 4a^2}\right.$ $\left.-4a^2 \log (z + \sqrt{z^2 - 4a^2})\right\} = 4(x + ay + b)$.

25. Form the P.D.E. by eliminating a and b from $3z = ax^3 + 2\sqrt{a-1}\,y^{3/2} + b$.

26. Find the P.D.E. of all planes which cut off equal intercepts on the x and y axes.

27. Find the P.D.E. of all planes passing through the origin.

28. Find the P.D.E. of all spheres whose centres lie on the z-axis.

29. Find the P.D.E. of all spheres of radius c having their centres on the xoy-plane.

30. Find the P.D.E. of all spheres of radius c having their centres on the zox-plane.

31. Form the P.D.E. by eliminating the arbitrary function 'f' from
(a) $z = f\left(\dfrac{xy}{z}\right)$; (b) $z = f(x^2 + y^2 + z^2)$

32. Form the P.D.E. by eliminating the arbitrary function f from
(a) $xyz = f(x + y + z)$; (b) $\dfrac{xy}{z} = f(x^2 - y + z)$

33. Form the P.D.E. by eliminating 'ϕ' from
(a) $\phi\left(\dfrac{1}{x} - \dfrac{1}{y}, \dfrac{1}{y} - \dfrac{1}{z}\right) = 0$; (b) $\phi(x^3 - y^3, x^2 - z^2) = 0$

34. Form the P.D.E. by eliminating 'ϕ' from
(a) $\phi\left(x^2 + y^2 + z^2, \dfrac{y}{z}\right) = 0$; (b) $\phi\left(x^2 - y^2 - 2z, \dfrac{y}{zx}\right) = 0$

35. Form the P.D.E. by eliminating 'ϕ' from

$$(a)\ \phi\left(\frac{x-y}{y-z},\ xy+yz+zx\right)=0; \quad (b)\ \phi\left(\frac{x+y+z}{z}, x^2-y^2\right)=0$$

Form the P.D.E.s by eliminating the arbitrary functions from the following relations.

36. $z = f(x+iy)+g(x-iy)$, where $i=\sqrt{-1}$ and $x+iy \neq z$.
37. $z = f(2y+3x)+g(y-3x)$.
38. $z = f_1(y-x)+f_2(y+x)+f_3(y+2x)$.
39. $z = xf(2x+3y)+g(2x+3y)$
40. $z = f(x+y)+yg(x+y)$
41. $z = (x-iy)f(x+iy)+g(x-iy)$, where $i=\sqrt{-1}$ and $x+iy \neq z$.
42. $z = f(\sqrt{x}+y)+g(\sqrt{x}-y)$
43. $z = f(x)\cdot\phi(y)$
44. $z = yf(x)+x\phi(y)$
45. $z = f(x+y+z)+\phi(x-y)$.

1.5 SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

The relation between the independent variables and the dependent variable (containing arbitrary constants or functions) from which a partial differential equation is formed is called the *primitive* or *solution* of the P.D.E.

In other words, a *solution* of a P.D.E. is a relation between the independent and the dependent variables, which satisfies the P.D.E. Solution of a P.D.E. is also called *integral* of the P.D.E.

As was seen in Section 1.2, the primitive of a P.D.E. may contain arbitrary constants or arbitrary functions. Accordingly, we have two types of solutions for a P.D.E.

A solution of a P.D.E. which contains as many arbitrary constants as the number of independent variables is called *the complete solution* or *complete integral* of the equation.

A solution of a P.D.E. which contains as many arbitrary functions as the order of the equation is called the *general solution* or *general integral* of the equation.

Both these types of solutions can be obtained for the same P.D.E. For example, the equation $z = px + qy$ is obtained when we eliminate the arbitrary constants a and b from $z = ax + by$ or the arbitrary function 'f' from $z = x \cdot f\left(\frac{y}{x}\right)$.

Thus $z = ax + by$ is the complete solution and $z = x \cdot f\left(\frac{y}{x}\right)$ is the general solution of the P.D.E. $z = px + qy$.

The complete solution $z = ax + by$ can be rewritten as $z = x\left\{a + b\left(\frac{y}{x}\right)\right\}$. Comparing this with the general solution $z = xf\left(\frac{y}{x}\right)$, we note that $a + b\left(\frac{y}{x}\right)$ is a particular case of $f(y/x)$. Hence the general solution of a P.D.E. is more general than the complete solution. Thus when the solution of a P.D.E. is required, we should try to give the general solution. However there are certain P.D.E.s for which methods are not available for finding the general solutions directly, but methods are available for finding the complete solutions only in other cases. In such cases, we indicate the procedure for finding the general solution from the complete solution as explained in Section 1.6.

1.6 PROCEDURE TO FIND GENERAL SOLUTION

Let
$$F(x, y, z, p, q) = 0 \tag{1}$$
be a first order P.D.E. Let its complete solution be
$$\phi(x, y, z, a, b) = 0 \tag{2}$$
where a and b are arbitrary constants.

Let $b = f(a)$ [or $a = g(b)$], where 'f' is an arbitrary function.

Then (2) becomes
$$\phi[x, y, z, a, f(a)] = 0 \tag{3}$$

Differentiating (2) partially with respect to a, we get
$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b} \cdot f'(a) = 0 \tag{4}$$

Theoretically, it is possible to eliminate 'a' between (3) and (4).

This eliminant, which contains the arbitrary function 'f', is the general solution of (1).

A solution obtained by giving particular values to the arbitrary constants in the complete solution or to the arbitrary functions in the general solution is called *a particular solution* or *particular integral* of the P.D.E.

Thus for the P.D.E. $z = px + qy$, for which the complete solution is $z = ax + by$ and the general solution is $z = x \cdot f\left(\frac{y}{x}\right)$, the following are particular solutions.

(i) $z = 2x + 3y$
(ii) $z = 3x - 4y$
(iii) $z = x \cdot e^{\frac{y}{x}}$
(iv) $z = x \sin\left(\frac{y}{x}\right)$

There is yet another type of solution of a P.D.E., called the *singular solution* or *singular integral*. Geometrically the singular solution of a P.D.E. represents the envelope of the family of surfaces represented by the complete solution of that P.D.E. The singular solution will neither contain arbitrary constants nor arbitrary functions but at the same time cannot be obtained as particular case of the complete or general solution.

1.7 PROCEDURE TO FIND SINGULAR SOLUTION

Let
$$F(x, y, z, p, q) = 0 \tag{1}$$
be a first order P.D.E.

Let its complete solution be
$$\phi(x, y, z, a, b) = 0 \tag{2}$$

Differentiating (2) partially with respect to a and then b, we have
$$\frac{\partial \phi}{\partial a} = 0 \tag{3}$$
and
$$\frac{\partial \phi}{\partial b} = 0 \tag{4}$$

The eliminant of a and b from equations (2), (3) and (4), if it exists, is the singular solution of the P.D.E. (1).

As pointed out earlier, P.D.E.s can be divided into two categories — one for which methods are readily available only for finding complete solutions and the other for which methods are available for finding general solutions. First order non-linear equations that belong to the first category will be discussed in Section 1.8.

1.8 COMPLETE SOLUTIONS OF FIRST ORDER NON-LINEAR P.D.E.S

A P.D.E., the partial derivatives occurring in which are of the first degree, is said to be *linear*; otherwise it is said to be *non-linear*.

First order non-linear P.D.E.s, for which complete solutions can be found out, are divided into four standard types. Some first-order non-linear P.D.E.s, which do not fall under any of the four standard types, can be transformed into one or the other of the standard types by suitable changes of variables. We shall discuss below the special methods of finding the complete solutions for these types of equations.

Type I

Equations of the form $f(p, q) = 0$, i.e. the P.D.E.s that contain p and q only explicitly.

For equations of this type, it is known that a solution will be of the following form,
$$z = ax + by + c \tag{1}$$
But this solution contains three arbitrary constants, whereas the number of independent variables is two. Hence if we can reduce the number of arbitrary constants in (1) by one, it becomes the complete solution of the equation $f(p, q) = 0$. Now from (1), $p = a$ and $q = b$. If (1) is to be a solution of $f(p, q) = 0$, the values of p and q obtained from (1) should satisfy the given equation.

i.e.
$$f(a, b) = 0$$

Solving this, we can get $b = \phi(a)$, where ϕ is a known function. Using this value of b in (1), the complete solutiion of the given P.D.E. is

$$z = ax + \phi(a)y + c \tag{2}$$

The general solution can be obtained from (2) by the method given earlier.

To find the singular solution, we have to eliminate a and c from

$$z = ax + \phi(a)y + c, \quad x + \phi'(a)y = 0 \quad \text{and} \quad 1 = 0$$

of which the last equation is absurd. Hence there is no singular solution for equations of type I.

Type II

Clairaut's type, i.e. the P.D.E.s of the form

$$z = px + qy + f(p, q) \tag{1}$$

For equations of this type also, it is known that a solution will be of the form

$$z = ax + by + c \tag{2}$$

If we can reduce the number of arbitrary constants in (2) by one, it becomes the complete solution of (1).

From (2) we get $p = a$ and $q = b$.

As before, $$z = ax + by + f(a, b) \tag{3}$$

From (2) and (3), we get $c = f(a, b)$

Thus the complete solution of (1) is given by (3).

Note

Without going through all these formalities, we can quickly write down the complete solution of a clairaut's type of P.D.E. by simply replacing p and q by a and b in it respectively.

The general and singular solutions of (1) can be found out by the usual methods. For clairaut's type of equations, singular solutions will normally exist.

Type III

Equations not containing x and y explicitly, i.e. equations of the form

$$f(z, p, q) = 0 \tag{1}$$

For equations of this type, it is known that a solution will be of the form

$$z = \phi(x + ay) \tag{2}$$

where 'a' is an arbitrary constant and ϕ is a specific function to be found out.

Putting $x + ay = u$, (2) becomes $z = \phi(u)$ or $z(u)$

$$\therefore \qquad p = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du}$$

and
$$q = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a\frac{dz}{du}$$

If (2) is to be a solution of (1), the values of p and q obtained should satisfy (1).

$$\text{i.e. } f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0 \tag{3}$$

From (3), we can get

$$\frac{dz}{du} = \psi(z, a) \tag{4}$$

Now (4) is an ordinary differential equation, which can be solved by the variable separable method.

The solution of (4), which will be of the form $g(z, a) = u + b$ or $g(z, a) = x + ay + b$, is the complete solution of (1).

The general and singular solutions of (1) can be found out by the usual methods.

Type IV

Equations of the form

$$f(x, p) = g(y, q) \tag{1}$$

that is equations which do not contain z explicitly and in which terms containing p and x can be separated from those containing q and y.

To find the complete solution of (1), we assume that $f(x, p) = g(y, q) = a$, where 'a' is an arbitrary constant.

Solving $f(x, p) = a$, we can get $p = \phi(x, a)$ and solving $g(y, q) = a$, we can get $q = \psi(y, a)$.

Now

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \text{ or } pdx + qdy$$

i.e.
$$dz = \phi(x, a)dx + \psi(y, a)dy$$

Integrating with respect to the concerned variables, we get

$$z = \int \phi(x, a)dx + \int \psi(y, a)dy + b \tag{2}$$

The complete solution of (1) is given by (2), which contains two arbitrary constants a and b.

The general and singular solutions of (1) are found out by the usual methods.

1.9 EQUATIONS REDUCIBLE TO STANDARD TYPES — TRANSFORMATION

Type A

Equations of the form $f(x^m p, y^n q) = 0$ or $f(x^m p, y^n q, z) = 0$,
where m and n are constants, each not equal to 1.
We make the transformations $x^{1-m} = X$ and $y^{1-n} = Y$.

Then $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X} \cdot \dfrac{\partial X}{\partial x} = (1-m)x^{-m}P$, where $P \equiv \dfrac{\partial z}{\partial X}$ and

$q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial Y} \cdot \dfrac{\partial Y}{\partial y} = (1-n)y^{-n}Q$, where $Q \equiv \dfrac{\partial z}{\partial Y}$.

Therefore the equation $f(x^m p, y^n q) = 0$ reduces to $f\{(1-m)P, (1-n)Q\} = 0$, which is a type I equation.

The equation $f(x^m p, y^n q, z) = 0$ reduces to $f\{(1-m)P, (1-n)Q, z\} = 0$, which is a type III equation.

Type B

Equations of the form $f(px, qy) = 0$ or $f(px, qy, z) = 0$

Note

These equations correspond to m = 1 and n = 1 of the type A equations.

The required transformations are

$$\log x = X \text{ and } \log y = Y$$

In this case, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X} \cdot \dfrac{\partial X}{\partial x} = \dfrac{\partial z}{\partial X} \cdot \dfrac{1}{x}$ or $px = P$ and $q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial Y} \cdot \dfrac{\partial Y}{\partial y} = \dfrac{\partial z}{\partial Y} \cdot \dfrac{1}{y}$ or $qy = Q$, where $P \equiv \dfrac{\partial z}{\partial X}$ and $Q \equiv \dfrac{\partial z}{\partial Y}$.

Therefore the equation $f(px, qy) = 0$ reduces to $f(P, Q) = 0$, which is a type I equation.

The equation $f(px, qy, z) = 0$ reduces to $f(P, Q, z) = 0$, which is a type III equation.

Type C

Equations of the form $f(z^k p, z^k q) = 0$ or $f(z^k p, z^k q, x, y) = 0$, where k is a constant $\neq -1$.

We make the transformation $Z = z^{k+1}$

Then
$$P = \frac{\partial Z}{\partial x} = (k+1)z^k p \text{ and}$$

$$Q = \frac{\partial Z}{\partial y} = (k+1)z^k q$$

Therefore the equation $f(z^k p, z^k q) = 0$ reduces to $f\left(\frac{P}{k+1}, \frac{Q}{k+1}\right) = 0$, which is a type I equation and the equation $f(z^k p, z^k q, x, y) = 0$ reduces to $f\left(\frac{P}{k+1}, \frac{Q}{k+1}, x, y\right) = 0$, which may be a type IV equation.

Type D

Equations of the form $f\left(\frac{p}{z}, \frac{q}{z}\right) = 0$ or $f(p/z, q/z, x, y) = 0$, which correspond to $k = -1$ of type C equations.

The required transformation is $Z = \log z$

Then $P = \frac{\partial Z}{\partial x} = \frac{1}{z}p$ and $Q = \frac{\partial Z}{\partial y} = \frac{1}{z}q$

Therefore the equations $f(p/z, q/z) = 0$ and $f(p/z, q/z, x, y) = 0$ reduce respectively to type I and type IV equations.

Type E

Equations of the form $f\left(x^m z^k p, y^n z^k q\right) = 0$ where $m, n \neq 1$; $k \neq -1$

We make the transformations

$$X = x^{1-m}, Y = y^{1-n} \text{ and } Z = z^{k+1}$$

Then
$$P = \frac{\partial Z}{\partial X} = \frac{dZ}{dz} \cdot \frac{\partial z}{\partial x} \cdot \frac{dx}{dX}$$
$$= (k+1)z^k p \cdot \frac{x^m}{1-m}$$

and
$$Q = (k+1)z^k q \frac{y^n}{1-n}$$

$\therefore$ The given equation reduces to

$$f\left\{\left(\frac{1-m}{k+1}\right)P, \left(\frac{1-n}{k+1}\right)Q\right\} = 0$$

which is a type I equation.

Type F

Equations of the form $f\left(\frac{px}{z}, \frac{qy}{z}\right) = 0$

By putting $X = \log x$, $Y = \log y$ and $Z = \log z$ the equation reduces to $f(P, Q) = 0$, where $P = \frac{\partial Z}{\partial X}$ and $Q = \frac{\partial Z}{\partial Y}$.

Worked Examples 1(b)

Example 1

Solve the equation $pq + p + q = 0$.

This equation contains only p and q explicitly.

$\therefore$ Let a solution of the equation be

$$z = ax + by + c \tag{1}$$

From (1), we get $p = a$ and $q = b$.

Since (1) is a solution of the given equation,

$$ab + a + b = 0$$

$$\therefore \quad b = -\frac{a}{a+1} \tag{2}$$

Using (2) in (1), the required complete solution of the equation

$$z = ax - \frac{a}{a+1}y + c \tag{3}$$

To find the general solution, we put $c = f(a)$ in (3), where 'f' is an arbitrary function.

i.e.
$$z = ax - \frac{a}{a+1}y + f(a) \tag{4}$$

Differentiating (4) partially with respect to a, we get

$$x - \frac{1}{(a+1)^2}y + f'(a) = 0 \tag{5}$$

Eliminating a between (4) and (5), we get the required general solution.

To find the singular solution, we have to differentiate (3) partially with respect to a and c.

When we differentiate (3) partially with respect to c, we get $0 = 1$, which is absurd.

Hence, no singular solution exists for the given equation.

Example 2

Solve the equation $p^2 + q^2 = 4pq$.

$$p^2 + q^2 - 4pq = 0 \tag{1}$$

As (1) contains only p and q, a solution of (1) will be of the form

$$z = ax + by + c \tag{2}$$

From (2), we get $p = a$ and $q = b$.

Since (2) is a solution of (1),

$$a^2+b^2-4ab=0$$

Solving for b, we get

$$b=\frac{4a\pm\sqrt{16a^2-4a^2}}{2}$$
$$=(2\pm\sqrt{3})a$$

Using in (2), the complete solution of (1) is

$$z=ax+(2\pm\sqrt{3})ay+c \tag{3}$$

There is no singular solution for (1), as in Example 1.

To get the general solution, we put $c=f(a)$ in (3), which becomes

$$z=ax+(2\pm\sqrt{3})ay+f(a) \tag{4}$$

where f is an arbitrary function.

Differentiating (4) partially with respect to a, we get

$$0=x+(2\pm\sqrt{3})y+f'(a) \tag{5}$$

The eliminant of 'a' between (4) and (5) gives the general solution of (1).

Example 3

Solve the equation $x^4p^2-yzq-z^2=0$.

As it is, the equation

$$x^4p^2-yzq-z^2=0 \tag{1}$$

does not belong to any of the four standard types.

Rewriting Eq. (1), we get

$$\left(\frac{x^2p}{z}\right)^2-\left(\frac{yq}{z}\right)=1 \tag{2}$$

As L.H.S. of (2) is a function of $\frac{x^2p}{z}$ and $\frac{yq}{z}$, we make the transformations

$X=x^{-1}$, $Y=\log y$ and $Z=\log z$

(by the transformation rules for type A and type F equations)

Then $$p=\frac{\partial z}{\partial x}=\frac{dz}{dZ}\cdot\frac{\partial Z}{\partial X}\cdot\frac{dX}{dx}=zP\left(-\frac{1}{x^2}\right)$$

$\therefore$ $$\frac{x^2p}{z}=-P$$

and $$q=\frac{\partial z}{\partial y}=\frac{dz}{dZ}\cdot\frac{\partial Z}{\partial Y}\cdot\frac{dY}{dy}=zQ.\frac{1}{y}$$

$\therefore$ $$\frac{yq}{z}=Q$$

Equation (2) becomes

$$P^2 - Q = 1 \tag{3}$$

Equation 3 contains only P and Q explicitly.

Therefore a solution of (3) will be of the form

$$Z = aX + bY + c \tag{4}$$

$\therefore$ $P = a$ and $Q = b$, obtained from (4), satisfy Eq. 3.

$$\therefore \quad a^2 - b = 1$$

$$\therefore \quad b = a^2 - 1$$

$\therefore$ The complete solution of (3) is

$$Z = aX + (a^2 - 1)Y + c$$

$\therefore$ The complete solution of (1) is

$$\log z = \frac{a}{x} + (a^2 - 1)\log y + c$$

Singular solution does not exist and general solution is found out as usual.

Example 4

Solve the equation $z^2\left(\dfrac{p^2}{x^2} + \dfrac{q^2}{y^2}\right) = 1$.

The given equation does not belong to any of the four standard types.

It can be rewritten as

$$(x^{-1}zp)^2 + (y^{-1}zq)^2 = 1 \tag{1}$$

which is of the form $(x^m z^k p)^2 + (y^n z^k q)^2 = 1$ [Refer to type E equations]

$\therefore$ We make the transformations

$$X = x^{1-m},\ Y = y^{1-n} \text{ and } Z = z^{k+1}$$

i.e.

$$X = x^2,\ Y = y^2 \text{ and } Z = z^2$$

Then

$$p = \frac{\partial z}{\partial x} = \frac{dz}{dZ}\cdot\frac{\partial Z}{\partial X}\cdot\frac{dX}{dx} = \frac{1}{2z}\cdot P \cdot 2x$$

$$\therefore \quad P = x^{-1}zp$$

Similarly, $Q = y^{-1}zq$.

Using these in (1), it becomes

$$P^2 + Q^2 = 1 \tag{2}$$

As (2) contains only P and Q explicitly, a solution of the equation will be of the form

$$Z = aX + bY + c \tag{3}$$

$\therefore P = a$ and $Q = b$, obtained from (3), satisfy Eq. 2.

i.e. $$a^2 + b^2 = 1.$$

$\therefore$ $$b = \pm\sqrt{1 - a^2}$$

$\therefore$ The complete solution of (2) is

$$Z = aX \pm \sqrt{1 - a^2}Y + c$$

$\therefore$ The complete solution of (1) is

$$z^2 = ax^2 \pm \sqrt{1 - a^2}y^2 + c$$

Singular solution does not exist and general solution is found out as usual.

Example 5

Solve the equation $pq\,xy = z^2$.

The equation

$$pq\,xy = z^2 \tag{1}$$

does not belong to any of the four standard types.

Rewriting (1),

$$\left(\frac{px}{z}\right)\left(\frac{qy}{z}\right) = 1 \tag{2}$$

As (2) contains $\frac{px}{z}$ and $\frac{qy}{z}$, we make the substitutions $X = \log x$, $Y = \log y$ and $Z = \log z$ [Refer to type F equations]

Then $$P = \frac{\partial z}{\partial x} = \frac{dz}{dZ} \cdot \frac{\partial Z}{\partial X} \cdot \frac{dX}{dx} = z \cdot P \cdot \frac{1}{x}$$

i.e. $$\frac{px}{z} = P$$

Similarly $$\frac{qy}{z} = Q$$

Using these in (2), it becomes

$$PQ = 1 \tag{3}$$

which contains only P and Q explicitly. A solution of (3) is of the form

$$Z = aX + bY + c \tag{4}$$

$\therefore$ $P = a$ and $Q = b$, obtained from 4, satisfy (3)

i.e. $$ab = 1 \quad \text{or} \quad b = \frac{1}{a}$$

$\therefore$ The complete solution of (3) is $Z = aX + \dfrac{1}{a}Y + c$

$\therefore$ The complete solution of (1) is

$$\log z = a \log x + \frac{1}{a} \log y + c \tag{5}$$

General solution of (1) is obtained as usual.

Note ✍

To find the singular solution of (1), we should not use the complete solution of (3). We should use only that of (1) given in (5).

If we put $c = \log k$, (5) becomes

$$\log z = \log (x^a y^{1/a} k)$$

i.e. $$z = x^a y^{1/a} k \tag{6}$$

Differentiating (6) partially with respect to a,

$$\log x - \frac{1}{a^2} \log y = 0 \tag{7}$$

Differentiating (6) partially with respect to k,

$$0 = x^a y^{1/a} \tag{8}$$

Eliminating a and k from (6), (7) and (8), that is using (8) in (6), the singular solution of equation (1) is $z = 0$.

Example 6

Solve the equation $z^4 q^2 - z^2 p = 1$.

The equation can be solved directly, as it contains p, q and z only explicitly. However we shall transform it into a simpler equation and solve it.

The equation can be rewritten as

$$(z^2 q)^2 - (z^2 p) = 1 \tag{1}$$

which contains $z^2 p$ and $z^2 q$.

Hence we make the transformation $Z = z^3$ [Refer to type C equations]

$$\therefore \qquad P = \frac{\partial Z}{\partial x} = 3z^2 p$$

$$\text{i.e.} \qquad z^2 p = \frac{P}{3}$$

Similarly $z^2 q = \dfrac{Q}{3}$

Using these values in (1), we get

$$Q^2 - 3P = 9 \tag{2}$$

As (2) is an equation containing P and Q only, a solution of (2) will be of the form

$$Z = ax + by + c \tag{3}$$

Now $P = a$ and $Q = b$, obtained from (3) satisfy Eq. 2.

$$\therefore \qquad b^2 - 3a = 9$$

$$\text{i.e.} \qquad b = \pm\sqrt{3a + 9}$$

$\therefore$ Complete solution of (2) is $Z = ax \pm \sqrt{3a + 9}y + c$, i.e. complete solution of (1) is $z^3 = ax \pm \sqrt{3a + 9}y + c$. Singular solution does not exist. General solution is found out as usual.

Example 7

Solve the equation $z = px + qy + p^2 + pq + q^2$

The given equation

$$z = px + qy + (p^2 + pq + q^2) \tag{1}$$

is a Clairaut's type equation.

$\therefore$ The complete solution of (1) is

$$z = ax + by + a^2 + ab + b^2 \tag{2}$$

[got by replacing p and q in (1) by a and b]

Let us now find the singular solution of (1).

Differentiating (2) partially with respect to a and then b, we get

$$x + 2a + b = 0 \tag{3}$$

$$\text{and} \qquad y + a + 2b = 0 \tag{4}$$

The eliminant of a and b from (2), (3) and (4) is the required singular solution. Solving (3) and (4) for a and b, we get

$$a = \frac{1}{3}(y - 2x) \quad \text{and} \quad b = \frac{1}{3}(x - 2y)$$

Using these values in (2), the singular solution is

$$z = \frac{x}{3}(y - 2x) + \frac{y}{3}(x - 2y) + \frac{1}{9}(y - 2x)^2$$

$$+\frac{1}{9}(y - 2x)(x - 2y) + \frac{1}{9}(x - 2y)^2$$

i.e. $$9z = 3x(y - 2x) + 3y(x - 2y)$$

$$+(y - 2x)^2 + (y - 2x)(x - 2y) + (x - 2y)^2$$

i.e. $$3z + x^2 - xy + y^2 = 0$$

General solution of (1) is found out as usual.

Example 8

Solve the equation $z = px + qy + \left(\frac{q}{p} - p\right)$.

The given equation

$$z = px + qy + \left(\frac{q}{p} - p\right) \tag{1}$$

is a Clairaut's type equation.

$\therefore$ The complete solution of (1) is

$$z = ax + by + \frac{b}{a} - a \tag{2}$$

The general solution of (1) is found out as usual.

To find the singular solution of (1), we differentiate (2) partially with respect to a and then b.

We get

$$0 = x - b/a^2 - 1 \tag{3}$$

and $$0 = y + 1/a \tag{4}$$

Using $a = -\frac{1}{y}$ got from (4) in (3), we get

$$x - by^2 - 1 = 0$$

i.e. $$b = \frac{x - 1}{y^2}$$

Using these values of a and b in (2), we get

$$z = -x/y + \frac{x - 1}{y} - \left(\frac{x - 1}{y}\right) + \frac{1}{y}$$

i.e. $yz = 1 - x$, which is the singular solution of (1).

Example 9

Solve the equation $Z = px + qy + c\sqrt{1+p^2+q^2}$.
The given equation

$$z = px + qy + c\sqrt{1+p^2+q^2} \tag{1}$$

is a Clairaut's type equation.
$\therefore$ Its complete solution is

$$z = ax + by + c\sqrt{1+a^2+b^2} \tag{2}$$

where a and b are arbitrary constants and c is a given constant.

The general solution of (1) is found out from (2) as usual.

To find the singular solution of (1), we differentiate (2) partially with respect to a and then b.

$$0 = x + \frac{ca}{\sqrt{1+a^2+b^2}} \tag{3}$$

and

$$0 = y + \frac{cb}{\sqrt{1+a^2+b^2}} \tag{4}$$

From (3) and (4), we get $\frac{a}{b} = \frac{x}{y}$ or $\frac{a}{x} = \frac{b}{y} = k$, say

$\therefore$ $a = kx$ and $b = ky$

Using these values in (3), we have

$$\frac{kc}{\sqrt{1+k^2(x^2+y^2)}} = -1$$

since k is negative.

i.e. $1 + k^2(x^2+y^2) = k^2c^2$

or $k^2(c^2 - x^2 - y^2) = 1$

i.e. $k = -\frac{1}{\sqrt{c^2-x^2-y^2}}$

$\therefore$ $a = -\frac{x}{\sqrt{c^2-x^2-y^2}}, \quad b = -\frac{y}{\sqrt{c^2-x^2-y^2}}$

and $\sqrt{1+a^2+b^2} = \frac{c}{\sqrt{c^2-x^2-y^2}}$

Using these values in (2), the singular solution of (1) is got as

$$z = -\frac{x^2}{\sqrt{c^2-x^2-y^2}} - \frac{y^2}{\sqrt{c^2-x^2-y^2}} + \frac{c^2}{\sqrt{c^2-x^2-y^2}}$$

i.e. $z = \sqrt{c^2-x^2-y^2}$ or

$$x^2 + y^2 + z^2 = c^2$$

Example 10

Solve the equation $(pq - p - q)(z - px - qy) = pq$.

Rewriting the given equation as

$$z = px + qy + \frac{pq}{pq - p - q} \tag{1}$$

we identify it as a Clairaut's type equation.
Hence its complete solution is

$$z = ax + by + \frac{ab}{ab - a - b} \tag{2}$$

The general solution of (1) is found out as usual from (2).

Let us now find the singular solution of (1).

Differentiating (2) partially with respect to a and then b, we get

$$0 = x + \frac{(ab - a - b)b - ab(b - 1)}{(ab - a - b)^2}$$

i.e.
$$0 = x - \frac{b^2}{(ab - a - b)^2} \tag{3}$$

and similarly
$$0 = y - \frac{a^2}{(ab - a - b)^2} \tag{4}$$

From (3) and (4), we get $\dfrac{a^2}{b^2} = y/x$ or

$$\frac{a}{\sqrt{y}} = \frac{b}{\sqrt{x}} = k, \text{ say}$$

$$\therefore \quad a = k\sqrt{y} \text{ and } b = k\sqrt{x}$$

Using these values in (3), we get

$$k^2x - (k^2\sqrt{xy} - k\sqrt{y} - k\sqrt{x})^2 x = 0$$

i.e.
$$(k\sqrt{xy} - \sqrt{x} - \sqrt{y}) = 1$$

$$\therefore \quad k = \frac{1 + \sqrt{x} + \sqrt{y}}{\sqrt{xy}}$$

Hence $a = \dfrac{1 + \sqrt{x} + \sqrt{y}}{\sqrt{x}}$ and $b = \dfrac{1 + \sqrt{x} + \sqrt{y}}{\sqrt{y}}$

Also

$$\frac{ab}{ab - a - b} = \frac{1}{1 - 1/b - 1/a} = \frac{1}{1 - \dfrac{\sqrt{y}}{1 + \sqrt{x} + \sqrt{y}} - \dfrac{\sqrt{x}}{1 + \sqrt{x} + \sqrt{y}}}$$

$$= 1 + \sqrt{x} + \sqrt{y}$$

Using these values in (2), the singular solution of (1) is

$$z = \sqrt{x}(1 + \sqrt{x} + \sqrt{y}) + \sqrt{y}(1 + \sqrt{x} + \sqrt{y}) + (1 + \sqrt{x} + \sqrt{y})$$

i.e.
$$z = (1 + \sqrt{x} + \sqrt{y})^2.$$

Example 11

Transform the equation $4xyz = pq + 2px^2y + 2qxy^2$ by means of the substitutions $X = x^2$ and $Y = y^2$ and hence solve it.

$$P = \frac{\partial z}{\partial X} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial X} = \frac{p}{2x}$$

and similarly $$Q = \frac{q}{2y}$$

Rewriting the given equation, we have

$$4z = \frac{pq}{xy} + 2px + 2qy \qquad (1)$$

Using the transformations in (1), it becomes

$$4z = 4PQ + 4PX + 4QY$$

i.e. $$z = PX + QY + PQ \qquad (2)$$

which is a Clairaut's type of equation.

The complete solution of (2) is

$$z = aX + bY + ab \qquad (3)$$

Therefore the complete solution (1) is

$$z = ax^2 + by^2 + ab \qquad (4)$$

The general solution of (1) is obtained from (4) as usual.

The singular solution of (1) is obtained as follows.

Differentiating (4) partially with, respect to a and then b, we get

$$0 = x^2 + b \qquad (5)$$

and $$0 = y^2 + a \qquad (6)$$

From (5) and (6), $a = -y^2$ and $b = -x^2$. Using these values in (4), the singular solution of (1) is

$$z = -x^2y^2 - x^2y^2 + x^2y^2$$

i.e. $$z + x^2y^2 = 0$$

Example 12

Solve the equation $z^2(p^2 + q^2 + 1) = c^2$, where c is a constant.

The given equation

$$z^2(p^2 + q^2 + 1) = c^2 \qquad (1)$$

does not contain x and y explicitly.

Therefore (1) has a solution of the form

$$z = y(u) = z(x + ay) \qquad (2)$$

where $z(u) = z(x + ay)$ is a function of $(x + ay)$, where a is an arbitrary constant.

From (2), we have $p = \dfrac{dz}{du}$ and $q = \dfrac{dz}{du} \cdot a$

Since (2) is a solution of (1), we get

$$z^2 \left\{ \left(\frac{dz}{du}\right)^2 + a^2 \left(\frac{dz}{du}\right)^2 + 1 \right\} = c^2$$

i.e. $$(1 + a^2)\left(\frac{dz}{du}\right)^2 = \frac{c^2}{z^2} - 1$$

i.e. $$\sqrt{1+a^2}\frac{dz}{du} = \frac{\sqrt{c^2 - z^2}}{z}$$

i.e. $$\sqrt{1+a^2}\frac{zdz}{\sqrt{c^2 - z^2}} = du \qquad (3)$$

Integrating (3), the complete solution of (1) is

$$-\frac{1}{2}\sqrt{1+a^2}\int \frac{-2zdz}{\sqrt{c^2 - z^2}} = u + b$$

i.e. $$-\sqrt{1+a^2}\sqrt{c^2 - z^2} = x + ay + b \text{ or}$$

$$(1 + a^2)(c^2 - z^2) = (x + ay + b)^2 \qquad (4)$$

The general and singular solutions of (1) are found out from (4) as usual.

Example 13

Solve the equation $p(1 - q^2) = q(1 - z)$

The given equation

$$p(1 - q^2) = q(1 - z) \qquad (1)$$

does not contain x and y explicitly.

Therefore (1) has a solution of the form

$$z = z(u) = z(x + ay) \qquad (2)$$

where a is an arbitrary constant.

From (2), $p = \dfrac{dz}{du}$ and $q = a\dfrac{dz}{du}$

Since (2) is a solution of (1), we get

$$\frac{dz}{du}\left\{1 - a^2\left(\frac{dz}{du}\right)^2\right\} = a\frac{dz}{du}(1 - z)$$

i.e. $$\frac{dz}{du}\left[1 - a^2\left(\frac{dz}{du}\right)^2 - a + az\right] = 0$$

As z is not a constant, $\frac{dz}{du} \neq 0$

$$\therefore \qquad 1 - a^2\left(\frac{dz}{du}\right)^2 - a + az = 0$$

i.e.
$$a^2\left(\frac{dz}{du}\right)^2 = az + 1 - a$$

or
$$a\frac{dz}{du} = \sqrt{az + 1 - a} \tag{3}$$

Solving (3), we get

$$a\int \frac{dz}{\sqrt{az+1-a}} = u + b$$

i.e.
$$2\sqrt{az+1-a} = x + ay + b \quad \text{or}$$

$$4(az + 1 - a) = (x + ay + b)^2 \tag{4}$$

which is the complete solution of (1).

The general and singular solutions of (1) are found out from (4) as usual.

Example 14

Solve the equation $\qquad 9pqz^4 = 4(1 + z^3)$.

The given equation

$$9pqz^4 = 4(1 + z^3) \tag{1}$$

does not contain x and y explicitly.

Therefore (1) has got a solution of the form

$$z = z(u) = z(x + ay) \tag{2}$$

where a is an arbitrary constant.

From (2), $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$.

Since (2) is a solution of (1), we get

$$9a\left(\frac{dz}{du}\right)^2 z^4 = 4(1 + z^3)$$

i.e.
$$3\sqrt{a}z^2\frac{dz}{du} = 2\sqrt{1+z^3} \tag{3}$$

Solving (3), we get

$$\frac{\sqrt{a}}{2}\int \frac{3z^2 dz}{\sqrt{1+z^3}} = u + b$$

i.e.
$$\sqrt{a}\cdot\sqrt{1+z^3} = x + ay + b \qquad \text{or}$$

$$a(1 + z^3) = (x + ay + b)^2 \tag{4}$$

which is the complete solution of (1).

The general and singular solutions of (1) are found out from (4) as usual.

Example 15

Solve the equation $\dfrac{x^2}{p} + \dfrac{y^2}{q} = z$.

The given equation does not belong to any of the standard types.

It can be rewritten as

$$\frac{1}{px^{-2}} + \frac{1}{qy^{-2}} = z \tag{1}$$

As equation (1) contains px^{-2} and qy^{-2}, we make the substitutions $X = x^3$ and $Y = y^3$. [Refer to type A equations]

Then $P = \dfrac{\partial z}{\partial X} = p \cdot \dfrac{1}{3x^2}$ or $px^{-2} = 3P$ and similarly $qy^{-2} = 3Q$.

Then (1) becomes

$$\frac{1}{P} + \frac{1}{Q} = 3Z \tag{2}$$

As (2) does not contain X and Y explicitly, it has a solution of the form

$$z = z(u) = z(X + aY) \tag{3}$$

From (3), $P = \dfrac{dz}{du}$ and $Q = a\dfrac{dz}{du}$

Since (3) is a solution of (2), we get

$$\frac{dz}{du}(1 + a) = 3az\left(\frac{dz}{du}\right)^2$$

$$\frac{dz}{du}\left(3az\frac{dz}{du} - a - 1\right) = 0$$

$$\text{As} \quad \frac{dz}{du} \neq 0, \qquad 3az\frac{dz}{du} = a + 1 \tag{4}$$

Solving (4), $\displaystyle\int 3az\, dz = (a + 1)u + b$

i.e. $\quad \frac{3}{2}az^2 = (a + 1)(X + aY) + b$

which is the complete solution of equation (2).

$\therefore$ The complete solution of equation (1) is

$$\frac{3}{2}az^2 = (a + 1)(x^3 + ay^3) + b$$

where a and b are arbitrary constants.

The general and singular solutions are found out as usual.

Example 16

Solve the equation

$$p^2 + x^2y^2q^2 = x^2z^2$$

The given equation does not belong to any of the standard types.
Rewriting it, we have

$$\left(x^{-1}p\right)^2 + (yq)^2 = z^2 \tag{1}$$

As equation (1) contains $x^{-1}p$ and yq, we make the transformations $X = x^2$ and $Y = \log y$ [Refer to type A and type B equations]

$$\therefore \qquad \frac{\partial z}{\partial X} = p \cdot \frac{1}{2x} \quad \text{and} \quad Q = \frac{\partial z}{\partial Y} = qy$$

i.e. $$x^{-1}p = 2P \quad \text{and} \quad yq = Q$$

Using these values in (1), it becomes

$$4P^2 + Q^2 = z^2 \tag{2}$$

As (2) does not contain X and Y explicitly, it has got a solution of the form

$$z = z(u) = z(X + aY) \tag{3}$$

From (3), we have

$$P = \frac{dz}{du} \text{ and } Q = a\frac{dz}{du}$$

Using these values in (2), we get

$$\left(\frac{dz}{du}\right)^2 (4 + a^2) = z^2$$

i.e. $$\sqrt{a^2 + 4}\ \frac{dz}{du} = z \tag{4}$$

Solving (4), we get

$$\sqrt{a^2 + 4}\log z = X + aY + b$$

which is the complete solution of (2).

$\therefore$ The complete solution of (1) is

$$\sqrt{a^2 + 4}\log z = x^2 + a\log y + b$$

where a and b are arbitrary constants.

The general and singular solutions are found out as usual.

Example 17

Solve the equation

$$x^2p^2 + xpq = z^2$$

The given equation can be rewritten as

$$(xp)^2 + (xp)q = z^2 \tag{1}$$

Putting $X = \log x$, we get $P = \dfrac{\partial z}{\partial X} = px$

Using this in (1), it becomes

$$P^2 + Pq = z^2 \tag{2}$$

As Eq. 2 does not contain X and y explicitly, it has a solution of the form

$$z = z(u) = z(X + ay) \tag{3}$$

From (3),

$$P = \frac{dz}{du} \quad \text{and} \quad q = a\frac{dz}{du}$$

Using these values in (2), we have

$$\left(\frac{dz}{du}\right)^2 + a\left(\frac{dz}{du}\right)^2 = z^2$$

i.e.

$$\sqrt{1+a}\,\frac{dz}{du} = z \tag{4}$$

Solving (4), we get $\sqrt{1+a}\log z = X + ay + b$, which is the complete solution of (2).

$\therefore$ The complete solution of (1) is

$$\sqrt{1+a}\log z = \log x + ay + b$$

The general and singular solutions are found out as usual.

Example 18

Solve the equation

$$q^2y^2 = z(z - px)$$

As the given equation contains px and qy, we make the following substitutions.

$$X = \log x \quad \text{and} \quad Y = \log y$$

$$\therefore \quad P = \frac{\partial z}{\partial X} = px \quad \text{and} \quad Q = \frac{\partial z}{\partial Y} = qy$$

Using these in the given equation, it becomes

$$Q^2 = z(z - P) \quad \text{or} \quad Pz + Q^2 = z^2 \tag{1}$$

As Eq. (1) does not contain X and Y explicitly, it has a solution of the form

$$z = z(u) = z(X + aY) \tag{2}$$

From (2),

$$P = \frac{dz}{du} \quad \text{and} \quad Q = a\frac{dz}{du}$$

Using these values in (1), it becomes

$$z\frac{dz}{du} + a^2\left(\frac{dz}{du}\right)^2 = z^2$$

or

$$a^2\left(\frac{dz}{du}\right)^2 + z\frac{dz}{du} - z^2 = 0 \tag{3}$$

Solving (3) for $\frac{dz}{du}$, we get

$$\frac{dz}{du} = \frac{-z \pm \sqrt{z^2 + 4a^2z^2}}{2a^2}$$
$$= \frac{(-1 \pm \sqrt{1 + 4a^2})z}{2a^2}$$

Solving this equation, we get

$$2a^2 \int \frac{dz}{z} = (-1 \pm \sqrt{1 + 4a^2})u + b$$

i.e.

$$2a^2 \log z = (-1 \pm \sqrt{1 + 4a^2})(X + aY) + b$$

which is the complete solution of (1).

$\therefore$ The complete solution of the given equation is

$$2a^2 \log z = (-1 \pm \sqrt{1 + 4a^2})(\log x + a \log y) + b$$

The general and singular solutions are found out as usual.

Example 19

Solve the equation

$$\sqrt{p} + \sqrt{q} = x + y$$

The given equation does not contain z explicitly and is variable separable.

That is the equation can be rewritten as

$$\sqrt{p} - x = y - \sqrt{q} = a, \text{ say} \tag{1}$$

$$\therefore \quad p = (x + a)^2 \quad \text{and} \quad q = (y - a)^2$$

Now $$dz = p dx + q dy$$

$$= (x + a)^2 dx + (y - a)^2 dy \tag{2}$$

Integrating both sides with respect to the concerned variables, we get

$$z = \frac{(x + a)^3}{3} + \frac{(y - a)^3}{3} + b \tag{3}$$

where a and b are arbitrary constants. Equation (3) is the complete solution of the given equation.

General solution is found out as usual. Singular solution does not exist.

Example 20

Solve the equation

$$yp = 2xy + \log q$$

The given equation, which does not contain z, can be rewritten as

$$p - 2x = \frac{1}{y} \log q = a, \text{ say} \tag{1}$$

$$\therefore \quad p = 2x + a \quad \text{and} \quad q = e^{ay}$$

Now $$dz = p dx + q dy$$

i.e. $$dz = (2x + a) dx + e^{ay} dy \tag{2}$$

Integrating (2), we get

$$z = x^2 + ax + \frac{1}{a} e^{ay} + b \tag{3}$$

where a and b are arbitrary constants.

Equation (3) is the complete solution of the given equation.

General solution is found out as usual.

Singular solution does not exist.

Example 21

Solve the equation

$$p^2(1 + x^2) y = q x^2$$

The given equation, which does not contain z, can be rewritten as

$$p^2 \frac{(1 + x^2)}{x^2} = \frac{q}{y} = a, \text{ say} \tag{1}$$

$$p = \frac{\sqrt{a} \cdot x}{\sqrt{1+x^2}} \quad \text{and} \quad q = ay$$

$$dz = pdx + qdy$$

$$= \sqrt{a} \cdot \frac{x}{\sqrt{1+x^2}} dx + aydy \tag{2}$$

Integrating (2), we get the complete solution of the given equation as

$$z = \sqrt{a(1+x^2)} + \frac{ay^2}{2} + b \tag{3}$$

where a and b are arbitrary constants.

From (3), we get the general solution as usual. Singular solution does not exist.

Example 22

Solve the equation $\quad z^2(p^2+q^2) = x + y$

The given equation

$$z^2(p^2+q^2) = x + y \tag{1}$$

does not belong to any of the standard types.

Equation (1) can be rewritten as

$$(zp)^2 + (zq)^2 = x + y$$

Since the equation contains zp and zq, we make the substitution $Z = z^2$

$$\therefore \quad P = \frac{\partial Z}{\partial x} = 2zp \quad \text{and} \quad Q = \frac{\partial Z}{\partial y} = 2zq$$

Using these in (1), it becomes

$$P^2 + Q^2 = 4x + 4y \tag{2}$$

which does not contain Z explicitly.

Rewriting (2), we get

$$P^2 - 4x = 4y - Q^2 = 4a, \text{ say} \tag{3}$$

$$\therefore \quad P = 2\sqrt{x+a} \quad \text{and} \quad Q = 2\sqrt{y-a}$$

$$dZ = Pdx + Qdy$$

$$= 2\sqrt{x+a}dx + 2\sqrt{y-a}dy$$

Integrating, we get

$$Z = \frac{4}{3}(x+a)^{3/2} + \frac{4}{3}(y-a)^{3/2} + b$$

i.e. $$z^2 = \frac{4}{3} \cdot (x+a)^{3/2} + \frac{4}{3} \cdot (y-a)^{3/2} + b$$

which is the complete solution of (1).

General solution is found out as usual.

Singular solution does not exist.

Example 23

Solve the equation

$$p^2 + q^2 = z^2(x^2 + y^2)$$

The given equation does not belong to any of the standard types.
It can be rewritten as

$$(z^{-1}p)^2 + (z^{-1}q)^2 = x^2 + y^2 \tag{1}$$

As the Eq. (1) contains $z^{-1}p$ and $z^{-1}q$, we make the substitution $Z = \log z$

$$\therefore \qquad P = \frac{p}{z} \quad \text{and} \quad Q = \frac{q}{z}$$

Using these values in (1), it becomes

$$P^2 + Q^2 = x^2 + y^2 \tag{2}$$

As Eq. 2 does not contain Z explicitly, we rewrite it as

$$P^2 - x^2 = y^2 - Q^2 = a^2, \text{ say} \tag{3}$$

From (3),

$$P = \sqrt{x^2 + a^2} \quad \text{and} \quad Q = \sqrt{y^2 - a^2}$$

$$dZ = Pdx + Qdy$$

$$= \sqrt{x^2 + a^2}dx + \sqrt{y^2 - a^2}dy$$

Integrating, we get

$$Z = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cosh^{-1}(y/a) + b$$

$\therefore$ The complete solution of (1) is

$$\log z = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cosh^{-1}(y/a) + b$$

where a and b are arbitrary constants.
General solution is found out as usual.
Singular solution does not exist.

Example 24

Solve the equation $(x + pz)^2 + (y + qz)^2 = 1$.

The given equation does not belong to any of the standard types.
But the equation contains pz and qz.

Therefore we make the substitution $Z = z^2$.

Then $P = \dfrac{\partial Z}{\partial x} = 2zp$ and $Q = 2zq$.

Using these values in the given equation, it becomes

$$\left(x + \frac{P}{2}\right)^2 + \left(y + \frac{Q}{2}\right)^2 = 1 \tag{1}$$

Equation (1) does not contain Z explicitly. Rewriting (1), we have

$$\left(x + \frac{P}{2}\right)^2 = 1 - \left(y + \frac{Q}{2}\right)^2 = a^2, \text{ say} \tag{2}$$

From (2), $x + \dfrac{P}{2} = a$ or $P = 2(a - x)$ and $y + \dfrac{Q}{2} = \sqrt{1 - a^2}$ or $Q = 2(\sqrt{1 - a^2} - y)$

Now $dZ = P dx + Q dy$

$$= 2(a - x)dx + 2(\sqrt{1 - a^2} - y)dy \tag{3}$$

Integrating (3) and replacing Z by z^2, the complete solution of the given equation is

$$z^2 = -(a - x)^2 + 2\sqrt{1 - a^2}y - y^2 + b$$

General solution is found out as usual. Singular solution does not exist.

Example 25

Solve the equation $pz^2 \sin^2 x + qz^2 \cos^2 y = 1$. The given equation does not belong to any of the standard types.

The given equation contains $(z^2 p)$ and $(z^2 q)$.

Therefore we make the substitution $Z = z^3$

$$\therefore \quad P = \frac{\partial Z}{\partial x} = 3z^2 p \text{ and } Q = 3z^2 q$$

Using these values in the given equation, it becomes

$$\frac{P}{3} \sin^2 x + \frac{Q}{3} \cos^2 y = 1 \tag{1}$$

Equation (1) does not contain Z explicitly. Rewriting (1), we have

$$\frac{P}{3} \sin^2 x = 1 - \frac{Q}{3} \cos^2 y = a, \text{ say} \tag{2}$$

From (2), $P = 3a \operatorname{cosec}^2 x$ and $Q = 3(1 - a) \sec^2 y$

Now $dZ = Pdx + Qdy$

$$= 3a \operatorname{cosec}^2 x dx + 3(1-a)\sec^2 y dy \tag{3}$$

Integrating (3) and replacing Z by z^3, the complete solution of the given equation is

$$z^3 = -3a \cot x + 3(1-a)\tan y + b$$

General solution is found out as usual. Singular solution does not exist.

Exercise 1(b)

Part A (Short-Answer Questions)

1. Define complete solution and general solution of a P.D.E.
2. How will you find the general solution of a P.D.E. from its complete solution?
3. What is the geometrical significance of the singular solution of a P.D.E.?
4. How will you find the singular solution of a P.D.E. from its complete solution?
5. Find the complete solution of the P.D.E. $q = f(p)$.
6. Find the complete solution of the P.D.E. $z = px + qy + f(p, q)$.

Find the complete solution of the following P.D.E.s.

7. $pq = k$
8. $p = e^q$
9. $p^2 + q^2 = 2$
10. $p + q = z$
11. $p^2 = qz$
12. $pq = z$
13. $pq = xy$
14. $px = qy$
15. $pe^y = qe^x$
16. Rewrite the equation $pqz = p^2(qx + p^2) + q^2(py + q^2)$ as a Clairaut's equation and hence write down its complete solution.

Part B

17. Solve the equation (a) $\sqrt{p} + \sqrt{q} = 1$; (b) $p^2 + q^2 = k^2$. Find the singular solutions, if they exist.
18. Solve the equation $3p^2 - 2q^2 = 4pq$. Find the singular solution, if it exists.

19. Solve the equation $p^2 - 2pq + 3q = 5$. Find the singular solution, if it exists.

Convert the following equations into equations of the form $f(p, q) = 0$ and hence solve them.

20. $p^2x^2 + q^2y^2 = z^2$
21. $p^2x + q^2y = z$
22. $px^2 + qy^2 = z^2$
23. $z^2(p^2 - q^2) = 1$
24. $2x^4p^2 - yzq - 3z^2 = 0$
25. $(y - x)(qy - px) = (p - q)^2$ [**Hint**: Put $x + y = X$ and $xy = Y$]

Find the singular solutions of the following partial differential equations.

26. $z = px + qy - 2\sqrt{pq}$
27. $\dfrac{z}{pq} = \dfrac{x}{q} + \dfrac{y}{p} - \sqrt{pq}$
28. $z = px + qy + p^2q^2$
29. $(p + q)(z - px - qy) = 1$
30. $z = px + qy + p^2 - q^2$
31. $z = px + qy + \sqrt{p^2 + q^2}$
32. $(1 - x)p + (2 - y)q = 3 - z$

Solve the following equations.

33. $p^2 + q^2 = z$
34. $1 + p^2 + q^2 = z^2$
35. (a) $pz = 1 + q^2$; (b) $qz = 1 + p^2$
36. $p(1 + q^2) = q(z - a)$
37. $9(p^2z + q^2) = 4$

Convert the following equations into equations of the form $f(p, q, z) = 0$ and hence solve them.

38. $\dfrac{p}{x^2} + \dfrac{q}{y^2} = z$
39. $(p^2x^2 + q^2)z^2 = 1$
40. $p^2x^4 + y^2zq = 2z^2$

Solve the following equations.

41. $q = px + p^2$

42. $yp + xq + pq = 0$
43. $yp - x^2q^2 = x^2y$
44. $q(p - \sin x) = \cos y$

Convert the following equations into equations of the form $f(p, q, x, y) = 0$ and hence solve them.

45. $(p^2 - q^2)z = x - y$
46. $(p^2 + q^2)z^2 = x^2 + y^2$
47. $p^2 + x^2y^2q^2 = x^2z^2$
48. $4z^2q^2 = y - x + 2zp$
49. $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$ [**Hint**: Put $x + y = X$ and $x - y = Y$]
50. $(p^2 + q^2)(x^2 + y^2) = 1$ [**Hint**: Put $x = r\cos\theta$ and $y = r\sin\theta$]

1.10 GENERAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations, for which the general solution can be obtained directly, can be divided into the following three categories

1. Equations that can be solved by direct (partial) integration. For example, consider the equation

$$\frac{\partial z}{\partial x} = a \tag{1}$$

If z were a function of x only, direct integration with respect to x will give the solution as

$$z = ax + b \tag{2}$$

If (2) is to be the general solution of (1), b need not be a constant, but it may be an arbitrary function of y, say $f(y)$. Then (2) becomes

$$z = ax + f(y) \tag{3}$$

When we differentiate (3) partially with respect to x, we get Eq. (1). As (3) contains an arbitrary function, it is the general solution.

Thus when we get the solution of an equation by partial integration with respect to x [or y], we should take an arbitrary function of y [or x] in the place of arbitrary constants taken when ordinary integration is performed.

Equations, in which the dependent variable occurs only in the partial derivatives, can be solved by this partial integration method.

2. Lagrange's linear equation of the first order, which will be discussed in Section 1.11.
3. Linear partial differential equations of higher order with constant coefficients, which will be discussed in Section 1.12.

1.11 LAGRANGE'S LINEAR EQUATION

A linear partial differential equation of the first order, which is of the form $Pp+Qq=R$ where P, Q, R are functions of x, y, z, is called *Lagrange's linear equation*.
We have already shown that the elimination of the arbitrary function 'f' from $f(u, v)=0$ leads to Lagrange's linear equation.

General solution of Lagrange's linear equation

The general solution of the equation $Pp+Qq=R$ is $f(u, v)=0$, where 'f' is an arbitrary function and $u(x, y, z)=a$ and $v(x, y, z)=b$ are independent solutions of the simultaneous differential equations $\dfrac{dx}{P}=\dfrac{dy}{Q}=\dfrac{dz}{R}$.

Proof

$$f(u, v)=0 \tag{1}$$

Differentiating (1) partially with respect to x and then y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0 \tag{2}$$

and

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0 \tag{3}$$

Eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ from (2) and (3), we get

$$\frac{u_x+u_zp}{u_y+u_zq}=\frac{v_x+v_zp}{v_y+v_zq}$$

i.e.

$$(u_yv_z-u_zv_y)p+(u_zv_x-u_xv_z)q=u_xv_y-u_yv_x \tag{4}$$

Taking $P=u_yv_z-u_zv_y$, $Q=u_zv_x-u_xv_z$ and $R=u_xv_y-u_yv_x$, Eq. (4) takes the form

$$Pp+Qq=R \tag{5}$$

Since the primitive of equation (5) is equation (1), that contains an arbitrary function 'f', we conclude that $f(u, v)=0$ is the general solution of the Lagrange's linear equation (5).

Now consider $u=a$ and $v=b$

$\therefore$ $du=0$ and $dv=0$

i.e.

$$u_x dx+u_y dy+u_z dz=0 \tag{6}$$

and

$$v_x dx+v_y dy+v_z dz=0 \tag{7}$$

Solving (6) and (7) for $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$, we get

$$\frac{\mathrm{d}x}{u_y v_z - u_z v_y} = \frac{\mathrm{d}y}{u_z v_x - u_x v_z} = \frac{\mathrm{d}z}{u_x v_y - u_y v_x}$$

i.e.
$$\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R} \tag{8}$$

When we eliminate a and b from $u = a$ and $v = b$, we get the simultaneous equations (8). In other words, the solutions of equations (8) are $u = a$ and $v = b$.

Therefore the general solution of $Pp + Qq = R$ is $f(u, v) = 0$, where $u = a$ and $v = b$ are independent solutions of $\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R}$.

Working rule to solve $Pp + Qq = R$

(i) To solve $Pp + Qq = R$, we form the corresponding subsidiary simultaneous equations $\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R}$.

(ii) Solving these equations, we get two independent solutions $u = a$ and $v = b$.

(iii) Then the required general solution is $f(u, v) = 0$ or $u = \phi(v)$ or $v = \psi(u)$.

1.12 SOLUTION OF THE SIMULTANEOUS EQUATIONS $\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R}$

Method of grouping

By grouping any two of three ratios, it may be possible to get an ordinary differential equation containing only two variables, eventhough P; Q; R are, in general, functions of x, y, z. By solving this equation, we can get a solution of the simultaneous equations. By this method, we may be able to get two independent solutions, by using different groupings.

Method of multipliers

If we can find a set of three quantities l, m, n, which may be constants or functions of the variables x, y, z, such that $l\mathrm{P} + m\mathrm{Q} + n\mathrm{R} = 0$, then a solution of the simultaneous equations is found out as follows.

$$\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R} = \frac{l\mathrm{d}x + m\mathrm{d}y + n\mathrm{d}z}{lP + mQ + nR}$$

Since $l\mathrm{P} + m\mathrm{Q} + n\mathrm{R} = 0$, $l\mathrm{d}x + m\mathrm{d}y + n\mathrm{d}z = 0$. If $l\mathrm{d}x + m\mathrm{d}y + n\mathrm{d}z$ is an exact differential of some function $u(x, y, z)$, then we get $\mathrm{d}u = 0$. Integrating this, we get $u = a$, which is a solution of $\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R}$.

Similarly, if we can find another set of independent multipliers l', m', n', we can get another independent solution $v = b$.

Note ✍

1. *We may use the method of grouping to get one solution and the method of multipliers to get the other solution of* $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
2. *The subsidiary equations are called Lagrange's subsidiary simultaneous equations.*
3. *The multipliers* l, m, n *are called Lagrange multipliers.*

Worked Examples 1(c)

Example 1

Solve the equations (i) $\frac{\partial^2 z}{\partial x^2} = xy$; (ii) $\frac{\partial^2 z}{\partial y^2} = \sin xy$

(i)
$$\frac{\partial^2 z}{\partial x^2} = xy \tag{1}$$

Integrating both sides of (1) partially with respect to x (i.e. treating y as a constant),

$$\frac{\partial z}{\partial x} = y\frac{x^2}{2} + \phi(y) \tag{2}$$

Integrating (2) partially with respect to x,

$$z = \frac{x^3}{6}y + f(y) + x \cdot \phi(y) \tag{3}$$

where $f(y)$ and $\phi(y)$ are arbitrary functions. Equation (3) is the required general solution of (1).

(ii)
$$\frac{\partial^2 z}{\partial y^2} = \sin xy \tag{4}$$

Integrating (4) partially with respect to y,

$$\frac{\partial z}{\partial y} = -\frac{1}{x}\cos xy + \phi(x) \tag{5}$$

Integrating (5) partially with respect to y,

$$z = -\frac{1}{x^2}\sin xy + f(x) + y \cdot \phi(x) \tag{6}$$

where $f(x)$ and $\phi(x)$ are arbitrary functions. Equation (6) is the required general solution of (4).

Example 2

Solve the equation $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$, if $u = 0$ when $t = 0$ and $\dfrac{\partial u}{\partial t} = 0$ when $x = 0$. Also show that $u \to \sin x$, when $t \to \infty$.

$$\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x \tag{1}$$

Integrating (1) partially with respect to x,

$$\frac{\partial u}{\partial t} = e^{-t}\sin x + f(t) \tag{2}$$

When $x = 0$, $\dfrac{\partial u}{\partial t} = 0$. (given)
Using this in (2), we get $f(t) = 0$.

$\therefore$ Equation (2) becomes $\dfrac{\partial u}{\partial t} = e^{-t}\sin x$ (3)

Integrating (3) partially with respect to t, we get

$$u = -e^{-t}\sin x + g(x) \tag{4}$$

Using the given condition, namely, $u = 0$ when $t = 0$, in (4), we get

$$0 = -\sin x + g(x) \text{ or } g(x) = \sin x$$

Using this value in (4), the required particular solution of (1) is $u = \sin x(1 - e^{-t})$.

$$\text{Now } \lim_{t\to\infty}(u) = \sin x\left[\lim_{t\to\infty}(1 - e^{-t})\right]$$
$$= \sin x$$

That is when $t \to \infty$, $u \to \sin x$.

Example 3

Solve the equation $\dfrac{\partial^2 z}{\partial x^2} + z = 0$, given that $z = e^y$ and $\dfrac{\partial z}{\partial x} = 1$ when $x = 0$.

$$\frac{\partial^2 z}{\partial x^2} + z = 0 \tag{1}$$

If z were a function of x alone, the equation (1) would have been the ordinary differential equation

$$\frac{d^2 z}{dx^2} + z = 0, \text{ i.e. } (D^2 + 1)z = 0 \tag{2}$$

The auxiliary equation of (2) is $m^2 + 1 = 0$. Its roots are $\pm i$. Hence the solution of (2) is

$$z = A\cos x + B\sin x \tag{3}$$

Solution (3) can be assumed to be obtained by integrating (2) ordinarily with respect to x.

If we replace A and B in (3) by arbitrary functions of y, the solution can be assumed to have been obtained by integrating (1) partially with respect to x.

Thus the general solution of (1) is

$$z = f(y) \cdot \cos x + g(y) \cdot \sin x \tag{4}$$

From (4),
$$\frac{\partial z}{\partial x} = -f(y)\sin x + g(y)\cos x \tag{5}$$

Using the condition that $z = e^y$ when $x = 0$ in (4), we get

$$f(y) = e^y \tag{6}$$

Using the condition that $\frac{\partial z}{\partial x} = 1$ when $x = 0$ in (5),

$$g(y) = 1 \tag{7}$$

Using (6) and (7) in (4), the required solution of (1) is $z = e^y \cos x + \sin x$.

Example 4

Solve the equations $\frac{\partial z}{\partial x} = 3x - y$ and $\frac{\partial z}{\partial y} = -x + \cos y$ simultaneously.

$$\frac{\partial z}{\partial x} = 3x - y \tag{1}$$

$$\frac{\partial z}{\partial y} = -x + \cos y \tag{2}$$

Integrating (1) partially with respect to x,

$$z = \frac{3x^2}{2} - yx + f(y) \tag{3}$$

Differentiating (3) partially with respect to y,

$$\frac{\partial z}{\partial y} = -x + f'(y) \tag{4}$$

Comparing (2) and (4), we get $f'(y) = \cos y$

$\therefore$
$$f(y) = \sin y + c \tag{5}$$

$\therefore$ The required solution is

$z = \frac{3}{2}x^2 - xy + \sin y + c$, where c is an arbitrary constant.

Example 5

By changing the independent variables by the transformations $u = x - y$ and $v = x + y$, show that the equation $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ can be transformed as $\frac{\partial^2 z}{\partial v^2} = 0$ and hence solve it.

$$u = x - y \text{ and } v = x + y$$

$$\therefore \qquad x = \frac{u+v}{2} \text{ and } y = \frac{v-u}{2}$$

If we express x and y in z in terms of u and v, z becomes a function of u and v.

$$z_x = \frac{\partial z}{\partial x} = z_u \cdot u_x + z_v \cdot v_x, \text{ where } z_u = \frac{\partial z}{\partial u} \text{ and } u_x = \frac{\partial u}{\partial x}, \text{ etc.}$$

$$= z_u + z_v$$

$$z_y = z_u \cdot u_y + z_v \cdot v_y = -z_u + z_v$$

$$z_{xx} = (z_{uu} + z_{uv}) + (z_{vu} + z_{vv}) = z_{uu} + 2z_{uv} + z_{vv}$$

$$z_{xy} = (-z_{uu} + z_{uv}) + (-z_{vu} + z_{vv}) = -z_{uu} + z_{vv}$$

$$z_{yy} = z_{uu} - z_{uv} + (-z_{vu} + z_{vv}) = z_{uu} - 2z_{uv} + z_{vv}$$

Using these values in the given equation $z_{xx}+2z_{xy}+z_{yy} = 0$, it becomes $4z_{vv} = 0$.

i.e.
$$\frac{\partial^2 z}{\partial v^2} = 0 \tag{1}$$

Integrating (1) partially with respect to v,

$$\frac{\partial z}{\partial v} = g(u) \tag{2}$$

Integrating (2) partially with respect to v,

$$z = v \cdot g(u) + f(u) \tag{3}$$

$\therefore$ The solution of the given equation is

$$z = f(x-y) + (x+y)g(x-y)$$

Example 6

By changing the independent variables by the transformations $u = x$ and $v = \dfrac{y}{x}$, transform the equation $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x \partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} = 0$ and hence solve it.

When $u = x$ and $v = y/x$, $\quad x = u$ and $y = uv$.

$\therefore$ $\quad z$, which is a function of x and y, can also treated as a function of u and v.

$$z_x = z_u \cdot u_x + z_v \cdot v_x = z_u - \frac{y}{x^2} z_v$$

$$z_y = z_u \cdot u_y + z_v \cdot v_y = \frac{1}{x} \cdot z_v$$

$$z_{xx} = z_{uu} + z_{uv}\left(-\frac{y}{x^2}\right) + \frac{2y}{x^3} z_v - \frac{y}{x^2}\left[z_{vu} + z_{vv}\left(\frac{-y}{x^2}\right)\right]$$

$$z_{xy} = z_v \cdot \left(-\frac{1}{x^2}\right) + \frac{1}{x}\left[z_{vu} + z_{vv}\left(-\frac{y}{x^2}\right)\right]; \; z_{yy} = \frac{1}{x}\left[z_{vv} \cdot \frac{1}{x}\right]$$

Using these values in the given equation, it becomes,

$$\left(x^2 z_{uu} - y z_{uv} + \frac{2y}{x} z_v - y z_{uv} + \frac{y^2}{x^2} z_{vv}\right)$$

$$+\left(-\frac{2y}{x} z_v + 2y z_{uv} - \frac{2y^2}{x^2} z_{vv}\right) + \left(\frac{y^2}{x^2} z_{vv}\right) = 0$$

i.e. $$x^2 z_{uu} = 0 \quad \text{or} \quad z_{uu} = 0 \tag{1}$$

Integrating (1) partially with respect to u,

$$z_u = \phi(v) \tag{2}$$

Integrating (2) partially with respect to u,

$$z = f(v) + u \cdot \phi(v) \tag{3}$$

$\therefore$ Solution of the given equation is

$$z = f(y/x) + x \cdot \phi(y/x)$$

Example 7

Transform the partial differential equation $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = 0$ to the form $\frac{\partial^2 z}{\partial u \partial v} = 0$ by using the substitutions $u = x + \alpha y$ and $v = x + \beta y$, where α and β are appropriate constants and hence solve the given equation.

Clearly z, which is a function of x and y, can also be treated as a function of u and v.

$$z_x = z_u + z_v; \quad z_y = \alpha z_u + \beta z_v$$

$$z_{xx} = z_{uu} + 2z_{uv} + z_{vv}; \quad z_{xy} = z_{uu} \cdot \alpha + z_{uv} \cdot \beta$$

$$+ z_{vu} \cdot \alpha + z_{vv} \cdot \beta \text{ or } \alpha z_{uu} + (\alpha + \beta) z_{uv} + \beta z_{vv} \cdot$$

$$z_{yy} = \alpha(z_{uu} \cdot \alpha + z_{uv} \cdot \beta) + \beta(z_{vu} \cdot \alpha + z_{vv} \cdot \beta)$$

$$= \alpha^2 z_{uu} + 2\alpha\beta z_{uv} + \beta^2 z_{vv}.$$

Using these values in the given equation, it becomes

$$(z_{uu} + 2z_{uv} + z_{vv}) - 5\left[\alpha z_{uu} + (\alpha + \beta) z_{uv} + \beta z_{vv}\right]$$

$$+ 6\left[\alpha^2 z_{uu} + 2\alpha\beta z_{uv} + \beta^2 z_{vv}\right] = 0$$

i.e. $(6\alpha^2 - 5\alpha + 1) z_{uu} + [2 - 5(\alpha + \beta) + 12\alpha\beta] z_{uv} + (6\beta^2 - 5\beta + 1) z_{vv} = 0$ (1)

Since (1) has to reduce to the form $z_{uv} = 0$, coefficient of $z_{uu} = 0 =$ coefficient of z_{vv}.

i.e. $$6\alpha^2 - 5\alpha + 1 = 0 \text{ and } 6\beta^2 - 5\beta + 1 = 0$$

i.e. $$\alpha = \frac{1}{2}, \frac{1}{3} \quad \text{and } \beta = \frac{1}{2}, \frac{1}{3}$$

If we choose equal values for α and β, coefficient of z_{uv} also becomes zero. Hence we choose $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$.

For these values of α and β, equation (1) becomes

$$-\frac{1}{6}z_{uv} = 0 \text{ or } \frac{\partial^2 z}{\partial u \partial v} = 0 \tag{2}$$

Integrating (2) partially with respect to u,

$$\frac{\partial z}{\partial v} = \phi(v) \tag{3}$$

Integrating (3) partially with respect to v,

$$z = \int \phi(v)\mathrm{d}v + f(u)$$

i.e. $$z = f(u) + g(v)$$

$\therefore$ The solution of the given equation is

$$z = f\left(x + \frac{1}{2}y\right) + g\left(x + \frac{1}{3}y\right)$$

$$\text{or } z = f(y + 2x) + g(y + 3x)$$

Example 8

Solve the equation $x^2p + y^2q + z^2 = 0$.
The given equation

$$x^2p + y^2q = -z^2 \tag{1}$$

is a Lagrange's linear equation with $P = x^2$, $Q = y^2$ and $R = -z^2$
The subsidiary equations are

$$\frac{\mathrm{d}x}{x^2} = \frac{\mathrm{d}y}{y^2} = \frac{\mathrm{d}z}{-z^2}$$

Taking the first two ratios, we get an ordinary differential equation in x and y, namely, $\frac{\mathrm{d}x}{x^2} = \frac{\mathrm{d}y}{y^2}$.

Integrating, we get $-\frac{1}{x} = -\frac{1}{y} - a$

i.e.
$$\frac{1}{x} - \frac{1}{y} = a \tag{2}$$

Taking the last two ratios, we get the equation $\frac{dy}{y^2} = \frac{-dz}{z^2}$

$$\frac{dy}{y^2} = \frac{-dz}{z^2}$$

Integrating, we get $\frac{-1}{y} = \frac{1}{z} - b$

Solving,

$$\frac{1}{y} + \frac{1}{z} = b \tag{3}$$

$\therefore$ The general solution of the given equation is $f\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} + \frac{1}{z}\right) = 0$, where '$f$' is an arbitrary function.

Example 9

Solve the equation $y^2 p - xyq = x(z - 2y)$.

The given equation is a Lagrange's linear equation with $P = y^2$, $Q = -xy$, $R = x(z - 2y)$. The subsidiary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Taking the first two ratios, we get

$$\frac{dx}{y} = \frac{dy}{-x} \text{ or } -x dx = y dy$$

Integrating, we get $\quad \frac{x^2}{2} + \frac{y^2}{2} = \frac{a}{2}$ or $x^2 + y^2 = a \quad (1)$

From the subsidiary equations, we have

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} = \frac{z dy + y dz}{-2xy^2}$$

From the first and last ratios, we get

$$\frac{dx}{1} = \frac{d(yz)}{-2x} \quad \text{or} \quad -2x dx = d(yz)$$

Integrating, we get $\quad x^2 + yz = b \quad (2)$

From (1) and (2) the general solution of the given equation is $f(x^2 + y^2, x^2 + yz) = 0$.

Example 10

Solve the equation $(p - q)z = z^2 + (x + y)$. This is a Lagrange's linear equation with $P = z$, $Q = -z$ and $R = z^2 + (x + y)$.

The subsidiary equations are

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x + y)}$$

From the first two ratios, we get $dx = -dy$

Integrating, we get $$x + y = a^2 \qquad (1)$$

Note ✍

Neither the method of grouping nor the method of multipliers can be used to get the second solution.

We make use of solution (1), i.e. we put $x + y = a^2$ in the third ratio.

From the first and third ratios, we get

$$\frac{dx}{z} = \frac{dz}{z^2 + a^2} \text{ or } 2dx = \frac{2zdz}{z^2 + a^2}$$

Integrating, we get $2x = \log (z^2 + a^2) + b$. Now using the value of a^2 from (1), the second solution is

$$2x - \log (z^2 + x + y) = b \qquad (2)$$

From (1) and (2), the general solution of the given equation is

$$f[x + y, 2x - \log (x + y + z^2)] = 0$$

Example 11

Solve the equation $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$.

This is a Lagrange's linear equation with $P = (z^2 - 2yz - y^2)$, $Q = xy + zx$ and $R = xy - zx$.

The subsidiary equations are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y + z)} = \frac{dz}{x(y - z)}$$

From the last two ratios, we have

$$(y - z)dy = (y + z)dz$$

i.e. $$ydy - (zdy + ydz) - zdz = 0$$

i.e. $$ydy - d(yz) - zdz = 0$$

Integrating, we get

$$\frac{y^2}{2} - yz - \frac{z^2}{2} = \frac{a}{2} \text{ or}$$

$$y^2 - 2yz - z^2 = a \tag{1}$$

Using the multipliers, x, y, z, each of the above ratios $= \dfrac{x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z}{0}$

$$\therefore \qquad x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z = 0$$

Integrating, we get $\qquad x^2 + y^2 + z^2 = b \qquad (2)$

Therefore the general solution of the given equation is $f(y^2 - 2yz - z^2, x^2 + y^2 + z^2) = 0$.

Example 12

Solve the equation $(x - 2z)p + (2z - y)q = y - x$. This is a Lagrange's linear equation with $P = x - 2z$, $Q = 2z - y$ and $R = y - x$.

The subsidiary equations are

$$\frac{\mathrm{d}x}{x - 2z} = \frac{\mathrm{d}y}{2z - y} = \frac{\mathrm{d}z}{y - x} \tag{1}$$

Using the multipliers 1, 1, 1, each ratio in (1) $= \dfrac{\mathrm{d}x + \mathrm{d}y + \mathrm{d}z}{0}$

$$\therefore \qquad \mathrm{d}x + \mathrm{d}y + \mathrm{d}z = 0$$

Integrating, we get, $\qquad x + y + z = a \qquad (2)$

Using the multipliers $y, x, 2z$, each ratio in (1) $= \dfrac{y\mathrm{d}x + x\mathrm{d}y + 2z\mathrm{d}z}{0}$

$$\therefore \qquad \mathrm{d}(xy) + 2z\mathrm{d}z = 0$$

Integrating, we get $\qquad xy + z^2 = b \qquad (3)$

Therefore the general solution of the given equation is $f(x + y + z,\ xy + z^2) = 0$

Example 13

Solve the equation $(x^2 - y^2 - z^2)p + 2xyq = 2zx$. This is a Lagrange's linear equation with $P = x^2 - y^2 - z^2$, $\quad Q = 2xy$, $\quad R = 2zx$.

The subsidiary equations are

$$\frac{\mathrm{d}x}{x^2 - y^2 - z^2} = \frac{\mathrm{d}y}{2xy} = \frac{\mathrm{d}z}{2zx} \tag{1}$$

Taking the last two ratios, we get

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get $\log y = \log z + \log a$

i.e. $$\frac{y}{z} = a \tag{2}$$

Using the multipliers $x, y, z,$ each of the ratios in (1) $= \dfrac{x dx + y dy + z dz}{x(x^2 + y^2 + z^2)}$ (3)

Taking the last ratio in (1) and the ratio in (3),

$$\frac{dz}{2zx} = \frac{\frac{1}{2} d(x^2 + y^2 + z^2)}{x(x^2 + y^2 + z^2)}$$

i.e. $$\frac{dz}{z} = \frac{d(x^2 + y^2 + z^2)}{x^2 + y^2 + z^2}$$

Integrating, we get $\log b + \log z = \log (x^2 + y^2 + z^2)$

i.e. $$\frac{x^2 + y^2 + z^2}{z} = b \tag{4}$$

Therefore the general solution of the given equation is $f\left(\dfrac{y}{z}, \dfrac{x^2 + y^2 + z^2}{z}\right) = 0.$

Example 14

Solve the equation $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$.
This is a Lagrange's linear equation with $P = x^2(y - z)$, $Q = y^2(z - x)$, $R = z^2(x - y)$.

The subsidiary equations are

$$\frac{dx}{x^2(y - z)} = \frac{dy}{y^2(z - x)} = \frac{dz}{z^2(x - y)} \tag{1}$$

Using the multipliers $\dfrac{1}{x^2}, \dfrac{1}{y^2}, \dfrac{1}{z^2}$, each of the ratios in (1) $= \dfrac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$

$\therefore$ $$\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

Integrating, we get $-\dfrac{1}{x} - \dfrac{1}{y} - \dfrac{1}{z} = -a$

or $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a \tag{2}$$

Using the multipliers $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$, each of the ratios in (1) $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$$\therefore \qquad \frac{1}{x}\mathrm{d}x + \frac{1}{y}\mathrm{d}y + \frac{1}{z}\mathrm{d}z = 0$$

Integrating, we get $\log x + \log y + \log z = \log b$

or $$xyz = b \tag{3}$$

Therefore the general solution of the given equation is $f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, \quad xyz\right) = 0.$

Example 15

Solve the equation $(mz - ny)p + (nx - lz)q = ly - mx$. Hence write down the solution of the equation $(2z - y)p + (x + z)q + 2x + y = 0$.
The equation $(mz - ny)p + (nx - lz)q = ly - mx$
is a Lagrange's linear equation with $P = mz - ny$, $Q = nx - lz$, $R = ly - mx$.
The subsidiary equations are

$$\frac{\mathrm{d}x}{mz - ny} = \frac{\mathrm{d}y}{nx - lz} = \frac{\mathrm{d}z}{ly - mx} \tag{1}$$

Using the two sets of multipliers l, m, n and x, y, z, each of the above ratios in (1)

$$= \frac{l\mathrm{d}x + m\mathrm{d}y + n\mathrm{d}z}{0} \quad \text{and also} \quad = \frac{x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z}{0}$$

$$\therefore \qquad l\mathrm{d}x + m\mathrm{d}y + n\mathrm{d}z = 0 \quad \text{and} \quad x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z = 0$$

Integrating both the equations, we get

$$lx + my + nz = a \text{ and } x^2 + y^2 + z^2 = b$$

Therefore the general solution of the given equation is $f(lx + my + nz, x^2 + y^2 + z^2) = 0$.
Comparing the equation

$$(2z - y)p + (x + z)q = -2x - y \tag{2}$$

with the previous equation (1), we get $l = -1$, $m = 2$, $n = 1$.
Therefore the solution of equation (2) is

$$f(-x + 2y + z, x^2 + y^2 + z^2) = 0$$

Example 16

Solve the equation $(y + z)p + (z + x)q = x + y$.
This is a Lagrange's linear equation with $P = y + z$, $Q = z + x$ and $R = x + y$.

The subsidiary equations are

$$\frac{\mathrm{d}x}{y+z} = \frac{\mathrm{d}y}{z+x} = \frac{\mathrm{d}z}{x+y} \tag{1}$$

Each of the ratios in (1) is equal to

$$\frac{\mathrm{d}(x-y)}{-(x-y)} = \frac{\mathrm{d}(y-z)}{-(y-z)} = \frac{\mathrm{d}(z-x)}{-(z-x)} \tag{2}$$

Taking the first two ratios in (2), we get

$$\frac{\mathrm{d}(x-y)}{x-y} = \frac{\mathrm{d}(y-z)}{y-z}$$

Integrating, we get $\log(x-y) = \log(y-z) + \log a$

i.e.
$$\frac{x-y}{y-z} = a \tag{3}$$

Note ✍

Taking the last two ratios in (2) and integrating, we get another solution, namely

$$\frac{z-x}{y-z} = b \tag{4}$$

But solution (4) is not independent of solution (3), since $-\left(1 + \frac{x-y}{y-z}\right) = -(1+a)$, *i.e.* $\frac{z-x}{y-z} = b$.

Hence we should use solution, (3) or (4) only to write down the general solution of the given equation.

Now each of the ratios in (1) is also equal to

$$\frac{\mathrm{d}(x+y+z)}{2(x+y+z)} \tag{5}$$

Taking the first ratio in (2) and the ratio (5), we have $\frac{\mathrm{d}(x+y+z)}{(x+y+z)} = -\frac{2\mathrm{d}(x-y)}{x-y}$

Integrating, we get $\log(x+y+z) = -2\log(x-y) + \log c$

i.e.
$$(x-y)^2(x+y+z) = c \tag{6}$$

Therefore the general solution of the given equation is $f\left\{\frac{x-y}{y-z}, (x-y)^2(x+y+z)\right\} = 0$

Example 17

Solve the equation $x(y^2+z^2)p + y(z^2+x^2)q = z(y^2-x^2)$.

This is a Lagrange's linear equation with $P = x(y^2+z^2)$, $Q = y(z^2+x^2)$ and $R = z(y^2-x^2)$.

The subsidiary equations are

$$\frac{dx}{x(y^2+z^2)} = \frac{dy}{y(z^2+x^2)} = \frac{dz}{z(y^2-x^2)} \tag{1}$$

Using the multipliers $-\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, each of the ratios in (1) $= \dfrac{-\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

Integrating, we get $-\log x + \log y + \log z = \log a$

i.e.
$$\frac{yz}{x} = a \tag{2}$$

Using the multipliers $x, -y, z$, each of the ratios in (1) $= \dfrac{xdx - ydy + zdz}{0}$

$\therefore$
$$xdx - ydy + zdz = 0$$

Integrating, we get
$$x^2 - y^2 + z^2 = b \tag{3}$$

Therefore the general solution of the given equation is $f\left(\frac{yz}{x},\ x^2 - y^2 + z^2\right) = 0$

Example 18

Find the integral surface of the equation $px + qy = z$, passing through $x + y = 1$ and $x^2 + y^2 + z^2 = 4$.

The general solution or integral of the Lagrange's linear equation

$$px + qy = z \tag{1}$$

represents a surface. This surface is called the integral surface of the equation.

Now the particular integral surface passing through the circle given by (2) and (3) is required.

$$x + y = 1 \tag{2}$$

$$x^2 + y^2 + z^2 = 4 \tag{3}$$

First let us find the general integral surface of equation (1).

The subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \tag{4}$$

Two independendent solutions of (4) are easily found as

$$\frac{x}{y} = a \tag{5}$$

and
$$\frac{y}{z} = b \tag{5$'$}$$

Therefore the general integral surface of (1) is

$$f\left(\frac{x}{y}, \frac{y}{z}\right) = 0 \tag{6}$$

Instead of finding the particular value of 'f' that satisfies (2) and (3), we proceed alternatively as follows.

We eliminate x, y, z from (2), (3), (5) and (5)′ and get a relation satisfied by a and b, which are then replaced by their equivalents, namely, $\frac{x}{y}$ and $\frac{y}{z}$ respectively.

Using (5)′ in (3), $$x^2 + y^2 + \frac{y^2}{b^2} = 4 \tag{7}$$

Using (5) in (2) and (7), we have

$$x\left(1 + \frac{1}{a}\right) = 1 \tag{8}$$

and $$x^2\left(1 + \frac{1}{a^2} + \frac{1}{a^2b^2}\right) = 4 \tag{9}$$

Eliminating x between (8) and (9), we get

$$\frac{(a^2b^2 + b^2 + 1)}{b^2(a+1)^2} = 4 \tag{10}$$

Substituting for a and b from (5) and (6) in (10), we get $\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1 = 4\frac{y^2}{z^2}\left(\frac{x+y}{y}\right)^2$.

viz., $x^2 + y^2 + z^2 = 4(x+y)^2$, which is the equation of the required integral surface.

Example 19

Show that the integral surface of the equation $2y(z-3)p + (2x-z)q = y(2x-3)$ that passes through the circle $x^2 + y^2 = 2x$, $z = 0$ is $x^2 + y^2 - z^2 - 2x + 4z = 0$.

The subsidiary equations of the given Lagrange's equation are

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)} \tag{1}$$

Taking the first and last ratios in (1), we have

$$\frac{dx}{2z-6} = \frac{dz}{2x-3} \text{ or } (2x-3)dx = (2z-6)dz$$

Integrating, we get $$x^2 - z^2 - 3x + 6z = a \tag{2}$$

Using the multipliers $1, 2y, -2$, each ratio in (1) $= \dfrac{dx + 2y\,dy - 2dz}{0}$

$$\therefore \qquad dx + 2y\,dy - 2dz = 0$$

Integrating, we get
$$x + y^2 - 2z = b \tag{3}$$

The required surface has to pass through
$$x^2 + y^2 = 2x \quad \text{and} \tag{4}$$
$$z = 0 \tag{5}$$

Using (5) in (2) and (3), we get
$$x^2 - 3x = a \tag{6}$$
and
$$x + y^2 = b \tag{7}$$
From (6) and (7), we get
$$x^2 + y^2 - 2x = a + b \tag{8}$$

Using (4) in (8), we have
$$a + b = 0 \tag{9}$$

Substituting for a and b from (2) and (3) in (9), we get $x^2 + y^2 - z^2 - 2x + 4z = 0$, which is the equation of the required integral surface.

Example 20

Show that the integral surface of the partial differential equation $x(y^2+z)p - y(x^2 + z)q = (x^2 - y^2)z$ which contains the straight line $x + y = 0$, $z = 1$ is $x^2 + y^2 + 2xyz - 2z + 2 = 0$.

The subsidiary equations of the given Lagrange's equation are
$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z} \tag{1}$$

Using the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, each of the ratios in (1) $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$$\therefore \qquad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, we get
$$xyz = a \tag{2}$$

Using the multipliers $x, y, -1$, each of the ratios in (1) $= \dfrac{x\,dx + y\,dy - dz}{0}$

$$\therefore \qquad x\,dx + y\,dy - dz = 0$$

Integrating, we get

$$x^2 + y^2 - 2z = b \tag{3}$$

The required surface has to pass through

$$x + y = 0 \tag{4}$$

and
$$z = 1 \tag{5}$$

Using (4) and (5) in (2) and (3), we have

$$-x^2 = a \tag{6}$$

and
$$2x^2 - 2 = b \tag{7}$$

Eliminating x between (6) and (7), we get

$$2a + b + 2 = 0 \tag{8}$$

Substituting for a and b from (2) and (3) in (8), we get $2xyz + x^2 + y^2 - 2z + 2 = 0$ or $x^2 + y^2 + 2xyz - 2z + 2 = 0$, which is the equation of the required surface.

Exercise 1(c)

Part A (Short-Answer Questions)

Solve the following equations.

1. $\dfrac{\partial^2 z}{\partial x^2} = 0$
2. $\dfrac{\partial^2 z}{\partial y^2} = 0$
3. $\dfrac{\partial^2 z}{\partial x \partial y} = 0$
4. $\dfrac{\partial^2 z}{\partial x^2} = e^{x+y}$
5. $\dfrac{\partial^2 z}{\partial y^2} = \cos(2x + 3y)$
6. $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{1}{xy}$
7. $\dfrac{\partial^2 z}{\partial x^2} = \sin y$

8. $\dfrac{\partial^2 z}{\partial y^2} = \cos y$

9. $\dfrac{\partial^2 z}{\partial x \partial y} = k$

10. $\dfrac{\partial^2 z}{\partial x \partial y} = x^2 + y^2$

11. Give the working rule to solve the Lagrange's linear equation.

Find the general solutions of the following Lagrange's linear equations.

12. $pyz + qzx = xy$
13. $yq - xp = z$
14. $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$
15. $p \tan x + q \tan y = \tan z$
16. $px^2 + qy^2 = z^2$

Part B

17. Solve the equation $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, given that $\dfrac{\partial z}{\partial y} = -2 \sin y$ when $x = 0$ and $z = 0$ when y is an odd multiple of $\dfrac{\pi}{2}$.

18. Solve the equation $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$, given that $\dfrac{\partial z}{\partial x} = a \sin y$ and $\dfrac{\partial z}{\partial y} = 0$ when $x = 0$.

19. Solve the equation $\dfrac{\partial^2 z}{\partial y^2} = z$, given that $z = e^x$ and $\dfrac{\partial z}{\partial y} = e^{-x}$ when $y = 0$.

20. Solve the equations $p = 6x + 3y, \quad q = 3x - 4y$ simultaneously.

21. Solve the equation $x\dfrac{\partial z}{\partial x} = 2x + y + 3z$.

22. Solve the equation $\dfrac{\partial^2 z}{\partial x \partial y} + 18xy^2 + \sin(2x - y) = 0$.

23. Solve the equation $\dfrac{\partial^2 z}{\partial y^2} - 5\dfrac{\partial z}{\partial y} + 6z = 12y$.

24. Solve the equations $\dfrac{\partial^2 z}{\partial x^2} = 0, \quad \dfrac{\partial^2 z}{\partial y^2} = 0$ simultaneously.

25. By changing the independent variables by the transformations $u = x + at$, $v = x - at$, show that the equation $\dfrac{\partial^2 z}{\partial t^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$ gets transformed into the equation $\dfrac{\partial^2 z}{\partial u \partial v} = 0$. Hence obtain the general solution of the equation.

26. By changing the independent variables by the transformations $z = x + iy$, $z^* = x - iy$, where $i = \sqrt{-1}$, show that the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ gets transformed into the equation $\frac{\partial^2 u}{\partial z \partial z^*} = 0$. Hence obtain the general solution of the equation.

27. Use the transformations $x = u + v$, $y = u - v$ to change the equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$ as $\frac{\partial^2 z}{\partial u \partial v} = 0$ and hence solve it.

28. Find the solution of the equation $y^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} \frac{\partial z}{\partial x} + \frac{x^2}{y} \frac{\partial z}{\partial y}$, by transforming it to a simpler form using the substitutions $u = x^2 + y^2$, $v = x^2 - y^2$.

29. Reduce the equation $4y^3 z_{xx} - y z_{yy} + z_y = 0$ to a simpler form by using the transformations $u = y^2 + x$ and $v = y^2 - x$ and hence solve it.

Find the general solutions of the following linear partial differential equations.

30. (i) $p \cot x + q \cot y = \cot z$
(ii) $(a - x)p + (b - y)q = (c - z)$

31. $\frac{y^2 z}{x} p + xzq = y^2$

32. (i) $x^2 p + y^2 q = (x + y)z$; (ii) $x^2 p - y^2 q = (x - y)z$

33. $(y^2 + z^2)p - xyq + xz = 0$

34. $(y^2 + z^2 - x^2)p - 2xyq + 2zx = 0$

35. $p - q = \log (x + y)$

36. $z(xp - yq) = y^2 - x^2$

37. (i) $(y - z)p + (z - x)q = x - y$; (ii)$(y - z)p + (x - y)q = z - x$

38. (i) $x(y - z)p + y(z - x)q = z(x - y)$
(ii) $\frac{y - z}{yz} p + \frac{z - x}{zx} \cdot q = \frac{x - y}{xy}$

39. $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

40. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$. [See example (16)]

41. (i) $(y + z)p - (x + z)q = x - y$ (ii) $(3z - 4y)p + (4x - 2z)q = 2y - 3x$

42. $(y^3 x - 2x^4)p + (2y^4 - x^3 y)q = (x^3 - y^3)z$.

43. Find the integral surface of the equation $px + qy = z$, that passes through the circle $x^2 + y^2 + z^2 = 4$, $x + y + z = 2$.

44. Find the integral surface of the equation $yp + xq + 1 = z$, that passes through the curve $z = x^2 + y + 1$ and $y = 2x$.

45. Show that the integral surface of the equation $(x^2-a^2)p+(xy-az\tan\alpha)q = xz - ay\cot\alpha$, that passes through the curve $x^2+y^2=a^2, z=0$ is $x^2+y^2-a^2=z^2\tan^2\alpha$.

1.13 LINEAR P.D.E.'S OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

Linear partial differential equations of higher order with constant coefficients may be divided into two categories as given below.

(i) Equations in which the partial derivatives occurring are all of the same order (of course, with degree 1 each) and the coefficients are constants. Such equations are called *homogeneous linear P.D.E.s* with constant coefficients.

(ii) Equations in which the partial derivatives occurring are not of the same order and the coefficients are constants are called *non-homogeneous linear P.D.E.s* with constant coefficients.

For example,

$$\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x\partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{x+y} \text{ and}$$

$$\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2\partial y} - 4\frac{\partial^3 z}{\partial x\partial y^2} + 12\frac{\partial^3 z}{\partial y^3} = x+2y$$

are equations of the first category.

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x\partial y} + \frac{\partial z}{\partial x} = x^2+y^2 \text{ and}$$

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} - 3\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = \cos(x+2y)$$

are equations of the second category.

The standard form of a homogeneous linear partial differential equation of the n^{th} order with constant coefficients is

$$a_0\frac{\partial^n z}{\partial x^n} + a_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \ldots + a_n\frac{\partial^n z}{\partial y^n} = R(x,y) \tag{1}$$

where a's are constants.

If we use the operators $D \equiv \dfrac{\partial}{\partial x}$ and $D' \equiv \dfrac{\partial}{\partial y}$, we can symbolically write equation (1) as

$$\left(a_0D^n + a_1D^{n-1}D' + a_2D^{n-2}D'^2 + \ldots + a_nD'^n\right)z = R(x,y) \tag{2}$$

i.e.

$$f(D,D')z = R(x,y) \tag{3}$$

where $f(D,D')$ is a homogeneous polynomial of the n^{th} degree in D and D'.

The method of solving (3) is similar to that of solving ordinary linear differential equations with constant coefficients.

The general solution of (3) is of the form z = (Complementary function) + (Particular integral), where the complementary function (C.F.) is the R.H.S. of the general solution of $f(D, D')z = 0$ and the particular integral (P.I.) is given symbolically by $\dfrac{1}{f(D, D')} R(x, y)$.

Complementary function of $f(D, D')z = R(x, y)$

C.F. of the solution of $f(D, D')z = R(x, y)$ is the R.H.S. of the solution of

$$f(D, D')z = 0 \tag{1}$$

Let us assume that

$$z = \phi(y + mx) \tag{2}$$

is a solution of equation (1), where ϕ is an arbitrary function.

Differentiating (2) partially with respect to x and then with respect to y, we have

$$Dz = \frac{\partial z}{\partial x} = m\phi'(y + mx)$$

$$D^2 z = \frac{\partial^2 z}{\partial x^2} = m^2\phi''(y + mx)$$

$$\vdots$$

$$D^n z = \frac{\partial^n z}{\partial x^n} = m^n\phi^{(n)}(y + mx)$$

Similarly, $$D'^n_z = \frac{\partial^n z}{\partial y^n} = \phi^{(n)}(y + mx) \text{ and}$$

$$D^{n-r} D'^r_y = \frac{\partial^n z}{\partial x^{n-r}\partial y^r} = m^{n-r}\phi^{(n)}(y + mx)$$

Since (2) is a solution of (1), we have

$$(a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n)\phi^{(n)}(y + mx) = 0 \tag{3}$$

Since ϕ is arbitrary, $\phi^{(n)}(y + mx) \not\equiv 0$

$\therefore$ (3) reduces to $a_0 m^n + a_1 m^{n-1} + \ldots + a_n = 0$ or $f(m, 1) = 0$ (4)

Thus $z = \phi(y + mx)$ will be a solution of (1), if m satisfies the algebraic equation (4) or m is a root of equation (4), which we get by replacing D by m and D' by 1 in the equation $f(D, D')z = 0$ and by dropping z from it.

The equation $f(m, 1) = 0$ is called *the auxiliary equation*, which is an algebraic equation of the n^{th} degree in m and hence will have n roots.

Case(i)

The roots of (4) are distinct (real or complex).

Let them be $m_1, m_2, \ldots m_n$.

The solutions of (1) corresponding to these roots are $z = \phi_1(y + m_1x)$, $z = \phi_2(y+m_2x), \ldots, z = \phi_n(y+m_nx)$. The general solution of (1) is given by a linear combination of these solutions.

That is the general solution of (1) is given by

$$z = \phi_1(y + m_1x) + \phi_2(y + m_2x) + \ldots + \phi_n(y + m_nx)$$

$\therefore$ C.F. of the solution of $f(D, D')z = R(x, y)$ is $\phi_1(y+m_1x)+\phi_2(y+m_2x)+\ldots+\phi_n(y + m_nx)$, where ϕ_r's are arbitrary functions.

Case(ii)

Two of the roots of (4) are equal and others are distinct.

Let them be $m_1, m_1, m_3, m_4, \ldots, m_n$.

Note

If we apply the rule arrived at in Case (i), the solution of (1) will be $z = [\phi_1(y + m_1x) + \phi_2(y + m_1x)] + \phi_3(y+m_3x)+\ldots+\phi_n(y+m_nx)$, *i.e.* $z = \phi(y+m_1x) + \phi_3(y + m_3x) + \ldots + \phi_n(y + m_nx)$, *which contains only* $(n-1)$ *arbitrary functions. Hence it cannot be the general solution of Equation (1).*

Then $f(m, 1) \equiv a_0(m - m_1)^2(m - m_3)\ldots(m - m_n)$

$$\therefore \qquad f(D, D') \equiv a_0(D - m_1D')^2(D - m_3D')\ldots(D - m_nD')$$

Hence solution of (1) will be a combination of the solutions of the component equations

$$(D - m_1D')^2z = 0, (D - m_3D')z = 0, \ldots, (D - m_nD')z = 0$$

Consider $(D - m_rD')z = 0$, i.e. $p - m_rq = 0$, which is a Lagrange's linear equation.

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_r} = \frac{dz}{0}$$

Solving, we get $y + m_rx = a$ and $z = b$.

$\therefore$ General solution of $(D - m_rD')z = 0$ is $f_r(y + m_rx, z) = 0$ or $z = \phi_r(y + m_rx)$.

Now consider $$(D - m_1D')^2z = 0 \qquad (5)$$

Let $$(D - m_1D')z = u \qquad (6)$$

$\therefore$ becomes $(D - m_1D')u = 0$ (7)

The solution of (7) is $u = \phi_1(y + m_1 x)$. Using this value of u in (6), it becomes

$$(D - m_1 D')z = \phi_1(y + m_1 x)$$

or
$$p - m_1 q = \phi_1(y + m_1 x) \tag{8}$$

which is a Lagrange's equation.

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{\phi_1(y + m_1 x)}$$

Solving, we get $y + m_1 x = a$ and $z - x\phi_1(y + m_1 x) = b$

$\therefore$ The solution of Eq. (8) and hence Eq. (5) is

$$f\,[y + m_1 x, z - x \cdot \phi_1(y + m_1 x)] = 0$$

or
$$z - x \cdot \phi_1(y + m_1 x) = \phi_2(y + m_1 x)$$

or
$$z = x \cdot \phi_1(y + m_1 x) + \phi_2(y + m_1 x)$$

$\therefore$ General solution of equation (1) is

$$z = x\phi_1(y + m_1 x) + \phi_2(y + m_1 x) + \phi_3(y + m_3 x) + \ldots + \phi_n(y + m_n x)$$

$\therefore$ C.F. of the solution of $f(D, D')z = R(x, y)$ is

$$x\phi_1(y + m_1 x) + \phi_2(y + m_1 x) + \phi_3(y + m_3 x) + \ldots + \phi_n(y + m_n x)$$

Case(iii)

'r' of the roots of Eq. (4) are equal and others distinct.

i.e.
$$m_1 = m_2 = m_3 = \ldots = m_r$$

Proceeding as in Case (ii), we can show that the part of the C.F. of the solution of $f(D, D')z = R(x, y)$ is

$$\phi_1(y + m_1 x) + x\phi_2(y + m_1 x) + x^2\phi_3(y + m_1 x) + \ldots + x^{r-1}\phi_r(y + m_1 x)$$

The Particular Integral of the solution of $f(D, D')z = R(x, y)$.

As in the case of ordinary differential equations, there are formulas/methods for finding particular integrals (P.I.) of the solution of homogeneous (and also non-homogeneous) linear P.D.E.s with constant coefficients. The formulas/methods are given below without proof.

1. $\dfrac{1}{f(D, D')} e^{ax+by} = \dfrac{1}{f(a, b)} e^{ax+by}$, if $f(a, b) \neq 0$

1(a). If $f(a, b) = 0$, $(D - \frac{a}{b}D')$ or its power will be a factor of $f(D, D')$. In this case we factorise $f(D, D')$ and proceed as in ordinary differential equations and use the following results.

$$\frac{1}{\left(D-\frac{a}{b}D'\right)}e^{ax+by}=xe^{ax+by};\ \frac{1}{\left(D-\frac{a}{b}D'\right)^2}e^{ax+by}=$$
$$\frac{x^2}{2!}e^{ax+by},\ \ldots,\ \frac{1}{\left(D-\frac{a}{b}D'\right)^r}e^{ax+by}=\frac{x^r}{r!}e^{ax+by}$$

The above results can be derived by using Lagrange's linear equation method.
For example, let $\frac{1}{D-\frac{a}{b}D'}e^{ax+by}=z.$

i.e. $p-\frac{a}{b}q=e^{ax+by}$

The subsidiary equations are

$$\frac{dx}{1}=\frac{bdy}{-a}=\frac{dz}{e^{ax+by}}$$

The solutions of these equations are $ax+by=c_1$ and $z=xe^{c_1}$ or $z=xe^{ax+by}$.

2. $\frac{1}{f(D^2, DD', D'^2)}\begin{matrix}\sin\\ \cos\end{matrix}(ax+by)$

$$=\frac{1}{f(-a^2,-ab,-b^2)}\begin{matrix}\sin\\ \cos\end{matrix}(ax+by)$$

provided $f(a^2, -ab, -b^2)\neq 0$.

2(a).

If $f(-a^2,-ab,-b^2) = 0$, then $\left(D^2-\frac{a^2}{b^2}D'^2\right)$ will be a factor of $f(D^2, DD', D'^2)$. In this case, we proceed as in ordinary differential equations and use the results

$$\frac{1}{D^2-\frac{a^2}{b^2}D'^2}\sin(ax+by)=-\frac{x}{2a}\cos(ax+by)\text{ and}$$
$$\frac{1}{D^2-\frac{a^2}{b^2}D'^2}\cos(ax+by)=\frac{x}{2a}\sin(ax+by)$$

which may be verified by the reader.

3. $\frac{1}{f(D, D')}x^m y^n=[f(D, D')]^{-1}x^m y^n$ where $[f(D, D')]^{-1}$ is to be expanded in series of powers of D and D'.

4. $\dfrac{1}{f(D, D')} e^{ax+by} F(x, y) = e^{ax+by} \cdot \dfrac{1}{f(D+a, D'+b)} F(x, y).$

5. $\dfrac{1}{D - mD'} F(x, y) = \left[\int F(x, a - x) \mathrm{d}x \right]_{a \to y+mx}$

This result can be derived by assuming that $\dfrac{1}{D - mD'} F(x, y) = z$ and solving for z by using Lagrange's linear equation method.

1.14 COMPLEMENTARY FUNCTION FOR A NON-HOMOGENEOUS LINEAR EQUATION

Let the non-homogeneous linear equation be $f(D, D') = 0$.
We resolve $f(D, D')$ into linear factors of the form $(D - aD' - b)$.
The C.F. is the linear combination or simply the sum of (the R.H.S. functions of) the solutions of the component equations $(D - a_r D' - b_r)z = 0$.

Now let us consider the equation $(D - aD' - b)z = 0$

i.e. $p - aq = bz$, which is a Lagrange's linear equation

The subsidiary equations are

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{-a} = \frac{\mathrm{d}z}{bz}$$

One solution of these equations $y + ax = c_1$. The other solution is $\log z = bx + \log c_2$

$$\text{or} \quad z = c_2 e^{bx}$$

$\therefore$ The general solution of the equation is

$$\phi\left(\frac{z}{e^{bx}}, y + ax\right) = 0 \quad \text{or} \quad z = e^{bx} f(y + ax)$$

Note ✍

The rules / methods for finding P.I.s are the same as those for homogeneous linear equations.

1.15 SOLUTION OF P.D.E.S BY THE METHOD OF SEPARATION OF VARIABLES

In the next few chapters on applications of partial differential equations, we will have to solve *boundary value problems*, i.e. partial differential equations that satisfy certain given conditions called boundary conditions.

When solving a boundary value problem, if we first find the general solution of the concerned partial differential equation, it will be very difficult to find particular values of the arbitrary functions involved in the general solution that satisfy the boundary conditions. Hence in such situations, we try to find particular solutions of the partial differential equation that satisfy the boundary conditions and then combine them to get the solution of the boundary value problem.

A simple but powerful method of obtaining such particular solution is the *method of separation of variables*. In this method of solving a P.D.E. with z as the dependent variable and x and y as independent variables, the solution is assumed to be of the form $z = f(x) \cdot g(y)$, where f is a function of x alone and g is a function of y alone.

This assumption makes the solution of the P.D.E. depend on solutions of ordinary differential equations.

The variable separable solution of a P.D.E. is called a particular solution, as it can be verified to be a particular form of the general solution of the P.D.E.

For example, consider the equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2} \tag{1}$$

A variable separable solution of (1) can be obtained as

$$z = (ae^{px} + be^{-px})(ce^{\text{pat}} + de^{-\text{pat}}) \tag{2}$$

where a, b, c, d, p are constants.

(2) can be rewritten as

$$z = \{ac\ e^{p(x+at)} + bd\ e^{-p(x+at)}\} + \{ad\ e^{p(x-at)} + bc\ e^{-p(x-at)}\} \tag{3}$$

(3) is a particular case of

$$z = f(x + at) + \phi(x - at)$$

which is the general solution of (1) [see Problem 25 in Exercise 1(c)].

Worked Examples 1(d)

Example 1

Solve the equation

$$(D^3 + 2D^2D' - DD'^2 - 2D'^3)z = 0$$

The auxiliary equation (got by replacing D by m and D' by 1 in the given P.D.E.) is

$$\begin{aligned} m^3 + 2m^2 - m - 2 &= 0 \\ \text{i.e.} \quad m^2(m+2) - (m+2) &= 0 \\ \text{i.e.} \quad (m-1)(m+1)(m+2) &= 0 \\ \therefore \quad m &= 1, -1, -2 \end{aligned}$$

$\therefore$ General solution of the given equation is

$$z = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y - 2x)$$

Note ✍

There is no particular integral in the general solution, since the R.H.S. member of the given P.D.E. is zero.

Example 2

Solve the equation

$$(D^3 - D^2D' - 8DD'^2 + 12D'^3)z = 0$$

The auxiliary equation is $m^3 - m^2 - 8m + 12 = 0$

$m = 2$ is a root of the auxiliary equation.

It is $(m-2)(m^2+m-6) = 0$ or $(m-2)(m-2)(m+3) = 0$

$$\therefore \quad m = 2, 2, -3$$

$\therefore$ The general solution of the given equation is

$$z = xf_1(y+2x) + f_2(y+2x) + f_3(y-3x)$$

Example 3

Solve the equation $\quad (D^2 - 3DD' + 2D'^2)z = 2\cosh(3x+4y)$

The auxiliary equation is $\quad m^2 - 3m + 2 = 0$

i.e. $\quad (m-1)(m-2) = 0$

$$\therefore \quad m = 1, 2$$

$\therefore$ The C.F. of the given P.D.E. = $f_1(y+x) + f_2(y+2x)$

$$\text{P.I.} = \frac{1}{D^2 - 3DD' + 2D'^2} 2\cosh(3x+4y)$$

$$= \frac{1}{D^2 - 3DD' + 2D'^2}\left[e^{3x+4y} + e^{-(3x+4y)}\right]$$

$$= \frac{1}{3^2 - 3.3.4 + 2.4^2}e^{3x+4y} + \frac{1}{(-3)^2 - 3(-3)(-4) + 2(-4)^2}e^{-(3x+4y)}$$

$$= \frac{1}{5}\left[e^{3x+4y} + e^{-(3x+4y)}\right]$$

$$= \frac{2}{5}\cosh(3x+4y)$$

$\therefore$ The general solution of the given equation is $z = f_1(y + x) + f_2(y + 2x) + \frac{2}{5}\cosh(3x + 4y)$.

Example 4

Solve the equation

$$(9D^2 + 6DD' + D'^2)z = (e^x + e^{-2y})^2$$

The auxiliary equation is

$$9m^2 + 6m + 1 = 0 \quad \text{i.e.} \quad (3m + 1)^2 = 0$$

$$\therefore \qquad m = -1/3, -1/3$$

$$\therefore \qquad \text{C.F.} = xf_1(y - \tfrac{1}{3} \cdot x) + f_2(y - \tfrac{1}{3} \cdot x) \quad \text{or}$$

$$xf_1(3y - x) + f_2(3y - x)$$

$$\text{P.I.} = \frac{1}{9D^2 + 6DD' + D'^2}(e^x + e^{-2y})^2$$

$$= \frac{1}{9D^2 + 6DD' + D'^2}\left(e^{2x} + e^{-4y} + 2e^{x-2y}\right)$$

$$= \frac{1}{(3D + D')^2}e^{2x} + \frac{1}{(3D + D')^2}e^{-4y} + 2 \cdot \frac{1}{(3D + D')^2}e^{x-2y}$$

$$= \frac{1}{36}e^{2x} + \frac{1}{16}e^{-4y} + 2e^{x-2y}$$

$\therefore$ The general solution of the given equation is

$$z = xf_1(3y - x) + f_2(3y - x) + \frac{1}{36}e^{2x} + \frac{1}{16}e^{-4y} + 2e^{x-2y}$$

Example 5

Solve the equation

$$(D^3 - 3DD'^2 + 2D'^3)z = e^{2x-y} + e^{x+y}$$

The auxiliary equation is $m^3 - 3m + 2 = 0$

i.e. $\qquad (m - 1)(m^2 + m - 2) = 0$

i.e. $\qquad (m - 1)^2(m + 2) = 0$

$$\therefore \qquad m = 1, 1, -2$$

$$\therefore \qquad \text{C.F.} = xf_1(y + x) + f_2(y + x) + f_3(y - 2x)$$

$$\text{P.I.} = \frac{1}{D^3 - 3DD'^2 + 2D'^3}(e^{2x-y} + e^{x+y})$$
$$= \frac{1}{(D+2D')(D-D')^2}e^{2x-y} + \frac{1}{(D-D')^2(D+2D')}e^{x+y}$$
$$= \frac{1}{9} \cdot \frac{1}{D+2D'}e^{2x-y} + \frac{1}{9} \cdot \frac{1}{(D-D')^2}e^{x+y}$$
$$= \frac{1}{9}\left[xe^{2x-y} + \frac{x^2}{2}e^{x+y}\right]$$

$\therefore$ The general solution of the given equation is

$$z = xf_1(y+x) + f_2(y+x) + f_3(y-2x) + \frac{x}{9}e^{2x-y} + \frac{x^2}{18}e^{x+y}$$

Example 6

Solve the equation

$$(D^3 - 6D^2D' + 12DD'^2 - 8D'^3)z = (1 + e^{2x+y})^2$$

The auxiliary equation is $m^3 - 6m^2 + 12m - 8 = 0$

i.e. $\quad (m-2)^3 = 0$

$\therefore \quad m = 2, 2, 2$

$$\therefore \quad \text{C.F.} = x^2 f_1(y+2x) + x \cdot f_2(y+2x) + f_3(y+2x)$$

$$\text{P.I.} = \frac{1}{(D-2D')^3}(1 + e^{2x+y})^2$$
$$= \frac{1}{(D-2D')^3}(1) + 2 \cdot \frac{1}{(D-2D')^3}e^{2x+y} + \frac{1}{(D-2D')^3}e^{4x+2y}$$
$$= \frac{x^3}{3!} + 2\frac{x^3}{3!}e^{2x+y} + \frac{x^3}{3!}e^{4x+2y}$$
$$\left[\text{since } \frac{1}{(D-2D')^3}(1) = \frac{1}{(D-2D')^3}e^{0.x+0.y} \text{ and } \frac{1}{\left(D - \frac{a}{b}D'\right)^3}e^{ax+by} = \frac{x^3}{3!}e^{ax+by}\right]$$
$$= \frac{x^3}{6}(1 + e^{2x+y})^2$$

$\therefore$ The general solution of the given equation is

$$z = x^2 f_1(y+2x) + xf_2(y+2x) + f_3(y+2x) + \frac{x^3}{6}(1 + e^{2x+y})^2.$$

Example 7

Solve the equation $(D^2 + 2DD' + D'^2)z = x^2y + e^{x-y}$

The auxiliary equation is $m^2 + 2m + 1 = 0$ or $(m+1)^2 = 0 \quad \therefore \quad m = -1, -1$

$$\therefore \quad \text{C.F.} = xf_1(y-x) + f_2(y-x)$$

$$
\begin{aligned}
(\text{P.I.})_1 &= \frac{1}{(D+D')^2}x^2y \\
&= \frac{1}{D^2\left(1+\dfrac{D'}{D}\right)^2}x^2y \\
&= \frac{1}{D^2}\left(1+\frac{D'}{D}\right)^{-2}(x^2y) \\
&= \frac{1}{D^2}\left(1-\frac{2D'}{D}+3\frac{D'^2}{D^2}\right)(x^2y) \\
&= \frac{1}{D^2}\left\{x^2y-\frac{2}{D}x^2\right\} \\
&= y\cdot\frac{1}{D^2}(x^2)-2\cdot\frac{1}{D^3}(x^2) \\
&= y\cdot\frac{x^4}{3.4}-2\cdot\frac{x^5}{3.4.5} \\
&= \frac{x^4y}{12}-\frac{x^5}{30} \\
(\text{P.I.})_2 &= \frac{1}{(D+D')^2}e^{x-y} = \frac{x^2}{2!}e^{x-y}
\end{aligned}
$$

$\therefore$ The general solution is

$$z = xf_1(y-x)+f_2(y-x)+\frac{x^4y}{12}-\frac{x^5}{30}+\frac{x^2}{2}e^{x-y}$$

Example 8

Solve the equation

$$(D^3-7DD'^2-6D'^3)z = x^2+xy^2+y^3$$

The auxiliary equation is

$$m^3-7m-6=0, \quad \text{i.e.} \quad (m+1)(m^2-m-6)=0$$

i.e. $$(m+1)(m+2)(m-3)=0$$

$\therefore$ $$m=-1,-2,3$$

$\therefore$ $$\text{C.F.} = f_1(y-x)+f_2(y-2x)+f_3(y+3x)$$

$$
\begin{aligned}
\text{P.I.} &= \frac{1}{D^3-7DD'^2-6D'^3}(x^2+xy^2+y^3) \\
&= \frac{1}{D^3}\left\{1-\frac{\left(7DD'^2+6D'^3\right)}{D^3}\right\}^{-1}(x^2+xy^2+y^3) \\
&= \frac{1}{D^3}\left[1+\frac{D'^2}{D^3}(7D+6D')+\cdots\right](x^2+xy^2+y^3)
\end{aligned}
$$

$$= \left[\frac{1}{D^3} + \frac{1}{D^6}(7DD'^2 + 6D'^3)\right](x^2 + xy^2 + y^3)$$
$$= \frac{1}{D^3}(x^2 + xy^2 + y^3) + \frac{1}{D^6}\{7D \cdot (2x + 6y) + 36\}$$
$$= \frac{1}{D^3}(x^2 + xy^2 + y^3) + \frac{1}{D^6}(50)$$
$$= \frac{x^5}{3.4.5} + y^2 \cdot \frac{x^4}{2.3.4} + y^3 \cdot \frac{x^3}{1.2.3} + 50 \cdot \frac{x^3}{1.2.3}$$
$$= \frac{1}{60}x^5 + \frac{25}{3}x^3 + \frac{1}{24}x^4y^2 + \frac{1}{6}x^3y^3$$

$\therefore$ The general solution is

$$z = f_1(y - x) + f_2(y - 2x) + f_3(y + 3x) + \frac{x^5}{60} + \frac{25}{3}x^3 + \frac{1}{24}x^4y^2 + \frac{1}{6}x^3y^3$$

Example 9

Solve the equation

$$(D^2 + 4DD' - 5D'^2)z = xy + \sin(2x + 3y)$$

The auxiliary equation is $m^2 + 4m - 5 = 0$

i.e. $(m + 5)(m - 1) = 0$

$\therefore$ $m = -5, 1$

$\therefore$ $\text{C.F.} = \phi_1(y - 5x) + \phi_2(y + x)$

$$(\text{P.I.})_1 = \frac{1}{D^2 + 4DD' - 5D'^2}(xy)$$
$$= \frac{1}{D^2\left\{1 + \frac{D'}{D^2}(4D - 5D')\right\}}(xy)$$
$$= \frac{1}{D^2}\left\{1 + \frac{D'}{D^2}(4D - 5D')\right\}^{-1}(xy)$$
$$= \frac{1}{D^2}\left\{1 - \frac{D'}{D^2}(4D - 5D') + \ldots\right\}(xy)$$
$$= \frac{1}{D^2}(xy) - \frac{1}{D^4} \cdot 4D'(xy)$$
$$= \frac{x^3y}{6} - \frac{1}{D^4}(4x)$$
$$= \frac{1}{6}x^3y - \frac{1}{30}x^5$$

$$(\text{P.I})_2 = \frac{1}{D^2 + 4DD' - 5D'^2} \sin(2x + 3y)$$
$$= \frac{1}{-2^2 + 4 \cdot (-2.3) - 5(-3^2)} \sin(2x + 3y)$$
$$= \frac{1}{17} \sin(2x + 3y)$$

$\therefore$ General solution is
$$z = \phi_1(y - 5x) + \phi_2(y + x) + \frac{1}{6}x^3y - \frac{1}{30}x^5 + \frac{1}{17}\sin(2x + 3y)$$

Example 10

Solve the equation
$$(D^2 + D'^2)z = \sin 2x \sin 3y + 2\sin^2(x + y)$$

The auxiliary equation is $m^2 + 1 = 0$

i.e. $\qquad m = \pm i$

$\therefore \qquad \text{C.F.} = \phi_1(y + ix) + \phi_2(y - ix)$

$$(\text{P.I.})_1 = \frac{1}{D^2 + D'^2} \sin 2x \sin 3y$$
$$= \frac{1}{D^2 + D'^2} \cdot \frac{1}{2}\left\{\cos(2x - 3y) - \cos(2x + 3y)\right\}$$
$$= \frac{1}{2}\left[\frac{1}{-4 - 9}\cos(2x - 3y) - \frac{1}{-4 - 9}\cos(2x + 3y)\right]$$
$$= -\frac{1}{13} \cdot \frac{1}{2}\{\cos(2x - 3y) - \cos(2x + 3y)\}\Big]$$
$$= -\frac{1}{13}\sin 2x \sin 3y$$

$$(\text{P.I.})_2 = \frac{1}{D^2 + D'^2} 2\sin^2(x + y)$$
$$= \frac{1}{D^2 + D'^2}\left\{1 - \cos(2x + 2y)\right\}$$
$$= \frac{1}{D^2}\left(1 + \frac{D'^2}{D^2}\right)^{-1}(1) - \frac{1}{D^2 + D'^2}\cos(2x + 2y)$$
$$= \frac{1}{D^2}(1) - \frac{1}{-4 - 4}\cos(2x + 2y)$$
$$= \frac{x^2}{2} + \frac{1}{8}\cos(2x + 2y)$$

$\therefore$ General solution is
$$z = \phi_1(y + ix) + \phi_2(y - ix) - \frac{1}{13}\sin 2x \sin 3y + \frac{x^2}{2} + \frac{1}{8}\cos(2x + 2y)$$

Example 11

Solve the equation

$$(16D^4 - D'^4)z = \cos(x + 2y)$$

The auxiliary equation is $16m^4 - 1 = 0$

i.e. $(m^2 - 1/4)(m^2 + 1/4) = 0$

$\therefore$ $m = \pm 1/2, \pm i/2$

$\therefore$ $\text{C.F.} = f_1\left(y + \frac{1}{2}x\right) + f_2\left(y - \frac{1}{2}x\right) + f_3\left(y + \frac{ix}{2}\right) + f_4\left(y - \frac{ix}{2}\right)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{16D^4 - D'^4}\cos(x + 2y) \\
&= \frac{1}{(4D^2 - D'^2)(4D^2 + D'^2)}\cos(x + 2y) \\
&= \frac{1}{(4D^2 - D'^2)} \cdot \frac{1}{4(-1) + (-4)}\cos(x + 2y) \\
&= -\frac{1}{8} \cdot \frac{1}{4D^2 - D'^2}\cos(x + 2y) \\
&= -\frac{1}{32} \cdot \frac{1}{D^2 - \frac{1}{4}D'^2}\cos(x + 2y) \\
&= -\frac{1}{32} \cdot \frac{x}{2}\sin(x + 2y) \quad \left[\because \frac{1}{D^2 - \frac{a^2}{b^2}D'^2}\cos(ax + by) = \frac{x}{2a}\sin(ax + by)\right] \\
&= -\frac{1}{64}x\sin(x + 2y)
\end{aligned}$$

$\therefore$ General solution is

$$z = f_1\left(y + \frac{x}{2}\right) + f_2\left(y - \frac{x}{2}\right) + f_3\left(y + \frac{ix}{2}\right) + f_4\left(y - \frac{ix}{2}\right) - \frac{1}{64}x\sin(x + 2y)$$

Example 12

Solve the equation

$$(D^3 + D^2D' - 4DD'^2 - 4D'^3)z = \cos(2x + y)$$

The auxiliary equation is $m^3 + m^2 - 4m - 4 = 0$

i.e. $m^2(m + 1) - 4(m + 1) = 0$

i.e. $(m + 1)(m + 2)(m - 2) = 0$

$\therefore$ $m = -1, -2, 2$

$\therefore$ C.F. $= \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+2x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D^2-4D'^2)(D+D')}\cos(2x+y) \\
&= \frac{1}{(D^2-4D'^2)}\cdot\frac{(D-D')}{D^2-D'^2}\cos(2x-y) \\
&= \frac{1}{(D^2-4D'^2)}\cdot\frac{1}{-4-(-1)}(D-D')\cos(2x-y) \\
&= -\frac{1}{3}\cdot\frac{1}{D^2-4D'^2}\{-2\sin(2x-y)-\sin(2x-y)\} \\
&= \frac{1}{D^2-4D'^2}\sin(2x-y) \\
&= -\frac{x}{4}\cos(2x-y)\left[\because \frac{1}{D^2-\frac{a^2}{b^2}D'^2}\sin(ax+by) = -\frac{x}{2a}\cos(ax+by)\right]
\end{aligned}$$

$\therefore$ General solution is

$$z = \phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+2x)-\frac{x}{4}\cos(2x-y)$$

Example 13

Solve the equation

$$(D^2-2DD'+D'^2)z = x^2y^2e^{x+y}$$

The auxiliary equation is $m^2-2m+1=0$

$\therefore \quad m = 1, 1$

$\therefore \quad$ C.F. $= xf_1(y+x)+f_2(y+x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D-D')^2}e^{x+y}(x^2y^2) \\
&= e^{x+y}\frac{1}{\{(D+1)-(D'+1)\}^2}x^2y^2 \\
&= e^{x+y}\frac{1}{(D-D')^2}x^2y^2 \\
&= e^{x+y}\frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}(x^2y^2) \\
&= e^{x+y}\frac{1}{D^2}\left(1+\frac{2D'}{D}+3\frac{D'^2}{D^2}\right)(x^2y^2) \\
&= e^{x+y}\frac{1}{D^2}\left\{x^2y^2+\frac{2}{D}(2x^2y)+\frac{3}{D^2}(2x^2)\right\}
\end{aligned}$$

$$= e^{x+y}\left[y^2 \cdot \frac{1}{D^2}(x^2) + 4y \cdot \frac{1}{D^3}(x^2) + 6 \cdot \frac{1}{D^4}(x^2)\right]$$

$$= \left(\frac{1}{12}x^4 y^2 + \frac{1}{15}x^5 y + \frac{1}{60}x^6\right) e^{x+y}$$

$\therefore$ General solution is

$$z = x f_1(y+x) + f_2(y+x) + \left(\frac{1}{12}y^2 + \frac{1}{15}xy + \frac{1}{60}x^2\right) x^4 e^{x+y}$$

Example 14

Solve the equation

$$(D^2 - D'^2)z = e^{x-y}\sin(2x+3y)$$

The auxiliary equation is $m^2 - 1 = 0$

$\therefore \quad m = \pm 1$

$\therefore$ C.F. $= f_1(y+x) + f_2(y-x)$

$$\text{P.I.} = \frac{1}{D^2 - D'^2} e^{x-y}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{(D+1)^2 - (D'-1)^2}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{D^2 - D'^2 + 2(D+D')}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{2(D+D')+5}\sin(2x+3y)$$

$$= e^{x-y}\frac{2(D+D')-5}{4(D+D')^2 - 25}\sin(2x+3y)$$

$$= e^{x-y}\{2(D+D')-5\} \cdot \frac{1}{4(D^2+2DD'+D'^2)-25}\sin(2x+3y)$$

$$= e^{x-y}\{2(D+D')-5\} \cdot \left(-\frac{1}{125}\right)\sin(2x+3y)$$

$$= -\frac{1}{125}e^{x-y}\{4\cos(2x+3y) + 6\cos(2x+3y) - 5\sin(2x+3y)\}$$

$$= \frac{1}{25}e^{x-y}\{\sin(2x+3y) - 2\cos(2x+3y)\}$$

$\therefore$ General solution is

$$z = f_1(y+x) + f_2(y-x) + \frac{1}{25}e^{x-y}\{\sin(2x+3y) - 2\cos(2x+3y)\}$$

Example 15

Solve the equation

$$(D^2 - 5\,DD' + 6\,D'^2)\,z = y\sin x$$

The auxiliary equation is $m^2 - 5m + 6 = 0$

i.e. $(m-2)(m-3)=0$

$\therefore$ $m = 2, 3$

$\therefore$ $\text{C.F.} = \phi_1(y+2x) + \phi_2(y+3x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D-2D')(D-3D')} y\sin x \\
&= \frac{1}{D-2D'}\left[\int (a-3x)\sin x\,dx\right]_{a\to y+3x} \\
&= \frac{1}{D-2D'}[(a-3x)(-\cos x)+3(-\sin x)]_{a\to y+3x} \\
&= \frac{1}{D-2D'}[-y\cos x - 3\sin x] \\
&= -\left\{\int [(a-2x)\cos x + 3\sin x]\,dx\right\}_{a\to y+2x} \\
&= -\left[(a-2x)\sin x + 2(-\cos x) - 3\cos x\right]_{a\to y+2x} \\
&= 5\cos x - y\sin x
\end{aligned}$$

$\therefore$ General solution is

$$z = \phi_1(y+2x) + \phi_2(y+3x) + 5\cos x - y\sin x$$

Example 16

Solve the equation

$$(4D^2 - 4DD' + D'^2)z = 16\log(x+2y)$$

The auxiliary equation is $4m^2 - 4m + 1 = 0$

i.e. $(2m-1)^2 = 0$

$\therefore$ $m = 1/2, 1/2$

$$\therefore \quad \text{C.F.} = xf_1\left(y+\frac{1}{2}x\right) + f_2\left(y+\frac{1}{2}x\right) \quad \text{or}$$

$$xf_1(2y+x) + f_2(2y+x)$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(2D-D')^2} 16\log(x+2y) \\
&= 4\cdot\frac{1}{(D-1/2D')}\cdot\frac{1}{D-1/2D'}\log(x+2y) \\
&= 4\cdot\frac{1}{D-1/2D'}\left\{\int \log\left[x+2\left(a-\frac{1}{2}x\right)\right]dx\right\}_{a\to y+\frac{1}{2}x}
\end{aligned}$$

$$= 4 \cdot \frac{1}{D - 1/2D'} \left[\int \log(2a) \mathrm{d}x \right]_{a \to y + \frac{1}{2}x}$$

$$= 4 \cdot \frac{1}{D - 1/2D'} \{x \log (x + 2y)\}$$

$$= 4 \left[\int x \log \left\{ x + 2\left(a - \frac{1}{2}x \right) \right\} \mathrm{d}x \right]_{a \to y + \frac{1}{2}x}$$

$$= 4 \left[\int x \log(2a) \mathrm{d}x \right]_{a \to y + \frac{1}{2}x}$$

$$= 2x^2 (\log 2a)_{a \to y + \frac{1}{2}x}$$

$$= 2x^2 \log(x + 2y)$$

$\therefore$ General solution is

$$z = x f_1(x + 2y) + f_2(x + 2y) + 2x^2 \log(x + 2y)$$

Example 17

Solve the equation

$$(D^2 + 2DD' + D'^2 - 2D - 2D')z = \cosh(x - y)$$

The given equation is a non-homogeneous linear equation

$$\begin{aligned} D^2 + 2DD' + D'^2 - 2D - 2D' &\equiv (D + D')^2 - 2(D + D') \\ &= (D + D')(D + D' - 2) \end{aligned}$$

$\therefore$ The given equation is

$$(D + D')(D + D' - 2)z = \cosh(x - y)$$

$\therefore$ C.F. $= f_1(y - x) + e^{2x} \cdot f_2(y - x)$ [$\because$ the part of C.F. corresponding to $(D - aD' - b)z = 0$ is $e^{bx} f(y + ax)$]

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D + D')(D + D' - 2)} \frac{1}{2} \{e^{x-y} + e^{-x+y}\} \\ &= \frac{1}{2} \cdot \frac{1}{D + D'} \cdot \frac{-1}{2} (e^{x-y} + e^{-x+y}) \\ &= -\frac{1}{4} \cdot (x e^{x-y} + x e^{-x+y}) \\ &= -\frac{x}{2} \cosh(x - y) \end{aligned}$$

$\therefore$ General solution is

$$z = f_1(y - x) + e^{2x} f_2(y - x) - \frac{x}{2} \cosh(x - y)$$

Example 18

Solve the equation $(D^2 - D'^2 - 3D + 3D')z = e^{x+2y} + xy$

$$\begin{aligned} D^2 - D'^2 - 3D + 3D' &\equiv (D + D')(D - D') - 3(D - D') \\ &= (D - D')(D + D' - 3) \end{aligned}$$

$\therefore$ The given equation is

$$(D - D')(D + D' - 3)z = e^{x+2y} + xy$$

$$\therefore \qquad \text{C.F.} = f_1(y + x) + e^{3x} f_2(y - x)$$

$$\begin{aligned} (\text{P.I.})_1 &= \frac{1}{(D - D')(D + D' - 3)} e^{x+2y} \\ &= \frac{1}{(D + D' - 3)} \cdot (-1)e^{x+2y} \\ &= -xe^{x+2y} \end{aligned}$$

$$(\text{P.I.})_2 = \frac{1}{(D - D')(D + D' - 3)} xy$$

$$= -\frac{1}{3D}\left(1 - \frac{D'}{D}\right)^{-1} \left\{1 - \frac{D + D'}{3}\right\}^{-1} xy$$

$$= \frac{1}{3D}\left(1 + \frac{D'}{D}\right)\left\{1 + \frac{1}{3}(D + D') + \frac{1}{9}(D + D')^2 + \frac{1}{27}(D + D')^3 + \ldots\right\} xy$$

$$= -\frac{1}{3}\left(\frac{1}{D} + \frac{D'}{D^2}\right)\left\{1 + \frac{1}{3}D + \frac{1}{3}D' + \frac{1}{9}D^2 + \frac{2}{9}DD' + \frac{1}{27}D^3 + \frac{1}{9}D^2D'\right\}(xy)$$

$$= -\frac{1}{3}\left[\frac{1}{D} + \frac{1}{3} + \frac{1}{3}\frac{D'}{D} + \frac{1}{9}D + \frac{2}{9}D' + \frac{1}{9}DD' + \frac{D'}{D^2} + \frac{1}{3}\frac{D'}{D} + \frac{1}{9}D' \right.$$
$$\left. + \frac{1}{27}DD'\right](xy)$$

$$= -\frac{1}{3}\left[\frac{D'}{D^2} + \frac{2}{3}\frac{D'}{D} + \frac{1}{D} + \frac{1}{3} + \frac{1}{9}D + \frac{1}{3}D' + \frac{4}{27}DD'\right] xy$$

$$= -\frac{1}{3}\left[\frac{x^3}{6} + \frac{x^2}{3} + \frac{x^2 y}{2} + \frac{1}{3}xy + \frac{1}{9}y + \frac{1}{3}x + \frac{4}{27}\right]$$

$\therefore$ General solution is

$$z = \text{C.F} + (\text{P.I.})_1 + (\text{P.I.})_2$$

Example 19

Solve the equation $(D^2 - 3DD' + 2D'^2 + 2D - 2D')z = x + y + \sin(2x + y)$

$$\begin{aligned} D^2 - 3DD' + 2D'^2 + 2D - 2D' &\equiv (D - D')(D - 2D') + 2(D - D') \\ &= (D - D')(D - 2D' + 2) \end{aligned}$$

$\therefore$ The given equation is

$$(D - D')(D - 2D' + 2)z = (x + y) + \sin(2x + y)$$

$$\text{C.F.} = f_1(y + x) + e^{-2x} f_2(y + 2x)$$

$$(\text{P.I.})_1 = \frac{1}{(D - D')(D - 2D' + 2)}(x + y)$$

$$= \frac{1}{2D}\left(1 - \frac{D'}{D}\right)^{-1}\left(1 + \frac{D - 2D'}{2}\right)^{-1}(x + y)$$

$$= \frac{1}{2}\left(\frac{1}{D} + \frac{D'}{D^2}\right)\left\{1 - \frac{1}{2}(D - 2D') + \frac{1}{4}(D - 2D')^2 + \cdots\right\}(x + y)$$

$$= \frac{1}{2}\left[\frac{1}{D} + \frac{D'}{D^2}\right]\left[1 - \frac{1}{2}D + D' + \frac{1}{4}D^2 - DD'\right](x + y)$$

$$= \frac{1}{2}\left[\frac{1}{D} - \frac{1}{2} + \frac{D'}{D} + \frac{1}{4}D - D' + \frac{D'}{D^2} - \frac{1}{2}\frac{D'}{D} + \frac{1}{4}D'\right](x + y)$$

$$= \frac{1}{2}\left(\frac{D'}{D^2} + \frac{1}{2}\cdot\frac{D'}{D} + \frac{1}{D} - \frac{1}{2} + \frac{1}{4}D - \frac{3}{4}D'\right)(x + y)$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + \frac{x}{2} + \frac{x^2}{2} + xy - \frac{1}{2}y - \frac{1}{2}x + 1/4 - 3/4\right]$$

$$= \frac{1}{2}x^2 + \frac{1}{2}xy - \frac{1}{4}y - 1/4$$

$$(\text{P. I.})_2 = \frac{1}{D^2 - 3DD' + 2D'^2 + 2D - 2D'}\sin(2x + y)$$

$$= \frac{1}{-4 + 6 - 2 + 2(D - D')}\sin(2x + y)$$

$$= \frac{(D + D')}{2(D^2 - D'^2)}\sin(2x + y)$$

$$= \frac{-1}{6}\{2\cos(2x + y) + \cos(2x + y)\}$$

$$= \frac{-1}{2}\cos(2x + y)$$

$\therefore$ General solution is

$$z = f_1(y + x) + e^{-2x}\ f_2(y + 2x) + \frac{1}{2}x^2 + \frac{1}{2}xy - \frac{1}{4}y - \frac{1}{4} - \frac{1}{2}\cos(2x + y)$$

Example 20

Solve the equation $(D^2 - DD' + D' - 1)z = e^{2x+3y} + \cos^2(x + 2y)$

$$D^2 - DD' + D' - 1 \equiv (D^2 - 1) - D'(D - 1)$$
$$= (D - 1)(D - D' + 1)$$

$\therefore$ The given equation is

$$(D-1)(D-D'+1)z = e^{2x+3y} + \cos^2(x+2y)$$

$$\therefore \qquad \text{C.F.} = e^x f_1(y) + e^{-x} f_2(y+x)$$

$$\begin{aligned}(\text{P.I.})_1 &= \frac{1}{(D-1)(D-D'+1)} e^{2x+3y} \\ &= \frac{1}{(2-1)(D-D'+1)} e^{2x+3y} \\ &= x e^{2x+3y}\end{aligned}$$

$$\begin{aligned}(\text{P.I.})_2 &= \frac{1}{(D-1)(D-D'+1)} \frac{1}{2}\{1+\cos(2x+4y)\} \\ &= \frac{1}{2}(-1) + \frac{1}{2} \cdot \frac{1}{(D^2 - DD' + D' - 1)} \cos(2x+4y) \\ &= -1/2 + \frac{1}{2} \cdot \frac{1}{-4+8+D'-1} \cos(2x+4y) \\ &= -1/2 + 1/2 . \frac{D'-3}{(D'^2-9)} \cos(2x+4y) \\ &= \frac{-1}{2} - \frac{1}{50}\{-4\sin(2x+4y) - 3\cos(2x+4y)\} \\ &= \frac{1}{50}\{4\sin(2x+4y) + 3\cos(2x+4y)\} - 1/2\end{aligned}$$

$\therefore$ General solution is

$$z = e^x f_1(y) + e^{-x} f_2(y+x) + x e^{2x+3y} - 1/2 + \frac{1}{50}\{4\sin(2x+4y) + 3\cos(2x+4y)\}$$

Example 21

Solve the equation $(2D^2 - DD' - D'^2 + 6D + 3D')z = xe^y + ye^x$

$$\begin{aligned}2D^2 - DD' - D'^2 + 6D + 3D' &\equiv (2D+D')(D-D') + 3(2D+D') \\ &= (2D+D')(D-D'+3)\end{aligned}$$

$\therefore$ The given equation is

$$(2D+D')(D-D'+3)z = xe^y + ye^x$$

$$\therefore \qquad \text{C.F.} = f_1\left(y - \frac{x}{2}\right) + e^{-3x} f_2(y+x)$$

$$\text{or} \qquad f_1(2y-x) + e^{-3x} \cdot f_2(y+x)$$

$$(\text{P.I.})_1 = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'}(xe^y)$$

$$= e^y \cdot \frac{1}{2D^2 - D(D'+1) - (D'+1)^2 + 6D + 3(D'+1)}(x)$$

$$= e^y \cdot \frac{1}{2 + 5D + D' + 2D^2 - DD' - D'^2}(x)$$

$$= \frac{e^y}{2}\{1 + \frac{1}{2}(5D + D' + 2D^2 - DD' - D'^2)\}^{-1}(x)$$

$$= \frac{e^y}{2}\{1 - \frac{5}{2} \cdot D\}(x)$$

$$= \frac{1}{4}(2x - 5)e^y$$

$$(\text{P.I.})_2 = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'}(ye^x)$$

$$= e^x \cdot \frac{1}{2(D+1)^2 - (D+1)D' - D'^2 + 6(D+1) + 3D'}(y)$$

$$= e^x \cdot \frac{1}{8 + 10D + 2D' + 2D^2 - DD' - D'^2}(y)$$

$$= \frac{e^x}{8}\left\{1 + \frac{1}{8}(10D + 2D' + 2D^2 - DD' - D'^2)\right\}^{-1}(y)$$

$$= \frac{e^x}{8}\left\{1 - \frac{1}{4}D'\right\}(y)$$

$$= \frac{1}{32}(4y - 1)e^x$$

$\therefore$ General solution is

$$z = f_1(2y - x) + e^{-3x} f_2(y + x) + \frac{1}{4}(2x - 5)e^y + \frac{1}{32}(4y - 1)e^x$$

Example 22

Solve the equation $2x\frac{\partial z}{\partial x} - 3y\frac{\partial z}{\partial y} = 0$, by the method of separation of variables.

Let $z = X(x) \cdot Y(y)$ be a solution of

$$2xz_x - 3yz_y = 0 \qquad (1)$$

Then $z_x = X'Y$ and $z_y = XY'$, where $X' = \frac{dX}{dx}$ and $Y' = \frac{dY}{dy}$ satisfy Eq. (1).

i.e. $$2xX'Y - 3yXY' = 0$$

i.e. $$2x\frac{X'}{X} = 3y\frac{Y'}{Y}$$

L.H.S. is a function of x alone and R.H.S. is a function of y alone. They are equal for all values of x and y. This is possible only if each is a constant.

$$\therefore \quad 2x\frac{X'}{X} = 3y\frac{Y'}{Y} = k$$

i.e.
$$2\frac{X'}{X} = \frac{k}{x} \tag{2}$$

and
$$\frac{3Y'}{Y} = \frac{k}{y} \tag{3}$$

Integrating both sides of (2) with respect to x,

$$2\log X = k\log x + \log A$$

i.e. $X^2 = Ax^k$ or $X = ax^{k/2}$ (4)

Similarly, from (3), $Y = by^{k/3}$

$\therefore$ Required solution of (1) is

$$z = abx^{k/2}y^{k/3} \quad \text{or} \quad z = cx^{k/2}y^{k/3}$$

Example 23

Solve the equation $\frac{\partial z}{\partial x} = 2\frac{\partial z}{\partial y} + z$, by the method of separation of variables, given that

$$z(x, 0) = 6e^{-3x}$$

Let

$$z = X(x) \cdot Y(y) \tag{1}$$

be a solution of

$$z_x = 2z_y + z \tag{2}$$

Then $z_x = X'Y$ and $z_y = XY'$ satisfy equation (2).

i.e. $X'Y = 2XY' + XY$

Dividing throughout by XY, we get

$$\frac{X'}{X} = 2\frac{Y'}{Y} + 1 = k$$

[$\because$ the L.H.S. is a function of x alone and the R.H.S. is a function of y alone]

$$\therefore \quad \frac{X'}{X} = k \tag{3}$$

and
$$\frac{Y'}{Y} = \frac{k-1}{2} \tag{4}$$

Integrating (3) and (4) with respect to x and y respectively, we get

$$\log X = kx + \log A \text{ and } \log Y = \left(\frac{k-1}{2}\right) y + \log B$$

i.e. $$X = Ae^{kx} \text{ and} Y = Be^{\left(\frac{k-1}{2}\right)y}$$

$\therefore$ Required solution is

$$z = ce^{kx} \cdot e^{\left(\frac{k-1}{2}\right)y} \tag{5}$$

Given that $z(x, 0) = 6e^{-3x}$

$\therefore$ $$ce^{kx} = 6e^{-3x}$$

$\therefore$ $$c = 6 \text{ and } k = -3$$

Using these values in (5), the required solution is $z = 6e^{-(3x+2y)}$.

Example 24

Solve the equation $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$, by the method of separation of variables.

Let $$z = X(x).Y(y) \tag{1}$$

be a solution of the equation

$$z_{xx} - 2z_x + z_y = 0 \tag{2}$$

Then $z_x = X'Y$, $z_{xx} = X''Y$ and $z_y = XY'$ satisfy (2).

i.e. $$X''Y - 2X'Y + XY' = 0$$
Dividing throughout by XY, we get

$$\frac{X''}{X} - 2\frac{X'}{X} + \frac{Y'}{Y} = 0$$

i.e. $$\frac{X'' - 2X'}{X} = -\frac{Y'}{Y} = k$$

i.e. $$X'' - 2X' - kX = 0 \tag{3}$$

and $$Y' + kY = 0 \tag{4}$$

i.e. $$(D^2 - 2D - k)X = 0 \tag{5}$$

where $D \equiv \dfrac{d}{dx}$ and

$$\frac{Y'}{Y} = -k \tag{6}$$

A.E. of (5) is $m^2 - 2m - k = 0$

$\therefore$ $$m = \frac{2 \pm \sqrt{4+4k}}{2} \text{ or } 1 \pm \sqrt{k+1}$$

$\therefore$ Solution of (5) is

$$X = Ae^{(1+\sqrt{k+1})x} + Be^{(1-\sqrt{k+1})x}$$

Solution of (6) is

$$Y = ce^{-ky}$$

Using these values in (1), the required solution is

$$z = \{Ae^{(1+\sqrt{k+1})x} + Be^{(1-\sqrt{k+1})x}\}ce^{-ky}$$

or $$z = \{c_1e^{(1+\sqrt{k+1})x} + c_2e^{(1-\sqrt{k+1})x}\}e^{-ky}$$

Example 25

Solve the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y} + 5u$, by the method of separation of variables, given that $u = 0$ and $\dfrac{\partial u}{\partial x} = e^{-3y}$ when $x = 0$ and for all values of y.

Let $$u(x, y) = X(x).Y(y) \tag{1}$$
be a solution of

$$u_{xx} = u_y + 5u \tag{2}$$

Then $u_{xx} = X''Y$ and $u_y = XY'$ satisfy (2)

i.e. $$X''Y = XY' + 5XY$$

Dividing throughout by XY, we get

$$\frac{X''}{X} = \frac{Y'}{Y} + 5 = k$$

$$X'' - kX = 0 \tag{3}$$

and $$\frac{Y'}{Y} = k - 5 \tag{4}$$

Assuming that k is positive, the solutions of (3) and (4) are

$$X = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$$

and $$Y = ce^{(k-5)y}$$

Using these values in (1), the required solution is

$$u(x, y) = (C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x})e^{(k-5)y} \tag{5}$$

Given: $u = 0$ when $x = 0$ and for all y

$\therefore$ $$(C_1 + C_2)e^{(k-5)y} = 0$$

i.e. $$C_1 + C_2 = 0 \tag{6}$$

Differentiating (5) partially with respect to x, we have

$$\frac{\partial u}{\partial x} = \sqrt{k}(C_1 e^{\sqrt{k}x} - C_2 e^{-\sqrt{k}x})e^{(k-5)y} \tag{7}$$

Given: $\dfrac{\partial u}{\partial x} = e^{-3y}$, when $x = 0$ and for all y.

$\therefore$ $$\sqrt{k}(C_1 - C_2)e^{(k-5)y} = e^{-3y}$$

$\therefore$ $$\sqrt{k}(C_1 - C_2) = 1 \tag{8}$$

and $$k - 5 = -3 \tag{9}$$

Solving (6), (8) and (9), we get

$$k = 2, C_1 = \frac{1}{2\sqrt{2}} \quad \text{and} \quad C_2 = -\frac{1}{2\sqrt{2}}$$

Using these values in (5), the required solution is

$$u(x, y) = \frac{1}{\sqrt{2}} \sinh x\sqrt{2} \cdot e^{-3y}.$$

Exercise 1(d)

Part A (Short-Answer Questions)

Solve the following equations.

1. $(D^3 - 3D^2D' - 4DD'^2 + 12D'^3)z = 0$
2. $(D - D')^3 z = 0$

3. $(D^2 + D'^2)^2 z = 0$
4. $(D^3 + 4D^2D' - 5DD'^2)z = 0$
5. $(2D^2D' - 5DD'^2 - 3D'^3)z = 0$
6. $(D + D' - 1)(D - D' + 1)z = 0$
7. $D(D - 2D' + 3)z = 0$
8. $D'(D + 3D' - 2)z = 0$
9. $(D + D')(D - D' - 1)z = 0$
10. $(D - D')(D + D' + 1)z = 0$

Find the particular integrals of the following equations.

11. $(D^2 + 2DD' + D'^2)z = e^{x-y}$
12. $(D^2 - DD' - 2D'^2)z = \sin(3x + 4y)$
13. $(D^2 - 4D'^2)z = \sin(2x + y)$
14. $\{(D - 1)^2 - D'^2\}z = e^{x+y}$
15. $(D^2 - D'^2 + D)z = \cos(x + y)$

Solve the following partial differential equations by the method of separation of variables.

16. $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0$, given that $u(x, 0) = 4e^{-x}$
17. $\dfrac{\partial u}{\partial x} = 4\dfrac{\partial u}{\partial y}$, given that $u(0, y) = 8e^{-3y}$
18. $\dfrac{\partial z}{\partial x} + 4z = \dfrac{\partial z}{\partial t}$, given that $z(x, 0) = 4e^{-3x}$
19. $x^2\dfrac{\partial z}{\partial y} + y^3\dfrac{\partial z}{\partial x} = 0$
20. $\dfrac{\partial u}{\partial y} = 2\dfrac{\partial^2 u}{\partial x^2}$

Part B

Solve the following partial differential equations.

21. $(D^2 + 3DD' - 4D'^2)z = (e^{2x} - e^{-y})^3$
22. $(D^3 - 7DD'^2 - 6D'^3)z = \sinh(2x - 3y)$
23. $(D^2 - 7DD' + 12D'^2)z = (e^{3x} + e^{4x})e^y$
24. $(D^2 + 2DD' + D'^2)z = x^2 + xy + y^2$
25. $(D^3 + 2D^2D')z = e^{2x} + 3x^2y$
26. $(D^2 - 3DD' + 2D'^2)z = e^{2x+3y} + \sin(x - 2y)$

27. $(D^2 - 6DD' + 9D'^2)z = x^2y^2 + \cos(3x + y)$
28. $(D^2 - DD')z = \cos x \cos 2y$
29. $(8D^3 - 4D^2D' - 18DD'^2 + 9D'^3)z = \sin(3x + 2y)$
30. $(D^2 - 3DD' + 2D'^2)z = (2 + 4x)e^{x+2y}$
31. $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y$
32. $(D^2 + DD' - 6D'^2)z = y \cos x$
33. $(D^2 + D'^2)z = \dfrac{8}{x^2 + y^2}$
34. $D(D^2 + 4DD' + 3D'^2 - 3D - 5D' + 2)z = e^x + e^y$
35. $(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = \cosh(2x + y)$
36. $(D^2 - DD' + D)z = x^2 + y^2$
37. $(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$
38. $(D^2 - D'^2 - 2D + 1)z = xy + e^{2x+3y}$
39. $(D^2 + DD' + D' - 1)z = \sinh(3x - 2y)$
40. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = \cos 2x \cos y$
41. $(2DD' + D'^2 - 3D')z = 4\sin^3(x + 2y)$
42. $(D^2 - D'^2 + D + 3D' - 2)z = xe^x + ye^y$
43. Solve equation $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u$, by the method of separation of variables, given that $u(0, y) = 3e^{-y} - e^{-5y}$ [**Hint**: Assume the R.H.S. of the solution as the sum of two terms of the form $Ce^{\frac{kx}{4}+(3-k)y}$ with different values for c and k]
44. Solve equation $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$, by the method of separation of variables, given that $z = 0$ and $\dfrac{\partial z}{\partial x} = 4e^{-3y} + 6e^{-8y}$ when $x = 0$.
45. Solve the equation $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial z}{\partial y} + 2z$, by the method of separation of variables, given that $z = 0$ and $\dfrac{\partial z}{\partial x} = 1 + e^{-3y}$ when $x = 0$.

Answers

Exercise 1(a)

2. $pq = z$

3. $pq = 4xyz$

4. $z = px + qy + pq$

5. $z = px + qy + p^2 + q^2$

6. $px + qy = 3z$

7. $p = q$

8. $px + qy = z - \dfrac{1}{z}$

9. $p^2 + q^2 = 1$

10. $py = qx$

11. $py^2 + qx^2 = 0$

12. $ap + bq = 0$

13. $px = qy$

14. $px + qy = 0$

15. $s = 0$

16. $s = a$

17. $r = 0$

18. $t = 0$

19. $r = \sin x$

20. $t = \cos y$

21. $px + qy = pq$

22. $pq = p + q$

23. $p^2 + q^2 = z$

24. $pz = 1 + q^2$

25. $yp - x^2q^2 = x^2y$

26. $p = q$

27. $z = px + qy$

28. $py = qx$

29. $z^2(p^2 + q^2 + 1) = c^2$

30. $(p^2 + q^2 + 1)y^2 = c^2q^2$

31. (a) $px = qy$; (b) $py = qx$

32. (a) $x(y - z)p + y(z - x)q = z(x - y)$;
(b) $x(y - z)p + y(z + 2x^2)q = z(y + 2x^2)$

33. (a) $px^2 + qy^2 = z^2$
(b) $y^2zp + x^2zq = xy^2$

34. (a) $(y^2 + z^2)p - xyq + xz = 0$;
(b) $x(y^2 + z)p + y(x^2 + z)q = z(x^2 - y^2)$

35. (a) $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$;
(b) $yp + xq = z$

36. $r + t = 0$

37. $2r + 3s - 9t = 0$

38. $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{(\partial x)^2 \partial y} - \dfrac{\partial^3 z}{\partial x(\partial y)^2} + 2\dfrac{\partial^3 z}{\partial y^3} = 0.$

39. $9r - 12s + 4t = 0.$

40. $r - 2s + t = 0.$

41. $(x - iy)(r - t) = 2(p - iq).$

42. $4xr - t + 2p = 0.$

43. $zs = pq.$

44. $xys = px + qy - z.$

45. $(1 + q)r + (q - p)s - (1 + p)t = 0.$

Exercise 1(b)

7. $z = ax + \dfrac{k}{a}y + b.$

8. $z = ax + y \log a + b.$

9. $z = ax \pm \sqrt{2 - a^2}\,y + b.$

10. $(1 + a)\log z = x + ay + b.$

11. $\log z = a(x + ay) + b.$

12. $4az = (x + ay + b)^2.$

13. $z = a\dfrac{x^2}{2} + \dfrac{y^2}{2a} + b.$

14. $z = a\log(xy) + b.$

15. $z = a(e^x + e^y) + b.$

16. $z = ax + by + \dfrac{a^3}{b} + \dfrac{b^3}{a}.$

17. (a) C.S. is $z = ax + (1 - \sqrt{a})^2 y$; No singular solution (S. S.).
(b) C.S. is $z = ax \pm \sqrt{k^2 - a^2}\,y + b$; No S.S.

18. C.S. is $z = ax + \frac{1}{2}(-2 \pm \sqrt{10})ay + b$; No S.S.

19. C.S. is $z = ax + \left(\frac{5 - a^2}{3 - 2a}\right) y + b$; No S.S.

20. $\log z = a \log x \pm \sqrt{1 - a^2} \log y + b$.

21. $\sqrt{z} = a\sqrt{x} \pm \sqrt{1 - a^2}\sqrt{y} + b$.

22. $\frac{1}{z} = \frac{a}{x} + \frac{(1 - a)}{y} + b$.

23. $z^2 = ax \pm \sqrt{a^2 - 4} \cdot y + b$.

24. $\log z = \frac{a}{x} + (2a^2 - 3) \log y + b$.

25. $z = a^2(x + y) + axy + b$.

26. $xy = 1$.

27. $729z^4 = 1024xy$.

28. $16z^3 + 27x^2y^2 = 0$.

29. $z^4 = 16xy$.

30. $4z = y^2 - x^2$.

31. $x^2 + y^2 = 1$.

32. $z = 3$.

33. $4(1 + a^2)z = (x + ay + b)^2$.

34. $\sqrt{1 + a^2} \log(z + \sqrt{z^2 - 1}) = x + ay + b$.

35. (a) $z^2 \pm z\sqrt{z^2 - 4a^2} - 4a^2 \log(z + \sqrt{z^2 - 4a^2}) = 4(x + ay + b)$

(b) $az^2 \mp z\sqrt{a^2z^2 - 4} \pm \frac{4}{a} \log(az + \sqrt{a^2z^2 - 4}) = 4(x + ay + b)$

36. $4(bz - ab - 1) = (x + by + c)^2.$

37. $(z + a^2)^3 = (x + ay + b)^2.$

38. $3(1 + a) \log z = x^3 + ay^3 + b.$

39. $\sqrt{a^2 + 1}z^2 = 2 (\log x + ay + b).$

40. $2 \log z = (a \pm \sqrt{a^2 + 8}) \left(\frac{1}{x} + \frac{a}{y} + b\right).$

41. $4z = -x^2 \pm \left\{x\sqrt{x^2 + 4a^2} + 4a^2 \log (x + \sqrt{x^2 + 4a^2}) + 4(a^2y + b)\right\}.$

42. $2z = ax^2 - \frac{a}{a + 1}y^2 + b.$

43. $3z = ax^3 + 2\sqrt{a - 1}y^{3/2} + b.$

44. $z = ax - \cos x + \frac{1}{a} \sin y + b.$

45. $z^{3/2} = (x + a)^{3/2} + (y + a)^{3/2} + b.$

46. $z^2 = x\sqrt{x^2 + a^2} + a^2 \sinh^{-1} \frac{x}{a} + y\sqrt{y^2 - a^2} - a^2 \cosh^{-1} \frac{y}{a} + b.$

47. $\log z = \frac{\sqrt{a}x^2}{2} + \sqrt{1 - a} \log y + b.$

48. $z^2 = x^2 + ax + \frac{2}{3}(y + a)^{3/2} + b.$

49. $z = \sqrt{a(x + y)} + \sqrt{(1 - a)(x - y)} + b.$

50. $z = \frac{a}{2} \log (x^2 + y^2) + \sqrt{1 - a^2} \tan^{-1} \left(\frac{y}{x}\right) + b.$

Exercise 1(c)

1. $z = xf(y) + \phi(y).$

2. $z = yf(x) + \phi(x).$

3. $z = f(x) + \phi(y)$.

4. $z = xf(y) + \phi(y) + e^{x+y}$.

5. $z = yf(x) + \phi(x) - \dfrac{1}{9}\cos(2x + 3y)$.

6. $z = f(x) + \phi(y) + \log x \cdot \log y$.

7. $z = xf(y) + \phi(y) + \dfrac{x^2}{2}\sin y$.

8. $z = yf(x) + \phi(x) - \cos y$.

9. $z = f(x) + \phi(y) + kxy$.

10. $z = f(x) + \phi(y) + \dfrac{xy}{3}\left(x^2 + y^2\right)$

12. $f(x^2 - y^2, y^2 - z^2) = 0$

13. $f\left(xy, \dfrac{y}{z}\right) = 0$.

14. $f(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$.

15. $f\left(\dfrac{\sin x}{\sin y}, \dfrac{\sin y}{\sin z}\right) = 0$.

16. $f\left(\dfrac{1}{x} - \dfrac{1}{y}, \dfrac{1}{y} - \dfrac{1}{z}\right) = 0$.

17. $z = (1 + \cos x)\cos y$.

18. $z = c\cosh ax + \sinh ax \sin y$.

19. $z = e^y \cosh x + e^{-y}\sinh x$.

20. $z = 3x^2 + 3xy - 2y^2 + c$

21. $z = x^3 f(y) - x - y/3$.

