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# 1 MULTIPLE INTEGRALS

## INTRODUCTION AND NOTATION

## 1.1 Double and Triple Integrals

The symbol $\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)dxdy$ denotes a double integral. It means that $f(x,y)$ is to be integrated first with respect to $x$, considering $y$ temporarily, as a constant, between the limits $x_1$ and $x_2$ and the resulting expression is then to be integrated with respect to $y$ between the limits $y_1$ and $y_2$. Here $x_1$ and $x_2$ may be constants or functions of $y$ alone.

(i.e.) $x_1 = f_1(y), x_2 = f_2(y)$, $y_1$ and $y_2$ are constants.

Similarly the symbol $\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)dxdydz$ denotes a triple integral. This is concluded as follows. First $f(x,y,z)$ is integrated with respect to $x$ between the limits $x_1$ and $x_2$, keeping $y$ and $z$ as constants temporarily. The resulting function is integrated with respect to $y$ between the limits $y_1$ and $y_2$, keeping $z$ constant. The result thus obtained is finally integrated with respect to $z$ between the limits $z_1$ and $z_2$. In this integral, $z_1$ and $z_2$ are constants, $y_1$ and $y_2$ are either constants or functions of $z$ alone and $x_1$, $x_2$ are either constants or functions of $y$ and $z$.

**Note:** In $\int\int f(x,y)dxdy$, if the right hand integral has limits involving $x$, then the first integral is with respect to $y$.

**Differentiation Formulas:**

(i) $\dfrac{d}{dx}(k) = 0$

(ii) $\dfrac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

(iii) $\dfrac{d}{dx}[k.f(x)] = k.f'(x)$

(iv) $\dfrac{d}{dx}[f(x).g(x)] = f(x)g'(x) + g(x)f'(x)$

(v) $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

(vi) $\dfrac{d}{dx}f(g(x)) = f'(g(x)).g'(x)$

(vii) $\dfrac{d}{dx}(x^n) = nx^{n-1}$

(viii) $\dfrac{d}{dx}(e^x) = e^x$

(ix) $\dfrac{d}{dx}(a^x) = a^x \log a,\ a > 0$

(x) $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x},\ x > 0$





(xi) $\dfrac{d}{dx}(\sin x) = \cos x$

(xii) $\dfrac{d}{dx}(\cos x) = -\sin x$

(xiii) $\dfrac{d}{dx}(\tan x) = \sec^2 x$

(xiv) $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

(xv) $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

(xvi) $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

(xvii) $\dfrac{d}{dx}(\sinh x) = \cosh x$

(xviii) $\dfrac{d}{dx}(\cosh x) = \sinh x$

(xix) $\dfrac{d}{dx}(\tanh x) = \sec h^2 x$

(xx) $\dfrac{d}{dx}(\coth x) = -\csc h^2 x$

(xxi) $\dfrac{d}{dx}(\sec hx) = -\sec hx \tanh x$

(xxii) $\dfrac{d}{dx}(\csc hx) = -\csc hx \coth x$

(xxiii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} = -\frac{d}{dx}(\cos^{-1} x)$

(xxiv) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} = -\frac{d}{dx}(\cot^{-1} x)$

(xxv) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} = -\frac{d}{dx}(\csc^{-1} x)$

(xxvi) $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

(xxvii) $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$

(xxviii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1-x^2}, |x| < 1$

(xxix) $\frac{d}{dx}(\coth^{-1} x) = -\frac{1}{x^2-1}, |x| > 1$

(xxx) $\frac{d}{dx}(\sec h^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$

(xxxi) $\frac{d}{dx}(\csc h^{-1} x) = \frac{1}{x\sqrt{x^2+1}}$

**Integration Formulas:**

1. $\int dx = x + C$
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$





3. $\int \frac{1}{x} dx = \log x + C$
4. $\int e^{ax} dx = \frac{e^{ax}}{a} + C$
5. $\int \sin x dx = -\cos x + C$
6. $\int \cos x dx = \sin x + C$
7. $\int \sec^2 x dx = \tan x + C$
8. $\int \csc^2 x dx = -\cot x + C$
9. $\int \sec x \tan x dx = \sec x + C$
10. $\int \csc x \cot x dx = -\csc x + C$
11. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
12. $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
13. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$
14. $\int \sinh x dx = \cosh x + C$
15. $\int \cosh x dx = \sinh x + C$
16. $\int \sec h^2 x dx = \tanh x + C$

17. $\int \operatorname{csc} h^2 x dx = -\coth x dx + C$

18. $\int \sec hx \tanh x dx = -\sec hx + C$

19. $\int \operatorname{csc} hx \coth x dx = -\operatorname{csc} hx + C$

20. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a} + C$

21. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x} + C$

22. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$

23. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$

24. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a} + C$ (or) $\log(x + \sqrt{x^2 + a^2})$

25. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} + C$ (or) $\log(x + \sqrt{x^2 - a^2})$

26. $\int \tan x dx = \log \sec x = -\log \cos x + C$

27. $\int \cot x dx = \log \sin x + C$

28. $\int \sec x dx = \log(\sec x + \tan x) + C$

29. $\int \csc x dx = \log(\csc x - \cot x) + C$





30. $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + C$

31. $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + C$

32. $\int \sqrt{a^2 - x^2} dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C$

33. $\int \sqrt{a^2 + x^2} dx = \frac{a^2}{2} \sinh^{-1} \frac{x}{a} + \frac{x}{2}\sqrt{a^2 + x^2} + C$

34. $\int \sqrt{x^2 - a^2} dx = -\frac{a^2}{2} \cosh^{-1} \frac{x}{a} + \frac{x}{2}\sqrt{x^2 - a^2} + C$

35. $\int a^x dx = \frac{a^x}{\log a} + C$

**Example 1.1.** *Evaluate* $\int_0^2 \int_0^1 4xydxdy$

**Solution:** Let $I = \int_0^2 \int_0^1 4xydxdy$

$$= 4 \int_0^2 \left(y\frac{x^2}{2}\right)_0^1 dy = 2\left(\frac{y^2}{2}\right)_0^2 = 4$$

**Example 1.2.** *Evaluate* $\int_0^a \int_0^b \int_0^c (x + y + z)dxdydz$

**Solution:** Let $I = \int_0^a \int_0^b \left(\frac{x^2}{2} + yx + zx\right)_{x=0}^{x=c} dydz$

$$= \int_0^a \int_0^b \left(\frac{c^2}{2} + yc + zc\right) dydz$$

$$= \int_0^a \left( \frac{c^2y}{2} + \frac{y^2c}{2} + zcy \right)_{y=0}^{y=b} dz$$

$$= \int_0^a \left( \frac{c^2b}{2} + \frac{b^2c}{2} + zcb \right) dz$$

$$= \left( \frac{c^2bz}{2} + \frac{b^2cz}{2} + \frac{z^2cb}{2} \right)_{z=0}^{z=a} = \frac{abc}{2}(a+b+c)$$

**Example 1.3.** *Evaluate* $\int_0^{2\pi} \int_0^{\pi} \int_0^a r^4 \sin\phi dr d\phi d\theta$

**Solution:** The given integral

$$\int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_0^a r^4 \sin\phi dr = \int_0^{2\pi} d\theta \int_0^{\pi} \left( \frac{r^5}{5} \right)_0^a \sin\phi d\phi$$

$$= \frac{a^5}{5} \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi d\phi = \frac{a^5}{5} \int_0^{2\pi} (-\cos\phi)_0^{\pi} d\theta$$

$$= \frac{2}{5} a^5 \int_0^{2\pi} d\theta \text{ as } \cos\pi = -1, \cos 0 = 1$$

$$= \frac{4}{5} \pi a^5$$

**Example 1.4.** *Evaluate* $\int_0^1 \int_0^x dx dy$

**Solution:** Let I be the given integral.

$\therefore I = \int_0^1 (y)_0^x dx$ as right hand integral has a limit which is function of x.





$$I = \int_0^1 x dx = \left( \frac{x^2}{2} \right)_0^1 = \frac{1}{2}.$$

**Example 1.5.** *Evaluate* $\int_0^{\pi} \int_0^{a\sin\theta} r d\theta dr$

**Solution:** Let $I = \int_0^{\pi} \left( \frac{r^2}{2} \right)_0^{a\sin\theta} = \frac{1}{2} \int_0^{\pi} a^2 \sin^2\theta d\theta$

$$= \frac{a^2}{2} \int_0^{\pi} \left( \frac{1 - \cos 2\theta}{2} \right) d\theta$$

$$= \frac{a^2}{2} . \frac{1}{2} \left( \theta - \frac{\sin 2\theta}{2} \right)_0^{\pi}$$

$$= \frac{\pi a^2}{4} \text{ as } \sin\pi = 0, \sin 0 = 0$$

**Example 1.6.** *Evaluate* $\int_0^1 \int_0^{\sqrt{1+y^2}} \frac{dx dy}{1 + x^2 + y^2}$

**Solution:** The given integral $= \int_0^1 \left[ \int_0^{\sqrt{1+y^2}} \frac{1}{(1+y^2)+x^2} dx \right] dy$

$$= \int_0^1 \left[ \frac{1}{\sqrt{1+y^2}} \tan^{-1} \left( \frac{x}{\sqrt{1+y^2}} \right) \right]_{x=0}^{x=\sqrt{1+y^2}} dy$$

$$= \int_0^1 \left[ \frac{1}{\sqrt{1+y^2}} (\tan^{-1}(1) - \tan^{-1} 0) \right] dy$$

$$= \frac{\pi}{4} \int_0^1 \frac{dy}{\sqrt{1+y^2}}$$

$= \frac{\pi}{4}\left[\log(y+\sqrt{1+y^2})\right]_0^1 = \frac{\pi}{4}\log(1+\sqrt{2})$

**Example 1.7.** *Evaluate* $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}dxdy$

**Solution:** Let $I = \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}dxdy$

$= \int_0^a \left[\frac{x}{2}\sqrt{(a^2-y^2)-x^2} + \left(\frac{a^2-y^2}{2}\right)\sin^{-1}\left(\frac{x}{\sqrt{a^2-y^2}}\right)\right]_0^{\sqrt{a^2-y^2}} dy$

$= \int_0^a \left(\frac{a^2-y^2}{2}\right)\frac{\pi}{2}y = \frac{\pi}{4}\left[a^2y - \frac{y^3}{3}\right]_0^a = \frac{\pi a^3}{6}$

**Example 1.8.** *Evaluate* $\int_0^1 \int_{y^2}^1 \int_0^{1-x} xdzdxdy$

**Solution:** Let $I = \int_0^1 \int_{y^2}^1 \int_0^{1-x} xdzdxdy$

$= \int_0^1 \int_{y^2}^1 x(z)_0^{1-x}dxdy = \int_0^1 \int_{y^2}^1 (x-x^2)dxdy$

$= \int_0^1 \left[\left(\frac{1}{2}-\frac{1}{3}\right) - \left(\frac{y^4}{2}-\frac{y^6}{3}\right)\right] dy$

$= \int_0^1 \left(\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}\right) dy$

$= \left(\frac{1}{6}y - \frac{1}{2}\frac{y^5}{5} + \frac{1}{3}\frac{y^7}{7}\right)_0^1$

$= \frac{1}{6} - \frac{1}{10} + \frac{1}{21} = \frac{4}{35}$





**Example 1.9.** *Evaluate* $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$

**Solution:**

Let $I = \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{(1-x^2-y^2)-z^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}} dxdy$

$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2}dxdy = \frac{\pi}{2}\int_0^1 (y)_0^{\sqrt{1-x^2}}dx = \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}dx$

$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = \frac{\pi^2}{8}$

**Example 1.10.** *Evaluate* $\int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} r\sqrt{a^2-r^2}d\theta dr$

**Solution:** Let $I = \int_0^{\frac{\pi}{2}} \left(-\frac{1}{3}\right)\left[(a^2-r^2)^{3/2}\right]_0^{a\cos\theta} d\theta$

