region above xy - plane bounded by the curve $z^2 = x^2 + y^2$ and the plane $z = 4$, if $\overrightarrow{F} = 4xz\overrightarrow{i} + xyz^2\overrightarrow{j} + 3z\overrightarrow{k}$.

[Ans: 320π]

3 LAPLACE TRANSFORM

3.1 Introduction

The Laplace transform plays a vital role in Mathematics required of Mathematicians, Physicists, Engineers and Scientists. This is due to the fact that Laplace transform methods provide easy and effective techniques for the solution of many problems arising in various fields of Science and Engineering. This chapter is devoted to the study of Laplace transforms of some elementary functions, some basic problems, convolution theorem, Inverse Laplace transforms and finding solutions to linear differential equations using Laplace transforms.

Definition: Let $f(t)$ be a function of t for $t > 0$. Then the Laplace transform of $f(t)$, denoted as $L[f(t)]$, is defined by

$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$, where 's' is real or complex. Thus $L[f(t)] = F(s)$. The Laplace transform of $f(t)$ is also denoted as $\overline{f}(s)$.

Laplace transform of some elementary functions:

S.No.	f(t)	L[f(t)]
1	1	$\frac{1}{s}$ $(s>0)$
2	t	$\frac{1}{s^2}$ $(s>0)$
3	t^n	$\frac{n!}{s^{n+1}}$ $(s>0)$
4	e^{at}	$\frac{1}{s-a}$ $(s>a)$
5	$\sin at$	$\frac{a}{s^2+a^2}$ $(s>0)$
6	$\cos at$	$\frac{s}{s^2+a^2}$ $(s>0)$
7	$\sinh at$	$\frac{a}{s^2-a^2}$ $(s>\lvert a\rvert)$
8	$\cosh at$	$\frac{s}{s^2-a^2}$ $(s>\lvert a\rvert)$
9	e^{-at}	$\frac{1}{s+a}$ $(s>-a)$

Remark 1: It may be noted that not all $f(t)$ are Laplace transformable.

Remark 2: The sufficient conditions for the existence of Laplace transform of $f(t)$ are

(i) it is piecewise continuous in every finite interval;

(ii) it is of exponential order ρ, for all $s > \rho$.

Remark 3: However, the above conditions are not necessary.

Illustration: Prove that $L[1] = \frac{1}{s}, (s > 0)$.

Proof:

$L[1] = \int_0^\infty e^{-st} dt \, (\because f(t) = 1))$

$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}(0-1) = \frac{1}{s}$

Note: Similarly using the definition we can establish the results of other functions.

3.2 Some basic operational properties of Laplace transforms

1. Linearity Property:

$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$ where a and b are any two constants.

Proof:

$L[af(t) + bg(t)] = \int_0^\infty e^{-st}[af(t) + bg(t)]dt$, by definition.

$= a \int_0^\infty e^{-st} f(t)dt + b \int_0^\infty e^{-st} g(t)dt$

$= aL[f(t)] + bL[g(t)]$

This result can be extended to a finite number of functions.

Example: Find: $L[2e^{2t}+3e^{-3t}]$

$$L[2e^{2t}+3e^{-3t}] = 2L[e^{2t}]+3L[e^{-3t}] = \frac{2}{s-2}+\frac{3}{s+3}$$

2. First Shifting Property:

If $L[f(t)] = F(s)$, then $L[e^{at}f(t)] = F(s-a)$

Proof: We know that $L[f(t)] = \int_0^\infty e^{-st}f(t)dt = F(s)$

$\therefore L[e^{at}f(t)] = \int_0^\infty e^{-st}[e^{at}f(t)]dt$

$= \int_0^\infty e^{-(s-a)t}f(t)dt, (s-a)>0$

$= F(s-a)$

Similarly, $L[e^{-at}f(t)] = F(s+a)$

Example: Find: $L[e^{-t}\sin 3t]$

We know that $L[\sin 3t] = \dfrac{3}{s^2+9}$

Then $L[e^{-t}\sin 3t] = \dfrac{3}{(s+1)^2+9} = \dfrac{3}{s^2+2s+10}$

3. Second Shifting Property:

If $L[f(t)] = F(s)$ and $g(t) = \begin{cases} \text{f(t-a), t>a} \\ \text{0, t<a} \end{cases}$ then

$L[g(t)] = e^{-as}F(s)$

Proof: By definition, $L[g(t)] = \int_0^\infty e^{-st}g(t)dt$

$= \int_0^a e^{-st}g(t)dt + \int_a^\infty e^{-st}g(t)dt = 0 + \int_a^\infty e^{-st}f(t-a)dt$

Put $t-a=u$, then $t=a \Rightarrow u=0$

$dt = du$ and $t=\infty \Rightarrow u=\infty$

$L[g(t)] = \int_0^\infty e^{-s(u+a)}f(u)du = e^{-sa}\int_0^\infty e^{-su}f(u)du$

$= e^{-as}L[f(u)] = e^{-as}F(s)$ (as 'u' is a dummy variable, L[f(u)]=L[f(t)])