22. $z = f(x) + \phi(y) - 3x^2y^3 - \dfrac{1}{2}\sin(2x - y)$

23. $z = e^{2y} f(x) + e^{3y} g(x) + 2y + 5/3$.

24. $z = Axy + Bx + Cy + D$.

25. $z = f(x + at) + \phi(x - at)$.

26. $z = f(x + iy) + \phi(x - iy)$.

27. $z = f(x + y) + \phi(x - y)$.

28. $z = (x^2 - y^2) f(x^2 + y^2) + \phi(x^2 + y^2)$.

29. $z = f(y^2 + x) + \phi(y^2 - x)$.

30. (i) $f\left(\dfrac{\sec x}{\sec y}, \dfrac{\sec y}{\sec z}\right) = 0$;
(ii) $f\left(\dfrac{a - x}{b - y}, \dfrac{b - y}{c - z}\right) = 0$.

31. $f(x^3 - y^3, x^2 - z^2) = 0$.

32. (i) $f\left(\dfrac{1}{x} - \dfrac{1}{y}, \dfrac{x - y}{z}\right) = 0$;
(ii) $f\left(\dfrac{1}{x} + \dfrac{1}{y}, \dfrac{x + y}{z}\right) = 0$.

33. $f(y/z, x^2 + y^2 + z^2) = 0$.

34. $f\left(\dfrac{y}{z}, \dfrac{x^2 + y^2 + z^2}{z}\right) = 0$.

35. $f\left[x \log(x + y) - z, x + y\right] = 0$.

36. $f(xy, x^2 + y^2 + z^2) = 0$.

37. (i) $f(x + y + z, x^2 + y^2 + z^2) = 0$;
(ii) $f(x + y + z, x^2 + 2yz) = 0$.

38. (i) $f(x + y + z, xyz) = 0$;
(ii) $f(x + y + z, xyz) = 0$.

39. $f(xyz, x^2+y^2+z^2)=0.$

40. $f\left(\dfrac{x-y}{y-z}, xy+yz+zx\right)=0.$

41. (i) $f(x+y+z, x^2+y^2-z^2)=0;$
(ii) $f(2x+3y+4z, x^2+y^2+z^2)=0.$

42. $f\left(x^3y^3z, \dfrac{x}{y^2}+\dfrac{y}{x^2}\right)=0.$

43. $xy+yz+zx=0.$

44. $3(2x+2y-3z+3)^2=(y-x)(x+y)^3.$

Exercise 1(d)

1. $z=f_1(y-2x)+f_2(y+2x)+f_3(y+3x).$

2. $z=f_1(y+x)+xf_2(y+x)+x^2f_3(y+x).$

3. $z=f_1(y+ix)+xf_2(y+ix)+f_3(y-ix)+xf_4(y-ix)$

4. $z=f_1(y)+f_2(y-5x)+f_3(y+x).$

5. $z=f_1(x)+f_2\left(y-\dfrac{x}{2}\right)+f_3(y+3x).$

6. $z=e^x f_1(y-x)+e^{-x}f_2(y+x).$

7. $z=f_1(y)+e^{-3x}f_2(y+2x).$

8. $z=f_1(x)+e^{2x}f_2(y-3x).$

9. $z=f_1(y-x)+e^x f_2(y+x).$

10. $z=f_1(y+x)+e^{-x}f_2(y-x).$

11. $\dfrac{x^2}{2}e^{x-y}.$

12. $\dfrac{1}{35}\sin(3x+4y).$

13. $-\frac{x}{4}\cos(2x+y)$.

14. $-e^{x+y}$.

15. $\sin(x+y)$.

16. $u=4e^{-x+\frac{3}{2}y}$.

17. $u=8e^{-12x-3y}$.

18. $z=4e^{-3x+t}$

19. $z=ce^{k(3y^4-4x^3)}$.

20. $u=e^{2ky}\left(Ae^{\sqrt{k}x}+Be^{-\sqrt{k}x}\right)$.

21. $z=f_1(y+x)+f_2(y-4x)+\frac{1}{36}e^{6x}-\frac{3}{5}xe^{4x-y}-\frac{1}{8}e^{2x-2y}-\frac{1}{36}e^{-3y}$.

22. $z=f_1(y-x)+f_2(y-2x)+f_3(y+3x)+\frac{1}{44}\cosh(2x-3y)$.

23. $z=f_1(y+3x)+f_2(y+4x)+x\left(e^{4x+y}-e^{3x+y}\right)$.

24. $z=f_1(y-x)+xf_2(y-x)+\frac{1}{4}\cdot\left(x^4-2x^3y+2x^2y^2\right)$.

25. $z=f_1(y)+xf_2(y)+f_3(y+2x)+\frac{1}{4}xe^{2x}+\frac{x^5}{60}\left(y+\frac{x}{3}\right)$

26. $z=f_1(y+x)+f_2(y+2x)+\frac{1}{4}e^{2x+3y}-\frac{1}{15}\sin(x-2y)$.

27. $z=f_1(y+3x)+xf_2(y+3x)+\frac{x^4}{60}(9x^2+12xy+5y^2)+\frac{x^2}{2}\cos(3x+y)$.

28. $z=f_1(y)+f_2(y+x)+\frac{1}{2}\cos(x+2y)-\frac{1}{6}\cos(x-2y)$.

29. $z=f_1(2y+x)+f_2(2y+3x)+f_3(2y-3x)-\frac{x}{96}\sin(3x+2y)$

30. $z=f_1(y+x)+f_2(y+2x)+\frac{2}{9}e^{x+2y}(11+6x)$.

31. $z = f_1(y-x) + xf_2(y-x) + f_3(y+x) + \dfrac{e^x}{25}(2\sin 2y + \cos 2y)$.

32. $z = f_1(y-3x) + f_2(y+2x) - y\cos x + \sin x$.

33. $z = f_1(y+ix) + f_2(y-ix) + \dfrac{1}{2}\left[\log(x^2+y^2)\right]^2 + 2\left(\tan^{-1}\dfrac{y}{x}\right)^2$.

34. $z = f_1(y) + e^x f_2(y-x) + e^{2x} f_3(y-3x) - xe^x + xye^y$.

35. $z = e^x f_1(y+x) + e^{2x} f_2(y+x) - \dfrac{x}{2}e^{2x+y} + \dfrac{1}{12}e^{-2x-y}$.

36. $z = f_1(y) + e^{-x} f_2(y+x) + \dfrac{x^3}{3} + xy^2 - x^2 + 2xy + 4x$.

37. $z = e^x f_1(y-x) + e^{3x} f_2(y-2x) + (x+2y+6)$

38. $z = e^x f_1(y+x) + e^x f_2(y-x) + (x+2)y - \dfrac{1}{8}e^{2x+3y}$.

39. $z = e^{-x} f_1(y) + e^x f_2(y-x) + \dfrac{1}{8}xe^{3x-2y} - \dfrac{1}{8}e^{2y-3x}$.

40. $z = f_1(y-x) + e^{-2x} f_2(y+2x) + \dfrac{1}{12}\sin(2x+y) + \dfrac{1}{4}\sin(2x-y) - \dfrac{1}{2}\cos(2x-y)$.

41. $z = f_1(x) + e^{3x/2}\cdot f_2\left(y - \dfrac{x}{2}\right) + \dfrac{3}{50}\{3\cos(x+2y) - 4\sin(x+2y)\}$
$+\dfrac{1}{306}\{4\sin(3x+6y) - \cos(3x+6y)\}$.

42. $z = e^x f_1(y-x) + e^{-2x} f_2(y+x) + \dfrac{e^x}{54}(9x^2 - 6x + 2) +$
$e^y\left(xy - \dfrac{x^2}{2} - y - 3\right)$.

43. $u = 3e^{x-y} - e^{2x-5y}$.

44. $z = \left(e^{3x} - e^{-x}\right)e^{-3y} + \left(e^{4x} - e^{-2x}\right)e^{-8y}$.

45. $z = \dfrac{1}{\sqrt{2}}\sinh x\sqrt{2} + e^{-3y}\sin x$.

Chapter 2

Fourier Series

2.1 INTRODUCTION

Periodic functions appear in a variety of physical problems, such as those containing vibrating springs and membranes, planetary motion, a swinging pendulum and musical sounds. In some of these problems, the periodic function may be quite complicated and hence in order to understand its basic nature better, it may be convenient to represent it in a series of simple periodic functions. Since trigonometric functions are the simplest examples of periodic functions, we usually look for series representation in terms of sines and cosines.

Originally Fourier series was applied in the study of vibration and heat diffusion. There are numerous problems in Science and Engineering in which sinusoidal signals and hence Fourier series play an important role. For example, sinusoidal signals are useful in describing the periodic behaviour of the earth's climate. Alternating current sources generate sinusoidal voltages and currents. Fourier analysis enables us to analyse the response of a Linear Time Invariant system, such as a circuit, to such sinusoidal inputs. Waves in the ocean consist of the linear combination of sinusoidal waves with different wavelengths. Signals transmitted by radio and television stations are sinusoidal in nature.

Many of the ordinary functions that occur frequently in Science and Engineering can be expressed in the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \tag{1}$$

or more generally in the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \tag{2}$$

Now $\cos n(2\pi + x) = \cos(2n\pi + nx) = \cos nx$, for $n = 1, 2, 3, \cdots$; and
$\sin n(2\pi + x) = \sin(2n\pi + nx) = \sin nx$, for $n = 1, 2, 3, \cdots$

Thus all the trigonometric functions in (1) are periodic with period 2π. The constant $\left(\frac{a_0}{2}\right)$ may be regarded as periodic with period 2π. Hence the infinite trigonometric series (1) is periodic with period 2π.

If a function $f(x)$ is to be expressed (or expanded) in the form of the series (1), as a prerequisite, $f(x)$ should be defined in an interval of length 2π and should satisfy certain conditions, known as *Dirichlet's conditions*, which are stated below.

The infinite trigonometric series (2) is periodic with period $2l$, since
$\cos \frac{n\pi}{l}(2l + x) = \cos \frac{n\pi x}{l}$; and
$\sin \frac{n\pi}{l}(2l + x) = \sin \frac{n\pi x}{l}$ and $\frac{a_0}{2}$ may be regarded as periodic with period $2l$.

If a function $f(x)$ is to be expressed (or expanded) in the form of the series (2), as a prerequisite, it should be defined in an interval of length $2l$ and should satisfy Dirichlet's conditions.

Note ✍

Since series (1) is only a particular case of series (2) when $l = \pi$, we shall develop the theory of Fourier series in the form (2) and obtain the derivations with reference to series (2). Whenever results are required relating to series (1), we simply replace l by π and obtain the required results.

2.2 DIRICHLET'S CONDITIONS

A function $f(x)$ defined in $c \le x \le c+2l$ can be expanded as an infinite trigonometric seris of the form $\frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{l} + \sum b_n \sin \frac{n\pi x}{l}$, provided

1. $f(x)$ is single-valued and finite in $(c, c + 2l)$
2. $f(x)$ is continuous or piecewise continuous with finite number of finite discontinuities in $(c, c + 2l)$.
3. $f(x)$ has no or finite number of maxima or minima in $(c, c + 2l)$.

Note ✍

All the functions that we deal with will satisfy the above Dirichlet's conditions and hence can be expanded in the form of the infinite trigonometric series given above.

2.3 EULER'S FORMULAS

If a function $f(x)$ defined in $(c, c + 2l)$ can be expanded as the infinite trigonometric series $\frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum\limits_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ then

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l} \, dx, n \ge 0 \text{ and}$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l} \, dx, n \ge 1$$

[Formulas given above for a_n and b_n are called *Euler's formulas for Fourier coefficients*]

Proof

Before we proceed to find the values of a_n and b_n, we shall obtain the values of certain definite integrals, which are required in the evaluation of a_n and b_n.

$$\begin{aligned}
\int_c^{c+2l} \cos\frac{n\pi x}{l}\,dx &= \frac{l}{n\pi}\left(\sin\frac{n\pi x}{l}\right)_c^{c+2l} \\
&= \frac{l}{n\pi}\left\{\sin\frac{n\pi}{l}(c+2l) - \sin\frac{n\pi c}{l}\right\} \\
&= \frac{l}{n\pi}\left\{\sin\frac{n\pi c}{l} - \sin\frac{n\pi c}{l}\right\} = 0 \qquad (1)
\end{aligned}$$

$$\begin{aligned}
\int_c^{c+2l} \sin\frac{n\pi x}{l}\,dx &= -\frac{l}{n\pi}\left(\cos\frac{n\pi x}{l}\right)_c^{c+2l} \\
&= -\frac{l}{n\pi}\left\{\cos\frac{n\pi}{l}(c+2l) - \cos\frac{n\pi c}{l}\right\} \\
&= -\frac{l}{n\pi}\left\{\cos\frac{n\pi c}{l} - \cos\frac{n\pi c}{l}\right\} = 0 \qquad (2)
\end{aligned}$$

$$\begin{aligned}
\int_c^{c+2l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx &= \frac{l}{2}\int_c^{c+2l}\left[\cos\frac{(m+n)\pi x}{l} + \cos\frac{(m-n)\pi x}{l}\right]dx \\
&= 0, \text{ if } m \neq n [\text{by}(1)] \qquad (3)
\end{aligned}$$

$$\begin{aligned}
\int_c^{c+2l} \cos^2\frac{n\pi x}{l}\,dx &= \frac{l}{2}\int_c^{c+2l}\left(1 + \cos\frac{2n\pi x}{l}\right)dx \\
&= \frac{1}{2}\times 2l\ [\because \text{ the second term vanishes as in (1)}] \\
&= l \qquad (4)
\end{aligned}$$

$$\begin{aligned}
\int_c^{c+2l} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx &= \frac{1}{2}\int_c^{c+2l}\left[\cos\frac{(m-n)\pi x}{l} - \cos\frac{(m+n)\pi x}{l}\right]dx \\
&= 0, \text{ if } m \neq n [\text{by}(1)] \qquad (5)
\end{aligned}$$

$$\begin{aligned}
\int_c^{c+2l} \sin^2\frac{n\pi x}{l}\,dx &= \frac{1}{2}\int_c^{c+2l}\left(1 - \cos\frac{2n\pi x}{l}\right)dx \\
&= \frac{1}{2}\times 2l\ [\because \text{ the second term vanishes as in (1)}] \\
&= l \qquad (6)
\end{aligned}$$

$$\int_c^{c+2l} \sin\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = \frac{1}{2}\int_c^{c+2l}\left[\sin\frac{(m+n)\pi x}{l} + \sin\frac{(m-n)\pi x}{l}\right]dx$$

$$= 0, \text{ when } m \neq n \text{ and also when } m = n \qquad (7)$$

[by (2)].

Now
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad (8)$$

Integrating both sides of (8) with respect to x between the limits c and $c + 2l$, we get

$$\int_c^{c+2l} f(x)\,dx = \frac{a_0}{2}\int_c^{c+2l} dx + \sum_{n=1}^{\infty} a_n \int_c^{c+2l} \cos\frac{n\pi x}{l}\,dx + \sum_{n=1}^{\infty} b_n \int_c^{c+2l} \sin\frac{n\pi x}{l}\,dx,$$

assuming that the term by term integration is possible.

$$= \frac{a_0}{2}[x]_c^{c+2l} + \sum_{n=1}^{\infty} a_n \times 0 + \sum_{n=1}^{\infty} b_n \times 0$$

[by (1) and (2)]

$$= a_0 \cdot l$$

$$\therefore\ a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx \qquad (9)$$

Multiplying both sides of (8) by $\cos\frac{m\pi x}{l}$ where m is a fixed positive integer and integrating term by term with respect to x between c and $c + 2l$, we get

$$\int_c^{c+2l} f(x)\cos\frac{m\pi x}{l}\,dx = \frac{a_0}{2}\int_c^{c+2l} \cos\frac{m\pi x}{l}\,dx + \sum_{n=1}^{\infty} a_n \int_c^{c+2l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_c^{c+2l} \sin\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx$$

$$= a_m \int_c^{c+2l} \cos^2\frac{m\pi x}{l}\,dx, \text{ [by (1), (3) and (7)]}$$

$$= a_m \cdot l, \text{ [by (4)]}$$

$$\therefore\ a_m = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{m\pi x}{l}\,dx, \text{ where } m = 1, 2, 3, \cdots \qquad (10)$$

Combining (9) and (10), we have,

$$a_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \cos \frac{n\pi x}{l}\, dx, \text{ for } n \geq 0 \tag{11}$$

Note ✍

Only if the constant term is taken as $\frac{a_0}{2}$, formula (11) is true for $n = 0$

Similarly, multiplying both sides of (8) by $\sin \frac{m\pi x}{l}$, integrating term by term with respect to x between c and $c + 2l$ and using (2), (5), (6) and (7), we get

$$b_m = \frac{1}{l} \int_{c}^{c+2l} f(x) \sin \frac{m\pi x}{l}\, dx \text{ or}$$

$$b_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \sin \frac{n\pi x}{l}\, dx, \text{ for } n \geq 1 \tag{12}$$

2.4 DEFINITION OF FOURIER SERIES

The infinite trigonometric series $\frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum\limits_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ is called *the Fourier series of* $f(x)$ in the interval $c \leq x \leq c + 2l$, provided the coefficients are given by the Euler's formulas. Very often, the Fourier series expansions of $f(x)$ are required in the intervals $(-l, l)$ and $(0, 2l)$ which are obtained by taking $c = -l$ and $c = 0$ respectively in the above discussions.

When we require Fourier series expansions of $f(x)$ in $(-\pi, \pi)$ and $(0, 2\pi)$, we simply put $l = \pi$ in all the assumptions and the results derived.

2.5 IMPORTANT CONCEPTS

1. We have already observed that if a function $f(x)$ is to be expanded in Fourier series of the form $\frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum\limits_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ which is of *period* $2l$, $f(x)$ should be defined in an interval of *length* $2l$ and should satisfy Dirichlet's Conditions in that interval. *Conversely*, if a function $f(x)$ is defined and satisfies Dirichlet's conditions in an interval of length $2l$, it can be expanded in Fourier series of *period* $2l$.
2. Since the Fourier series of $f(x)$ in $(0, 2l)$ [or $(-l, l)$], i.e. $\frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum\limits_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ is periodic with period $2l$, we may expect $f(x)$ also to be periodic with period $2l$. In fact, $f(x)$ is periodic with period $2l$, in the sense that the Fourier series $\frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum\limits_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ represents (or converges to) $f(x)$ in $(0, 2l)$ and its periodic extensions outside $(0, 2l)$.

3. We recall that the function $f(x)$ is said to be *periodic* with period $2l$, if the graphs of $y = f(x)$ in the intervals $(c-4l, c-2l)$, $(c-2l, c)$, $(c+2l, c+4l)$, $(c+4l, c+6l)$ etc. are periodic repetitions of the graph of $y = f(x)$ in $(c, c+2l)$ as given in Figs 2.1, 2.2 and 2.3.

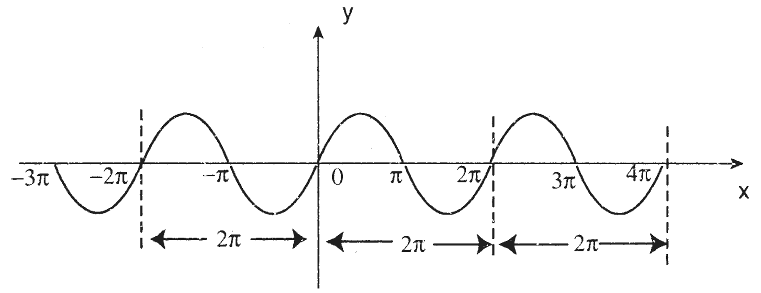


Fig. 2.1

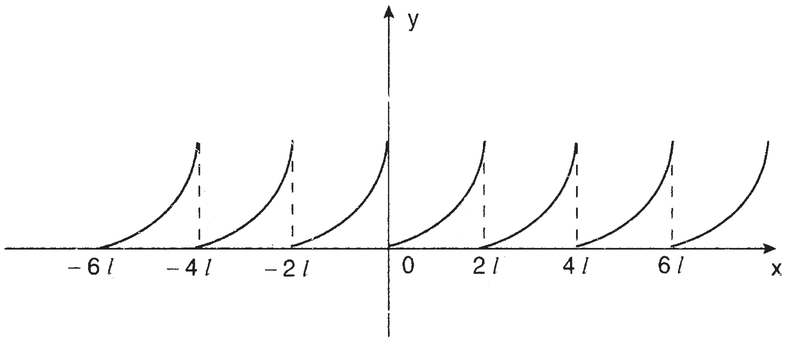


Fig. 2.2

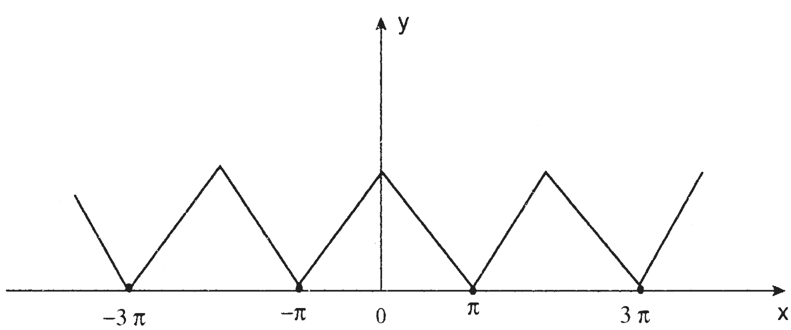


Fig. 2.3

The functions represented by the graphs in Fig. 2.1 and 2.3 are periodic with period 2π, whereas the function represented by the graph in Fig. 2.2 is periodic with period $2l$. The function represented by Fig. 2.1 takes the same value $f(x) = \sin x$ in $(-\infty, \infty)$. The function represented by Fig. 2.1 assumes different values in $(-4l, -2l)$, $(-2l, 0)$, $(0, 2l)$, $(2l, 4l)$ etc. namely,

$$f(x) = \begin{cases} (x+4l)^2 & \text{in } (-4l, -2l) \\ (x+2l)^2 & \text{in } (-2l, 0) \\ x^2 & \text{in } (0, 2l) \\ (x-2l)^2 & \text{in } (2l, 4l) \\ (x-4l)^2 & \text{in } (4l, 6l) \end{cases}$$

The function represented by Fig. 2.3 assumes different values in $(-3\pi, -\pi)$, $(-\pi, \pi)$, $(\pi, 3\pi)$, etc. namely,

$$f(x) = \begin{cases} x + 3\pi, & \text{in } (-3\pi, -2\pi) \\ -x - \pi, & \text{in } (-2\pi, -\pi) \end{cases}$$

$$f(x) = \begin{cases} x + \pi, & \text{in } (-\pi, 0) \\ -x + \pi, & \text{in } (0, \pi) \end{cases}$$

$$f(x) = \begin{cases} x - \pi, & \text{in } (\pi, 2\pi) \\ 3\pi - x, & \text{in } (2\pi, 3\pi), \text{ etc.} \end{cases}$$

4. From the examples given above, a periodic function can be defined analytically as follows.

   (a) If $f(x) = \phi(x)$ in $(-\infty, \infty)$, i.e. $f(x)$ assumes the same value in $(-\infty, \infty)$, then $f(x)$ is said to be periodic with period $2l$, if

   $$\phi(x + 2l) = \phi(x), \text{ for } -\infty < x < \infty$$

   The Fourier series of $f(x)$ of period $2l$, in this case, will represent $\phi(x)$ everywhere.

   (b) If $f(x)$ assumes different values in different intervals of length $2l$, i.e. if

   $$f(x) = \begin{cases} \cdots \cdots\cdots & \\ \phi_{-2}(x) & \text{in } (c - 4l, c - 2l) \\ \phi_{-1}(x) & \text{in } (c - 2l, c) \\ \phi_1(x) & \text{in } (c, c + 2l) \\ \phi_2(x) & \text{in } (c + 2l, c + 4l) \\ \phi_3(x) & \text{in } (c + 4l, c + 6l) \\ \cdots \cdots\cdots & \end{cases}$$

   then $f(x)$ is said to be periodic with period $2l$, if

   $$\phi_{-2}(x) = \phi_1(x + 4l), \phi_{-1}(x) = \phi_1(x + 2l),$$
   $$\phi_2(x) = \phi_1(x - 2l), \phi_3(x) = \phi_1(x - 4l), \text{ etc}$$

   In this case, the Fourier series of $f(x)$ of period $2l$ will represent $\phi_1(x)$ in $(c, c + 2l)$, $\phi_2(x)$ in $(c + 2l, c + 4l)$, etc.
   In other words, the Fourier series of $\phi_{-1}(x)$ in $(c - 2l, c)$, that of $\phi_1(x)$ in $(c, c + 2l)$, that of $\phi_2(x)$ in $(c + 2l, c + 4l)$, etc. will be identical.

5. Examples

(a) The Fourier series of $f(x) = \sin^4 x \cdot \cos^3 x$ in $(0, 2\pi)$ or $(-2\pi, 0)$ or in $(2\pi, 4\pi)$ etc. will be $\frac{3}{64}\cos x - \frac{3}{64}\cos 3x - \frac{1}{64}\cos 5x + \frac{1}{64}\cos 7x$.
In other words, the Fourier series $\frac{3}{64}\cos x - \frac{3}{64}\cos 3x - \frac{1}{64}\cos 5x + \frac{1}{64}\cos 7x$ will represent $\sin^4 x \cos^3 x$ in $(-2\pi, 0)$ and in $(2\pi, 4\pi)$ etc., since $\sin^4(2\pi + x)\cdot\cos^3(2\pi + x) = \sin^4 x \cdot \cos^3 x$ for all x and $f(x)$ assumes the same value $\sin^4 x \cdot \cos^3 x$ for all x in $(-\infty, \infty)$.

(b) The Fourier series of $f(x) = x^2$ in $(0, 2l)$ is

$$\frac{4l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} - \frac{4l^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}$$

The same will be the Fourier series of $f(x) = (x + 2l)^2$ in $(-2l, 0)$ and $f(x) = (x - 2l)^2$ in $(2l, 4l)$, etc.
In other words, the above Fourier series represent x^2 in $(0, 2l)$, $(x + 2l)^2$ in $(-2l, 0)$, $(x - 2l)^2$ in $(2l, 4l)$, etc. This is because $(x + 2l)^2$ and $(x - 2l)^2$ are periodic extensions in $(-2l, 0)$ and $(2l, 4l)$ respectively of x^2 in $(0, 2l)$.

(c) The Fourier series of $f(x) = \phi_0(x) = \begin{cases} x + \pi, & \text{in } (-\pi, 0) \\ -x + \pi, & \text{in } (0, \pi) \end{cases}$ is

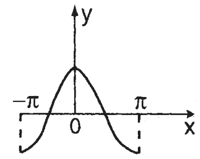
$\frac{\pi}{2} + \frac{4}{\pi}\sum\limits_{n=1,3,\cdots}^{\infty}\frac{1}{n^2}\cos nx$. The same will be the Fourier series of its periodic extensions in $(-3\pi, -\pi)$ and $(\pi, 3\pi)$, etc., i.e. the above Fourier series will represent $f(x) = \phi_{-1}(x) = \begin{cases} x + 3\pi, & \text{in } (-3\pi, -2\pi) \\ -x - \pi, & \text{in } (-2\pi, -\pi) \end{cases}$

and

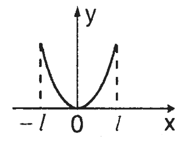
$$f(x) = \phi_1(x) = \begin{cases} x - \pi, & \text{in } (\pi, 2\pi) \\ 3\pi - x, & \text{in } (2\pi, 3\pi) \end{cases}, \text{ etc}$$

2.6 FOURIER SERIES OF EVEN AND ODD FUNCTIONS

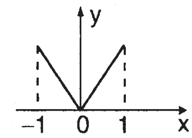
Certain functions defined in symmetric ranges of the form $(-l, l), (-\pi, \pi)$ or $(-\infty, \infty)$ can be classified as even and odd functions. If the graph of $y = f(x)$ in $(-l, l)$ is symmetric about the y-axis, then the function $f(x)$ is said to be an *even function* in $(-l, l)$.



Graph of y = cos x
Fig. 2.4



Graph of y = x^2
Fig. 2.5



Graph of y = | x |
Fig. 2.6

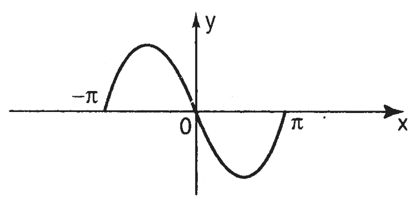
Analytically, an even function can be defined as follows.

If $f(x) = \phi(x)$ in $(-l, l)$ such that $\phi(-x) = \phi(x)$, then $f(x)$ is said to be an even function of x in $(-l, l)$. [Refer to Fig. 2.4 and 2.5]

If $f(x) = \begin{cases} \phi_1(x) & \text{in } (-l, 0) \\ \phi_2(x) & \text{in } (0, l) \end{cases}$

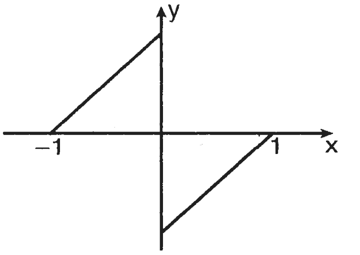
such that $\phi_1(-x) = \phi_2(x)$ or $\phi_2(-x) = \phi_1(x)$, then $f(x)$ is said to be an even function of x in $(-l, l)$. [Refer to Fig. 2.6]

If the graph of $y = f(x)$ in $(-l, l)$ is symmetric about the origin, then the function $f(x)$ is said to be an *odd function* of x in $(-l, l)$.



Graph of y = sin x

Fig. 2.7



Graph of $y = \begin{cases} x+1, \text{ in } (-1, 0) \\ x-1, \text{ in } (0, 1) \end{cases}$

Fig. 28

Analytically, an odd function can be defined as follows.

If $f(x) = \phi(x)$ in $(-l, l)$ such that $\phi(-x) = -\phi(x)$, then $f(x)$ is said to be an odd function of x in $(-l, l)$ [Refer to Fig. 2.7]

If $f(x) = \begin{cases} \phi_1(x) & \text{in } (-l, 0) \\ \phi_2(x) & \text{in } (0, l), \end{cases}$

such that $\phi_1(-x) = -\phi_2(x)$ or $\phi_2(-x) = -\phi_1(x)$, then $f(x)$ is said to be an odd function of x in $(-l, l)$. [Refer to Fig. 2.8].

Note ✍

1. *Functions defined in $(-l, l)$ may be neither even nor odd.*
2. *The question of a function, defined in a non-symmetric range like $(0, 2l)$, being even or odd does not arise at all.*

2.7 THEOREM

(i) The Fourier series of an even function $f(x)$ in $(-l, l)$ contains only cosine terms (constant term included), i.e. the Fourier series of an even function $f(x)$ in $(-l, l)$ is given by $f(x) = \frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{l}$

where $a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$.

(ii) The Fourier series of an odd function $f(x)$ in $(-l, l)$ contains only sine terms, i.e. the Fourier series of an odd function $f(x)$ in $(-l, l)$ is given by

$f(x) = \sum b_n \sin \frac{n\pi x}{l}$, where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$.

Proof

Since $f(x)$ is defined in an interval of length $2l$, it can be expanded as a Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Case (i) $f(x)$ is even in $(-l, l)$.

Since $f(x)$ is even and $\sin \frac{n\pi x}{l}$ is odd in $(-l, l)$, $f(x) \cdot \sin \frac{n\pi x}{l}$ is an odd function of x in $(-l, l)$.

$$\therefore\ b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx$$

$= 0$, by the property of the definite integral of an odd function in a symmetric range.

Since $\cos \frac{n\pi x}{l}$ is even in $(-l, l)$, $f(x) \cos \frac{n\pi x}{l}$ is an even function of x in $(-l, l)$
$\therefore$ By the property of the definite integral of an even function in a symmetric range,

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l} \int_{0}^{l} f(x) \cos \frac{n\pi x}{l}\, dx,\ n \geq 0.$$

Case(ii) $f(x)$ is odd in $(-l, l)$

$\therefore$ $f(x) \cos \frac{n\pi x}{l}$ is an odd function of x and $f(x) \sin \frac{n\pi x}{l}$ is an even function of x in $(-l, l)$

$$\therefore\ a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx = 0 \text{ and}$$

$$b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n\pi x}{l}\, dx,\ n \geq 1,$$

by the properties mentioned above. Hence the results.

2.8 CONVERGENCE OF FOURIER SERIES AT SPECIFIC POINTS

When $f(x)$ is expandable as a Fourier series of the form $\frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{l} + \sum b_n \sin \frac{n\pi x}{l}$ in $(c, c+2l)$, $f(x)$ is either continuous in $(c, c+2l)$ or discontinuous with a finite number of finite discontinuities in $(c, c+2l)$ [by Dirichlet's condition]. In both the cases, we say that the Fourier series represents or converges to $f(x)$ in $(c, c+2l)$. Let us now consider a specific point $x = \alpha$ in $(c, c+2l)$.

(i) If $x = \alpha$ is a point of continuity of $f(x)$ in $(c, c+2l)$, then the Fourier series of $f(x)$ at $x = \alpha$ converges to $f(\alpha)$, since $f(\alpha)$ assumes a unique value.

i.e. $$[\text{the sum of the Fourier series of } f(x)]_{x=\alpha} = f(\alpha) \tag{1}$$

Note ✍

If $x = \alpha$ is a point of discontinuity of $f(x)$ in $(c, c+2l)$, the above result does not hold good, since $f(\alpha)$ is not uniquely defined.]

(ii) If $x = \alpha$ is an interior point of discontinuity of $f(x)$ in $(c, c+2l)$, i.e. $c < \alpha < c+2l$, then the Fourier series of $f(x)$ at $x = \alpha$ converges to $\frac{1}{2} \lim_{h\to 0} [f(\alpha - h) + f(\alpha + h)]$. (Proof assumed),

i.e. $$[\text{Sum of the Fourier series of } f(x)]_{x=\alpha} = \frac{1}{2} \lim_{h\to 0} [f(\alpha - h) + f(\alpha + h)] \tag{2}$$

(iii) If α coincides with the left extremity c of the interval $(c, c+2l)$, $(\alpha + h)$ lies within $(c, c+2l)$, but $(\alpha - h)$ lies within $(c - 2l, c)$. We have already observed that the Fourier series of $f(x)$ in $(c, c+2l)$ represents $f(x)$ in this interval but it represents $f(x+2l)$ in $(c-2l, c)$.

$\therefore$ Formula (ii) gets modified as follows:

$$[\text{Sum of the Fourier series of } f(x)]_{x=\alpha=c}$$
$$= \frac{1}{2} \lim_{h\to 0} [f(\alpha - h + 2l) + f(\alpha + h)] \tag{3}$$

(iv) If α coincides with the right extremity $(c+2l)$ of the interval $(c, c+2l)$, $(\alpha - h)$ lies within $(c, c+2l)$, but $(\alpha + h)$ lies within $(c+2l, c+4l)$. As observed already, the Fourier series of $f(x)$ in $(c, c+2l)$ represents $f(x)$ in this interval, but it represents $f(x-2l)$ in $(c+2l, c+4l)$.

$\therefore$ Formula (ii) gets modified as follows:

$$[\text{Sum of the Fourier series of } f(x)]_{x=\alpha=c+2l}$$
$$= \frac{1}{2} \lim_{h\to 0} [f(\alpha - h) + f(\alpha + h - 2l)] \tag{4}$$

Worked Examples 2(a)

Example 1

Find the Fourier series of period 2l for the function $f(x) = x(2l - x)$ in $(0, 2l)$. Deduce the sum of $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{ in } (0, 2l) \qquad (1)$$

$$a_n = \frac{1}{l}\int_0^{2l} x(2l-x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(2lx - x^2)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (2l - 2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right.$$

$$\left. +(-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}, \text{ using Bernoulli's formula.}$$

$$= \frac{l}{n^2\pi^2}[-2l\cos 2n\pi - 2l] = -\frac{4l^2}{n^2\pi^2}$$

Note ✍

Though Euler's formula for a_0 is a particular case of that for a_n, corresponding to $n = 0$, the value of a_0 cannot be deduced from that of a_n by putting $n = 0$ in this example. In some problems, a_0 can be deduced from a_n. Hence in all problems we shall first find a_n and if possible deduce the value of a_0 from it.

$$a_0 = \frac{1}{l}\int_0^{2l} x(2l-x)\,dx = \frac{1}{l}\left[lx^2 - \frac{x^3}{3}\right]_0^{2l} = \frac{4}{3}l^2.$$

$$b_n = \frac{1}{l}\int_0^{2l} x(2l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(2lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (2l - 2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right.$$

$$\left. +(-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= 0$$

Using these values in (1), we have

$$x(2l-x)=\frac{2}{3}l^2-\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}\text{ in }(0,2l) \qquad (2)$$

The required series $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\infty$ can be obtained by putting $x=l$ in the Fourier series in (2).

$x=l$ lies in $(0,2l)$ and is a point of continuity of the function $f(x)=x(2l-x)$.

$\therefore$ $$[\text{Sum the Fourier series in}(2)]_{x=l}=f(l)$$

i.e. $$\frac{2}{3}l^2-\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi=l(2l-l)$$

i.e. $$-\frac{4l^2}{\pi^2}\left\{-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\cdots\infty\right]=\frac{l^2}{3}$$

$\therefore$ $$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\infty=\frac{\pi^2}{12}$$

Example 2

Find the Fourier series expansion of the function $f(x)=\begin{cases}0, & \text{in } -\pi\le x\le 0\\ \sin x, & \text{in } 0\le x\le\pi\end{cases}$

Hence find the values of

1. $\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\cdots\infty$
2. $\frac{1}{1\cdot3}-\frac{1}{3\cdot5}+\frac{1}{5\cdot7}-\cdots\infty$

Since $f(x)$ is defined in a range of length 2π, it can be expanded as a Fourier series of period 2π.

Let $$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx\text{ in }(-\pi,\pi) \qquad (1)$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\cdot\cos nx\,dx+\int_{0}^{\pi}\sin x\cos nx\,dx\right]$$

$$=\frac{1}{2\pi}\int_{0}^{\pi}[\sin(n+1)x-\sin(n-1)x]\,dx$$

$$=\frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right]_0^{\pi},\text{ if } n\neq1$$

$$= \frac{1}{2\pi}\left[\left\{-\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}\right\}-\left\{-\frac{1}{n+1}+\frac{1}{n-1}\right\}\right],$$

if $n \neq 1$ [$\because \cos n\pi = (-1)^n$]

$$= \frac{1}{2\pi}\left[\left(\frac{1}{n-1}-\frac{1}{n+1}\right)\left\{(-1)^{n-1}-1\right\}\right],$$

if $n \neq 1$ $\left[\because (-1)^{n+1} = (-1)^{n-1}\right]$

$$= \frac{1}{\pi(n^2-1)}\left\{(-1)^{n-1}-1\right\}$$

$$= \begin{cases} -\frac{2}{\pi(n^2-1)}, & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd, but } \neq 1 \end{cases}$$

Putting $n = 0$ in the value of a_n, we get $a_0 = \frac{2}{\pi}$.

$$a_1 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cdot \cos x \, dx + \int_{0}^{\pi} \sin x \cos x \, dx\right], \text{ by Euler's formula.}$$

$$= \frac{1}{2\pi}(\sin^2 x)_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cdot \sin nx \, dx + \int_{0}^{\pi} \sin x \sin nx \, dx\right]$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} [\cos(n-1)x - \cos(n+1)x] \, dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi}, \text{ if } n \neq 1$$

$= 0$, if $n \neq 1$.

$$b_1 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cdot \sin x \, dx + \int_{0}^{\pi} \sin^2 x \, dx\right], \text{ by Euler's formula.}$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}(1 - \cos 2x) \, dx = \frac{1}{2\pi}\left(x - \frac{\sin 2x}{2}\right)_0^{\pi} = \frac{1}{2}$$

Using these values in (1),

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi} \sum_{n=2,4,6\cdots}^{\infty} \frac{1}{(n-1)(n+1)} \cos nx + \frac{1}{2} \sin x \text{ in } (-\pi, \pi) \qquad (2)$$

Putting $x = 0$ in the Fourier series in (2), we get the series $\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \cdots \infty$.

The value of $f(x)$ at $x = 0$ is uniquely found as 0, both from the value of $f(x)$ in $-\pi \le x \le 0$ and from the value of $f(x)$ in $0 \le x \le \pi$.

$\therefore$ $x = 0$ is a point of continuity of $f(x)$.

$\therefore$ [Sum of the Fourier series of $f(x)]_{x=0} = f(0)$.

i.e.
$$\frac{1}{\pi} - \frac{2}{\pi}\left\{\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \cdots\infty\right] + \frac{1}{2} \times 0 = 0.$$

$$\therefore \quad \frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \cdot\infty = \frac{1}{2}.$$

Now putting $x = \frac{\pi}{2}$, which is a point of continuity of $f(x)$, in the Fourier series in (2), we get

$$\frac{1}{\pi} - \frac{2}{\pi}\left\{-\frac{1}{1\cdot3} + \frac{1}{3\cdot5} - \frac{1}{5\cdot7} + \cdots\infty\right\} + \frac{1}{2} = 1$$

i.e.
$$\frac{2}{\pi}\left\{\frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \cdots\infty\right] = \frac{1}{2} - \frac{1}{\pi}$$

$$\therefore \quad \frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \cdots\infty = \frac{\pi-2}{4}$$

Example 3

Find the Fourier series of period 2 for the function

$$f(x) = \begin{cases} k, & \text{in } -1 < x < 0 \\ x, & \text{in } 0 < x < 1 \end{cases}$$

Hence find the sum of

(i) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\infty$

(ii) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\infty$

Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \text{ in } (-1, 1)$$

$$[\because \ 2l = 2 \text{ and hence } l = 1] \qquad (1)$$

$$a_n = \frac{1}{1}\int_0^1 f(x) \cos n\pi x \, dx$$

$$= \int_{-1}^{0} k \cos n\pi x \, dx + \int_{0}^{1} x \cos n\pi x \, dx$$

$$= k \left(\frac{\sin n\pi x}{n\pi} \right)_{-1}^{0} + \left[x \left(\frac{\sin n\pi x}{n\pi} \right) - 1 \left(\frac{-\cos n\pi x}{n^2\pi^2} \right) \right]_0^1$$

$$= \frac{1}{n^2\pi^2} \left\{ (-1)^n - 1 \right\}, \text{ if } n \neq 0$$

$$= \begin{cases} \dfrac{-2}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$a_0 = \frac{1}{1} \left[\int_{-1}^{0} k \, dx + \int_0^1 x \, dx \right] = k(x)_{-1}^{0} + \frac{1}{2}(x^2)_0^1 = k + \frac{1}{2}$$

$$b_n = \frac{1}{1} \int_{-1}^{1} f(x) \sin n\pi x \, dx$$

$$= \int_{-1}^{0} k \sin n\pi x \, dx + \int_{0}^{1} x \sin n\pi x \, dx$$

$$= k \left(-\frac{\cos n\pi x}{n\pi} \right)_{-1}^{0} + \left[x \left(-\frac{\cos n\pi x}{n\pi} \right) - 1 \cdot \left(-\frac{\sin n\pi x}{n^2\pi^2} \right) \right]_0^1$$

$$= -\frac{k}{n\pi} \left\{ 1 - (-1)^n \right\} - \frac{1}{n\pi} (-1)^n$$

Using these values in (1), we have

$$f(x) = \left(\frac{k}{2} + \frac{1}{4} \right) - \frac{2}{\pi^2} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^2} \cos n\pi x - \frac{1}{\pi} \sum \left[k \left\{ 1 - (-1)^n \right\} \right.$$

$$\left. + (-1)^n \right] \frac{1}{n} \sin n\pi x \text{ in } (-1, 1) \qquad (2)$$

By putting $x = 0$ in (2), we get the series $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty$

$\therefore$ We require the sum of the Fourier series of $f(x)$ at $x = 0$.

Since $f(0-) = k$ and $f(0+) = 0$, as per the definition of $f(x)$.

$\therefore \quad x = 0$ is a point of discontinuity of $f(x)$.

$$\therefore \quad [\text{Sum of the Fourier series of } f(x)]_{x=0} = \frac{1}{2} \lim_{h \to 0} [f(0-h) + f(0+h)]$$

$$= \frac{1}{2} \lim_{h \to 0} [k + h] = \frac{k}{2}$$

i.e. $$\frac{k}{2}+\frac{1}{4}-\frac{2}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty\right)=\frac{k}{2}$$

$\therefore$ $$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty=\frac{\pi^2}{8}$$

By putting $x=\frac{1}{2}$ in (2), we get the series $1-\frac{1}{3}+\frac{1}{5}+\cdots$
$x=\frac{1}{2}$ is a point of continuity for $f(x)$.

$\therefore$ $$[\text{Sum of the Fourier series of } f(x)]_{x=\frac{1}{2}}=f\left(\frac{1}{2}\right)$$

i.e. $$\left(\frac{k}{2}+\frac{1}{4}\right)-\frac{1}{\pi}(2k-1)\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}=\frac{1}{2}$$

i.e. $$\frac{(2k-1)}{\pi}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\infty\right)=\frac{k}{2}-\frac{1}{4}\text{ or }\frac{(2k-1)}{4}$$

$\therefore$ $$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\infty=\frac{\pi}{4}$$

Example 4

Find the Fourier series of $f(x)=x^2$ in $(0, 2l)$.
Hence deduce that

(i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\infty=\frac{\pi^2}{6}$
(ii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\infty=\frac{\pi^2}{12}$
(iii) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty=\frac{\pi^2}{8}$

Since $f(x)$ is defined in a range of length $2l$, it can be expanded as a Fourier series of period $2l$.

Let $$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\text{ in }(0,2l) \qquad (1)$$

$$a_n=\frac{1}{l}\int_0^{2l}x^2\cos\frac{n\pi x}{l}\,dx$$

$$=\frac{1}{l}\left[x^2\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-2x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)+2\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$=\frac{4l^2}{n^2\pi^2},\text{ if } n\neq 0.$$

$$a_0=\frac{1}{l}\int_0^{2l}x^2\,dx=\frac{1}{l}\left(\frac{x^3}{3}\right)_0^{2l}=\frac{8l^2}{3}$$

$$b_n = \frac{1}{l}\int_0^{2l} x^2 \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[x^2\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2x\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= -\frac{4l^2}{n\pi}$$

Using these values in (1), we have

$$x^2 = \frac{4l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right) - \frac{4l^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{l}\right) \text{ in } (0, 2l) \tag{2}$$

Putting $x = 0$ in the R.H.S. of (2), we get the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\infty$.

$x = 0$ is the left extremity of the range $(0, 2l)$. Since the Fourier series in (2) represents x^2 in $(0, 2l)$ and $(x + 2l)^2$ in $(-2l, 0)$, $x = 0$ is a point of discontinuity.

$\therefore$ [Sum of the Fourier series of $f(x)]_{x=0} = \frac{1}{2}\lim_{h\to 0}[f(0-h) + f(0+h)]$

$$= \frac{1}{2}\lim_{h\to 0}\left[(-h+2l)^2 + h^2\right] [\because\ x = -h \text{ lies in} (-2l, 0) \text{ and } x = h \text{ lies in } (0, 2l)]$$

$$= 2l^2$$

i.e. $$\frac{4l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = 2l^2$$

$\therefore$ $$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\infty = \frac{\pi^2}{6} \tag{3}$$

Putting $x = l$ in the R.H.S. of (2), we get the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\infty$

$x = l$ is a point of continuity of the function $f(x) = x^2$.

$\therefore$ [Sum of the Fourier series of $f(x)]_{x=l} = f(l)$

i.e. $$\frac{4l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n = l^2$$

i.e. $$\frac{4l^2}{\pi^2}\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\infty\right) = -\frac{l^2}{3}$$

$\therefore$ $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\infty = \frac{\pi^2}{12} \tag{4}$$

Adding (3) and (4), we get

$$2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty\right)=\frac{\pi^2}{6}+\frac{\pi^2}{12}=\frac{\pi^2}{4}$$

$$\therefore \qquad \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty=\frac{\pi^2}{8}$$

Example 5

Find the Fourier series expansion of $f(x)=x^2+x$ in $(-2,2)$. Hence find the sum of the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\infty$.

Since $f(x)$ is defined in a range of length 4, it can be expanded as a Fourier series of period 4.

$$\text{Let}\quad f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{2}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{2}[\text{since } 2l=4] \tag{1}$$

$$a_n=\frac{1}{2}\int_{-2}^{2}(x^2+x)\cos\frac{n\pi x}{2}\,dx$$

$$=\frac{1}{2}\int_{-2}^{2}x^2\cos\frac{n\pi x}{2}\,dx+\frac{1}{2}\int_{-2}^{2}x\cos\frac{n\pi x}{2}\,dx$$

$$=\int_{0}^{2}x^2\cos\frac{n\pi x}{2}\,dx+0,\left[\because\ x^2\cos\frac{n\pi x}{2}\text{ is an even function and } x\cos\frac{n\pi x}{2}\text{ is an odd function of } x\right]$$

$$=\left[x^2\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)-2x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)+2\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}\right)\right]_0^2$$

$$=\frac{16}{n^2\pi^2}(-1)^n,\ \text{if } n\neq 0.$$

$$a_0=\frac{1}{2}\int_{-2}^{2}(x^2+x)\,dx=\int_{0}^{2}x^2\,dx=\frac{8}{3}$$

$$\left(\because\ x^2\text{ is an even function and } x \text{ is an odd function of } x\right)$$

$$b_n = \frac{1}{2}\int_{-2}^{2}(x^2+x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{2}x^2\sin\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{-2}^{2}x\sin\frac{n\pi x}{2}\,dx$$

$$= \int_{0}^{2}x\sin\frac{n\pi x}{2}\,dx$$

$$\left[\because x^2\sin\frac{n\pi x}{2} \text{ is an odd function and } x\sin\frac{n\pi x}{2} \text{ is an even function of } x \text{ in } (-2,2)\right]$$

$$= \left[x\cdot\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\cdot\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^2$$

$$= -\frac{4}{n\pi}(-1)^n$$

Using these values in (1), we have

$$x^2+x = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2} \text{ in } (-2,2) \quad (2)$$

Putting $x = -2$ or 2 in the R.H.S. of (2), we get the required series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty$.

Let us consider $x = 2$, which is the right extremity of the range $(-2, 2)$.

The Fourier series of $f(x)$ represents $f(x)$ in $(-2, 2)$ and $f(x-4)$ in the next period $(2, 6)$, i.e. The Fourier series in the R.H.S. of (2) represents $x^2 + x$ in $(-2, 2)$ and $\left\{(x-4)^2 + (x-4)\right\}$ in $(2, 6)$.

Evidently $x = 2$ is a point of discontinuity of $f(x)$.

$$\therefore \text{ [Sum of the Fourier series of } f(x)]_{x=2} = \frac{1}{2}\lim_{h\to 0}\Big[\{(2-h)^2+(2-h)\} + \{(2+h-4)^2+(2+h-4)\}\Big] = 4$$

i.e.
$$\frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cdot(-1)^n = 4$$

i.e.
$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\infty = \frac{\pi^2}{6}.$$

Example 6

Find the Fourier series expansion of $f(x) = x(1-x)(2-x)$ in $(0, 2)$. Deduce the sum of the series $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots\infty$.

Since the function $f(x)$ (1) is defined in a range of length 2, it can be expanded as a Fourier series of period 2.

$\therefore$ Let $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$ [since $2l = 2$]

$$a_n = \frac{1}{1}\int_0^2 x(1-x)(2-x)\cos n\pi x \, dx$$

$$= \left[(2x - 3x^2 + x^3)\left(\frac{\sin n\pi x}{n\pi}\right) - (2 - 6x + 3x^2)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right.$$

$$\left. +(-6+6x)\left(\frac{-\sin n\pi x}{n^3\pi^3}\right) - 6\cdot\left(\frac{\cos n\pi x}{n^4\pi^4}\right)\right]_0^2$$

$$= 0, \text{ if } n \neq 0.$$

$$a_0 = \frac{1}{1}\int_0^2 (2x - 3x^2 + x^3)\, dx = \left(x^2 - x^3 + \frac{x^4}{4}\right)_0^2 = 0.$$

$$b_n = \frac{1}{1}\int_0^2 (2x - 3x^2 + x^3)\sin n\pi x dx$$

$$= \left[(2x - 3x^2 + x^3)\left(\frac{-\cos n\pi x}{n\pi}\right) - (2 - 6x + 3x^2)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right.$$

$$\left. +(-6+6x)\left(\frac{\cos n\pi x}{n^3\pi^3}\right) - 6\cdot\left(\frac{\sin n\pi x}{n^4\pi^4}\right)\right]_0^2$$

$$= \frac{12}{n^3\pi^3}$$

Using these values in (1), we have

$$x(1-x)(2-x) = \frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin n\pi x \tag{2}$$

Putting $x = \frac{1}{2}$ in the R.H.S. of (2), we get the series $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots \infty$.
$x = \frac{1}{2}$ is a point of continuity of $f(x)$.

$\therefore$ $\qquad$ [Sum of the Fourier series of $f(x)]_{x=\frac{1}{2}} = f\left(\frac{1}{2}\right)$

i.e. $$\frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}$$

i.e. $$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}.$$

Example 7

Find the Fourier series of period 2π for the function $f(x) = x\cos x$ in $0 < x < 2\pi$.

Let $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx + \sum\limits_{n=1}^{\infty} b_n \sin nx$ (1)

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos x \cos nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n+1)x + \cos(n-1)x\right] dx$$

$$= \frac{1}{2\pi}\left[\left\{x \cdot \frac{\sin(n+1)x}{n+1} + \frac{\cos(n+1)x}{(n+1)^2}\right\}_0^{2\pi} + \left\{x \cdot \frac{\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2}\right\}_0^{2\pi}\right], \text{ if } n \neq 1$$

$$= 0, \text{ if } n \neq 1$$

$$a_0 = 0$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\cos^2 x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x(1+\cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} + x\frac{\sin 2x}{2} + \frac{\cos 2x}{4}\right]_0^{2\pi} = \pi.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\cos x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x + \sin(n-1)x\right] dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1}\right\} + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{2\pi}$$

$$+ \frac{1}{2\pi}\left[x\left\{\frac{-\cos(n-1)x}{n-1}\right\} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}, \text{ if } n \neq 1$$

$$= -\frac{1}{n+1} - \frac{1}{n-1} = -\left\{\frac{1}{n+1} + \frac{1}{n-1}\right\} = -\frac{2n}{n^2-1}, \text{ if } n \neq 1.$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\cos x \sin x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x \, dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) + \frac{\sin 2x}{4}\right]_0^{2\pi} = -\frac{1}{2}$$

Using these values in (1), we get

$$f(x) = \pi\cos x - \frac{1}{2}\sin x - 2\sum_{n=2,3,\cdots}^{\infty} \frac{n}{n^2-1}\sin nx$$

Example 8

Find the Fourier series of period 2π for the function $f(x) = \sqrt{1-\cos x}$ in $-\pi < x < \pi$.

$$f(-x) = \sqrt{1-\cos(-x)} = \sqrt{1-\cos x} = f(x)$$

$\therefore$ $f(x) = \sqrt{1-\cos x}$ is an even function of x in $-\pi < x < \pi$.

Note

Since $\sqrt{1-\cos x} = \pm\sqrt{2}\sin\frac{x}{2}$, we should not conclude that $\sqrt{1-\cos x}$ is an odd function of x in $-\pi < x < \pi$. If we note the values of $\sqrt{1-\cos x}$ and $\sqrt{2}\sin\frac{x}{2}$, we can find that

$$\sqrt{1-\cos x} = \begin{cases} -\sqrt{2}\sin\frac{x}{2}, & \text{in } (-\pi, 0) \\ \sqrt{2}\sin\frac{x}{2}, & \text{in } (0, \pi) \end{cases} \qquad (1)$$

From (1) also, it is evident that $\sqrt{1-\cos x}$ is an even function of x in $(-\pi, \pi)$.

$\therefore$ Fourier series of $f(x)$ will not contain sine terms.

Let $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ in $(-\pi, \pi)$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sqrt{1-\cos x}\cos nx \, dx$$

$$= \frac{2\sqrt{2}}{\pi}\int_0^{\pi} \sin\frac{x}{2}\cos nx \, dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi}\left[\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x\right] dx$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-\cos\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}} + \frac{\cos\left(n-\frac{1}{2}\right)x}{n-\frac{1}{2}}\right]_0^{\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1}{n+\frac{1}{2}} - \frac{1}{n-\frac{1}{2}}\right]$$

$$\left[\because \cos\left(n \pm \frac{1}{2}\right)\pi = \cos n\pi \cos\frac{\pi}{2} \mp \sin n\pi \sin\frac{\pi}{2} = 0\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right)$$

$$= \frac{-4\sqrt{2}}{\pi(4n^2-1)} \qquad (2)$$

$$\text{and } a_0 = -\frac{4\sqrt{2}}{(-\pi)} \qquad [\text{by putting } n = 0 \text{ in } (2)]$$

$$= \frac{4\sqrt{2}}{\pi}$$

Using these values in (1), we have

$$\sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx \text{ in } (-\pi, \pi)$$

Example 9

Find the Fourier series of period 2π for the function $f(x) = |\cos x|$ in $-\pi \leq x \leq \pi$

$f(-x) = |\cos(-x)| = |\cos x| = f(x)$

$\therefore f(x)$ is an even function of x in $-\pi \leq x \leq \pi$.

$\therefore$ Fourier series of $f(x)$ will not contain sine terms.

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ in } -\pi \leq x \leq \pi \qquad (1)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x \cos nx \, dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)\cos nx \, dx\right]$$

$$\left[\because \cos x > 0 \text{ in } \left(0, \frac{\pi}{2}\right) \text{ and } < 0 \text{ in } \left(\frac{\pi}{2}, \pi\right) \text{ and } |\cos x| \text{ is positive}\right]$$

$$= \frac{1}{\pi}\left\{\int_0^{\frac{\pi}{2}} [\cos(n+1)x + \cos(n-1)x] \, dx \right.$$

$$\left. - \int_{\frac{\pi}{2}}^{\pi} [\cos(n+1)x + \cos(n-1)x] \, dx\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_0^{\frac{\pi}{2}}$$

$$- \frac{1}{\pi}\left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_{\frac{\pi}{2}}^{\pi}, \text{ if } n \neq 1$$

$$= \frac{1}{\pi}\left\{\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right\} + \frac{1}{\pi}\left\{\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right\}$$

$$= \frac{2}{\pi}\left\{\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\cos\frac{n\pi}{2}\right\},$$

$$\left[\text{Since } \sin(n \pm 1)\frac{\pi}{2} = \sin\frac{n\pi}{2}\cos\frac{\pi}{2} \pm \cos\frac{n\pi}{2}\cdot\sin\frac{\pi}{2} = \pm\cos\frac{n\pi}{2}\right]$$

$$= -\frac{4}{\pi(n^2-1)}\cos\frac{n\pi}{2}, \text{ if } n \neq 1$$

$$a_0 = \frac{4}{\pi}; \quad a_1 = \frac{2}{\pi}\int_0^{\pi/2} |\cos x| \cos x \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos^2 x \, dx - \int_{\frac{\pi}{2}}^{\pi} \cos^2 x \, dx\right]$$

$$= \frac{2}{\pi}\cdot\frac{1}{2}\left[\int_0^{\frac{\pi}{2}} (1+\cos 2x)\, dx - \int_{\frac{\pi}{2}}^{\pi} (1+\cos 2x)\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(x + \frac{\sin 2x}{2}\right)_0^{\frac{\pi}{2}} - \left(x + \frac{\sin 2x}{2}\right)_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \left(\pi - \frac{\pi}{2}\right)\right] = 0.$$

Using these values in (1), we get

$$|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2-1}\cos\frac{n\pi}{2}\cos nx$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2,4,6\cdots}^{\infty} \frac{1}{n^2-1}\cos\frac{n\pi}{2}\cos nx$$

$$\left[\because \cos\frac{n\pi}{2} = 0 \text{ when n is odd}\right]$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos n\pi \cos 2nx$$

$$= \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{4n^2 - 1} \cos 2nx \text{ in } (-\pi \ \pi)$$

Example 10

Find the Fourier series expansion of $f(x) = \sin ax$ in $(-l, l)$.

Since $f(x)$ is defined in a range of length $2l$, we can expand $f(x)$ in Fourier series of period $2l$.

Also $f(-x) = \sin[a(-x)] = -\sin ax = -f(x)$

$\therefore$ $f(x)$ is an odd function of x in $(-l, l)$.

Hence Fourier series of $f(x)$ will not contain cosine terms.

Let $$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l} \int_0^l \sin ax \cdot \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{1}{l} \int_0^l \left[\cos\left(\frac{n\pi}{l} - a\right) x - \cos\left(\frac{n\pi}{l} + a\right) x\right] dx$$

$$= \frac{1}{l} \left[\frac{\sin\left(\frac{n\pi}{l} - a\right) x}{\frac{n\pi}{l} - a} - \frac{\sin\left(\frac{n\pi}{l} + a\right) x}{\frac{n\pi}{l} + a}\right]_0^l$$

$$= \frac{1}{n\pi - la} \sin\left(\frac{n\pi}{l} - a\right) l - \frac{1}{n\pi + la} \sin\left(\frac{n\pi}{l} + a\right) l$$

$$= \frac{1}{n\pi - la} \sin(n\pi - al) - \frac{1}{n\pi + la} \sin(n\pi + al)$$

$$= \frac{1}{n\pi - al} \left\{-(-1)^n \sin al\right\} - \frac{1}{n\pi + al} \left\{(-1)^n \sin al\right\}$$

$$= (-1)^{n+1} \sin al \left\{\frac{1}{n\pi - al} + \frac{1}{n\pi + al}\right\}$$

$$= \frac{(-1)^{n+1} 2n\pi \sin al}{n^2\pi^2 - a^2 l^2}$$

Using this value in (1), we get

$$\sin ax = 2\pi \sin al \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2\pi^2 - a^2 l^2} \sin \frac{n\pi x}{l}$$

Example 11

Find the Fourier series expansion of $f(x) = e^{-x}$ in $(-\pi, \pi)$. Hence obtain a series for cosec π.

Though the range $(-\pi, \pi)$ is symmetric about the origin, e^{-x} is neither an even function nor an odd function.

$\therefore$ Let
$$f(x) = \tfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \tag{1}$$
in $(-\pi, \pi)$ [$\because$ the length of the range is 2π]

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\cos nx \,dx \\ &= \frac{1}{\pi}\left\{\frac{e^{-x}}{n^2+1}(-\cos nx + n\sin nx)\right\}_{-\pi}^{\pi} \\ &= -\frac{1}{\pi(n^2+1)}\left\{e^{-\pi}(-1)^n - e^{\pi}(-1)^n\right\} \\ &= \frac{2(-1)^n}{\pi(n^2+1)}\sinh \pi \end{aligned}$$

and
$$a_0 = \frac{2\sinh \pi}{\pi}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\sin nx \,dx \\ &= \frac{1}{\pi}\left\{\frac{e^{-x}}{n^2+1}(-\sin nx - n\cos nx)\right\}_{-\pi}^{\pi} \\ &= -\frac{n}{\pi(n^2+1)}\left\{e^{-\pi}(-1)^n - e^{\pi}(-1)^n\right\} \\ &= \frac{2n(-1)^n}{\pi(n^2+1)}\sinh \pi \end{aligned}$$

Using these values in (1), we get

$$e^{-x} = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos nx + \frac{2\sinh \pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}\sin nx \text{ in } (-\pi, \pi) \tag{1}$$

[Sum of the Fourier series of $f(x)]_{x=0} = f(0)$,

[Since $x = 0$ is a point of continuity of $f(x)$]

i.e. $$\frac{\sinh\pi}{\pi}\left[1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\right]=e^{-0}=1$$

i.e. $$\pi \operatorname{cosech}\pi=1+2\times\left(\frac{-1}{2}\right)+2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

i.e. $$\operatorname{cosech}\pi=\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Example 12

Find the Fourier series of period 2π for the function $f(x)=\sinh\alpha x$ in $(-\pi,\pi)$.

$$f(-x)=\sinh(-\alpha x)=-\sinh\alpha x=-f(x) \text{ in } (-\pi,\pi)$$

$\therefore$ $\sinh\alpha x$ is an odd function of x in $(-\pi,\pi)$.
$\therefore$ Fourier series of $\sinh\alpha x$ in $(-\pi,\pi)$ will not contain the constant term and the cosine terms.

Let $$f(x)=\sum_{n=1}^{\infty}b_n\sin nx \text{ in } (-\pi,\pi) \qquad (1)$$

$$b_n=\frac{2}{\pi}\int_0^{\pi}\sinh\alpha x\sin nx\,dx$$
$$=\frac{1}{\pi}\int_0^{\pi}\left(e^{\alpha x}-e^{-\alpha x}\right)\sin nx\,dx$$
$$=\frac{1}{\pi}\left[\left\{\frac{e^{\alpha x}}{n^2+\alpha^2}(\alpha\sin nx-n\cos nx)\right\}_0^{\pi}-\left\{\frac{e^{-\alpha x}}{n^2+\alpha^2}(-\alpha\sin nx-n\cos nx)\right\}_0^{\pi}\right]$$
$$=\frac{1}{\pi(n^2+\alpha^2)}\left[-n(-1)^ne^{\alpha\pi}+n+ne^{-\alpha\pi}(-1)^n-n\right]$$
$$=\frac{-n(-1)^n}{\pi(n^2+\alpha^2)}\left(e^{\alpha\pi}-e^{-\alpha\pi}\right)=\frac{2n(-1)^{n-1}\sinh\alpha\pi}{\pi(n^2+\alpha^2)}$$

Using this value of b_n in (1), we get

$$\sinh\alpha x=\frac{2\sinh\alpha\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{n^2+\alpha^2}\sin nx$$

Example 13

Find the Fourier series expansion of period 2 for the function

$$f(x)=\begin{cases}\pi x, & \text{in } 0\le x\le 1\\ \pi(2-x), & \text{in } 1\le x\le 2\end{cases}$$

Deduce the sum of $\sum\limits_{n=1,3,\cdots}^{\infty}\frac{1}{n^2}$.

Let
$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\pi x+\sum_{n=1}^{\infty}b_n\sin n\pi x \qquad (1)$$

$$a_n=\frac{1}{1}\int_0^2 f(x)\cos n\pi x\,\mathrm{d}x \quad [\because\ 2l=2 \text{ or } l=1]$$

$$=\int_0^1 \pi x\cos n\pi x\,\mathrm{d}x+\int_1^2 \pi(2-x)\cos n\pi x\,\mathrm{d}x$$

$$=\pi\left[x\left(\frac{\sin n\pi x}{n\pi}\right)-1\cdot\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$
$$+\pi\left[(2-x)\left(\frac{\sin n\pi x}{n\pi}\right)+1\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$=\frac{1}{n^2\pi}\left\{(-1)^n-1\right\}+\frac{1}{n^2\pi}\left\{(-1)^n-1\right\},\ \text{if } n\neq 0$$

$$=\begin{cases}0, & \text{if } n \text{ is even, } \neq 0\\ -\frac{4}{n^2\pi}, & \text{if } n \text{ is odd}\end{cases}$$

$$a_0=\frac{1}{1}\int_0^2 f(x)\,\mathrm{d}x=\int_0^1 \pi x\,\mathrm{d}x+\int_1^2 \pi(2-x)\,\mathrm{d}x$$

$$=\pi\left[\frac{x^2}{2}\right]_0^1+\pi\left[\frac{(2-x)^2}{-2}\right]_1^2$$

$$=\frac{\pi}{2}+\frac{\pi}{2}=\pi$$

$$b_n=\frac{1}{1}\int_0^2 f(x)\sin n\pi x\,\mathrm{d}x$$

$$=\int_0^1 \pi x\sin n\pi x\,\mathrm{d}x+\int_1^2 \pi(2-x)\sin n\pi x\,\mathrm{d}x$$

$$= \pi\left[x\left(\frac{-\cos n\pi x}{n\pi}\right) - 1\cdot\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$+\pi\left[(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) + 1\cdot\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= -\frac{1}{n}(-1)^n + \frac{1}{n}(-1)^n = 0$$

Using these values in (1), we get

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,\cdots}^{\infty} \frac{1}{n^2}\cos n\pi x \text{ in } (0 \le x \le 2) \qquad (2)$$

$x = 1$ is a point of continuity of $f(x)$.

$\therefore$ $\quad$ [Sum of Fourier series of $f(x)]_{x=1} = f(1)$

i.e. $$\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\infty\right) = \pi$$

i.e. $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\infty = \frac{\pi^2}{8}$$

Example 14

Find the Fourier series of period $\frac{\pi}{2}$ for the function $f(x) = \begin{cases}\sin x, & \text{in } 0 \le x \le \frac{\pi}{4}\\ \cos x, & \text{in } \frac{\pi}{4} \le x \le \frac{\pi}{2}\end{cases}$

Here $2l = \frac{\pi}{2}$ $\quad \therefore\ l = \frac{\pi}{4}$

Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 4nx + \sum_{n=1}^{\infty} b_n \sin 4nx \text{ in } (0, \frac{\pi}{2}) \qquad (1)$$

$$a_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(x)\cos 4nx\, dx$$

$$= \frac{4}{\pi}\left[\int_0^{\frac{\pi}{4}} \sin x\cos 4nx\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\cos 4nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{4}} \{\sin(4n+1)x - \sin(4n-1)x\}\, dx\right.$$

$$\left.+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \{\cos(4n+1)x + \cos(4n-1)x\}\, dx\right]$$

$$= \frac{2}{\pi}\left[\left[\left\{\frac{-\cos(4n-1)x}{4n+1}+\frac{\cos(4n-1)x}{4n-1}\right\}\right]^{\frac{\pi}{4}}\right.$$

$$\left.+\left\{\frac{\sin(4n+1)x}{4n+1}+\frac{\sin(4n-1)x}{4n-1}\right\}_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos\left(n\pi+\frac{\pi}{4}\right)}{4n+1}+\frac{\cos\left(n\pi-\frac{\pi}{4}\right)}{4n-1}+\frac{1}{4n+1}-\frac{1}{4n-1}+\frac{\sin\left(2n\pi+\frac{\pi}{2}\right)}{4n+1}\right.$$

$$\left.+\frac{\sin\left(2n\pi-\frac{\pi}{2}\right)}{4n-1}-\frac{\sin\left(n\pi+\frac{\pi}{4}\right)}{4n+1}-\frac{\sin\left(n\pi-\frac{\pi}{4}\right)}{4n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{-(-1)^n}{(4n+1)\sqrt{2}}+\frac{(-1)^n}{(4n-1)\sqrt{2}}+\frac{1}{4n+1}-\frac{1}{4n-1}+\frac{1}{4n+1}-\frac{1}{4n-1}\right.$$

$$\left.-\frac{(-1)^n}{(4n+1)\sqrt{2}}+\frac{(-1)^n}{(4n-1)\sqrt{2}}\right]$$

$$= \frac{4}{\pi}\left[\frac{(-1)^n}{\sqrt{2}}\left(\frac{1}{4n-1}-\frac{1}{4n+1}\right)+\left(\frac{1}{4n+1}-\frac{1}{4n-1}\right)\right]$$

$$= \frac{8}{\pi\left(16n^2-1\right)}\left[\frac{(-1)^n}{\sqrt{2}}-1\right]$$

$$a_0 = \frac{8}{\pi}\left\{1-\frac{1}{\sqrt{2}}\right\}$$

$$b_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(x)\sin 4nx \, dx$$

$$= \frac{4}{\pi}\left[\int_0^{\frac{\pi}{4}} \sin x \sin 4nx \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \sin 4nx \, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{4}} \{\cos(4n-1)x - \cos(4n+1)\}\, dx\right.$$

$$\left.+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \{\sin(4n+1)x + \sin(4n-1)x\}\, dx\right]$$

$$= \frac{2}{\pi}\left[\left\{\frac{\sin(4n-1)x}{4n-1}-\frac{\sin(4n+1)x}{4n+1}\right\}_0^{\frac{\pi}{4}}\right.$$

$$\left.-\left\{\frac{\cos(4n+1)x}{4n+1}+\frac{\cos(4n-1)x}{4n-1}\right\}_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin\left(n\pi - \frac{\pi}{4}\right)}{4n-1} - \frac{\sin\left(n\pi + \frac{\pi}{4}\right)}{4n+1} - \frac{\cos\left(2n\pi + \frac{\pi}{2}\right)}{4n+1} - \frac{\cos\left(2n\pi - \frac{\pi}{2}\right)}{4n-1}\right.$$

$$\left. + \frac{\cos\left(n\pi + \frac{\pi}{4}\right)}{4n+1} + \frac{\cos\left(n\pi - \frac{\pi}{4}\right)}{4n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{-(-1)^n}{(4n-1)\sqrt{2}} - \frac{(-1)^n}{(4n+1)\sqrt{2}} + \frac{(-1)^n}{(4n+1)\sqrt{2}} + \frac{(-1)^n}{(4n-1)\sqrt{2}}\right]$$

$$= 0.$$

Using these values in (1), we get

$$f(x) = \frac{4}{\pi}\left(1 - \frac{1}{\sqrt{2}}\right) + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{16n^2-1}\left\{\frac{(-1)^n}{\sqrt{2}} - 1\right\}\cos 4nx, \text{ in } \left(0, \frac{\pi}{2}\right)$$

Example 15

Find the Fourier series expansion of $f(x)$ given by $f(x) = \begin{cases} x, & \text{in } 0 < x < 2 \\ 0, & \text{in } 2 < x < 4 \end{cases}$

Since $f(x)$ is defined in a range of length 4, we can expand it as a Fourier series of period 4.

i.e. $2l = 4$

$\therefore \quad l = 2$

Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \text{ in } (0, 4) \tag{1}$$

$$a_n = \frac{1}{2}\int_0^4 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[x\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) + \left(\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^2, \text{ if } n \neq 0$$

$$= \frac{2}{n^2\pi^2}\left\{(-1)^n - 1\right\}$$

$$= \begin{cases} -\frac{4}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even and } \neq 0 \end{cases}$$

$$a_0 = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{4}(x^2)_0^2 = 1$$

$$b_n = \frac{1}{2}\int_0^2 x \sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) + \left(\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^2$$

$$= -\frac{2}{n\pi}(-1)^n$$

Using these values in (1), we get

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2} \text{ in } (0,4)$$

Example 16

Find the Fourier series expansion of $f(x)$ given that $f(x) = \begin{cases} 1, & \text{for } 0 < x < 1 \\ 2, & \text{for } 1 < x < 3 \end{cases}$

Since the function is defined in a range of length 3, it can be expanded as a Fourier series period 3.
i.e. $2l = 3$

$$\therefore \quad l = \frac{3}{2}$$

Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{3} \text{ in } (0,3) \tag{1}$$

$$a_n = \frac{2}{3}\int_0^3 f(x)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[\int_0^1 1\cdot\cos\frac{2n\pi x}{3}\,dx + \int_1^3 2\cos\frac{2n\pi x}{3}\,dx\right]$$

$$= \frac{2}{3}\left[\frac{3}{2n\pi}\left(\sin\frac{2n\pi x}{3}\right)_0^1 + 2\cdot\frac{3}{2n\pi}\left(\sin\frac{2n\pi x}{3}\right)_1^3\right]$$

$$= \frac{1}{n\pi}\left\{\sin\frac{2n\pi}{3} - 2\sin\frac{2n\pi}{3}\right\} = -\frac{1}{n\pi}\sin\frac{2n\pi}{3},\ n \neq 0.$$

$$a_0 = \frac{2}{3}\left[\int_0^1 1\,dx + \int_1^3 2\,dx\right]$$

$$= \frac{2}{3}[1+4] = \frac{10}{3}$$

$$b_n = \frac{2}{3}\int_0^3 f(x)\sin\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[\int_0^1 1\cdot\sin\frac{2n\pi x}{3}\,dx + \int_1^3 2\cdot\sin\frac{2n\pi x}{3}\,dx\right]$$

$$\frac{2}{3}\left[\left(\frac{-\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right)_0^1 - 2\left(\frac{\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right)_1^3\right]$$

$$= -\frac{1}{n\pi}\left[\left\{\cos\frac{2n\pi}{3} - 1\right\} + 2\left\{1 - \cos\frac{2n\pi}{3}\right\}\right]$$

$$= -\frac{1}{n\pi}\left(1 - \cos\frac{2n\pi}{3}\right)$$

Using these values in (1), we have

$$f(x) = \frac{5}{3} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi}{3}\cos\frac{2n\pi x}{3} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\cos\frac{2n\pi}{3}\right)\sin\frac{2n\pi x}{3} \text{ in } (0, 3)$$

Example 17

Find the Fourier series expansion of period 2π for the function

$$f(x) = \begin{cases} x(\pi - x), & \text{in } -\pi \le x \le 0 \\ x(\pi + x), & \text{in } 0 \le x \le \pi \end{cases}$$

Since the range $(-\pi, \pi)$ is symmetrically divided into two subranges and $f(x)$ assumes the values $\phi_1(x) = x(\pi - x)$ in $(-\pi, 0)$ and $\phi_2(x) = x(\pi + x)$ in $(0, \pi)$, the function $f(x)$ may be odd or even. Let us first test for the oddness or evenness of $f(x)$.

$$\phi_1(-x) = -x(\pi + x)$$
$$= -\phi_2(x)$$

$\therefore$ $f(x)$ is an odd function in $(-\pi, \pi)$.

$\therefore$ The Fourier series of $f(x)$ will contain only sine terms.

Let $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \text{ in } (-\pi, \pi) \qquad (1)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \ \mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^{\pi} x(\pi + x)\sin nx \ \mathrm{d}x$$

$$= \frac{2}{\pi}\left[(\pi x + x^2)\left(\frac{-\cos nx}{n}\right) - (\pi + 2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{-2}{\pi n}\cdot 2\pi^2(-1)^n + \frac{2\cdot 2}{\pi\cdot n^3}\{(-1)^n - 1\}$$

$$= -\frac{4\pi}{n}(-1)^n + \frac{4}{\pi n^3}\{(-1)^n - 1\}$$

Using this value in (1), we get

$$f(x) = \sum_{n=1}^{\infty}\left[-\frac{4\pi}{n}(-1)^n + \frac{4}{\pi n^3}\{(-1)^n - 1\}\right]\sin nx \text{ in } (-\pi, \pi)$$

Example 18

Obtain the Fourier series for the function given by $f(x) = \begin{cases} 1+\frac{2x}{l}, & \text{in} -l \le x \le 0 \\ 1-\frac{2x}{l}, & \text{in} \quad 0 \le x \le l \end{cases}$.

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\infty = \frac{\pi^2}{8}$:

The range is symmetrically divided into two subranges and

$$f(x) = \phi_1(x) = 1 + \frac{2x}{l} \text{ in } -l \le x \le 0$$

$$= \phi_2(x) = 1 - \frac{2x}{l} \text{ in } 0 \le x \le l$$

$$\phi_1(-x) = 1 - \frac{2x}{l} = \phi_2(x)$$

$\therefore$ $f(x)$ is an even function of x in $(-l, l)$. $\therefore$ The Fourier series will not contain sine terms and will be of period $2l$.

Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \text{ in } (-l, l)$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l} \ \mathrm{d}x$$

$$= \frac{2}{l}\int_0^{l}\left(1 - \frac{2x}{l}\right)\cos\frac{n\pi x}{l} \ \mathrm{d}x$$

$$= \frac{2}{l}\left[\left(1-\frac{2x}{l}\right)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)+\frac{2}{l}\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{4}{n^2\pi^2}\{1-(-1)^n\}, \text{ if } n\neq 0$$

$$= \begin{cases} 0, & \text{if } n \text{ is even and } \neq 0 \\ \frac{8}{n^2\pi^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$a_0 = \frac{2}{l}\int_0^l \left(1-\frac{2x}{l}\right) dx = \frac{2}{l}\left[x-\frac{x^2}{l}\right]_0^l = 0$$

Using these values in (1), we get

$$f(x) = \frac{8}{\pi^2}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} \qquad (1)$$

$x = 0$ is a point of continuity of $f(x)$.

$\therefore$ [Sum of the Fourier series of $f(x)]_{x=0} = f(0)$

$$i.e. \frac{8}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\infty\right] = 1$$

$$\therefore \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\infty = \frac{\pi^2}{8}.$$

Example 19

Find the Fourier series expansion of $f(x)$ in $(-2, 2)$ which is defined as follows:

$$f(x) = \begin{cases} 0, & \text{in } (-2,-1) \\ x+x^2, & \text{in } (-1,0) \\ x-x^2, & \text{in } (0,1) \\ 0, & \text{in } (1,2) \end{cases}$$

The symmetric range $(-2, 2)$ is symmetrically divided into 4 subranges.

$$\text{Let} f(x) = \begin{cases} \phi_1(x) = 0, & \text{in } (-2,-1) \\ \phi_2(x) = x+x^2, & \text{in } (-1,0) \\ \phi_3(x) = x-x^2, & \text{in } (0,1) \\ \phi_4(x) = 0, & \text{in } (1,2) \end{cases}$$

We note that $\phi_1(x) = -\phi_4(x)$

and $\phi_2(-x) = -\phi_3(x)$

$\therefore$ $f(x)$ is an odd function in $(-2, 2)$

This can also be graphically verified as shown in Fig. 2.9

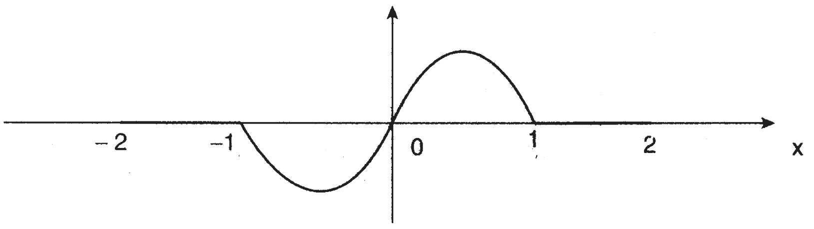


Fig. 2.9

The Fourier series of $f(x)$ will be of period 4 and will contain only the sine terms

Let $$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} \text{ in } (-2, 2) \tag{1}$$

$$b_n = \frac{2}{2} \int_0^2 f(x) \sin \frac{n\pi x}{2}\, dx$$

$$= \int_0^1 (x - x^2) \sin \frac{n\pi x}{2}\, dx$$

$$= \left[(x - x^2) \left(\frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) - (1 - 2x) \left(\frac{-\sin \frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}} \right) + (-2) \left(\frac{\cos \frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}} \right) \right]_0^1$$

$$= \frac{-4}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{16}{n^3\pi^3} \left\{ 1 - \cos \frac{n\pi}{2} \right\}$$

Using this value in (1), we get

$$f(x) = \sum_{n=1}^{\infty} \left[\frac{-4}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{16}{n^3\pi^3} \left\{ 1 - \cos \frac{n\pi}{2} \right\} \right] \sin \frac{n\pi x}{2} \text{ in } (-2, 2)$$

Example 20

Find the Fourier series expansion of $f(x)$ in $(-\pi, \pi)$, when $f(x)$ is defined as follows:

$$f(x) = \begin{cases} \pi + x, & \text{in } -\pi \le x \le -\frac{\pi}{2} \\ -x, & \text{in } -\frac{\pi}{2} \le x \le 0 \\ x, & \text{in } 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \text{in } \frac{\pi}{2} \le x \le \pi \end{cases} \tag{1}$$

The symmetric range $(-\pi, \pi)$ is symmetrically divided into 4 subranges.

$$f(x) = \begin{cases} \phi_1(x) = \pi + x, \text{ in } (-\pi, -\pi/2) \\ \phi_2(x) = -x, \text{ in } (-\pi/2, 0) \\ \phi_3(x) = x, \text{ in } (0, \pi/2) \\ \phi_4(x) = \pi - x, \text{ in } (\pi/2, \pi) \end{cases} \tag{2}$$

We note that $\phi_1(-x) = \phi_4(x)$ and $\phi_2(-x) = \phi_3(x)$.

$\therefore f(x)$ is an even function of x in $(-\pi, \pi)$. This is verified graphically also as shown in Fig 2.10.

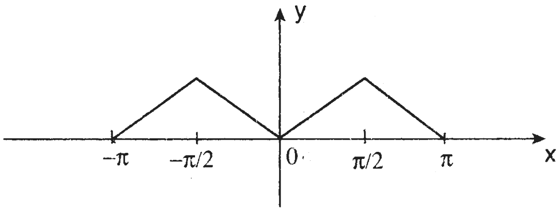


Fig. 2.10

The Fourier series of $f(x)$ will be of period 2π and will not contain sine terms.

Let $f(x) = \dfrac{a_0}{2} + \sum a_n \cos nx$ in $(-\pi, \pi)$ (3)

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\, dx + \int_{\pi/2}^{\pi}(\pi - x)\cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\left(x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right)_0^{\pi/2} + \left\{(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left\{\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\right\}\right.$$
$$\left. + \left\{-\frac{1}{n^2}(-1)^n - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\{1 + (-1)^n\}\right], \text{ if } n \neq 0$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{2}{\pi}\left[\frac{1}{2m^2}(\cos m\pi - 1)\right], & \text{if } n \text{ is even and } = 2m \end{cases}$$

$$= \begin{cases} 0, & \text{if } m \text{ is even} \\ -\frac{2}{\pi m^2}, & \text{if } m \text{ is odd} \end{cases}$$

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} x\, dx + \int_{\pi/2}^{\pi}(\pi - x)\, dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \left\{\frac{(\pi - x)^2}{-2}\right\}_{\pi/2}^{\pi}\right] = \frac{\pi}{2}$$

Using these values in (1), we have

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{m^2} \cos 2mx, \text{ in } (-\pi, \pi)$$

Exercise 2 (a)

Part A (Short-Answer Questions)

1. State the Dirichlet's conditions that a function $f(x)$ should satisfy so that it may be expanded in the form $\frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$ in$(c, c+2\pi)$.
2. State Euler's formulas for the Fourier coefficients.
3. Define Fourier series of $f(x)$ in $(c, c+2l)$.
4. If $f(x)$ is to be expanded as a Fourier series of the form $\frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{l} + \sum b_n \sin \frac{n\pi x}{l}$, in what range is $f(x)$ to be defined?
5. If the Fourier series of $f(x)$ in $(0, 2\pi)$ is $\frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$, what are the functions represented by the same series in $(-2\pi, 0)$ and $(2\pi, 4\pi)$?
6. If the Fourier series of $f(x)$ in $(-l, l)$ is $\frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{l} + \sum b_n \sin \frac{n\pi x}{l}$, what is the Fourier series of $f(x-2l)$ in $(l, 3l)$?
7. If the Fourier series of $f(x)$ in $(-\pi, \pi)$ is $\frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$, what is the Fourier series of $f(x+2\pi)$ in $(-3\pi, -\pi)$?
8. Give the complete definition of a periodic function.
9. The Fourier series of $\sin^3 x \cos^4 x$ in $(-\pi, \pi)$, that in $(-3\pi, -\pi)$ and that in $(\pi, 3\pi)$ are identical. Support or refute this statement with reason.
10. The Fourier series of x^2 in $(0, 2)$, that of $(x+2)^2$ in $(-2, 0)$ and that of $(x-2)^2$ in $(2, 4)$ are identical. Support or refute this statement with reason.
11. Only if $f(x+2l) = f(x)$, $f(x)$ can be expanded as a Fourier series of period $2l$. Support or refute the above statement with reason.
12. Define even and odd functions graphically.
13. Since $x^2 = (-x)^2$ in $(0, 2)$, x^2 is an even function of x in $(0, 2)$. Support or refute the above statement with reason.
14. Since $-x^3 = (-x)^3$ in $(0, 2\pi)$, x^3 is an odd function of x in $(0, 2\pi)$. Support or refute the above statement with reason.
15. Write down the form of the Fourier series of an even function in $(-\pi, \pi)$ and the associated Euler's formulas for the Fourier coefficients.
16. Write down the form of the Fourier series of an odd function in $(-l, l)$ and the associated Euler's formulas for the Fourier coefficients.

17. Write down the formula for the sum of the Fourier series of $f(x)$ at the point $x = \alpha$, if
(i) $x = \alpha$ is a point of continuity of $f(x)$
(ii) $x = \alpha$ is an interior point of discontinuity of $f(x)$
18. Write down the formula for the sum of the Fourier series of $f(x)$ in $(c, c+2\pi)$ at the point of discontinuity $x = \alpha$, if
(i) it coincides with the left end c
(ii) it coincides with the right end $c + 2\pi$
19. Find the Fourier series of $f(x) = \sin^3 x + \cos^3 x$ in $(-\pi, \pi)$.
20. Find the Fourier series of $f(x) = \cos^4 x$ in $(0, 2\pi)$.

Part B

21. Find the Fourier series of period 2π for the function $f(x) = x(2\pi - x)$ in $(0, 2\pi)$. Deduce the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \infty$.
22. Find the Fourier series of period $2l$ for the function $f(x) = (l - x)^2$ in $(0, 2l)$. Deduce the sum of the series $\sum \frac{1}{n^2}$.
23. Find the Fourier series expansion of $f(x) = \pi^2 - x^2$ in $-\pi < x < \pi$.
24. Obtain the Fourier expension of $f(x) = 1 - x$ in $-1 < x < 1$. Deduce the sum of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \infty$.
25. Obtain the Fourier series of period $2l$ for the function

$$\begin{aligned} f(x) &= l - x, \text{in } 0 < x \leq l \\ &= 0, \text{in } l \leq x < 2l \end{aligned}$$

Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \infty = \frac{\pi}{4}$ and $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = \frac{\pi^2}{8}$.
26. Find the Fourier series of period 2π for the function
$f(x) = \begin{cases} 0, & \text{in } (-\pi, 0) \\ \frac{\pi x}{4}, & \text{in } (0, \pi) \end{cases}$. Deduce the sum of the series $\sum\limits_{n=0}^{\infty} \frac{1}{(2n+1)^2}$
27. Find the Fourier series expansion of the function $f(x) = \begin{cases} \cos \pi x, & \text{in } (-1, 0) \\ 0, & \text{in } (0, 1) \end{cases}$
28. Find the Fourier series expansion of the function $f(x) = \begin{cases} x, & \text{in } (0, \pi) \\ 2\pi - x, & \text{in } (\pi, 2\pi) \end{cases}$
Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty$.
29. Find the Fourier series expansion of the function

$$\begin{aligned} f(x) &= x, \text{ when } -l < x < 0 \\ &= k, \text{ when } 0 < x < l \end{aligned}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = \frac{\pi^2}{8}$.
30. Find the Fourier series of $f(x) = x^2$ in $-\pi \leq x \leq \pi$ and hence prove that (i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \infty = \frac{\pi^2}{6}$, (ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \ldots \infty = \frac{\pi^2}{12}$; and (iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = \frac{\pi^2}{8}$.

31. Find the Fourier series expansion of $f(x) = x^2 - x$ in $(-l, l)$. Deduce the values of (i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\infty$; and (ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\infty$.
32. Find the Fourier series expansion of period 2π for the function $f(x) = x\sin x$ in $0 < x < 2\pi$. Deduce the sum of the series $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\infty$.
33. Find the Fourier series of period 2π for the function $f(x) = x\cos x$ in $-\pi < x < \pi$.
34. Find the Fourier series of period 2 for the function $f(x) = x\sin\pi x$ in $-1 \leq x \leq 1$. Deduce the value of $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\infty$.
35. Find the Fourier series of period 2π for the function $f(x) = \sqrt{1+\cos x}$ in $-\pi < x < \pi$.
36. Find the Fourier series of period 2π for the function $f(x) = \frac{1}{12}x(\pi - x)(2\pi - x)$ in $(0, 2\pi)$. Deduce the sum of the series $1^{-3} - 3^{-3} + 5^{-3} - 7^{-3} + \ldots$
37. Find the Fourier series of period $2l$ for the function $f(x) = |x|$ in $(-l, l)$. Hence find the value of $1^{-2} + 3^{-2} + 5^{-2} + \ldots\infty$.
38. Find the Fourier series of period 2π for the function $f(x) = |\sin x|$ in $(-\pi, \pi)$.
39. Find the Fourier series of period 2π for the function $f(x) = \cos ax$ in $-\pi \leq x \leq \pi$, when 'a' is not an integer. Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{9n^2-1}$.
40. Find the Fourier series of period $2l$ for the function $f(x) = e^{ax}$ in $(0, 2l)$.
41. Find the Fourier series expansion for the function $f(x) = \cosh\alpha x$ in $(-\pi, \pi)$.
42. Find the Fourier series of period 4 for the function $f(x)$ defined as follows in $(-2, 2)$:

$$f(x) = \begin{cases} -2, & \text{in } -2 < x < 1 \\ -1, & \text{in } -1 < x < 0 \\ 1, & \text{in } 0 < x < 1 \\ 2, & \text{in } 1 < x < 2 \end{cases}$$

43. Find the Fourier series of period 2π for the function

$$f(x) = \begin{cases} \cos x - \sin x, & \text{in } (-\pi, 0) \\ \cos x + \sin x, & \text{in } (0, \pi) \end{cases}$$ Hence deduce the sum of the series $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \ldots$

44. Find the Fourier series of period 6 for the function

$$f(x) = \begin{cases} 2x + x^2 & \text{in } (-3, 0) \\ 2x - x^2 & \text{in } (0, 3) \end{cases}$$

45. Find the Fourier series of period 2π for the function

$$f(x) = \begin{cases} -\pi x - x^2, & \text{in } (-\pi, 0) \\ \pi x - x^2, & \text{in } (0, \pi) \end{cases}$$

Deduce the sum of the series (i) $\sum \frac{1}{n^2}$ and (ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$.

46. Find the Fourier series of period 4 for the function

$$f(x) = \begin{cases} 0, & \text{in } (-2, -1) \\ x + x^3, & \text{in } (-1, 1) \\ 0, & \text{in } (1, 2) \end{cases}$$

47. Find the Fourier series of period $2l$ for the function

$$f(x) = \begin{cases} l + x, & \text{in } (-l, -l/2) \\ 0, & \text{in } (-l/2, l/2) \\ l - x, & \text{in } (l/2, l) \end{cases}$$

48. Find the Fourier series of period 2π for the function

$$f(x) = \begin{cases} x - 1, & \text{in } -\pi < x < 0 \\ x + 1, & \text{in } 0 < x < \pi \end{cases}$$

49. Find the Fourier series of period 2π for the function

$$f(x) = \begin{cases} -(\pi + x), & \text{in } (-\pi, -\pi/2) \\ x, & \text{in } (-\pi/2, \pi/2) \\ \pi - x, & \text{in } (\pi/2, \pi) \end{cases}$$

50. Find the Fourier series of period 6 for the function

$$f(x) = \begin{cases} 0, & \text{in } -3 < x < -1 \\ 1 + \cos \pi x, & \text{in } -1 < x < 1 \\ 0, & \text{in } 1 < x < 3 \end{cases}$$

2.9 HALF-RANGE FOURIER SERIES AND PARSEVAL'S THEOREM

Introduction

If a function $f(x)$ is to be expanded as a Fourier series of period $2l$, $f(x)$ should be defined in a range of length $2l$, in particular, in the range $(-l, l)$ or $(0, 2l)$. But in some situations, the value of $f(x)$ will be available only in a range of length l, in particular in the range $(0, l)$. Without knowing the value of $f(x)$ in the full range, i.e., either in $(-l, l)$ or in $(0, 2l)$, we cannot expand $f(x)$ as a Fourier series of period $2l$, since the Fourier coefficients cannot be found out.

In such situations, i.e., when the value of $f(x)$ is given in $(0, l)$, we assign some value for $f(x)$ in $(-l, 0)$ [or in $(l, 2l)$], so that $f(x)$ is defined completely in the full range $(-l, l)$ [or in $(0, 2l)$]. If we assign an arbitrary value for $f(x)$ in $(-l, 0)$, the Fourier series of $f(x)$ will contain both cosine and sine terms. This kind of Fourier series of period $2l$, resulting from arbitrary assignment of value for $f(x)$ in $(-l, 0)$ is not of interest.

If we assign a suitable value for $f(x)$ in $(-l, 0)$ so that the given value of $f(x)$ in $(0, l)$ and the assigned value of $f(x)$ in $(-l, 0)$ together make $f(x)$ even or odd in $(-l, l)$, then the Fourier series of $f(x)$ will be of period $2l$ and will contain only cosine terms or sine terms respectively. Such series are called *Fourier half-range cosine series* or *sine series* respectively and will represent the given value of $f(x)$ in $(0, l)$.

Note ✍

The term 'half-range series' is used because the Fourier series is of period $2l$, even though the function is defined in a range of length l.

Theorem

A function $f(x)$ defined in $(0, l)$ can be expanded as a Fourier series of period $2l$ containing (i) only cosine terms and (ii) only sine terms, by extending $f(x)$ suitably in $(-l, 0)$.

Proof

Let $$f(x) = \phi(x) \text{ in } (0, l)$$

(i) Let us assign the value $f(x) = \phi(-x)$ in $(-l, 0)$. By the definition of an even function given in Section 2.6, $f(x)$ is even in $(-l, l)$.
$\therefore$ Fourier series of $f(x)$ will be of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}, \text{ where}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx, \quad n \geq 0$$

(ii) Let us assign the value $f(x) = -\phi(-x)$ in $(-l, 0)$. By the definition of an odd function given in the previous section, $f(x)$ is odd in $(-l, l)$.
$\therefore$ Fourier series of $f(x)$ will be of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \text{ where}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx.$$

Note ✍

1. *The values $\phi(-x)$ and $-\phi(-x)$ assigned to $f(x)$ in $(-l, 0)$ in order to make $f(x)$ even and odd respectively in $(-l, l)$ are called the even and odd extensions of $f(x)$ in $(-l, 0)$.*
2. *The evaluation of a_n and b_n by the modified Euler's formulas requires only the given value of $f(x)$ in $(0, l)$.*

Theorem

A function $f(x)$ defined in $(0, l)$ can be expanded as a Fourier series of period $2l$ containing (i) only cosine terms and (ii) only sine terms, by extending $f(x)$ suitably in $(l, 2l)$.

Proof

Let $$f(x) = \phi(x) \text{ in } (0, l)$$

(i) Let us assign the value $f(x) = \phi(2l - x)$ in $(l, 2l)$. Let the Fourier series of $f(x)$ be given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Now $b_n = \dfrac{1}{l}\displaystyle\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$

$$= \frac{1}{l}\left[\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx + \int_l^{2l} \phi(2l - x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{l}\left[\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx + \int_l^{0} \phi(y)\sin\frac{n\pi}{l}(2l - y)(-\,dy)\right],$$

on putting $2l - x = y$ in the second integral.

$$= \frac{1}{l}\left[\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx - \int_0^{l} \phi(y)\sin\frac{n\pi y}{l}\,dy\right] = 0$$

This means that the Fourier series will contain only cosine terms.

i.e. $f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

Now $$a_n = \frac{1}{l}\left[\int_0^1 \phi(x)\cos\frac{n\pi x}{l}\,dx + \int_l^{2l} \phi(2l - x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{l}\left[\int_0^l \phi(x)\cos\frac{n\pi x}{l}\,dx + \int_l^{0} \phi(y)\cos\frac{n\pi}{l}(2l - y)(-\,dy)\right],$$

on putting $2l - x = y$ in the second integral.

$$= \frac{1}{l}\left[\int_0^l \phi(x)\cos\frac{n\pi x}{l}\,dx + \int_0^{l} \phi(y)\cos\frac{n\pi y}{l}\,dy\right]$$

i.e. $$a_n = \frac{2}{l}\int_0^l \phi(x)\cos\frac{n\pi x}{l}\,dx, \quad n \geq 0$$

(ii) Let us assign the value $f(x) = -\phi(2l - x)$ in $(l, 2l)$.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$. Proceeding as in (i), we can prove that

$$a_n = 0, \quad n \geq 0$$

$\therefore$ The Fourier series of $f(x)$ will contain only sine terms.

i.e. $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$.

Proceeding as in (i), we can prove that

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

Note ✍

1. *The extended values of $f(x)$ in $(l, 2l)$, namely $\phi(2l - x)$ and $-\phi(2l - x)$ are only the periodic extensions in $(l, 2l)$ of $\phi(-x)$ and $-\phi(-x)$, that are the even and odd extensions of $f(x)$ in $(-l, 0)$.*
2. *Even in this case, the evaluation of a_n and b_n requires only the given value of $f(x)$ in $(0, l)$*

2.10 ROOT-MEAN SQUARE VALUE OF A FUNCTION

Definition

If a function $y = f(x)$ is defined in $(c, c + 2l)$, then $\sqrt{\frac{1}{2l}\int_c^{c+2l} y^2\,dx}$ is called *the root mean-square(R. M. S.)value of y in* $(c, c + 2l)$ and is denoted by $\bar{y}$.

Thus $\bar{y}^2 = \frac{1}{2l}\int_c^{c+2l} y^2\,dx$.

If $y = f(x)$ can be expanded as a Fourier series in $(c, c + 2l)$, then $\bar{y}^2$ can be expressed in terms of Fourier coefficients a_0, a_n and b_n. The formula that expresses $\bar{y}^2$ in terms of a_0, a_n and b_n is known as *Parseval's formula* which is stated as a theorem.

Parseval's theorem

If $y = f(x)$ can be expanded as Fourier series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ in $(c, c + 2l)$, then the root-mean square value $\bar{y}$ of $y = f(x)$ in $(c, c + 2l)$ is given by

$$\bar{y}^2 = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2$$

Proof

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{in } (c, c+2l) \tag{1}$$

$\therefore$ By Euler's formulas for the Fourier coefficients,

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\, dx, n \geq 0 \tag{2}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\, dx, n \geq 1 \tag{3}$$

Now, by definition,

$$\begin{aligned}
\overline{y}^2 &= \frac{1}{2l}\int_c^{c+2l} y^2\, dx = \frac{1}{2l}\int_c^{c+2l} [f(x)]^2\, dx \\
&= \frac{1}{2l}\int_c^{c+2l} f(x)\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\right] dx, \qquad \text{using (1)} \\
&= \frac{a_0}{4}\left[\frac{1}{l}\int_c^{c+2l} f(x)\, dx\right] + \sum_{n=1}^{\infty}\frac{a_n}{2}\left\{\frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\, dx\right\} \\
&\qquad + \sum_{n=1}^{\infty}\frac{b_n}{2}\left\{\frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\, dx\right\} \\
&= \frac{a_0}{4}\cdot a_0 + \sum_{n=1}^{\infty}\frac{a_n}{2}\cdot a_n + \sum_{n=1}^{\infty}\frac{b_n}{2}\cdot b_n, \qquad \text{by using (2) and (3)} \\
&= \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2.
\end{aligned}$$

Corollary 1

If the Fourier half-range cosine series of $y = f(x)$ in $(0, l)$ is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$, then

$$\overline{y}^2 = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2, \quad \text{where } \overline{y}^2 = \frac{1}{l}\int_0^l y^2\, dx$$

Corollary 2

If the Fourier half-range sine series of $y = f(x)$ in $(0, l)$ is $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$, then

$$\overline{y}^2 = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2, \quad \text{where } \overline{y}^2 = \frac{1}{l}\int_0^l y^2\, dx$$

Worked Examples 2(b)

Example 1

Find the half-range (i) cosine series and (ii) sine series for $f(x) = x^2$ in $(0, \pi)$

(i) To get the half-range cosine series for $f(x)$ in $(0, \pi)$, we should give an even extension for $f(x)$ in $(-\pi, 0)$.
i.e. put $f(x) = (-x)^2 = x^2$ in $(-\pi, 0)$
Now $f(x)$ is even in $(-\pi, \pi)$

$$\therefore \qquad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx \\
&= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx \\
&= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right)\right. \\
&\qquad \left. +2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi} \\
&= \frac{4}{\pi n^2}\cdot \pi(-1)^n = \frac{4(-1)^n}{n^2}, n \neq 0 \\
a_0 &= \frac{2}{\pi}\int_0^{\pi} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi} x^2\, dx = \frac{2}{3}\pi^2
\end{aligned}$$

$\therefore$ The Fourier half-range cosine series of x^2 is given by

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \quad \text{in } (0, \pi).$$

(ii) To get the half-range sine series of $f(x)$ in $(0, \pi)$, we should give an odd extension for $f(x)$ in $(-\pi, 0)$.

$$\text{i.e.} \qquad \text{put } f(x) = -(-x)^2 \text{ in } (-\pi, 0)$$
$$= -x^2 \text{ in } (-\pi, 0)$$

Now $f(x)$ is odd in $(-\pi, \pi)$.

$$\therefore \qquad f(x) = \sum_{n=1}^{\infty} b_n \sin nx \tag{2}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{n}(-1)^{n+1} + \frac{2}{n^3}\{(-1)^n - 1\}\right]$$

$$= \begin{cases} \frac{2}{\pi}\left[\frac{\pi^2}{n} - \frac{4}{n^3}\right], & \text{if } n \text{ is odd} \\ -\frac{2\pi}{n}, & \text{if } n \text{ is even} \end{cases}$$

Using this value in (2), we get the half-range sine series of x^2 in $(0, \pi)$.

Example 2

Find (i) the Fourier half-range cosine series and (ii) the Fourier half-range sine series of $f(x) = \begin{cases} x, & \text{in } 0 < x < 1 \\ 2 - x, & \text{in } 1 < x < 2 \end{cases}$

(i) To get the half-range cosine series, we give an even extension for $f(x)$ in $(-2, 0)$.

i.e. we put $f(x) = \begin{cases} 2 + x, & \text{in } -2 < x < -1 \\ -x, & \text{in } -1 < x < 0 \end{cases}$

Now $f(x)$ has been made an even function in $(-2, 2)$. Here $2l = 4$.

Let the half-range cosine series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} \quad (1)$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 x\cos\frac{n\pi x}{2}\,dx + \int_1^2 (2 - x)\cos\frac{n\pi x}{2}\,dx$$

$$= \left[\left\{x\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) + \frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right\}_0^1 + \left\{(2 - x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - \frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right\}_1^2\right]$$

$$= \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos n\pi$$
$$- \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}$$
$$= \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\{1+(-1)^n\}$$
$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{2}{m^2\pi^2}[(-1)^m - 1], & \text{if } n \text{ is even and } = 2m \end{cases}$$
$$= \begin{cases} 0, & \text{if } m = \frac{n}{2} \text{ is even} \\ \frac{-4}{m^2\pi^2}, & \text{if } m = \frac{n}{2} \text{ is odd} \end{cases}$$
$$= \begin{cases} 0, & \text{if } n \text{ is a multiple of 4} \\ \frac{-16}{n^2\pi^2}, & \text{if } n \text{ is even, but not a multiple of 4.} \end{cases}$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^1 x\,dx + \int_1^2 (2-x)\,dx$$
$$= \left(\frac{x^2}{2}\right)_0^1 + \left[\frac{(2-x)^2}{-2}\right]_1^2$$
$$= \frac{1}{2} + \frac{1}{2} = 1.$$

Using these values in (1), the required cosine series is given by

$$f(x) = \frac{1}{2} - \frac{16}{\pi^2}\left[\frac{1}{2^2}\cos\pi x + \frac{1}{6^2}\cos 3\pi x + \frac{1}{10^2}\cos 5\pi x + \cdots \infty\right]$$

(ii) To get the half-range sine series of $f(x)$, we give an odd extension for $f(x)$ in $(-2, 0)$.

i.e. we put $f(x) = \begin{cases} -(2+x), & \text{in } -2 < x < -1 \\ x, & \text{in } -1 < x < 0. \end{cases}$

Now $f(x)$ has been made an odd function in $(-2, 2)$. Here $2l = 4$.

Let the half-range sine series be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 x\sin\frac{n\pi x}{2}\,dx + \int_1^2 (2-x)\sin\frac{n\pi x}{2}\,dx$$

$$= \left[\left\{x\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) + \frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right\}_0^1 \right.$$

$$\left. + \left\{(2-x)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - \frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right\}_1^2\right]$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$+ \frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \begin{cases} \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using this value in (2), the required sine series is given by

$$f(x) = \frac{8}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{2} - \frac{1}{3^2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi x}{2} - \cdots\infty\right]$$

Note ✍

From the above two examples, it is clear that any function defined in $(0, l)$ can be expanded as a cosine series and also as a sine series. Depending on the nature of the Fourier series required, we give the corresponding extension for the function in $(-l, 0)$

Example 3

Find the Fourier half-range cosine series of the function $f(x) = (x+1)^2$ in $(-1, 0)$. Hence find the value of $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\infty$.

To get the half-range cosine series of $f(x)$, we give an even extension to $f(x)$ in $(0, 1)$, i.e. we put $f(x) = (-x+1)^2$ in $(0, 1)$

Now $f(x)$ is even in $(-1, 1)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x, \quad \text{since } 2l = 2$$

$$a_n = \frac{2}{1}\int_0^1 f(x)\cos n\pi x \, dx$$

$$= 2\int_0^1 (1-x)^2 \cos n\pi x \, dx$$

Note ✍

We do not use the given value of $f(x)$ in $(-1, 0)$ for evaluating a_n, but use the assigned value of $f(x)$ in $(0, 1)$. Hence extra care should be taken while assigning the value of $f(x)$ in $(0, 1)$. However, a_n can also be found out by using the formula $a_n = \frac{2}{1}\int_{-1}^{0} (x+1)^2 \cos n\pi x \, dx$

$$= 2\left[(1-x)^2\left(\frac{\sin n\pi x}{n\pi}\right) - \{-2(1-x)\}\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= \frac{4}{n^2\pi^2}, \text{ if } n \neq 0.$$

$$a_0 = \frac{2}{1}\int_0^1 f(x)\, dx = 2\int_0^1 (1-x)^2\, dx = 2\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= 2/3$$

$\therefore$ The required half-range cosine series is

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x.$$

This series represents $(x+1)^2$ in $(-1, 0)$ and $(1-x)^2$ in $(0, 1)$.
$x = 0$ is a point of continuity for $f(x)$.

$\therefore$ [Sum of the Fourier series of $f(x)]_{x=0} = f(0)$

i.e.
$$\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = 1$$

$\therefore$
$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Example 4

Find the half-range sine series of the function $f(x) = \pi - x$ in $(\pi, 2\pi)$, by suitably extending $f(x)$ in $(0, \pi)$. Deduce the sum of the series $1 - 1/3 + 1/5 - 1/7 + \cdots \infty$.
If $f(x) = \phi(x)$ in $(0, l)$, we should assign $f(x) = -\phi(2l - x)$ in $(l, 2l)$ in order to get a sine series.
Hence if $f(x) = \psi(x)$ in $(l, 2l)$, we should assign $f(x) = -\psi(2l - x)$ in $(0, l)$ in order to get a sine series. This is obtained by putting $-\phi(2l - x) = \psi(x)$ and by making the transformation $2l - x = u$.

Since $f(x) = \pi - x$ in $(\pi, 2\pi)$, we put $f(x) = -\{\pi - (2\pi - x)\}$ in $(0, \pi)$ i.e. we put $f(x) = \pi - x$ in $(0, \pi)$ to get sine series.
Let the Fourier sine series of $f(x)$ be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\sin nx \, dx \\ &= \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - \left(\frac{\sin nx}{n^2}\right)\right]_0^{\pi} \\ &= \frac{2}{n} \end{aligned}$$

Hence

$$f(x) = 2\sum_{n=1}^{\infty} \frac{1}{n}\sin nx \tag{1}$$

Putting $x = \pi/2$, we can get the series, whose sum is required.

$x = \pi/2$ is a point of continuity for $f(x)$.

$\therefore$ [Sum of the Fourier series of $f(x)]_{x=\pi/2} = f(\pi/2)$

i.e. $2\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \infty\right] = \pi - \pi/2$

$\therefore$ $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \infty = \frac{\pi}{4}$

Note

If the specific value of the extension of $f(x)$ in $(0, \pi)$ is not required, we can also evaluate b_n by using the formula $b_n = \frac{2}{\pi}\int_{\pi}^{2\pi} (\pi - x)\sin nx \, dx$

Example 5

Find the half-range cosine series of $f(x) = x(l - x)$ in $(0, l)$. How should $f(x)$ be extended in order to get this cosine series (i) in the range $(-l, 0)$ and (ii) in the range $(l, 2l)$?

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$, since the length of the given half-range $= l$.

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right.$$

$$\left. + (-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2l}{n^2\pi^2}\left[-l\cos n\pi - l\right]$$

$$= -\frac{2l^2}{n^2\pi^2}\{(-1)^n + 1\}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ -\frac{4l^2}{n^2\pi^2}, & \text{if } n \text{ is even and} \neq 0 \end{cases}$$

$$a_0 = \frac{2}{l}\int_0^l (lx - x^2)\,dx = \frac{2}{l}\left[\frac{lx^2}{2} - \frac{x^3}{3}\right]_0^l = \frac{l^2}{3}$$

$\therefore$ Required half-range cosine series is given by

$$f(x) = \frac{l^2}{6} - \frac{4l^2}{\pi^2}\sum_{n=2,4,6..}^{\infty}\frac{1}{n^2}\cos nx \quad \text{or}$$

$$= \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos 2nx.$$

To get this half-range cosine series, we should assign $f(x) = -(lx + x^2)$ in $(-l, 0)$ and assign $f(x) = (2l - x)(x - l)$ in $(l, 2l)$.

Example 6

Find the half-range sine series of $f(x)$ in $(0, \lambda)$, given that

$$f(x) = \begin{cases} (\lambda - c)x, & \text{in } (0, c) \\ (\lambda - x)c, & \text{in } (c, \lambda) \end{cases}$$

We give an odd extenstion to $f(x)$ in $(-\lambda, 0)$.

i.e. we put $f(x) = \begin{cases} -(\lambda + x)c, & \text{in } (-\lambda, -c) \\ (\lambda - c)x, & \text{in } (-c, 0) \end{cases}$

Now $f(x)$ is odd in $(-\lambda, \lambda)$.

Let $$f(x) = \sum b_n \sin\frac{n\pi x}{\lambda} \tag{1}$$

$$b_n = \frac{2}{\lambda}\int_0^{\lambda} f(x)\sin\frac{n\pi x}{\lambda}\,dx$$

$$= \frac{2}{\lambda}\left[\int_0^{c}(\lambda - c)x\sin\frac{n\pi x}{\lambda}\,dx + \int_c^{\lambda}(\lambda - x)c\sin\frac{n\pi x}{\lambda}\,dx\right]$$

$$= \frac{2}{\lambda}(\lambda - c)\left[x\left(\frac{-\cos\frac{n\pi x}{\lambda}}{\frac{n\pi}{\lambda}}\right) + \frac{\sin\frac{n\pi x}{\lambda}}{n^2\frac{\pi^2}{\lambda^2}}\right]_0^c$$

$$+ \frac{2c}{\lambda}\left[(\lambda - x)\left(\frac{-\cos\frac{n\pi x}{\lambda}}{\frac{n\pi}{\lambda}}\right) - \frac{\sin\frac{n\pi x}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}}\right]_c^{\lambda}$$

$$= \frac{2(\lambda - c)}{\lambda}\left[-\frac{c\lambda}{n\pi}\cos\frac{n\pi c}{\lambda} + \frac{\lambda^2}{n^2\pi^2}\sin\frac{n\pi c}{\lambda}\right]$$

$$+ \frac{2c}{\lambda}\left[\frac{\lambda(\lambda - c)}{n\pi}\cos\frac{n\pi c}{\lambda} + \frac{\lambda^2}{n^2\pi^2}\sin\frac{n\pi c}{\lambda}\right]$$

$$= \frac{2\lambda^2}{n^2\pi^2}\sin\frac{n\pi c}{\lambda}$$

Using in (1), we get the required half-range sine series as

$$f(x) = \frac{2\lambda^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi c}{\lambda}\sin\frac{n\pi x}{\lambda}$$

Example 7

Find the half-range cosine series of $f(x) = \sin x$ in $(0, \pi)$.
We give an even extension for $f(x)$ in $(-\pi, 0)$.
i.e. we put $f(x) = -\sin x$ in $(-\pi, 0)$.
Now $f(x)$ is even in $(-\pi, \pi)$.

Let $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\left\{1 - (-1)^{n-1}\right\}\right]$$

$$= -\frac{2}{\pi(n^2-1)}\left\{1 - (-1)^{n-1}\right\}$$

$$= \begin{cases} -\frac{4}{\pi(n^2-1)}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd and} \neq 1 \end{cases}$$

$$a_0 = \frac{4}{\pi}, \text{ on putting } n = 0 \text{ in } a_n.$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{1}{\pi}\left(\frac{-\cos 2x}{2}\right)_0^{\pi} = 0.$$

Using these values in (1), the required half-range cosine series is obtained as

$$\sin x = \frac{4}{\pi}\left[\frac{1}{2} - \sum_{n=2,4,6..}^{\infty} \frac{1}{n^2-1} \cdot \cos nx\right]$$

$$= \frac{4}{\pi}\left[\frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos 2nx\right]$$

Example 8

Find the half-range sine series of $f(x) = \sin ax$ in $(0, l)$.

We give an odd extension for $f(x)$ in $(-l, 0)$.

i.e. we put $f(x) = -\sin[a(-x)] = \sin ax$ in $(-l, 0)$

$\therefore$ $f(x)$ is odd in $(-l, l)$

Let $$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^{l} \sin ax \cdot \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{1}{l}\int_0^{l} \left[\cos\left(\frac{n\pi}{l} - a\right)x - \cos\left(\frac{n\pi}{l} + a\right)x\right] dx$$

$$= \frac{1}{l}\left[\frac{\sin\left(\left(\frac{n\pi}{l}-a\right)x\right)}{\left(\frac{n\pi}{l}-a\right)} - \frac{\sin\left(\frac{n\pi}{l}+a\right)x}{\left(\frac{n\pi}{l}+a\right)}\right]_0^l$$

$$= \frac{1}{n\pi - al}\sin(n\pi - al) - \frac{1}{n\pi + al}\sin(n\pi + al)$$

$$= \frac{1}{n\pi - al}(-1)^{n+1}\sin al + \frac{1}{n\pi + al}(-1)^{n+1}\sin al$$

$$= (-1)^{n+1}\sin al \cdot \frac{2n\pi}{n^2\pi^2 - a^2l^2}$$

Using this value in (1), we get the half-range sine series as

$$\sin ax = 2\pi \sin al \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cdot n}{n^2\pi^2 - a^2l^2}\sin\frac{n\pi x}{l}$$

Example 9

Find the half-range cosine series of $f(x) = x\sin x$ in $(0, \pi)$. Deduce the sum of the series $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \ldots \infty$.

We give an even extension for $f(x)$ in $(-\pi, 0)$

i.e. we put $f(x) = -x\sin(-x)$

$$= x\sin x \text{ in } (-\pi, 0)$$

Now $f(x)$ is even in $(-\pi, \pi)$.

$\therefore$ Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ (1)

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$= \frac{1}{\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1}\right\} + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{\pi}$$

$$- \frac{1}{\pi}\left[x\left\{\frac{-\cos(n-1)x}{n-1}\right\} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{\pi}$$

$$= -\frac{1}{n+1}(-1)^{n+1} + \frac{1}{n-1}(-1)^{n-1}$$

$$= (-1)^{n-1}\left\{\frac{1}{n-1} - \frac{1}{n+1}\right\} = \frac{2(-1)^{n-1}}{n^2-1}, \quad \text{if} n \neq 1.$$

$$a_0 = 2$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) + \frac{\sin 2x}{4}\right]_0^{\pi}$$

$$= \frac{-1}{2}$$

Using these values, we get the required cosine series as

$$x \sin x = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^2-1} \cos nx \;\text{ in } (0, \pi) \tag{2}$$

$$x = \frac{\pi}{2} \text{ is a point of continuity of } x \sin x$$

$\therefore$ [Sum of the Fourier series of $f(x)]_{x=\frac{\pi}{2}} = f(\frac{\pi}{2})$

i.e. $$1 + 2\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^2-1} \cos\frac{n\pi}{2} = \frac{\pi}{2}$$

i.e. $$1 + 2\left\{\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots \infty\right\} = \frac{\pi}{2}$$

$\therefore$ $$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots \infty = \frac{\pi - 2}{4}$$

Example 10

Find the half-range sine series of $f(x) = \dfrac{\sinh ax}{\sinh a\pi}$ in $(0, \pi)$.

We give an odd extension for $f(x)$ in $(-\pi, 0)$.

i.e. we put $f(x) = \dfrac{-\sinh(-x)}{\sinh a\pi} = \dfrac{\sinh ax}{\sin a\pi}$ in $(-\pi, 0)$

Now $f(x)$ is odd in $(-\pi, \pi)$.

$\therefore$ Let $$f(x) = \sum b_n \sin nx \tag{1}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \frac{\sinh ax}{\sinh a\pi} \sin nx dx$$

$$= \frac{1}{\pi \sinh a\pi}\left[\int_0^\pi e^{ax}\sin nx\,dx - \int_0^\pi e^{-ax}\sin nx\,dx\right]$$

$$= \frac{1}{\pi \sinh a\pi}\left[\left\{\frac{e^{ax}}{n^2+a^2}(a\sin nx - n\cos nx)\right\}_0^\pi - \left\{\frac{e^{-ax}}{n^2+a^2}(-a\sin nx - n\cos nx)\right\}_0^\pi\right]$$

$$= \frac{1}{\pi \sinh a\pi}\left[\frac{-n(-1)^n e^{a\pi}}{n^2+a^2} + \frac{n(-1)^n e^{-a\pi}}{n^2+a^2}\right]$$

$$= \frac{1}{\pi \sinh a\pi}\cdot\frac{2(-1)^{n-1} n\sinh a\pi}{n^2+a^2}$$

$$= \frac{2}{\pi}\cdot\frac{(-1)^{n-1}\cdot n}{n^2+a^2}$$

Using this value in (1), we get the required half-range sine series as

$$\frac{\sinh ax}{\sinh a\pi} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{n^2+a^2}\sin nx \text{ in } (0,\pi)$$

Example 11

Find the Fourier series of period 2π for the function $f(x) = x^2 - x$ in $(-\pi,\pi)$. Hence deduce the sum of the series $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\infty$, assuming that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

Let $\quad x^2 - x = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ in } (-\pi,\pi) \qquad (1)$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2 - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x^2\cos nx\,dx \;[\because x\cos nx \text{ is odd in } (-\pi,\pi)]$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{4}{n^2}(-1)^n, \text{ if } n \neq 0.$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2 - x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{3}\pi^2$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2 - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx \left[\because\ x^2\sin nx \text{is odd in } (-\pi, \pi)\right]$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{-2}{n}(-1)^n$$

Using these values in (1), we get

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx \text{ in } (-\pi, \pi)$$

Now the terms of the series $\sum\limits_{n=1}^{\infty}\frac{1}{n^4}$, whose sum is required, are the squares of the Fourier coefficients a_n multiplied by a constant. Whenever this situation arises, we apply Parseval's theorem, which states that

$\frac{1}{4}a_0^2 + \frac{1}{2}\sum\limits_{n=1}^{\infty}a_n^2 + \frac{1}{2}\sum\limits_{n=1}^{\infty}b_n^2 = \bar{y}^2$, the square of the R. M. S. value of $y = f(x)$ in $(-\pi, \pi)$

Thus
$$\frac{1}{4}\cdot\frac{4}{9}\pi^4 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16\cdot(-1)^{2n}}{n^4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{4\cdot(-1)^{2n+2}}{n^2}$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(x^2 - x)^2)\,dx$$

i.e.
$$\frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4} + 2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{2\pi}\cdot 2\int_0^{\pi}(x^4 + x^2)\,dx$$
$$= \frac{1}{5}\pi^4 + \frac{1}{3}\pi^2$$

i.e.
$$8\sum_{n=1}^{\infty}\frac{1}{n^4} = \left(\frac{1}{5} - \frac{1}{9}\right)\pi^4 + \frac{1}{3}\pi^2 - 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

i.e.
$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}\left(\because\ \sum\frac{1}{n^2} = \frac{\pi^2}{6}\right)$$

Example 12

Find the Fourier series expansion of period 21 for the function

$$f(x) = \begin{cases} x, & \text{in } (0, l) \\ 0, & \text{in } (l, 2l) \end{cases}$$

Hence deduce the sum of the series $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots \infty$, assuming that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty = \frac{\pi^2}{6}$.

Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{in } (0, 2l)$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l} \, dx$$

$$= \frac{1}{l} \cdot \int_0^{l} x \cos \frac{n\pi x}{l} \, dx$$

$$= \frac{1}{l} \left[x \left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) + \left(\frac{\cos \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right]_0^l$$

$$= \frac{l}{n^2\pi^2} \left\{ (-1)^n - 1 \right\}$$

$$= \begin{cases} \frac{-2l}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even and } \neq 0 \end{cases}$$

$$a_0 = \frac{1}{l} \int_0^{l} x \, dx = \frac{l}{2}$$

$$b_n = \frac{1}{l} \int_0^{l} x \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{1}{l} \left[x \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) + \left(\frac{\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right]_0^l$$

$$= \frac{-l}{n\pi} (-1)^n$$

Using these values in (1), we get

$$f(x) = \frac{l}{4} - \frac{2l}{\pi^2} \sum_{n=1,3,5..}^{\infty} \frac{1}{n^2} \cos \frac{n\pi x}{l} + \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \text{ in } (0, 2l)$$

Now the series to be summed up contains constant multiples of squares of a_n. Hence we apply Parseval's theorem.

$$\frac{1}{4}a_0^2 + \frac{1}{2}\sum a_n^2 + \frac{1}{2}\sum b_n^2 = \frac{1}{2l}\int_0^{2l} [f(x)]^2 \, dx$$

i.e. $$\frac{l^2}{16} + \frac{1}{2}\sum_{n=1,3,5\ldots}^{\infty} \frac{4l^2}{n^4\pi^4} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{l^2}{n^2\pi^2} = \frac{1}{2l}\int_0^{l} x^2 \, dx$$

i.e. $$\frac{l^2}{16} + \frac{2l^2}{\pi^4}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} + \frac{l^2}{2\pi^2}\cdot\frac{\pi^2}{6} = \frac{l^2}{6}$$

$$(\because \sum \frac{1}{n^2} = \frac{\pi^2}{6})$$

i.e. $$\frac{2}{\pi^4}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{1}{6} - \frac{1}{16} - \frac{1}{12} = \frac{1}{48}$$

$\therefore$ $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

Example 13

Find the Fourier series expansion of period l for the function $f(x) = \begin{cases} x, & \text{in } (0, \frac{l}{2}) \\ l - x, & \text{in } (\frac{l}{2}, l) \end{cases}$. Hence deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$.

Here the length of the full range = period of the Fourier series required $= l$.

$\therefore$ The Fourier series of $f(x)$ is of the form

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{2n\pi x}{l} + \sum b_n \sin\frac{2n\pi x}{l} \quad \text{in } (0, l) \tag{1}$$

$$a_n = \frac{1}{l/2}\int_0^{l} f(x)\cos\frac{2n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} x\cos\frac{2n\pi x}{l}\,dx + \int_{l/2}^{l} (l-x)\cos\frac{2n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\left\{x\left(\frac{\sin\frac{2n\pi x}{l}}{\frac{2n\pi}{l}}\right) + \frac{\cos\frac{2n\pi x}{l}}{\frac{4n^2\pi^2}{l^2}}\right\}_0^{\frac{l}{2}} + \left\{(l-x)\left(\frac{\sin\frac{2n\pi x}{l}}{\frac{2n\pi}{l}}\right) - \left(\frac{\cos\frac{2n\pi x}{l}}{\frac{4n^2\pi^2}{l^2}}\right)\right\}_{\frac{l}{2}}^{l}\right]$$

$$= \frac{l}{n^2\pi^2}\{(-1)^n - 1\} = \begin{cases} -\frac{2l}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even and } \neq 0. \end{cases}$$

$$a_0 = \frac{1}{\frac{l}{2}} \int_0^l f(x)dx$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{2}} x\,\mathrm{d}x + \int_{\frac{l}{2}}^{l} (l-x)\,\mathrm{d}x\right]$$

$$= \frac{2}{l}\left[\left(\frac{x^2}{2}\right)_0^{\frac{l}{2}} + \left\{\frac{(l-x)^2}{-2}\right\}_{\frac{l}{2}}^{l}\right] = \frac{l}{2}$$

$$b_n = \frac{1}{\frac{l}{2}} \int_0^l f(x)\sin\frac{2n\pi x}{l}\,\mathrm{d}x$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{2}} x\sin\frac{2n\pi x}{l}\,\mathrm{d}x + \int_{\frac{l}{2}}^{l} (l-x)\sin\frac{2n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\left\{x\left(\frac{-\cos\frac{2n\pi x}{l}}{\frac{2n\pi}{l}}\right) + \frac{\sin\frac{2n\pi x}{l}}{\frac{4n^2\pi^2}{l^2}}\right\}_0^{\frac{l}{2}} + \left\{(l-x)\left(\frac{-\cos\frac{2n\pi x}{l}}{\frac{2n\pi}{l}}\right) - \frac{\sin\frac{2n\pi x}{l}}{\frac{4n^2\pi^2}{l^2}}\right\}_{\frac{l}{2}}^{l}\right] = 0$$

Using these values in (1), we get

$$f(x) = \frac{l}{4} - \frac{2l}{\pi^2}\sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2}\cos\frac{2n\pi x}{l} \quad \text{in } (0, l).$$

Since the series to be summed up contains constant multiples of squares of a_n, we apply Parseval's theorem.

$$\frac{1}{4}a_0^2 + \frac{1}{2}\sum a_n^2 + \frac{1}{2}\sum b_n^2 = \frac{1}{l}\int_0^l [f(x)]^2\,\mathrm{d}x$$

i.e.

$$\frac{l^2}{16} + \frac{1}{2}\cdot\frac{4l^2}{\pi^4}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^4} = \frac{l}{l}\left[\int_0^{\frac{l}{2}} x^2\,\mathrm{d}x + \int_{\frac{l}{2}}^{l} (l-x)^2\,\mathrm{d}x\right]$$

i.e. $$\frac{l^2}{16}+\frac{2l^2}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{1}{l}\left[\frac{l^3}{24}+\frac{l^3}{24}\right]$$
$$=\frac{l^2}{12}$$

$\therefore$ $$\frac{2}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{1}{48}$$

$\therefore$ $$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{\pi^4}{96}.$$

Example 14

Find the Fourier series of period 2π for the function
$$f(x)=\begin{cases}1, & \text{in } (0,\pi)\\ 2, & \text{in } (\pi,2\pi).\end{cases}$$
Hence find the sum of the series $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty$.

Let $$f(x)=\frac{a_0}{2}+\sum a_n\cos nx+\sum b_n\sin nx \quad \text{in } (0,2\pi) \tag{1}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x$$
$$=\frac{1}{\pi}\left[\int_0^{\pi} 1\cdot\cos nx\,\mathrm{d}x+\int_{\pi}^{2\pi} 2\cdot\cos nx\,\mathrm{d}x\right]$$
$$=\frac{1}{\pi}\left[\left(\frac{\sin nx}{n}\right)_0^{\pi}+2\left(\frac{\sin nx}{n}\right)_{\pi}^{2\pi}\right],\quad \text{if } n\neq 0$$
$$=0,\quad \text{if } n\neq 0$$
$$a_0=\frac{1}{\pi}\left[\int_0^{\pi} 1\,\mathrm{d}x+\int_{\pi}^{2\pi} 2\,\mathrm{d}x\right]=3$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$
$$=\frac{1}{\pi}\left[\int_0^{\pi} 1\cdot\sin nx\,\mathrm{d}x+\int_{\pi}^{2\pi} 2\sin nx\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[-\left(\frac{\cos nx}{n}\right)_0^{\pi} - 2\left(\frac{\cos nx}{n}\right)_{\pi}^{2\pi}\right]$$

$$= -\frac{1}{n\pi}\{1-(-1)^n\}$$

$$= \begin{cases} \frac{-2}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using these values in (1), we get

$$f(x) = \frac{3}{2} - \frac{2}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n}\sin nx \text{ in } (0, 2\pi).$$

Since the series to be summed up contains constant multiples of squares of b_n, we apply Parseval's theorem.

$$\frac{1}{4}a_0^2 + \frac{1}{2}\sum a_n^2 + \frac{1}{2}\sum b_n^2 = \frac{1}{2\pi}\int_0^{2\pi} [f(x)]^2 \, dx.$$

i.e. $$\frac{9}{4} + \frac{1}{2}\cdot\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{1}{2\pi}\left[\int_0^{\pi} 1^2\cdot dx + \int_{\pi}^{2\pi} 2^2 \cdot dx\right]$$

i.e. $$\frac{9}{4} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{5}{2}$$

$\therefore$ $$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{2}\left(\frac{5}{2}-\frac{9}{4}\right) = \frac{\pi^2}{8}$$

Example 15

Find the half-range sine series of $f(x) = a$ in $(0, l)$. Deduce the sum of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty$.

Giving an odd extension for $f(x)$ in $(-l, 0)$, $f(x)$ is made an odd function in $(-l, l)$.

$\therefore$ Let $$f(x) = \sum b_n \sin\frac{n\pi x}{l} \tag{1}$$

$$b_n = \frac{2}{l}\int_0^l a\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2a}{l}\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}_0^l = \frac{2a}{n\pi}\{1-(-1)^n\}$$

$$= \begin{cases} \frac{4a}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using this value in (1), we get

$$a = \frac{4a}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l} \text{ in } (0, l)$$

Since the series whose sum is required contains constant multiples of squares of b_n, we apply Parseval's theorem.

$$\frac{1}{2} \sum b_n^2 = \frac{1}{l} \int_0^l [f(x)]^2 \, dx$$

i.e.

$$\frac{1}{2} \cdot \frac{16a^2}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{(2n-1)^2} = a^2$$

i.e.

$$\frac{8a^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = a^2$$

$\therefore$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Example 16

Find the half-range cosine series of $f(x) = x$ in $(0, 1)$. Deduce the sum of the series $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots \infty$.

Giving an even extension for $f(x)$ in $(-1, 0)$, $f(x)$ is made an even function in $(-1, 1)$.

$\therefore$ Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \tag{1}$$

$$\begin{aligned} a_n &= \frac{2}{1} \int_0^1 f(x) \cos n\pi x \, dx \\ &= 2 \left[x \left(\frac{\sin n\pi x}{n\pi} \right) + \left(\frac{\cos n\pi x}{n^2\pi^2} \right) \right]_0^1 \\ &= \frac{2}{n^2\pi^2} \{(-1)^n - 1\} \\ &= \begin{cases} \frac{-4}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even and} \neq 0. \end{cases} \end{aligned}$$

$$a_0 = \frac{2}{1} \int_0^1 x \, dx = 1$$

Using these values in (1), we get

$$x = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1,3,5..}^{\infty} \frac{1}{n^2} \cos n\pi x \text{ in } (0, 1)$$

Since the series to be summed up contains constant multiples of squares of a_n, we apply Parseval's theorem.

$$\frac{1}{4}a_0^2 + \frac{1}{2}\sum a_n^2 = \frac{1}{1}\int_0^1 x^2\,dx$$

i.e. $$\frac{1}{4} + \frac{1}{2}\cdot\frac{16}{\pi^4}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^4} = \frac{1}{3}$$

i.e. $$\frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{1}{12}$$

$\therefore$ $$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}.$$

Example 17

Find the half-range sine series of $f(x) = l - x$ in $(0, l)$. Hence prove that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty = \frac{\pi^2}{6}$.

Giving an odd extension for $f(x)$ in $(-l, 0)$, $f(x)$ is made an odd function in $(-l, l)$.

$\therefore$ Let $$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \tag{1}$$

$$b_n = \frac{2}{l}\int_0^l (l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2l}{n\pi}$$

Using this value in (1), we get

$$l - x = \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l} \quad \text{in } (0, l)$$

Since the series to be summed up contains constant multiples of squares of b_n, we apply Parseval's theorem.

$$\frac{1}{2}\sum b_n^2 = \frac{1}{l}\int_0^l (l-x)^2\,dx$$

i.e. $$\frac{1}{2}\cdot\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{l}\left[\frac{(l-x)^3}{-3}\right]_0^l=\frac{l^2}{3}$$

$\therefore$ $$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Example 18

Find the half-range cosine series of $f(x)=(\pi-x)^2$ in $(0,\pi)$. Hence find the sum of the series $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\infty$.

Giving an even extention for $f(x)$ in $(-\pi,0)$, the function $f(x)$ is made an even function in $(-\pi,\pi)$.

$\therefore$ Let $$f(x)=\frac{a_0}{2}+\sum a_n\cos nx \tag{1}$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi}(\pi-x)^2\cos nx\,\mathrm{d}x \\ &= \frac{2}{\pi}\left[(\pi-x)^2\left(\frac{\sin nx}{n}\right)-\{-2(\pi-x)\}\left(\frac{-\cos nx}{n^2}\right)\right. \\ &\qquad \left. +2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi} \\ &= \frac{4}{n^2},\quad \text{if } n\neq 0 \end{aligned}$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}(\pi-x)^2\,\mathrm{d}x=\frac{2}{\pi}\left\{\frac{(\pi-x)^3}{-3}\right\}_0^{\pi}=\frac{2}{3}\pi^2$$

Using these values in (1), we get $(\pi-x)^2=\frac{\pi^2}{3}+16\sum\limits_{n=1}^{\infty}\frac{1}{n^2}\cos nx$ in $(0,\pi)$. Since the series to be summed up contains constant multiples of squares of a_n, we apply Parseval's theorem.

$$\frac{a_0^2}{4}+\frac{1}{2}\sum a_n^2=\frac{1}{\pi}\int_0^{\pi}(\pi-x)^4\,\mathrm{d}x$$

i.e. $$\frac{\pi^4}{9}+\frac{1}{2}\cdot 16\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{1}{\pi}\left\{\frac{(\pi-x)^5}{-5}\right\}_0^{\pi}=\frac{\pi^4}{5}$$

i.e. $$8\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{4}{45}\pi^4$$

$\therefore$ $$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$$

Example 19

Find the half-range sine series of $f(x) = \begin{cases} x & \text{in } (0, \frac{\pi}{2}) \\ \pi - x, & \text{in } (\frac{\pi}{2}, \pi) \end{cases}$ in $(0, \pi)$. Hence find the sum of the series $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots \infty$.

Giving an odd extension for $f(x)$ in $(-\pi, 0)$, the function $f(x)$ is made an odd function in $(-\pi, \pi)$.

$\therefore$ Let $\quad f(x) = \sum b_n \sin nx \qquad (1)$

$$\begin{aligned}
b_n &= \frac{2}{\pi} \int_o^{\pi} f(x) \sin nx \, dx \\
&= \frac{2}{\pi} \left[\int_o^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx \right] \\
&= \frac{2}{\pi} \left[\left\{ x \left(-\frac{\cos nx}{n} \right) + \frac{\sin nx}{n^2} \right\}_0^{\pi/2} \right. \\
&\quad \left. + \left\{ (\pi - x) \left(-\frac{\cos nx}{n} \right) - \frac{\sin nx}{n^2} \right\}_{\pi/2}^{\pi} \right] \\
&= \frac{2}{\pi} \left[\frac{-\pi}{2n} \cos \frac{n\pi}{2} + \frac{1}{n^2} \sin \frac{n\pi}{2} \right. \\
&\quad \left. + \frac{\pi}{2n} \cos \frac{n\pi}{2} + \frac{1}{n^2} \sin \frac{n\pi}{2} \right] \\
&= \frac{4}{\pi n^2} \sin \frac{n\pi}{2}, \text{ which becomes zero for even values of } n.
\end{aligned}$$

Using this value in (1), we get

$$f(x) = \frac{4}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin nx \text{ in } (0, \pi)$$

Since the series whose sum is required contains constant multiples of squares of b_n, we apply Parseval's theorem.

$$\frac{1}{2} \sum b_n^2 = \frac{1}{\pi} \int_0^{\pi} [f(x)]^2 \, dx$$

i.e. $$\frac{1}{2} \cdot \frac{16}{\pi^2} \sum_{n=1,3,5..}^{\infty} \frac{1}{n^4} \sin^2 \frac{n\pi}{2} = \frac{1}{\pi} \left[\int_0^{\pi/2} x^2 \, dx + \int_{\pi/2}^{\pi} (\pi - x)^2 \, dx \right]$$

i.e. $$\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{\pi^2}{12}$$

$$\left(\because \sin\frac{n\pi}{2}=\pm 1,\ \text{when n is odd and} \sin^2\frac{n\pi}{2}=1\right)$$

$\therefore$ $$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{\pi^4}{96}.$$

Example 20

Find the half-range cosine series of $f(x)=x(\pi-x)$ in $(0,\pi)$. Hence find the sum of the series $1/1^4+1/2^4+1/3^4+\ldots\infty$.
Giving an even extension for $f(x)$ in $(-\pi,0)$, the function $f(x)$ is made an even function in $(-\pi,\pi)$

$$\therefore \text{ Let } f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \tag{1}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi}x(\pi-x)\cos nx\ \mathrm{d}x\\
&= \frac{2}{\pi}\left[(\pi x-x^2)\left(\frac{\sin nx}{n}\right)-(\pi-2x)\left(-\frac{\cos nx}{n^2}\right)+2\frac{\sin nx}{n^3}\right]_\theta^\pi\\
&= -\frac{2}{n^2}\{(-1)^n+1\}\\
&= \begin{cases}0, & \text{if } n \text{ is odd}\\ -\frac{4}{n^2}, & \text{if } n \text{ is even and} \neq 0\end{cases}
\end{aligned}$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}(\pi x-x^2)\ \mathrm{d}x=\frac{2}{\pi}\left\{\pi\frac{x^2}{2}-\frac{x^3}{3}\right\}_0^{\pi}=\frac{\pi^2}{3}$$

Using these values in (1), we get

$$x(\pi-x)=\frac{\pi^2}{6}-4\cdot\sum_{n=2,4,6..}^{\infty}\frac{1}{n^2}\cos nx \text{ in } (0,\pi)$$

or $$x(\pi-x)=\frac{\pi^2}{6}-\sum_{n=1}^{\infty}\frac{1}{n^2}\cos 2nx \text{ in } (0,\pi)$$

Since the series to be summed up contains constant multiples of squares of a_n, we apply Parseval's theorem.

$$\frac{1}{4}a_0^2+\frac{1}{2}\sum a_n^2=\frac{1}{\pi}\int_0^{\pi}x^2(\pi-x)^2\ \mathrm{d}x$$

i.e. $$\frac{\pi^4}{36}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{1}{\pi}\int_0^{\pi}(\pi^2x^2-2\pi x^3+x^4)\ \mathrm{d}x$$

$$= \frac{1}{\pi}\left(\pi^2\frac{x^3}{3} - 2\pi\frac{x^4}{4} + \frac{x^5}{5}\right)_0^\pi$$

$$= \frac{\pi^4}{30}$$

i.e. $$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{180}$$

$\therefore$ $$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

Exercise 2(b)

Part A (Short-Answer Questions)

1. Why is Fourier half-range series called so?
2. When a function is defined in $(0, 2l)$, is it possible to expand it as a Fourier half-range series? How?
3. If $f(x)$ is defined in $(0, l)$, how should $f(x)$ be defined in $(-l, 0)$, so that the Fourier half-range series of $f(x)$ may contain (i) only cosine terms and (ii) only sine terms?
4. If $f(x)$ is defined in $(0, l)$, how should $f(x)$ be defined in $(l, 2l)$, so that the Fourier half-range series of $f(x)$ may contain (i) only cosine terms and (ii) only sine terms?
5. When $f(x)$, defined in $(-\pi, 0)$, is expanded as a Fourier half-range cosine series, write down the formula for the Fourier coefficients.
6. When $f(x)$ defined in $(-l, 0)$ is expanded as a Fourier half- range sine series, write down the formula for the Fourier coefficients.
7. Write down the even and odd extensions of $f(x)$ in $(-l, 0)$, if $f(x) = x^2+x$ in $(0, l)$.
8. Write down the extensions of $f(x)$ in $(l, 2l)$, if $f(x) = x(l-x)$ in $(0, l)$ so as to get cosine and sine series..
9. Define the root-mean square value of a function $f(x)$ in $(0, 2\pi)$.
10. State Parseval's theorem.
11. If the impressed voltage E at time t is given by the series $E = \sum_{n=1,3,5..}^{\infty} E_n \sin(n\omega t + \alpha_n)$, find the effective value of E.

Note

The R. M. S. value is also called the effective value. Rewrite E as $E = \sum_{n=1,3,5..}^{\infty}(E_n\sin\alpha_n)\cos n\omega t + \sum_{n=1,3,5..}^{\infty}(E_n\cos\alpha_n)\sin n\omega t$ *and use Parseval's theorem*

12. If an alternating current I is represented by the series $I = \sum_{n=1,3,5..}^{\infty} I_n \sin(n\omega t + \alpha_n)$, find the effective value of I.

13. If the half-range series of $f(x) = 1$ in $(0, l)$ is given by $1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\frac{(2n-1)\pi x}{l}$, find the value of $1^{-2}+3^{-2}+5^{-2}+\ldots\infty$.
14. If the half-range cosine series of $(x^2 - x + 1/6)$ in $0 \le x \le 1$ is $\frac{a_0}{2} + \sum a_n \cos n\pi x$, find the value of $a_0^2 + 2\sum a_n^2$.
15. If the half-range sine series of $x(\pi - x)$ in $0 \le x \le \pi$ is $\sum b_n \sin nx$, find the value of $\sum b_n^2$.

Part B

16. Obtain the half-range cosine series of $f(x) = \pi^2 - x^2$ in $(0, \pi)$. Deduce the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2}\ldots\infty$.
17. Find the half-range sine series of $f(x)$ in $(0, 2l)$, given that

$$f(x) = \begin{cases} kx, & \text{in } (0, l) \\ k(2l - x), & \text{in } (l, 2l). \end{cases}$$

Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\infty$.
18. Find the half-range cosine series of the function $f(x) = (x+2)^2$ in $(-2, 0)$. Hence find the value of $\sum_{n=1}^{\infty}\frac{1}{n^2}$.
19. Find the half-range sine series of the function $f(x) = l - x$ in $(l, 2l)$. Deduce the sum of the series $1 - 1/3 + 1/5 - 1/7 + \ldots\infty$.
20. Find the half-range sine series of $f(x) = x(\pi - x)$ in $(0, \pi)$. How should $f(x)$ be extended in order to get this sine series in $(-\pi, 0)$ and in $(\pi, 2\pi)$? Also find the sum of the series $1 - 1/3^3 + 1/5^3\ldots\infty$.
21. Find the half-range sine series of $f(x)$ in $(0, l)$, given that

$$f(x) = \begin{cases} \frac{b}{a}x & \text{in } (0, a) \\ \frac{b}{l-a}(l - x) & \text{in } (a, l) \end{cases}$$

22. Find the half-range sine series of $f(x) = \cos x$ in $0 < x < \pi$. How should $f(x)$ be defined at $x = 0$ and $x = \pi$, so that the series converges to $f(x)$ in $0 \le x \le \pi$?
23. Find the half-range cosine series of $f(x) = \cos ax$ in $(0, \pi)$, where a is neither zero nor an integer.
24. Find the half-range sine series of

$$f(x) = \begin{cases} \sin x, & \text{in } 0 \le x \le \pi/4 \\ \cos x, & \text{in } \pi/4 \le x \le \pi/2 \end{cases}$$

25. Find the half-range sine series of $f(x) = x\cos\pi x$ in $(0, 1)$. Deduce the sum of the series $\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4}\ldots\infty$.
26. Find the half-range sine series of $f(x) = x\sin x$ in $(0, \pi)$.
27. Find the half-range cosine series of $f(x) = 6x^2 - 6x + 1$ in $(0, 1)$. Deduce the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\infty$.
28. Find the Fourier sine series of $f(x) = e^{ax}$ in $(0, \pi)$.
29. Find the Fourier series of period 2π for the function $f(x) = x^2$ in $(-\pi, \pi)$. Hence find the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^4}$.

30. Find the Fourier series of period 2 for the function $f(x) = x^2$ in $(0, 2)$. Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$, assuming that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

31. Find the Fourier series of period 2 for the function $f(x) = x^2 + x$ in $(-1, 1)$. Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$, given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

32. Find the Fourier series of period 3 for the function $f(x) = 2x - x^2$ in $(0, 3)$. Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$, given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$.

33. Find the Fourier series of period π for the function $f(x) = \begin{cases} x, & \text{in } (0, \pi/2) \\ \frac{\pi}{2} - x, & \text{in } (\pi/2, \pi) \end{cases}$ Hence find the sum of $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$, given that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

34. Find the Fourier series of period 4 for the function $f(x) = \begin{cases} 2, & \text{in } (-2, 0) \\ x, & \text{in } (0, 2) \end{cases}$

    Hence find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$, assuming that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

35. Find the Fourier series of period 2π for the function $f(x) = \begin{cases} 0, & \text{in } (0, \pi) \\ a, & \text{in } (\pi, 2\pi) \end{cases}$

    Hence deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

36. Find the half-range cosine series of $f(x) = \begin{cases} 1, & \text{in } (0, 1) \\ 2, & \text{in } (1, 2) \end{cases}$ in (0,2) Hence find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

37. Find the half-range sine series of $f(x) = \frac{\pi}{2} - x$ in $(0, \pi)$. Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

38. Find the half-range cosine series of $f(x) = 1 + x$ in $(0, 1)$. Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$.

39. Find the half-range sine series of $f(x) = \begin{cases} 2x, & \text{in } (0, 1) \\ 4 - 2x, & \text{in } (1, 2) \end{cases}$

    Hence deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$.

40. Find the half-range cosine series of $f(x) = \begin{cases} x, & \text{in } (0, \pi/2) \\ \pi - x, & \text{in } (\pi/2, \pi) \end{cases}$

    Hence deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$.

2.11 HARMONIC ANALYSIS

Introduction

We know that the Fourier series of $f(x)$ in $(0, 2l)$ or $(-l, l)$ is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

i.e.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left\{\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos\frac{n\pi x}{l} + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin\frac{n\pi x}{l}\right\} \tag{1}$$

Let $A_n = \sqrt{a_n^2 + b_n^2}$ and $\alpha_n = \tan^{-1}\frac{b_n}{a_n}$, so that

$$\cos\alpha_n = \frac{a_n}{\sqrt{a_n^2 + b_n^2}} \text{ and } \sin\alpha_n = \frac{b_n}{\sqrt{a_n^2 + b_n^2}}$$

Using these in (1), we get the Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \left(\cos\frac{n\pi x}{l} \cos\alpha_n + \sin\frac{n\pi x}{l} \sin\alpha_n\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l} - \alpha_n\right) \tag{2}$$

If we assume $A_n = \sqrt{a_n^2 + b_n^2}$ and $\beta_n = \tan^{-1}\frac{a_n}{b_n}$,
(1) will take the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l} + \beta_n\right) \tag{3}$$

$A_n \cos\left(\frac{n\pi x}{l} - \alpha_n\right)$ or $A_n \sin\left(\frac{n\pi x}{l} + \beta_n\right)$ is called the n^{th} *harmonic* in the Fourier expansion of $f(x)$.

The first harmonic $A_1 \cos\left(\frac{\pi x}{l} - \alpha_1\right)$ or $A_1 \sin\left(\frac{\pi x}{l} + \beta_1\right)$ is also called *the fundamental term* in the Fourier expansion of $f(x)$.

The second harmonic $A_2 \cos\left(\frac{2\pi x}{l} - \alpha_2\right)$ or $A_2 \sin\left(\frac{2\pi x}{l} + \beta_2\right)$ is also called *the octave* in the Fourier expansion of $f(x)$.

It is clear that we require the values of a_n and b_n to calculate the n^{th} harmonic. When $f(x)$ is defined by one or more mathematical expressions, the Fourier coefficients a_n and b_n are found out by integration using Euler's formulas. But in some practical problems, $f(x)$ will be defined by means of its values at equally spaced values of x

in the given interval. In such problems, $f(x)$ will be defined in $(0, 2l)$ in a tabular form as given below:

x	x_0	x_1	x_2	$-$	x_{k-1}
$y = f(x)$	y_0	y_1	y_2	$-$	y_{k-1}

Here $x_1 - x_0 = x_2 - x_1 = \ldots = x_k - x_{k-1} = \frac{2l}{k}$ and $x_0 = 0$ and $x_k = 2l$.

When $y = f(x)$ is defined in a tabular form as given above, a_n and b_n cannot be evaluated exactly by mathematical integration, but are evaluated approximately by numerical integration as explained below:

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= 2 \times \left[\frac{1}{2l}\int_0^{2l} f(x).\cos\frac{n\pi x}{l}\,dx\right]$$

$$= 2 \times \text{ Mean value of } f(x)\cos\frac{n\pi x}{l} \text{ over } (0, 2l)$$

Note ✍

We recall that the mean square value of $y = f(x)$ over $(0, 2l)$ was defined as

$$\bar{y}^2 = \frac{1}{2l}\int_0^{2l} y^2\,dx \text{ or } \frac{1}{2l}\int_0^{2l} [f(x)]^2\,dx$$

$$a_n \simeq 2 \times \textit{ statistical average value of } f(x)\cos\frac{n\pi x}{l} \textit{ or } y\cos\frac{n\pi x}{l} \textit{ over } (0, 2l)$$

$$\simeq 2 \times \frac{1}{k}\sum_{r=0}^{k-1} y_r\cos\frac{n\pi x_r}{l},\ n = 0, 1, 2, \ldots$$

In particular, $a_0 \simeq 2 \times \frac{1}{k}\sum_{r=0}^{k-1} y_r$

Similarly $b_n \simeq 2 \times \frac{1}{k}\sum_{r=0}^{k-1} y_r \sin\frac{n\pi x_r}{l},\ n = 1, 2, \ldots$

Note ✍

1. *When the interval $(0, 2l)$ is divided into k equal sub-intervals, each of length $\frac{2l}{k}$, only k values of $y = f(x)$ are taken into consideration for numerical computation of a_n and b_n.*
   i.e. either the values $y_0, y_1, \ldots, y_{k-1}$ corresponding to the left ends of the various sub-intervals, namely $x_0, x_1, \ldots, x_{k-1}$ are considered or the values

$y_1, y_2, \ldots, y_k$ *corresponding to the right ends of the various sub-intervals, namely* $x_1, x_2, \ldots, x_k$ *are considered, where* $x_0 = 0$ *and* $x_k = 2l$.

2. *The process of finding the harmonics in the Fourier expansion of a function numerically is known as harmonic analysis.*
3. *In most situations, the amplitudes of the successive harmonics* $A_1, A_2, A_3, \ldots$ *will decrease very rapidly. Hence in most harmonic analysis problems, we may have to find the first few harmonics only.*
4. *Though* $A_n \cos\left(\frac{n\pi x}{l} - \alpha_n\right)$ *or* $A_n \sin\left(\frac{n\pi x}{l} + \beta_n\right)$ *is called the* n^{th} *harmonic, it need not be put in either of these forms. It is enough if we give the* n^{th} *harmonic in the form* $\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$.

2.12 COMPLEX FORM OF FOURIER SERIES

The Fourier series of $f(x)$ in $(c, c+2l)$ can also be put in the exponential form with complex coefficients as explained below:
The trigonometric form of the Fourier series of $f(x)$ defined in $(c, c+2l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \tag{1}$$

Using the exponential values of $\cos\frac{n\pi x}{l}$ and $\sin\frac{n\pi x}{l}$, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right) + b_n\left(\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{in\pi x/l} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{\frac{-in\pi x}{l}} \tag{2}$$

If we put $\frac{a_0}{2} = c_0$, $\frac{a_n - ib_n}{2} = c_n$ and $\frac{a_n + ib_n}{2} = c_{-n}$, then (2) can be put as

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/l} + \sum_{n=1}^{\infty} c_{-n} e^{-in\pi x/l}$$

i.e. $$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/l} + \sum_{n=-\infty}^{-1} c_n e^{in\pi x/l}$$

i.e. $$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} \tag{3}$$

Equation (3) is called *the complex form* or *exponential form* of the Fourier series of $f(x)$ in $(c, c+2l)$. The coefficient c_n in (3) is given by

$$c_n = \frac{1}{2}(a_n - ib_n)$$

$$= \frac{1}{2}\left[\frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx - \frac{i}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx\right],$$

by Euler's formulas for Fourier Coefficients.

$$= \frac{1}{2l}\int_c^{c+2l} f(x)\left[\cos\frac{n\pi x}{l} - i\,\sin\frac{n\pi x}{l}\right]dx$$

$$= \frac{1}{2l}\int_c^{c+2l} f(x)e^{-in\pi x/l}\,dx \tag{4}$$

This formula for c_n holds good for positive and negative integral values of n and for $n = 0$

Note

When $l = \pi$, the complex form of Fourier series of $f(x)$ in $(c, c + 2\pi)$ takes the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad \textit{where}$$

$$c_n = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)e^{-inx}\,dx.$$

Worked Examples 2(c)

Example 1

Obtain the first three harmonics in the Fourier series expansion in (0, 12) for the function $y = f(x)$ defined by the table given below:

x:	0	1	2	3	4	5	6	7	8	9	10	11
y:	1.8	1.1	0.3	0.16	0.5	1.5	2.16	1.88	1.25	1.30	1.76	2.00

The length of the interval $= 2l = 12$ $\therefore l = 6$.
$\therefore$ The Fourier series of $y = f(x)$ is of the form

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{6} + \sum b_n \sin\frac{n\pi x}{6}$$

The interval (0, 12) is divided into 12 subintervals, each of length 1.
The values of y at the left end-points of the 12 sub-intervals, namely at $x = 0, 1, 2, \cdots 11$, are given.

$$\therefore \quad a_0 = 2 \times \frac{1}{12} \sum_{r=0}^{11} y_r$$

$$a_n = 2 \times \frac{1}{12} \sum_{r=0}^{11} y_r \cos \frac{n\pi x_r}{6}; \; b_n = 2 \times \frac{1}{12} \sum y_r \sin \frac{n\pi x_r}{6}$$

To compute $a_0, a_1, a_2, a_3, b_1, b_2, b_3$, we tabulate the values of x_r, y_r, $\cos \frac{n\pi x_r}{6}$ $(n = 1, 2, 3)$ and $\sin \frac{n\pi x_r}{6}$ $(n = 1, 2, 3)$ as shown below:

x_r	y_r	$\cos \frac{\pi x_r}{6}$	$\sin \frac{\pi x_r}{6}$	$\cos \frac{\pi x_r}{3}$	$\sin \frac{\pi x_r}{3}$	$\cos \frac{\pi x_r}{2}$	$\sin \frac{\pi x_r}{2}$
0	1.8	1	0	1	0	1	0
1	1.1	0.866	0.5	0.5	0.866	0	1
2	0.3	0.5	0.866	−0.5	0.866	−1	0
3	0.16	0	1	−1	0	0	−1
4	0.5	−0.5	0.866	−0.5	−0.866	1	0
5	0.15	−0.866	0.5	0.5	−0.866	0	1
6	2.16	−1	0	1	0	−1	0
7	1.88	−0.866	−0.5	0.5	0.866	0	−1
8	1.25	−0.5	−0.866	−0.5	0.866	1	0
9	1.30	0	−1	−1	0	0	1
10	1.76	0.5	−0.866	−0.5	−0.866	−1	0
11	2.00	0.866	0.5	0.5	−0.866	0	−1

$$a_0 = \frac{1}{6} \sum y_r = \frac{1}{6} \times 14.36 = 2.393$$

$$\begin{aligned} a_1 &= \frac{1}{6} \sum y_r \cos \frac{\pi x_r}{6} \\ &= \frac{1}{6} [(1.8 - 2.16) + (1.1 + 2.00 - 0.15 - 1.88) \times 0.866 \\ &\quad +(0.3 + 1.76 - 0.5 - 1.25) \times 0.5] \\ &= 0.120 \end{aligned}$$

$$\begin{aligned} b_1 &= \frac{1}{6} \sum y_r \sin \frac{\pi x_r}{6} \\ &= \frac{1}{6} [(0.16 - 1.30) + (0.3 + 0.5 - 1.25 - 1.76) \times 0.866 \\ &\quad +(1.1 + 0.15 - 1.88 - 2.00) \times 0.5] \\ &= -0.728 \end{aligned}$$

$$a_2 = \frac{1}{6}\sum y_r \cos\frac{2\pi x_r}{6} \text{ or } \frac{1}{6}\sum y_r \cos\frac{\pi x_r}{3}$$
$$= \frac{1}{6}[(1.8 - 0.16 + 2.16 - 1.30) + (1.1 - 0.3 - 0.5 + 0.15$$
$$+1.88 - 1.25 - 1.76 + 2.00) \times 0.5]$$
$$= 0.527$$
$$b_2 = \frac{1}{6}\sum y_r \sin\frac{\pi x_r}{3}$$
$$= \frac{1}{6}[(1.1 + 0.3 - 0.5 - 0.15 + 1.88 + 1.25 - 1.76 - 2.00) \times .866]$$
$$= 0.104$$
$$a_3 = \frac{1}{6}\sum y_r \cos\frac{3\pi x_r}{6} \text{ or } \frac{1}{6}\sum y_r \cos\frac{\pi x_r}{2}$$
$$= \frac{1}{6}(1.8 - 0.3 + 0.5 - 2.16 + 1.25 - 1.76)$$
$$= -0.112$$
$$b_3 = \frac{1}{6}\sum y_r \sin\frac{\pi x_r}{2}$$
$$= \frac{1}{6}[1.1 - 0.16 + 0.15 - 1.88 + 1.30 - 2.00]$$
$$= -0.248$$

$\therefore$ The Fourier series of $f(x)$ in (0, 12) upto the third harmonic is

$$f(x) = 1.197 + \left(0.120\cos\frac{\pi x}{6} - 0.728\sin\frac{\pi x}{6}\right)$$
$$+ \left(0.527\cos\frac{\pi x}{3} + 0.104\sin\frac{\pi x}{3}\right) + \left(-0.112\cos\frac{\pi x}{2} - 0.248\sin\frac{\pi x}{2}\right)$$

Example 2

The following are 12 values of y corresponding to equidistant values of the angles x° in the range 0° to 360°. Find the first three harmonics in the Fourier series expansion of y in $(0, 2\pi)$

x°:	0	30	60	90	120	150	180	210	240
y:	10.5	20.2	26.4	29.3	27.0	21.5	12.8	1.6	−11.2
x°:	270	300	330						
y:	−18.0	−15.8	−3.5						

Since $f(x)$ is defined in $(0, 2\pi)$, the Fourier series is of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$.

$$a_0 = 2 \times \frac{1}{12} \sum_{r=0}^{11} y_r = \frac{1}{6} \times 100.8 = 16.8$$

To compute $a_1, a_2, a_3, b_1, b_2, b_3$, we may use the following graphical method, known as *Harrison's method*, instead of the tabulation method.