$= -\frac{1}{3}\int_0^{\frac{\pi}{2}} \left[(a^2-a^2\cos^2\theta)^{3/2} - (a^2)^{3/2}\right]_0^{a\cos\theta} d\theta$

$= -\frac{1}{3}\int_0^{\frac{\pi}{2}} (a^3\sin^3\theta - a^3)d\theta$

$= \frac{a^3}{3}\int_0^{\frac{\pi}{2}} (1-\sin^3\theta)d\theta$

$= \frac{a^3}{3}\left[\theta + \frac{1}{4}\left(-\frac{1}{3}\cos 3\theta + 3\cos\theta\right)\right]_0^{\frac{\pi}{2}}$

$= \frac{a^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right)$

**Example 1.11.** *Evaluate* $\int_0^1 \int_0^y xye^{-x^2}dxdy$

**Solution:** Let $I = \int_0^1 y\left(\frac{e^{-x^2}}{2}\right)_0^y dy$

$= -\frac{1}{2}\int_0^1 (ye^{-y^2} - y)dy$

$= -\frac{1}{2}\left[\frac{e^{-y^2}}{-2} - \frac{y^2}{2}\right]_0^1$

$= \frac{1}{4}[(e^{-1} + 1) - (1)] = \frac{1}{4e}$

**Example 1.12.** *Evaluate* $\int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2 dxdydz$

**Solution:** Let $I = \int_0^a \int_0^{a-x} [x^2z]_0^{a-x-y} dydx$

$= \int_0^a \int_0^{a-x} x^2(a-x-y)dydx$

$= \int_0^a \left[x^2(a-x)y - x^2.\frac{y^2}{2}\right]_0^{a-x} dx$

$= \frac{1}{2}\int_0^a x^2(a-x)^2 dx$

$= \frac{1}{2}[a^2\frac{x^3}{3} - 2a\frac{x^4}{4} + \frac{x^5}{5}]_0^a = \frac{a^5}{60}$





**Example 1.13.** *Evaluate* $\int_0^a \int_0^x \int_0^y xyzdzdydx$

**Solution:** Let $I = \int_0^a \int_0^x \left[\frac{xyz^2}{2}\right]_0^y dydx$

$= \frac{1}{2}\int_0^a \int_0^x xy^3 dydx = \frac{1}{2}\int_0^a \left[\frac{xy^4}{4}\right]_0^x dx$

$= \frac{1}{8}\left[\frac{x^6}{6}\right]_0^a = \frac{a^6}{48}$

**Example 1.14.** *Evaluate* $\int_0^{1/2} \int_0^1 \frac{x}{\sqrt{1-x^2y^2}}dydx$

**Solution:** Let $I = \int_0^{1/2} x\left[\frac{\sin^{-1} xy}{x}\right]_0^1$

$= \int_0^{1/2} \sin^{-1} xdx$

$= \left[x\sin^{-1} x + \sqrt{1-x^2}\right]_0^{1/2} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$

**Example 1.15.** *Evaluate* $\int_0^1 \int_0^{1-z} \int_0^{1-y-z} xyzdxdydz$

**Solution:** Let $I = \int_0^1 \int_0^{1-z} yz\left(\frac{x^2}{2}\right)_0^{1-y-z} dydz$

$= \frac{1}{2}\int_0^1 \int_0^{1-z} yz(1-y-z)^2 dydz$

$= \frac{1}{2}\int_0^1 yz[(1-z)^2 - 2(1-z)y + y^2]dydz$

$$= \frac{1}{2}\int_0^1 \left[z(1-z)^2\frac{y^2}{2} - 2z(1-z)\frac{y^3}{3} + z\frac{y^4}{4}\right]_0^{1-z} dz$$

$$= \frac{1}{2}\int_0^1 \left(\frac{1}{2}z(1-z)^4 - \frac{2}{3}z(1-z)^4 + \frac{1}{4}z(1-z)^4\right) dz$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)\int_0^1 z(1-z)^4 dz$$

$$= \frac{1}{24}\int_0^1 [1-(1-z)](1-z)^4 dz$$

$$= \frac{1}{24}\left[\frac{(1-z)^5}{-5} + \frac{(1-z)^6}{6}\right]$$

$$= \frac{1}{24}\left(\frac{1}{5} - \frac{1}{6}\right) = \frac{1}{720}$$

**Example 1.16.** *Evaluate* $\int_0^{\log 2}\int_0^x\int_0^{x-y} e^{x+y+z}dxdydz$

**Solution:** Let $I = \int_0^{\log 2}\int_0^x\int_0^{x-y} e^{x+y+z}dxdydz$

$$= \int_0^{\log 2} dx \int_0^x dy e^{x+y}(e^z)_{z=0}^{z=x+y}$$

$$= \int_0^{\log 2} dx \int_0^x \left(e^{2x+2y} - e^{x+y}\right) dy$$

$$= \int_0^{\log 2}\left[e^{2x}\frac{e^{2y}}{2} - e^x e^y\right]_{y=0}^{y=x} dx$$

$$= \int_0^{\log 2}\left(\frac{e^{4x}}{2} - \frac{3e^{2x}}{2} + e^x\right) dx$$





$$= \left[\frac{1}{8}e^{4x} - \frac{3e^{2x}}{4} + e^x\right]_0^{\log 2} = \frac{5}{8}$$

**Example 1.17.** *Evaluate* $\int\int xydxdy$ *over the region bounded by the x-axis, ordinate at* $x = 2a$ *and the parabola* $x^2 = 4ay$.

**Solution:**

In the region $y$ varies from 0 to $\dfrac{x^2}{4a}$ and $x$ varies from 0 to $2a$.

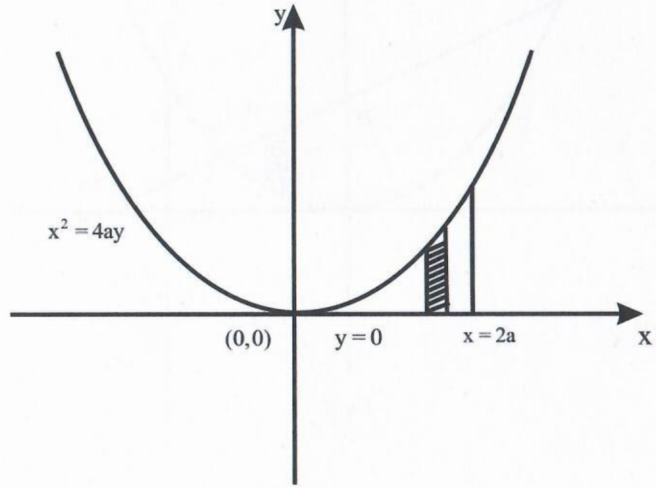



Fig. 1

$$\therefore I = \int_0^{2a}\int_0^{x^2/4a} xydydx = \int_0^{2a}\left[x\frac{y^2}{2}\right]_0^{x^2/4a}$$

$$= \int_0^{2a}\frac{x}{2}.\frac{x^4}{16a^2}dx = \frac{1}{32a^2}\int_0^{2a} x^5 dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{a^4}{3}.$$

**Example 1.18.** *Evaluate* $\int\int y dxdy$ *over the area bounded by* $x = 0, y = x^2, x + y = 2$ *in the first quadrant.*

**Solution:** We have to find the integral over the region OAB.

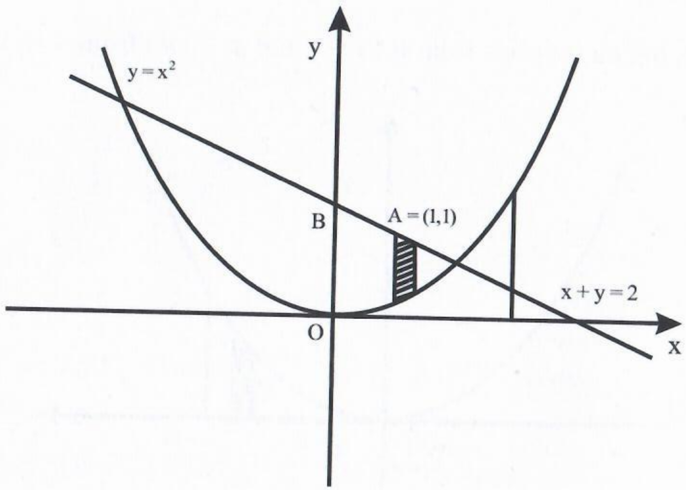



Fig. 2

$$I = \int_0^1 \int_{x^2}^{2-x} ydydx = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 [(2-x)^2 - x^4]dx$$

$$= \frac{1}{2}\int_0^1 [4 - 4x + x^2 - x^4]dx$$





$$= \frac{1}{2}\left[4x - 2x^2 + \frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{2}\left[4 - 2 + \frac{1}{3} - \frac{1}{5}\right] = \frac{16}{15}$$

**Example 1.19.** *Evaluate* $\int\int (x^2 + y^2)dxdy$ *over the area of the triangle whose vertices are* $(0,1), (1,1), (1,2)$.

**Solution:**

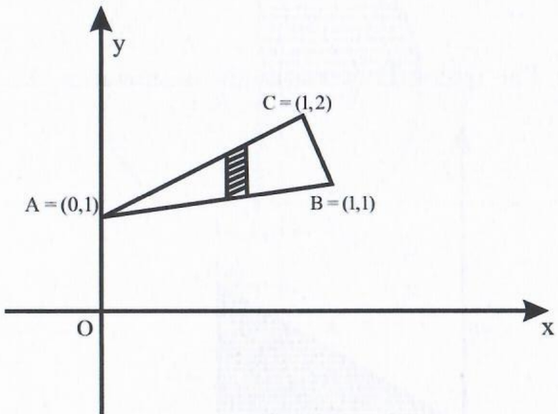



Fig. 3

The equation of the line AC is $\frac{y-2}{1} = \frac{x-1}{1}$

(i.e.) $y = x + 1$.

$$\therefore I = \int_0^1 dx \int_1^{x+1} (x^2 + y^2)dy = \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_1^{x+1} dx$$

$$= \int_0^1 \left\{ \left( x^2(x+1) + \frac{(x+1)^3}{3} \right) - \left( x^2 + \frac{1}{3} \right) \right\} dx$$

$$= \frac{1}{3} \int_0^1 (4x^3 + 3x^2 + 3x) dx = \frac{1}{3} \left[ x^4 + x^3 + \frac{3x^2}{2} \right]_0^1$$

$$= \frac{1}{3} \left[ 1 + 1 + \frac{3}{2} \right] = \frac{7}{6}$$

**Example 1.20.** *Evaluate $\int \int e^{y/x} dxdy$ where $D$ is the region bounded by the straight lines $y = x, y = 0$ and $x = 1$.*

**Solution:** The region D is a triangle as shown in the figure.

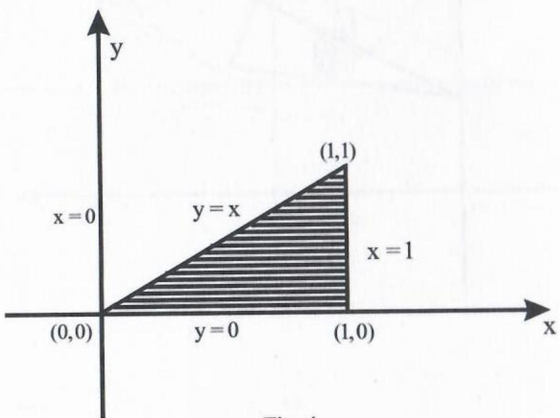



**Fig. 4**

In this region $x$ varies from 0 to 1. For each fixed $x, y$ varies from 0 to $x$.

$$\therefore I = \int_0^1 \int_0^x e^{y/x} dydx = \int_0^1 [xe^{y/x}]_0^x$$





$$= \int_0^1 x(e-1)dx = \frac{1}{2}(e-1)$$

**Example 1.21.** *Evaluate $\int \int_D x^2y^2 dxdy$ where $D$ is the circular disc $x^2 + y^2 \leq 1$.*

**Solution:** In D, $x$ varies from -1 to 1. For a fixed $x, y$ varies from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$.