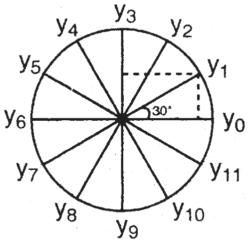


Fig. 2.11

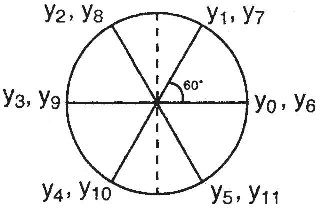


Fig. 2.12

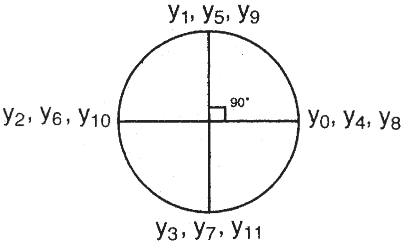


Fig. 2.13

We draw a circle of convenient radius and divide the central angle into 12 equal parts each of magnitude 30° by means of radii vectors. The radii vectors which measure the angles 0°, 30°, 60°, 90°, ... 330° are supposed to be of lengths $y_0, y_1, y_2, \ldots, y_{11}$ (not geometrically) and this is indicated near the ends of corresponding radii vectors. [Fig.(2.11)]

Now $\quad a_1 = \frac{1}{6} \sum y_r \cos x_r$

$$= \frac{1}{6} \left[y_0 \cos 0^\circ + y_1 \cos 30^\circ + y_2 \cos 60^\circ + \ldots + y_{11} \cos 330^\circ \right]$$

$$= \frac{1}{6} \times \text{sum of the horizontal projections of the various radii vectors}$$

in the Harrison's circle for 30°.

While computing the sum, those horizonal projections that lie on the right of the vertical are taken to be positive and those on the left are taken to be negative. Also those horizontal projections that contain cos 30° are grouped separately and so are those that contain cos 60°:

Thus

$$a_1 = \frac{1}{6}\left[(y_0 - y_6) + (y_1 + y_{11} - y_5 - y_7)\cos 30^\circ + (y_2 + y_{10} - y_4 - y_8)\cos 60^\circ\right]$$

Note ✍

The horizontal projections of y_3 and y_9 are zero each.

$$\begin{aligned} a_1 &= \frac{1}{6}[(10.5 - 12.8) + (20.2 - 3.5 - 21.5 - 1.6) \times 0.866 \\ &\quad +(26.4 - 15.8 - 27.0 + 11.2) \times 0.5] \\ &= -1.740 \\ b_1 &= \frac{1}{6}\sum y_r \sin x_r \\ &= \left[y_0 \sin 0^\circ + y_1 \sin 30^\circ + y_2 \sin 60^\circ + \ldots + y_{11} \sin 330^\circ\right] \\ &= \frac{1}{6} \times \text{Sum of the vertical projections of the various radii vectors} \\ &\quad \text{in the Harrison's circle for } 30^\circ. \end{aligned}$$

While computing the sum, those vertical projections that lie above the horizontal line are taken to be positive and those below the horizontal line are taken to be negative. As before the terms with sin 30° are grouped together and those with sin 60° are grouped separately. The vertical projections of the horizonal radii vectors (i.e. y_0 and y_6) are taken as zero each.

Thus

$$\begin{aligned} b_1 &= \frac{1}{6}\left[(y_3 - y_9) + (y_1 + y_5 - y_7 - y_{11})\sin 30^\circ + (y_2 + y_4 - y_8 - y_{10})\sin 60^\circ\right] \\ &= \frac{1}{6}[(29.3 + 18.0) + (20.2 + 21.5 - 1.6 + 3.5) \times 0.5 \\ &\quad +(26.4 + 27.0 + 11.2 + 15.8) \times 0.866] \\ &= 23.121 \end{aligned}$$

To compute a_2 and b_2, we use Harrison's circle for 60° [Fig.(2.12)]

$$\begin{aligned} a_2 &= \frac{1}{6}\sum y_r \cos 2x_r \\ &= \frac{1}{6}[(y_0 + y_6 - y_3 - y_9) + (y_1 + y_7 + y_5 + y_{11} - y_2 - y_8 - y_4 - y_{10}) \times 0.5] \\ &= 3.117 \end{aligned}$$

$$b_2 = \frac{1}{6}\sum y_r \sin 2x_r$$

$$= \frac{1}{6}[y_1 + y_7 + y_2 + y_8 - y_4 - y_{10} - y_5 - y_{11}) \times 0.866]$$

$$= 1.126$$

To compute a_3 and b_3, we use Harrison's circle for 90° [Fig. (2.13)]

$$a_3 = \frac{1}{6}\sum y_r \cos 3x_r$$

$$= \frac{1}{6}(y_0 + y_4 + y_8 - y_2 - y_6 - y_{10}) = 0.483$$

$$b_3 = \frac{1}{6}\sum y_r \sin 3x_r = \frac{1}{6}(y_1 + y_5 + y_9 - y_3 - y_7 - y_{11}) = -0.617$$

$\therefore$ The required Fourier series is

$$f(x) = 8.4 + (-1.740\cos x + 23.121\sin x) + (3.117\cos 2x + 1.126\sin 2x) + (0.483\cos 3x - 0.617\sin 3x) + \ldots$$

Example 3

A function $y = f(x)$ is given by the following table of values. Make a harmonic analysis of the function upto the third harmonic.

$x°$:	45	90	135	180	225	270	315	360	405
y:	1.5	1.0	0.5	0	0.5	1.0	1.5	2.0	1.5
$x°$:	450	495	540	585	630	675	720		
y:	1.0	0.5	0	0.5	1.0	1.5	1.0		

We note that $f(2\pi + x) = f(x)$

$\therefore$ $f(x)$ is periodic with period 2π.

$\therefore$ It is enough we consider the values of $f(x)$ in one period, say $(0, 2\pi)$. We also note that

$$f(360° - 45°) = 1.5 = f(45°)$$

$$f(360° - 90°) = 1.0 = f(90°), \text{ etc.}$$

i.e. $$f(2\pi - x) = f(x)$$

Hence the Fourier series of $f(x)$ will contain only cosine terms, i.e. $b_1 = b_2 = b_3 = 0$. The interval $(0, 2\pi)$ is divided into sub-intervals, each of length $\frac{\pi}{4}$, i.e. it is divided into 8 sub-intervals.

Hence we should consider only 8 values of $y = f(x)$ for harmonic analysis, i.e. the values of $y = f(x)$ at the right ends of various sub-intervals, namely, $45°, 90°, 135°, \ldots 360°$. We shall call the values of y as $y_1, y_2, y_3, \ldots, y_8$.

$$a_0 = 2 \times \frac{1}{8}\sum_{r=1}^{8} y_r = \frac{1}{4} \times 8.0 = 2.0$$

The values of a_1, a_2 and a_3 are found out using Harrison's circles for $45°$, $90°$ and $135°$ as shown in Figs 2.14, 2.15 and 2.16 respectively.

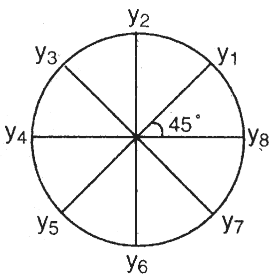


Fig. 2.14

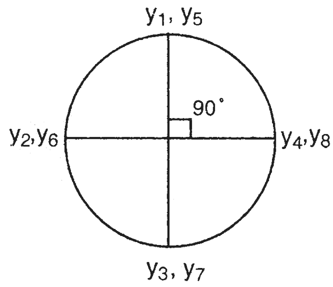


Fig. 2.15

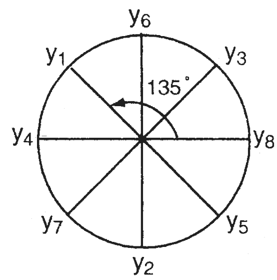


Fig. 2.16

$$a_1 = \frac{1}{4}\left[(y_8 - y_4) + (y_1 + y_7 - y_3 - y_5)\cos 45°\right] = 0.854$$

$$a_2 = \frac{1}{4}(y_4 + y_8 - y_2 - y_6) = 0$$

$$a_3 = \frac{1}{4}\left[(y_8 - y_4) + (y_3 + y_5 - y_1 - y_7)\cos 45°\right] = 0.147$$

$\therefore$ The required Fourier series is

$$f(x) = 1.0 + 0.854\cos x + 0.147\cos 3x.$$

Example 4

A function $y = f(x)$ is given by the following table of values. Make a harmonic analysis of the function in $(0, T)$ upto the second harmonic.

x:	0	T/6	T/3	T/2	2T/3	5T/6	T
y:	0	9.2	14.4	17.8	17.3	11.7	0

The interval $(0, T)$ is divided into sub-intervals each of length $T/6$, i.e., it is divided into 6 sub-intervals.

Hence we consider only 6 values of $y = f(x)$ i.e., $y_0, y_1, \ldots y_5$ corresponding to $x = 0, T/6, \ldots, \frac{5T}{6}$. Since $2l = T$, the Fourier series is of the form $y = \frac{a_0}{2} +$

$$\cos \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{l}$$

$$a_0 = 2 \times \frac{1}{6}\sum y_r = \frac{1}{3} \times 70.4 = 23.47$$

Since $a_1 = \frac{1}{3}\sum y_r \cos\frac{2\pi x_r}{T}$ and $b_1 = \frac{1}{3}\sum y_r \sin\frac{2\pi}{T}x_r$ and hence the arguments of cosine and sine functions increase by $\pi/3$, we use a Harrison circle for $60°$ [Fig. 2.17].

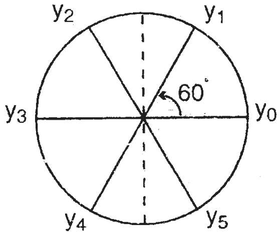


Fig. 2.17

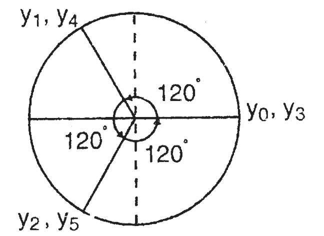


Fig. 2.18

$$a_1 = \frac{1}{3}\left[(y_0 - y_3) + (y_1 + y_5 - y_2 - y_4)\cos 60^\circ\right]$$
$$= \frac{1}{3}[-17.8 + (-10.8) \times 0.5]$$
$$= -7.733$$
$$b_1 = \frac{1}{3}(y_1 + y_2 - y_4 - y_5)\sin 60^\circ$$
$$= \frac{1}{3} \times (-5.4) \times 0.866 = -1.559$$

As the arguments of the cosine and sine functions in the formulas $a_2 = \frac{1}{3}\sum y_r \cos\frac{4\pi}{T}x_r$ and $b_2 = \frac{1}{3}\sum y_r \sin\frac{4\pi}{T}x_r$ increase by $\frac{2\pi}{3}$, we use a Harrison's circle for 120° [Fig.2.18].

$$a_2 = \frac{1}{3}\left[(y_0 + y_3) - (y_1 + y_4 + y_2 + y_5)\cos 60^\circ\right]$$
$$= \frac{1}{3}[17.8 - 52.6 \times 0.5] = -2.833$$
$$b_2 = \frac{1}{3}\left[(y_1 + y_4 - y_2 - y_5)\sin 60^\circ\right]$$
$$= \frac{1}{3} \times 0.4 \times 0.866 = 0.115$$

$\therefore$ The Fourier series upto the second harmonic is

$$f(x) = 11.735 - 7.733\cos\frac{2\pi x}{T} - 1.559\sin\frac{2\pi x}{T}$$
$$- 2.833\cos\frac{4\pi x}{T} + 0.115\sin\frac{4\pi x}{T}$$

Example 5

The turning moment T units of the crank shaft of a steam engine is given for a series of values of the crank-angle θ in degrees in the following table:

θ:	0	30	60	90	120	150	180
T:	0	5224	8097	7850	5499	2626	0

Find the first three terms in a series of sines to represent T. Also find T when $\theta = 75°$.
The half-range sine series of $T = f(\theta)$ in $(0, \pi)$ is required. Let it be

$$f(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta$$

$$\text{Then } b_n = \frac{2}{\pi}\int_o^{\pi} f(\theta)\sin n\theta d\theta$$

$$= 2 \times \text{Mean value of} f(\theta)\sin n\theta \text{over}(0, \pi)$$

$$= 2 \times \frac{1}{k}\sum_{r=0}^{k-1} T_r \sin n\theta_r.$$

Since the interval $(0, \pi)$ is divided into sub-intervals, each of length $\frac{\pi}{6}$, we consider only 6 values of T, namely $T_0, T_1, T_2, \ldots T_5$, corresponding to $\theta = 0, 30°, 60°, \ldots, 150°$

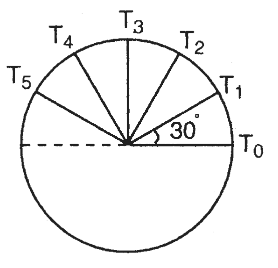


Fig. 2.19

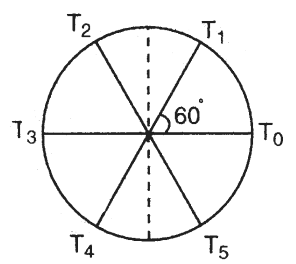


Fig. 2.20

$$b_1 = 2 \times \frac{1}{6}\sum_{r=0}^{5} T_r \sin\theta_r$$

$$= \frac{1}{3}\left[T_3 + (T_1 + T_5)\sin 30° + (T_2 + T_4)\sin 60°\right]$$

(from the Harrison's circle for 30°,Fig. 2.19)

$$= \frac{1}{3}[7850 + 3925 + 11774] = 7850$$

$$b_2 = 2 \times \frac{1}{6}\sum_{r=0}^{5} T_r \sin 2\theta_r$$

$$= \frac{1}{3}\left[(T_1 + T_2 - T_4 - T_5)\sin 60°\right]$$

(from the Harrison's circle for 60°,Fig. 2.20)

$$= \frac{1}{3} \times 5196 \times 0.866 = 1500$$

$$b_3 = 2 \times \frac{1}{6}\sum_{r=0}^{5} T_r \sin 3\theta_r$$
$$= \frac{1}{3}(T_1 + T_5 - T_3), \text{ (from the Harrisons's circle for } 90^\circ\text{,Fig. 2.21)}$$
$$= \frac{1}{3}(7850 - 7850) = 0$$

∴ The required Fourier sine series upto the third harmonic is

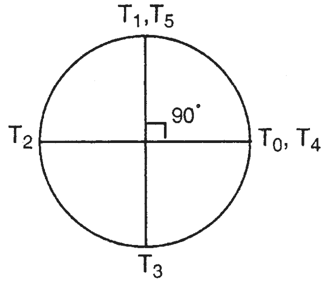


Fig. 2.21

$$T = 7850 \sin\theta + 1500 \sin 2\theta + 0.\sin 3\theta, \text{ in } (0, \pi)$$
$$[T]_{\theta=75^\circ} = 7850 \sin 75^\circ + 1500 \sin 150^\circ$$
$$= 8332.5 \text{ units}$$

Example 6

Obtain the constant term and the first three harmonics in the Fourier cosine series of $y = f(x)$ in (0, 6) using the following table:

x:	0	1	2	3	4	5
y:	4	8	15	7	6	2

The interval (0, 6) is divided into 6 sub-intervals each of length 1. Hence we consider the 6 values of y, namely $y_0, y_1, \ldots, y_5$, corresponding to $x = 0, 1, \ldots, 5$ for harmonic analysis. As the half-range cosine series is required in (0, 6), $l = 6$.

∴ Fourier cosine series of $f(x)$ in (0, 6) is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{6} \text{in}(0, 6)$$
$$a_0 = \frac{2}{6}\int_0^6 f(x)\, dx = 2 \times \frac{1}{6}\sum_{r=0}^{5} y_r = 14$$

$$a_1 = 2 \times \frac{1}{6} \sum_{r=0}^{5} y_r \cos \frac{\pi x_r}{6}$$

$$= \frac{1}{3} \left[y_0 + (y_1 - y_5) \cos 30^\circ + (y_2 - y_4) \cos 60^\circ \right],$$

(from the Harrison's circle for 30°, Fig. 2.22)

$$= 4.565$$

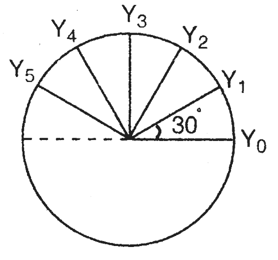


Fig. 2.22

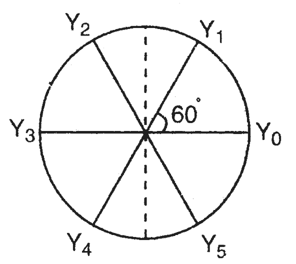


Fig. 2.23

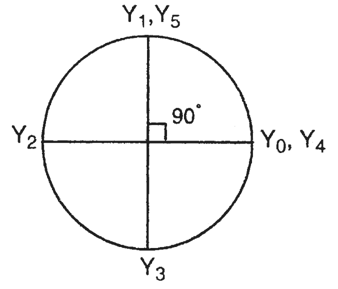


Fig. 2.24

$$a_2 = 2 \times \frac{1}{6} \sum_{r=0}^{5} y_r \cos \frac{\pi x_r}{3}$$

$$= \frac{1}{3} \left[(y_0 - y_3) + (y_1 + y_5 - y_2 - y_4) \cos 60^\circ \right]$$

(from the Harrison's circle for 60°, Fig. 2.23)

$$= -2.833$$

$$a_3 = 2 \times \frac{1}{6} \sum_{r=0}^{5} y_r \cos \frac{\pi x_r}{2}$$

$= \frac{1}{3}(y_0 + y_4 - y_2)$(from the Harrison's circle for 90°, Fig. 2.24)

$$= -1.667$$

$\therefore$ The required half-range cosine series is

$$f(x) = 7 + 4.565\cos\frac{\pi x}{6} - 2.833\cos\frac{\pi x}{3} - 1.667\cos\frac{\pi x}{2}\text{in } (0, 6)$$

Example 7

Find the complex form of the Fourier series of $f(x) = e^x \text{in}(0, 2)$.

Since $2l = 2$ or $l = 1$, the complex form of the Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x} \tag{1}$$

$$\begin{aligned}
c_n &= \frac{1}{2}\int_0^2 f(x)e^{-in\pi x}\,dx \\
&= \frac{1}{2}\int_0^2 e^x e^{-in\pi x}\,dx \\
&= \frac{1}{2}\left[\frac{e^{(1-in\pi)x}}{1-in\pi}\right]_0^2 \\
&= \frac{1}{2(1-in\pi)}\left\{e^{2(1-in\pi)} - 1\right\} \\
&= \frac{(1+in\pi)}{2(1+n^2\pi^2)}\left\{e^2(\cos 2n\pi - i\sin 2n\pi) - 1\right\} \\
&= \frac{(e^2-1)(1+in\pi)}{2(1+n^2\pi^2)}
\end{aligned}$$

Using this value in (1), we get

$$e^x = \left(\frac{e^2-1}{2}\right)\sum_{n=-\infty}^{\infty}\frac{(1+in\pi)}{(1+n^2\pi^2)}e^{in\pi x}$$

Example 8

Find the complex form of the Fourier series of $f(x) = e^{-ax}\text{in } (-l, l)$.

Let the complex form of the Fourier series be

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} \tag{1}$$

$$\begin{aligned}
c_n &= \frac{1}{2l}\int_{-l}^{l} e^{-ax}e^{-in\pi x/l}\,dx \\
&= \frac{1}{2l}\int_{-l}^{l} e^{-(al+in\pi)x/l}\,dx
\end{aligned}$$

$$= \frac{1}{2l}\left[\frac{e^{-(al+in\pi)x/l}}{-(al+in\pi)/l}\right]_{-l}^{l}$$

$$= -\frac{1}{2(al+in\pi)}\left[e^{-(al+in\pi)} - e^{(al+in\pi)}\right]$$

$$= \frac{1}{2(al+in\pi)}\left[e^{al}(-1)^n - e^{-al}(-1)^n\right]$$

$$\left[\because\ e^{\pm in\pi} = \cos n\pi \pm i\sin n\pi = (-1)^n\right]$$

$$= \frac{\sinh\ al(-1)^n}{al+in\pi}$$

$$= \frac{\sinh\ al\cdot(al-in\pi)(-1)^n}{a^2l^2+n^2\pi^2}$$

Using this value in (1), we have

$$e^{-ax} = \sinh\ al\sum_{n=-\infty}^{\infty}\frac{(-1)^n(al-in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l}\ \text{ in } (-l,l)$$

Example 9

Find the complex form of the Fourier series of $f(x) = \sin x$ in $(0,\pi)$.
Here $2l = \pi$ or $l = \pi/2$.

$\therefore$ The complex form of Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2nx} \tag{1}$$

$$c_n = \frac{1}{\pi}\int_0^{\pi}\sin x e^{-i2nx}\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-i2nx}}{1-4n^2}\{-i2n\sin x - \cos x\}\right]_0^{\pi}$$

$$= \frac{1}{\pi(4n^2-1)}\left[-e^{i2n\pi} - 1\right] = -\frac{2}{\pi(4n^2-1)}$$

Using this value in (1), we get

$$\sin x = -\frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{4n^2-1}\cdot e^{i2nx}\ \text{ in } (0,\pi)$$

Example 10

Find the complex form of the Fourier series of $f(x) = \cos\ ax$ in $(-\pi,\pi)$, where a is neither zero nor an integer.
Here $2l = 2\pi$ or $l = \pi$.

$\therefore$ The complex form of Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{1}$$

$$\begin{aligned}
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos ax . e^{-inx}\, dx \\
&= \frac{1}{2\pi} \left[\frac{e^{-inx}}{a^2 - n^2} \{-in \cos ax + a \sin ax\} \right]_{-\pi}^{\pi} \\
&= \frac{1}{2\pi(a^2 - n^2)} \Big[e^{-in\pi} (-in \cos a\pi + a \sin a\pi) \\
&\qquad - e^{in\pi} (-in \cos a\pi - a \sin a\pi) \Big] \\
&= \frac{1}{2\pi(a^2 - n^2)} (-1)^n 2a \sin a\pi
\end{aligned}$$

Using this value in (1), we get

$$\cos ax = \frac{a \sin a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 - n^2} e^{inx} \text{ in } (-\pi, \pi)$$

Exercise 2(c)

Part A (Short-Answer Questions)

1. What do you mean by harmonics and harmonic analysis in Fourier series?
2. Give the formula used for computing a_n numerically in the Fourier half-range cosine series of $f(x)$ in $(0, l)$.
3. Give the formula used for computing b_n numerically in the Fourier half-range sine series of $f(x)$ in $(0, \pi)$.
4. Write down the complex form of the Fourier series of $f(x)$ in $(0, 2l)$ and the Euler's formula for the associated Fourier coefficient.
5. If the trigonometric and complex forms of Fourier series of $f(x)$ in $(0, 2\pi)$ are respectively $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ and $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, how are a_0, a_n, b_n and c_n related?

Part B

6. Find the Fourier series of period 2π as far as the third harmonic to represent the function $y = f(x)$ defined by the following table.

$x°$:	0	30	60	90	120	150	180	210	240
y:	2.34	3.01	3.69	4.15	3.69	2.20	0.83	0.51	0.88

x°:	270	300	330	360
y:	1.09	1.19	1.64	2.34

7. Obtain the first three harmonics in the Fourier series of $y = f(x)$ which is defined by means of the table given below in (0, 12).

x :	0	1	2	3	4	5	6	7
y :	6.824	7.976	8.026	7.204	5.676	3.674	1.764	0.552
x :	8	9	10	11				
y :	0.262	0.904	2.492	4.736				

8. Obtain the first three harmonics in the Fourier series of $y = f(x)$ which is defined by means of the following table in $(0, 2\pi)$.

x° :	0	45	90	135	180	225	270	315
y :	6.824	8.001	7.204	4.675	1.764	0.407	0.904	3.614

9. Find the first three harmonics in the Fourier series of period 8 for the function $y = f(x)$ which is defined by means of the following table.

x :	1	2	3	4	5	6	7	8
y :	365	337	205	80	56	93	184	298

10. Find the Fourier series of $y = f(x)$ in $(0, 2\pi)$ upto the third harmonic, using the definition of y given by the following table.

x :	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y :	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

11. Find the first three harmonics in the Fourier series of $y = f(x)$, which is defined in the following table, in (0, 6) .

x :	0	1	2	3	4	5	6
y :	1.0	1.4	1.9	1.7	1.5	1.2	1.0

12. Find the first three harmonics in the Fourier series of $y = f(x)$ in $(0, 2\pi)$, using the following table of values of x and y.

x :	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7/6\pi$	$4/3\pi$
y :	0	0.26	0.52	0.79	1.05	1.31	0	-1.31	-1.05
x :	$3\pi/2$	$5\pi/3$	$11\pi/6$						
y :	-0.79	-0.52	-0.26						

[**Hint**: $f(2\pi - x) = -f(x)$. Hence the Fourier series of $f(x)$ in $(0, 2\pi)$ will not contain cosine terms]

13. Analyse the current i given by the following table into its constituent harmonics as far as the third harmonic.

θ° :	0	30	60	90	120	150	180	210
i (amp) :	0	24.0	32.5	27.5	18.2	13.0	0	-24.0
θ° :	240	270	300	330				
i (amp) :	-32.5	-27.5	-18.2	-13.0				

[**Hint** $f(\pi + x) = -f(x)$. Hence $a_0, a_2, a_4, \ldots, b_2, b_4, \ldots$ are all zero. It is enough to compute a_1, a_3, b_1 and b_3]

14. Find the constant term and the first three harmonics in the Fourier cosine series of $y = f(x)$ in $(0, \pi)$ using the following table.

x :	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$
y :	10	12	15	20	17	11

15. Find the first three harmonics in the Fourier sine series of $y = f(x)$ in $(0, 180°)$ using the following table.

$x°$:	0	15	30	45	60	75	90	105	120	135
y :	0	2.7	5.2	7.0	8.1	8.3	7.9	6.8	5.5	4.1
$x°$	150	165	180							
y :	2.6	1.2	0							

16. Find the complex form of the Fourier series of $f(x) = e^{-x}$ in $(-\pi, \pi)$.
17. Find the complex form of the Fourier series of $f(x) = e^{ax}$ in $(0, 2l)$.
18. Find the complex form of the Fourier series of $f(x) = \cos x$ in $(0, \pi)$.
19. Find the complex form of the Fourier series of $f(x) = \sin 2x$ in $(0, 1)$.
20. Find the complex form of the fourier series of $f(x) = \sin ax$ in $(-\pi, \pi)$.

Answers

Exercise 2(a)

19. $f(x) = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x + \frac{3}{4}\sin x - \frac{1}{4}\sin 3x.$

20. $f(x) = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos 4x.$

21. $f(x) = \frac{2}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx;\ \frac{\pi^2}{12}.$

22. $f(x) = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l};\ \frac{\pi^2}{6}.$

23. $f(x) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx.$

24. $f(x) = 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x;\ \frac{\pi}{4}.$

25. $f(x) = \frac{l}{4} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{l} + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}.$

26. $f(x) = \frac{\pi^2}{16} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x + \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx;\ \frac{\pi^2}{8}.$

27. $f(x) = \frac{1}{2}\cos\pi x - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2n\pi x.$

28. $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x;\ \frac{\pi^2}{8}.$

29. $f(x) = \left(\frac{k}{2} - \frac{l}{4}\right) + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{l}$

$+\frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{l}{n}(-1)^{n+1} + \frac{k}{n}\{1-(-1)^n\}\right]\sin\frac{n\pi x}{l};\ \frac{\pi^2}{8}.$

30. $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx.$

31. $f(x) = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{l} + \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l};\ \frac{\pi^2}{12};\ \frac{\pi^2}{6}.$

32. $f(x) = -1 + \pi\sin x - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{1}{n^2-1}\cos nx;\ \pi/4 - 1/2$

33. $f(x) = -\frac{1}{2}\sin x + 2\sum_{n=1}^{\infty}\frac{(-1)^n.n}{n^2-1}\sin nx.$

34. $x\sin\pi x = \frac{1}{\pi} - \frac{1}{2\pi}\cos\pi x - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos n\pi x;\ \frac{\pi}{4} - \frac{1}{2}.$

35. $\sqrt{1+\cos x} = \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\cos nx.$

36. $\frac{1}{12}x(\pi - x)(2\pi - x) = \sum_{n=1}^{\infty}\frac{1}{n^3}\sin nx;\ \pi^3/32$

37. $|x| = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l};\ \frac{\pi^2}{8}.$

38. $|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cdot\cos 2nx.$

39. $\cos ax = \frac{\sin a\pi}{a\pi} + \frac{2a\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos nx}{n^2-a^2};\ \frac{1}{2}\left(1 - \frac{\pi}{3\sqrt{3}}\right).$

40. $e^{ax} = \frac{e^{2al}-1}{2al} + al(e^{2al}-1)\sum_{n=1}^{\infty}\frac{1}{l^2a^2+n^2\pi^2}\cos\frac{n\pi x}{l}$

$-\pi\left(e^{2al}-1\right)\sum_{n=1}^{\infty}\frac{n}{l^2a^2+n^2\pi^2}\sin\frac{n\pi x}{l}.$

41. $\cosh \alpha x = \dfrac{2\alpha^2}{\pi} \sinh \alpha\pi \left[\dfrac{1}{2\alpha^2} + \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2+\alpha^2} \cos nx \right]$.

42. $f(x) = \sum_{n=1}^{\infty} \left\{ \dfrac{2}{n\pi} \left(1 + \cos \dfrac{n\pi}{2}\right) - \dfrac{4}{n\pi}(-1)^n \right\} \sin \dfrac{n\pi x}{2}$.

43. $f(x) = \dfrac{2}{\pi} - \dfrac{4}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{4n^2-1} \cos 2nx;\ \dfrac{\pi}{4} - \dfrac{1}{2}$.

44. $f(x) = \sum_{n=1}^{\infty} \left[\dfrac{6}{n\pi}(-1)^n - \dfrac{36}{n^3\pi^3} \left\{(-1)^n - 1\right\} \right] \sin \dfrac{n\pi x}{3}$.

45. $f(x) = \dfrac{\pi^2}{6} - \sum_{n=1}^{\infty} \dfrac{1}{n^2} \cos 2nx;\ \dfrac{\pi^2}{6};\ \dfrac{\pi^2}{12}$.

46. $f(x) = \sum_{n=1}^{\infty} \left\{ -\dfrac{4}{n\pi} \cos \dfrac{n\pi}{2} + \dfrac{16}{n^2\pi^2} \sin \dfrac{n\pi}{2} + \dfrac{48}{n^3\pi^3} \cos \dfrac{n\pi}{2} - \dfrac{96}{n^4\pi^4} \sin \dfrac{n\pi}{2} \right\} \sin \dfrac{n\pi x}{2}$.

47. $f(x) = \dfrac{l}{8} + 2l \sum_{n=1}^{\infty} \left\{ -\dfrac{1}{n^2\pi^2} \cos n\pi - \dfrac{1}{2n\pi} \sin \dfrac{n\pi}{2} + \dfrac{1}{n^2\pi^2} \cos \dfrac{n\pi}{2} \right\} \cos \dfrac{n\pi x}{l}$.

48. $f(x) = \dfrac{2}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{n} \left\{1 - (\pi + 1)(-1)^n\right\} \sin nx$.

49. $f(x) = \dfrac{4}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{n^2} \sin \dfrac{n\pi}{2} \sin nx$.

50. $f(x) = \dfrac{1}{3} + \sum_{\substack{n=1 \\ (n \neq 3)}}^{\infty} \left\{ \dfrac{2}{n\pi} \sin \dfrac{n\pi}{3} - \dfrac{2n}{\pi(n^2-9)} \sin \dfrac{n\pi}{3} \right\} \cos \dfrac{n\pi x}{3} + \dfrac{1}{3} \cos \pi x$.

Exercise 2(b)

7. $x^2 - x;\ x - x^2$.

8. $-x(l+x);\ x(l+x)$.

11. $\dfrac{1}{2} \sum_{n=1}^{\infty} E_{2n-1}^2$

12. $\dfrac{1}{2} \sum_{n=1}^{\infty} I_{2n-1}^2$.

13. $\dfrac{\pi^2}{8}$.

14. $\dfrac{1}{45}$.

15. $\dfrac{\pi^4}{15}$.

16. $\pi^2 - x^2 = \dfrac{2}{3}\pi^2 + 4\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^2} \cos nx;\ \dfrac{\pi^2}{12}$.

17. $f(x) = \dfrac{8kl}{\pi^2} \sum_{n=1,3,5..}^{\infty} \dfrac{1}{n^2} \sin\dfrac{n\pi}{2} \sin\dfrac{n\pi x}{2l};\ \dfrac{\pi^2}{8}$.

(18) $(x+2)^2 = \dfrac{4}{3} + \dfrac{16}{\pi^2}\sum_{n=1}^{\infty} \dfrac{1}{n^2} \cos\dfrac{n\pi x}{2};\ \dfrac{\pi^2}{6}$.

19. $l - x = \dfrac{2l}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{n} \sin\dfrac{n\pi x}{l}$ in $(l, 2l)$; $\dfrac{\pi}{4}$.

20. $x(\pi - x) = \dfrac{8}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^3} \sin(2n-1)x$ in $(0, \pi)$;

$x(\pi + x)$; $(\pi - x)(2\pi - x)$; $\dfrac{\pi^3}{32}$.

21. $f(x) = \dfrac{2bl^2}{a(l-a)\pi^2} \cdot \sum_{n=1}^{\infty} \dfrac{1}{n^2} \sin\dfrac{n\pi a}{l} \sin\dfrac{n\pi x}{l}$.

22. $\cos x = \dfrac{8}{\pi}\sum_{n=1}^{\infty} \dfrac{n}{4n^2-1} \sin 2nx$; $f(0)$ and $f(\pi)$ must be defined as 0 each.

23. $\cos ax = \dfrac{\sin a\pi}{a\pi} + \dfrac{2a\sin a\pi}{\pi}\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^2-a^2} \cos nx$.

24. $f(x) = \dfrac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{4n^2-1} \sin\dfrac{n\pi}{2} \sin 2nx$.

25. $x\cos\pi x = -\dfrac{1}{2\pi}\sin\pi x + \dfrac{2}{\pi}\sum_{n=2}^{\infty} \dfrac{(-1)^n}{n^2-1} \sin n\pi x;\ 1$.

26. $f(x) = (\pi/2)\sin x + \dfrac{4}{\pi}\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1} \cdot n}{(n^2-1)^2} \sin nx$.

27. $6x^2 - 6x + 1 = \dfrac{6}{\pi^2}\sum_{n=1}^{\infty} \dfrac{1}{n^2} \cos 2n\pi x;\ \dfrac{\pi^2}{12}$.

28. $e^{ax} = \dfrac{2}{\pi}\sum_{n=1}^{\infty} \dfrac{n}{n^2+a^2} \left\{1 + (-1)^{n-1}e^{a\pi}\right\} \sin nx$.

29. $x^2 = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos nx;\ \dfrac{\pi^4}{90}.$

30. $x^2 = \dfrac{4}{3} + \dfrac{4}{\pi^2}\sum_{n=1}^{\infty} \dfrac{1}{n^2} \cos n\pi x - \dfrac{4}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{n} \sin n\pi x;\ \dfrac{\pi^4}{90}.$

31. $x^2 + x = \dfrac{1}{3} + \dfrac{4}{\pi^2}\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos n\pi x - \dfrac{2}{\pi}\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n} \sin n\pi x;\ \dfrac{\pi^4}{90}.$

32. $2x - x^2 = -\dfrac{9}{\pi^2}\sum_{n=1}^{\infty} \dfrac{1}{n^2} \cos \dfrac{2n\pi x}{3} + \dfrac{3}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{n} \sin \dfrac{2n\pi x}{3};\ \dfrac{\pi^4}{90}.$

33. $f(x) = -\dfrac{2}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^2} \cos 2(2n-1)x + \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)} \sin 2(2n-1)x;\ \dfrac{\pi^4}{96}.$

34. $f(x) = \dfrac{3}{2} - \dfrac{4}{\pi^2}\sum_{n=1,3,5..}^{\infty} \dfrac{1}{n^2} \cos \dfrac{n\pi x}{2} - \dfrac{2}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{n} \sin \dfrac{n\pi x}{2};\ \dfrac{\pi^4}{96}.$

35. $f(x) = \dfrac{a}{2} - \dfrac{2a}{\pi}\sum_{n=1,3,5..}^{\infty} \dfrac{1}{n} \sin nx;\ \dfrac{\pi^2}{8}.$

36. $f(x) = \dfrac{3}{2} - \dfrac{2}{\pi}\sum_{n=1,3,5..}^{\infty} \dfrac{1}{n} \sin \dfrac{n\pi}{2} \cos \dfrac{n\pi x}{2};\ \dfrac{\pi^2}{8}.$

37. $\dfrac{\pi}{2} - x = \sum_{n=1}^{\infty} \dfrac{1}{n} \sin 2nx;\ \dfrac{\pi^2}{6}.$

38. $1 + x = \dfrac{3}{2} - \dfrac{4}{\pi^2}\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^2} \cos (2n-1)\pi x.$

39. $f(x) = \dfrac{16}{\pi^2}\sum_{n=1}^{\infty} \dfrac{1}{n^2} \sin \dfrac{n\pi}{2} \sin \dfrac{n\pi x}{2};\ \dfrac{\pi^4}{96}.$

40. $f(x) = \dfrac{\pi}{4} - \dfrac{2}{\pi}\sum_{n=1,3,5..}^{\infty} \dfrac{1}{n^2} \cos 2nx;\ \dfrac{\pi^4}{96}.$

Exercise 2(c)

6. $f(x) = 2.102 + 0.559\cos x + 1.535\sin x - 0.519\cos 2x - 0.091\sin 2x + 0.20\cos 3x + 0\sin 3x.$

7. $f(x) = 4.174 + 2.450\cos\dfrac{\pi x}{6} + 3.160\sin\dfrac{\pi x}{6} + 0.120\cos\dfrac{\pi x}{3} + 0.034\sin\dfrac{\pi x}{3} + 0.080\cos\dfrac{\pi x}{2} + 0.010\sin\dfrac{\pi x}{2}.$

8. $f(x) = 4.174 + 2.420\cos x + 3.105\sin x + 0.12\cos 2x + 0.03\sin 2x + 0.110\cos 3x - 0.045\sin 3x.$

9. $f(x) = 202 + \left(159\cos\dfrac{\pi x}{4} + 10\sin\dfrac{\pi x}{4}\right) + \left(-21\cos\dfrac{\pi x}{2} + 13\sin\dfrac{\pi x}{2}\right) + \left(-4\cos\dfrac{3\pi x}{4} - \sin\dfrac{3\pi x}{4}\right)$

10. $f(x) = 0.75 + 0.373\cos x + 1.005\sin x + 0.890\cos 2x - 0.110\sin 2x - 0.067\cos 3x.$

11. $f(x) = 1.45 - 0.367\cos\dfrac{\pi x}{3} + 0.173\sin\dfrac{\pi x}{3} - 0.1\cos\dfrac{2\pi x}{3} - 0.05\sin\dfrac{2\pi x}{3} + 0.033\cos\pi x.$

12. $f(x) = 0.978\sin x - 0.456\sin 2x + 0.26\sin 3x.$

13. $i = 5.559\cos\theta + 29.969\sin\theta - 4.767\cos 3\theta + 3.167\sin 3\theta.$

14. $f(x) = 14.167 + 3.289\cos x - 4.833\cos 2x + 4\cos 3x.$

15. $f(x) = 7.837\sin x + 1.484\sin 2x - 0.028\sin 3x.$

16. $e^{-x} = \dfrac{\sinh\pi}{\pi}\displaystyle\sum_{n=-\infty}^{\infty}\dfrac{(-1)^n(1-in)}{1+n^2}e^{inx}.$

17. $e^{ax} = \left(\dfrac{e^{2al}-1}{2}\right)\displaystyle\sum_{n=-\infty}^{\infty}\dfrac{(al+in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l}.$

18. $\cos x = \dfrac{4i}{\pi}\displaystyle\sum_{n=-\infty}^{\infty}\dfrac{n}{1-4n^2}e^{i2nx}.$

19. $\sin 2x = \dfrac{1}{2}\displaystyle\sum_{n=-\infty}^{\infty}\dfrac{1}{n^2\pi^2-1}(\cos 2 - 1 + in\pi\sin 2)\; e^{i2n\pi x}.$

20. $\sin ax = \dfrac{i\sin a\pi}{\pi}\displaystyle\sum_{n=-\infty}^{\infty}\dfrac{(-1)^n}{n^2-a^2}\, ne^{inx}.$

Chapter 3

Applications of Partial Differential Equations—

Part A

Vibration of Strings

3A.1 INTRODUCTION

Partial differential equations occur in connection with various physical and engineering problems when the physical quantity involved depends on two or more independent variables such as time and one or more coordinates in space. In this chapter and in the following chapters we shall study some of the most important partial differential equations occurring in engineering applications. We shall derive these equations as models of physical systems and consider methods for obtaining solutions of those equations corresponding to the given physical situations.

One of the most fundamental and common phenomena that is found in nature is the phenomenon of wave motion. When a stone is dropped into a pond, the surface of water is disturbed and waves of displacement travel radially outward. When a bell or tuning fork is struck, sound waves are propagated from the source of sound. The electrical oscillations of a radio antenna generate electromagnetic waves that are propagated through space. Whatever be the nature of wave phenomenon, whether it be the displacement of a tightly stretched string, the deflection of a stretched membrane, the propagation of currents and potentials along an electrical transmission line or the propagation of electromagnetic waves in free space, these entities are governed by a certain partial differential equation, known as the wave equation.

The present chapter will be devoted to the one dimensional wave equation that governs the small transverse vibrations of a stretched string and that is also similar to the radio equations occurring in the field of electrical engineering. The one dimensional variable heat flow equation, that is similar to the telegraph equations occurring in the field of electrical engineering, will be discussed in Chapter 4(B) Chapter 4(C) will be devoted respectively to two dimensional Laplace equation in Cartesian form and in Polar form which govern steadystate heat flow in two dimensions.

Classification of Partial Differential Equations of the Second Order

The most general form of linear partial differential equation of the second order in two independent variables is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \qquad (1)$$

where A, B, C, D, E, F, and G are functions of x and y only.

An equation of the from

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + f(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0 \qquad (2)$$

in which the terms involving the second order partial derivatives alone are linear, is known as a *quasi-linear* P.D.E. of the second order.

The equation (1) or (2) is said to be of

1. The *elliptic type*, if $B^2 - 4\,AC < 0$
2. The *parabolic type*, if $B^2 - 4\,AC = 0$
3. The *hyperbolic type*, if $B^2 - 4\,AC > 0$

For example, let us consider the following equations:

(i) $$\frac{\partial^2 u}{\partial x^2} - 2x\frac{\partial^2 u}{\partial x\,\partial y} + x^2\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial y} = 0$$

Or $\quad u_{xx} - 2xu_{xy} + x^2 u_{yy} - 2u_y = 0$

Here $A = 1$, $B = -2x$, $C = x^2$

$$B^2 - 4\,AC = (-2x)^2 - 4x^2 = 0$$

$\therefore$ The equation is parabolic at all points.

(ii) $y^2 u_{xx} + x^2 u_{yy} = 0$

Here $A = y^2$, $B = 0$, $C = x^2$

$$B^2 - 4\,AC = 0 - 4x^2y^2 < 0, \text{ for all } x \text{ and } y\ (\neq 0)$$

$\therefore$ The equation is elliptic at all points.

(iii) $x^2 u_{xx} - y^2 u_{yy} = 0$

Here $A = x^2$, $B = 0$, $C = -y^2$

$$B^2 - 4\,AC = 0 + 4x^2y^2 > 0, \text{ for all } x \text{ and } y\ (\neq 0)$$

$\therefore$ The equation is hyperbolic at all points.

(iv) $y_{xx} - x\,u_{yy} = 0$

Here $A = 1$, $B = 0$, $C = -x$

$$B^2 - 4\,AC = 0 + 4x$$

$\therefore$ The equation is elliptic, parabolic and hyperbolic, according as $x < 0$, $x = 0$ and $x > 0$ respectively.

(v) $u_{xx} + 4u_{xy} + (x^2 + 4y^2)u_{yy} = \sin(x + y)$

Here $A = 1$, $B = 4$, $C = x^2 + 4y^2$

$$B^2 - 4\,AC = 16 - 4\,(x^2 + 4y^2)$$

$$= 4(4 - x^2 - 4y^2)$$

$\therefore$ The equation is of the elliptic type if $4 - x^2 - 4y^2 < 0$

i.e., $x^2 + 4y^2 > 4$

i.e., $\dfrac{x^2}{4} + \dfrac{y^2}{1} > 1$

i.e., outside the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1.$

The given P.D.E. is of the parabolic type on the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$ and of the hyperbolic type inside the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$.

A liner or *quasi-linear* P.D.E. of the second order can be converted into simple forms, *canonical forms* or *normal forms* by effecting suitable transformations of independent variables, the discussion of which is beyond the scope of this book.

The frequently occurring canonical forms of P.D.E's are

(i) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ (Laplace equation)

(ii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = -f(x, y)$ (Poisson equation)

(iii) $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$ (One dimensional heat flow equation)

(iv) $\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}$ (One dimensional wave equation)

It can be easily verified that equations (i) and (ii) are of the elliptic type, equation (iii) is of the parabolic type and equation (iv) is of the hyperbolic type.

3A.2 TRANSVERSE VIBRATIONS OF A STRETCHED STRING — ONE-DIMENSIONAL WAVE EQUATION

Let us derive the partial differential equation governing small transverse vibrations of an elastic string which is stretched to a length l and then fixed at its two ends 0 and A (Fig. 3A-1). The end 0 of the string is taken as the origin, the position $0A$ of the string at equilibrium as the x-axis and the line through 0 perpendicular to the x-axis and lying in the plane of motion of the string as the y-axis.

By disturbing the equilibrium of the string at a certain instant, say $t = 0$, it is allowed to vibrate transversely, i.e. at right angles to the equilibrium position of the string, in the xy-plane. Our aim is to study the vibrations of the string, that is, to find the deflection (displacement) of the string $y(x, t)$ at any point x and at any time $t > 0$. In order to derive the partial differential equation satisfied by $y(x, t)$ in the simplest form, we make the following assumptions.

1. The tension T caused by stretching the string before fixing it at the end points is constant at all points of the deflected string and at all times.
2. T is so large that other external forces such as weight of the string and friction may be considered negligible.
3. The string is homogeneous (i.e. the mass of the string per unit length is constant) and perfectly elastic and so does not offer resistance to bending.
4. Deflection y and the slope $\dfrac{\partial y}{\partial x}$ at every point of the string are small, so that their higher powers may be neglected.

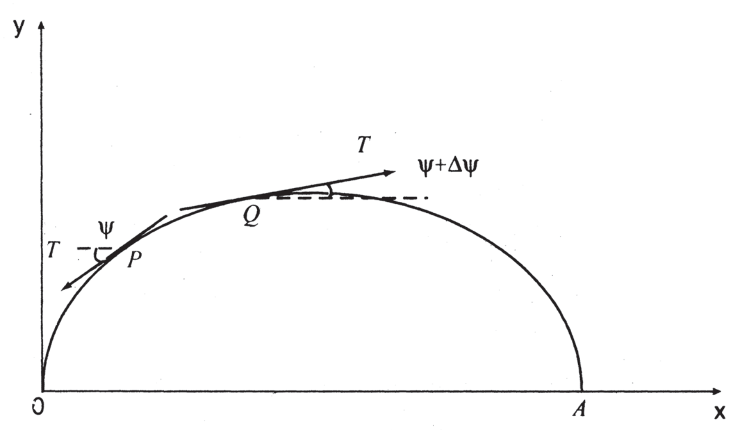


Fig. 3A.1

Let us consider the motion of an element PQ of the string, where $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ are two neighbouring points. Let ψ and $\psi + \Delta\psi$ be the angles made by the tangents at P and Q respectively with the x-axis and $\widehat{PQ} = \Delta s$.

The theoretical expression for the acceleration of PQ in the positive y direction $= \dfrac{\partial^2 y}{\partial t^2}$.

Note

We use partial derivative because y is a function of x and t.

Therefore the theoretical expression for the force acting on PQ in the positive y direction

$$= m\Delta s \frac{\partial^2 y}{\partial t^2} \tag{1}$$

by Newton's second law, where m is the mass per unit length of the string.

The actual external force acting on PQ in the positive y direction

$$\begin{aligned} &= T\sin(\psi + \Delta\psi) - T\sin\psi \\ &= T[(\psi + \Delta\psi) - \psi] \qquad \text{by Assumption 4} \\ &= T\Delta\psi \end{aligned} \tag{2}$$

Equating (1) and (2), we get the equation of motion of the element PQ as

$$m\frac{\partial^2 y}{\partial t^2} = T\frac{\Delta\psi}{\Delta s} \tag{3}$$

Taking limits on both sides of (3) as $Q \to P$, i.e. $\Delta s \to 0$, we get

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left(\frac{\mathrm{d}\psi}{\mathrm{d}s}\right) \tag{4}$$

Now $\dfrac{\mathrm{d}\psi}{\mathrm{d}s}$ = curvature at P of the deflection curve

$$\begin{aligned} &= \frac{\partial^2 y}{\partial x^2} \Big/ \left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{3/2} \\ &= \frac{\partial^2 y}{\partial x^2} \end{aligned} \tag{5}$$

$\left[\because \left(\dfrac{\partial y}{\partial x}\right)^2 \text{ is negligible, by Assumption 4}\right]$

Since T and m are both positive, $\dfrac{T}{m}$ is positive and hence $\dfrac{T}{m}$ can be taken as a^2 (6)

Using (5) and (6) in (4), we get the partial differential equation of the vibrating string as

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \tag{7}$$

Note

Equation (7) is also known as the one dimensional wave equation.

3A.3 TRANSMISSION LINE EQUATIONS

Let us consider the flow of electricity in a long cable or telephone wire which is imperfectly insulated so that leakage occurs along the entire length of the cable. Let R, L, C, G be respectively the resistance, inductance, capacitance and leakance to the ground per unit length of the cable, each assumed to be constant.

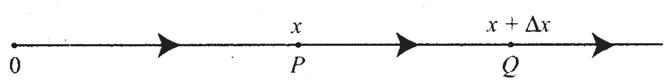


Fig. 3A.2

Let e be the potential and i the current at time t at a point P of the cable at a distance x from a given point 0. Let $e + \Delta e$ and $i + \Delta i$ be the potential and the current at the point $Q(x + \Delta x)$ at time t (Fig.3A.2)

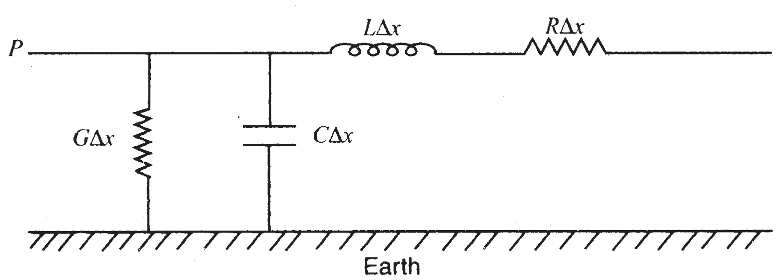


Fig. 3A.3

By Kirchhoff's voltage law, the potential drop across the segment PQ = potential drop due to resistance + that due to inductance (Fig. 3A.3)

i.e. $$-\Delta e = (R\Delta x)i + (L\Delta x)\frac{\partial i}{\partial t}$$

i.e. $$-\frac{\Delta e}{\Delta x} = Ri + L\frac{\partial i}{\partial t}$$

Taking limits as $\Delta x \to 0$, we get

$$-\frac{\partial e}{\partial x} = Ri + L\frac{\partial i}{\partial t} \tag{1}$$

which is called *the first transmission line equation.*

By Kirchhoff's current law, loss of current while crosssing the segment PQ = loss of current due to leakance + that due to capacitance.

i.e. $$-\Delta i = (G\Delta x)e + (C\Delta x)\frac{\partial e}{\partial t}$$

i.e. $$-\frac{\Delta i}{\Delta x} = Ge + C\frac{\partial e}{\partial t}$$

Taking limits as $\Delta x \to 0$, we get

$$-\frac{\partial i}{\partial x} = Ge + C\frac{\partial e}{\partial t} \tag{2}$$

which is called *the second transmission line equation.*

Differentiating (1) and (2) partially with respect x and t respectively, we have

$$-\frac{\partial^2 e}{\partial x^2} = R\frac{\partial i}{\partial x} + L\frac{\partial^2 i}{\partial x \partial t} \tag{3}$$

and

$$-\frac{\partial^2 i}{\partial x \partial t} = G\frac{\partial e}{\partial t} + C\frac{\partial^2 e}{\partial t^2} \tag{4}$$

Eliminating $\frac{\partial^2 i}{\partial x \partial t}$ from (3) and (4), we have

$$\frac{\partial^2 e}{\partial x^2} = -R\frac{\partial i}{\partial x} + LG\frac{\partial e}{\partial t} + LC\frac{\partial^2 e}{\partial t^2}$$

$$= R\left(Ge + C\frac{\partial e}{\partial t}\right) + LG\frac{\partial e}{\partial t} + LC\frac{\partial^2 e}{\partial t^2}, \qquad \text{using(2)}$$

i.e.

$$\frac{\partial^2 e}{\partial x^2} = LC\frac{\partial^2 e}{\partial t^2} + (RC + LG)\frac{\partial e}{\partial t} + RGe \tag{5}$$

Now differentiating (1) and (2) partially with respect to t and x respectively, we have

$$-\frac{\partial^2 e}{\partial x \partial t} = R\frac{\partial i}{\partial t} + L\frac{\partial^2 i}{\partial t^2} \tag{6}$$

and

$$-\frac{\partial^2 i}{\partial x^2} = G\frac{\partial e}{\partial x} + C\frac{\partial^2 e}{\partial x \partial t} \tag{7}$$

Eliminating e from (1), (6) and (7), we get

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (RC + LG)\frac{\partial i}{\partial t} + RGi \tag{8}$$

Equations (5) and (8) are called *the telephone equations.*

Corollary 1

If $R = G = 0$, Eq. (5) and (8) become $\frac{\partial^2 e}{\partial x^2} = LC\frac{\partial^2 e}{\partial t^2}$ and $\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2}$, which are known as *the radio equations*. These equations are similar to the one dimensional wave equation.

Corollary 2

If $L = G = 0$, Equation(5) and (8) become $\frac{\partial^2 e}{\partial x^2} = RC\frac{\partial e}{\partial t}$ and $\frac{\partial^2 i}{\partial x^2} = RC\frac{\partial i}{\partial t}$, which are known as *the telegraph equations*. These equations are similar to the one dimensional heat flow equation which will be derived in Chapter 3(B)

3A.4 VARIABLE SEPARABLE SOLUTIONS OF THE WAVE EQUATION $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

Let

$$y(x, t) = X(x) \cdot T(t) \tag{1}$$

be a solution of the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{2}$$

where $X(x)$ is a function of x alone and $T(t)$ is a function of t alone.

Then $\frac{\partial^2 y}{\partial t^2} = XT''$ and $\frac{\partial^2 y}{\partial x^2} = X''T$, where $T'' = \frac{d^2 T}{dt^2}$ and $X'' = \frac{d^2 X}{dx^2}$ satisfy equation (2)

i.e. $$XT'' = a^2 X'' T$$

i.e. $$\frac{X''}{X} = \frac{T''}{a^2 T} \tag{3}$$

The L.H.S. of (3) is a function of x alone and the R.H.S. is a function of t alone. They are equal for all values of the independent variables x and t. This is possible only if each is a constant.

$\therefore$ $$\frac{X''}{X} = \frac{T''}{a^2 T} = k$$

where k is a constant.

$\therefore$ $$X'' - kX = 0 \tag{4}$$

and

$$T'' - ka^2 T = 0 \tag{5}$$

The nature of the solutions of (4) and (5) depends on the nature of values of k. Hence the following three cases arise.

Case 1

k is positive. Let $k = p^2$

Then equations (4) and (5) become

$$(D^2 - p^2)X = 0$$

and

$$(D'^2 - p^2 a^2)T = 0$$

where

$$D \equiv \frac{\mathrm{d}}{\mathrm{d}x} \qquad \text{and} \qquad D' \equiv \frac{\mathrm{d}}{\mathrm{d}t}$$

The solutions of these equations are

$$X = Ae^{px} + Be^{-px}$$

and

$$T = Ce^{pat} + De^{-pat}$$

Case 2

k is negative. Let $k = -p^2$

Then equations (4) and (5) become

$$(D^2 + p^2)X = 0$$

and

$$(D'^2 + p^2a^2)T = 0$$

The solutions of these equations are $X = A\cos px + B\sin px$ and $T = C\cos pat + D\sin pat$.

Case 3

$k = 0$. Then equations (4) and (5) become

$$\frac{d^2X}{dx^2} = 0$$

and

$$\frac{d^2T}{dt^2} = 0$$

The solutions of these equations (found by integrating them) are $X = Ax + B$ and $T = Ct + D$. Since $y(x,t) = X.T$ is the solution of the wave equation, the three mathematically possible solutions of the wave equation are

$$y(x,t) = (Ae^{px} + Be^{-px})(Ce^{pat} + De^{-pat}) \tag{6}$$

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \tag{7}$$

and

$$y(x,t) = (Ax + B)(Ct + D) \tag{8}$$

3A.5 CHOICE OF PROPER SOLUTION

Out of the three mathematically possible solutions derived, we have to choose that solution which is consistent with the physical nature of the problem and the given boundary conditions. When we deal with the vibration of an elastic string, $y(x, t)$, representing the displacement of the string at any point x, must be periodic in t. Hence solution (7), which consists of periodic functions in t is the proper solution of the problems on vibration of strings. The arbitrary constants in the suitable solution are found out by using the boundary conditions of the problem. In problems, we shall directly assume that (7) is the proper solution of vibration of string problems.

Note ✍

The proper solution (7) which is periodic in 't' is incidentally periodic in x also.

3A.6 SOLUTION OF A DAMPED VIBRATING STRING EQUATION

The one dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ represents the transverse vibration of a string in vacuum or in a free medium where there is no resistance to the motion of the string.

If the string is made to vibrate in a damped medium which offers resistance to the motion of the string proportional to its velocity, the above equation no longer represents the damped vibration of the string.

In this case, the vibration of the string is represented by the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} - 2b \frac{\partial y}{\partial t} \tag{1}$$

We shall find the appropriate solution of Eq. (1) by the method of separation of variables.

Let

$$y(x, t) = X(x) \cdot T(t) \tag{2}$$

be a solution of equation (1), where $X(x)$ is a function of x alone and $T(t)$ is a function of t alone.

Then $\frac{\partial y}{\partial t} = XT'$, $\frac{\partial^2 y}{\partial t^2} = XT''$ and $\frac{\partial^2 y}{\partial x^2} = X''T$ (where the dashes denote ordinary derivatives with respect to the concerned variables) satisfy Eq. (1).

i.e.
$$XT'' = a^2 X''T - 2bXT'$$

Dividing this equation throughout by XT, we get

$$\frac{T''}{T} = a^2 \frac{X''}{X} - 2b \frac{T'}{T}$$

i.e.
$$\frac{X''}{X} = \frac{1}{a^2}\left(\frac{T'' + 2bT'}{T}\right) \tag{3}$$

The L.H.S. of (3) is a function of x alone and the R.H.S. is a function of t alone. They are equal for all values of the independent variables x and t. This is possible only if each is a constant.

$$\therefore \qquad \frac{X''}{X} = \frac{T'' + 2bT'}{a^2 T} = k$$

where k is a constant.

$$\therefore \qquad X'' - kX = 0 \tag{4}$$

and

$$T'' + 2bT' - ka^2 T = 0 \tag{5}$$

The appropriate solution of the vibration (though damped) of the string should be periodic in both t and x.

The solution of (4) will contain periodic functions in x, only if k is negative.

Hence we assume that $k = -p^2$.

Therefore Eq. (4) becomes $X'' + p^2 X = 0$. Solving this equation, we get

$$X = (A \cos px + B \sin px) \tag{6}$$

When $k = -p^2$, Eq. (5) becomes

$$T'' + 2bT' + p^2 a^2 T = 0$$

or

$$(D^2 + 2bD + p^2 a^2)T = 0, \text{ where } D \equiv \frac{\mathrm{d}}{\mathrm{d}t}$$

The auxiliary equation of this equation is

$$m^2 + 2bm + p^2 a^2 = 0$$

$$\therefore \qquad m = \frac{-2b \pm \sqrt{4b^2 - 4p^2 a^2}}{2}$$

$$= -b \pm \sqrt{b^2 - p^2 a^2} \tag{7}$$

As the solution of (5) has to contain periodic functions in t, the values of m in (7) should be complex.

Therefore (7) is rewritten as

$$m = -b \pm i\sqrt{p^2 a^2 - b^2}$$

assuming that $b < pa$.

Therefore the solution of Eq. (5) is

$$T = e^{-bt}(C \cos \sqrt{p^2 a^2 - b^2}t + D \sin \sqrt{p^2 a^2 - b^2}t) \tag{8}$$

Using (6) and (8) in (2), the required solution of equation (1) is

$$y(x,t) = e^{-bt}(A\cos px + B\sin px)\cdot(C\cos\sqrt{p^2a^2-b^2}t + D\sin\sqrt{p^2a^2-b^2}t)$$

Worked Examples 3A

Example 1

A uniform string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form of the curve (i) $y = k\sin^3\left(\frac{\pi x}{l}\right)$ and (ii) $y = kx(l-x)$ and then releasing it from this position at time $t = 0$. Find the displacement of the point of the string at a distance x from one end at time t.

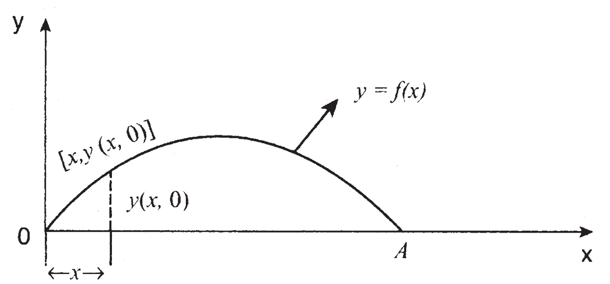


Fig. 3A.4

The displacement $y(x,t)$ of the point of the string at a distance x from the left end 0 at time t is given by the equation (Fig. 3A.4)

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \tag{1}$$

Since the ends of the string $x = 0$ and $x = l$ are fixed, they do not undergo any displacement at any time.
Hence

$$y(0,t) = 0, \quad \text{for} \quad t \geq 0 \tag{2}$$

and

$$y(l,t) = 0, \quad \text{for} \quad t \geq 0 \tag{3}$$

Since the string is released from rest initially, that is, at $t = 0$, the initial velocity of every point of the string in the y-direction (the direction of motion of the string) is zero.

Hence

$$\frac{\partial y}{\partial t}(x, 0) = 0, \quad \text{for} \quad 0 \leq x \leq l \tag{4}$$

Since the string is initially displaced into the form of the curve $y = f(x)$, the coordinates $[x, y(x, 0)]$ satisfy the equation $y = f(x)$, where $y(x, 0)$ is the initial displacement of the point 'x' in the y-direction.

Hence

$$y(x, 0) = f(x) \quad \text{for} \quad 0 \leq x \leq l \tag{5}$$

where $f(x) = k \sin^3\left(\frac{\pi x}{l}\right)$ in (i) and $= kx(l - x)$ in (ii). Conditions (2), (3), (4) and (5) are collectively called *boundary conditions* of the problem. We have to get the solution of Eq. (1), that satisfies the boundary conditions. Of the three mathematically possible solutions of Eq. (1), the appropriate solution, consistent with the vibration of the string is

$$y(x, t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \tag{6}$$

where A, B, C, D and p are arbitrary constants that are to be found out by using the boundary conditions.

Using boundary condition (2) in (6), we have

$$A(C \cos pat + D \sin pat) = 0 \qquad \text{for all } t \geq 0$$

$$\therefore \quad A = 0$$

Using boundary condition (3) in (6), we have

$$B \sin pl(C \cos pat + D \sin pat) = 0 \quad \text{for all} \quad t \geq 0$$

$$\therefore \quad B \sin pl = 0$$

$$\therefore \quad \text{Either } B = 0 \text{ or } \sin pl = 0.$$

If $B = 0$, the solution becomes $y(x, t) = 0$, which is meaningless.

$$\therefore \quad \sin pl = 0$$

$$\therefore \quad pl = n\pi$$

or

$$p = \frac{n\pi}{l}, \quad \text{where} \quad n = 0, 1, 2, 3, \ldots \infty$$

Differentiating both sides of (6) partially with respect to t, we have

$$\frac{\partial y}{\partial t}(x, t) = B \sin px \cdot pa(-C \sin pat + D \cos pat) \tag{7}$$

where

$$p = \frac{n\pi}{l}$$

Using boundary condition (4) in (7), we have

$$B \sin px \cdot pa \cdot D = 0, \qquad \text{for} \qquad 0 \le x \le l$$

As $B \neq 0$ and $p \neq 0$, we get $D = 0$

Using these values of A, p and D in (6), the solution reduces to

$$y(x,t) = BC \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

where $n = 1, 2, 3, \ldots \infty$

Note

$n = 0$ is omitted, since the solution corresponding to $n = 0$ is meaningless.

Taking $BC = k$, Eq. (1) has infinitely many solutions given below.

$$y(x,t) = k \sin \frac{\pi x}{l} \cos \frac{\pi at}{l}$$
$$y(x,t) = k \sin \frac{2\pi x}{l} \cos \frac{2\pi at}{l}$$
$$y(x,t) = k \sin \frac{3\pi x}{l} \cos \frac{3\pi at}{l}, \qquad \text{etc.}$$

Since Eq. (1) is linear, a linear combination of the R.H.S. members of all the above solutions is the general solution of Eq. (1). Thus the most general solution of Eq. (1) is

$$y(x,t) = \sum_{n=1}^{\infty} (c_n k) \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

or

$$y(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \tag{8}$$

where λ_n is yet to be found out.

Using boundary conditions (5) in (8) we have

$$\sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} = f(x), \ \text{for } 0 \le x \le l \tag{9}$$

If we can express $f(x)$ in a series comparable with the L.H.S. series of (9), we can get the values of λ_n.

(i) $f(x) = k \sin^3 \frac{\pi x}{l}$

$$= \frac{k}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right)$$

Using this form of $f(x)$ in (9) and comparing like terms, we get

$$\lambda_1 = \frac{3k}{4}, \lambda_3 = -\frac{k}{4}, \lambda_2 = 0 = \lambda_4 = \lambda_5 = \dots$$

Using these values in (8), the required solution is

$$y(x,t) = \frac{3k}{4} \sin\frac{\pi x}{l} \cos\frac{\pi a t}{l} - \frac{k}{4} \sin\frac{3\pi x}{l} \cos\frac{3\pi a t}{l}$$

(ii) $f(x) = kx(l-x)$.

If we expand $f(x)$ as Fourier half-range sine series in $(0, l)$, that is, in the form

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

it is comparable with the L.H.S. series of (9).

Thus

$$\begin{aligned}
\lambda_n = b_n &= \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx, \text{ by Euler's formula} \\
&= \frac{2k}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx \\
&= \frac{2k}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right. \\
&\qquad \left. + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l \\
&= \frac{4kl^2}{n^3\pi^3}\{1-(-1)^n\} \\
&= \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{8kl^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

Using this value of λ_n in (8), the required solution is

$$y(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin\frac{(2n-1)\pi x}{l} \cos\frac{(2n-1)\pi a t}{l}$$

Example 2

A tightly stretched string of length $2l$ is fastened at both ends. The midpoint of the string is displaced by a distance b transversely and the string is released from rest in this position. Find the displacement of any point of the string at any subsequent time.

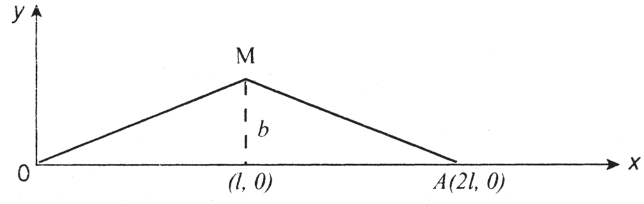


Fig. 3A.5

Let us find the analytic equation of the curve which the string assumes initially. The initial curve consists of the two lines 0M and MA, where the origin 0 represents one end of the string and 0A is the equilibrium position of the string (Fig. 3A.5).

Equation of OM is $y = \frac{b}{l}x$.

Equation of MA is $\frac{y-0}{b-0} = \frac{x-2l}{l-2l}$ or $y = \frac{b}{l}(2l - x)$

The displacement $y(x, t)$ of any point 'x' of the string at any time 't' is given by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$y(0, t) = 0, \text{ for } t \geq 0 \tag{2}$$

$$y(2l, t) = 0, \text{ for } t \geq 0 \tag{3}$$

since the ends of the string are fixed.

$$\frac{\partial y}{\partial t}(x, 0) = 0, \text{ for } 0 \leq x \leq 2l \tag{4}$$

since the string is released from rest initially.

$$y(x, 0) = \begin{cases} \frac{b}{l}x, & \text{for } \quad 0 \leq x \leq l \\ \frac{b}{l}(2l - x), & \text{for } \quad l \leq x \leq 2l \end{cases} \tag{5}$$

since $y(x, 0)$ is given by $f(x)$, where $y = f(x)$ is the equation of the initial position curve of the string.

The suitable solution of Eq. (1), consistent with the vibration of the string, is

$$y(x, t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \tag{6}$$

Using boundary condition (2) in (6), we have

$$A(C \cos pat + D \sin pat) = 0, \text{ for all } t \geq 0$$

$$\therefore A = 0$$

Using boundary condition (3) in (6), we have

$$B \sin 2pl(C \cos pat + D \sin pat) = 0, \text{ for all } t \geq 0$$

$\therefore$ Either $B = 0$ or $\sin 2pl = 0$

If we assume that $B = 0$, it results in a trivial solution.

$\therefore$ $\sin 2pl = 0$

$\therefore$ $2pl = n\pi$ or $p = \dfrac{n\pi}{2l}$, where $n = 0, 1, 2, \ldots \infty$.

Differentiating both sides of (6) partially with respect to t, we have

$$\frac{\partial y}{\partial t}(x, t) = B \sin px \cdot pa(-C \sin pat + D \cos pat) \tag{7}$$

where $p = \dfrac{n\pi}{2l}$.

Using boundary condition (4) in (7), we have

$$B \sin px \cdot pa \cdot D = 0, \quad \text{for } 0 \leq x \leq 2l$$

As $B \neq 0$ and $p \neq 0$, we get $D = 0$.

Using these values of A, p and D in (6), it reduces to

$$y(x, t) = k \sin \frac{n\pi x}{2l} \cos \frac{n\pi at}{2l} \tag{8}$$

where $n = 1, 2, \ldots \infty$.

Therefore the most general solution of Eq. (1) [got as a linear combination of the R.H.S. members of solutions in (8)] is

$$y(x, t) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{2l} \cos \frac{n\pi at}{2l} \tag{9}$$

Using boundary condition (5) in (9), we have

$$\sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{2l} = f(x) \tag{10}$$

where $f(x) = \begin{cases} \dfrac{b}{l}x, & \text{in } (0, l) \\ \dfrac{b}{l}(2l - x), & \text{in } (l, 2l) \end{cases}$

$f(x)$ can be expressed in the form of a series similar to the L.H.S. of (10) by means of Fourier sine series in $(0, 2l)$.

Let it be $\sum\limits_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{2l}$.

$$\text{Thus } \lambda_n = b_n = \frac{2}{2l}\int_0^{2l} f(x)\sin\frac{n\pi x}{2l}dx$$

$$= \frac{1}{l}\left[\int_0^l \frac{b}{l}x\sin\frac{n\pi x}{2l}dx + \int_l^{2l}\frac{b}{l}(2l-x)\sin\frac{n\pi x}{2l}dx\right]$$

$$= \frac{b}{l^2}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - \left(\frac{-\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)\right\}_0^l\right.$$

$$\left.+\left\{(2l-x)\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)\right\}_l^{2l}\right]$$

$$= \frac{b}{l^2}\left[\left\{-\frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} + \left\{\frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \right.\right.$$

$$\left.\left.\frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right]$$

$$= \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Using this value of λ_n in (9), the required solution is

$$y(x,t) = \frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\cos\frac{n\pi a t}{2l} \tag{11}$$

Example 3

Solve the one-dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$ in $-l \le x \le l, t \ge 0$, given that $y(-l,t) = 0$, $y(l,t) = 0$, $\frac{\partial y}{\partial t}(x,0) = 0$ and $y(x,0) = \frac{b}{l}(l - |x|)$.

Shifting the origin to the point $(-l, 0)$, we get $x = X - l$ and $y = Y$, where (X, Y) are the coordinates of the point (x, y) with reference to the new system of coordinate axes. The differential equation in the new system is

$$\frac{\partial y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial X^2} \tag{1}$$

$$0 \le X \le 2l, t \ge 0$$

The boundary conditions become

$$Y(0,t) = 0 \tag{2}$$

$$Y(2l,t) = 0 \tag{3}$$

for all $t \ge 0$.

$$\frac{\partial Y}{\partial t}(X, 0) = 0 \tag{4}$$

and

$$Y(X, 0) = \begin{cases} \frac{b}{l}X, \text{ in } 0 \le X \le l \\ \frac{b}{l}(2l - X), \text{ in } l \le X \le 2l \end{cases} \tag{5}$$

Since the last boundary condition in the old system is

$$y(x, 0) = \begin{cases} \frac{b}{l}(l + x), & \text{in } -l \le x \le 0 \\ \frac{b}{l}(l - x), & \text{in } 0 \le x \le l \end{cases}$$

Note ✍

$y(x, t) \equiv Y(X, t)$, where x varies from $-l$ to l and X varies from 0 to $2l$.

The transformed version of the given problem is identical with that of Example 2. Proceeding as in Example 2, the required solution of Equation (1) is

$$Y(X, t) = \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{2} \sin\frac{n\pi X}{2l} \cos\frac{n\pi at}{2l}$$

Since $\sin\frac{n\pi}{2} = 0$, when n is an even integer, the solution can be rewritten as

$$Y(X, t) = \frac{8b}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{2} \sin\frac{n\pi X}{2l} \cos\frac{n\pi at}{2l}$$

where $0 \le X \le 2l$, $t \ge 0$.

Changing over to the old system of coordinates, the solution becomes

$$y(x, t) = \frac{8b}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{2} \sin\frac{n\pi}{2l}(x + l) \cos\frac{n\pi at}{2l}$$

Now $\sin\frac{n\pi}{2l}(x + l) = \sin\left(\frac{n\pi}{2} + \frac{n\pi x}{2l}\right)$

$$= \sin\frac{n\pi}{2} \cos\frac{n\pi x}{2l} + \cos\frac{n\pi}{2} \sin\frac{n\pi x}{2l}$$

$$= \sin\frac{n\pi}{2} \cos\frac{n\pi x}{2l}, \quad \text{since } n \text{ is odd.}$$

$\therefore$ The required solution is

$$y(x,t) = \frac{8b}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2} \sin^2 \frac{n\pi}{2} \cos \frac{n\pi x}{2l} \cos \frac{n\pi at}{2l} \text{ or}$$

$$y(x,t) = \frac{8b}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2} \cos \frac{n\pi x}{2l} \cos \frac{n\pi at}{2l}$$

where $-l \le x \le l$ and $t \ge 0$.

Example 4

A uniform string of line density ρ stretched to tension ρa^2 executes small transverse vibration in a plane through the undisturbed line of the string. The ends $x = 0$ and $x = l$ of the string are fixed. The string is at rest, with the point $x = b$ drawn aside through a small distance d and released at time $t = 0$. Find the transverse displacement of any point of the string at any subsequent time.

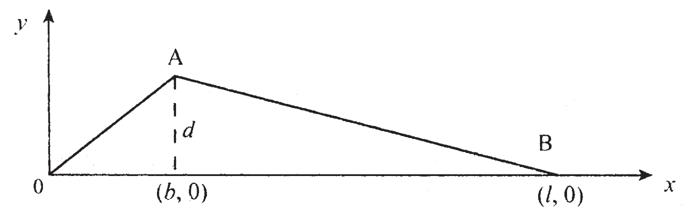


Fig. 3A.6

The curve, which the string assumes initially, consists of two broken lines 0A and AB (Fig. 3A.6).

Equation of 0A is $y = \frac{d}{b}x$.

Equation of AB is $\frac{y-0}{d-0} = \frac{x-l}{b-l}$ or $y = \frac{d}{l-b}(l-x)$.

The displacement $y(x,t)$ of any point 'x' of the string at any time 't is given by

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} \text{ or}$$

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

since $T = \rho a^2$.

We have to solve Eq. (1), satisfying the following boundary conditions.

$$y(0,t) = 0, \text{ for } t \ge 0 \tag{2}$$

$$y(l,t) = 0, \text{ for } t \ge 0 \tag{3}$$

$$\frac{\partial y}{\partial t}(x,0) = 0, \text{ for } 0 \le x \le l \tag{4}$$

$$y(x,0) = f(x), \text{ where } f(x) = \begin{cases} \dfrac{d}{b}x, & \text{in } 0 \le x \le b \\ \dfrac{d}{l-b}(l-x), & \text{in } b \le x \le d \end{cases} \tag{5}$$

The appropriate solution of Eq. (1), consistent with the vibration of the string, is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \tag{6}$$

Using boundary condition (2) in (6), we have

$$A(C\cos pat + D\sin pat) = 0, \text{ for all } t \ge 0 \ \therefore A = 0$$

Using boundary condition (3) in (6), we have

$$B\sin pl(C\cos pat + D\sin pat) = 0, \text{ for all } t \ge 0$$

$\therefore$ either $B = 0$ or $\sin pl = 0$.

If we assume that $B = 0$, it results in a trivial solution.

$\therefore$ $\qquad \sin pl = 0$

$\therefore$ $\qquad pl = n\pi$ or $p = \dfrac{n\pi}{l}$, where $\quad n = 0, 1, 2, \ldots \infty$

Differentiating both sides of (6) partially with respect to t, we have

$$\frac{\partial y}{\partial t}(x,t) = B\sin px \cdot pa(-C\sin pat + D\cos pat) \tag{7}$$

where $\qquad p = \dfrac{n\pi}{l}$

Using boundary condition (4) in (7), we have

$$B\sin px \cdot pa \cdot D = 0, \text{ for } 0 \le x \le l$$

As $B \ne 0$ and $p \ne 0$, we get $D = 0$.

Using these values of A, p and D in (6), it reduces to

$$y(x,t) = k\sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi at}{l} \tag{8}$$

where $n = 0, 1, 2, \ldots \infty$.

Therefore the most general solution of equation (1) is

$$y(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \tag{9}$$

Using boundary condition (5) in (9), we have

$$\sum_{n=1}^{\infty} \lambda_n \sin\frac{n\pi x}{l} = f(x) \quad \text{in} \quad 0 \le x \le l$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

which is the Fourier half-range sine series of $f(x)$ in $(0, l)$. Comparing coefficients of $\sin \frac{n\pi x}{l}$, we get

$$\lambda_n = b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$= \frac{2}{l}\left[\int_0^b \frac{d}{b}x\sin\frac{n\pi x}{l}\mathrm{d}x + \int_b^l \frac{d}{l-b}(l-x)\sin\frac{n\pi x}{l}\mathrm{d}x\right]$$

$$= \frac{2d}{lb}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^b +$$

$$\frac{2d}{l(l-b)}\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_b^l$$

$$= \frac{2d}{lb}\left[-\frac{lb}{n\pi}\cos\frac{n\pi b}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi b}{l}\right]$$

$$+ \frac{2d}{l(l-b)}\left[\frac{l(l-b)}{n\pi}\cos\frac{n\pi b}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi b}{l}\right]$$

$$= \frac{2dl}{n^2\pi^2}\left(\frac{1}{b} + \frac{1}{l-b}\right)\sin\frac{n\pi b}{l}$$

$$= \frac{2dl^2}{n^2\pi^2 b(l-b)}\sin\frac{n\pi b}{l}$$

Using this value of λ_n in (9), the required solution is

$$y(x,t) = \frac{2dl^2}{\pi^2 b(l-b)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi b}{l}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

Example 5

The points of trisection of a tightly stretched string of length 30 cm with fixed ends pulled aside through a distance of 1 cm on opposite sides of the position of equilibrium and the string is released from rest. Find an expression for the displacement of the string at any subsequent time. Show also that the midpoint of the string remains always at rest.

The curve, which the string assumes initially consists of three broken lines 0A, AB and BC (Fig. 3A.7).

Equation of 0A is $y = \frac{1}{10}x$.

Equation of AB is $\frac{y+1}{2} = \frac{x-20}{-10}$ or $y = \frac{1}{5}(15 - x)$

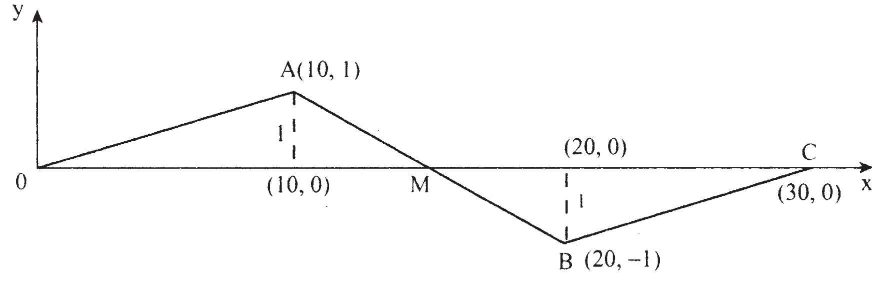


Fig. 3A.7

Equation of BC is $\dfrac{y-0}{-1}=\dfrac{x-30}{-10}$ or $y=\dfrac{1}{10}(x-30)$

The displacement $y(x,t)$ of any point 'x' of the string at any time 't' is given by

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2} \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$y(0,t)=0,\ \text{for}\ t\geq 0 \tag{2}$$

$$y(30,t)=0,\ \text{for}\ t\geq 0 \tag{3}$$

$$\frac{\partial y}{\partial t}(x,0)=0,\ \text{for}\ 0\leq x\leq 30 \tag{4}$$

$$y(x,0)=f(x),\ \text{where}\ f(x)=\begin{cases}\dfrac{1}{10}x, & \text{in}\ 0\leq x\leq 10\\[2mm] \dfrac{1}{10}(30-2x), & \text{in}\ 10\leq x\leq 20\\[2mm] \dfrac{1}{10}(x-30), & \text{in}\ 20\leq x\leq 30\end{cases} \tag{5}$$

The appropriate solution of equation (1), consistent with the vibration of the string, is

$$y(x,t)=(A\cos px+B\sin px)(C\cos pat+D\sin pat) \tag{6}$$

Using boundary condition (2) in (6), we have

$$A(C\cos pat+D\sin pat)=0,\ \text{for all}\ t\geq 0$$

$$\therefore \qquad A=0$$

Using boundary condition (3) in (6), we have

$$B\sin 30p(C\cos pat+D\sin pat)=0,\ \text{for all}\ t\geq 0$$

$$\therefore \qquad \text{Either}\ B=0\ \text{or}\ \sin 30p=0$$

If we assume that $B=0$, it results in a trivial solution.

$\therefore \quad \sin 30p = 0$

$\therefore \quad 30p = n\pi$ or $p = \dfrac{n\pi}{30}$ where $n = 0, 1, 2, \ldots \infty$.

Differentiating both sides of (6) partially with respect to t, we have

$$\frac{\partial y}{\partial t}(x, t) = B \sin px \cdot pa(-C \sin pat + D \cos pat) \tag{7}$$

where $p = \dfrac{n\pi}{30}$

Using boundary condition (4) in (7), we have

$$B \sin px \cdot pa \cdot D = 0, \text{ for } 0 \le x \le 30.$$

As $B \neq 0$ and $p \neq 0$, we get $D = 0$.

Using these values of A, p and D in (6), it reduces to

$$y(x, t) = k \sin \frac{n\pi x}{30} \cos \frac{n\pi at}{30} \tag{8}$$

where $n = 0, 1, 2, \ldots \infty$.

Therefore the most general solutions of Eq. (1) is

$$y(x, t) = \sum_{n-1}^{\infty} \lambda_n \sin \frac{n\pi x}{30} \cos \frac{n\pi at}{30} \tag{9}$$

Using boundary condition (5) in (9), we have

$$\sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{30} = f(x) \text{ in } 0 \le x \le 30$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{30}$$

which is the Fourier half-range sine series of $f(x)$ in (0, 30).

$$\therefore \quad \lambda_n = b_n = \frac{2}{30} \int_0^{30} f(x) \sin \frac{n\pi x}{30} dx$$

$$= \frac{1}{15} \left[\int_0^{10} \frac{1}{10} x \sin \frac{n\pi x}{30} dx + \int_{10}^{20} \frac{1}{10}(30 - 2x) \sin \frac{n\pi x}{30} dx \right.$$

$$\left. + \int_{20}^{30} \frac{1}{10}(x - 30) \sin \frac{n\pi x}{30} dx \right]$$

$$= \frac{1}{150}\left[\left\{x\left(\frac{-\cos\dfrac{n\pi x}{30}}{\dfrac{n\pi}{30}}\right) - \left(\frac{-\sin\dfrac{n\pi x}{30}}{\dfrac{n^2\pi^2}{30^2}}\right)\right\}_0^{10}\right.$$

$$+\left\{(30-2x)\left(\frac{-\cos\dfrac{n\pi x}{30}}{\dfrac{n\pi}{30}}\right) - (-2)\left(\frac{-\sin\dfrac{n\pi x}{30}}{\dfrac{n^2\pi^2}{30^2}}\right)\right\}_{10}^{20}$$

$$\left.+\left\{(x-30)\left(\frac{-\cos\dfrac{n\pi x}{30}}{\dfrac{n\pi}{30}}\right) - \left(\frac{-\sin\dfrac{n\pi x}{30}}{\dfrac{n^2\pi^2}{30^2}}\right)\right\}_{20}^{30}\right]$$

$$= \frac{1}{150}\left[\left\{-\frac{300}{n\pi}\cos\frac{n\pi}{3} + \frac{900}{n^2\pi^2}\sin\frac{n\pi}{3}\right\} + \left\{\frac{300}{n\pi}\cos\frac{2n\pi}{3} - \right.\right.$$

$$\left.\frac{1800}{n^2\pi^2}\sin\frac{2n\pi}{3} + \frac{300}{n\pi}\cos\frac{n\pi}{3} + \frac{1800}{n^2\pi^2}\sin\frac{n\pi}{3}\right\} +$$

$$\left.\left\{-\frac{300}{n\pi}\cos\frac{2n\pi}{3} - \frac{900}{n^2\pi^2}\sin\frac{2n\pi}{3}\right\}\right]$$

$$= \frac{1}{150}\cdot\frac{2700}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right)$$

$$= \frac{18}{n^2\pi^2}\left\{\sin\frac{n\pi}{3} - \sin\left(n\pi - \frac{n\pi}{3}\right)\right\}$$

$$= \frac{18}{n^2\pi^2}\left\{\sin\frac{n\pi}{3} + \cos n\pi\cdot\sin\frac{n\pi}{3}\right\}$$

$$= \frac{18}{n^2\pi^2}\left\{1 + (-1)^n\right\}\sin\frac{n\pi}{3}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{36}{n^2\pi^2}\sin\dfrac{n\pi}{3} & \text{if } n \text{ is even} \end{cases}$$

Using this value of λ_n in (9), the required solution is

$$y(x,t) = \frac{36}{\pi^2}\sum_{n=2,4,6\ldots}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\cdot\sin\frac{n\pi x}{30}\cos\frac{n\pi at}{30} \quad \text{or}$$

$$y(x,t) = \frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{2n\pi}{3}\sin\frac{n\pi x}{15}\cdot\cos\frac{n\pi at}{15}$$

The displacement of the mid-point (15, 0) of the string is given by

$$\begin{aligned} y(15, t) &= \frac{9}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{2n\pi}{3} \sin n\pi \cos \frac{n\pi at}{15} \\ &= 0, \quad \text{for all } t \geq 0 \end{aligned}$$

This means that the midpoint of the string remains at rest always.

Example 6

A tightly stretched string with fixed end points $x = 0$ and $x = 50$ is initially at rest in its equilibrium position. If it is set to vibrate by giving each point a velocity (i) $v = v_0 \sin^3 \dfrac{\pi x}{50}$ and (ii) $v = v_0 \sin \dfrac{\pi x}{50} \cos \dfrac{2\pi x}{50}$, find the displacement of any point of the string at any subsequent time.

The displacement $y(x, t)$ of any point 'x' of the string at any time 't' is given by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

We have to solve Eq. (1) satisfying the following boundary conditions.

$$y(0, t) = 0, \quad \text{for} \quad t \geq 0 \tag{2}$$

$$y(50, t) = 0, \quad \text{for} \quad t \geq 0 \tag{3}$$

$$y(x, 0) = 0, \quad \text{for} \quad 0 \leq x \leq 50 \tag{4}$$

since the string is in its equilibrium position initially and so the initial displacement of every point of the string is zero.

$$\frac{\partial y}{\partial t}(x, 0) = f(x), \quad \text{for} \quad 0 \leq x \leq 50 \tag{5}$$

$$\begin{aligned} \text{where} \qquad f(x) &= v_0 \sin^3 \frac{\pi x}{50} \text{ for (i) and} \\ f(x) &= v_0 \sin \frac{\pi x}{50} \cos \frac{2\pi x}{50} \quad \text{for } (ii) \end{aligned}$$

Note ✍

The boundary condition with non zero value on the R.H.S. should be taken as the last boundary condition.

The suitable solution of Eq. (1), consistent with the vibration of the string, is

$$y(x, t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \tag{6}$$

Using boundary condition (2) in (6), we have

$$A(C \cos pat + D \sin pat) = 0, \quad \text{for all} \quad t \geq 0$$

$$\therefore \qquad A = 0.$$

Using boundary condition (3) in (6), we have

$$B \sin 50p(C \cos pat + D \sin pat) = 0, \quad \text{for all } t \geq 0$$

$\therefore$ Either $B = 0$ or $\sin 50p = 0$

If we assume that $B = 0$, we get a trivial solution.

$\therefore \quad \sin 50p = 0$

$\therefore \quad 50p = n\pi$ or $p = \dfrac{n\pi}{50}$, where $n = 0, 1, 2, \ldots \infty$.

Using boundary condition (4) in (6), we have

$B \sin px \cdot C = 0$, where $p = \dfrac{n\pi}{50}$, for $0 \leq x \leq 50$

As $B \neq 0$, we get $C = 0$.

Using these values of A, p and C in (6), it reduces to

$$y(x, t) = k \sin \frac{n\pi x}{50} \sin \frac{n\pi at}{50} \tag{7}$$

where $k = BD$ and $n = 0, 1, 2, \ldots \infty$.

$\therefore$ The most general solution of Eq. (1) is

$$y(x, t) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi}{50} \sin \frac{n\pi at}{50} \tag{8}$$

Note ✍

Only after getting the most general solution, we should use the non zero boundary condition.

Differentiating both sides of (8) partially with respect to t, we have

$$\frac{\partial y}{\partial t}(x, t) = \sum_{n=1}^{\infty} \left(\frac{n\pi a}{50} \cdot \lambda n\right) \sin \frac{n\pi x}{50} \cos \frac{n\pi at}{50} \tag{9}$$

Using boundary condition (5) in (9), we have

$$\sum_{n=1}^{\infty} \left(\frac{n\pi a}{50} \lambda_n\right) \sin \frac{n\pi x}{50} = v, \quad \text{since } v = \frac{\partial y}{\partial t}(x, 0)$$

(i) $$v = v_0 \sin^3 \frac{\pi x}{50}$$

$$= \frac{v_0}{4}\left(3 \sin \frac{\pi x}{50} - \sin \frac{3\pi x}{50}\right)$$

$$\therefore \quad \sum_{n=1}^{\infty} \left(\frac{n\pi a}{50} \lambda_n\right) \sin \frac{n\pi x}{50} = \frac{v_0}{4}\left(3 \sin \frac{\pi x}{50} - \sin \frac{3\pi x}{50}\right)$$

Comparing like terms, we get

$$\frac{\pi a}{50}\lambda_1 = \frac{3v_0}{4}, \frac{3\pi a}{50}\lambda_3 = -\frac{v_0}{4} \quad \text{and} \quad \frac{n\pi a}{50}\lambda_n = 0, \text{ for } n = 2, 4, 5, 6 \ldots \infty.$$

$$\therefore \qquad \lambda_1 = \frac{75v_0}{2\pi a}, \lambda_3 = -\frac{25v_0}{6\pi a} \text{ and } \lambda_2 = 0 = \lambda_4 = \lambda_5 = \ldots$$

Using these values in (8), the required solution is

$$y(x,t) = \frac{75v_0}{2\pi a}\sin\frac{\pi x}{50}\sin\frac{\pi at}{50} - \frac{25v_0}{6\pi a}\sin\frac{3\pi x}{50}\sin\frac{3\pi at}{50}$$

(ii)

$$v = v_0 \sin\frac{\pi x}{50}\cos\frac{2\pi x}{50}$$

$$= \frac{v_0}{2}\left(\sin\frac{3\pi x}{50} - \sin\frac{\pi x}{50}\right)$$

$$\therefore \qquad \sum_{n=1}^{\infty}\left(\frac{n\pi a}{50}\lambda_n\right)\sin\frac{n\pi x}{50} = \frac{v_0}{2}\left(\sin\frac{3\pi x}{50} - \sin\frac{\pi x}{50}\right)$$

Comparing like terms, we get

$$\frac{\pi a}{50}\lambda_1 = -\frac{v_0}{2}, \frac{3\pi a}{50}\lambda_3 = \frac{v_0}{2} \text{ and } \frac{n\pi a}{50}\lambda_n = 0, \text{ for } n = 2, 4, 5, 6, \ldots \infty.$$

$$\therefore \qquad \lambda_1 = -\frac{25v_0}{\pi a}, \lambda_3 = \frac{25v_0}{3\pi a} \text{ and } \lambda_2 = 0 = \lambda_4 = \lambda_5 = \ldots$$

Using these values in (8), the required solution is

$$y(x,t) = -\frac{25v_0}{\pi a}\sin\frac{\pi x}{50}\sin\frac{\pi at}{50} + \frac{25v_0}{3\pi a}\sin\frac{3\pi x}{50}\sin\frac{3\pi at}{50}$$

Example 7

A taut string of length $2l$, fastened at both ends, is disturbed from its position of equilibrium by imparting to each of its points an initial velocity of magnitude $k(2lx - x^2)$. Find the displacement function $y(x,t)$.

The displacement $y(x,t)$ of any point 'x' of the string at any time 't' is given by

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$y(0,t) = 0, \qquad \text{for } t \geq 0 \tag{2}$$

$$y(2l,t) = 0, \qquad \text{for } t \geq 0 \tag{3}$$

$$y(x,0) = 0, \qquad \text{for } 0 \leq x \leq 2l \tag{4}$$

$$\frac{\partial y}{\partial t}(x,0) = k(2lx - x^2), \qquad \text{for } 0 \leq x \leq 2l \tag{5}$$

The appropriate solution of Eq. 1, consistent with the vibration of the string, is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \tag{6}$$

Using boundary condition (2) in (6), we have

$$A(C\cos pat + D\sin pat) = 0, \quad \text{for all} \quad t \geq 0$$

$$\therefore \qquad A = 0$$

Using boundary condition (3) in (6), we have

$$B\sin 2lp(C\cos pat + D\sin pat) = 0, \text{ for all } t \geq 0.$$

$$\therefore \qquad \text{Either } B = 0 \text{ or } \sin 2lp = 0.$$

If we assume that $B = 0$, it leads to a trivial solution.

$$\therefore \qquad \sin 2lp = 0$$

$$\therefore \qquad 2lp = n\pi \text{ or } p = \frac{n\pi}{2l}, \text{ where } n = 0, 1, 2, \ldots \infty.$$

Using boundary condition (4) in (6), we have

$$B\sin px \cdot C = 0, \quad \text{for } 0 \leq x \leq 2l, \quad \text{where } p = \frac{n\pi}{2l}$$

As $B \neq 0$, we get $C = 0$.

Using these values of A, p and C in (6), it reduces to

$$y(x,t) = k\sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l} \tag{7}$$

where $n = 0, 1, 2, \ldots \infty$.

Therefore the most general solution of Eq. (1) is

$$y(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l} \tag{8}$$

Differentiating both sides of (8) partially with respect to t, we have

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty}\left(\frac{n\pi a}{2l}\cdot\lambda_n\right)\sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l} \tag{9}$$

Using boundary condition (5) in (9), we have

$$\sum_{n=1}^{\infty}\left(\frac{n\pi a}{2l}\cdot\lambda_n\right)\sin\frac{n\pi x}{2l} = k(2lx - x^2), \quad \text{for } 0 \leq x \leq 2l$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2l}$$

which is Fourier half-range sine series of $k(2lx - x^2)$ in $(0, 2l)$,

Comparing like terms, we get

$$\frac{n\pi a}{2l} \cdot \lambda_n = b_n = \frac{2}{2l}\int_0^{2l} k(2lx - x^2)\sin\frac{n\pi x}{2l}\mathrm{d}x$$

$$\therefore \quad \lambda_n = \frac{2k}{n\pi a}\left[(2lx - x^2)\left(-\frac{\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (2l - 2x)\left(\frac{-\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{2l}}{\frac{n^3\pi^3}{8l^3}}\right)\right]_0^{2l}$$

$$= \frac{32kl^3}{n^4\pi^4 a}\left\{1 - (-1)^n\right\}$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{64kl^3}{n^4\pi^4 a}, & \text{if } n \text{ is odd} \end{cases}$$

Using this value of λ_n in (8), the required solution is

$$y(x,t) = \frac{64kl^3}{\pi^4 a}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{2l}\sin\frac{(2n-1)\pi at}{2l}$$

Example 8

A string is stretched between two fixed points at a distance of 60 cm and the points of the string are given initial velocities v, where

$$v = \frac{\lambda x}{30}, \qquad \text{in } 0 < x < 30$$

$$= \frac{\lambda}{30}(60 - x), \text{ in } 30 < x < 60$$

x being the distance from an end point. Find the displacement of the string at any time.

The displacement $y(x,t)$ of any point 'x' of the string at any time 't' is given by

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditons.

$$y(0, t) = 0, \text{ for } t \geq 0 \tag{2}$$

$$y(60, t) = 0, \text{ for } t \geq 0 \tag{3}$$

$$y(x, 0) = 0, \text{ for } 0 \leq x \leq 60 \tag{4}$$

$$\frac{\partial y}{\partial t}(x, 0) = v, \text{ where } v = \begin{cases} \dfrac{\lambda x}{30}, & \text{in } 0 < x < 30 \\ \dfrac{\lambda}{30}(60 - x), & \text{in } 30 < x < 60 \end{cases} \tag{5}$$

The proper solution of Eq. (1), consistent with the vibration of the string is

$$y(x, t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \tag{6}$$

Using boundary condition (2) in (6), we have

$$A(C \cos pat + D \sin pat) = 0, \text{ for all } \quad t \geq 0$$

$$\therefore \quad A = 0$$

Using boundary condition (3) in (6), we have

$$B \sin 60p \cdot (C \cos pat + D \sin pat) = 0, \text{ for all } t \geq 0$$

$$\therefore \quad \text{Either } B = 0 \text{ or } \sin 60p = 0$$

If we assume that $B = 0$, it leads to a trivial solution.

$$\therefore \quad \sin 60p = 0$$

$$\therefore \quad 60p = n\pi \text{ or } p = \frac{n\pi}{60}, \text{ where } n = 0, 1, 2, \ldots \infty.$$

Using boundary condition (4) in (6), we have

$$B \sin px \cdot C = 0, \quad \text{for } 0 \leq x \leq 60, \text{ where } p = \frac{n\pi}{60}$$

As $B \neq 0$, we get $C = 0$.
Using these values of A, p and C in (6), it reduces to

$$y(x, t) = k \sin \frac{n\pi x}{60} \sin \frac{n\pi at}{60} \tag{7}$$

where $n = 0, 1, 2, \ldots \infty$.
$\therefore$ The most general solution of Eq. (1) is

$$y(x, t) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{60} \cdot \sin \frac{n\pi at}{60} \tag{8}$$

Differentiating both sides of (8) partially with respect to t, we have

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty}\left(\frac{n\pi a}{60}\lambda_n\right)\sin\frac{n\pi x}{60}\cos\frac{n\pi at}{60} \tag{9}$$

Using boundary condition (5) in (9), we have

$$\sum_{n=1}^{\infty}\left(\frac{n\pi a}{60}\lambda_n\right)\sin\frac{n\pi x}{60} = v = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{60}$$

which is Fourier half-range sine series of v in (0, 60).

Comparing like terms, we get

$$\frac{n\pi a}{60}\lambda_n = b_n = \frac{2}{60}\int_0^{60} v\sin\frac{n\pi x}{60}dx$$

$$\therefore \qquad \lambda_n = \frac{2}{n\pi a}\left[\int_0^{30}\frac{\lambda x}{30}\sin\frac{n\pi x}{60}dx + \int_{30}^{60}\frac{\lambda}{30}(60-x)\sin\frac{n\pi x}{60}dx\right]$$

$$= \frac{\lambda}{15n\pi a}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{60}}{\frac{n\pi}{60}}\right) - \left(\frac{-\sin\frac{n\pi x}{60}}{\frac{n^2\pi^2}{60^2}}\right)\right\}_0^{30} + \left\{(60-x)\left(\frac{-\cos\frac{n\pi x}{60}}{\frac{n\pi}{60}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{60}}{\frac{n^2\pi^2}{60^2}}\right)\right\}_{30}^{60}\right]$$

$$= \frac{\lambda}{15n\pi a}\left[\left\{-\frac{1800}{n\pi}\cos\frac{n\pi}{2} + \frac{3600}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} + \left\{\frac{1800}{n\pi}\cos\frac{n\pi}{2} + \frac{3600}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right]$$

$$= \frac{480\lambda}{n^3\pi^3 a}\sin\frac{n\pi}{2}$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{480\lambda}{n^3\pi^3 a}\sin\dfrac{n\pi}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Using this value of λ_n in (8), the required solution is

$$y(x,t) = \frac{480\lambda}{\pi^3 a}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{60}\sin\frac{n\pi at}{60}$$

or

$$y(x,t) = \frac{480\lambda}{\pi^3 a}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{60}\sin\frac{(2n-1)\pi at}{60}$$

Example 9

A uniform string of length l is struck in such a way that an initial velocity v_0 (constant) is imparted to the portion of the string between $\frac{l}{4}$ and $\frac{3l}{4}$, while the string is in its equilibrium position. Find the subsequent displacement of the string as a function of x and t.

The displacement $y(x, t)$ of any point 'x' of the string at any time 't' is given by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$y(0, t) = 0, \text{ for } t \geq 0 \tag{2}$$

$$y(l, t) = 0, \text{ for } t \geq 0 \tag{3}$$

$$y(x, 0) = 0, \text{ for } 0 \leq x \leq l \tag{4}$$

$$\frac{\partial y}{\partial t}(x, 0) = v, \text{ where } v = \begin{cases} 0, & \text{in } 0 < x < \frac{l}{4} \\ v_0, & \text{in } \frac{l}{4} < x < \frac{3l}{4} \\ 0, & \text{in } \frac{3l}{4} < x < l \end{cases} \tag{5}$$

The appropriate solution of Eq. (1), consistent with the vibration of the string, is

$$y(x, t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \tag{6}$$

Using boundary condition (2) in (6), we have

$$A(C \cos pat + D \sin pat) = 0, \quad \text{for all } t \geq 0$$

$$\therefore \quad A = 0.$$

Using boundary condition (3) in (6), we have

$$B \sin pl(C \cos pat + D \sin pat) = 0, \quad \text{for all } t \geq 0$$

$$\therefore \quad \text{Either } B = 0 \text{ or } \sin pl = 0$$

If we assume that $B = 0$, we get a trivial solution.

$\therefore \quad \sin pl = 0$

$\therefore \quad pl = n\pi \text{ or } p = \frac{n\pi}{l}, \text{ where } n = 0, 1, 2, \ldots \infty.$

Using boundary condition (4) in (6), we have

$$B \sin px \cdot C = 0, \quad \text{for } 0 \le x \le l, \quad \text{where } p = \frac{n\pi}{l}$$

As $B \neq 0$, we get $C = 0$.
Using these values of A, p and C in (6), it reduces to

$$y(x,t) = k \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l} \tag{7}$$

where $n = 0, 1, 2, \ldots \infty$.
Therefore the most general solution of Eq. (1) is

$$y(x,t) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l} \tag{8}$$

Differentiating both sides of (8) partially with respect to t, we have

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} \left(\frac{n\pi a}{l} \cdot \lambda_n \right) \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \tag{9}$$

Using boundary condition (5) in (9), we have

$$\sum_{n=1}^{\infty} \left(\frac{n\pi a}{l} \lambda_n \right) \sin \frac{n\pi x}{l} = v = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

which is Fourier half-range sine series of v in $(0, l)$
Comparing like terms, we get

$$\frac{n\pi a}{l} \lambda_n = b_n = \frac{2}{l} \int_0^l v \sin \frac{n\pi x}{l} dx$$

$$\therefore \qquad \lambda_n = \frac{2}{n\pi a} \int_{l/4}^{3l/4} v_0 \sin \frac{n\pi x}{l} dx$$

$$= \frac{2v_0}{n\pi a} \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right)_{l/4}^{3l/4}$$

$$= \frac{2v_0 l}{n^2\pi^2 a} \left(\cos \frac{n\pi}{4} - \cos \frac{3n\pi}{4} \right)$$

$$= \frac{2v_0 l}{n^2\pi^2 a} \left[\cos \frac{n\pi}{4} - \cos \left(n\pi - \frac{n\pi}{4} \right) \right]$$

$$= \frac{2v_0 l}{n^2\pi^2 a} \left\{ 1 - (-1)^n \right\} \cos \frac{n\pi}{4}$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{4v_0 l}{n^2\pi^2 a} \cos \dfrac{n\pi}{4}, & \text{if } n \text{ is odd} \end{cases}$$

Using this value of λ_n in (8), the required solution is

$$y(x,t) = \frac{4v_0 l}{\pi^2 a} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2} \cos\frac{n\pi}{4} \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l}$$

Example 10

Solve the problem of the vibrating string for the following boundary conditions.
(i) $y(0,t) = 0$, (ii) $y(l,t) = 0$, (iii) $\frac{\partial y}{\partial t}(x,0) = v_0 \sin\frac{\pi x}{l}$ and (iv) $y(x,0) = y_0 \sin\frac{2\pi x}{l}$.

The displacement $y(x,t)$ of any point 'x' of the string at any time 't' is given by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

We have to solve Eq. (1), satisfying the given boundary conditions.
The proper solution of Eq. (1), consistent with the vibration of the string, is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \tag{2}$$

Using boundary condition (i) in (2), we have

$$A(C\cos pat + D\sin pat) = 0, \quad \text{for all } t \geq 0$$

$$\therefore \quad A = 0$$

Using boundary condition (ii) in (2), we have

$$B\sin pl(C\cos pat + D\sin pat) = 0, \quad \text{for all } t \geq 0$$

$$\therefore \quad \text{Either } B = 0 \text{ or } \sin pl = 0$$

If we assume that $B = 0$, we get a trivial solution.

$$\therefore \quad \sin pl = 0$$

$$\therefore \quad pl = n\pi \text{ or } p = \frac{n\pi}{l}, \quad \text{where } n = 0, 1, 2, \ldots \infty.$$

Note ✍

The next two boundary conditions contain non zero values on the R.H.S. Hence we should proceed to use them, only after getting the general solution of Eq. (1).

Using the values of A and p in (2), it reduces to

$$y(x,t) = a' \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l} + b\sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l} \tag{3}$$

where $BC = a'$, $BD = b$ and $n = 0, 1, 2, \ldots \infty$.

The most general solution of (1) is got by superposing the above infinitely many solutions. That is

$$y(x,t)=\sum_{n=1}^{\infty}\left[(c_n a')\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}+(c_n b)\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}\right]$$

Taking $c_n a' = \lambda_n$ and $c_n b = \mu_n$, the most general solution of (1) becomes

$$y(x,t)=\sum_{n=1}^{\infty}\lambda_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}+\sum_{n=1}^{\infty}\mu_n\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \qquad (4)$$

Differentiating both sides of (4) partially with respect to t, we have

$$\frac{\partial y}{\partial t}(x,t)=\sum_{n=1}^{\infty}\left(-\frac{n\pi a}{l}\cdot\lambda_n\right)\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$+\sum_{n=1}^{\infty}\left(\frac{n\pi a}{l}\mu_n\right)\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \qquad (5)$$

Using boundary condition (iii) in (5), we have

$$\sum_{n=1}^{\infty}\left(\frac{n\pi a}{l}\mu_n\right)\sin\frac{n\pi x}{l}=v_0\sin\frac{\pi x}{l}$$

Comparing like terms, we get

$$\frac{\pi a}{l}\mu_1=v_0, \qquad \frac{n\pi a}{l}\mu_n=0, \text{ for } n=2,3,4,\ldots\infty$$

$$\therefore \qquad \mu_1=\frac{lv_0}{\pi a} \quad\text{and}\quad \mu_2=0=\mu_3=\mu_4=\ldots$$

Using boundary condition (iv) in (4), we have

$$\sum_{n=1}^{\infty}\lambda_n\sin\frac{n\pi x}{l}=y_0\sin\frac{2\pi x}{l}$$

Comparing like terms, we get

$$\lambda_2=y_0 \quad\text{and}\quad \lambda_1=0=\lambda_3=\lambda_4=\ldots$$

Using these values of λ_n and μ_n in (4), the required solution is

$$y(x,t)=y_0\sin\frac{2\pi x}{l}\cos\frac{2\pi at}{l}+\frac{lv_0}{\pi a}\sin\frac{\pi x}{l}\sin\frac{\pi at}{l}$$

Example 11

A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form of the curve $y=x(l-x)$ and also by imparting a constant velocity k to every point of the string in this position at time $t=0$. Determine the displacement function $y(x,t)$.

The displacement $y(x, t)$ of any point 'x' of the string at any time 't' is given by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$y(0, t) = 0, \quad \text{for } t \geq 0 \tag{2}$$

$$y(l, t) = 0, \quad \text{for } t \geq 0 \tag{3}$$

$$y(x, 0) = x(l - x), \quad \text{for } 0 \leq x \leq l \tag{4}$$

$$\frac{\partial y}{\partial t}(x, 0) = k, \quad \text{for } 0 < x < l \tag{5}$$

The appropriate solution of Eq. (1), consistent with the vibration of the string, is

$$y(x, t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat) \tag{6}$$

Using boundary condition (2) in (6), we have

$$A(C \cos pat + D \sin pat) = 0, \text{ for all } t \geq 0$$

$$\therefore \quad A = 0$$

Using boundary condition (3) in (6), we have

$$B \sin pl(C \cos pat + D \sin pat) = 0, \text{ for all } t \geq 0$$

$$\therefore \quad \text{Either } B = 0 \text{ or } \sin pl = 0$$

If we assume that $B = 0$, we get a trivial solution.

$$\therefore \quad \sin pl = 0$$

$$\therefore \quad pl = n\pi \text{ or } p = \frac{n\pi}{l}, \text{ where } n = 0, 1, 2, \ldots \infty$$

Using these values of A and p in (6), it reduces to

$$y(x, t) = b \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} + c \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l} \tag{7}$$

where $n = 0, 1, 2, \ldots \infty$

Therefore the most general solution of Eq. (1) is

$$y(x, t) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} + \sum_{n=1}^{\infty} \mu_n \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l} \tag{8}$$

Using boundary condition (4) in (8), we have

$$\sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} = x(l - x) \text{ (for } 0 \leq x \leq l) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

which is the Fourier half-range sine series of $x(l - x)$ in $(0, l)$.

Comparing like terms, we get

$$\lambda_n = b_n = \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right.$$

$$\left. + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{4l^2}{n^3\pi^3}\{1 - (-1)^n\}$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{8l^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \end{cases}$$

Differentiating both sides of (8) partially with respect to t, we have

$$\frac{\partial y}{\partial t}(x, t) = \sum_{n=1}^{\infty}\left(-\frac{n\pi a}{l}\lambda_n\right)\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} +$$

$$\sum_{n=1}^{\infty}\left(\frac{n\pi a}{l}\mu_n\right)\sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi at}{l} \tag{9}$$

Using boundary condition (5) in (9), we have

$$\sum_{n=1}^{\infty}\left(\frac{n\pi a}{l}\mu_n\right)\sin\frac{n\pi x}{l} = k, \text{ (for } 0 < x < l) = \sum_{n=1}^{\infty} b_n' \sin\frac{n\pi x}{l}$$

which is the Fourier half-range sine series of k in $(0, l)$.
Comparing like terms, we get

$$\frac{n\pi a}{l}\mu_n = b_n' = \frac{2}{l}\int_0^l k\sin\frac{n\pi x}{l}\,dx$$

$$\therefore \qquad \mu_n = \frac{2k}{n\pi a}\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^l$$

$$= \frac{2kl}{n^2\pi^2 a}\{1 - (-1)^n\}$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{4kl}{n^2\pi^2 a}, & \text{if } n \text{ is odd} \end{cases}$$

Using these values of λ_n and μ_n in (8), the required solution is

$$y(x,t) = \frac{8l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi at}{l}$$
$$+\frac{4kl}{\pi^2 a}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{l}\sin\frac{(2n-1)\pi at}{l}$$

Example 12

A string is stretched tightly between $x = 0$ and $x = l$ and both its ends are given a displacement $y = a\sin\omega t$ perpendicular to the string. If the string statisfies the differential equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$, show that the oscillations of the string are given by

$$y(x,t) = a\sec\frac{\omega l}{2c}\cdot\cos\left(\frac{\omega x}{c} - \frac{\omega l}{2c}\right)\sin\omega t$$

The appropriate solution of the given equation, consistent with the vibration of the string, is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pct + D\sin pct) \tag{1}$$

We have to evaluate the constants A, B, C, D and p in (1) by using the following boundary conditions.

$$y(0,t) = a\sin\omega t \tag{2}$$
$$y(l,t) = a\sin\omega t \tag{3}$$

Using boundary condition (2) in (1), we have $A(C\cos pct + D\sin pct) = a\sin\omega t$

i.e. $$AC\cos pct + AD\sin pct = a\sin\omega t \tag{4}$$

Comparing sine terms on both sides of (4), we have

$$AD = a \quad \text{and } pc = \omega \quad \text{or} \quad p = \frac{\omega}{c}$$

Comparing cosine terms on both sides of (4) we have

$$AC = 0$$
$$A \neq 0, \quad \text{as} \quad AD = a$$
$$\therefore \quad C = 0$$

Using these values in (1), it reduces to

$$y(x,t) = \left(a\cos\frac{\omega x}{c} + b\sin\frac{\omega x}{c}\right)\sin\omega t \tag{5}$$

where $b = BD$.

Using boundary condition (3) in (5), we have

$$\left(a\cos\frac{\omega l}{c}+b\sin\frac{\omega l}{c}\right)\sin\omega t=a\sin\omega t$$

$\therefore$
$$a\cos\frac{\omega l}{c}+b\sin\frac{\omega l}{c}=a$$

i.e.
$$b\sin\frac{\omega l}{c}=a\left(1-\cos\frac{\omega l}{c}\right)$$

i.e.
$$b=\frac{a\cdot 2\sin^2\dfrac{\omega l}{2c}}{2\sin\dfrac{\omega l}{2c}\cdot\cos\dfrac{\omega l}{2c}}=\frac{a\sin\dfrac{\omega l}{2c}}{\cos\dfrac{\omega l}{2c}}$$

Using this value of b in (5), the required solution is

$$y(x,t)=a\left[\cos\frac{\omega x}{c}+\frac{\sin\dfrac{\omega l}{2c}\cdot\dfrac{\sin\omega x}{c}}{\cos\dfrac{\omega l}{2c}}\right]\sin\omega t$$

i.e.
$$y(x,t)=a\sec\frac{\omega l}{2c}\left[\cos\frac{\omega x}{c}\cos\frac{\omega l}{2c}+\sin\frac{\omega x}{c}\sin\frac{\omega l}{2c}\right]\sin\omega t$$

i.e.
$$y(x,t)=a\sec\frac{\omega l}{2c}\cdot\cos\left(\frac{\omega x}{c}-\frac{\omega l}{2c}\right)\sin\omega t$$

Example 13

The differential equation of a vibrating string that is viscously damped is

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}-2b\frac{\partial y}{\partial t}$$

If the string is fastened at both ends and given zero initial velocity, find the displacement function $y(x,t)$, when

(i) the length of the string is l and initial displacement is $\sin\dfrac{\pi x}{l}$; and

(ii) the length of the string is π and initial displacement is $(\pi x-x^2)$.

The appropriate solution of the given equation, consistent with vibration of the string, is

$$y(x,t)=e^{-bt}(A\cos px+B\sin px)\left(C\cos\sqrt{p^2a^2-b^2}\,t+D\sin\sqrt{p^2a^2-b^2}\,t\right) \quad (1)$$

[Refer to section 3A.6]

We have to find the values of the arbitrary constants in the solution (1), by using the following boundary conditions.

$$y(0,t)=0, \qquad \text{for } t\ge 0 \quad (2)$$

$$y(l,t)=0, \qquad \text{for } t\ge 0 \quad (3)$$

$$\frac{\partial y}{\partial t}(x,0) = 0, \qquad \text{for } 0 \le x \le l \tag{4}$$

$$y(x,0) = f(x), \qquad \text{for } 0 \le x \le l \tag{5}$$

Note ✍

In (i), $f(x) = \sin\dfrac{\pi x}{l}$ *and in (ii),* $l = \pi$ *and* $f(x) = \pi x - x^2$.

Using boundary condition (2) in (1), we have

$$Ae^{-bt}\left(C\cos\sqrt{p^2a^2 - b^2}t + D\sin\sqrt{p^2a^2 - b^2}t\right) = 0, \quad \text{for all } t \ge 0$$

$$\therefore \qquad A = 0$$

Using boundary condition (3) in (1), we have

$$B\sin pl \cdot e^{-bt}\left(C\cos\sqrt{p^2a^2 - b^2}t + D\sin\sqrt{p^2a^2 - b^2}t\right) = 0, \quad \text{for all } t \ge 0$$

$$\therefore \qquad \text{Either } B = 0 \text{ or } \sin pl = 0$$

If we assume that $B = 0$, we get a trivial solution

$$\therefore \qquad \sin pl = 0$$

$$\therefore \qquad pl = n\pi \quad \text{or} \quad p = \frac{n\pi}{l}, \quad \text{where} \quad n = 0, 1, 2, \ldots \infty$$

Differentiating both sides of (1) partially with respect to t, we have

$$\frac{\partial y}{\partial t}(x,t) = B\sin px\left[-be^{-bt}\left(C\cos\sqrt{p^2a^2 - b^2}t + D\sin\sqrt{p^2a^2 - b^2}t\right)\right.$$
$$\left. + e^{-bt}\sqrt{p^2a^2 - b^2}\left(-C\sin\sqrt{p^2a^2 - b^2}t + D\cos\sqrt{p^2a^2 - b^2}t\right)\right] \tag{6}$$

Using boundary condition (4) in (6), we have

$$B\sin px\left[-bC + \sqrt{p^2a^2 - b^2}\cdot D\right] = 0, \quad \text{for} \quad 0 \le x \le l$$

As $B \ne 0$, we get $D = \dfrac{b}{\sqrt{p^2a^2 - b^2}}C$

Using the values of A, p and D in (1), it reduces to

$$y(x,t) = ke^{-bt}\sin\frac{n\pi x}{l}\left[\cos\sqrt{\frac{n^2\pi^2a^2}{l^2} - b^2}t\right.$$
$$\left. + \frac{b}{\sqrt{\dfrac{n^2\pi^2a^2}{l^2} - b^2}}\sin\sqrt{\frac{n^2\pi^2a^2}{l^2} - b^2}t\right], \quad \text{where} \quad k = BC.$$

or $y(x,t) = ke^{-bt}\sin\frac{n\pi x}{l}\left(\cos c_n t + \frac{b}{c_n}\sin c_n t\right)$, where

$$n = 0, 1, 2, \ldots\infty \text{ and } c_n = \sqrt{\frac{n^2\pi^2a^2}{l^2} - b^2}$$

Therefore the most general solution of the given equation

$$y(x,t) = e^{-bt}\sum_{n=1}^{\infty}\lambda_n \sin\frac{n\pi x}{l}\left(\cos c_n t + \frac{b}{c_n}\sin c_n t\right) \tag{7}$$

Using boundary condition (5) in (7), we have

$$\sum_{n=1}^{\infty}\lambda_n \sin\frac{n\pi x}{l} = f(x) \quad \text{in} \quad 0 \le x \le l$$

(i) $$f(x) = \sin\frac{\pi x}{l}$$

$\therefore$ $$\sum_{n=1}^{\infty}\lambda_n \sin\frac{n\pi x}{l} = \sin\frac{\pi x}{l}$$

Comparing like terms, we get

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = 0 = \lambda_3 = \lambda_4 = \ldots$$

Using these values of λ_n in (7), the required solution is

$$y(x,t) = e^{-bt}\sin\frac{\pi x}{l}\left(\cos c_1 t + \frac{b}{c_1}\sin c_1 t\right), \quad \text{where} \quad c_1 = \sqrt{\frac{\pi^2 a^2}{l^2} - b^2}$$

(ii) $$l = \pi \text{ and } f(x) = \pi x - x^2$$

In this case, the most general solution the given equation becomes

$$y(x,t) = e^{-bt}\sum_{n=1}^{\infty}\lambda_n \sin nx\left(\cos c_n t + \frac{b}{c_n}\sin c_n t\right) \tag{8}$$

where $c_n = \sqrt{n^2a^2 - b^2}$

Using boundary condition (5) in (8), we have

$$\sum_{n=1}^{\infty}\lambda_n \sin nx = \pi x - x^2, \quad \text{for} \quad 0 \le x \le \pi$$

$$= \sum_{n=1}^{\infty} b_n \sin nx$$

which is the Fourier half-range sine series of $(\pi x - x^2)$ in $(0, \pi)$.

Comparing like terms, we have

$$\begin{aligned}
\lambda_n = b_n &= \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx \, dx \\
&= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right)\right. \\
&\qquad \left. +(-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi} \\
&= \frac{4}{\pi n^3}\{1 - (-1)^n\} \\
&= \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{8}{\pi n^3}, & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

Using this value of λ_n in (8), the required solution is

$$y(x,t) = \frac{8}{\pi}e^{-bt}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^3}\sin nx\left(\cos c_n t + \frac{b}{c_n}\sin c_n t\right)$$

where $c_n = \sqrt{n^2a^2 - b^2}$.

PROBLEMS ON TRANSMISSION LINE EQUATION

Example 14

Neglecting the resistance R and leakance G, find the e.m.f. $v(x,t)$ and the current $i(x,t)$ in a line l km long t second after the ends are suddenly grounded, if initially $i(x,0) = i_0$ and $v(x,0) = E\sin\dfrac{\pi x}{l}$.

[Refer to Section 3A.3]

When $R = G = 0$, the telephone equation for the potential v reduces to the radio equation

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}$$

or

$$\frac{\partial^2 v}{\partial t^2} = a^2\frac{\partial^2 v}{\partial x^2} \tag{1}$$

where $a^2 = \dfrac{1}{LC}$.

We have to solve (1), that is similar to the one dimensional wave equation, satisfying the following boundary conditions.

$$v(0,t) = 0, \quad \text{for } t \geq 0 \tag{2}$$

$$v(l,t) = 0, \quad \text{for } t \geq 0 \tag{3}$$

since the ends of the line are grounded.

$$\frac{\partial v}{\partial t}(x, 0) = 0, \quad \text{for } 0 \le x \le l \tag{4}$$

Since the second transmission line equation reduces to $C\dfrac{\partial v}{\partial t} = -\dfrac{\partial i}{\partial x}$, when $G = 0$ and hence

$$\begin{aligned}\frac{\partial v}{\partial t}(x, 0) &= -\frac{1}{C}\frac{\partial i}{\partial x}(x, 0)\\ &= -\frac{1}{C}\frac{\partial i_0}{\partial x}\\ &= 0\end{aligned}$$

$$v(x, 0) = E\sin\frac{\pi x}{l}, \quad \text{for } 0 \le x \le l \tag{5}$$

The appropriate solution of Eq. (1), consistent with the flow of electricity in a transmission line is

$$v(x, t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \tag{6}$$

Using boundary condition (2) in (6), we have

$$A(C\cos pat + D\sin pat) = 0, \quad \text{for all } t \ge 0$$

$\therefore$ $A = 0$

Using boundary condition (3) in (6), we have

$$B\sin pl(C\cos pat + D\sin pat) = 0, \quad \text{for all } t \ge 0$$

$\therefore$ Either $B = 0$ or $\sin pl = 0$

If we assume that $B = 0$, we get a trivial solution.

$\therefore$ $\sin pl = 0$

$\therefore$ $pl = n\pi$ or $p = \dfrac{n\pi}{l}$, where $n = 0, 1, 2, \ldots \infty$

Differentiating both sides of (6) partially with respect to t, we have

$$\frac{\partial v}{\partial t}(x, t) = B\sin px \cdot pa(-C\sin pat + D\cos pat) \tag{7}$$

where $p = \dfrac{n\pi}{l}$

Using boundary condition (4) in (7), we have

$$B\sin px \cdot pa \cdot D = 0, \quad \text{for } 0 \le x \le l$$

As $B \neq 0$ and $p \neq 0$, we get $D = 0$.

Using these values of A, p and D in (6), it reduces to

$$v(x,t) = k\sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi at}{l}, \quad \text{where } n = 1, 2, \ldots\infty.$$

Therefore the most general solution of equation (1) is

$$v(x,t) = \sum_{n=1}^{\infty}\lambda_n \sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi at}{l} \tag{8}$$

Using boundary condition (5) in (8), we have

$$\sum_{n=1}^{\infty}\lambda_n \sin\frac{n\pi x}{l} = E\sin\frac{\pi x}{l} \tag{9}$$

Comparing like terms in (9), we get

$$\lambda_1 = E \quad \text{and} \quad \lambda_2 = 0 = \lambda_3 = \lambda_4 = \ldots.$$

Using these values of λ_n in (8), the required solution is

$$v(x,t) = E\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} \quad \text{or}$$

$$v(x,t) = E\sin\frac{\pi x}{l}\cos\frac{\pi t}{l\sqrt{LC}} \tag{10}$$

Differentiating (10) partially with respect to x, we have

$$\frac{\partial v}{\partial x} = \frac{E\pi}{l}\cos\frac{\pi x}{l}\cos\frac{\pi t}{l\sqrt{LC}} \tag{11}$$

The first transmission line equation reduces to $\dfrac{\partial i}{\partial t} = -\dfrac{1}{L}\dfrac{\partial v}{\partial x}$, when

$$R = 0 \tag{12}$$

From (11) and (12), we have

$$\frac{\partial i}{\partial t} = -\frac{E\pi}{lL}\cos\frac{\pi x}{l}\cos\frac{\pi t}{l\sqrt{LC}} \tag{13}$$

Integrating (13) partially with respect to t, we get

$$i(x,t) = -\frac{E\pi}{lL}\cdot\frac{l\sqrt{LC}}{\pi}\cos\frac{\pi x}{l}\sin\frac{\pi t}{l\sqrt{LC}} + f(x)$$

$$= -E\sqrt{\frac{C}{L}}\cos\frac{\pi x}{l}\sin\frac{\pi t}{l\sqrt{LC}} + f(x) \tag{14}$$

Using the given condition $i(x,0) = i_0$, we get $f(x) = i_0$.

Using this value of $f(x)$ in (14), we get

$$i(x,t) = i_0 - E\sqrt{\frac{C}{L}}\cos\frac{\pi x}{l}\sin\frac{\pi t}{l\sqrt{LC}}$$

Example 15

Neglecting R and G, find the e.m.f. $v(x, t)$ in a line l km long t seconds after the ends were suddenly grounded, if initially $i(x, 0) = i_0$, so that

$$\frac{\partial v}{\partial t}(x, 0) = 0 \quad \text{and} \quad v(x, 0) = \frac{Ex}{l} \tag{1}$$

This problem is similar to the previous example with the only change in boundary condition (2).

The boundary condition (2) for this problem is

$$v(x, 0) = \frac{Ex}{l}, \quad \text{for} \quad 0 \leq x < l \tag{2}$$

Proceeding as in the previous example, the most general solution of Eq. (1) is

$$v(x, t) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \tag{3}$$

Using boundary condition (2) in (3), we have

$$\sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} = \frac{Ex}{l} \text{ in } (0, l) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

which is Fourier half-range sine series of $\frac{Ex}{l}$ in $(0, l)$

$$\therefore \qquad \lambda_n = b_n = \frac{2}{l} \int_0^l \frac{Ex}{l} \sin \frac{n\pi x}{l} dx$$

$$= \frac{2E}{l^2} \left[x \left(-\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right]_0^l$$

$$= -\frac{2E}{n\pi} \cos n\pi \quad \text{or} \quad \frac{2E}{n\pi}(-1)^{n+1}$$

Using this value of λ_n in (3) the required solution is

$$v(x, t) = \frac{2E}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}, \quad \text{where} \quad a = \frac{1}{\sqrt{LC}}$$

Example 16

A line of length l is initially uncharged so that $i(x, 0) = 0$, $e(x, 0) = 0$ and $\frac{\partial e}{\partial t}(x, 0) = 0$. At $t = 0$ the end $x = l$ is suddenly connected with a constant potential E, while the other end is grounded. Neglecting R and G, find $e(x, t)$ and $i(x, t)$.

When $R = G = 0$, the potential $e(x, t)$ is given by the equation

$$\frac{\partial^2 e}{\partial t^2} = a^2 \frac{\partial^2 e}{\partial x^2} \tag{1}$$

where $a^2 = \dfrac{1}{LC}$.

We have to solve (1), satisfying the following boundary conditions.

$$e(0, t) = 0, \quad \text{for} \quad t \geq 0 \tag{2}$$

$$e(l, t) = E, \quad \text{for} \quad t \geq 0 \tag{3}$$

$$\frac{\partial e}{\partial t}(x, 0) = 0, \quad \text{for} \quad 0 < x < l \tag{4}$$

$$e(x, 0) = 0, \quad \text{for} \quad 0 < x < l \tag{5}$$

From the previous problems, we observe that Eq. (1) can be solved by the method of separation of variables subject to given boundary conditions, if and only if the end values in (2) and (3) are zero each.

In this problem, $e(l, t) = E, \quad \text{for} \quad t \geq 0$.

Hence we adopt a slightly different procedure in this case as given below.

We assume that

$$e(x, t) = e_1(x) + e_2(x, t) \tag{6}$$

Using (6) in (1), we get

$$\frac{\partial^2}{\partial t^2}(e_1 + e_2) = a^2 \frac{\partial^2}{\partial x^2}(e_1 + e_2)$$

This gives rise to the two equations

$$\frac{\partial^2 e_1}{\partial t^2} = a^2 \frac{\partial^2 e_1}{\partial x^2} \quad \text{or} \quad \frac{d^2 e_1}{dx^2} = 0 \tag{7}$$

[since $e_1(x)$ is a function of x only]

and

$$\frac{\partial^2 e_2}{\partial t^2} = a^2 \frac{\partial^2 e_2}{\partial x^2} \tag{8}$$

Since $e_1(x)$ is independent of t and the end values do not change with t, we assume that $e_1(x)$ governs the end points $x = 0$ and $x = l$ and $e_2(x, t)$ governs the interior points $0 < x < l$. Thus we have to solve Eq. 7, satisfying the conditions

$$e_1(0) = 0 \tag{9}$$

and

$$e_1(l) = E \tag{10}$$

Solving Eq. (7), we get

$$e_1(x) = ax + b \tag{11}$$

Using boundary condition (9) in (11), we get $b = 0$.

Using boundary condition (10) in (11), we get $a = \frac{E}{l}$.

$$\therefore \qquad e_1(x) = \frac{Ex}{l} \tag{12}$$

Now we have to solve Eq. (8), satisfying the following boundary conditions which are obtained using (6).

$$e_2(0, t) = e(0, t) - e_1(0) = 0, \quad \text{for all } t \geq 0 \tag{13}$$

[from (2) and (9)]

$$e_2(l, t) = e(l, t) - e_1(l) = 0, \quad \text{for all } \geq 0 \tag{14}$$

[from (3) and (10)]

$$\frac{\partial e_2}{\partial t}(x, 0) = \frac{\partial e}{\partial t}(x, 0) = 0, \quad \text{for } 0 < x < l \tag{15}$$

[from (4) and since $e_1(x)$ is a function of x]

$$\begin{aligned} e_2(x, 0) &= e(x, 0) - e_1(x) \\ &= -\frac{Ex}{l}, \quad \text{for } 0 < x < l \end{aligned} \tag{16}$$

[from (5) and (12)]

Note

Equation (8) is readily solved, as the end conditions, (13) and (14) for $e_2(x, t)$ contain only zero values

Proceeding as in Example (14), the most general solution of Eq. (8) is

$$e_2(x, t) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \tag{17}$$

Using boundary condition (16) in (17), we have

$$\sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} = -\frac{Ex}{l} \quad \text{in} \quad (0 < x < l) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

which is the Fourier half-range sine series of $\left(-\frac{Ex}{l}\right)$ in $(0, l)$.

$$\begin{aligned} \therefore \qquad \lambda_n = b_n &= \frac{2}{l} \int_0^l -\frac{Ex}{l} \sin \frac{n\pi x}{l} dx \\ &= -\frac{2E}{l^2} \left[x \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) \right]_0^l \\ &= \frac{2E}{n\pi} \cos n\pi \quad \text{or} \quad \frac{2E}{n\pi}(-1)^n \end{aligned}$$

Using this value of λ_n in (17), we get

$$e_2(x,t) = \frac{2E}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \tag{18}$$

Using (12) and (18) in (6), the required solution is

$$e(x,t) = \frac{Ex}{l} + \frac{2E}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \tag{19}$$

Differentiating (19) partially with respect to x,

$$\frac{de}{dx} = -L\frac{di}{dt} = \frac{E}{l} + \frac{2E}{\pi}\cdot\frac{\pi}{l}\sum_{n=1}^{\infty}(-1)^n\cos\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \tag{20}$$

[from the first transmission line equation for $R = 0$]

Integrating (20) partially with respect to t,

$$i(x,t) = -\frac{E}{lL}t - \frac{2E}{lL}\cdot\frac{l}{\pi a}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos\frac{n\pi x}{l}\sin\frac{n\pi at}{l} + f(x) \tag{21}$$

Using the condition $i(x,0) = 0$ in (21), we get

$$f(x) = 0$$

Using this value of $f(x)$ and putting $a = \frac{1}{\sqrt{LC}}$, we get

$$i(x,t) = -\frac{Et}{lL} - \frac{2E}{\pi}\sqrt{\frac{C}{L}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos\frac{n\pi x}{l}\sin\frac{n\pi t}{l\sqrt{LC}}$$

Example 17

Find *the D'Alembert's solution* of the differential equation governing the propagation of current and potential along a dissipationless transmission line $R = 0, G = 0$, given that $e(x,0) = f(x)$ and $\frac{\partial e}{\partial t}(x,0) = 0$.

When $R = 0, G = 0$, the differential equation governing the propagation of potential along a transmission line is

$$\frac{\partial^2 e}{\partial t^2} = a^2\frac{\partial^2 e}{\partial x^2} \tag{1}$$

where $a^2 = \frac{1}{LC}$.

We have to solve Eq. (1), satisfying the initial conditions

$$e(x,0) = f(x) \tag{2}$$

$$\frac{\partial e}{\partial t}(x,0) = g(x) \tag{3}$$

Note ✍

In the present problem, $g(x) = 0$.

Denoting $\dfrac{\partial}{\partial t}$ by D and $\dfrac{\partial}{\partial x}$ by D', Eq. (1) becomes $D^2 - a^2 D'^2 = 0$, which is a homogeneous linear equation.

The auxiliary equation is $m^2 = a^2$.

$$\therefore \quad m = \pm a$$

Hence the general solution of Eq. (1) is

$$e(x, t) = \phi_1(x + at) + \phi_2(x - at) \tag{4}$$

where ϕ_1 and ϕ_2 are arbitrary functions.

Using (2) in (4), we get

$$\phi_1(x) + \phi_2(x) = f(x) \tag{5}$$

Differentiating (4) partially with respect to t, we have

$$\frac{\partial e}{\partial t}(x, t) = a\phi_1'(x + at) - a\phi_2'(x - at) \tag{6}$$

Using (3) in (6), we get

$$a[\phi_1'(x) - \phi_2'(x)] = g(x) \tag{7}$$

Integrating (6) with respect to x, we have

$$\phi_1(x) - \phi_2(x) = \frac{1}{a}\int g(x)\mathrm{d}x + c \quad \text{or}$$

$$= \frac{1}{a}\int_k^x g(u)\mathrm{d}u \tag{8}$$

Solving (5) and (8), we get

$$\phi_1(x) = \frac{1}{2}f(x) + \frac{1}{2a}\int_k^x g(u)\mathrm{d}u \tag{9}$$

and

$$\phi_2(x) = \frac{1}{2}f(x) - \frac{1}{2a}\int_k^x g(u)\mathrm{d}u \tag{10}$$

Using (9) and (10) in (4), the required solution is

$$e(x, t) = \frac{1}{2}f(x + at) + \frac{1}{2a}\int_k^{x+at} g(u)\mathrm{d}u + \frac{1}{2}f(x - at)$$

$$- \frac{1}{2a}\int_k^{x-at} g(u)\mathrm{d}u$$

i.e.

$$e(x, t) = \frac{1}{2}[f(x + at) + f(x - at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(u)\mathrm{d}u \tag{11}$$

Equation (11) is called *D'Alembert's solution* of Eq. (1). Since $g(x) = 0$ in the present problem, the required solution reduces to

$$e(x,t) = \frac{1}{2}\left[f\left(x + \frac{t}{\sqrt{LC}}\right) + f\left(x - \frac{t}{\sqrt{LC}}\right)\right] \tag{12}$$

since $a^2 = \dfrac{1}{\sqrt{LC}}$.

Differentiating (12) partially with respect to x,

$$\frac{\partial e}{\partial x} = -L\frac{\partial i}{\partial t} = \frac{1}{2}\left[f'\left(x + \frac{t}{\sqrt{LC}}\right) + f'\left(x - \frac{t}{\sqrt{LC}}\right)\right] \tag{13}$$

Integrating (13) partially with respect to t,

$$i(x,t) = -\frac{1}{2L}\cdot\sqrt{LC}\left[f\left(x + \frac{t}{\sqrt{LC}}\right) - f\left(x - \frac{t}{\sqrt{LC}}\right)\right] + \phi(x) \tag{14}$$

Assuming that $i(x, 0) = 0$ and using it in (14),

we get $$\phi(x) = 0$$

Hence the current is given by

$$i(x,t) = \frac{1}{2}\sqrt{\frac{C}{L}}\left[f\left(x - \frac{t}{\sqrt{LC}}\right) - f\left(x + \frac{t}{\sqrt{LC}}\right)\right]$$

Example 18

Solve the telephone equation $\dfrac{\partial^2 e}{\partial x^2} = LC\dfrac{\partial^2 e}{\partial t^2} + RC\dfrac{\partial e}{\partial t}$ given that $e(0,t) = E_0 \sin pt$ and $e = 0$, when $x \to \infty$, assuming that $\dfrac{pL}{R}$ is large compared with unity.

The method of separation of variables will not yield the solution in the required form.

Since $e \to 0$ as $x \to \infty$, there should be a factor of the form $e^{-ax}(a > 0)$ in the required solution.

Also since $e(0,t) = E_0 \sin pt$, let us assume the solution of the given equation as

$$e(x,t) = E_0 e^{-ax} \sin(bx + pt) \tag{1}$$

where a and b are unknown constants.

From (1),

$$\frac{\partial e}{\partial t} = E_0 p e^{-ax}\cos(bx + pt)$$

$$\frac{\partial^2 e}{\partial t^2} = -E_0 p^2 e^{-ax}\sin(bx + pt)$$

$$\frac{\partial^2 e}{\partial x^2} = E_0\Big[a^2 e^{-ax}\sin(bx + pt) - b^2 e^{-ax}\sin(bx + pt)$$
$$-2abe^{-ax}\cos(bx + pt)\Big]$$

Since (1) is a solution of the given equation, the values of these derivatives satisfy the given equation.

$$\therefore \quad (a^2 - b^2)e^{-ax}\sin(bx + pt) - 2abe^{-ax}\cos(bx + pt) = -LCp^2e^{-ax}\sin(bx + pt) + RCpe^{-ax}\cos(bx + pt) \quad (2)$$

Equating like terms on both sides of (2), we get

$$a^2 - b^2 = -LCp^2 \quad (3)$$

$$2ab = -RCp \quad (4)$$

Eliminating b from (3) and (4), we have

$$a^2 - \frac{R^2C^2p^2}{4a^2} = -LCp^2$$

i.e. $$4a^4 + 4LCp^2a^2 - R^2C^2p^2 = 0$$

$$\therefore \quad a^2 = \frac{-4LCp^2 \pm \sqrt{16L^2C^2p^4 + 16R^2C^2p^2}}{8}$$

$$= \frac{1}{2}\left[-LCp^2 + LCp^2\left(1 + \frac{R^2}{p^2L^2}\right)^{1/2}\right], \quad \text{since } a^2 > 0$$

$$= \frac{1}{2}\left[-LCp^2 + LCp^2\left\{1 + \frac{R^2}{2p^2L^2}\right\}\right], \quad \text{omitting}$$

higher powers of $\frac{R}{pL}$, as $\frac{R}{pL}$ is small.

$$= \frac{1}{4}R^2\frac{C}{L}$$

Hence $a = \frac{R}{2}\sqrt{\frac{C}{L}}$, as $a > 0$.

Using this value of a in (4), we have

$$b = -\frac{RCp}{R}\sqrt{\frac{L}{C}} = -p\sqrt{LC}$$

Hence the required solution is

$$e(x, t) = E_0 e^{-R/2\sqrt{\frac{C}{L}}x} \cdot \sin(pt - \sqrt{LC}px)$$

Note ✍

For problems on telegraph equation, see worked examples (18), (19) and (20) and also questions (32), (33), (34) and (35) in the excercise in Chapter 3(B).

PROBLEMS ON VIBRATION OF MEMBRANES

Example 19

Given that the differential equation of a vibrating membrane (such as the membrane of a drum) is $\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$, find the deflection $z(x, y, t)$ of the rectangular membrane of sides 1 and 2, if it is fastened along the edges and excited from rest with an initial displacement $f(x, y) = k \sin \pi x \sin \pi y$.

The equation

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) \tag{1}$$

which represents the vibration of a membrane is assumed.

We have to solve Eq. (1), satisfying the following boundary conditions.

$$z(0, y, t) = 0, \quad \text{for} \quad 0 < y < 2 \quad \text{and} \quad t \geq 0 \tag{2}$$

$$z(1, y, t) = 0, \quad \text{for} \quad 0 < y < 2 \quad \text{and} \quad t \geq 0 \tag{3}$$

$$z(x, 0, t) = 0, \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad t \geq 0 \tag{4}$$

$$z(x, 2, t) = 0, \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad t \geq 0 \tag{5}$$

$$\frac{\partial z}{\partial t}(x, y, 0) = 0, \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad 0 < y < 2 \tag{6}$$

$$z(x, y, 0) = f(x, y) = k \sin \pi x \sin \pi y, \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad 0 < y < 2 \tag{7}$$

Let

$$z(x, y, t) = X(x) \cdot Y(y) \cdot T(t) \tag{8}$$

be a solution of the Eq. (1).

From (8),

$$\frac{\partial^2 z}{\partial x^2} = X''YT, \quad \frac{\partial^2 z}{\partial y^2} = XY''t \quad \text{and} \quad \frac{\partial^2 z}{\partial t^2} = XYT''$$

where the dashes denote ordinary derivatives with respect to the corresponding variables.

Then $XYT'' = c^2(X''YT + XY''T)$

Dividing throughout by XYT, we have

$$\frac{T''}{T} = c^2\left(\frac{X''}{X} + \frac{Y''}{Y}\right)$$

i.e.

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} - \frac{Y''}{Y} = -p^2$$

Note ✍

The negative constant is assumed, since it is the case of vibration and so X, Y and T should be periodic in x, y and t respectively.

$$\therefore \qquad X'' + p^2 X = 0 \tag{9}$$

and

$$\frac{Y''}{Y} = \frac{1}{c^2}\frac{T''}{T} + p^2 = -q^2$$

$$\therefore \qquad Y'' + q^2 Y = 0 \tag{10}$$

and

$$T'' + (p^2 + q^2)c^2 T = 0$$

or

$$T'' + r^2 T = 0 \tag{11}$$

where $r^2 = (p^2 + q^2)c^2$

Solving equations (9), (10) and (11), we get

$$X = A\cos px + B\sin px$$

$$Y = C\cos qy + D\sin qy$$

and $\qquad T = E\cos rt + F\sin rt$

Using these values in (8), the appropriate solution of Eq. (1) is

$$z(x, y, t) = (A\cos px + B\sin px)(C\cos qy + D\sin qy)(E\cos rt + F\sin rt) \tag{12}$$

Using boundary condition (2) in (12), we have

$$A(C\cos qy + D\sin qy)(E\cos rt + F\sin rt) = 0, \text{ for } \quad 0 < y < 2 \text{ and } t \geq 0$$

$$\therefore \qquad A = 0$$

Using boundary condition (3) in (12), we have

$$B\sin p(C\cos qy + D\sin qy)(E\cos rt + F\sin rt) = 0,$$

$$\text{for} \quad 0 < y < 2 \quad \text{and} \quad t \geq 0$$

$\therefore$ Either $B = 0$ or $\sin p = 0$

The value $B = 0$ leads to a trivial solution.

$\therefore \qquad \sin p = 0$

$\therefore \qquad p = m\pi, \quad$ where $\quad m = 0, 1, 2, \ldots \infty.$

Using boundary condition (4) in (12), we have

$$B\sin px \cdot C(E\cos rt + F\sin rt) = 0 \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad t \geq 0$$

As $B \neq 0$, we get $C = 0$.

Using boundary condition (5) in (12), we have

$$B\sin px \cdot D \cdot \sin 2q(E\cos rt + F\sin rt) = 0,$$
$$\text{for} \quad 0 < x < 1 \quad \text{and} \quad t \geq 0$$

$\therefore$ Either $D = 0$ or $\sin 2q = 0$

The value $D = 0$ leads to a trivial solution.

$$\therefore \qquad \sin 2q = 0$$

$$\therefore \qquad 2q = n\pi \quad \text{or} \quad q = \frac{n\pi}{2}, \quad \text{where} \quad n = 0, 1, 2, \ldots \infty.$$

Differentiating (12) (with $A = 0$ and $C = 0$) partially with respect to t, we have

$$\frac{\partial z}{\partial t}(x, y, t) = BD\sin px \sin qy \cdot r(-E\sin rt + F\cos rt) \tag{13}$$

Using boundary condition (6) in (13), we have

$$BD\sin px \sin qy \cdot rF = 0, \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad 0 < y < 2$$

As $B \neq 0, D \neq 0$ and $r = c\sqrt{p^2 + q^2} \neq 0$, we get $F = 0$. Using the values of A, C, F, p, q and r in (12), it reduces to

$$z(x, y, t) = k\sin m\pi x \cdot \sin\frac{n\pi y}{2}\cos\left\{\sqrt{m^2 + \frac{n^2}{4}}\pi ct\right\} \tag{14}$$

where $\qquad k = \text{BDE}$

$$m = 0, 1, 2, \ldots .\infty$$

and $\qquad n = 0, 1, 2, \ldots .\infty$

The most general solution of Eq. (1) is obtained by superposing infinitely many solutions given in (14). It is

$$z(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}\sin m\pi x \sin\frac{n\pi y}{2}\cos\left(\sqrt{m^2 + \frac{n^2}{4}}\pi ct\right) \tag{15}$$

Using boundary condition (7) in (15), we have

$\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}\lambda_{mn}\sin m\pi x \sin\dfrac{n\pi y}{2} = f(x, y) = k\sin\pi x \sin\pi y$, in $0 < x < 1$ and $0 < y < 2$.

Comparing like terms, we get $\lambda_{12} = k$ and $\lambda_{mn} = 0$, for all other values of m and n. Using these values of λ_{mn} in (15), the required solution is

$$z(x, y, t) = k\sin\pi x \sin\pi y \cos\sqrt{2}\pi ct$$

Example 20

Find the deflection $z(x, y, t)$ of a rectangular membrane $(0 \leq x \leq a, 0 \leq y \leq b)$ whose boundary is fixed, given that it starts from rest and

$$z(x, y, 0) = kxy(a-x)(b-y)$$

We have to solve the equation

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) \tag{1}$$

that represents the vibration of membrane, satisfying the following boundary conditions.

$$z(0, y, t) = 0, \text{ for } 0 < y < b \text{ and } t \geq 0 \tag{2}$$

$$z(a, y, t) = 0, \text{ for } 0 < y < b \text{ and } t \geq 0 \tag{3}$$

$$z(x, 0, t) = 0, \text{ for } 0 < x < a \text{ and } t \geq 0 \tag{4}$$

$$z(x, b, t) = 0, \text{ for } 0 < x < a \text{ and } t \geq 0 \tag{5}$$

$$\frac{\partial z}{\partial t}(x, y, 0) = 0, \text{ for } 0 < x < a \text{ and } 0 < y < b \tag{6}$$

$$z(x, y, 0) = kxy(a-x)(b-y), \text{ for } 0 < x < a \text{ and } 0 < y < b \tag{7}$$

Proceeding as in Example 19, the most general solution of Eq. (1) is

$$z(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \left(\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \pi ct \right) \tag{8}$$

Using boundary condition (7) in (8), we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = kxy(a-x)(b-y) \text{ in } 0 < x < a \text{ and } 0 < y < b$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

which is Fourier double sine series of $kxy(a-x)(b-y)$ in $0 < x < a$ and $0 < y < b$.

Comparing like terms, we get

$$\lambda_{mn} = b_{mn} = \frac{4}{ab} \int_0^b \int_0^a kxy(a-x)(b-y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx\, dy$$

$$= \frac{4k}{ab} \int_0^a (ax - x^2) \sin \frac{m\pi x}{a} dx \cdot \int_0^b (by - y^2) \sin \frac{n\pi y}{b} dy$$

$$= \frac{4k}{ab}\left[\left[\left\{(ax - x^2)\left(\frac{-\cos\frac{m\pi x}{a}}{\frac{m\pi}{a}}\right) - (a - 2x)\left(\frac{-\sin\frac{m\pi x}{a}}{\frac{m^2\pi^2}{a^2}}\right)\right.\right.\right.$$

$$\left.\left. + (-2)\left(\frac{\cos\frac{m\pi x}{a}}{\frac{m^3\pi^3}{a^3}}\right)\right\}\right]_0^a \times \left\{(by - y^2)\left(\frac{-\cos\frac{n\pi y}{b}}{\frac{n\pi}{b}}\right)\right.$$

$$\left.\left. - (b - 2y)\left(\frac{-\sin\frac{n\pi y}{b}}{\frac{n^2\pi^2}{b^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi y}{b}}{\frac{n^3\pi^3}{b^3}}\right)\right\}_0^b\right]$$

$$= \frac{4k}{ab}\left[\frac{2a^3}{m^3\pi^3}\left\{1 - (-1)^m\right\} \cdot \frac{2b^3}{n^3\pi^3}\left\{1 - (-1)^n\right\}\right]$$

$$= \begin{cases} \dfrac{64ka^2b^2}{m^3n^3\pi^6}, & \text{if both } m \text{ and } n \text{ are odd} \\ 0, & \text{otherwise} \end{cases}$$

Using this value of λ_{mn} in (8), the required solution is

$$z(x, y, t) = \frac{64ka^2b^2}{\pi^6}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2m-1)^3(2n-1)^3}\sin\frac{(2m-1)\pi x}{a}.$$

$$\sin\frac{(2n-1)\pi y}{b}\cos\left\{\sqrt{\frac{(2m-1)^2}{a^2} + \frac{(2n-1)^2}{b^2}}\cdot\pi ct\right\}.$$

Exercise 3(a)

Part A (Short-Answer Questions)

1. When do you say that a *quasi-linear* P.D.E. is (i) elliptic, (ii) parabolic, (iii) hyperbolic?
2. Classify the following P.D.E.s as to whether they are elliptic, parabolic or hyperbolic.
   (i) $u_{xx} + 2u_{xy} + u_{yy} = 0$
   (ii) $u_{xy} - u_x = 0$
   (iii) $x^2u_{xx} + (1 - y^2)\,u_{yy} = 0; -\infty < x < \infty; -1 < y < 1.$
   (iv) $xf_{xx} + yf_{yy} = 0; x > 0; y > 0.$
   (v) $f_{xx} - 2f_{xy} = 0$

(vi) $f_{xx} + 2f_{xy} + 4f_{yy} = 0$

(vii) $(x + 1)f_{xx} + 2(x + 2) f_{xy} + (x + 3)f_{yy} = 0$

3. State the assumptions made while deriving the equation of vibration of a string in the simple form.
4. What does a^2 represent in the equation $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$?
5. Write down the two transmission line equations.
6. Write down the telephone equations for potential and current.
7. Write down the radio equations for potential and current.
8. Write down the telegraph equations for potential and current.
9. When do the telephone equations reduces to the form of one dimensional wave equation?
10. When do the telephone equations reduce to the form of one dimensional heat flow equation?
11. Write down the three mathematically possible solutions of one dimensional wave equation.
12. Write down the appropriate solution of the vibration of string equation. How is it chosen?
13. Write down the differential equation that represents damped vibration of a string.
14. Write down the suitable variable separable solution of the equation that represents damped vibration of a string.
15. Write down the differential equation that represents the vibration of a membrane.
16. Write down the appropriate solution of the equation representing the vibration of a membrane.
17. Write down the form of the general solution of the vibration of string equation, if the string fixed at the ends, is given a non zero initial displacement and zero initial velocity.
18. Write down the form of the general solution of the vibration of string equation, if the string, fixed at the ends, is given a non zero initial velocity from its equilibrium position.
19. Write down the form of the general solution of the vibration of string equation, if the string is fixed at its ends.

Part B

20. A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form of the curve (a) $y = 2\sin\dfrac{2\pi x}{l} + 3\sin\dfrac{3\pi x}{l}$ and (b) $y = 2\sin\dfrac{3\pi x}{l}\cos\dfrac{2\pi x}{l}$ and then releasing it from this position at time $t = 0$. Find the displacement function $y(x, t)$.

21. The ends of a uniform string of length $2l$ are fixed. The initial displacement is $y(x, 0) = kx(2l - x)$, $0 < x < 2l$, while the initial velocity is zero. Find the displacement at any distance x from the end $x = 0$ at any time t.

22. A tightly stretched string of length π is fastened at both ends. The mid-point of the string is displaced by a distance d transversely and the string is released from rest in this position. Find the displacement of any point of the string at any subsequent time.

23. Solve Problem 22, if length of the string is 50 cm and $d = 1$ cm.

24. A tightly stretched string of length l has its ends $x = 0$ and $x = l$ fixed. The point $x = l/3$ is drawn aside by a small distance h and released from rest at time $t = 0$. Find $y(x, t)$ at any subsequent time t.

25. An elastic string is stretched between two points at a distance l. One end is taken as the origin and at a distance $\frac{2l}{3}$ from this end, the string is displaced a distance d transversely and is released from rest when it is in this position. Find the displacement function $y(x, t)$.

26. The points of trisection of a string of length π are pulled aside through a distance b on opposite sides of the position of equilibrium and the string is released from rest. Find the subsequent displacement of the string.

27. A tightly stretched string with fixed end $x = 0$ and $x = \pi$ is initially at rest in its equilibrium position. If it is set in motion by giving each point a velocity $v = 16 \sin^5 x$, find the displacement of any point of the string at any time t.

28. A taut string of length 50 cm fastened at both ends, is disturbed from its position of equilibrium by imparting to each of its points an initial velocity of magnitude kx for $0 < x < 50$. Find the displacement function $y(x, t)$.

29. A string is stretched between two fixed points at a distance of l cm and the points of the string are given initial velocity $v = \lambda(lx - x^2)$, for $0 < x < l$. Find the displacement function $y(x, t)$.

30. A string is stretched between two fixed points at a distance of $2l$ cm and the points of the string are given initial velocities v, where

$$v = \frac{kx}{l} \text{ in } 0 < x < l$$

$$= \frac{k}{l}(2l - x) \text{ in } l < x < 2l$$

x being the distance from an end point. Find the displacement of the string at any time.

31. A string is stretched and fastened to two point l apart. Motion is started by displacing the string into the form of the curve $y = y_0 \sin \frac{n\pi}{l}$ and also by imparting a constant velocity v_0 to every point of the string in this position at time $t = 0$. Determine the displacement function $y(x, t)$.

32. A string is stretched between two fixed points at a distance of π cm and the points of the string are given initial velocities v, where

$$v = x, \text{ in } 0 < x < \frac{\pi}{2}$$

$$= \pi - x, \text{ in } \frac{\pi}{2} < x < \pi$$

after displacing it to the position $y = x(\pi - x)$ at $t = 0$. Find the displacement of the string at any time.

33. A tightly stretched string of length l is fastened at both ends. The mid-point of the string is taken to a distance h and the various points of the string are given (different) velocities v, where $v = kx(l - x)$, in this position at $t = 0$. Find the subsequent displacement of the string.

34. Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$, if $u(0, t) = 0$, $u(l, t) = A \sin \omega t$ and $u(x, 0) = 0$.

35. The differential equation of a vibrating string that is viscously damped is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} - 2b \frac{\partial y}{\partial t}$$

If the string is fastened at both ends and given zero initial velocity, find the displacement function $y(x, t)$ when the length of the string is l and initial displacement is $\sin^3\left(\frac{\pi x}{l}\right)$.

36. Solve the viscously damped vibrating string problem, if the mid-point of the string of length l is taken to a distance h and the string is released from rest from this position at time $t = 0$. Assume that the ends of the string are fisec.

37. Neglecting the resistance R and leakage G, find the e.m.f. $v(x, t)$ in a line a km long t seconds after the ends were suddenly grounded, if initially $i(x, 0) = i_0$ so that $\frac{\partial v}{\partial t}(x, 0) = 0$ and $v(x, 0) =$ $v(x, 0) = E_1 \sin \frac{\pi x}{a} + E_7 \sin \frac{7\pi x}{a}$.

38. Neglecting R and G, find e.m.f. $e(x, t)$ in a line of length 1 km, t seconds after the ends were suddenly grounded, if initially $i(x_0) = i_0$ and $\left(\frac{\partial e}{\partial t}\right)_{t=0} = 0$ and $e(x, 0) = 2E_0 \sin 3\pi x(1 + \cos \pi x)$.

39. Show that the solution of the equation $\frac{\partial^2 e}{\partial x^2} = LC \frac{\partial^2 e}{\partial t^2}$ appropriate to the case when a periodic e.m.f. $v_0 \cos pt$ is applied at the end $x = 0$ of a line is given by $e = v_0 \cos(pt - p\sqrt{LC}x)$.

[**Hint:** Assume the solution in the form $e(x, t) = k \cos(ax + bt)$ and find the values of k, a, b, such that the assumed solution satisfies the given equation and the given boundary condition.]

40. A tightly stretched unit square membrane with sides $x = 0$, $x = 1$, $y = 0$ and $y = 1$ starts vibrating from rest and its initial displacement is (a) $k \sin 2\pi x \sin \pi y$, (b) $k \sin \pi x \sin 2\pi y$. Find the deflection of any point (x, y) of the membrane at any time t.

41. Find the deflection $z(x, y, t)$ of a rectangular membrane ($0 < x < 2$, $0 < y < 1$) whose boundary is fixed, given that it starts from rest and $z(x, y, 0) = \lambda xy(2 - x)(1 - y)$.

42. Find the deflection $z(x, y, t)$ of a rectangular membrane ($0 < x < a$, $0 < y < b$) whose boundary is fixed, given that its starts from rest and $z(x, y, 0) = xy(a^2 - x^2)(b^2 - y^2)$.

Answers

Exercise 3(a)

2. (i) Parabolic (ii) Hyperbolic (iii) Elliptic (iv) Elliptic (v) Hyperbolic (vi) Elliptic (vii) Hyperbolic

20. (a) $y(x, t) = 2\sin\frac{2n\pi}{l}\cos\frac{2\pi at}{l} + 3\sin\frac{3\pi x}{l}\cdot\cos\frac{3\pi at}{l}$

(b) $y(x, t) = \sin\frac{n\pi}{l}\cos\frac{\pi at}{l} + \sin\frac{5\pi x}{l}\cos\frac{5\pi at}{l}$.

21. $y(x, t) = \frac{32kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2l}\cos\frac{(2n-1)\pi at}{2l}$.

22. $y(x, t) = \frac{8d}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin(2n-1)x\cdot\cos(2n-1)at$

23. $y(x, t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{60}\cdot\cos\frac{(2n-1)\pi at}{60}$.

24. $y(x, t) = \frac{9h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$.

25. $y(x, t) = \frac{9d}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{2n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$.

26. $y(x, t) = \frac{9b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{2n\pi}{3}\sin 2nx\cos 2nat$.

27. $y(x, t) = \frac{10}{a}\sin x\sin at - \frac{5}{3a}\sin 3x\sin 3at + \frac{1}{a}\sin 5x\sin 5at$.

28. $y(x, t) = \frac{5000\,k}{\pi^2 a}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^{n+1}\sin\frac{n\pi x}{50}\cos\frac{n\pi at}{50}$.

29. $y(x, t) = \frac{8\lambda l^3}{\pi^4 a}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{2l}\cdot\sin\frac{(2n-1)\pi\,\text{at}}{l}$.

30. $y(x, t) = \frac{16kl}{\pi^3 a}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2l}\sin\frac{(2n-1)\pi at}{2l}$

31. $y(x,t) = y_0 \sin\frac{\pi x}{l}\cos\frac{\pi at}{l} + \frac{4v_0 l}{\pi^2 a}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$

32. $y(x,t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin(2n-1)x\cdot\cos(2n-1)at$

$+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin(2n-1)x\cdot\sin(2n-1)at$.

33. $y(x,t) = \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi at}{l}$

$\frac{8kl^3}{\pi^4 a}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{l}\cdot\sin\frac{(2n-1)\pi at}{l}$.

34. $u(x,t) = A\,\text{cosec}\frac{\omega l}{a}\cdot\sin\frac{\omega x}{a}\cdot\sin\omega t$.

35. $y(x,t) = \frac{3}{4}e^{-bt}\sin\frac{\pi x}{l}(\cos c_1 t + \frac{b}{c_1}\sin c_1 t) - \frac{1}{4}e^{-bt}\sin\frac{3\pi x}{l}(\cos c_3 t +$

$\frac{b}{c_3}\sin c_3 t)$, where $c_1 = \sqrt{\frac{\pi^2 a^2}{l^2} - b^2}$ and $c_3 = \sqrt{\frac{9\pi^2 a^2}{l^2} - b^2}$.

36. $y(x,t) = \frac{8h}{\pi^2}e^{-bt}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cdot(\cos c_n t + \frac{b}{c_n}\sin c_n t)$,

where $c_n = \sqrt{\frac{n^2\pi^2 a^2}{l^2} - b^2}$.

37. $v(x,t) = E_1\sin\frac{\pi x}{a}\cos\frac{\pi t}{a\sqrt{LC}} + E_7\sin\frac{7\pi x}{a}\cos\frac{7\pi t}{a\sqrt{LC}}$.

38. $e(x,t) = E_0\left\{\sin 2\pi x\cos\frac{2\pi t}{\sqrt{LC}} + 2\sin 3\pi x\cos\frac{3\pi t}{\sqrt{LC}} + \sin 4\pi x\cos\frac{4\pi t}{\sqrt{LC}}\right\}$.

40. (a) $z(x,y,t) = k\sin 2\pi x\sin\pi y\cos\sqrt{5}\pi at$;

(b) $z(x,y,t) = k\sin\pi x\sin 2\pi y\cos\sqrt{5}\pi at$.

41. $z(x,y,t) = \frac{256\lambda}{\pi^6}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{1}{(2m-1)^3(2n-1)^3}\sin\frac{(2m-1)\pi x}{2}\right.$

$\left.\sin(2n-1)\pi y\cdot\cos\sqrt{\frac{(2\text{m}-1)^2}{4} + (2n-1)^2}\,\pi at\right\}$.

42. $z(x,y,t) = \frac{144a^3b^3}{\pi^6}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{1}{m^3n^3}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\,\pi ct\right\}$.

Part B

One-Dimensional Heat Flow

3B.1 INTRODUCTION

The partial differential equation $\frac{\partial u}{\partial t} = \alpha^2 \nabla^2 u$ governs the distribution of temperature u in homogeneous solids. As a consequence of Maxwell's electromagnetic equations, the current density J satisfies the equation $\nabla^2 J = \mu\sigma \frac{\partial J}{\partial t}$. If U is the concentration of a certain material in gms/cc in a certain homogeneous medium of diffusivity constant k measured in sq cm/sec, U satisfies the equation $\nabla^2 U = \frac{1}{k}\frac{\partial U}{\partial t}$.

In the theory of consolidation of soil, it is shown that, if U is the excess hydrostatic pressure at any point, at any time t and C_v is the coefficient of consolidation, U satisfies the equation $\nabla^2 U = \frac{1}{C_v}\frac{\partial U}{\partial t}$. All these equations are of the heat flow equation form.

In this chapter, we shall derive and discuss the equation of heat flow in one dimension.

3B.2 EQUATION OF VARIABLE HEAT FLOW IN ONE DIMENSION

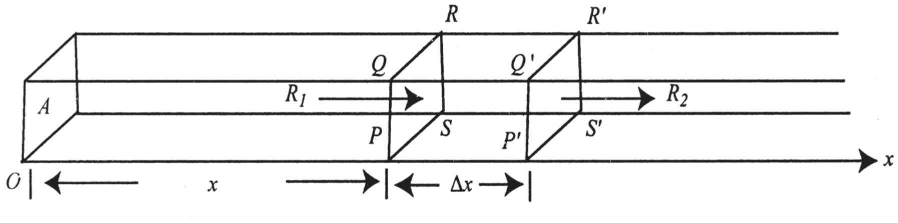


Fig. 3B.1

Consider a homogeneous bar or rod of constant cross-sectional area A made up of conducting material of density ρ, specific heat c and thermal conductivity k. It is assumed that the surface of the bar is insulated in order to make heat flow along parallel lines perpendicular to the area A.

Take one end of the bar as the origin and the direction of heat flow as the positive x-axis.

Let us now consider the heat flow in an element of the bar contained between two parallel sections $PQRS$ and $P'Q'R'S'$ which are at distances x and $x + \Delta x$ from the origin as shown in Fig.3B.1.

Let u and $u + \Delta u$ be the temperatures of this element at times t and $t + \Delta t$ respectively.

$\therefore$ Increase in temperature in the element in Δt time $= \Delta u$

$\therefore$ Increase of heat in the element in Δt time

= (specific heat) · (mass of the element) · (increase in temperature) [by a law of thermodynamics]

$= c(\rho A \Delta x)\Delta u$

$\therefore$ Rate of increase of heat in the element at time t

$$= c\rho A \Delta x \cdot \frac{\partial u}{\partial t} \tag{1}$$

Let R_1 and R_2 be the rate of inflow through the section $PQRS$ and rate of outflow through the section $P'Q'R'S'$ of the element.

Now
$$R_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x \text{ and } R_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} \tag{2}$$

Since the rate of flow of heat across any area A is proportional to A and the temperature gradient normal to the area, that is, $\frac{\partial u}{\partial x}$, by a law of thermodynamics. The constant of proportionality is the thermal conductivity.

Note ✍

The negative sign is taken in (2), since R_1 and R_2 are positive but $\frac{\partial u}{\partial x}$ is negative. $\frac{\partial u}{\partial x}$ is negative, since u is a decreasing function of x, as heat flows from a higher to lower temperature.

$\therefore$ Rate of increase of heat in the element at time t

$$= R_1 - R_2$$

$$= kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \tag{3}$$

From (1) and (3), we have

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right] \tag{4}$$

Equation (4) gives the temperature distribution at time t in the element of the bar.

Taking limits of Eq. (4) as $\Delta x \to 0$, we get the equation of one dimensional heat flow as

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho}\frac{\partial^2 u}{\partial x^2} \tag{5}$$

This equation gives the temperature $u(x, t)$ at any point of the bar at a distance x from one end of the bar at time t.

Let $\frac{k}{c\rho}$, a positive constant depending on the material of the bar, be denoted as α^2 or K. α^2 is called the *diffusivity* of the material of the bar.

Thus the equation takes the form

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \tag{6}$$

Note ✍

Equation (6) is also called one dimensional diffusion equation.

3B.3 VARIABLE SEPARABLE SOLUTIONS OF THE HEAT EQUATION

The one dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \tag{1}$$

Let
$$u(x, t) = X(x) \cdot T(t) \tag{2}$$

be a solution of Eq. (1), where $X(x)$ is a function of x alone and $T(t)$ is a function of t alone.

Then $\frac{\partial u}{\partial t} = XT'$ and $\frac{\partial^2 u}{\partial x^2} = X''T$, where $T' = \frac{\mathrm{d}T}{\mathrm{d}t}$ and $X'' = \frac{\mathrm{d}^2X}{\mathrm{d}x^2}$, satisfy Eq. (1).

i.e.
$$XT' = \alpha^2 X''T$$

i.e.
$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} \tag{3}$$

The L.H.S. of (3) is a function of x alone and the R.H.S. is a function of t alone.

They are equal for all values of independent variables x and t. This is possible only if each is a constant.

$\therefore$
$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = k, \quad \text{where } k \text{ is a constant.}$$

$\therefore$
$$X'' - kX = 0 \tag{4}$$

and
$$T' - k\alpha^2 T = 0 \tag{5}$$

The nature of the solutions of (4) and (5) depends on the nature of the values of k. Hence the following three cases come into being.

Case 1 k is positive. Let $k = p^2$.

Then equations (4) and (5) become

$(D^2 - p^2)X = 0$ and $(D' - p^2\alpha^2)T = 0$, where

$$D \equiv \frac{\mathrm{d}}{\mathrm{d}x} \quad \text{and} \quad D' \equiv \frac{\mathrm{d}}{\mathrm{d}t}.$$

The solutions of these equations are

$$X = C_1 e^{px} + C_2 e^{-px} \quad \text{and} \quad T = C_3 e^{p^2\alpha^2 t}$$

Case 2 k is negative. Let $k = -p^2$.

Then equations (4) and (5) become

$$(D^2 + p^2)X = 0 \quad \text{and} \quad (D' + p^2\alpha^2)T = 0$$

The solutions of these equations are

$$X = C_1 \cos px + C_2 \sin px \quad \text{and} \quad T = C_3 e^{-p^2\alpha^2 t}$$

Case 3 $k = 0$.

Then equations (4) and (5) become

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = 0 \quad \text{and} \quad \frac{\mathrm{d}T}{\mathrm{d}t} = 0$$

The solutions of these equations are

$$X = C_1 x + C_2 \quad \text{and} \quad T = C_3$$

Since $u(x,t) = X \cdot T$ is the solution of Eq. (1), the three mathematically possible solutions of Eq. (1) are

$$u(x,t) = (Ae^{px} + Be^{-px})e^{p^2\alpha^2 t} \tag{6}$$

$$u(x,t) = (A \cos px + B \sin px)e^{-p^2\alpha^2 t} \tag{7}$$

and

$$u(x,t) = Ax + B \tag{8}$$

where C_1C_3 and C_2C_3 have been taken as A and B.

Choice of proper solution

Out of the three mathematically possible solutions derived, we have to choose that solution which is consistent with the physical nature of the problem and the given boundary conditions. As we are dealing with heat conduction, $u(x,t)$, representing the temperature at any point at time t, must decrease when t increases. In other words, $u(x,t)$ cannot be infinite as $t \to \infty$. Hence solution (7) is the proper solution in all variable (transient) heat flow problems.

When heat flow is under steadystate conditions, the temperature at any point does not vary with time, that is it is independent of time. Hence the proper solution in steadystate heat flow problems is solution (8).

In problems, we may directly assume that (7) or (8) is the proper solution, according to whether the temperature distribution in the bar is under transient or steadystate conditions. Of course, the arbitrary constants in the suitable solution are to be found out by using the boundary conditions of the problem.

Worked Examples 3B

PROBLEMS WITH ZERO BOUNDARY VALUES (TEMPERATURES OR TEMPERATURE GRADIENTS)

Example 1

A uniform bar of length l through which heat flows is insulated at its sides. The ends are kept at zero temperature. If the initial temperature at the interior points of the bar is given by (i) $k\sin^3\dfrac{\pi x}{l}$, (ii) $k(lx - x^2)$, for $0 < x < l$, find the temperature distribution in the bar after time t.

The temperature $u(x, t)$ at a point of the bar, which is at a distance x from one end, at time t is given by the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

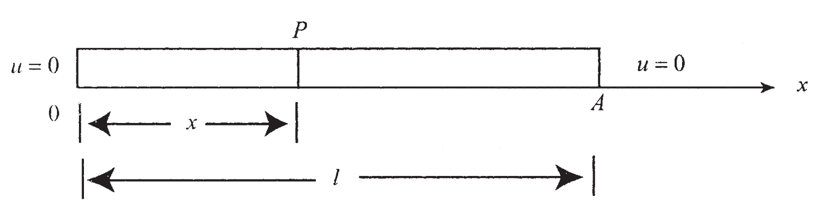


Fig. 3B.2

Since the ends $x = 0$ and $x = l$ are kept at zero temperature, that is, the ends are maintained at zero temperature at all times (Fig. 3B.2)we have

$$u(0, t) = 0, \text{ for all } t \geq 0 \tag{2}$$

$$u(l, t) = 0, \text{ for all } t \geq 0 \tag{3}$$

Since the initial temperature at the interior points of the bar is $f(x)$, we have

$$u(x, 0) = f(x), \text{ for } 0 < x < l \tag{4}$$

where $f(x) = k\sin^3\dfrac{\pi x}{l}$ in (i) and $= k(lx - x^2)$ in (ii).

We have to get the solution of Eq. (1) that satisfies the boundary conditions (2), (3) and (4).

Of the three mathematically possible solutions of Eq. (1), the appropriate solution that satisfies the condition $u \neq \infty$ as $t \to \infty$ is

$$u(x,t) = (A\cos px + B\sin px)e^{-p^2\alpha^2 t} \tag{5}$$

where A, B and p are arbitrary constants that are to be found out by using the boundary conditions.

Using boundary condition (2) in (5), we have

$$\therefore \quad Ae^{-p^2\alpha^2 t} = 0, \text{ for all } t \geq 0$$

$$\therefore \quad A = 0$$

Using boundary condition (3) in (5), we have

$$B\sin pl\, e^{-p^2\alpha^2 t} = 0, \text{ for all } t \geq 0$$

$$\therefore \quad B\sin pl = 0$$

i.e. $\quad$ either $B = 0$ or $\sin pl = 0$

If we assume that $B = 0$, the solution becomes $u(x,t) = 0$, which is meaningless.

$$\therefore \quad \sin pl = 0$$

$$\therefore \quad pl = n\pi$$

or $\quad p = \dfrac{n\pi}{l}$, where $n = 0, 1, 2, \ldots \infty$.

Using these values of A and p in (5), the solution reduces to

$$u(x,t) = B\sin\frac{n\pi x}{l} \cdot e^{-\frac{n^2\pi^2\alpha^2 t}{l^2}} \tag{6}$$

where $n = 1, 2, \ldots \infty$.

Note ✍

$n = 0$ is omitted, since the solution corresponding to $n = 0$ is meaningless.

Superposing the infinitely many solutions contained in Step (6), we get the most general solution of Eq. (1) as

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-n^2\pi^2\alpha^2 t/l^2} \tag{7}$$

Using the boundary condition (4) in (7), we have

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = f(x), \text{ for } 0 < x < l \tag{8}$$

If we can express $f(x)$ in a series comparable with the L.H.S. series of (8), we can get the values of B_n. Since $\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$ is of form of Fourier half-range sine series of a function, in most situations we may have to expand $f(x)$ as a Fourier half-range sine series.

(i) $f(x) = k\sin^3\left(\frac{\pi x}{l}\right)$

$$= \frac{k}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$$

Using this form of $f(x)$ in (8) and comparing like terms, we get

$$B_1 = \frac{3k}{4}, \quad B_3 = -\frac{k}{4}, \quad B_2 = B_4 = B_5 = \ldots = 0$$

Using these values in (7), the required solution is

$$u(x,t) = \frac{3k}{4}\sin\frac{\pi x}{l}e^{-\pi^2\alpha^2 t/l^2} - \frac{k}{4}\sin\frac{3\pi x}{l}e^{-9\pi^2\alpha^2 t/l^2}$$

(ii) $f(x) = k(lx - x^2)$ in $0 < x < l$

Let the Fourier half-range sine series of $f(x)$ in $(0, l)$ be $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Using this form of $f(x)$ in (8) and comparing like terms, we get

$$B_n = b_n = \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$= \frac{2k}{l}\left[(lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{4kl^2}{n^3\pi^3}\{1 - (-1)^n\}$$

$$= \begin{cases} \frac{8kl^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using this value of B_n in (7), the required solution is

$$u(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cdot e^{\frac{-(2n-1)^2\pi^2\alpha^2 t}{l^2}}$$

Example 2

Solve the equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$, satisfying the following conditions.

(i) u remains finite as $t \to \infty$
(ii) $u = 0$, when $x = \pm\, a$, for all $t > 0$
(iii) $u = x$, when $t = 0$ and $-a < x < a$

We have to solve the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

satisfying the following boundary conditions.

$$u(-a, t) = 0, \text{ for all } t \geq 0 \tag{2}$$
$$u(a, t) = 0, \text{ for all } t \geq 0 \tag{3}$$
$$u(x, 0) = x, \text{ for } -a < x < a \tag{4}$$

We have observed in Example 1 that the arbitrary constant A in the proper solution of Eq. (1) was easily calculated, when the left boundary condition was of the form $u(0, t) = 0$, for all $t \geq 0$. Using the boundary condition (2), namely, $u(-a, t) = 0$, for all $t \geq 0$ in the proper solution, the constant A cannot be immediately calculated.

Hence, to bring the left boundary condition to the required form, we shift the origin to the point $-a$, so that we have $x = X - a$, where X is the coordinate of the point x with reference to the new origin.

With reference to the new origin, Eq.(1) becomes

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial X^2} \tag{1$'$}$$

and the boundary conditions become

$$u(0, t) = 0, \quad \text{for all } t \geq 0 \tag{2$'$}$$
$$u(2a, t) = 0, \quad \text{for all } t \geq 0 \tag{3$'$}$$
$$u(X, 0) = X - a, \quad \text{for } 0 < X < 2a \tag{4$'$}$$

The appropriate solution of Eq. (1′), that satisfies the condition $u \neq \infty$ as $t \to \infty$ is

$$u(X, t) = (A \cos pX + B \sin pX)e^{-p^2\alpha^2 t} \tag{5}$$

Using boundary condition (2)′ in (5), we have

$$A \cdot e^{-p^2\alpha^2 t} = 0, \text{ for all } t \geq 0$$

$$\therefore \qquad A = 0$$

Using boundary condition (3)′ in (5), we have

$$B \sin 2ap\, e^{-p^2\alpha^2 t} = 0, \quad \text{for all } t \geq 0$$

$$\therefore \quad B \sin 2ap = 0$$

$$\therefore \quad \text{Either } B = 0 \text{ or } \sin 2ap = 0$$

But $B = 0$ leads to a trivial solution

$$\therefore \quad \sin 2ap = 0$$

$$\therefore \quad 2ap = n\pi$$

or

$$p = \frac{n\pi}{2a}, \text{ where } n = 0, 1, 2, \ldots \infty$$

Using these values of A and p in (5), it reduces to

$$u(X, t) = B \sin \frac{n\pi X}{2a} \cdot e^{-n^2\pi^2\alpha^2 t/4a^2} \tag{6}$$

where $n = 1, 2, \ldots \infty$.

Therefore the most general solution of Eq. (1′)

$$u(X, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi X}{2a} e^{-n^2\pi^2\alpha^2 t/4a^2} \tag{7}$$

Using boundary condition (4)′ in (7), we have

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi X}{2a} = X - a \text{ in } 0 < X < 2a$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi X}{2a},$$

which is Fourier half-range sine series of $(X - a)$ in $(0, 2a)$.
Comparing like terms, we get

$$B_n = b_n = \frac{2}{2a} \int_0^{2a} (X - a) \sin \frac{n\pi X}{2a} dX$$

$$= \frac{1}{a} \left[(X - a) \left(\frac{-\cos \frac{n\pi X}{2a}}{\frac{n\pi}{2a}} \right) - \left(-\frac{\sin \frac{n\pi X}{2a}}{\frac{n^2\pi^2}{4a^2}} \right) \right]_0^{2a}$$

$$= -\frac{2a}{n\pi} \left\{ (-1)^n + 1 \right\}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{-4a}{n\pi}, & \text{if } n \text{ is even} \end{cases}$$

Using this value of B_n in (7), we have

$$u(X,t) = \frac{-4a}{\pi} \sum_{n=2,4,6\ldots}^{\infty} \frac{1}{n} \sin\frac{n\pi X}{2a}\, e^{-n^2\pi^2\alpha^2 t/4a^2}$$

Noting that $u(x,t) \equiv u(X,t)$, the required solution of Eq. (1), with reference to the old origin, is

$$u(x,t) = -\frac{4a}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n}\sin\left\{\frac{2n\pi(x+a)}{2a}\right\} e^{-4n^2\pi^2\alpha^2 t/4a^2}$$

i.e. $$u(x,t) = \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n+1}\sin\frac{n\pi x}{a}\cdot e^{-n^2\pi^2\alpha^2 t/a^2}$$

Example 3

Find the temperature distribution in a homogeneous bar of length π which is insulated laterally, if the ends are kept at zero temperature and if, initially, the temperature is k at the centre of the bar and falls uniformly to zero at its ends.

Figure 3B.3 represents the graph of the initial temperature in the bar.

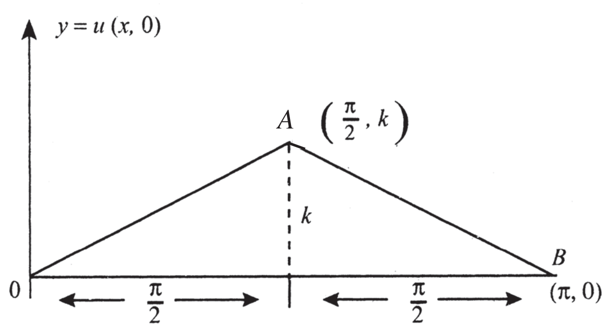


Fig. 3B.3

Equation of OA is $y = \dfrac{2k}{\pi}x$ and the equation of AB is $\dfrac{y-0}{k-0} = \dfrac{x-\pi}{\frac{\pi}{2}-\pi}$

i.e. $$y = \frac{2k}{\pi}(\pi - x)$$

Hence $$u(x,0) = \begin{cases} \dfrac{2k}{\pi}x, & \text{in } 0 \le x \le \dfrac{\pi}{2} \\ \dfrac{2k}{\pi}(\pi - x), & \text{in } \dfrac{\pi}{2} \le x \le \pi \end{cases}$$

The temperature distribution $u(x,t)$ in the bar is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial t^2} \tag{1}$$

We have to solve Eq. (1) satisfying the following boundary conditions.

$$u(0, t) = 0, \text{ for all } t \geq 0 \tag{2}$$

$$u(\pi, t) = 0, \text{ for all } t \geq 0 \tag{3}$$

$$u(x, 0) = \begin{cases} \dfrac{2k}{\pi}x, & \text{in } 0 \leq x \leq \dfrac{\pi}{2} \\ \dfrac{2k}{\pi}(\pi - x), & \text{in } \dfrac{\pi}{2} \leq x \leq \pi \end{cases} \tag{4}$$

As $u(x, t)$ has to remain finite when $t \to \infty$, the proper solution of Eq. (1) is

$$u(x, t) = (A \cos px + B \sin px)e^{-p^2\alpha^2 t} \tag{5}$$

Using boundary condition (2) in (5), we have

$$A \cdot e^{-p^2\alpha^2 t} = 0, \text{ for all } t \geq 0$$

$$\therefore \quad A = 0$$

Using boundary condition (3) in (5), we have

$$B \sin p\pi \cdot e^{-p^2\alpha^2 t} = 0, \text{ for all } t \geq 0$$

$$\therefore \quad B = 0 \quad \text{or} \quad \sin p\pi = 0$$

$B = 0$ leads to a trivial solution.

$$\therefore \quad \sin p\pi = 0$$

$$\therefore \quad p\pi = n\pi \text{ or } p = n, \text{ where } n = 0, 1, 2, \ldots \infty$$

Using these values of A and p in (5), it reduces to

$$u(x, t) = B \sin nx \, e^{-n^2\alpha^2 t} \tag{6}$$

where $n = 1, 2, 3, \ldots \infty$.

Therefore the most general solution of Eq. (1) is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin nx \, e^{-n^2\alpha^2 t} \tag{7}$$

Using boundary condition (4) in (7), we have

$$\sum_{n=1}^{\infty} B_n \sin nx = f(x) \text{ in } (0, \pi), \text{ where}$$

$$f(x) = \begin{cases} \dfrac{2k}{\pi}x, & \text{in } 0 \leq x \leq \pi/2 \\ \dfrac{2k}{\pi}(\pi - x), & \text{in } \pi/2 \leq x \leq \pi \end{cases}$$

If the Fourier half-range sine series of $f(x)$ in $(0, \pi)$ is $\sum_{n=1}^{\infty} b_n \sin nx$, it is comparable with $\sum_{n=1}^{\infty} B_n \sin nx$.

$$\text{Hence} \quad B_n = b_n = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \frac{2k}{\pi} x \sin nx \mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} \frac{2k}{\pi}(\pi - x) \sin nx \mathrm{d}x\right]$$

$$= \frac{4k}{\pi^2}\left[\left\{x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right\}_0^{\frac{\pi}{2}}\right.$$

$$\left. + \left\{(\pi - x)\left(\frac{-\cos nx}{n}\right) + \left(\frac{-\sin nx}{n^2}\right)\right\}_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{8k}{n^2\pi^2} \sin \frac{n\pi}{2}$$

Using this value of B_n in (7), the required solution is

$$u(x, t) = \frac{8k}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin nx e^{-n^2\alpha^2 t}$$

or

$$u(x, t) = \frac{8k}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin(2n-1)x e^{-(2n-1)^2\alpha^2 t}$$

Example 4

A rod of length 20 cm has its ends A and B kept at 30°C and 90°C respectively, until steadystate conditions prevail. If the temperature at each end is then suddenly reduced to 0°C and maintained so, find the temperature $u(x, t)$ at a distance x from A at time t.

When steadystate conditions prevail, the temperature at any point of the bar does not depend on t, but only on x. Hence when steadystate conditions prevail in the bar, the temperature distribution is given by

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0 \tag{1}$$

$$\left[\because \frac{\partial u}{\partial t} = 0 \text{ and } \frac{\partial^2 u}{\partial x^2} \text{ becomes } \frac{\mathrm{d}u^2}{\mathrm{d}x^2}\right]$$

We have to solve (1) satisfying the following boundary conditions

$$u(0) = 30 \tag{2}$$

and

$$u(20) = 90 \tag{3}$$

Solving Eq. (1), we get

$$u(x) = C_1 x + C_2 \tag{4}$$

Using (2) in (4), we get $C_2 = 30$
Using (3) in (5), we get $C_1 = 3$
Using these values in (4), the solution of Eq. (1) is

$$u(x) = 3x + 30 \tag{5}$$

That is, as long as the steadystate conditions prevail in the bar, the temperature distribution in it is given by (5).

Once we alter the end temperatures (or the end conditions), the heat flow or the temperature distribution in the bar will not be under steadystate conditions and hence will depend on time also. However the temperature distribution at the interior points of the bar in the steadystate will be the initial temperature distribution in the transient state.

In the transient state, the temperature distribution in the bar is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

The corresponding boundary conditions are

$$u(0, t) = 0, \text{for all } t \geq 0 \tag{7}$$

$$u(20, t) = 0, \text{for all } t \geq 0 \tag{8}$$

$$u(x, 0) = 3x + 30, \text{for } 0 < x < 20 \tag{9}$$

As $u \neq \infty$ when $t \to \infty$, the proper solution of Eq. (6) is

$$u(x, t) = (A \cos px + B \sin px) e^{-p^2 \alpha^2 t} \tag{10}$$

Using boundary condition (7) in (10), we have

$$A \cdot e^{-p^2 \alpha^2 t} = 0, \text{ for all } t \geq 0$$

$$\therefore \quad A = 0$$

Using boundary condition (8) in (10), we have

$$B \sin 20p \cdot e^{-p^2 \alpha^2 t} = 0, \text{ for all } t \geq 0$$

$$\therefore \quad B = 0 \text{ or } \sin 20p = 0$$

$B = 0$ leads to a trivial solution.

$$\therefore \quad \sin 20p = 0$$

$$\therefore \quad 20p = n\pi \text{ or } p = \frac{n\pi}{20}, \text{ where } n = 0, 1, 2, \ldots \infty$$

Using these values of A and p in (10), it reduces to

$$u(x,t) = B \sin \frac{n\pi x}{20} e^{-n^2\pi^2\alpha^2 t/20^2} \tag{11}$$

where $n = 1, 2, 3, \ldots \infty$.

Therefore the most general solution of Eq. (6) is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{20} e^{-n^2\pi^2\alpha^2 t/400} \tag{12}$$

Using boundary condition (9) in (12), we have

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{20} = 3x + 30 \text{ in } (0, 20)$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

which is Fourier half-range sine series of $(3x + 30)$ in $(0, 20)$.
Comparing like terms,

$$B_n = b_n = \frac{2}{20} \int_0^{20} (3x + 30) \sin \frac{n\pi x}{20} \mathrm{d}x$$

$$= \frac{3}{10} \left[(x + 10) \left(\frac{-\cos \frac{n\pi x}{20}}{\frac{n\pi}{20}} \right) - \left(-\frac{\sin \frac{n\pi x}{20}}{\frac{n^2\pi^2}{20^2}} \right) \right]_0^{20}$$

$$= -\frac{6}{n\pi} \left\{ 30(-1)^n - 10 \right\} = \frac{60}{n\pi} \left\{ 1 - 3(-1)^n \right\}$$

Using this value of B_n in (12), the required solution is

$$u(x,t) = \frac{60}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ 1 - 3(-1)^n \right\} \sin \frac{n\pi x}{20} \cdot e^{-n^2\pi^2\alpha^2 t/400}$$

Example 5

Solve the one dimensional heat flow equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

satisfying the following boundary conditions.

(i) $\frac{\partial u}{\partial x}(0, t) = 0$, for all $t \geq 0$

(ii) $\frac{\partial u}{\partial x}(\pi, t) = 0$, for all $t \geq 0$; and

(iii) $u(x, 0) = \cos^2 x, 0 < x < \pi$

Note ✍

When conditions (i) and (ii) are satisfied, it means that the ends $x = 0$ and $x = \pi$ of the bar are thermally insulated, so that heat cannot flow in or out through these ends.

The appropriate solution of the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

satisfying the condition that $u \neq \infty$ when $t \to \infty$ is

$$u(x, t) = (A \cos px + B \sin px)e^{-p^2\alpha^2 t} \tag{2}$$

Differentiating (2) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, t) = p(-A \sin px + B \cos px)e^{-p^2\alpha^2 t} \tag{3}$$

Using boundary condition (i) in (3), we have

$$p \cdot B \cdot e^{-p^2\alpha^2 t} = 0, \text{ for all } t \geq 0$$

$\therefore$ $B = 0$ [$\because$ if $p = 0, u(x, t) = A$, which is meaningless]

Note ✍

When the zero left end temperature condition was used in the proper solution, we got $A = 0$ in all the earlier examples. When the zero left end temperature gradient condition is used, we get $B = 0$.

Using boundary condition (ii) in (3), we have

$$-pA \sin p\pi \cdot e^{-p^2\alpha^2 t} = 0, \text{ for all } t \geq 0$$

$\therefore$ Either $A = 0$ or $\sin p\pi = 0$

$A = 0$ leads to a trivial solution.

$\therefore$ $\sin p\pi = 0$

$\therefore$ $p\pi = n\pi$ or $p = n$, where $n = 0, 1, 2, \ldots \infty$

Using these values of B and p in (2), it reduces to

$$u(x, t) = A \cos nx \cdot e^{-n^2\alpha^2 t} \tag{4}$$

where $n = 0, 1, 2, \ldots \infty$.

Note

$n = 0$ gives $u(x, t) = A$, which cannot be omitted.

Therefore the most general solution of Eq. (1) is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos nx \, e^{-n^2\alpha^2 t} \tag{5}$$

Using boundary condition (iii) in (5), we have

$$\sum_{n=0}^{\infty} A_n \cos nx = \cos^2 x \text{ in } (0, \pi) \tag{6}$$

In general, we have to expand the function in the R.H.S. as a Fourier half-range cosine series in $(0, \pi)$ so that it may be compared with L.H.S. series.

In this problem, it is not necessary. We can rewrite $\cos^2 x$ as $\frac{1}{2}(1 + \cos 2x)$, so that comparison is possible.

Thus
$$\sum_{n=0}^{\infty} A_n \cos nx = \frac{1}{2} + \frac{1}{2}\cos 2x$$

Comparing like terms, we have

$$A_0 = \frac{1}{2}, \; A_2 = 1/2, \; A_1 = A_3 = A_4 = \ldots = 0$$

Using these values of $A_n's$ in (5), the required solution is

$$u(x, t) = \frac{1}{2} + \frac{1}{2}\cos 2x \, e^{-4\alpha^2 t}$$

Example 6

The temperature at one end of a bar 20 cm long and with insulated sides is kept at 0°C and that at the other end is kept at 60°C until steady state conditions prevail. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter. Find the temperature distribution in the bar.

Show also that the sum of the temperatures at any two points equidistant from the centre of the bar is 60°C.

When steadystate conditions prevail in the bar, the temperature distribution is given by

$$\frac{d^2u}{dx^2} = 0 \tag{1}$$

The corresponding boundary conditions are

$$u(0) = 0 \tag{2}$$

and
$$u(20) = 60 \tag{3}$$

Solving the Eq. (1), we get

$$u(x) = C_1 x + C_2 \tag{4}$$

Using (2) and (3) in (4), we get

$$C_1 = 3 \text{ and } C_2 = 0$$

$$\therefore \qquad u(x) = 3x \tag{5}$$

Once the ends are insulated, the heat flow is under transient state and the subsequent temperature distribution is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

The corresponding boundary conditions are

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad \text{for all } t \geq 0 \tag{7}$$

$$\frac{\partial u}{\partial x}(20, t) = 0, \quad \text{for all } t \geq 0 \tag{8}$$

$$u(x, 0) = 3x, \quad \text{for } 0 < x < 20 \tag{9}$$

As $u \neq \infty$ when $t \to \infty$, the appropriate solution of Eq. (6) is

$$u(x, t) = (A \cos px + B \sin px)\, e^{-p^2\alpha^2 t} \tag{10}$$

Differentiating (10) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, t) = p(-A \sin px + B \cos px) e^{-p^2\alpha^2 t} \tag{11}$$

Using boundary condition (7) in (11), we have

$$p \cdot B \cdot e^{-p^2\alpha^2 t} = 0, \text{ for all } t \geq 0$$

$$\therefore \qquad \text{Either } p = 0 \text{ or } B = 0$$

But $p = 0$ makes $u(x, t) = A$, which is meaningless.

$$\therefore \qquad B = 0$$

Using boundary condition (8) in (11), we have

$$-pA \sin 20p \cdot e^{-p^2\alpha^2 t} = 0, \text{ for all } t \geq 0$$

$$\therefore \qquad \text{Either } A = 0 \text{ or } \sin 20p = 0$$

$A = 0$ leads to a trivial solution

$$\therefore \qquad \sin 20p = 0$$

$$\therefore \qquad 20p = n\pi \text{ or } p = \frac{n\pi}{20}, \text{ where } n = 0, 1, 2, \ldots \infty$$

Using these values of B and p in (10), it reduces to

$$u(x,t) = A\cos\frac{n\pi x}{20}\cdot e^{-n^2\pi^2\alpha^2 t/20^2}$$

where $n = 0, 1, 2, \ldots \infty$

Therefore the most general solution of Eq. (6) is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{20} e^{-n^2\pi^2\alpha^2 t/400} \tag{12}$$

Using boundary condition (9) in (12), we have

$$\sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{20} = 3x \text{ in } 0 < x < 20$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{20}$$

which is the Fourier half-range cosine series of $3x$ in $(0, 20)$.
Comparing like terms, we get

$$A_0 = \frac{a_0}{2} = \frac{1}{2}\cdot\frac{2}{20}\int_0^{20} 3x\, dx$$

$$= \frac{3}{20}\left(\frac{x^2}{2}\right)_0^{20} = 30$$

and

$$A_n = a_n = \frac{2}{20}\int_0^{20} 3x\cos\frac{n\pi x}{20}\, dx$$

$$= \frac{3}{10}\left[x\left(\frac{\sin\frac{n\pi x}{20}}{\frac{n\pi}{20}}\right) - \left(\frac{-\cos\frac{n\pi x}{20}}{\frac{n^2\pi^2}{20^2}}\right)\right]_0^{20}$$

$$= \frac{120}{n^2\pi^2}\left\{(-1)^n - 1\right\}$$

$$= \begin{cases} -\dfrac{240}{n^2\pi^2}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

Using these values of A_0 and A_n in (12), the required solution is

$$u(x,t) = 30 - \frac{240}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{20}\cdot e^{\frac{-(2n-1)^2\pi^2\alpha^2 t}{400}} \tag{13}$$

Points P and Q which are equidistant from the centre of the bar can be assumed to have the x coordinates x and $20 - x$ [Fig. 3B.4]

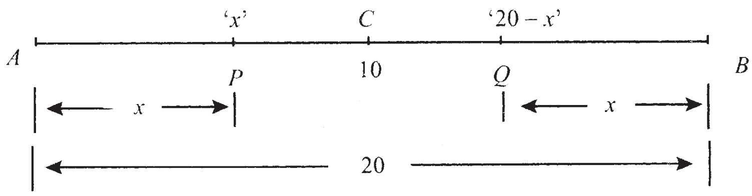


Fig. 3B.4

Temperature at P is given by (13).
Temperature at Q is given by

$$u(20-x,t) = 30 - \frac{240}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi(20-x)}{20}\cdot e^{\frac{-(2n-1)^2\pi^2\alpha^2 t}{400}}$$

Now
$$\cos\frac{(2n-1)\pi(20-x)}{20} = \cos\left\{(2n-1)\pi - \frac{(2n-1)\pi x}{20}\right\}$$
$$= (-1)^{2n-1}\cos\frac{(2n-1)\pi x}{20}$$
$$= -\cos\frac{(2n-1)\pi x}{20}$$

$$\therefore \quad u(20-x,t) = 30 + \frac{240}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{20}\, e^{\frac{-(2n-1)^2\pi^2\alpha^2 t}{400}} \qquad (14)$$

Adding (13) and (14), we get

$$u_P + u_Q = u(x,t) + u(20-x,t) = 60$$

Example 7

Solve the equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$ satisfying the following conditions.

(i) u is finite when $t \to \infty$.
(ii) $\frac{\partial u}{\partial x} = 0$ when $x = 0$, for all values of t
(iii) $u = 0$ when $x = l$, for all values of t
(iv) $u = u_0$ when $t = 0$, for $0 < x < l$.

We have to solve the equation

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \qquad (1)$$

satisfying the following boundary conditions.

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad \text{for all } t \geq 0 \tag{2}$$

$$u(l, t) = 0, \quad \text{for all } t \geq 0 \tag{3}$$

$$u(x, 0) = u_0, \quad \text{for } 0 < x < l \tag{4}$$

Since u is finite as $t \to \infty$, the proper solution of Eq. (1) is

$$u(x, t) = (A \cos px + B \sin px)\, e^{-p^2\alpha^2 t} \tag{5}$$

Differentiating (5) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, t) = p(-A \sin px + B \cos px)\, e^{-p^2\alpha^2 t} \tag{6}$$

Using boundary condition (2) in (6), we have

$$pB\, e^{-p^2\alpha^2 t} = 0, \text{ for all values of } t \geq 0$$

$\therefore$ Either $p = 0$ or $B = 0$

$p = 0$ makes $u(x, t) = A$, which is meaningless.

$$\therefore \quad B = 0$$

Using boundary condition (3) in (5), we have

$$A \cos pl \cdot e^{-p^2\alpha^2 t} = 0 \text{ for all } t \geq 0$$

$\therefore$ Either $A = 0$ or $\cos\ pl = 0$

$A = 0$ leads to a trivial solution.

$\therefore \quad \cos\ pl = 0$

$\therefore \quad pl =$ an odd multiple of $\frac{\pi}{2}$ or $(2n-1)\frac{\pi}{2}$

$\therefore \quad p = \frac{(2n-1)\pi}{2l}$, where $n = 1, 2, 3, \ldots \infty$.

Note

In all the problems considered so far, we had $p = \frac{n\pi}{l}$, on using the second boundary condition; but in this problem, we have $p = \frac{(2n-1)\pi}{2l}$.

Using these values of B and p in (5), it reduces to

$$u(x, t) = A \cos \frac{(2n-1)\pi x}{2l} \cdot e^{-(2n-1)^2\pi^2\alpha^2 t/4l^2} \tag{7}$$

where $n = 1, 2, 3, \ldots \infty$.

Therefore the most general solution of Eq. (1) is

$$u(x, t) = \sum_{n=1}^{\infty} A_{2n-1} \cos \frac{(2n-1)\pi x}{2l} \cdot e^{-(2n-1)^2\pi^2 t/4l^2} \tag{8}$$

Note

While superposing the solutions in (7), the unknown constants have been assumed as A_{2n-1} instead of the usual A_n, just to have one-to-one correspondence between the suffix of A and the arguments of the cosine and exponential functions in all the terms of the solution (8).

Using boundary condition (4) in (8), we have

$$\sum_{n=1}^{\infty} A_{2n-1} \cos \frac{(2n-1)\pi x}{2l} = u_0 \text{ in } (0, l) \tag{9}$$

The series in the L.H.S. of (9) is not in the form of the Fourier half-range cosine series of any function in $(0, l)$, that is, $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$. Hence, to find A_{2n-1}, we proceed as in the derivation of Euler's formula for the Fourier coefficients.

Multiplying both sides of (9) by $\cos \frac{(2n-1)\pi x}{2l}$ and integrating with respect to x between 0 and l, we get

$$A_{2n-1} \int_0^l \cos^2 \frac{(2n-1)\pi x}{2l} \mathrm{d}x = u_0 \int_0^l \cos \frac{(2n-1)\pi x}{2l} \mathrm{d}x$$

[$\because$ All other integrals in the L.H.S. vanish]

i.e. $$A_{2n-1} \cdot \frac{1}{2} \left[x + \frac{\sin \frac{(2n-1)\pi x}{l}}{\frac{(2n-1)\pi}{l}} \right]_0^l = u_0 \left[\frac{\sin \frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)\pi}{2l}} \right]_0^l$$

i.e. $$A_{2n-1} \cdot \frac{l}{2} = u_0 \cdot \frac{2l}{(2n-1)\pi} \sin \frac{(2n-1)\pi}{2}$$

$\therefore$ $$A_{2n-1} = \frac{4u_0}{(2n-1)\pi} (-1)^{n+1}$$

Using this value of A_{2n-1} in (8), the required solution is

$$u(x, t) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \cos \frac{(2n-1)\pi x}{2l} \cdot e^{-(2n-1)^2\pi^2\alpha^2 t/4l^2}$$

Example 8

An insulated metal rod of length 100 cm has one end A kept at 0°C and the other end B at 100°C until steady state conditions prevail. At time $t = 0$, the end B is suddenly insulated while the temperature at A is maintained at 0°C. Find the temperature at any point of the rod at any subsequent time.

When steady state conditions prevail in the rod, the temperature distribution is given by

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0 \tag{1}$$

The corresponding boundary conditions are

$$u(0) = 0 \tag{2}$$

and
$$u(100) = 100 \tag{3}$$

Solving the Eq. (1), we get

$$u(x) = c_1 x + c_2 \tag{4}$$

Using (2) and (3) in (4), we get $c_1 = 1$ and $c_2 = 0$

$$\therefore \quad u(x) = x \tag{5}$$

Once end B is insulated, though the temperature at A is not altered, the heat flow is under transient conditions and the subsequent temperature distribution in the rod is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

The corresponding boundary conditions are

$$u(0, t) = 0, \text{ for all } t \geq 0 \tag{7}$$

$$\frac{\partial u}{\partial x}(l, t) = 0, \text{ for all } t \geq 0 \tag{8}$$

$$u(x, 0) = x, \text{ for } 0 < x < l \tag{9}$$

where $l = 100$.

As $u \neq \infty$ when $t \to \infty$, the appropriate solution of Eq. (6) is

$$u(x, t) = (A \cos px + B \sin px)\, e^{-p^2\alpha^2 t} \tag{10}$$

Using boundary condition (7) in (10), we have

$$A \cdot e^{-p^2\alpha^2 t} = 0, \text{ for all } t \geq 0$$

$$\therefore \quad A = 0$$

Differentiating (10) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, t) = Bp \cos px \cdot e^{-p^2\alpha^2 t} \tag{11}$$

Using boundary condition (8) in (11), we have

$$Bp \cos\ pl\ e^{-p^2\alpha^2 t} = 0$$

$$\therefore \quad \text{Either } B = 0,\ p = 0 \text{ or } \cos\ pl = 0$$

$B = 0$ and $p = 0$ lead to meaningless solutions.

$$\therefore \quad \cos\ pl = 0$$

$$\therefore \quad pl = \frac{(2n-1)\pi}{2}$$

or
$$p = \frac{(2n-1)\pi}{2l}, \text{ where } n = 1, 2, 3, \ldots \infty$$

Using these values of A and p in (10), it reduces to

$$u(x,t) = B \sin \frac{(2n-1)\pi x}{2l} e^{-(2n-1)^2\pi^2\alpha^2 t/4l^2} \tag{12}$$

where $n = 1, 2, 3, \ldots \infty$.

Therefore the most general solution of Eq. (6) is

$$u(x,t) = \sum_{n=1}^{\infty} B_{2n-1} \sin \frac{(2n-1)\pi x}{2l} e^{-(2n-1)^2\pi^2\alpha^2 t/4l^2} \tag{13}$$

Using boundary condition (9) in (13), we have

$$\sum_{n=1}^{\infty} B_{2n-1} \sin \frac{(2n-1)\pi x}{2l} = x \text{ in } (0, l)$$

Proceeding as in Example 7, we get

$$\begin{aligned}
B_{2n-1} &= \frac{2}{l} \int_0^l x \sin \frac{(2n-1)\pi x}{2l} dx \\
&= \frac{2}{l} \left[x \left\{ \frac{-\cos \frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)\pi}{2l}} \right\} - \left\{ \frac{-\sin \frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)^2\pi^2}{4l^2}} \right\} \right]_0^l \\
&= \frac{8l}{(2n-1)^2\pi^2} \sin \frac{(2n-1)\pi}{2} \\
&= \frac{8l(-1)^{n+1}}{(2n-1)^2\pi^2}
\end{aligned}$$

Using this value of B_{2n-1} in (13), the required solution is

$$u(x,t) = \frac{8l}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{2l} \cdot e^{-(2n-1)^2\pi^2\alpha^2 t/4l^2}$$

where $l = 100$.

PROBLEMS ON TEMPERATURE IN A SLAB WITH FACES WITH ZERO TEMPERATURE

Example 9

Faces of a slab of width c are kept at temperature zero. If the initial temperature in the slab is $f(x)$, determine the temperature formula. If $f(x) = u_0$, a constant, find the flux $-k\frac{\partial u}{\partial x}(x_0, t)$ across any plane $x = x_0 (0 \leq x_0 \leq c)$ and show that no heat flows across the central plane $x_0 = \frac{c}{2}$, where k^2 is the diffusivity of the substance.

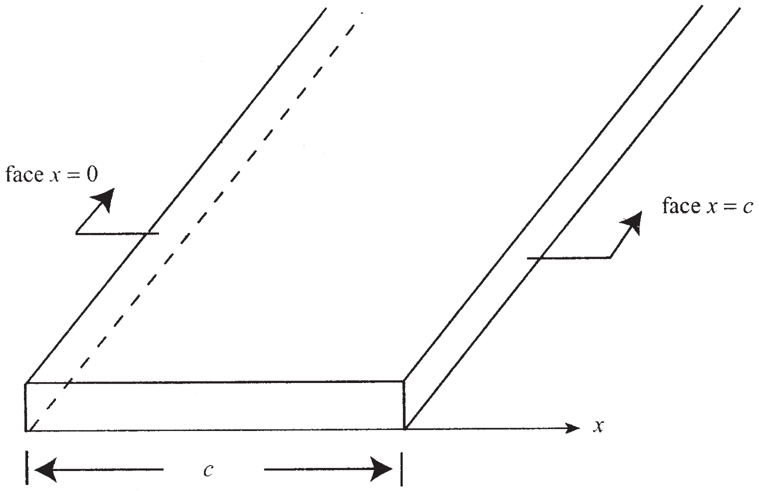


Fig. 3B.5

Though the slab is a three dimensional solid (Fig. 3B.5 it is assumed that the temperature in it at a given time t depends only on and varies with respect to x, the distance measured from one face along the width of the slab. Hence the temperature function $u(x, t)$ at any interior point of the slab is given by

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Note ✍

The problem of temperature distribution in a slab is exactly similar to that in a homogeneous bar.

We have to solve Eq. (1) satisfying the following boundary conditions.

$$u(0, t) = 0, \qquad \text{for all } t \geq 0 \tag{2}$$

$$u(c, t) = 0, \qquad \text{for all } t \geq 0 \tag{3}$$

$$u(x, 0) = f(x), \quad \text{for } 0 < x < c \tag{4}$$

Proceeding as in Example 1, the most general solution of Eq. (1) is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{c} e^{-n^2\pi^2 k^2 t/c^2} \tag{5}$$

Using boundary condition (4) in (5), we have

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{c} = f(x) \text{ in } (0, c) = \sum b_n \sin \frac{n\pi x}{c}$$

which is the Fourier half-range sine series of $f(x)$ in $(0, c)$.

Comparing like terms, we get

$$B_n = b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\mathrm{d}x \tag{6}$$

Using this value of B_n given by (6) in (5), the required solution is

$$u(x,t) = \frac{2}{c}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{c}\exp\left(\frac{-n^2\pi^2k^2t}{c^2}\right)\int_0^c f(\theta)\sin\frac{n\pi\theta}{c}\mathrm{d}\theta \tag{7}$$

When $f(x) = u_0$, from (6), we get

$$B_n = \frac{2}{c}\int_0^c u_0\sin\frac{n\pi x}{c}\mathrm{d}x$$

$$= \frac{2u_0}{c}\left(-\frac{\cos\frac{n\pi x}{c}}{\frac{n\pi}{c}}\right)_0^c$$

$$= \frac{2u_0}{n\pi}\{1 - \cos n\pi\}$$

$$= \begin{cases}\frac{4u_0}{n\pi}, & \text{if } n \text{ is odd}\\ 0, & \text{if } n \text{ is even}\end{cases}$$

Therefore the required solution in this case is

$$u(x,t) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{c}\cdot\exp\left\{\frac{-(2n-1)^2\pi^2k^2t}{c^2}\right\} \tag{8}$$

Differentiating (8) partially with respect to x,

$$\frac{\partial u}{\partial x}(x,t) = \frac{4u_0}{c}\sum_{n=1}^{\infty}\cos\frac{(2n-1)\pi x}{c}\exp\left\{\frac{-(2n-1)^2\pi^2k^2t}{c^2}\right\}$$

Therefore the flux across the plane $x = x_0$ is given by

$$-k\frac{\partial u}{\partial x}(x_0,t) = -\frac{4ku_0}{c}\sum_{n=1}^{\infty}\cos\frac{(2n-1)\pi x_0}{c}\exp\left\{\frac{-(2n-1)^2\pi^2k^2t}{c^2}\right\}$$

Therefore the flux across the central plane $x = \frac{c}{2}$ is given by

$$-k\frac{\partial u}{\partial x}\left(\frac{c}{2},t\right) = -\frac{4ku_0}{c}\sum_{n=1}^{\infty}\cos\frac{(2n-1)\pi}{2}\exp\left\{\frac{-(2n-1)^2\pi^2k^2t}{c^2}\right\}$$

$$= 0, \text{ since } \cos\frac{(2n-1)\pi}{2} = 0$$

That is no heat flows across the central plane of the slab.

Example 10

Two slabs of the same material, one 60 cm thick and the other 30 cm thick are placed face to face in perfect contact. The thicker slab is initially at temperature 100°C, the thinner one initially at zero. The outer faces are kept at zero temperature for $t > 0$. Find the temperature at the centre of the thicker slab (Fig. 3B.6)

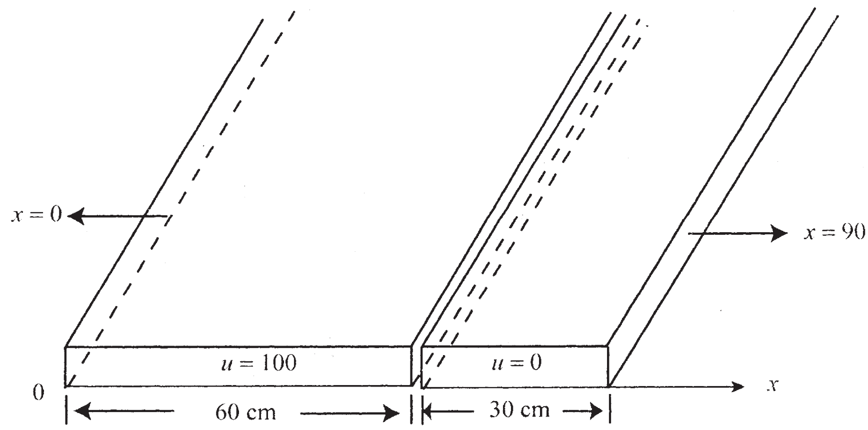


Fig. 3B.6

$u(x, t)$, the temperature function at any point of the slab at time t is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

The corresponding boundary conditions are the following.

$$u(0, t) = 0, \text{ for all } t \geq 0 \tag{2}$$

$$u(90, t) = 0, \text{ for all } t \geq 0 \tag{3}$$

$$u(x, 0) = \begin{cases} 100, & \text{in } 0 < x < 60 \\ 0, & \text{in } 60 < x < 90 \end{cases} \tag{4}$$

Proceeding as in Example 1, the most general solution of Eq. (1) is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{90} e^{-n^2\pi^2\alpha^2 t/90^2} \tag{5}$$

Using boundary condition (4) in (5), we have

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{90} = f(x) \text{ in } (0, 90), \text{ where}$$

$$f(x) = \begin{cases} 100, & \text{in } 0 < x < 60 \\ 0, & \text{in } 60 < x < 90 \end{cases}$$

If the Fourier half-range sine series of $f(x)$ in $(0, 90)$ is $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{90}$, then comparison of like terms gives

$$B_n = b_n = \frac{2}{90} \int_0^{60} 100 \sin \frac{n\pi x}{90} dx$$

$$= \frac{20}{9} \left(\frac{-\cos \frac{n\pi x}{90}}{\frac{n\pi}{90}} \right)_0^{60}$$

$$= \frac{200}{n\pi} \cdot \left\{ 1 - \cos \frac{2n\pi}{3} \right\}$$

$$= \frac{400}{n\pi} \sin^2 \left(\frac{n\pi}{3} \right)$$

Using this value of B_n in (5), the required solution is

$$u(x, t) = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin^2 \left(\frac{n\pi}{3} \right) \sin \frac{n\pi x}{90} \exp \left\{ \frac{-n^2\pi^2\alpha^2 t}{90^2} \right\}$$

Therefore the temperature at the centre $(x = 30)$ of the slab is given by

$$u(30, t) = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin^3 \left(\frac{n\pi}{3} \right) \exp \left\{ \frac{-n^2\pi^2\alpha^2 t}{90^2} \right\}.$$

PROBLEMS WITH NON-ZERO BOUNDARY VALUES (TEMPERATURES OR TEMPERATURE GRADIENTS)

Example 11

A bar 10 cm long has originally a temperature of 0°C throughout its length. At time $t = 0$ sec, the temperature at the end $x = 0$ is raised to 20°C, while that at the end $x = 10$ is raised to 40°C. Determine the resulting temperature distribution in the bar.
The temperature distribution $u(x, t)$ in the bar is given by the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

We have to solve Eq. (1) satisfying the following boundary conditions.

$$u(0, t) = 20, \quad \text{for all } t \geq 0 \tag{2}$$

$$u(10, t) = 40, \quad \text{for all } t \geq 0 \tag{3}$$

$$u(x, 0) = 0, \quad \text{for } 0 < x < 10 \tag{4}$$

In all the earlier problems, the boundary values in (2) and (3) were zero each and hence we were able to get the values of two of the unknown constants in the proper solution easily. The usual procedure will not give the values of unknown constants in the proper solution in this example, since we have non-zero values in the boundary conditions (2) and (3). Hence we adopt a slightly different procedure, similar to the one used in Example 16 of Chapter 3(A).

Let
$$u(x,t) = u_1(x) + u_2(x,t) \tag{5}$$

Using (5) in (1), we get

$$\frac{\partial}{\partial t}(u_1 + u_2) = \alpha^2 \frac{\partial^2}{\partial x^2}(u_1 + u_2)$$

This gives rise to the two equations

$$\frac{\partial u_1(x)}{\partial t} = \alpha^2 \frac{\partial^2}{\partial x^2} u_1(x) \text{ or } \frac{d^2 u_1}{dx^2} = 0 \tag{6}$$

[$\because$ $u_1(x)$ is a function of x only]

and
$$\frac{\partial u_2}{\partial t} = \alpha^2 \frac{\partial^2 u_2}{\partial x^2} \tag{7}$$

Since $u_1(x)$ is independent of t and the end values at $x = 0$ and $x = 10$ do not change with t, we assume that $u_1(x)$ corresponds to the end points and $u_2(x,t)$ corresponds to the interior points $0 < x < 10$.

Note ✍

$u_1(x)$ is referred to as the steadystate part and $u_2(x,t)$ as the transient part of $u(x,t)$.

Thus we have to solve Eq. (6) satisfying the end conditions

$$u_1(0) = 20 \tag{8}$$

and
$$u_1(10) = 40 \tag{9}$$

Solving Eq. (6), we get

$$u_1(x) = c_1 x + c_2 \tag{10}$$

Using boundary conditions (8) and (9) in (10), we get $c_1 = 2$ and $c_2 = 20$.

$\therefore$
$$u_1(x) = 2x + 20 \tag{11}$$

Now we have to solve Eq. (7), satisfying the following boundary conditions which are obtained by using (5) and the boundary conditions (2), (3), (4), (8), (9) and Step (11).

$$u_2(0,t) = u(0,t) - u_1(0) = 0, \quad \text{for all } t \geq 0 \tag{12}$$

$$u_2(10,t) = u(10,t) - u_1(10) = 0, \quad \text{for all } t \geq 0 \tag{13}$$

$$u_2(x,0) = u(x,0) - u_1(x) = -(2x + 20), \quad \text{for } 0 < x < 10 \tag{14}$$

Note ✍

Equation 7 is readily solvable, as the boundary conditions (12) and (13) have zero values in the R.H.S.

Proceeding as in Example 1, we get the most general solution of Equation (7) as

$$u_2(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} e^{-n^2\pi^2\alpha^2 t/10^2} \tag{15}$$

Using boundary condition (14) in (15), we get

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} = -(2x+20) \text{ in } (0, 10)$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}$$

which is the Fourier half-range sine series of $-(2x+20)$ in $(0, 10)$.

Comparing like terms, we have

$$B_n = b_n = \frac{2}{10}\int_0^{10} \{-(2x+20)\} \sin\frac{n\pi x}{10} dx$$

$$= -\frac{2}{5}\left[(x+10)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - \left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{10^2}}\right)\right]_0^{10}$$

$$= \frac{4}{n\pi}\left\{20(-1)^n - 10\right\} \text{ or } \frac{40}{n\pi}\left\{2(-1)^n - 1\right\}$$

Using this value of B_n in (15), we get

$$u_2(x,t) = \frac{40}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left\{2(-1)^n - 1\right\} \sin\frac{n\pi x}{10} e^{-n^2\pi^2\alpha^2 t/10^2} \tag{16}$$

Using (11) and (16) in (5), the required solution is

$$u(x,t) = (2x+20) + \frac{40}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left\{2(-1)^n - 1\right\} \sin\frac{n\pi x}{10} \exp\left\{\frac{-n^2\pi^2\alpha^2 t}{100}\right\}$$

Example 12

The ends A and B of a rod 40 cm long have their temperatures kept at 0°C and 80°C respectively, until steadystate conditions prevail. The temperature of the end B is then suddenly reduced to 40°C and kept so, while that of the end A is kept at 0°C. Find the subsequent temperature distribution $u(x,t)$ in the rod.

When steadystate conditions prevail, the temperature distribution is given by

$$\frac{d^2u}{dx^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the boundary conditions

$$u(0) = 0 \tag{2}$$

$$u(40) = 80 \tag{3}$$

Solving Eq. (1), we get

$$u(x) = ax + b \tag{4}$$

Using the boundary conditions (2) and (3) in (4), we get $a = 2$ and $b = 0$

Therefore the solution of Eq. (1) is

$$u(x) = 2x \tag{5}$$

In the transient state, the temperature distribution in the rod is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

The corresponding boundary conditions are

$$u(0, t) = 0, \quad \text{for all } t \geq 0 \tag{7}$$

$$u(40, t) = 40, \quad \text{for all } t \geq 0 \tag{8}$$

$$u(x, 0) = 2x, \quad \text{for } 0 < x < 40 \tag{9}$$

Since one of the end values is non-zero, we adopt the modified procedure explained in Example 11.

Let
$$u(x, t) = u_1(x) + u_2(x, t) \tag{10}$$

where $u_1(x)$ is given by

$$\frac{d^2u_1}{dx^2} = 0 \tag{11}$$

and $u_2(x, t)$ is given by

$$\frac{\partial u_2}{\partial t} = \alpha^2 \frac{\partial^2 u_2}{\partial x^2} \tag{12}$$

The boundary conditions for Eq. (11) are

$$u_1(0) = 0 \tag{13}$$

and
$$u_1(40) = 40 \tag{14}$$

Solving Eq. (11), we get

$$u_1(x) = c_1x + c_2 \tag{15}$$

Using boundary conditions (13) and (14) in (15), we get $c_1 = 1$ and $c_2 = 0$.

$$\therefore \qquad u_1(x) = x \tag{16}$$

The boundary conditions for Eq. (12) are

$$u_2(0, t) = u(0, t) - u_1(0) = 0, \quad \text{for all } t \geq 0 \tag{17}$$

$$u_2(40, t) = u(40, t) - u_1(40) = 0, \text{ for all } t \geq 0 \tag{18}$$

$$u_2(x, 0) = u(x, 0) - u_1(x) = x, \quad \text{for } 0 < x < 40 \tag{19}$$

Proceeding as in Example 1, we get the most general solution of Equation (12) as

$$u_2(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{40} e^{-n^2\pi^2\alpha^2 t/40^2} \tag{20}$$

Using boundary condition (19) in (20), we get

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{40} = x \text{ in } (0, 40)$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{40}$$

which is the Fourier half-range sine series of x in $(0, 40)$. Comparing like terms in the two series, we have

$$B_n = b_n = \frac{2}{40}\int_0^{40} x \sin\frac{n\pi x}{40} \mathrm{d}x$$

$$= \frac{1}{20}\left[x\left(-\frac{\cos\frac{n\pi x}{40}}{\frac{n\pi}{40}}\right) - \left(-\frac{\sin\frac{n\pi x}{40}}{\frac{n^2\pi^2}{40^2}}\right)\right]_0^{40}$$

$$= \frac{-2}{n\pi} \times 40\cos n\pi = \frac{80}{n\pi}(-1)^{n+1}$$

Using this value of B_n in (20), we get

$$u_2(x, t) = \frac{80}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n+1} \sin\frac{n\pi x}{40} \cdot e^{-n^2\pi^2\alpha^2 t/40^2} \tag{21}$$

Using (16) and (21) in (10), the required solution is

$$u(x, t) = x + \frac{80}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n+1} \sin\frac{n\pi x}{40} \cdot e^{-n^2\pi^2\alpha^2 t/40^2}$$

Example 13

A bar of length 10 cm has its ends A and B kept at 50°C and 100°C until steadystate conditions prevail. The temperature at A is then suddenly raised to 90°C and at the same instant, that at B is lowered to 60°C and the end temperatures are maintained thereafter. Find the temperature at distance x from the end A at time t.

When steadystate conditions prevail, the temperature distribution is given by

$$\frac{d^2u}{dx^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(0) = 50 \tag{2}$$

and
$$u(10) = 100 \tag{3}$$

Solving Eq. (1), we get

$$u(x) = ax + b \tag{4}$$

Using the boundary conditions (2) and (3) in (4), we get $a = 5$ and $b = 50$.

Therefore the solution of Eq. (1) is

$$u(x) = 5x + 50 \tag{5}$$

In the transient state, the temperature distribution in the bar is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

The corresponding boundary conditions are

$$u(0, t) = 90, \quad \text{for all } t \geq 0 \tag{7}$$

$$u(10, t) = 60, \quad \text{for all } t \geq 0 \tag{8}$$

$$u(x, 0) = 5x + 50, \quad \text{for } 0 < x < 10 \tag{9}$$

Since the end values in (7) and (8) are non-zero each, we adopt the modified procedure as in Examples 11 and 12.

Let
$$u(x, t) = u_1(x) + u_2(x, t) \tag{10}$$

where $u_1(x)$ is given by

$$\frac{d^2u_1}{dx^2} = 0 \tag{11}$$

and $u_2(x, t)$ is given by

$$\frac{\partial u_2}{\partial t} = \alpha^2 \frac{\partial^2 u_2}{\partial x^2} \tag{12}$$

The boundary conditions for Eq. (11) are

$$u_1(0) = 90 \tag{13}$$

and

$$u_1(10) = 60 \tag{14}$$

Solving Eq. (11), we get

$$u_1(x) = c_1 x + c_2 \tag{15}$$

Using boundary conditions (13) and (14) in (15), we get $c_1 = -3$ and $c_2 = 90$

$$\therefore \quad u_1(x) = 90 - 3x \tag{16}$$

The boundary conditions for Eq. (12) are

$$u_2(0, t) = u(0, t) - u_1(0) = 0, \qquad \text{for all } t \geq 0 \tag{17}$$

$$u_2(10, t) = u(10, t) - u_1(10) = 0, \qquad \text{for all } t \geq 0 \tag{18}$$

$$u_2(x, 0) = u(x, 0) - u_1(x) = 8x - 40, \quad \text{for } 0 < x < 10 \tag{19}$$

Proceeding as in Example 1, we get the most general solution of Eq. (12) as

$$u_2(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{10} \cdot e^{-n^2\pi^2\alpha^2 t/10^2} \tag{20}$$

Using boundary condition (19) in (20), we get

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{10} = 8x - 40 \text{ in } (0, 10)$$

$$= \sum b_n \sin \frac{n\pi x}{10}$$

which is the Fourier half-range sine series of $(8x - 40)$ in $(0, 10)$.
Comparing like terms in the two series, we have

$$B_n = b_n = \frac{2}{10} \int_0^{10} (8x - 40) \sin \frac{n\pi x}{10} dx$$

$$= \frac{8}{5} \left[(x - 5) \left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}} \right) - \left(-\frac{\sin \frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}} \right) \right]_0^{10}$$

$$= \frac{-16}{n\pi} \{5 \cos n\pi + 5\}$$

$$= -\frac{80}{n\pi} \{(-1)^n + 1\}$$

$$= \begin{cases} -\dfrac{160}{n\pi}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

Using this value of B_n in (20) and then using (16) and (20) in (10), the required solution is

$$u(x,t) = 90 - 3x - \frac{160}{\pi}\sum_{n=2,4,6\ldots}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{10}\exp\left\{\frac{-n^2\pi^2\alpha^2 t}{100}\right\}$$

i.e. $$u(x,t) = 90 - 3x - \frac{80}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{5}\exp\left(\frac{-n^2\pi^2\alpha^2 t}{25}\right)$$

Example 14

A bar AB with insulated sides is initially at temperature 0°C throughout. Heat is suddenly applied at the end $x = l$ at a constant rate A, so that $\frac{\partial u}{\partial x} = A$ for $x = l$, while the end A is not disturbed. Find the subsequent temperature distribution in the bar.

The temperature distribution $u(x,t)$ in the bar is given by the equation

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \tag{1}$$

We have to solve Eq. (1) satisfying the following boundary conditions.

$$u(0,t) = 0, \quad \text{for all } t \geq 0 \tag{2}$$

$$\frac{\partial u}{\partial x}(l,t) = A, \quad \text{for all } t \geq 0 \tag{3}$$

$$u(x,0) = 0, \quad \text{for } 0 < x < l \tag{4}$$

Since condition (3) has a non-zero value on the right side, we adopt the modified procedure.

Let $$u(x,t) = u_1(x) + u_2(x,t) \tag{5}$$

where $u_1(x)$ is given by

$$\frac{d^2 u_1}{dx^2} = 0 \tag{6}$$

and $u_2(x,t)$ is given by

$$\frac{\partial u_2}{\partial t} = \alpha^2\frac{\partial^2 u_2}{\partial x^2} \tag{7}$$

The boundary conditions for Eq. (6) are

$$u_1(0) = 0 \tag{8}$$

and $$\frac{du_1}{dx}(l) = A \tag{9}$$

Solving Eq. (6), we get

$$u_1(x) = C_1 x + C_2 \tag{10}$$

Using boundary condition (8) in (10), we get $C_2 = 0$

From (10), we have

$$\frac{du_1}{dx}(x) = C_1 \tag{11}$$

Using boundary condition (9) in (11), we get $C_1 = A$.

$$\therefore \qquad u_1(x) = Ax \tag{12}$$

The boundary conditions for Eq. (7) are

$$u_2(0,t) = u(0,t) - u_1(0) = 0, \qquad \text{for all } t \geq 0 \tag{13}$$

$$\frac{\partial u_2}{\partial x}(l,t) = \frac{\partial u}{\partial x}(l,t) - \frac{du_1}{dx}(l) = 0, \qquad \text{for all } t \geq 0 \tag{14}$$

$$u_2(x,0) = u(x,0) - u_1(x) = -Ax, \quad \text{for } 0 < x < l \tag{15}$$

Proceeding as in Example 8, we get the most general solution of Eq. (7) as

$$u_2(x,t) = \sum_{n=1}^{\infty} B_{2n-1} \sin\frac{(2n-1)\pi x}{2l} \exp\left\{\frac{-(2n-1)^2\pi^2\alpha^2 t}{4l^2}\right\} \tag{16}$$

Using boundary condition (15) in (16), we have

$$\sum_{n=1}^{\infty} B_{2n-1} \sin\frac{(2n-1)\pi x}{2l} = -Ax \text{ in } (0,l)$$

$$\therefore \qquad B_{2n-1} = \frac{2}{l}\int_0^l -Ax \sin\frac{(2n-1)\pi x}{2l}\,dx$$

$$= -\frac{2A}{l}\left[x\left\{\frac{-\cos\frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)\pi}{2l}}\right\} - \left\{\frac{-\sin\frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)^2\pi^2}{4l^2}}\right\}\right]_0^l$$

$$= -\frac{8Al}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi}{2}$$

$$= \frac{8Al\cdot(-1)^n}{(2n-1)^2\pi^2}$$

Using this value of B_{2n-1} in (16) and then using (12) and (16) in (5), the required solution is

$$u(x,t) = Ax + \frac{8Al}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2l}\exp\left\{\frac{-(2n-1)^2\pi^2\alpha^2 t}{4l^2}\right\}$$

Example 15

An insulated metal rod of length 100 cm has one end A kept at 0°C and the other end B at 100°C until steady state conditions prevail. At time $t = 0$, the temperature at B is suddenly reduced to 50°C and thereafter maintained, while at the same time $t = 0$, the end A is insulated. Find the temperature at any point of the rod at any subsequent time.

When steadystate conditions prevail, the temperature distribution in the rod is given by

$$\frac{d^2u}{dx^2} = 0 \tag{1}$$

We have to solve Eq. (1) satisfying the boundary conditions

$$u(0) = 0 \tag{2}$$

and

$$u(100) = 100 \tag{3}$$

Solving Eq. (1), we get

$$u(x) = ax + b \tag{4}$$

Using boundary conditions (2) and (3) in (4), we get $a = 1$ and $b = 0$.

Therefore the solution of Eq. (1) is

$$u(x) = x \tag{5}$$

In the transient state, the temperature distribution in the rod is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

The corresponding boundary conditions are

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad \text{for all } t \geq 0 \tag{7}$$

$$u(100, t) = 50, \quad \text{for all } t \geq 0 \tag{8}$$

$$u(x, 0) = x, \quad \text{for } 0 < x < 100 \tag{9}$$

Since the boundary value in (8) is non-zero, we adopt the modified procedure.

Let

$$u(x, t) = u_1(x) + u_2(x, t) \tag{10}$$

where $u_1(x)$ is given by

$$\frac{d^2u_1}{dx^2} = 0 \tag{11}$$

and $u_2(x, t)$ is given by

$$\frac{\partial u_2}{\partial t} = \alpha^2 \frac{\partial^2 u_2}{\partial x^2} \tag{12}$$

The boundary conditions for Eq. (11) are

$$\frac{du_1}{dx}(0) = 0 \tag{13}$$

$$u_1(100) = 50 \tag{14}$$

Solving Eq. (11), we get

$$u_1(x) = c_1 x + c_2 \tag{15}$$

From (15), we have

$$\frac{du_1}{dx}(x) = c_1 \tag{16}$$

Using boundary condition (13) in (16), we get $c_1 = 0$.
Using boundary condition (14) in (16), we get $c_2 = 50$.

$$\therefore \qquad u_1(x) = 50 \tag{17}$$

The boundary conditions for Eq. 12 are

$$\frac{\partial u_2}{\partial x}(0, t) = \frac{\partial u}{\partial x}(0, t) - \frac{du_1}{dx}(0) = 0, \quad \text{for all } t \geq 0 \tag{18}$$

$$u_2(100, t) = u(100, t) - u_1(100) = 0, \quad \text{for all } t \geq 0 \tag{19}$$

$$u_2(x, 0) = u(x, 0) - u_1(x) = x - 50, \quad \text{for } 0 < x < 100 \tag{20}$$

Proceeding as in Example 7, we get the most general solution of Eq. (12) as

$$u_2(x, t) = \sum_{n=1}^{\infty} A_{2n-1} \cos \frac{(2n-1)\pi x}{200} e^{-(2n-1)^2\pi^2\alpha^2 t/4\times 100^2} \tag{21}$$

Using boundary condition (20) in (21), we get

$$\sum_{n=1}^{\infty} A_{2n-1} \cos \frac{(2n-1)\pi x}{200} = x - 50 \text{ in } (0, 100)$$

$$\therefore \quad A_{2n-1} = \frac{2}{100} \int_0^{100} (x - 50) \cos \frac{(2n-1)\pi x}{200} dx$$

$$= \frac{1}{50} \left[(x - 50) \left\{ \frac{\sin \frac{(2n-1)\pi x}{200}}{\frac{(2n-1)\pi}{200}} \right\} - \left\{ \frac{-\cos \frac{(2n-1)\pi x}{200}}{\frac{(2n-1)^2\pi^2}{200^2}} \right\} \right]_0^{100}$$

$$= \frac{200}{(2n-1)\pi} \sin \frac{(2n-1)\pi}{2} - \frac{200^2}{(2n-1)^2\pi^2}$$

$$= \frac{200(-1)^{n+1}}{(2n-1)\pi} - \frac{200^2}{(2n-1)^2\pi^2}$$

Using this value of A_{2n-1} in (21) and then using (17) and (21) in (10), the required solution is

$$u(x, t) = 50 + \frac{200}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{2n-1} - \frac{200}{(2n-1)^2\pi^2}\right\}\cos\frac{(2n-1)\pi x}{200}$$
$$\exp\left\{-\frac{(2n-1)^2\pi^2\alpha^2 t}{40000}\right\}$$

Example 16

Solve the equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$ with the boundary conditions

$$u = u_0 e^{-wt}(w > 0) \text{ at } x = 0 \text{ and } u = 0 \text{ at } x = l$$

using the method of separation of variables. Show that the temperature at the mid-point of the rod is $\frac{1}{2}u_0 e^{-wt}\sec\frac{l}{2\alpha}\sqrt{w}$.

We have to solve the equation

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \tag{1}$$

Satisfying the boundary conditions

$$u(0, t) = u_0 e^{-wt}, \quad \text{for all } t \geq 0 \tag{2}$$

and

$$u(l, t) = 0, \quad \text{for all } t \geq 0 \tag{3}$$

As $u \neq \infty$, when $t \to \infty$, the appropriate solution of Eq. (1) is

$$u(x, t) = (A\cos px + B\sin px)e^{-p^2\alpha^2 t} \tag{4}$$

Using boundary condition (2) in (4), we have

$$Ae^{-p^2\alpha^2 t} = u_0 e^{-wt}$$

$$\therefore \quad A = u_0 \text{ and } p^2 = \frac{w}{\alpha^2} \text{ or } p = \frac{\sqrt{w}}{\alpha}$$

Using boundary condition (3) in (4), we have

$$\left(u_0\cos\frac{\sqrt{w}}{\alpha}l + B\sin\frac{\sqrt{w}}{\alpha}l\right)e^{-wt} = 0, \quad \text{for all } t \geq 0$$

$$\therefore \quad u_0\cos\frac{\sqrt{w}}{\alpha}l + B\sin\frac{\sqrt{w}}{\alpha}l = 0$$

$$\therefore \quad B = -\frac{u_0\cos\frac{\sqrt{w}}{\alpha}l}{\sin\frac{\sqrt{w}}{\alpha}l}$$

Using the values of A, B and p in (4), the required solution is

$$u(x,t) = u_0 \left\{ \cos \frac{\sqrt{w}}{\alpha} x - \frac{\cos \frac{\sqrt{w}}{\alpha} l}{\sin \frac{\sqrt{w}}{\alpha} l} \sin \frac{\sqrt{w}}{\alpha} x \right\} e^{-wt}$$

i.e.
$$u(x,t) = u_0 \frac{\sin \frac{\sqrt{w}}{\alpha}(l-x)}{\sin \frac{\sqrt{w}}{\alpha} l} e^{-wt} \tag{5}$$

The temperature at the mid point of the rod is given by $u\left(\frac{l}{2}, t\right)$.

From (5), on putting $x = \frac{l}{2}$, we get

$$u\left(\frac{l}{2}, t\right) = u_0 \frac{\sin \frac{\sqrt{w}}{2\alpha} l}{\sin \frac{\sqrt{w}}{\alpha} l} e^{-wt}$$
$$= \frac{u_0}{2} \sec \frac{\sqrt{wl}}{2\alpha} \cdot e^{-wt}$$

Example 17

The end $x = 0$ of a very long homogeneous rod is maintained at a temperature $u = u_0 \sin wt$. If $u \to 0$ as $x \to \infty$, find an expression giving u at any time, at any point of the bar.

The temperature distribution in the rod is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

We have to solve Eq. (1) satisfying the boundary conditions

$$u(0,t) = u_0 \sin wt, \qquad \text{for all } t \geq 0 \tag{2}$$
$$u(x,t) \to 0, \quad \text{as } x \to \infty, \text{ for all } t \geq 0 \tag{3}$$

The variable separable solution of (1) will not give the required solution.

In order that the solution of Eq. (1) may satisfy the boundary conditions (2) and (3), let us assume the solution of (1) as

$$u(x,t) = Ae^{-Bx} \sin (Ct - Dx) \tag{4}$$

From (4), $\dfrac{\partial u}{\partial t} = ACe^{-Bx}\cos(Ct - Dx)$

$$\frac{\partial u}{\partial x} = -ABe^{-Bx}\sin(Ct - Dx) - ADe^{-Bx}\cos(Ct - Dx)$$

$$\frac{\partial^2 u}{\partial x^2} = AB^2e^{-Bx}\sin(Ct - Dx) + ABDe^{-Bx}\cos(Ct - Dx)$$
$$+ ABDe^{-Bx}\cos(Ct - Dx) - AD^2e^{-Bx}\sin(Ct - Dx)$$

Since (4) is a solution of (1), we have

$$C\cos\theta = \alpha^2(B^2 - D^2)\sin\theta + 2BD\alpha^2\cos\theta$$

where $\theta = Ct - Dx$.
Equating like terms, we get

$$\alpha^2(B^2 - D^2) = 0 \tag{5}$$

and
$$2BD\alpha^2 = C \tag{6}$$

From (5),

$$D = B > 0 \tag{7}$$

[$\because e^{-Bx}$ and hence $u(x,t) \to 0$ as $x \to \infty$]
Using (7) in (6), we get

$$2B^2\alpha^2 = C \tag{8}$$

Using boundary condition (2) in (4), we have

$$A\sin Ct = u_0 \sin wt$$

$\therefore$
$$A = u_0 \text{ and } C = w$$

From (8), $B^2 = \dfrac{w}{2\alpha^2}$ or $B = \dfrac{1}{\alpha}\sqrt{\dfrac{w}{2}}$

From (7), $D = \dfrac{1}{\alpha}\sqrt{\dfrac{w}{2}}$

Note ✍

Had we assumed the solution as $u(x,t) = Ae^{-Bx}\sin(Ct + Dx)$, (6) would have been $-2BD\alpha^2 = C$ and hence (8) would have been $-2B^2\alpha^2 = w$ or $B^2 = -\dfrac{w}{2\alpha^2}$ which is absurd. Hence the assumption of the solution in the form (4) is justified.

Using the values of A, B, C and D in (4), the required solution is

$$u(x,t) = u_0 e^{-\frac{x}{\alpha}\sqrt{\frac{w}{2}}} \cdot \sin\left(wt - \frac{x}{\alpha}\sqrt{\frac{w}{2}}\right)$$

PROBLEMS ON TRANSMISSION LINE EQUATIONS

Example 18

A telegraph cable is a km long. Initially the line is uncharged so that $V(x,0)=0$. If, at $t=0$, the end $x=a$ is connected to a constant e.m.f. E, find $V(x,t)$ and $i(x,t)$. In particular, show that the current at the end $x=0$ is given by

$$-\frac{E}{aR}+\frac{2E}{aR}\sum_{n=1}^{\infty}(-1)^{n+1}\exp\left(-\frac{n^2\pi^2 t}{a^2RC}\right)$$

(Refer to the discussion on transmission line equations in Chapter 3(A)

The potential at any point at time t in a telegraph cable is given by

$$\frac{\partial^2 V}{\partial x^2}=RC\frac{\partial V}{\partial t}$$

or
$$\frac{\partial V}{\partial t}=\alpha^2\frac{\partial^2 V}{\partial x^2} \tag{1}$$

where $\alpha^2=\dfrac{1}{RC}$

We have to solve Eq. (1) satisfying the following boundary conditions.

$$V(0,t)=0,\quad \text{for all } t\geq 0 \tag{2}$$

$$V(a,t)=E,\quad \text{for all } t\geq 0 \tag{3}$$

$$V(x,0)=0,\quad \text{for all } 0<x<a \tag{4}$$

Since condition (3) contains a non-zero boundary value, we adopt the modified procedure.

Let
$$V(x,t)=V_1(x)+V_2(x,t) \tag{5}$$

where $V_1(x)$ is given by

$$\frac{d^2V_1}{dx^2}=0 \tag{6}$$

and $V_2(x,t)$ is given by

$$\frac{\partial V_2}{\partial t}=\alpha^2\frac{\partial^2 V_2}{\partial x^2} \tag{7}$$

The boundary conditions for Eq. (6) are

$$V_1(0)=0 \tag{8}$$

$$V_1(a)=E \tag{9}$$

Solving Eq. (6), we get

$$V_1(x)=C_1x+C_2 \tag{10}$$

Using boundary conditions (8) and (9) in (10), we get $C_1 = \frac{E}{a}$ and $C_2 = 0$

$$\therefore \quad V_1(x) = \frac{Ex}{a} \tag{11}$$

The boundary conditions for Eq. (7) are

$$V_2(0, t) = V(0, t) - V_1(0) = 0, \quad \text{for all } t \geq 0 \tag{12}$$

$$V_2(a, t) = V(a, t) - V_1(a) = 0, \quad \text{for all } t \geq 0 \tag{13}$$

$$V_2(x, 0) = V(x, 0) - V_1(x) = -\frac{Ex}{a}, \quad \text{for } 0 < x < a \tag{14}$$

Proceeding as in Example 1, we get the most general solution of Eq. (7) as

$$V_2(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a} e^{-n^2\pi^2\alpha^2 t/a^2} \tag{15}$$

Using boundary condition (14) in (15), we have

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a} = -\frac{Ex}{a} \quad \text{in } (0, a)$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}$$

which is the Fourier half-range sine series of $\left(-\frac{Ex}{a}\right)$ in $(0, a)$.

Comparing like terms in the two series, we have

$$B_n = b_n = \frac{2}{a}\int_0^a -\frac{Ex}{a}\sin\frac{n\pi x}{a}\,dx$$

$$= -\frac{2E}{a^2}\left[x\left(-\frac{\cos\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right) - \left(-\frac{\sin\frac{n\pi x}{a}}{\frac{n^2\pi^2}{a^2}}\right)\right]_0^a$$

$$= \frac{2E}{n\pi}(-1)^n$$

Using this value of B_n in (15) and using (11) and (15) in (5), the required solution is

$$V(x, t) = \frac{Ex}{a} + \frac{2E}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^n \sin\frac{n\pi x}{a} e^{-n^2\pi^2\alpha^2 t/a^2} \tag{16}$$

For the telegraph equation ($L = G = 0$),

$$Ri = -\frac{\partial V}{\partial x} \quad \text{or} \quad i(x, t) = -\frac{1}{R}\frac{\partial V}{\partial x}(x, t)$$

Differentiating (16) partially with respect to x, we have

$$i(x,t) = -\frac{E}{aR} + \frac{2E}{aR}\sum_{n=1}^{\infty}(-1)^{n+1}\cos\frac{n\pi x}{a}e^{-n^2\pi^2\alpha^2 t/a^2}$$

$$\therefore \qquad i(0,t) = -\frac{E}{aR} + \frac{2E}{aR}\sum_{n=1}^{\infty}(-1)^{n+1}\exp\left(-\frac{n^2\pi^2 t}{a^2RC}\right)$$

since $\alpha^2 = \dfrac{1}{RC}$.

Example 19

A transmission line 1000 km long is initially under steadystate conditions with potential 1200 volts at the sending end and 1100 volts at the load ($x = 1000$). The terminal end of the line is suddenly grounded, reducing its potential to zero, but the potential at the sending end is kept at 1200 volts. Find the potential function $e(x,t)$. Assume that $L = G = 0$.

When $L = G = 0$, the potential function $e(x,t)$ in the transmission line is given by

$$\frac{\partial e}{\partial t} = \alpha^2\frac{\partial^2 e}{\partial x^2} \tag{1}$$

where $\alpha^2 = \dfrac{1}{RC}$.

When steadystate conditions prevail, the potential function is given by

$$\frac{d^2 e}{dx^2} = 0 \tag{2}$$

The boundary conditions for Eq. (2) are

$$e(0) = 1200 \tag{3}$$

and

$$e(1000) = 1100 \tag{4}$$

Solving Eq. (2), we get

$$e(x) = ax + b \tag{5}$$

Using boundary conditions (3) and (4) in (5), we have $a = -0.1$ and $b = 1200$.

Therefore the solution of Eq. (2) is

$$e(x) = 1200 - 0.1x \tag{6}$$

In the transient state, the potential function is given by Eq. (1).

The corresponding boundary conditions are

$$e(0,t) = 1200, \qquad \text{for all } t \geq 0 \tag{7}$$

$$e(1000,t) = 0, \qquad \text{for all } t \geq 0 \tag{8}$$

$$e(x,0) = 1200 - 0.1x, \quad \text{for } 0 < x < 1000 \tag{9}$$

Since the boundary value in (7) is non-zero, we adopt the modified procedure.

Let $$e(x,t) = e_1(x) + e_2(x,t) \tag{10}$$

where $e_1(x)$ is given by

$$\frac{d^2 e_1}{dx^2} = 0 \tag{11}$$

and $e_2(x,t)$ is given by

$$\frac{\partial e_2}{\partial t} = \alpha^2 \frac{\partial^2 e_2}{\partial x^2} \tag{12}$$

The boundary conditions for Eq. (11) are

$$e_1(0) = 1200 \tag{13}$$

and $$e_1(1000) = 0 \tag{14}$$

Solving Eq. (11), we get

$$e_1(x) = C_1 x + C_2 \tag{15}$$

Using boundary conditions (13) and (14) in (15), we get $C_1 = -1.2$ and $C_2 = 1200$

$\therefore$ $$e_1(x) = 1200 - 1.2x \tag{16}$$

The boundary conditions for Eq. (12) are

$$e_2(0,t) = e(0,t) - e_1(0) = 0, \quad \text{for all } t \geq 0 \tag{17}$$

$$e_2(1000,t) = e(1000,t) - e_1(1000) = 0, \text{for all } t \geq 0 \tag{18}$$

$$e_2(x,0) = e(x,0) - e_1(x) = 1.1x, \quad \text{for } 0 < x < 1000 \tag{19}$$

Proceeding as in Example 1, we get the most general solution of Eq. (12) as

$$e_2(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{1000} \exp\left\{\frac{-n^2\pi^2\alpha^2 t}{1000^2}\right\} \tag{20}$$

Using boundary condition (19) in (20), we get

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{1000} = 1.1x \text{ in } (0, 1000)$$

$$= \sum b_n \sin\frac{n\pi x}{1000}$$

which is Fourier half-range sine series of $1.1x$ in $(0, 1000)$.

Comparing like terms, we get

$$B_n = b_n = \frac{2}{1000}\int_0^{1000} 1.1x \sin\frac{n\pi x}{1000}\mathrm{d}x$$

$$= \frac{2.2}{1000}\left[x\left(\frac{-\cos\dfrac{n\pi x}{1000}}{\dfrac{n\pi}{1000}}\right)-\left(\frac{-\sin\dfrac{n\pi x}{1000}}{\dfrac{n^2\pi^2}{1000^2}}\right)\right]_0^{1000}$$

$$= \frac{2200}{n\pi}(-1)^{n+1}$$

Using this value of B_n in (20) and then using (16) and (20) in (10), the required solution is

$$e(x,t) = 1200 - 1.2x + \frac{2200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{1000}\right)\cdot\exp\left\{\frac{-n^2\pi^2 t}{10^6 RC}\right\}$$

Example 20

A submarine cable $\left(L = G = 0 \text{ and } a = \dfrac{1}{RC}\right)$ of length l has zero initial current and charge. The end $x = 0$ is insulated and a constant voltage E is applied at $x = l$. Show that the voltage at any point is given by

$$v(x,t) = E + \frac{4E}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-a(2n-1)^2\pi^2 t/4l^2}\cos\left\{\frac{(2n-1)\pi x}{2l}\right\}$$

The voltage function $v(x,t)$ is given by the equation

$$\frac{\partial v}{\partial t} = \frac{1}{RC}\frac{\partial^2 V}{\partial x^2} \quad \text{or} \quad \frac{\partial v}{\partial t} = a\frac{\partial^2 v}{\partial x^2} \tag{1}$$

We have to solve Eq. (1) satisfying the following boundary conditions.

$$\frac{\partial v}{\partial x}(0,t) = 0, \quad \text{for all } t \geq 0 \tag{2}$$

$$v(l,t) = E, \quad \text{for all } t \geq 0 \tag{3}$$

$$v(x,0) = 0, \quad \text{for } 0 < x < l \tag{4}$$

Since the boundary value in (3) is non-zero, we adopt the modified procedure.

Let

$$v(x,t) = v_1(x) + v_2(x,t) \tag{5}$$

where $v_1(x)$ is given by

$$\frac{\mathrm{d}^2 v_1}{\mathrm{d}x^2} = 0 \tag{6}$$

and $v_2(x, t)$ is given by

$$\frac{\partial v_2}{\partial t} = \alpha^2 \frac{\partial^2 v_2}{\partial x^2} \tag{7}$$

The boundary conditions for Eq. (6) are

$$\frac{\partial v_1}{\partial x}(0) = 0 \tag{8}$$

and
$$v_1(l) = E \tag{9}$$

Solving Eq. (6), we get

$$v_1(x) = C_1 x + C_2 \tag{10}$$

From (10), we have

$$\frac{\partial v_1(x)}{\partial x} = C_1 \tag{11}$$

Using boundary condition (8) in (11), we get $C_1 = 0$
Using boundary condition (9) in (10), we get $C_2 = E$.

$$\therefore \qquad v_1(x) = E \tag{12}$$

The boundary conditions for Eq. (7) are

$$\frac{\partial v_2}{\partial x}(0, t) = \frac{\partial v}{\partial x}(0, t) - \frac{\mathrm{d}v_1}{\mathrm{d}x}(0) = 0, \text{for all } t \geq 0 \tag{13}$$

$$v_2(l, t) = v(l, t) - v_1(l) = 0, \qquad \text{for all } t \geq 0 \tag{14}$$

$$v_2(x, 0) = v(x, 0) - v_1(x) = -E, \text{ for } 0 < x < l \tag{15}$$

Proceeding as in Example 7, with u_0 replaced by $-E$, the solution of Eq. (7) is

$$v_2(x, t) = -\frac{4E}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)} \cos\frac{(2n-1)\pi x}{2l} \exp\left\{\frac{-(2n-1)^2\pi^2 at}{4l^2}\right\} \tag{16}$$

Using (12) and (16) in (5), the required solution is

$$v(x, t) = E + \frac{4E}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \frac{\cos(2n-1)\pi x}{2l} \cdot \exp\left\{\frac{-(2n-1)^2\pi^2 at}{4l^2}\right\}$$

Exercise3B

Part A (Short-Answer Questions)

1. State the two laws of thermodynamics used in the derivation of one dimensional heat flow equation.
2. What does α^2 represent in the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}?$$

3. Write down the three mathematically possible solutions of one dimensional heat flow equation.
4. Write down the appropriate solution of the one dimensional heat flow equation. How is it chosen?
5. Write down the form of the general solution of one dimensional heat flow equation, when both the ends of the bar are kept at zero temperature.
6. Write down the form of the general solution of one dimensional heat flow equation, when both the ends of the rod are insulated?
7. In what type of one dimensional heat flow problems, will neither the Fourier sine series nor cosine series be useful?
8. Write down the form of the temperature function, when heat flow in a bar is under steadystate conditions.
9. Explain briefly the procedure used to solve $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ satisfying the conditions

$$u(0, t) = A, u(l, t) = B \text{ and } u(x, 0) = f(x)$$

Part B

10. A uniform bar of length 10 cm through which heat flows is insulated at its sides. The ends are kept at zero temperature. If the initial temperature at the interior points of the bar is given by (i) $3 \sin \frac{\pi x}{5} + 2 \sin \frac{2\pi x}{5}$ (ii) $2 \sin \frac{\pi x}{5} \cos \frac{2\pi x}{5}$ find the temperature distribution in the bar
11. Obtain the solution of the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

satisfying the following conditions: (i) $u \neq \infty$, as $t \to \infty$; (ii) $u = 0$ for $x = 0$ and $x = \pi$ for any value of t; (iii) $u = \pi x - x^2$, when $t = 0$ in the range $(0, \pi)$.

12. Find the solution of the equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

that satisfies the condition $u(0, t) = 0$ and $u(l, t) = 0$ for $t \geq 0$ and

$$u(x, 0) = \begin{cases} x, & \text{for } 0 < x < \dfrac{l}{2} \\ l - x, & \text{for } \dfrac{l}{2} < x < l \end{cases}$$

13. A rod of length l has its ends A and B kept at 0°C and T°C respectively, until steadystate conditions prevail. If the temperature at B is reduced suddenly to 0°C and kept so, while that of A is maintained, find the temperature $u(x, t)$ at a distance x from the end A at time t.

14. Solve the one dimensional heat flow equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

satisfying the following boundary conditions.

(i) $\dfrac{\partial u}{\partial x}(0, t) = 0$, for all $t \geq 0$

(ii) $\dfrac{\partial u}{\partial x}(l, t) = 0$, for all $t \geq 0$

(iii) (a) $u(x, 0) = 2\cos\dfrac{3\pi x}{l}\cos\dfrac{2\pi x}{l}$, for $0 < x < l$

(b) $u(x, 0) = \cos^4\dfrac{\pi x}{l}$ in $(0, l)$;

(c) $u(x, 0) = lx - x^2$ in $(0, l)$

15. The temperature at one end of a bar, 50 cm long and with insulated sides, is kept at 0°C and that at the other end is kept at 100°C until steadystate conditions prevail. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter. Find the temperature distribution. Show also that the sum of the temperature at any two points equidistant from the centre of the bar is always 100°C.

16. A uniform rod of length a whose surface is thermally insulated is initially at temperature $\theta = \theta_0$. At time $t = 0$, one end is suddenly cooled to temperature $\theta = 0$ and subsequently maintained at this temperature. The other end remains thermally insulated. Show that the temperature at this end at time t is given by

$$\theta = \frac{4\theta_0}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \exp\left\{-(2n+1)^2\alpha^2\pi^2 t/4a^2\right\}$$

17. An insulated metal rod of length 20 cm has one end A kept at 0°C and the other end B at 60°C until steadystate conditions prevail. At time $t = 0$, the end B is suddenly insulated while the temperature at A is maintained at 0°C. Find the temperature at any point of the rod at any subsequent time.

18. Two slabs of iron each 20 cm thick, one at temperature 100°C and the other at temperature 0°C throughout, are placed face to face in perfect contact and their outer faces are kept at 0°C. Find the temperature 10 minutes after contact was made, at a point on their common face.
19. Find the temperature in a flat slab of unit width such that (i) its initial temperature varies uniformly from zero at one face to u_0 at the other, (ii) the temperature of the face initially at zero remains at zero for $t > 0$ and (iii) the face initially at temperature u_0 is insulated for $t > 0$.
20. Find the temperature $\theta(x, t)$ in an infinite slab of thickness l, if the faces $x = 0$ and $x = l$ are kept at a constant temperature T°, the initial temperature of the slab being 0°.
21. A bar 40 cm long has originally a temperature of 0°C along all its length. At time $t = 0$ sec, the temperature at the end $x = 0$ is raised to 50°C, while that at the end $x = 40$ is raised to 100°C. Determine the resulting temperature distribution.
22. The ends A and B of a rod 10 cm long have their temperatures kept at 0°C and 20°C respectively, until steadystate conditions prevail. The temperature of the end B is then suddenly raised to 60°C and kept so while that of the end A is kept at 0°C. Find the temperature $u(x, t)$.
23. A rod l cm long with insulated lateral surface is initially at the temperature 100°C throughout. If the temperatures at the ends are suddenly reduced to 25°C and 75°C respectively, find the temperature distribution in the rod at any subsequent time.
24. The ends A and B of a bar 50 cm long are kept at 0°C and 100°C respectively, until steadystate conditions prevail. The temperatures at A and B are then suddenly raised to 50°C and 150°C respectively and they are maintained thereafter. Find an expression for the temperature at a distance x from A at any time t subsequent to the changes in the end temperatures.
25. A rod AB of length 10 cm has the ends A and B kept at temperatures 40°C and 100°C respectively, until the steadystate is reached. At some time thereafter the temperatures at A and B are lowered to 10°C and 50°C and they are maintained thereafter. Find the subsequent temperature distribution.
26. The ends A and B of a rod 20 cm long have the temperatures at 30°C and 80°C until steadystate prevails. The temperatures of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t.
27. A bar 25 cm long with its sides impervious to heat, has its ends A and B kept at 100°C and 200°C respectively. After the temperature distribution becomes steady, the end A is suddenly cooled to 50°C and at the same instant, the end B is warmed to 300°C. Find an expression for the temperature at a distance x from A at any time t subsequent to the changes in the end temperatures.
28. A bar with insulated sides is initially at temperature 0°C throughout. Heat is suddenly applied at the end $x = 0$ at a constant rate A, so that $\dfrac{\partial u}{\partial x} = A$ for

$x = 0$, while the end $x = l$ is maintained at 0°C temperature. Find the temperature in the bar at a subsequent time.

29. An insulated metal rod of length 60 cm has one end A kept at 0°C and the other end B at 60°C until steadystate conditions prevail. At time $t = 0$, the temperature at A is suddenly increased to 30°C and thereafter maintained, while at the same time $t = 0$ the end B is insulated. Find the subsequent temperature distribution in the rod.

30. Solve the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, satisfying the conditions: (i) $u(0, t) = 0$; (ii) $u(1, t) = t$ and (iii) $u(x, 0) = 0$.

    [**Hint**: Assume $u(x, t) = x\left\{a(x^2 - 1) + t\right\} + u_2(x, t)$. When $u(x, t)$ satisfies the given equation and boundary conditions, $a = \dfrac{1}{6}$ and $u_2(x, t)$ satisfies $\dfrac{\partial u_2}{\partial t} = \dfrac{\partial^2 u_2}{\partial x^2}$ such that $u_2(0, t) = 0$, $u_2(1, t) = 0$ and $u_2(x, 0) = \dfrac{1}{6}x(1 - x^2)$].

31. Solve the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, satisfying the conditions: (i) $u(0, t) = t$, (ii) $u(1, t) = 0$ and (iii) $u(x, 0) = 0$. [**Hint:** Assume $u(x, t) = \left\{ax(x^2 - 1) + (x - 1)\left(\dfrac{x}{2} - t\right)\right\} + u_2(x, t)$. When $u(x, t)$ satisfies the given equation and the boundary conditions, $a = -1/6$ and $u_2(x, t)$ satisfies $\dfrac{\partial u_2}{\partial t} = \dfrac{\partial^2 u_2}{\partial x^2}$ such that $u_2(0, t) = 0$, $u_2(1, t) = 0$ and $u_2(x, 0) = \dfrac{1}{6}x(x^2 - 3x + 2)$].

32. A transmission line 1000 km long is initially under steadystate conditions with potential 1300 volts at the sending end ($x = 0$) and 1200 volts at the receiving end ($x = 1000$). The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the inductance and leakage to be negligible, find the potential $e(x, t)$.

33. A steady voltage ditribution of 20 volts at the sending end and 12 volts at the receiving end is maintained in a telephone wire of length l. At time $t = 0$, the receiving end is grounded. Find the voltage and current t secs. later. Neglect leakage and inductance.

34. In a telegraph wire, the sending end of the line is at potential e_0, the far end being earthed until steadystate conditions prevail. The sending end is suddenly earthed. Show that the potential at a point distant x from the sending end at time t is given by $e(x, t) = \dfrac{2e_0}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\sin\dfrac{n\pi x}{l}\exp\left\{-n^2\pi^2 t/CRl^2\right\}$, where l is the length of the wire and C, R have their usual meanings.

35. A submarine cable of length l has the end $x = l$ grounded and constant voltage E is applied at the end $x = 0$ with zero initial conditions. Find the expression for the current at $x = 0$.

Answers

Exercise 3B

10. (i) $u(x,t) = 3\sin\dfrac{\pi x}{5}e^{-4\pi^2\alpha^2 t/100} + 2\sin\dfrac{2\pi x}{5}e^{-16\pi^2\alpha^2 t/100}$

(ii) $u(x,t) = -\sin\dfrac{\pi x}{5}e^{-4\pi^2\alpha^2 t/100} + \sin\dfrac{3\pi x}{5}e^{-36\pi^2\alpha^2 t/100}$

11. $u(x,t) = \dfrac{8}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^3}\sin(2n-1)x\ \exp\left\{-(2n-1)^2\alpha^2 t\right\}$

12. $u(x,t) = \dfrac{4l}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{(2n-1)^2}\sin\dfrac{(2n-1)\pi x}{l}\ \exp\left\{-(2n-1)^2\pi^2\alpha^2 t/l^2\right\}.$

13. $u(x,t) = \dfrac{2T}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n}\sin\dfrac{n\pi x}{l}\exp(-n^2\pi^2\alpha^2 t/l^2).$

14. (a) $u(x,t) = \cos\dfrac{\pi x}{l}\exp(-\pi^2\alpha^2 t/l^2) + \cos\dfrac{5\pi x}{l}\exp(-25\pi^2\alpha^2 t/l^2).$

(b) $u(x,t) = \dfrac{3}{8} + \dfrac{1}{2}\cos\dfrac{2\pi x}{l}\exp(-4\pi^2\alpha^2 t/l^2)$

$+\dfrac{1}{8}\cos\dfrac{4\pi x}{l}\exp(-16\pi^2\alpha^2 t/l^2).$

(c) $u(x,t) = \dfrac{l^2}{6} - \dfrac{l^2}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\cos\dfrac{2n\pi x}{l}\exp(-4n^2\pi^2\alpha^2 t/l^2).$

15. $u(x,t) = 50 - \dfrac{400}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}\cos\dfrac{(2n-1)\pi x}{50}.$

$\exp\left\{-(2n-1)^2\pi^2\alpha^2 t/2500\right\}.$

17. $u(x,t) = \dfrac{480}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{(2n-1)^2}\sin\dfrac{(2n-1)\pi x}{40}\ \exp\left\{-(2n-1)^2\pi^2\alpha^2 t/1600\right\}.$

18. $u(20, 600) = \dfrac{400}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\sin^2\left(\dfrac{n\pi}{4}\right)\sin\left(\dfrac{n\pi}{2}\right).\exp\left(-3n^2\pi^2\alpha^2/8\right).$

19. $u(x,t) = \dfrac{4u_0}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{(2n-1)^2}\sin\dfrac{(2n-1)\pi x}{2}.\exp\left\{\dfrac{-(2n-1)^2\pi^2\alpha^2 t}{4}\right\}.$

20. $u(x,t) = T - \dfrac{4T}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{2n-1}\sin\dfrac{(2n-1)\pi x}{l}.\exp\left\{-(2n-1)^2\pi^2\alpha^2 t/l^2\right\}.$

21. $u(x,t)=\dfrac{5x}{4}+50+\dfrac{100}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}(2\cos n\pi-1).\sin\dfrac{n\pi x}{40}.\exp\left\{-n^2\pi^2\alpha^2 t/1600\right\}.$

22. $u(x,t)=6x+\dfrac{80}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}(-1)^n\sin\dfrac{n\pi x}{10}.\exp\left(-n^2\pi^2\alpha^2 t/100\right).$

23. $u(x,t)=\dfrac{50}{l}x+25+\dfrac{50}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\{3-\cos n\pi\}\sin\dfrac{n\pi x}{l}.\exp\left\{-n^2\pi^2\alpha^2 t/l^2\right\}.$

24. $u(x,t)=(2x+50)-\dfrac{200}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{2n-1}\sin\dfrac{(2n-1)\pi x}{50}.$

$\exp\left\{-(2n-1)^2\pi^2\alpha^2 t/2500\right\}.$

25. $u(x,t)=(4x+10)+\dfrac{20}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}(3-5\cos n\pi)\sin\dfrac{n\pi x}{10}.\exp\left(-n^2\pi^2\alpha^2 t/100\right).$

26. $u(x,t)=x+40-\dfrac{20}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}(2\cos n\pi+1)\sin\dfrac{n\pi x}{20}.\exp\left(-n^2\pi^2\alpha^2 t/400\right).$

27. $u(x,t)=10x+50+\dfrac{100}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}(2\cos n\pi+1)\sin\dfrac{n\pi x}{25}.\exp\left(-n^2\pi^2\alpha^2 t/625\right).$

28. $u(x,t)=A(x-l)+\dfrac{8Al}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}\cos\dfrac{(2n-1)\pi x}{2l}.$

$\exp\left\{-(2n-1)^2\pi^2\alpha^2 t/4l^2\right\}.$

29. $u(x,t)=30+\dfrac{120}{\pi}\sum_{n=1}^{\infty}\left\{-\dfrac{1}{2n-1}+\dfrac{4(-1)^{n+1}}{(2n-1)^2\pi}\right\}\sin\dfrac{(2n-1)\pi x}{120}.$

$\exp\left\{-(2n-1)^2\pi^2\alpha^2 t/120^2\right\}.$

30. $u(x,t)=\dfrac{1}{6}(x^3-x+6xt)+\dfrac{2}{\pi^3}\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n^3}\sin n\pi x.\exp\left(-n^2\pi^2 t\right).$

31. $u(x,t)=-\dfrac{1}{6}(x^3-3x^2+2x+6xt-6t)+\dfrac{2}{\pi^3}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^3}\sin n\pi x.\exp(-n^2\pi^2 t).$

32. $e(x,t)=1300-1.3x+\dfrac{2400}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n}\sin\dfrac{n\pi x}{1000}\exp\left\{-n^2\pi^2 t/1000^2 RC\right\}.$

33. $e(x,t) = \dfrac{20}{l}(l-x) - \dfrac{24}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin\dfrac{n\pi x}{l}.\exp\left(\dfrac{-n^2\pi^2 t}{l^2 RC}\right)$

$$i(x,t) = \frac{20}{lR} + \frac{24}{lR}\sum_{n=1}^{\infty}(-1)^n\cos\frac{n\pi x}{l}\exp\left(-n^2\pi^2 t/l^2 RC\right).$$

35. $i = \dfrac{E}{lR}\left\{-1 + 2\displaystyle\sum_{n=1}^{\infty}\exp\left(-n^2\pi^2 t/l^2 RC\right)\right\}.$

Part C

Steady State Heat Flow in Two Dimensions [Cartesian Coordinates]

3C.1 INTRODUCTION

When the heat flow is along plane curves, lying in the same or parallel planes, instead of along straight lines, then the heat flow is said to be two dimensional. When we consider heat flow or temperature distribution in thin uniform plate or sheet made of conducting material, the heat flow is assumed to be two dimensional.

When all the edges of the plate are straight lines, that is, when the plate is in the form of a rectangle or square, cartesian coordinates will be used to discuss the temperature distribution in the plate, as the straight edges can be easily represented in the cartesian system. When one or more edges of the plate are circular arcs, that is, when the plate is in the form of a circle, semicircle, sector of a circle or circular ring, polar coordinates will be used to discuss the temperature distribution in the plate, as the circular edges can be easily represented in the polar system.

In this chapter, we shall first derive the partial differential equation of variable heat flow in two dimensional cartesians and then deduce the equation of steadystate heat flow.

3C.2 EQUATION OF VARIABLE HEAT FLOW IN TWO DIMENSIONS IN CARTESIAN COORDINATES

Let us consider heat flow in a thin plate or sheet, of thickness h, which is made up of conducting material of density ρ, thermal conductivity k and specific heat c. Let the xoy-plane be taken in one face of plate. Let us assume that the surfaces of the plate are insulated, so that heat flow takes place only in the xoy-plane and not along the normal to xoy-plane.

Let us now consider the heat flow in an element of the plate in the form of a small rectangle $ABCD$, the coordinates of the vertices of which are shown in Fig.3C.1. Let u and $u + \Delta u$ be the temperatures of this element at times t and $t + \Delta t$ respectively.

Therefore increase in temperature in the element in Δt time $= \Delta u$.

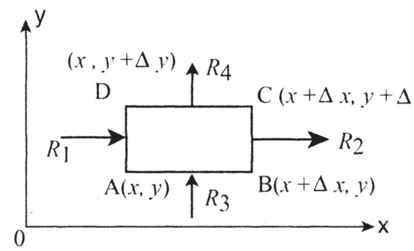


Fig. 3C.1

Therefore increase of heat in the element in Δt time = (specific heat) (mass of the element) (increase in temperature) [by a law of thermodynamics] = $c(\rho h \Delta x \Delta y)\Delta u$.

Therefore rate of increase of heat in the element at time t is

$$= h\rho c \Delta x \Delta y \cdot \frac{\partial u}{\partial t} \tag{1}$$

Let R_1 and R_3 be the rates of inflow of heat into the element through the sides AD and AB respectively at time t.

Let R_2 and R_4 be the rates of outflow of heat from the element through the sides BC and DC respectively at time t.

Therefore rate of increase of heat in the element at time t is

$$\begin{aligned} &= R_1 - R_2 + R_3 - R_4 \\ &= \left[-k(h\Delta y)\left(\frac{\partial u}{\partial x}\right)_x\right] - \left[-k(h\Delta y)\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}\right] \\ &\quad + \left[-k(h\Delta x)\left(\frac{\partial u}{\partial y}\right)_y\right] - \left[-k(h\Delta x)\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y}\right] \end{aligned}$$

by a law of thermodynamics. (For explanation, see the derivation of one dimensional heat flow equation in Chapter 3(B)

$$= hk\Delta x \Delta y \left[\left\{\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right\} + \left\{\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\Delta y}\right\}\right] \tag{2}$$

Equating (1) and (2), we get

$$\frac{\partial u}{\partial t} = \frac{k}{\rho c}\left[\left\{\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right\} + \left\{\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\Delta y}\right\}\right] \tag{3}$$

Equation (3) gives the temperature distribution at time t in the element $ABCD$ of the plate.

Taking limits as $\Delta x \to 0$ and $\Delta y \to 0$ in (3), we get the equation that gives the temperature at the point $A(x, y)$ at time t.

Thus the partial differential equation, representing variable temperature distribution in a two dimensional plate or variable heat flow in two dimensions is

$$\frac{\partial u}{\partial t} = \frac{k}{\rho c}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Since $\frac{k}{\rho c}$ depends on the material of the plate and positive, we denote it by α^2, which is called *the diffusivity* of the material of the plate.

Thus the equation of variable heat flow in two dimensional cartesians is

$$\frac{\partial u}{\partial t} = \alpha^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{4}$$

Deduction

When steadystate conditions prevail in the plate, the temperature at any point of the plate does not depend on t, but depends on x and y only.

i.e. $$\frac{\partial u}{\partial t} = 0 \text{ in Eq. (4)}$$

Thus steadystate temperature distribution in a two dimensional plate is given by $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, which is the familiar *Laplace equation* in two dimensional cartesians.

Note ✍

1. *If the surfaces of the plate are not insulated, heat flow will be along non-planar curves, so that heat flow is three dimensional. In this case, the equation of heat flow will take the form*

$$\frac{\partial u}{\partial t} = \alpha^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

2. *If heat flows along straight lines all parallel to x-axis, then $R_3 = 0 = R_4$. In this case, heat flow is one dimensional and Eq. (4) reduces to $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$, which has been directly derived in Chapter 3(B).*
3. *The following functions which occur in various branches of Applied Mathematics and Engineering satisfy Laplace equation $\nabla^2 u = 0$.*

   (i) the temperature in the theory of thermal equilibrium of solids.
   (ii) the gravitational potential in regions not occupied by attracting matter.

(iii) the electrostatic potential in a uniform dielectric, in the theory of electrostatics.

(iv) the magnetic potential in free space, in the theory of magnetostatics.

(v) the electric potential, in the theory of the steady flow of electric current in solid conductors.

(vi) the velocity potential at points of a homogeneous liquid moving irrotationally in hydrodynamic problems.

3C.3 VARIABLE SEPARABLE SOLUTIONS OF LAPLACE EQUATION

Laplace equation in two dimensional cartesians is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

Let $$u(x, y) = X(x) \cdot Y(y) \tag{2}$$

be a solution of Eq. (1), where $X(x)$ is a function of x alone and $Y(y)$ is a function o y alone.

Then $\frac{\partial^2 u}{\partial x^2} = X''Y$ and $\frac{\partial^2 u}{\partial y^2} = XY''$, where $X'' = \frac{d^2X}{dx^2}$ and $Y'' = \frac{d^2X}{dy^2}$, satisfy Eq. (1).

i.e. $$X''Y + XY'' = 0$$

i.e. $$\frac{X''}{X} = -\frac{Y''}{Y} \tag{3}$$

The L.H.S. of (3) is a function of x alone and the R.H.S. is a function of y alone. They are equal for all values of the independent variables x and y. This is possible only if each is a constant.

$\therefore$ $$\frac{X''}{X} = -\frac{Y''}{Y} = k, \text{ where } k \text{ is a constant.}$$

$\therefore$ $$X'' - kX = 0 \tag{4}$$

and $$Y'' + ky = 0 \tag{5}$$

The nature of the solutions of (4) and (5) depends on the nature of values of k. Hence the following three cases arise:

Case (1)

k is positive. Let $k = p^2$.

Then Eq. (4) and (5) become

$$(D^2 - p^2)X = 0 \text{ and } (D_1^2 + p^2)Y = 0$$

where $$D \equiv \frac{d}{dx} \quad \text{and} \quad D_1 \equiv \frac{d}{dy}$$

The solutions of these equations are $X = Ae^{px} + Be^{-px}$ and $Y = C\cos py + D\sin py$.

Case (2)

k is negative. Let $k = -p^2$.

Then Eq. (4) and (5) become

$$(D^2 + p^2)X = 0 \quad \text{and} \quad (D_1^2 - p^2)Y = 0$$

The solutions of these equations are $X = A\cos px + B\sin px$ and $Y = ce^{py} + De^{-py}$.

Case (3)

$k = 0$.

Then Eq. (4) and (5) become

$$\frac{d^2X}{dx^2} = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} = 0.$$

The solutions of these equations are

$$X = Ax + B \quad \text{and} \quad Y = Cy + D$$

Since $u(x, y) = X(x)\cdot Y(y)$ is the solution of Eq. (1), the three mathematically possible solutions of Eq. (1) are

$$u(x, y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py) \tag{6}$$
$$u(x, y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \tag{7}$$

and $$u(x, y) = (Ax + B)(Cy + D) \tag{8}$$

3C.4 CHOICE OF PROPER SOLUTION

Out of the three mathematically possible solutions derived, we have to choose that solution which is consistent with the given boundary conditions. We have already observed that Laplace equation represents steadystate heat flow in two dimensional plates in the form of rectangles or squares whose sides are parallel to the coordinate axes, that is, whose sides are $x = 0$, $x = a$, $y = 0$ and $y = b$.

Laplace equation is readily solvable, that is, the arbitrary constants in the solutions can be easily found out, if three of the boundary values (either temperatures or gradients) prescribed on any three sides of the rectangle are zero each and the fourth boundary value is non-zero.

If the non-zero boundary value is prescribed either on $x = 0$ or on $x = a$ (in which y is varying), that solution in which periodic functions in y occur will be the proper solution. That is, (6) will be the proper solution. It can be verified in individual problems that solutions (7) and (8) become trivial in such situations.

If the non-zero boundary value is prescribed either on $y = 0$ or $y = b$ (in which x is varying), that solution in which periodic functions in x occur will be the proper solution. That is, (7) will be the proper solution. It can be verified in individual problems that solutions (6) and (8) become trivial in such situations.

Thus we cannot choose a single solution as the appropriate solution in all situations. Invariably, solution (8) need not be considered, as it will result in a trivial solution. Solution (6) or (7) will be the suitable solution, according as non-zero boundary value is prescribed on the side $x = k$ or $y = k$.

Worked Examples 3C

PROBLEMS ON TEMPERTURE DISTRIBUTION IN VERY LONG PLATES

Example 1

A rectangular plate with insulated surfaces is a cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the two long edges $x = 0$ and $x = a$ and the short edge at infinity are kept at temperature 0°C, while the other short edge $y = 0$ is kept at temperature (i) $u_0 \sin^3 \dfrac{\pi x}{a}$ and (ii) T (constant). Find the steadystate temperature at any point (x, y) of the plate (Fig. 3C.2)

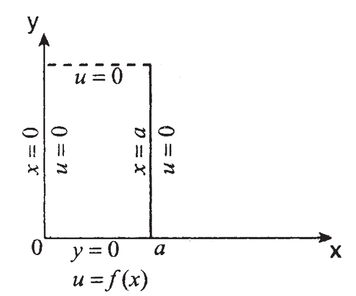


Fig. 3C.2

The temperature $u(x, y)$ at any point (x, y) of the plate in the steadystate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1) satisfying the following boundary conditions.

$$u(0, y) = 0, \qquad \text{for all } y > 0 \tag{2}$$

$$u(a, y) = 0, \qquad \text{for all } y > 0 \tag{3}$$

$$u(x, \infty) = 0, \qquad \text{for } 0 \le x \le a \tag{4}$$

$$u(x, 0) = f(x), \quad \text{for } 0 \le x \le a \tag{5}$$

where $f(x) = u_0 \sin^3 \dfrac{\pi x}{a}$ for (i) and $f(x) = T$ for (ii).

The three possible solutions of Eq. (1) are

$$u(x, y) = (Ae^{px} + Be^{-px})(C \cos py + D \sin py) \tag{6}$$

$$u(x, y) = (A \cos px + B \sin px)(Ce^{py} + De^{-py}) \tag{7}$$

and

$$u(x, y) = (Ax + B)(Cy + D) \tag{8}$$

By boundary condition (4), $u \to 0$ when $y \to \infty$. Of the three possible solutions, only Solution (7) can satisfy this condition. Hence we reject the other two solutions.

Rewriting (7), we have

$$u(x, y)e^{-py} = (A \cos px + B \sin px)(C + De^{-2py}) \tag{7$'$}$$

Using boundary condition (4) in (7)$'$, we have $(A \cos px + B \sin px)C = 0$, for $0 \leq x \leq a$.

$$\therefore \qquad C = 0$$

Using boundary condition (2) in (7), we have

$$A.D.e^{-py} = 0, \quad \text{for all} \quad y > 0$$

$$\therefore \qquad \text{Either } A = 0 \text{ or } D = 0$$

If we assume that $D = 0$, we get a trivial solution.

$$\therefore \qquad A = 0$$

Using boundary condition (3) in (7), we have

$$B \sin pa.De^{-py} = 0, \quad \text{for all } y > 0$$

The assumption that $B = 0$ leads to a trivial solution.

$$\therefore \qquad \sin pa = 0$$

$$\therefore \qquad pa = n\pi \qquad \text{or} \quad p = \frac{n\pi}{a}$$

where $n = 0, 1, 2, \ldots \infty$.

Using these values of A, C and p in (7), it reduces to

$$u(x, y) = \lambda \sin \frac{n\pi x}{a} \cdot e^{-n\pi y/a} \tag{9}$$

where $n = 0, 1, 2, \ldots, \infty$.

The most general solution of Eq. (1) [got by superposing all the solutions in (9) except the one corresponding to $n = 0$] is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{a} e^{\frac{-n\pi y}{a}} \tag{10}$$

Using boundary condition (5) in (10), we have

$$\sum \lambda_n \sin \frac{n\pi x}{a} = f(x) \quad \text{in} \quad (0, a) \tag{11}$$

(i) $f(x) = u_0 \sin^3 \dfrac{\pi x}{a}$

$$= \frac{u_0}{4}\left(3 \sin \frac{\pi x}{a} - \sin \frac{3\pi x}{a}\right)$$

Using this form of $f(x)$ in (11) and comparing like terms, we get

$$\lambda_1 = \frac{3u_0}{4}, \lambda_3 = -\frac{u_0}{4} \quad \text{and} \quad \lambda_2 = 0 = \lambda_4 = \lambda_5 = \ldots$$

Using these values of λ_n in (10), the required solution is

$$u(x, y) = \frac{3u_0}{4} \sin \frac{\pi x}{a} e^{-\pi y/a} - \frac{u_0}{4} \sin \frac{3\pi x}{a} e^{-3\pi y/a}$$

(ii) $f(x) = T$ in $(0, a)$

Let the Fourier half-range sine series of $f(x)$ in $(0, a)$ be $\sum\limits_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{a}$.

Using this form of $f(x)$ in (11) and comparing like terms, we get

$$\lambda_n = b_n = \frac{2}{a}\int_0^a T \sin \frac{n\pi x}{a} dx$$

$$= \frac{2T}{a}\left(-\frac{\cos \frac{n\pi x}{a}}{\frac{n\pi}{a}}\right)_0^a$$

$$= \frac{2T}{n\pi}\left\{1 - (-1)^n\right\}$$

$$= \begin{cases} \dfrac{4T}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using this value of λ_n in (10), the required solution is

$$u(x, y) = \frac{4T}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{a} \exp\{-(2n-1)\pi y/a\}$$

Example 2

An infinitely long metal plate in the form of an area is enclosed between the lines $y = 0$ and $y = \pi$ for positive values of x. The temperature is zero along the edges $y = 0$, $y = \pi$ and the edge at infinity. If the edge $x = 0$ is kept at temperature ky, find the steadystate temperature distribution in the plate (Fig. 3C.3)

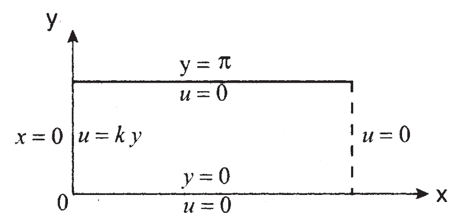


Fig. 3C.3

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1) satisfying the following boundary conditions.

$$u(x, 0) = 0, \quad \text{for all } x > 0 \tag{2}$$

$$u(x, \pi) = 0, \quad \text{for all } x > 0 \tag{3}$$

$$u(\infty, y) = 0, \quad \text{for } 0 \le y \le \pi \tag{4}$$

$$u(0, y) = ky, \quad \text{for } 0 \le y \le \pi \tag{5}$$

Of the three possible solutions of Eq. (1), the solution

$$u(x, y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py) \tag{6'}$$

can satisfy the boundary condition (4). Rewriting (6), we have

$$u(x, y)e^{-px} = (A + Be^{-2px})(C\cos py + D\sin py) \tag{6)'}$$

Using boundary condition (4) in (6'), we have

$$A(C\cos py + D\sin py) = 0, \quad \text{for} \quad 0 \le y \le \pi$$

$$\therefore \qquad A = 0$$

Using boundary condition (2) in (6), we have

$$B.Ce^{-px} = 0, \quad \text{for all} \quad x > 0$$

$$\therefore \qquad \text{Either } B = 0 \text{ or } C = 0$$

If we assume that $B = 0$, we get a trivial solution.

$$\therefore \qquad C = 0$$

Using boundary condition (3) in (6), we have

$$Be^{-px} \cdot D \sin p\pi = 0, \quad \text{for all} \quad x > 0$$

$$\therefore \quad B = 0,\ D = 0 \text{ or } \sin p\pi = 0$$

The values $B=0$ and $D=0$ lead to trivial solution.

$$\therefore \quad \sin p\pi = 0$$

$$\therefore \quad p = n$$

where $\quad n = 0, 1, 2, 3, \ldots \infty.$

Using these values of A, C and p in (6), it reduces to

$$u(x, y) = \lambda e^{-nx} \cdot \sin ny$$

where $\quad n = 0, 1, 2, \ldots, \infty$

Therefore the most general solution of Eq. 1 is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_n e^{-nx} \sin ny \tag{7}$$

Using boundary condition (5) in (7), we have

$$\sum_{n=1}^{\infty} \lambda_n \sin ny = ky \text{ in } (0, \pi)$$

$$= \sum b_n \sin ny$$

which is the Fourier half-range sine series of ky in $(0, \pi)$. Comparing like terms in the two series, we get

$$\lambda_n = b_n = \frac{2}{\pi} \int_0^{\pi} ky \sin ny \, dy$$

$$= \frac{2k}{\pi} \left[y \left(\frac{-\cos ny}{n} \right) - \left(\frac{-\sin ny}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2k}{n} (-1)^{n+1}$$

Using this value of λ_n in (7), the required solution is

$$u(x, y) = 2k \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-nx} \sin ny$$

Example 3

A long rectangular plate with insulated surface is l cm wide. If the temperature along one short edge $(y=0)$ is $u(x,0)=k(lx-x^2)$ degrees, for $0<x<l$, while the two long edges $x=0$ and $x=l$ as well as the other short edge are kept of 0°C, find the steadystate temperature function $u(x,y)$.

The steadystate temperature $u(x,y)$ at any point (x,y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(0,y)=0, \qquad \text{for all } y>0 \tag{2}$$

$$u(l,y)=0, \qquad \text{for all } y>0 \tag{3}$$

$$u(x,\infty)=0, \qquad \text{for } 0\le x\le l \tag{4}$$

$$u(x,0)=k(lx-x^2), \quad \text{for } 0\le x\le l \tag{5}$$

Proceeding as in Example 1, the most general solution of Eq. (1) can be obtained as

$$u(x,y)=\sum_{n=1}^{\infty}\lambda_n \sin\frac{n\pi x}{l}\cdot e^{-n\pi y/l} \tag{6}$$

Using boundary condition (5) in (6), we have

$$\sum_{n=1}^{\infty}\lambda_n \sin\frac{n\pi x}{l}=k(lx-x^2) \text{ in } (0,l)$$

$$=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

which is the Fourier half-range sine series of $k(lx-x^2)$ in $(0,l)$.
Comparing like terms, we have

$$\lambda_n=b_n=\frac{2}{l}\int_0^l k(lx-x^2)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$=\frac{2k}{l}\left[(lx-x^2)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(l-2x)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right.$$

$$\left.+(-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{4kl^2}{n^3\pi^3}\{1-(-1)^n\}$$

$$= \begin{cases} \dfrac{8kl^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using this value of λ_n in (6), the required solution is

$$u(x, y) = \frac{8kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\exp\{-(2n-1)\pi y/l\}$$

Example 4

A rectangular plate with insulated surfaces is 20 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $x = 0$ is given by

$$\begin{aligned} u &= 10y, && \text{for } 0 \le y \le 10 \\ &= 10(20-y), && \text{for } 10 \le y \le 20 \end{aligned}$$

and the two long edges as well as the other short edge are kept at 0°C, find the steadystate temperature distribution in the plate.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(x, 0) = 0, \quad \text{for all } x > 0 \tag{2}$$

$$u(x, 20) = 0, \quad \text{for all } x > 0 \tag{3}$$

$$u(\infty, y) = 0, \quad \text{for } 0 \le y \le 20 \tag{4}$$

$$u(0, y) = f(y), \quad \text{for } 0 \le y \le 20 \tag{5}$$

where $$f(y) = \begin{cases} 10y, & \text{in } 0 \le y \le 10 \\ 10(20-y), & \text{in } 10 \le y \le 20 \end{cases}$$

Proceeding as in Example 2, the most general solution of Eq. (1) can be obtained as

$$u(x, y) = \sum_{n=1}^{\infty}\lambda_n e^{-n\pi x/20}\sin\frac{n\pi y}{20} \tag{6}$$

Using boundary condition (5) in (6), we have in $(0, 20) = \sum b_n \sin\dfrac{n\pi y}{20}$

$$\sum_{n=1}^{\infty}\lambda_n \sin\frac{n\pi y}{20} = f(y)$$

which is the Fourier half-range sine series of $f(y)$ in $(0, 20)$.

Comparing like terms, we get

$$\lambda_n = b_n = \frac{2}{20}\int_0^{20} f(y)\sin\frac{n\pi y}{20}dy$$

$$= \int_0^{10} y\sin\frac{n\pi y}{20}dy + \int_{10}^{20}(20-y)\sin\frac{n\pi y}{20}dy$$

$$= \left[\left\{y\left(\frac{-\cos\frac{n\pi y}{20}}{\frac{n\pi}{20}}\right) - \left(\frac{-\sin\frac{n\pi y}{20}}{\frac{n^2\pi^2}{20^2}}\right)\right\}_0^{10}\right.$$

$$\left. + \left\{(20-y)\left(\frac{-\cos\frac{n\pi y}{20}}{\frac{n\pi}{20}}\right) - (-1)\left(\frac{-\sin\frac{n\pi y}{20}}{\frac{n^2\pi^2}{20^2}}\right)\right\}_{10}^{20}\right]$$

$$= \left[\left\{-\frac{200}{n\pi}\cos\frac{n\pi}{2} + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right.$$

$$\left. + \left\{\frac{200}{n\pi}\cos\frac{n\pi}{2} + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right]$$

$$= \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Using this value of λ_n in (6), the required solution is

$$u(x, y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\exp(-n\pi x/20)\sin\frac{n\pi y}{20} \text{ or}$$

$$u(x, y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\exp\{-(2n-1)\pi x/20\}\sin\frac{(2n-1)\pi y}{20}$$

Example 5

A plate is in the form of the semi-infinite strip $0 \le x \le l, 0 \le y < \infty$. The edges $x = 0$ and $x = l$ are insulated. The edge $y = 0$ is kept at temperature

(i) $2\cos\frac{3\pi x}{l} + 3\cos\frac{4\pi x}{l}$ and (ii) $kx, 0 \le x \le l$. Find the steadystate temperature distribution in the plate (Fig. 3C.4).

The temperature $u(x, y)$ at any point (x, y) of the plate in the steadystate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1) satisfying the following boundary conditions.

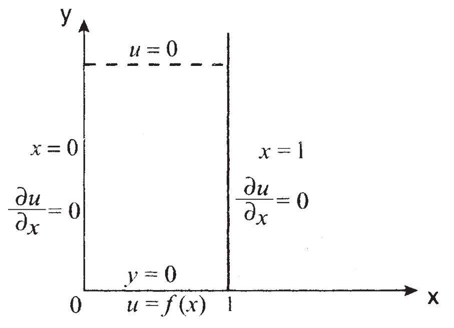


Fig. 3C.4

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \text{for all } y > 0 \tag{2}$$

$$\frac{\partial u}{\partial x}(l, y) = 0, \quad \text{for all } y > 0 \tag{3}$$

$$u(x, \infty) = 0, \quad \text{for } 0 \le x \le l \tag{4}$$

$$u(x, 0) = f(x), \quad \text{for } 0 \le x \le l \tag{5}$$

where $f(x) = 2\cos\dfrac{3\pi x}{l} + 3\cos\dfrac{4\pi x}{l}$ for (i) and

$f(x) = kx$ for (ii)

Note ✍

When an edge is insulated, the temperature gradient at all points on that edge is zero, that is, the derivative of u with respect to the variable along the perpendicular to that edge is zero.

Though the boundary condition in the edge at infinity is not specified, we assume that the temperature in that edge is kept at zero.

Of the three mathematically possible solutions of Eq. (1), the solution

$$u(x, y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \tag{6}$$

is the proper solution, as it alone can satisfy the boundary condition (4).

Rewriting (6), we have

$$u(x, y)e^{-py} = (A\cos px + B\sin px)(C + De^{-2py}) \tag{6$'$}$$

Using boundary condition (4) in (6)$'$, we have

$$C = 0$$

Using this value of C in (6), it reduces to

$$u(x, y) = (A\cos px + B\sin px)De^{-py} \tag{7}$$

Differentiating (7) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, y) = p(-A \sin px + B \cos px)De^{-py} \tag{8}$$

Using boundary condition (2) in (8), we have $p.B.D\, e^{-py} = 0$, for all $y > 0$.

$$\therefore \qquad p = 0 \text{ or } B = 0 \text{ or } D = 0$$

If we assume that $p = 0$ and $D = 0$, we get trivial solutions.

$$\therefore \qquad B = 0$$

Using boundary condition (3) in (8), we have

$$-p.A \sin pl.De^{-py} = 0, \text{ for all } y > 0$$

The values $p = 0$, $A = 0$ and $D = 0$ lead to trivial solutions.

$$\therefore \qquad \sin pl = 0$$

$$\therefore \qquad pl = n\pi \text{ or } p = \frac{n\pi}{l}, \text{ where } n = 0, 1, 2, \ldots, \infty.$$

Using the values of B, C and p in (6), it reduces to

$$u(x, y) = \lambda \cos \frac{n\pi x}{l} e^{-n\pi y/l}, \text{ where } \lambda = AD \text{ and } n = 0, 1, 2, \ldots \infty.$$

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=0}^{\infty} \lambda_n \cos \frac{n\pi x}{l} e^{-n\pi y/l} \tag{9}$$

Note ✍

The solution corresponding to $n = 0$ is non-trivial and hence it is to be included in the general solution.

Using boundary condition (5) in (9), we have

$$\sum_{n=0}^{\infty} \lambda_n \cos \frac{n\pi x}{l} = f(x) \text{ in } 0 \leq x \leq l \tag{10}$$

(i) $f(x) = 2 \cos \frac{3\pi x}{l} + 3 \cos \frac{4\pi x}{l}$

Using this value of $f(x)$ in (10) and comparing like terms, we get

$$\lambda_3 = 2, \lambda_4 = 3 \text{ and } \lambda_0 = 0 = \lambda_1 = \lambda_2 = \lambda_5 = \lambda_6 = \ldots$$

Using these values of λ_n in (9), the required solution is

$$u(x, y) = 2 \cos \frac{3\pi x}{l} e^{-3\pi y/l} + 3 \cos \frac{4\pi x}{l} e^{-4\pi y/l}$$

(ii) $f(x) = kx \text{in}(0, l)$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}, \text{ which is the Fourier half-range cosine series of } kx$$

in $(0, l)$.

Using this form of $f(x)$ in (10) and comparing like terms, we get

$$\lambda_n = a_n = \frac{2}{l}\int_0^l kx \cos \frac{n\pi x}{l} dx$$

$$= \frac{2k}{l}\left[x\left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \left(-\frac{\cos \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2kl}{n^2\pi^2}\left\{(-1)^n - 1\right\}$$

$$= \begin{cases} -\frac{4kl}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Also $\lambda_0 = \frac{a_0}{2} = \frac{1}{2} \times \frac{2}{l}\int_0^l kx dx = \frac{kl}{2}$.

Using these values of λ_0 and λ_n in (9), the required solution is

$$u(x, y) = \frac{kl}{2} - \frac{4kl}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} \cdot \exp\{-(2n-1)\pi y/l\}$$

Example 6

A plate is in the form of the semi-infinite strip $0 \leq x < \infty, 0 \leq y \leq l$. The surface of the plate and the edge $y = l$ are insulated. If the temperatures along the edge $y = 0$ and the short edge at infinity are kept at temperature 0°C, while the temperature along the other short edge is kept at temperature T°C, find the steadystate temperature distribution in the plate (Fig. 3C.5).

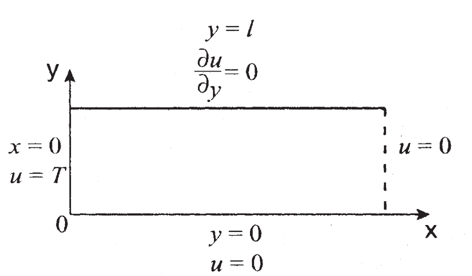


Fig. 3C.5

The temperature $u(x, y)$ at any point (x, y) of the plate in the steadystate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(x, 0) = 0, \quad \text{for all } x > 0 \tag{2}$$

$$\frac{\partial u}{\partial y}(x, l) = 0, \quad \text{for all } x > 0 \tag{3}$$

$$u(\infty, y) = 0, \quad \text{for } 0 \le y \le l \tag{4}$$

$$u(0, y) = T, \quad \text{for } 0 \le y \le l \tag{5}$$

As $u(x, y) = 0$ when $x \to \infty$, as per boundary condition (4), the proper solution of Eq. (1) is

$$u(x, y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py) \tag{6}$$

Rewriting (6), we have

$$u(x, y) \cdot e^{-px} = (A + Be^{-2px})(C\cos py + D\sin py) \tag{6$'$}$$

Using boundary condition (4) in (6)$'$, we have

$$A \cdot (C\cos py + D\sin py) = 0, \text{ for } 0 \le y \le l$$

$$\therefore \qquad A = 0$$

Using this value of A in (6), it reduces to

$$u(x, y) = Be^{-px}(C\cos py + D\sin py) \tag{7}$$

Using boundary condition (2) in (7), we have

$$Be^{-px} \cdot C = 0, \text{ for all } x > 0$$

$$\therefore \qquad \text{Either } B = 0 \text{ or } C = 0$$

If we assume that $B = 0$, we get a trivial solution.

$$\therefore \qquad C = 0$$

Using this value of C in (7), it reduces to

$$u(x, y) = BDe^{-px}\sin py \tag{8}$$

Differentiating (8) partially with respect to y, we get

$$\frac{\partial u}{\partial y}(x, y) = BDp.e^{-px}.\cos py \tag{8$'$}$$

Using boundary condition (3) in (8)′, we have

$$BDpe^{-px} \cos pl = 0$$

As $B \neq 0$, $D \neq 0$, $p \neq 0$, $\cos pl = 0$

$$\therefore \qquad pl = (2n - 1)\frac{\pi}{2}$$

$$\therefore \qquad p = \frac{(2n-1)\pi}{2l}$$

where $\qquad n = 1, 2, 3, \ldots \infty.$

Using this value of p in (8), it becomes

$$u(x, y) = \lambda e^{-(2n-1)\pi x/2l} \cdot \sin \frac{(2n-1)\pi y}{2l}$$

where $n = 1, 2, 3, \ldots \infty$.

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_{2n-1} e^{-(2n-1)\pi x/2l} \cdot \sin \frac{(2n-1)\pi y}{2l} \tag{9}$$

Using boundary condition (5) in (9), we have

$$\sum_{n=1}^{\infty} \lambda_{2n-1} \sin \frac{(2n-1)\pi y}{2l} = T \text{ in } (0, l)$$

$$\therefore \qquad \lambda_{2n-1} = \frac{2}{l} \int_0^l T \sin \frac{(2n-1)\pi y}{2l} dy$$

$$= \frac{2T}{l} \left[-\frac{\cos \frac{(2n-1)\pi y}{2l}}{\frac{(2n-1)\pi}{2l}} \right]_0^l$$

$$= \frac{4T}{(2n-1)\pi}$$

Using this value of λ_{2n-1} in (9), the required solution is

$$u(x, y) = \frac{4T}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \exp\{-(2n-1)\pi x/2l\} \sin \frac{(2n-1)\pi y}{2l}$$

PROBLEMS ON TEMPERTURE DISTRIBUTION IN FINITE PLATES

Example 7

Find the steadystate temperature distribution in a rectangular plate of sides a and b, which is insulated on the lateral surface and three of whose edges $x = 0$, $x = a$, $y = b$ are kept at zero temperature, if the temperature in the edge $y = 0$ is (i) $3 \sin \frac{2\pi x}{a} + 2 \sin \frac{3\pi x}{a}$ and (ii) $kx(a - x)$ (Fig. 3C.6).

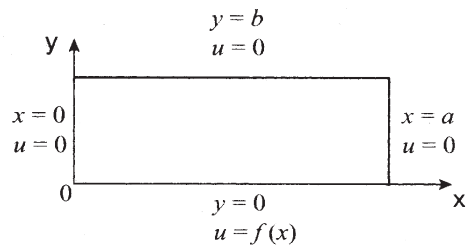


Fig. 3C.6

The temperature $u(x, y)$ at any point (x, y) of the plate in the steadystate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(0, y) = 0, \quad \text{for } 0 \le y \le b \qquad (2)$$

$$u(a, y) = 0, \quad \text{for } 0 \le y \le b \qquad (3)$$

$$u(x, b) = 0, \quad \text{for } 0 \le x \le a \qquad (4)$$

$$u(x, 0) = f(x), \quad \text{for } 0 \le x \le a \qquad (5)$$

where $f(x) = 3 \sin \frac{2\pi x}{a} + 2 \sin \frac{3\pi x}{a}$ for (i) and $f(x) = kx(a - x)$ for (ii)

The three mathematically possible solutions of Eq. (1) are

$$u(x, y) = (Ae^{px} + Be^{-px})(C \cos py + D \sin py) \qquad (6)$$

$$u(x, y) = (A \cos px + B \sin px)(Ce^{py} + De^{-py}) \qquad (7)$$

$$u(x, y) = (Ax + B)(Cy + D) \qquad (8)$$

Using boundary conditions (2) and (3) in solution (6), we get

$$A + B = 0$$

and $$Ae^{pa} + Be^{-pa} = 0$$

Solving these equations we get $A = 0 = B$, which lead to a trivial solution. Similarly, we will get a trivial solution if we use the boundary conditions in (8). Hence the suitable solution for the present problem is solution (7).

Note ✍

This conclusion is in accordance with the discussion on the choice of proper solution seen already.

Using boundary condition (2) in (7), we have

$$A(Ce^{py} + De^{-py}) = 0, \text{ for } 0 \le y \le b$$

$$A = 0$$

Using boundary condition (3) in (7), we have

$$B \sin pa(Ce^{py} + De^{-py}) = 0, \quad \text{for } 0 \leq y \leq b$$

$\therefore$ Either $B = 0$ or $\sin pa = 0$

If B is taken as zero, we get a trivial solution

$$\therefore \quad \sin pa = 0$$

$$\therefore \quad pa = n\pi \text{ or } p = \frac{n\pi}{a}$$

where $n = 0, 1, 2, \ldots \infty$

Using boundary condition (4) in (7), we have

$$B \sin px(Ce^{pb} + De^{-pb}) = 0, \quad \text{for} \quad 0 \leq x \leq a$$

As $B \neq 0$, $\quad Ce^{pb} + De^{-pb} = 0$

or $\quad D = -Ce^{2pb}$

Using these values of A, D and p in (7), it reduces to

$$\begin{aligned}
u(x, y) &= BC \sin \frac{n\pi x}{a} \left\{ e^{\frac{n\pi y}{a}} - e^{\frac{2n\pi b}{a}} \cdot e^{-\frac{n\pi y}{a}} \right\} \\
&= \left(BCe^{n\pi b/a} \right) \sin \frac{n\pi x}{a} \left\{ e^{\frac{n\pi}{a}(y-b)} - e^{-\frac{n\pi}{a}(y-b)} \right\} \\
&= \lambda_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi}{a}(y - b) \text{ where } n = 0, 1, 2, \ldots \infty \text{ and}
\end{aligned}$$

$$\lambda_n = 2BCe^{n\pi b/a}$$

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi}{a}(y - b) \tag{9}$$

Using boundary condition (5) in (9), we have

$$\sum_{n=1}^{\infty} \left(-\lambda_n \sinh \frac{n\pi b}{a} \right) \sin \frac{n\pi x}{a} = f(x) \tag{10}$$

(i) $f(x) = 3 \sin \dfrac{2\pi x}{a} + 2 \sin \dfrac{3\pi x}{a}$

Using this value of $f(x)$ in (10) and comparing like terms, we get

$$-\lambda_2 \sinh \frac{2\pi b}{a} = 3, \; -\lambda_3 \sinh \frac{3\pi b}{a} = 2 \text{ and } \lambda_1 = \lambda_4 = \lambda_5 = \ldots = 0$$

Using these values of λ_n in (9), the required solution is

$$u(x,y) = -3\,\text{cosech}\frac{2\pi b}{a}\sin\frac{2\pi x}{a}\sinh\frac{2\pi}{a}(y-b)$$
$$-2\,\text{cosech}\frac{3\pi b}{a}\sin\frac{3\pi x}{a}\sinh\frac{3\pi}{a}(y-b)$$

or
$$u(x,y) = 3\,\text{cosech}\frac{2\pi b}{a}\sin\frac{2\pi x}{a}\sinh\frac{2\pi}{a}(b-y)$$
$$+2\,\text{cosech}\frac{3\pi b}{a}\sin\frac{3\pi x}{a}\sinh\frac{3\pi}{a}(b-y)$$

(ii) $f(x) = kx(a-x)$ in $(0,a)$

Let the Fourier half-range sine series of

$$f(x) \text{ in } (0,a) \text{ be } \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}$$

Using this form of $f(x)$ in (10) and comparing like terms, we get

$$-\lambda_n \sinh\frac{n\pi b}{a} = b_n = \frac{2}{a}\int_0^a kx(a-x)\sin\frac{n\pi x}{a}\,dx$$

$$= \frac{2k}{a}\left[(ax-x^2)\left(-\frac{\cos\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right) - (a-2x)\left(-\frac{\sin\frac{n\pi x}{a}}{\frac{n^2\pi^2}{a^2}}\right)\right.$$
$$\left. +(-2)\left(\frac{\cos\frac{n\pi x}{a}}{\frac{n^3\pi^3}{a^3}}\right)\right]_0^a$$

$$= \frac{4ka^2}{n^3\pi^3}\left\{1-(-1)^n\right\}$$

$$\lambda_n = \begin{cases} -\dfrac{8ka^2}{n^3\pi^3}\,\text{cosech}\dfrac{n\pi b}{a}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Using this value of λ_n in (9), the required solution is

$$u(x,y) = \frac{8ka^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\text{cosech}\frac{(2n-1)\pi b}{a}\cdot\sin\frac{(2n-1)\pi x}{a}$$
$$\sinh\frac{(2n-1)\pi(b-y)}{a}$$

Example 8

A square plate of length 20 cm has its faces insulated and its edges along $x=0$, $x=20$, $y=0$ and $y=20$. If the temperature along the edge $x=20$ is given by

$$u = \frac{T}{10}y, \qquad \text{for } 0 \le y \le 10$$
$$= \frac{T}{10}(20-y), \quad \text{for } 10 \le y \le 20$$

while the other three edges are kept at 0°C, find the steadystate temperature distribution in the plate (Fig. 3C.7).

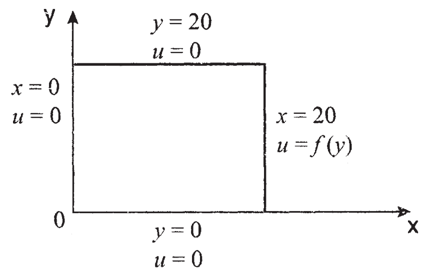


Fig. 3C.7

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(x, 0) = 0, \qquad \text{for } 0 \le x \le 20 \tag{2}$$
$$u(x, 20) = 0, \qquad \text{for } 0 \le x \le 20 \tag{3}$$
$$u(0, y) = 0, \qquad \text{for } 0 \le y \le 20 \tag{4}$$
$$u(20, y) = f(y), \qquad \text{for } 0 \le y \le 20 \tag{5}$$

Since non-zero temperature is prescribed on the edge $x = 20$ in which y is varying, the proper solution of Eq. (1) is

$$u(x, y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py) \tag{6}$$

Using boundary condition (2) in (6), we have

$$(Ae^{px} + Be^{-px})C = 0, \text{ for } 0 \le x \le 20$$
$$\therefore \qquad C = 0$$

Using boundary condition (3) in (6), we have

$$(Ae^{px} + Be^{-px}) \cdot D\sin 20p = 0, \text{ for } 0 \le x \le 20$$
$$\therefore \qquad \text{Either } D = 0 \text{ or } \sin 20p = 0$$

If $D = 0$, we get a trivial solution.

$\therefore$ $\sin 20p = 0$

$\therefore$ $20p = n\pi$ or $p = \dfrac{n\pi}{20}$

where $n = 0, 1, 2, \ldots \infty.$

Using boundary condition (4) in (6), we have

$$(A + B)D \sin py = 0, \quad \text{for} \quad 0 \le y \le 20$$

As $D \neq 0$, $A + B = 0$ or $B = -A$

Using these values of B, C and p in (6), it reduces to

$$u(x, y) = AD\left(e^{\frac{n\pi x}{20}} - e^{-\frac{n\pi x}{20}}\right)\sin\frac{n\pi y}{20}$$

or $$u(x, y) = \lambda \sinh\frac{n\pi x}{20}\sin\frac{n\pi y}{20}, \text{ where } n = 0, 1, 2, \ldots \infty.$$

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_n \sinh\frac{n\pi x}{20}\sin\frac{n\pi y}{20} \tag{7}$$

Using boundary condition (5) in (7), we have

$$\sum_{n=1}^{\infty}(\lambda_n \sinh n\pi)\sin\frac{n\pi y}{20} = f(y) \text{ in } (0, 20) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{20}$$

which is the Fourier half-range sine series of $f(y)$ in (0, 20).

Comparing like terms, we get

$$\begin{aligned}
\lambda_n \sinh n\pi = b_n &= \frac{2}{20}\int_0^{20} f(y)\sin\frac{n\pi y}{20}\,dy \\
&= \frac{T}{100}\left[\int_0^{10} y \sin\frac{n\pi y}{20}\,dy + \int_{10}^{20}(20 - y)\sin\frac{n\pi y}{20}\,dy\right] \\
&= \frac{T}{100}\left[\left\{y\left(-\frac{\cos\frac{n\pi y}{20}}{\frac{n\pi}{20}}\right) - \left(-\frac{\sin\frac{n\pi y}{20}}{\frac{n^2\pi^2}{20^2}}\right)\right\}_0^{10}\right. \\
&\quad \left. + \left\{(20 - y)\left(-\frac{\cos\frac{n\pi y}{20}}{\frac{n\pi}{20}}\right) - (-1)\left(-\frac{\sin\frac{n\pi y}{20}}{\frac{n^2\pi^2}{20^2}}\right)\right\}_{10}^{20}\right]
\end{aligned}$$

$$= \frac{T}{100}\left[\left(-\frac{200}{n\pi}\cos\frac{n\pi}{2} + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right.$$

$$\left. + \left(\frac{200}{n\pi}\cos\frac{n\pi}{2} + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{8T}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore \qquad \lambda_n = \frac{8T}{n^2\pi^2}\sin\frac{n\pi}{2}\operatorname{cosech} n\pi$$

Using this value of λ_n in (7), the required solution is

$$u(x, y) = \frac{8T}{\pi^2}\sum \frac{1}{n^2}\sin\frac{n\pi}{2}\operatorname{cosech} n\pi \sinh\frac{n\pi x}{20}\cdot\sin\frac{n\pi y}{20}$$

or
$$u(x, y) = \frac{8T}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\operatorname{cosech}(2n-1)\pi$$

$$\sinh\frac{(2n-1)\pi x}{20}\sin\frac{(2n-1)\pi y}{20}$$

Example 9

If a square plate is bounded by the lines $x = \pm a$ and $y = \pm a$ and three of its edges are kept at temperature 0°C, while the temperature along the edge $y = a$ is kept at $u = x + a$, $-a \le x \le a$, find the steadystate temperature in the plate (Fig. 3C.8).