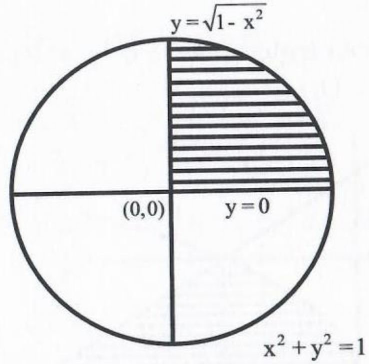



**Fig. 4a**

$$\therefore \int_D x^2y^2 dxdy = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2y^2 dydx$$

$= 4 \int_0^1 \int_0^{\sqrt{1-x^2}} x^2y^2 dydx$ [As integrand is even $\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx$ if f(x) is even]

$$= 4 \int_0^1 \left[ \frac{x^2y^3}{3} \right]_0^{\sqrt{1-x^2}}$$

$$= \frac{4}{3} \int_0^1 x^2(1-x^2)^{3/2} dx$$

$$= \frac{4}{3}\int_0^{\pi/2} \sin^2\theta\cos^4\theta d\theta \text{ put } x = \sin\theta, dx = \cos\theta d\theta$$

$$= \frac{4}{3}\left(\frac{1.3.1}{2.4.6}\right)\left(\frac{\pi}{2}\right) = \frac{\pi}{24}$$

**Example 1.22.** *Evaluate* $I = \int\int_D xydxdy$ *where* $D$ *is the region bounded by the curve* $x = y^2, x = 2 - y, y = 0$ *and* $y = 1$.

**Solution:** The given region bounded by the curves is shown in the figure.

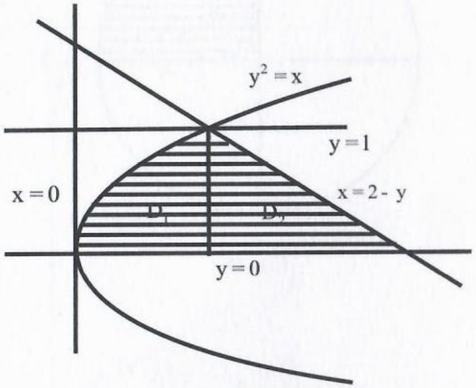



**Fig. 5**

In this region $x$ varies from 0 to 2 when $0 \le x \le 1$ for fixed $x, y$ varies from 0 to $\sqrt{x}$. When $1 \le x \le 2$, $y$ varies from 0 to $2 - x$.

$\therefore$ The region D can be subdivided into two regions $D_1$ and $D_2$ as shown in the figure.

$$I = \int\int_D xydxdy = \int\int_{D_1} xydydx + \int\int_{D_2} xydydx$$





$$= \int_0^1\int_0^{\sqrt{x}} xydydx + \int_1^2\int_0^{2-x} xydydx$$

$$= \int_0^1 \left[\frac{xy^2}{2}\right]_0^{\sqrt{x}} + \int_1^2 \left[\frac{xy^2}{2}\right]_0^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 x^2dx + \frac{1}{2}\int_1^2 x(2-x)^2dx$$

$$= \frac{1}{2}\int_0^1 x^2dx + \frac{1}{2}\int_1^2 (x^3 - 4x^2 + 4x)dx$$

$$= \left[\frac{x^3}{6}\right]_0^1 + \frac{1}{2}\left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{4x^2}{2}\right]_1^2 = \frac{3}{8}$$

**Example 1.23.** *Evaluate* $I = \int\int\int_D xyzdxdydz$ *where* $D$ *is the region bounded by the positive octant of the sphere* $x^2 + y^2 + z^2 = a^2$.

**Solution:**

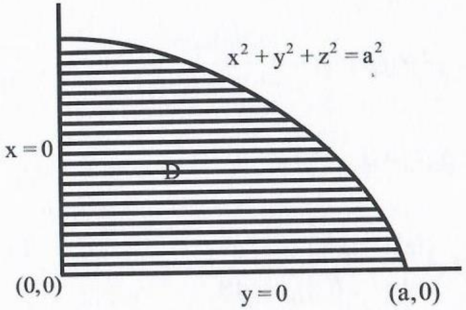



**Fig. 6**

The projection of the given region in the $xy$ plane ($z = 0$) is the region bounded by the circle $x^2 + y^2 = a^2$ and lying in the 1st quadrant as shown in the figure.

In the given region $x$ varies 0 to $a$. For a fixed $x, y$ varies from 0 to $\sqrt{a^2 - x^2}$. For a fixed $(x, y), z$ varies from 0 to $\sqrt{a^2 - x^2 - y^2}$

$$\therefore I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz dz dy dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy \left(\frac{z^2}{2}\right)_0^{\sqrt{a^2-x^2-y^2}} dy dx$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} xy(a^2 - x^2 - y^2) dy dx$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} x(a^2 y - x^2 y - y^3) dy dx$$

$$= \frac{1}{2} \int_0^a x \left[a^2 \frac{y^2}{2} - x^2 \frac{y^2}{2} - \frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{8} \int_0^a x(a^2 - x^2)^2 dx$$

$$= \frac{1}{8} \int_0^a (a^4 x - 2a^2 x^3 + x^5) dx$$

$$= \frac{1}{8} \left[a^4 \frac{x^2}{2} - 2a^2 \frac{x^4}{4} + \frac{x^6}{6}\right]_0^a = \frac{a^6}{48}$$

**Example 1.24.** *Evaluate* $I = \int \int \int_D \dfrac{dx dy dz}{(1 + x + y + z)^3}$ *where* $D$ *is the region bounded by the planes* $x = 0, y = 0, z = 0$ *and* $x + y + z = 1$.





**Solution:** The given region is a tetrahedron. The projection of the given region in the $xy$ plane is the triangle bounded by the lines $x = 0, y = 0$ and $x + y = 1$ as shown in the figure. In the

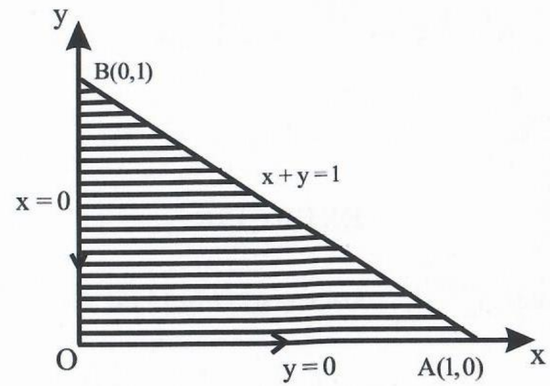



**Fig. 7**

given region $x$ varies 0 to 1. For each fixed $x, y$ varies from 0 to $1 - x$. For each fixed $(x, y), z$ varies from 0 to $1 - x - y$.

$$\therefore I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dx dy dz}{(1 + x + y + z)^3} dy dx$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-x} [(1 + x + y + z)^{-2}]_0^{1-x-y} dy dx$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-x} \left[\frac{1}{4} - (x + y + 1)^{-2}\right] dy dx$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{1}{4} y + (x + y + 1)^{-1}\right]_0^{1-x} dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{1}{4}(1-x) + \frac{1}{2} - (x+1)^{-1}\right] dx$$

$$= -\frac{1}{2}\left[\frac{1}{4}x - \frac{1}{8}x^2 + \frac{1}{2}x - \log(x+1)\right]$$

$$= \frac{1}{2}\log 2 - \frac{5}{16}$$

## EXERCISE

1. Evaluate $\int_0^{\pi/2}\int_{\pi/2}^{\pi} \cos(x+y)dxdy$ [Ans: -2]

2. Evaluate $\int_0^1\int_0^x e^{x+y}dydx$ $\left[\text{Ans:}\frac{1}{2}(e-1)^2\right]$

3. Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} rd\theta dr$ $\left[\text{Ans:}\frac{3}{4}\pi a^2\right]$

4. Evaluate $\int_0^1 dx\int_0^2 dy\int_1^2 x^2yzdz$ [Ans: 1]

5. Find the value of $\int_2^3\int_1^2 \frac{dxdy}{xy}$ [Ans: $\log 2\log\frac{3}{2}$]

6. Evaluate $\int_0^1\int_0^2 xy^2dydx$. [Ans: 4/3]

7. Find $I = \int_0^{\pi}\int_0^{\pi/2}\int_0^1 r^2\sin\theta drd\theta d\phi$ $\left[\text{Ans:}\frac{\pi}{3}\right]$

8. Find the value of $\int_0^2\int_1^3\int_1^2 xy^2zdzdydx$ [Ans: 26]

9. Evaluate $\int_0^1\int_0^z\int_0^{y+z} dzdydx$ [Ans: 1/2]



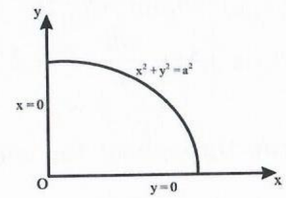


10. Find the limits of $\int\int_R f(x,y)dxdy$, where R is the 1st quadrant bounded by (a) $x=0, y=0, x+y=1$ (b) $x=0, x=y, y=1$.

    $\left[\text{Ans:}a)\int_0^1\int_0^{1-x} f(x,y)dxdy\ b)\int_0^1\int_0^{1-y} f(x,y)dydx\right]$

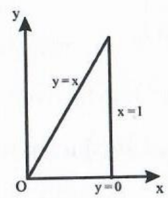
11. Sketch the region by integration to $\int_0^a\int_0^{\sqrt{a^2-x^2}} f(x,y)dxdy$.

    Ans:




12. Sketch the region by integration to $\int_0^1\int_0^x f(x,y)dxdy$.

    Ans:




13. Evaluate $\int_0^1\int_0^x dxdy$

14. Evaluate $\int_1^2\int_0^x \frac{1}{x^2+y^2}dxdy$ $\left[\text{Ans:}\frac{\pi}{4}\log 2\right]$

15. Evaluate $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{(a^2-r^2)/a} r d\theta dr dz$ $\left[\text{Ans:}\frac{5}{64}\pi a^3\right]$

16. Evaluate $\int_0^2\int_0^x\int_0^{x+y} e^{x+y+z} dxdydz$ $\left[\text{Ans:}\frac{e^8-3}{8}-\frac{3}{4}e^4+e^2\right]$

17. Evaluate $\int\int(x^2-y^2)xdxdy$ over the positive quadrant of the circle $x^2+y^2=a^2$ $\left[\text{Ans:}\frac{a^5}{15}\right]$

18. Evaluate $\int\int(x+y)^2xdxdy$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\left[\text{Ans:}\frac{\pi ab}{4}(a^2+b^2)\right]$

19. Evaluate $\int\int dxdy$ throughout the area bounded by

$y=x^2, x+y=2$ $\left[\text{Ans:}-\frac{10}{3}\right]$

20. Evaluate $\int\int x^2y^2dxdy$ over the area of the circle

$x^2+y^2=1$ $\left[\text{Ans:}\frac{\pi}{24}\right]$

21. Evaluate $\int\int(x^2+y^2)dxdy$ over the area bounded by the parabola $y^2=4x$ and its lactusrectum. $\left[\text{Ans:}\frac{344}{105}\right]$

22. Evaluate $\int_0^1\int_x^1\frac{ydxdy}{x^2+y^2}$ and also sketch the region of integration. $\left[\text{Ans:}\frac{\pi}{4}\right]$