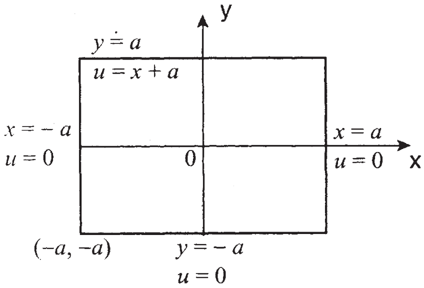


Fig. 3C.8

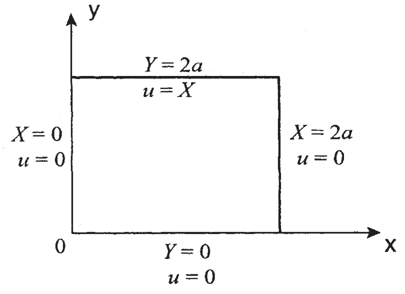


Fig. 3C.9

In Examples 7 and 8, we have observed that the arbitrary constants in the appropriate solution of the Laplace equation can be readily found out, only if two adjacent edges of the square (or rectangle) are taken as coordinate axes. As this condition is not satisfied in the present problem, we shift the origin to the point $(-a, -a)$, so that two adjacent edges may lie along the coordinate axes in the new system (Fig. 3C.9).

The transformation equations are

$$x = X - a$$
$$y = Y - a$$

The equations of the edges are $X = 0$, $X = 2a$, $Y = 0$ and $Y = 2a$ in the new system.

Let us work out the problem with reference to the new system. The steadystate temperature $u(X, Y)$ at any point (X, Y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(0, Y) = 0, \quad \text{for } 0 \le Y \le 2a \tag{2}$$
$$u(2a, Y) = 0, \quad \text{for } 0 \le Y \le 2a \tag{3}$$
$$u(X, 0) = 0, \quad \text{for } 0 \le X \le 2a \tag{4}$$
$$u(X, 2a) = X, \quad \text{for } 0 \le X \le 2a \tag{5}$$

Since non-zero temperature is prescribed on the edge $Y = 2a$, in which X is varying, the proper solution of equation (1) is

$$u(X, Y) = (A \cos pX + B \sin pX)(Ce^{pY} + De^{-pY}) \tag{6}$$

Using boundary conditions (2) in (6), we have

$$A(Ce^{pY} + De^{-pY}) = 0, \text{ for } 0 \le Y \le 2a$$

$\therefore$ $$A = 0$$

Using boundary condition (3) in (6), we have

$$B \sin 2pa(Ce^{pY} + De^{-pY}) = 0, \text{ for } 0 \le Y \le 2a$$

$\therefore$ Either $B = 0$ or $\sin 2pa = 0$

If $B = 0$, we get a trivial solution and so $B \neq 0$

$\therefore$ $\sin 2pa = 0$

$\therefore$ $2pa = n\pi$ or $p = \dfrac{n\pi}{2a}$, where $n = 0, 1, 2, \ldots \infty$.

Using boundary condition (4) in (6), we have

$$B \sin pX(C + D) = 0, \text{ for } 0 \le X \le 2a$$

As $B \neq 0$, we get $C + D = 0$ or $D = -C$.

Using these values of A, D and p in (6), it reduces to

$$u(X, Y) = BC \sin \frac{n\pi X}{2a} \left\{ e^{n\pi Y/2a} - e^{-n\pi Y/2a} \right\}$$

$$= \lambda \sin \frac{n\pi X}{2a} \sinh \frac{n\pi Y}{2a}, \text{ where } \lambda = 2BC$$

and $\quad n = 0, 1, 2, \ldots \infty.$

Therefore the most general solution of Eq. (1) is

$$u(X, Y) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi X}{2a} \sinh \frac{nxY}{2a} \tag{7}$$

Using boundary condition (5) in (7), we have

$$\sum_{n=1}^{\infty} (\lambda_n \sinh n\pi) \sin \frac{n\pi X}{2a} = X \text{ in } (0, 2a)$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi X}{2a}$$

which is the Fourier half-range sine series of X in $(0, 2a)$.

Comparing like terms in the two series, we get

$$\lambda_n \sinh n\pi = b_n = \frac{2}{2a} \int_0^{2a} X \sin \frac{n\pi X}{2a} \, dX$$

$$= \frac{1}{a} \left[X \left(\frac{-\cos \frac{n\pi X}{2a}}{\frac{n\pi}{2a}} \right) - \left(\frac{-\sin \frac{n\pi X}{2a}}{\frac{n^2\pi^2}{4a^2}} \right) \right]_0^{2a}$$

$$= -\frac{4a}{n\pi} \cos n\pi$$

$$\therefore \quad \lambda_n = \frac{4a}{n\pi} (-1)^{n+1} \operatorname{cosech} n\pi$$

Using this value of λ_n in (7), the required solution with reference to the new system is

$$u(X, Y) = \frac{4a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \operatorname{cosech} n\pi \cdot \sin \frac{n\pi X}{2a} \sinh \frac{n\pi Y}{2a}$$

With reference to the old system, the required solution is

$$u(x, y) = \frac{4a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \operatorname{cosech} n\pi \cdot \sin \frac{n\pi}{2a}(x + a) \cdot \sinh \frac{n\pi}{2a}(y + a)$$

Example 10

A rectangular plate is bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$. Its surfaces are insulated. The temperatures along $x=0$ and $y=0$ are kept at 0°C and the others at 100°C. Find the steadystate temperature at any point of the plate (Figs 3C.10, 3C.11 and 3C.12.

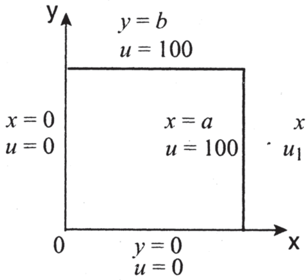


Fig. 3C.10

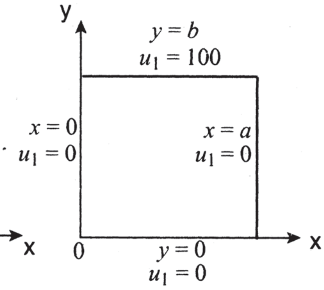


Fig. 3C.11

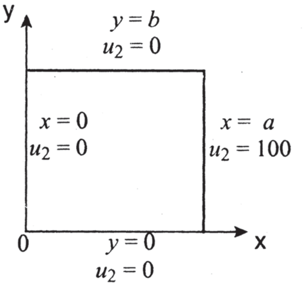


Fig. 3C.12

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

The corresponding boundary conditions are

$$u(0, y) = 0, \quad \text{for } 0 \le y \le b \tag{2}$$

$$u(a, y) = 100, \quad \text{for } 0 \le y \le b \tag{3}$$

$$u(x, 0) = 0, \quad \text{for } 0 < x < a \tag{4}$$

$$u(x, b) = 100, \quad \text{for } 0 < x < a \tag{5}$$

From previous examples, it is obvious that Eq. (1) is readily solvable, that is, the arbitrary constants in the proper solution of Eq. (1) can be easily found out, only if three of the boundary values (temperatures along three of the edges) are zero each and the fourth boundary value (temperature along the fourth edge) is non-zero.

As two boundary values are non-zero each in this problem, we adopt a slightly modified procedure as explained below.

Let
$$u(x, y) = u_1(x, y) + u_2(x, y) \tag{6}$$

Using (6) in (1), we get

$$\frac{\partial^2}{\partial x^2}(u_1 + u_2) + \frac{\partial^2}{\partial y^2}(u_1 + u_2) = 0$$

Separating the derivatives of u_1 and those of u_2 we have

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0 \tag{7}$$

and

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0 \tag{8}$$

We assume convenient boundary conditions for Eq.(7)[i.e. three zero and one non-zero boundary conditions] which are given below.

$$u_1(0, y) = 0, \quad \text{for } 0 \le y \le b \tag{9}$$

$$u_1(a, y) = 0, \quad \text{for } 0 \le y \le b \tag{10}$$

$$u_1(x, 0) = 0, \quad \text{for } 0 < x < a \tag{11}$$

$$u_1(x, b) = 100, \text{ for } 0 < x < a \tag{12}$$

The boundary conditions for Eq.(8) are obtained by using (6) and the boundary conditions (2), (3), (4), (5) for $u(x, y)$ and the boundary conditions (9), (10), (11), (12) for $u_1(x, y)$.

Thus

$$u_2(0, y) = 0, \quad \text{for } 0 \le y \le b \tag{13}$$

$$u_2(a, y) = 100, \quad \text{for } 0 \le y \le b \tag{14}$$

$$u_2(x, 0) = 0, \quad \text{for } 0 < x < a \tag{15}$$

$$u_2(x, b) = 0, \quad \text{for } 0 < x < a \tag{16}$$

The appropriate solution of Eq.(7) consistent with the given boundary conditions for $u_1(x, y)$ is

$$u_1(x, y) = (A \cos px + B \sin px)(Ce^{py} + De^{-py}) \tag{17}$$

Using boundary conditions (9), (10) and (11) in (17) and proceeding as in Example 9, the most general solution of Eq.(7) can be obtained as

$$u_1(x, y) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \tag{18}$$

Using boundary condition (12) in (18), we have

$$\sum_{n=1}^{\infty} \left(\lambda_n \sinh \frac{n\pi b}{a}\right) \sin \frac{n\pi x}{a} = 100, \text{ in } (0, a)$$

$$= \sum b_n \sin \frac{n\pi x}{a}$$

which is the Fourier half-range sine series of 100 in $(0, a)$.

Comparing like terms in the two series, we get

$$\lambda_n \sinh \frac{n\pi b}{a} = b_n = \frac{2}{a}\int_0^a 100 \sin \frac{n\pi x}{a} \mathrm{d}x$$

$$= \frac{200}{a}\left(\frac{-\cos \frac{n\pi x}{a}}{\frac{n\pi}{a}}\right)_0^a$$

$$= \frac{200}{n\pi}\left\{1-(-1)^n\right\}$$

$$= \begin{cases} \frac{400}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using this value of λ_n in (18), the required solution of Eq. (7) is

$$u_1(x, y) = \frac{400}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \operatorname{cosech} \frac{(2n-1)\pi b}{a} \sin \frac{(2n-1)\pi x}{a} \sinh \frac{(2n-1)\pi y}{a} \tag{19}$$

Now solving Eq. (8) subject to the boundary conditions (13), (14), (15) and (16) [proceeding as in Example 8] or by interchanging x and y and also a and b in (19), we get

$$u_2(x, y) = \frac{400}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \operatorname{cosech} \frac{(2n-1)\pi a}{b} \sinh \frac{(2n-1)\pi x}{b} \sin \frac{(2n-1)\pi y}{b} \tag{20}$$

Using (19) and (20) in (6), the required solution of Eq. (1) is

$$u(x, y) = \frac{400}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\left[\operatorname{cosech} \frac{(2n-1)\pi b}{a} \cdot \sin \frac{(2n-1)\pi x}{a}\right.$$
$$\left.\sinh \frac{(2n-1)\pi y}{a} + \operatorname{cosech} \frac{(2n-1)\pi a}{b} \cdot \sin \frac{(2n-1)\pi y}{b} \cdot \sinh \frac{(2n-1)\pi x}{b}\right]$$

Note ✍

If non-zero temperatures are prescribed on all the four sides of the rectangle (or square), the concept used in the previous example is extended by assuming that $u(x, y) = \sum_{r=1}^{4} u_r(x, y)$. *Three of the boundary values of each of the* *equations* $\frac{\partial^2 u_r}{\partial x^2}+\frac{\partial^2 u_r}{\partial y^2}=0$ *are assumed to be zero and the fourth one non-zero in such a way that we get the given boundary values of* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ *from those of* $\frac{\partial^2 u_r}{\partial x^2}+\frac{\partial^2 u_r}{\partial y^2}=0$ *by superposition.*

Example 11

A square plate has its faces and its edge $y=0$ insulated. Its edges $x=0$ and $x=10$ are kept at temperature zero and its edge $y=10$ at temperature 100°C. Find the steadystate temperature distribution in the plate.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(0, y)=0, \qquad \text{for } 0 \leq y \leq 10 \tag{2}$$

$$u(10, y)=0, \qquad \text{for } 0 \leq y \leq 10 \tag{3}$$

$$\frac{\partial u}{\partial y}(x, 0)=0, \qquad \text{for } 0 < x < 10 \tag{4}$$

$$u(x, 10)=100, \qquad \text{for } 0 < x < 10 \tag{5}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y)=(A\cos px+B\sin px)(Ce^{py}+De^{-py}) \tag{6}$$

Using boundary conditions (2) and (3) in (6), we can get, as usual,

$$A=0$$

and $$p=\frac{n\pi}{10}$$

where $$n=0, 1, 2, \ldots \infty.$$

Differentiating (6) partially with respect to y,

$$\frac{\partial u}{\partial y}(x, y)=Bp\sin px(Ce^{py}-De^{-py}) \tag{7}$$

Using boundary condition (4) in (7), we have

$$Bp\sin px(C-D)=0, \text{ for } 0 < x < 10$$

As $B \neq 0$ and $p \neq 0$, we get $D=C$

Using these values of A, D and p in (6), it reduces to

$$u(x, y) = BC \sin \frac{n\pi x}{10} \cdot \left(e^{n\pi y/10} + e^{-n\pi y/10}\right)$$
$$= \lambda \sin \frac{n\pi x}{10} \cosh \frac{n\pi y}{10}, \text{ where } \lambda = 2BC \text{ and } n = 0, 1, 2, \ldots \infty.$$

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{10} \cosh \frac{n\pi y}{10} \tag{8}$$

Using boundary condition (5) in (8), we have

$$\sum_{n=1}^{\infty} (\lambda_n \cosh n\pi) \sin \frac{n\pi x}{10} = 100 \text{ in } (0, 10)$$
$$= \sum b_n \sin \frac{n\pi x}{10}$$

which is the Fourier half-range sine series of 100 in (0, 10).
Comparing like terms in the two series, we get

$$\lambda_n \cosh n\pi = b_n = \frac{2}{10} \int_0^{10} 100 \sin \frac{n\pi x}{10} dx$$
$$= 20 \left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}} \right)_0^{10}$$
$$= \frac{200}{n\pi} \left\{1 - (-1)^n\right\}$$
$$= \begin{cases} \dfrac{400}{n\pi}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using this value of λ_n in (8), the required solution is

$$u(x, y) = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \operatorname{sech}(2n-1)\pi \cdot \sin \frac{(2n-1)\pi x}{10} \cdot \cosh \frac{(2n-1)\pi y}{10}$$

Example 12

A rectangular plate of sides 20 cm and 10 cm has its faces and the edge $x = 20$ insulated. Its edges $y = 0$ and $y = 10$ are kept at temperature zero, while the edge $x = 0$ is kept at temperature ky. Find the steadystate temperature distribution in the plate.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(x, 0) = 0, \quad \text{for } 0 \le x \le 20 \tag{2}$$

$$u(x, 10) = 0, \quad \text{for } 0 \le x \le 20 \tag{3}$$

$$\frac{\partial u}{\partial x}(20, y) = 0, \quad \text{for } 0 < y < 10 \tag{4}$$

$$u(0, y) = ky, \quad \text{for } 0 < y < 10 \tag{5}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py) \tag{6}$$

Using boundary conditions (2) and (3) in (6), we can get, as usual,

$$C = 0 \text{ and } p = \frac{n\pi}{10}, \text{ where } n = 0, 1, 2, \ldots \infty$$

Differentiating (6) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, y) = p(Ae^{px} - Be^{-px}) \cdot D\sin py \tag{7}$$

Using boundary condition (4) in (7), we have

$$p(Ae^{20p} - Be^{-20p})D\sin py = 0, \text{ for } 0 < y < 10$$

As $p \neq 0$ and $D \neq 0$, $Ae^{20p} - Be^{-20p} = 0$

$$\therefore \qquad B = Ae^{40p}$$

Using these values of B, C and p in (6), it reduces to

$$\begin{aligned} u(x, y) &= AD\left\{e^{n\pi x/10} + e^{40n\pi/10} \cdot e^{-n\pi x/10}\right\}\sin\frac{n\pi y}{10} \\ &= (2ADe^{2n\pi})\cosh\frac{n\pi(x-20)}{10} \cdot \sin\frac{n\pi y}{10} \\ &= \lambda_n \cosh\frac{n\pi(20-x)}{10} \cdot \sin\frac{n\pi y}{10} \end{aligned}$$

[$\because$ cosh θ is an even function]
where $n = 1, 2, 3, \ldots \infty$.

$\therefore$ The most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_n \cosh \frac{n\pi(20 - x)}{10} \sin \frac{n\pi y}{10} \tag{8}$$

Using boundary condition (5) in (8), we have

$$\sum_{n=1}^{\infty} (\lambda_n \cosh 2n\pi) \sin \frac{n\pi y}{10} = ky \text{ in } (0, 10)$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{10}$$

which is the Fourier half-range sine series of ky in (0, 10).
Comparing like terms in the two series, we have

$$\lambda_n \cosh 2n\pi = b_n = \frac{2}{10} \int_0^{10} ky \sin \frac{n\pi y}{10} dy$$

$$= \frac{2k}{10} \left[y \left(\frac{-\cos \frac{n\pi y}{10}}{\frac{n\pi}{10}} \right) - \left(\frac{-\sin \frac{n\pi y}{10}}{\frac{n^2\pi^2}{100}} \right) \right]_0^{10}$$

$$= \frac{20k}{n\pi} (-1)^{n+1}$$

Using this value of λ_n in (8), the required solution is

$$u(x, y) = \frac{20k}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \operatorname{sech} 2n\pi \cdot \cosh \frac{n\pi(20 - x)}{10} \sin \frac{n\pi y}{10}$$

Example 13

A square plate has its faces and its edges $x = 0$ and $x = a$ insulated. If the edge $y = a$ is kept at temperature zero, while the edge $y = 0$ is kept at temperature $4 \cos^3 \left(\frac{\pi x}{a} \right)$, find the steadystate temperature distribution in the plate.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$\frac{\partial u}{\partial x}(0, y) = 0, \qquad \text{for } 0 \le y \le a \qquad (2)$$

$$\frac{\partial u}{\partial x}(a, y) = 0, \qquad \text{for } 0 \le y \le a \qquad (3)$$

$$u(x, a) = 0, \qquad \text{for } 0 < x < a \qquad (4)$$

$$u(x, 0) = 4\cos^3\left(\frac{\pi x}{a}\right), \quad \text{for } 0 < x < a \qquad (5)$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (A\cos px + B\cos px)(Ce^{py} + De^{-py}) \qquad (6)$$

Differentiating (6) partially with respect to x,

$$\frac{\partial u}{\partial x}(x, y) = p(-A\sin px + B\cos px)(Ce^{py} + De^{-py}) \qquad (7)$$

Using boundary conditions (2) and (3) in (7), we can get, as usual,

$$B = 0 \text{ and } p = \frac{n\pi}{a}, \text{ where } n = 0, 1, 2, \ldots \infty$$

Using boundary conditions (4) in (6), we have
$A\cos px(Ce^{pa} + De^{-pa}) = 0$, for $0 < x < a$
As $A \neq 0$, $D = -Ce^{2pa}$.
Using these values of B, D and p in (6), we get

$$\begin{aligned} u(x, y) &= AC\cos\frac{n\pi x}{a}\left\{e^{n\pi y/a} - e^{2n\pi a/a} \cdot e^{-n\pi y/a}\right\} \\ &= (2ACe^{n\pi})\cos\frac{n\pi x}{a}\sinh\frac{n\pi(y-a)}{a} \end{aligned}$$

or
$$u(x, y) = \lambda_n \cos\frac{n\pi x}{a}\sinh\frac{n\pi(y-a)}{a}$$

where $n = 0, 1, 2, \ldots \infty$.
Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_n \cos\frac{n\pi x}{a}\sinh\frac{n\pi(y-a)}{a} \qquad (8)$$

Using boundary condition (5) in (8), we have

$$\sum_{n=1}^{\infty}(-\lambda_n \sinh n\pi)\cos\frac{n\pi x}{a} = 4\cos^3\frac{\pi x}{a} \text{in } (0, a)$$

$$= 3\cos\frac{n\pi}{a} + \cos\frac{3\pi x}{a}$$

Comparing like terms, we get

$$-\lambda_1 \sinh \pi = 3; -\lambda_3 \sinh 3\pi = 1; \lambda_2 = 0 = \lambda_4 = \lambda_5 = \dots$$

$$\therefore \quad \lambda_1 = -3 \operatorname{cosech} \pi; \lambda_3 = -\operatorname{cosech} 3\pi; \lambda_2 = 0 = \lambda_4 = \lambda_5 = \dots$$

Using these values in (8), the required solution is

$$u(x, y) = 3 \operatorname{cosech} \pi \cos\frac{\pi x}{a} \sinh\frac{\pi(a-y)}{a} + \operatorname{cosech} 3\pi \cos\frac{3\pi x}{a} \sinh\frac{3\pi(a-y)}{a}$$

Example 14

A rectangular plate of sides a and b has its faces and the edges $y = 0$ and $y = b$ insulated. If the edge $x = 0$ is kept at temperature zero, while the edge $x = a$ is kept at temperature $k(2y - b)$, find the steadystate temperature distribution in the plate.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve the Eq. (1), satisfying the following boundary conditions.

$$\frac{\partial u}{\partial y}(x, 0) = 0, \qquad \text{for } 0 \le x \le a \tag{2}$$

$$\frac{\partial u}{\partial y}(x, b) = 0, \qquad \text{for } 0 \le x \le a \tag{3}$$

$$u(0, y) = 0, \qquad \text{for } 0 < y < b \tag{4}$$

$$u(a, y) = k(2y - b), \text{ for } 0 < y < b \tag{5}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py) \tag{6}$$

Differentiating (6) partially with respect to y, we have

$$\frac{\partial u}{\partial y}(x, y) = (Ae^{px} + Be^{-px})p(-C\sin py + D\cos py) \tag{7}$$

Using boundary conditions (2) and (3) in (7), we can get, as usual,

$$D = 0 \text{ and } p = \frac{n\pi}{b}, \text{ where } n = 0, 1, 2, \ldots \infty$$

Using boundary condition (4) in (6), we have

$$(A + B)C \cos py = 0, \text{ for } 0 < y < b$$

As $C \neq 0$, we get $B = -A$.

Using these values of B, D and p in (6), it reduces to

$$u(x, y) = AC\left(e^{\frac{n\pi x}{b}} - e^{\frac{-n\pi x}{b}}\right)\cos\frac{n\pi y}{b}$$

$$= \lambda \sinh\frac{n\pi x}{b} \cdot \cos\frac{n\pi y}{b}$$

where $\lambda = 2AC$ and $n = 0, 1, 2, \ldots \infty$

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_n \sinh\frac{n\pi x}{b} \cos\frac{n\pi y}{b} \tag{8}$$

Using boundary condition (5) in (8) we have

$$\sum_{n=1}^{\infty} \lambda_n \sinh\frac{n\pi a}{b} \cdot \cos\frac{n\pi y}{b} = k(2y - b)\text{in}(0, b)$$

$$= \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi y}{b},$$

which is Fourier half-range cosine series of $k(2y - b)$ in $(0, b)$.

Comparing like terms in the two series, we get

$$\lambda_n \sinh\frac{n\pi a}{b} = a_n = \frac{2}{b}\int_0^b k(2y - b)\cos\frac{n\pi y}{b} dy$$

$$= \frac{2k}{b}\left[(2y - b)\left(\frac{\sin\frac{n\pi y}{b}}{\frac{n\pi}{b}}\right) - 2\left(\frac{-\cos\frac{n\pi y}{b}}{\frac{n^2\pi^2}{b^2}}\right)\right]_0^b$$

$$= \frac{4kb}{n^2\pi^2}\left\{(-1)^n - 1\right\}$$

$$= \begin{cases} \dfrac{-8kb}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Also
$$a_0 = \frac{2}{b}\int_0^b k(2y-b)\mathrm{d}y$$
$$= \frac{2k}{b}(y^2 - by)_0^b = 0$$

We note that the constant term in the R.H.S. series is also zero.

Using this value of λ_n in (8), the required solution is

$$u(x, y) = -\frac{8kb}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\operatorname{cosech}\frac{(2n-1)\pi a}{b}$$
$$\sinh\frac{(2n-1)\pi x}{b}\cos\frac{(2n-1)\pi y}{b}$$

Example 15

Find the steadystate temperature distribution on a square plate of side a insulated along three of its sides and with the side $y=0$ kept at temperature zero for $0 < x < \frac{a}{2}$ and at temperature T for $\frac{a}{2} < x < a$.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve the Eq. (1), satisfying the following boundary conditions.

$$\frac{\partial u}{\partial x}(0, y) = 0, \qquad \text{for } 0 \le y \le a \tag{2}$$
$$\frac{\partial u}{\partial x}(a, y) = 0, \qquad \text{for } 0 \le y \le a \tag{3}$$
$$\frac{\partial u}{\partial y}(x, a) = 0, \qquad \text{for } 0 < x < a \tag{4}$$
$$u(x, 0) = f(x), \text{ for } 0 < x < a \tag{5}$$

where
$$f(x) = \begin{cases} 0, & \text{in } \left(0, \frac{a}{2}\right) \\ T, & \text{in } \left(\frac{a}{2}, a\right) \end{cases}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \tag{6}$$

Differentiating (6) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, y) = p(-A\sin px + B\cos px)(Ce^{py} + De^{-py}) \tag{7}$$

Using boundary conditions (2) and (3) in (7), we can get, as usual,

$$B = 0 \text{ and } p = \frac{n\pi}{a}, \text{ where } n = 0, 1, 2, \ldots \infty$$

Differentiating (6) partially with respect to y, we have

$$\frac{\partial u}{\partial y}(x, y) = A \cos px \cdot p(Ce^{py} - De^{-py}) \tag{8}$$

Using boundary condition (4) in (7), we have

$$A \cdot \cos px . p(Ce^{pa} - De^{-pa}) = 0, \text{ for } 0 < x < a$$

As $A \neq 0$ and $p \neq 0$, $Ce^{pa} - De^{-pa} = 0$

$$\therefore \qquad D = Ce^{2pa}$$

Using these values of B, D, and p in (6), we have

$$\begin{aligned} u(x, y) &= AC \cos \frac{n\pi x}{a} \left\{ e^{n\pi y/a} + e^{2n\pi a/a} \cdot e^{-n\pi y/a} \right\} \\ &= (2ACe^{n\pi}) \cos \frac{n\pi x}{a} \cosh \frac{n\pi(y - a)}{a} \\ &= \lambda_n \cos \frac{n\pi x}{a} \cosh \frac{n\pi(a - y)}{a} \end{aligned}$$

where $n = 0, 1, 2, \ldots \infty$ ($\because \cosh\theta$ is even)

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=0}^{\infty} \lambda_n \cos \frac{n\pi x}{a} \cosh \frac{n\pi(a - y)}{a} \tag{9}$$

Using boundary condition (5) in (9), we have

$$\begin{aligned} \sum_{n=0}^{\infty} (\lambda_n \cosh n\pi) \cos \frac{n\pi x}{a} &= f(x) \text{ in } (0, a) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{a} \end{aligned}$$

which is the Fourier half-range cosine series of $f(x)$ in $(0, a)$.

Equating like terms in the two series, we get

$$\lambda_n \cosh n\pi = a_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}\mathrm{d}x$$

$$= \frac{2}{a}T\int_{a/2}^{a}\cos\frac{n\pi x}{a}\mathrm{d}x$$

$$= \frac{2T}{a}\cdot\left(\frac{\sin\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right)_{a/2}^{a}$$

$$= \frac{2T}{n\pi}\left(-\sin\frac{n\pi}{2}\right)$$

$$\lambda_0 = \frac{a_0}{2} = \frac{1}{2}\cdot\frac{2}{a}\int_{a/2}^{a} T\mathrm{d}x = \frac{T}{2}$$

Using these values of λ_n in (9), the required solution is

$$u(x, y) = \frac{T}{2} - \frac{2T}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\text{sech}\, n\pi\cos\frac{n\pi x}{a}\cdot\cosh\frac{n\pi(a-y)}{a}$$

$$i.e.\ u(x, y) = \frac{T}{2} + \frac{2T}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\text{sech}(2n-1)\pi\cdot\cos\frac{(2n-1)\pi x}{a}\cdot\cosh\frac{(2n-1)\pi(a-y)}{a}$$

Example 16

Find the steadystate temperature distribution $u(x, y)$ in the uniform square $0 \le x \le \pi$; $0 \le y \le \pi$, when the edge $x = \pi$ is maintained at temperature $(2\cos 3y - 5\cos 4y)$, the other three edges being thermally insulated.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

We have to solve the Eq. (1), satisfying the following boundary conditions.

$$\frac{\partial u}{\partial y}(x, 0) = 0, \text{ for } 0 \leq x \leq \pi \tag{2}$$

$$\frac{\partial u}{\partial y}(x, \pi) = 0, \text{ for } 0 \leq x \leq \pi \tag{3}$$

$$\frac{\partial u}{\partial x}(0, y) = 0, \text{ for } 0 < y < \pi \tag{4}$$

$$u(\pi, y) = 2\cos 3y - 5\cos 4y, \text{ for } 0 < y < \pi \tag{5}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py) \tag{6}$$

Differentiating (6) partially with respect to y, we have

$$\frac{\partial u}{\partial y}(x, y) = p(Ae^{px} + Be^{-px})(-C\sin py + D\cos py) \tag{7}$$

Using boundary conditions (2) and (3) in (7), we can get, as usual,

$$D = 0 \text{ and } p = n, \text{ where } n = 0, 1, 2, \ldots \infty$$

Differentiating (6) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, y) = p(Ae^{px} - Be^{-px}) \cdot C\cos py \tag{8}$$

Using boundary condition (4) in (8), we have

$$p(A - B)C\cos py = 0, \text{ for } 0 < y < \pi$$

As $p \neq 0$ and $C \neq 0$, we get $B = A$.
Using these values of B, D, and p in (6), it reduces to

$$\begin{aligned} u(x, y) &= AC(e^{nx} + e^{-nx})\cos ny \\ &= \lambda \cosh nx \cos ny \end{aligned}$$

where $n = 0, 1, 2, \ldots \infty$ and $\lambda = 2AC$
Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=0}^{\infty} \lambda_n \cosh nx \cos ny \tag{9}$$

Using boundary condition (5) in (9), we have

$$\sum_{n=0}^{\infty} (\lambda_n \cosh n\pi)\cos ny = 2\cos 3y - 5\cos 4y \text{ in } (0, \pi)$$

Comparing like terms, we get

$$\lambda_3 \cosh 3\pi = 2;\ \lambda_4 \cosh 4\pi = -5 \text{ and } \lambda_0 = 0 = \lambda_1 = \lambda_2 = \lambda_5 \ldots$$

i.e. $\lambda_3 = 2 \operatorname{sech} 3\pi, \lambda_4 = -5 \operatorname{sech} 4\pi$

and $\lambda_0 = 0 = \lambda_1 = \lambda_2 = \lambda_5 = \ldots$

Using these values in (9), the required solution is

$$u(x, y) = 2 \operatorname{sech} 3\pi \cosh 3x \cos 3y - 5 \operatorname{sech} 4\pi \cdot \cosh 4x \cos 4y$$

Example 17

A square metal plate of side a is bounded by the lines $x=0$, $x=a$, $y=0$ and $y=a$. The edges $x=0$ and $y=a$ are kept at zero temperature, the edge $x=a$ is insulated and the edge $y=0$ is kept at temperature kx. Find the steadystate temperature distribution in the plate.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve the Eq. (1), satisfying the following boundary conditions.

$$u(0, y) = 0, \quad \text{for } 0 \le y \le a \tag{2}$$

$$\frac{\partial u}{\partial x}(a, y) = 0, \quad \text{for } 0 \le y \le a \tag{3}$$

$$u(x, a) = 0, \quad \text{for } 0 < x < a \tag{4}$$

$$u(x, 0) = kx, \quad \text{for } 0 < x < a \tag{5}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (A \cos px + B \sin px)(Ce^{py} + De^{-py}) \tag{6}$$

Using boundary condition (2) in (6), we can get $A = 0$.
Differentiating (6) partially with respect to x, we have

$$\frac{\partial u}{\partial x}(x, y) = Bp \cos px(Ce^{py} + De^{-py}) \tag{7}$$

Using boundary conditions (3) in (7), we have

$$Bp \cos pa(Ce^{py} + De^{-py}), \text{ for } 0 \le y \le a$$

$$\text{Either } B = 0,\ p = 0 \text{ or } \cos pa = 0$$

If $B = 0$ and $p = 0$, we get trivial solutions

$$\therefore \qquad \cos pa = 0$$

$$\therefore \qquad pa = \frac{(2n-1)\pi}{2} \quad \text{or} \quad p = \frac{(2n-1)\pi}{2a}$$

where $n = 1, 2, \ldots \infty$.

Using boundary condition (4) in (6), we have

$$B \sin px(Ce^{pa} + De^{-pa}) = 0$$

As $B \neq 0$, $Ce^{pa} + De^{-pa} = 0$

$$\therefore \qquad D = -Ce^{2pa}$$

Using these values of A, D, and p in (6), it reduces to

$$\begin{aligned} u(x, y) &= BC \sin \frac{(2n-1)\pi x}{2a} \left\{ e^{(2n-1)\pi y/2a} - e^{2(2n-1)\pi a/2a} e^{-(2n-1)\pi y/2a} \right\} \\ &= \left\{ (2BCe^{(2n-1)\pi/2} \right\} \sin \frac{(2n-1)\pi x}{2a} \sinh \frac{(2n-1)\pi(y-a)}{2a} \\ &= \lambda_{2n-1} \sin \frac{(2n-1)\pi x}{2a} \sinh \frac{(2n-1)\pi(y-a)}{2a} \end{aligned}$$

where $n = 1, 2, \ldots \infty$.

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_{2n-1} \sin \frac{(2n-1)\pi x}{2a} \cdot \sinh \frac{(2n-1)\pi(y-a)}{2a} \tag{8}$$

Using boundary condition (5) in (8), we have

$$\sum_{n=1}^{\infty} -\lambda_{2n-1} \sinh \frac{(2n-1)\pi}{2} \sin \frac{(2n-1)\pi x}{2a} = kx \text{ in } (0, a)$$

$$\begin{aligned} \therefore \qquad -\lambda_{2n-1} \sinh \frac{(2n-1)\pi}{2} &= \frac{2}{a} \int_0^a kx \sin \frac{(2n-1)\pi x}{2a} dx \\ &= \frac{2k}{a} \left[x \left\{ \frac{-\cos \frac{(2n-1)\pi x}{2a}}{\frac{(2n-1)\pi}{2a}} \right\} - \left\{ \frac{-\sin \frac{(2n-1)\pi x}{2a}}{\frac{(2n-1)^2\pi^2}{4a^2}} \right\} \right]_0^a \\ &= \frac{8ka}{(2n-1)^2\pi^2} \sin \frac{(2n-1)\pi}{2} \end{aligned}$$

$$\therefore \qquad \lambda_{2n-1} = \frac{8ka}{(2n-1)^2\pi^2} \cdot (-1)^n \operatorname{cosech} \frac{(2n-1)\pi}{2}$$

Using these values of λ_{2n-1} in (8), the required solution is

$$u(x, y) = \frac{8ka}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \operatorname{cosech} \frac{(2n-1)\pi}{2} \sin \frac{(2n-1)\pi x}{2a}$$

$$\sinh \frac{(2n-1)\pi(a-y)}{2a}$$

Example 18

A rectangular plate of sides a and b is bounded by the lines $x=0, x=a, y=0$ and $y=b$. The edges $x=0$ and $y=b$ are kept at zero temperature, while the edge $y=0$ is kept insulated. If the temperature along the edge $x=a$ is kept at $T°\mathrm{C}$, find the steadystate temperature distribution in the plate.

The steadystate temperature distribution in the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve the Eq. (1), satisfying the following boundary conditions.

$$\frac{\partial u}{\partial y}(x, 0) = 0, \quad \text{for } 0 \le x \le a \tag{2}$$

$$u(x, b) = 0, \quad \text{for } 0 \le x \le a \tag{3}$$

$$u(0, y) = 0, \quad \text{for } 0 < y < b \tag{4}$$

$$u(a, y) = T, \quad \text{for } 0 < y < b \tag{5}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (Ae^{px} + Be^{-px})(C \cos py + D \sin py) \tag{6}$$

Differentiating (6) partially with respect to y and then using boundary condition (2), we can get $D = 0$.

Using boundary conditions (3) in (6), we can get

$$p = \frac{(2n-1)\pi}{2b}, \quad \text{where } n = 1, 2, 3, \ldots \infty$$

Using boundary condition (4) in (6), we can get $B = -A$.

Using these values of B, D and p in (6), it reduces to

$$u(x, y) = \lambda \sinh \frac{(2n-1)\pi x}{2b} \cdot \cos \frac{(2n-1)\pi y}{2b}$$

where $\qquad \lambda = 2AC$ and $n = 1, 2, \ldots \infty$

Therefore the most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_{2n-1} \sinh \frac{(2n-1)\pi x}{2b} \cos \frac{(2n-1)\pi y}{2b} \tag{7}$$

Using boundary condition (5) in (7), we have

$$\sum_{n=1}^{\infty} \left\{ \lambda_{2n-1} \sinh \frac{(2n-1)\pi a}{2b} \right\} \cos \frac{(2n-1)\pi y}{2b} = T \text{ in } (0, b)$$

$$\therefore \quad \lambda_{2n-1} \sinh \frac{(2n-1)\pi a}{2b} = \frac{2}{b} \int_0^b T \cos \frac{(2n-1)\pi y}{2b} dy$$

$$= \frac{2T}{b} \left\{ \frac{\sin \frac{(2n-1)\pi y}{2b}}{\frac{(2n-1)\pi}{2b}} \right\}_0^b$$

$$= \frac{4T}{(2n-1)\pi} \sin \frac{(2n-1)\pi}{2}$$

$$\therefore \quad \lambda_{2n-1} = \frac{4T}{(2n-1)\pi} \cdot \operatorname{cosech} \frac{(2n-1)\pi a}{2b} \cdot (-1)^{n+1}$$

Using this value of λ_{2n-1} in (7), the required solution is

$$u(x, y) = \frac{4T}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \operatorname{cosech} \frac{(2n-1)\pi a}{2b}$$

$$\sinh . \frac{(2n-1)\pi x}{2b} \cdot \cos \frac{(2n-1)\pi y}{2b}$$

Example 19

A square metal plate of side 10 cm has the edges represented by the lines $x = 10$ and $y = 10$ insulated. The edge $x = 0$ is kept at a temperature of zero degree and the edge $y = 0$ at a temperature of 100°C. Find the steadystate temperature distribution in the plate.

The steadystate temperature $u(x, y)$ at any point (x, y) of the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve Eq. (1), satisfying the following boundary conditions.

$$u(0, y) = 0, \quad \text{for } 0 \le y \le 10 \tag{2}$$

$$\frac{\partial u}{\partial x}(10, y) = 0, \quad \text{for } 0 \le y \le 10 \tag{3}$$

$$\frac{\partial u}{\partial y}(x, 10) = 0, \quad \text{for } 0 < x < 10 \tag{4}$$

$$u(x, 0) = 100, \text{ for } 0 < x < 10 \tag{5}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (A \cos px + B \sin px)(Ce^{py} + De^{-py}) \tag{6}$$

Using boundary condition (2) in (6), we can get $A = 0$.

Differentiating (6) partially with respect to x and then using the boundary condition (3), we can get

$$p = \frac{(2n-1)\pi}{20}, \text{ where } n = 1, 2, \ldots \infty$$

Differentiating (6) partially with respect to y and then using the boundary condition (4), we can get $D = Ce^{20p}$.

Using these values of A, D and p in (6), it reduces to

$$u(x, y) = \left[2BCe^{(2n-1)\pi/2}\right] \sin \frac{(2n-1)\pi x}{20} \cosh \frac{(2n-1)\pi(y-10)}{20}$$

or $$u(x, y) = \lambda_{2n-1} \sin \frac{(2n-1)\pi x}{20} \cosh \frac{(2n-1)\pi(10-y)}{20}$$

where $n = 1, 2, \ldots \infty$ [$\because \cosh\theta$ is even]

$\therefore$ The most general solution of Eq. (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_{2n-1} \sin \frac{(2n-1)\pi x}{20} \cosh \frac{(2n-1)\pi(10-y)}{20} \tag{7}$$

Using boundary condition (5) in (7), we have

$$\sum_{n=1}^{\infty} \lambda_{2n-1} \cosh \frac{(2n-1)\pi}{2} \sin \frac{(2n-1)\pi x}{20} = 100 \text{ in } (0, 10)$$

$$\therefore \quad \lambda_{2n-1} \cosh \frac{(2n-1)\pi}{2} = \frac{2}{10} \int_0^{10} 100 \sin \frac{(2n-1)\pi x}{20} dx$$

$$= 20 \left\{ \frac{-\cos \frac{(2n-1)\pi x}{20}}{\frac{(2n-1)\pi}{20}} \right\}_0^{10}$$

$$= \frac{400}{(2n-1)\pi}$$

$$\lambda_{2n-1} = \frac{400}{(2n-1)\pi} \operatorname{sech} \frac{(2n-1)\pi}{2}$$

Using this value of λ_{2n-1} in (7), the required solution is

$$u(x, y) = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \operatorname{sech} \frac{(2n-1)\pi}{2} \sin \frac{(2n-1)\pi x}{20}$$

$$\cosh \frac{(2n-1)\pi(10-y)}{20}$$

Example 20

A square metal plate of side a has the edges $x = 0$ and $y = 0$ insulated. The edge $y = a$ is kept at temperature 0°C and the edge $x = a$ is kept at temperature ky. Find the steadystate temperature distribution in the plate.

The steadystate temperature distribution in the plate is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

We have to solve the Eq. (1), satisfying the following boundary conditions.

$$\frac{\partial u}{\partial y}(x, 0) = 0, \ \text{for}\, 0 \le x \le a \tag{2}$$

$$u(x, a) = 0, \ \text{for}\, 0 \le x \le a \tag{3}$$

$$\frac{\partial u}{\partial x}(0, y) = 0, \ \text{for}\, 0 < y < a \tag{4}$$

$$u(a, y) = ky, \text{for}\, 0 < y < a \tag{5}$$

Consistent with the non-zero boundary condition (5), the proper solution of Eq. (1) is

$$u(x, y) = (Ae^{px} + Be^{-px})(C \cos py + D \sin py) \tag{6}$$

Differentiating (6) partially with respect to y and then using boundary condition (2), we can get $D = 0$.

Using boundary conditions (3) in (6), we can get

$$p = \frac{(2n-1)\pi}{2a}, \ \text{where}\, n = 1, 2, 3, \ldots \infty.$$

Differentiating (6) partially with respect to x and then using boundary condition (4), we can get $B = A$.

Using these values of B, D and p in (6), it reduces to

$$u(x, y) = AC \left\{ e^{(2n-1)\pi x/2a} + e^{-(2n-1)\pi x/2a} \right\} \cos \frac{(2n-1)\pi y}{2a}$$

or $$u(x, y) = \lambda \cosh \frac{(2n-1)\pi x}{2a} \cos \frac{(2n-1)\pi y}{2a} \text{where}\, n = 1, 2, 3, \ldots \infty.$$

Therefore the most general solution of Eq. 1 is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_{2n-1} \cosh \frac{(2n-1)\pi x}{2a} \cos \frac{(2n-1)\pi y}{2a} \tag{7}$$

Using boundary condition (5) in (7), we have

$$\sum_{n=1}^{\infty} \left[\lambda_{2n-1} \cosh \frac{(2n-1)\pi}{2} \right] \cos \frac{(2n-1)\pi y}{2a} = ky \text{ in}(0, a)$$

$$\therefore \; \lambda_{2n-1} \cosh \frac{(2n-1)\pi}{2} = \frac{2}{a} \int_0^a ky \cos \frac{(2n-1)\pi y}{2a} dy$$

$$= \frac{2k}{a} \left[y \left\{ \frac{\sin \frac{(2n-1)\pi y}{2a}}{\frac{(2n-1)\pi}{2a}} \right\} - \left\{ \frac{-\cos \frac{(2n-1)\pi y}{2a}}{\frac{(2n-1)^2\pi^2}{4a}} \right\} \right]_0^a$$

$$= \frac{4ka}{\pi^2} \left\{ \frac{(-1)^{n+1}\pi}{2n-1} - \frac{2}{(2n-1)^2} \right\}$$

$$\therefore \quad \lambda_{2n-1} = \frac{4ka}{\pi^2} \left\{ \frac{(-1)^{n+1}\pi}{2n-1} - \frac{2}{(2n-1)^2} \right\} \operatorname{sech} \frac{(2n-1)\pi}{2}$$

Using this value of λ_{2n-1} in (7), the required solution is

$$u(x, y) = \frac{4ka}{\pi^2} \sum_{n=1}^{\infty} \left\{ \frac{(-1)^{n+1}\pi}{2n-1} - \frac{2}{(2n-1)^2} \right\} \operatorname{sech} \frac{(2n-1)\pi}{2} \cosh \frac{(2n-1)\pi x}{2a} \cdot \cos \frac{(2n-1)\pi y}{2a} .$$

Exercise 3C

Part A (Short-Answer Questions)

1. State the two laws of thermodynamics used in the derivation of two dimensional heat flow equation.
2. Write down the partial differential equation that represents variable heat flow in two dimensions. Deduce the equation of steadystate heat flow in two dimensions.
3. Write down the three mathematically possible solutions of Laplace equation in two dimensions.

4. Given the boundary conditions on a square or rectangular plate, how will you identify the proper solution?

5. Explain why $u(x, y) = (Ax + B)(Cy + D)$ cannot be the proper solution of Laplace equation in boundary value problems, by taking an example.

Part B

6. A rectangular plate with insulated surfaces is a cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the two long edges $x = 0$ and $x = a$ and the short edge at infinity are kept at temperature 0°C, while the other short edge $y = 0$ is kept at temperature (i) $u = 2k \sin \dfrac{\pi x}{2} \cos \dfrac{3\pi x}{a}$, (ii) $u = kx$ and (iii) $u = \begin{cases} kx, & \text{for } 0 \le x \le \dfrac{a}{2} \\ k(a - x), & \text{for } \dfrac{a}{2} \le x \le a \end{cases}$
Find the steadystate temperature at any point (x, y) of the plate.

7. An infinitely long metal plate in the form of an area is enclosed between the lines $y = 0$ and $y = 10$ for positive values of x. The temperature is zero along the edges $y = 0$, $y = 10$ and the edge at infinity. If the edge $x = 0$ is kept at temperature (i) $u = 4k \sin^3 \dfrac{\pi y}{10}$, (ii) $u = T$ and (iii) $u = ky(10 - y)$, find the steadystate temperature at any point (x, y) of the plate.

8. A plate is in the form of the semi-infinite strip $0 \le x \le \pi, 0 \le y \le \infty$. The edges $x = 0$ and $x = \pi$ are insulated. The edge at infinity is kept at temperature 0°C, while the edge $y = 0$ is kept at temperature
$$u = \begin{cases} x, & \text{in } 0 \le x \le \dfrac{\pi}{2} \\ \pi - x, & \text{in } \dfrac{\pi}{2} \le x \le \pi \end{cases}$$
Find the steadystate temperature distribution in the plate.

9. The two long edges $y = 0$ and $y = l$ of a long rectangular plate are insulated. If the temperature in the short edge at infinity is kept at 0°C, while that in the short edge $x = 0$ is kept at $ky(l - y)$, find the steadystate temperature distribution in the plate.

10. A plate is in the form of the semi-infinite strip $0 \le x < \infty, 0 \le y \le l$. The surface of the plate and the edge $y = 0$ are insulated. If the temperature along the edge $y = l$ and the short edge at infinity are kept temperature 0°C, while the temperature along the other short edge is kept at temperature T°C, find the steadystate temperature distribution in the plate.

11. If the temperatures along the long edge $x = 0$ and the short edge at infinity of a long plate kept at 0°C, the other long edge $x = 10$ is insulated and the other short edge $y = 0$ is kept at temperature kx, find the steadystate temperature distribution in the plate.

12. Find the steadystate temperature distribution in a square plate of side a, which is insulated on the lateral surface and three of whose edges $x = a$, $y = 0$, $y = a$

are kept at zero temperature, if the temperature in the edge $x = 0$ is (i) $k \sin^3 \dfrac{\pi y}{a}$ and (ii) $ky(a - y)$.

13. A rectangular plate of sides a and b has its faces insulated and its edges along $x = 0, x = a, y = 0$ and $y = b$. If the temperature along the edge $y = b$ is given by (i) $u = 3 \sin \dfrac{4\pi x}{a} + 5 \sin \dfrac{6\pi x}{a}$ and (ii) $u = x$ in $0 \le x \le \dfrac{a}{2}$ and $u = a - x$ in $\dfrac{a}{2} \le x \le a$, while the other three edges are kept at 0°C, find the steadystate temperature in the plate.

14. If a square plate is bounded by the lines $x = \pm\pi$ and $y = \pm\pi$ and three of its edges are kept at temperature 0°C, while the edge $x = \pi$ is kept at temperature $u = y + \pi, -\pi \le y \le \pi$, find the steadystate temperature in the plate.

15. Find the electrostatic potential in the rectangle, $0 \le x \le 20, 0 \le y \le 40$, whose upper edge is kept at potential 110 volts and whose other edges are grounded.

Note ✍

The electrical force of attraction or repulsion between charged particles (governed by Coulomb's law) is the gradient of a function u, *called* electrostatic potential *and at any point, free of charges.* u *is a solution of Laplace equation* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. *Hence we have to solve the Laplace equation satisfyng the given boundary conditions, to get the potential distribution in the rectangle.*

16. A square plate has its faces and its edge $x = 0$ insulated. Its edges $y = 0$ and $y = a$ are kept at temperature zero, while its edge $x = a$ is kept at temperature T°C. Find the steadystate temperature distribution in the plate.

17. A rectangular plate of sides a and b has its faces and the edge $y = b$ insulated. Its edges $x = 0$ and $x = a$ are kept at temperature zero, while the edge $y = 0$ i kept at temperature kx. Find the steadystate temperature distribution in the plate.

18. A square plate of side 20 cm has its faces and its edges $x = 0$ and $x = 20$ insulated. If the edge $y = 0$ is kept at temperature zero, while the edge $y = 20$ is kept at temperature $u = (10 - x)$, find the steadystate temperature distribution in the plate.

19. A square plate of side π has its faces and the edges $y = 0$ and $y = \pi$ insulated. If the edge $x = \pi$ is kept at temperature zero, while the edge $x = 0$ is kept at temperature $(2 \cos 3y + 3 \cos 4y)$, find the steadystate temperature distribution in the plate.

20. Find the steadystate temperature distribution on a rectangular plate of sides a and b, insulated along three of its sides $x = 0, x = a$ and $y = 0$ and with side $y = b$ kept at temperature $kx, 0 \le x \le a$.

21. Find the steadystate temperature distribution in a square plate of side a, insulated along three of its sides $y=0$, $y=a$ and $x=a$ and with the side $x=0$ kept at temperature $0°$ for $0 < x < \frac{a}{2}$ and $100°$ for $\frac{a}{2} < x < a$.

22. A square metal plate of side 10 cm is bounded by the lines $x=0$, $x=10$, $y=0$ and $y=10$. The edges $y=0$ and $x=10$ are kept at zero temperature, the edge $x=0$ is kept insulated and the edge $y=10$ is kept at temperature T°C. Find the steadystate temperature distribution in the plate.

23. A square metal plate of side a is bounded by the lines $x=0$, $x=a$, $y=0$ and $y=a$. The edges $x=a$ and $y=0$ are kept at zero temperature, while the edge $y=a$ is kept insulated. If the temperature along the edge $x=0$ is ky, find the steadystate temperature distribution in the plate.

24. A rectangular metal plate of sides a and b has the edges $x=0$ and $y=0$ insulated. The edge $x=a$ is kept at a temperature of 0°C and the edge $y=b$ is kept at a temperature 100°C. Find the steadystate temperature distribution in the plate.

25. A square metal plate of side 10 cm has the edges $x=10$ and $y=10$ insulated. The edge $y=0$ is kept at temperature zero and the edge $x=0$ is kept at temperature ky. Find the steadystate temperature distribution in the plate.

26. If the faces of a thin square plate of side π are perfectly insulated, the edges are kept at zero temperature and the initial temperature at any point (x, y) of the plate is $u(x, y, 0) = f(x, y)$, show that the temperature in the plate at any subsequent time is given by

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \lambda_{mn} \sin mx \sin ny e^{-\alpha^2(m^2+n^2)t}$$

where $\lambda_{mn} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(x, y) \sin mx \sin ny \, dx \, dy.$

Find the temperature in the plate at time t, if $f(x, y) = xy(\pi - x)(\pi - y)$.

[**Hint** Solve $\frac{\partial u}{\partial t} = \alpha^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ by the method of separation of variables. Proceed as in worked Example 19 of Chapter 3(A)]

Answers

***Exercise* 3(C)**

6. (i) $u(x, y) = -k \sin \frac{2\pi x}{a} e^{-2\pi y/a} + k \sin \frac{4\pi x}{a} e^{-4\pi y/a}$.

(ii) $u(x, y) = \frac{2ka}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{a} \cdot \exp(-n\pi y/a)$.

(iii) $u(x, y) = \dfrac{4ka}{\pi^2} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(2n-1)^2} \sin \dfrac{(2n-1)\pi x}{a} \exp\{-(2n-1)\pi y/a\}.$

7. (i) $u(x, y) = 3k \exp(-\pi x/10) \sin \dfrac{\pi y}{10} - k \exp\left(-\dfrac{3\pi x}{10}\right) \sin \dfrac{3\pi y}{10}.$

(ii) $u(x, y) = \dfrac{4T}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{2n-1} \exp\{-(2n-1)\pi x/10\} \sin\left\{\dfrac{(2n-1)\pi y}{10}\right\}.$

(iii) $u(x, y) = \dfrac{800k}{\pi^3} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^3} \exp\{-(2n-1)\pi x/10\} \sin\left\{\dfrac{(2n-1)\pi y}{10}\right\}.$

8. $u(x, y) = \dfrac{\pi}{4} - \dfrac{2}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^2} \cos\{2(2n-1)x\} \exp\{-2(2n-1)y\}.$

9. $u(x, y) = \dfrac{kl^2}{6} - \dfrac{kl^2}{\pi^2} \sum_{n=1}^{\infty} \dfrac{1}{n^2} \cos \dfrac{2n\pi y}{l} \cdot \exp(-2n\pi x/l).$

10. $u(x, y) = \dfrac{4T}{\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{2n-1} \exp\{-(2n-1)\pi x/2l\} \cos\{(2n-1)\pi y/2l\}.$

11. $u(x, y) = \dfrac{80k}{\pi^2} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(2n-1)^2} \sin \dfrac{(2n-1)\pi x}{20} \exp\{-(2n-1)\pi y/20\}.$

12. (i) $\frac{3k}{4} \operatorname{cosech} \pi \cdot \sinh \frac{\pi(a-x)}{a} \cdot \sin \frac{\pi y}{a} - \frac{k}{4} \operatorname{cosech} 3\pi \cdot \sinh \cdot \frac{3\pi(a-x)}{a} \cdot \sin \frac{3\pi y}{a}.$

(ii) $u(x, y) = \dfrac{8ka^2}{\pi^3}$

$\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^3} \operatorname{cosech}(2n-1)\pi \sinh \dfrac{(2n-1)\pi(a-x)}{a} \sin \dfrac{(2n-1)\pi y}{a}.$

13. (i) $u(x, y) = 3 \operatorname{cosech} \dfrac{4\pi b}{a} \sin \dfrac{4\pi x}{a} \sinh \dfrac{4\pi y}{a} + 5 \operatorname{cosech} \dfrac{6\pi b}{a} \cdot \sin \dfrac{6\pi x}{a} \cdot$
$\sinh \dfrac{6\pi y}{a}.$

(ii) $u(x, y) = \dfrac{4a}{\pi^2}$

$\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(2n-1)^2} \operatorname{cosech} \dfrac{(2n-1)\pi b}{a} \sin \dfrac{(2n-1)\pi x}{a} \sinh \dfrac{(2n-1)\pi y}{a}.$

14. $u(x, y) = 4 \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n} \operatorname{cosech} n\pi \sinh \dfrac{n}{2}(\pi + x) \cdot \sin \dfrac{n}{2}(\pi + y).$

15. $u(x, y) = \dfrac{440}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{2n-1} \operatorname{cosech} 2(2n-1)\pi . \sin \dfrac{(2n-1)\pi x}{20} \cdot \sinh \dfrac{(2n-1)\pi y}{20}.$

16. $u(x, y) = \dfrac{4T}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{2n-1} \operatorname{sech}(2n-1)\pi \cosh \dfrac{(2n-1)\pi x}{a} \cdot \sin \dfrac{(2n-1)\pi y}{a}$.

17. $u(x, y) = \dfrac{2ka}{\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n} \operatorname{sech} \dfrac{n\pi b}{a} \cdot \sin \dfrac{n\pi x}{a} \cdot \cosh \dfrac{n\pi(b-y)}{a}$.

18. $u(x, y) = \dfrac{80}{\pi^2} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^2} \operatorname{cosech}(2n-1)\pi \cdot \cos \dfrac{(2n-1)\pi x}{20} \sinh \dfrac{(2n-1)\pi y}{20}$.

19. $u(x, y) = 2 \operatorname{cosech} 3\pi \cdot \sinh 3(\pi - x) \cdot \cos 3y + 3 \operatorname{cosech} 4\pi \sinh 4(\pi - x) \cdot \cos 4y$.

20. $u(x, y) = \dfrac{ka}{2} - \dfrac{4k}{\pi^2} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^2} \operatorname{sech} \dfrac{(2n-1)\pi b}{a} . \cos \dfrac{(2n-1)\pi x}{a} \cosh \dfrac{(2n-1)\pi y}{a}$.

21. $u(x, y) = 50 + \dfrac{200}{\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^n}{2n-1} \operatorname{sech}(2n-1)\pi \cosh \dfrac{(2n-1)\pi(a-x)}{a} \cos \dfrac{(2n-1)\pi y}{a}$.

22. $u(x, y) = \dfrac{4T}{\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{2n-1} \operatorname{cosech} \dfrac{(2n-1)\pi}{2} \cdot \cos \dfrac{(2n-1)\pi x}{20} \cdot \sinh \dfrac{(2n-1)\pi y}{20}$.

23. $u(x, y) = \dfrac{8ka}{\pi^2} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(2n-1)^2} \operatorname{cosech} \dfrac{(2n-1)\pi}{2} . \sinh \dfrac{(2n-1)\pi(a-x)}{2a} \cdot \sin \dfrac{(2n-1)\pi y}{2a}$.

24. $u(x, y) = \dfrac{400}{\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{2n-1} \operatorname{sech} \dfrac{(2n-1)\pi b}{2a} \cdot \cos \dfrac{(2n-1)\pi x}{2a} \cosh \dfrac{(2n-1)\pi y}{2a}$.

25. $u(x, y) = \dfrac{80k}{\pi^2} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(2n-1)^2} \operatorname{sech} \dfrac{(2n-1)\pi}{2} \cosh \dfrac{(2n-1)\pi(10-x)}{20} \cdot \sin \dfrac{(2n-1)\pi y}{20}$.

26. $u(x, y) = \dfrac{64}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \dfrac{1}{(2m-1)^3 (2n-1)^3} \sin(2m-1)x \cdot \sin(2n-1)y . \exp[-\{(2m-1)^2 + (2n-1)^2\}\alpha^2 t]$.

Chapter 4

Fourier Transforms

4.1 INTRODUCTION

The development of mathematical representation of periodic phenomena using complex numbers leads to complex form of the Fourier series representation of periodic function. The representation of periodic signals as a linear combination of harmonically related complex exponentials can be extended to develop a representation of aperiodic signals as linear combination of complex exponentials. This leads to Fourier Transforms.

Fourier transform is widely used in the theory of communication engineering, wave propagation and other branches of applied mathematics.

We have already discussed Laplace transforms and its applications for the solution of ordinary differential equations. In this Chapter, we shall discuss three other integral transforms and their applications for the solution of partial differential equations.

The effect of applying an integral transform to a partial differential equation is to reduce the number of independent variables by one, as will be seen in the solution of problems towards the end. The choice of a particular transform is decided by the nature of the boundary conditions and the convenience of inverting the transform function $\bar{f}(s)$ to give $f(x)$.

The three integral transforms, namely, Complex Fourier transform, Fourier Cosine transform and Fourier Sine transform and their inverse transforms are defined by means of a powerful theorem, known as Fourier Integral Theorem, which is given in Section 4.2.

4.2 FOURIER INTEGRAL THEOREM

If $f(x)$ is piecewise continuous, has piecewise continuous derivatives in every finite interval in $(-\infty, \infty)$ and absolutely integrable in $(-\infty, \infty)$, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cdot e^{is(x-t)}\, dt\, ds \text{ or equivalently}$$

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos\{s(x-t)\}\, dt\, ds.$$

Proof

When $f(x)$ satisfies the conditions given in the theorem, we can prove that $f(x)$ can be expanded as an infinite series of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} \tag{1}$$

in $(-l, l)$ however large l may be, where

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(t) e^{-in\pi t/l}\, dt \tag{2}$$

Note

The series (1) is the Complex form of the Fourier series of $f(x)$ in $(-l, l)$.

Putting $s_n = \dfrac{n\pi}{l}$ and inserting (2) in (1), we have

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} \frac{1}{2l} \int_{-l}^{l} f(t) e^{is_n(x-t)}\, dt \\ &= \frac{1}{2\pi} \int_{-l}^{l} \left[\sum_{n=-\infty}^{\infty} f(t) e^{is_n(x-t)} \cdot \frac{\pi}{l} \right] dt \end{aligned}$$

on interchanging summation and integration.

$$= \frac{1}{2\pi} \int_{-l}^{l} \left[\sum_{n=-\infty}^{\infty} f(t) e^{is_n(x-t)} \Delta S_n \right] dt, \text{ since}$$

$$\Delta s_n = s_{n+1} - s_n = \frac{(n+1)\pi}{l} - \frac{n\pi}{l} = \frac{\pi}{l}.$$

Taking limits as $\Delta S_n \longrightarrow 0$ or equivalently $l \longrightarrow \infty$, we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{is(x-t)}\, ds \right] dt \tag{3}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{is(x-t)}\, dt\, ds \tag{4}$$

[$\because$ the limits of integration are constants]

From (3),
$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\int_{-\infty}^{\infty}[\cos s(x-t)+i\sin s(x-t)]\,ds\,dt$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\cdot 2\int_{0}^{\infty}\cos s(x-t)\,ds\,dt$$

[$\because$ $\cos s(x-t)$ is an even function and $\sin s(x-t)$ is an odd function of s in $(-\infty,\infty)$]

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos s(x-t)\,dt\,ds \tag{5}$$

[$\because$ the limits of the integration are constants]

Note ✍

1. *In (5), the limits for t are $-\infty$ and ∞ and those for s are 0 and ∞.*
2. *The R.H.S. of (4) is called the Fourier Complex integral or Fourier Complex integral representation of $f(x)$. The R.H.S. of (5) is called the Fourier integral or Fourier integral representation of $f(x)$.*
3. *At a point of discontinuity, the value of the integral in the R.H.S. of (3) or (5) $=\frac{1}{2}\{f(x-0)+f(x+0)\}$.*

From (5), we have

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)[\cos sx\cos st+\sin sx\cdot\sin st]\,dt\,ds$$
$$=\frac{1}{\pi}\int_{0}^{\infty}\cos sx\left(\int_{-\infty}^{\infty} f(t)\cos st\,dt\right)ds+\frac{1}{\pi}\int_{0}^{\infty}\sin sx\left(\int_{-\infty}^{\infty} f(t)\sin st\,dt\right)ds \tag{6}$$

If $f(x)$ [or $f(t)$] is even,

$f(t)\cos st$ is an even function of t and $f(t)\sin st$ is an odd function of t. Hence, by the property of definite integrals, we get the following from (6),

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty} f(t)\cos sx\cos st\,dt\,ds \tag{7}$$

The R.H.S. of (7) is called *the Fourier Cosine integral* of $f(x)$, provided $f(x)$ is even.

If $f(x)$ [or $f(t)$] is odd, $f(t)\cos st$ is an odd function of t and $f(t)\sin st$ is an even function of t.

Hence, by the property of definite integrals, we get the following from (6),

$$f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\sin sx \sin st \; dt \; ds \tag{8}$$

The R.H.S. of (8) is called *the Fourier sine integral* of $f(x)$, provided $f(x)$ is odd.

4.3 FOURIER TRANSFORMS

Definition

$\int_{-\infty}^{\infty} f(x)e^{-isx}\,dx$ is called *the Fourier transform of* $f(x)$ and is denoted by $\bar{f}(s)$ or $F\{f(x)\}$. F is the Fourier transform operator. Thus $\bar{f}(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-isx}\,dx$, where s is used as the transform variable. Sometimes the letter p or ω is used as the transform variable.

From Step (4) of Fourier integral theorem, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\left[\int_{-\infty}^{\infty} f(t)e^{-ist}\,dt\right] ds \tag{4$'$}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)e^{ixs}\,ds$$

Now $\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)e^{ixs}\,ds$ is called *the inverse Fourier Transform of* $\bar{f}(s)$ and denoted by $F^{-1}\{\bar{f}(s)\}$.

Thus, once the Fourier transform of $f(x)$ is defined as given above, the inverse Fourier transform of $\bar{f}(s)$ is provided by Fourier Integral theorem.

(4)$'$ may be rewritten as

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ist}\,dt\right] ds$$

Accordingly, some authors define $F\{f(x)\}$ and $F^{-1}\{\bar{f}(s)\}$ (Fourier transform pair) as follows:

$$F\{f(x)\} = \bar{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx}\, dx \text{ and}$$

$$F^{-1}\{\bar{f}(s)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(s) \cdot e^{ixs}\, ds$$

Definition

$\int_0^\infty f(x) \cos sx \, dx$ is called *Fourier Cosine transform* of $f(x)$ and is denoted by $\bar{f}_C(s)$ or $F_C\{f(x)\}$. F_C is the Fourier cosine transform operator.

From Step (7) of Fourier integral theorem we have

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos sx \left[\int_0^\infty f(t) \cos st \, dt \right] ds \qquad (7)'$$

$$= \frac{2}{\pi} \int_0^\infty \bar{f}_C(s) \cos xs \, ds$$

$\frac{2}{\pi} \int_0^\infty \bar{f}_C(s) \cos xs \, ds$ is called the *inverse Fourier Cosine transform* of $\bar{f}_C(s)$ and denoted by $F_C^{-1}\{\bar{f}_C(s)\}$. Thus, once the Fourier cosine transform of $f(x)$ is defined, the inverse Fourier cosine transform of $\bar{f}_C(s)$ is provided by the Fourier cosine integral formula.

(7)′ may be rewritten as

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos sx \left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos st \, dt \right] ds$$

Accordingly, some authors define the Fourier cosine transform pair as follows,

$$F_C\{f(x)\} = \bar{f}_C(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx \text{ and}$$

$$F_C^{-1}\{\bar{f}_C(s)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_C(s) \cos xs \, ds$$

Definition

$\int_0^\infty f(x) \sin sx \, dx$ is called *Fourier Sine transform* of $f(x)$ and is denoted by $\bar{f}_S(s)$ or $F_S\{f(x)\}$.

F_S is the Fourier sine transform operator. Thus

$$\bar{f}_S(s) = F_S\{f(x)\} = \int_0^\infty f(x)\sin sx\, dx$$

From Step (8) of Fourier integral theorem, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin sx\left[\int_0^\infty f(t)\sin st\, dt\right] ds \tag{8$'$}$$

$$= \frac{2}{\pi}\int_0^\infty \bar{f}_S(s)\sin xs\, ds$$

$\frac{2}{\pi}\int_0^\infty \bar{f}_S(s)\sin xs \;\; ds$ is called the *inverse Fourier sine transform* of $\bar{f}_S(s)$ and denoted by $F_S^{-1}\{\bar{f}_s(s)\}$. Thus once the Fourier sine transform of $f(x)$ is defined, the inverse Fourier sine transform of $\bar{f}_S(s)$ is provided by the Fourier sine integral formula.

(8)$'$ may be rewritten as

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin\, sx\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin\, st\;\, dt\right] ds$$

Accordingly, some authors define the Fourier sine transform pair as follows.

$$F_S\{f(x)\} = \bar{f}_S(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\, dx \text{ and}$$

$$F_S^{-1}\{\bar{f}_S(s)\} = f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}_s(s)\sin xs\;\, ds$$

4.4 ALTERNATIVE FORM OF FOURIER COMPLEX INTEGRAL FORMULA

The Fourier integral formula for $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos s(x-t)\;\, dt\;\, ds$$

$$= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos s(t-x)\;\, dt\;\, ds$$

$$= \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\left[e^{is(t-x)} + \bar{e}^{is(t-x)}\right] dt\ ds$$

$$= \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)e^{is(t-x)}\ dt\ ds + \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\bar{e}^{is(t-x)}\ dt\ ds.$$

Putting $s = -s'$ in the second integral, we get

$$f(x) = \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)e^{is(t-x)}\ dt\ ds + \frac{1}{2\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty} f(t)e^{is'(t-x)}\ dt\ ds'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{is(t-x)}\ dt\ ds$$

[on changing s' into s and combining the two integrals] (1)

(1) provides an alternative formula for $f(x)$. Comparing this with the Fourier Complex integral formula derived in (4) of Fourier integral theorem, we note that x and t can be interchanged in the exponential function.

Note ✍

Equation (1) can be re-written as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\left[\int_{-\infty}^{\infty} f(t)e^{ist}\ dt\right] ds$$

Based on this, some authors define Fourier transform pair as follows:

$$\bar{f}(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{isx}\ dx \text{ and}$$

$$F^{-1}\{\bar{f}(s)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)e^{-ixs}\ ds$$

We shall follow the definitions which were given earlier.

4.5 RELATIONSHIP BETWEEN FOURIER TRANSFORM AND LAPLACE TRANSFORM

Let $f(t)$ be defined as $f(t) = \begin{cases} e^{-xt}\phi(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$

Then $$F\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-iyt}\,dt$$

where y is the Fourier transform variable.

$$= \int_{-\infty}^{0} 0 \cdot e^{-iyt}\,dt + \int_{0}^{\infty} e^{-xt}\,\phi(t)e^{-iyt}\,dt$$

$$= \int_{0}^{\infty} e^{-st}\phi(t)\,dt, \text{ where } s = x + iy$$

i.e. $$F\{f(t)\} = L\{\phi(t)\}$$

Worked Examples 4(a)

Example 1

Find the Fourier integral representation of $f(x)$ defined as

$$f(x) = \begin{cases} 0, & \text{for } x < 0 \\ 1/2, & \text{for } x = 0 \\ e^{-x}, & \text{for } x > 0 \end{cases}$$

Verify the representation at $x = 0$.

Fourier (complex) integral representation is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{-ist} \cdot e^{isx}\,dt\,ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\left[\int_{-\infty}^{0} + \int_{0}^{\infty} f(t)e^{-ist}\,dt\right] ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\left[\int_{0}^{\infty} e^{-(1+is)t} \cdot dt\right] ds,$$

on using the given values of $f(t)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\left\{\frac{e^{-(1+is)t}}{-(1+is)}\right\}_{t=0}^{t=\infty} ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx} \cdot \frac{1}{1+is}\,ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(1 - is)}{1 + s^2} (\cos xs + i \sin xs) \, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + s^2} [\{\cos xs + s \sin xs\} + i\{\sin xs - s \cos xs\}] \, ds$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \left(\frac{\cos xs + s \sin xs}{1 + s^2} \right) ds \tag{1}$$

by property of definite integrals, as the real part is even and the imaginary part is odd.

Putting $x = 0$ in the integral representation (1), we get $f(0) = \frac{1}{\pi} \int_{0}^{\infty} \frac{ds}{1 + s^2} = \frac{1}{\pi} [\tan^{-1} s]_0^{\infty} = \frac{1}{2}$. Thus the integral representation (1) holds good for $x = 0$ also.

Example 2

Using Fourier integral formula, prove that

$$e^{-x} \cos x = \frac{2}{\pi} \int_{0}^{\infty} \frac{(\lambda^2 + 2) \cos x\lambda}{\lambda^4 + 4} \, d\lambda.$$

Note ✍

λ has been used instead of 's'

The presence of $\cos x\lambda$ in the integral suggests that the Fourier cosine integral formula for $e^{-x} \cos x$ has been used.

Fourier cosine integral representation is given by

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(t) \cos \lambda t \cos \lambda x \, dt \, d\lambda$$

$$\therefore \; e^{-x} \cos x = \frac{2}{\pi} \int_{0}^{\infty} \cos x\lambda \, d\lambda \left[\int_{0}^{\infty} e^{-t} \cos t \cos \lambda t \, dt \right]$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \cos x\lambda \, d\lambda \cdot \left[\frac{1}{2} \int_{0}^{\infty} e^{-t} \{\cos(\lambda + 1)t + \cos(\lambda - 1)t\} \, dt \right]$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \cos x\lambda \, d\lambda \left[\frac{e^{-t}}{(\lambda + 1)^2 + 1} \{-\cos(\lambda + 1)t + (\lambda + 1) \sin(\lambda + 1)t\} \right.$$

$$\left. + \frac{e^{-t}}{(\lambda - 1)^2 + 1} \{-\cos(\lambda - 1)t + (\lambda - 1) \sin(\lambda - 1)t\} \right]_0^{\infty}$$

$$= \frac{1}{\pi}\int_0^\infty \left\{\frac{1}{(\lambda+1)^2+1} + \frac{1}{(\lambda-1)^2+1}\right\} \cos x\lambda \, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{(\lambda^2+2)\cos x\lambda}{\lambda^4+4}\, d\lambda.$$

Example 3

Using Fourier integral formula, prove that

$$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{u \sin xu}{(u^2+a^2)(u^2+b^2)}\, du (a, b > 0)$$

Note ✍

The letter 'u' has been used instead of 's'

The presence of $\sin xu$ in the integral suggests that the Fourier sine integral formula has been used.

Fourier sine integral representation is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t) \sin ut \sin xu \; dt \; du$$

$$\therefore \quad e^{-ax} - e^{-bx} = \frac{2}{\pi}\int_0^\infty \sin x\, u \; du \left[\int_0^\infty (e^{-at} - e^{-bt}) \sin ut \; dt\right]$$

$$= \frac{2}{\pi}\int_0^\infty \sin xu \; du \left[\frac{e^{-at}}{a^2+u^2}\{-a \sin ut - u \cos ut\}\right.$$

$$\left. - \frac{e^{-bt}}{b^2+u^2}\{-b \sin ut - u \cos ut\}\right]_0^\infty$$

$$= \frac{2}{\pi}\int_0^\infty \sin xu \; du \left[\frac{u}{a^2+u^2} - \frac{u}{b^2+u^2}\right]$$

$$= \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{u \sin xu}{(u^2+a^2)(u^2+b^2)}\, du$$

Example 4

Find the Fourier transform of the unit step function and unit impulse function.

(i) The unit step function is defined as

$$u_a(x) = \begin{cases} 0, & \text{for } x < a \\ 1, & \text{for } x \geq a \end{cases}$$

$$\therefore \qquad F\{u_a(x)\} = \int_a^{\infty} e^{-isx}\, dx = \left[\frac{e^{-isx}}{-is}\right]_a^{\infty} = \frac{1}{is}e^{-ias}$$

In particular $\qquad F\{u_0(x)\} = \frac{1}{is} \quad \text{or} \quad -\frac{i}{s}$

(ii) The unit impulse function or Dirac Delta function $\delta_a(x)$ is defined as $\lim\limits_{\in \to 0}[f(x)]$, where

$$f(x) = \begin{cases} \frac{1}{\in}, & \text{for } a - \frac{\in}{2} \leq x \leq a + \frac{\in}{2} \\ 0, & \text{elsewhere} \end{cases}$$

$$\therefore \qquad F\{f(x)\} = \int_{a-\in/2}^{a+\in/2} \frac{1}{\in} e^{-isx}\, dx$$

$$= \frac{1}{\in}\left[\frac{e^{-isx}}{-is}\right]_{a-\in/2}^{a+\in/2}$$

$$= \frac{1}{i \in s}\left\{e^{-is(a-\in/2)} - e^{-is(a+\in/2)}\right\}$$

$$= e^{-ias} \cdot \left[\frac{\sin\left(\frac{\in S}{2}\right)}{\left(\frac{\in S}{2}\right)}\right]$$

$$\therefore \qquad F\{\delta_a(x)\} = \lim_{\in \to 0}\left[e^{-ias} \cdot \left\{\frac{\sin\left(\frac{\in S}{2}\right)}{\frac{\in S}{2}}\right\}\right]$$

$$= e^{-ias}$$

In particular, $\qquad F\{\delta_0(x)\} = 1$

Example 5

Find the Fourier transform of $f(x)$, defined as

$$f(x) = \begin{cases} 1, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}$$

and hence find the value of $\int\limits_0^{\infty} \frac{\sin x}{x}\, dx$

$$F\{f(x)\} = \int_{-a}^{a} e^{-isx}\ dx = \int_{-a}^{a} (\cos sx - i\sin sx)\ dx$$

$$= 2\int_{0}^{a} \cos sx\ dx, \text{ by the property of definite integrals}$$

$$= \frac{2}{s}\sin as.$$

Taking Fourier inverse transforms,

$$F^{-1}\left\{\frac{2}{s}\sin as\right\} = f(x)$$

i.e. $$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{s}\sin\ a\ s\ e^{ixs}\ ds = f(x)$$

i.e. $$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{s}\sin\ a\ s(\cos\ xs + i\sin xs)\ ds = f(x)$$

i.e. $$\frac{2}{\pi}\int_{0}^{\infty} \frac{1}{s}\sin as\ \cos\ xs\ ds = f(x)\left[\because \frac{1}{s}\sin\ as\sin xs \text{ is odd}\right]$$

i.e. $$\int_{0}^{\infty} \frac{1}{s}\sin\ as\ \cos\ xs\ ds = \begin{cases} \pi/2, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}$$

Putting $a = 1$ and $x = 0$,
so that $|0| < 1$, we get

$$\int_{0}^{\infty} \frac{\sin s}{s}\ ds = \frac{\pi}{2}$$

Changing the dummy variable s into x, we get

$$\int_{0}^{\infty} \frac{\sin x}{x}\ dx = \frac{\pi}{2}$$

Example 6

Find the inverse Fourier transform of $\bar{f}(s)$ given by $\bar{f}(s) = \begin{cases} a - |s|, & \text{for } |s| \leq a \\ 0, & \text{for } |s| > a \end{cases}$

Hence show that $\displaystyle\int_{0}^{\infty} \frac{\sin^2 x}{x^2}\ dx = \frac{\pi}{2}$

$$F^{-1}\{\bar{f}(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)e^{ixs}\ ds$$

$$= \frac{1}{2\pi}\int_{-a}^{a} \{a - |s|\}\ (\cos xs + i \sin xs)\ ds$$

$$= \frac{1}{\pi}\int_{0}^{a} (a - s)\ \cos xs\ ds \qquad [\because \{a - |s|\} \sin xs \text{ is odd}]$$

$$= \frac{1}{\pi}\left[(a - s)\frac{\sin xs}{x} - \frac{\cos xs}{x^2}\right]_0^a$$

$$= \frac{1}{\pi x^2}(1 - \cos ax)$$

$$= \frac{a^2}{2\pi}\left\{\frac{\sin \frac{ax}{2}}{\frac{ax}{2}}\right\}^2$$

$$\therefore \qquad F\left[\frac{a^2}{2\pi}\left\{\frac{\sin \frac{ax}{2}}{\frac{ax}{2}}\right\}^2\right] = \bar{f}(s)$$

i.e.
$$\frac{a^2}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{\sin \frac{ax}{2}}{\frac{ax}{2}}\right\}^2 e^{-isx}\ dx = \begin{cases} a - |s|, & \text{for } |s| \le a \\ 0, & \text{for } |s| > a \end{cases}$$

Taking $a = 2$ and letting $s \longrightarrow 0$, we get

$$\int_{-\infty}^{\infty}\left\{\frac{\sin x}{x}\right\}^2\ dx = \pi$$

Since the integrand is an even function of x,
we get

$$\int_{0}^{\infty}\left\{\frac{\sin x}{x}\right\}^2\ dx = \frac{\pi}{2}$$

Example 7

Find the Fourier transform of $f(x)$ given by

$$f(x) = \begin{cases} 1 - x^2, & \text{for } |x| < 1 \\ 0, & \text{for } |x| > 1 \end{cases}$$

Hence evaluate $\displaystyle\int_{0}^{\infty}\left\{\frac{\sin x - x\cos x}{x^3}\right\}\cos\frac{x}{2}\ dx$

$$F\{f(x)\} = \int_{-1}^{1} (1 - x^2)e^{-isx}\ dx$$

$$= 2\int_{0}^{1} (1 - x^2)\cos\ sx\ dx \quad [\because (1 - x^2)\sin\ sx \text{ is an odd function of } x]$$

$$= 2\left[(1 - x^2)\frac{\sin sx}{s} - 2x\frac{\cos sx}{s^2} + 2\frac{\sin sx}{s^3}\right]_0^1$$

$$= \frac{4}{s^3}(\sin\ s - s\cos\ s)$$

Taking inverse Fourier transform, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{s^3}(\sin s - s\cos s)(\cos xs + i\sin xs)\ ds = f(x)$$

i.e.
$$\int_{0}^{\infty} \frac{(\sin s - s\cos s)}{s^3}\cos xs\ ds = \pi/4.\begin{cases} 1 - x^2, & \text{for } |x| < 1 \\ 0, & \text{for } |x| > 1 \end{cases}$$

Taking $x = \dfrac{1}{2}$,

we have,
$$\int_{0}^{\infty} \left\{\frac{\sin s - s\cos s}{s^3}\right\}\cos\frac{s}{2}\ ds = \frac{3\pi}{16}$$

Replacing the dummy variables s by x, we get the required result.

Example 8

Find the Fourier transform of $e^{-a^2x^2}$. Hence

(i) prove that $e^{-x^2/2}$ is self-reciprocal with respect to Fourier transforms; and
(ii) find the Fourier cosine transform of e^{-x^2}.

$$F\left\{e^{-a^2x^2}\right\} = \int_{-\infty}^{\infty} e^{-a^2x^2}\cdot e^{-isx}\ dx$$

$$= \int_{-\infty}^{\infty} e^{-\left(ax + \frac{is}{2a}\right)^2}\cdot e^{-\frac{s^2}{4a^2}}\ dx$$

$$= e^{-s^2/4a^2} \cdot \frac{1}{a} \int_{-\infty}^{\infty} e^{-t^2} \, dt, \text{ on putting } ax + \frac{is}{2a} = t$$

$$= \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2} \tag{1}$$

(i) Had we assumed the definition of the Fourier transform as

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx} \, dx$$

(1) would have become

$$F\left\{e^{-a^2x^2}\right\} = \frac{1}{a\sqrt{2}} e^{-s^2/4a^2}$$

Putting $a = \frac{1}{\sqrt{2}}$ in (2), we get

$$F\left\{e^{-x^2/2}\right\} = e^{-s^2/2} \text{ and so } F^{-1}\left\{e^{-s^2/2}\right\} = e^{-x^2/2}$$

i.e. $e^{-x^2/2}$ is self-reciprocal with respect to Fourier transforms.
(ii) From (1), we have

$$\int_{-\infty}^{\infty} e^{-a^2x^2} (\cos sx - i \sin sx) \, dx = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$$

Equating the real parts on both sides, we get

$$\int_{0}^{\infty} e^{-a^2x^2} \cos sx \, dx = \frac{\sqrt{\pi}}{2a} e^{-s^2/4a^2}$$

or

$$F_C\left\{e^{-a^2x^2}\right\} = \frac{\sqrt{\pi}}{2a} e^{-s^2/4a^2}$$

Example 9

Find the Fourier cosine transform of e^{-ax} and use it to find the Fourier transform of $e^{-a|x|} \cos bx$.

$$F_C(e^{-ax}) = \int_{0}^{\infty} e^{-ax} \cos sx \, dx$$

$$= \left[\frac{e^{-ax}}{s^2 + a^2} (-a \cos sx + s \sin sx) \right]_0^\infty$$

$$= \frac{a}{s^2 + a^2}, \ a > 0 \qquad (1)$$

Now
$$F(e^{-a|x|} \cos bx) = \int_{-\infty}^{\infty} e^{-a|x|} \cos bx \cdot e^{-isx} \, dx$$

$$= 2 \int_0^\infty e^{-ax} \cos bx \cos sx \, dx,$$

(since the integral of the odd part of the integrand vanishes)

$$= \int_0^\infty e^{-ax} \{\cos (s+b)x + \cos (s-b)x\} \, dx$$

$$= F_C\{e^{-ax}\}_{s \to s+b} + F_C\{e^{-ax}\}_{s \to s-b}$$

$$= a \left\{ \frac{1}{(s+b)^2 + a^2} + \frac{1}{(s-b)^2 + a^2} \right\}, \text{ by } (1)$$

Example 10

Find the Fourier cosine transform of $f(x)$ defined as $f(x) = \begin{cases} 1, & \text{for } 0 < x < a \\ 0, & \text{for } x \geq a \end{cases}$

Hence find the inverse Fourier cosine transform of $\left(\frac{\sin as}{s} \right)$. Verify your answer by directly finding $F_C^{-1} \left(\frac{\sin as}{s} \right)$.

$$F_C\{f(x)\} = \int_0^\infty f(x) \cos sx \, dx = \int_0^a \cos sx \, dx$$

$$= \frac{\sin as}{s}$$

$\therefore$
$$F_C^{-1} \left\{ \frac{\sin as}{s} \right\} = f(x) = \begin{cases} 1, & \text{for } 0 < x < a \\ 0, & \text{for } x \geq a \end{cases}$$

Now
$$F_C^{-1} \left(\frac{\sin as}{s} \right) = \frac{2}{\pi} \int_0^\infty \frac{\sin as}{s} \cos xs \, ds$$

$$= \frac{1}{\pi} \int_0^\infty \frac{\sin(a+x)s + \sin(a-x)s}{s} \, ds$$

$$= \frac{1}{\pi}\left\{\int_0^\infty \frac{\sin(a+x)s}{s}\,ds + \int_0^\infty \frac{\sin(a-x)s}{s}\,ds\right]$$

$$= \begin{cases} \frac{1}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right), \text{if } x < a \\ \frac{1}{\pi}\left(\frac{\pi}{2}-\frac{\pi}{2}\right), \text{if } x > a \end{cases} \qquad \left(\because \int_0^\infty \frac{\sin ms}{s}\,ds = \frac{\pi}{2}, \text{ when } m > 0\right)$$

$$= \begin{cases} 1, \text{if } 0 < x < a \\ 0, \text{if } x \geq a \end{cases}$$

Example 11

Find the Fourier sine transform of $f(x)$ defined as $f(x) = \begin{cases} \sin x, & \text{when } 0 < x < a \\ 0, & \text{when } x > a \end{cases}$

$$F_S\{f(x)\} = \int_0^\infty f(x)\sin sx\,dx$$

$$= \int_0^a \sin x \sin sx\,dx$$

$$= \frac{1}{2}\int_0^a [\cos(s-1)x - \cos(s+1)x]\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(s-1)x}{s-1} - \frac{\sin(s+1)x}{s+1}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1}\right]$$

Example 12

Find the Fourier cosine transform of $f(x)$ defined as $f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2-x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2 \end{cases}$

$$F_C\{f(x)\} = \int_0^\infty f(x)\cos sx\,dx$$

$$= \int_0^1 x\cos sx\,dx + \int_1^2 (2-x)\cos sx\,dx$$

$$= \left[x\frac{\sin sx}{s} + \frac{\cos sx}{s^2}\right]_0^1 + \left[(2-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_1^2$$

$$= \frac{2\cos s}{s^2} - \frac{1}{s^2}(1+\cos 2s)$$
$$= \frac{2\cos s(1-\cos s)}{s^2}$$

Example 13

Find the Fourier sine transform of e^{-ax} $(a > 0)$. Hence find $F_S\{xe^{-ax}\}$ and $F_S\left\{\frac{e^{-ax}}{x}\right\}$.

Deduce the value of $\int\limits_0^\infty \frac{\sin sx}{x}\,dx$.

$$F_S(e^{-ax}) = \int\limits_0^\infty e^{-ax}\sin sx\,dx$$
$$= \left[\frac{e^{-ax}}{s^2+a^2}(-a\sin sx - s\cos sx)\right]_0^\infty$$
$$= \frac{s}{s^2+a^2}$$

i.e. $$\int\limits_0^\infty e^{-ax}\sin sx\,dx = \frac{s}{s^2+a^2} \qquad (1)$$

Differentiating both sides of (1) with respect to 'a', we get,

$$\int\limits_0^\infty -xe^{-ax}\sin sx\,dx = -\frac{2as}{(s^2+a^2)^2}$$

i.e. $$F_S(xe^{-ax}) = \frac{2as}{(s^2+a^2)^2}$$

Integrating both sides of (1) with respect to 'a' between a and ∞,

$$\int\limits_0^\infty \left(\frac{e^{-ax}}{-x}\right)_a^\infty \sin sx\,dx = \left[-\cot^{-1}\left(\frac{a}{s}\right)\right]_a^\infty$$

i.e. $$\int\limits_0^\infty \left(\frac{e^{-ax}}{x}\right)\sin sx\,dx = \cot^{-1}\left(\frac{a}{s}\right)$$

i.e. $$F_S\left(\frac{e^{-ax}}{x}\right) = \cot^{-1}\left(\frac{a}{s}\right), \quad a > 0 \qquad (2)$$

Taking limits on both sides of (2) as $a \to 0$,

we get $\quad F_S\left(\frac{1}{x}\right) = \cot^{-1}(0) = \frac{\pi}{2}$

Thus $\quad \int_0^\infty \frac{\sin sx}{x}\,dx = \frac{\pi}{2}, \quad s > 0.$

Example 14

Find the Fourier cosine transform of $\frac{1}{x^2 + a^2}$

$$F_C\left\{\frac{1}{x^2 + a^2}\right\} = \int_0^\infty \frac{\cos sx}{x^2 + a^2}\,dx = I, \text{ say}$$

$$I = \int_0^\infty \frac{\cos sx}{x^2 + a^2}\,dx \tag{1}$$

Differentiating both sides of (1) with respect to 's', we get

$$\frac{dI}{ds} = \int_0^\infty -\frac{x}{x^2 + a^2}\sin sx\ dx\left[= -F_S\left(\frac{x}{x^2 + a^2}\right)\right] \tag{2}$$

$$= \int_0^\infty \frac{a^2 - (x^2 + a^2)}{x(x^2 + a^2)}\sin sx\,dx$$

$$= a^2\int_0^\infty \frac{\sin sx}{x(x^2 + a^2)}\,dx - \int_0^{\pi/2} \frac{\sin sx}{x}\,dx$$

$$= a^2\int_0^\infty \frac{\sin sx}{x(x^2 + a^2)}\,dx - \pi/2 \tag{3}$$

Differentiating both sides of (3) with respect to 's' again, we get

$$\frac{d^2I}{ds^2} = a^2\int_0^\infty \frac{\cos sx}{x^2 + a^2}\,dx$$

i.e. $$\frac{d^2I}{ds^2} - a^2I = 0 \tag{4}$$

Solving the differential equation (4), we get

$$I = Ae^{as} + Be^{-as} \tag{5}$$

From (1), when $\quad s = 0, \quad I = \left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right)_0^\infty = \frac{\pi}{2a}$

Using this in (5), we have $A + B = \dfrac{\pi}{2a}$ (6)

From (3), when $s = 0$, $\dfrac{dI}{ds} = -\dfrac{\pi}{2}$.

Using this in (5), we have $A - B = -\dfrac{\pi}{2a}$ (7)

Solving (6) and (7), we get $\quad A = 0$ and $B = \dfrac{\pi}{2a}$

Using this in (5), we have

$$I = F_C\left\{\frac{1}{x^2+a^2}\right\} = \frac{\pi}{2a}e^{-as}$$

Also from (2), we have $\quad \dfrac{dI}{ds} = -F_S\left(\dfrac{x}{x^2+a^2}\right) = -\dfrac{\pi}{2}e^{-as}$

$$\therefore \qquad F_S\left(\frac{x}{x^2+a^2}\right) = \frac{\pi}{2}e^{-as}.$$

Example 15

Find $f(x)$, if its Fourier sine transform is $\dfrac{s}{s^2+1}$.

Given $\quad F_S\{f(x)\} = \dfrac{s}{s^2+1}$

$$\therefore \qquad f(x) = F_S^{-1}\left\{\frac{s}{s^2+1}\right\} = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+1}\sin xs \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{(s^2+1-1)}{s(s^2+1)}\sin xs \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin xs}{s}\, ds - \frac{2}{\pi}\int_0^\infty \frac{\sin xs}{s(s^2+1)}\, ds$$

$$= \frac{2}{\pi}\cdot\frac{\pi}{2} - \frac{2}{\pi}\int_0^\infty \frac{\sin xs}{s(s^2+1)}\, ds \tag{1}$$

Differentiating (1) with respect to x, we get

$$\frac{df}{dx} = -\frac{2}{\pi}\int_0^\infty \frac{\cos xs}{s^2+1}\, ds \left[= -F_C^{-1}\left(\frac{1}{s^2+1}\right)\right] \tag{2}$$

and

$$\frac{d^2f}{dx^2} = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+1}\sin xs \, ds = f$$

i.e. $$\frac{d^2 f}{dx^2} - f = 0 \tag{3}$$

Solving (3), we get $f = Ae^{x} + Be^{-x}$ (4)

From (1), when $x = 0$, $f = 1$. Using this in (4)

we have $$A + B = 1 \tag{5}$$

From (2), when $x = 0$, $\frac{df}{dx} = -\frac{2}{\pi}(\tan^{-1} s)_0^{\infty} = -1$. Using this in (4),

we have $$A - B = -1 \tag{6}$$

Solving (5) and (6), we get $A = 0$ and $B = 1$

$$\therefore \quad f(x) = F_S^{-1}\left\{\frac{1}{s^2+1}\right\} = e^{-x}$$

Also from (2), we have $f_c^{-1}\left\{\frac{s}{s^2+1}\right\} = e^{-x}$

Example 16

Find the Fourier sine and cosine transforms of x^{n-1}. Hence deduce that $\frac{1}{\sqrt{x}}$ is self-reciprocal under both the transforms. Also find $F\left\{\frac{1}{\sqrt{|x|}}\right\}$.

Consider $$\int_0^{\infty} x^{n-1} e^{-isx}\, dx = \int_0^{\infty} \left(\frac{t}{is}\right)^{n-1} e^{-t}\left(\frac{dt}{is}\right)$$

on putting $isx = t$

$$= \left(\frac{-i}{s}\right)^n \int_0^{\infty} t^{n-1} e^{-t}\, dt$$

$$= \frac{\overline{|(n)}}{s^n} \cdot (e^{-i\pi/2})^n \quad \left[\because e^{-i\pi/2} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i\right]$$

i.e. $$\int_0^{\infty} x^{n-1}(\cos sx - i\sin sx)\, dx = \left\{\frac{\overline{|(n)}}{s^n}\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)\right\} \tag{1}$$

Equating real parts on both sides of (1) we get

$$\int_0^{\infty} x^{n-1}\cos sx\, dx = \frac{\overline{|(n)}}{s^n}\cos\frac{n\pi}{2}$$

i.e. $$F_C(x^{n-1}) = \frac{\overline{|(n)}}{s^n}\cos\frac{n\pi}{2} \tag{2}$$

Similarly, equating imaginary parts on both sides of (1), we get,

$$F_S(x^{n-1}) = \frac{\Gamma(n)}{s^n} \sin \frac{n\pi}{2} \tag{3}$$

Assuming that $F_C\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$ and

$$F_S\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx \text{ and taking } n = \frac{1}{2}$$

we get
$$F_C\left(\frac{1}{\sqrt{x}}\right) = F_S\left(\frac{1}{\sqrt{x}}\right) = \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} \cdot \sin \frac{\pi}{4}$$
$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{s}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{s}}$$

Thus $\frac{1}{\sqrt{x}}$ is self-reciprocal under both Fourier cosine and sine transforms.

Now
$$F\left\{\frac{1}{\sqrt{|x|}}\right\} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}} e^{-isx} \, dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}} (\cos sx - i \sin sx) \, dx$$
$$= 2 \int_0^\infty \frac{1}{\sqrt{x}} \cos sx \, dx \quad \text{[by the property of even and odd functions]}$$
$$= 2 \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} \cos \frac{\pi}{4} \qquad \left[\text{by putting } n = \frac{1}{2} \text{ in (2)}\right]$$
$$= \sqrt{\frac{2\pi}{s}}$$

Example 17

Solve, for $f(x)$, the integral equation

$$\int_0^\infty f(x) \cos sx \, dx = \begin{cases} 1 - s, & \text{for } 0 < s < 1 \\ 0, & \text{for } s > 1 \end{cases}$$

Given
$$F_C\{f(x)\} = \begin{cases} 1 - s, & \text{for } 0 < s < 1 \\ 0, & \text{for } s > 1 \end{cases}$$
$$= \bar{f}_C(s), \text{ say.}$$

$$\therefore \quad f(x) = F_C^{-1}\{\bar{f}_C(s)\}$$
$$= \frac{2}{\pi}\int_0^1 (1-s)\cos xs \; ds$$

$$= \frac{2}{\pi}\left[(1-s)\frac{\sin xs}{x} - \frac{\cos xs}{x^2}\right]_0^1$$
$$= \frac{2}{\pi x^2}(1-\cos x)$$

Example 18

Solve, for $f(x)$, the integral equation

$$\int_0^\infty f(x)\sin xt \; dx = \begin{cases} 1, & \text{for } 0 \le t < 1 \\ 2, & \text{for } 1 \le t < 2 \\ 0, & \text{for } t \ge 2 \end{cases}$$

Note ✍

(Instead of the usual transform variable s, the letter 't' is used)

$$\text{Given} \quad F_S\{f(x)\} = \bar{f}_S(t) = \begin{cases} 1, & \text{for } 0 \le t \le 1 \\ 2, & \text{for } 1 \le t < 2 \\ 0, & \text{for } t \ge 2 \end{cases}$$

$$\therefore \quad f(x) = F_S^{-1}\{\bar{f}_S(t)\}$$
$$= \frac{2}{\pi}\left[\int_0^1 1\cdot\sin xt \; dt + \int_1^2 2\cdot\sin xt \; dt\right]$$
$$= \frac{2}{\pi}\left[\left(\frac{-\cos xt}{x}\right)_0^1 + 2\left(\frac{-\cos xt}{x}\right)_1^2\right]$$
$$= \frac{2}{\pi x}[(1-\cos x) + 2(\cos x - \cos 2x)]$$
$$= \frac{2}{\pi x}(1+\cos x - 2\cos 2x).$$

Exercise 4(a)

Part A (Short Answer Questions)

1. State both the forms of Fourier integral theorem.
2. Write down the Fourier cosine and sine integral representations of $f(x)$.
3. Write down the complex Fourier transform pair.
4. Write down the Fourier cosine transform pair.

5. Write down the Fourier sine transform pair.

6. Solve, for $f(x)$, the equation $\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x)e^{-isx}\,dx = \phi(s)$.

7. Find $f(x)$, if $\sqrt{\frac{2}{\pi}} \int\limits_{0}^{\infty} f(x)\cos sx\ \ dx = \phi(s)$.

8. Find $f(x)$, if $\sqrt{\frac{2}{\pi}} \int\limits_{0}^{\infty} f(x)\sin sx\ \ dx = \phi(s)$.

9. Find $f(x)$, if $\int\limits_{-\infty}^{\infty} f(x)e^{isx}\,dx = \phi(s)$.

10. How are Fourier and Laplace transforms related?
11. Find the Fourier cosine transform of $e^{-ax}(a > 0)$.
12. Find the Fourier sine transform of $e^{-ax}(a > 0)$.
13. Find the Fourier exponential transform of $e^{-a|x|}$, $a > 0$.
14. Find the Fourier cosine transform of $2e^{-5x} + 5e^{-2x}$.
15. Find the Fourier sine transform of $e^{-2x} + 4e^{-3x}$.
16. Find the Fourier complex transform of $f(x)$,

if $\quad f(x) = \begin{cases} k, & \text{in } |x| \le l \\ 0, & \text{in } |x| > l \end{cases}$

Part B

17. Find the Fourier integral representation of

$$f(x) = \begin{cases} 1, & \text{for } |x| \le 1 \\ 0, & \text{for } |x| > 1 \end{cases}$$

Hence evaluate $\int\limits_{0}^{\infty} \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$.

18. Use the appropriate Fourier integral formula to prove that

$$e^{-ax} = \frac{2a}{\pi}\int_0^{\infty} \frac{\cos x\lambda}{\lambda^2 + a^2}d\lambda.$$

19. Find the Fourier sine integral formula for $\frac{\pi}{2}e^{-x}$.
20. Use Fourier integral formula to prove that

$$\int\limits_{0}^{\infty}\left(\frac{1-\cos\pi\lambda}{\lambda}\right)\sin x\lambda\ \,d\lambda = \begin{cases} \pi/2, & \text{when}\quad 0 < x < \pi \\ 0, & \text{when}\quad x > \pi \end{cases}$$

21. Use Fourier integral formula to prove that

$$\int_0^\infty \frac{\sin \pi\lambda \sin x\lambda}{1-\lambda^2}\, d\lambda = \begin{cases} \dfrac{\pi}{2}\sin x, & \text{when } 0 \le x \le \pi \\ 0, & \text{when } x > \pi \end{cases}$$

22. Find the Fourier transform of $f(x)$, given by

$$f(x) = \begin{cases} x, & \text{for } |x| \le a \\ 0, & \text{for } |x| > a \end{cases}$$

23. Find the Fourier transform of $f(x)$, given by

$$f(x) = \begin{cases} 1-|x|, & \text{for } |x| < 1 \\ 0, & \text{for } |x| > 1 \end{cases}$$

Hence show that $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$.

24. Find the inverse Fourier transform of $\phi(s)$, defined as

$$\phi(s) = \begin{cases} 1+s^2, & \text{for } |s| < 1 \\ 0, & \text{for } |s| > 1 \end{cases}$$

25. Find the inverse Fourier transform of $\phi(s)$, defined as

$$\phi(s) = \begin{cases} 1, & \text{for } |s| < s_0 \\ 0, & \text{for } |s| > s_0 \end{cases}$$

Hence evaluate $\displaystyle\int_0^\infty \frac{\sin mx}{x}\, dx, m > 0$.

26. Find the Fourier cosine transform of e^{-ax}; $a > 0$. Hence find $F_C\{xe^{-ax}\}$ and $F\{|x|e^{-a|x|}\}$.
27. Find the Fourier sine transform of e^{-ax}; $a > 0$. Hence find $F_S\{xe^{-ax}\}$ and $F\{xe^{-a|x|}\}$.
28. Find the Fourier transform of $e^{-ax^2}\cos bx$.
29. Find the inverse transform of e^{-s^2}.
30. Find the inverse Fourier transform of $\dfrac{1}{(1+s^2)^2}$.

(**Hint:** Use contour integration).

31. Find the Fourier sine transform of $f(x)$, given by

$$f(x) = \begin{cases} 1, & \text{for } 0 < x < 1 \\ 0, & \text{for } x > 1 \end{cases}$$

32. Find the Fourier sine transform of $f(x)$, if

$$f(x)=\begin{cases}0, & 0<x<a\\ x, & a\le x\le b\\ 0, & x>b\end{cases}$$

33. Find the Fourier cosine transform of $f(x)$, if

$$f(x)=\begin{cases}\cos x, & \text{for } 0<x<a\\ 0, & \text{for } x>a\end{cases}$$

34. Find the Fourier sine transform of $\dfrac{x}{1+x^2}$.

35. Find the Fourier cosine transform of e^{-x^2}, using the definition directly.

36. Find the Fourier sine transform of $\dfrac{e^{-ax}}{x}(a>0)$, using the definition directly.

37. Find the inverse Fourier cosine transform of $\dfrac{1}{1+s^2}$.

38. Find the inverse Fourier sine transform of $\dfrac{e^{-as}}{s}(a>0)$, using the definition directly.

39. Find $f(x)$, if $\displaystyle\int_0^\infty f(x)\cos sx\ \mathrm{d}x=\frac{\sin s}{s}$.

40. Find $f(x)$, if $\displaystyle\int_0^\infty f(x)\sin sx\ \mathrm{d}x=e^{-as}$.

4.6 PROPERTIES OF FOURIER TRANSFORMS

1. Linearity property

F is a linear operator,i.e. $F[(c_1f_1(x)+c_2f_2(x)]=c_1F\{f_1(x)\}+c_2F\{f_2(x)\}$, where c_1 and c_2 are constants.

Proof

$$F\left[c_1f_1(x)+c_2f_2(x)\right]=\int_{-\infty}^{\infty}\left[c_1f_1(x)+c_2f_2(x)\right]e^{-isx}\,\mathrm{d}x$$

$$=c_1\int_{-\infty}^{\infty}f_1(x)e^{-isx}\,\mathrm{d}x+c_2\int_{-\infty}^{\infty}f_2(x)e^{-isx}\,\mathrm{d}x$$

$$=c_1F\{f_1(x)\}+c_2F\{f_2(x)\}.$$

2. Change of scale property

If $$F\{f(x)\} = \bar{f}(s), \quad \text{then}$$
$$F\{f(ax)\} = \frac{1}{|a|}\bar{f}\left\{\frac{s}{a}\right\}$$

Proof

$$F\{f(ax)\} = \int_{-\infty}^{\infty} f(ax)e^{-isx}\,dx$$
$$= \int_{-\infty}^{\infty} f(t)e^{-ist/a}\cdot\frac{dt}{a}, \text{ on putting } ax = t \text{ and assuming that } a > 0.$$
$$= \frac{1}{a}\bar{f}\left\{\frac{s}{a}\right\}$$

But $$F\{f(ax)\} = \int_{\infty}^{-\infty} f(t)e^{-ist/a}\cdot\frac{dt}{a}, \text{ if } a < 0$$
$$= -\frac{1}{a}\bar{f}\left\{\frac{s}{a}\right\}$$

$\therefore$ $$F\{f(ax)\} = \frac{1}{|a|}\bar{f}\left\{\frac{s}{a}\right\}$$

Similarly, $$F_C\{f(ax)\} = \frac{1}{a}\cdot\bar{f}_C\left\{\frac{s}{a}\right\} \text{ and } F_S\{f(ax)\} = \frac{1}{a}\cdot\bar{f}_S\left\{\frac{s}{a}\right\}.$$

3. Shifting property (Shifting in x)

If $F\{f(x)\} = \bar{f}(s)$, then $F\{f(x-a)\} = e^{-ias}\bar{f}(s)$.

Proof

$$F\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a)e^{-isx}\,dx$$
$$= \int_{-\infty}^{\infty} f(t)e^{-is(t+a)}\,dt, \text{ on putting } t = x - a$$
$$= e^{-ias}\bar{f}(s).$$

4. Shifting in respect of s

If $F\{f(x)\} = \bar{f}(s)$, then $F\{e^{-iax}f(x)\} = \bar{f}(s+a)$

Proof

$$F\{e^{-iax} f(x)\} = \int_{-\infty}^{\infty} e^{-iax} f(x) e^{-isx} \, dx$$
$$= \int_{-\infty}^{\infty} f(x) e^{-i(s+a)x} \, dx$$
$$= \bar{f}(s+a)$$

Similarly $\qquad F\{e^{iax} f(x)\} = \bar{f}(s-a).$

5. Modulation theorem

If $\quad F\{f(x)\} = \bar{f}(s), \quad$ then $\quad F\{f(x)\cos ax\} = \frac{1}{2}[\bar{f}(s+a) + \bar{f}(s-a)].$

Proof

$$F\{f(x)\cos ax\} = \frac{1}{2} F[f(x)(e^{iax} + e^{-iax})]$$
$$= \frac{1}{2}[F\{f(x)e^{iax}\} + F\{f(x)e^{-iax}\}]$$
$$= \frac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)].$$

Corollaries

(i) $F_C\{f(x)\cos ax\} = \frac{1}{2}\{\bar{f}_C(s+a) + \bar{f}_C(s-a)\}$

(ii) $F_C\{f(x)\sin ax\} = \frac{1}{2}\{\bar{f}_S(a+s) + \bar{f}_S(a-s)\}$

(iii) $F_S\{f(x)\cos ax\} = \frac{1}{2}\{\bar{f}_S(s+a) + \bar{f}_S(s-a)\}$

(iv) $F_S\{f(x)\sin ax\} = \frac{1}{2}\{\bar{f}_C(s-a) - \bar{f}_C(s+a)\}$

6. Conjugate symmetry property

If $F\{f(x) = \bar{f}(s)$, then $F\{f^*(-x)\} = [\bar{f}(s)]^*$, where * denotes complex conjugate.

Proof

$$\bar{f}(s) = \int_{-\infty}^{\infty} f(x) e^{-isx} dx$$

$$\therefore \qquad [\bar{f}(s)]^* = \int_{-\infty}^{\infty} f^*(x)e^{isx}dx$$

$$= \int_{-\infty}^{\infty} f^*(-t)e^{-ist}dt, \text{ on putting } x = -t$$

$$= F\{f^*(-x)\}$$

Note ✍

1. $F\{f^*(x)\} = [\bar{f}(-s)]^*$.
2. If $f(x)$ is a real valued function, then $F\{f(-x)\} = [\bar{f}(s)]^*$.

5. Transform of derivatives

If $f(x)$ is continuous, $f'(x)$ is piecewise continuously differentiable, $f(x)$ and $f'(x)$ are absolutely integrable in $(-\infty, \infty)$ and $\lim\limits_{x\to\pm\infty} [f(x)] = 0$, then

$$F\{f'(x) = is\bar{f}(s), \text{ where } \bar{f}(s) = F\{f(x)\}$$

Proof

By the first three conditions given, $F\{f(x)$ and $F\{f'(x)\}$ exist.

$$F\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)e^{-isx}\,dx$$

$$= [e^{-isx}f(x)]_{-\infty}^{\infty} + is\int_{-\infty}^{\infty} e^{-isx}f(x)\,dx, \text{ on integrating by parts.}$$

$$= 0 + \text{is}F\{f(x)\}, \text{ by the given condition.}$$

$$= \text{is }\bar{f}(s).$$

The theorem can be extended as follows.

If $f, f', f'', \ldots, f^{(n-1)}$ are continuous, $f^{(n)}$ is piecewise continuous, $f, f' f'' \ldots, f^{(n)}$ are absolutely integrable in $(-\infty, \infty)$ and $f, f' f'', \ldots, f^{(n-1)} \to 0$ as $x \to \pm\infty$, then

$$F\{f^{(n)}(x)\} = (is)^n \bar{f}(s)$$

where $\bar{f}(s) = F\{f(x)\}$.

Corollaries

(i) $F_C\{f'(x) = sF_S\{f(x)\} - f(0)$.
(ii) $F_S\{f'(x)\} = -sF_C\{f(x)\}$.
(iii) $F_C\{f''(x)\} = -s^2F_C\{f(x)\} - f'(0)$.
(iv) $F_S\{f''(x)\} = -s^2F_S\{f(x)\} + sf(0)$.

Note ✍

1. *When we solve boundary value problems (partial differential equations) by using Fourier cosine and sine transforms, the modified forms of the above formulas given below will be used. It is assumed that transforms are taken with respect to the variable x. We use the following notations.*

$$F_C\{f(x, y)\} = \bar{f}_C(s, y) \text{ and } F_S\{f(x, y)\} = \bar{f}_S(s, y)$$

1. $F_C\left\{\dfrac{\partial f}{\partial x}\right\} = s\bar{f}_s(s, y) - f(0, y).$

2. $F_S\left\{\dfrac{\partial f}{\partial x}\right\} = -s\bar{f}_C(s, y).$

3. $F_C\left\{\dfrac{\partial^2 f}{\partial x^2}\right\} = -s^2\bar{f}_C(s, y) - \dfrac{\partial f}{\partial x}(0, y).$

4. $F_S\left\{\dfrac{\partial^2 f}{\partial x^2}\right\} = -s^2\bar{f}_S(s, y) + sf(0, y).$

2. *If $f(0, y)$ is given but $\dfrac{\partial f}{\partial x}(0, y)$ is not known in a boundary value problem, Fourier sine transform is used. On the other hand, if $\dfrac{\partial f}{\partial x}(0, y)$ is given but $f(0, y)$ is not known, Fourier cosine transform is used.*
3. *When the transforms are taken with respect to x,*

$$F_C\left\{\frac{\partial f}{\partial y}\right\} = \int_0^\infty \frac{\partial f}{\partial y}\cos sx\,dx = \frac{\partial}{\partial y}\int_0^\infty f(x, y)\cos sx\,dx$$
$$= \frac{\partial}{\partial y}\bar{f}_C(s, y)$$

Extending, we get $\quad F_C\left\{\dfrac{\partial^r f}{\partial y^r}\right\} = \dfrac{\partial^r}{\partial y^r}\bar{f}_C(s, y).$

Similarly, $\quad F_S\left\{\dfrac{\partial^r f}{\partial y^r}\right\} = \dfrac{\partial^r}{\partial y^r}\bar{f}_S(s, y).$

8. Derivatives of the transform

If $\quad F\{f(x)\} = \bar{f}(s), \quad$ then $\quad -iF\{xf(x)\} = \dfrac{d}{ds}\bar{f}(s)$

Proof

$$\bar{f}(s) = \int_{-\infty}^{\infty} e^{-isx} f(x)\,dx$$

$$\therefore \qquad \frac{d}{ds}\bar{f}(s) = \int_{-\infty}^{\infty} \frac{d}{ds}[e^{-isx} f(x)]\, dx$$

$$= (-i) \int_{-\infty}^{\infty} e^{-isx}[x f(x)]\, dx = -i \cdot F\{x f(x)\}$$

Extending, we get, $\dfrac{d^r}{ds^r}\bar{f}(s) = (-i)^r F\{x^r f(x)\}$

Definition

$\int_{-\infty}^{\infty} f(x-u)g(u)\, du$ is called *the convolution product* or simply *the convolution* of the functions $f(x)$ and $g(x)$ and is denoted by $f(x)^* g(x)$.

9. Convolution theorem

The Fourier transform of the convolution of two functions is the product of their Fourier transforms.

i.e if $F\{f(x)\} = \bar{f}(s)$ and $F\{g(x)\} = \bar{g}(s)$, then

$$F\{f(x)^* g(x)\} = \bar{f}(s) \cdot \bar{g}(s).$$

Proof

$$F\{f(x)^* g(x)\} = \int_{-\infty}^{\infty} f(x)^* g(x) e^{-isx}\, dx$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x-u)g(u)\, du\right] e^{-isx}\, dx$$

$$= \int_{-\infty}^{\infty} g(u)\left[\int_{-\infty}^{\infty} f(x-u)e^{-isx}\, dx\right] du,$$

on changing the order of integration.

$$= \int_{-\infty}^{\infty} g(u)\left[e^{-ius}\bar{f}(s)\right] du, \text{ by the shifting property.}$$

$$= \bar{f}(s) \cdot \int_{-\infty}^{\infty} g(u) e^{-isu}\, du$$

$$= \bar{f}(s) \cdot \bar{g}(s)$$

Inverting, we get $F^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = f(x)^{*}g(x)$

$$= F^{-1}\{\bar{f}(s)\}^{*}F^{-1}(\bar{g}(s)\}$$

10. Parseval's identity (or) energy theorem

If $F\{f(x)\} = \bar{f}(s)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\bar{f}(s)|^2\, ds$$

By convolution theorem,

$$f(x)^{*}g(x) = F^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\}$$

i.e. $$\int_{-\infty}^{\infty} f(u)\cdot g(x-u)\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\bar{g}(s)e^{ixs}\, ds \tag{1}$$

Putting $x = 0$ in (1), we get,

$$\int_{-\infty}^{\infty} f(u)g(-u)\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\bar{g}(s)\, ds \tag{2}$$

(2) is true for any $g(u)$; take $g(u) = [f(-u)]^{*}$ and hence $g(-u) = [f(u)]^{*}$, where $[f(u)]^{*}$ is the complex conjugate of $f(u)$.

Also $\bar{g}(s) = F\{g(x)\} = F\{f(-x)\}^{*} = [F\{f(x)\}]^{*} = [\bar{f}(s)]^{*}$

(by the conjugate symmetry property)

Using these in (2), we get

$$\int_{-\infty}^{\infty} f(u)[f(u)]^{*}\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)[\bar{f}(s)]^{*}\, ds$$

i.e. $$\int_{-\infty}^{\infty} |f(u)|^2\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\bar{f}(s)|^2\, ds.$$

or $$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\bar{f}(s)|^2\, ds,$$ on changing the dummy variable.

Note ✍

Had we assumed that $F\{f(x)\} = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-isx}\,\mathrm{d}u$, *the Parseval's identity would have assumed the form* $\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x = \int_{-\infty}^{\infty} |\bar{f}(s)|^2\,\mathrm{d}s$.

Property 11

If $\bar{f}_C(s)$, $\bar{g}_C(s)$ are the Fourier cosine transforms and $\bar{f}_S(s)$, $\bar{g}_S(s)$ are the Fourier sine transforms of $f(x)$ and $g(x)$ respectively, then

(i) $\displaystyle\int_0^{\infty} f(x)g(x)\,\mathrm{d}x = \int_0^{\infty} \bar{f}_C(s)\bar{g}_C(s)\,\mathrm{d}s = \int_0^{\infty} \bar{f}_S(s)\bar{g}_S(s)\,\mathrm{d}s$

(ii) $\displaystyle\int_0^{\infty} |f(x)|^2\,\mathrm{d}x = \int_0^{\infty} |\bar{f}_C(s)|^2\,\mathrm{d}s = \int_0^{\infty} |\bar{f}_S(s)|^2\,\mathrm{d}s,$

which is Parseval's identity for Fourier cosine and sine transforms.

Proof

(i)
$$\int_0^{\infty} \bar{f}_C(s)\bar{g}_C(s)\,\mathrm{d}s = \int_0^{\infty} \bar{f}_C(s)\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} g(x)\cos sx\,\mathrm{d}x\right]\mathrm{d}s$$
$$= \int_0^{\infty} g(x)\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} \bar{f}_C(s)\cos xs\,\mathrm{d}s\right]\mathrm{d}x,$$
changing the order of integration
$$= \int_0^{\infty} f(x)g(x)\,\mathrm{d}x.$$

Note ✍

Had we used the definition $\bar{f}_C(s) = \int_0^{\infty} f(x)\cos sx\,\mathrm{d}x$ *and* $f(x) = \dfrac{2}{\pi}\int_0^{\infty} \bar{f}_C(s)\cos xs\,\mathrm{d}s$, *this result would have been* $\int_0^{\infty} f(x)g(x)\,\mathrm{d}x = \dfrac{2}{\pi}\int_0^{\infty} \bar{f}_C(s)\bar{g}_C(s)\,\mathrm{d}s$.

Similarly we can prove the other part of the result.

(ii) Replacing $g(x) = f^*(x)$ in (i) and noting that $F_C\{f^*(x)\} = \{\bar{f}_C(s)\}^*$ and

$F_S\{f^*(x)\} = \{\bar{f}_S(s)\}^*$, we get $\int_0^\infty f(x) f^*(x)\, dx = \int_0^\infty \bar{f}_C(s)\{\bar{f}_C(s)\}^*\, ds$

$= \int_0^\infty \bar{f}_S(s)\{\bar{f}_S(s)\}^*\, ds$ i.e. $\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |\bar{f}_C(s)|^2\, ds = \int_0^\infty |\bar{f}_S(s)|^2\, ds$

Note ✍

Had we adopted the other definition, the result would have been

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{2}{\pi}\int_{-\infty}^{\infty} |\bar{f}_C(s)|^2 ds = \frac{2}{\pi}\int_{-\infty}^{\infty} |\bar{f}_S(s)|^2 ds.]$$

Property 12

If $F_C\{f(x)\} = \bar{f}_C(s)$ and $F_S\{f(x)\} = \bar{f}_S(s)$, then

(i) $\dfrac{d}{ds}\{\bar{f}_C(s)\} = -F_S\{xf(x)\}$; and

(ii) $\dfrac{d}{ds}\{\bar{f}_S(s)\} = F_C\{xf(x)\}$.

Proof

$$\bar{f}_C(s) = \int_0^\infty f(x)\cos sx\, dx$$

$$\therefore \quad \frac{d}{ds}\{\bar{f}_C(s)\} = \int_0^\infty f(x)\{-x\sin sx\}\, dx$$

$$= -\int_0^\infty \{xf(x)\}\sin sx\, dx$$

$$= -F_S\{xf(x)\}$$

Similarly the result (ii) follows.

Worked Examples 4(b)

Example 1

Find the function $f(x)$ for which the Fourier transform is

$$2\sin[3(s - 2\pi)]/(s - 2\pi)]$$

Let us first find $F^{-1}\left\{\frac{2\sin 3s}{s}\right\}$

$$F^{-1}\left\{\frac{2\sin 3s}{s}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin 3s}{s}e^{ixs}\,ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin 3s}{s}(\cos xs + i\sin xs)\,ds$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\frac{2\sin 3s\cos xs}{s}\,ds$$

{by the property of odd and even functions}

$$= \frac{1}{\pi}\int_{0}^{\infty}\left\{\frac{\sin(3+x)s}{s} + \frac{\sin(3-x)s}{s}\right\}ds$$

$$= \begin{cases} \frac{1}{\pi}\left\{\frac{\pi}{2}+\frac{\pi}{2}\right\}, & \text{if } 3+x>0 \text{ and } 3-x>0 \\ 0, & \text{if } 3+x>0 \text{ and } 3-x<0 \text{ or} \\ & 3+x<0 \text{ and } 3-x>0 \end{cases}$$

$$\left[\because \int_{0}^{\infty}\frac{\sin mx}{x}\,dx = \frac{\pi}{2} \text{ or } -\frac{\pi}{2} \text{ according as } m>0 \text{ or } <0\right]$$

$$= \begin{cases} 1, & \text{if } -3<x<3 \\ 0, & \text{if } x<-3 \text{ or } x>3 \end{cases}$$

$$= \begin{cases} 1, & \text{if } |x|<3 \\ 0, & \text{if } |x|>3 \end{cases} \qquad (1)$$

By the shifting property, $F\{e^{iax}f(x)\} = \bar{f}(s-a)$

$$\therefore \qquad F^{-1}\{\bar{f}(s-a)\} = e^{iax}\cdot F^{-1}\{\bar{f}(s)\}$$

Thus,

$$F^{-1}\left[\frac{2\sin\{3(s-2\pi)\}}{s-2\pi}\right] = e^{-i2\pi x}\cdot F^{-1}\left[\frac{2\sin 3s}{s}\right]$$

$$= e^{i2\pi x}\times\begin{cases} 1, & \text{if } |x|<3 \\ 0, & \text{if } |x|>3 \end{cases} \qquad \text{from (1)}$$

$$= \begin{cases} e^{i2\pi x}, & \text{if } |x|<3 \\ 0, & \text{if } |x|>3 \end{cases}$$

Example 2

Find the Fourier transform of $f(x)$, defined as

$$f(x) = \begin{cases} 1, & \text{for} \quad |x| < a \\ 0, & \text{for} \quad |x| > a \end{cases}$$

Hence find $\quad F\left[f(x)\left\{1+\cos\dfrac{\pi x}{a}\right\}\right]$

By Example (5) of Section 2(a),

$$F\{f(x)\} = \frac{2}{s}\sin as = \bar{f}(s), \text{ say} \tag{1}$$

By Modulation theorem,

$$\begin{aligned} F\left\{f(x)\cos\frac{\pi x}{a}\right\} &= \frac{1}{2}\left[\bar{f}\left(s+\frac{\pi}{a}\right)+\bar{f}\left(s-\frac{\pi}{a}\right)\right] \\ &= \frac{1}{s+\frac{\pi}{a}}\sin a\left(s+\frac{\pi}{a}\right)+\frac{1}{s-\frac{\pi}{a}}\sin a\left(s-\frac{\pi}{a}\right) \\ &= \frac{a}{as+\pi}\{-\sin as\}+\frac{a}{as-\pi}\{-\sin as\} \\ &= \frac{2a^2 s\sin as}{\pi^2-a^2s^2} \end{aligned} \tag{2}$$

Adding (1) and (2), we get

$$\begin{aligned} F\left[f(x)\left\{1+\cos\frac{\pi x}{a}\right\}\right] &= 2\sin as\left[\frac{1}{s}+\frac{a^2 s}{\pi^2-a^2s^2}\right] \\ &= \frac{2\pi^2\sin as}{s(\pi^2-a^2s^2)} \end{aligned}$$

Example 3

Find the Fourier transform of $f(x) = x^n e^{-ax}U(x)$, where $U(x)$ is the unit step function.

$$\begin{aligned} F\{e^{-ax}U(x)\} &= \int_{-\infty}^{0} e^{-ax}\cdot 0\, e^{-isx}\,dx + \int_{0}^{\infty} e^{-ax}\cdot 1\cdot e^{-isx}\,dx \\ &= \left[\frac{e^{-(a+is)x}}{-(a+is)}\right]_0^{\infty} = \frac{1}{a+is} \end{aligned}$$

By property (8),

$$\begin{aligned} F\{x^n e^{-ax}U(x)\} &= \frac{1}{(-i)^n}\frac{d^n}{ds^n}\left\{\frac{1}{a+is}\right\} \\ &= \frac{1}{(-i)^n}\frac{(-1)^n i^n n!}{(a+is)^{n+1}} = \frac{n!}{(a+is)^{n+1}} \end{aligned}$$

Example 4

Find the Fourier transform of $\left\{\dfrac{\sin ax}{x}\right\}$ and hence prove that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2} = a\pi.$

$$F\left\{\frac{\sin ax}{x}\right\} = \int_{-\infty}^{\infty} \left\{\frac{\sin ax}{x}\right\} e^{-isx}\, dx$$

$$= 2\int_0^{\infty} \frac{\sin ax \cos sx}{x}\, dx$$

$$= \int_0^{\infty} \left\{\frac{\sin (a+s)x}{x} + \frac{\sin (a-s)x}{x}\right\} dx$$

$$= \begin{cases} \pi, & \text{if } |s| < a \\ 0, & \text{if } |s| > a \end{cases} \quad \text{[proceeding as in Example (1)]}$$

Now $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\bar{f}(s)|^2\, ds$, by Parseval's identity.

$$\therefore \quad \int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2} dx = \frac{1}{2\pi}\int_{-a}^{a} \pi^2 ds = \frac{1}{2\pi}\cdot \pi^2 \cdot 2a = a\pi.$$

Example 5

Find the Fourier transform of $f(x)$, if

$$f(x) = \begin{cases} 1-|x|, & \text{for } |x| < 1 \\ 0, & \text{for } |x| > 1 \end{cases}$$

Hence prove that $\displaystyle\int_0^{\infty} \frac{\sin^4 x}{x^4}\, dx = \frac{\pi}{3}$

$$F\{f(x)\} = \int_{-1}^{1} \{1-|x|\}\, e^{-isx}\, dx$$

$$= 2\int_0^1 (1-x)\cos sx\;\, dx \quad \text{by property of even and odd functions.}$$

$$= 2\left[(1-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_0^1$$

$$= 2\frac{(1-\cos s)}{s^2} = \frac{4\sin^2\frac{s}{2}}{s^2}$$

By Parseval's identity,

$$\int_{-1}^{1} \{1-|x|\}^2 \, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{16\sin^4\left(\frac{s}{2}\right)}{s^4} \, ds$$

i.e.
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{16\sin^4\left(\frac{s}{2}\right)}{s^4} \, ds = 2\int_0^1 (1-x)^2 \, dx = \frac{2}{3}$$

Putting $\frac{s}{2} = t$, we get,

$$\frac{16}{\pi}\int_0^{\infty} \frac{\sin^4 t}{16t^4} \cdot 2 \, dt = \frac{2}{3}$$

$$\therefore \qquad \int_0^{\infty} \frac{\sin^4 t}{t^4} dt = \frac{\pi}{3} \text{ or } \int_0^{\infty} \frac{\sin^4 x}{x^4} \, dx = \frac{\pi}{3}.$$

Example 6

Using Parseval's identity for Fourier cosine and sine transforms of e^{-ax}, evaluate

(i) $\displaystyle\int_0^{\infty} \frac{dx}{(a^2+x^2)^2}$ and (ii) $\displaystyle\int_0^{\infty} \frac{x^2}{(a^2+x^2)^2} \, dx.$

(i) $$F_C(e^{-ax}) = \int_0^{\infty} e^{-ax} \cos sx \, dx = \frac{a}{s^2+a^2}$$

By Parseval's identity,

$$\int_0^{\infty} |f(x)|^2 \, dx = \frac{2}{\pi}\int_0^{\infty} |\bar{f}_c(s)|^2 \, ds$$

$$\therefore \qquad \int_0^\infty e^{-2ax}\,\mathrm{d}x = \frac{2}{\pi}a^2\int_0^\infty \frac{\mathrm{d}s}{(s^2+a^2)^2}$$

$$\text{i.e.} \qquad \int_0^\infty \frac{\mathrm{d}s}{(s^2+a^2)^2} = \frac{\pi}{2a^2}\cdot\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$= \frac{\pi}{4a^3}, \quad \text{if} \quad a > 0$$

Changing the dummy variable s into x, we get the first result.

(ii) Now $\qquad F_S(e^{-ax}) = \int_0^\infty e^{-ax}\sin sx\;\mathrm{d}x = \frac{s}{s^2+a^2}.$

By Parseval's identity,

$$\int_0^\infty |f(x)|^2\,\mathrm{d}x = \frac{2}{\pi}\int_0^\infty |\bar{f}_s(s)|^2\,\mathrm{d}s$$

$$\text{i.e.} \qquad \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\,\mathrm{d}s = \int_0^\infty e^{-2ax}\,\mathrm{d}x$$

$$\therefore \qquad \int_0^\infty \frac{x^2\,\mathrm{d}x}{(x^2+a^2)^2} = \frac{\pi}{4a}, \quad \text{if } a > 0, \text{ on changing the dummy variables.}$$

Example 7

Use transfrom methods to evaluate

$$\text{(i)} \quad \int_0^\infty \frac{\mathrm{d}x}{(x^2+1)(x^2+4)} \qquad \text{and} \qquad \text{(ii)} \int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,\mathrm{d}x.$$

(i) Let $\qquad f(x) = e^{-x}$ and $g(x) = e^{-2x}$

Then $\qquad \bar{f}_C(s) = \dfrac{1}{s^2+1}$ and $\bar{g}_C(s) = \dfrac{2}{s^2+4}$

By property (11),

$$\int_0^\infty f(x)g(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^\infty \bar{f}_C(s)\bar{g}_C(s)\,\mathrm{d}s$$

$$\therefore \qquad \int_0^\infty e^{-3x}\,dx = \frac{2}{\pi}\int_0^\infty \frac{2}{(s^2+1)(s^2+4)}\,ds$$

$$\text{i.e.} \qquad \int_0^\infty \frac{ds}{(s^2+1)(s^2+4)} = \frac{\pi}{4}\left[\frac{e^{-3x}}{-3}\right]_0^\infty = \frac{\pi}{12}$$

Changing 's' into x, we get

$$\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{12}$$

(ii) Let $\quad f(x) = e^{-ax}$ and $g(x) = e^{-bx}$; $\; a, b > 0$

Then $\quad \bar{f}_S(s) = \dfrac{s}{s^2+a^2} \quad$ and $\quad \bar{g}_S(s) = \dfrac{s}{s^2+b^2}$

By property (11)

$$\int_0^\infty f(x)g(x)\,dx = \frac{2}{\pi}\int_0^\infty \bar{f}_S(s)\cdot\bar{g}_S(s)\,ds$$

$$\therefore \qquad \int_0^\infty e^{-(a+b)x}\,dx = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\,ds$$

$$\text{i.e.} \qquad \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\,ds = \frac{\pi}{2}\cdot\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = \frac{\pi}{2(a+b)}$$

Changing 's' into x, we get

$$\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{2(a+b)}$$

Example 8

(i) Find $F_S(e^{-ax})$ and hence find $F_C(xe^{-ax})$

(ii) Find $F_C\left(e^{-a^2x^2}\right)$ and hence find $F_S\left(xe^{-a^2x^2}\right)$

(i) $F_S(e^{-ax}) = \dfrac{s}{s^2+a^2}, \quad$ if $a > 0$.

By property (12), $\quad \dfrac{d}{ds}\{F_S(e^{-ax})\} = F_C(xe^{-ax})$

$$\therefore \qquad F_C(xe^{-ax}) = \frac{d}{ds}\left\{\frac{s}{s^2+a^2}\right\} = \frac{a^2-s^2}{(s^2+a^2)^2}.$$

(ii) In Worked Example 8 of Section 4(a), we have already found out $F_C\left(e^{-a^2x^2}\right)$ through $F\left(e^{-a^2x^2}\right)$. However, we shall find $F_C\left(e^{-a^2x^2}\right)$ by an alternative direct method.

Let $$I = F_C\left(e^{-a^2x^2}\right) = \int_0^\infty e^{-a^2x^2} \cos sx \, dx$$

$$\therefore \quad \frac{dI}{ds} = -\int_0^\infty xe^{-a^2x^2} \sin sx \, dx$$

$$= \int_0^\infty \sin sx \, d\left[\frac{e^{-a^2x^2}}{2a^2}\right]$$

$$= \frac{1}{2a^2}\left[\left(e^{-a^2x^2} \sin sx\right)_0^\infty - s\int_0^\infty e^{-a^2x^2} \cos sx \, dx\right]$$

i.e. $$\frac{dI}{ds} = -\frac{s}{2a^2} I$$

Solving $$\log I = -\frac{s^2}{4a^2} + \log C$$

i.e., $$I = Ce^{-s^2/4a^2} \tag{1}$$

When $$s = 0, I = \int_0^\infty e^{-a^2x^2} \, dx = \frac{1}{a}\int_0^\infty e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2a}$$

Using in (1), $$C = \frac{\sqrt{\pi}}{2a}$$

$$\therefore \quad F_C\left(e^{-a^2x^2}\right) = \frac{\sqrt{\pi}}{2a} e^{-s^2/4a^2}$$

By property (12), $$\frac{d}{ds}\left\{F_C\left(e^{-a^2x^2}\right)\right\} = -F_S\left(xe^{-a^2x^2}\right)$$

$$\therefore \quad F_S\left(xe^{-a^2x^2}\right) = -\frac{d}{ds}\left[\frac{\sqrt{\pi}}{2a} e^{-s^2/4a^2}\right]$$

$$= -\frac{\sqrt{\pi}}{2a}\left(-\frac{s}{2a^2}\right) e^{-s^2/4a^2}$$

$$= \frac{\sqrt{\pi}}{4a^3} s e^{-s^2/4a^2}$$

Example 9

Solve the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{-x}, x > 0$, using Fourier transforms, given that $y(0) = 0$ and $y'(0) = 0$.

Taking Fourier complex transforms on both sides of the given differential equation, we have

$$(is)^2\bar{y}(s) + 3(is)\bar{y}(s) + 2\bar{y}(s) = F(e^{-x}), x > 0$$

$$= F\{U(x)\cdot e^{-x}\}, \quad \text{where} \quad U(x) \text{ is the unit step function}$$

i.e.
$$[(is)^2 + 3(is) + 2]\bar{y}(s) = \int_0^\infty e^{-(1+is)x}\,dx = \frac{1}{1+is}$$

$\therefore$
$$\bar{y}(s) = \frac{1}{(is+1)^2(is+2)}$$

$$= -\frac{1}{is+1} + \frac{1}{(is+1)^2} + \frac{1}{is+2}, \text{ by partial fractions.}$$

$\therefore$
$$y = -F^{-1}\left\{\frac{1}{is+1}\right\} + F^{-1}\left\{\frac{1}{(is+1)^2}\right\} + F^{-1}\left\{\frac{1}{is+2}\right\}$$

$$= -U(x)e^{-x} + U(x)\cdot xe^{-x} + U(x)\cdot e^{-2x},$$

since
$$F\{U(x)xe^{-x}\} = \int_0^\infty xe^{-(1+is)x}\,dx$$

$$= \left[x\left\{\frac{e^{-(1+is)x}}{-(1+is)}\right\} - \left\{\frac{e^{-(1+is)x}}{(1+is)^2}\right\}\right]_0^\infty$$

$$= \frac{1}{(1+is)^2}$$

i.e.
$$y = -e^{-x} + xe^{-x} + e^{-2x}, x > 0$$

Note ✍

We have got the solution valid in $x > 0$, i.e. y and its derivatives are assumed to be zero in $x \le 0$. Though we have not explicitly used the conditions $y(0) = 0$ and $y'(0) = 0$, they have been taken care of, as the solution obtained satisfies both the conditions. However the Fourier complex transform method will fail if $y(0)$ and $y'(0)$ are prescribed non-zero values. In such situations, the Fourier sine and cosine transforms are used as explained in the following problem.

Example 10

Solve the equation $y'' + 3y' + 2y = e^{-x}, x > 0$, using transform method, given that $y(0) = 1, y'(0) = 2$.

The given conditions must be interpreted as $y(0+) = 1$ and $y'(0+) = 2$.

Taking Fourier cosine transforms on both sides of the given equation, we get

$$[-s^2\bar{y}_C(s) - y'(0)] + 3[s\bar{y}_S(s) - y(0)] + 2\bar{y}_C(s) = \frac{1}{s^2+1}$$

[Refer to Corollaries under property 7]

i.e. $$(2-s^2)\bar{y}_C(s)+3s\bar{y}_S(s)=\frac{1}{s^2+1}+5 \tag{1}$$

Taking Fourier sine transforms of the given equation, we get

$$[-s^2\bar{y}_S(s)+sy(0)]+3[-s\bar{y}_C(s)]+2\bar{y}_S(s)=\frac{s}{s^2+1}$$

i.e. $$-3s\bar{y}_C(s)+(2-s^2)\bar{y}_S(s)=\frac{s}{s^2+1}-s \tag{2}$$

Solving (1) and (2), we get

$$\begin{aligned}\bar{y}_S(s)&=\frac{5s-s^3}{(s^2+1)^2(s^2+4)}+\frac{s^3+13s}{(s^2+1)(s^2+4)}\\&=\frac{3s}{s^2+1}+\frac{2s}{(s^2+1)^2}-\frac{2s}{s^2+4},\ \text{by partial fractions.}\end{aligned}$$

Taking inverse sine transforms, we get

$$y(x)=3e^{-x}-2e^{-2x}+xe^{-x},\ x>0$$

Example 11

Solve the equation $(D^2-4D+4)y=xe^{-x}, x>0$, given that $y(0)=0$ and $y'(0)=0$.

Taking Fourier complex transforms of the given equation, we get,

$$\begin{aligned}(is-2)^2\bar{y}(s)&=F\{U(x)\cdot xe^{-x}\}\\&=\int_0^\infty xe^{-(1+is)x}\,dx\\&=\left[x\left\{\frac{e^{-(1+is)x}}{-(1+is)}\right\}-\frac{e^{-(1+is)x}}{(1+is)^2}\right]_0^\infty\\&=\frac{1}{(1+is)^2}\end{aligned}$$

$$\begin{aligned}\therefore\quad \bar{y}(s)&=\frac{1}{(is-2)^2(is+1)^2}\\&=\frac{-2/27}{is-2}+\frac{1/9}{(is-2)^2}+\frac{2/27}{is+1}+\frac{1/9}{(is+1)^2}\end{aligned}$$

Inverting, we get

$$y=\frac{-2}{27}e^{2x}+\frac{1}{9}xe^{2x}+\frac{2}{27}e^{-x}+\frac{1}{9}xe^{-x};\quad x>0$$

Example 12

Solve the equation $y'' - 4y' + 5y = 1, \quad x > 0$,
given that $y(0) = 0$ and $y'(0) = 0$.

Taking complex Fourier transforms of the given equation, we get

$$[(is)^2 - 4is + 5]\bar{y}(s) = F\{1 \cdot U(x)\}$$

$$= \int_0^\infty e^{-isx}\,dx$$

$$= \left[\frac{e^{-isx}}{-is}\right]_0^\infty$$

$$= \frac{1}{is}$$

$$\therefore \quad \bar{y}(s) = \frac{1}{(is)(is-2-i)(is-2+i)}$$

$$= \frac{1/5}{is} + \frac{1/(-2+4i)}{is-2-i} + \frac{1/(-2-4i)}{is-2+i}, \quad \text{by partial fractions}$$

$$= \frac{1}{5}\cdot\frac{1}{is} - \frac{1}{10}(1+2i)\cdot\frac{1}{is-2-i} - \frac{1}{10}(1-2i)\cdot\frac{1}{is-2+i}$$

Taking inverse transforms, we get

$$y = \frac{1}{5} - \frac{1}{10}(1+2i)e^{(2+i)x} - \frac{1}{10}(1-2i)e^{(2-i)x}$$

$$= \frac{1}{5} - \frac{1}{5}e^{2x}\cos x + \frac{2}{5}e^{2x}\sin x;\ x > 0$$

Example 13

Solve the equation $(D^2 - 4D + 3)y = \cos 3x, x > 0$, given that $y(0) = 0$ and $y'(0) = 0$.

Taking complex Fourier transforms of the given equation, we get,

$$[(is)^2 - 4is + 3]\bar{y}(s) = F[U(x)\cos 3x]$$

$$= \int_0^\infty e^{-isx}\cos 3x\,dx = \frac{is}{9+(is)^2}$$

$$\therefore \quad \bar{y}(s) = \frac{is}{(is-1)(is-3)[(is)^2+9]}$$

$$= \frac{-1/20}{is-1} + \frac{1/12}{is-3} - \frac{(1/30)is}{(is)^2+9} - \frac{1/5}{(is)^2+9}$$

by partial fractions.

Inverting, we get

$$y = \frac{-1}{20}e^{x} + \frac{1}{12}e^{3x} - \frac{1}{30}\cos 3x - \frac{1}{15}\sin 3x; \qquad x > 0,$$

since $\qquad F[U(x)\sin 3x] = \dfrac{3}{(is)^2 + 9}$

Example 14

Solve the equation $y'' + \omega_0^2 y = \sin \omega x$, $x > 0$, given that $y(0) = 0$ and $y'(0) = 0$, when

$$\text{(i)}\quad \omega \neq \omega_0 \ \text{ and } \qquad \text{(ii)}\ \omega = \omega_0$$

Taking complex Fourier transforms of the given equation, we get,

$$[(is)^2 + \omega_0^2]\bar{y}(s) = F\{U(x)\cdot \sin \omega x\}$$

$$= \int_0^\infty e^{-isx} \sin \omega x \ \mathrm{d}x$$

$$= \frac{\omega}{\omega^2 - s^2}$$

$$\therefore \qquad \bar{y}(s) = \frac{\omega}{(\omega_0^2 - s^2)(\omega^2 - s^2)}, \quad \text{when } \omega \neq \omega_0$$

$$= \frac{\left[\dfrac{\omega}{\omega^2 - \omega_0^2}\right]}{\omega_0^2 - s^2} - \frac{\left[\dfrac{\omega}{\omega^2 - \omega_0^2}\right]}{\omega^2 - s^2}, \text{ by partial fractions.}$$

Inverting, we get

$$y = \frac{\omega}{\omega_0(\omega^2 - \omega_0^2)}\sin \omega_0 x - \frac{1}{\omega^2 - \omega_0^2}\sin \omega x.$$

When $\qquad \omega = \omega_0, \quad \bar{y}(s) = \dfrac{\omega_0}{(\omega_0^2 - s^2)^2}$

Inverting, $\qquad y = F^{-1}\left\{\dfrac{\omega_0}{(\omega_0^2 - s^2)^2}\right\} \qquad (1)$

Consider $\qquad \displaystyle\int_0^\infty (\sin ax)e^{-isx}\,\mathrm{d}x = \frac{a}{a^2 - s^2}$

Differentiating both sides with respect to ‘a’,

$$\int_0^\infty (x\cos ax)e^{-isx}\,\mathrm{d}x = \frac{1}{a^2 - s^2} - \frac{2a^2}{(a^2 - s^2)^2}$$

$$\therefore \quad \frac{a}{(a^2-s^2)^2} = \frac{1}{2a}\cdot\frac{1}{a^2-s^2} - \frac{1}{2a}\int_0^\infty (x\cos ax)e^{-isx}\,\mathrm{d}x$$

$$\therefore \quad F^{-1}\left\{\frac{a}{(a^2-s^2)^2}\right\} = \frac{1}{2a^2}U(x)\sin ax - \frac{1}{2a}U(x)x\cos ax \qquad (2)$$

Using (2) in (1), we get,

$$y = \frac{1}{2\omega_0^2}\sin\omega_0 x - \frac{1}{2\omega_0}x\cos\omega_0 x,\ x > 0$$

Example 15

Solve the wave equation $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$, subject to the initial conditions $y(x,0) = f(x)$, $-\infty < x < \infty$, $\frac{\partial y}{\partial t}(x,0) = g(x)$ and the boundary conditions $y(x,t) \longrightarrow 0$, as $x \longrightarrow \pm\infty$.

Taking Fourier transforms of the equation with respect to x, we get,

$$\int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial t^2}(x,t)\cdot e^{-isx}\,\mathrm{d}x = a^2(is)^2\bar{y}(s,t)$$

i.e. $$\frac{\partial^2}{\partial t^2}\bar{y}(s,t) + a^2s^2\bar{y}(s,t) = 0 \qquad (1)$$

Taking Fourier transforms of the initial conditions, we have

$$\bar{y}(s,0) = \bar{f}(s) \qquad (2)$$

and $$\frac{\partial\bar{y}}{\partial t}(s,0) = \bar{g}(s) \qquad (3)$$

(1) is an ordinary differential equation in $\bar{y}(s,t)$. Solving (1), we get

$$\bar{y}(s,t) = A\cos ast + B\sin ast \qquad (4)$$

Using (2) in (4), we get $\quad A = \bar{f}(s)$

Using (3) in (4), we get $\quad B = \frac{1}{as}\bar{g}(s)$

Inserting these values in (4), we get

$$\begin{aligned}\bar{y}(s,t) &= \bar{f}(s)\cos ast + \frac{1}{as}\bar{g}(s)\sin ast \\ &= \frac{1}{2}\{\bar{f}(s)e^{iast} + \bar{f}(s)e^{-iast}\} + \frac{1}{2a}\left\{\frac{\bar{g}(s)}{is}e^{iast} - \frac{\bar{g}(s)}{is}e^{-iast}\right\}\end{aligned}$$

Taking inverse Fourier transforms, we get

$$y(x,t) = \frac{1}{4\pi}\left[\int_{-\infty}^{\infty} \bar{f}(s)e^{i(x+at)s}\,\mathrm{d}s + \int_{-\infty}^{\infty} \bar{f}(s)e^{i(x-at)s}\,\mathrm{d}s\right] +$$

$$\frac{1}{4\pi a}\left[\int_{-\infty}^{\infty} \frac{\bar{g}(s)}{is}e^{i(x+at)s}\,\mathrm{d}s - \int_{-\infty}^{\infty} \frac{\bar{g}(s)}{is}e^{i(x-at)s}\,\mathrm{d}s\right] \qquad (5)$$

Now $\quad F\{\phi'(x)\} = is\phi(s)$, by property (7)

Putting $\quad \phi'(x) = g(x), \quad$ we get $\phi(x) = \int_{c}^{x} g(u)\,\mathrm{d}u$

and $\quad \frac{\bar{g}(s)}{is} = \bar{\phi}(s)$

$$\therefore \quad F^{-1}\left\{\frac{\bar{g}(s)}{is}\right\} = \phi(x) = \int_{c}^{x} g(u)\,\mathrm{d}u$$

i.e. $\quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\bar{g}(s)}{is}e^{ixs}\mathrm{d}s = \int_{c}^{x} g(u)\,\mathrm{d}u$

Using (6) in (5), we get

$$y(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\left[\int_{c}^{x+at} g(u)\,\mathrm{d}u - \int_{c}^{x-at} g(u)\,\mathrm{d}u\right]$$

i.e. $\quad y(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(u)\,\mathrm{d}u$

Example 16

Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $y \geq 0$ subject to the boundary conditions $u(x,0) = f(x)$, $-\infty < x < \infty$ and $u(x,y) \to 0$ as $y \to \infty$.

Taking Fourier transforms of the Laplace equation with respect to x, we get

$$(is)^2\bar{u}(s,y) + \frac{\partial^2}{\partial y^2}\bar{u}(s,y) = 0$$

i.e. $$\frac{\partial^2}{\partial y^2}\bar{u}(s,y) - s^2\bar{u}(s,y) = 0 \qquad (1)$$

The transforms of the boundary conditions are

$$\bar{u}(s,0) = \bar{f}(s) \qquad (2)$$

and $$\bar{u}(s, y) \to 0 \quad \text{as} \quad y \to \infty \tag{3}$$

Solving the ordinary differential equation (1), we get

$$\bar{u}(s, y) = Ae^{sy} + Be^{-sy} \tag{4}$$

For the solution (4) to be consistent with (3), $A = 0$

$\therefore$ $\bar{u}(s, y) = Be^{-|s|y}$, as the coefficient of y cannot be positive

Using (2), we get $B = \bar{f}(s)$.

$$\therefore \quad \bar{u}(s, y) = \bar{f}(s)e^{-|s|y} \tag{5}$$

Consider $$F^{-1}\{e^{|s|y}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|s|y}e^{isx}\, ds$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{(y+ix)s}\, ds + \int_{0}^{\infty} e^{-(y-ix)s}\, ds\right]$$

[since $|s| = -s$, when $s < 0$ and $|s| = s$, when $s > 0$]

$$= \frac{1}{2\pi}\left[\left\{\frac{e^{(y+ix)s}}{y+ix}\right\}_{-\infty}^{0} + \left\{\frac{e^{-(y-ix)s}}{-(y-ix)}\right\}_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{y+ix} + \frac{1}{y-ix}\right] = \frac{1}{\pi}\cdot\frac{y}{x^2+y^2} \tag{6}$$

Taking inverse transforms on both sides of (5), we have

$$u(x, y) = F^{-1}\{\bar{f}(s)\bar{g}(s)\}, \quad \text{where } \bar{g}(s) = e^{-|s|y}$$

$$= \int_{-\infty}^{\infty} f(t)\cdot g(x-t)\, dt, \text{ by convolution theorem}$$

$$= \int_{-\infty}^{\infty} f(t)\cdot\frac{1}{\pi}\cdot\frac{y}{(x-t)^2+y^2}\, dt, \text{ by 6} \tag{7}$$

i.e. $$u(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(t)dt}{(x-t)^2+y^2}$$

Example 17

Solve the one-dimensional heat flow equation $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$ for a rod with insulated sides extending from $-\infty$ to ∞ and with initial temperature distribution given by $u(x, 0) = f(x)$.

Assume that $u(x, t) \to 0$ as $x \to \pm\infty$.

Taking Fourier transforms of the given equation with respect to x, we get

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x, t) e^{-isx}\, dx = \alpha^2 (is)^2 \bar{u}(s, t)$$

i.e.
$$\frac{\partial}{\partial t} \bar{u}(s, t) = -\alpha^2 s^2 \bar{u}(s, t) \tag{1}$$

The Fourier transform of the initial condition is

$$\bar{u}(s, 0) = \bar{f}(s) \tag{2}$$

Solving the ordinary differential equation (1), we get

$$\bar{u}(s, t) = A e^{-\alpha^2 s^2 t} \tag{3}$$

Using (2) in (3), $\quad A = \bar{f}(s).$

$$\therefore \quad \bar{u}(s, t) = \bar{f}(s) \cdot e^{-\alpha^2 s^2 t} \tag{4}$$

From Example (8) in Section 2(a), we have

$$F(e^{-a^2 x^2}) = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2} \quad \text{or}$$

$$F^{-1}(e^{-s^2/4a^2}) = \frac{a}{\sqrt{\pi}} e^{-a^2 x^2}$$

Putting $\dfrac{1}{4a^2} = \alpha^2 t$, we get $a^2 = \dfrac{1}{4\alpha^2 t}$ or $a = \dfrac{1}{2\alpha\sqrt{t}}$ and hence

$$F^{-1}(e^{-\alpha^2 s^2 t}) = \frac{1}{2\alpha\sqrt{\pi t}} e^{-x^2/4\alpha^2 t} \tag{5}$$

Taking Fourier inverse transforms of (4) by using convolution theorem and using (5), we get

$$u(x, t) = \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} f(z) e^{-(x-z)^2/4\alpha^2 t}\, dz$$

$$= \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} f(z) e^{-(z-x)^2/4\alpha^2 t}\, dz \tag{6}$$

Putting $\dfrac{z-x}{2\alpha\sqrt{t}} = \omega$ in (6), the solution takes the following form:

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2\sqrt{t}\alpha\omega) e^{-\omega^2} d\omega.$$

Example 18

Solve the equation $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$, satisfying the boundary conditions $u(0,t) = k, t \geq 0$ and $u(x,t) \longrightarrow 0$ as $x \longrightarrow \infty$ and the initial condition $u(x,0) = 0$.

Refer to Note (2) under Property (7). Since $x > 0$ and $u(0,t)$ is given, we take Fourier sine transforms of the equation with respect to x.

Thus $$\frac{\partial}{\partial t}\bar{u}_S(s,t) = \alpha^2[-s^2\bar{u}_S(s,t) + su(0,t)]$$

i.e. $$\frac{\partial}{\partial t}\bar{u}\ (s,t) + \alpha^2 s^2 \bar{u}_S(s,t) = k\alpha^2 s \tag{1}$$

Transform of the initial condition is

$$\bar{u}_S(s,0) = 0 \tag{2}$$

Solving the ordinary differential equation (1), we get

$$\bar{u}_S(s,t) = Ae^{-\alpha^2 s^2 t} + \frac{1}{D + \alpha^2 s^2}\left(k\alpha^2 s\right), \left\{D \equiv \frac{d}{dt}\right\}$$

$$= Ae^{-\alpha^2 s^2 t} + \frac{k}{s} \tag{3}$$

Using (2) in (3), we get

$$A = -\frac{k}{s}$$

$$\bar{u}_S(s,t) = \frac{k}{s}(1 - e^{-\alpha^2 s^2 t})$$

Taking the inverse sine transforms, we get,

$$u(x,t) = \frac{2k}{\pi}\int_0^{\infty} \frac{1}{s}\left(1 - e^{-\alpha^2 s^2 t}\right) \sin xs\ ds$$

Example 19

Solve the equation $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$, satisfying the boundary conditions $\dfrac{\partial u}{\partial x}(0,t) = k, t \geq 0$ and $u(x,t) \longrightarrow 0$ as $x \longrightarrow \infty$ and the initial condition $u(x,0) = 0$.

Refer to Note (2) under Property (7). Since $x > 0$ and $\frac{\partial u}{\partial x}(0, t)$ is given, we take Fourier cosine transforms of the equation with respect to x.

Thus $$\frac{\partial}{\partial t}\bar{u}_C(s,t) = \alpha^2\left[-s^2\bar{u}_C(s,t) - \frac{\partial u}{\partial x}(0,t)\right]$$

i.e. $$\frac{\partial}{\partial t}\bar{u}_C(s,t) + \alpha^2 s^2 \bar{u}_C(s,t) = -k\alpha^2 \tag{1}$$

Transform of the initial condition is

$$\bar{u}_C(s,0) = 0 \tag{2}$$

Solving (1) and using (2), we get, as in the previous example

$$\bar{u}_C(s,t) = Ae^{-\alpha^2 s^2 t} - \frac{k}{s^2} \tag{3}$$

Using (2) in (3), we get $\quad A = \frac{k}{s^2}$

$\therefore \quad \bar{u}_C(s,t) = \frac{k}{s^2}\left(e^{-\alpha^2 s^2 t} - 1\right)$ Taking the inverse cosine transforms, we get

$$u(x,t) = \frac{2k}{\pi}\int_0^\infty \frac{1}{s^2}\left(e^{-\alpha^2 s^2 t} - 1\right)\cos xs \ \ ds$$

Example 20

Solve the equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$, satisfying the boundary conditions $u(0,t) = 0$ and $u(x,t) \longrightarrow 0$ as $x \longrightarrow \infty$ and the initial condition $u(x,0) = f(x), x > 0$.

Refer to Note (2) under Property (7). Since $x > 0$ and $u(0,t)$ is given, we take Fourier sine transforms of the equation with respect to x.

Thus $\quad \frac{\partial}{\partial t}\bar{u}_S(s,t) = \alpha^2[-s^2\bar{u}_S(s,t) + su(0,t)]$

i.e. $$\frac{\partial}{\partial t}\bar{u}_S(s,t) + \alpha^2 s^2\bar{u}_S(s,t) = 0 \tag{1}$$

Sine transforms of the initial condition is

$$\bar{u}_S(s,0) = \bar{f}_S(s) \tag{2}$$

Solving (1), we get $$\bar{u}_S(s,t) = Ae^{-\alpha^2 s^2 t} \tag{3}$$

Using (2) in (3), $$A = \bar{f}_S(s)$$

$\therefore$ $$\bar{u}_S(s,t) = \bar{f}_S(s)e^{-\alpha^2 s^2 t} \tag{4}$$

From Example (8), we have

$$F_C\left(e^{-a^2x^2}\right) = \frac{\sqrt{\pi}}{2a}e^{-s^2/4a^2}$$

Taking $$\frac{1}{4a^2} = \alpha^2 t \quad \text{or} \quad a = \frac{1}{2\alpha\sqrt{t}}$$

$$F_C\left\{e^{-x^2/4\alpha^2 t}\right\} = \alpha\sqrt{\pi t}\,e^{-\alpha^2 s^2 t}$$

or $$F_C^{-1}\left\{e^{-\alpha^2 s^2 t}\right\} = \frac{1}{\alpha\sqrt{\pi t}}e^{-x^2/4\alpha^2 t} = g(x,t), \text{ say.} \tag{5}$$

Then $$\bar{g}_C(s,t) = e^{-\alpha^2 s^2 t} \tag{6}$$

Using (6) in (4), we have

$$\bar{u}_S(s,t) = \bar{f}_S(s)\bar{g}_C(s,t)$$

Taking inverse sine transforms, we get

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \bar{f}_S(s)\cdot\bar{g}_C(s,t)\sin xs\ \mathrm{d}s$$

$$= \frac{2}{\pi}\int_0^\infty \bar{g}_C(s,t)\sin xs\ \mathrm{d}s\int_0^\infty f(z)\sin sz\ \mathrm{d}z,$$

on using the definition for $\bar{f}_S(s)$

$$= \int_0^\infty f(z)\,\mathrm{d}z\cdot\frac{2}{\pi}\int_0^\infty \bar{g}_C(s,t)\sin xs\sin zs\ \mathrm{d}s,$$

on changing the order of integration.

$$= \frac{1}{2}\int_0^\infty f(z)\,\mathrm{d}z\cdot\frac{2}{\pi}\int_0^\infty \bar{g}_C(s,t)\{\cos(x-z)s - \cos(x+z)s\}\,\mathrm{d}s$$

$$= \frac{1}{2}\int_0^\infty f(z)\ \mathrm{d}z[g(x-z,t) - g(x+z,t)]$$

$$= \frac{1}{2\alpha\sqrt{\pi t}}\int_0^\infty f(z)\left[e^{-(x-z)^2/4\alpha^2 t} - e^{-(x+z)^2/4\alpha^2 t}\right]\mathrm{d}z, \text{ from(5).}$$

Exercise 4(b)

Part A (Short-Answer Questions)

1. State the relation between $F\{f(x)\}$ and $F\{f(ax)\}$.
2. Derive the change of scale property for Fourier sine and cosine transforms.
3. Obtain $F\{f(x-a)\}$ in terms of $F\{f(x)\}$.
4. Find $F\{e^{iax}f(x)\}$ in terms of $F\{f(x)\}$.
5. State and prove the modulation theorem.
6. State and derive the conjugate symmetry property.
7. Express $F\{f'(x)\}$ in terms of $F\{f(x)\}$. State the conditions under which this relation holds good.
8. Express $F_S\{f''(x)\}$ and $F_C\{f''(x)\}$ in terms of $F_S\{f(x)\}$ and $F_C\{f(x)\}$ respectively.
9. Prove that $F\{xf(x)\} = i\dfrac{\mathrm{d}}{\mathrm{d}s}\bar{f}(s)$.
10. State convolution theorem in Fourier transforms.
11. State Parseval's identity in Fourier transforms. Also state the corresponding result in Fourier cosine and sine transforms.
12. Express $\dfrac{\mathrm{d}}{\mathrm{d}s}\{\bar{f}_C(s)\}$ and $\dfrac{\mathrm{d}}{\mathrm{d}s}\{\bar{f}_S(s)\}$ in terms of Fourier sine and cosine transforms respectively.
13. Prove that $F\{e^{-ax}U(x)\} = \dfrac{1}{a+is}$, where $U(x)$ is the unit step function.
14. Prove that $F\{xU(x)\} = \dfrac{1}{(is)^2}$.
15. Prove that $F\{xe^{-ax}U(x)\} = \dfrac{1}{(a+is)^2}$.
16. Prove that $F\{\sin ax \cdot U(x)\} = \dfrac{a}{a^2-s^2}$.
17. Prove that $F\{\cos ax \cdot U(x)\} = \dfrac{is}{a^2-s^2}$.

Part B

18. Prove that $F(1) = 2\pi\delta(s)$, where $\delta(s)$ is the unit impulse function. Hence find $F(\cos ax)$ and $F(\sin ax)$, using modulation theorem.
 [**Hint:** Use the definition of $\delta(s)$, proceed as in example 4(ii) of Section 4(a) and prove that $F^{-1}\{\delta(s)\} = \dfrac{1}{2\pi}\cdot 1$].
19. Find $F\{e^{-ax}\sin bxU(x)\}$ and $F\{e^{-ax}\cos bxU(x)\}$.
20. Find $F\{e^{-3|x|}\sin 2x\}$ and $F\{xe^{-2x}\sin 3xU(x)\}$.
21. Find $F(e^{-x^2})$ and hence find $F(e^{-x^2}\cos x)$, using modulation theorem.

22. Find the Fourier transform of $f(x)$ given by

$$f(x) = \begin{cases} 1, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}$$

Hence prove that $$\int_0^\infty \frac{\sin^2 dx}{x^2}\, dx = \frac{\pi}{2}.$$

23. Find the Fourier transform of $f(x)$ given by

$$f(x) = \begin{cases} a^2 - x^2 & \text{for } |x| < a \\ 0, & \text{for } |x| > a > 0. \end{cases}$$

Hence prove that

(i) $$\int_0^\infty \frac{\sin x - x\cos x}{x^3}\, dx = \frac{\pi}{4} \text{ and}$$

(ii) $$\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{15}.$$

24. Find the Fourier cosine transform of (xe^{-ax}) and hence find the value of

$$\int_0^\infty \frac{(x^2 - a^2)^2}{(x^2 + a^2)^4}.$$

[**Hint:** See Problem (26) in Exercise 4(a).]

25. Find the Fourier sine transform of (xe^{-ax}) and hence evaluate

$$\int_0^\infty \frac{x^2\, dx}{(x^2 + a^2)^4}.$$

[**Hint:** See Example (13) in Section 4(a).]

26. Use transform methods to evaluate

(i) $\displaystyle\int_0^\infty \frac{dx}{(x^2 + a^2)(x^2 + b^2)}$; $a, b > 0$ and (ii) $\displaystyle\int_0^\infty \frac{x^2\, dx}{(x^2 + 4)(x^2 + 9)}$.

27. Use $F_C(e^{-ax})$ to find $F_S(xe^{-ax})$ and use the latter to find $F_C(x^2 e^{-ax})$.

Solve the following initial value problems using Fourier transforms.

28. $(D^2 - D - 6)y = e^{3x}$; $\quad y(0) = 0, \quad y'(0) = 0;\ x > 0.$
29. $(D^2 + 6D + 9)y = e^{-3x}$; $\quad y(0) = 0, \quad y'(0) = 0;\ x > 0.$
30. $(D^2 - 2D + 5)y = xe^{2x}$; $\quad y(0) = 0, \quad y'(0) = 0;\ x > 0.$

31. $(D^2-3D+2)y = 1+x;\quad y(0)=0,\quad y'(0)=0; x>0.$
32. $(D^2+2D+2)y = \sin 2x;\quad y(0)=0,\quad y'(0)=0; x>0.$
33. $(D^2+1)y = \cos x + \sin x;\quad y(0)=0,\quad y'(0)=0; x>0.$
34. $(D-1)^2 y = \cos x;\quad y(0)=0,\quad y'(0)=0; x>0.$
35. Solve the equation $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$, satisfying the initial conditions $y(x,0) = f(x)$, $\frac{\partial y}{\partial t}(x,0) = 0$ and the boundary conditions $y(0,t)=0$, $y(x,t)\longrightarrow 0$ as $x\longrightarrow\infty$. Use transform method.
36. Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$; $x\geq 0$, using transform method, given that $u(0,y)=f(y)$, $-\infty<y<\infty$ and $u(x,y)\longrightarrow 0$ as $x\longrightarrow\infty$.
37. Solve the equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$, $-\infty<x<\infty$, using transform method, given that $u(x,0) = \begin{cases} k, & \text{for } x>0 \\ 0, & \text{for } x<0\end{cases}$ and $u(x,t)\longrightarrow 0$ as $x\longrightarrow\pm\infty$.
38. Solve the equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$, $x\geq 0$, using transform method, given that $u(0,t)=f(t)$, $t\geq 0$, $u(x,t)\to 0$ as $x\to 0$ and $u(x,0)=0$.
39. Using transform method, solve the equation $\frac{\partial u}{\partial x} = \alpha^2\frac{\partial^2 u}{\partial x^2}$, $x\geq 0$, subject to the boundary conditions $\frac{\partial u}{\partial x}(0,t) = f(t)$, $t\geq 0$ and $u(x,t)\longrightarrow 0$ as $x\longrightarrow\infty$ and the initial condition $u(x,0)=0$.
40. Using transform method, solve the equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$, $x\geq 0$, subject to the boundary conditions $u(0,t)=0$ and $u(x,t)\longrightarrow 0$ as $x\longrightarrow\infty$ and the initial condition

$$u(x,0) = \begin{cases} 1, & \text{for } 0\leq x\leq 1 \\ 0, & \text{for } x>1\end{cases}$$

4.7 FINITE FOURIER TRANSFORMS

Definitions

1. If the function $f(x)$ is piecewise continuous in the interval $(0,l)$, then $\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$, where n is an integer, is called the *Finite Fourier Sine Transform of* $f(x)$ in $(0,l)$ and denoted by $F_S\{f(x)\}$ or $\bar{f}_S(n)$.

i.e. $$F_S\{f(x)\} = \bar{f}_S(n) = \int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

2. If the function $f(x)$ is piecewise continuous in the interval $(0, l)$, then $\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$, where n is an integer, is called *the Finite Fourier Cosine Transform of* $f(x)$ in $(0, l)$ and denoted by $F_C\{f(x)\}$ or $\bar{f}_C(n)$.

i.e.
$$F_C\{f(x)\} = \bar{f}_C(n) = \int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$$

Inversion formulas

1. If $\bar{f}_S(n)$ is the finite Fourier sine transform of $f(x)$ in $(0, l)$, then $f(x) = \frac{2}{l}\sum_{n=1}^{\infty}\bar{f}_S(n)\sin\frac{n\pi x}{l}$ is called *the inverse finite Fourier sine transform* of $\bar{f}_S(n)$ and denoted as $F_S^{-1}\{\bar{f}_S(n)\}$. Once we assume the definition of $F_S\{f(x)\}$, the inversion formula is derived as follows:
Since $f(x)$ is piecewise continuous in $(0, l)$, it can be expanded as an infinite trigonometric series of the form $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ (i.e., Fourier half-range sine series)

i.e.
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad (1)$$

Multiplying both sides of (1) by $\sin\frac{n\pi x}{l}$ and integrating with respect to x between the limits 0 and l, we get

$$\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = b_1\int_0^l \sin\frac{\pi x}{l}\sin\frac{n\pi x}{l}\,dx + b_2\int_0^l \sin\frac{2\pi x}{l}\sin\frac{n\pi x}{l}\,dx$$
$$+\ldots + b_n\int_0^l \sin^2\frac{n\pi x}{l}\,dx + \ldots$$
$$= b_n\int_0^l \sin^2\frac{n\pi x}{l}\,dx \qquad [\because \text{ all other integrals vanish}]$$
$$= \frac{b_n}{2}\int_0^l \left[1 - \cos\frac{2n\pi x}{l}\right] dx$$
$$= \frac{b_n}{2}\left\{x - \frac{\sin\left[\frac{2n\pi x}{l}\right]}{\left[\frac{2n\pi}{l}\right]}\right\}_0^l$$
$$= \frac{b_n}{2}\cdot l$$

$$\therefore \quad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\bar{f}_S(n), \text{ by definition} \tag{2}$$

Inserting (2) in (1), we get the following inversion formula.

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty}\bar{f}_S(n)\sin\frac{n\pi x}{l}$$

2. If $\bar{f}_C(n) = F_C\{f(x)\}$ in $(0, l)$, then $f(x) = \frac{1}{l}\bar{f}_C(0) + \frac{2}{l}\sum_{n=1}^{\infty}\bar{f}_C(n)\cos\frac{n\pi x}{l}$ is called the *inverse finite Fourier cosine transform of* $\bar{f}_C(n)$ and denoted as $F_C^{-1}\{\bar{f}_C(n)\}$.

As before, once the definition of $F_C\{f(x)\}$ is assumed, the inversion formula is derived as follows:

Since $f(x)$ is piecewise continuous in $(0, l)$, it can be expanded as an infinite trigonometric series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$ (i.e., Fourier half-range cosine series).

i.e.
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} \tag{3}$$

Integrating both sides of (3) with respect to x between 0 and l, we get

$$\int_0^l f(x)\,dx = \frac{a_0}{2}\int_0^l dx = \frac{a_0}{2}\cdot l$$

$$\therefore \quad \frac{a_0}{2} = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{l}\bar{f}_C(0) \text{ (by definition)} \tag{4}$$

Multiplying both sides of (3) by $\cos\frac{n\pi x}{l}$ and integrating with respect to x between the limits 0 and l, we get

$$\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{a_0}{2}\int_0^l \cos\frac{n\pi x}{l}\,dx + a_1\int_0^l \cos\frac{\pi x}{l}\cos\frac{n\pi x}{l}\,dx$$
$$+ a_2\int_0^l \cos\frac{2\pi x}{l}\cos\frac{n\pi x}{l}\,dx + \ldots + a_n\int_0^l \cos^2\frac{n\pi x}{l}\,dx + \ldots$$

$$= a_n \int_0^l \cos^2 \frac{n\pi x}{l}\, dx \qquad [\because \text{ all other integrals vanish}]$$

$$= \frac{a_n}{2} \int_0^l \left[1 + \cos \frac{2n\pi x}{l}\right] dx$$

$$= \frac{a_n}{2} \left\{ x + \frac{\sin\left(\frac{2n\pi x}{l}\right)}{\left(\frac{2n\pi}{l}\right)} \right\}_0^l = \frac{a_n}{2} \cdot l$$

$$\therefore \qquad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l} \bar{f}_C(n) \quad \text{(by definition)} \tag{5}$$

Inserting (4) and (5) in (3), we get the inversion formula

$$f(x) = \frac{1}{l} \bar{f}_C(0) + \frac{2}{l} \sum_{n=1}^{\infty} \bar{f}_C(n) \cos \frac{n\pi x}{l}$$

Finite Fourier transforms of derivatives

(i) $F_S\{f'(x)\} = -\frac{n\pi}{l} \bar{f}_C(n)$

(ii) $F_C\{f'(x)\} = (-1)^n f(l) - f(0) + \frac{n\pi}{l} \bar{f}_S(n)$

(iii) $F_S\{f''(x)\} = -\frac{n^2\pi^2}{l^2} \bar{f}_S(n) + \frac{n\pi}{l} \{f(0) - (-1)^n f(l)\}$

(iv) $F_C\{f''(x)\} = -\frac{n^2\pi^2}{l^2} \bar{f}_C(n) + (-1)^n f'(l) - f'(0).$

Proof

(i) $$F_S\{f'(x)\} = \int_0^l f'(x) \sin \frac{n\pi x}{l} dx$$

$$= \int_0^l \sin \frac{n\pi x}{l} d\{f(x)\}$$

$$= \left\{ f(x) \sin \frac{n\pi x}{l} \right\}_0^l - \frac{n\pi}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$

$$= -\frac{n\pi}{l} \bar{f}_C(n).$$

(iii) $$F_S\{f''(x)\} = \int_0^l \sin\frac{n\pi x}{l}\, d\{f'(x)\}$$

$$= \left\{f'(x)\sin\frac{n\pi x}{l}\right\}_0^l - \frac{n\pi}{l}\int_0^l f'(x)\cos\frac{n\pi x}{l}\,dx$$

$$= -\frac{n\pi}{l}\int_0^l \cos\frac{n\pi x}{l}\, d\{f(x)\}$$

$$= -\frac{n\pi}{l}\left[\left\{f(x)\cos\frac{n\pi x}{l}\right\}_0^l + \frac{n\pi}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= -\frac{n^2\pi^2}{l^2}\bar{f}_S(n) + \frac{n\pi}{l}\{f(0) - (-1)^n f(l)\}$$

Similarly the results (ii) and (iv) may be proved.

Note ✍

Similar formulas hold good for finite Fourier transforms of partial derivatives, with minor changes.

Worked Examples 4(c)

Example 1

Find the finite Fourier sine and cosine transforms of $\left(\frac{x}{\pi}\right)$ in $(0, \pi)$.

$$F_S\left(\frac{x}{\pi}\right) = \int_0^\pi \frac{x}{\pi}\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{x\left(\frac{-\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right\}_0^\pi$$

$$= \frac{1}{\pi}\left\{-\frac{\pi}{n}(-1)^n\right\} = \frac{(-1)^{n+1}}{n}$$

$$F_C\left(\frac{x}{\pi}\right) = \frac{1}{\pi}\left\{x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right\}_0^\pi = \frac{1}{\pi n^2}\{(-1)^n - 1\}, n \neq 0$$

Example 2

Find the finite Fourier cosine and sine transforms of $f(x)$, if

$$f(x) = \begin{cases} 1, & \text{for } 0 < x < \pi/2 \\ -1, & \text{for } \pi/2 < x < \pi \end{cases}$$

$$F_C\{f(x)\} = \int_0^{\pi/2} 1 \cdot \cos nx \, dx - \int_{\pi/2}^{\pi} 1 \cdot \cos nx \, dx$$

$$= \left[\frac{\sin nx}{n}\right]_0^{\pi/2} - \left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{n}\left\{\sin\frac{n\pi}{2} - 0 - \sin n\pi + \sin\frac{n\pi}{2}\right\}$$

$$= \frac{2}{n}\sin\frac{n\pi}{2}, \quad n \neq 0.$$

$$F_S\{f(x)\} = \int_0^{\pi/2} \sin nx \, dx - \int_{\pi/2}^{\pi} \sin nx \, dx$$

$$= -\frac{1}{n}(\cos nx)_0^{\pi/2} + \frac{1}{n}(\cos nx)_{\pi/2}^{\pi}$$

$$= \frac{1}{n}\left\{1 - 2\cos\frac{n\pi}{2} + (-1)^n\right\}$$

Example 3

Find the finite Fourier sine and cosine transforms of $\left(1 - \frac{x}{\pi}\right)^2$ in $(0, \pi)$

$$F_S\left\{\left(1 - \frac{x}{\pi}\right)^2\right\} = \int_0^{\pi} \left(1 - \frac{x}{\pi}\right)^2 \sin nx \, dx$$

$$= \left[\left(1 - \frac{x}{\pi}\right)^2\left(\frac{-\cos nx}{n}\right) - \left(\frac{-2}{\pi}\right)\left(\frac{1-x}{\pi}\right)\left(\frac{-\sin nx}{n^2}\right) + \frac{2}{\pi^2}\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{1}{n} + \frac{2}{\pi^2 n^3}\{(-1)^n - 1\}$$

$$F_C\left\{\left(1 - \frac{x}{\pi}\right)^2\right\} = \int_0^{\pi} \left(1 - \frac{x}{\pi}\right)^2 \cos nx \, dx$$

$$= \left[\left(1 - \frac{x}{\pi}\right)^2 \frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\left(1 - \frac{x}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right) + \frac{2}{\pi^2}\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}, \quad n \neq 0.$$

Example 4

Find the finite Fourier sine and cosine transforms of e^{ax} in $(0, l)$.

$$F_S(e^{ax}) = \int_0^l e^{ax} \sin \frac{n\pi x}{l}\, dx$$

$$= \left[\frac{e^{ax}}{a^2 + \frac{n^2\pi^2}{l^2}} \left\{ a \sin \frac{n\pi x}{l} - \frac{n\pi}{l} \cos \frac{n\pi x}{l} \right\} \right]_0^l$$

$$= \frac{e^{al}}{a^2 + \frac{n^2\pi^2}{l^2}} \cdot \frac{n\pi}{l}(-1)^{n+1} + \frac{\frac{n\pi}{l}}{a^2 + \frac{n^2\pi^2}{l^2}}$$

$$= \frac{n\pi l}{n^2\pi^2 + l^2a^2} \left\{ (-1)^{n+1} e^{al} + 1 \right\}$$

$$F_C(e^{ax}) = \int_0^l e^{ax} \cos \frac{n\pi x}{l}\, dx$$

$$= \left[\frac{e^{ax}}{a^2 + \frac{n^2\pi^2}{l^2}} \left\{ a \cos \frac{n\pi x}{l} + \frac{n\pi}{l} \sin \frac{n\pi x}{l} \right\} \right]_0^l$$

$$= \frac{al^2}{n^2\pi^2 + a^2l^2} \left\{ (-1)^n e^{al} - 1 \right\}$$

Example 5

Find the finite Fourier sine transform of $\cos ax$ and finite Fourier cosine transform of $\sin ax$ in $(0, \pi)$.

$$F_S\,(\cos ax) = \int_0^\pi \cos ax \sin nx\ dx$$

$$= \frac{1}{2} \int_0^\pi [\sin\,(n + a)x + \sin\,(n - a)x]\, dx$$

$$= -\frac{1}{2} \left[\frac{\cos\,(n + a)x}{n + a} + \frac{\cos\,(n - a)x}{n - a} \right]_0^\pi$$

$$= \frac{1}{2} \left[\frac{1}{n + a}\{1 - \cos\,(n + a)\pi\} + \frac{1}{n - a}\{1 - \cos\,(n - a)\pi\} \right]$$

$$= \frac{1}{2} \left[\left\{ \frac{1}{n + a} + \frac{1}{n - a} \right\} - \left\{ \frac{1}{n + a} + \frac{1}{n - a} \right\} \cos n\pi \cos a\pi \right]$$

$$= \frac{n}{n^2 - a^2}[1 - (-1)^n \cos a\pi].$$

$$F_C(\sin ax) = \int_0^{\pi} \sin ax \cos nx \ \mathrm{d}x$$

$$= \frac{a}{a^2 - n^2}[1 - (-1)^n \cos a\pi], \text{ on interchanging } n \text{ and } a \text{ in (1).}$$

Example 6

Find $f(x)$, if its finite sine transform is given by

$$\bar{f}_S(n) = \frac{1 - \cos n\pi}{n^2\pi^2} \text{ in } 0 < x < \pi$$

The inverse finite Fourier sine transform is given by

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \bar{f}_S(n) \sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1 - \cos n\pi}{n^2\pi^2}\right\}\sin nx$$

$$= \frac{2}{\pi^3}\sum_{n=1}^{\infty}\left\{\frac{1 - (-1)^n}{n^2}\right\}\sin nx$$

$$= \frac{4}{\pi^3}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\sin nx$$

$$= \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin (2n-1)x.$$

Example 7

Find $f(x)$, if its finite cosine transform is given by $\bar{f}_C(n) = \dfrac{1}{(2n+1)^2}\cos\dfrac{2n\pi}{3}$ in $0 < x < 1$.

The inverse finite Fourier cosine transform in $(0, l)$ is given by

$$f(x) = \frac{1}{l}\bar{f}_C(0) + \frac{2}{l}\sum_{n=1}^{\infty}\bar{f}_C(n)\cos\frac{n\pi x}{l}$$

Here $\qquad l = 1 \quad \text{and} \quad \bar{f}_C(0) = 1$

$\therefore \qquad f(x) = 1 + 2\sum_{n=1}^{\infty}\dfrac{1}{(2n+1)^2}\cos\dfrac{2n\pi}{3}\cos n\pi x$

Example 8

Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, 0 < x < \pi, t > 0$, using finite Fourier transforms, given that $u(0, t) = 0, u(\pi, t) = 0$ for $t > 0$ and $u(x, 0) = 4\sin^3 x$.

Since $u(0, t)$ and $u(\pi, t)$ are given, we take finite Fourier sine transforms on both sides of the given equation, with respect to x in $(0, \pi)$.

Then $$\frac{\partial}{\partial t}\bar{u}_S(n,t) = -n^2\bar{u}_S(n,t) + n\{u(0,t) - (-1)^n u(\pi,t)\}$$

i.e. $$\frac{\partial}{\partial t}\bar{u}_S = -n^2\bar{u}_S \tag{1}$$

on using the given boundary conditions.
Solving (1), we get

$$\bar{u}_S(n,t) = Ae^{-n^2 t} \tag{2}$$

Taking the finite sine transform of the initial condition $u(x,0) = 4\sin^3 x$, we get

$$\bar{u}_s(n,0) = \int_0^\pi 4\sin^3 x \sin nx \ \mathrm{d}x$$

$$= \int_0^\pi (3\sin x - \sin 3x)\sin nx \ \mathrm{d}x$$

$$= 0, \quad \text{when } n \neq 1 \text{ and } n \neq 3.$$

When $$n = 1, \bar{u}_S(1,0) = \int_0^\pi 3\sin^2 x \ \mathrm{d}x$$

$$= \frac{3}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{3\pi}{2} \tag{3}$$

When $$n = 3, \bar{u}_S(3,0) = \int_0^\pi (-\sin^2 3x)\ \mathrm{d}x$$

$$= \frac{1}{2}\left[\frac{\sin 6x}{6} - x\right]_0^\pi = -\frac{\pi}{2} \tag{4}$$

Using (3) and (4) in (2), we get, $A = \frac{3\pi}{2}$, when $n = 1$, $A = -\frac{\pi}{2}$, when $n = 3$ and $A = 0$, for all other values of n.

By inversion formula,

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\bar{u}_S(n,t)\sin nx$$

$$= \frac{2}{\pi}\left\{\frac{3\pi}{2}\sin x \cdot e^{-t} - \frac{\pi}{2}\sin 3x \cdot e^{-9t}\right\}$$

$$= 3\sin xe^{-t} - \sin 3x \cdot e^{-9t}.$$

Example 9

Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 10$, using finite Fourier transforms, given that $u(0,t) = 0$, $u(10,t) = 0$, for $t > 0$ and $u(x,0) = 10x - x^2$ for $0 < x < 10$.
Since $u(0,t)$ and $u(10,t)$ are given, we take finite Fourier sine transforms on both sides of the given equation with respect to x in (0, 10).

Then $$\frac{\partial}{\partial t}\bar{u}_S(n,t) = -\frac{n^2\pi^2}{10^2}\bar{u}_S(n,t) + \frac{n\pi}{10}\{u(0,t) - (-1)^n u(10,t)\}$$

i.e. $$\frac{d}{dt}\bar{u}_S(n,t) = -\frac{n^2\pi^2}{100}\bar{u}_S(n,t) \tag{1}$$

on using the given boundary conditions.

Solving (1), we get

$$\bar{u}_S(n,t) = Ae^{-n^2\pi^2 t/100} \tag{2}$$

Taking the finite sine transform of the initial condition $u(x,0) = 10x - x^2$ in (0,10), we get

$$\bar{u}_S(n,0) = \int_0^{10} (10x - x^2)\sin\frac{n\pi x}{10}\,dx$$

$$= \left[(10x - x^2)\left\{\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right\} - (10-2x)\left\{\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{10^2}}\right\} + (-2)\cdot\left\{\frac{\cos\frac{n\pi x}{10}}{\frac{n^3\pi^3}{10^3}}\right\}\right]_0^{10}$$

by Bernoulli's integration formula.

$$= \frac{2000}{n^3\pi^3}\{1-(-1)^n\} = \begin{cases} \frac{4000}{n^3\pi^3}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases} \tag{3}$$

Using (3) in (2), $A = \begin{cases} \frac{4000}{n^3\pi^3}, & \text{if n is odd} \\ 0, & \text{if n is even} \end{cases}$

By inversion formula,

$$u(x,t) = \frac{2}{10}\sum_{n=1}^{\infty} \bar{u}_S(n,t)\sin\frac{n\pi x}{10}$$

$$= \frac{800}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3}\sin\frac{n\pi x}{10}e^{-n^2\pi^2 t/100}$$

Example 10

Solve the equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$, $0 < x < l$, using finite Fourier transforms, given that $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(l,t) = 0$ for $t > 0$ and $u(x,0) = kx$, for $0 < x < l$.

Since $\dfrac{\partial u}{\partial x}(0,t)$ and $\dfrac{\partial u}{\partial x}(l,t)$ are given, we take finite Fourier cosine transforms on both sides of the given equation with respect to x in $(0,l)$.

Then $$\frac{\partial}{\partial t}\bar{u}_C(n,t) = \alpha^2\left[-\frac{n^2\pi^2}{l^2}\bar{u}_C(n,t) + (-1)^n\frac{\partial u}{\partial x}(l,t) - \frac{\partial u}{\partial x}(0,t)\right]$$

i.e. $$\frac{d}{dt}\bar{u}_C(n,t) = \frac{-n^2\pi^2\alpha^2}{l^2}\bar{u}_C(n,t) \tag{1}$$

on using the given boundary conditions.

Solving (1), we get

$$\bar{u}_C(n,t) = Ae^{-n^2\pi^2\alpha^2 t/l^2} \tag{2}$$

Taking the finite cosine transform of the initial conditon $u(x,0) = \kappa x$, in $(0,l)$, we get

$$\bar{u}_C(n,0) = \int_0^l kx\cos\frac{n\pi x}{l}\,\mathrm{d}x$$

$$= k\left[x\left\{\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\} + \frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]_0^l, \text{ by}$$

Bernoulli's integration formula,

$$= \frac{kl^2}{n^2\pi^2}\{(-1)^n - 1\},\ \text{when } n\neq 0 \tag{3}$$

When $n = 0$,

$$\bar{u}_C(0,0) = \int_0^l kx\ \mathrm{d}x = \frac{kl^2}{2} \tag{4}$$

Using (4) in (2), $\quad A = \dfrac{kl^2}{2},\quad$ when $n = 0$

Using (3) in (2), $\quad A = \dfrac{kl^2}{n^2\pi^2}, \{(-1)^n - 1\},\quad$ when $n \neq 0$.

By inversion formula,

$$u(x,t) = \frac{1}{l}\bar{u}_C(0,t) + \frac{2}{l}\sum_{n=1}^{\infty}\bar{u}_C(n,t)\cos\frac{n\pi x}{l}$$

$$= \frac{kl}{2} + \frac{2kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\{(-1)^n - 1\}\cos\frac{n\pi x}{l}\cdot e^{-n^2\pi^2\alpha^2 t/l^2}$$

$$= \frac{kl}{2} - \frac{4kl}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}e^{-n^2\pi^2\alpha^2 t/l^2}$$

Exercise 4 (c)

Part A (Short Answer Questions)

1. Define finite Fourier sine and cosine transforms of $f(x)$.
2. Define the inverse finite fourier sine and cosine transforms.
3. Find the finite Fourier sine transform of $f'(x)$.
4. Find the finite Fourier cosine transform of $f'(x)$.
5. Write down the formulas for $F_S\{f''(x)\}$ and $F_C\{f''(x)\}$.
6. Write down the formulas for $F_S\left\{\frac{\partial u}{\partial x}(x,t)\right\}$ and $F_C\left\{\frac{\partial u}{\partial x}(x,t)\right\}$, where the transforms are taken with respect to x.
7. Write down the formulas for $F_S\left\{\frac{\partial^2 u}{\partial x^2}(x,t)\right\}$ and $F_C\left\{\frac{\partial^2 u}{\partial x^2}(x,t)\right\}$, where the transforms are taken with respect to x.

Part B

8. Find the finite Fourier sine and cosine transforms of $f(x) = 1$ in $(0, l)$.
9. Find the finite Fourier sine and cosine transforms of $f(x) = x$ in $(0, \pi)$.
10. Find the finite Fourier sine and cosine transforms of $f(x) = x^2$ in $(0, 1)$.
11. Find the finite Fourier sine and cosine transforms of $f(x) = x^3$ in $(0, 2)$.
12. Find the finite Fourier sine and cosine transforms of $f(x) = x(\pi - x)$ in $(0, \pi)$.
13. Find the finite Fourier sine and cosine transforms of
$$f(x) = \begin{cases} x, & \text{in } (0, l/2) \\ l - x & \text{in } (l/2, l) \end{cases} \quad \text{in the interval } (0, 1).$$
14. Find the finite Fourier cosine transform of $f(x) = \frac{\pi}{3} - x + \frac{x^2}{2\pi}$ in $(0, \pi)$.
15. Find $f(x)$, if (i) $\bar{f}_S(n) = \frac{16(-1)^{n-1}}{n^3}$, if $0 < x < 8$; and (ii) $\bar{f}_S(n) = \frac{2\pi(-1)^{n-1}}{n^3}$, if $0 < x < \pi$, where $n = 1, 2, 3, \ldots$
16. Find $f(x)$, if $$\bar{F}_C(n) = \frac{\sin\left(\frac{n\pi}{2}\right)}{2n}, \; n = 1, 2, 3, \ldots$$ $$= \pi/4, \quad n = 0;$$ where $0 < x < 2\pi$.
17. Solve $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, $0 < x < 1, t > 0$, using finite Fourier transforms, given that $u(0, t) = 0$, $u(1, t) = 0$ for $t > 0$ and $u(x, 0) = 2\sin 2\pi x \cos \pi x$.

18. Solve $\dfrac{\partial u}{\partial t} = K\dfrac{\partial^2 u}{\partial x^2}$, $0 < x < 2, t > 0$, using finite Fourier transforms, given that

$$u(0,t) = 0, \quad u(2,t) = 0, \text{ for } t > 0 \text{ and}$$

$$u(x,0) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 < x < 2. \end{cases}$$

19. Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $0 < x < \pi, t > 0$, using finite Fourier transforms, given that $\dfrac{\partial u}{\partial x}(0,t) = 0$, $\dfrac{\partial u}{\partial x}(\pi,t) = 0$, for $t > 0$ and $u(x,0) = 2\cos^2 x$, for $0 < x < \pi$.

20. Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $0 < x < l, t > 0$ using finite Fourier transforms, given that $\dfrac{\partial u}{\partial x}(0,t) = 0$, $\dfrac{\partial u}{\partial x}(l,t) = 0$ for $t > 0$ and $u(x,0) = lx - x^2$ for $0 < x < l$.

Answers

Exercise 4 (a)

11. $\dfrac{a}{s^2 + a^2}$

12. $\dfrac{s}{s^2 + a^2}$

13. $\dfrac{2a}{s^2 + a^2}$

14. $10\left\{\dfrac{1}{s^2+4} + \dfrac{1}{s^2+25}\right\}$

15. $\dfrac{5s(s^2+5)}{(s^2+4)(s^2+9)}$

16. $\dfrac{2k \sin ls}{s}$

17. $\dfrac{2}{\pi}\displaystyle\int_0^\infty \dfrac{\sin s \cos xs}{s}\, ds$; $\qquad I = \begin{cases} \pi/2, & \text{for } |x| \le 1 \\ 0, & \text{for } |x| > 1. \end{cases}$

19. $\dfrac{\pi}{2} e^{-x} = \displaystyle\int_0^\infty \dfrac{s \sin xs}{s^2 + 1}\, ds$

22. $\dfrac{2i}{s^2}(as \cos as - \sin as)$

23. $\frac{2}{s^2}(1 - \cos as)$

24. $\frac{1}{\pi x^3}(x^2 \sin x + x \cos x - \sin x)$

25. $\frac{1}{\pi x} \sin s_0 x;\ \frac{\pi}{2}$

26. $\frac{a}{s^2 + a^2};\ \frac{a^2 - s^2}{(a^2 + s^2)^2};\ \frac{2(a^2 - s^2)}{(a^2 + s^2)^2}$

27. $\frac{s}{s^2 + a^2};\ \frac{2as}{(s^2 + a^2)^2};\ -\frac{4ias}{(s^2 + a^2)^2}$

28. $\sqrt{\frac{\pi}{4a}}\left\{e^{-(s+b)^2/4a} + e^{-(s-b)^2/4a}\right\}$

29. $\frac{1}{2\sqrt{\pi}}e^{-x^2/4}$

30. $\frac{1}{4}(1 + x)e^{-x}$

31. $\frac{1 - \cos s}{s}$

32. $\frac{1}{s}(a \cos as - b \cos bs) + \frac{1}{b^2}(\sin bs - \sin as)$

33. $\frac{1}{2}\left\{\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s-1}\right\}$

34. $\frac{\pi}{2}e^{-s}$

35. $\frac{\sqrt{\pi}}{2}e^{-s^2/4}$

36. $\tan^{-1}\left(\frac{s}{a}\right)$

37. e^{-x}

38. $\frac{2}{\pi}\tan^{-1}\left(\frac{x}{a}\right)$

39. $f(x) = \begin{cases} 1, & \text{for } 0 < x < 1 \\ 0, & \text{for } x \geq 1. \end{cases}$

40. $\frac{2}{\pi} \cdot \frac{x}{a^2 + x^2}$

Exercise 4(b)

18. $\pi\{\delta(s+a)+\delta(s-a)\}$; $\pi\left\{\delta(s+a)e^{i\pi/2}+\delta(s-a)e^{-\frac{i\pi}{2}}\right\}$

19. $\dfrac{b}{(a+is)^2+b^2}$; $\dfrac{a+is}{(a+is)^2+b^2}$

20. $\dfrac{-24is}{s^4+10s^2+169}$; $\dfrac{6(2+is)}{(2+is)^2+9}$

21. $\dfrac{\sqrt{\pi}}{2}\left\{e^{-(s-1)^2/4}+e^{-(s+1)^2/4}\right\}$

23. $\dfrac{4}{s^3}(\sin as - as\cos as)$

24. $\dfrac{\pi}{8a^3}$

25. $\dfrac{\pi}{32a^5}$

26. (i) $\pi/2ab(a+b)$; (ii) $\dfrac{\pi}{10}$

27. $\dfrac{2as}{(s^2+a^2)^2}$; $\dfrac{2a(a^2-3s^2)}{(s^2+a^2)^3}$

28. $y=\dfrac{1}{25}e^{-2x}-\dfrac{1}{25}e^{3x}+\dfrac{1}{5}xe^{3x}$

29. $y=\dfrac{1}{2}x^2e^{-3x}$

30. $y=\dfrac{2}{25}e^{x}\cos 2x-\dfrac{3}{50}e^{x}\sin 2x-\dfrac{2}{25}e^{2x}+\dfrac{1}{5}xe^{2x}$

31. $y=\dfrac{5}{4}+\dfrac{1}{2}x-2e^{x}+\dfrac{3}{4}e^{2x}$

32. $y=e^{-x}\left[\dfrac{1}{5}\cos x+\dfrac{2}{5}\sin x\right]-\dfrac{1}{10}(\sin 2x+2\cos 2x)$

33. $y=\dfrac{1}{2}\sin x+\dfrac{1}{2}x\sin x-\dfrac{1}{2}x\cos x$

34. $y=\dfrac{1}{2}xe^{x}-\dfrac{1}{2}\sin x$

35. $y(x,t)=\dfrac{1}{2}[f(x+ct)+f(x-ct)]$

36. $u(x, y) = \dfrac{x}{\pi}\displaystyle\int_{-\infty}^{\infty} f(z)\cdot\dfrac{1}{x^2+(y-z)^2}\,dz\,(x>0).$

37. $u(x, t) = \dfrac{k}{2} + \dfrac{k}{\sqrt{\pi}}\displaystyle\int_{0}^{x/2c\sqrt{t}} e^{-\omega^2}\,d\omega$

38. $u(x, t) = \dfrac{x}{2\alpha\sqrt{\pi}}\displaystyle\int_{0}^{t} \dfrac{f(z)}{(t-z)^{3/2}} e^{-x^2/4\alpha^2(t-z)}\,dz.$

39. $u(x, t) = -\dfrac{\alpha}{\sqrt{\pi}}\displaystyle\int_{0}^{t} \dfrac{f(z)}{\sqrt{t-z}} e^{-x^2/4\alpha^2(t-z)}\,dz.$

40. $u(x, t) = \dfrac{2}{\pi}\displaystyle\int_{0}^{\infty} \left[\dfrac{1-\cos s}{s}\right] e^{-s^2 t}\sin xs\,ds.$

Exercise 4(c)

8. $\dfrac{l}{n\pi}\{1-(-1)^n\}$; 0, if $n \neq 0$ and l, if $n = 0$.

9. $\dfrac{\pi}{n}(-1)^{n+1}$; $\dfrac{1}{n^2}\{(-1)^n - 1\}$, if $n \neq 0$ and $\dfrac{\pi^2}{2}$, if $n = 0$.

10. $\dfrac{(-1)^{n+1}}{n\pi} + \dfrac{2}{n^3\pi^3}\{(-1)^n - 1\}$, $\dfrac{2(-1)^n}{n^2\pi^2}$ if $n \neq 0$ and $\dfrac{1}{3}$, if $n = 0$.

11. $\dfrac{(-1)^n}{n\pi} + \dfrac{6(-1)^n}{n^3\pi^3}$; $\dfrac{3(-1)^n}{n^2\pi^2} - \dfrac{6}{n^4\pi^4}$, if $n \neq 0$ and 4, if $n = 0$.

12. $\dfrac{2}{n^3}\{1-(-1)^n\}$; $-\dfrac{\pi}{n^2}\{1+(-1)^n\}$, if $n \neq 0$ and $\dfrac{\pi^3}{6}$, if $n = 0$.

13. $\dfrac{2l^2}{n^2}\sin\dfrac{n\pi}{2}$; $\dfrac{l^2}{n^2\pi^2}\left[\cos\dfrac{n\pi}{2} - 1\right]$, if $n \neq 0$ and $\dfrac{l^2}{4}$, if $n = 0$.

14. $\dfrac{1}{n^2}$, if $n \neq 0$ and 0, if $n = 0$

15. (i) $4\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^3}\sin\dfrac{n\pi x}{8}$; (ii) $4\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^3}\sin nx.$

16. $\dfrac{1}{4} + \dfrac{1}{2\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n}\sin\dfrac{n\pi}{2}\cos\dfrac{nx}{2}$

17. $u(x, t) = e^{-2\pi^2 t}\sin\pi x + e^{-18\pi^2 t}\sin 3\pi x.$

18. $u(x,t) = \dfrac{8}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\sin\dfrac{n\pi}{2}\sin\dfrac{n\pi x}{2}e^{-Kn^2\pi^2 t/4}$

19. $u(x,t) = 1 + \cos 2x \cdot e^{-4t}$

20. $u(x,t) = \dfrac{l^2}{6} - \dfrac{l^2}{\pi^2}\sum\dfrac{1}{n^2}\cos\dfrac{2n\pi x}{l}e^{-4n^2\pi^2\alpha^2 t/l^2}$

Chapter 5

Z-Transforms and Difference Equations

5.1 INTRODUCTION

In Communication Engineering, two basic types of signals (that are represented mathematically as functions of one or more independent variables) are encountered. They are continuous time signals and discrete time signals. Continuous time signals are defined for a continuum of values of the independent variable, namely time and are denoted by a function $\{f(t)\}$. On the other hand, discrete time signals are defined only at discrete set of values of the independent variable and are denoted by a sequence $\{f(n)\}$. A discrete time signal $\{f(n)\}$ may represent a phenomenon for which the independent variable is inherently discrete or successive samples of the underlying phenomenon (for which the independent variable is continuous) at the sampling instants 0, T, $2T$, ... Here T is called the *sampling period*.

Laplace transform and Fourier transform play important roles in the study of continuous time signals. Z-transform which is the discrete time counterpart of the Laplace transform plays an important role in discrete time signal analysis.

Signal Analysis is an engineering discipline of broad scope. It is used not only in communication technology, but also in the fields of astronomy, oceanography, crystallography, bio-engineering, antenna design, system theory, computer sciences and in many other fields.

Definition of the Z-transform

If $\{f(n)\}$ is a sequence defined for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$, then $\sum_{n=-\infty}^{\infty} f(n)z^{-n}$ is called the *two-sided* or *bilateral Z-transform* of $\{f(n)\}$ and denoted by $Z\{f(n)\}$ or $\bar{f}(z)$, where z is a complex variable in general.

If $\{f(n)\}$ is a causal sequence, i.e. if $f(n) = 0$ for $n < 0$, then the Z-transform is called *one-sided* or *unilateral Z-transform* of $\{f(n)\}$ and is defined as

$$Z\{f(n)\} = \bar{f}(z) = \sum_{n=0}^{\infty} f(n)z^{-n}$$

We shall mostly deal with one sided Z-transform which will be hereafter referred to as Z-transform.

Note ✍

The series $\sum_{n=0}^{\infty} f(n)z^{-n}$ will be convergent only for certain values of z i.e. only in a certain region of the z-plane (called the region of convergence of the z-transform) depending on the sequence $\{f(n)\}$.

The inverse Z-transform of $\bar{f}(z) = Z\{f(n)\}$ is defined as $Z^{-1}\{\bar{f}(z)\} = \{f(n)\}$.

If the function $f(t)$ is defined by means of a sequence of its sampled values at 0, T, $2T$, ..., then the Z-transform of $f(t)$ is defined as

$$Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)z^{-n}$$

We shall denote both $Z\{f(n)\}$ and $Z\{f(t)\}$ by $\bar{f}(z)$.

5.2 PROPERTIES OF Z-TRANSFORMS

1. Linearity

The Z-transform is linear

i.e. $$Z\{af(n) + bg(n)\} = aZ\{f(n)\} + bZ\{g(n)\}.$$

Proof

$$\begin{aligned} Z\{af(n) + bg(n)\} &= \sum_{n=0}^{\infty} \{af(n) + bg(n)\}z^{-n} \\ &= a\sum_{n=0}^{\infty} f(n)z^{-n} + b\sum_{n=0}^{\infty} g(n)z^{-n} \\ &= aZ\{f(n)\} + bZ\{g(n)\} \end{aligned}$$

Similarly, $$Z\{af(t) + bg(t)\} = aZ\{f(t)\} + bZ\{g(t)\}$$

2. Time Shifting

(i) $Z\{f(n - n_0)\} = z^{-n0} \cdot Z\{f(n)\}$ or $z^{-n0}\,\bar{f}(z)$

(ii) $Z\{f(t + T)\} = z\{\bar{f}(z) - f(0)\}$

Proof

(i) $$\begin{aligned} Z\{f(n - n_0)\} &= \sum_{n=0}^{\infty} f(n - n_0)z^{-n} \\ &= \sum_{m=-n_0}^{\infty} f(m)\, z^{-(m+n0)} \\ &= z^{-n0}\sum_{m=0}^{\infty} f(m)\, z^{-m} \quad \{\because f(n) \text{ is causal}\} \\ &= z^{-n0}\,\bar{f}(z) \text{ if } n \geq n_0 \end{aligned}$$

(ii)
$$Z\{f(t+T)\} = \sum_{n=0}^{\infty} f(nT+T)z^{-n} \quad [\because Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)z^{-n}]$$
$$= \sum_{n=0}^{\infty} f\{(n+1)T\}z^{-n} = \sum_{m=1}^{\infty} f(mT)z^{-(m-1)}$$
$$= z\left[\sum_{m=0}^{\infty} f(mT)z^{-m} - f(0)\right]$$
$$= z\{\bar{f}(z) - f(0)\}$$

Extending this result, we get

$$Z\{f(t+kT)\} = z^k\left[\bar{f}(z) - f(0) - \frac{f(T)}{z} - \frac{f(2T)}{z^2} - \ldots - \frac{f\{(k-1)T\}}{z^{k-1}}\right]$$

3. Frequency shifting

(i) $Z\{a^n f(n)\} = \bar{f}\left(\dfrac{z}{a}\right)$

(ii) $Z\{a^n f(t)\} = \bar{f}(z/a)$

Proof

(i)
$$Z\{a^n f(n)\} = \sum_{n=0}^{\infty} a^n f(n)z^{-n}$$
$$= \sum_{n=0}^{\infty} f(n)\left(\frac{z}{a}\right)^{-n} = \bar{f}(z/a)$$

Similarly (ii) can be proved.

Corollary

If $Z\{f(t)\} = \bar{f}(z)$, then $Z\{e^{-at} f(t)\} = \bar{f}(ze^{aT})$

This result follows, if we replace a^n by e^{-anT} or $(e^{-aT})^n$, i.e. 'a' by e^{-aT} in (ii).

4. Time reversal for bilateral Z-transform

If $Z\{f(n)\} = \bar{f}(z)$, then $Z\{f(-n)\} = \bar{f}\left(\frac{1}{z}\right)$

Proof

$$Z\{f(-n)\} = \sum_{n=-\infty}^{\infty} f(-n)z^{-n}$$
$$= \sum_{m=\infty}^{-\infty} f(m)z^{m}$$

$$= \sum_{m=-\infty}^{\infty} f(m) \left(\frac{1}{z}\right)^{-m}$$

$$= \bar{f}\left(\frac{1}{z}\right)$$

5. Differentiation in the Z-domain

(i) $Z\{nf(n)\} = -z \dfrac{d}{dz} \bar{f}(z)$

(ii) $Z\{nf(t)\} = -z \dfrac{d}{dz} \bar{f}(z)$

Proof

(i) $$\bar{f}(z) = Z\{f(n) = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$\therefore$ $$\frac{d}{dz}\bar{f}(z) = \sum_{n=0}^{\infty} -n f(n) z^{-n-1}$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty} \{n f(n)\} z^{-n}$$

$\therefore$ $$Z\{nf(n)\} = -z \frac{d}{dz} \bar{f}(z)$$

Similarly, (ii) can be proved.

6. Initial value theorem

(i) If Z$\{f(n)\} = \bar{f}(z)$, then $f(0) = \lim\limits_{z \to \infty} \bar{f}(z)$.

(ii) If $Z\{f(t)\} = \bar{f}(z)$, then $f(0) = \lim\limits_{z \to \infty} \bar{f}(z)$.

Proof

(i) $$\bar{f}(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$= f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \ldots \infty$$

$\therefore$ $$\lim_{z \to \infty} \left[\bar{f}(z)\right] = f(0)$$

Similarly, (ii) can be proved.

7. Final value theorem

(i) If $Z\{f(n)\} = \bar{f}(z)$, then $\lim\limits_{n \to \infty} [f(n)] = \lim\limits_{z \to 1} \{(z-1)\, \bar{f}(z)\}$.

(ii) If $Z\{f(t)\} = \bar{f}(z)$, then $\lim\limits_{t \to \infty} [f(t)] = \lim\limits_{z \to 1} \{(z-1)\, \bar{f}(z)\}$.

Proof

(i) $$Z\{f(n+1)\} = \sum_{n=0}^{\infty} f\,(n+1)z^{-n}$$

$$= \sum_{m=1}^{\infty} f\,(m)z^{-m+1} = z\,[\,\bar{f}\,(z) - f(0)]$$

$\therefore$ $$z\,\bar{f}\,(z) - zf(0) - \bar{f}\,(z) = Z\{f(n+1)\} - Z\{f(n)\}$$

i.e. $$(z-1)\;\bar{f}\,(z) - zf(0) = \sum_{n=0}^{\infty} \{f(n+1) - f(n)\}\, z^{-n}$$

Taking limits as $z \to 1$,

$$\lim_{z\to 1} [(z-1)\;\bar{f}\;(z)] - f(0) = \sum_{n=0}^{\infty} \{f(n+1) - f(n)\}$$

$$= \lim_{n\to\infty} \{f(1) - f(0)\} + \{f(2) - f(1)\}$$

$$+ \{f(3) - f(2)\} + \cdots + \{f(n+1) - f(n)\}]$$

$$= \lim_{n\to\infty} [f(n+1) - f(0)]$$

$$= \lim_{n\to\infty} [f(n) - f(0)]$$

$\therefore$ $$\lim_{n\to\infty} [f(n)] = \lim_{z\to 1} [(z-1)\;\bar{f}\,(z)]$$

Similarly, (ii) can be proved, starting with Property 2(ii).

8. Convolution theorem

Definitions

The convolution of the two sequences $\{f(n)\}$ and $\{g(n)\}$ is defined as (i) $\{f(n)^*\, g(n)\}$

$= \sum_{r=-\infty}^{\infty} f\,(r)\cdot g(n-r)$ if the sequences are non-causal and

(ii) $\{f(n)^*gn)\} = \sum_{r=0}^{n} f\,(r)\; g(n-r)$, if the sequences are causal.

The convolution of two functions $f(t)$ and $g(t)$ is defined as $f(t)^*g(t) = \sum_{r=0}^{n} f\,(rT)\cdot g(n - r)T$, where T is the sampling period.

Statement of the theorem

(i) If $Z\{f(n)\} = \bar{f}\,(z)$ and $Z\{g(n)\} = \bar{g}\,(z)$, then
$Z\{f(n)^*g(n)\} = \bar{f}\,(z)\cdot\,\bar{g}\,(z)$.

(ii) If $Z\{f(t)\} = \bar{f}\,(z)$ and $Z\{g(t)\} = \bar{g}\,(z)$, then
$Z\{f(t)^*g(t)\} = \bar{f}\,(z)\cdot\,\bar{g}\,(z)$.

Proof

(for the bilateral Z-transform)

(i)
$$Z\{f(n)*g(n)\} = Z\left[\sum_{r=-\infty}^{\infty} f(r)\cdot g(n-r)\right]$$

$$= \sum_{n=-\infty}^{\infty}\left[\sum_{r=-\infty}^{\infty} f(r)g(n-r)\right]z^{-n}$$

$$= \sum_{r=-\infty}^{\infty} f(r)\left[\sum_{n=-\infty}^{\infty} g(n-r)\,z^{-n}\right],$$

by changing the order of summation.

$$= \sum_{r=-\infty}^{\infty} f(r)\left[\sum_{m=-\infty}^{\infty} g(m)\,z^{-(m+r)}\right], \text{ by putting } n-r=m$$

$$= \sum_{r=-\infty}^{\infty} f(r)z^{-r}\left[\sum_{m=-\infty}^{\infty} g(m)\,z^{-m}\right]$$

$$= \sum_{r=-\infty}^{\infty} f(r)z^{-r}\cdot \bar{g}(z)$$

$$= \bar{g}(z)\cdot \sum_{r=-\infty}^{\infty} f(r)z^{-r} = \bar{f}(z)\cdot\bar{g}(z)$$

(ii)
$$\bar{f}(z)\cdot\bar{g}(z) = \left[\sum_{r=0}^{\infty} f(rT)z^{-r}\right]\left[\sum_{s=0}^{\infty} g(sT)z^{-s}\right]$$

$$= \sum_{n=0}^{\infty} h(nT)z^{-n} \tag{1}$$

say, where

$$h(nT) = f(0\cdot T)\,g(nT) + f(1T)\cdot g\{(n-1)T\} + f(2T)\cdot g\{(n-2)T\} + \cdots + f(nT)\cdot g(0\cdot T)$$

$$= \sum_{r=0}^{n} f(rT)\cdot g\{(n-r)T\} \tag{2}$$

Using (2) in (1), we get

$$\bar{f}(z)\cdot\bar{g}(z) = \sum_{n=0}^{\infty}\left[\sum_{r=0}^{n} f(rT)g\{(n-r)T\}\right]z^{-n}$$

$$= \sum_{n=0}^{\infty}[f(t)*g(t)]\,z^{-n}$$

$$= Z\,[f(t)*g(t)]$$

Note ✍

Proof of (i) in the unilateral Z-transform case can be given as in the proof of (ii).

5.3 Z-TRANSFORMS OF SOME BASIC FUNCTIONS

1. **$Z\{\delta(n)\}$,** where $\delta(n)$ is the unit impulse sequence defined by

$$\delta(n) = \begin{cases} 1, & \text{for } n=0 \\ 0, & \text{for } n \neq 0 \end{cases}$$

$$Z\{\delta(n)\} = \sum_{n=0}^{\infty} \delta(n) z^{-n} = 1$$

2. **$Z(k)$ and $Z\{U(n)\}$,** where k is a constant and $U(n)$ is the unit step sequence defined by

$$U(n) = \begin{cases} 1, & \text{for } n \geq 0, \text{ i.e. for } n = 0, 1, 2, \ldots \\ 0, & \text{for } n < 0 \end{cases}$$

(i)
$$Z(k) = \sum_{n=0}^{\infty} kz^{-n} = k\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots \infty\right)$$

$$= k \cdot \frac{1}{1 - \frac{1}{z}} = \frac{kz}{z-1}$$

where the region of convergence (ROC) is $\left|\frac{1}{z}\right| < 1$ or $|z| > 1$

(ii) In particular,

$$Z\{U(n)\} = Z(1) = \frac{z}{z-1}, \text{ if } |z| > 1 \text{ and}$$

$$Z\{U(t)\} = \frac{z}{z-1}$$

3. **$Z\{a^n\}$, $Z\{(-1)^n\}$, $Z\{e^{at}\}$, $Z\{e^{-at}\}$ and $Z(a^{n-1})$**

(i)
$$Z\{a^n\} = \sum_{n=0}^{\infty} a^n \cdot z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n, \text{ if } n \geq 0$$

$$= \frac{1}{1 - \frac{a}{z}} \text{ or } \frac{z}{z-a}, \text{ where the ROC is}$$

$$\left|\frac{a}{z}\right| < 1 \text{ or } |z| > |a|$$

(ii) In particular, $Z\{(-1)^n\} = \dfrac{z}{z+1}$, where the ROC is $|z| > 1$.

(iii) $$Z\{e^{at}\} = Z\{e^{anT}\} = Z\{(e^{aT})^n\}$$

$$= \frac{z}{z-e^{aT}}, \text{ where the ROC is } |z| > \left|e^{aT}\right|$$

Alternatively, by Property 3,

$$Z\{e^{at}\} = Z\{e^{at}\, U(t)\} = Z\{U(t)\}_{z \to ze-aT}$$

$$= \left(\frac{z}{z-1}\right)_{z \to ze^{-aT}} = \frac{ze^{-aT}}{ze^{-aT}-1} = \frac{z}{z-e^{aT}}$$

(iv) Similarly, $Z\{e^{-at}\} = \dfrac{z}{z-e^{-aT}}$

(v) $Z\{a^{n-1}) = z^{-1} Z(a^n)$, by Property 2, if $n \geq 1$

$$= z^{-1} \cdot \frac{z}{z-a}$$

$$= \frac{1}{z-a}, \text{ if } n \geq 1$$

4. $Z(n)$, $Z(na^n)$, $Z(n^2)$, $Z\{n(n-1)\}$, $z\{t\}$

(i) $$Z(n) = \sum_{n=0}^{\infty} nz^{-n} = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \ldots$$

$$= \frac{1}{z}\left\{1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \ldots\right\}$$

$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2}, \text{ if } \left|\frac{1}{z}\right| < 1 \text{ or } |z| > 1$$

$$= \frac{z}{(z-1)^2}, \text{ where the ROC is } |z| > 1$$

Alternatively $Z(n) = Z\{nU(n)\}$

$$= -z\frac{d}{dz}\bar{U}(z), \text{ by Property 5}$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{z}{(z-1)^2}$$

(ii) $$Z\{na^n\} = -z\frac{d}{dz}\{Z(a^n)\}, \text{ by Property 5}$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-a}\right)$$

$$= \frac{az}{(z-a)^2}, \text{ where the ROC is } |z| > |a|$$

(iii) $$Z\{n^2\} = z\{n \cdot n\} = -z\frac{d}{dz}\{Z(n)\}$$

$$= -z\frac{d}{dz}\left\{\frac{z}{(z-1)^2}\right\}$$

$$= -z\left[\frac{(z-1)^2 - 2z(z-1)}{(z-1)^4}\right]$$

$$= \frac{z(z+1)}{(z-1)^3}$$

(iv) $$Z\{n(n-1)\} = Z(n^2) - Z(n) \text{ [by Property 1]}$$

$$= \frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2}$$

$$= \frac{2z}{(z-1)^3}$$

(v) $$Z(t) = Z(nT)$$

$$= T \cdot Z(n) = \frac{Tz}{(z-1)^2}$$

5. $Z(n^k)$ and $Z(t^k)$

(i) $$Z(n^k) = Z\{n \cdot n^{k-1}\}$$

$$= -z\frac{d}{dz}\{Z(n^{k-1})\}, \text{ which is a recurrence formula.}$$

In particular, $$Z(n) = -z\frac{d}{dz}\{Z(1)\} = \frac{z}{(z-1)^2}$$

$$Z(n^2) = -z\frac{d}{dz}\{Z(n)\} = -z\frac{d}{dz}\left\{\frac{z}{(z-1)^2}\right\}$$

$$= \frac{z(z+1)}{(z-1)^3}$$

Similarly, $$Z(n^3) = \frac{z(z^2+4z+1)}{(z-1)^4} \text{ and so on.}$$

(ii) $$Z(t^k) = Z\{(nT)^k\}$$

$$= T^k Z(n^k)$$

$$= -T^k \cdot z\frac{d}{dz}\{Z(n^{k-1})\} \text{ from (i) above}$$

$$= -Tz\frac{d}{dz}\{Z(nT)^{k-1}\}$$

$$= -Tz\frac{d}{dz}\{Z(t^{k-1})\}, \text{ which is a recurrence formula.}$$

In particular, $Z(t) = -Tz \dfrac{d}{dz}\{Z(1)\}$

$$= -Tz \frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{Tz}{(z-1)^2}$$

Similarly, $Z(t^2) = \dfrac{T^2 z(z+1)}{(z-1)^3}$ and

$$Z(t^3) = \frac{T^3 z(z^2+4z+1)}{(z-1)^4} \text{ and so on.}$$

6. $\boldsymbol{Z\left(\frac{1}{n}\right)}$ **and** $\boldsymbol{Z\left(\frac{1}{n+1}\right)}$

(i) $$Z\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$$

$$= \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \ldots\infty$$

$$= -\log\left(1-\frac{1}{z}\right), \text{ where the ROC is } \left|\frac{1}{z}\right| < 1 \text{ or } |z| > 1$$

$$= \log\left(\frac{z}{z-1}\right)$$

(ii) $$Z\left(\frac{1}{n+1}\right) = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n}$$

$$= 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \ldots\infty$$

$$= z\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \ldots\infty\right]$$

$$= -z \log\left(1-\frac{1}{z}\right) = z\log\left(\frac{z}{z-1}\right)$$

7. $Z\left\{\boldsymbol{\frac{a^n}{n!}}\right\}$ **and** $\boldsymbol{Z\left\{\frac{1}{n!}\right\}}$

(i) $$Z\left\{\frac{a^n}{n!}\right\} = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{a}{z}\right)^n$$

$$= 1 + \frac{1}{1!}\left(\frac{a}{z}\right) + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \ldots\infty = e^{a/z}$$

(ii) Putting $a = 1$, we get $Z\left\{\dfrac{1}{n!}\right\} = e^{1/z}$.

8. $\boldsymbol{Z(r^n \cos n\theta), Z(r^n \sin n\theta), Z(\cos n\theta), Z(\sin n\theta)}$

(i) We know that $Z(a^n) = \dfrac{z}{z-a}$, if $|z| > a$

Putting $a = re^{iq}$, we have

$$Z\{(re^{iq})^n\} = \frac{z}{z-re^{i\theta}}, \text{ if } |z| > |r|$$

i.e. $Z\{r^n (\cos n\theta + i \sin n\theta)\} = \dfrac{z}{z-r(\cos\theta + i\sin\theta)}$

$$= \frac{z\{(z-r\cos\theta)+ir\sin\theta\}}{(z-r\cos\theta)^2 + r^2\sin^2\theta}$$

$$= \frac{z(z-r\cos\theta)+izr\sin\theta\}}{z^2-2zr\cos\theta)^2+r^2}$$

Equating the real parts on both sides, we get

$$Z(r^n \cos n\theta) = \frac{z(z-r\cos\theta)}{z^2-2zr\cos\theta+r^2}, \text{ if } |z| > |r| \tag{1}$$

(ii) Equating the imaginary parts on both sides, we get

$$Z(r^n \sin n\theta) = \frac{zr\sin\theta}{z^2-2zr\cos\theta+r^2}, \text{ if } |z| > r \tag{2}$$

(iii) Putting $r = 1$ in Eq. (1), we have

$$Z(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2-2zr\cos\theta+1}, \text{ if } |z| > 1 \tag{3}$$

In particular, $Z\left(\cos\dfrac{n\pi}{2}\right) = \dfrac{z^2}{z^2+1}$

(iv) Putting $r = 1$ in Eq. (2), we have

$$Z(\sin n\theta) = \frac{z\sin\theta}{z^2-2z\cos\theta+1}, \text{ if } |z| > 1 \tag{4}$$

In particular, $Z\left(\sin\dfrac{n\pi}{2}\right) = \dfrac{z}{z^2+1}$

9. $\boldsymbol{Z(\cos \omega t), Z(\sin \omega t), Z(e^{-at} \cos bt), Z(e^{-at} \sin bt)}$

(i) $Z(\cos \omega t) = Z(\cos \omega nT)$ or $Z\{\cos n(\omega T)\}$

$$= \frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1},$$

if $|z| > 1$ [on using Eq. (3) of result 8]

(ii) $$Z(\sin \omega t) = \frac{z\sin\omega T}{z^2-2z\cos\omega T+1}, \text{ if } |z| > 1$$

[on using Eq. (4) of Function 8],

$$Z(\cos \omega t) = \frac{1}{2} Z(e^{i\omega t} + e^{-i\omega t})$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{i\omega T}} + \frac{z}{z-e^{-i\omega T}}\right] \quad \text{[by steps (iii) and (iv) of result (3)]}$$

$$= \frac{z}{2}\left[\frac{2z-(e^{i\omega T}+e^{-i\omega T})}{z^2-(e^{i\omega T}+e^{-i\omega T})z+1}\right]$$

$$= \frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1}$$

$$Z(\sin \omega t) = \frac{1}{2i} Z(e^{i\omega T} - e^{-i\omega T})$$

$$= \frac{z}{2i}\left[\frac{1}{z-e^{i\omega T}} - \frac{1}{z-e^{-i\omega T}}\right]$$

$$= \frac{z}{2i}\left[\frac{e^{i\omega T}-e^{-i\omega T}}{z^2-(e^{i\omega T}+e^{-\omega T})z+1}\right]$$

$$= \frac{z\sin\omega T}{z^2-2z\cos\omega T+1}$$

(iii) $Z\{e^{-at}\cos bt\} = Z(\cos bt)_{z \to ze^{aT}}$, by corollary under Property 3

$$= \left[\frac{z(z-\cos bT)}{z^2-2z\cos bT+1}\right]_{z\to ze^{aT}}$$

$$= \frac{ze^{aT}(ze^{aT}-\cos bT)}{z^2e^{2aT}-2ze^{aT}\cos bT+1}$$

(iv) $Z\{e^{-at}\sin bt\} = Z(\sin bt)_{z \to ze^{aT}}$, by the same rule

$$= \left(\frac{z\sin bT}{z^2-2z\cos bT+1}\right)_{z\to ze^{aT}}$$

$$= \frac{ze^{aT}\sin bT}{z^2e^{2aT}-2ze^{aT}\cos bT+1}$$

Worked Examples 5(a)

Example 1

Find the bilateral Z-transforms of
(i) $a^n\,\delta(n-k)$, (ii) $-\alpha^n\,U(-n-1)$; and (iii) $-n\alpha^n\,U(-n-1)$

(i) $$\delta(n-k) = \begin{cases} 1, & \text{for } n=k \\ 0, & \text{for } n\neq k \end{cases}$$

$$Z\{\delta(n-k)\} = \sum_{n=-\infty}^{\infty} \delta(n-k)z^{-n} = z^{-k}$$

By Property 3, $Z\{a^n\ \delta(n-k)\} = \left(\frac{z}{a}\right)^{-k}$

$$U(-n-1) = \begin{cases} 1, & \text{if } -n-1 \geq 0, \text{ ie. if } n \leq -1 \\ 0, & \text{if } -n-1 < 0, \text{ i.e. if } n > -1 \end{cases}$$

$$Z\{U(-n-1)\} = \sum_{n=-\infty}^{\infty} U(-n-1)z^{-n}$$

$$= \sum_{n=-\infty}^{-1} z^{-n} \text{ or } \sum_{n=1}^{\infty} z^n$$

$$= z(1 + z + z^2 + \ldots\infty)$$

$$= \frac{z}{1-z}$$

(ii) $Z\{a^n f(n)\} = \bar{f}\ (z/a)$, by Property 3, which is true for bilateral Z-transform also.

$\therefore$ $$Z\{a^n\ U(-n-1)\} = \frac{z/\alpha}{1-z/\alpha} \text{ or } \frac{z}{\alpha-z}$$

$\therefore$ $$Z\{-a^n\ U(-n-1)\} = \frac{z}{z-\alpha}$$

(iii) $Z\{nf(n)\} = -z\frac{\mathrm{d}}{\mathrm{d}z}\{\ \bar{f}\ (z)\}$, by Property 5.

$\therefore$ $$Z\{-n\alpha^n\ U(-n-1)\} = -z\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z}{z-\alpha}\right) = \frac{\alpha z}{(z-\alpha)^2}$$

Example 2

Find the bilateral Z-transform of $f(n)$, if

(i) $f(n) = \begin{cases} a^n, & \text{for } 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$

(ii) $f(n) = b^{|n|}, b > 0$

(i) $$Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n)z^{-n}$$

$$= \sum_{n=0}^{N-1} a^n\ z^{-n}$$

$$= 1+\frac{a}{z}+\left(\frac{a}{z}\right)^2+\ldots+\left(\frac{a}{z}\right)^{N-1}$$

$$= \frac{1-\left(\frac{a}{z}\right)^N}{1-\left(\frac{a}{z}\right)}$$

(ii) $$f(n) = b^{|n|}$$

i.e. $$f(n) = \begin{cases} b^n, & \text{if } n \geq 0 \\ b^{-n}, & \text{if } n < 0 \end{cases}$$

$\therefore$ $$Z\{b^{|n|}\} = \sum_{n=-\infty}^{-1} b^{-n} z^{-n} + \sum_{n=0}^{\infty} b^n z^{-n}$$

$$= \sum_{n=1}^{\infty} (bz)^n + \sum_{n=0}^{\infty} \left(\frac{b}{z}\right)^n$$

$$= \frac{bz}{1-bz} + \frac{1}{1-\frac{b}{z}} = \frac{bz}{1-bz} + \frac{z}{z-b}$$

Example 3

(In what follows, Z-transform refers to one sided Z-transform, unless otherwise specified.) Find the Z-transforms of

(i) $a^n \cosh an$; and

(ii) $2^n \sinh 3n$

(i) $$Z\{a^n \cosh an\} = Z\left\{a^n\left(\frac{e^{an}+e^{-an}}{2}\right)\right\}$$

$$= \frac{1}{2}\ [Z\{ae^a)^n\} + Z\{(ae^{-a})^n\}]$$

$$= \frac{1}{2}\left[\frac{z}{z-ae^a} + \frac{z}{z-ae^{-a}}\right]\left[(\because z(b^n)) = \frac{z}{z-b}\right]$$

$$= \frac{z}{2}\left[\frac{2z-a(e^a+e^{-a})}{z^2-a(e^a+e^{-a})z+a^2}\right]$$

$$= \frac{z(z-a\cosh a)}{z^2-2a(\cosh a)z+a^2}$$

(ii) $$Z\{2^n \sinh 3n\} = Z\left\{2^n\left(\frac{e^{3n}-e^{-3n}}{2}\right)\right\}$$

$$= \frac{1}{2}\ [Z\{(2e^3)^n\} - Z\{(2e^{-3})^n\}]$$

$$= \frac{1}{2}\left[\frac{z}{z-2e^3} - \frac{z}{z-2e^{-3}}\right]$$

$$= \frac{z}{2}\left[\frac{2(e^3 - e^{-3})}{z^2 - 2(e^3 + e^{-3})z + 4}\right]$$

$$= \frac{2z \sinh 3}{z^2 - 4z \cosh 3 + 4}$$

Example 4

Find the Z-transforms of (i) $t^2 e^{-t}$; and (ii) $(t + T)e^{-(t + T)}$

(i) $$Z(t^2) = \frac{T^2 z(z+1)}{(z-1)^3} \quad \text{[Refer to basic transforms (5)]}$$

$$Z\{e^{-at} f(t)\} = \bar{f}\,(ze^{aT}), \text{ by corollary under Property 3.}$$

$\therefore$ $$Z\{t^2 \times e^{-t}\} = \left[\frac{T^2 z(z+1)}{(z-1)^3}\right]_{z \to ze^T}$$

$$= \frac{T^2 ze^T (ze^T + 1)}{(ze^T - 1)^3}$$

(ii) $$Z(t) = \frac{Tz}{(z-1)^2} \quad \text{[Refer to basic transform (4)]}$$

$$Z\{te^{-t}\} = \left[\frac{Tz}{(z-1)^2}\right]_{z \to ze^T} = \frac{Tze^T}{(ze^T - 1)^2}$$

Now $$Z\{f(t + T)\} = z\bar{f}\,(z) - f(0), \text{ by Property 2}$$

$\therefore$ $$Z\{(t + T)e^{-(t + T)}\} = \frac{Tz^2 e^T}{(ze^T - 1)^2} - 0 = \frac{Te^T z^2}{(ze^T - 1)^2}$$

Example 5

Find the Z-transforms of (i) $f(n) = an^2 + bn + c$; and (ii) $f(n) = \frac{1}{2}\,(n + 1)(n + 2)$.

(i) $$f(n) = an^2 + bn + c$$

$\therefore$ $$Z\{ f(n)\} = aZ(n^2) + bZ(n) + cZ(1)$$

$$= a \cdot \frac{z(z+1)}{(z-1)^3} + b\frac{z}{(z-1)^2} + c\frac{z}{z-1},$$

by Z-transforms of basic functions

$$= \frac{z}{(z-1)^3}\,[a(z + 1) + b(z - 1) + c(z - 1)^2]$$

$$= \frac{z}{(z-1)^3}\,[cz^2 + (a + b - 2c)z + (a - b + c)]$$

(ii) $$f(n) = \frac{1}{2}(n + 1)(n + 2) = \frac{1}{2}\,(n^2 + 3n + 2)$$

$\therefore$ $$Z\{ f(n)\} = \frac{1}{2}\,\{Z(n^2) + 3Z(n) + 2Z(1)\}$$

$$= \frac{1}{2}\left[\frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{z-1}\right]$$

$$\frac{z}{2(z-1)^3}\left[(z+1) + 3(z-1) + 2(z-1)^2\right]$$

$$= \frac{z^3}{(z-1)^3}$$

Example 6

Find Z-transforms of (i) $f(n) = \dfrac{1}{n(n-1)}$; and

(ii) $f(n) = \dfrac{2n+3}{(n+1)(n+2)}$

(i) $$f(n) = \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$$

$$Z\left\{\frac{1}{n-1}\right\} = \sum_{n=2}^{\infty}\left(\frac{1}{n-1}\right) z^{-n}$$

$$= \frac{1}{1}\cdot\left(\frac{1}{z}\right)^2 + \frac{1}{2}\cdot\left(\frac{1}{z}\right)^3 + \frac{1}{3}\cdot\left(\frac{1}{z}\right)^4 + \cdots$$

$$= \frac{1}{z}\left[-\log\left(1-\frac{1}{z}\right)\right] = \frac{1}{z}\log\left(\frac{z}{z-1}\right)$$

$$Z\left\{\frac{1}{n}\right\} = \log\left(\frac{z}{z-1}\right) \quad \text{[Refer to basic transform (6)]}$$

$\therefore$ $$Z\{f(n)\} = Z\left\{\frac{1}{n-1}\right\} - Z\left\{\frac{1}{n}\right\}$$

$$= \left(\frac{1}{z} - 1\right)\log\left(\frac{z}{z-1}\right) = \left(\frac{z-1}{z}\right)\log\left(\frac{z-1}{z}\right)$$

(ii) $$f(n) = \frac{2n+3}{(n+1)(n+2)} = \frac{1}{n+1} + \frac{1}{n+2}, \text{ by partial fractions.}$$

$$Z\left\{\frac{1}{n+1}\right\} = z\log\left(\frac{z}{z-1}\right) \quad \text{[Refer to basic transform (6)]}$$

$$Z\left\{\frac{1}{n+2}\right\} = \sum_{n=0}^{\infty}\frac{1}{n+2}\cdot z^{-n}$$

$$= \frac{1}{2} + \frac{1}{3}\cdot\frac{1}{z} + \frac{1}{4}\cdot\left(\frac{1}{z}\right)^2 + \cdots$$

$$= z^2\left[\frac{1}{2}\cdot\left(\frac{1}{z}\right)^2 + \frac{1}{3}\cdot\left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$= z^2\left[-\log\left(1-\frac{1}{z}\right)-\frac{1}{z}\right]$$

$$= z^2 \log\left(\frac{z}{z-1}\right) - z$$

$$\therefore \quad Z\{f(n)\} = Z\left(\frac{1}{n+1}\right) + Z\left(\frac{1}{n+2}\right)$$

$$= z(z+1)\log\left(\frac{z}{z-1}\right) - z$$

Example 7

Given that $\bar{f}(z) = \log(1 + az^{-1})$, for $|z| > |a|$, find $f(n)$ and also $Z\{nf(n)\}$.

$$\bar{f}(z) = \log(1 + az^{-1})$$

$$= az^{-1} - \frac{1}{2}(az^{-1})^2 + \frac{1}{3}(az^{-1})^3 - \cdots + (-1)^{n-1} \times \frac{1}{n}(az^{-1})^n + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}a^n}{n} z^{-n}, \text{ for } |z| > |a|$$

$$\therefore \quad f(n) = \text{coefficient of } z^{-n} \text{ in the R.H.S.}$$

$$= -\frac{(-a)^n}{n}$$

Now $$Z\{nf(n)\} = -z\frac{d}{dz}\bar{f}(z), \text{ by Property 5.}$$

$$= -z\frac{d}{dz}\log(1+az^{-1})$$

$$= -z\frac{1}{1+az^{-1}}(-az^{-2})$$

$$= \frac{az^{-1}}{1+az^{-1}}$$

Example 8

Find the Z-transforms of

(i) $\sin^2\left(\frac{n\pi}{4}\right)$

(ii) $\sin^3\left(\frac{n\pi}{6}\right)$; and

(iii) $\cos(n\pi/2 + \pi/4)$

(i) Let $$f(n) = \sin^2\left(\frac{n\pi}{4}\right) = \frac{1}{2}\left(1-\cos\frac{n\pi}{2}\right)$$

$$\therefore \qquad Z\{f(n)\} = Z(1/2) - \frac{1}{2} Z\left(\cos\frac{n\pi}{2}\right)$$

$$= \frac{z}{2(z-1)} - \frac{1}{2}\frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2 - 2z\cos\frac{\pi}{2}+1}$$

[Refer to basic transform (8)]

$$= \frac{1}{2}\left[\frac{z}{z-1} - \frac{z^2}{z^2+1}\right]$$

(ii) Let $f(n) = \sin^3 \dfrac{n\pi}{6}$

$$= \frac{3}{4}\sin\frac{n\pi}{6} - \frac{1}{4}\sin\frac{n\pi}{2}$$

$$\therefore \qquad Z\{ f(n)\} = \frac{3}{4} Z\left(\sin\frac{n\pi}{6}\right) - \frac{1}{4} Z\left(\sin\frac{n\pi}{2}\right)$$

$$= \frac{3}{4}\frac{z\sin\pi/6}{z^2 - 2z\cos\pi/6 + 1} - \frac{1}{4}\cdot\frac{z\sin\pi/2}{z^2 - 2z\cos\pi/2+1}$$

by basic transform (8)

$$= \frac{3}{8}\cdot\frac{z}{z^2 - z\sqrt{3}+1} - \frac{1}{4}\cdot\frac{z}{z^2+1}$$

(iii) Let $f(n) = \cos\,(np/2 + p/4)$

$$= \cos\frac{n\pi}{2}\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\sin\frac{\pi}{4}$$

$$\therefore \qquad Z\{ f(n)\} = \frac{1}{\sqrt{2}}\left\{Z\left(\cos\frac{n\pi}{2}\right) - Z\left(\sin\frac{n\pi}{2}\right)\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right\}$$

$$= \frac{1}{\sqrt{2}}\frac{z(z-1)}{z^2+1}$$

Example 9

Find the Z-transforms of (i) $\alpha^n \sin \alpha n$; and (ii) $n \cos n\theta$.

(i)
$$Z(\sin \alpha n) = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$$

$$\therefore \qquad Z\{a^n \sin \alpha n\} = \{Z(\sin \alpha n)\}_{z \to z/a}, \text{ by Property 3}$$

$$= \frac{\frac{z}{\alpha}\sin\alpha}{\frac{z^2}{\alpha^2} - 2\frac{z}{\alpha}\cos\alpha + 1} = \frac{\alpha z\sin\alpha}{z^2 - 2\alpha z\cos\alpha + \alpha^2}$$

(ii) $$Z(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$$

$\therefore$ $$Z(n\cos n\theta) = -z \cdot \frac{d}{dz}\{Z(\cos n\theta)\}$$

$$= -z\frac{d}{dz}\left[\frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}\right]$$

$$= -z\left[\frac{(z^2 - 2z\cos\theta + 1)(2z - \cos\theta) - (z^2 - z\cos\theta)(2z - 2\cos\theta)}{(z^2 - 2z\cos\theta + 1)^2}\right]$$

$$= \frac{z(z^2\cos\theta - 2z + \cos\theta)}{(z^2 - 2z\cos\theta + 1)^2}$$

Example 10

Find the Z-transforms of

(i) $\cos^2 t$

(ii) $\cos^3 t$; and

(iii) cosh at sin bt

(i) $$Z(\cos^2 t) = \frac{1}{2}Z(1) + \frac{1}{2}Z(\cos 2t)$$

$$= \frac{z}{2(z-1)} + \frac{1}{2}\frac{z(z - \cos 2T)}{(z^2 - 2z\cos 2T + 1)}$$

(ii) $$Z(\cos^3 t) = Z\left(\frac{3}{4}\cos t + \frac{1}{4}\cos 3t\right)$$

$$= \frac{3}{4}\frac{z(z - \cos T)}{(z^2 - 2z\cos T + 1)} + \frac{1}{4}\frac{z(z - \cos 3T)}{z^2 - 2z\cos 2T + 1}$$

(iii) $$Z(\cosh at \sin bt) = \frac{1}{2}Z\{(e^{at} + e^{-at})\sin bt\}$$

$$= \frac{1}{2}[Z(e^{at}\sin bt) + Z(e^{-at}\sin bt)]$$

$$= \frac{1}{2}\left[\frac{ze^{-aT}\sin bT}{z^2e^{-2aT} - 2ze^{-aT}\cos bT + 1} + \frac{ze^{aT}\sin bT}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}\right]$$

Example 11

(i) use initial value theorem to find $f(0)$ when

$$\bar{f}(z) = \frac{ze^{aT}(ze^{aT} - \cos bT)}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}$$

(ii) Use final value theorem to find $f(\infty)$, when

$$\bar{f}(z) = \frac{Tze^{aT}}{(ze^{aT} - 1)^2}$$

(i) By initial value theorem,

$$f(0) = \lim_{z\to\infty}\left[\bar{f}(z)\right]$$

$$= \lim_{z\to\infty}\left[\frac{e^{aT}\left(e^{aT} - \frac{1}{z}\cos bT\right)}{e^{2aT} - \frac{2}{z}e^{aT}\cos bT + \frac{1}{z^2}}\right]$$

$$= 1$$

(ii) By final value theorem,

$$f(\infty) = \lim_{z\to 1}\{(z-1)\,\bar{f}(z)\}$$

$$= \lim_{z\to 1}\left[\frac{(z-1)Tze^{aT}}{(ze^{aT}-1)^2}\right] = 0$$

Example 12

Find the Z-transform of $f(n) * g(n)$, where

(i) $f(n) = U(n)$ and $g(n) = \delta(n) + \left(\frac{1}{2}\right)^n U(n)$

(ii) $f(n) = nU(n)$ and $g(n) = 2^n\, U(-n-1)$

(i) By convolution theorem,

$$Z\{f(n)*g(n)\} = \bar{f}(z)\cdot\bar{g}(z)$$

$$\bar{f}(z) = Z\{U(n)\} = \frac{z}{z-1}$$

$$\bar{g}(z) = Z\{\delta(n)\} + Z\left\{\left(\frac{1}{2}\right)^n U(n)\right\}$$

$$= 1 + \frac{z}{z-1/2} = 1 + \frac{2z}{2z-1} \text{ or } \frac{4z-1}{2z-1}$$

$$\therefore \quad Z\{f(n)*g(n)\} = \frac{z(4z-1)}{(z-1)(2z-1)}$$

$$\bar{f}(z) = Z\{nU(n)\} = \frac{z}{(z-1)^2}$$

$$\bar{g}(z) = Z\{2^n\, U(-n-1)\} = \sum_{n=-\infty}^{\infty} 2^n\, U(-n-1) z^{-n}$$

(bilateral Z-transform taken)

$$= \sum_{n=-\infty}^{-1} 2^n z^{-n} \quad \left[\because U(-n-1) = \begin{cases}1, & \text{if } n \le -1\\ 0, & \text{if } n > -1\end{cases}\right]$$

$$= \sum_{n=1}^{\infty} 2^{-n} z^n$$

$$= (z/2) + (z/2)^2 + (z/2)^3 + \cdots \infty$$

$$= \frac{z/2}{1-z/2} = \frac{z}{2-z}$$

$$\therefore \qquad Z\{f(n)\}* g(n)\} = \frac{z^2}{(z-1)^2(2-z)}$$

Example 13

Find the Z-transform of $f(n)* g(n)$, where

(i) $f(n) = \left(\frac{1}{2}\right)^n$ and $g(n) = \cos n\pi$

(ii) $f(n) = \begin{cases} (1/3)^n, & \text{for } n \geq 0 \\ (1/2)^{-n}, & \text{for } n < 0; \text{ and } g(n) = \left(\frac{1}{2}\right)^n U(n) \end{cases}$

(i)
$$\bar{f}(z) = Z\left\{\left(\frac{1}{2}\right)^n\right\} = \frac{z}{z-1/2} = \frac{2z}{2z-1}$$

$$\bar{g}(z) = Z\{\cos n\pi\} = Z\{(-1)^n\} = \frac{z}{z+1}$$

By convolution theorem

$$Z\{f(n)* g(n)\} = \bar{f}(z)\,\bar{g}(z)$$

$$= \frac{2z}{2z-1}\cdot\frac{z}{z+1} = \frac{2z^2}{2z^2+z-1}$$

(ii)
$$\bar{f}(z) = \sum_{n=-\infty}^{-1}\left(\frac{1}{2}\right)^{-n} z^{-n} + \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n z^{-n}$$

$$= \sum_{n=1}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{3z}\right)^n$$

$$= \frac{z}{2}(1 - z/2)^{-1} + \left(1-\frac{1}{3z}\right)^{-1}$$

$$= \frac{z}{2-z}\cdot\frac{3z}{3z-1}$$

$$\bar{g}(z) = Z\{(1/2)^n U(n)\} = \frac{2z}{2z-1}$$

$$\therefore \qquad Z\{f(n)* g(n)\} = \frac{6z^3}{(2-z)(2z-1)(3z-1)}$$

Example 14

Use convolution theorem to find the sum of the first n natural numbers.

$$1 + 2 + 3 + \cdots + n = \sum_{k=0}^{n} k$$

$$= \sum_{r=0}^{n} rU(r)\, U(n-r) \quad [\because U(r) = 1, \text{ as } r \geq 0$$

and $U(n - r) = 1$, as $r \leq n$]

$$= \{nU(n)\} * U(n)$$

$\therefore$ By convolution theorem,

$$Z\left\{\sum_{k=0}^{n} k\right\} = Z\{nU(n)\}\, Z\{U(n)\}$$

$$= \frac{z}{(z-1)^2} \cdot \frac{z}{z-1} = \frac{z^2}{(z-1)^3}$$

Taking Inverse Z-transforms,

$$\sum_{k=0}^{n} k = Z^{-1}\left\{\frac{z^2}{(z-1)^3}\right\}$$

$$= Z^{-1}\left[\frac{1}{2}\left\{\frac{z(z+1) + z(z-1)}{(z-1)^3}\right\}\right]$$

$$= \frac{1}{2}\left[Z^{-1}\left\{\frac{z(z+1)}{(z-1)^3}\right\} + Z^{-1}\left\{\frac{z}{(z-1)^2}\right\}\right]$$

$$= \frac{1}{2}\,(n^2 + n)$$

$$= \frac{1}{2}\, n(n+1)$$

Example 15

Use convolution theorem to find the inverse Z-transform of

(i) $\dfrac{8z^2}{(2z-1)(4z+1)}$ and (ii) $\dfrac{z^2}{(z+a)^2}$

(i) $$Z^{-1}\left\{\frac{8z^2}{(2z-1)(4z+1)}\right\} = Z^{-1}\left\{\frac{z}{z-1/2} \cdot \frac{z}{z+1/4}\right\}$$

$$= Z^{-1}\left\{\frac{z}{z-1/2}\right\} * Z^{-1}\left\{\frac{z}{z+1/4}\right\}$$

$$= \left(\frac{1}{2}\right)^n * (-1/4)^n$$

$$= \sum_{r=0}^{n} \left(\frac{1}{2}\right)^{n-r} (-1/4)^r$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n} (1/2)^{-r} (-1/4)^r$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n} (-1/2)^r = \left(\frac{1}{2}\right)^n \left\{\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-(-1/2)}\right\}$$

$$= \frac{2}{3}\left\{\left(\frac{1}{2}\right)^n + \frac{1}{2}\cdot(-1/4)^n\right\}$$

$$= \frac{2}{3}(1/2)^n + \frac{1}{3}(-1/4)^n$$

(ii) $$Z^{-1}\left\{\frac{z^2}{(z+a)^2}\right\} = Z^{-1}\left\{\frac{z}{z+a}\cdot\frac{z}{z+a}\right\}$$

$$= Z^{-1}\left(\frac{z}{z+a}\right) * Z^{-1}\left(\frac{z}{z+a}\right)$$

$$= (-a)^n * (-a)^n$$

$$= \sum_{r=0}^{n} (-a)^r \cdot (-a)^{n-r}$$

$$= \sum_{r=0}^{n} (-a)^n = (n+1)(-a)^n$$

Exercise 5(a)

Part A (Short-Answer Questions)

1. Define unilateral and bilateral Z-transforms of $\{f(n)\}$.
2. Define one sided Z-transform of $f(t)$.
3. Prove that $Z\{f(n-m)\} = z^{-m}\,\overline{f}(z)$.
4. Prove that $Z\{f(t+T)\} = z\{\overline{f}(z) - f(0)\}$.
5. Express $Z\{a^n f(n)\}$ in terms of $Z\{f(n)\}$.
6. Prove that $Z\{f(t)\,e^{-at}\} = Z\{f(t)\}_{z \to ze^{aT}}$.
7. Derive the time-reversal property of a bilateral Z-transform.
8. How are $Z\{f(n)\}$ and $Z\{nf(n)\}$ related?
9. State initial value theorem in Z-transforms.
10. State final value theorem in Z-transforms.
11. Define convolution of two sequences with respect to unilateral and bilateral Z-transforms.
12. State convolution theorem in Z-transforms.

Find the Z-transforms of the following sequences/functions:

13. $2^n\,\delta(n-1)$
14. $a^n\,U(n) + b^n\,U(-n-1)$
15. $ab^n\ (a, b \neq 0)$
16. $a(-a)^n\,U(n)$
17. $\left(\frac{1}{2}\right)^n U(n) + \left(\frac{1}{3}\right)^n U(n)$
18. $3 \cdot 2^n + 4 \cdot (-1)^n$
19. e^{at+b}
20. $e^{2(t+T)}$
21. $(n-1)a^{n-1}$
22. nC_2
23. $\dfrac{1}{n(n+1)}$
24. $\sin \dfrac{n\pi}{3}$
25. $\cos \dfrac{n\pi}{4}$
26. $\sin(t+T)$
27. $\cos(2t+T)$
28. $\cosh t$
29. $\sinh 2t$
30. $\sinh(t+T)$

Part B

Find the Z-transforms of the following sequences/functions:

31. $2^{n-1} + \frac{1}{2} \cdot 4^n - 3^n$.
32. $\frac{1}{12} 6^n - \frac{1}{3} \cdot 3^n + \frac{1}{4} \cdot 2^n$
33. $\dfrac{1}{\sqrt{5}}\left\{\left(\dfrac{\sqrt{5}+1}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right\}$
34. $\left(\dfrac{1-3i}{4}\right)(1+i)^n + \left(\dfrac{1+3i}{4}\right)(1-i)^n$
35. $-\dfrac{1}{4}(-2)^n + \dfrac{1}{4(1-i)}(-2i)^n + \dfrac{1}{4(1+i)}(2i)^n$
36. $\frac{1}{5} 2^n - \frac{1}{5} \cdot (-3)^n - 5n(-3)^{n-1}$
37. $2(-1)^n + 2^n(n+2)$
38. $a^n \sinh an$
39. $2^n \cosh 5n$
40. $t^3 e^{-2t}$

41. $n(n-1)2^n$
42. nC_3
43. $\dfrac{1}{n(n+1)(n+2)}$
44. $\dfrac{n-2}{n(n-1)}$
45. $\dfrac{1}{n!}(a^n + a^{-n})$
46. $\cos\left(\dfrac{n\pi}{8}+\alpha\right)$
47. $\cos^2 \dfrac{n\pi}{6}$
48. $\cos^3 \dfrac{n\pi}{4}$
49. $\alpha^n \cos \alpha n$
50. $n \sin n\theta$
51. $na^n \cos n\theta$
52. $na^n \sin n\theta$
53. $r^n \cos(n\theta + \phi)$
54. $r^t \sin(\omega t + \phi)$
55. $\sin^2 2t$
56. $\sin^3 t$
57. $t \sin t$
58. $t \cos t$
59. $e^t \sin 2t$
60. $e^{-2t} \cos 3t$
61. Find the initial and final values of $f(n)$, if

$$\bar{f}(z) = \frac{0.4z^2}{(z-1)(z^2-0.736z+0.136)}$$

62. Find the Z-transform of $f(n) * g(n)$, when
 (i) $f(n) = a^n U(n)$ and $g(n) = a^n U(n)$
 (ii) $f(n) = U(n)$ and $g(n) = 2^n U(n)$
63. Find the Z-transform of $f(n) * g(n)$, if
 (i) $f(n) = a^n U(n)$ and $g(n) = b^n U(n)$
 (ii) $f(n) = \left(\dfrac{1}{4}\right)^n U(n-1)$ and $g(n) = 1 + \left(\dfrac{1}{2}\right)^n$
64. Express (i) $(n+1)$ and (ii) $\sum\limits_{k=0}^{n} n^2$ as convolution products and hence find their Z-transforms.
65. Use convolution theorem to find the inverse Z-transform of
 (i) $\dfrac{z^2}{(z+a)(z+b)}$ (ii) $\dfrac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$

5.4 INVERSE Z-TRANSFORMS

The inverse Z-transform of $\bar{f}(z)$ has been already defined as $Z^{-1}\{\bar{f}(z)\} = f(n)$, when $Z\{f(n)\} = \bar{f}(z)$.

$Z^{-1}\{\bar{f}(z)\}$ can be found out by any one of the following methods:

Method 1 (Expansion Method)

If $\bar{f}(z)$ can be expanded in a series of ascending powers of z^{-1}, i.e. in the form $\sum_{n=0}^{\infty} f(n)z^{-n}$, by binomial, exponential and logarithmic theorems, the coefficient of z^{-n} in the expansion gives $Z^{-1}\{\bar{f}(z)\}$.

Method 2 (Long Division Method)

When the usual methods of expansion of $\bar{f}(z)$ fail and if $\bar{f}(z) = \dfrac{g(z^{-1})}{h(z^{-1})}$, then $g(z^{-1})$ is divided by $h(z^{-1})$ in the classical manner and hence the expansion $\sum_{n=0}^{\infty} f(n)z^{-n}$ is obtained in the quotient.

Method 3 (Partial Fractions Method)

When $\bar{f}(z)$ is a rational function in which the denominator can be factorised, $\bar{f}(z)$ is resolved into partial fractions and then $Z^{-1}\{\bar{f}(z)\}$ is derived as the sum of the inverse Z-transforms of the partial fractions.

Method 4 (By Cauchy's Residue Theorem)

By using the relation between the Z-transform and Fourier transform of a sequence, it can be proved that

$$f(n) = \frac{1}{2\pi i}\oint_C \bar{f}(z)\, z^{n-1}\, dz,$$

where C is a circle whose centre is the origin and radius is sufficiently large to include all the isolated singularities of $\bar{f}(z)$. C may also be a closed contour including the origin and all the isolated singularities of $\bar{f}(z)$.

By Cauchy's residue theorem,

$\oint_C \bar{f}(z)\, z^{n-1} dz = 2\pi i \times$ sum of the residues of $\bar{f}(z)z^{n-1}$ at the isolated singularities.

$\therefore\ f(n) =$ Sum of the residues of $\bar{f}(z)z^{n-1}$ at the isolated singularities.

5.5 FORMATION OF DIFFERENCE EQUATIONS AND USE OF *Z*-TRANSFORM TO SOLVE THEM

Z-transforms can be used to solve finite difference equations of the form $ay(n+2) + by(n+1) + cy(n) = \phi(n)$ with given values of $y(0)$ and $y(1)$.

Taking *Z*-transforms on both sides of the given difference equation and using the values of $y(0)$ and $y(1)$, we will get $\bar{y}(z)$. Then

$$Z^{-1}\{\bar{y}(z)\} \text{ will give } y(n)$$

To express $Z\{y(n+1)\}$ and $Z\{y(n+2)\}$ in terms of $\bar{y}(z)$, the following results may be noted.

(i)
$$\begin{aligned} Z\{y(n+1)\} &= \sum_{n=0}^{\infty} y(n+1)z^{-n} = z \sum_{n=0}^{\infty} y(n+1)z^{-(n+1)} \\ &= z[y(1)z^{-1} + y(2)z^{-2} + \ldots] \\ &= z[y(0) + y(1)z^{-1} + y(2)z^{-2} + \ldots] - zy(0) \\ &= z\bar{y}(z) - zy(0) \end{aligned}$$

(ii) Similarly,

$$Z\{y(n+2)\} = z^2\bar{y}(z) - z^2 y(0) - zy(1)$$

and
$$Z\{y(n+3) = z^3\bar{y}(z) - z^3 y(0) - z^2 y(1) - zy(2) \text{ and so on.}$$

Worked Examples 5(b)

Example 1

Find the inverse *Z*-transform of $\dfrac{1+2z^{-1}}{1-z^{-1}}$, by the long division method.

$$\begin{array}{l} (1 - z^{-1})\ 1 + 2z^{-1}\ (1 + 3z^{-1} + 3z^{-2} \\ \quad\ \underline{1 - z^{-1}} \\ \quad\quad 3z^{-1} \\ \quad\quad \underline{3z^{-1} - 3z^{-2}.} \\ \quad\quad\quad 3z^{-2} \\ \quad\quad\quad \underline{3z^{-2} - 3z^{-3}} \\ \quad\quad\quad\quad 3z^{-3.} \end{array}$$

Thus $$\frac{1+2z^{-1}}{1-z^{-1}} = \sum_{n=0}^{\infty} f(n)z^{-n} = 1 + 3z^{-1} + 3z^{-2} + \ldots + 3z^{-n} + \ldots$$

$\therefore$
$$f(n) = \begin{cases} 1, & \text{for } n = 0 \\ 3, & \text{for } n \geq 1 \end{cases}$$

or
$$f(n) = 1 + 2U(n-1)$$

Example 2

Find $Z^{-1}\left\{\dfrac{4z}{(z-1)^3}\right\}$ by the long division method.

$$\bar{f}(z) = \frac{4z}{(z-1)^3} = \frac{4z^{-2}}{(1-z^{-1})^3} = \frac{4z^{-2}}{1-3z^{-1}+3z^{-2}-z^{-3}}$$

$$
\begin{array}{l}
1 - 3z^{-1} + 3z^{-2} - z^{-3}\,)\; 4z^{-2}\; (4z^{-2} + 12z^{-3} + 24z^{-4} \\
\qquad \underline{4z^{-2} - 12z^{-3} + 12z^{-4} - 4z^{-5}} \\
\qquad\qquad 12z^{-3} - 12z^{-4} + 4z^{-5} \\
\qquad\qquad \underline{12z^{-3} - 36z^{-4} + 36z^{-5} - 12z^{-6}} \\
\qquad\qquad\qquad 24z^{-4} - 32z^{-5} + 12z^{-6} \\
\qquad\qquad\qquad \underline{24z^{-4} - 72z^{-5} + 72z^{-6} - 24z^{-7}} \\
\qquad\qquad\qquad\qquad 40z^{-5} - 60z^{-6} + 24z^{-7}
\end{array}
$$

Thus $$\frac{4z}{(z-1)^3} = \sum_{n=0}^{\infty} f(n)z^{-n} = 4z^{-2} + 12z^{-3} + 24z^{-4} + \ldots$$

$$= 2[1.2z^{-2} + 2.3z^{-3} + 3.4z^{-4} + \ldots + (n-1)\cdot nz^{-n} + \ldots]$$

$$\therefore\; Z^{-1}\left\{\frac{4z}{(z-1)^3}\right\} = \begin{cases} 0, & \text{for } n = 0, 1 \\ 2(n-1)n, & \text{for } n \geq 2 \end{cases}$$

$$= 2(n-1)nU(n)$$

Example 3

Find $Z^{-1}\left\{\dfrac{z^2+z}{(z-1)^3}\right\}$ by the long division method.

$$\bar{f}(z) = \frac{z^2+z}{(z-1)^3} = \frac{z^{-1}+z^{-2}}{(1-z^{-1})^3}$$

$$
\begin{array}{l}
1 - 3z^{-1} + 3z^{-2} - z^{-3}\,)\; z^{-1} + z^{-2}\; (z^{-1} + 4z^{-2} + 9z^{-3} + 16z^{-4} \\
\qquad \underline{z^{-1} - 3z^{-2} + 3z^{-3} - z^{-4}} \\
\qquad\qquad 4z^{-2} - 3z^{-3} + z^{-4} \\
\qquad\qquad \underline{4z^{-2} - 12z^{-3} + 12z^{-4} - 4z^{-5}} \\
\qquad\qquad\qquad 9z^{-3} - 11z^{-4} + 4z^{-5} \\
\qquad\qquad\qquad \underline{9z^{-3} - 27z^{-4} + 27z^{-5} - 9z^{-6}} \\
\qquad\qquad\qquad\qquad 16z^{-4} - 23z^{-5} + 9z^{-6} \\
\qquad\qquad\qquad\qquad \underline{16z^{-4} - 48z^{-5} + 48z^{-6} - 16z^{-7}} \\
\qquad\qquad\qquad\qquad\qquad 25z^{-5} - 39z^{-6} + 16z^{-7}
\end{array}
$$

Thus $$\left\{\frac{z^2+z}{(z-1)^3}\right\} = z^{-1} + 4z^{-2} + 9z^{-3} + 16z^{-4} + \ldots\ldots \infty$$

i.e. $$\sum_{n=0}^{\infty} f(n)z^{-n} = \sum_{n=1}^{\infty} n^2 z^{-n}$$

$$\therefore \qquad f(n) = \begin{cases} 0, & \text{for } n = 0 \\ n^2, & \text{for } n \geq 1 \end{cases}$$

or $\qquad f(n) = n^2\, U(n-1)$ or $n^2\, U(n)$

Example 4

Find $Z^{-1}\left\{\dfrac{1}{1+4z^{-2}}\right\}$ by the long division method.

$$\begin{array}{l} 1 + 4z^{-2})\ 1\ (1 - 4z^{-2} + 16z^{-4} - 64z^{-6} \\ \quad\ \underline{1 + 4z^{-2}} \\ \qquad -4z^{-2} \\ \qquad \underline{-4z^{-2} - 16z^{-4}} \\ \qquad\qquad 16z^{-4} \\ \qquad\qquad \underline{16z^{-4} + 64z^{-6}} \\ \qquad\qquad\qquad -64z^{-6} \\ \qquad\qquad\qquad \underline{-64z^{-6} - 256z^{-8}} \\ \qquad\qquad\qquad\qquad \underline{256z^{-8}} \end{array}$$

Thus
$$\frac{1}{1+4z^{-2}} = 1 - 4z^{-2} + 16z^{-4} - 64z^{-6} + \ldots$$
$$= \sum_{n=0}^{\infty} 2^n \cos\frac{n\pi}{2}\, z^{-n}$$

$$\therefore \qquad f(n) = 2^n \cos\frac{n\pi}{2}$$

Example 5

Find $Z^{-1}\left\{\dfrac{3z^2-18z+26}{(z-2)(z-3)(z-4)}\right\}$ by the partial fractions method.

Let
$$\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{z-4}$$

$$\therefore \qquad 3z^2 - 18z + 26 = A(z-3)(z-4) + B(z-2)(z-4) + C(z-2)(z-3)$$
$$A = 1 = B = C$$

$$\therefore\ Z^{-1}\left\{\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}\right\} = Z^{-1}\left\{\frac{1}{z-2}\right\} + Z^{-1}\left\{\frac{1}{z-3}\right\} + Z^{-1}\left\{\frac{1}{z-4}\right\}$$

$$= 2^{n-1} + 3^{n-1} + 4^{n-1}$$
$$= \frac{1}{2}\cdot 2^n + \frac{1}{3}3^n + \frac{1}{4}4^n, \text{ where } n \geq 1$$

Example 6

Find $Z^{-1}\left\{\dfrac{4z^3}{(2z-1)^2(z-1)}\right\}$, by the method of partial fractions.

Let $$= \frac{\bar{f}(z)}{z} = \frac{4z^2}{(2z-1)^2(z-1)} = \frac{A}{(2z-1)} + \frac{B}{(2z-1)^2} + \frac{C}{z-1}$$

$\therefore$ $$4z^2 = A(2z-1)(z-1) + B(z-1) + C(2z-1)^2$$

$\therefore$ $$\frac{\bar{f}(z)}{z} = -\frac{6}{2z-1} - \frac{2}{(2z-1)^2} + \frac{4}{z-1}$$

$\therefore$ $$\bar{f}(z) = -\frac{3z}{z-1/2} - \frac{1}{2}\cdot\frac{z}{(z-1/2)^2} + \frac{4z}{z-1}$$

$\therefore$ $$Z^{-1}\{\bar{f}(z)\} = -3Z^{-1}\left\{\frac{z}{z-\frac{1}{2}}\right\} - Z^{-1}\left\{\frac{\frac{1}{2}z}{\left(z-\frac{1}{2}\right)^2}\right\} + 4Z^{-1}\left\{\frac{z}{z-1}\right\}$$

$$= -3(1/2)^n - n\left(\frac{1}{2}\right)^n + 4$$

$$\left[\because Z(a^n) = \frac{z}{z-a} \text{ and } Z(na^n) = \frac{az}{(z-a)^2}\right]$$

$$= 4 - (n+3)\left(\frac{1}{2}\right)^n$$

Example 7

Find $Z^{-1}\left\{\dfrac{4-8z^{-1}+6z^{-2}}{(1+z^{-1})(1-2z^{-1})^2}\right\}$ by the method of partial fractions.

Let $$\bar{f}(z) = \frac{4-8z^{-1}+6z^{-2}}{(1+z^{-1})(1-2z^{-1})^2} = \frac{4z^3-8z^2+6z}{(z+1)(z-2)^2}$$

Let $$\frac{\bar{f}(z)}{z} = \frac{4z^2-8z+6}{(z+1)(z-2)^2} = \frac{A}{z+1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$$

$\therefore$ $$4z^2 - 8z + 6 = A(z-2)^2 + B(z+1)(z-2) + C(z+1)$$

$$A = B = C = 2$$

$\therefore$ $$\bar{f}(z) = 2\frac{z}{z+1} + 2\cdot\frac{z}{z-2} + \frac{2z}{(z-2)^2}$$

$\therefore$ $$Z^{-1}\{\bar{f}(z)\} = 2Z^{-1}\left\{\frac{z}{z+1}\right\} + 2Z^{-1}\left\{\frac{z}{z-2}\right\} + Z^{-1}\left\{\frac{2z}{(z-2)^2}\right\}$$

$$= 2(-1)^n + 2\cdot 2^n + n\cdot 2^n$$
$$= 2(-1)^n + (n+2)\cdot 2^n$$

Example 8

Find $Z^{-1}\left\{\dfrac{z^2+2z}{z^2+2z+4}\right\}$ by the method of partial fractions.

Let
$$\frac{\bar{f}(z)}{z} = \frac{z+2}{z^2+2z+4} = \frac{z+2}{(z+1-i\sqrt{3})(z+1+i\sqrt{3})}$$
$$= \frac{A}{(z+1)-i\sqrt{3}} + \frac{B}{z+1+i\sqrt{3}}$$

$\therefore$
$$z+2 = A(z+1+i\sqrt{3}) + B(z+1-i\sqrt{3})$$
$$A = \frac{1}{2\sqrt{3}}(\sqrt{3}-i) \text{ and } B = \frac{1}{2\sqrt{3}}(\sqrt{3}+i)$$

$$\therefore \quad \bar{f}(z) = \frac{1}{2\sqrt{3}}(\sqrt{3}-i)\cdot\frac{z}{z+1-i\sqrt{3}} + \frac{1}{2\sqrt{3}}(\sqrt{3}+i)\frac{z}{z+1+i\sqrt{3}}$$

$$\therefore \quad Z^{-1}\{\bar{f}(z)\} = \frac{1}{2\sqrt{3}}[(\sqrt{3}-i)(-1+i\sqrt{3})^n + (\sqrt{3}+i)(-1-i\sqrt{3})^n]$$
$$= \frac{1}{2\sqrt{3}}\left[(\sqrt{3}-i)2^n\left(\cos\frac{2\pi}{3}n + i\sin\frac{2\pi}{3}n\right) + (\sqrt{3}+i)2^n\left(\cos\frac{2\pi}{3}n - i\sin\frac{2\pi}{3}n\right)\right]$$

[$\because$ the modulus-amplitude form of $(-1\pm i\sqrt{3})$ is $2\left(\cos\dfrac{2\pi}{3} \pm i\sin\dfrac{2\pi}{3}\right)$]

$$= \frac{1}{2\sqrt{3}}\cdot 2^n\cdot 2\left[\sqrt{3}\cos\frac{2\pi n}{3} + \sin\frac{2\pi n}{3}\right]$$
$$= 2^n\left[\cos\frac{2\pi n}{3} + \frac{1}{\sqrt{3}}\sin\frac{2\pi n}{3}\right]$$

Example 9

Find $Z^{-1}\left\{\dfrac{z^2}{(z+2)(z^2+4)}\right\}$ by the method of partial fractions.

Let
$$\frac{\bar{f}(z)}{z} = \frac{z}{(z+2)(z^2+4)} = \frac{A}{z+2} + \frac{Bz+C}{z^2+4}$$
$$= \frac{-\frac{1}{4}}{z+2} + \frac{\frac{1}{4}z+\frac{1}{2}}{z^2+4}$$

$$\therefore \quad \bar{f}(z) = -\frac{1}{4}\frac{z}{z+2} + \frac{1}{4}\cdot\frac{z^2}{z^2+4} + \frac{1}{4}\frac{2z}{z^2+4}$$

$$\therefore \qquad Z^{-1}\{\bar{f}(z)\} = -\frac{1}{4}(-2)^n + \frac{1}{4}\cdot 2^n \cos\frac{n\pi}{2} + \frac{1}{4}\cdot 2^n \sin\frac{n\pi}{2}$$

$$\left[\because Z\left(a^n \cos\frac{n\pi}{2}\right) = \frac{z^2}{z^2+a^2} \text{ and } Z\left(a^n \sin\frac{n\pi}{2}\right) = \frac{az}{z^2+a^2}\right]$$

Example 10

Find $Z^{-1}\left\{\dfrac{z^2}{(z-a)(z-b)}\right\}$, by using Residue theorem.

$Z^{-1}\left\{\dfrac{z^2}{(z-a)(z-b)}\right\} = \dfrac{1}{2\pi i}\displaystyle\oint_C \dfrac{z^{n-1}\cdot z^2}{(z-a)(z-b)}\,dz$, where C is the circle whose centre is the origin and which includes the singularities $z = a$ and $z = b$.

$$= [(\text{Res.})_{z=a} + (\text{Res.})_{z=b}] \text{ of } \frac{z^{n+1}}{(z-a)(z-b)}, \text{ by}$$

Cauchy's residue theorem

$z = a$ and $z = b$ are simple poles.

$$\therefore \qquad (\text{Res.})_{z=a} = \left(\frac{z^{n+1}}{z-b}\right)_{z=a} = \frac{a^{n+1}}{a-b} \text{ and } (\text{Res.})_{z=b} = \frac{b^{n+1}}{b-a}$$

$$\therefore \qquad Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = \frac{1}{a-b}(a^{n+1} - b^{n+1}) \text{ or } \frac{a}{a-b}\cdot a^n - \frac{b}{a-b}\cdot b^n$$

Example 11

Find $Z^{-1}\left\{\dfrac{z(z^2-z+2)}{(z+1)(z-1)^2}\right\}$, by using Residue theorem.

By residue theorem,

$Z^{-1}\left\{\dfrac{z(z^2-z+2)}{(z+1)(z-1)^2}\right\}$ = Sum of the residue of $\left[z^{n-1}\bar{f}(z) = \dfrac{z^n(z^2-z+2)}{(z+1)(z-1)^2}\right]$

at the poles. $z = -1$ is a simple pole of $z^{n-1}\bar{f}(z)$

$$\therefore \qquad R_{(z=-1)} = \frac{(-1)^n\cdot(1+1+2)}{(-1-1)^2} = (-1)^n$$

$z = 1$ is a double pole of $z^{n-1}\bar{f}(z)$.

$$\therefore \quad R_{(z=1)} = \left[\frac{d}{dz}\left\{\frac{z^n(z^2-z+2)}{z+1}\right\}\right]_{z=1}$$

$$= \left[\frac{(z+1)\{(n+2)z^{n+1} - (n+1)z^n + 2nz^{n-1}\} - (z^{n+2} - z^{n+1} + 2z^n)}{(z+1)^2}\right]_{z=1}$$

$$= \frac{1}{2}\ \{n + 2 - (n + 1) + 2n - 1\} = n$$

$$Z^{-1}\left\{\frac{z(z^2 - z + 2)}{(z+1)(z-1)^2}\right\} = (-1)^n + n$$

Example 12

Find $Z^{-1}\left\{\frac{2z^2 + 4z}{(z-2)^3}\right\}$, by using Residue theorem.

By residue theorem,

$$Z^{-1}\left\{\frac{2z^2 + 4z}{(z-2)^3}\right\} = \text{the residue of } \left\{\frac{2z^{n+1} + 4z^n}{(z-2)^3}\right\} \text{ at the only triple pole } (z = 2).$$

$$R_{z=3} = \frac{1}{2!}\left[\frac{d^2}{dz^2}\left\{\frac{2z^{n+1} + 4z^n}{(z-2)^3}(z-2)^3\right\}\right]$$

$$= \frac{1}{2}\ \{2(n + 1)nz^{n-1} + 4n(n - 1)z^{n-1}\}_{z=2}$$

$$= \frac{1}{2}\ [2(n + 1)\ n \cdot 2^{n-1} + 4n(n - 1)2^{n-2}\}$$

$$= n^2\ 2^n$$

$$\therefore \qquad Z^{-1}\left\{\frac{2z^2 + 4z}{(z-2)^3}\right\} = n^2\ 2^n$$

Example 13

Find $Z^{-1}\left\{\frac{z^2}{(z+2)(z^2+4)}\right\}$, by the method of residues.

By residue theorem, $Z^{-1}\left\{\frac{z^2}{(z+2)(z^2+4)}\right\}$ = Sum of the residues of $\frac{z^{n+1}}{(z+2)(z^2+4)}$ at the singularities.

$z = -2, \pm 2i$ are simple poles.

$$R_{(z=-2)} = \left(\frac{z^{n+1}}{z^2+4}\right)_{z=-2} = \frac{1}{8}\ (-2)^{n+1}$$

$$R_{(z=2i)} = \left\{\frac{z^{n+1}}{(z+2)(z+2i)}\right\}_{z=2i} = \left(\frac{(2i)^{n+1}}{4i(2+2i)}\right)$$

$$= \frac{2^n i^n}{4(1+i)} = \frac{2^n(1-i)i^n}{8}$$

Similarly $\qquad R_{(z=-2i)} = \frac{2^n(1+i)(-i)^n}{8}$

$$\therefore \quad Z^{-1}\left\{\frac{z^2}{(z+2)(z^2+4)}\right\} = \frac{1}{8}\,(-2)^{n+1} + \frac{2^n}{8}\,(1-i)\left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right)$$

$$+ \frac{2^n}{8}\,(1+i)\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)$$

$$= \frac{1}{8}\,(-2)^{n+1} + \frac{2^{n+1}}{8}\left(\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right)$$

$$= -\frac{1}{4}\,(-2)^n + \frac{1}{4}\cdot 2^n\left(\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right)$$

Example 14

Find $Z^{-1}\left\{\dfrac{z}{z^2+2z+2}\right\}$, by the method of residues.

By residue theorem, $Z^{-1}\left\{\dfrac{z}{z^2+2z+2}\right\}$ = sum of the residues of $\dfrac{z^n}{z^2+2z+2}$ at its singularities.

The singularities of $\dfrac{z^n}{z^2+2z+2}$ are given by $z^2 + 2z + 2 = 0$, i.e. $z = -1 + i$ and $-1 - i$ which are simple poles.

$$R_{(z=-1+i)} = \left(\frac{z^n}{z+1+i}\right)_{z=-1+i} = \frac{1}{2i}\,(-1+i)^n$$

$$R_{(z=-1-i)} = \left(\frac{z^n}{z+1-i}\right)_{z=-1-i} = -\frac{1}{2i}\,(-1-i)^n$$

$$\therefore \quad Z^{-1}\left\{\frac{z}{z^2+2z+2}\right\} = \frac{1}{2i}\,(-1+i)^n - \frac{1}{2i}\,(-1-i)^n$$

$$= \frac{(\sqrt{2})^n}{2i}\left\{\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^n - \left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)^n\right\}$$

$$= \frac{(\sqrt{2})^n}{2i}\cdot 2i\sin\frac{3n\pi}{4}$$

$$= (\sqrt{2})^n \sin\frac{3n\pi}{4}$$

Example 15

Form the difference equation by eliminating the arbitrary constants A and B from the relations (i) $y_x = A\cdot 3^x + B\cdot 4^x$ and (ii) $y_x = (Ax + B)\,(-2)^x$

(i) $$y_x = A\cdot 3^x + 1 + B\cdot 4^x \qquad (1)$$

$\therefore$ $$y_{x+1} = A\cdot 3^{x+1} + B\cdot 4^{x+1} \qquad (2)$$

and $$y_{x+2} = A\cdot 3^{x+2} + B\cdot 4^{x+2} \qquad (3)$$

Eliminating A and B from equations (1), (2) and (3), we get

$$\begin{vmatrix} y_x & 3^x & 4^x \\ y_{x+1} & 3^{x+1} & 4^{x+1} \\ y_{x+2} & 3^{x+2} & 4^{x+2} \end{vmatrix} = 0$$

i.e. $$3^x \times 4^x \begin{vmatrix} y_x & 1 & 1 \\ y_{x+1} & 3 & 4 \\ y_{x+2} & 9 & 16 \end{vmatrix} = 0 \text{ (or) } \begin{vmatrix} y_x & 1 & 0 \\ y_{x+1} & 3 & 1 \\ y_{x+2} & 9 & 7 \end{vmatrix} = 0$$

i.e. $$y_x(21-9) - (7y_{x+1} - y_{x+2}) = 0$$

i.e. $$y_{x+2} - 7y_{x+1} + 12y_x = 0$$

(ii) $$y_x = (Ax + B)(-2)^x \quad (1)$$

$\therefore$ $$y_{x+1} = (Ax + A + B)(-2)^{x+1}$$

or $$\frac{y_{x+1}}{-2} = (Ax + A + B)(-2)^x \quad (2)$$

and $$y_{x+2} = (Ax + 2A + B)(-2)^{x+2}$$

or $$\frac{y_{x+2}}{4} = (Ax + 2A + B)(-2)^x \quad (3)$$

Now, (1) + (3) – 2 × (2) gives

$$y_x + \frac{y_{x+2}}{4} - 2 \times \frac{y_{x+1}}{-2} = 0$$

i.e., $$y_{x+2} + 4y_{x+1} + 4y_x = 0$$

Example 16

From the difference equation by eliminating the arbitrary constants a and b from the relations (i) $y_n = a \cos n\theta + b \sin n$ and (ii) $y_n = an^2 + bn$

(i) $$y_n = a \cos n\theta + b \sin n\theta \quad (1)$$

$\therefore$ $$y_{n+1} = a \cos (n+1)\theta + b \sin (n+1)\theta \quad (2)$$

and $$y_{n+2} = a \cos (n+2)\theta + b \sin (n+2)\theta \quad (3)$$

$$\begin{aligned} y_{n+2} + y_n &= a[\cos (n+2)\theta + \cos n\theta] + b[\sin (n+2)\theta + \sin n\theta] \\ &= 2a \cos (n+1)\theta \cos \theta + 2b \sin (n+1)\theta \cos \theta \\ &= 2 \cos \theta \cdot y_{n+1} \end{aligned}$$

$\therefore$ The required D.E, is $y_{n+2} - 2y_{n+1} \cos \theta + y_n = 0$

(ii) $$y_n = an^2 + bn \quad (1)$$

$$\begin{aligned} \Delta y_n = y_{n+1} - y_n &= a\{(n+1)^2 - n^2\} + b \\ &= a(2n+1) + b \quad (2) \end{aligned}$$

$$\Delta^2 y_n = 2a \tag{3}$$

From (3), $$a = \frac{1}{2}\Delta^2 y_n \tag{4}$$

Using (4) in (2), $$b = \Delta y_n - \frac{(2n+1)}{2}\Delta^2 y_n \tag{5}$$

Using (4) and (5) in (1), we get

$$y_n = \frac{n^2}{2}\Delta^2 y_n + n\left\{\Delta y_n - \frac{(2n+1)}{2}\Delta^2 y_n\right\}$$

i.e. $$2y_n = \{n^2 - n(2n+1\}\Delta^2 y_n + 2n\,\Delta y_n$$

i.e. $$n(n+1)\Delta^2 y_n - 2n\,\Delta y_n + 2y_n = 0$$

Example 17

Show that n straight lines, no two of which are parallel and no three of which meet in a point, divide a plane into $\frac{1}{2}(n^2+n+2)$ parts.

Let y_n denote the number of sub-regions formed by n straight lines.

When the $(n + 1)^{\text{th}}$ lines is drawn, it will intersect each of the previous n lines at n points and hence generate $(n + 1)$ more sub-regions in addition to the previous y_n sub-regions.

∴ $$y_{n+1} = y_n + (n+1)$$

i.e. $$\Delta y_n = n + 1 = n^{[1]} + 1$$

∴ $$y_n = \Delta^{-1} n^{[1]} + \Delta^{-1}(1)$$

$$= \frac{n^{[2]}}{2} + n^{[1]} + c$$

i.e. $$y_n = \frac{1}{2}n(n-1) + n + c \tag{1}$$

Clearly, when $n = 1$, $y_n = 2$

Using this in (1), we get $2 = 1 + c$ ∴ $c = 1$

∴ $$y_n = \frac{1}{2}(n^2+n+2)$$

Example 18

Form the difference equation satisfied by the nth order determinant

$$D_n = \begin{vmatrix} 1+x^2 & x & 0 & 0 & \cdots & 0 \\ x & 1+x^2 & x & 0 & \cdots & 0 \\ 0 & x & 1+x^2 & x & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

Expanding D_n in terms of elements of the first row,

$$D_n = (1+x^2)\begin{vmatrix} 1+x^2 & x & 0 & 0 & \cdots & 0 \\ x & 1+x^2 & x & 0 & \cdots & 0 \\ 0 & x & 1+x^2 & x & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} - x\begin{vmatrix} x & x & 0 & \cdots & 0 \\ 0 & 1+x^2 & x & \cdots & 0 \\ 0 & x & 1+x^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

$$= (1+x^2)D_{n-1} - x^2\begin{vmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1+x^2 & x & \cdots & 0 \\ 0 & x & 1+x^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

$$= (1+x^2)D_{n-1} - x^2 D_{n-2}; \text{ viz., } D_n - (1+x^2)D_{n-1} + x^2 D_{n-2} = 0$$

Changing n into $(n+2)$, the required difference equation becomes

$$D_{n+2} - (1+x^2)D_{n+1} + x^2 D_n = 0$$

Example 19

If $I_n = \int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\alpha} d\theta$, where n is an integer and $0 < \alpha < \pi$, form the difference equation satisfied by I_n.

$$I_n = \int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\alpha} d\theta$$

$$I_{n+2} = \int_0^\pi \frac{\cos (n+2)\theta}{\cos\theta - \cos\alpha} d\theta$$

Now $$I_{n+2} + I_n = \int_0^\pi \frac{\cos (n+2)\theta + \cos n\theta}{\cos\theta - \cos\alpha} d\theta$$

$$= \int_0^\pi \frac{2\cos (n+1)\theta \cos n\theta}{\cos\theta - \cos\alpha} d\theta$$

$$= \int_0^\pi \frac{2\cos (n+1)\theta \{(\cos\theta - \cos\alpha) + \cos\alpha\}}{\cos\theta - \cos\alpha} d\theta$$

$$= 2\int_0^\pi \cos (n+1)\,\theta\, d\theta + 2\cos\alpha \int_0^\pi \frac{\cos (n+1)\theta}{\cos\theta - \cos\alpha} d\theta$$

$$= 2\left\{\frac{\sin (n+1)\theta}{n+1}\right\}_0^\pi + 2\cos\alpha \cdot I_{n+1}$$

$\therefore$ The required difference equation is $I_{n+2} - 2\cos\alpha \cdot I_{n+1} + I_n = 0$.

Example 20

Solve the difference equation $y(n + 3) - 3y(n + 1) + 2y(n) = 0$, given that

$$y(0) = 4,\ y(1) = 0 \text{ and } y(2) = 8$$

Taking Z-transforms on both sides of the given equation, we have $Z\{y(n + 3)\} - 3Z\{y(n + 1) + 2Z\{y(n)\} = 0$

i.e. $\{z^3\ \bar{y}(z) - z^3 y(0) - z^2 y(1) - zy(2)] - 3\{z\bar{y}(z) - zy(0)] + 2\bar{y}(z) = 0$

i.e. $(z^3 - 3z + 2)\bar{y}(z) = 4z^3 - 4z$

$$\therefore \quad \frac{\bar{y}(z)}{z} = \frac{4z^2-4}{(z-1)^2(z+2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+2}$$

$$= \frac{8/3}{z-1} + \frac{4/3}{z+2}$$

$$\therefore \quad \bar{y}(z) = \frac{8}{3}\frac{z}{z-1} + \frac{4}{3}\frac{z}{z+2}$$

Inverting, we get $y(n) = \frac{8}{3} + \frac{4}{3}(-2)^n$

Example 21

Solve the equation $f(n) + 3f(n - 1) - 4f(n - 2) = 0$, $n \geq 2$, given that $f(0) = 3$ and $f(1) = -2$.

Changing n into $(n + 2)$ in the given equation, it becomes $f(n + 2) + 3f(n + 1) -4f(n) = 0$, $n \geq 0$. Taking Z-transforms of this equation, we have

$$[z^2\ \bar{f}(z) - z^2 f(0) - zf(1)] + 3[z\ \bar{f}(z) - zf(0)] - 4\bar{f}(z) = 0$$

i.e. $(z^2 + 3z - 4)\ \bar{f}(z) = 3z^2 + 7z$

$$\therefore \quad \frac{\bar{f}(z)}{z} = \frac{3z+7}{(z+4)(z-1)} = \frac{1}{z+4} + \frac{2}{z-1}$$

$$\therefore \quad \bar{f}(z) = \frac{z}{z+4} + \frac{2z}{z-1}$$

$$\therefore \quad f(n) = (-4)^n + 2$$

Example 22

Solve the equation $y_{n+2} - 7y_{n+1} + 12y_n = 2^n$, given that $y_0 = y_1 = 0$.

Note ✍

y_n means $y(n)$.

Taking Z-transforms of the given equation,

$$[z^2\ \bar{y}(z) - z^2 y(0) - zy(1)] - 7[z\bar{y}(z) - zy(0)] + 12\ \bar{y}(z) = \frac{z}{z-2}$$

i.e. $$(z^2 - 7z + 12)\bar{y}(z) = \frac{z}{z-2}$$

$$\therefore \qquad \frac{\bar{y}(z)}{z} = \frac{1}{(z-2)(z-3)(z-4)} = \frac{1/2}{z-2} - \frac{1}{z-3} + \frac{1/2}{z-4}$$

$$\therefore \qquad \bar{y}(z) = \frac{1}{2}\frac{z}{z-2} - \frac{z}{z-3} + \frac{1}{2}\frac{z}{z-4}$$

Inverting, we get $y_n = 1/2 \cdot 2^n - 3^n + 1/2 \cdot 4^n$

Example 23

Solve the equation $x_{n+2} - 5x_{n+1} + 6x_n = 36$, given that $x_0 = x_1 = 0$. Taking Z-transforms of the given equation,

$$[z^2\, \bar{x}(z) - z^2 x(0) - zx(1)] - 5[z\, \bar{x}(z) - zx(0)] + 6\bar{x}(z) = 36\frac{z}{z-1}$$

i.e. $$(z^2 - 5z + 6)\bar{x}(z) = \frac{36z}{z-1}$$

i.e. $$\frac{\bar{x}(z)}{z} = \frac{36}{(z-1)(z-2)(z-3)} = \frac{18}{z-1} - \frac{36}{z-2} + \frac{18}{z-3}$$

$$\therefore \qquad \bar{x}(z) = 18\frac{z}{z-1} - 36\frac{z}{z-2} + 18\frac{z}{z-3}$$

Inverting, we get $$x_n = 18 - 36 \cdot (2)^n + 18 \cdot (3)^n$$

Example 24

Solve the equation $y(x+2) + 4y(x+1) + 4y(x) = x$, given that $y(0) = 0$ and $y(1) = 1$.

Note ✍

The letter 'x' is used instead of 'n'

Taking Z-transforms of the given equation,

$$[z^2\, \bar{y}(z) - z^2 y(0) - zy(1)] + 4[z\bar{y}(z) - zy(0)] + 4\,\bar{y}(z) = \frac{z}{(z-1)^2}$$

$$\left[\because Z(x) \equiv Z(n) = \frac{z}{(z-1)^2}\right]$$

i.e. $$(z^2 + 4z + 4)\,\bar{y}(z) = z + \frac{z}{(z-1)^2}$$

$$\therefore \qquad \frac{\bar{y}(z)}{z} = \frac{1}{(z+2)^2} + \frac{1}{(z-1)^2(z+2)^2}$$

$$= \frac{1}{(z+2)^2} + \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+2} + \frac{D}{(z+2)^2}$$

$$\therefore \qquad \bar{y}(z) = -\frac{2}{27}\frac{z}{z-1} + \frac{1}{9}\cdot\frac{z}{(z-1)^2} + \frac{2}{27}\cdot\frac{z}{z+2} + \frac{10}{9}\frac{z}{(z+2)^2}$$

Inverting, we get

$$y(n) = -\frac{2}{27} + \frac{1}{9}n + \frac{2}{27}(-2)^n - \frac{5}{9}n(-2)^n$$

Example 25

Solve the equation $y_{n+2} + y_n = 2^n \cdot n$

Taking Z-transforms of the given equation,

$$[z^2\,\bar{y}(z) - z^2 y(0) - zy(1)] + \bar{y}(z) = Z\{n \cdot 2^n\}$$

$$= \frac{2z}{(z-2)^2} = \left[\because Z(na^n) = \frac{az}{(z-a)^2}\right]$$

Since $y(0)$ and $y(1)$ are not given, we assume that $y(0) = A$ and $y(1) = B$.

$$\therefore \quad (z^2+1)\,\bar{y}(z) = Az^2 + Bz + \frac{2z}{(z-2)^2}$$

$$\therefore \quad \bar{y}(z) = A\frac{z^2}{z^2+1} + B\frac{z}{z^2+1} + \frac{2z}{(z^2+1)(z-2)^2}$$

Taking inverses

$$y(n) = A\cos\frac{n\pi}{2} + B\sin\frac{n\pi}{2} + Z^{-1}\left\{\frac{2z}{(z^2+1)(z-2)^2}\right\} \qquad (1)$$

$$Z^{-1}\left\{\frac{2z}{(z^2+1)(z-2)^2}\right\} = \text{Sum of the residues of } \left[\frac{2z^n}{(z^2+1)(z-2)^2}\right] \text{ at its poles.}$$

$$R_{(z=i)} = \left[\frac{2z^n}{(z+i)(z-2)^2}\right]_{z=i} = \frac{i^{n-1}}{3-4i} = \frac{(3+4i)i^{n-1}}{25}$$

$$= \frac{3}{25}\left[\cos(n-1)\frac{\pi}{2} + i\sin(n-1)\frac{\pi}{2}\right] + \frac{4}{25}\left[\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right]$$

Similarly

$$R_{(z=-i)} = \frac{3}{25}\,[\cos(n-1)\pi/2 - i\sin(n-1)\pi/2] + \frac{4}{25}\left[\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right]$$

$$R_{(z=2)} = \left[\frac{d}{dz}\left\{\frac{2z^n}{z^2+1}\right\}\right]_{z=2} \qquad (\because z = 2 \text{ is a double pole})$$

$$= \left[\frac{(z^2+1)2nz^{n-1} - 2z^n.2z}{(z^2+1)^2}\right]_{z=2}$$

$$= \frac{5.2n.2^{n-1} - 4.2^{n+1}}{25} = \frac{5n-8}{25}.2^n$$

Using these values in (1), we get

$$y(n) = A\cos\frac{n\pi}{2} + B\sin\frac{n\pi}{2} + \frac{6}{25}\cos(n-1)\frac{\pi}{2} + \frac{8}{25}\cos\frac{n\pi}{2} + \frac{(5n-8)}{25}\,2^n$$

$$= A\cos\frac{n\pi}{2} + B\sin\frac{n\pi}{2} + \frac{6}{25}\sin\frac{n\pi}{2} + \frac{8}{25}\cos\frac{n\pi}{2} + \frac{(5n-8)}{25}\,2^n$$

$$= C_1 \cos \frac{n\pi}{2} + C_2 \sin \frac{n\pi}{2} + \frac{(5n-8)}{25} 2^n$$

Example 26

Solve the simultaneous difference equations

$x_{n+1} = 7x_n + 10y_n$; $y_{n+1} = x_n + 4y_n$ given that $x_0 = 3$ and $y_0 = 2$.

Taking Z-transforms of both the equations,

$$z \cdot \bar{x}(z) - z \cdot x(0) = 7\,\bar{x}(z) + 10\,\bar{y}(z)$$

and $$z\,\bar{y}(z) - zy(0) = \bar{x}(z) + 4\,\bar{y}(z)$$

i.e. $$(z-7)\,\bar{x} - 10\,\bar{y} = 3z \quad (1)$$

and $$\bar{x} - (z-4)\,\bar{y} = 2z \quad (2)$$

Solving (1) and (2), we get

$$\frac{\bar{x}(z)}{z} = \frac{3z+8}{(z-2)(z-9)} \text{ and } \frac{\bar{y}(z)}{z} = \frac{2z-11}{(z-2)(z-9)}$$

$$\therefore \quad \bar{x}(z) = \frac{-2z}{z-2} + \frac{5z}{z-9} \text{ and } \bar{y}(z) = \frac{z}{z-2} + \frac{z}{z-9}$$

Inverting, we get $x(n) = -2^{n+1} + 5 \cdot 9^n$ and $y(n) = 2^n + 9^n$

Exercise 5(b)

Part A (Short-Answer Questions)

1. Write down the formula for finding $Z^{-1}\{\bar{f}(z)\}$ using Cauchy's residue theorem.
2. Express $Z\{f(n+2)\}$ and $Z\{f(n+3)$ in terms of $\bar{f}(z)$.
3. Find $Z^{-1}\left\{\frac{z}{z-a}\right\}$ by expansion method.
4. Find $Z^{-1}\left\{\frac{1}{z+2}\right\}$ by expansion method.
5. Find $Z^{-1}(e^{a/z})$ by expansion method.
6. Find $Z^{-1}\{e^{-2z}\}$ by expansion method.
7. Find $Z^{-1}\{\log(1 - z^{-1} - 6z^{-2})\}$ by expansion method.
8. Find $Z^{-1}\left\{\log\frac{z-a}{z+b}\right\}$ by expansion method.
9. Find $\bar{f}(z)$ from the equation $f(n+2) - 3f(n+1) + 2f(n) = 0$, if $f(0) = 0$ and $f(1) = 1$.
10. Find $\bar{f}(z)$ from the equation $f(n+1) - 2f(n) = U(n)$, if $f(0) = 0$.

Part B

Find the inverse Z-transforms of the following functions by the long division method.

11. $\dfrac{4z}{(z-1)^2}$

12. $\dfrac{z^2+z}{(z-1)^2}$

13. $\dfrac{z}{(z-1)(z-2)}$

14. $\dfrac{2z^2+4z}{(z-2)^3}$

Find the inverse Z-transforms of the following functions by the partial fractions method.

15. $\dfrac{8z^2}{8z^2-6z+1}$

16. $\dfrac{z+2}{z^2-5z+6}$

17. $\dfrac{z^2-3z}{z^2-3z+10}$

18. $\dfrac{z^2+2z}{(z-1)(z-2)(z-3)}$

19. $\dfrac{z^{-2}}{(1-z^{-1})(1-2z^{-1})(1-3z^{-1})}$

20. $\dfrac{2z^2-z}{(z-1)(z-2)^2}$

21. $\dfrac{z}{(z+1)(z-1)^2}$

22. $\dfrac{z^{-2}}{(1+z^{-1})^2(1-z^{-1})}$

23. $\dfrac{z}{z^2-2z+2}$

24. $\dfrac{z^2+z}{(z-1)(z^2+1)}$

25. $\dfrac{z^2-3}{(z+2)(z^2+1)}$

Find the inverse Z-transforms of the following functions by the method of residues.

26. $\dfrac{z}{z^2+7z+10}$

27. $\dfrac{z-4}{z^2+5z+6}$

28. $\dfrac{z^2-4z}{(z-2)^2}$

29. $\dfrac{4z^2-2z}{(z-1)(z-2)^2}$

30. $\dfrac{2z}{(z-1)(z^2+1)}$

31. $\dfrac{z^2+1}{z^2-2z+2}$

32. $\dfrac{z(z^2-1)}{(z^2+1)^2}$

33. Form the difference equation by eliminating the constants a and b from the relation $y_x = \alpha \cdot 2^x + b \cdot (-2)^x$.
34. Form the difference equation by eliminating the constants a and b from the relation $y_n = (a - bn) \cdot 3^n$
35. Form the difference equation by eliminating a and b from $y_n = an^2 - bn$.
36. Form the difference equation by eliminating A and B from $y_n = (\sqrt{2})^n \left(A \cos \dfrac{n\pi}{4} + B \sin \dfrac{n\pi}{4} \right)$.
37. Form the difference equation by eliminating a and b from $y_x = 2^x \left(a \cos \dfrac{\pi x}{3} + b \sin \dfrac{\pi x}{3} \right)$
38. n circles are drawn in a plane so that each circle intersects all others and no three meet in a point. Prove that the plane is divided into $(n^2 - n + 2)$ parts, by forming the difference equation satisfied by the number y_n of sub-regions obtained.
39. A solid is so constructed that every face is a triangle. Form the difference equation satisfied by y_n, the number of faces for n vertices.
40. Form the difference equation satisfied by D_n, where D_n is the nth order determinant given by

$$D_n = \begin{vmatrix} 2\cos\theta & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2\cos\theta & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2\cos\theta & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 2\cos\theta \end{vmatrix}$$

and hence find the value of D_n.

41. Find the difference equation satisfied by

$$I_n = \int_0^{\pi} \frac{\cos n\, x}{5 - 3\cos x}\, dx$$

and hence find I_n.

42. Find the difference equation satisfied by

$$I_n = \int_0^{\pi} \frac{\cos n\theta - \cos n\alpha}{\cos\theta - \cos\alpha}\, d\theta$$

and hence find I_n.

Solve the following difference equation, using Z-transforms.

43. $y(n+1) - y(n) = 3^n$, given that $y(0) = 0$.
44. $y(n+2) + y(n) = 0$.
45. $y(n+3) - 6y(n+2) + 12y(n+1) - 8y(n) = 0$, given that $y(0) = -1$, $y(1) = 0$ and $y(2) = 1$.
46. $y(n) + y(n-1) - 8y(n-2) - 12y(n-3) = 0$, $n \geq 3$, given that $y(0) = 1$, $y(1) = -6$, $y(2) = 12$.
47. $y(n+2) - 4y(n+1) + 4y(n) = 0$, given that $y(0) = 1$ and $y(1) = 0$.
48. $y(n+2) - y(n+1) - 2y(n) = 4$, given that $y(0) = -1$ and $y(1) = 3$.
49. $y(n+2) - 4y(n+1) + 4y(n) = \pi$, given that $y(0) = 0$ and $y(1) = 0$.
50. $y(n+2) - 3y(n+1) + 2y(n) = 2^n$, given that $y(0) = 3$ and $y(1) = 6$.
51. $y(n+2) + 6y(n+1) + 9y(n) = 2^n$, given that $y(0) = 0$ and $y(1) = 0$.
52. $y(n+2) - y(n) = 2^n$, given that $y(0) = 0$ and $y(1) = 1$.
53. $y(n+2) - 5y(n+1) + 6y(n) = 4^n$, given that $y(0) = 0$ and $y(1) = 1$.
54. $y(n+2) - 4y(n) = 2^n$, given that $y(0) = 0$ and $y(1) = 0$.
55. $y(n+2) - 5y(n+1) + 6y(n) = 5^n$, given that $y(0) = 0$ and $y(1) = 0$.
56. $y(n+2) + 2y(n+1) + y(n) = n$, given that $y(0) = 0$ and $y(1) = 0$.
57. $y(n+2) - 5y(n+1) + 6y(n) = n$, given that $y(0) = 0$ and $y(1) = 0$.
58. $y(n+2) - 5y(n+1) + 6y(n) = n(n-1)$, given that $y(0) = 0$ and $y(1) = 0$.
59. $y(n+2) - 4y(n+1) + 3y(n) = 2^n \cdot n^2$, given that $y(0) = 0$ and $y(1) = 0$.
60. $x(n+2) - y(n) = 1$ and $y(n+2) - x(n) = 1$, given that $x(0) = 0$ and $y(0) = -1$.

Answers

Exercise 5(a)

13. $\frac{2}{z}$;

14. $\frac{z}{z-a} + \frac{z}{z-b}$;

15. $\frac{az}{z-b}$;

16. $\dfrac{az}{z+a}$;

17. $z\left\{\dfrac{2}{2z-1}+\dfrac{3}{3z-1}\right\}$;

18. $z\left\{\dfrac{3}{z-2}+\dfrac{4}{z+1}\right\}$;

19. $e^{b}\cdot\dfrac{z}{z-e^{aT}}$;

20. $e^{2T}\cdot\dfrac{z}{z-e^{2T}}$;

21. $\dfrac{a}{(z-a)^2}$;

22. $\dfrac{z}{(z-1)^3}$;

23. $(1-z)\log\left(\dfrac{z}{z-1}\right)$;

24. $\dfrac{\sqrt{3}z/2}{z^2-z+1}$;

25. $\dfrac{z\left(z-1/\sqrt{2}\right)}{z^2-z\sqrt{2}+1}$;

26. $\dfrac{z^2\sin T}{z^2-2z\cos T+1}$;

27. $\dfrac{z^2(z-\cos 2T)}{z^2-2z\cdot\cos 2T+1}-1$;

28. $\dfrac{z}{2}\left(\dfrac{1}{z-e^{T}}+\dfrac{1}{z-e^{-T}}\right)$;

29. $\dfrac{z}{2}\left(\dfrac{1}{z-e^{2T}}-\dfrac{1}{z-e^{-2T}}\right)$;

30. $\dfrac{z^2}{2}\left(\dfrac{1}{z-e^{T}}-\dfrac{1}{z-e^{-T}}\right)$;

31. $\dfrac{z(z^2-6z+6)}{2(2-z)(3-z)(4-z)}$;

32. $\dfrac{z}{(z-2)(z-3)(z-6)}$;

33. $\dfrac{z}{z^2 - z - 1}$;

34. $\dfrac{z(z+2)}{2(z^2 - 2z + 2)}$;

35. $\dfrac{z^2}{(z+2)(z^2+4)}$;

36. $\dfrac{5z}{(z-2)(z+3)^2}$;

37. $\dfrac{4z^3 - 8z^2 + 6z}{(z+1)(z-2)^2}$;

38. $\dfrac{az \sinh a}{z^2 - 2az \cosh a + a^2}$;

39. $\dfrac{z(z - 2\cosh 5)}{z^2 - 2z\cosh 5 + 4}$;

40. $\dfrac{T^3 z e^{2T}\left(z^2 e^{4T} + 4ze^{2T} + 1\right)}{\left(ze^{2T} - 1\right)^4}$;

41. $\dfrac{8z}{(z-2)^3}$;

42. $\dfrac{z}{(z-1)^4}$;

43. $\dfrac{1}{2}\left[(1-z)^2 \log\left(\dfrac{z}{z-1}\right) - z\right]$;

44. $\left(2 - \dfrac{1}{z}\right) \log \dfrac{z}{z-1}$;

45. $e^{a/z} + e^{1/az}$;

46. $\dfrac{z^2 \cos\alpha - z\cos(\alpha/8 - \alpha)}{z^2 - 2z\cos\dfrac{\pi}{8} + 1}$;

47. $\dfrac{z}{2}\left[\dfrac{1}{z-1} + \dfrac{z - 1/2}{z^2 - z + 1}\right]$;

48. $\dfrac{3}{4}\left[\dfrac{z(z - 1/\sqrt{2})}{z^2 - z\sqrt{2} + 1}\right] + 1/4\left[\dfrac{z(z + 1/\sqrt{2})}{z^2 + z\sqrt{2} + 1}\right]$;

49. $\dfrac{z(z - \alpha\cos\alpha)}{z^2 - 2z\alpha\cos\alpha + \alpha^2}$;

50. $\dfrac{(z^3 - z)\sin\theta}{(z^2 - 2z\cos\theta + 1)^2}$;

51. $\dfrac{a(z^3\cos\theta - 2az^2 + a^2 z\cos\theta)}{(z^2 - 2az\cos\theta + a^2)^2}$;

52. $\dfrac{az(z^2 - a^2)\sin\theta}{(z^2 - 2az\cos\theta + a^2)^2}$;

53. $\dfrac{z^2\cos\phi - zr\cos(\theta - \phi)}{z^2 - 2zr\cos\theta + r^2}$;

54. $\dfrac{z^2\sin\phi + r^T z\sin(\omega T - \phi)}{z^2 - 2r^T z\cos\omega T + r^{2T}}$;

55. $\dfrac{z}{2(z-1)} + \dfrac{1}{2}.\dfrac{z(z - \cos 4T)}{z^2 - 2z\cos 4T + 1}$;

56. $\dfrac{3}{4}.\dfrac{z\sin T}{z^2 - 2z\cos T + 1} - \dfrac{1}{4}.\dfrac{z\sin 3T}{z^2 - 2z\cos 3T + 1}$;

57. $\dfrac{Tz(z^2 - 1)\sin T}{(z^2 - 2z\cos T + 1)^2}$;

58. $\dfrac{T(z^3\cos T - 2z^2 + z\cos T)}{(z^2 - 2z\cos T + 1)^2}$;

59. $\dfrac{ze^{-T}\sin 2T}{z^2 e^{-2T} - 2ze^{-T}\cos 2T + 1}$;

60. $\dfrac{ze^{2T}(ze^{2T} - \cos 3T)}{z^2 e^{4T} - 2ze^{2T}\cos 3T + 1}$;

61. 0; 1

62. (a) $\dfrac{z^2}{(z-a)^2}$

(b) $\dfrac{z^2}{(z-1)(z-2)}$;

63. (a) $\dfrac{z^2}{(z-a)(z-b)}$;

(b) $\dfrac{4z^2 - 3z}{(z-1)(2z-1)(4z-1)}$;

64. (i) $\dfrac{z^2}{(z-1)^2}$;

(ii) $\dfrac{z^3+z^2}{(z-1)^4}$;

65. (i) $\dfrac{(-1)^n}{b-a}$ $\{b^{n+1} - a^{n+1}$;

(ii) $\left(\dfrac{1}{2}\right)^{n-1} - \left(\dfrac{1}{4}\right)^n$.

Exercise 5(b)

3. a^n;
4. $(-2)^{n-1}$;
5. $\dfrac{a^n}{n!}$;
6. $\dfrac{(-2)^n}{n!}$;
7. $\dfrac{1}{n}$ $[(-1)^n 2^n - 3^n]$;
8. $\dfrac{1}{n}$ $[(-b)^n - a^n]$;
9. $\bar{f}(z) = \dfrac{z}{z^2-3z+2}$;
10. $\bar{f}(z) = \dfrac{z}{(z-1)(z-2)}$;
11. $4n$;
12. $(2n + 1)$;
13. $2^n - 1$;
14. $n^2\, 2^n$;
15. $2^{1-n} - 4^{-n}$;
16. $5 \cdot 3^{n-1} - 4.2^{n-1}$;
17. $\dfrac{2}{7} \cdot 5^n + \dfrac{5}{7} \cdot (-2)^n$;
18. $\dfrac{3}{2} - 4 \cdot 2^n + \dfrac{5}{2} 3^n$;
19. $\dfrac{1}{2} - 2^n + \dfrac{1}{2} 3^n$;
20. $1 - 2^n + \dfrac{3}{2}\, n \cdot 2^n$;

21. $\frac{1}{4}(-1)^n - \frac{1}{4} + \frac{n}{2}$;

22. $\frac{1}{4} - \frac{1}{4}(-1)^n + \frac{n}{2}(-1)^n$;

23. $\left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}$;

24. $1 - \cos\frac{n\pi}{2}$;

25. $-\frac{1}{10}(-2)^n + \frac{1}{10}\cos\frac{n\pi}{2} - \frac{1}{20}\sin\frac{n\pi}{2}$;

26. $\frac{1}{3}[(-2)^n - (-5)^n]$;

27. $3 \cdot (-2)^n - \frac{7}{3} \cdot (-3)^n$

28. $(1 - n) \cdot 2^n$;

29. $2 + (3n - 2) \cdot 2^n$;

30. $1 - \left(\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right)$

31. $\left(\frac{1-3i}{4}\right)(1+i)^n + \left(\frac{1+3i}{4}\right)(1-i)^n$;

32. $\frac{n}{2}[i^{n-1} + (-i)^{n-1}]$;

33. $y_{x+2} - 4y_x = 0$

34. $y_{n+2} - 6y_{n+1} + 9y_n = 0$

35. $(n^2 + n)\,\Delta^2 y_n - 2n\Delta y_n + 2y_n = 0$

36. $y_{n+2} - 2y_{n+1} + 2y_n = 0$

37. $y_{x+2} - 2y_{x+1} + 4y_x = 0$

38. $y_{n+1} = y_n + 2n$

39. $y_{n+1} = y_n + 2$

40. $D_{n+2} - 2\cos\theta\, D_{n+1} + D_n = 0;\ D_n = \frac{\sin(n+1)\theta}{\sin\theta}$

41. $I_n = \frac{\pi}{4} \cdot \left(\frac{1}{3}\right)^n$

42. $I_n = \frac{\pi \sin n\alpha}{\sin\alpha}$

43. $y(n) = \frac{1}{2}(3^n - 1)$;

44. $y(n) = A\cos\frac{n\pi}{2} + B\sin\frac{n\pi}{2}$;

45. $y(n) = 2^n \left\{-1+\frac{11}{8}n-\frac{3}{8}n^2\right\}$;

46. $y(n) = -\frac{8}{25}.3^n + \frac{33}{25}.(-2)^n + \frac{6}{5}n(-2)^n$;

47. $y(n) = 2^n (1 - n)$;

48. $y(n) = -2 - (-1)^n + 2 \cdot 2^n$;

49. $y(n) = \pi [1 + (n - 2) \cdot 2^{n-1}]$;

50. $y(n) = 1 + 2^{n+1} + n \cdot 2^{n-1}$;

51. $y(n) = \frac{1}{25}.2^n - \frac{1}{25}.(-3)^n + \frac{1}{5}n \cdot (-3)^n$;

52. $y(n) = \frac{1}{3} [2^n - (-1)^n]$

53. $y(n) = \frac{1}{2} (4^n - 3^n)$;

54. $y(n) = \frac{1}{16} \{(-2)^n - 2^n + n \cdot 2^{n+1}\}$

55. $y(n) = \frac{1}{3}.2^n - \frac{1}{2}.3^n$;

56. $y(n) = \frac{1}{4} (n - 1) \{1 + (-1)^{n-1}\}$;

57. $y(n) = \frac{1}{4} \cdot 3^n - 2^n + \left(\frac{1}{2}\right)^n + \frac{3}{4}$

58. $y(n) = \frac{1}{2} . 3^n - 2^{n+1} + 3/2 + n$;

59. $y(n) = 3 + 5 \cdot 3^n - 2^n (n^2 + 8)$;

60. $x(n) = \frac{1}{2}\left(1-\cos\frac{n\pi}{2}-\sin\frac{n\pi}{2}\right)$ and $y(n) = \frac{1}{2}\left(\sin\frac{n\pi}{2}-\cos\frac{n\pi}{2}-1\right)$

Appendix A

E 0016
B.E./B.Tech. Degree Examinations, November/December 2009
Third Semester
MA 1201
Transforms and Partial Differential Equations
(Common to all branches)
(Regulations 2008)

Time: Three hours | Maximum: 100 marks

Answer ALL questions.
Part A – (10 × 2 = 20 marks)

1. Find the sum of the Fourier series of $f(x) = x + x^2$ in $-\pi < x < \pi$ at $x = \pi$.
2. What is known as harmonic analysis?
3. State inverse theorem for complex Fourier transform.
4. If $f(x) = e^{-\alpha x}$, $\alpha > 0$, find Fourier sine transform of $f(x)$.
5. Form the p.d.e. by eliminating f from $z = f(y/x)$.
6. Find the complete integral of $pq = xy$.
7. Write any two solutions of one dimensional wave equation.
8. Express the two solutions of the Laplace equation $u_{xx} + u_{yy} = 0$ involving exponential terms in x or y.
9. Find the z-transform of $\{n\}$.
10. State convolution theorem on z-transform.

Part B—(5 × 16 = 80 marks)

11. (a) Find the fourier series for $f(x) = \begin{bmatrix} -K, -\pi < x < 0 \\ K, 0 < x < \pi \end{bmatrix}$

Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$. (8)

(b) Find a cosine series for $f(x) = x$ in $0 < x < 1$. (8)

Or

12. (a) Find a fourier series for $f(x) = 2x - x^2$ in $0 < x < 2$. (8)

(b) Derive the complex form of the fourier series for $f(x) = e^{\alpha x}, -\pi < x < \pi$.

(8)

13. (a) Find the fourier transform of e^{-x^2} (8)

(b) Find the Fourier cosine transform of

$$f(x)\begin{bmatrix} 1-x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{bmatrix}$$

Hence find $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx.$ (8)

Or

14. (a) State and prove convolution theorem on fourier transform.

(b) Find the Fourier sine transform of e^{-2x}, $x > 0$, Hence evaluate

$$\int_0^\infty \frac{x^2}{(x^2+4)^2}\,dx.$$ (8)

15. (a) Find the partial differential equation of all planes which are at a constant distance k from the origin. (6)

(b) Solve: $(D^2 + 3DD' + 2D'^2)\,z = \sin(2x + y) + x + y.$ (10)

Or

16. (a) Solve: $x(z^2 - y^2)p + y(x^2 - z^2)q = z\,(y^2 - x^2).$ (8)

(b) Solve: $p\,(1 + q^2) = q\,(z - a).$ (8)

17. If a string of length l is initially at rest in its equilibrium position and each of its points is given a velocity v such that

$$v = \begin{cases} kx & \text{for} \quad 0 < x < 1/2 \\ k(l-x) & \text{for} \quad 1/2 < x < 1 \end{cases}$$

Find the displacement function $y(x, t)$. (16)

Or

18. An infinitely long metal plate in the form of a^n area is enclosed between the lines $y = 0$ and $y = \pi$ for $x > 0$. The temperature is zero along the edges $y = 0$ and $y = \pi$ and at infinity. If the edge $x = 0$ is kept at a constant temperature T°C, find the steady state temperature at any point of the plate. (16)

19. (a) Find the z-transform of (i) $2n \cos \frac{n\pi}{2}$ and (ii) $\frac{a^n}{n!}$ (8)

(b) Find the inverse z-transform of $\frac{z(z+1)}{(z-1)^3}$, using the method of residues. (8)

Or

20. (a) Find the z-transform of $n(n-1)\,u(n)$. (8)

(b) Solve using z-transform, $y_{n+2} - 3y_{n+1} - 10y_n = 0$ given $y_0 = 1$ and $y_1 = 0$. (8)

SOLUTIONS

Part–A

1. $x = \pi$ (the right end of the given interval) is a point of discontinuity of $f(x) = x + x^2$.

$\therefore$ [Sum of the F.S. of $f(x)]_{x=\pi} = \dfrac{1}{2} \lim\limits_{h \to 0} [f(-\pi + h) + f(\pi - h)]$

$$= \frac{1}{2} \lim_{h \to 0} \left[\left\{ (-\pi + h) + (-\pi + h)^2 \right\} + \left\{ (\pi - h) + (\pi - h)^2 \right\} \right]$$

$$= \frac{1}{2} \left[\left\{ (-\pi + \pi^2) + (\pi + \pi^2) \right\} \right] = \pi^2.$$

2. The process of finding the constant term and the required harmonics, namely, $A_n \cos \left(\dfrac{n\pi x}{l} - \alpha_n \right)$ or $A_n \sin \left(\dfrac{n\pi x}{l} + \beta_n \right)$; $n = 1, 2, 3, \ldots$ in the Fourier expansion of a function $f(x)$ in $(c, c + 2l)$ is known as harmonic analysis.

3. When $\bar{f}(s) = F\{f(x)\} = \displaystyle\int_{-\infty}^{\infty} f(x) e^{-isx}\, dx$ is the complex Fourier transform of $f(x)$, then the inverse complex Fourier transform of $\bar{f}(s)$ is given by

$$F^{-1}\{\bar{f}(s)\} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(s)\, e^{ixs}\, ds.$$

4. $$F_s\{e^{-ax}\} = \int_0^{\infty} e^{-ax} \sin sx\, dx = \left[\frac{e^{-ax}}{s^2 + a^2} (-a \sin sx - s \cos sx) \right]_0^{\infty}$$

$$= \frac{s}{s^2 + a^2} \quad (\because a > 0)$$

5. $z = f(y/x)$; $p = f' \cdot (-y/x^2)$; $q = f' \cdot \left(\dfrac{1}{x} \right)$

$\therefore \dfrac{p}{q} = -y/x$; Required P.D.E. is $px + qy = 0$.

6. The given equation is $\frac{p}{x} = \frac{y}{q}$ = a, say $\therefore$ $p = ax$ and $q = \frac{1}{a}\ y$.

Now $dz = p\ dx + q\ dy = ax\ dx + \frac{1}{a}\ y\ dy$

$\therefore$ Complete solution is $z = \frac{a}{2}x^2 + \frac{1}{2a}y^2 + b$

7. Two solutions of one-dimensional wave equation, viz., $\frac{\partial y}{\partial t^2} = a^2\frac{\partial^2 y}{dx^2}$ are (i) $y(x, t) = (Ae^{px} + Be^{-px})(Ce^{\text{pat}} + De^{-\text{pat}})$ and (ii) $y(x, t) = (A\cos px + B\sin px)(C\text{ Cos pat} + D\sin\text{pat})$.

8. Two solutions of Laplace equation $\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 u}{dy^2} = 0$ are

(i) $u(x, y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py)$ and

(ii) $u(x, y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py})$.