23. Evaluate $\int\int xydxdy$ over the region in the positive quadrant bounded by the line $2x+3y=6$ $\left[\text{Ans:}\frac{3}{2}\right]$

24. Evaluate $\int\int\int_V(xy+yz+zx)dxdydz$, where V is the region of sphere bounded by $x=0, x=1, y=0, y=2, z=0, z=3$ $\left[\text{Ans:}\frac{33}{2}\right]$

25. Evaluate $\int\int\int_D x^2yzdxdydz$, where D is the tetrahedron bounded by the planes $x=0, y=0, z=0$ and

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$ $\left[\text{Ans:}\frac{a^2b^2c^2}{2520}\right]$

26. Evaluate $\int_0^{\pi/4}\int_0^{a\sin\theta}\frac{rdrd\theta}{\sqrt{a^2-r^2}}$ $\left[\text{Ans:}\frac{a(\pi-3)}{6}\right]$

27. Evaluate $\int_0^4\int_{y^2/4}^y\frac{ydxdy}{x^2+y^2}$ and also sketch the region of integration. [Ans: 2 log 2]

28. Evaluate $\int\int_R x^2dxdy$, where R is the region bounded by the hyperbola $xy=4, y=0, x=1$ and $x=2$. [Ans: 6]

29. Evaluate $\int\int\int_V dxdydz$, where V is the finite region of space formed by the planes $x=0, y=0, z=0$ and $2x+3y+4z=12$. [Ans: 12]

30. Evaluate $\int\int_R xydxdy$, where R is the region bounded by the parabola $y^2=x$ and the lines $y=0$ and $x+y=2$, lying in the 1st quadrant. [Ans: 3/8]

## 1.2 Change of Order of Integration

Sometimes a double integral may be conveniently evaluated by changing the order of integration. When the limits of the integration are variables, the change of order of integration will involve a change in the limit also. In such cases, it will be convenient to draw a figure indicating the original region of integration.

**Example 1.25.** *Evaluate* $\int_0^a \int_y^a \dfrac{xdxdy}{x^2+y^2}$ *by changing the order of integration.*

**Solution:** The region D of integration is bounded by the lines $x=y, x=a, y=0$ and $y=a$ as shown in the figure.

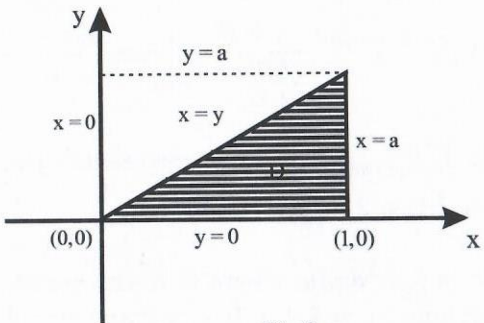



**Fig. 8**

We have to change the order of integration as $dydx$.

Here in the region D, $x$ varies from 0 to $a$ and for fixed $x, y$ varies from 0 to $x$.

$$\therefore \int_0^a \int_y^a \frac{xdxdy}{x^2+y^2} = \int_0^a \int_0^x \frac{xdxdy}{x^2+y^2} = \int_0^a x\left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x$$





$$= \int_0^a x\left[\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx = \int_0^a[\tan^{-1}1 - \tan^{-1}0]dx$$

$$= \int_0^a[\tan^{-1}1 - \tan^{-1}0]dx = \frac{\pi}{4}\int_0^a dx$$

$$= \frac{\pi}{4}[x]_0^a = \frac{\pi a}{4}$$

**Example 1.26.** *Evaluate* $I = \int_0^\infty \int_x^\infty \dfrac{e^{-y}}{y}dydx$ *by changing the order of integration.*

**Solution:** The region of integration is as shown. When are change the order $x$ varies from 0 to $y$ and $y$ varies from 0 to $\infty$.

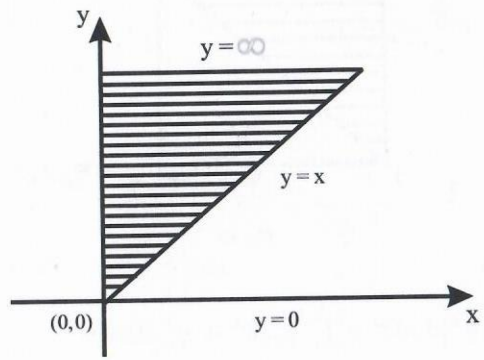



**Fig. 9**

$$I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}dydx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}dxdy$$

$$= \int_0^\infty \left[\frac{e^{-y}}{y}x\right]_0^y = \int_0^\infty e^{-y}dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^{\infty} = 1\left[\frac{e^{-\infty} - e^{-0}}{-1}\right] = \frac{0-1}{-1} = 1$$

**Example 1.27.** *Change the order of integration in* $\int_0^a \int_x^a (x^2 + y^2)dxdy$ *and hence evaluate the same.*

**Solution:** The region of integration is $OAB$ bounded by $x = 0, x = y, y = a$ and $x = a$.

On changing the order of integration $x = 0, x = y$. Then the strip slides from $y = 0$ to $y = a$.

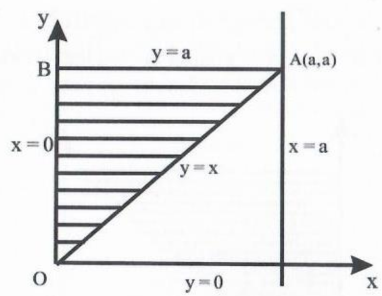



**Fig. 10**

$$\therefore \int_0^a \int_x^a (x^2 + y^2)dydx = \int_0^a \int_0^y (x^2 + y^2)dxdy$$

$$= \int_0^a \left[\frac{x^3}{3} + y^2 x\right]_0^y = \int_0^a \left[\frac{y^3}{3} + y^3\right] dy$$

$$= \frac{4}{3}\int_0^a y^3 dy = \frac{1}{3}[y^4]_0^a = \frac{a^4}{3}$$

**Example 1.28.** *By changing the order of integration, prove that* $\int_0^{\infty} \int_0^y ye^{-y^2/x}dxdy = \frac{1}{2}$





**Solution:** The region of integration is as shown:

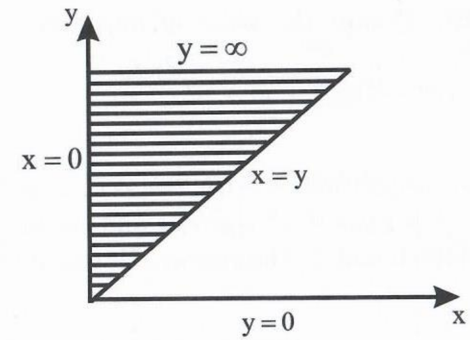



**Fig. 11**

It is bounded by $x = 0, x = y, y = 0, y = \infty$. On changing the order of integration $y$ ranges from $y = x, y = \infty$ and $x$ ranges from $x = 0$ to $x = \infty$.

$$\int_0^{\infty} \int_0^y ye^{-y^2/x}dxdy = \int_{x=0}^{\infty} \int_{y=x}^{\infty} ye^{-y^2/x}dydx$$

$$= \int_{x=0}^{\infty} \left[\int_0^{\infty} e^{-y^2/x} d\left(\frac{y^2}{2}\right)\right] dx$$

$$= \int_0^{\infty} \frac{1}{2}\left[\frac{e^{-y^2/x}}{-1/x}\right]_x^{\infty} = \int_0^{\infty} \frac{-x}{2}(0 - e^{x})dx$$

$$= \frac{1}{2}\int_0^{\infty} xe^{-x}dx$$

$$= \frac{1}{2}\left[x\left(\frac{e^{-x}}{-1}\right) - 1.\frac{e^{-x}}{1}\right]_0^{\infty}, \text{ on integrating by parts}$$

$$= \frac{1}{2}(0+1) = \frac{1}{2}$$

**Example 1.29.** *Change the order of integration and evaluate* $I = \int_0^1 \int_0^{\sqrt{1-x^2}} \dfrac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}} dxdy$

**Solution:** The integration is with respect to $y$ between $y = 0, y = \sqrt{1-x^2}$. (i.e.) $y = 0, x^2+y^2 = 1$ and the next integration is w.r.to $x$ between 0 and 1. The region of integration is as shown here as $OAB$.

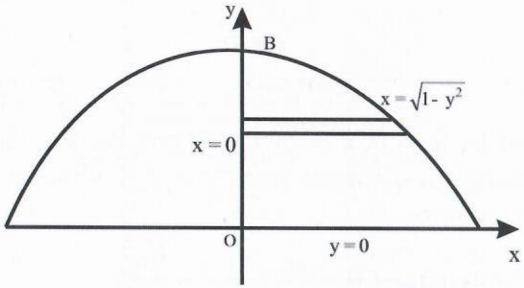



Fig. 12

We now change the order of integration. Integration I w.r.to $x$ between $x = 0, x = \sqrt{1-y^2}$. To cover $OAB$, $y$ slides from 0 to 1.

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}} dxdy$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}} dydx$$





$$= \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}} dxdy$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} \left[ \frac{e^y}{(e^y+1)} \frac{dx}{\sqrt{(1-y^2)-x^2}} \right] dy$$

$$= \int_0^1 \frac{e^y}{(e^y+1)} \left[ \sin^{-1} \frac{x}{\sqrt{1-y^2}} \right]_0^{\sqrt{1-y^2}} dy = \int_0^1 \frac{e^y}{(e^y+1)} \frac{\pi}{2} dy$$

$$= \frac{\pi}{2} \left[\log(e^y+1)\right]_0^1 = \frac{\pi}{2} \log\left(\frac{e+1}{2}\right)$$

**Example 1.30.** *By changing the order of integration, evaluate* $\int_0^3 \int_1^{4-y} (x+y) dxdy$.

**Solution:** The region D of integration is bounded by the line $y = 0, y = 3, x = 1$ and by the parabola $x^2 = 4 - y$ as shown in the figure.

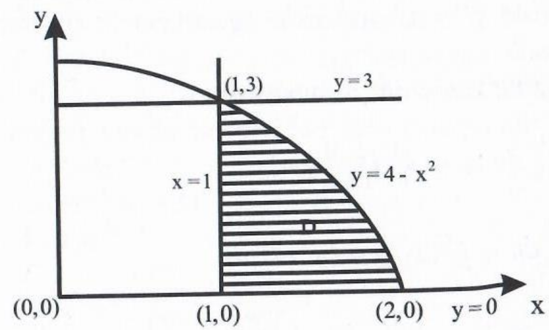



Fig. 13

In this region, $x$ varies from 1 to 2 and $y$ varies from 0 to $4-x^2$.

$$\int_0^3\int_1^{\sqrt{4-y}}(x+y)dxdy = \int_1^2\int_0^{4-x^2}(x+y)dydx$$

$$= \int_1^2\left[xy+\frac{y^2}{2}\right]_0^{4-x^2} = \int_1^2\left[x(4-x^2)+\frac{(4-x^2)^2}{2}\right]dx$$

$$= \int_1^2\left[\frac{x^4}{2}-x^3-4x^2+4x+8\right]dx$$

$$= \left[\frac{x^5}{10}-\frac{x^4}{4}-4\frac{x^3}{3}+2x^2+8x\right]_1^2 = \frac{241}{60}$$

**Example 1.31.** *Change the order of integration and evaluate* $I = \int_0^4\int_{x^2/4}^{2\sqrt{x}} dydx$

**Solution:** The region of integration is bounded by $y = \dfrac{x^2}{4}, y = 2\sqrt{x}$. (i.e) $x^2 = 4y$ and $y^2 = 4x$ which are parabolas and $x = 0$ and 4. Solving $y^2 = 4x$ and $x^2 = 4y$, we get $(0,0),(4,4)$.