9. $Z(n) = \sum_{n=0}^{\infty} nz^{-n} = \frac{1}{z}\{1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \cdots\infty$

$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2} = \frac{z}{(z-1)^2},\ if\ |z| > 1$$

10. If $Z\{f(n)\} = \bar{f}(z)$ and $Z\{g(n)\} = \bar{g}(z)$ are one-sided z-transforms of $\{f(n)\}$ and $\{g(n)\}$, then

$Z\{f(n) * g(n)\} = \bar{f}(z)\cdot\bar{g}(z)$, where the convolution

$$\{f(n) * g(n)\} = \sum_{r=0}^{n} f(r)\cdot g(n-r)$$

Part–B

11. (a) $f(x) = \begin{cases} -k, in -\pi < x < 0 \text{ (is an odd function of } x \text{ in)} \\ k, \text{ in } 0 < x < \pi\ (-\pi < x < \pi) \end{cases}$

$\therefore$ Fourier series of $f(x) \equiv \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} k \sin nx\, dx = -\frac{2k}{\pi n}(\cos nx)_0^{\pi}$$

$$= -\frac{2k}{\pi n}\{(-1)^n - 1\} = \begin{cases} \frac{4k}{\pi n}, \text{ when } n \text{ is odd} \\ 0, \text{ when } n \text{ is even} \end{cases}$$

$$\therefore f(x) = \frac{4k}{\pi}\sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin nxin(-\pi, \pi) \quad (1)$$

Putting x = $\frac{\pi}{2}$, which is a point of continuity of $f(x)$ in (1), we get

$$\frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots \infty\right) = k$$

$$\therefore 1 - \frac{1}{3} + \frac{1}{5} \cdots \infty = \frac{\pi}{4}$$

(b) [This is the worked example (16) in page 2–65 of the book]

12. (a) Let the F.S. of $f(x) = 2x - x^2$ be $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$ in $(0, 2)$

$$a_n = \frac{1}{1}\int_0^2 (2x - x^2)\cos n\pi x\, d\, x$$

$$= \left[(2x - x^2)\frac{\sin n\pi x}{n\pi} - (2 - 2x)\left(\frac{-\cos n\pi x}{n^2 x^2}\right) + (-2)\left(\frac{-\sin n\pi x}{n^3 \pi^3}\right)\right]_0^2$$

$$= \frac{1}{n^2\pi^2}\{-2 - 2\} = -\frac{4}{n^2\pi^2}$$

$$a_0 = \frac{1}{1}\int_0^2 (2x - x^2)\, dx = \left(x^2 - \frac{x^3}{2}\right)_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

$$b_n = \frac{1}{1}\int_0^2 (2x - x^2)\sin n\pi\, x\, d\, x$$

$$= \left[(2x - x^2)\left(\frac{-\cos n\pi x}{n\pi}\right) - (2 - 2x)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= -\frac{2}{n^3\pi^3}\{1 - 1\} = 0$$

$$\therefore \text{F.S. of } f(x) \sim \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x \text{ in } (0, 2)$$

(b) Let the complex form of the F.S. of $f(x) = e^{ax}$ in $(-\pi, \pi)$ be

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\, e^{-inx}\, dx = 2\,\frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{a-in}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{a-in}\left\{e^{(a-in)\pi} - e^{-(a-in)\pi}\right\}$$

$$= \frac{1}{2\pi(a-in)}\left\{e^{a\pi}(-1)^n - e^{-a\pi}(-1)^n\right\}'$$

$$= \frac{(-1)^n}{\pi}\,\frac{\sinh a\pi}{a-in} = \frac{\sinh a\pi}{\pi}\,\frac{(-1)^n(a+in)}{a^2+n^2}$$

$$\therefore \text{F.S.of } e^{ax} \sim \frac{\sinh a\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(a+in)}{a^2+n^2}\, e^{inx}\ in\ (-\pi, \pi)$$

13. (a) [This is the same as the worked example (8) in page 4–14 of the book, if we put $a = 1$. $F\left\{e^{-x^2}\right\} = \sqrt{\pi}\, e^{-s^2/4}$]

(b) $F_c\{f(x)\} = \int_0^{\infty} f(x)\cos sx\, dx$

$$\therefore F_c\{1-x\} = \int_0^{1}(1-x)\cos sx\, dx$$

$$= \left[(1-x)\frac{\sin sx}{s} - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1$$

$$= \frac{1-\cos s}{s^2} = \frac{2\sin^2\frac{s}{2}}{s^2}$$

i.e., $\dfrac{2}{\pi}\cdot\dfrac{1}{2}\displaystyle\int_0^{\infty}\frac{\sin^2\left(\frac{s}{2}\right)}{\left(\frac{s}{2}\right)^2}\cos xs\, ds = 1-x \quad (1)$

Putting x $= 0$ in (1) we get $\displaystyle\int_0^{\infty}\frac{\sin^2\left(\frac{s}{2}\right)}{\left(\frac{s}{2}\right)^2}\, ds = \pi$ (2)

Putting $\frac{s}{2} = x$ and $ds = 2dx$ in (2), we get

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$$

14. (a) [This is a standard theorem, the statement and proof of which are availale in page 4–31 of the book]

(b) This is the worked example 2.(ii) in page 4.38 of the book. We have to take $a = 2$.

$$\left.\int_0^\infty \frac{x^2}{(x^2+4)^2}\, dx = \frac{\pi}{8}\right]$$

15. (a) [This is the worked example (5) in page 4–7 of the book]

(b) $(D^2 + 3DD' + 2D'^2)\, z = \sin(2x + y) + x + y$

A.E. is $m^2 + 3m + 2 = 0$ or $(m + 1)(m + 2) = 0$; $m = -1, -2$

$\therefore$ C.F. $= f_1(y - x) + f_2(y - 2x)$

$$\text{P.I.}_1, = \frac{1}{D^2 + 3DD' + 2D'^2} \sin(2x+y) = \frac{1}{-4+3(-2)+2(-1)} \sin(2x + y)$$

$$= -\frac{1}{12} \sin(2x + y)$$

$$\text{P.I.}_2 = \frac{1}{D^2\left\{1+\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)\right\}}(x + y) = \frac{1}{D^2}\left(1+\frac{3D'}{D}\right)^{-1}(x + y)$$

$$= \frac{1}{D^2}\left(1-\frac{3D'}{D}\right)(x + y) = \frac{1}{D^2}\left\{x + y - \frac{3.1}{D}\right\}$$

$$= \frac{x^3}{6} + \frac{x^2 y}{2} - \frac{x^3}{2} = \frac{1}{2}x^2 y - \frac{1}{3}x^3$$

$\therefore$ G.S. is $z = f_1(y - x) + f_2(y - 2x) - \frac{1}{12}\sin(2x + y) + \frac{1}{2}x^2 y - \frac{1}{3}x^3$

16. (a) $x(z^2 - y^2)\, p + y\,(x^2 - z^2) q = z\,(y^2 - x^2)$

L.S.S.E.'s are $\dfrac{dx}{x(z^2 - y^2)} = \dfrac{dy}{y(x^2 - z^2)} = \dfrac{dz}{z(y^2 - x^2)}$ (1)

Each ratio in (1) $= \dfrac{x\,dx + y\,dy + z\,dz}{0}$

$\therefore\ x\,dx + y\,dy + z\,dz = 0$

Solving, we get $x^2 + y^2 + z^2 = a$ (2)

Also each ratio in (1) $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$\therefore\ \dfrac{1}{x}dx + \dfrac{1}{y}dy + \dfrac{1}{z}dz = 0$

Solving, we get $xyz = b$ (3)

$\therefore$ G.S. of the given *PDE* is $f(x^2 + y^2 + z^2, xyz) = 0$

(b) $p(1 + q^2) = q(z - a)$. This is a I order PDE independent of x and y explicitly

Let $z = z(u) = z\,(x + by)$ be a solution (1) of the given equation, where b is an arbitrary constant

From (1), $p = \dfrac{dz}{du}$ and $q = b\dfrac{dz}{du}$

Since (1) is a solution of the given PDE, we get

$$\frac{dz}{du}\left\{1 + b^2\left(\frac{dz}{du}\right)^2\right\} = b\,\frac{dz}{du}\,(z - a)$$

i.e., $b^2\left(\dfrac{dz}{du}\right)^2 = bz - (ab + 1)$

$\therefore\ b\,\dfrac{dz}{du} = \sqrt{bz - (ab + 1)}$

i.e., $\displaystyle\int \frac{b\,dz}{\sqrt{bz - (ab + 1)}} = \int du + c$

i.e., $2\sqrt{bz - (ab + 1)} = x + by + c$

or $4\{bz - (ab + 1)\} = (x + by + c)^2$ (2) is the C.S., where b and c are arbitrary constants

To find the G.S. put $c = f(b)$

$\therefore\ 4\ \{bz - (ab + 1)\} = \{x + by + f(b)\}^2$ (4)

Differentiating (3) partially w.r.t. 'b', we get

$4\,(z - a) = 2\{x + by + f(b)\}\ \{y + f'(b)\}$ (5)

The eliminant of 'b' between (4) and (5) is the G.S.

17. [***Note***: There is an error in the problem. Since the length of the string is l, the initial velocity function must have been given as

$$v = \begin{cases} kx, \text{ for } 0 < x < \dfrac{l}{2} \\ k(l-x), \text{ for } \dfrac{l}{2} < x < l \end{cases}$$

Making these correction,the solution of the problem is given below].

The boundary conditions for the given problem are

(5) $y(0, t) = 0, t \geq 0$; (2) $y(l, t) = 0, t \geq 0$; (3) $y(x, 0) = 0, 0 < x < l$; (4) $\geq \dfrac{\partial y}{\partial x}$

$$(x, 0) = v = \begin{cases} kx, \left(0, \dfrac{l}{2}\right) \\ k(l-x), \text{ in } \left(\dfrac{l}{2}, l\right) \end{cases}$$

The proper solution of the vibration of string eqn. is $y(x, t) = (A \cos px + B \sin px)$ $(c \cos pat + D \sin pat)$ (6)

Using (1) in (6), $A = 0$

Using (2) in (6), $p = \dfrac{n\pi}{l}$;n = 0, 1, 2, 3, . . .

Using (3) in (6), $C = 0$

Using there values in (6), the proper solution becomes

$$y(x,t) = \lambda \sin \frac{n\pi x}{l} \cdot \sin \frac{n\pi at}{l} \quad (n = 0, 1, 2, \cdots \infty)$$

The most general form of the required solution becomes

$$y(x, t) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} \cdot \sin \frac{n\pi at}{l} \tag{7}$$

Differentiating (7) partially w.r.t. 't', we get

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} \left(\frac{n\pi a}{l} \lambda_n\right) \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi at}{l} \tag{7$'$}$$

Using (5) in (7)′ we have

$$\sum_{n=1}^{\infty} \left(\frac{n\pi a}{l} \lambda_n\right) \sin \frac{n\pi x}{l} = v \; in(0, l) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

(Fourier half-range sine series)

$$\therefore \frac{n\pi a}{l} \lambda_n = b_n = \frac{2}{l} \int_0^l v \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2k}{l} \left[\int_0^{l/2} x \sin \frac{n\pi x}{l} \, dx + \int_{l/2}^{l} (l - x) \sin \frac{n\pi x}{l} \, dx \right]$$

$$= \frac{2k}{l}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right) - 1\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)\right\}_0^{\frac{l}{2}} + \right.$$

$$\left.\left\{(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)\right\}_{\frac{l}{2}}^{l}\right]$$

$$= \frac{2k}{l}\left[\left\{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right.$$

$$\left. + \left\{\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right] = \frac{4kl}{n^2\pi^2}\sin\frac{n\pi}{2} \quad \therefore\ \lambda_n = \frac{4kl^2}{n^3\pi^3 a}\sin\frac{n\pi}{2}$$

Substituting this value of λ_n in (6), the required solution is

$$y(x,t) = \frac{4kl^2}{\pi^3 a}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

18. [This problem is more or less the same as the worked example (2) in page 3–121 of the book. The required solution may be got by incorporating the changes indicated below]

Step (5) must be taken as "$u(0, y) = T$, for $0 \le y \le \pi$ (5)

Except this change, all the steps are the same upto step (7) [We continue after step (7)].

Using boundary condition (5) in (7), we have

$$\sum_{n=1}^{\infty}\lambda_n \sin ny = T\ in(0,\pi) = \sum_{n=1}^{\infty} b_n \sin ny,$$

which is the Fourier half-range sine series of T in $(0, \pi)$.

Comparing like terms in the two series, we get

$$\lambda_n = b_n = \frac{2}{\pi}\int_0^{\pi} T\sin ny\, dy = \frac{2T}{\pi}\left[\frac{-\cos ny}{n}\right]_0^{\pi}$$

$$= -\frac{2T}{n\pi}\left\{(-1)^n - 1\right\} = \begin{cases} \dfrac{4T}{n\pi}, \text{ if } n \text{ is odd} \\ 0, \text{if } n \text{ is even} \end{cases}$$

Using this value of λ_n in (7), the required solution is

$$u(x, y) = \frac{4T}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} e^{-nx} \sin ny$$

19. (a) (i) We know that $Z(a^n) = \dfrac{z}{z-a}$ [got by using the definition]

$$\therefore \text{Z}(e^{in\theta}) = \text{Z}\{(e^{i\theta})^n\} = \frac{Z}{z - e^{i\theta}}$$

$$= \frac{z}{(z-\cos\theta) - i\sin\theta} = \frac{z\left\{(z-\cos\theta) + i\sin\theta\right\}}{(z-\cos\theta)^2 + \sin^2\theta}$$

$$= = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} + i\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating the real parts,

$$Z(\cos n\theta) = = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$$

$$\therefore Z\left(\cos\frac{n\pi}{2}\right) = \frac{z^2}{z^2+1}$$

Since $Z\{a^n f(n)\} = \bar{f}\left(\dfrac{z}{a}\right)$, we have

$$Z\left\{2^n \cos\frac{n\pi}{2}\right\} = \frac{\left(\dfrac{z}{2}\right)^2}{\left(\dfrac{z}{2}\right)^2 + 1} = \frac{z^2}{z^2+4}$$

(ii) $Z\left\{\dfrac{a^n}{n!}\right\} = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{a}{z}\right)^n = e^{a/z}$

(b) By Cauchy's residue method,

$f(n) = \dfrac{1}{2\pi i}\displaystyle\int_c \bar{f}(z)\cdot z^{n-1}\,dz$, where c is $|z| = R$, where

R is sufficiently large

i.e., $f(n) = \text{Residue of } \dfrac{z^n (z+1)}{(z-1)^3}$ at the triple pole $z = 1$

$$(\text{Res.})_{z=1} = \lim_{z \to 1} \frac{1}{\lfloor 2} \cdot \frac{d^2}{dz^2} \left\{ \frac{(z-1)^3 \, z^n \, (z+1)}{(z-1)^3} \right\} \Bigg]$$

$$= \frac{1}{2} \lim_{z \to 1} \left[(n+1) n \, z^{n-1} + n(n-1) \, z^{n-2} \right] = \frac{1}{2} \times 2n^2 = n^2$$

$\therefore \; f(n) = n^2.$

20. (a) $Z\{n\ (n-1)\ u(n) = Z\ \{n(n-1)$, which is one-sided Z-transform, as

$$u(n) = \begin{cases} 1, \text{if } n \geq 0 \\ 0, \text{if n} < 0 \end{cases}$$

$$\text{Now } z(n) = -z \cdot \frac{d}{dz} \tag{1}$$

$$= -z \cdot \frac{d}{dz} \left(\frac{z}{z-1} \right) = \frac{z}{(z-1)^2}$$

$$z(n^2) = -z(n \cdot n) \quad = -z \cdot \frac{d}{dz} \left\{ \frac{z}{(z-1)^2} \right\}$$

$$= -z \left[\frac{(z-1)^2 - 2z(z-1)}{(z-1)^4} \right] = \frac{z(z+1)}{(z-1)^3}$$

$$\therefore \; Z\left\{ n(n-1) \right\} = \frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2} = \frac{2z}{(z-1)^3}$$

(b) $y_{n+2} - 3y_{n+1} - 10y_n = 0$ (1)

Taking z-transforms on both sides of (1), we have

$$\left[z^2 \bar{y}(z) - z^2 y(0) - zy(1) \right] - 3\left[z\,\bar{y}(z) - zy(0) \right] - 10\,\bar{y}(z) = 0$$

i.e., $(z^2 - 3z - 10)\ \bar{y}(z) = z^2 - 3z$

$$\therefore \; \bar{y}\,(z) = \frac{z^2 - 3z}{z^2 - 3z - 10}$$

$$\therefore \quad \frac{\bar{y}(z)}{z} = \frac{z-3}{(z-5)(z+2)} = \frac{2/7}{z-5} + \frac{5/7}{z+2}$$

(by splitting into partial fractions)

$$\therefore \quad \bar{y}(z) = \frac{2}{7} \cdot \frac{z}{z-5} + \frac{5}{7} \cdot \frac{z}{z+2}$$

Inverting, we get

$$y(n) = \frac{2}{7} \times 5^n + \frac{5}{7} \times (-2)^n$$

Appendix B

Anna University Coimbatore
B.E./B.Tech. Degree Examinations, December 2009
Regulations: 2008
Third Semester
080100008 - Transforms and Partial Differential Equations
(Common Aeronautical/Automobile/Biomedical/Civil/CSE/IT/EEE/EIE/ECE/ICE/Mechanical/Biotech/Chemical/Fashion Tech./Textile Tech./Textile Chemistry)

Time: Three hours | Maximum: 100 marks

PART-A

Answer ALL questions.

($20 \times 2 = 20$ marks)

1. State Dirichlet's conditions for Fourier series.
2. Determine the value of b_n in the Fourier series expansion of xsinx in $(-\pi, \pi)$.
3. Find the root mean square value of $f(x) = x^2$ in the interval $(0, \pi)$.
4. Define Harmonic analysis.
5. State the fourier integral theorem.
6. Write down the fourier cosine transform pair formulae.
7. Find the fourier sine transform of e^{-x}
8. Prove that $F[f(ax)] = \dfrac{1}{a} F\left(\dfrac{s}{a}\right), a > 0$
9. Form the partial differential equation by eliminating the arbitrary constants a and b from $(x - a)^2 + (y - b^2) + z^2 = 1$.
10. Form the partial differential equation by eliminating the arbitrary function from $\phi\left[z^2 - xy, \dfrac{x}{z}\right] = 0$
11. Find the singular integral of the partial differential equation $z = px + qy + p^2 - q^2$
12. Find the general solution of $(D^2 - 5DD' + 6D^2)\, z = 0$
13. Classify the partial differential equation $y^2\, u_{xx} - 2xy\, u_{xy} + x^2\, u_{yy} + 2_{ux} - 3_{uy} = 0$
14. What are the possible solutions of one dimensional wave equation.

15. State Fourier law of heat condution.
16. In steady state conditions derive the solution of one dimensional heat flow equation.
17. Find $Z[na^n]$
18. Find the Z-transform of $(n+1)(n+2)$
19. State and prove initial value theorem in Z-transform.
20. Define convolution of two sequences in Z-transform.

Part B

(5 × 12 = 60 marks)

Answer any five questions

21. (a) Obtain the Fourier series expansion of $f(x)$ if, $f(x) =$

$$\left.\begin{matrix} 1+\dfrac{2x}{\pi}, -\pi \le x \le 0 \\ 1-\dfrac{2x}{\pi}, 0 \le x \le \pi \end{matrix}\right] \quad (8)$$

(b) Obtain the sine series for the function $f(x) = \begin{bmatrix} x & \text{in } 0 \le x \le \dfrac{\lambda}{2} \\ \lambda - x \text{ in } & \dfrac{\lambda}{2} \le x \le \pi \end{bmatrix}$ (4)

22. (a) Find the fourier series of $f(x) = x^2$ in $-x < x < \pi$. Hence show that (6)

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots=\frac{\pi^4}{90}$$

(b) Find the fourier series upto second harmonic for the following data: (6)

x:	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y:	1.0	1.4	1.9	1.7	1.5	1.2	1.0

23. (a) Find the fourier transform of $f(x) = \begin{bmatrix} 1-x^2 & in\,|x| \le 1 \\ 0 & in\,|x| > 1 \end{bmatrix}$ (8)

Hence prove that $\displaystyle\int_0^\infty \frac{\sin s - s\cos s}{s^2}\cos\frac{s}{2}\,ds = \frac{3\pi}{16}$

(b) Using Parseval's identity, evaluate $\int_0^{\infty} \frac{x^2}{(a^2+x^2)^2}\, dx$ if $a > 0$ (4)

24. (a) Prove that $e^{-x^2/2}$ is self reciprocal under Fourier cosine transform.(6)

(b) Find the fourier sine transform of the function $f(x) = \frac{e^{-ax}}{x}$ (6)

25. (a) Find the singular solution of $z = px + qy + \sqrt{1 + p^2 + q^2}$ (8)

(b) Solve: $x(z^2 - y^2)\, p + y(x^2 - z^2)\, q = z(y^2 - x^2)$ (4)

26. (a) Solve: $p(1 - q^2) = q(1 - z)$ (6)

(b) Solve: $(D^2 - DD' - 20D^2)\, z = e^{5x+y} \sin(4x - y)$ (6)

27. (a) If a string of length λ is initially at rest in its equilibrium position and each of its points is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{x-0} = v_0 \sin^3 \frac{\pi x}{\lambda}$, $0 < x < \lambda$

Determine the displacement function $y(x, t)$ (8)

(b) Solve the equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ subject to the boundary conditions $u(0, t) = 0$, $u(l, t) = 0$ and $u(x, 0) = x$ (4)

28. (a) Using Z-transform, solve the difference equation (8)
$y(n + 2) - 3y(n + 1) + 2y(n) = 2^n$, given that $y(0) = y(1) = 0$

(b) Using convolution theorem, evalutate $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]$ (4)

SOLUTIONS

PART-A

1. Dirichlet's conditions given in 1.2 in page 1.2 of the book-III edition.
2. $f(x) = x \sin x$ is an even function of x in $(-\pi, \pi)$.
   $\therefore$ Fourier series of $f(x)$ will contain only cosine terms
   terms $\therefore$ $b_n = 0$
3. R.M.S. value of $f(x)$ in $(c, c + 2l)$ $= \sqrt{\frac{1}{2l} \int_c^{c+2l} [f(x)]^2\, dx}$

$\therefore$ R.M.S.V. of x^2 in $(0, \pi)$ $= \sqrt{\frac{1}{\pi}\int_0^{\pi} x^4\, dx} = \sqrt{\frac{1}{\pi}\cdot\frac{\pi^5}{5}} = \frac{\pi^2}{\sqrt{5}}$

4. The process of finding the constant term and the required harmonics, namely, $A_n \cos\left(\frac{n\pi x}{l} - \alpha_n\right)$ or $A_n \sin\left(\frac{n\pi x}{l} + \beta_n\right)$; n = 1, 2, 3, . . . in the fourier series expansion of a function $f(x)$ in $(c, c + 2l)$ in called Harmonic Analysis.
5. Statement of Fourier Integral Theorem is available in 2.2 in page 2.1 of the book
6. $F_c\ \{f(x)\} = \bar{f}_c(s) = \int_0^{\infty} f(x) \cos sx\, dx$ and

$$F_c^{-1}\{\bar{f}_c(s)\} = f(x) = \frac{2}{\pi}\int_0^{\infty} \bar{f}_c(s) \cos xs\, ds$$

7. $\text{Fs}(e^{-x}) = \int_0^{\infty} e^{-x} \sin sx\, dx = \left[\frac{e^{-x}}{s^2+1}(-\sin sx - s\cos sx)\right]_0^{\infty} = \frac{s}{s^2+1}$
8. $F\{f(ax)\} = \int_{-\infty}^{\infty} f(ax)\, e^{-isx} dx$

$$= \frac{1}{a}\int f(t) e^{-i\left(\frac{s}{a}\right)t} dt, \text{on putting } ax = t\ (\because a > 0) = \frac{1}{a} F\left(\frac{s}{a}\right)$$

9. $(x - a)^2 + (y - b)^2 + z^2 = 1$ (1)
   Differentiating (i) w.r.t. x; $2(x - a) + 2z\, p = 0$ (2)
   Differentiating (i) w.r.t.y; $2(y - b) + 2zq = 0$ (3)
   Using (2) and (3) in (i); we get $(zp)^2 + (zq)^2 + z^2 = 1$
   i.e., $z^2\,(p^2 + q^2 + 1) = 1$.
10. $\phi\left(z^2 - xy, \frac{x}{z}\right) = 0$. i.e., $\phi(u, v) = 0$ (i), where $u = z^2 - xy$ and $v = \frac{x}{z}$ (1)

   Differentiating (i) w.r.t.x;

   $$\phi_u\ (2zp - y) + \phi_v\left\{\frac{z.1 - xp}{z^2}\right\} = 0 \qquad (2)$$

   Differentiating (i) w.r.t.y;

   $$\phi_u\ (2zq - x) + \phi_v\left\{-\frac{x}{z^2}q\right\} = 0 \qquad (3)$$

Eliminating ϕ_u and ϕ_v from (2) and (3), we have

$$\frac{2zp-y}{2zq-x}=\frac{z-xp}{-xq}$$

i.e., $-2zxpq + xy\,q = 2z^2\,q - zx - 2zxpq + x^2p$

i.e., $x^2p + (2z^2 - xy)q = zx$

11. C.S. of the given equation is $z = ax + by + a^2 - b^2$ (1)

Differentiating (i) w.r.t.a; $0 = x + 2a$ (2)

Differentiating (i) w.r.t.b; $0 = y - 2b$ (3)

From (2) and (3), we get a $= -\dfrac{x}{2}$ and $b = \dfrac{y}{2}$ (4)

Using (4) in (1), the required S. S. is

$$z=-\frac{-x^2}{2}+\frac{y^2}{2}+\frac{x^2}{4}-\frac{y^2}{4}$$

i.e., $z=\dfrac{-x^2}{4}+\dfrac{y^2}{2}$ or $4z=y^2-x^2$

12. A.E. is $m^2 - 5m + 6 = 0$; i.e., $(m - 2)(m - 3) = 0$; $\therefore m = 2, 3$

$\therefore$ G.S. required is $z = f_1(y + 2x) + f_2(y + 3x)$, where f_1 and f_2 are arbitrary functions.

13. $y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} + 2u_x - 3u_y = 0$

$A = y^2$; $B = -2xy$; $C = x^2$

Now $B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$

$\therefore$ The given P.D.E. is parabolic.

14. The possible solutions of $y_{tt} = a^2y_{xx}$ are

(i) $y(x, t) = (Ae^{px} + Be^{-px})(Ce^{\text{pat}} + D\,e^{-\text{pat}})$

(ii) $y(x, t) = (A\cos px + B\sin px)(C\cos \text{pat} + D\sin \text{pat})$

(iii) $y(x, t) = (Ax + B)(Ct + D)$

15. Fourier law of heat conduction states that the rate of flow of heat across any area A is proportional to A and the temperature gradient normal to the area, i.e., $R\propto A\dfrac{\partial u}{\partial x}$ or $R=-kA\dfrac{\partial u}{\partial x}$, where k is the thermal conductivity of the conducting material.

16. The D.E. aof one dimensional heat flow under steady state conditions is

$$\frac{\partial^2 u}{\partial x^2}=0$$

Solving we get $\dfrac{\partial u}{\partial x} = A$ and $u = Ax + B$ where A and B are arbitrary constants

17. $Z(a^n) = \dfrac{z}{z-a} \quad \therefore Z\{na^n\} = -z\dfrac{d}{dz}\{Z(a^n)\} \quad = -z\dfrac{d}{dz}\left(\dfrac{z}{z-a}\right) = \dfrac{az}{(z-a)^2}$

18. $Z\{(n+1)(n+2)\} = Z(n^2+3n+2) = \; = \dfrac{z(z+1)}{(z-1)^3} + 3\dfrac{z}{(z-1)^2} + 2\dfrac{z}{z-1}$

$$= \frac{z}{(z-1)^3}\left\{(z+1) + 3(z-1) + 2(z-1)^2\right\} = \frac{2z^3}{(z-1)^3}$$

19. (Statement and proof of the initial value theorem in Z-transforms are available in 6(i) in page 5.4 of the book.)

20. Convolution of two sequences $\{f(n)\}$ and $\{g(n)\}$ is defined as $\{f(n)*g(n)\}$ $= \; = \sum_{r=0}^{n} f(r)\cdot g(n-r)$, if the sequences are causal or $\{f(n) * g(n)\} =$

$= \sum_{r=-\infty}^{\infty} f(r)\cdot g(n-r)$ if the sequences are non-causal.

PART-B

21. (a) This is the worked example (18) in page 2.35 of the book. We have to take $l = \pi$.

(b) Giving an odd extension for $f(x)$ in $(-\lambda, 0)$, the function $f(x)$ is made an odd function in $(-\lambda, \lambda)$

$\therefore$ Let F.S. of $f(x)$ be $= \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{\lambda}$

$$b_n = \frac{2}{\lambda}\int_0^{\lambda} f(x)\sin\frac{n\pi x}{\lambda}\,dx$$

$$= \frac{2}{\lambda}\left[\int_0^{\lambda/2} x\sin\frac{n\pi x}{\lambda}\,dx + \int_{\lambda/2}^{\lambda}(\lambda - x)\sin\frac{n\pi x}{\lambda}\,dx\right]$$

$$= \frac{2}{\lambda}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{\lambda}}{n\pi/\lambda}\right) - 1\left(\frac{-\sin\frac{n\pi x}{\lambda}}{n^2\pi^2/\lambda^2}\right)\right\}_0^{\frac{\lambda}{2}}\right.$$

$$\left. + \left\{(\lambda - x)\left(\frac{-\cos\frac{n\pi x}{\lambda}}{n\pi/\lambda}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{\lambda}}{n^2\pi^2/\lambda^2}\right)\right\}^{\lambda/2}\right]$$

$$= \frac{2}{\lambda}\left[\left\{-\frac{\lambda^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{\lambda^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} + \left\{\frac{\lambda^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{\lambda^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right]$$

$$= \frac{4\lambda}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore f(x) \sim \frac{4\lambda}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{\lambda}\ in\ (0, \lambda)$$

22. (a) $f(x) = x^2$ is an even function of x in $-\pi < x < \pi$

$\therefore$ Let the F.S. of $f(x)$ be $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ in $(-\pi, \pi)$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos n\,x\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4}{\pi n^2} \times \pi(-1)^n = 4\frac{(-1)^n}{n^2}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{3\pi}(x^3)_0^{\pi} = \frac{2}{3}\pi^2$$

$$\therefore f(x) \equiv \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)}{n^2}\cos nx \text{ in } (-\pi, \pi)$$

By Parseval's theorem,

$$\frac{1}{4}\cdot\frac{4}{9}\pi^4 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{1}{\pi}\left(\frac{x^5}{5}\right)_0^{\pi} = \frac{\pi^4}{5}$$

$$\therefore\ 8\sum_{n=1}^{\infty}\frac{1}{n^4} = \left(\frac{1}{5} - \frac{1}{9}\right)\pi^4$$

i.e., $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$

22. (b) As the interval $(0, 2\pi)$ is divided into sub-intervals of length $\frac{\pi}{3}$, we take only 6 values of $y = f(x)$,i.e., $y_0, y_1, \ldots y_5$ corresponding to $x = 0$, $\frac{\pi}{3}, \frac{2\pi}{3}, \ldots \frac{5\pi}{3}$

The required Fourier series is of the form

$$y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = 2 \times \frac{1}{6} \sum y_r = \frac{1}{3} \times 8.7 = 2.9$$

Since $a_1 = \frac{1}{3} \sum y_r \cos x_r$ and $b_1 = \frac{1}{3} \sum y_r \sin x_r$, a Harrison circle for $\frac{\pi}{3}$ is used

$$a_1 = \frac{1}{3}\left[(y_0 - y_3) + (y_1 + y_5 - y_2 - y_4) \cos \frac{\pi}{3}\right]$$

$$= \frac{1}{3}[-0.7 - 0.8 \times 0.5] = -0.367$$

$$b_1 = = \frac{1}{3}\left[(y_1 + y_2 - y_4 - y_5) \sin \frac{\pi}{3}\right] = \frac{1}{3} \times 0.6 \times 0.866 = 0.173$$

Since $a_2 = \frac{1}{3} \cdot \sum y_r \cos 2x_r$ and $b_2 = \frac{1}{3} \sum y_r \sin 2x_r$, a

Harrison circle for $\frac{2\pi}{3}$ is used.

$$a_2 = \frac{1}{3}\left[(- y_1 - y_4 - y_2 - y_5) \cos \frac{\pi}{3} + (y_0 + y_3)\right]$$

$$= \frac{1}{3} \times [-6.0 \times 0.5 + 2.7] = - 0.1$$

$$b_2 = \frac{1}{3}\left(y_1 + y_4 - y_2 - y_5\right) \sin \frac{\pi}{3}$$

$$= \frac{1}{3}(-0.2 \times 0.866) = -0.058$$

$\therefore y = 1.45 + (-0.367 \cos x + 0.173 \sin x) + (-0.1 \cos 2x - 0.058 \sin 2x)$

23. (a) (This problem is the worked example (7) in page 4-13 of the book)

(b) This problem is the worked example (6)(ii) in page 4-38 of the book)

24. (a) [Refer to the worked example (8) in page 4-14 of the book]

In that worked example, we have proved in step (i) that

$$F\left\{e^{-a^2x^2}\right\} = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$$

Taking $a = \dfrac{1}{\sqrt{2}}$, This result becomes

$$F\left\{e^{-x^2/2}\right\} = \sqrt{2\pi}\, e^{-s^2/2}$$

i.e., $\displaystyle\int_{-\infty}^{\infty} e^{-x^2/2} (\cos sx - i \sin xs)\, dx = \sqrt{2\pi}\, e^{s^2/2}$

i.e., $\displaystyle 2\int_0^{\infty} e^{-x^2/2} \cos sx\, dx = \sqrt{2\pi}\, e^{-s^2/2}$

i.e., $\displaystyle \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x^2/2} \cos sx\, dx = e^{-s^2/2}$

i.e., $F_C\,(e^{-x^2/2}) = e^{-s^2/2}$

$\therefore$ For the definition of $F_C\{f(x)\}$ used above, $e^{-x2/2}$ is self-reciprocal under Fourier cosine transform.

(b) (The solution of this problem is available in the worked example (13) in page 4-18 of the book)

25. (a) (This problem is the worked example 9 in page 4-35 of the book, if we take c = 1 in the worked example)

(b) L.S.S.E's are $\dfrac{dx}{x(z^2 - y^2)} = \dfrac{dy}{y(x^2 - z^2)} = \dfrac{dz}{z(y^2 - x^2)}$ (1)

Each ratio in (1) = $\dfrac{x\,dx + y\,dy + z\,d\,z}{0}$

$\therefore$ $xdx + ydy + zdz = 0$

Solving, we get $x^2 + y^2 + z^2 = a$ (2)

Also each ratio in (1) $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$\therefore$ $\dfrac{1}{x}dx + \dfrac{1}{y}dy + \dfrac{1}{z}dz = 0$

Solving, we get $xyz = b$ (3)

$\therefore$ G.S. of the given P.D.E. is $f(x^2 + y^2 + z^2, xyz) = 0$

26. (a) (This problem is the worked example (13) in page 4-38 of the book)

(b) A.E. is $m^2 - m - 20 = 0$; i.e., $(m - 5)(m + 4) = 0$

$\therefore$ $m = -4, 5$

$\therefore$ C.F. $= f_1(y - 4x) + f_2(y + 5x)$, where f_1 and f_2 are arbitrary functions

$$\text{P.I.}_1 = \frac{1}{(D-5D')(D+4D')} e^{5x+y} = \frac{1}{9} \cdot \frac{1}{D-5D'} e^{5x+y} = \frac{1}{9} x e^{5x+y}$$

$$\text{P,I.}_2 = \frac{1}{D^2 - DD' - 20D'^2} \sin(4x-y)$$

$$= \frac{1}{-16+4+20} \sin(4x-y) = \frac{1}{8} \sin(4x-y)$$

$\therefore$ G.S. is z = C.F. + P.I.$_1$ + P.I.$_2$

27. (a) This problem is the same as the worked example 6(i) in page 3-24 of the book. Instead of $l = 50$ of the worked example, it is given as λ. The following changes are to be made.

$$\frac{\pi a}{\lambda} \lambda_1 = \frac{3v_0}{4} \text{ and } \frac{3\pi a}{\lambda} \lambda_3 = -\frac{v_0}{4}; \frac{n\pi a}{\lambda} \lambda_n = 0 \text{ for } n = 2, 4, 5 \ldots$$

$$\therefore \lambda_1 = \frac{3\lambda v_0}{4\pi a} \text{ and } \lambda_3 = -\frac{\lambda v_0}{12\pi a}; \lambda_n = 0, \text{for } n = 2, 4, 5, \ldots$$

$\therefore$ The required solution is

$$y(x, t) = = \frac{3\lambda v_0}{4\pi a} \sin\frac{\pi x}{\lambda} \sin\frac{\pi at}{\lambda} - \frac{\lambda v_0}{12\pi a} \sin\frac{3\pi x}{\lambda} \sin\frac{3\pi at}{\lambda}$$

(b) This problem is similar to the worked example (1) in page 3-65 of the book, except for a minor change.
We have to take $f(x) = x$ in step (4)
Steps upto (8) are identical with those in the worked example.

We have $\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = x$ in $(0, l) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$, which is

Fourier half-range sine series of x in (0, l).
Comparing like terms, we get

$$B_n = b_n = \frac{2}{l} \int_0^l x \sin\frac{n\pi x}{l}\, dx = \frac{2}{l}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - 1 \left(\frac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2} \right) \right]_0^l$$

$$= -\frac{2l}{n\pi}(-1)^n$$

Using this value of B_n in step (7), the required solution is

$$u(x,t) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{l} \cdot e^{-n^2\pi^2\alpha^2 t/l^2}$$

28. (a) $y_{n+2}-3y_{n+1}+2y_n=2^n$, given $y_0=y(1)=0$

Taking Z-transform of the given equation is

$$[z^2\,\bar{y}\,(z)-z^2y(0)-zy(1)]-3[z\;\bar{y}\,(z)-z\,\bar{y}\,(0)]+2\,\bar{y}\,(z)=\frac{z}{z-2}$$

Using the given conditions, we get

$$(z^2-3z+2)\;\bar{y}\,(\mathrm{z})=\frac{z}{z-2}$$

$$\therefore\;\bar{y}(z)=\frac{z}{(z-2)^2\;(z-1)}$$

$$\therefore\;\frac{\bar{y}(z)}{z}\cdot=\frac{1}{(z-2)^2\;(z-1)}=\frac{A}{z-2}+\frac{B}{(z-2)^2}+\frac{C}{z-1}$$

$$=-\frac{1}{z-2}+\frac{1}{(z-2)}+\frac{1}{z-1}$$

$$\therefore\;\bar{y}\,(z)=-\frac{z}{z-2}+\frac{z}{(z-2)^2}+\frac{z}{z-1}$$

Inverting, we get

$$y(n)=-2^n+n.2^{n-1}+1$$

or $y(n)=1+2^{n-1}\,(n-2)$

(b) $$Z^{-1}\left\{\frac{z^2}{(z-a)^2}\right\}=Z^{-1}\left\{\left(\frac{z}{z-a)}\right)\left(\frac{z}{z-a)}\right)\right\}$$

$$=Z^{-1}\left(\frac{z}{z-a}\right)*Z^{-1}\left(\frac{z}{z-a}\right)$$

$$=\left(a^n\right)*\left(a^n\right)$$

$$=\sum_{r=0}^{n}a^r\,.a^{n-r}$$

$$=\sum_{r=0}^{n}a^n=(n+1)\,a^n$$

Appendix C

T 3048
B.E./B.Tech. Degree Examinations, November/December 2009
Third Semester
Civil Engineering
MA 2211 Transforms and Partial Differential Equations
(Common to all branches)
(Regulations 2008)

Time: Three hours Maximum: 100 marks

Answer ALL questions.
Part A – (10 × 2 = 20 marks)

1. State the sufficient condition of a function $f(x)$ to be expressed as a fourier series
2. Obtain the first term of the Fourier series for the function
   $f(x) = x^2$, $-x < x < \pi$.
3. Find the Fourier transform of
$$f(x) = \begin{cases} e^{ikx}, a < x < b \\ 0 \quad x < a \text{ and } x > b \end{cases}$$
4. Find the Fourier sine transform of $\dfrac{1}{x}$
5. Find the partial differential equation of all planes cutting equal intercepts from the x and y axes.
6. Solved $(D^3 - 2D^2D')z = 0$
7. Classify the partial differential equation $4\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$
8. Write down all possible solutions of one dimensional wave equation.
9. If $F(z) = \dfrac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)}$, find $f(0)$.

10. Find the Z-transform of

$$f(n) = \begin{cases} \dfrac{a^n}{n!} & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Part B—(5 × 16 = 80 Marks)

11. (a) (i) Obtaint the Fourier series of the periodic function defined by

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots + \infty = \dfrac{\pi^2}{8}$ (10)

(ii) Compute upto first harmonic of the Fourier series of (x) given by the following table

x:	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
$f(x)$	1.98	1.30	1.05	1.30	–0.88	–0.25	1.98

(6)

Or

(b) (i) Expand $f(x) = x - x^2$ as a Fourier series in $-L < x < L$ and using this series find the root mean square value of $f(x)$ in the interval. (10)

(ii) Find the complex form of the Fourier series of

$f(x) = e^{-x}$ in $-1 < x < 1$. (6)

12. (a) (i) Find the Fourier transform of $f(x) \begin{cases} = 1 - |x| \text{ if } |x| < 1 \\ = 0 \text{ if } |x| \geq 1 \end{cases}$ and hence

find the value of $\displaystyle\int_0^\infty \frac{\sin^4 t}{t^4}\, dt$. (8)

(ii) Evaluate $\displaystyle\int_0^\infty \frac{dx}{(4 + x^2)\left(25 + x^2\right)}$ using transform methods. (8)

(b) (i) Find the Fourier cosine transform of e^{-x^2}.

(ii) Prove that $\dfrac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine and cosine transforms. (8)

13. (a) A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point an initial velocity $3x(l - x)$, find the displacement. (16)

Or

(b) A rod, 30 cm long has its ends A and B kept at 20°C and 80°C respectively, until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0°C and kept so. Find the resulting temperature function $U(x, t)$ taking $x = 0$ at A. (16)

14. (a) (i) Find the inverse Z-transform of $\dfrac{10z}{z^2 - 3z + 2}$ (8)

(ii) Solve the equation $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$ given $u_0 = u_1 = 0$. (8)

Or

(b) (i) Using convolution theorem, find the Z^{-1} of $\dfrac{z^2}{(z-4)(z-3)}$. (8)

(ii) Finc the inverse z-transform of $\dfrac{z^3 - 20z}{(z-2)^3 (z-4)}$. (8)

15. (a) (i) Solve $z = px + qy + p^2q^2$ (8)

(ii) Solve $(D^2 + 2DD' + 2D'^2)$ $(z = \sinh (x + y) + e^{x+2y}$. (8)

Or

(b) (i) Sovle $(y - xz)\, p + (yz - x)\, q = (x + y)(x - y)$. (8)

(ii) Solve $(D^2 - D'^2 - 3D + 3D')z = xy + 7$. (8)

SOLUTIONS

PART-A

1. Dirichlet's conditions given in 2.2 in page 2.2 of the book
2. $f(x) = x^2$ is even in $(-\pi, \pi)$

$\therefore$ F.S. of $f(x) \sim \dfrac{a_0}{2} + \sum a_n \cos nx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\, dx = \frac{2}{3}\pi^2,\ a_1 = \frac{2}{\pi}\int_0^{\pi} x^2 \cos x\, dx = \frac{2}{\pi}\left\{x^2 \sin x - 2x(-\cos x)\right.$$

$$\left. + 2(-\sin x)\right\}_0^{\pi} = \frac{2}{\pi} \times 2\pi(-1) = -4$$

$\therefore\ f(x) \sim \dfrac{\pi^2}{3} - 4\cos x + \cdots$

3. $F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-isx}\,dx = \int_a^b e^{ikx}\,e^{-isx}\,dx = \left\{\dfrac{e^{i(k-s)x}}{i(k-s)}\right\}_a^b$

$$= \frac{-i}{k-s}\left\{e^{i(k-s)b} - e^{i(k-s)a}\right\}$$

4. $F_S\left(\dfrac{1}{x}\right) = \int_0^{\infty} \dfrac{1}{x}\sin sx\,dx = \dfrac{\pi}{2}$, if $s > 0$.

5. The analytic equation of the given family of planes is

$\dfrac{x}{a} + \dfrac{y}{a} + \dfrac{z}{b} = 1$ or $A(x+y) + Bz = 1$ (1)

From (1) $A + Bp = 0$(2) and $A + Bq = 0$ (3)

From (2) and (3), we get $p = q$, which is the rquired P.D.E.

6. $(D^3 - 2D^2D')\,z = 0$ The A.E. is $m^3 - 2m^2 = 0$ $\therefore$ $m = 0, 0, 2$.

$\therefore$ Required solutin is $z = f_1(y) + x\,f_2(y) + f_3(y + 2x)$

7. For the given equation, $A = 4$, $B = 0$; $C = 0$

$\therefore$ $B^2 - 4AC = 0$ $\therefore$ The given equation is parabolic

8. The possible solutions of $\dfrac{\partial^2 y}{\partial t^2} = a^2\dfrac{\partial^2 y}{\partial x^2}$ are

(i) $y(x,t) = (Ae^{px} + Be^{-px})\,(C\,e^{\text{pat}} + De^{-\text{pat}})$

(ii) $y(x, t) = (A\cos px + B\sin px)\,(C\cos pat + D\sin \text{pat})$

(iii) $y(x, t) = (Ax + B)\,(Ct + D)$

9. $\dfrac{F(z)}{z} = \dfrac{-8}{z-\frac{1}{2}} + \dfrac{2}{z-\frac{1}{4}} + \dfrac{6}{z-\frac{3}{4}}$ $\therefore F(z) = -8\dfrac{z}{z-\frac{1}{2}} + 2\cdot\dfrac{z}{z-\frac{1}{4}} + 6\dfrac{z}{z-\frac{3}{4}}$

$\therefore\ f(n) = -8\cdot\left(\dfrac{1}{2}\right)^n + 2\cdot\left(\dfrac{1}{4}\right)^n + 6\cdot\left(\dfrac{3}{4}\right)^n$ $\therefore f(0) = 0$

10. $Z\left(\dfrac{a^n}{\lfloor n}\right) = \sum_{n=0}^{\infty}\dfrac{\left(\frac{a}{z}\right)^n}{\lfloor n} = e^{a/z}$

PART-B

11. (a) (i) Let the F.S. of $f(x)$ be

$$\frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx \text{ in } (-\pi, \pi)$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx\, dx + \int_{0}^{\pi} x \cos nx dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} \left\{x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi n^2}\left\{(-1)^n - 1\right\} = \begin{cases} -\dfrac{2}{\pi n^2}, \text{ if } n \text{ is odd} \\ 0, \quad \text{if } n \text{ is even} \end{cases}$$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\, dx + \int_{0}^{\pi} x\, dx\right] = \frac{1}{\pi}\left[-\pi(x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_{0}^{\pi}\right] = \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = -\frac{\pi}{2}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx\, dx + \int_{0}^{\pi} x \sin n\, x\, dx\right]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left\{x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\left\{1 - (-1)^n\right\} - \frac{\pi}{n}\left(-1)^n\right)\right] = \frac{1}{n}\left\{1 - 2\,(-1)^n\right\}$$

$$\therefore\ f(x) \sim -\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}\left\{1 - 2\,(-1)^n\right\}\sin nx \text{ in } (-\pi, \pi)$$

Putting $x = 0$, which is a point of discontinuity of $f(x)$,

$$-\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2} = \frac{1}{2}(-\pi + 0) = -\frac{\pi}{2}$$

$$\therefore \frac{2}{\pi} S = \frac{\pi}{4} \therefore S = \frac{\pi^2}{8}$$

(ii) As the internal $(0, T)$ is divided into sub-intervals of length $\frac{T}{6}$, we take only 6 values of $y = f(x)$, i.e., $y_0, y_1, y_2, y_3, y_4, y_5$ corresponding to x $= 0, \frac{T}{6}, \ldots, \frac{5T}{6}$.

Since $2l = T$, the required Fourier series is of the form

$$y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi \text{x}}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{T}$$

$$a_0 = 2 \times \frac{1}{6}\sum y_n = \frac{1}{3} \times 4.5 = 1.5$$

Since $a_1 = 1/3\ \Sigma y_r \cos\frac{2\pi x_r}{T}$ and $b_1 = 1/3 \sum y_r \sin\frac{2n\pi_r}{T}$, we use a Harrison circle for $\pi/3$.

$$\therefore a_1 = \frac{1}{3}\left[(y_0 - y_3) + (y_1 + y_5 - y_2 - y_4)\cos 60°\right]$$

$$= \frac{1}{3}[0.68 + 0.88 \times 0.5] = 0.3733$$

$$b_1 = \frac{1}{3}[(y_1 + y_2 - y_4 - y_5)\sin 60°] = \frac{1}{3}(3.48 \times 0.866) = 1.0046$$

$$\therefore f(x) \sim 0.75 + (0.373.3 \cos\ (2\pi x/T) + 1.0046 \sin\frac{2\pi x}{T}$$

11. (b) (i) Let the F.S. of $f(x)$ be $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

$$a_n \frac{1}{L} = \int_{-L}^{L} (x - x^2)\cos\frac{n\pi x}{L}\,dx = \frac{-2}{L}\int_0^L x^2 \cos\frac{n\pi x}{L}\,dx$$

(since the integrand of the other part of the integral is odd)

$$= -\frac{2}{L}\left[x^2\left(\frac{\sin\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - 2x\left(\frac{-\cos\frac{n\pi x}{L}}{n^2\pi^2/L^2}\right) + 2\left(\frac{-\sin\frac{n\pi x}{L}}{n^3\pi^3/L^3}\right)\right]_0^L$$

$$= -\frac{2}{L} \times \frac{2L^2}{n^2\pi^2} \times L(-1)^n = \frac{4L^2}{n^2\pi^2}(-1)^{n+1}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L}(x - x^2)\,dx = -\frac{2}{L}\int_0^L x^2 dx = -\frac{2}{3}L^2$$

$$b_n = \frac{1}{L}\int_{-L}^{L}(x - x^2)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left[x\left(\frac{-\cos n\pi x/L}{n\pi/L}\right) - 1\left(\frac{-\sin n\pi x/L}{n^2\pi^2/L^2}\right)\right]_0^L = -\frac{2}{L}\cdot\frac{L}{n\pi}\cdot L(-1)^n = \frac{2L}{n\pi}(-1)^{n+1}$$

$$\therefore f(x) = -\frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{L} + \frac{2L}{\pi}\sum\frac{1}{n}(-1)^{n+1}\sin\frac{n\pi x}{L}$$

$$\overline{f(x)}^2 = \frac{1}{4}a_0^2 + \frac{1}{2}\sum a_n^2 + \frac{1}{2}\sum b_n^2$$

$$= \frac{1}{4}\times\frac{4}{9}L^4 + \frac{1}{2}\sum\frac{16L^4}{n^4\pi^4} + \frac{1}{2}\sum\frac{4L^2}{n^2\pi^2}$$

$$\therefore \bar{f}(x) = \sqrt{\frac{L^4}{9} + \frac{8L^4}{\pi^4}\sum\frac{1}{n^4} + \frac{2L^2}{\pi^2}\sum\frac{1}{n^2}}$$

(ii) (This problem is the same as the worked examples 8 in page 1.87 of the book. We have to put $a = 1$ and $l = l$) The required complex form of the Fourier series is $e^{-x} = (\sinh 1)\displaystyle\sum_{n=-\infty}^{\infty}\frac{(-1)^n(1 - i n\pi)}{1 + n^2\pi^2}e^{in\pi x}$ in$(-1, 1)$.

12. (a) (i) This problem is the same as worked example (5) in page 2.37 of the book)

(ii) $\displaystyle\int_0^{\infty}\frac{dx}{(x^2+4)(x^2+25)}$. Let $f(x) = e^{-2x}$ and $g(x) = e^{-5x}$

Then $\bar{f}_c(s) = \dfrac{2}{s^2+4}$ and $\bar{g}_c(s) = \dfrac{5}{s^2+25}$

By a property of Fourier cosine transforms.

$$\int_0^\infty f(x)\cdot g(x)\,dx = \frac{2}{\pi}\int_0^\infty \bar{f}_c(s)\,\bar{g}_c(s)\,ds$$

$$\therefore \int_0^\infty \frac{10}{(s^2+4)(s^2+25)}\,ds = \frac{\pi}{2}\int_0^\infty e^{-7x}\,dx = -\frac{\pi}{14}\left(e^{-7x}\right)_0^\infty = \frac{\pi}{14}$$

$$\therefore \int_0^\infty \frac{ds}{(s^2+4)(s^2+25)} = \frac{\pi}{140}$$

(b) (i) (This problem is the same as the worked example (8) in page 4.14 of the book. We have to take a = 1)

$$F_c\,\{e^{-x^2}\} = \frac{\sqrt{\pi}}{2}\,e^{-s^2/4}.$$

(ii) This problem is the same as the worked example (16) in page 4-21 of the book)

13. (a) This problem is the same as the worked example (7) in page 4-26 of the book. We have to replace l by $l/2$ and put $k = 3$

(b) (This problem is the same as the worked example (4) in page 4.72 of the book. In the worked example, we have to replace 20 cm by 30 cm and 30°C and 90°C by 20°C and 80°C respectively)

14. (a) (i) $\bar{f}(z) = \dfrac{10z}{z^2-3z+2}$; $\dfrac{\bar{f}(z)}{z} = \dfrac{10}{(z-1)(z-2)} = \dfrac{10}{z-2} - \dfrac{10}{z-1}$

$$\therefore \bar{f}(z) = 10\cdot\frac{z}{z-2} - 10\,\frac{z}{z-1} \quad \therefore f(n) = 10\,(2^n - 1)$$

(ii) $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$

Taking z-transforms on both sides, we get

$$\left[z^2\,\bar{u}(z) - z^2\,u(0) - zu(1)\right] + 6\left[z\,\bar{u}(z) - zu(0)\right] + 9\bar{u}(z) = \frac{z}{z-2}$$

$$\therefore \frac{\bar{u}(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{1/25}{z-2} - \frac{1/25}{z+3} - \frac{1/5}{(z+3)^2}$$

$$\therefore \bar{u}(z) = \frac{1}{25}\cdot\frac{z}{z-2} - \frac{1}{25}\cdot\frac{z}{z+3} - \frac{1}{5}\,\frac{z}{(z+3)^2}$$

Inverting

$$u(n) = \frac{1}{25}.2^n - \frac{1}{25}\cdot(-3)^n - \frac{1}{5} \times -\frac{1}{3}\, n\,(-3)^n$$

i.e., $u(n) = \frac{1}{25}.2^n - \frac{1}{25}\cdot(-3)^n + \frac{1}{15}\, n\,(-3)^n$

14. (b) (i) $Z^{-1}\left\{\frac{z^2}{(z-4)(z-3)}\right\} = Z^{-1}\left\{\frac{z}{z-4}\cdot\frac{z}{z-3}\right\}$

$$= Z^{-1}\left(\frac{z}{z-4}\right)^{*} Z^{-1}\left(\frac{z}{z-3}\right)$$

$$= 4^n * 3^n$$

$$= \sum_{r=0}^{n} 4^{n-r}3^r = 4^n \sum_{r=0}^{n}\left(\frac{3}{4}\right)^r$$

$$= 4^n\left[\frac{1-\left(\frac{3}{4}\right)^{n+1}}{1-\frac{3}{4}}\right] = 4^{n+1}\left[1-\left(\frac{3}{4}\right)^{n+1}\right] = 4^{n+1} - 3^{n+1}.$$

(ii) $\frac{\bar{f}(z)}{z} = \frac{A}{z-2} + \frac{B}{(z-2)^2} + \frac{C}{(z-2)^3} + \frac{D}{z-4}$

$\therefore A(z-1)^2(z-4) + B(z-2)(z-4) + C(z-4) + D(z-2)^3 = z^2 - 20$

Putting $z = 2$; $-2C = -16$ $\therefore$ $C = 8$

Putting $z = 4$; $8D = -4$ $\therefore$ $D = -\frac{1}{2}$

Coeffts of z^2; $A + D = 0$ $\therefore$ $A = \frac{1}{2}$

Constants ; $-16A + 8B - 4C - 8D = -20$

in $\quad -8 + 8B - 32 + 4 = -20$

$\therefore$ $B = 2$

$$\therefore\ \bar{f}(z) = \frac{1}{2}\cdot\frac{z}{z-2} + 2\cdot\frac{z}{(z-2)^2} + 8\cdot\frac{z}{(z-2)^3} - \frac{1}{2}\cdot\frac{z}{z-4}$$

Inverting, we get

$$f(n) = \frac{1}{2}\cdot 2^n + n\cdot 2^n + (n^2 - n)2^n - \frac{1}{2}\cdot 4^n$$

i.e., $f(n) = \frac{1}{2} \cdot (2^n - 4^n) + n^2 2^n$

15. (a) (i) $z = px + qy + p^2q^2$ – This is a Clairaut's type equation

C.S. is $z = ax + by + a^2b^2$ (1)

Diffg. (1) partially w.r.t. 'a', $x + 2ab^2 = 0$ (2)

Diffg. (1) partially w.r.t. 'b', $y + 2a^2b = 0$ (3)

From (2) and (3), we get $\frac{x}{y} = \frac{b}{a}$ or $\frac{b}{x} = \frac{a}{y} = k$, say

$\therefore$ $b = kx$ and $a = ky$

Using in (2); $x + 2k^3x^2y = 0$ or $k = -\frac{1}{(2xy)^{1/3}}$

$\therefore$ $a = -\frac{y}{(2xy)^{1/3}}$ and $b = -\frac{x}{(2xy)^{1/3}}$

Using these values in (1), we get

$$z = -\frac{xy}{(2xy)^{1/3}} - \frac{xy}{(2xy)^{1/3}} + \frac{x^2y^2}{(2xy)^{4/3}}$$

i.e., $z = -(2xy)^{2/3} + \frac{1}{4}(2xy)^{2/3} = -\frac{3}{4}(2xy)^{2/3}$

$\therefore$ $64z^3 = -27 \times 4x^2y^2$ or $16z^3 + 27x^2y^2 = 0$,

which is the singular solution of the given P.D.E.

To get the general solution, we put $b = f(a)$ in (1), where f is an arbitrary function.

Then $z = ax + f(a) \cdot y + a^2[f(a)]^2$ (4)

Diffg. (4) partially w.r.t. 'a', we get

$0 = x + y f'(a) + 2a[f(a)]^2 + 2a^2 f(a).f'(a)$ (5)

The eliminant of 'a' from (4) and (5) gives the G.S. of the given P.D.E.

(ii) $(D^2 + 2DD' + D'^2)z = \sinh(x + y) + e^{x+2y}$

A.E. is $(m + 1)^2 = 0$ $\therefore$ $m = -1, -1$

$\therefore$ C.F $= f_1(y - x) + xf_2(y - x)$

$$\text{P.I.}_1 = \frac{1}{(D+D')^2}\left\{\frac{e^{x+y} - e^{-(x+y)}}{2}\right\} = \frac{1}{2}\left[\frac{1}{4}e^{x+y} - \frac{1}{4}e^{-(x+y)}\right]$$

$$= \frac{1}{4}\sinh(x + y)$$

$$\text{P.I.}_2 = \frac{1}{(D+D')^2}e^{x+2y} = \frac{1}{9}e^{x+2y}$$

$\therefore$ G.S. is z = C.F. + P.I.$_1$ + P.I.$_2$.

15. (b) (i) $(y - zx)p + (yz - x)q = (x + y)(x - y)$

This is a Lagrange Linear equation

L.S.S.E's are

$$\frac{dx}{y - zx} = \frac{dy}{yz - x} = \frac{dz}{x^2 - y^2}$$

Each ratio $= \dfrac{x\,dx + y\,dy + zdz}{xy - x^2z + y^2z - xy + x^2z - y^2z} = \dfrac{x\,dx + y\,dy + z\,dz}{0}$

$\therefore$ One solution is given by

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{a}{2} \text{ or } x^2 + y^2 + z^2 = a$$

Also each ratio $= \dfrac{y\,dx + x\,dy + dz}{y^2 - xyz + xyz - x^2 + x^2 - y^2} = \dfrac{ydx + xdy + dz}{0}$

$\therefore$ Another solution is given by $d(xy) + dz = 0$ i.e., $xy + z = b$

$\therefore$ G.S. of the given equation is $f(x^2 + y^2 + z^2, xy + z) = 0$.

(ii) $(D^2 - D'^2 - 3D + 3D')z = xy + 7$

i.e., $[(D - D'(D + D') - 3(D - D')]\, z = xy + 7$

i.e., $(D - D')(D + D' - 3)\, z = xy + 7$

C.F. $= f_1(y + x) + e^{3x} \cdot f_2(y - x)$

$$\text{P.I.} = \frac{1}{D\left(1 - \dfrac{D'}{D}\right)(-3)\left\{1 - \dfrac{D + D'}{3}\right\}}(xy + 7)$$

$$= -\frac{1}{3D}\left(1 - \frac{D'}{D}\right)^{-1}\left\{1 - \left(\frac{D + D'}{3}\right)\right\}^{-1}(xy + 7)$$

$$= -\frac{1}{3D}\left(1 + \frac{D'}{D}\right)\left\{1 + \frac{1}{3}(D + D') + \frac{1}{9}(D + D')^2 + \frac{1}{27}(D + D')^3 + \right\}(xy + 7)$$

$$= -\frac{1}{3}\left[\frac{D^1}{D^2} + \frac{2}{3}\frac{D^1}{D} + \frac{1}{D} + \frac{1}{3} + \frac{1}{9}D + \frac{1}{3}D' + \frac{4}{27}DD'\right](xy + 7)$$

$$= -\frac{1}{3}\left(\frac{x^3}{6} + \frac{x^2}{3} + \frac{x^2y}{2} + \frac{1}{3}xy + \frac{1}{9}y + \frac{1}{3}x + \frac{4}{27}\right) + \left(7x + \frac{7}{3}\right)$$

$$= -\frac{1}{3}\left(\frac{x^3}{6} + \frac{x^2}{3} + \frac{x^3y}{2} + \frac{1}{3}xy + \frac{1}{9}y + \frac{22}{3}x + \frac{67}{27}\right)$$

$\therefore$ G.S.is $z =$ C.F + P.I.

Appendix D

E 203
B.E./B.Tech. Degree Examinations, November/December 2009
Third Semester
Civil Engineering
MA 31 Transforms and Partial Differential Equations
(Common to all branches)
(Regulations 2008)

Time: Three hours | Maximum: 100 marks

Answer ALL questions.
Part A – (10 × 2 = 20 marks)

1. State the Dirichlet's conditions for the function on Fourier series expansion.
2. If $f(x) = 3x - 4x^3$, defined in the interval $(-2, 2)$, then find the value of a_1 in the Fourier series expansion.
3. State Fourier integral theorem.
4. Let $F_C(s)$ be the Fourier cosine transform of $f(x)$. Prove that $F_C\,[f(x)\cos ax] = \dfrac{1}{2}\,[F_C(s + a) + F_C(s - a)]$.
5. Form the partial differential equation by eliminating the arbitrary constants from $z = (x + a)^3 + (y - b)^2$.
6. Find the complete integral of $p - 3q = 6$.
7. What are the possible solutions for Laplace quation $U_{xx} + U_{yy} = 0$ by method of separation of variables?
8. A rod 20 cm long with insulated sides has its ends A and B kept at 30°C and 90°C respectively. Find the steady sate temperature distribution of the rod.
9. Define Z-transform of the sequence $\{f(n)\}$
10. Form a difference equation by eliminating the arbitrary constants a and b from $y_n = a - b3^n$.

PART B

(5 × 16 = 80 marks)

11. (a) (i) Find the Fourier series exapansion of the periodic function $f(x)$ of the period 4 defined by $f(x) = \begin{cases} 1 + x & -2 \le x \le 0 \\ 1 - x & 0 \le x \le 2 \end{cases}$ Hence deduce that

$$\sum_{1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$ (8)

(ii) Obtain a Fourier series upto the second harmonic from the data: (8)

t:	0	$\frac{T}{6}$	$\frac{T}{3}$	$\frac{T}{2}$	$\frac{2T}{6}$	$\frac{5T}{6}$	T
y:	1.98	1.30	1.05	1.30	–0.88	–0.25	1.98

Or

(b) (i) Show that the complex form of the Fourier series of the periodic function $f(x) = e^{-x}$, $-1 < x < 1$ and $f(x+2) = f(x)$ is

$$f(x) = \sum_{-\infty}^{\infty} \frac{(-1)^n (1 - in\pi)}{1 + n^2 \pi^2} (\sinh 1) e^{in\pi x}$$ (8)

(ii) Find the half range cosine series of $f(x) = x$ in $(0, \pi)$ andhence prove that $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots \frac{\pi^4}{96}$ (8)

12. (a) (i) Find the Fourier Cosine Transform of $f(x) = e^{-a^2x^2}$ and hence find theFourier Cosine Transform of $e^{\frac{-x^2}{2}}$ and Fourier Sine Transform of $xe^{\frac{-x^2}{2}}$ (10)

(ii) Find Fourier Sine Transform of e^{-ax}, $a > 0$. Hence evaluate

$$\int_0^{\infty} \frac{s(\sin sx)\, ds}{a^2 + s^2}.$$ (6)

(b) Find the Fourier Transform of

$$f(x) = \begin{Bmatrix} 2 - |x|, \text{ in } |x| \le 2 \\ 0, \quad \text{ in } |x| > 2 \end{Bmatrix}.$$

Hence find the value $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^4.$ (16)

Or

13. (a) (i) Find the general solution of $x(y - z)p + y(z - x)q = z(x - y)$ (8)

(ii) Solve: $(D^3 + D^2D' - D'^3)\, z = \sin 2x \cos y$ (8)

Or

(b) (i) Find the singular integral of $z = px + qy + \sqrt{1 + p^2 + q^2}$ (8)

(ii) Solve: $(D^2 + 2DD' + D'^2 - 2D - 2D')\, z = e^{2x-y} + 3$ (8)

14. (a) A square plate is bounded by the lines $x = 0, y = 0, x = 20$ and $y = 20$. Its faces are insulated. The temperature along the edge $y = 20$ is given by $x(20 - x)$ while the other three edges are kept at 0°C. Find the steady-state temperature distribution on the plate. (16)

Or

(b) A string is stretched tightly between $x = 0$ and $x = 10$ and is fastened at both ends. At time $t = 0$, the string is given a shape defined by $f(x) = kx(10 - x)$ and then released. Find the displacement of any point x of the string at any time t.

15. (a) (i) Find the Z-transform of $\left\{\frac{1}{n!}\right\}$ and $\left\{\frac{1}{(n+1)!}\right\}$. (6)

(ii) Find the inverse Z-transform of $\frac{z^3 + 3z}{(z-1)^2 (z^2+1)}$. (10)

Or

(b) (i) Using convolution theorem find the inverse Z-transform of $\frac{14z^2}{(7z-1)(2z-1)}$. (8)

(ii) Solve the difference equation by using Z-transform $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$ given that $u_0 = u_1 = 1$. (8)

SOLUTIONS

PART-A

1. Dirichlet's conditions given in 2.2 in page 2.2 of the book

2. $a_1 = \frac{1}{2}\int_{-2}^{2} (3x - 4x^3) \cos \frac{\pi x}{2} dx = 0$, since the integrand is an odd function of x.

3. Statement of Fourier integral theorem given in 4.2 in page 4.1 of the book.

4. $F_c\,[f(x)\cos ax] = \int_0^\infty f(x) \cos ax \cos sx \, dx$

$$= \frac{1}{2}\int_0^\infty f(x)\{\cos(s+a)x + \cos(s-a)x\}dx = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

5. $z = (x + a)^3 + (y - b)^2$ (1); $p = 3(x + a)^2$ (2); $q = 2(y - b)$ (3). Using (2) and (3) in (1), we get $z = \left(\frac{p}{3}\right)^{3/2} + \left(\frac{q}{2}\right)^2$, which is the required P.D.E.

6. $p - 3q = 6$. If $z = ax + by + c$ is the required C.S., $p = a$; $q = b$ Then $a - 3b = 6$; $a = 3b + 6$.

   $\therefore$ The required C.S. is $z = (3b + 6)\,x + by + c$, where b and c are arbitrary constants.

7. The possible solutins of $U_{xx} + U_{yy} = 0$ are
   (i) $U(x, y) = (Ae^{px} + Be^{-px})\,(C \cos py + D \sin py)$

   (ii) $U(x, y) = (A \cos px + B \sin px)\,(Ce^{py} + De^{-py})$ and

   (iii) $U(x, y) = (Ax + B)\,(Cy + D)$

8. S.S.T. distribution in the rod is given by $\frac{d^2u}{dx^2} = 0$ or $u(x) = Ax + B$;

   Using $u(0) = 30$, we get $B = 30$.

   Using $u(20) = 90$, we get $20A + 30 = 90$ $\therefore$ $A = 3$

   $\therefore$ The required S.S.T. distribution is $u(x) = 3x + 30$

9. The unilateral Z-transform of $\{f(n)\}$ is defined as

   $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$. The bilateral Z-transform of $\{f(n)\}$ is defined

   as $Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$

10.

$y_n = a - b.3^n$ (1)	$y_n - y_{n+1} = 2 \times 3^n b$ (4)	$\frac{y_{n+1} - y_{n+2}}{y_n - y_{n+1}} = 3$ (6)
$y_{n+1} = a - b.3^{n+1}$ (2)	$y_{n+1} - y_{n+2} = 2 \times 3^{n+1} b$ (5)	
$y_{n+2} = a - b.3^{n+2}$ (3)		

   i.e., $y_{n+1} - y_{n+2} = 3y_n - 3y_n - 3y_{n+1}$ (or) $y_{n+2} - 4y_{n+1} + 3y_n = 0$, which is the required difference equation.

PART B

11. (a) (i) (This problem is the same as the worked example (18) in page 2.35 of the book. In the worked example, we have to put $l = 2$).

   (ii) (***Note***: There is an error in the question. The 5th values of t must have been $(4T/6)$ or $(2T/3)$. Correcting the entry with $(2T/3)$, the solution is given below)

As the interval (0, T) is divided into sub-intervals of length (T/6), we take only 6 values of y, i.e., $y_0, y_1, y_2, y_3, y_4, y_5$ corresponding to $t = 0$, (T/6), . . . (5T/6).

Since $2l = T$, the required Fourier series is the form

$$y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi t}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi t}{T}$$

$$a_0 = 2 \times \frac{1}{6}\,\Sigma\, y_n = \frac{1}{3} \times 4.5 = 1.5$$

Since $a_1 = \frac{1}{3}\Sigma y_n \cos\frac{2\pi t_r}{T}$ and $b_1 = \frac{1}{3}\Sigma y_n \sin\frac{2\pi t_n}{T}$, we use a Harrison circle for $\frac{\pi}{3}$.

$$\therefore\ a_1 = \frac{1}{3}\left[(y_0 - y_3) + (y_1 + y_5 - y_2 - y_4)\cos 60°\right]$$

$$= \frac{1}{3}\left[0.68 + 0.88 \times 0.5\right] = 0.3733$$

$$b_1 = \frac{1}{3}[(y_1 + y_2 - y_4 - y_5)\sin 60° = \frac{1}{3}[3.48 \times 0.866] = 1.0046$$

Since $a_2 = \frac{1}{3}\,\Sigma y_n \cos\frac{4\pi t_n}{T}$ and $b_2 = \frac{1}{3}\Sigma y_n \sin\frac{4\pi t_{nr}}{T}$, we use a H'circle for angle 120°

$$a_2 = \frac{1}{3}\left[(y_0 + y_3) - (y_1 + y_4 + y_2 + y_5)\cos 60°\right]$$

$$= \frac{1}{3}\left[3.28 - 1.22 \times 0.5\right] = 0.89$$

$$b_2 = \frac{1}{3}[(y_1 + y_4 - y_2 - y_5)\sin 60°] = \frac{1}{3}[-0.38 \times 0.866] = -\,0.1097$$

$$\therefore\ f(x) \sim 0.75 + \left(0.3733 \cos\frac{2\pi t}{T} + 1.0046 \sin\frac{2\pi t}{T}\right)$$
$$+ \left(0.89 \cos\frac{4\pi t}{T} - 0.1097 \sin\frac{4\pi t}{T}\right)$$

11. (b) (i) The complex form of the Fourier series of e^{-ax} in $(-l, l)$ is worked out in the worked example 8 in page 2.87 of the book. If we put $a = 1$ and $l = 1$, we have the solution for this problem)

(ii) Let the half range cosine series of $f(x)$ be $f(x) = \frac{a_0}{2} + \sum a_n \cos nx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx dx = \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} = \frac{2}{n^2\pi}\{(-1)^n - 1\}$$

$$= \begin{cases} -4/n^2\pi, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x dx = \pi \quad \therefore \quad x = \frac{\pi}{2} - \frac{4}{\pi^2}\sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2}\cos nx \sin(0,\pi)$$

By Parseval's theorem, $\frac{1}{4}a_0^2 + \frac{1}{2}\Sigma a_n^2 = \frac{1}{\pi}\int_0^{\pi} x^2 dx$

i.e., $\frac{\pi^2}{4} + \frac{1}{2}\sum_{n=1,3,5\ldots}^{\infty} \frac{16}{n^4\pi^2} = \frac{\pi^2}{3}$

i.e., $\frac{8}{\pi^2}\sum_{n=1,3\ldots}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{12} \qquad \therefore \sum_{n=1,3\ldots}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{96}$

12. (a) (i) $F\{e^{-a^2x^2}\} = \int_{-\infty}^{\infty} e^{-a^2x^2} \cdot e^{-isx} dx = \int_{-\infty}^{\infty} e^{-\left\{ax + \frac{is}{2a}\right\}^2} \cdot e^{-\frac{s^2}{4a^2}} dx$

$$= e^{-\frac{s^2}{4a^2}} \cdot \frac{1}{a}\int_{-\infty}^{\infty} e^{-t^2} dt, \left(\text{on putting } ax + \frac{is}{2a} = t\right) = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$$

i.e., $\int_{-\infty}^{\infty} e^{-a^2x^2} (\cos sx - i \sin sx) dx = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$

Equating R.P.'s $\int_{-\infty}^{\infty} e^{-a^2x^2} \cos sx\, dx = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$

i.e., $F_c\{e^{-a^2x^2}\} = \frac{\sqrt{\pi}}{2a} e^{-s^2/4a^2}$

Putting $a = \frac{1}{\sqrt{2}}$, we get, $F_c\left(e^{-\frac{x^2}{2}}\right) = \sqrt{\frac{\pi}{2}} e^{-s^2/2} \left\{= \overline{f}_c(s)\right\}$

Since $F_s\left\{xe^{-\frac{x^2}{2}}\right\} = -\frac{d}{d_s}\left(\sqrt{\frac{\pi}{2}} \cdot e^{-\frac{s^2}{2}}\right) = \sqrt{\frac{\pi}{2}}\, e^{-s^2/2} s$

(ii) $F_s\,(e^{-ax}) = \int_0^{\infty} e^{-ax} \sin sx\, dx = \left[\frac{e^{-ax}}{s^2 + a^2}(-a \sin - sx \cos sx\right]_0^{\infty}$

$$= \frac{s}{s^2 + a^2}$$

$\therefore\ F_s^{-1}\left(\frac{s}{s^2 + a^2}\right) = e^{-ax}$

$$\text{i.e., } \frac{2}{\pi}\int_0^{\infty}\frac{s}{s^2+a^2}\sin sx\,ds = e^{-ax} \text{ (OR) } \int_0^{\infty}\frac{s\sin xs}{s^2+a^2}\,ds = \frac{\pi}{2}e^{-ax}$$

12. (b) $F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx = \int_{-2}^{2}\{2-|x|\}\,(\cos sx - i\sin sx)\,dx$

$$= 2\int_0^2 (2-x)\cos sx\,dx = 2\left[(2-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_0^2$$

$$= 2\left(\frac{1-\cos 2s}{s^2}\right) = 4\,\frac{\sin^2 s}{s^2} \qquad (1)$$

$$\therefore\; F^{-1}\left\{4\left(\frac{\sin s}{s}\right)^2\right\} = 2-|x| \text{ in } |x|<2$$

$$\text{i.e., } \frac{4}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s}{s}\right)^2 e^{ixs}\,ds = 2-|x| \text{ in } |x|<2$$

Putting $x = 0$, $\int_{-\infty}^{\infty}\left(\frac{\sin s}{s}\right)^2 ds = \pi$ or $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$

By Parseval's identity, from step (1);

$$\int_{-2}^{2}\{2-|x|\}^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{16\sin^4 s}{s^4}\,ds$$

$$\text{i.e., } \frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin s}{s}\right)^4 ds = 2\int_0^2 (2-x)^2\,dx = \frac{2}{-3}\left[(2-x)^3\right]_0^2 = \frac{16}{3}$$

$$\therefore\; \int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$$

13. (a) (i) L.S.S.E's are $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$

Each ratio $= \dfrac{dx+dy+dz}{0}$ $\quad\therefore\; dx + dy\; dz = 0$

i.e., $x+y+z=c_1$

Also each ratio $= \dfrac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$

$\therefore\ \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$

Solving, $xyz=c_2$

$\therefore$ Gen. solution of the given equation is $f(x+y+z, x\,y\,z)=0$

(ii) $(D^3+D^2D'-D'^2)z=\sin 2x\cos y$

A.E. is $m^3+m^2-m-1=0$ i.e. $(m+1)(m^2-1)=0$

i.e. $m=-1,-1,1$

$\therefore$ C.F. $=f_2(y-x)+xf_2(y-x)+f_3(y+x)$

$$\text{P.I.} = \frac{1}{(D+D')(D^2-D'^2)}\frac{1}{2}\{\sin(2x+y)+\sin(2x-y)\}$$

$$=\frac{1}{2}\cdot\frac{1}{D+D'}\frac{1}{-4+1}\{\sin(2x+y)+\sin(2x-y)\}$$

$$=-\frac{1}{6}\cdot\frac{(D-D')}{D^2-D'^2}\{\sin(2x+y)+\sin(2x-y)\}$$

$$=\frac{1}{18}\{\cos(2x+y)+3\cos(2x+y)\}$$

$\therefore$ G.S. is z = C.F. + P.I.

13. (b) (i) (This is the same as the worked example (9) in page 1.35 in the book. We have to simply put $C=1$)

$\therefore$ The required singular solution is $x^2+y^2+z^2=1$

(ii) The given equation is $\{(D+D')^2-2(D+D')\}\,z=e^{2x-y}+3$

i.e., $(D+D')(D+D'-2)\,z=e^{2x-y}+3$

$\therefore$ C.F. $=\phi_1(y-x)+e^{2x}\,\phi_2(y-x)$

$$\text{P.I}_1=\frac{1}{(D+D')(D+D'-2)}e^{2x-y}=\frac{1}{1\cdot(-1)}e^{2x-y}=-e^{2x-y}$$

$$\text{P.I}_2=\frac{1}{(D+D')(D+D'-2)}3e^{0x+0y}=3\cdot\frac{1}{D+D'}\frac{1}{(-2)} \qquad (1)$$

$$= -\frac{3}{2}\frac{1}{D+D'}(1) = -\frac{3}{2}x$$

$\therefore$ The G.S. is z = C.F. + P.I + P.I$_2$

14. (a) The S.S.T $u(x, y)$ at any point of the plate is given by $u_{xx} + u_{yy} = 0$ (1). We have to solve (1), satisfying the B.C.'s (2) $u(0, y) = 0$, (3) $u(20, y) = 0$, (4) $u(x, 0) = 0$ and (5) $u(x, 20) = x(20 - x)$. Since the non-zero temp is prescribed along $y = 20$ in which x is varying the proper solution of (1) is

$u(x, y) = (A \cos px + B \sin px)(Ce^{py} + De^{-py})$ (6)

Using B.C. (2) in (6), we get $A = 0$

Using B.C (3) in (6), we get $p = \dfrac{n\pi}{20}$ $(n = 0, 1, 2, \ldots \infty)$

Using B.C (4) in (6), we get $D = -C$

Using these values in (6), reduces to

$$u(x, y) = 2\text{BC} \sin\frac{n\pi x}{20}\left(\frac{e^{n\pi y/20} - e^{-n\pi y/2}}{2}\right)$$

$$= \lambda \sin\frac{n\pi x}{20}\sinh\frac{n\pi y}{20}, \text{ where } n = 0, 1, 2, \ldots \infty$$

$\therefore$ The most general form of the solution (1) is

$$u(x, y) = \sum_{n=1}^{\infty}\lambda_n \sin\frac{n\pi x}{20}\sinh\frac{n\pi y}{20} \qquad (1)$$

Using B.C. (5) in (7), we have

$$\sum_{n=1}^{\infty}(\lambda_n \sinh n\pi)\sin\frac{n\pi x}{20} = x(20 - x) \text{ in } 0 \le x \le 20$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{20} \text{ (Fourier half range series)}$$

$$\therefore \lambda_n \sinh n\pi = b_n = \frac{2}{20}\int_0^{20}(20x - x^2)\sin\frac{n\pi x}{20}\,dx$$

$$= \frac{1}{10}\left[(20x - x^2)\left(\frac{-\cos\frac{n\pi x}{20}}{\frac{n\pi}{20}}\right) - (20 - 2x)\left(\frac{-\sin\frac{n\pi x}{20}}{n^2\pi^2/20^2}\right) + (-2)\frac{\cos\frac{n\pi x}{20}}{n^3\pi^3/20^3}\right]_0^{20}$$

$$= -\frac{2}{10} \times \frac{20^3}{n^3 \pi^3} \{(-1)^n - 1\} = \begin{cases} \dfrac{3200}{n^3 \pi^3}, \text{ if } n \text{ is odd} \\ 0, \text{ if } n \text{ is even} \end{cases}$$

Using this value of λ_n in (7), the required solution is

$$u(x, y) = \frac{3200}{\pi^3} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \text{ cosech } n\pi. \sin \frac{n\pi x}{20} \sinh \frac{n\pi y}{20}$$

14. (b) (This problem is the worked example 1(ii) in page 3.10 of the book. We have to put $l = 10$)

15. (a) (i) $Z\left(\dfrac{1}{n!}\right) = \displaystyle\sum_{n=0}^{\infty} \frac{z^{-n}}{\lfloor n} = 1 + \frac{z^{-1}}{\lfloor 1} + \frac{z^{-2}}{\lfloor 2} + \cdots \infty = e^{1/z}$

$$Z\left(\frac{1}{\lfloor n+1}\right) = \sum_{n=0}^{\infty} \frac{z^{-n}}{\lfloor n+1} = \frac{1}{\lfloor 1} + \frac{z^{-1}}{\lfloor 2} + \frac{z^{-2}}{\lfloor 3} + \cdots \infty = z\left[\frac{z^{-1}}{\lfloor 1} + \frac{z^{-2}}{\lfloor 2} + \frac{z^{-3}}{\lfloor 3} + \cdots \infty\right]$$

$$= y\,(e^{1/z} - 1)$$

$$= z\,(e^{1/z} - 1)$$

(ii) $\dfrac{\bar{f}(z)}{z} = \dfrac{z+3}{(z-1)^2\,(z^2+1)} = \dfrac{A}{z-1} + \dfrac{B}{(z-1)^2} + \dfrac{Cz+D}{z^2+1}$

$$= \frac{-3/2}{z-1} + \frac{2}{(z-1)^2} + \frac{3/2\,z}{z^2+1} - \frac{1/2}{z^2+1}$$

$$\therefore\ \bar{f}(z) = -\frac{3}{2} \cdot \frac{z}{z-1} + 2 \cdot \frac{z}{(z-1)^2} + \frac{3}{2} \cdot \frac{z^2}{z^2+1} - \frac{1}{2} \cdot \frac{z}{z^2+1}$$

Taking inverses

$$f(n) = -\frac{3}{2} + 2n + \frac{3}{2} \cos \frac{n\pi}{2} - \frac{1}{2} \sin \frac{n\pi}{2}$$

15. (b) (i) $\bar{f}(z) = \dfrac{14 z^2}{(7z-1)(2z-1)} = \dfrac{z}{z - \frac{1}{7}} \cdot \dfrac{z}{z - \frac{1}{2}}$

$$\therefore\ Z^{-1}\{\bar{f}(z)\} = Z^{-1}\left\{\frac{z}{z - \frac{1}{7}}\right\} * Z^{-1}\left\{\frac{z}{z - \frac{1}{2}}\right\}$$

$$= \left(\frac{1}{7}\right)^n * \left(\frac{1}{2}\right)^n = \sum_{r=0}^{n} \left(\frac{1}{7}\right)^{n-r} \cdot \left(\frac{1}{2}\right)^r = \frac{1}{7^n} \sum_{r=0}^{n} \left(\frac{7}{2}\right)^n$$

$$= \frac{1}{7^n} \left\{ \frac{\left(7/2\right)^{n+1} - 1}{7/2 - 1} \right\} = \frac{2}{5} \left\{ \frac{7}{2^{n+1}} - \frac{1}{7^n} \right\} = \frac{1}{5} \left\{ \frac{7}{2^n} - \frac{2}{7^n} \right\}$$

(ii) $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$

Taking z-transforms, we get

$$[z^2\, \bar{u}(z) - z^2\, u(0) - zu(1)] + 6\, [z\, \bar{u}(z) - zu(0)] + 9\bar{u}(z) = \frac{z}{z-2}$$

$$\therefore\ \frac{\bar{u}(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{1/25}{z-2} - \frac{1/25}{z+3} - \frac{1/5}{(z+3)^2}$$

$$\therefore\ \bar{u}(z) = \frac{1}{25} \cdot \frac{z}{z-2} - \frac{1}{25} \cdot \frac{z}{z+3} - \frac{1}{5} \cdot \frac{z}{(z+3)^2}$$

Inverting

$$u(n) = \frac{1}{25} \cdot 2^n - \frac{1}{25} \cdot (-3)^n - \frac{1}{5} \times -\frac{1}{3}\, n\, (-3)^n$$

$$= \frac{1}{25} \cdot 2^n - \frac{1}{25} \cdot (-3)^n + \frac{1}{15}\, n\, (-3)^n$$

Question Paper Code: **53185**

B.E./B.Tech. Degree Examinations, November/December 2010
Regulations 2008

Third Semester

Common to all branches

MA 2211 Transforms and Partial Differential Equations

Time: Three Hours **Maximum: 100 Marks**

Answer ALL Questions

Part A – (10 × 2 = 20 Marks)

1. Find the constant term in the expansion of $\cos^2 x$ as a Fourier series in the interval $(-\pi, \pi)$.
2. Find the root mean square value of $f(x) = x^2$ in (0, l).
3. Write the Fourier transform pair.
4. Find the Fourier sine transform of $f(x) = e^{-ax}$, $a > 0$.
5. Form the partial differential equation by eliminating the arbitrary function from $z^2 - xy = f\left(\frac{x}{z}\right)$.
6. Find the particular integral of $(D^2 - 2DD' + D'^2)\, z = e^{x-y}$.
7. Write down the three possible solutions of one dimensional heat equation.
8. Give three possible solutions of two dimensional steady state heat flow equation.
9. Define the unit step sequence. Write its Z-transform.
10. Form a difference equation by eliminating the arbitrary constant A from $y_n = A \cdot 3^n$.

Part B – (5 × 16 = 80 Marks)

11. (a) (i) Find the Fourier series expansion of $f(x) = \begin{cases} x & \text{for } 0 \le x \le \pi \\ 2\pi - x & \text{for } \pi \le x \le 2\pi \end{cases}$

Also, deduce that $= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = \frac{\pi^2}{8}$ (10)

(ii) Find the Fourier series expansion of $f(x) = 1 - x^2$ in the interval $(-1, 1)$. (6)

OR

11. (b) (i) Obtain the half range cosine series for $f(x) = x$ in $(0, \pi)$. (8)

(ii) Find the Fourier series as far as the second harmonic to represent the function $f(x)$ with period 6, given in the following table.

x	0	1	2	3	4	5
$f(x)$	9	18	24	28	26	20

(8)

12. (a) (i) Derive the Parseval's identity for Fourier Transforms. (8)

(ii) Find the Fourier integral representation of $f(x)$ defined as

$$f(x) = \begin{cases} 0 \text{ for } x < 0 \\ \dfrac{1}{2} \text{ for } x = 0 \\ e^{-x} \text{for } x > 0 \end{cases} \quad (8)$$

OR

12. (b) (i) Find the Fourier sine transform of

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x & 1 < x < 2 \\ 0, & x > 2 \end{cases} \quad (8)$$

(ii) Evaluate $\int\limits_0^\infty \dfrac{dx}{(x^2 + a^2)(x^2 + b^2)}$ using Fourier cosine transforms of e^{-ax} and e^{-bx}. (8)

13. (a) (i) Form the PDE by eliminating the arbitrary function ϕ from $\phi(x^2 + y^2 + z^2, ax + by + cz) = 0$. (8)

(ii) Solve the partial differential equation $x^2 (y - z) p + y^2 (z - x) q = z^2 (x - y)$. (8)

OR

13. (b) (i) Solve the equation $[D^3 + D^2D' - 4DD'^2 - 4D'^3]z = \cos(2x + y)$. (8)

(ii) Solve $[2D^2 - DD' - D'^2 + 6D + 3D']z = xe^y$. (8)

14. (a) A tightly stretched string of length $2l$ is fastened at both ends. The midpoint of the string is displaced by a distance 'b' transversely and the string is released from rest in this position. Find an expression for the transverse displacement of the string at any time during the subsequent motion. (16)

OR

14. (b) A square plate is bounded by the lines $x = 0, y = 0, x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by

$$u(x, 20) = x(20 - x),\ 0 < x < 20$$

while the other two edges are kept at 0°C. Find the steady state temperature distribution in the plate. (16)

15. (a) (i) Find the Z-transform of $\cos n\theta$ and $\sin n\theta$. Hence deduce the Z-transforms of $\cos (n + 1)\theta$ and $a^n \sin n\theta$. (10)

(ii) Find the inverse Z-transform of $\dfrac{z(z+1)}{(z-1)^3}$ by residue method. (6)

OR

15. (b) (i) Form the difference equation from the relation $y_n = a + b \cdot 3^n$. (8)

(ii) Solve $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0$ and $y_1 = 1$, using Z-transform. (8)

Solutions

Part – A

1. $$a_0 = \frac{2}{\pi}\int_0^{\pi} \cos^2 x \, dx = \frac{1}{\pi}\int_0^{\pi} (1+\cos 2x)\, dx = \frac{1}{\pi}\left\{x + \frac{\sin 2x}{2}\right\}_0^{\pi} = 1$$

$\therefore$ constant term in the F.S. expansion $= \dfrac{1}{2}$.

2. $$\bar{f}(x) = \sqrt{\frac{1}{l}\int_0^{l} x^4 \, dx} = \sqrt{\frac{1}{l} \times \frac{l^5}{5}} = \frac{l^2}{\sqrt{5}}.$$

3. $$F\{f(x)\} = \bar{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx \text{ and}$$

$$F^{-1}\left\{\bar{f}(s)\right\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\; e^{ixs}\, ds$$

4. $$F_s(e^{-ax}) = \int_0^{\infty} e^{-ax} \sin sx \, dx = \left[\frac{e^{-ax}}{s^2 + a^2}(-a \sin sx - s \cos sx\right]_0^{\infty}$$

$$= \frac{s}{s^2 + a^2}$$

5. $$z^2 - xy = f\left(\frac{x}{z}\right) \tag{1}$$

Differentiating (1) partially w.r.t. x,

$$2zp - y = f'\left(\frac{x}{z}\right) \cdot \left\{\frac{z - xp}{z^2}\right\} \tag{2}$$

Differentiating (1) partially w.r.t. y;

$$2zq - x = f'\left(\frac{x}{z}\right) \cdot \left\{-\frac{x}{z^2} \cdot q\right\} \tag{3}$$

(2) ÷ (3) gives $\dfrac{2zp - y}{2zq - x} = \dfrac{z - xp}{-xq}$

i.e., $-2pqxz + xyq = 2z^2q - 2pqxz - xz + x^2p$
i.e., $x^2p + (2z^2 - xy)q = xz$

6. P.I. $\dfrac{1}{(D-D')^2}\, e^{x-y} = \dfrac{1}{4}\, e^{x-y}$

7. Three possible solutions of the equations $u_t = \alpha^2 u_{xx}$ are
   (i) $u(x, t) = (Ae^{px} + Be^{-px})\, e^{p^2 \alpha^2 t}$
   (ii) $u(x, t) = (A \cos px + B \sin px)\, e^{-p^2 \alpha^2 t}$ and
   (iii) $u(x, t) = Ax + B$, where A, B, p are arbitrary constants

8. Three possible solutions of the equation $u_{xx} + u_{yy} = 0$ are
   (i) $u(x, y) = (Ae^{px} + Be^{-px})\,(C \cos py + D \sin py)$
   (ii) $u(x, y) = (A \cos px + B \sin px)\,(Ce^{py} + De^{-py})$
   (iii) $u(x, y) = (Ax + B)\,(Cy + D)$, where A, B, C, D, p are arbitrary constants

9. The unit step sequence u_n is defined as $u_n = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$

   $$Z\{u_n\} = \sum_{n=0}^{\infty} 1 \cdot z^{\,n} = \frac{1}{1-\frac{1}{z}} \text{ (or) } \frac{z}{z-1}$$

10. $y_n = A \cdot 3^n$ ...(1); $y_{n+1} = 3A \cdot 3^n$ ...(2)

    (2) ÷ (1) gives $\dfrac{y_n + 1}{y_n} = 3$ (or) $y_{n+1} - 3y_n = 0.$

Part – B

11. (a) (i) Let the F.S. of $f(x)$ be $\dfrac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$, in $(0, 2\pi)$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} x \cos nx\, dx + \int_{\pi}^{2\pi} (2\pi - x) \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right\}_0^{\pi} + \left\{(2\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}\left\{(-1)^n - 1\right\} - \frac{1}{n^2}\left\{1 - (-1)^n\right\}\right] = \frac{2}{\pi n^2}\left\{(-1)^n - 1\right\}$$

$$= \begin{cases} 0, \text{ if } n \text{ is even} \\ \dfrac{-4}{\pi n^2}, \text{ if } n \text{ is odd} \end{cases}$$

$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi} x\, dx + \int_{\pi}^{2\pi} (2\pi - x)\, dx\right] = \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} - \left\{\frac{(2\pi - x)^2}{2}\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right] = \pi$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} x \sin nx\, dx + \int_{\pi}^{2\pi} (2\pi - x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\left(\frac{-\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right\}_0^{\pi} + \left\{(2\pi - x)\left(\frac{-\cos nx}{n}\right) - \frac{\sin nx}{n^2}\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}(-1)^n + \frac{\pi}{n}(-1)^n\right] = 0$$

$\therefore$ F.S. of $f(x) \sim \dfrac{\pi}{2} - \dfrac{4}{\pi}\displaystyle\sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2}\cos nx$ in $(0, 2\pi)$

Putting $x = 0$, that is a point of continuity of $f(x)$, we get

$$\frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\} = 0$$

$$\therefore \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty = \frac{\pi^2}{8}$$

11. (a) (ii) $f(x) = 1 - x^2$ is an even function in $(-1, 1)$.

$\therefore$ F.S. of $f(x)$ will be of period 2 and will not contain sine terms.

Let the F.S. of $f(x)$ be $\dfrac{a_0}{2} + \sum a_n \cos n\pi x$

$$a_n = 2\int_0^1 (1 - x^2)\cos n\pi x\, dx = 2\left[(1 - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) - 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2 \times \left(\frac{-2}{n^2\pi^2}\right)(-1)^n \text{ or } \frac{-4(-1)^n}{n^2\pi^2}$$

$$a_0 = 2\int_0^1 (1 - x^2)\, dx = 2\left(x - \frac{x^3}{3}\right)_0^1 = \frac{4}{3}.$$

$\therefore$ Required F.S. is $\dfrac{2}{3} - \dfrac{4}{\pi^2}\sum \dfrac{(\ 1)^n}{n^2}\cos n\pi x$ in $(-1, 1)$

11. (b) (i) Given an even extension for $f(x)$ in $(-\pi, 0)$.

i.e. put $f(x) = -x$ in $(-\pi, 0)$

Then $f(x)$ is even in $(-\pi, \pi)$

F.S. of $f(x) \sim \dfrac{a_0}{2} + \sum a_n \cos nx$ in $(-\pi, \pi)$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

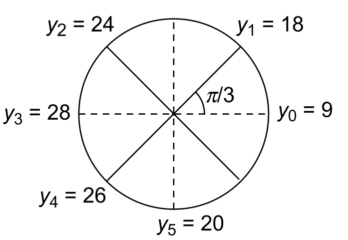
$$= \frac{2}{\pi n^2}\left\{(-1)^n - 1\right\} = \begin{cases} 0, \text{if } n \text{ even} \\ \dfrac{-4}{n^2\pi^2}, \text{if } n \text{ odd} \end{cases}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx\left(\frac{1}{\pi}[x^2]_0^{\pi}\right) = \pi$$

$\therefore$ The required Fourier half-range cosine series of

x is $\dfrac{\pi}{2} - \dfrac{4}{\pi^2}\sum\limits_{n=1,3,5\ldots}^{\infty}\dfrac{1}{n^2}\cos nx$ in $(0, \pi)$

11. (b) (ii) $x : x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5$
$y = f(x) : y_0 = 9, y_1 = 18, y_2 = 24, y_3 = 28, y_4 = 26, y_5 = 20$
$2l = 6 \therefore l = 3$



Let the required F.S. be $y = \dfrac{a_0}{2} + \left(a_1\cos\dfrac{\pi x}{3} + b_1\sin\dfrac{\pi x}{3}\right) + \left(a_2\cos\dfrac{2\pi x}{3} + b_2\sin\dfrac{2\pi x}{3}\right)$

$$a_0 = 2\times\frac{1}{6}\sum_{r=0}^{5} y_r = \frac{1}{3}\times 125 = 41.67$$

$$a_1 = 2\times\frac{1}{6}\sum_{r=0}^{5} y_r\cos\frac{\pi x_r}{3}$$

$$= \frac{1}{3}\left[(9-28) + (18+20-24-26)\cos\frac{\pi}{3}\right]$$

$$= \frac{1}{3}(-19 - 12\times 0.5) = -8.33$$

$$b_1 = 2\times\frac{1}{6}\sum_{r=0}^{5} y_n\sin\frac{\pi x_r}{3} = \frac{1}{3}\left[(18+24-26-20)\times\sin\frac{\pi}{3}\right]$$

$$= \frac{1}{3}\times -4\times 0.866 = -1.15$$

$$a_2 = \frac{1}{3}\sum_{r=0}^{5} y_r \cos\frac{2\pi x_r}{3}$$

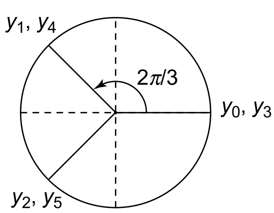
$$= \frac{1}{3}\left[(9+28)-(18+26+24+20)\cos\frac{\pi}{3}\right]$$

$$= \frac{1}{3}[37 - 88 \times 0.5] = -2.33$$

$$b_2 = \frac{1}{3}\sum_{r=0}^{5} y_r \sin\frac{2\pi x_r}{3}$$

$$= \frac{1}{3}\left[(18+26-24-20)\sin\frac{\pi}{3}\right] = 0$$

The required F.S. is $20.84 - \left(8.33\cos\frac{\pi x}{3} + 1.15\sin\frac{\pi x}{3}\right) - 2.33$ cos $\frac{2\pi x}{3}$.



12. (a) (i) (The derivation is available in Q.10 in page 4.32 of the book.)

12. (a) (ii) (This problem is the worked example (1) in page 4.8 of the book.)

12. (b) (i) $F_s\{f(x)\} = \int_0^\infty f(x)\sin sx\,dx = \int_0^1 x\sin sx\,dx + \int_1^2 (2-x)\sin sx\,dx$

$$= \left\{x\left(\frac{-\cos sx}{s}\right) + \frac{\sin sx}{s^2}\right\}_0^1 + \left\{(2-x)\left(\frac{-\cos sx}{s}\right) + \frac{-\sin sx}{s^2}\right\}_1^2$$

$$= \left(-\frac{\cos s}{s} + \frac{\sin s}{s^2}\right) + \left\{-\frac{\sin 2s}{s^2} + \frac{\cos s}{s} + \frac{\sin s}{s^2}\right\}$$

$$= \frac{1}{s^2}(2\sin s - \sin 2s) \text{ (or) } \frac{2\sin s\,(1-\cos s)}{s^2}$$

12. (b) (ii) Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$

Then $\overline{f_c}(s) = \frac{a}{s^2 + a^2}$ and $\overline{g}_c(s) = \frac{b}{s^2 + b^2}$

$$\int_0^\infty f(x)\, g(x)\, dx = \frac{2}{\pi}\int_0^\infty \bar{f}_c(x)\cdot \bar{g}_c(s)\, ds$$

$$\therefore \int_0^\infty e^{-(a+b)x}\, dx = \frac{2}{\pi}\int_0^\infty \frac{ab}{(s^2+a^2)(s^2+b^2)}\, ds$$

$$\therefore \int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2ab}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = \frac{\pi}{2ab(a+b)}$$

13. (a) (i) This problem is the same as the worked example 11-(ii) given in page 1.12 of the book]

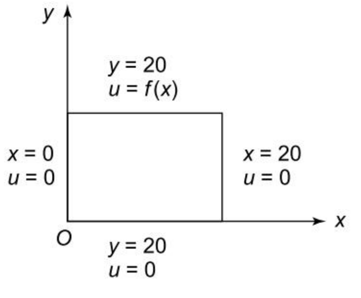
13· (a) (ii) This problem is the worked example (14) given in page 1.62 of the book]

13. (b) (i) This problem is the worked example (12) given in page 1.84 of the book]

13. (b) (ii) This problem is the worked example (21) given in page 1.91 of the book. In the R.H.S. of the worked example, 2 terms, namely $xe^y + ye^x$ are given. The part of the P.I. corresponding to xe^y alone should be included in the required G.S.]

14. (a) This is the worked example (2) given in page 3.13 of the book.]

14. (b) The S.S.T. $u(x, y)$ is given by $u_{xx} + u_{yy} = 0$ (1)



We have to solve (1) satisfying the following boundary conditions:

$$u(0, y) = 0,\ 0 \le y \le 20 \quad (2)$$

$$u(20, y) = 0,\ 0 \le y \le 20 \quad (3)$$

$$u(x, 0) = 0,\ 0 \le x \le 20 \quad (4)$$

$$u(x, 20) = x(20 - x),\ 0 \le x \le 20 \quad (5)$$

Since non-zero temperature is prescribed on the edge $y = 20$, the proper solution of (1) is

$$u(x, y) = (A \cos px + B \sin px)\,(C \cosh py + D \sinh py) \quad (6)$$

Using (2) in (6); $A = 0$

Using (3) in (6); $B \sin 20\, p = 0$

$$B \neq 0 \quad \therefore p = \frac{n\pi}{20} \;\; (n = 0, 1, 2, \ldots \infty)$$

Using (4) in (6); $C = 0$

$\therefore$ The required solution becomes

$$u(x, y) = \lambda \sin \frac{n\pi x}{20} \sin \frac{n\pi y}{20}, \; n = 0, 1, 2, \ldots \infty$$

$\therefore$ The general solution is $u(x, y) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{20} \sin h \frac{n\pi y}{20}$ (7)

Using (5) in (7), we have

$$\sum_{n=1}^{\infty} (\lambda_n \sinh n\pi) \sin \frac{n\pi x}{20} = x(20 - x) \text{ in } 0 \leq x \leq 20$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} \;\; \text{(Fourier H.R. series)}$$

$\therefore \lambda_n \sinh n\pi = b_n$

$$= \frac{2}{20} \int_0^{20} (20x - x^2) \sin \frac{n\pi x}{20} dx$$

$$= \frac{1}{10}\left[(20x - x^2)\left(\frac{-\cos \frac{n\pi x}{20}}{\frac{n\pi}{20}}\right) - (20 - 2x)\left(\frac{-\sin \frac{n\pi x}{20}}{\frac{n^2\pi^2}{20^2}}\right) + (-2)\left(\frac{\cos \frac{n\pi x}{20}}{\frac{n^3\pi^3}{20^3}}\right)\right]_0^{20}$$

$$= \frac{-2}{10} \times \frac{20^3}{n^3\pi^3} [(-1)^n - 1]$$

$$= \begin{cases} \frac{3200}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \lambda_n = \begin{cases} \frac{3200}{n^3\pi^3} & \text{cosech } n\pi, \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$\therefore$ The final solution is

$$u(x, y) = \frac{3200}{\pi^3} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^3} \text{ cosech } n\pi \sin \frac{n\pi x}{20} \sinh \frac{n\pi y}{20}$$

15. (a) (i) Consider $Z(e^{in\theta}) = Z\{(e^{i\theta})^n\} = \dfrac{z}{z - e^{i\theta}}$

i.e., $Z(\cos n\theta + i \sin n\theta) = \dfrac{z\{(z - \cos\theta) + i\sin\theta\}}{(z - \cos\theta)^2 + \sin^2\theta}$

$$= \frac{z(z - \cos\theta)}{z^2 - (2\cos\theta)z + 1} + i\,\frac{z\sin\theta}{z^2 - (2\cos\theta)z + 1}$$

Equating the R.P.'s; $Z(\cos n\theta) = \dfrac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$

Equating the I.P.'s; $Z(\sin n\theta) = \dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

Now $Z\{\cos(n+1)\theta] = \cos\theta\, Z(\cos n\theta) - \sin\theta\, Z(\sin n\theta)$

$$= \frac{z\cos\theta\,(z - \cos\theta) - z\sin^2\theta}{z^2 - 2z\cos\theta + 1} = \frac{z^2\cos\theta - z}{z^2 - 2z\cos\theta + 1}$$

$Z\{a^n f(n)\} = \bar{f}\left(\dfrac{z}{a}\right)$

$$\therefore Z\{a^n \sin n\theta\} = \left\{\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}\right\}_{y \to \frac{z}{a}}$$

$$= \frac{\frac{z}{a}\sin\theta}{\frac{z^2}{a^2} - \frac{2z}{a}\cos\theta + 1} = \frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$$

15. (a) (ii) $\bar{f}(z) = \dfrac{z(z+1)}{(z-1)^3}$

$f(n)$ = Sum of the residues of $\left[\bar{f}(z)\cdot z^{n-1}\right]$ at the isolated singularities.

$= \text{Res. of } \dfrac{z^n(z+1)}{(z-1)^3}$ at $z = 1$, which is a triple pole

$$= \frac{1}{2}D^2\left[\frac{z^n(z+1)}{(z-1)^3} \times (z-1)^3\right]_{z=1}$$

$$= \frac{1}{2}\left\{(n+1)nz^{n-1} + n(n-1)z^{n-2}\right\}_{z=1}$$

$$= \frac{1}{2}\{(n+1)n + n(n-1)\}$$

$= n^2$

15. (b) (i) $y_n = a + b \cdot 3^n$ (1); $y_{n+1} = a + 3b \cdot 3^n$ (2); $y_{n+2} = a + 9b \cdot 3^n$ (3); $y_{n+1} - y_n = 2b \cdot 3^n$ (4); $y_{n+2} - y_{n+1} = 6b \cdot 3^n$ (5);

From (4) and (5); $\dfrac{y_{n+2} - y_{n+1}}{y_{n+1} - y_n} = 3$ i.e., $y_{n+2} - 4y_{n+1} + 3y_n = 0$ is the required difference equation.

15. (b) (i) Taking Z-transforms of the given equation,

$[z^2 \ \bar{y}(z) - z^2 \ y(0) - z \cdot y(1)] + 4 \ [z \ \bar{y}(z) - z \cdot y \ (0)] + 3 \ \bar{y}(z) = \dfrac{z}{z-2}$

i.e., $(z^2 + 4z + 3) \ \bar{y} \ (z) = \dfrac{z}{z-2} + z$ or $\dfrac{z(z-1)}{z-2}$

$\therefore \ \bar{y}(z) = \dfrac{z(z-1)}{(z+3)(z+1)(z-2)}$

Now $\dfrac{\bar{y}(z)}{z} = -\dfrac{2}{5} \cdot \dfrac{1}{z+3} + \dfrac{1}{3} \cdot \dfrac{1}{z+1} + \dfrac{1}{15} \cdot \dfrac{1}{z-2}$

$\therefore \ \bar{y}(z) = -\dfrac{2}{5} \cdot \dfrac{z}{z+3} + \dfrac{1}{3} \cdot \dfrac{z}{z+1} + \dfrac{1}{15} \cdot \dfrac{z}{z-2}$

Inverting, the required solution is $y_n = -\dfrac{2}{5}(-3)^n + \dfrac{1}{3}(-1)^n + \dfrac{1}{15} \cdot 2^n$.

Question Paper Code: **K 406**

B.E./B.Tech. Degree Examinations, November/December 2010

Third Semester

Civil Engineering

MA 31—Transforms and Partial Differential Equations

Common to all branches

Regulations 2008

Time: Three Hours **Maximum: 100 Marks**

Answer ALL Questions

Part A – (10 × 2 = 20 Marks)

1. State Dirichlet's conditions for a function $f(x)$ to be represented as a Fourier series in an interval $(0, 2\pi)$.
2. Write down the form of the Fourier series of an odd function in $(-l, l)$ and the associated Euler's formulae for the Fourier coefficients.
3. State the Fourier integral theorem.
4. Let $F(s)$ denote the Fourier transform of $f(x)$. Show that $F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right)$.
5. Form the partial differential equation by eliminating the arbitrary function f from $z = f(xy)$.
6. Find the complete integral of $z = px + qy + \sqrt{pq}$.
7. What does a^2 represent in one dimensional heat flow equation $u_t = a^2 u_{xx}$?
8. The ends A and B of a rod of length 10 cm long have their temperature kept 20°C and 80°C. Find the steady state temperature distribution on the rod.
9. Find the Z-transform of n^2.
10. Form a difference equation by eliminating arbitrary constants from $y_n = A + B \cdot 2^n$.

Part B – (5 × 16 = 80 marks)

11. (a) (i) If $f(x) = \begin{cases} 1 - x, & -\pi \le x \le 0 \\ 1 + x, & 0 \le x \le \pi, \end{cases}$ find the Fourier series for $f(x)$ and hence deduce the value of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ (8)

(ii) Find the constant term and the coefficient of the first sine and cosine terms in the Fourier series of $y = f(x)$ which is defined by the following data in $(0, 2\pi)$: (8)

x:	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$:	1.98	1.30	1.05	1.30	–0.88	–0.25	1.98

Or

(b) (i) Find the Fourier series for the function $f(x) = |x|, -1 < x < 1$. Hence find the value of $1^{-2} + 3^{-2} + 5^{-2} + \ldots$ (8)

(ii) Find the half-range sine series for a function $f(x) = x(\pi - x)$, $0 < x < \pi$. Hence deduce $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \pi^3/32$. (8)

12. (a) Find the Fourier transform of $f(x) = \begin{cases} 1 - |x| & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ and also find the inverse transform. Hence deduce that

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2} \text{ and } \int_0^{\infty} \left(\frac{\sin x}{x}\right)^4 dx = \frac{\pi}{3}. \quad (16)$$

Or

(b) Find the Fourier sine and cosine transform of e^{-x}. Hence evaluate

(i) $\int_0^{\infty} \frac{x^2}{(x^2+1)^2} dx$ and

(ii) $\int_0^{\infty} \frac{x^2}{(x^2+1)^2} dx$. (16)

13. (a) (i) Find the general solution of $x(y - z)p + y(z - x)q - z(x - y) = 0$. (8)

(ii) Solve $(D^2 + DD' - 6D'^2)z = \cos(x + 2y)$. (8)

Or

(b) (i) Find the singular solution of $z = px + qy + p^2 - q^2$. (8)

(ii) Solve $(D^2 + 2DD' + D'^2)z = -x \sin y$. (8)

14. (a) A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ and then released it from this position at time $t = 0$. Find the displacement of the point of the string at a distance x from one end at any time t. (16)

Or

(b) A long rectangular plate with insulated surface is l cm wide. If the temperature along one short edge $y = 0$ is $u(x, 0) = k\left[\sin\left(\frac{\pi x}{l}\right) + 3\sin\left(\frac{3\pi x}{l}\right)\right]$ for $0 < x < 1$, while the two long edges

$x = 0$ and $x = l$ as well as the other short edge are kept of 0°C, find the steady state temperature function $u(x, y)$. (16)

15. (a) (i) Find the Z-transform of $(n + 1)^2$ and $\sin(3n + 5)$. (4 + 4)

(ii) Find the inverse Z-transform of $\dfrac{2z^2 + 3z}{(z+2)(z-4)}$. (8)

Or

(b) (i) Using convolution theorem find the inverse Z-transform of $\dfrac{z^2}{(z-4)(z-5)}$. (6)

(ii) Solve, by using Z-transform, $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$, given that $y_0 = 0$ and $y_1 = 1$. (10)

Solutions

Part – A

1. [Answer is available in page 2.2 of the book]
   Simply change the interval $c \le x \le c + 2l$ into $(0, 2\pi)$).
2. [Answer is available in page 2.10 of the book the first 3 lines of the page starting with (ii)]
3. [Answer is available in 4.2 given in page 4.1 of the book.]
4. [Answer is available in page 4.27 of the book under (2). Change of Scale Property. In the book, $\bar{f}(s)$ is used instead of $F(s)$]
5. [This is the same as the problem (13) in page 1.20 of the book.] $z = f(xy)$;
   Differentiating partially w.r.t. x, we get $p = f'(xy) \cdot y$ (1)
   Differentiating partially w.r.t. y, we get $q = f'(xy) \cdot x$ (2)
   (1) ÷ (2) gives $\dfrac{p}{q} = \dfrac{y}{x}$ or $px = qy$
6. As the given equation is a Clairant's equation, the complete integral is $Z = ax + by + \sqrt{ab}$.
7. In the equation $u_t = \alpha^2 u_{xx}$, α^2 represents the diffusivity of the material of the bar, equal to $\dfrac{k}{cp}$, where k is the thermal conductivity, c is the specific heat and ρ is the density of the material of the bar.
8. The steady-state distribution is given by $\dfrac{d^2u}{dx^2} = 0$ subject to $u(0) = 20$ and $u(10) = 80$.

   The general solution of the equation is $u(x) = Ax + B$. Using the given boundary conditions, we get $A = 6$ and $B = 20$. $\therefore$ The required temperature distribution is given by $u(x) = 6x + 20$.

9. $Z(n^2) = Z(n \cdot n) = -z \cdot \dfrac{d}{dz}\{Z(n)\} = -z \cdot \dfrac{d}{dz}\left\{\dfrac{z}{(z-1)^2}\right\}$
$= \dfrac{z(z+1)}{(z-1)^3}$

10. $y_n = A + B \cdot 2^n$ (1); $y_{n+1} = A + 2B \cdot 2^n$ (2); $y_{n+2} = A + 4B \cdot 2^n$ (3)
Eliminating A and $(B \cdot 2n)$ from (1), (2), (3) the required equation is

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 1 & 2 \\ y_{n+2} & 1 & 4 \end{vmatrix} = 0. \text{ i.e., } y_{n+2} - 3y_{n+1} + 2y_n = 0.$$

Part – B

11. (a) (i) $f(x) = \begin{cases} \phi_1(x) = 1 - x, & \text{in } -\pi \le x \le 0 \\ \phi_2(x) = 1 + x, & \text{in } 0 \le x \le \pi \end{cases}$

$\phi_1(-x) = 1 + x = \phi_2(x)$

$\therefore f(x)$ is even in $(-\pi, \pi)$ and so the Fourier series of $f(x)$ will not contain sine terms and will be of period 2π. Let F.S. of $f(x)$ be $\dfrac{a_0}{2} + \sum a_n$ an cos nx in $(-\pi, \pi)$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (1+x)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[(1+x)\left(\frac{\sin nx}{n}\right) - 1\cdot\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{n^2\pi}\left\{(-1)^n - 1\right\}$$

$$= \begin{cases} 0, \text{ when } n \text{ is even} \\ -\dfrac{4}{n^2\pi}, \text{ when } n \text{ is odd} \end{cases}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (1+x)\, dx = \frac{2}{\pi}\left(\pi + \frac{\pi^2}{2}\right) = 2 + \pi$$

The required F.S. of $f(x)$ is $\left(1 + \dfrac{\pi}{2}\right) - \dfrac{4}{\pi}\displaystyle\sum_{n=1,3,5\ldots}^{\infty} \dfrac{1}{n^2}\cos nx$ in $(-\pi, \pi)$

Putting $x = 0$, that is a point of continuity of $f(x)$,

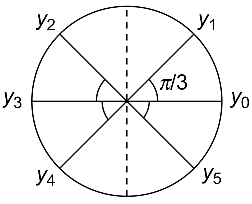
$$1 + \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\infty\right) = 1$$

$$\therefore\ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\infty = \frac{\pi^2}{8}.$$

11. (a) (ii) The length of the interval $= 2\pi$ and that of the sub-interval $= \dfrac{\pi}{3}$. $\therefore$ The 6 values of $y = f(x)$ should be considered at the left/right end points of the sub-intervals.

Accordingly, $x : x_0 = 0, x_1 = \frac{\pi}{3}, x_2 = \frac{2\pi}{3}, x_3 = \pi, x_4 = \frac{4\pi}{3}, x_5 = \frac{5\pi}{3}$

$y : y_0 = 1.98, y_1 = 1.30, y_2 = 1.05, y_3 = 1.30, y_4 = -0.88, y_5 = -0.25$



Let the required F.S. be $y = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + \ldots$

$$a_0 = 2 \times \frac{1}{6} \sum_{r=0}^{5} y_r = \frac{1}{3} \times 4.50 = 1.50$$

$$a_1 = 2 \times \frac{1}{6} \sum_{r=0}^{5} y_r \cos x_r$$

$$= \frac{1}{3}\left\{(y_0 - y_3) + (y_1 + y_5 - y_2 - y_4) \cos \frac{\pi}{3}\right\}$$

$$= \frac{1}{3}\{0.68 + 0.88 \times 0.5\} = 0.373$$

$$b_1 = 2 \times \frac{1}{6} \sum_{r=0}^{5} y_r \sin x_r = \frac{1}{3}\left\{(y_1 + y_2 - y_4 - y_5) \sin \frac{\pi}{3}\right\}$$

$$= \frac{1}{3} \times 3.48 \times 0.866 = 1.005$$

The required F.S. of $y = f(x)$ is $0.75 + 0.373 \cos x + 1.005 \sin x$

11. (b) (i) $f(x) = |x|$ is even in $-1 < x < 1$.

$\therefore$ The F.S. of $f(x)$ will not contain sine terms and will be of period 2l.

Let the F.S. of $f(x)$ be $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$ in $(-l, l)$

$$a_n = \frac{2}{l}\int_0^l |x| \cos \frac{n\pi x}{l}\, dx \text{ (or) } \frac{2}{l}\int_0^l x \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[x \frac{\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)} - 1 \cdot \frac{-\cos \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]_0^l = \frac{2l}{n^2\pi^2}[(-1)^n - 1]$$

$$= \begin{cases} \frac{-4l}{n^2\pi^2}, \text{ when } n \text{ is odd} \\ 0, \text{ when } n \text{ is even} \end{cases}$$

$$a_0 = \frac{2}{l}\int_0^l |x|\,dx \text{ (or) } \frac{2}{l}\int_0^l x\,dx = \frac{1}{l}[x^2]_0^l = l$$

$$\therefore |x| \sim \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} \text{ in } (-l, l) \qquad (1)$$

Putting $x = 0$, that is a point of continuity of $|x|$, in (1),

$$\frac{4l}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty\right] = \frac{l}{2} \quad \therefore\ 1^{-2} + 3^{-2} + 5^{-2} + \cdots \infty = \frac{\pi^2}{8}.$$

11. (b) (ii) Give an odd extension for $f(x)$ in $(-\pi, 0)$

i.e., put $f(x) = x(\pi + x)$ in $(-\pi, 0)$.

Then $f(x)$ has become an odd function in $(-\pi, \pi)$

$\therefore$ F.S. of $f(x)$ will contain only sine terms and will be of period 2π.

Let the required half-range F.S.S of $f(x)$ be $\sum_{n=1}^{\infty} b_n \sin nx$ in $(0, \pi)$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4}{\pi n^3}[1-(-1)^n] = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{8}{\pi n^3}, & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore x(\pi - x) \sim \frac{8}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^3}\sin nx \text{ in } (0, \pi) \qquad (1)$$

Putting $x = \dfrac{\pi}{2}$, that is a point of continuity for $x(\pi - x)$ in (1),

$$\frac{8}{\pi}\left[\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}\cdots\infty\right] = \frac{\pi^2}{4}$$

$$\therefore\ \frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}\cdots\infty \text{ (or) } \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$

12. (a) [Major part of this problem is the Worked example (5) in page 4.37 of the book.]

It is proved in the W.E. that $F\{f(x)\} = \left(\dfrac{\sin\frac{s}{2}}{\frac{s}{2}}\right)^2$

$$\therefore\ F^{-1}\left[\left\{\frac{\sin\left(\frac{s}{2}\right)}{\left(\frac{s}{2}\right)}\right\}^2\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{\sin\left(\frac{s}{2}\right)}{\left(\frac{s}{2}\right)}\right\}^2 e^{ixs}\,ds$$

$$= f(x) = \begin{cases} 1-|x| & \text{for } |x| \le 1 \\ 0, & \text{for } |x| > 1 \end{cases}$$

Putting $x = 0$ (or) $f(0) = 1$, we get

$$\frac{2}{2\pi}\int_0^{\infty}\left\{\frac{\sin\left(\frac{s}{2}\right)}{\left(\frac{s}{2}\right)}\right\}^2 ds = 1 \text{ i.e., } \int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 \cdot 2\,dt = \pi, \text{ on putting } \frac{s}{2} = t$$

$$\therefore\ \int_0^{\infty}\left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

The deduction $\int_0^{\infty}\left(\frac{\sin x}{x}\right)^4 dx = \frac{\pi}{3}.$ is a part of the W.E.

12. (b) [This problem is the Worked example (6) in page 4.38 of the book, in which we have to put $a = 1$.]

13. (a) (i) $x(y-z)\,p + y(z-x)\,q = z(x-y)$, which is a Lagrange Linear equation with $P = x(y-z)$, $Q = y(z-x)$ and $R = z(x-y)$.

The subsidiary equations are $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$ (1)

Using the multipliers 1, 1, 1, each ratio in (1) $= \dfrac{d(x+y+z)}{0}$

$\therefore\ d(x+y+z) = 0$ i.e., $x + y + z = a$ is one independent solution of Lagrange's subsidiary equations

Using the multipliers $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$; each ratio in (1) $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$\therefore\ \dfrac{1}{x}dx + \dfrac{1}{y}dy + \dfrac{1}{z}dz = 0$

Integrating, we get $\log(xyz) = \log b$ or $xyz = b$ is the second independent solution of Lagrange's subsidiary equations

$\therefore$ General solution of the given equation is

$f(x + y + z, xyz) = 0$, where f is an arbitrary function.

13. (a) (ii) $(D^2 + DD' - 6D'^2)\,Z = \cos(x + 2y)$. This is a homogeneous linear equation of the second order.

A.E. is $m^2 + m - 6 = 0$ i.e., $(m + 3)(m - 2) = 0$ $\therefore\ m = -3, 2$.

C.F. = $\phi_1(y - 3x) + \phi_2(y + 2x)$, where ϕ_1, ϕ_2 are arbitrary functions.

P.I. = $\dfrac{1}{D^2 + DD' - 6D'^2} \cos(x + 2y) = \dfrac{1}{-1 - 2 + 24} \cos(x + 2y)$

or $\dfrac{1}{21} \cos(x + 2y)$

$\therefore$ G.S. of the given equation is Z = C.F. + P.I.

13. (b) (i) $z = px + qy + p^2 - q^2$. This is a Clairaut's type equation.

$\therefore$ C.S. is $z = ax + by + a^2 - b^2$ (1)

Differentiating (1) partially w.r.t. 'a', $x + 2a = 0$ (2)

Differentiating (1) partially w.r.t. 'b', $y - 2b = 0$ (3)

From (2), $a = -\dfrac{x}{2}$; From (3), $b = \dfrac{y}{2}$

Using these values of a and b in (1), we get

$z = -\dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{x^2}{4} - \dfrac{y^2}{4}$, i.e., $4z = y^2 - x^2$. This is the required singular solution.

13. (b) (ii) $(D^2 + 2DD' + D'^2)\, Z = -x \sin y$. This is a homogeneous linear equation of second order.

A.E. is $m^2 + 2m + 1 = 0$ or $(m + 1)^2 = 0 \setminus m = -1, -1$.

C.F. = $\phi_1(y - x) + x\, \phi_2(y - x)$, where ϕ_1 and ϕ_2 are arbitrary functions.

P.I. = $\dfrac{1}{(D + D')^2}$ I.P. of $(-xe^{iy})$ = I.P. of $(-e^{iy}) \cdot \dfrac{1}{(D + D' + i)^2}\,(x)$

$= -$ I.P. of $\left[e^{iy} \cdot \dfrac{1}{i^2\, \{1 - i\,(D + D')\}^2}\,(x) \right]$

= I.P. of $[e^{iy}\, \{1 + 2i\,(D + D') + \cdots\}\,(x)]$

= I.P. of $[(\cos y + i \sin y)\,(x + 2i)]$

$= x \sin y + 2 \cos y$

$\therefore$ G.S. of the given equation is Z = C.F. + P.I.

14. (a) [This problem is the W.E. (1) in page 3.10 of the book. Only part (ii) of this W.E. provides the required solution.]

14. (b) This problem is the W.E. (1) in page 3.119 of the book. We have to change a into l and put $f(x) = k\left[\sin\left(\dfrac{\pi x}{l}\right) + 3 \sin\left(\dfrac{3\pi x}{l}\right)\right]$.

Steps (1) to (10) are the same as in the W.E.

Using the boundary condition (5), namely,

$u(x, 0) = k\left[\sin\left(\dfrac{\pi x}{l}\right) + 3 \sin\left(\dfrac{3\pi x}{l}\right)\right]$ in step (10), we have

$$\sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{l} = k\left[\sin\left(\frac{\pi x}{l}\right) + 3 \sin\left(\frac{3\pi x}{l}\right)\right] \qquad (11)$$

Comparing like terms in (11), we have

$$\lambda_1 = k;\ \lambda_3 = 3k;\ \lambda_2 = \lambda_4 = \lambda_5 = \lambda_6 = \ldots = 0.$$

Using these values of λ_n in (10), the required solution is

$$u(x, y) = k \sin \frac{\pi x}{l}\ e^{-\pi y/l} + 3k \sin \frac{3\pi x}{l}\ e^{-3\pi y/l}.$$

15. (a) (i) $Z\{(n+1)^2\} = Z(n^2 + 2n + 1)$

$$= \frac{z(z+1)}{(z-1)^3} + 2\frac{z}{(z-1)^2} + \frac{z}{z-1} = \frac{z^2(z+1)}{(z-1)^3}$$

$Z\{\sin(3n+5)\} = Z\{\sin 3n \cdot \cos 5 + \cos 3n \cdot \sin 5\}$

$$= (\cos 5) \cdot \frac{z \sin 3}{z^2 - 2z\cos 3 + 1} + (\sin 5) \cdot \frac{z(z - \cos 3)}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z^2 \sin 5 - z \sin 2}{z^2 - 2z\cos 3 + 1}$$

15. (a) (ii) Let $\bar{f}(z) = \dfrac{2z^2 + 3z}{(z+2)(z-4)}$; $\therefore \dfrac{\bar{f}(z)}{(z)} = \dfrac{2z+3}{(z+2)(z-4)} = \dfrac{\left(\frac{1}{6}\right)}{z+2} + \dfrac{\left(\frac{11}{6}\right)}{z-4}$

$$\therefore\ \bar{f}(z) = \frac{1}{6} \cdot \frac{z}{z+2} + \frac{11}{6} \cdot \frac{z}{z-4}$$

$$\therefore\ Z^{-1}\left\{\bar{f}(z)\right\} = \frac{1}{6} \cdot Z^{-1}\left\{\frac{z}{z+2}\right\} + \frac{11}{6} \cdot Z^{-1}\left\{\frac{z}{z-4}\right\}$$

i.e., $f(n) = \dfrac{1}{6} \times (-2)^n + \dfrac{11}{6} \times 4^n.$

15. (b) (i) $Z^{-1}\left\{\dfrac{z}{z-4} \cdot \dfrac{z}{z-5}\right\} = Z^{-1}\left\{\dfrac{z}{z-4}\right\} * Z^{-1}\left\{\dfrac{z}{z-5}\right\}$

$$= (4^n) * (5^n) = \sum_{r=0}^{n} 4^r \cdot 5^{n-r} = 5^n \cdot \sum_{r=0}^{n} \left(\frac{4}{5}\right)^r$$

$$= 5^n \cdot \left\{\frac{1 - \left(\frac{4}{5}\right)^{n+1}}{1 - \frac{4}{5}}\right\} = 5^{n+1} - 4^{n+1}$$

15. (b) (ii) $y_{n+2} + 4y_{n+1} + 3y_n = 3n$

Taking Z-transforms of the given equation

$Z(y_{n+2}) + 4Z(y_{n+1}) + 3Z(y_n) = Z(3^n)$

i.e., $[z^2\ \bar{y}(z) - z^2 y(0) - z \cdot y(1)] + 4[z\,\bar{y}(z) - zy(0)] + 3\,\bar{y}(z) = \dfrac{z}{z-3}$

i.e., $(z^2 + 4z + 3)\ \bar{y}(z) = \dfrac{z}{z-3} + z$ or $\dfrac{z(z-2)}{z-3}$

$$\therefore\ \bar{y}(z) = \frac{z(z-2)}{(z+3)(z+1)(z-3)} \text{ or } \frac{\bar{y}(z)}{z} = \frac{z-2}{(z+3)(z+1)(z-3)}$$

Now $\dfrac{\bar{y}(z)}{z} = \dfrac{\left(\dfrac{-5}{12}\right)}{z+3} + \dfrac{\left(\dfrac{3}{8}\right)}{z+1} + \dfrac{\dfrac{1}{24}}{z-3}$

$$\therefore\ \bar{y}(z) = -\frac{5}{12}\cdot\frac{z}{z+3} + \frac{3}{8}\cdot\frac{z}{z+1} + \frac{1}{24}\cdot\frac{z}{z-3}$$

Inverting, we get

$y(n) = -\dfrac{5}{12}\cdot(-3)^n + \dfrac{3}{8}\cdot(-1)^n + \dfrac{1}{24}\cdot 3^n$, which is the required solution.

Anna University of Technology, Coimbatore

B.E./B.Tech. Degree Examinations : Nov/Dec 2010
Regulations : 2008
Third Semester

080100008-Transforms and Partial Differential Equations
(Common to Civil/EEE/EIE/ICE/ECE/Biomedical/Biotech/ AERO/AUTO/CSE/IT/Mechanical/Chemical/FT/II/TC)

Time 3 Hours **Max. Marks: 100**

Part-A

Part A – (20 × 2 = 40 Marks)

Answer All Questions

1. State the conditions for $f(x)$ to have Fourier series expansion.
2. Write a_0, a_n in the expansion of $x + x^3$ as Fourier Series in $(-\pi, \pi)$.
3. Expand $f(x) = 1$ in a sine series in $0 < x < \pi$
4. Find Root Mean Square value of the function $f(x) = x$ in the interval (0, l).
5. Define Fourier Transform Pair.
6. Find Fourier Cosine transform of e^{-2x}.
7. If F(S) is the Fourier Transform of $f(x)$, show that the Fourier Transform of $e^{iax} f(x)$ is $F(S - a)$.
8. State Parseval's Identity for Fourier Transform.
9. Eliminate the arbitrary constants a and b from $z = (x^2 + a)(y^2 + b)$.
10. Form the PDE by eliminating the functions from $z = f(x + t) + g(x - t)$.
11. Find the complete integral $q = 2px$.
12. Solve $(D^3 - 3DD'^2 + 2D'^3)z = 0$
13. Find the nature of PDE $4u_{xx} + 4u_{xy} + u_{yy} + 2u_x - u_y = 0$.
14. What are the various solutions of one dimensional Wave Equation?
15. A string is stretched and fastened to two points 'l' apart. Motion is started by displacing the string into the form $y = y_0 \sin\left(\frac{\pi x}{l}\right)$ from which it is released at time $t = 0$. Formulate this problem as the boundary value problem.
16. A rod of length 20 cm whose one end is kept at 30°C and the other end is kept at 70°C is maintained so until steady state prevails. Find the steady state temperature.
17. Find $Z[e^{-an}]$.

18. Prove that $Z[n] = \dfrac{z}{(z-1)^2}$
19. Prove that $Z[f(n+1)] = zF(z) - zf(0)$
20. State Initial and Final value theorem on Z-transform.

Part – B (5 × 12 = 60 Marks)

Answer any Five Questions

21 (a). If $f(x) = \left(\dfrac{\pi - x}{2}\right)$ find the Fourier Series of the period 2π in the interval $(0, 2\pi)$.

Hence deduce that $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + ... = \dfrac{\pi}{4}$ (8)

(b) Find the Fourier expansion of $f(x) = x$ in the interval $(-\pi, \pi)$ (4)

22. Show that the Fourier Transform of $f(x) = \begin{cases} a^2 - x^2 & |x| \le a \\ 0 & otherwise \end{cases}$ is $2\sqrt{\dfrac{2}{\pi}}\left(\dfrac{\sin as - as\cos as}{s^3}\right)$. Hence deduce that $\displaystyle\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt = \frac{\pi}{4}$. Using Parseval's Identify show that $\displaystyle\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$

23. (a) Solve $(mz - ny)p + (nx - lz)q = ly - mx$

(b) Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y$

24. A string of length l is initially at rest in its equilibrium position and motion is started by giving each of its points is given a velocity $V = \begin{cases} cx & ,0 \le x \le l/2 \\ c(1-x), & l/2 \le x \le l \end{cases}$. Find the displacement function $y(x, t)$.

25. (a) Evaluate $Z^{-1}\left[\dfrac{z}{z^2 + 7z + 10}\right]$ (6)

(b) Using z-transforms solve $u(n + 2) - 5u(n + 1) + 6u(n) = 4^n$ given that $u(0) = 0, u(1) = 1$ (6)

26. (a) Find the constant term and the coefficient of the first sine and cosine terms in the Fourier expansion of, $y = f(x)$ as given in the following table: (6)

x	0	1	2	3	4	5	6
$f(x)$	9	18	24	28	26	20	9

26. (b) Find the Fourier transform of $f(x) = \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & otherwise \end{cases}$

hence find the value of $\int_0^\infty \frac{\sin^4 x}{x^4} dx$ dx (6)

27. A metal bar 30 cm long has its ends A and B kept at 20°C and 80°C respectively, until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0°C and kept so. Find the resulting temperature $u(x, t)$ taking $x = 0$ at A.
28. (a) Solve $p(1 + q) = qz$ (6)

    (b) Using Convolution theorem, evaluate $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$ (6)

Solutions

Part – A

1. [Answer is available in page 2.2 of the book under the heading "2.2 Dirichlet's Conditions).
2. The interval $(-\pi, \pi)$ is symmetric about the origin and $f(x) = x + x^3$ is an odd function in $(-\pi, \pi)$.
   The Fourier series of $f(x)$ in $(-\pi, \pi)$ will contain only sine terms. $\therefore a_0 = 0$ and $a_n = 0$.
3. Give an odd extension for $f(x)$ in $-\pi < x < 0$; i.e., put $f(x) = -1$ in $-\pi < x < 0$

   $\therefore f(x)$ has been made an odd function in $-\pi, < x < \pi$

   $\therefore$ F.S. of $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

   $$b_n = \frac{2}{\pi}\int_0^\pi 1 \sin nx\, dx$$

   $$= \frac{2}{\pi}\left(\frac{-\cos nx}{n}\right)_0^\pi = \frac{2}{\pi n}\{1-(-1)^n\}$$

   $$= \begin{cases} \frac{4}{\pi n}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

   $\therefore$ Required sine series is

   $$f(x) = \frac{4}{p}\sum_{n=1,3,5...}^{\infty} \frac{1}{n} \sin nx$$

4. R.M.S. value of $f(x) = x$ in $(0, l)$ is given by

   $$\overline{f(x)} = \sqrt{\frac{1}{l}\int_0^l x^2 dx} = \sqrt{\frac{1}{l}\cdot\frac{l^3}{3}} = \frac{l}{\sqrt{3}}$$

5. $F\{f(x)\} = \overline{f}(s) = \int_{-\infty}^{\infty} f(x)e^{-isx}\, dx$ and

$F^{-1}\left\{\overline{f}(s)\right\} = f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \overline{f}(s)e^{ixs}$ ds.

6. $F_C\{e^{-2x}\} = \displaystyle\int_0^{\infty} e^{-2x}\cos sx\, dx = \left[\dfrac{e^{-2x}}{s^2+4}(-2\cos sx + s\sin sx)\right]_0^{\infty}$

$= \dfrac{2}{s^2+4}$

7. $F\{f(x)\} = F(s) = \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$

$\therefore\ F\{e^{iax} f(x)\} = \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{-i(s-a)x}\, dx = F(s-a)$

8. Parseval's identity: If $\overline{f}(s) = F\{f(x)\} = \displaystyle\int_{-\infty}^{\infty} f(x)e^{-isx}dx$, then $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2\, dx$

$= \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \left|\overline{f}(s)\right|^2$ ds.

9. $z = (x^2 + a)(y^2 + b)$ ...(1); $p = 2x(y^2 + b)$ ...(2); $q = 2y(x^2 + a)$ ...(3)

Using the values of $x^2 + a$ and $y^2 + b$ from (2) and (3) in (1), $z = \dfrac{q}{2y}\cdot\dfrac{p}{2x}$

i.e., $pq = 4xyz$ is the required P.D.E.

10. $z = f(x+t) + g(x-t)$ (1); $p = f' + g'$ (2); $q = \dfrac{\partial z}{\partial t} = f' - g'$ (3);

$r = f'' + g''$ ...(4); $s = f'' - g''$ ...(5); $t = f'' + g''$ (6)

From (4) and (6), the eliminant is $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial t^2}$, which is the required P.D.E.

11. $q = 2px = a$, where a is an arbitrary constant

$\therefore\ p = \dfrac{a}{2x}$ and $q = a$; $dz = p\, dx + q$ dy

i.e., $dz = \dfrac{a}{2x}\, dx + a dy$

Integrating, the C.S. is $z = \dfrac{a}{2}\log x + ay + b$.

12. $(D^3 - 3DD'^2 + 2D'^3)z = 0$ is a homogeneous linear P.D.E.

A.E. is $m^3 - 3m + 2 = 0$; i.e., $(m-1)(m^2 + m - 2) = 0$

or $(m-1)^2(m+2) = 0$

$\therefore \qquad m = 1, 1, -2$

$\therefore$ Solution of the P.D.E. is

$$z = \phi_1(y+x) + x\,\phi_2(y+x) + \phi_3(y-2x)$$

13. $4u_{xx} + 4u_{xy} + u_{yy} + 2u_x - u_y = 0$

    A = 4, B = 4, C = 1. $B_2 - 4AC = 0$

    $\therefore$ The given equation is of the parabolic type.

14. The possible solutions of $y_{tt} = a^2 y_{xx}$ are

    (i) $y(x, t) = (Ae^{px} + Be^{-px})(Ce^{pat} + De^{-pat})$

    (ii) $y(x, t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$

    (iii) $y(x, t) = (Ax + B)(Ct + D)$, where A, B, C, D and p are arbitrary constants

15. To solve the equation $y_{tt} = a^2 y_{xx}$, satisfying the following boundary conditions:

    (i) $y(0, t) = 0, t \geq 0$, (ii) $y(l, t) = 0, t \geq 0$, (iii) $\frac{\partial y}{\partial t}(x, 0) = 0, 0 < x < l$ and

    (iv) $y(x, 0) = y_0 \sin\left(\frac{\pi x}{l}\right), 0 < x < l.$

16. The steady-state temperature distribution is given by $\frac{d^2u}{dx^2} = 0$, whose solution is $u(x) = Ax + B$.

    The boundary conditions are $u(0) = 30$, using which we get $B = 30$ and $u(20) = 70$, using which we get $20A + 30 = 70 \therefore A = 2$

    $\therefore$ The S.S.T. is given by $u(x) = 2x + 30$.

17. $$Z(e^{-an}) = \sum_{n=0}^{\infty}(e^a)^{-n}, z^{-x} = \sum_{n=0}^{\infty}\left(\frac{1}{ze^a}\right)^n = \frac{1}{1-\left(\frac{1}{ze^a}\right)} = \frac{ze^a}{ze^a - 1}$$

18. $$Z(n) = \sum_{n=0}^{\infty} nz^{-n} = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots \infty = \frac{1}{z}\left(1 + \frac{2}{z} + \frac{3}{z^2} + \cdots\infty\right)$$

    $$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2} = \frac{z}{(z-1)^2}$$

19. $$Z\{f(n+1)\} = \sum_{n=0}^{\infty} f(n+1)\, z^{-n} = z\sum_{n=0}^{\infty} f(n+1) \cdot z^{-(n+1)}$$

    $$= z\sum_{n=1}^{\infty} f(n)\, z^{-n} = z\,\{F(z) - f(0)\} = zF(z) - zf(0).$$

20. Initial value theorem: If $Z\{f(n)\} = \overline{f}(z)$, then $f(0) = \lim_{z\to\infty} \overline{f}(z)$

    Final value theorem: If $Z\{f(n)\} = \overline{f}(z)$, then $\lim_{n\to\infty}[f(n)] = \lim_{z\to 1}\left\{(z-1)\overline{f}(z)\right\}$

Part-B

21. (a) Let the F.S. of $f(x) = \frac{\pi - x}{2}$ be $\frac{a_0}{2} + \Sigma a_n \cos nx + \Sigma b_n \sin nx$ in $(0, 2\pi)$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)\cos nx\,dx = \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^{2\pi} = 0$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\,dx = \frac{1}{-4\pi}\left[(\pi - x)^2\right]_0^{2\pi} = -\frac{1}{4\pi}(\pi^2 - \pi^2) = 0$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\sin nx\,dx = \frac{1}{2\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= -\frac{1}{2n\pi}\{-\pi - \pi\} = \frac{1}{n}$$

$$\therefore\ \frac{\pi - x}{2} \sim \sum_{n=1}^{\infty}\frac{1}{n}\ \sin nx \qquad (1)$$

Putting $x = \frac{\pi}{2}$, which is a point of continuity of $\left(\frac{\pi - x}{2}\right)$, in (1), we get

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\infty = \frac{\pi}{4}$$

21. (b) $f(x) = x$ is an odd function in the symmetric interval $(-\pi, \pi)$
$\therefore$ F.S. of $f(x)$ will contain only sine terms.

Let $x \sim \sum_{n=1}^{\infty} b_n\ \sin nx$ in $(-\pi, \pi)$

$$bn = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

i.e., $bn = \frac{2}{\pi}\cdot\frac{(-\pi)}{n}(-1)^n = \frac{2(-1)^{n+1}}{n}$

$\therefore$ Required F.S. of x is $2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\ \sin nx$

22. $F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2 - x^2)\,e^{-isx}\,dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^a (a^2 - x^2)\cos sx\,dx = \sqrt{\frac{2}{\pi}}\left[\left(a^2 - x^2\right)\frac{\sin sx}{s} - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left\{\frac{2\sin as}{s^3} - \frac{2a\cos as}{s^2}\right\}$$

$$= 2\sqrt{\frac{2}{\pi}}\left\{\frac{\sin as - as\cos as}{s^3}\right\}$$

Taking inverse transforms,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)e^{ixs}\,ds = f(x)$$

i.e., $\frac{4}{\pi}\int_0^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\cos xs\,ds = \begin{cases} a^2 - x^2, \text{ in} |x| \le a \\ 0 \quad ,\textit{elsewhere}\end{cases}$

i.e., $\frac{4}{\pi}a^2\int_0^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)\cos\left(\frac{xt}{a}\right)dt = \begin{cases} a^2 - x^2, \text{ in} |x| \le a \\ 0 \quad ,\textit{elsewhere}\end{cases}$

(on putting $as = t$)

Putting $x = 0$, we get

$$\int_0^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)dt = \frac{\pi}{4}$$

By Parseval's identity, $\int_{-\infty}^{\infty}|f(x)|^2\,dx = \int_{-\infty}^{\infty}\left|\overline{f}(x)\right|^2 ds$

$$\therefore \int_{-a}^{a}(a^2 - x^2)dx = \int_{-\infty}^{\infty} 4\times\frac{2}{\pi}\left(\frac{\sin as - as\cos as}{s^3}\right)^2 ds$$

i.e., $2\int_0^{a}(a^4 - 2a^2x^2 + x^4)dx = \frac{8}{\pi}\times 2\int_0^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)^2 a^5\,dt$

i.e., $\frac{8}{15} = \frac{8}{\pi}\int_0^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt$ (or) $\int_0^{\infty}\left(\frac{\sin t - t\cos t}{b^3}\right)^2 dt = \frac{\pi}{15}$

23. (a) $(mz - ny)\,p + (nx - lz)\,q = ly - mx$

This is a Lagrange's linear equation.

The subsidiary equations are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} \tag{1}$$

Choosing the multipliers l, m, n, each ratio in (1) = $\frac{ldx + mdy + ndz}{0}$

$\therefore$ $ldx + mdy + ndz = 0$. $\therefore$ $lx + my + nz = a$ is one solution of the equations.

Choosing the multipliers x, y, z each ratio in (1) = $\frac{x\,dx + y\,dy + z\,dz}{0}$

$\therefore$ $xdx + ydy + zdz = 0$ $\therefore$ $x^2 + y^2 + z^2 = b$ is another solution of the equations.

$\therefore$ G.S. of the L.L. Eqn. is $f(lx + my + nz, x^2 + y^2 + z^2) = 0$, where f is an arbitrary function.

23. (b) $(D^3 + D^2D' - DD'^2 - D'^3)\, z = e^x \cos 2y$, which is a linear homogeneous P.D.E.

A.E. is $m^3 + m^2 - m - 1 = 0$, i.e., $(m^2 - 1)(m + 1) = 0$

(or) $(m - 1)(m + 1)^2 = 0$

$\therefore\ m = 1, -1, -1.$

$\therefore$ C.F. $= \phi_1(y + x) + \phi_2(y - x) + x\,\phi_3(y - x)$

$$\text{P.I.} = \frac{1}{(D-D')(D+D')^2} e^x \cos 2y = e^x \cdot \frac{1}{(D+1-D')(D+1+D')^2} \cos 2y$$

$$= e^x \frac{1}{(D-D'+1)(D^2+D'^2+2DD'+2D+2D'+1)} \cos 2y$$

$$= e^x \frac{1}{(D-D'+1)(2D+2D'-3)} \cos 2y = e^x \frac{1}{2D^2-2D'^2-D+5D'-3} \cos 2y$$

$$= e^x \frac{1}{5-(D-5D')} \cos 2y = e^x \frac{\{5+(D-5D')\}}{25-(D^2-10DD'+25D^{12})} \cos 2y$$

$$= e^x \frac{1}{125}(5 \cos 2y + 10 \sin 2y) = \frac{1}{25} e^x (\cos 2y + 2 \sin 2y)$$

G.S. in z = C.F. + P.I.

24. [This problem is the same as W.E. (8) in Page 3.28 of the book.

In the W.E. l is taken as 60 and c is equal to $\dfrac{\lambda}{30}$.

In the end, we will get $\lambda_n = \begin{cases} 0, \text{ if } n \text{ is even} \\ \dfrac{4cl^2}{n^3\pi^3} \sin \dfrac{n\pi}{2}, \text{if } n \text{ is odd} \end{cases}$

25. (a) Let $\overline{f}(z) = \dfrac{z}{z^2+7z+10}\ \therefore \dfrac{\overline{f}(z)}{z} = \dfrac{1}{(z+2)(z+5)} = \dfrac{1/3}{z+2} - \dfrac{1/3}{z+5}$

$$\therefore Z^{-1}\left\{\overline{f}(z)\right\} = \frac{1}{3}\left[Z^{-1}\left[\frac{z}{z+2}\right] - z^{-1}\left(\frac{z}{z+5}\right)\right]$$

$$= \frac{1}{3}\left[(-2)^n - (-5)^n\right]$$

25. (b) Taking Z-transforms of the equations

$$[z^2\,\overline{u}(z) - z^2u(0) - zu(1)] - 5\,[z\,\overline{u}(z) - zu(0)] + 6\,\overline{u}(z) = \frac{z}{z-4}$$

i.e. $(z^2 - 5z + 6)\,\overline{u}(z) = \dfrac{z}{z-4} + z$

$\therefore\ \bar{u}\ (z) = \dfrac{z}{(z-2)(z-4)} \qquad \therefore\ \dfrac{\bar{u}(z)}{z} = 1/2\left(\dfrac{1}{z-4} - \dfrac{1}{z-2}\right)$

$\therefore\ \bar{u}(z) = \dfrac{1}{2}\left(\dfrac{z}{z-4} - \dfrac{z}{z-2}\right)$

Inverting, $u(n) = \dfrac{1}{2}\ (4^n - 2^n)$

26. (a) $2l = 6$ or $l = 3$; length of sub-interval = 1

$\therefore$ 6 values of $y = f(x)$ should be considered at the left/right end points of the sub-intervals.

Accordingly, $x : x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5$

$y = f(x) : y_0 = 9, y_1 = 18, y_2 = 24, y_3 = 28, y_4 = 26, y_5 = 20$

Let the required F.S. be $y = \dfrac{a_0}{2} + a_1 \cos\dfrac{\pi x}{3} + b_1 \sin\dfrac{\pi x}{3}\ + ...$

$$a_0 = 2\ \times \frac{1}{6}\sum_{n=0}^{5} y_r = \frac{1}{3}\times 125\ = 41.67$$

$$a_1 = 2\ \times \frac{1}{6}\sum_{r=0}^{5} y_r \cos\frac{\pi x_r}{3}$$

$$= \frac{1}{3}\left[(9-28) + (18+20-24-26)\times\cos\frac{\pi}{3}\right]$$

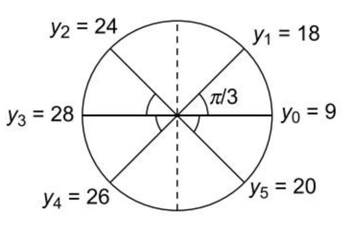
$$= \frac{1}{3}\ [-19 - 12\times 0.5] = -8.33$$

$$b_1 = 2\times \frac{1}{6}\sum_{r=0}^{5} y_r \sin\frac{\pi x_r}{3}$$

$$= \frac{1}{3}[(18+24-26-20)]\times\sin\frac{\pi}{3}$$

$$= \frac{1}{3}\ \times -4\times -0.866 = -1.15$$

The required F.S. is $20.84 - 8.33\cos\dfrac{\pi x}{3} - 1.15\sin\dfrac{\pi x}{3}$



26. (b) [This problem is the worked example (5) in page 4.37 of the book.]
27. (Refer to the worked example (4) in page 3.72 of the book. The key steps only are given below with necessary changes)

Step (2) : $u(0) = 20$; Step (3) : $u(30) = 80$

Step (5) : $u(x) = 2x + 20$

Step (8) : $u(30, t) = 0$, $t \geq 0$; Step (9) : $u(x, 0) = 2x + 20$, for $0 < x < 30$

Step (11) : $u(x, t) = \text{B} \sin \dfrac{n\pi x}{30} \cdot e - n^2\pi^2\alpha\, t/30^2$, since $p = \dfrac{n\pi}{30}$

Step (12) : $u(x, t) = \displaystyle\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30}\ e^{-n^2 \pi^2 \alpha^2 t/900}$

$$B_n = b_n = \frac{2}{30}\int_0^{30} (2x+20)\sin\frac{n\pi x}{30}\,dx$$

$$= \frac{2}{15}\left[(x+10)\left(-\frac{\cos\frac{n\pi x}{30}}{n\pi/30}\right) + \frac{\sin\frac{n\pi x}{30}}{n^2\pi^2/30^2}\right]_0^{30}$$

$$= \frac{4}{n\pi}\left[10 - 40(-1)^n\right] = \frac{40}{n\pi}\left[1 - 4(-1)^n\right]$$

The required solution is $u(x, t) = \dfrac{40}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\left\{1 - 4(-1)^n\right\}\sin\frac{n\pi x}{30}\cdot e^{-n^2\pi^2 a^2 t/900}$

28. $p(1 + q) = qz$ (1)

Let a solution of (1) be $z = z(u) = z(x + ay)$ ...(2), where a is an arbitrary constant.

Then $p = \dfrac{dz}{du}$ and $q = a\dfrac{dz}{du}$

Since (2) is a solution of (1), we get $\dfrac{dz}{du}\left(1 + a\dfrac{dz}{du}\right) = a\dfrac{dz}{du}\cdot z$

i.e., $1 + a\dfrac{dz}{du} = az$ or $a\dfrac{dz}{du} = az - 1$

i.e., $\dfrac{adz}{az-1} = du \cdot \quad \therefore \displaystyle\int\frac{a\,dz}{az-1} = u + b$

$\therefore$ The required complete solution is $\log(az - 1) = u + b$ (or) $\log(az - 1) = x + ay + b$, where a and b are arbitrary constants.

To find the G.S., we take $\log(az - 1) = x + ay + f(a)$ ...(3)

Differentiating (3) partially w.r.t. a, we get

$\dfrac{z}{az-1} = y + f'(a)$ ...(4)

The eliminant of 'a' from (3) and (4) gives the G.S.

28. (b) $Z^{-1}\left\{\dfrac{z^2}{(z-1)(z-3)}\right\} = Z^{-1}\left\{\dfrac{z}{z-1}\right\} * Z^{-1}\left\{\dfrac{z}{z-3}\right\}$

$$= (1^n) * 3^n$$

$$= \sum_{r=0}^{n} 1^n \cdot 3^{n-r} = 3^n \cdot \sum_{r=0}^{n} \left(\frac{1}{3}\right)^r$$

$$= 3^n \left\{ \frac{1-\left(\frac{1}{3}\right)^{n+1}}{1-\frac{1}{3}} \right\} = \frac{1}{2}\left(3^{n+1}-1\right)$$

Question Paper Code: **G 0353**

B.E./B.Tech. Degree Examinations, November/December 2010
Regulations 2008

Third Semester

Common to all branches

MA1201/CK201 Transforms and Partial Differential Equations

Time: Three Hours **Maximum: 100 Marks**

Answer ALL Questions

Part A – (10 × 2 = 20 Marks)

1. Find the sum of the Fourier series for $f(x) = \begin{cases} x, 0 ,< x < 1 \\ 2, 1 < x < 2 \end{cases}$ at $x = 1$.
2. Write down the complex form of the Fourier series for $f(x)$ in $(c, c + 2\pi)$.
3. State inversion theorem for a complex Fourier transform.
4. Find the Fourier cosine transform of e^{-2x}, $x \geq 0$.
5. Find the complete integral of $z = px + qy + \sqrt{pq}$.
6. Find the general solution of $\dfrac{\partial^2 z}{\partial x^2} = 0$.
7. What are the various solutions of $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$.
8. A rod 30 cm long has its end A and B kept at 20°C and 80°C respectively, until steady state conditions prevail. Determine the temperature at steady state.
9. If $X(z)$ is the z-transform of $x(n)$, find $Z[x(n - m)]$.
10. Find the z-transform of $\left(\dfrac{1}{2}\right)^{n-1}$.

Part B – (5 × 16 = 80 Marks)

11. (a) Find the Fourier series of $f(x) = x + x^2$ in the interval $(-\pi, \pi)$. (8)
    (b) Expand $f(x) = \pi x - x^2$, $0 < x < \pi$ as a Fourier cosine series. (8)

OR

12. (a) If $f(x) = \begin{cases} x, 0 < x < l \\ 2l - x, l < x < 2l, \end{cases}$ expand $f(x)$ as a Fourier series. (8)

(b) Expand $f(x) = x$ in a half range sine series in the ragen $0 < x < 1$ and hence find $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (8)

13. (a) Show that $e^{-x^2/2}$ is self reciprocal under Fourier transform. (8)
    (b) Find the Fourier cosine transform of e^{-ax}. Hence evaluate $\int_0^{\infty} \frac{dx}{\left(x^2 + a^2\right)\left(x^2 + b^2\right)}$ (8)

OR

14. (a) Find the Fourier transform of $f(x) = \begin{cases} 1 - x^2 & if\ |x| < 1 \\ 0 & if\ |x| > 1 \end{cases}$. Hence evaluate $\int_0^{\infty} \left(\frac{x \cos x - \sin x}{x^3}\right) dx$. (8)
    (b) Find the function if its sine transform is $\frac{e^{-as}}{s}$. Hence deduce $F_s^{-1}\left(\frac{1}{s}\right)$. (8)
15. (a) Find the singular integral of $z = px + qy + p^2 - q^2$. (6)
    (b) Solve $(D^2 - 2DD' + D'^2)z = e^{x+2y} + \sin(2x - 3y)$. (10)

OR

16. (a) Solve $(3z - 4y)p + (4x - 2z)q = 2y - 3x$. (8)
    (b) Solve $(D^2 + 3DD' + 2D'^2)z = x + y$ (8)
17. If a string of length l is initially at rest in its equilibrium position and each of its points is given a velocity v such that $v = \begin{cases} kx, \text{ for } 0 < x\, l/2 \\ k(l - x), \text{ for } \frac{l}{2} < x < l. \end{cases}$.
    Find the displacement function $y(x, t)$. (16)

OR

18. A square plate is bounded by the lines $x = 0$, $y = 0$, $x = l$ and $y = l$ and its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, l) = lx - x^2$ for $0 < x < l$, while the other three edges are maintained at 0°C. Find the steady-state temperature distribution in the plate. (16)
19. (a) Find the Z transform of $\frac{1}{n+1}$ and $2^n \cos \frac{n\pi}{2}$. (8)
    (b) Using convolution theorem, find the inverse z-transform of $\frac{8z^2}{(2z-1)(4z-1)}$. (8)

OR

20. (a) Find $Z^{-1}\left[\dfrac{z^2-3z}{(z+2)(z-5)}\right]$ by residue method. (8)

(b) Solve, using z-transform, $y_{n+2}-4y_n=3^n$ given $y_0=0, y_1=0$. (8)

Solutions

Part – A

1. $x = 1$ is a point of discontinuity of $f(x)$.

$\therefore$ [Sum of the F.S. of $f(x)$ at $z = 1$] $= \dfrac{1}{2}\lim\limits_{h\to 0}[f(1-h)+f(1+h)]$

$$= \frac{1}{2}\lim_{h\to 0}[(1-h)+2] = \frac{3}{2}$$

2. The complex form of the F.S. of $f(x)$ in $(c, c+2\pi)$ is

$$f(x) \approx \sum_{n=-\infty}^{\infty} c_n e^{inx}\text{, where}$$

$$c_n = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)e^{-inx}\,dx.$$

3. The inverse Fourier transform of $\overline{f}(s)$ is given by

$$F^{-1}\{\overline{f}(s)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f}(s)e^{ixs}\,ds \text{ or } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{f}(s)e^{ixs}\,ds,$$

according as the Fourier transform of $f(x)$ is defined as

$$\overline{f}(s) = \int_{-\infty}^{\infty} f(x)e^{-isx}\,dx \text{ or } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}\,a.$$

4. $F_c\{e^{-2x}\} = \displaystyle\int_0^{\infty} e^{-2x}\cos sx\,dx$

$$= \left[\frac{e^{-2x}}{s^2+4}(-2\cos sx + s\sin sx)\right]_0^{\infty} = \frac{2}{s^2+4}$$

5. The given equation is a Clairaut's type equation.
$\therefore$ The C.S. of the equation is $z = ax + by + \sqrt{ab}$, where a and b are arbitrary constants.