After changing the order of integration

$$I = \int_0^4\int_{x^2/4}^{2\sqrt{x}} dydx = \int_0^4\int_{y^2/4}^{2\sqrt{y}} dxdy$$

$$= \int_0^4[x]_{y^2/4}^{2\sqrt{y}}dy = \int_0^4(2\sqrt{y}-(y^2/4))dy$$

$$= \left[\frac{4}{3}y^{3/2}-\frac{y^3}{12}\right]_0^4 = \frac{32}{3}-\frac{16}{3} = \frac{16}{3}$$

**Example 1.32.** *Change the order of integration and evaluate*





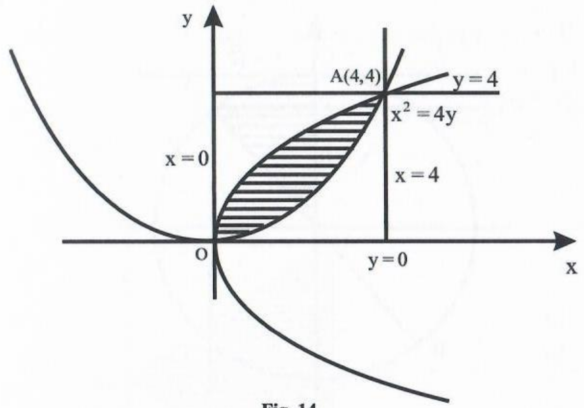



Fig. 14

$$I = \int_0^1 dx\int_x^{\sqrt{2-x^2}}\frac{x}{\sqrt{x^2+y^2}}dy$$

**Solution:** The limits for $y$ are $x$ and $\sqrt{2-x^2}$ and those for $x$ are 0 and 1. We, therefore, draw the curve $y = x$ which is a straight line and $y = \sqrt{2-x^2}$ which is the upper half of the circle $x^2+y^2 = 2$. The region of integration is OACD. Solving the equation $y = x$ and $x^2+y^2 = 2$, we get the points of intersection $A(1,1)$ and $B(-1,-1)$. If we consider the strip parallel to the x-axis the region has to be divided into two points OAD and ADC. In the region ODA, $x$ varies from 0 to $y$ and $y$ varies from 0 to 1. In the region ADC, $x$ varies from 0 to $\sqrt{2-y^2}$ and $y$ varies from 1 to $\sqrt{2}$.

$$\therefore I = \int_0^1 dx\int_x^{\sqrt{2-x^2}}\frac{x}{\sqrt{x^2+y^2}}dy$$

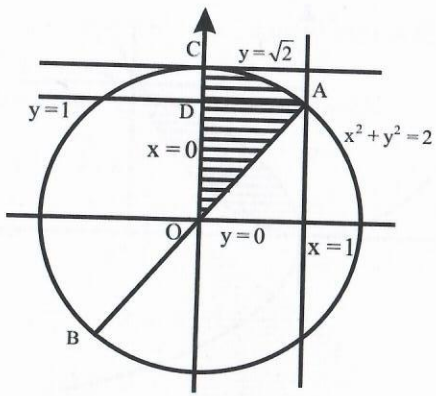



Fig. 15

$$= \int_0^1 dy \int_0^y \frac{x}{\sqrt{x^2+y^2}}dx + \int_1^{\sqrt{2}} dy \int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}dx = I_1 + I_2$$

Now $I_1 = \int_0^1 dy[\sqrt{x^2+y^2}]_0^y = \int_0^1 (\sqrt{2}y - y)dy$

(i.e.) $I_1 = (\sqrt{2}-1)\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}(\sqrt{2}-1)$

$I_2 = \int_1^{\sqrt{2}} dy[\sqrt{x^2+y^2}]_0^{\sqrt{2-y^2}} = \int_1^{\sqrt{2}} (\sqrt{2}y - y)dy$

$$= \left[\sqrt{2}y - \frac{y^2}{2}\right]_1^{\sqrt{2}} = \frac{3}{2} - \sqrt{2}$$

$$I = I_1 + I_2 = 1 - \frac{1}{\sqrt{2}}$$

**Example 1.33.** *Change the order of integration and evaluate* $I = \int_0^5 dx \int_{2-x}^{2+x} dxdy$





**Solution:** The I integration is w.r.to y, where $y = 2 - x$ and $y = 2 + x$. The II integration is w.r.to x where $x = 0$ to 5. The region integration is as shown in the figure as ABC. Split ABC as ADC, ADB.

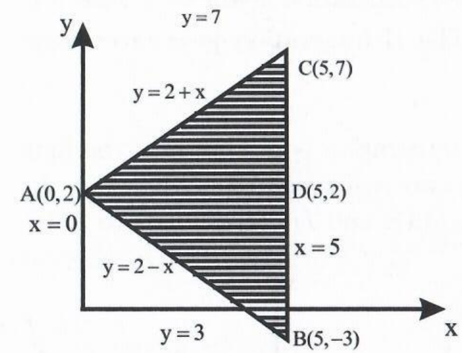



Fig. 16

$I = I_1 + I_2$, where $I_1 = \int_{-3}^{2} \int_{2-y}^{5} dxdy$ and $I_2 = \int_2^7 \int_{y-2}^{5} dxdy$

Now $I_1 = \int_{-3}^{2} [x]_{2-y}^{5} dy = \int_{-3}^{2} (3+y)dy$

$$= \left[3y + \frac{y^2}{2}\right]_{-3}^{2} = 17 - \frac{9}{2} = \frac{25}{2}$$

and $I_2 = \int_2^7 [x]_{y-2}^{5} dy = \int_2^7 (7-y)dy$

$$= \left[7y - \frac{y^2}{2}\right]_2^7 = 37 - \frac{49}{2} = \frac{25}{2}$$

$$\therefore I = I_1 + I_2 = \frac{25}{2} + \frac{25}{2} = 25$$

**Example 1.34.** *Change the order of integration and evaluate* $\int_0^a \int_{x^2/a}^{2a-x} xy\,dx\,dy$

**Solution:** The I integration is w.r.to y with limits $y = \dfrac{x^2}{2}$ and $y = 2a - x$. The II integration is w.r.to x between the limits $x = 0, x = a$.

The region of integration is as shown in the figure as OAC. We split this into two regions OAB, ABC. $I = I_1 + I_2$ where $I_1$ corresponds to OAB and $I_2$ corresponds to ABC.

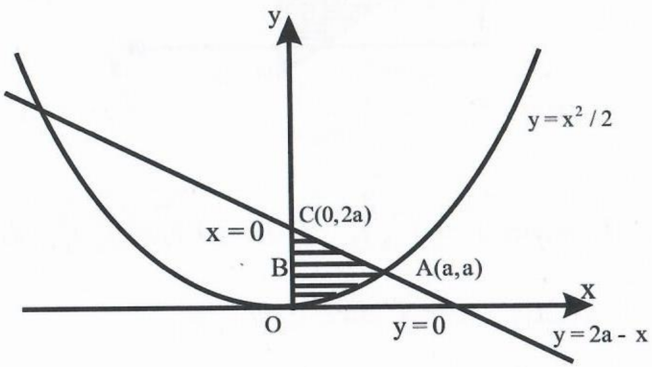



**Fig. 17**

$$I = \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy + \int_a^{2a} \int_0^{2a-y} xy\,dx\,dy$$

$$= \int_0^a y \left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} dy + \int_a^{2a} y \left(\frac{x^2}{2}\right)_0^{2a-y} dy$$





$$= \int_0^a \frac{a}{2} y^2 dy + \frac{1}{2} \int_a^{2a} (4a^2 y - 4ay^2 + y^3) dy$$

$$= \frac{a}{2} \left(\frac{y^3}{3}\right)_0^a + \frac{1}{2} \left[2a^2y^2 - 4a\frac{y^3}{3} + \frac{y^4}{4}\right]_0^a$$

$$= \frac{a^4}{6} + \frac{5a^4}{24} = \frac{3a^4}{8}$$

## EXERCISE

1. Change the order of integration in $\int_{-a}^a \int_0^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$

   $\left[\text{Ans: } \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx\right]$

2. Change the order of integration in $\int_0^a \int_y^a \dfrac{x}{x^2+y^2} dx\,dy$

   $\left[\text{Ans: } \int_0^a \int_0^x \dfrac{x}{x^2+y^2} dy\,dx\right]$

3. Change the order of integration in $\int_0^\infty \int_0^y f(x,y)\,dx\,dy$

   $[\text{Ans: } \int_0^\infty \int_x^\infty f(x,y)\,dx\,dy]$

4. Change the order of integration in $\int_0^1 \int_0^{2\sqrt{x}} \phi(x,y)\,dy\,dx$

   $\left[\text{Ans: } \int_0^2 \int_{y^2/4}^1 \phi(x,y)\,dx\,dy\right]$

5. Change the order of integration in $\int_0^1 \int_0^x dy\,dx$

$\left[\text{Ans: } \int_0^1 \int_y^1 dxdy\right]$

6. Change the order of integration in $\int_0^a \int_{x^2/a}^x f(x,y)dydx$

   $\left[\text{Ans: } \int_0^a \int_y^{\sqrt{ay}} f(x,y)dxdy\right]$

7. Change the order of integration in $\int_0^\infty \int_0^{1/y} f(x,y)dxdy$

   $\left[\text{Ans: } \int_0^\infty \int_0^{1/y} f(x,y)dxdy\right]$

8. Change the order of integration and show that $\int_0^1 \int_0^x dydx = \frac{1}{2}$.

9. Change the order of integration and hence evaluate $\int_1^4 \int_{\sqrt{y}}^2 (x^2+y^2)dxdy$ $\left[\text{Ans:}\frac{1006}{105}\right]$

10. Change the order of integration and hence evaluate

    $\int_0^a \int_0^{\sqrt{ax}} x^2 dydx$ $\left[\text{Ans:}\frac{2a^4}{7}\right]$

11. Evaluate $\int_0^a \int_0^{\sqrt{1-x^2}} y^2 dydx$ by interchanging the order of integration. $\left[\text{Ans:}\frac{\pi}{16}\right]$

12. Change the order of integration in $\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} ydxdy$ and then evaluate it. $\left[\text{Ans:}\frac{a^3}{6}\right]$





13. Change the order of integration in $\int_0^{2a} \int_{x^2/4a}^a (x+y)dydx$ and hence evaluate it. $\left[\text{Ans:}\frac{9a^3}{5}\right]$

14. Evaluate $\int_1^2 \int_0^{4/x} xydydx$ by changing the order of integration. [Ans:8 log 2]

15. Change the order of integration in $\int_0^1 \int_y^{2-y} xydxdy$ and then evaluate it. $\left[\text{Ans:}\frac{1}{3}\right]$

16. Change the order of integration in $\int_1^2 \int_1^{x^2} \frac{x^2}{y} dxdy$ and hence evaluate the same. $\left[\text{Ans:}\frac{2}{9}\log 2 - 7\right]$

17. Change the order of integration in $\int_0^a \int_{y^2/a}^y \frac{ydydx}{(a-x)\sqrt{ax-y^2}}$

    $\left[\text{Ans:}\frac{\pi a}{2}\right]$

18. Evaluate $\int_3^5 \int_0^{4/x} xydxdy$ after reversing the order of integration. $\left[\text{Ans:}8\log\frac{5}{3}\right]$

19. Change the order of integration in $\int_0^\infty \int_0^\infty e^{-x^2(1+t^2)} xdxdt$ and then evaluate it. $\left[\text{Ans:}\frac{\pi}{4}\right]$

20. Change the order of integration in each of the double integrals $\int_0^1 \int_1^2 \frac{dxdy}{x^2+y^2}$ and $\int_1^2 \int_y^2 \frac{dxdy}{x^2+y^2}$ and hence express

their sum as one double integral and evaluate it. $\left[\text{Ans:} \frac{\pi}{4} \log 2\right]$

21. Evaluate $\int_0^1 \int_{\sqrt{x}}^1 e^{x/y} dxdy$ by interchanging the order of integration. [Ans: 1/2]

## 1.3 Plane Area as Double Integral

Plane area enclosed by one or more curves can be expressed as a double integral. Let R be the plane region, the area of which is required. Let us divide the area into a large number of elemental areas ABCD, by dividing lines parallel to the y-axis of intervals $\Delta x$ and the lines parallel to the $x$-axis, at intervals of $\Delta y$.

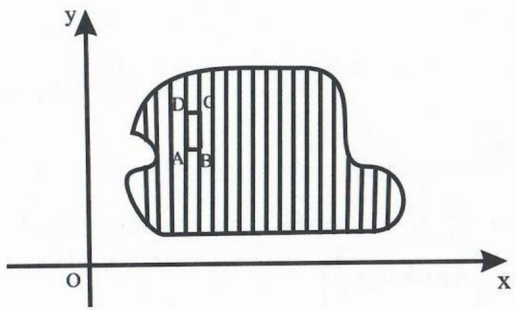



**Fig. 18**

Area of the rectangle $= \Delta x \Delta y$

Rectangle of area $\Delta = \lim_{\Delta x \to 0} [\sum \sum \Delta x \Delta y] = \int \int_R dxdy$

**Note:** In polar system $A = \int \int_D r dr d\theta$





**Example 1.35.** *Show, by double integration, that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.*

**Solution:** The region by integration is as shown.

Solving $y^2 = 4ax$ and $x^2 = 4ay$ are set $(0,0),(4a,4a)$. Take a strip parallel to $y$ axis implies limits for $y = \frac{x^2}{4a}$ and $y = 2\sqrt{ax}$ and then $x$ varies from 0 to $4a$.

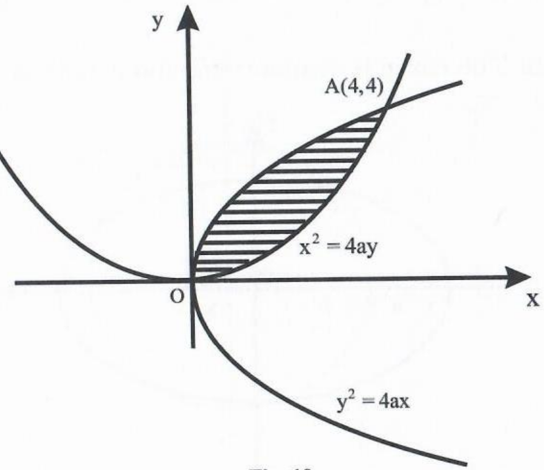



**Fig. 19**

$\therefore$ Area $= \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dx = \int_0^{4a} [y]_{x^2/4a}^{2\sqrt{ax}} dx$

$$= \int_0^{4a} \left(2\sqrt{a}x^{1/2} - \frac{x^2}{4a}\right) dx = \left[2\sqrt{a}\frac{x^{1/2}+1}{\frac{1}{2}+1} - \frac{1}{4a}\frac{x^3}{3}\right]_0^{4a}$$

$$= \left[\frac{4}{3}\sqrt{a}x^{3/2} - \frac{1}{12a}x^3\right]_0^{4a} = \frac{4}{3}\sqrt{a}(4a^{3/2}) - \frac{1}{12a}(4a)^3$$

$$= \frac{4}{3}a^2(4 \times 2) - \frac{1}{12a} \times 64a^3$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2 \text{ Sq.units.}$$

**Example 1.36.** *Find by double integration, the area enclosed by the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

**Solution:** The curve is symmetrical about both axes.

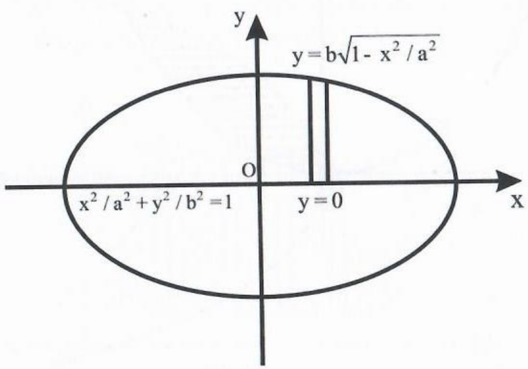



**Fig. 20**

$\therefore$ Area = 4 area in I quadrant.

$$= 4\int\int_A dydx = 4\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dydx$$

$$= 4\int_0^a [y]_0^{b\sqrt{1-\frac{x^2}{a^2}}} = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}dx$$





$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a} \times \frac{a^2}{2} \times \frac{\pi}{2} = \pi ab \text{ Sq.units.}$$

**Example 1.37.** *Find by double integration, smaller of the areas bounded by the circle* $x^2 + y^2 = 9$ *and* $x + y = 3$.

**Solution:** The region of integration is as shown $y$ varies from $3 - x$ to $\sqrt{9 - x^2}$ and $x$ varies from 0 to 3.

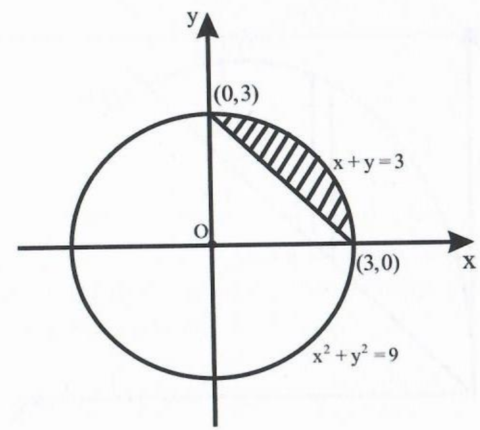



**Fig. 21**

Area $= \int\int dydx$

$$= \int_0^3 \int_{3-x}^{\sqrt{9-x^2}} dydx = \int_0^3 [\sqrt{9 - x^2} - (3 - x)]dx$$

$$= \left[\frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3 - 3[x]_0^3 + \left[\frac{x^2}{2}\right]_0^3$$

$$= \frac{9}{2}\frac{\pi}{2} - 9 + \frac{9}{2} = \frac{9\pi}{4} - \frac{9}{2} = \frac{9}{4}(\pi - 2) \text{ sq.units.}$$

**Example 1.38.** *Find the area between the parabola $y = 4x - x^2$ and the line $y = x$.*

**Solution:**

The two curves intersect at points whose abscissa are given by $4x - x^2 = x$ or $x^2 - 3x = 0$ (i.e.) $x = 0, 3$. Using vertical strips, the required area lies between $x = 0, x = 3$ and $y = x, y = 4x - x^2$.

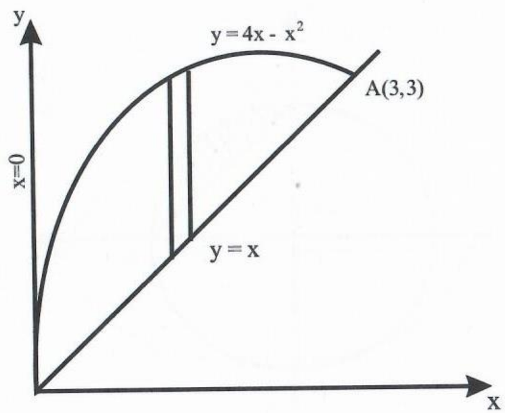



**Fig. 22**

$\therefore$ Required area $= \int_0^3 \int_x^{4x-x^2} dydx = \int_0^3 [y]_x^{4x-x^2} dx$

$= \int_0^3 (4x - x^2 - x)dx = \int_0^3 (3x - x^2)dx$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = 4.5 \text{ Sq.units.}$$





**Example 1.39.** *Find by double integration, the area bounded by the parabola $y = x^2$ and the line $y = 2x + 3$.*

**Solution:** The region of integration is as shown.

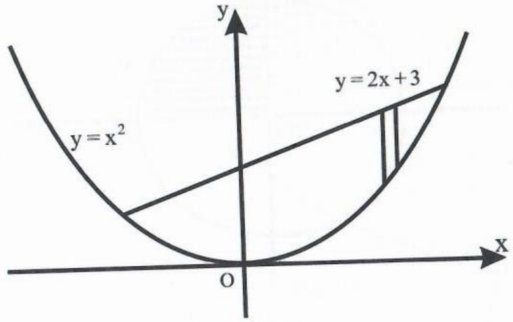



**Fig. 23**

Solving $y = 2x + 3$ we get $x^2 - 2x - 3 = 0$. (i.e.) $x = 3, -1$.

Required area $= \int\int dydx$ where $y$ varies from $y = x^2$ and $y = 2x + 3$. Further $x$ varies from -1 to 3.

$\therefore$ Required area $= \int_{-1}^3 \int_{x^2}^{2x+3} dydx = \int_{-1}^3 [y]_{x^2}^{2x+3} dx$

$$= \int_{-1}^3 (2x + 3 - x^2)dx = \left[2\left(\frac{x^2}{2}\right) + 3x - \frac{x^3}{3}\right]_{-1}^3$$

$$= \frac{32}{3} = 10\frac{2}{3} \text{ Sq.units.}$$

**Example 1.40.** *Find the area of the circle using double integral.*

**Solution:** Area of the circle $A = 4\int\int_D dxdy$ where D is the bounded region in the I quadrant of $xy$-plane. In D, $x$ varies from 0 to $r$, and for fixed $x$, $y$ varies from 0 to $\sqrt{r^2-x^2}$.

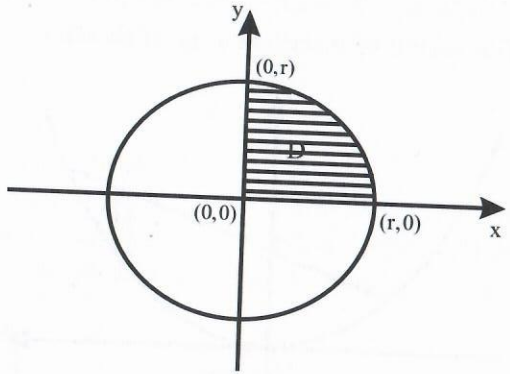



Fig. 24

$A = 4\int_0^r \int_0^{\sqrt{r^2-x^2}} dydx = 4\int_0^r \sqrt{r^2-x^2}$ put $x = r\sin\theta$

$= 4r^2 \int_0^{\pi/2} \cos^2\theta d\theta = 4r^2 \int_0^{\pi/2} \frac{1+\cos 2\theta}{2} d\theta$

$= 2r^2 \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \pi r^2$ as $\sin\pi = 0, \sin 0 = 0$

**Example 1.41.** *Find the area of the region D bounded by the parabolas $y = x^2$ and $x = y^2$.*

**Solution:** The point of intersection of the curve are given by $(0,0)$ and $(1,1)$. The required region D is as shown below.

The required area $A = \int\int_D dxdy = \int_0^1 \int_{x^2}^{\sqrt{x}} dydx$





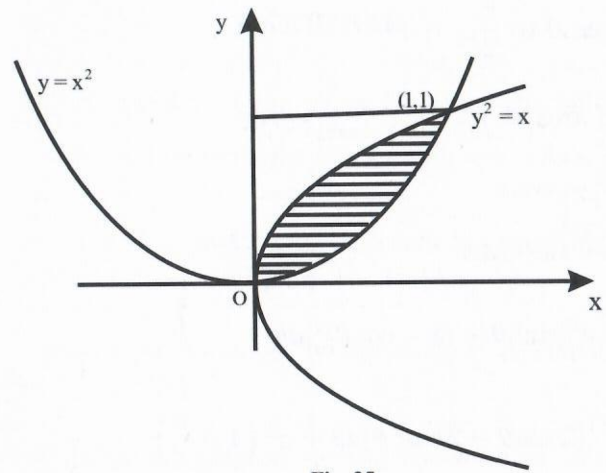



Fig. 25

$= \int_0^1 [y]_{x^2}^{\sqrt{x}} dx = \int_0^1 (\sqrt{x} - x^2) dx$

$= \left[2\frac{x^{3/2}}{3} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

**Example 1.42.** *Find by double integration, the area lying inside the circle $r = a\sin\theta$ and outside the cordioid $r = a(1-\cos\theta)$.*

**Solution:** Eliminating $r$ between the equations of two curves $\sin\theta = 1 - \cos\theta$ or $\sin\theta + \cos\theta = 1$.