6. $\dfrac{\partial^2 z}{\partial x^2} = 0$ (1); Integrating (1) partially w.r.t. x,

$\dfrac{\partial z}{\partial x} = f(y)$ (2); Integrating (2) partially w.r.t. x,

$z = xf(y) + g(y)$ (3). (3) is the required G.S. where f and g are arbitrary functions.

7. The various solutions of $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$ are

(i) $y(x, t) = (Ae^{px} + Be^{-px})(Ce^{pat} + De^{-pat})$
(ii) $y(x, t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat)$ and
(iii) $y(x, t) = (Ax + B)(Ct + D)$, where A, B, C, D, p are constants

8. The steady-state temperature in the rod is given by

$\dfrac{d^2u}{dx^2} = 0$ or $u(x) = Ax + B$, that satisfies $u(0) = 20$ and $u(30) = 80$.

Using the B.C.'s, $B = 20$ and $A = 2$.
$\therefore$ The required solution is $u(x) = 2x + 20$.

9. $$Z\{x(n-m)\} = \sum_{n=0}^{\infty} x(n-m)z^{-n} = \sum_{p=-m}^{\infty} x(p)z^{-(m+p)}$$

$$= z^{-m} \sum_{p=0}^{\infty} x(p)z^{-p} \quad \text{(since } x(p) \text{ is causal)}$$

$$= z^{-m} \times x(z), \text{ if } n \geq m.$$

10. $$Z\left\{\left(\frac{1}{2}\right)^{n-1}\right\} = \left(\frac{1}{2}\right)^{-1} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} = 2 \cdot \frac{z}{z - 1/2} = \frac{4z}{2z-1}, \text{ if } n \geq 0.$$

(or) $Z\left\{\left(\frac{1}{2}\right)^{n-1}\right\} = z^{-1} \cdot Z\left(\frac{1}{2}\right)^{n}$, by the property proved in (9) above

$$= z^{-1} \cdot \frac{z}{z-1/2} = \frac{2}{2z-1}, \text{ if } n \geq 1.$$

Part – B

11. (a) Let the F.S. of $f(x) = x + x^2$ in $(-\pi, \pi)$ be

$$\frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx, \text{ by the property of odd and even functions.}$$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{4}{n^2}\left\{(-1)^n\right\}, \text{ if } n \neq 0.$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + x^2\right) dx = \frac{2}{\pi}\int_0^\pi x^2\, dx = \frac{2}{3}\pi^2$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + x^2\right)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^\pi x \sin nx\, dx = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\cdot\left(\frac{-\sin nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{-2}{n}(-1)^n$$

$\therefore$ Required F.S. of $f(x)$ is

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

11. (b) Give an even extension for $f(x)$ in $(-\pi, 0)$; i.e., put $f(x) = -\pi x - x^2$. Then $f(x)$ has become an even function in $(-\pi, \pi)$.

Let the F.S. of $f(x)$ be $\dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx$ in $(-\pi, \pi)$

$$a_n = \frac{2}{\pi}\int_0^\pi\left(\pi x - x^2\right)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\left(\pi x - x^2\right)\frac{\sin nx}{n} - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{2}{\pi n^2}\left\{-\pi(-1)^n - \pi\right\} = -\frac{2}{n^2}\left\{(-1)^n + 1\right\} = \begin{cases} -\dfrac{4}{n^2}, \text{ if } n \text{ is even} \\ \quad 0, \text{ if } n \text{ is odd}\end{cases}$$

$$a_0 = \frac{2}{\pi}\int_0^\pi\left(\pi x - x^2\right)dx = \frac{2}{\pi}\left\{\frac{\pi x^2}{2} - \frac{x^3}{3}\right\}_0^\pi = \frac{\pi^2}{3}$$

$\therefore$ Required F.S. is $f(x) = \dfrac{\pi^2}{6} - 4\sum\limits_{n=2,4,6\ldots}^{\infty}\dfrac{1}{n^2}\cos nx$.

12. (a) Let the F.s. of $f(x)$ be $\dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos\dfrac{n\pi x}{l} + \sum\limits_{n=1}^{\infty} b_n \sin\dfrac{n\pi x}{l}$ in $(0, 2l)$

$$a_n = \frac{1}{l}\left[\int_0^l x\cos\frac{n\pi x}{l}\,dx + \int_l^{2l}(2l-x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{l}\left[\left\{x\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^l + \left\{(2l-x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) + \left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_l^{2l}\right]$$

$$= \frac{1}{l}\left[\frac{l^2}{n^2\pi^2}\left\{(-1)^n - 1\right\} + \frac{l^2}{n^2\pi^2}\left\{(-1)^n - 1\right\}\right]$$

$$= \frac{2l}{n^2\pi^2}\left\{(-1)^n - 1\right\} = \begin{cases} \dfrac{-4l}{n^2\pi^2}, \text{ if } n \text{ is odd} \\ 0, \text{ if } n \text{ is even} \end{cases}$$

$$a_0 = \frac{1}{l}\left[\int_0^l x\,dx + \int_l^{2l}(2l-x)\,dx\right] = \frac{1}{l}\left[\frac{l^2}{2} + \frac{l^2}{2}\right] = l$$

$$b_n = \frac{1}{l}\left[\int_0^l x\sin\frac{n\pi x}{l}\,dx + \int_l^{2l}(2l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{l}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^l + \left\{(2l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) + \left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_l^{2l}\right]$$

$$= \frac{1}{l}\left[-\frac{l^2}{n\pi}\cos n\pi + \frac{l^2}{n\pi}\cos n\pi\right] = 0$$

$\therefore$ Required F.S. of $f(x)$ is

$$\frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} \text{ in } (0, 2l)$$

12. (b) Give an odd extension for $f(x)$ in $(-l, 0)$ i.e., put $f(x) = -(-x) = x$ in $(-l, 0)$.

$\therefore f(x)$ has become an odd function of x in $(-l, l)$

Let the F.S. of $f(x)$ be $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ in $(-l, l)$

$$b_n = \frac{2}{l}\int_0^l x \sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2l}{n\pi}\left\{(-1)^n\right\}$$

$\therefore$ Required Fourier half-range sine series of

$$x \approx \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l} \text{ in } (0, 1)$$

By Parseval's theorem

$$\frac{1}{2}\sum b_n^2 \text{ or } \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{l}\int_0^l x^2 dx \text{ or } \frac{l^2}{3}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

13. (a) (This problem is the worked example 8 (i) in page 4.14 of the book.)

13. (b) $F_c\,(e^{-ax}) = \int_0^{\infty} e^{-ax}\cos sx\,dx = \left\{\frac{e^{-ax}}{s^2 + a^2}\left(-a\cos sx + s\sin sx\right)\right\}_0^{\infty}$

$$= \frac{a}{s^2 + a^2}$$

Similarly, $F_c(e^{-bx}) = \frac{b}{s^2 + b^2}$

By a property of Fourier cosine transforms (Refer to page 4.33 of the book).

$$\int_0^{\infty} f(x)g(x)\,dx = \frac{2}{\pi}\int_0^{\infty}\overline{f}_C(s)\cdot\overline{g}_C(s)\,ds \qquad (1)$$

Putting $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$ in (1),

$$\frac{2}{\pi}\int_0^{\infty}\frac{a}{s^2 + a^2}\cdot\frac{b}{s^2 + b^2}\,ds = \int_0^{\infty} e^{-(a+b)x}dx$$

$$= \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = \frac{1}{a+b}$$

$$\therefore \quad \int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

Changing the dummy variable s into x, we get

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

14. (a) [This problem is the same as the worked example (7) given in page 4.13 of the book, except for a minor change towards the end of the solution.

After taking the inverse Fourier transforms, we will have

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)\cos xs\, ds = \frac{\pi}{4}\cdot\begin{cases}1-x^2, & \text{for } |x|<1\\ 0, & \text{for } |x|>1\end{cases}$$

Taking $x = 0$, we get

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) ds = \frac{\pi}{4}$$

Changing the dummy variable s into x, we get

$$\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right) dx = -\frac{\pi}{4}$$

14. (b) Consider $\text{Fs}(e^{-ax}) = \int_0^\infty e^{-ax}\sin sx\, dx = \dfrac{s}{s^2+a^2}$ (1)

Integrating both sides of (1) w.r.t. a between a and ∞,

$$\int_0^\infty \left(\frac{e^{-ax}}{x}\right)\sin sx\, dx = \left\{s\times -\frac{1}{s}\cot^{-1}\frac{a}{s}\right\}_a^\infty$$

i.e., $\displaystyle\int_0^\infty \left(\frac{e^{-ax}}{x}\right)\sin sx\, dx = \cot^{-1}\frac{a}{s}\,(a>0)$

Interchanging x and s and then multiplying by $\dfrac{2}{\pi}$,

$$\frac{2}{\pi}\int_0^\infty \left(\frac{e^{-as}}{s}\right)\sin xs\, ds = \cot^{-1}\frac{a}{x} \text{ i.e., } F_S^{-1}\left(\frac{e^{-as}}{s}\right) = \cot^{-1}\left(\frac{a}{x}\right) \quad (2)$$

Taking limits of (2) as $a \to 0$; $F_S^{-1}\left(\dfrac{1}{s}\right) = \cot^{-1}(0) = \dfrac{\pi}{2}$

15. (a) $z = px + qy + p^2 - q^2$ (1) is a Clairaut's type equation.

$\therefore$ Its C.S. is $z = ax + by + a^2 - b^2$ (2)

Differentiating (2) partially w.r.t. a; $x + 2a = 0$ (3)

Differentiating (2) partially w.r.t. b; $y - 2b = 0$ (4)

From (3) and (4), we have $a = -\dfrac{x}{2}$ and $b = \dfrac{y}{2}$. Using these values in (2), we get

$$z = -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$$

i.e.,
$$z = -\frac{x^2}{4} + \frac{y^2}{4}$$

$\therefore$ The required singular integral is $4z = y^2 - x^2$.

15. (b) $(D^2 - 2DD' + D'^2)\, z = e^{x+2y} + \sin(2x - 3y)$.

This is a homogeneous linear equation of the second order.

A.E. is $m^2 - 2m + 1 = 0$ or $(m - 1)^2 = 0$ $\quad\therefore m = 1, 1$

$\therefore$ C.F. $= \phi_1(y + x) + x\phi_2(y + x)$, where ϕ_1 and ϕ_2 are arbitrary functions.

$$\text{P.I.}_1 = \frac{1}{(D - D')^2} e^{x+2y} = \frac{1}{(1-2)^2} e^{x+2y} \text{ or } e^{x+2y}$$

$$\text{P.I.}_2 = \frac{1}{D^2 - 2DD' + D'^2} \sin(2x - 3y)$$

$$= \frac{1}{-4 - 2\times 6 + (-9)} \sin(2x - 3y) \text{ or } -\frac{1}{25} \sin(2x - 3y)$$

G.S. is $\quad z = \text{C.F.} + \text{P.I.}_1 + \text{P.I.}_2$

16. $(3z - 4y)p + (4x - 2z)q = 2y - 3x$ (1)

This is a Lagrange's Linear equation

L.S. S.E.'s are $\dfrac{dx}{3z - 4y} = \dfrac{dy}{4x - 2z} = \dfrac{dz}{2y - 3x}$ (2)

Using the multipliers 2, 3, 4, each ratio in (2) $= \dfrac{2dx + 3dy + 4dz}{0}$

$\therefore$ $2dx + 3dy + 4dz = 0$

$\therefore$ One solution of (2) is $2x + 3y + 4z = a$ (3)

Using the multiplier's x, y, z;

each ratio in (2) $= \dfrac{x\,dx + y\,dy + z\,dz}{0}$

$\therefore$ $x\,dx + y\,dy + z\,dz = 0$

$\therefore$ Another solution of (2) is $x^2 + y^2 + z^2 = b$ (4)

$\therefore$ G.S. of (1) is $f(2x + 3y + 4z, x^2 + y^2 + z^2) = 0$, where f is an arbitrary function.

16. (b) $(D^2 + 3DD' + 2D'^2)\, z = x + y$

This is a homogeneous linear equation of the second order.

A.E. is $m^2 + 3m + 2 = 0$ or $(m + 1)(m + 2) = 0$ $\quad \therefore m = -1, -2$.

$\therefore$ C.F. $= \phi_1(y - x) + \phi_2(y - 2x)$

$$\text{P.I.} = \frac{1}{(D+D')(D+2D')}(x+y)$$

$$= \frac{1}{D^2\left(1+\frac{D'}{D}\right)\left(1+\frac{2D'}{D}\right)}(x+y)$$

$$= \frac{1}{D^2\left(1+\frac{D'}{D}\right)}\left(1+\frac{2D'}{D}\right)^{-1}(x+y)$$

$$= \frac{1}{D^2\left(1+\frac{D'}{D}\right)}\left(1-\frac{2D'}{D}\right)(x+y)$$

$$= \frac{1}{D^2\left(1+\frac{D'}{D}\right)}(x+y-2x)$$

$$= \frac{1}{D^2}\left(1-\frac{D'}{D}\right)(y-x)$$

$$= \frac{1}{D^2}(y-x-x) = \frac{yx^2}{2} - \frac{x^3}{3}$$

$\therefore$ G.S. required is $z =$ C.F. + P.I.

17. [This problem is the same as the worked example (8) in page 4.38 of the book, with 60 replaced by l and $\frac{\lambda}{30}$ replaced by k. However the key steps relevant to the problem alone are given below without explanation].

Step (2) : $y(l, t) = 0$, for $t \geq 0$

Step (5) : $\frac{\partial y}{\partial t}(x, 0) = v$, where $v = \begin{cases} kx, \text{ in } 0 < x\,\frac{l}{2} \\ k(l-x), \text{ in } \frac{l}{2} < x < l \end{cases}$

Step (7) : $y(x, t) = \lambda \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$, since $p = \frac{n\pi}{l}$

Step (8) : $y(x, t) = \sum_{n=1}^{\infty}\lambda_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$

Step (9) : $\sum_{n=1}^{\infty}\left(\frac{n\pi a}{l}\lambda_n\right)\sin\frac{n\pi x}{l} = v = \sum b_n \sin\frac{n\pi x}{l}$

Comparing like terms, we get

$$\frac{n\pi a}{l}\lambda_n = b_n = \frac{2}{l}\left[\int_0^{l/2} kx\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} k(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$\therefore \qquad \lambda_n = \frac{2k}{n\pi a}\left[\left\{x\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)+\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right\}_0^{l/2} + \left\{(l-x)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right\}_{l/2}^{l}\right]$$

$$= \frac{2k}{n\pi a}\left[\left\{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} + \left\{\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right]$$

$$= \frac{4kl^2}{n^3\pi^3 a}\sin\frac{n\pi}{2}$$

$\therefore$ The required solution is

$$y(x,t) = \frac{4kl^2}{\pi^3 a}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cdot\sin\frac{n\pi at}{l}$$

18. The steady-state temperature distribution $u(x, y)$ at any point (x, y) of the plate is given by $u_{xx} + u_{yy} = 0$ (1)
We have to solve (1) satisfying the boundary conditions
$u(0, y) = 0$ (2); $u(l, y) = 0$ (3); $u(x, 0) = 0$ (4) and $u(x, 1) = lx - x^2$ for $0 < x < l$ (5)
The proper solution of equation (1) consistent with B.C. (5) is $u(x, y) = (A\cos px + B\sin px)(C\cos h\,py + D\sin h\,py)$ (6)
Using B.C. (2) in (6), we get $A = 0$
Using B.C. (3) in (6), we get $p = \frac{n\pi}{l}$ $(n = 0, 1, 2, \cdots \infty)$
Using B.C. (4) in (6), we get $C = 0$
$\therefore$ Required solution of (1) is

$$u(x, y) = \lambda \sin\frac{n\pi x}{l} \sin h\,\frac{n\pi}{l};\ n = 0, 1, 2, \cdots \infty$$

The most general solution of (1) is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_n \sin\frac{n\pi x}{l} \sin h\frac{n\pi y}{l} \tag{7}$$

Using B.C. (5) in (7), we get

$$\sum_{n=1}^{\infty} (\lambda_n \sin h\, n\pi) \sin\frac{n\pi x}{l} = lx - x^2 \text{ in } (0, l)$$

$$= \sum b_n \sin\frac{n\pi x}{l} \quad \text{(F.H.R.S.S.)}$$

Comparing like terms,

$$\lambda_n \sin h\, n\pi = b_n = \frac{2}{l}\int_0^l (lx - x^2) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{4l^2}{n^3\pi^3}\{1 - (-1)^n\}$$

$$= \begin{cases} \dfrac{8l^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Using this value of λ_n in (7), the required solution is

$$u(x, y) = \frac{8l^2}{\pi^3} \sum \frac{1}{n^3} \operatorname{cosec} h\, n\pi \sin\frac{n\pi x}{l} \sin h\frac{n\pi y}{l}.$$

19. (a) $Z\left\{\dfrac{1}{n+1}\right\} = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n} = z\sum_{n=0}^{\infty} \frac{1}{n+1} z^{-(n+1)}$

$$= z\left\{\frac{z^{-1}}{1}+\frac{z^{-2}}{2}+\frac{z^{-3}}{3}+\cdots\infty\right\} = -z\log\left(1-z^{-1}\right)$$

$$= z\log\left(\frac{z}{z-1}\right)$$

$$Z\left(\cos\frac{n\pi}{2}\right) = \frac{z^2}{z^2+1}$$

$$\therefore\ Z\left\{2^n\cos\frac{n\pi}{2}\right\} = Z\left\{\cos\frac{n\pi}{2}\right\}_{z\to\frac{z}{2}}, \text{ by frequency shifting property}$$

$$= \frac{z^2/4}{z^2/4+1} = \frac{z^2}{z^2+4}$$

19. (b) $Z^{-1}\left\{\dfrac{8z^2}{(2z-1)(4z-1)}\right\} = Z^{-1}\left\{\dfrac{2z}{2z-1}\cdot\dfrac{4z}{4z-1}\right\}$ or

$$= Z^{-1}\left\{\left(\frac{z}{z-1/2}\right)\cdot\left(\frac{z}{z-1/4}\right)\right\}$$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n, \text{ by convolution theorem}$$

$$= \sum_{r=0}^{n}\left(\frac{1}{2}\right)^{n-r}\cdot\left(\frac{1}{4}\right)^r = \left(\frac{1}{2}\right)^n\sum_{r=0}^{n}\left(\frac{1}{2}\right)^{-r}\cdot\left(\frac{1}{4}\right)^r$$

$$= \left(\frac{1}{2}\right)^n\sum_{r=0}^{n}\left(\frac{1}{2}\right)^r = \left(\frac{1}{2}\right)^n\left\{\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}\right\}$$

$$= 2\left\{\left(\frac{1}{2}\right)^n-\left(\frac{1}{2}\right)^{2n+1}\right\} \quad\text{or}\quad 2\cdot\left(\frac{1}{2}\right)^n-\left(\frac{1}{4}\right)^n$$

20. (a) $Z^{-1}\left\{\dfrac{z^2-3z}{(z+2)(z-5)}\right\}$ = Sum of the residues of $\left[\dfrac{z^{n+1}-3z^n}{(z+2)(z-5)}\right]$ at the isolated singularities. In this problem, the singularities are the simple poles $z = -2$ and $z = 5$.

$$(\text{Res.})_{z=-2} = \left[\frac{z^{n+1}-3z^n}{z-5}\right]_{z=-2} = -\frac{1}{7}\left[(-2)^{n+1}-3(-2)^n\right]$$

$$= -\frac{1}{7}\times(-2)^n\{-2-3\} = \frac{5}{7}(-2)^n$$

$$(\text{Res.})_{z=5} = \left(\frac{z^{n+1} - 3z^n}{z+2}\right)_{z=5} = \frac{5^{n+1} - 3.5^n}{7} = \frac{2}{7} \cdot 5^n$$

$$\therefore \quad Z^{-1}\left\{\frac{z^2 - 3z}{(z+2)(z-5)}\right\} = \frac{2}{7} \cdot 5^n + \frac{5}{7} \cdot (-2)^n$$

20. (b) $y_{n+2} - 4y_n = 3^n$

Taking Z-transforms on both sides of the equation,

$$\left[z^2 \bar{y}(z) - z^2 y(0) - zy(1)\right] - 4\bar{y}(z) = \frac{z}{z-3}$$

i.e., $\bar{y}(z) = \dfrac{z}{(z-3)(z^2-4)}$, since $y(0) = y(1) = 0$

Let $\dfrac{\bar{y}(z)}{z} = \dfrac{1}{(z+2)(z-2)(z-3)}$

$= \dfrac{1/20}{z+2} - \dfrac{1/4}{z-2} + \dfrac{1/5}{z-3}$, by splitting into partial fractions

$\therefore \quad \bar{y}(z) = \dfrac{1}{20} \cdot \dfrac{z}{z+2} - \dfrac{1}{4} \cdot \dfrac{z}{z-2} + \dfrac{1}{5} \cdot \dfrac{z}{z-3}$

Inverting, we get

$$y(n) = \frac{1}{20} \cdot (-2)^n - \frac{1}{4} \cdot 2^n + \frac{1}{5} \cdot 3^n$$

B.E./B.Tech. Degree Examinations, November/December 2010
Regulations 2008

Third Semester

Civil Engineering

Common to all branches

MA2211 Transforms and Partial Differential Equations

Time: Three Hours **Maximum: 100 Marks**

Answer ALL Questions

Part A – (10 × 2 = 20 Marks)

1. Write the conditions for a function $f(x)$ to satisfy for the existence of a Fourier series.
2. If $x^2 = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\cos nx$, deduce that $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots = \dfrac{\pi^2}{6}$.
3. Find the Fourier cosine transform of e^{-ax}, $x \geq 0$.
4. If $F(x)$ is the Fourier transform of $f(x)$, show that $F[f(x-a)] = e^{ias} F(s)$.
5. Form the partial differential equation by eliminating the constants a and b from $z = (x^2 + a^2)(y^2 + b^2)$.
6. Solve the partial differential equation $pq = x$.
7. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y(x, 0) = y_0 \sin^3\left(\dfrac{\pi x}{l}\right)$. If it is released from rest in this position, write the boundary conditions.
8. Write all three possible solutions of steady state two-dimensional heat equation.
9. Find the Z-transform of $\sin \dfrac{n\pi}{2}$.
10. Find the difference equation generated by $y_n = a_n + b^{2n}$

Part B – (5 × 16 = 80 marks)

11. (a) (i) Find the fourier series for $f(x) = 2x - x^2$ in the interval $0 < x < 2$.

(ii) Find the half range cosine series of the function $f(x) = x(\pi - x)$ in the interval $0 < x < \pi$. Hence deduce that $\dfrac{1}{1^4} + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \cdots = \dfrac{\pi^4}{90}$.

Or

(b)(i) Find the complex form of the Fourier series of $f(x) = e^{ax}$, $-\pi < x < \pi$.

(ii) Find the first two harmonics of the Fourier eries from the following table:

x:	0	1	2	3	4	5
y:	9	18	24	28	26	20

12. (a) (i) Find the Fourier transform of $f(x)=\begin{cases}1-|x| & \text{if } |x|<1\\ 0 & \text{if } |x|\ge 1\end{cases}$. Hence deduce the value of $\int_0^\infty \frac{\sin^4 t}{t^4}\,dt$. (10)

(ii) Show that the Fourier transform of $e^{\frac{-x^2}{2}}$ is $e^{\frac{-s^2}{2}}$. (6)

Or

(b) (i) Find the Fourier sine and cosine transforms of

$$f(x)=\begin{cases}\sin x, & 0<x<a\\ 0, & x>a\end{cases}$$

(ii) Using Fourier cosine transform method, evaluate

$$\int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)}$$

13. (a) Solve:

(i) $(x^2-yz)\,p+(y^2-zx)q=z^2-xy$ (8)

(ii) $p(1+q)=qz$ (4)

(iii) $p^2+q^2=x^2+y^2$ (4)

Or

(b)(i) Find the partial differential equation of all planes which are at a constant distance 'a' from the origin.

(ii) Solve $(D^2+2DD'+D'^2-2D-2D')\,z=\sin(x+2y)$ where $D=\frac{\partial}{\partial x}$ and $D'=\frac{\partial}{\partial y}$.

14. (a)(i) A tightly stretched string of length l has its ends fastened at $x=0$ and $x=l$. The midpoint of the string is then taken to height 'b' and released from rest in that position. Find the lateral displacement of a point of the string at time 't' from the instant of release.

Or

(ii) A rectangular plate with insulated surface is 10 cm wide and so long compared to its width that may be considered infinite in length without introducing appreciable error. The temperature at short edge $y=0$ is given by $u=\begin{cases}20x & \text{for } 0\le x\le 5\\ 20(10-x) & \text{for } 5\le x\le 10\end{cases}$ and the other three edges are kept at 0°C. Find the steady state temperature at any point in the plate.

15. (a)(i) Solve by Z-transform $u_{n+2} + u_n = 2^n$ with $u_0 = 2$ and $u_1 = 1$.

(ii) Using convolution theorem, find the inverse Z – transform of $\left(\frac{z}{z-4)}\right)^3$.

(b)(i) Find $Z^{-1}\left[\frac{z(z^2-z+2)}{(z+1)(z-1)^2}\right]$ and $Z^{-1}\left[\frac{z}{(z-1))z-2)}\right]$ (6 + 4)

(ii) Find $Z\,(na^n \sin n\theta)$. (6)

Solutions

Part A

1. What are required are the Dirichlet's conditions that are available in the book.
2. Given: $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$. (*Note*: The question is incomplete) We shall assume that the given series is the Fourier half-range cosine series of $f(x) = x^2$ in $0 \le x \le \pi$.
Putting $x = \pi$, a point of continuity of x^2 in the R.S. of the given equality, we have $\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2} = \pi^2 \quad \therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.
3. $$F_C\,(e^{-ax}) = \int_0^{\infty} e^{-ax}\cos sx\,dx = \left[\frac{e^{-ax}}{s^2+a^2}(-a\cos sx + s\sin sx)\right]_0^{\infty}$$
$$= \frac{a}{s^2+a^2}\ (a > 0)$$
4. $$F\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a)\,e^{-isx}\,dx = \int_{-\infty}^{\infty} f(t)\,e^{-is(t+a)}dt, \text{ on putting } t = x-a$$
$$= e^{-ias}\int_{-\infty}^{\infty} f(t)e^{-ist}\,dt = e^{-ias}\,F(s)$$
[Note: If the definition is taken as $F(s) = F\{f(x) = \int_{-\infty}^{\infty} e^{-isx} f(x)\,dx$, the resulting given in the problem would have been obtained.]
5. $z = (x^2 + a^2)(y^2 + b^2)$ .. (1) Differing (1) partially w.r.t. x and then w.r.t. y, we get $p = 2x(y^2 + b^2)$ ... (2) and $q = 2y(x^2 + a^2)$ ... (3)
Substituting for $(x^2 + a^2)$ and $(y^2 + b^2)$ obtained from (3) and (2) in (1), we get the required P.D.E. as $z = \frac{q}{2y}\cdot\frac{p}{2x}$ or $pq = 4xyz$
6. $pq = x$; i.e., $\frac{p}{x} = \frac{1}{q} = a$, say, $\therefore p = ax$ and $q = \frac{1}{a}$

$dz = p\,dx + q\,dy = ax\,dx + \frac{1}{a}dy$; Integrating both sides, we get the complete solution as $z = \frac{ax^2}{2} + \frac{1}{a}y + b \cdots (1)$

Putting $b = f(a)$, (1) becomes = $z = \frac{a}{2}x^2 + \frac{1}{a}y + f(a) \cdots$ (2)

Differentiating (2) w.r.t. a partially, $0 = \frac{x^2}{2} - \frac{1}{a^2}y + f'(a) \cdots (3)$

Eliminant of a from (2) and (3) is the G.S. of the given equation.

7. Boundary condition are (i) $y(0, t) = 0$, $t \geq 0$; (ii) $y(l, t) = 0$, $t \geq 0$; (iii) $\frac{\partial y}{\partial t}(x, 0) = 0$, $0 < x < l$ and (iv) $y(x, 0) = y_0 \sin^3\left(\frac{\pi x}{l}\right), 0 < x < l.$

8. The three possible solutions of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ are

   (i) $u(x, y) = (Ae^{px} + Be^{-px})\,(C\cos py + D\sin py)$

   (ii) $u(x, y) = (A\cos px + B\sin px)\,(Ce^{py} + De^{-py})$

   (iii) $u(x, y) = (Ax + B)\,(Cy + D)$, where A, B, C, D are arbitrary constants.

9. $Z(a^n) = \frac{z}{z-a}$, if $|z| > a$; $Z(e^{in\pi/2}) = Z(e^{i\pi/2})^n = \frac{z}{z - e^{i\pi/2}} = \frac{z}{z-i} = \frac{z(z+i)}{z^2+1}$

   i.e. $Z\left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right) = \frac{z^2}{z^2+1} + i\,\frac{z}{z^2+1} \therefore Z\left(\sin\frac{n\pi}{2}\right) = \frac{z}{z^2+1}$

10. $y_n = a + b_n \cdot 2^n$ (1)

    [Note the question is wrong. It has been corrected]

    $y_{n+1} = a + 2(b \cdot 2^n)$ ... (2); $y_{n+2} = a + 4(b \cdot 2^n)$ ...(3)

    (2) – (1) gives; $y_{n+1} - y_n = b \cdot 2^n$ ... (4); (3) – (2) gives; $y_{n+2} - y_{n+1} = 2b \cdot 2^n$ .. (5)

    From (4) and (5); $y_{n+2} - y_{n+1} = 2(y_{n+1} - y_n)$; i.e., $y_{n+2} - 3y_{n+1} + 2y_n = 0$ is the difference equation required.

Part - B

11. (a)(i) We assume that the Fourier series of period 2 is required for $f(x) = 2x - x^2$ in $0 < x < 2$.

    The problem in which F.S. of period $2l$ for $f(x) = 2lx - x^2$ in $(0, 2l)$ is a worked example in the book. We simply put $l = 1$.

    Ans. $2x - x^2 = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(n\pi x)$ in $(0,2)$

    (ii) This problem is a worked example in the book.

11. (b)(i) Let the complex form of the Fourier series of $f(x) = e^{ax}$ in $-\pi < x < \pi$

be $f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{inx}$.

Then $c_n = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\cdot e^{-inx}\, dx = \dfrac{1}{2\pi}\left[\dfrac{e^{(a-in)x}}{a-in}\right]_{-\pi}^{\pi}$

$$= \frac{1}{2\pi(a-in)}\{e^{(a-in)\pi} - ^{-(a-in)\pi}\}$$

$$= \frac{1}{2\pi(a-in)}\{e^{a\pi}(-1)^n - e^{-a\pi}(-1)^n\} = \frac{\sinh a\pi}{\pi}\,\frac{(-1)^n}{a-in}$$

$$= \frac{\sinh a\pi}{\pi}\,\frac{(-1)^n\,(a+in)}{n^2+a^2}$$

$\therefore$ Required F.S. is $f(x) = \dfrac{\sinh a\pi}{\pi}\sum_{n=-\infty}^{\infty}\dfrac{(-1)^n\,(a+in)}{n^2+a^2}\, e^{inx}\ in\ (-\pi, \pi)$

(ii) The internal (0, 6) is divided into 6 sub-intervals each of length 1.
$2l = 6\ \therefore\ l = 3$; $y_0 = 9$, $y_1 = 18$, $y_2 = 24$, $y_3 = 28$, $y_4 = 26$, $y_5 = 20$.

Let the required F.S. be $y = \dfrac{a_0}{2} + \sum a_n \cos\dfrac{n\pi x}{3} + b_n \sin\dfrac{n\pi x}{3}\cdots$ (1)

$$a_0 = 2\times\frac{1}{6}\sum_{r=0}^{5} y_r = \frac{125}{3} = 41.67;\ a_1 = \frac{1}{3}\sum y_r \cos\frac{\pi}{3}$$

$$= \frac{1}{3}\left[(y_0 - y_3) + (y_1 + y_5 - y_2 - y_4\right]\times\cos\frac{\pi}{3}$$

$$= \frac{1}{3}[(9-28) + (18+20-24-26)\times 0.5] = \frac{25}{3} = -8.33$$

$$b_1 = \frac{1}{3}\sum y_r \sin\frac{\pi x_r}{3} = \frac{1}{3}(y_1 + y_2 - y_4 - y_5)\sin\frac{\pi}{3}$$

$$= -\frac{4}{3}\times 0.866 = -1.15$$

$$a_2 = \frac{1}{3}\sum y_r \cos\frac{2\pi x_r}{3} = \frac{1}{3}[(y_0 + y_3) - (y_1 + y_2 + y_4 + y_5)\cos 60°]$$

$$= -\frac{7}{3} = -2.33$$

$$b_2 = \frac{1}{3}\sum y_r \sin\frac{2\pi}{3}x_r = \frac{1}{3}(y_1 + y_4 - y_2 - y_5)\sin 60° = 0.$$

Required F.S. is

$$y = 20.84 + \left(-8.33\cos\frac{\pi x}{3} - 1.15\sin\frac{\pi x}{3}\right) - 2.33\cos\frac{2\pi x}{3}.$$

12. (a)(i) The problem is a worked example in the book.

(ii) The problem of finding $F(e^{-a^2x^2})$ is a worked example in the book.

It is derived that $F(e^{-a^2x^2}) = \dfrac{1}{a\sqrt{2}}\, e^{-s^2/4a^2}$.

Putting $a^2 = \dfrac{1}{2}$ or taking $a = \dfrac{1}{\sqrt{2}}$ in this result, we get

$$F(e^{-x^2/2}) = e^{-s^2/2}$$

12. (b)(i) $F_S\{f(x)\} = \int_0^\infty f(x)\sin sx\,dx$ and $F_C\{f(x)\} = \int_0^\infty f(x)\cos sx\,dx$

$$\therefore\ F_s\{\sin x\} = \int_0^\infty \sin x\cdot\sin sx\,dx = \frac{1}{2}\int_0^a [\cos(s-1)x - \cos(s+1)x]\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(s-1)x}{s-1} - \frac{\sin(s+1)x}{s+1}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1}\right]_0^a$$

$$F_c\{\sin x\} = \int_0^a \cos sx\cdot\sin x\,dx = \frac{1}{2}\int_0^a [\sin(s+1)x - \sin(s-1)x]\,dx$$

$$= \frac{1}{2}\left[-\frac{\cos(s+1)x}{s+1} + \frac{\cos(s-1)x}{s-1}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{1}{s+1}\{1-\cos(s+1)a\} - \frac{1}{s-1}\{1-\cos(s-1)a\}\right]$$

(ii) Let $f(t) = e^{-at}$ and $g(t) = e^{-bt}$ $\quad\therefore\quad \bar{f}_C(s) = \dfrac{a}{s^2+a^2}$; $\bar{g}_C(s) = \dfrac{b}{s^2+b^2}$

By the property of F' cosine transforms,

$$\int_0^\infty f(t)\,g(t)\,dt = \frac{2}{\pi}\int_0^\infty \bar{f}_C(s)\cdot\bar{g}_C(s)\,ds$$

$$\therefore\quad \int_0^\infty e^{-(a+b)t}\,dt = \frac{2}{\pi}\int_0^\infty \frac{ab}{(s^2+a^2)(s^2+b^2)}\,ds$$

$$\therefore \quad \int_0^\infty \frac{ds}{(s^2+a^2)(s^2 \mid b^2} = \frac{\pi}{2ab}\left\{\frac{e^{-(a+b)t}}{-(a+b)}\right\}_0^\infty = \frac{\pi}{2ab\,(a+b)}$$

Changing s as t, the required result follows

13. (a)(i) $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

This is a Lagrange's liner equation $P_p + Q_q = R$.

This subsidiary equation are $\dfrac{dx}{x^2 - yz} = \dfrac{dy}{y^2 - zx} = \dfrac{dz}{z^2 - xy}$ (1)

Each ratio in (1) is equal to

$$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(x+y+z)} = \frac{d(z-x)}{(z-x)(x+y+z)} \qquad (2)$$

From the first two ratios in (2), we get $\log(x-y) = \log(y-z) + \log a$

viz., $\dfrac{x-y}{y-z} = a$ (3)

Also each ratio is (1) = $\dfrac{d(x+y+z)}{(\Sigma x^2 - \Sigma yz)} = \dfrac{x\,dx + y\,dy + z\,dx}{\Sigma x^3 - 3xyz}$

i.e., $\dfrac{d(\Sigma x)}{\Sigma x^2 - \Sigma yz} = \dfrac{\frac{1}{2} d\,\Sigma x^2}{(\Sigma x)\,(\Sigma x^2 - \Sigma yz)}$;i.e., $2\Sigma x d(\Sigma x) = d(\Sigma x^2)$

Solving, we get $(\Sigma x)^2 - (\Sigma x^2) = 2b$; i.e. $xy + yz + zx = b$ (4)

$\therefore$ G.S of the given L.L.E. is $f\left\{\dfrac{x-y}{y-z)}, xy + yz + zx\right\} = 0.$

13. (a)(ii) $p(1 + q) = qz$... (1). Let a solution of (1) be $z = z(x + ay)$ (2)

From (2), $p = \dfrac{dz}{du}$ and $q = a\dfrac{dz}{du}$.

Since (2) is a solution of (1), $\dfrac{dz}{du}\left\{1 + a\dfrac{dz}{du}\right\} = a\dfrac{dz}{du}\cdot z$

Since $\dfrac{dz}{du} \neq 0$, we get $1 + a\dfrac{dz}{du} = az$ or $\displaystyle\int \frac{a\,d\,z}{az-1} = \int du + b$

$\therefore$ C.S. of(1) is $\log(az - 1) = x + ay + b$...(3)

To get G.S., we put $b = f(a)$ in (3), where f is an arbitrary function i.e., (3) becomes $\log(az - 1) = x + ay + f(a)$ (4)

Differentiating (4) partially w.r.t. a; $\dfrac{z}{az-1} = y + f'(a) \cdots (5)$

Eliminating a between (4) and (5), the G.S. of (1) is obtained.

13. (a)(iii) $p^2 + q^2 = x^2 + y^2$ ... (1); i.e. $p^2 - x^2 = y^2 - q^2 = a^2$...(2) where a is an arbitrary constant.

From (2); $p = \sqrt{x^2 + a^2}$ and $q = \sqrt{y^2 - a^2}$

Now $dz = pdx + q\,dy$

$\therefore$ C.S. of (1) is $z = \int \sqrt{x^2 + a^2}\, dx + \int \sqrt{y^2 - a^2}\, dy + b$

i.e., $2z = x\sqrt{x^2 + a^2} + a^2 \sinh^{-1}\left(\dfrac{x}{a}\right) + y\sqrt{y^2 - a^2} - a^2 \cosh^{-1} \dfrac{y}{a} + b$

(3) is the C.S. Putting $b = f(a)$ and differentiating. (3)w.r.t. a, we get a result (4)

The eliminant of a between (3) and (4) is the G.S. of (1)

13. (b)(i) This problem is a worked example in the book.

13. (b)(ii) $(D^2 + 2DD' + D'^2 - 2D - 2D')\, z = \sin(x + 2y)$ (1)

i.e., $(D + D')(D + D' - 2)z = \sin(x + 2y)$

Since the part of the C.F. Corresponding to $(D - aD' - b)z = 0$ is $e^{bx} f(y + ax)$, the C.F. of the solution of (1)

$$= f_1(y - x) + e^{2x} f_2(y - x) \qquad (2)$$

$$\text{P.I.} = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} \sin(x + 2y)$$

$$= \frac{1}{-1 - 4 - 4 - 2D - 2D'} \sin(x + 2y)$$

$$= \frac{(2D + 2D' - 9)}{81 - (2D + 2D')^2} \sin(x + 2y)$$

$$= \frac{1}{117}[2\cos(x + 2y) + 4\cos(x + 2y) - 9\sin(x + 2y)]$$

$$= \frac{1}{39}[2\cos(x + 2y) - 3\sin(x + 2y)] \qquad (3)$$

$\therefore$ G.S. of (1) is z = C.F. + P.I, which are given by (2) and (3)

14. (a) This problem is the same as a worked example in the book. In the W.E., the length of the string is taken as $(2l)$ instead of l.

$\therefore$ In the W.E., wherever we get $2l$, it is to be replaced by b or l should be replaced by $l/2$. Finally the required solution will become

$$y(x, t) = \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{2} \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}.$$

14. (b) In steady-state temperature at any point (x, y) in the plate is given by $u_{xx} + u_{yy} = 0$... (1). We have to solve (1) satisfying the following boundary conditions.

$u(0, y) = 0$ ... (2); $u(10, y) = 0$ ... (3); $u(x, \infty) = 0$... (4) and $u(x, 0)$

$$u(x, 0) = \begin{cases} 20x, & \text{in } 0 \le x \le 5 \\ 20(10 - x), & \text{in } 5 \le x \le 10 \end{cases} \qquad (5)$$

The 3 possible solutions of (1) are $u(x, y) = (A\, e^{px} + Be^{-px})\,(C \cos py + D \sin py)$ ... (6); $u(x, y) = (A \cos px + B \sin px)\,(C\, e^{py} + De^{-py})$ ... (7) and $u(x, y) = Ax + B)\,(Cy + D)$ ... (8).

Since $u \to 0$ as $y \to \infty$ by B.C.(4), (7) is the proper solution with $C = 0$.

Using B.C. (2) in (7) (with C deleted), we get $A = 0$

Using B.C. (3) in (7), we get $\sin 10p = 0 \;\therefore\; p = \dfrac{n\pi}{10}$, $n = 0, 1, 2, \ldots \infty$

$\therefore$ Required solution is $u(x, y) = \lambda \sin \dfrac{n\pi x}{10}\, e^{-n\pi y/10}$, where λ = B.D.

The general form of the required solution is

$$u(x, y) = \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{10}\, e^{-n\pi y/10}, \tag{9}$$

Using B.C. (5) in (9), we have $\displaystyle\sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{10} = u(x, 0)$ as given.

Let the Fourier half-range sine series of $u(x, 0)$ in (0, 10) be

$\displaystyle\sum b_n \sin \frac{n\pi x}{10}$ ... (10), where $b_n = \lambda_n = \dfrac{2}{10}\displaystyle\int_0^{10} u(x, 0) \sin \frac{n\pi x}{10}\, dx$

i.e.,
$$\lambda_n = \frac{1}{5}\left[\int_0^5 20x \sin \frac{n\pi x}{10}\, dx + \int_5^{10} 20(10 - x) \sin \frac{n\pi x}{10}\, dx\right]$$

$$= 4\left[\left\{x \cdot \left(\frac{-\cos \frac{n\pi x}{10}}{n\pi/10}\right)^{-1} - \left(\frac{-\sin \frac{n\pi x}{10}}{n^2\pi^2/10^2}\right)\right\}_0^5\right.$$

$$\left. + \left\{(10 - x)\left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) + \left(\frac{-\sin \frac{n\pi x}{10}}{\frac{n^2\pi^2}{10^2}}\right)\right\}_5^{10}\right]$$

$$= 4\left[\left\{-\frac{50}{n\pi} \cos \frac{n\pi}{2} + \frac{100}{n^2\pi^2} \sin \frac{n\pi}{2}\right\}\right.$$

$$\left. + \left\{\frac{50}{n\pi} \cos \frac{n\pi}{2} + \frac{100}{n^2\pi^2} \sin \frac{n\pi}{2}\right\}\right] = \frac{800}{n^2\pi^2} \sin \frac{n\pi}{2}$$

Using this value of λ_n in (9), the required solution is

$$u(x, y) = \frac{800}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{10}\, e^{-n\pi y/10}$$

15. (a)(i) $u_{n+2} - 2u_{n+1} + u_n = 2^n$; $u_0 = 2$ and $u_1 = 1$

Taking z-transforms of the given equation,

$$[z^2\,\bar{u}(z) - z^2\,u(0) - z\,u(1)] - 2[z\bar{u}(z) - zu(0)] + \bar{u}(z) = \frac{z}{z-2}$$

i.e., $(z^2 - 2z + 1)\,\bar{u}(z) - 2z + 4z = \dfrac{z}{z-2}$

i.e., $\dfrac{\bar{u}(z)}{z} = \dfrac{1}{(z-1)^2(z-2)} + \dfrac{2z}{(z-1)^2} - \dfrac{3}{(z-1)^2} = \dfrac{1}{z-2} + \dfrac{1}{z-1} - \dfrac{2}{(z-1)^2}$

$\therefore\ \bar{u}\,(z) = \dfrac{z}{z-2} + \dfrac{z}{z-1} - \dfrac{2z}{(z-1)^2}$

Inverting, we get $u_n = 2^n + 1 - 2n$.

(ii) $Z^{-1}\left\{\dfrac{z^2}{(z-4)^2}\right\} = Z^{-1}\left\{\left(\dfrac{z}{z-4)}\right) * \left(\dfrac{z}{z-4}\right)\right\}$ by convolution theorem

$$= 4^n * 4^n = \sum_{r=0}^{n} 4^r \cdot 4^{n-r} = (n+1)\,4^n \qquad (1)$$

Now $Z^{-1}\left\{\dfrac{z^3}{(z-4)^3}\right\} = Z^{-1}\left\{\dfrac{z^2}{(z-4)^2}\right\} * Z^{-1}\left\{\dfrac{z}{z-4}\right\}$

$= (n + 1)4^n * 4^n$ by (1)

$= \displaystyle\sum_{r=0}^{n} (r+1)4^r \times 4^{n-r}$, by convolution theorem

$= \dfrac{1}{2}\,4^n (n+1)(n+2)$ or $\dfrac{1}{2}(n^2 + 3n + 2)\cdot 4^n$

15. (b)(i) (1) $Z^{-1}\left\{\dfrac{z\,(z^2 - z + 2}{(z+1)(z-1)^2}\right\} = z^1\,\{\bar{f}(z)\}$, say

Then $\dfrac{\bar{f}(z)}{z} = \dfrac{z^2 - z + 2}{(z+1)(z-1)^2} = \dfrac{A}{z+1} + \dfrac{B}{z-1} + \dfrac{C}{(z-1)^2}$

$$= \frac{1}{z+1} + \frac{1}{(z-1)^2}$$

$\therefore\ \bar{f}(z) = \dfrac{z}{z+1} + \dfrac{z}{(z-1)^2}$; Inverting, $f(n) = (-1)^n + n$

(Note: This problem is worked out in the book, by Residue theorem method)

(2) $Z^{-1}\left\{\dfrac{z}{(z-1)(z-2)}\right\} = Z^{-1}\,\{\bar{f}\,(z)\}$, say

Then $\dfrac{\overline{f}(z)}{z} = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1} \therefore \overline{f}(z) = \dfrac{z}{z-2} - \dfrac{z}{z-1}$

Inverting, we get, $f(n) = 2^n - 1$.

(ii) $Z(na^n \sin n\,\theta)$.

$$Z(\sin n\,\theta) = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}; \therefore z(a^n \sin\theta) = \frac{z/a\sin\theta}{z^2/a^2 - 2z/a\cos\theta + 1}$$

i.e., $Z\,(a^n \sin n\,\theta) = \dfrac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$

$$\therefore \quad z\,(n\,a^n \sin n\,\theta) = -z\frac{d}{dy}\left\{\frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}\right\}$$

$$= -(a\sin\theta)\,z\left[\frac{(z^2 - 2az\cos\theta + a^2)\cdot 1 - z(2z - 2a\cos\theta)}{(z^2 - 2az\cos\theta + a^2)^2}\right]$$

$$= \frac{(a\sin\theta)\,z(z^2 - a^2)}{(z^2 - 2az\cos\theta + a^2)^2}$$

B.E./B.Tech. Degree Examinations, April/May 2011
Regulations 2008

Third Semester

Common to all branches

MA2211 Transforms and Partial Differential Equations

Time: Three Hours **Maximum: 100 Marks**

Answer ALL Questions

Part A – (10 × 2 = 20 Marks)

1. Give the expression for the Fourier Series co-efficient b_n for the function $f(x)$ defined in (–2, 2).

2. Without finding the values of a_0, a_n and b_n, the Fourier coefficients of Fourier series, for the function $F(x) = x^2$ in the interval $(0, \pi)$ find the value of $\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2\right]$

3. State and prove the change of scale property of Fourier Transform.
4. If $F_c(s)$ is the Fourier cosine transform of $f(x)$, prove that the Fourier cosine transform of $f(ax)$ is $\frac{1}{a}F_c\left(\frac{s}{a}\right)$.
5. From the partial differential equation by eliminating the arbitrary constants a and b from $z = (x^2 + a)(y^2 + b)$.
6. Solve the equation $(D - D')^3 z = 0$.
7. A rod 40 cm long with insulated sides has its ends A and B kept at 20°C and 60°C respectively. Find the steady state temperature at a location 15 cm from A.
8. Write down the three possible solutions of Laplace equation in two dimensions.

9. Find the Z-transform of a^n.

10. What advantage is gained when Z-transform is used to solve difference equation?

Part B – (5 × 16 = 80 marks)

11. (a) (i) Expand $f(x) = x(2\pi - x)$ as Fourier series in $(0, 2\pi)$ and hence deduce that the sum of $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

(ii) Obtain the Fourier series for the function $f(x)$ given by

$f(x)=\begin{cases}1-x, & -\pi<x<0\\ 1+x, & 0<x<\pi\end{cases}$. Hence deduce that

$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}.$

Or

(b)(i) Obtain the sine series for $f(x)=\begin{cases}x & \text{in } 0\le x\le \frac{l}{2}\\ l-x & \text{in } \frac{l}{2}\le x\le l\end{cases}.$

(ii) Find the Fourier series up to second harmonic for $y = f(x)$ from the following values.

x:	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y:	1.0	1.4	1.9	1.7	1.5	1.2	1.0

12. (a)(i) Find the Fourier transform of $f(x)=\begin{cases}1-x^2 & \text{if } |x|<1\\ 0 & \text{if } |x|>1\end{cases}$. Hence evaluate $\int_0^\infty \left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\,dx.$

(ii) Find the Fourier transform of $f(x)$ given by $f(x)=\begin{cases}1 & \text{for } |x|<a\\ 0 & \text{for } |x|>a>0\end{cases}$ and using Parseval's identity prove that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}.$

Or

(b) (i) Find the Fourier sine transform of $f(x)=\begin{cases}x, & 0<x<1\\ 2-x, & 1<x<2\\ 0, & x>2\end{cases}.$

(ii) Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using Fourier cosine transforms of e^{-ax} and e^{-bx}.

13. (a)(i) Form the partial differential equation by eliminating arbitrary functions f and ϕ from $z = f(x + ct) + \phi(x - ct)$.

(ii) Solve the partial differential equation $(mz - ny)p + (nx - lz)q = ly - mx$.

Or

(b)(i) Solve $(D^2 - D'^2)\, z = e^{x-y}\sin(2x + 3y)$.

(ii) Solve $(D^2 - 3DD' + 2D'^2 - 2D')\, z = x + y + \sin(2x + y)$.

14. (a) A uniform string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form of the curve $y = kx(l - x)$ and then released from this position at time $t = 0$. Derive the expression for the displacement of any point of the string at a distance x from one end at time t.

Or

(b) A rectangular plate with insulated surface is 20 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $x = 0$ is given by $u = \begin{cases} 10y & \text{for } 0 \le y \le 10 \\ 10(20 - y) & \text{for } 10 \le y \le 20 \end{cases}$ and the two long edges as well as the other short edge are kept at 0°C. Find the steady state temperature distribution in the plate.

15. (a)(i) Using convolution theorem, find inverse Z-transform of $\dfrac{z^2}{(z-1)(z-3)}$.

(ii) Find the Z-transforms of an cos $n\theta$ and $e^{-at}\sin bt$.

(b)(i) Solve the difference equation $y(n + 3) - 3y(n + 1) + 2y(n) = 0$, given that $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$.

(ii) Derive the difference equation from $y^n = (A + Bn)(-3)^n$. (6)

Solutions

Part A

1. If the function $f(x)$ is an *odd* function in (–2, 2), then the Fourier series will be of the form $\sum b_n \sin \dfrac{n\pi x}{2}$, where

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx \text{ or } \int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$$

If the function $f(x)$ is an even function in (–2, 2), then the F.S. will be of the form $\dfrac{a_0}{2} + \sum a_n \cos\dfrac{n\pi x}{2}$ and hence $b_n = 0$.

2. $\dfrac{a_0^2}{2} + \displaystyle\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = 2\overline{y}^2 = 2\times\frac{1}{\pi}\int_0^{\pi} y^2\,dx = \frac{2}{\pi}\int_0^{\pi} x^4\,dx$, since the given function is $y = f(x) = x^2$ in $(0, \pi)$

$\therefore$ Required value = $\dfrac{2}{\pi}\times\dfrac{\pi^5}{5} = \dfrac{2}{5}\pi^4$.

3. This is a standard result, worked out in the book.

4. $\int_0^\infty f(x) \cos sx\, dx = F_c(s)$

$\therefore \quad \int_0^\infty f(ax) \cos sx\, dx = \int_0^\infty f(t) \cos\left(\frac{s}{a}\right) t \cdot \frac{1}{a}\, dt,$ on putting $ax = t (a > 0)$

i.e., $\quad F_c\{f(ax)\} = \frac{1}{a} F_c\left(\frac{s}{a}\right)$

5. $z = (x^2 + a)(y^2 + b)$ ... (1); Differentiating (1) partially w.r.t. x and then w.r.t. y, we get $p = 2x(y^2 + b)$ ... (2) and $q = 2y(x^2 + a)$ ... (3).
   Using the values of $(x^2 + a)$ and $(y^2 + b)$ obtained from (3) and (2) in (1), we get, $z = \frac{q}{2y} \cdot \frac{p}{2x}$ or $pq = 4xyz$, which is the P.D.E. required.

6. $(D - D')^3 z = 0$; The A.E. is $(m - 1)^3 = 0 \quad \therefore \; m = 1, 1, 1.$
   $\therefore$ the G.S. of the given PDE is $z = f_1(y + x) + x f_2(y + x) + x^2 f_3(y + x)$, where f_1, f_2, f_3 are arbitrary functions.

7. The steady-state temperature in the rod is given by $\frac{d^2u}{dx^2} = 0$ ... (1) with the B.C′s $u(0) = 20$ ... (2) and $u(40) = 60$... (3).
   Solving (1), we get $u(x) = Ax + B$ ... (4); Using (2) in (4), $B = 20$;
   Using (3) in (4); $40A + 20 = 60 \;\; \therefore \; A = 1.$
   $\therefore \quad$ Solution of (1) is $u(x) = x + 20$ ... (5)
   $\therefore \quad [u(x)]_{x = 15} = 35°\text{C}$

8. The three possible solutions of the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ are
   (i) $u(x, y) = (Ae^{px} + Be^{-px})(C \cos py + D \sin py)$
   (ii) $u(x, y) = (A \cos px + B \sin px)(Ce^{py} + De^{-py})$ and
   (iii) $u(x, y) = (Ax + B)(Cy + D)$, where A, B, C, D and p are arbitary constants

9. $Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z - a}$, if $|z| > |a|$

10. When we solve a difference equation, by other methods, we first get the G.S. involving arbitrary constants. Then the arbitrary constants are evaluated by using the given boundary conditions. Thus we get the required P.S.
    When the Z-transform method is used, the given boundary conditions are absorbed in the beginning of the problem and the required P.S. is directly obtained.

PART - B

11. (a)(i) This problem is worked out in the book taking the interval as $(0, 2l)$. We have to simply change l as π.

Then $a_n = -\dfrac{4}{n^2}$; $a_0 = \dfrac{4}{3}\pi^2$; $b_n = 0$

$\therefore$ The required F.S. of

$$f(x) = x(2\pi - x) = \frac{2}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx \; ... (1) \quad \text{in } (0, 2\pi)$$

The required sum of $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots$ can be obtained by putting $x = 0$ or $x = 2\pi$ in (1), that are points of continuity for $f(x)$.

$\therefore$ [sum of the F.S]

i.e., $\dfrac{2}{3}\pi^2 - 4\sum_{n=1}^{\infty}\dfrac{1}{n^2} = 0 \quad \therefore \sum_{n=1}^{\infty}\dfrac{1}{n^2} = \dfrac{\pi^2}{6}$

11. (a)(ii) The given function is an even function in $(-\pi, \pi)$

Let the F.S. of $f(x)$ be $\dfrac{a_0}{2} + \sum a_n \cos n\,x \; in \; (-\pi, \pi)$

$$\text{Then } a_n = \frac{2}{\pi}\int_0^{\pi}(x+1)\cos n\,x\,dx$$

$$= \frac{2}{\pi}\left[(x+1)\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\{(-1)^n - 1\} = \begin{cases} -\dfrac{4}{\pi n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$= a_0 = \frac{2}{\pi}\int_0^{\pi}(x+1)\,dx = \frac{1}{\pi}\{(x+1)\}_0^{\pi} = \frac{1}{\pi}\{(\pi+1)^2 - 1\} = \pi + 2.$$

$\therefore$ Required F.S. of $f(x)$

$$= \frac{\pi}{2} + 1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx \; in(-\pi, \pi) \qquad (1)$$

Putting $x = 0$, which is a point of continuity of $f(x)$, in (1), we get

$$\frac{\pi}{2} + 1 - \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\left(\frac{1}{n^2}\right) = f(0) = 1$$

$$\therefore \qquad \sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8}.$$

11. (b)(i) Let the F.H.R. sine series of $f(x)$ be $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ in $(0, l)$

Then $b_n = \frac{2}{l}\left[\int_0^{l/2} x \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} (l - x) \sin \frac{n\pi x}{l}\, dx\right]$

$$= \frac{2}{l}\left[\left\{x\left(-\frac{\cos n\pi x/l}{n\pi/l}\right) - \left(\frac{-\sin n\pi x/l}{n^2\pi^2/l^2}\right)\right\}_0^{l/2} + \left\{(l-x)\left(\frac{-\cos n\pi x/l}{n\pi/l}\right) + \left(-\frac{\sin n\pi x/l}{n^2\pi^2/l^2}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\left\{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} + \left\{\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right]$$

$$= \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$\therefore$ Required F.S. of $f(x)$

$$\frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l} \quad \text{in } (0, l)$$

11. (b)(ii) Let the required F.S. be $\frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$ in $(0, 2\pi)$

$y_0 = y(0) = 1.0$; $y_1 = y(\pi/3) = 1.4$, $y_2 = 1.9$, $y_3 = 1.7$, $y_4 = 1.5$, $y_5 = 1.2$

$$a_0 = 2 \times \frac{1}{6}\sum_{r=0}^{5} y_r = \frac{1}{3}\times 8.7 = 2.9$$

$$a_1 = 2 \times \frac{1}{6}\sum y_r \cos\frac{\pi x_r}{3} \text{ and } b_1 = 1 \times \frac{1}{6}\sum y_r \sin\frac{\pi x_r}{3}$$

$$= \frac{1}{3}\left[(y_0 - y_3) + (y_1 + y_5 - y_2 - y_4)\cos\frac{\pi}{3}\right]$$

$$= \frac{1}{3}[-0.7 + (-0.8)\times 0.5] = -0.367$$

$$b_1 = \frac{1}{3}[y_1 + y_2 - y_4 - y_5)\sin\frac{\pi}{3}$$

$$= \frac{1}{3}[0.6 \times 0.866] = 0.173$$

$$a_2 = 2 \times \frac{1}{6}\sum y_r \cos\frac{2\pi x_r}{3}$$

$$= \frac{1}{3}\left[(y_0 + y_3) - (y_1 + y_2 - y_4 - y_5)\cos\frac{\pi}{3}\right]$$

$$= \frac{1}{3}[2.7 - 7.0 \times 0.5] = -0.1$$

$$b_2 = 2 \times \sum y_r \sin\frac{2\pi x_r}{3}$$

$$= \frac{1}{3}\left[y_1 + y_4 - y_2 - y_5)\sin\frac{\pi}{3}\right]$$

$$= -0.058$$

$\therefore$ Required F.S. is $f(x) \sim 1.45 - 0.367 \cos x + 0.173 \sin x - 0.1 \cos 2x - 0.058 \sin 2x$.

12. (a)(i) This problem is a worked example in the book.

12. (a)(ii) $F\{f(x)\} = \int_{-a}^{a} e^{-isx}\, dx = \int_{-a}^{a} (\cos sx - i \sin sx)\, dx$

$$= 2\int_0^a \cos sx\, dx = \frac{2}{s}(\sin sx)_0^a = \frac{2}{s}\sin as.$$

By Parseval's identity, $\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\overline{f}(s)|^2\, ds$

$$\therefore\ \frac{2}{\pi} \times 2\int_0^{\infty} \left(\frac{\sin as}{s}\right)^2 ds = 2a$$

i.e., $\frac{4}{\pi} \times 2\int_0^{\infty} \left(\frac{\sin t}{t/a}\right)^2 \frac{dt}{a} = 2a,$

i.e., $\frac{4}{\pi} \int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 a\, dt = 2a$, on putting as $= t$

$$\therefore\ \int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

12. (b)(i) $F_s\{f(x)\} = \int_0^1 x \sin sx\, dx + \int_1^2 (2 - x) \sin sx\, dx$

$$\therefore\ F_s\{\sin x\} = \left[\left\{x\left(\frac{-\cos sx}{s}\right) + \left(\frac{-\sin sx}{s^2}\right)\right\}_0^1 \left\{(2-x)\left(\frac{-\cos sx}{s}\right) + \left(\frac{-\sin sx}{s^2}\right)\right\}_1^2\right]$$

$$= \left\{\left(-\frac{\cos s}{s}\right) + \left(-\frac{\sin s}{s^2}\right)\right\} + \left(\frac{\sin 2s}{s^2} + \frac{\cos s}{s} + \frac{\sin s}{s^2}\right)$$

$$= \frac{2 \sin s}{s^2} - \frac{\sin 2s}{s^2} = \frac{2 \sin s}{s^2}(1 - \cos s)$$

12. (b)(ii) Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$ $\therefore \overline{f}_c(s) = \frac{a}{s^2 + a^2}$ and $\overline{g}_c(s) = \frac{b}{s^2 + b^2}$

By the property of F' cosine transforms.

$$\int_0^{\infty} f(x)\, g(x)\, dx = \frac{2}{\pi}\int_0^{\infty} \overline{f}_c(s)\, \overline{g}_c(s)\, ds$$

i.e., $\int_0^{\infty} e^{-(a+b)x}\, dx = \frac{2}{\pi}\int_0^{\infty} \frac{ab}{(s^2+a^2)(s^2+b^2)}\, ds$

$$\therefore \quad \int_0^{\infty} \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2ab}\left\{\frac{e^{-(a+b)x}}{-(a+b)}\right\}_0^{\infty} \frac{\pi}{2ab\,(a+b)}$$

Changing s into x, we get the required result.

13. (a)(i) $z = f(x + cy) + \phi(x - cy)$ ... (1) changing t as y.
$p = f' + \phi'$ ... (2); $q = C f' - C\,\phi'$ ... (3)

Differentiating (2); w.r.t. x; $r = f'' + \phi''$ ... (4)

Differentiating (3) w.r.t. y; $t = C^2 f'' + C^2\,\phi''$ .... (5)

From (4) and (5), we get, $t = C^2 r$; viz., $\frac{\partial^2 z}{\partial y^2} - c^2\frac{\partial^2 z}{\partial x^2} = 0$

Again changing y into t, the required P.D.E. is $\frac{\partial^2 z}{\partial x^2} - \frac{1}{C^2}\frac{\partial^2 z}{\partial t^2} = 0.$

13. (a)(ii) This problem is a worked example in the book.

13. (b)(i) This problem is a worked example in the book.

13. (b)(ii) This problem is a worked example in the book.

14. (a) This problem is a worked example in the book.

14. (b) This problem is a worked example in the book.

15. (a)(i) $Z^{-1}\left\{\frac{z^2}{(z-1)(z-3)}\right\} = Z^{-1}\left\{\left(\frac{z}{z-1}\right)\left(\frac{z}{z-3}\right)\right\}$

$$= Z^{-1}\left\{\frac{z}{z-1}\right\} * Z^{-1}\left\{\frac{z}{z-3}\right\}$$

$$= 1 * 3^n = \sum_{r=0}^{n} 1^{n-r}\cdot 3^r = 1 + 3 + 3^2 + \cdots + 3^n$$

$$= \frac{3^{n+1}-1}{3-1} = \frac{1}{2}(3^{n+1}-1)$$

15. (a)(ii) (1) $z\,\{a^n\ f(n)\} = \bar{f}\left(\frac{z}{a}\right)$; $\therefore$ $Z\{a^n \cos n\theta\} = z(\cos n\theta)_{z\to z/a}$

$$= \left\{\frac{z\,(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}\right\}_{z\to z/a} = \frac{z\,(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$$

(2) $Z\{e^{-at}\sin bt\} = Z(\sin bt)_{z \to ze^{aT}}$

$$= \left\{\frac{z \sin bT}{z^2 - 2z\cos bT + 1}\right\}_{z \to ze^{aT}} = \frac{ze^{aT}\sin bT}{z^2 e^{2aT} - 2ze^{aT}\cos bT + 1}$$

15. (b)(i) This problem is a worked example in the book

(ii) $y_n = (A + Bn)(-3)^n$ ... (1); $y_{n+1}) = \{A + B(n+1)\}(-3)^{n+1}$ ... (2) $y_{n+2} = \{A + B(n+2)\}(-3)^{n+2}$ ... (3).

Putting $\dfrac{y_n}{(-3)^n} = z_n$; we get $z_n = A + Bn$ ...(1)′; $Z_{n+1} = A + Bn + B$... (2)′ and $Z_{n+2} = A + Bn + 2B$... (3)′

From (2)′ and (3)′; $Z_{n+2} - 2Z_{n+1} = -(A + Bn) = -Z_n$, using (1)′

i.e., $\dfrac{y_{n+2}}{(-3)^{n+2}} - 2\dfrac{y_{n+1}}{(-3)^{n+1}} + \dfrac{y_n}{((-3)^n} = 0$

$\therefore$ The required D.E. is $y_{n+2} + 6y_{n+1} + 9y_n = 0$

B.E./B.Tech. Degree Examinations, May/June 2012
Regulations 2008

Third Semester

Common to all branches

MA2211 Transforms and Partial Differential Equations

Time: Three Hours **Maximum: 100 Marks**

Answer ALL Questions

Part A – (10 × 2 = 20 Marks)

1. Find the constant term in the expansion of $\cos^2 x$ as a Fourier series in the interval $(-\pi, \pi)$.

2. Define Root Mean square value of a function $f(x)$ over the interval (a, b).

3. What is the Fourier transform of $f(x - a)$, if the Fourier transform of $f(x)$ is $F(s)$?
4. Find the Fourier since transform of $f(x) = e^{-ax}$, $a > 0$.

5. Form the partial differential equation by eliminating the arbitrary function from $z^2 - xy = f\left(\frac{x}{z}\right)$.
6. Solve $(D^2 - 7DD' + 6D'^2)\, z = 0$.
7. What is the basic difference between the solution of one dimensional wave equation and one dimensional heat equation with respect to the time?
8. Write down the partial differential equation that represents steady state heat flow in two dimensions and name variables involved.

9. Find the Z-transform of $x(n) = \begin{cases} \dfrac{a^n}{n!} & \text{for } n \ge 0 \\ 0 & \text{otherwise} \end{cases}$

10. Solve $y_{n+1} - 2y_n = 0$ given $y_0 = 3$.

Part B – (5 × 16 = 80 marks)

11. (a) (i) Find the Fourier series of $f(x) = (\pi - x)^2$ in $(0, 2\pi)$ of periodicity 2π.

(ii) Obtain the Fourier series to represent the function $f(x) = |x|$, $-\pi < x < \pi$ and deduce $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Or

(b)(i) Find the half-range Fourier cosine series of $f(x) = (\pi - x)^2$ in the interval $(0, \pi)$. Hence find the sum of the series $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \infty$

(ii) Find the Fourier series up to second harmonic for $y = f(x)$ from the following values.

x:	0	1	2	3	4	5
y:	9	18	24	28	26	20

12. (a)(i) Derive the Parseval's identity for Fourier Transforms.

(ii) Find the Fourier integral representation of $f(x)$ defined as

$$f(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{1}{2} & \text{for } x = 0 \\ -e^{-x} & \text{for } x > 0 \end{cases}$$

Or

(b) (i) State and prove convolution theorem on Fourier transform.

(ii) Find Fourier sine and cosine transform of x^{n-1} and hence prove $1/\sqrt{x}$ is self reciprocal under Fourier sine and cosine transforms.

13. (a)(i) From the PDE by eliminating the arbitrary functions ϕ from $\phi(x^2 + y^2 + z^2, ax + by + cz) = 0$.

(ii) Solve the partial differential equation $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$.

Or

(b)(i) Solve the equation $(D^3 + D^2D' - 4DD'^2 - 4D'^3)z = \cos(2x + y)$.

(ii) Solve $(2D^2 - DD' - D'^2 + 6D = 3D')\, z = xe^y$.

14. (a) The ends A and B of a rod 40 cm long have their temperatures kept at 0°C and 80°C respectively, until steady state condition prevails. The temperature of the end B is then suddenly reduced to 40°C and kept so, while that of the end A is kept at 0°C. Find the subsequent temperature distribution $u(x, t)$ in the rod.

Or

(b) A long rectangular plate with insulated surfaceis 1 cm wide. If the temperature along one short edge $(y = 0)$ is $u(x, 0) = k(lx - x^2)$ degrees, for $0 < x < 1$, while the other two long edges $x = 0$ and $x = l$ as well as the other short edge are kept at 0°C, find the steady state temperature function $u(x, y)$.

15. (a)(i) Find $Z[n(n-1)(n-2)]$.

(ii) Using Convolution theorem, find the inverse Z-transform of $\dfrac{8z^2}{(2z-1)(4z-1)}$

(b)(i) Solve the difference equation $y(k+2)+y(k)=1$, $y(0)=y(1)=0$, using Z-transform.

(ii) Solve $y_{n+2}+y_n = 2^n \cdot n$, using Z-transform

Solutions

Part A

1. $a_0 = \dfrac{2}{\pi}\int_0^{\pi} \cos^2 x\, dx = \dfrac{2}{\pi}\times\dfrac{1}{2}\int_0^{\pi}(1+\cos 2x)\, dx = 1 \therefore$ Constant Term in F.S. $= \dfrac{1}{2}$

2. If $\bar{y}$ is the RMS value of $y = f(x)$ in (a, b), then $\bar{y} = \sqrt{\dfrac{1}{b-a}\int_a^b [f(x)]^2\, dx.}$

3. $F(s) = F(\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-isx}\, dx$

$\therefore F\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a)e^{-isx}\, dx = \int_{-\infty}^{\infty} f(t)e^{-is(t+a)}\, dt$, on putting $x - a = t$

$= e^{-ias} F(s)$.

4. $F_s(e^{-ax}) = \int_0^{\infty} e^{-ax}\sin sx\, dx = \left\{\dfrac{e^{-ax}}{s^2+a^2}(-a\sin sx - s\cos sx)\right\}_0^{\infty} = \dfrac{s}{s^2+a^2}$

5. $z^2 - xy = f\left(\dfrac{x}{z}\right)\cdots(1)$; Differentiating (1) partially w.r.t. x; we get

$$2zp - y = f'(u)\cdot\left\{\frac{z\cdot 1 - xp}{z^2}\right\}\cdots(2), \text{ where } u = \frac{x}{z}.$$

Differentiating (1) partially w.r.t. y, we get

$$2z\,q - x = f'(u)\cdot\left\{-\frac{x}{z^2}q\right\}\cdots(3)$$

(2) ÷ (3) gives; $\dfrac{2zp-y}{2zq-x} = \dfrac{z-xp}{-xq}$; viz., $qxy = 2z^2q - xz + x^2p$

i.e., the required P.D.E. is $x^2p + (2z^2 - xy)\,q = xz$.

6. $(D^2 - 7DD' + 6D'^2)z = 0$; The A.E. is $(m^2 = 7m + 6 = 0$; $\therefore m = 1, 6$

$\therefore$ Solution of the P.D.E. is $z = f_1(y+x) + f_2(y+6x)$, where f_1 and f_2 are arbitrary functions.

7. The proper solution of one dimensional wave equation will have cos pat or sin pat in the value of $y(x, t)$, depending on the initial condition. The proper solution of one dimensional heat flow equation will have $e^{-p^2\alpha^2 t}$ in the value of $u(x, t)$, irrespective of the end conditions.

8. The P.D.E. that represents steady-state heat flow in two dimensions in $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$; x, y that are the distances of the typical point (x, y) in the plate from the y- and x-axes are independent variables and u representing the S.S. temperature at the point (x, y) of the plate is the dependent variable.

9. $Z\left(\frac{a^n}{\lfloor n}\right) = \sum_{n=0}^{\infty} \frac{1}{\lfloor n} \cdot \left(\frac{a}{z}\right)^n = e^{a/z}$

10. $y_{n+1} - 2y_n = 0$; $y_0 = 3$

Taking Z-transforms of the given equation, we have

$$[z\,\bar{y}\,(z) - z\,y\,(0)] - 2\,\bar{y}\,(z) = 0$$

i.e. $(z-2)\,\bar{y}\,(z) = 3z$ or $\bar{y}(z) = 3 \cdot \frac{z}{z-2} \cdots (1)$

Inverting (1); $y(n) = 3Z^{-1}\left(\frac{z}{z-2)}\right) = 3 \times 2^n$ or $y_n = 3 \times 2^n$ is the required solution.

PART - B

11. (a)(i) Let the F.S. of $f(x) = (\pi - x)^2$ in $(0, 2\pi)$ be

$$\frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \cos n\,x\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2 \frac{\sin nx}{n} - \{-2(\pi - x)\}\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi} = \frac{4}{n^2}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2\,dx = -\frac{1}{3\pi}\{(\pi - x)^3\}_0^{2\pi} = \frac{2\pi^2}{3}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(-\frac{\cos nx}{n}\right) - \{-2\,(\pi - x)\}\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi} = 0$$

$\therefore$ Required F.S. of $f(x)$ is $\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$ in $(0, 2\pi)$

11. (a)(ii) $f(x) = |x|$ is an even function in $= \pi < x < \pi$.

$\therefore$ Let the F.S. of $f(x)$ be $\frac{a_0}{2} + \sum a_n \cos nx$ in $(-\pi, \pi)$

$$a_n = \frac{2}{\pi}\int_0^\pi |x| \cos nx\, dx = \frac{2}{\pi}\int_0^\pi x \cos nx\, dx$$

$$= \frac{2}{\pi}\left\{x\left(\frac{\sin nx}{n}\right) - 1\left(-\frac{\cos nx}{n^2}\right)\right\}_0^\pi = \frac{2}{n^2\pi}\{(-1)^n - 1\}$$

$$= \begin{cases} 0, & \text{If } n \text{ even} \\ -\frac{4}{n^2\pi}, & \text{If } n \text{ odd} \end{cases}$$

$$a_0 = \frac{2}{\pi}\int_0^\pi |x|\, dx = \frac{1}{\pi}(x^2)_0^\pi = \pi.$$

$$\therefore \quad = |x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3\ldots}^{\infty}\frac{1}{n^2}\cos nx \text{ in } (-\pi, \pi) \cdots (1)$$

Putting $x = 0$ in the R.S. of (1), $\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = f(0) = 0;$

$$\therefore \quad \sum_{n=1}^{\infty}\frac{1}{(2n-1)} = \frac{\pi^2}{8}$$

11. (b)(i) This problem is a worked example in the book.

11. (b)(ii) $2l = 6 \therefore l = 3$; The required F.S. of y is of the form

$$\frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{3} + \sum b_n \sin\frac{n\pi x}{3} \text{ in } (0,6)$$

Given: $y_0 = 9$, $y_1 = 18$, $y_2 = 24$, $y_3 = 28$, $y_4 = 26$ and $y_5 = 20$.

$$a_0 = 2\times\frac{1}{6}\sum y_r = \frac{125}{3} = 41.67; a_1 = \frac{1}{3}\sum y_r \cos\frac{\pi x_r}{3}$$

i.e., $a_1 = \frac{1}{3}\left[(y_0 - y_3) + (y_1 + y_5 - y_2 - y_4)\cos\frac{\pi}{3}\right] = -\frac{25}{3} = -8.33$

$$b_1 = \frac{1}{3}\sum y_r \sin\frac{\pi x_r}{3} = \frac{1}{3}(y_1 + y_2 - y_4 - y_5)\sin\frac{\pi}{3} = -\frac{4}{3}\times 0.866 = -1.15$$

$$a_2 = \frac{1}{3}\sum y_r \cos\frac{2\pi x_r}{3} = \frac{1}{3}\left[(y_0 + y_3) - (y_1 + y_2 + y_4 + y_5)\cos\frac{\pi}{3}\right]$$

$$= -\frac{7}{3} = -2.33$$

$$b_2 = \frac{1}{3}\sum y_r \sin\frac{2\pi}{3}x_r = \frac{1}{3}(y_1 + y_4 - y_2 - y_5)\sin\frac{\pi}{3} = 0.$$

$\therefore$ Required F.S. is

$$y = 20.84 + \left(-8.33\cos\frac{\pi x}{3} - 1.15\sin\frac{\pi x}{3}\right) - 2.33\cos\frac{2\pi x}{3}.$$

12. (a)(i) The derivation of Parseval's identity for Fourier transforms is a standard book-work, available in the book.

12. (a)(ii) This problem is a worked example in the book

12. (b)(i) Statement and proof of convolution theorem in Fourier transforms is a standard result, that is available in the book

12. (b)(ii) This problem is a worked example in the book

13. (a)(i) This problem is a worked example in the book. In the book, the function is taken as f instead of ϕ.

13. (a)(ii) This problem is a worked example in the book.

13. (b)(i) This problem is a worked example in the book.

13. (b)(ii) This problem is a worked example in the book. In the worked example, there are 2 terms in the R.S. of the P.D.E., namely $xe^y + ye^x$.

As only one term, namely xe^y, is given in the question paper problem in the R.S.

So it is enough we find $(\text{P.I.})_1$. The general solution of the given P.D.E. is $z = f_1(2y - x) + e^{-3x} f_2(y + x) + 1/4\ (2x - 5)e^y$.

14. (a) This problem is a worked example in the book.

14. (b) This problem is a worked example in the book.

15. (a)(i) $Z\{n(n-1)(n-2)\} = Z\{n^3 - 3n^2 + 2n\}$

$$= \frac{z(z^2 + 4z + 1)}{(z-1)^4} - 3\frac{z(z+1)}{(z-1)^3} + 2\frac{z}{(z-1)^2}$$

$$= \frac{z}{(z-1)^4}\left[(z^2 + 4z + 1) - 3(z^2 - 1) + 2(z^2 - 2z + 1)\right]$$

$$= 6\frac{z}{(z-1)^4}$$

15. (a)(ii) $Z^{-1}\left\{\frac{8z^2}{(2z-1)(4z-1)}\right\} = Z^{-1}\left\{\frac{z}{z-\frac{1}{2}}\cdot\frac{z}{z-\frac{1}{4}}\right\} = Z^{-1}\left\{\frac{z}{z-1/2}\right\} * Z^{-1}\left\{\frac{z}{z-\frac{1}{4}}\right\}$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n = \sum_{r=0}^{n}\left(\frac{1}{2}\right)^{n-r}\cdot\left(\frac{1}{4}\right)^r$$

$$= \left(\frac{1}{2}\right)^n \left\{ \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} \right\} = \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{2n}$$

15. (b)(i) $y_{k+2} + y_k = 1$; $y_0 = 0$ and $y_1 = 0$

Taking Z-transforms of the equation,

$$[z^2\, \overline{y}\,(z) - z^2 y(0) - z\; y(1)] + \overline{y}(z) = \frac{z}{z-1}$$

i.e., $$(z^2 + 1)\, \overline{y}(z) = \frac{z}{(z-1)}$$

$$\therefore \quad \overline{y}\,(z) = \frac{z}{(z-1)(z^2+1)}; \;\; \therefore \;\; \frac{\overline{y}(z)}{z} = \frac{1}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz + C}{z^2 + 1}$$

$$\therefore \quad \frac{\overline{y}(z)}{z} = \frac{1}{2}\left[\frac{1}{z-1} - \frac{z}{z^2+1} - \frac{1}{z^2+1}\right]$$

$$\therefore \quad \overline{y}(z) = \frac{1}{2}\left[\frac{z}{z-1} - \frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right]$$

Inverting, $y(k) = \dfrac{1}{2}\left[1 - \cos\dfrac{k\pi}{2} - \sin\dfrac{k\pi}{2}\right]$

(ii) This problem is a worked example in the book.

Question Paper Code 21522

B.E./B.Tech. Degree Examinations, May/June 2013
Regulations 2008

Third Semester

Common to all branches

MA2211/MA 31/MA 1201 A/CK 201/10177 MA 301/09010008/080210001
Transforms and Partial Differential Equations

Time: Three Hours **Maximum: 100 Marks**

Answer ALL Questions

Part A – (10 × 2 = 20 Marks)

1. State the Dirichlet's conditions for Fourier series.

2. What is meant by Harmonic Analysis?

3. Find the Fourier Sine Transform of e^{-ax}

4. If $F\{f(x)\} = F(s)$, prove that $F\{f(ax)\} = \frac{1}{2} \cdot F\left(\frac{s}{\alpha}\right)$

5. Form the PDE from $(x - a)^2 + (y - b)^2 + z^2 = r^2$.

6. Find the complete integral of $p + q = pq$.

7. In the one dimensional heat equation $u_t = c^2 \cdot u_{xx}$, what is c^2?

8. Write down the two dimensional heat equation both in transient and steady states.

9. Find $Z(n)$.

10. Obtain $Z^{-1}\left[\frac{z}{(z+1)(z+2)}\right]$

Part B – (5 × 16 = 80 marks)

11. (a) (i) Find the Fourier series of x^2 in $(-\pi, \pi)$ and hence deduce that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots \frac{\pi^4}{90}$ (8)

(ii) Obtain the Fourier cosine series of

$$f(x) = \begin{cases} kx, & 0 < x < l/2 \\ k(l - x) & l/2 < x < 1 \end{cases}$$ (8)

Or

11. (b)(i) Find the complex form of Foureir series of cos ax in $(-\pi, \pi)$, where a is not an integer.

(ii) Obtain the Fourier cosine series of $(x-1)^2$, $0 < x < 1$ and hence show that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$.

12. (a)(i) Find the Fourier transform of $f(x)=\begin{cases}1, & |x|<\alpha \\ 0, & |x|>\alpha\end{cases}$ and hence find $\int_0^\infty \frac{\sin x}{x}\,dx$

(ii) Verify the convolution theorem under Fourier transform, for $f(x) = g(x) = e^{-x^2}$

Or

(b)(i) Obtain the fourier transform of $e^{-x^2/2}$. (8)

(ii) Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$ using Parseval's identity

13. (a)(i) Solve: $x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$

(ii) Solve: $(D^2 + DD' - 6D'^2)z = y \cdot \cos x$.

Or

(b)(i) Solve: $z = px + qy + \sqrt{p^2+q^2+1}$

(ii) Solve: $(D^3 - 7DD'^2 - 6D'^3)\, z = \sin(2x + y)$.

14. (a) A tightly stretched string between the fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If each of its points is given a velocity $kx(l-x)$, find the displacement $y(x, t)$ of the string.

Or

(b) An infinitely long rectangular plate is of width 10 cm. The temperature along the short edge $y = 0$ is given by

$u\begin{cases}20x, & 0<x<5 \\ 20(10-x), & 5<x<10\end{cases}$. If all the other edges are kept at zero temperature, find the steady state temperature at any point on it.

15. (a)(i) Find $Z(\cos n\theta)$ and hence deduce $Z\left(\cos\frac{n\pi}{2}\right)$. (8)

(ii) Using Z-transform solve: $y_{n+2} - 3y_{n+1} - 10y_n = 0$; $y_0 = 1$ and $y_1 = 0$. (8)

Or

(b)(i) State and prove the second shifting property of Z-transform. (6)

(ii) Using convolution theorem, find $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$. (10)

Solutions

Part A

1. Statement of Dirichlet's conditions for Fourier series is available in the book.
2. If the F.S. of $f(x)$ in $(0, 2l)$ is $\frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{l} + \sum b_n \sin \frac{n\pi x}{l}$, it can be put as $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l} - \alpha_n\right)$ or as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l} - \beta_n\right)$$

where $A_n = \sqrt{a_n^2 + b_n^2}$, $\alpha_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$ and $\beta_n = \tan^{-1}\left(\frac{a_n}{b_n}\right)$. Computing a_n and b_n that will help us to find the nth harmonic $n = 1, 2, 3, \ldots$ is known as harmonic analysis. In other words, the process of finding the harmonics in the Fourier expansion of a function (defined in a tabulated form) numerically is known as harmonic analysis.

3. $F_S(e^{-ax}) = \int_0^{\infty} e^{-ax} \sin sx \, dx = \left[\frac{e^{-ax}}{s^2 + a^2}(-a \sin sx - s \cos sx)\right]_0^{\infty} = \frac{s}{s^2 + a^2}$
4. This is change of scale property of Fourier transforms. Proof is availabe in the book
5. $(x - a)^2 + (y - b)^2 + z^2 = r^2$ ... (1); Differentiation (1) partially w.r.t. x, we get $2(x - a) + 2zp = 0$ ... (2); Similarly, differentiating (1) partially w.r.t. y, we get $2(y - b) + 2zq = 0$ ... (3)
From (2) and (3); $x - a = -zp$ and $y - b = -zq$ ... (4)
Using (4) in (1); $z^2 p^2 + z^2q^2 + z^2 = r^2$ or $(p^2 + q^2 + 1)z^2 = r^2$ is the required P.D.E.

6. Let the C.S. of $p + q = pq$ ... (1) be $z = ax + by + c$ ...(2)
From (2), $p = a$ and $q = b$.
Since (2) is a solution of (1), $a + b = ab$ or $b = \frac{a}{a-1}$
Using this value of b in (2), the required complete integral is $z = ax + \frac{a}{a-1} \cdot y + c$, where a and c are arbitrary constants.

7. $c^2 = \frac{k}{sp}$ is the diffusivity of the material of the rod or bar through which heat flows, where k, s and p are respectively the thermal conductivity, specific heat and density of the heat conducting material.

8. The two-dimensional heat flow equation is (i) $\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$ in the transient state and (ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in steady state.

9. $Z(n) = \sum_{n=0}^{\infty} n \left(\frac{1}{z} \right)^n = \frac{1}{z} \left(1 - \frac{1}{z} \right)^{-2} = \frac{1}{z} \cdot \left(\frac{z}{z-1} \right)^2 = \frac{z}{(z-1)^2}$, when $\left| \frac{1}{z} \right| < 1$ or $|z| > 1$.

10. $Z^{-1} \left\{ \frac{z}{(z+1)(z+2)} \right\} \equiv Z^{-1} \{ \overline{f}(z) \}$, say

$$\therefore \frac{\overline{f}(z)}{z} = \frac{1}{(z+1)(x+2)} = \frac{1}{z+1} - \frac{1}{z+2}; \overline{f}(z) = \frac{z}{z+1} - \frac{z}{z+2}$$

$$\therefore Z^{-1} \{ \overline{f}(z) \} = (-1)^n - (-2)^n$$

PART - B

11. (a)(i) Since $f(x) = x^2$ is an even function in $(-\pi, \pi)$, let the F.S. of $f(x) = x^2$ be $\frac{a_0}{2} + \sum a_n \cos nx$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx = \frac{2}{\pi} \left[x^2 \frac{\sin nx}{n} - 2x \left(-\frac{\cos nx}{n^2} \right) + 2 \left(-\frac{\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{4}{n^2} (-1)^n$$

$$a_0 = \frac{2}{\pi} \int_0^{\infty} x^2 \, dx = \frac{2}{3\pi} \times \pi^3 = \frac{2}{3} \pi^2.$$

$$\therefore x^2 \sim \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \text{ in } (-\pi, \pi).$$

By Parseval's theorem,

$$\frac{1}{4} a_0^2 + \frac{1}{2} \sum a_n^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^4 \, dx$$

i.e., $\frac{1}{4} \times \frac{4}{9} 4\pi + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{\pi} \cdot \left(\frac{x^5}{5} \right)_0^{\pi} = \frac{1}{5} \pi^4$

$$\therefore 8 \sum_{n=1}^{\infty} \frac{1}{n^4} = \left(\frac{1}{5} - \frac{1}{9} \right) \pi^4 = \frac{4}{45} \pi^4$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{4}{45 \times 8} \pi^4 = \frac{1}{90} \pi^4$$

11. (a)(ii) Give an even extension for $f(x)$ in $(-l, 0)$, so that $f(x)$ is made an even function in $(-l, b)$.

Let the F.C. series of $f(x)$ in $(0, l)$ be $\dfrac{a_0}{2} + \sum a_n \cos \dfrac{n\pi x}{l}$.

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\left[\int_0^{l/2} x\cos\frac{n\pi x}{l}\,dx + \int_0^l (l-x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2k}{l}\left[\left\{x\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^{\frac{l}{2}}\right.$$

$$\left. + \left\{(l-x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) + \left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_{\frac{l}{2}}^{l}\right]$$

$$= \frac{2k}{l}\left[\left\{\frac{l^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi}\left(\cos\frac{n\pi}{2} - 1\right)\right\}\right.$$

$\therefore$

$$\left. - \frac{l^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\left\{(-1)^n - \cos\frac{n\pi}{2}\right\}\right]$$

$$= \begin{cases} 0, & \text{If } n \text{ is odd} \\ \dfrac{4\,kl}{n^2\pi^2}\{(-1)^{n/2} - 1, & \text{If } n \text{ is even} \end{cases}$$

$$a_0 = \frac{2}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right]$$

$$= \frac{2k}{l}\left[\left(\frac{x^2}{2}\right)_0^{l/2} + \left\{\frac{(l-x)}{-2}\right\}_{l/2}^{l}\right] = \frac{2k}{l}\left[\frac{l^2}{8} + \frac{l^2}{8}\right] = \frac{kl}{2}$$

$\therefore$ Required F.H.R. Cosine series of

$f(x)$ is $\dfrac{kl}{4} + \dfrac{4kl}{\pi^2}\sum_{n\text{ even}}\dfrac{1}{n^2}\{(-1)^{n/2} - 1\}\cos\dfrac{n\pi x}{l}$

11. (b)(i) This problem is a worked example in the book.

11. (b)(ii) Giving an even extension for $f(x)$ in $(-1, 0)$, the function $f(x)$ is made an even function in $(-1, 1)$

Let the Fourier series of $f(x)$ be $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$ in $(-1, 1)$

$$a_n = \frac{2}{1}\int_0^1 (x-1)^2 \cos n\pi x \, dx$$

$$= 2\left[(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= \frac{4}{n^2\pi^2}$$

$$a_0 = \frac{2}{1}\int_0^1 (x-1)^2 \, dx = \frac{2}{3}\left[(x-1)^3\right]_0^1 = \frac{2}{3}$$

$\therefore$ F.H.R.C. series of $f(x) \sim \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$ in $(0, 1)$ (1)

Putting $x = 0$ in (1), which is a point of continuity of $f(x) = (x-1)^2$,

we get $\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 \quad \therefore \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{4}\left(1 - \frac{1}{3}\right) = \frac{\pi^2}{6}.$

12. (a)(i) This problem is a worked example in the book

12. (a)(ii) Convolution theorem in Fourier transform is

$$F[f(x) * g(x)] = \bar{f}(s)\,\bar{g}(s) \ldots (1)$$

If we assume that $f(x) = g(x) = e^{-x^2}$, then $\bar{f}(s) = \bar{g}(s) = \sqrt{\pi}\, e^{-s^2/4}$

$$\text{L.S. of } (1) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x-u)\cdot g(u)\,du\right] e^{-isx}\,dx$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-(x-u)^2} e^{-u^2}\,du\right) e^{-isx}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-u^2}\,du \int_{-\infty}^{\infty} e^{-(x-u)^2} e^{-isx}\,dx = \int_{-\infty}^{\infty} e^{-u^2}\cdot e^{-ius} F(e^{-x^2}),$$

by shifting property

$$= F(e^{-x^2})\cdot\int_{-\infty}^{\infty} e^{-(u^2+ius)}\,du = F(e^{-x^2})\cdot\int_{-\infty}^{\infty} e^{-\left[\left(u+\frac{is}{2}\right)^2 - \frac{i^2 s^2}{4}\right]}\,du$$

$$= F(e^{-x^2}) \times e^{-s^2/4} \int_{-\infty}^{\infty} e^{-v^2}\, dv, \text{ where } v = u + \frac{is}{2}$$

$$= F(e^{-x^2}) \cdot \left(\sqrt{\pi}\, e^{-s^2/4}\right) = \overline{f}\,(s) \cdot \overline{g}\,(s) = \text{R.S. of (1)}$$

Hence convolution theorem is verified

12. (b)(i) $F(e^{-a^2x^2}) = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$. This problem is a worked example in the book.

Putting $a = \frac{1}{\sqrt{2}}$ in that worked example, we get

$$F\,(e^{-x^2/2}) = \sqrt{2\pi}\; e^{-s^2/2}$$

12. (b)(ii) This problem is a worked example in the book. We have to apply Parseval's identity to $F_C(e^{-ax})$.

13. (a)(i) $x(y^2 - z^2)\, p + y(z^2 - x^2)\, q = z(x^2 - y^2)$

This is a Lagrange's equation with $P = x(y^2 - z^2)$ etc.

The LSSE's are $\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}$ (1)

Using the multiplies $1/x$, $1/y$, $1/z$ in (1),

Each ratio in (1) = $\frac{1/x\; dx + 1/y\; dy + 1/x\; dz}{0}$

$\therefore$ $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$; Integrating, log xyz = log a or $xyz = a$ ...(2)

Using the multipliers x, y, z in (1),

Each ratio = $\frac{x\,dx + y\,dy + z\,dz}{0}$; $\therefore$ $x\,dx + y\,dy + z\,dz = 0$

Integrating, $x^2 + y^2 + z^2 = b$ ... (3).

$\therefore$ G.S. of the given equation is $f\left(xyz, \sum x^2\right) = 0$

13. (a)(ii) $(D^2 + DD' - 6D'^2)\, z = y \cos x$

The A.E. is $m^2 + m - 6 = 0$, i.e. $(m + 3)(m - 2) = 0$ $\therefore$ $m = -3, 2$.

$\therefore$ C.F. $= f_1(y - 3x) + f_2\,(y + 2x)$

P.I. = $\frac{1}{(D + 3D')(D - 2D')}\, y \times$ R.P. of e^{ix}

$$= \text{R.P. of } e^{ix}\left[\frac{1}{\{(D+i) + 3D'\}\{(D+i) - 2D'\}}\, y\right]$$

$= \text{R.P. of } e^{ix} \cdot \left(\dfrac{1}{-1 + iD'}\right) y = \text{R.P. of } [-e^{ix} (1 + iD')y]$

$= \text{R.P of } [-(\cos x + i \sin x)(y + i] = -y \cos x + \sin x$

$\therefore$ G.S. of the given equation is z = C.F. + P.I.

13. (b)(i) $z = px + qy + \sqrt{p^2 + q^2 + 1}$

This is a worked example in the book. We have to put $c = 1$ in the worked example

13. (b)(ii) $(D^3 - 7DD'^2 - 6D'^3)\, z = \sin(2x + y)$

A.E. is $m^2 - 7m - 6 = 0$, viz., $(m + 1)(m + 2)(m - 3) = 0$

$\therefore$ $m = -1, -2, 3$

$\therefore$ C.F. $f_1(y - x) + f_2(y - 2x) + f_3(y + 3x)$, where f_1, f_2, f_3 are arbitrary functions.

$$\text{P.I.} = \frac{1}{D^3 - 7DD'^2 - 6D'^3} \sin(2x + y) = \frac{-1}{4(D - 5D')} \sin(2x + y)$$

$$= -\frac{1}{4} \cdot \frac{(D + 5D')}{D^2 - 25D'^2} \sin(2x + y)$$

$$= -\frac{1}{84} \times 7 \cos(2x + y) = -\frac{1}{12} \cos(2x + y)$$

$\therefore$ G.S is z = C.F + P.I.

14. (a) This problem is a worked example in the book.

In the worked example, the length of the string is taken as 2 l.

If we replace 2 l by L or l by $(L/2)$ and work out the problem, the required solution will be obtained as

$$y(x, t) = \frac{8kL^3}{\pi^4 a} \sum_{n=1}^{\infty} \frac{1}{(2n - 1)^4} \sin \frac{(2n - 1)\, \pi x}{L} \cdot \sin \frac{2n\, \pi\, at}{L}$$

$\therefore$ The required solution is

$$y(x, t) = \frac{8kl^3}{\pi^4 a} \sum_{n=1}^{\infty} \frac{1}{(2n - 1)^4} \sin \frac{(2n - \pi)\, \pi x}{l} \sin \frac{(2n - 1)\pi\, at}{l}$$

14. (b) This problem is the same as the problem given in question 14(b) of the April/May 2010 question paper. The solution is given under April/May 2010 question paper.

15. (a)(i) $Z(a^n) = \dfrac{z}{(z - a)}$; $\therefore$ $Z(e^{in\theta}) = Z\{(e^{i\theta})^n\} = \dfrac{z}{z - e^{i\theta}}$

i.e., $Z(\cos n\,\theta + i \sin n\theta)$

$$= \frac{z}{(z - \cos\theta) - i \sin\theta} = \frac{z\{(z - \cos\theta) + i \sin\theta\}}{(z - \cos\theta)^2 + \sin^2\theta}$$

Equating the real parts,

$$Z(\cos n\theta) = = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}, \text{ if } |z| > 1$$

$$\therefore \quad Z\left(\cos\frac{n\pi}{2}\right) = \frac{z^2}{z^2+1}$$

15. (a)(ii) $y_{n+2} - 3y_{n+1} - 10y_n = 0$; $y_0 = 1$ and $y_1 = 0$

Taking z-transform of the given equation,

$$[z^2\,\bar{y}(z) - z^2 y(0) - z\,y(1)] - 3[z\,\bar{y}(z) - z\,y(0)] - 10\,\bar{y}(z) = 0$$

i.e., $[z^2 - 3z - 10)\,\bar{y}(z) = z^2 - 3z$

i.e., $\bar{y}(z) = \dfrac{z^2 - 3z}{(z-5)(z+2)} \quad \therefore \quad \dfrac{\bar{y}(z)}{z} = \dfrac{z-3}{(z-5)(z+2)} = \dfrac{2}{7}\cdot\dfrac{1}{z-5} + \dfrac{5}{7}\cdot\dfrac{1}{z+2}$

$$\therefore \quad \bar{y}(z) = \frac{2}{7}\cdot\frac{z}{z-5} + \frac{5}{7}\cdot\frac{z}{z+2}$$

Inverting; we get $y_n = \dfrac{2}{7}\times 5^n + \dfrac{5}{7}\times(-2)^n$

15. (b)(i) Time shifting property:

If $Zf(n) = \bar{f}(z)$, then $Z\{f(n-n_0)\} = z^{-n_0}\cdot\bar{f}(z)$

Frequency shifting property: $Z\{a^n\ f(n)\} = \bar{f}\left(\dfrac{z}{a}\right)$.

The proofs are available in the book

15. (b)(ii) $Z^{-1}\left\{\dfrac{z^2}{(z-a)(z-b)}\right\} = Z^{-1}\left\{\dfrac{z}{(z-a)}\right\} * Z^{-1}\left\{\dfrac{z}{z-b}\right\}$,

by convolution theorem

$$= a^n * b^n$$

$$= \sum_{r=0}^{n} a^r \cdot b^{n-r}$$

$$= b^n \sum_{r=-}^{n}\left(\frac{a}{b}\right)^n = b^n\cdot\left\{\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b} - 1}\right\}$$

$$= b^n\left[\frac{a^{n+1} - b^{n+1}}{(a-b)\,b^n}\right]$$

$$= \frac{1}{a-b}[a^{n+1} - b^{n+1}]$$

B.E./B.Tech. Degree Examinations, November/December 2014
Question paper code: 97107

Third Semester/ Civil Engineering

Common to all branches

MA6351 Transforms and Partial Differential Equations
(Regulation - 2013)

Time: Three hours **Maximum: 100 marks**

Answer ALL Questions

Part A – (10 × 2 = 20 marks)

1. Form the partial differential equation by eliminating the arbitrary function f from $z = f\left(\frac{y}{x}\right)$.
2. Find the complete solution of $p + q = 1$
3. State the sufficient conditions for existence of Fourier series.
4. If $(\pi - x)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ in $0 < x < 2\pi$, then deduce that value of $\sum_{n=1}^{\infty}\frac{1}{n^2}$
5 Classify the partial differential equation
$$(1-x^2)z_{xx} - 2xy\,z_{xy} + (1-y^2)z_{yy} + xz_x + 3x^2yz_y - 2z = 0$$
6. Write down the various possible solutions of one dimensional heat flow equation.
7. State and prove modulation theorem on Fourier transforms.
8. If $F\{f(x)\} = F(s)$, then find $F\{e^{iax}f(x)\}$.
9. Find the z-transform of n.
10. State initial value theorem on z-transforms.

Part B – (5 × 16 = 80 marks)

11. (a) (i) Find the singular solution of $z = px + qy + p^2 - q^2$ (8)
(ii) Solve $(D^2 - 2DD')\, z - x^3y + e^{2x-y}$ (8)

Or

11. (b) (i) Solve $x(y - z)p + y(z - x)q = z(x - y)$. (8)
11. (b) (ii) Solve $(D^3 - 7DD'^2 - 6D'^3)\, z - \sin(x + 2y)$ (8)

12. (a) (i) Find the Fourier series of $f(x) = x^2$ in $-\pi < x < \pi$. Hence, deduce the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (8)

(ii) Find the half range cosine series expansion of $(x - 1)^2$ in $0 < x < 1$. (8)

(b) (i) Compute the first two harmonics of the Fourier series of $f(x)$ from the table given. (8)

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
$f(x)$	1.0	1.4	1.9	1.7	1.5	1.2	1.0

(ii) Obtain the Fourier cosine series expansion of $f(x) = x$ in $0 < x < 4$. Hence, deduce the value of $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} \cdots$ to ∞. (8)

13. (a) If a tightly stretched string of length l is initially at rest in equilibrium position and each point of it is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l}$, $0 < x, < l$, determine the transverse displacement $y(x, t)$ (16)

Or

(b) A square plate is bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$. Its surface are insulated and the temperature along $y = b$ is kept at 100 °C. while the temperature along other three edges are at 0 °C. Find the steady-state temperature at any point in the plate. (16)

14. (a) Find the Fourier transform of $f(x) = \begin{cases} 1 - |x|, & \text{if } |x| < 1 \\ 0, & \text{otherwise} \end{cases}$. Hence deduce the values $(i) \int_0^{\infty} \frac{\sin^2 t}{t^2} dt$ (ii) $\int_0^{\infty} \frac{\sin^4 t}{t^4} dt$. (16)

Or

(b) (i) Find the Fourier transform of $e^{-a^2x^2}$, $a > 0$. Hence show that $e^{-x^2/2}$ is self reciprocal under the Fourier transform. (8)

(ii) Evaluate $\int_0^{x} \frac{dx}{(x^2 + a^2)(x^2 + b^2)}$, using Fourier transforms (8)

15. (a) (i) Find $Z(\cos n\theta)$ and $Z(\sin n\theta)$. (8)

(ii) Using z-transforms, solve $u_{n+2} - 3u_{n+1} + 2u_n = 0$ given that $u_0 = 0$, $u_1 = 1$. (8)

Or

(b) (i) Find the z-transforms of $\frac{1}{n(n+1)}$, for $n \geq 1$ (8)

(ii) Find the inverse Z-transforms of $\dfrac{z^2+z}{(z-1)(z^2+1)}$, using partial fraction (8)

Solutions

Part A

1. $z = f\left(\dfrac{y}{x}\right)$ (1)

Differentiating (1) partially w.r.t. x, we get $p = f'(u).\left(-\dfrac{y}{x^2}\right)$ (2),

where $u = \dfrac{y}{x}$.

Similarly, differentiating (1) partially w.r.t. y, we get $q = f'(u)\left(\dfrac{1}{x}\right)$ (3)

(2) ÷ (3) gives $\left(\dfrac{p}{q}\right) = -\left(\dfrac{y}{x}\right)$; i.e. $px + qy = 0$ is the required P.D.E.

2. $p + q = 1$ (1)

Let the C.S. be $z = ax + by + c$ (2)

From (2), $p = a$ and $q = b$.

Since (2) is a solution of (1), $a + b = 1$ $\therefore$ $b = 1 - a$.

$\therefore$ The required C.S. is $z = ax + (1 - a)\,y + c$

3. The conditions are available in Chapter 2, page 2-2 (Dirichlet's Conditions).

4. $(\pi - x)^2 = \dfrac{\pi^2}{3} + 4\displaystyle\sum_{n=1}^{\infty}\dfrac{\cos nx}{n^2}$ in $0 < x < 2\pi$ (1)

Putting $x = 0$ in the R.H.S. of (1), we get $\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n^2}$

But $x = 0$ is a point of discontinuity at the left extremity.

$\therefore$ (Sum of the R.H.S. series)$_{x=0} = \dfrac{1}{2}\lim\limits_{h\to 0}\,[f(0-h) + f(0+h)]$

i.e. $\dfrac{\pi^2}{3} + 4\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n^2} = \dfrac{1}{2}\lim_{h\to 0}\,\left[f(2\pi + h) + f(h)\right]$

$$= \frac{1}{2}\lim\left[(-\pi - h)^2 + (\pi - h)^2\right] = \pi^2$$

$\therefore$ $\displaystyle\sum\frac{1}{n^2} = \frac{\pi^2}{6}$

5. $A = 1 - x^2$, $B = -2xy$, $C = 1 - y^2$

   $B^2 - 4AC = 4[x^2y^2 - (1 - x^2)(1 - y^2)] = 4(x^2 + y^2 - 1)$

   $\therefore$ The given P.D.E. is of the elliptic type inside the circle $x^2 + y^2 = 1$, parabolic type on the circle $x^2 + y^2 = 1$and hyperbolic type outside the circle $x^2 + y^2 = 1$.

6. The possible solutions of the equation $u_t = \alpha^2 u_{xx}$ are

   (i) $u(x, t) = (Ae^{px} + Be^{-px})e^{p^2\alpha^2 t}$

   (ii) $u(x, t) = (A \cos px + B \sin px)e^{-p^2\alpha^2 t}$ and

   (iii) $u(x, t) = (Ax + B)$

7. This is a standard property of Fourier transforms, explained in Chapter 4 on page 4-28.

8. $$F\{e^{iax} f(x)\} = \int_{-\infty}^{\infty} e^{iax} f(x) e^{-isx} dx$$

   $$= \int_{-\infty}^{\infty} f(x) e^{-i(s-a)x} dx = \overline{f}(s-a)$$

9. This is a standard result given Chapter 5 on page 5-8.
10. This is a standard property of z-transform given in Chapter 5 on page 5-4.

PART - B

11. (a)(i) $z = px + qy + p^2 - q^2$ (1)

   The C.S. of (1) is $z = ax + by + (a^2 - b^2)$ (2)

   Differentiating (2) w.r.t. a, we get $x + 2a = 0$ (3)

   Differentiating (2) w.r.t. b, we get $y - 2b = 0$ (4)

   From (3) and (4), we get $a = -\dfrac{x}{2}$ and $b = \dfrac{y}{2}$ (5)

   Using (5) in (2), the required singular solution is

   $$z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$$

   i.e. $z = \dfrac{y^2}{4} - \dfrac{x^2}{4}$ or $4z = y^2 - x^2$

   (ii) $(D^2 - 2DD')z = x^3y + e^{2x-y}$

   A.E. is $m^2 - 2m = 0$; i.e. $m(m-2) = 0$ or $m = 0, 2$

   $\therefore$ C.F. $= f_1(y) + f_2(y + 2x)$

   $$\text{P.I.}_1 = \frac{1}{D^2 - 2DD'} x^3 y = \frac{1}{D^2}\left(1 - \frac{2D'}{D}\right)^{-1} x^3 y$$

$$= \frac{1}{D^2}\left(1 + \frac{2D'}{D}\right)x^3 y$$

$$= \left(\frac{1}{D^2}1 + \frac{2D'}{D^3}\right)x^3 y = \frac{1}{20}x^5 y + \frac{1}{60}x^6$$

$$\text{P.I.}_2 = \frac{1}{D^2 - 2DD'}e^{2x-y} = \frac{1}{4 - 2\times 2\times(-1)}e^{2x-y} = \frac{1}{8}e^{2x-y}$$

$\therefore$ G.S. is z = C.F. + P.I.$_1$ + P.I.$_2$

11. (b) (i) $x(y-z)\,p + y\,(z-x)\,q = z(x-y)$

LSSE'S are $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)} \cdots (1)$

Using the multipliers 1, 1, 1, each ratio in (1) = $\dfrac{dx + dy + dz}{0}$

$\therefore$ $dx + dy\ dz = 0$, hence, one solution is $x + y + z = a$

Using the multipliers $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$,

Each ratio in (1) = $\dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$\therefore$ $\dfrac{1}{x}dx + \dfrac{1}{y}dy + \dfrac{1}{z}dz = 0$

Integrating, we get $\log x + \log y + \log z = \log b$

$\therefore$ Second solution is $xy\,z = b$

$\therefore$ The G.S. of the given equation is f $(x + y + z, xyz) = 0$

(ii) $(D^3 - 7DD'^2 - 6D'^3)\,z = \sin(x + 2y)$

A.E. is $m^3 - 7m - 6 = 0$; i.e. $(m + 1)(m + 2)(m - 3) = 0$

$\therefore$ $m = -1, -2, 3$

$\therefore$ C.F. $= f_1(y - x) + f_2(y - 2x) + f_3(y + 3x)$

P.I. = $\dfrac{1}{D^3 - 7DD'^2 - 6D'^3}\ \sin(x + 2y)$

$$= \frac{1}{38D' - D}\sin(x + 2y) = \frac{38D' + D}{1444D'^2 - D^2}\sin(x + 2y)$$

$$= \frac{1}{-5775}(76 + 1)\cos(x + 2y) = -\frac{74}{5775}\cos(x + 2y)$$

$\therefore$ G.S. is z = CF + P.I.

12. (a) (i) $f(x) = x^2$ is an even function of x in $-\pi < x < \pi$.

Let the F.S. of $f(x)$ be $\frac{a_0}{2} + \Sigma a_n \cos nx$ in $[-\pi, \pi]$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi} \times \frac{2}{n^2}\{\pi(-1)^n\} = \frac{4(-1)^n}{n^2}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\, dx = \frac{2}{\pi}, \frac{\pi^3}{3} = \frac{2}{3}\pi^2$$

$\therefore$ Required Fourier series is $x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \cdots (1)$

Putting $x = \pm\pi$, in (1), $\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \times (-1)^n = \pi^2$

i.e., $4\sum \frac{1}{n^2} = \frac{2\pi^2}{3} \quad \therefore \quad \sum \frac{1}{n^2} = \frac{\pi^2}{6}$

(ii) Given an even extension to $f(x)$ in $(-1, 0)$; i.e. put $f(x) = (x + 1)^2$ in $(-1, 0)$

$\therefore$ $f(x)$ is even in $(-1, 1)$

Let the F.S. of $f(x)$ be $\frac{a_0}{2} + \sum a_n \cos n\pi x.(\because l = 1)$

$$a_n = \frac{2}{1}\int_0^1 (x-1)^2 \cos n\pi x\, dx$$

$$= 2\left[(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= \frac{4}{n^2\pi^2}\{0-(-1)\} = \frac{4}{n^2\pi^2}$$

$$a_0 = \frac{2}{1}\int_0^1 (x-1)^2\, dx = \frac{2}{3}\{(x-1)^3\}_0^1 = \frac{2}{3}$$

$\therefore$ Required H.R.F.S. is $(x-1)^2 \sim \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi x \text{ in } 0 < x < 1$

12. (b)(i)

x	:	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$
		x_0	x_1	x_2	x_3	x_4	x_5
y	:	1.0	1.4	1.9	1.7	1.5	1.2
		y_0	y_1	y_2	y_3	y_4	y_5

Since $f(x)$ is defined in $(0, 2\pi)$, the F.S. is of the form

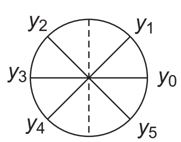
$$\frac{a_0}{2}+\sum a_n \cos nx+\sum b_n \sin nx$$

$$a_0 = 2\times\frac{1}{6}\sum_{r=0}^{5} y_r = \frac{1}{3}\times 8.7 = 2.9$$

$$a_0 = \frac{1}{3}\sum y_r \cos x_r$$

$$= \frac{1}{3}\left[\begin{array}{r} y_0 \cos 0 + y_1 \cos 60° + y_2 \cos 120° + y_3 \cos 180° + y_4 \cos 240° \\ + y_5 \cos 300° \end{array}\right]$$

$$= \frac{1}{3}\left[(y_0 - y_3) + (y_1 + y_5 - y_2 - y_4)\cos 60°\right]$$

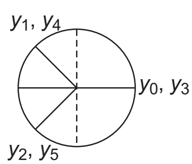


$$= \frac{1}{3}\left[(-0.7) + (1.4 + 1.2 - 1.9 - 1.5)\times 0.5\right]$$

$$= \frac{1}{3}\left[(-0.7) + (-0.4) = -0.367\right]$$

$$b_1 = \frac{1}{3}\left[(y_1 + y_2 - y_4 - y_5)\sin 60°\right] = \frac{1}{3}\times 0.6\times 0.866 = 0.173$$

$$a_2 = \frac{1}{3}\left[(y_0 + y_3) - (y_1 + y_4 + y_2 + y_5)\cos 60°\right]$$



$$= \frac{1}{3}\left[2.7 - (1.4 + 1.5 + 1.9 + 1.2)\times 0.5\right]$$

$$= \frac{1}{3}\times -0.3 = -0.1$$

$$b_2 = \frac{1}{3}(y_1 + y_4 - y_2 - y_5)\sin 60°$$

$$= \frac{1}{3}\times -0.2\times 0.866 = -0.058$$

$\therefore$ Required F.S. in

$$y = 1.45 - 0.367 \cos x + 0.173 \sin x - 0.1 \cos 2x - 0.058 \sin 2x$$

(ii) Give an even extension to $f(x)$ in $-4 < x < 0$; i.e. put $f(x) = -x$ in $(-4, 0)$

$\therefore$ $f(x)$ is even in $(-4, 4)$.

Let the F.S. be $f(x)$ is $\frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{4}$

$$a_n = \frac{2}{4}\int_0^4 x\cos\frac{n\pi x}{4}\,dx = \frac{1}{2}\left[x\left(\frac{\sin\frac{n\pi x}{4}}{\frac{n\pi}{4}}\right) + \frac{\cos\frac{n\pi x}{4}}{\frac{n^2\pi^2}{4^2}}\right]_0^4$$

$$= \frac{1}{2}\times\frac{16}{n^2\pi^2}\times\{(-1)^n - 1\}$$

$$= \begin{cases} -\dfrac{16}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$a_0 = \frac{2}{4}\int_0^4 x\,dx = 4$$

$\therefore$ Required H.R.C series is $x \sim 2 - \frac{16}{\pi^2}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{4}$ in $(0, 4)$

By Parseval's theorem, $\frac{1}{4}{a_0}^2 + \frac{1}{2}\sum {a_n}^2 = \frac{1}{4}\int_0^4 x^2\,dx$

i.e. $4 + \frac{1}{2}\sum_{\text{odd}}\frac{256}{n^4\pi^4} = \frac{1}{4}\times\frac{64}{3}$

i.e. $1 + \sum_{\text{odd}}\frac{32}{n^4\pi^4} = \frac{4}{3}$ $\because$ $\sum_{n=1,3,\cdots}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{96}$

13. (a) This problem has been worked out in Chapter 3, page 3.26, worked example 6. Instead of l, the value 50 is used in the look.

The required solution will be

$$y(x,t) = \frac{3lv_0}{4\pi a}\sin\frac{\pi x}{l}\sin\frac{\pi at}{l} - \frac{lv_0}{12\pi a}\sin\frac{3\pi x}{l}\sin\frac{3\pi at}{l}$$

(b) The SST in given by $u_{xx} + u_{yy} = 0$ (1)

B,C's are (2) $u(0, y) = 0$, (3) $u(a, y) = 0$, (4) $u(x, 0) = 0$ and (5) $u(x, b) = 100$

The proper solution of (1) is $u(x, y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py})$ (6)

Using the B,C's (2), (3), and (4) in (6), the general soln. of (1) will be obtained as $u(x, y) = \sum_{n=1}^{\infty} \lambda_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$ (7)

Using B.C. (5) in (7) and F.H.R series, we can get

$$\lambda_n \sinh\frac{n\pi b}{a} = \begin{cases} \dfrac{400}{n\pi}, & \text{if } n \text{ odd} \\ 0, & \text{if } n \text{ even} \end{cases}$$

$\therefore$ Required soln. is

$$u(x, y) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\text{cosech}\,\frac{(2n-1)\pi b}{a}\cdot\sin\frac{(2n-1)\pi x}{a}\cdot\sin h\frac{(2n-1)\pi y}{a}$$

14. (a) This problem has been given in Chapter 4, page 4-37 and worked example (5).

14. (b) (i) This problem has been given in Chapter 4, page 4-14 and worked example 8.

(b) (ii) This problem is has been given in Chapter 4, page 4-39 and worked example 6 (ii).

15. (a) (i) This problem has been given in Chapter 5, page 5-11 and worked example 8.

(ii) Taking Z-Transforms on both sides of the difference equation we have

$$\left[z^2\,\bar{u}(z) - z^2u(0) - zu(1)\right] - 3\left[z\,\bar{u}(z) - zu(0)\right] + 2\bar{u}(z) = 0$$

i.e. $z^2 - 3z + 2)\,\bar{u}(z) = z$

$$\therefore\quad \frac{\bar{u}(z)}{z} = \frac{1}{z^2-3z+2} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\therefore\quad \bar{u}(z) = \frac{z}{z-2} - \frac{z}{z-1}$$

$\therefore\ u(n) = 2^n - 1$

15. (b)(i) $Z\left\{\dfrac{1}{n(n+1)}\right\} = Z\left\{\dfrac{1}{n} - \dfrac{1}{n+1}\right\}$

$$= \log\left(\frac{z}{z-1}\right) - z\log\left(\frac{z}{z-1}\right) = (1-z)\log\left(\frac{z}{z-1}\right)$$

(ii) $$\frac{\bar{f}(z)}{z} = \frac{z+1}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz+c}{z^2+1} = \frac{1}{z-1} - \frac{z}{z^2+1}$$

$$\therefore\quad \bar{f}(z) = \frac{z}{z-1} - \frac{z^2}{z^2+1}$$

$$\therefore\quad f(n) = 1 - \cos\frac{n\pi}{2}$$

B.E./B.Tech. Degree Examinations, April/May 2015
Question paper code: 77191

Third Semester/ Civil Engineering

Common to all branches

MA6351 Transforms and Partial Differential Equations
(Regulation - 2013)

Time: Three hours **Maximum: 100 marks**

Answer ALL Questions

Part A – (10 × 2 = 20 marks)

1. Form the partial differential equation by eliminating the arbitrary constants a and b from $\log(az - 1) = x + ay + b$.
2. Find the complete solution of $q = 2px$.
3. The instantaneous current 'i' at time t of an alternating current wave is given by $i = I_1 \sin(\omega t + \alpha_1) + I_3 \sin(3\omega t + \alpha_3) + I_5 \sin(5\omega t + \alpha_5) + \ldots$. Find the effective value of the current 'i'.
4. If the Fourier series of the function $f(x) = x$, $-\pi < x < \pi$ with period 2π is given by $f(x) = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right)$, then find the sum of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$
5. Classify the partial differential equation $(1 - x^2)z_{xx} - 2xyz_{xy} + (1 - y^2)z_{yy} + xz_x + 3x^2yz_y - 2z = 0$.
6. A rod 30 cm long has its ends A and B kept at 20°C and 80°C respectively until steady state conditions prevail. Find this steady state temperature in the rod.
7. If the Fourier transform of $f(x)$ is $\Im(f(x)) = F(s)$, then show that $\Im(f(x - a)) = e^{-ias} F(s)$.
8. Find the Fourier sine transform of $1/x$.
9. If $Z(x(n)) = X(z)$, then show that $Z(a^n x(n)) = X\left(\frac{z}{a}\right)$.
10. State the convolution theorem of Z-transforms.

Part B – (5 × 16 = 80 marks)

11. (a) (i) Solve: $(x^2 - yz)p + (y^2 - xz)q = (z^2 - xy)$. (8)

(ii) Solve: $(D^2 - 3DD' + 2D'^2)z = (2 + 4x)e^{x+2y}$ (8)

Or

(b) (i) Obtain the complete solution of $p^2 + x^2y^2q^2 = x^2z^2$. (8)

(ii) Solve $z = px + qy + p^2q^2$ and obtain its singular solution. (8)

12. (a)(i) Find the half-range sine series of $f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 < x < \pi \end{cases}$. Hence deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$. (10)

(ii) Find the complex form of the Fourier series of $f(x) = e^{-x}$ in $-1 < x < 1$. (6)

Or

12. (b) (i) Find the Fourier series of $f(x) = |\sin x|$ in $-\pi < x < \pi$ of periodicity 2π. (8)

(ii) Compute upto the first three harmonics of the Fourier series of $f(x)$ given by the following table: (8)

x	0	$\pi/2$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$	1.0	1.4	1.9	1.7	1.5	1.2	1.0

13. (a) Solve $\frac{\partial u}{\partial t} = a^2 \frac{d^2u}{\partial x^2}$ subject to the conditions: $u(0, t) = 0 = u(l, t), t \geq 0$;

$$u(x, 0) = \begin{cases} x, & 0 \leq x \leq l/2 \\ l - x, & l/2 \leq x \leq l \end{cases},.$$ (16)

Or

(b) A string is stretched and fastened to two points that are distant l apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time $t = 0$. Find the displacement at any point of the string at a distance x from one end at any time t. (16)

14. (a) Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$ and hence evaluate $\int_0^{\infty} \frac{\sin x}{x} dx$. Using Parseval's identity, prove that $\int_0^{\infty} \frac{\sin^2 \cdot t}{t^2} dt = \frac{\pi}{2}$. (16)

Or

(b)(i) Show that the function $e^{-x^2/2}$ is self-reciprocal under Fourier transform by finding the Fourier transform of $e^{-a^2x^2}$, $a > 0$. (10)

(ii) Find the Fourier cosine transform of x^{n-1} (6)

15. (a)(i) Find $Z(r^n \cos n\theta)$ and $Z^{-1}\left[(1-az^{-1})^{-2}\right]$. (8)

(ii) Using convolution theorem, find $Z^{-1}\left[\dfrac{z^2}{(z-1/2)(z-1/4)}\right]$. (8)

(b)(i) Using Z-transform, solve the difference equation

$x(n+2) - 3x(n+1) + 2x(n) = 0$ given that $x(0) = 0$, $x(1) = 1$. (8)

(ii) Using residue method, find $Z^{-1}\left[\dfrac{z}{z^2-2z+2}\right]$. (8)

Solutions

Part A

1. $\log(az-1) = x + ay + b$ (1)

Differentiating (1) p.w.r.t x, $\dfrac{ap}{az-1} = 1$ (2)

Differentiating (1) p.w.r. t. y, $\dfrac{aq}{az-1} = a$ (3)

(3) ÷ (2) gives $\dfrac{q}{p} = a$ (4)

Using (4) in (2), we get $\dfrac{q}{\dfrac{qz}{p}-1} = 1$ or $\dfrac{pq}{qz-p} = 1$ or $z = p + \dfrac{p}{q}$ is the required P.D.E.

2. $q = 2px = a \quad \therefore \quad p = \dfrac{a}{2x}$ and $q = a$

$$dz = p\,dx + q\,dy = \frac{a}{2x}dx + a\,dy$$

$\therefore$ C.S. is $z = \dfrac{a}{2}\log x + ay + b$

3. $$I = \sum_{n=1,3,5\cdots}^{\infty} I_n \sin(n\omega t + \alpha_n) = \sum_{n=1,3,\cdots}^{\infty} I_n(\cos\alpha_n)\sin n\omega t + (I_n \sin\alpha_n)\cos n\omega t$$

Effective value of $I = \dfrac{1}{2}\sum_{n\text{ odd}} I_n^2(\cos^2\alpha_n + \sin^2\alpha_n)$

$$= \frac{1}{2}\sum_{n=1}^{\infty} I_{2n-1}^2$$

4. $x \sim 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \sin\frac{4x}{4} + \cdots\right)$ in $-\pi < x < \pi$ (1)

Put $x = \frac{\pi}{2}$ in (1); $2\left(1 - \frac{1}{3} + \frac{1}{5}\cdots\right) = \frac{\pi}{2}$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \infty = \frac{\pi}{4}$$

5. $A = 1 - x^2, \quad B = -2xy, \quad C = 1 - y^2$
$B^2 - 4\,AC = 4\,(x^2 + y^2 - 1)$
The given P.D.E. is of the elliptic or parabolic or hyperbolic type, according as $x^2 + y^2 - 1 < 0, = 0$ or > 0, viz., inside, on or outside the circle $x^2 + y^2 = 1$.

6. The SST is given by $\frac{d^2u}{dx^2} = 0$ or $u(x) = Ax + B$

Using $u(0) = 20$, $B = 20$; using $u(30) = 80$, $A = 2$.

$\therefore$ SST is $u(x) = 2x + 20$

7. This is a standard property of F' transforms, covered in Chapter 4, page 4.27.

8. The solution to this problem is provided in Chapter 4. It is proved in the example that $\int_0^\infty \frac{\sin sx}{x}\,dx = \frac{\pi}{2}\,(s > 0)$. The L.H.S is the F' sine transform of $\frac{1}{x}$.

9. This is a standard property of z-transform, given in 3(i) Chapter 5, page 5-3.

10. This is a standard theorem, given in Chapter 5, page 5-5.

PART - B

11. (a)(i) LSSE's are $\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \cdots (1)$ (1)

$$\text{viz., } \frac{d(x-y)}{(x-y)\,(x+y+z)} = \frac{d(y-z)}{(y-z)\,(x+y+z)} = \frac{d(z-x)}{(z-x)(x+y+z)}$$

viz., $\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$; Solving, $\log(x-y) - \log(y-z) = \log a$

i.e., $\frac{x-y}{y-z} = a$

Also each ratio in (1) is equal to

$$\frac{d(x+y+z)}{\Sigma x^2 - \Sigma yz} = \frac{x\,dx + y\,dy + z\,dz}{(x+y+x)\,(\Sigma x^2 - \Sigma yz)}$$

i.e. $$\frac{(x+y+z)^2}{2} = \frac{1}{2}(x^2+y^2+z^2)+b$$

or $xy + yz + zx = b$ $\therefore$ Reqd. G.S. is $f\left(\frac{x-y}{y-z}, xy+yz+zx\right) = 0$

11. (a)(ii) A.E. is $m^2 - 3m + 2 = 0$ $\therefore$ $m = 1, 2$

C.F. $= f_1(y+x) + f_2(y+2x)$

$$\text{P.I.} = \frac{1}{(D-D')(D-2D')}(2+4x)\,e^{x+2y}$$

$$= e^{x+2y}\frac{1}{\{D+1-(D'+2)\}\{D+1-2(D'+2)\}}(2+4x)$$

$$= e^{x+2y}\frac{1}{(D-D'-1)(D-2D'-3)}(2+4x)$$

$$= \frac{1}{3}e^{x+2y}\frac{1}{(1-D)\left(1-\frac{D}{3}\right)}(2+4x) = \frac{1}{3}e^{x+2y}\left(1+\frac{4}{3}D\right)(2+4x)$$

$$= \frac{1}{3}e^{x+2y}\left(4x+\frac{22}{3}\right)$$

$\therefore$ G.S is z = C.F. + P.I

11. (b)(i) This problem has been worked out in Chapter 1, page 1-41, worked example 16.

11. (b)(ii) The C.S. of the given PDE is $z = ax + by + a^2b^2$ (1)

Differentiating (1) p.w.r.t, a, we get $x + 2ab^2 = 0$ (2)

Differentiating (1) p.w.r.t b, we get $y + 2a^2 b = 0$ (3)

From (2) and (3), we get $\frac{a}{b} = \frac{y}{x}; \frac{a}{y} = \frac{b}{x} = k$

Using this in (2), $k = -\frac{1}{(2xy)^{1/3}}$ $\therefore$ $\left.\begin{aligned} a &= -y^{2/3}/2^{1/3}\,x^{1/3} \\ \text{and}\quad b &= -x^{2/3}/2^{1/3}\,y^{1/3}\end{aligned}\right\}$ (4)

Using (3) and (4) in (1), we get $16z^3 + 27x^2y^2 = 0$, which is the required S.S.

12. (a)(i) Given an odd extension for $f(x)$ in $(-\pi, 0)$

$\therefore$ $f(x)$ is odd in $(-\pi, \pi)$

Let the F.H.R.S.S. of $f(x)$ be $\sum_{\pi} b_n \sin nx$ is $(0, \pi)$

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx\, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right\}_0^{\pi/2} + \left\{(\pi - x)\left(-\frac{\cos nx}{n}\right) - \frac{\sin nx}{n^2}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{4}{n^2\pi} \sin \frac{n\pi}{2} = \begin{cases} 0, & \text{if } n \text{ is even} \\ \pm\dfrac{4}{n^2\,\pi'} & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \quad f(x) \sim \frac{4}{\pi}\left\{\frac{1}{1^2}\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \cdots \infty\right\} \text{ in } (0, \pi)$$

Putting $x = \dfrac{\pi}{2}$, we get $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots \infty = \dfrac{\pi}{4} \times \dfrac{\pi}{2} = \dfrac{\pi^2}{8}$

12. (a)(ii) Refer to the worked example (8) in Chapter 2 on page 2-87.

In the example, put $a = 1$ and $l = 1$
Required F.S. is

$$e^{-x} = (\sinh 1) \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1 - i n \pi)}{1 + n^2 \pi^2} - e^{in\pi x} \text{ in } (-1, 1)$$

12. (b)(i) $F(x) = |\sin x|; f(-x) = |-\sin x| = f(x)$

$\therefore$ $f(x)$ is an even function in $(-\pi, \pi)$

Let the F.S. be $f(x) \sim \dfrac{a_0}{2} + \Sigma a_n \cos nx \; in(-\pi, \pi)$

$$a_n = \frac{2}{\pi}\int_0^{\pi} |\sin x| \cos nx\, dx = \frac{1}{\pi}\int_0^{\pi} 2 \sin x \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} [\sin (n+1)x - \sin (n-1)x]\, dx$$

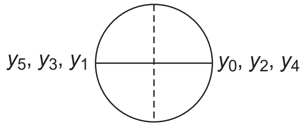
$$= \frac{1}{\pi} \cdot \frac{2}{n^2 - 1}\{(-1)^{n-1} - 1\} = \begin{cases} -\dfrac{4}{\pi(n^2 - 1)}, & \text{if } n \text{ even} \\ 0 \quad , & \text{if } n \text{ odd} \end{cases}$$

$$a_0 = \frac{2}{\pi}\int_0^{\infty} |\sin x|\, dx = \frac{2}{\pi}(-\cos x)_0^{\pi} = \frac{4}{\pi}$$

$$\therefore \quad |\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2,4,6}^{\infty} \frac{1}{n^2 - 1} \cos nx \text{ in } (-\pi, \pi)$$

12. (b)(ii) Refer to question 12 (b) (i) of Solved Question Paper 7 (Nov-Dec 2014) for the solution.

The same problem has been worked out in 12 (b) (i)



$$\begin{aligned} a_3 &= (y_0 + y_2 + y_4) - (y_1 + y_3 + y_5) \\ &= (1 + 1.9 + 1.5) - (1.4 + 1.7 + 1.2) \\ &= 4.4 - 4.3 = 0.1 \end{aligned}$$

and $b_3 = 0$

$\therefore$ Regd. F.S. is $y = 1.45 - 0.367 \cos x + 0.173 \sin x - 0.1 \cos 2x - 0.058 \sin 2x + 0.1 \cos 3x$

13. (a) B.C.'s (2) $u(0, t) = 0$; (3) $u(l, t) = 0$; (4) $u(x, 0) = \begin{cases} x, & \text{in } 0 \le x \le l/2 \\ l - x & \text{in } l/2 \le x \le l \end{cases}$

Proper solution of $u_{(t)} = a^2 u_{xx}$ (1) is $u(x, t)$

$$= (A \cos px + B \sin px)\, e^{-p^2 \alpha^2 t} \quad (5)$$

Using (2) in (5), $A = 0$; Using (3) in (5), $p = \dfrac{n\pi}{l}$; $n = 0, 1, 2, \dots \infty$

$\therefore$ Given solution is $u(x, t) = \displaystyle\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2\alpha^2}{l^2} l}$ (6)

Using (4) in (6);

$$B_n = \frac{2}{l}\left[\int_0^{l/2} x \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} (l - x) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{4l}{n^2\pi^2} \sin \frac{n\pi}{2} = \left\{\frac{4l\,(-1)^{n+1}}{\pi^2\,(2n-1)^2};\ n = 1, 2, 3, \cdots\right\}$$

$$\therefore\ u(x, t) = \frac{4l}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{l}\, e^{-(2n-1)^2 \pi^2 \alpha^2 t/l^2}$$

13. (b) This problem has been worked out in Chapter 3, page 3-12 as worked example 1 (ii).

14. (a) This problem has been worked out in Chapter 4, page 4-11 in worked example 5.

By Parseval's identity, $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\overline{f}(s)|^2 ds$

i.e. $\displaystyle\int_{-a}^{a} 1^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{s^2} \sin^2 as\, ds$

i.e. $$a = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 as}{s^2}\, ds;\ \text{Putting } a = 1,\ \int_{-\infty}^{\infty} \frac{\sin^2 s}{s^2}\, ds = \pi;$$

$$\therefore \quad \int_0^{\infty} \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}$$

14. (b)(i) This problem has been worked out in Chapter 4, page 4-14 in worked example 8.

14. (b)(ii) This problem has been worked out in Chapter 4, page 4.21 as worked example 16.

15. (a)(i) $Z(r^n \cos n\,\theta)$ – This problem has been worked out in Chapter 5, page 5-11.

$$Z^{-1}\,[(1 - az^{-1})^{-2}] = Z^{-1}\left[\left(1 - \frac{a}{z}\right)^{-2}\right] = z^{-1}\left\{1 + \frac{2a}{a} + \frac{3a^2}{z^2} + \cdots\right\}$$

$$= \sum_{n=0}^{\infty} (n+1)\, a^n\, z^{-n}$$

$= \text{Coefficient of } z^{-n} \text{ in } \Sigma = (n + 1)\, a^n$

15. (a)(ii) $$Z^{-1}\left[\frac{z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}\right] = Z^{-1}\left[\left\{\frac{z}{(z - 1/2}\right\} * \left\{\frac{z}{(z - 1/4}\right\}\right]$$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n$$

$$= \sum_{r=0}^{\infty}\left(\frac{1}{2}\right)^{n-r}\left(\frac{1}{4}\right)^r = \left(\frac{1}{2}\right)^n \sum_{r=0}^{n}\left(\frac{1}{2}\right)^r \text{ by Convolution}$$

$$= \left(\frac{1}{2}\right)^n \left\{\frac{1 - (1/2)^{n+1}}{1 - 1/2}\right\} = 2\left\{\left(\frac{1}{2}\right)^n - \frac{1}{2}\left(\frac{1}{4}\right)^n\right\}$$

$$= 2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n$$

15. (b)(i) Taking z-transform on both sides of the difference equation,
we have $[z^2\, \overline{x}\,(z) - z^2\, x\,(0) - z\, x(1)] - 3\,[z\,\overline{x}(z) - z\, x(0)] + 2\,\overline{x}(z) = 0$

i.e., $(z^2 - 3z + 2)\,\overline{x}\,(z) = z$

$$\therefore \quad \frac{\overline{x}\,(z)}{z} = \frac{1}{z^2 - 3z + 2} = \frac{1}{z-2} - \frac{1}{z-1}$$

$\therefore \qquad \bar{x}(z) = \dfrac{z}{z-2} - \dfrac{z}{z-1}$

$\therefore \qquad x(n) = 2^n - 1$

15. (b)(ii) $Z^{-1}\left\{\dfrac{z}{z^2-2z+2}\right\}$ = sum of the residues of $\dfrac{z^n}{z^2-2z+2}$ at its singularities.

The singularities of $\dfrac{z^n}{z^2-2z+2}$ are given by $z^2 - 2z + 2 = 0$,

i.e. $z = 1 + i$ and $z = 1 - i$ which are simple poles

$$\text{Res } z = 1 + i = \left(\frac{z^n}{z-1+i}\right)_{z=1+i} = \frac{1}{2i}(1+i)^n$$

$$\text{Res } z = 1 - i = \left(\frac{z^n}{z-1-i}\right)_{z=1-i} = -\frac{1}{2i}(1-i)^n$$

$$\therefore \quad z^{-1}\left\{\frac{z}{z^2-2z+2}\right\} = \frac{1}{2i}\{(1+i)^n - (1-i)^n\}$$

$$= \frac{(\sqrt{2})^n}{2i}\left\{\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) - \left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)\right\}$$

$$= \frac{(\sqrt{2})^n}{2i} \times 2i\sin\frac{n\pi}{4}$$

$$= (\sqrt{2})^n \cdot \sin\frac{n\pi}{4}$$

B.E./B.Tech. Degree Examinations, November/December 2015
Question paper code: 27327

Third Semester/ Civil Engineering

Common to all branches

MA6351 Transforms and Partial Differential Equations
(Regulation - 2013)

Time: Three hours **Maximum: 100 marks**

Answer ALL Questions

Part A – (10 × 2 = 20 marks)

1. Construct the partial differential equation of all spheres whose centers lie on the Z – axis, by the elimination of arbitrary constants.
2. Solve $(D + D' - 1)(D - 2D' + 3)z = 0$.
3. Find the root mean square value of $f(x) = x(l - x)$ in $0 \le x \le l$.
4. Find the sine series of function $f(x) = 1, 0 \le x \le \pi$.
5. Solve $3x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$ by method of separation of variables.
6. Write all possible solutions of two dimensional heat equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
7. If $F(s)$ is the Fourier Transform of $f(x)$, prove that $F\{f(ax)\} = \frac{1}{|a|}F\left(\frac{s}{a}\right), a \ne 0$
8. Evaluate $\int_0^\infty \frac{s^2}{(s^2 + a^2)(s^2 + b^2)}$ using Fourier Transforms.
9. Find the Z-transform of $\frac{1}{n+1}$.
10. State the final value theorem in Z-transform.
11. (a)(i) Find complete solution of $z^2(p^2 + q^2) = (x^2 + y^2)$ (8)

(ii) Find the general solution of $(D^2 + 2DD' + D'^2)z = 2\cos y - x\sin y$. (8)

Or

(b)(i) Find the general solution of $(z^2 - y^2 - 2yz)p + (xy + zx)q = (xy - zx)$. (8)

(ii) Find the general solution of $(D^2 + D'^2)\, z = x^2y^2$. (8)

12. (a)(i) Find the Fourier series expansion the following periodic function of period 4.

$$f(x)=\begin{cases}2+x & -2\le x\le 0\\ 2-x & 0\le x\le 2\end{cases}\text{ Hence deduce that }\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}\qquad (8)$$

(ii) Find the complex form of Fourier series of $f(x) = e^{ax}$ in the interval $(-\pi, \pi)$ where a is a real constant. Hence, deduce that $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}=\frac{\pi}{a\sinh a\pi}$

Or

(b)(i) Find the half range cosine series of $f(x) = (\pi - x)^2$, $0 < x < \pi$. Hence find the sum of series $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$ (8)

(ii) Determine the first two harmonics of Fourier series for the following data.

x:	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$
	y_0	y_1	y_2	y_3	y_4	y_5
$f(x)$:	1.98	1.80	1.05	1.30	−0.88	−0.25

13. (a) A tightly stretched string with fixed end points $x = 0$ and $x = 1$ is initially at rest in its equilibrium position. If it is vibrating by giving to each of its points a velocity

$$v=\begin{cases}\frac{2kx}{l} & \text{in } 0<x<\frac{1}{2}\\ \frac{2k(l-x)}{l} & \text{in } \frac{1}{2}<x<1\end{cases}$$

Find the displacement of the string at any distance x from one end at any time t.

Or

(b) A bar 10 cm long with insulated sides has its ends A and B maintained at temperature 50°C and 100°C, respectively, until steady state conditions prevails. The temperature at A is suddenly raised to 90°C and at the same time lowered to 60°C at B. Find the temperature distributed in the bar at time t. (16)

14. (a) (i) Find the Fourier sine integral representation of the function $f(x) = e^{-x}\sin x$. (8)

(ii) Find the Fourier cosine transform of the function $f(x)=\frac{e^{-ax}-e^{-bx}}{x}$.

Or

(b) (ii) Find the Fourier transform of the function $f(x)\begin{cases}1-|x|, & x \le 1\\ 0, & |x|>1\end{cases}$.

Hence deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$. (8)

(ii) Verify the convolution theorem for Fourier transform if $f(x) = g(x) = e^{-x^2}$.

15. (a)(i) If $\bar{u}(z) = \frac{z^3+z}{(z-1)^3}$ find the value of u_0, u_1, and u_2. (8)

(ii) Use convolution theorem to evaluate $Z^{-1}\left\{\frac{z^2}{(z-3)(z-4)}\right\}$

(b) (i) Using the inversion integral method (Residue Theorem), find the inverse Z-transform of $u(z) = \frac{z^2}{(z+2)(z^2+4)}$

(ii) Using the z-transform solve the difference equations $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$, given $u_0 = 0$, $u_1 = 1$.

Solutions

PART - A

1. The equation of any sphere whose centre is (0, 0, a) [viz. a point on the z-axis] and whose radius is b is

$$x^2 + y^2 + (z-a)^2 = b^2 \quad (1)$$

If a and b are treated as arbitrary constants, (1) represents the family of the given spheres

Differentiating (1) p.w.r. *t. x*; $2x + 2(z-a)p = 0$ (2)

Differentiating (1) p.w r *t x*; $2y + 2(z-a)q = 0$ (3)

From (2) and (3), we get $\frac{x}{y} = \frac{p}{q}$ or $qx - py = 0$, which is the required PDE.

2. Since the solutions of $(D - aD' - b)z = 0$ is $z = e^{bx} f(y + ax)$, the solution of the given equation is $z = e^x f_1(y - x) + e^{-3x} f_2(y + 2x)$

3. R.M.S. Value = $\sqrt{\frac{1}{l}\int_0^l x^2(l-x)^2\,dx} = \sqrt{\frac{1}{l}\int_0^l (l^2x^2 - 2lx^3 + x^4)\,dx}$

$$= \sqrt{\frac{1}{l}\left(\frac{l^5}{3} - \frac{l^5}{2} + \frac{l^5}{5}\right)} = \sqrt{\frac{1}{30}l^4}\ \frac{l^2}{\sqrt{30}}$$

4. Give an odd extension for $f(x)$ in $-\pi \le x \le 0$; i.e. put $f(x) = -1$ in $(-\pi, 0)$ $f(x)$ is odd in $(-\pi, \pi)$ Let the F.S.S. be $\sum b_n \sin nx$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1.\sin nx\, dx = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{4}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

$\therefore$ Required sine series is $1 = \dfrac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \dfrac{1}{n}\sin nx \text{ in } (0, \pi)$

5. Let $u = X(x), Y(y)$ be a solution of the given equation

   Then $3x\dfrac{X'}{X} = 2y\dfrac{Y'}{Y} = k$, say $\quad \therefore \quad X^3 = Ax^k$ or $X = ax^{k/3}$ and $Y = by^{k/2}$

   $\therefore$ Required solution is $y = ab\,x^{k/3}y^{k/2}$ or $y = c\,x^{k/3}y^{k/2}$.

6. All possible solution are given in Chapter 3, page 3-120.
7. This is the standard change of scale property of F' transforms that is given in Chapter 4, page 4-27.
8. This problem is worked out in Chapter 4, page 4-39 as worked example 7 (ii).
9. This problem is worked out in Chapter 5, page 5-10 as worked example (Z-transform of basic functions) 6 (ii).
10. Final value theorem in Z-transform is available as Property (7) in in Chapter 5, page 5-4.

PART - B

11. (a) (i) Put $Z = y^2$ $\therefore$ $P = \dfrac{\partial Z}{\partial x} = 2yp$ and $Q = \dfrac{\partial Z}{\partial y} = 2yq$

Using these in the given equation, it becomes

$P^2 + Q^2 = 4\,(x^2 + y^2)$

i.e. $P^2 - 4x^2 = 4y^2 - Q^2 = 4a^2$, say

$\therefore \quad P = 2\sqrt{x^2 + a^2}$ and $Q = 2\sqrt{y^2 - a^2}$

$dZ = Pdx + Qdy$

$= 2\sqrt{x^2 + a^2}\,dx + 2\sqrt{y^2 - a^2}\,dy$

Integrating,

$$(Z =)\,y^2 = x\sqrt{x^2 + a^2} + a^2\sinh^{-1}\frac{x}{a} + y\sqrt{y^2 - a^2} - a^2\cosh^{-1}\frac{y}{a} + b$$

(ii) A.E is $m^2 + 2m + 1$, i.e. $m = -1, -1$ $\therefore$ C.F $= x\,f_1(y - x) + f_2(y - x)$

$$\text{P.I.}_1 = \frac{1}{D^2 + 2DD' + D'^2}2\cos y = \frac{2}{0 + 0 - 1}\cos y = -2\cos y$$

$$\text{P.I.}_2 = \frac{1}{(D + D')^2}\text{I.P. of } x e^{iy} = \text{I.P. of } e^{iy}\frac{1}{(D + D' + i)^2}x$$

$= -\text{I.P. of } e^{iy}(x + 2i) = -x\sin y - 2\cos y$

$\therefore$ G.S is $Z = \text{C.F} + \text{P.I}_1 - \text{P.I}_2$

11. (b) (i) This problem is given in Chapter 1, page 1-60 as worked example 11.

(ii) $(D^2 + D'2)z = x^2y^2$.

A.E is $m^2 + 1 = 0 \therefore m = \pm i$, $\therefore$ C.F $= f_1(y + ix) + f_2(y - ix)$

$$\text{PI} = \frac{1}{D^2\left(1+\frac{D'^2}{D^2}\right)}x^2y^2 = \frac{1}{D^2}\left(1-\frac{D'^2}{D^2}\right)x^2y^2$$

$$= \frac{1}{D^2}(x^2y^2) - \frac{1}{D^4}2x^2 = \frac{x^4}{12}\left(y^2 + \frac{x^2}{15}\right)$$

$\therefore$ G.S. is z = C.F + P.I

12. (a) (i) $f(x) = \begin{cases} \phi_1(x) & = 2 + x, \quad \text{in} \quad (-2 \le x \le 0) \\ \phi_2(x) & = 2 - x, \quad \text{in} \quad (0 \le x \le 2) \end{cases}$ $\quad 2l = 4 \quad \therefore\ l = 2$

Since $\phi_1(-x) = \phi_2(x)$, the function $f(x)$ is even in $(-2 \le x \le 2)$

Let the F.S. of $f(x)$ be $\frac{a0}{2} + \sum a_n \cos\frac{n\pi x}{2}$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}dx = \int_0^2 (2-x)\cos\frac{n\pi x}{2}dx$$

$$= \left\{(2-x)\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)} - \frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n^2\pi^2}{4}\right)}\right\}_0^2$$

$$= -\frac{4}{n^2\pi^2}\{(-1)^n - 1\} = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{8}{n^2\pi^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$a_0 = \frac{2}{2}\int_0^2 (2-x)dx = \left(2x - \frac{x^2}{2}\right)_0^2 = 2$$

$\therefore$ Required F.S. is $f(x) = 1 + \frac{8}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos\left(\frac{n\pi x}{2}\right) \text{ in } -2 \le x \le 2$ (1)

Putting $x = 0$ in (1), which is a point of continuity of $f(x)$,

We get $1 + \frac{8}{\pi^2}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\infty\right\} = f(0) = 2$

$$\therefore \qquad \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\infty=\frac{\pi^2}{8}$$

(ii) The complex form of the *F.S.* of $f(x) = e^{-ax}$ in $(-l, l)$ is worked out in Chapter 2 on page 2.87.

Simply change a to $-a$ and l to π in that W.E. and obtain

$$e^{ax} = \sinh(-a\pi)\sum_{n=-\infty}^{\infty}\frac{(-1)^n(-a\pi-\text{in}\,\pi)}{a^2\pi^2+n^2\pi^2}e^{\text{inx}} \text{ in}(-\pi,\pi)$$

i.e.
$$e^{ax} = \left(\frac{\sinh a\pi}{\pi}\right)\sum_{n=-\infty}^{\infty}\frac{(-1)^n(a+in)}{a^2+n^2}e^{inx} \text{ in}(-\pi,\pi) \qquad (1)$$

Putting $x = 0$ in (1); we get

$$\left(\frac{\pi}{\sinh a\pi}\right) = \sum_{n=-\infty}^{\infty}\frac{(-1)^n(a+\text{in})}{a^2+n^2}$$

$\therefore$ Equating the R.P.'s; we get $\displaystyle\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}=\frac{\pi}{a\sinh a\pi}$

12. (b)(i) This problem has been worked out in Chapter 2, page 2-67 in worked example 18.

12. (b)(ii)

$$x:\begin{cases} 0 & \pi/3 & 2\pi/3 & \pi & 4\pi/3 & 5\pi/3 \\ x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{cases}$$

$$y:\begin{cases} 1.98 & 1.80 & 1.05 & 1.30 & -0.88 & -0.25 \\ y_0 & y_1 & y_2 & y_3 & y_4 & y_5 \end{cases}$$

Since $f(x)$ is defined in $(0, 2\pi)$, the F.S. is of the form

$$\frac{a_0}{2}+\sum a_n\cos nx+\sum b_n\sin nx$$

$$a_0 = 2\times\frac{1}{6}+\sum_{r=0}^{5}y_r = \frac{1}{3}\times 5 = 1.67$$

$$a_1 = \frac{1}{3}\sum y_r\cos x_r = \frac{1}{3}\left[(y_0-y_3)+(y_1+y_5-y_2-y_4)\cos 60°\right]$$

$$= \frac{1}{3}\left[0.68-(1.80-0.25-1.05+0.88)\times 0.5\right] = -0.033$$

$$b_1 = \frac{1}{3}(y_1+y_2-y_4-y_5)\sin 60°$$

$$= \frac{1}{3}\left[(1.80-1.05+0.88-0.25)\times 0.866\right] = 1.149$$

$$a_2 = \frac{1}{3}\left[(y_0 + y_3) - (y_1 + y_4 + y_2 + y_5)\cos 60°\right]$$

$$= \frac{1}{3}\left[(1.98 + 1.30) - (1.80 - 0.88 + 1.05 - 0.25) \times 0.5\right]$$

$$= \frac{1}{3}\left[3.28 - 0.86\right] = 0.807$$

$$b_2 = \frac{1}{3}(y_1 + y_4 - y_2 - y_5)\sin 60°$$

$$= \frac{1}{3}\left\{1.80 - 0.88 - 1.05 + 0.25\right\} \times 0.866 = 0.035$$

$\therefore$ Required F.S. is

$y = 0.835 + (-0.033\cos x + 1.149 \sin x) + (0.807 \cos 2x + 0.035 \sin 2x)$

13. (a) This problem has been worked out in Chapter 3, page 3-30 as worked example 8. We have to replace λ by k and 60 is given instead of l.
The required solution will then be

$$y(x,t) = \frac{8kl}{\pi^3 a}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}\ \sin\frac{(2n-1)\pi x}{l} \times \sin\frac{(2n-1)\pi at}{l}$$

13. (b) This problem has been worked out in Chapter 3, page 3-94 in worked example 13.

14. (a) (i) Fourier sine integral of $f(x) = e^{-x}$ sin is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\sin \mathrm{st}.\sin sx\, dt\, ds$$

$$\therefore\ e^{-x}\sin x = \frac{2}{\pi}\int_0^\infty \sin x s\, ds\left[\int_0^\infty e^{-t}\sin t \sin st\, dt\right]$$

$$= \frac{2}{\pi}\int_0^\infty \sin x s\, ds \cdot \frac{1}{2}\int_0^\infty e^{-t}\left\{\cos(s-1)t - \cos(s+1)t\right\}dt$$

$$= \frac{1}{\pi}\int_0^\infty \sin x s\, ds\left[\frac{e^{-t}}{1+(s-1)^2}\left\{-\cos(s-1)t + (s-1)\sin(s-1)t\right\}\right.$$

$$\left. - \frac{e^{-t}}{1+(s+1)^2}\left\{-\cos(s+1)t + (s+1)\sin(s+1)t\right\}\right]_0^\infty$$

i.e., $e^{-x}\sin x = \dfrac{4}{\pi}\displaystyle\int_0^\infty \frac{s\sin x s}{s^4+4}ds$

(ii) $f_c(e^{-ax}) = \int_0^\infty e^{-ax} \cos sx\, dx = \dfrac{a}{s^2 + a^2}$

$\therefore \quad f_c(e^{-\lambda x}) = \int_0^\infty e^{-\lambda x} \cos sx\, dx = \dfrac{\lambda}{s^2 + \lambda^2}$

Integrating both sides w.r.t. λ between a and b, we have

$$\int_0^\infty \left(\frac{e^{-\lambda x}}{-x} \right)_a^b \cos sx\, dx = \left[\frac{1}{2} \log (s^2 + \lambda^2) \right]_a^b \tag{1}$$

i.e., $$\int_0^\infty \left(\frac{e^{-ax} - e^{-bx}}{x} \right) \cos sx\, dx = \frac{1}{2} \log \left(\frac{s^2 + b^2}{s^2 + a^2} \right) \tag{2}$$

i.e., $F_c \left(\dfrac{e^{-ax} - e^{-bx}}{x} \right) = \dfrac{1}{2} \log \left(\dfrac{s^2 + b^2}{s^2 + a^2} \right)$

14. (b) (i) This problem has been worked out in Chapter 4, example 5, page 4.37.

(ii) $F(e^{-a^2x^2}) = \dfrac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$ (Refer to Worked Example 8 in Chapter 4. page 4-14)

$\therefore \quad F(e^{-x^2}) = \sqrt{\pi} e^{-s^2/4}$

$$F\{e^{-x^2} * e^{-x^2}\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-(x-u)^2} e^{-u^2} du \right] e^{-isx} dx$$

$$= \int_{-\infty}^{\infty} e^{-u^2} \left[\int_{-\infty}^{\infty} e^{-(x-u)^2} e^{-isx} dx \right] du$$

$$= \int_{-\infty}^{\infty} e^{-u^2} e^{-ius} F(e^{-x^2})\, dx, \text{ by shifting property}$$

$$= \sqrt{\pi} e^{-s^2/4} \sqrt{\pi} e^{-s^2/4}$$

$$= F(e^{-x^2}) F(e^{-x^2})$$

Thus $F\left[f(x) * g(x)\right] = \overline{f}(s).\overline{g}(s)$ is verified

15. (a) (i) $\overline{u} = \dfrac{z^3 + z}{(z-1)^3} = \dfrac{1 + z^{-2}}{(1 - z^{-1})^3} = \dfrac{1 + z^{-2}}{1 - 3z^{-1} + 3z^{-2} - z^{-3}}$

We apply the long division method to obtain the quotient

$$
\begin{array}{l}
1-3z^{-1}+3z^{-2}-z^{-3}\big)\,1+z^{-2}\,\big(1+3z^{-1}+7z^{-2}\\
\qquad\qquad\qquad\quad 1-3z^{-1}+3z^{-2}-z^{-3}\\
\qquad\qquad\qquad\quad \overline{3z^{-1}-2z^{-2}+z^{-3}}\\
\qquad\qquad\qquad\quad 3z^{-1}-9z^{-2}+9z^{-3}-3z^{-4}\\
\qquad\qquad\qquad\qquad\quad \overline{7z^{-2}-8z^{-3}+3z^{-4}}\\
\qquad\qquad\qquad\qquad\quad 7z^{-2}-21z^{-3}+21z^{-4}-7z^{-5}\\
\qquad\qquad\qquad\qquad\qquad\quad \overline{13z^{-3}-18z^{-4}+7z^{-5}}
\end{array}
$$

$$\therefore \quad \sum_{n=0}^{\infty} u(n)z^{-n} = 1+3z^{-1}+7z^{-2}+\cdots$$

$\therefore$ $u(0) = 1$, $u(1) = 3$ and $u_2 = (7)$

(ii) $$Z^{-1}\left\{\frac{z^2}{(z-3)(z-4)}\right\} = Z^{-1}\left\{\frac{z}{z-3}\cdot\frac{z}{z-4}\right\} = Z^{-1}\left(\frac{z}{z-3}\right) * Z^{-1}\left(\frac{z}{z-4}\right)$$

$$= 3^n * 4^n = \sum_{r=0}^{n} 3^r.4^{n-r}$$

$$= 4^n \sum_{r=0}^{n}\left(\frac{3}{4}\right)^r = 4^n\left[\frac{1-\left(\frac{3}{4}\right)^{n+1}}{1-\frac{3}{4}}\right]$$

$$= 4^{n+1} - 3^{n+1}$$

15.(b) (i) This problem has been worked out in Chapter 5, page 5-33 in worked example 13.

(ii) $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$; $u(0) = 0$ and $u(1) = 1$

Taking Z-Transform

$$\left[z^2\bar{u}(z) - z^2u(0) - zu(1)\right] + 4\left[z\bar{u}(z) - zu(0)\right] + 3\bar{u}(z) = \frac{z}{z-3}$$

i.e. $$(z^2+4z+3)\bar{u}(z) = \frac{z}{z-3} + z$$

$$\therefore \quad \bar{u}(z) = \frac{z^2-2z}{(z+1)(z+3)(z-3)} \qquad \therefore \frac{\bar{u}(z)}{z} = \frac{\frac{3}{8}}{z+1} - \frac{\frac{5}{12}}{z+3} + \frac{\frac{1}{24}}{z-3}$$

$$\therefore \quad \bar{u}(z) = \frac{3}{8}\cdot\frac{z}{z+1} - \frac{5}{12}\cdot\frac{z}{z+3} + \frac{1}{24}\cdot\frac{z}{z-3}$$

Inverting, $$u_n = \frac{3}{8}(-1)^n - \frac{5}{12}\cdot(-3)^n + \frac{1}{24}\cdot 3^n$$

Index