Squaring $1 + \sin 2\theta = 1$ or $\sin 2\theta = 0 \therefore 2\theta = 0$ or $\pi$

(i.e.) $\theta = 0$ or $\frac{\pi}{2}$.

For the required area, r varies from $a(1-\cos\theta)$ to $a\sin\theta$ and $\theta$

varies from 0 to $\frac{\pi}{2}$.

Required area $= \int_0^{\pi/2} \int_{a(1-\cos\theta)}^{a\sin\theta} r dr d\theta$

$= \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta}$

$= \frac{1}{2} \int_0^{\pi/2} a^2[\sin^2\theta - (1-\cos\theta)^2] d\theta$

$= \frac{a^2}{2} \int_0^{\pi/2} (2\cos\theta - 2\cos^2\theta) d\theta = a^2\left(1 - \frac{\pi}{4}\right)$

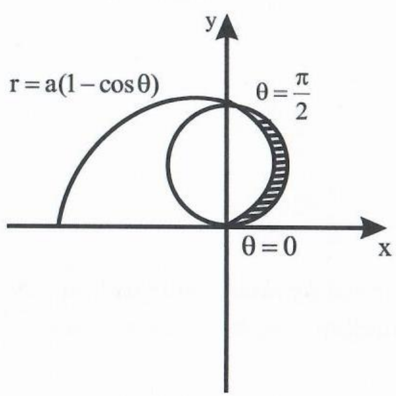



**Fig. 26**





## EXERCISE

1. Find the smaller of the areas bounded the ellipse $4x^2 + 9y^2 = 36$ and the straight line $2x+3y=6$. $\left[\text{Ans:} \frac{3}{2}(\pi-2)\right]$

2. Find by double integration, the area bounded by the parabolas $y^2 = 4 - x$ and $y^2 = 4 - 4x$ [Ans: 8]

3. Find the area bounded by the circle $r = 2\sin\theta$ and $r = 4\sin\theta$ [Ans: $3\pi$]

4. Find the area enclosed by the curve $r^2 = a^2\cos 2\theta$, by double integration. [Ans: $a^2$]

5. Find the area bounded by the parabola $y^2 = 4 - x$ and $y^2 = x$ by double integration. $\left[\text{Ans:} \frac{16}{3}\sqrt{2}\right]$

6. Find the area common to the parabola $y^2 = x$ and $x^2+y^2 = 2$ by double integration. $\left[\text{Ans:} \frac{\pi}{2} + \frac{1}{3}\right]$

7. Find the area that lies outside the circle $r = a\cos\theta$ and inside the circle $r = 2a\cos\theta$ $\left[\text{Ans:} \frac{3}{4}\pi a^2\right]$

8. Find the area common to the two circles $r = a$ and $r = 2a\cos\theta$. $\left[\text{Ans:} a^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)\right]$

## 1.4 Change of Variable from Cartesian to Polar Co-ordinates

Let $\int\int_R f(x,y)dxdy$ be the double integral. $x = r\cos\theta, y = r\sin\theta$ is the transformation from Cartesian to Polar coordinates.

Then $dxdy = |J|drd\theta$ where $J = \dfrac{\partial(x,y)}{\partial(r,\theta)}$ is the Jacobian of the transformation and

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r(\cos^2\theta + \sin^2\theta) = r$

$\therefore \int\int_R f(x,y)dxdy = \int\int_R f(r,\theta)rdrd\theta$

**Example 1.43.** *Change to polar coordinates and evaluate* $\int_0^a \int_y^a xdxdy$

**Solution:**

The region of integration is $x = y, x = a, y = 0, y = a$.

(i.e.) The triangle OAB putting $x = r\cos\theta, y = r\sin\theta$, the line $x = y$ becomes $r\cos\theta = r\sin\theta$

(i.e.) $\tan\theta = 1 \Rightarrow \theta = \dfrac{\pi}{4}$





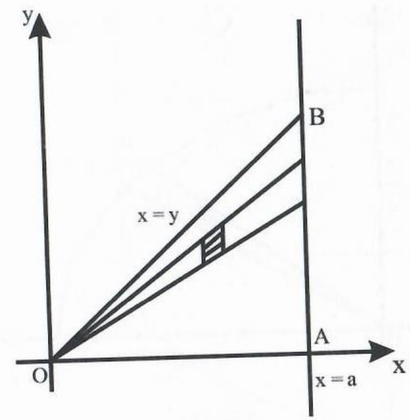



**Fig. 27**

Hence in polar form $I = \int_0^{\frac{\pi}{4}} \int_0^{a/\cos\theta} r^2 \cos\theta drd\theta$

$= \int_0^{\frac{\pi}{4}} \cos\theta \left(\dfrac{r^3}{3}\right)_0^{a/\cos\theta} d\theta = \dfrac{a^3}{3} \int_0^{\frac{\pi}{4}} \sec^2\theta d\theta$

$= \dfrac{a^3}{3} [\tan\theta]_0^{\frac{\pi}{4}} = \dfrac{a^3}{3}$

**Example 1.44.** *Change to polar coordinates and evaluate*

$\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)} dxdy$

**Solution:**

Putting $x = r\cos\theta, y = r\sin\theta$, the given limit $y^2 = a^2 - x^2$.

(i.e) The circle $x^2 + y^2 = a^2$ changes to $r = a$ and $y = 0$.

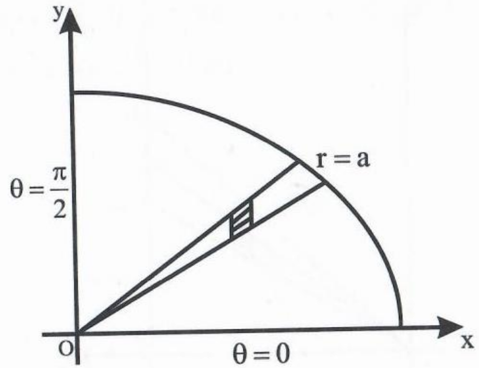



Fig. 28

(i.e.) The x-axis changes to initial line $\theta = 0$. Hence, in the given region $r$ changes from 0 to $a$ and $\theta$ changes from 0 to $\frac{\pi}{2}$.

$I = \int_0^{\frac{\pi}{2}} \int_0^a e^{-r^2} r dr d\theta$

$= \int_0^{\frac{\pi}{2}} \left(-\frac{1}{2}e^{-r^2}\right)_0^a d\theta$

$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} (e^{-a^2} - 1)d\theta = \frac{\pi}{4}(1 - e^{-a^2})$

**Example 1.45.** *Evaluate* $\int_0^{4a} \int_{y^2/4a}^{y} dydx$ *by changing to polar coordinates.*

**Solution:**

The region of integration is bounded by the parabola $x = y^2/4a$.





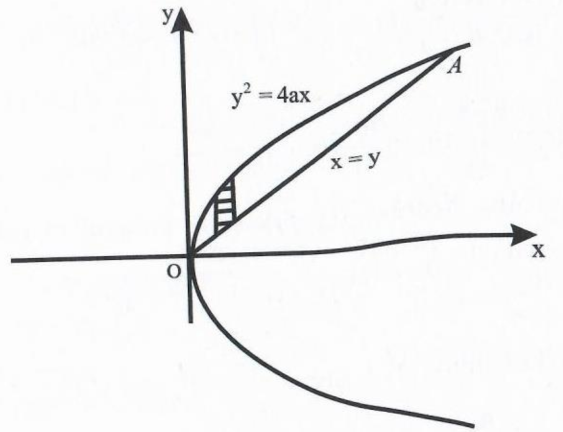



Fig. 29

(i.e.) $y^2 = 4ax$ and the line $x = y$.

By putting $x = r\cos\theta, y = r\sin\theta$, the parabola becomes $r^2\sin^2\theta = 4ar\cos\theta$.

(i.e.) $r = \dfrac{4a\cos\theta}{\sin^2\theta}$ and the line becomes $x\cos\theta = r\sin\theta$

(i.e.) $\theta = \dfrac{\pi}{4}$

Hence $r$ varies from 0 to $\dfrac{4a\cos\theta}{\sin^2\theta}$ and $\theta$ varies from $\dfrac{\pi}{4}$ to $\dfrac{\pi}{2}$.

$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{4a\cos\theta}{\sin^2\theta}} r d\theta dr = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{r^2}{2}\right)_0^{\frac{4a\cos\theta}{\sin^2\theta}}$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{16a^2\cos^2\theta}{\sin^4\theta}\right] d\theta = 8a^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^2\theta \csc^2\theta d\theta$$

$$= 8a^2\left[\frac{-\cot^3\theta}{3}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\frac{8a^2}{3}[0-1] = \frac{8a^2}{3}$$

**Example 1.46.** *Express the following integral in polar coordinates and evaluate* $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dxdy}{\sqrt{a^2-x^2-y^2}}$

**Solution:** The limits of $y$ are $\sqrt{ax-x^2}$ and $\sqrt{a^2-x^2}$.

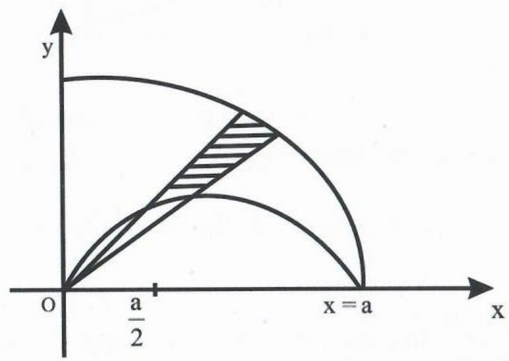



Fig. 30

(i.e.) Upper half of the circles i) $x^2+y^2-ax=0$; $\left(x-\frac{a}{2}\right)^2 + (y-0)^2 = \left(\frac{a}{2}\right)^2$ and ii) $x^2+y^2=a^2$

To change the given integral to polar coordinates, we put $x = r\cos\theta, y = r\sin\theta$ and $dxdy = rdrd\theta$.

The equations of the circles now become





i) $r^2 - ar\cos\theta = 0$ (i.e.) $r = a\cos\theta$

ii) $r^2 = a^2$ (i.e.) $r = a$

Hence $r$ changes from $r = a\cos\theta$ to $a$ and $\theta$ changes from 0 to $\frac{\pi}{2}$

$$\therefore I = \int_0^{\frac{\pi}{2}} \int_{a\cos\theta}^{a} \frac{rdrd\theta}{\sqrt{a^2-r^2}} = \int_0^{\frac{\pi}{2}} [-\sqrt{a^2-r^2}]_{a\cos\theta}^{a} d\theta$$

$$= \int_0^{\frac{\pi}{2}} a\sin\theta d\theta = [a\cos\theta]_0^{\frac{\pi}{2}} = a$$

**Example 1.47.** *Evaluate* $\int\int_R \frac{xydxdy}{\sqrt{x^2+y^2}}$ *where R is the region in the first quadrant enclosed by the circles* $x^2+y^2=a^2$ *and* $x^2+y^2=4a^2$*, by converting into polar coordinates.*

**Solution:** We put $x = r\cos\theta, y = r\sin\theta \therefore dxdy = rdrd\theta$

$r$ varies from $r = a$ to $r = 2a$ and $\theta$ varies from 0 to $\frac{\pi}{2}$

Hence $I = \int_0^{\frac{\pi}{2}} \int_a^{2a} \frac{r^3\sin\theta\cos\theta drd\theta}{\sqrt{r^2}}$

$$= \left(\int_a^{2a} r^2 dr\right)\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta d\theta\right)$$

$$= \frac{1}{2}\left(\frac{r^3}{3}\right)_0^{2a} \left(\frac{-\cos 2\theta}{2}\right)_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{12}(8a^3 - a^3)(\cos\pi - \cos 0)$$

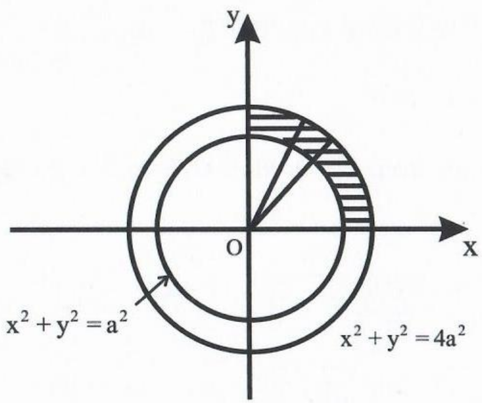



**Fig. 31**

$$= -\frac{1}{12}(7a^3)(-2) = \frac{7}{6}a^3$$

**Example 1.48.** *Evaluate* $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy$ *by using polar coordinates and hence evaluate* $\int_0^\infty e^{-(x^2)}dx$.

**Solution:**

$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy = \int\int_A e^{-(x^2+y^2)}dA$, where A is the positive quadrant.

Put $x = r\cos\theta, y = r\sin\theta \therefore dxdy = rdrd\theta$

In the positive quadrant A, given the limits for $r$ varies from 0 to $\infty$ and $\theta$ varies from 0 to $\frac{\pi}{2}$

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-(r^2)}rd\theta dr$$





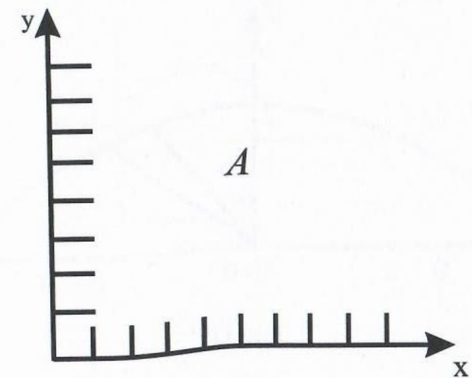



**Fig. 32**

Put $r^2 = R \Rightarrow 2rdr = dR$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\frac{e^{-R}}{-1}\right)_0^\infty d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$$

Since $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy = \int_0^\infty e^{-x^2}dx \ast \int_0^\infty e^{-y^2}dy$

$= I \times I = I^2$

$I^2 = \frac{\pi}{4}$

$\therefore I = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

**Example 1.49.** *Evaluate* $\int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)dxdy$ *by changing to polar coordinates.*

**Solution:**

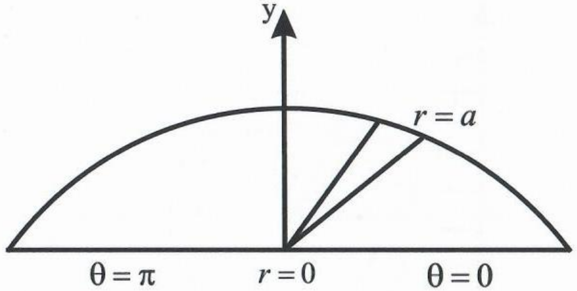



**Fig. 33**

Put $x = r\cos\theta, y = r\sin\theta \therefore dxdy = rdrd\theta$

$I = \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}(x^2+y^2)dxdy$

$= \int_0^{\pi}\int_0^{a} r^2.rdrd\theta = \left(\frac{r^4}{4}\right)_0^a [\theta]_0^{\pi} = \frac{\pi a^4}{4}$

**Example 1.50.** *Evaluate* $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)dydx.$

**Solution:**

Here $y$ varies from 0 to $\sqrt{2ax-x^2}$

(i.e.) $y = 0$ to $x^2+y^2 = 2ax$ which is a circle centre $(a,0)$ and radius $a$. Also the circle passes through $(0,0)$. Since $x$ varies from 0 to $2a$, the area of integration is the area above the x-axis and inside the circle. The polar equation of the circle is $x = 2a\cos\theta$.

Changing to polar coordinates $x = r\cos\theta, y = r\sin\theta$





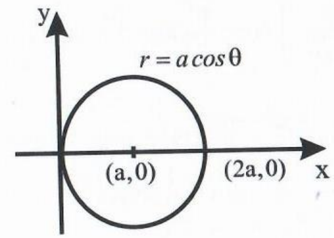



**Fig. 34**

$\therefore dxdy = rdrd\theta$

$I = \int_0^{\pi/2}\int_0^{2a\cos\theta} r^2 rdrd\theta = \int_0^{\pi/2}\left(\frac{r^4}{4}\right)_0^{2a\cos\theta} d\theta$

$= 4a^4\int_0^{\pi/2}\cos^4\theta d\theta = 4a^4\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{3}{4}\pi a^4$

$$\left[\int_0^{\pi/2}\cos^n\theta d\theta = \begin{cases}\frac{n-1}{n}\times\frac{n-3}{n-2}\times\cdots\times\frac{\pi}{2}, \text{n is even}\\ \frac{n-1}{n}\times\frac{n-3}{n-2}\times\cdots\times 1, \text{n is odd}\end{cases}\right]$$

**EXERCISE**

Changing to polar coordinates evaluate the following:

1. $\int\int x^2y^2dxdy$ over the circle $x^2+y^2 = a^2$ $\left[\text{Ans:}\frac{\pi}{24}a^6\right]$

2. $\int_0^a\int_y^a \frac{xdxdy}{x^2+y^2}$ $\left[\text{Ans:}\frac{\pi a}{4}\right]$

3. $\int_0^a \int_0^x \frac{x^3 dxdy}{\sqrt{x^2+y^2}}$ $\left[\text{Ans:}\frac{a^4}{4}\log(1+\sqrt{2})\right]$

4. $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}dxdy$ over the region bounded by $y=0, x=y, x=1$ $\left[\text{Ans:}\frac{1}{20}\pi a^5\right]$

5. $\int\int \frac{dxdy}{(x^2+y^2)^{3/2}}$ over the region bounded by $y=0, x=y, x=1$ $\left[\text{Ans:}\frac{\pi}{12}\right]$

## 1.5 Volume as a Triple Integral

Triple integrals can be used to evaluate volume V of a finite region D in space.

The volume $V = \int\int\int_D dxdydz$

**Example 1.51.** *Find the volume of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *by triple integrals.*

**Solution:**

Volume $= 8\times$ volume in the first octant.

$$V = 8\times\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dzdydx$$

$$= 8\times\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} [z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}$$





$$= 8c\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \left[\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right]$$

Put $r^2 = \left(1-\frac{x^2}{a^2}\right)b^2$

$$= \frac{8c}{b}\int_0^a \int_0^r \sqrt{r^2-y^2}dydx$$

$$= \frac{8c}{b}\int_0^a \left[\frac{r^2}{2}\sin^{-1}\frac{y}{r}+\frac{y}{2}\sqrt{r^2-y^2}\right]_0^r$$

$$= \frac{2c\pi}{b}\int_0^a r^2dx = \frac{2c\pi}{b}\int_0^a \left(1-\frac{x^2}{a^2}\right)b^2dx$$

$$= 2cb\pi\left(x-\frac{x^3}{3a^2}\right)_0^a = \frac{4\pi}{3}abc$$

**Example 1.52.** *Find the volume bounded by the cylinder* $x^2+y^2=4$ *and the planes* $y+z=4$ *and* $z=0$.

**Solution:**

z varies from $z=0$ to $z=4-y$ and x, y varies over all points of the circle $x^2+y^2=4$.

Volume $V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-y} dzdydx$

$$= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [z]_0^{4-y}dydx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)dydx$$

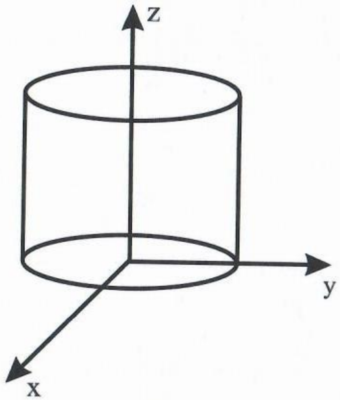



**Fig. 35**

$$= \int_{-2}^{2} \left[4y - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = 8 \times 2 \int_0^2 \sqrt{4-x^2}dx$$

$$V = 16 \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right] = 16\pi$$

**Example 1.53.** *Find the volume of the tetrahedron bounded by the coordinate planes and* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

**Solution:**

Volume required $= \int\int\int dxdydz$ with limits.

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dzdydx$$

$$= c\int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \left(1 - \frac{x}{a} - \frac{y}{b}\right) dydx$$





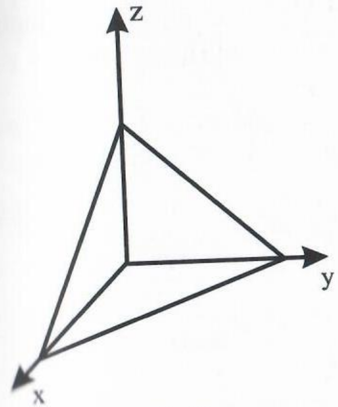



**Fig. 36**

$$= c\int_0^a \left[\left(1 - \frac{x}{a}\right)y - \frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= c\int_0^a \left[b\left(1 - \frac{x}{a}\right)^2 - \frac{b^2}{2b}\left(1 - \frac{x}{a}\right)^2\right] dx$$

$$= \frac{bc}{2}\int_0^a \left(1 - \frac{x}{a}\right)^2 dx = \frac{bc}{2}\left[\frac{\left(1 - \frac{x}{a}\right)^2}{3} \times \left(\frac{-a}{1}\right)\right]$$

$$= \frac{abc}{6}$$

**Example 1.54.** *Find the volume of the sphere* $x^2 + y^2 + z^2 = a^2$ *using the triple integrals.*

**Solution:**

Since the sphere $x^2+y^2+z^2=a^2$ is symmetric about the coordinate planes, the volume of the sphere $=8\times$ volume of the first octant.

$$=8\times\int\int\int_V dxdydz$$

$$=8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dzdydx$$

$$=8\int_0^a\int_0^{\sqrt{a^2-x^2}}[z]_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$=8\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2-y^2}dydx$$

$$\because\left[\int\sqrt{(a^2-x^2)}dx=\frac{x}{2}\sqrt{(a^2-x^2)}+\frac{a^2}{2}\sin^{-1}\frac{x}{2}\right]$$

$$=8\int_0^a\left[\frac{y}{2}\sqrt{(a^2-x^2)-y^2}+\frac{a^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$$

$$=2\pi\int_0^a(a^2-x^2)dx$$

$$=2\pi\left[a^2x-\frac{x^3}{3}\right]=\frac{4\pi}{3}a^3$$

## EXERCISE

1. Find the volume bounded by $xy$-plane, the cylinder $x^2+y^2=4$ and the plane $x+y+z=3$ $\left[\text{Ans:}\frac{1}{3\pi}(9\pi-4)\right]$
2. Find the volume in the positive octant bounded by the





plane $x+2y+3z=4$ and the coordinate planes. $\left[\text{Ans:}\frac{16}{9}\right]$

3. Evaluate $\int\int\int_V dxdydz$, where V is the volume enclosed by the cylinder $x^2+y^2=1$ and the planes $z=0, z=2-x$ $\left[\text{Ans:}2\pi-\frac{4}{3}\right]$