Also $L[f(t-a).H(t-a)] = \int_0^\infty L[f(t-a).H(t-a)]dt$

Put $t-a=u \Rightarrow dt=du$

$t=0 \Rightarrow u=-a; t=\infty \Rightarrow u=\infty$

$L[f(t-a).H(t-a)] = \int_{-a}^\infty e^{-s(u+a)}f(u)H(u)du$

$= \int_{-a}^0 e^{-s(u+a)}f(u)H(u)du + \int_0^\infty e^{-s(u+a)}f(u)H(u)du$

$= 0 + \int_0^\infty e^{-s(u+a)}f(u)du = e^{-as}\int_0^\infty e^{-su}f(u)du$

$= e^{-as}L[f(u)] = e^{-as}F(s)$ (as 'u' is a dummy variable, L[f(u)]=L[f(t)])

It can also be expressed as $L[f(t-a).H(t-a)] = e^{-as}F(s)$, where $H(t-a) = \begin{cases} 1, \text{if } t \ge a \\ 0, \text{if } t < a \end{cases}$, $(a>0)$ is called the unit step function or Heaviside's unit step function.

Example: we have that $L[t^3] = \frac{3!}{s^4} = \frac{6}{s^4}$

Then, $L[g(t)]$, where $g(t) = \begin{cases} (t-2)^3, t > 2 \\ 0, t < 2 \end{cases}$ is given by $\frac{6e^{-2s}}{s^4}$

4. Change of Scale Property:

If $L[f(t)] = F(s)$ then $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

Proof: We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s)$

$L[f(at)] = \int_0^\infty e^{-st} f(at)dt$

Put $at = u \Rightarrow adt = du$ when $t = 0 \Rightarrow u = 0$; when $t = \infty \Rightarrow t = \infty$

$\therefore L[f(at)] = \int_0^\infty e^{-s(u/a)} f(u)\frac{du}{a} = \frac{1}{a}\int_0^\infty e^{-(s/a)u} f(u)du$

$= \frac{1}{a}F\left(\frac{s}{a}\right)$

Example: Find $L[\cos 2t]$

As $L[\cos t] = \frac{s}{s^2+1}$

We have $L[\cos 2t] = \frac{1}{2}.\frac{s/2}{(s/2)^2+1} = \frac{s}{s^2+4}$

3.2.1 Laplace transform of Derivatives:

I. If $L[f(t)] = F(s)$ then $L[f'(t)] = sF(s) - f(0)$ provided $f(t)$ satisfies the sufficient conditions.

Proof: Let $L[f'(t)] = \int_0^\infty e^{-st} f'(t)dt$

$= \int_0^\infty e^{-st}\frac{df}{dt}dt = \int_0^\infty e^{-st}df(t)$

$= [e^{-s}f(t)]_0^\infty - \int_0^\infty f(t)e^{-st}(-s)dt$

$= -f(0) + sL[f(t)] = sF(s) - f(0)$

Example: For $f(t) = \sin 2t$, we have $L[f(t)] = \frac{2}{s^2+4}$.

Then we get $L[f'(t)] = L[2\cos 2t] = s\left(\frac{2}{s^2+4}\right) - 0 = \frac{2s}{s^2+4}$

II. If $L[f(t)] = F(s)$ then $L[f''(t)] = s^2F(s) - sf(0) - f'(0)$

Example: $L[f(t)] = \cos 3t$, find $L[f''(t)]$

$L[f(t)] = \frac{s}{s^2+9}$ (i.e.) $F(s) = \frac{s}{s^2+9}$

We know that $L[f''(t)] = s^2F(s) - sf(0) - f'(0)$

$= s^2.\frac{s}{s^2+9} - s(0) - 0 = \frac{s^3}{s^2+9} - s = \frac{-9s}{s^2+9}$

Laplace transform of integrals:

I. If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(u)du\right] = \dfrac{F(s)}{s}$

Proof: Let $\int_0^t f(u)du = F(t)$

$\therefore F'(t) = f(t)$ and $F(0) = 0$

We know that $L[F'(t)] = sL[F(t)] - F(0) = s.L[F(t)]$

$= s.L\left[\int_0^t f(u)du\right]$

(or) $L\left[\int_0^t f(u)du\right] = \dfrac{1}{s}L[F'(t)] = \dfrac{1}{s}L[f(t)] = \dfrac{F(s)}{s}$

Example: Find $L\left[\dfrac{1}{4}\sin 4t\right]$

We know that $L[\cos 4t] = \dfrac{s}{s^2+16}$

(i.e.) $L\left[\int_0^t \cos 4u du\right] = \dfrac{s}{s(s^2+16)} = \dfrac{1}{s^2+16}$, which can be directly verified.

For, $\int_0^t \cos 4u du = \dfrac{1}{4}[\sin 4u]_0^t = \dfrac{1}{4}\sin 4t$

$\therefore L\left[\dfrac{1}{4}\sin 4t\right] = \dfrac{1}{4}.\dfrac{4}{s^2+16} = \dfrac{1}{s^2+16}$

II. If $L[f(t)] = F(s)$ then $L[tf(t)] = -\dfrac{d}{ds}[F(s)]$

Proof: We have that $L[f(t)] = F(s)$

$$\text{(i.e.) } F(s) = L[f(t)] \tag{3.1}$$

Taking derivatives on both sides of (3.1) w.r.to 's', we get

$\dfrac{d}{ds}[F(s)] = \dfrac{d}{ds}\{L[f(t)]\} = \dfrac{d}{ds}\left[\int_0^\infty e^{-st}f(t)dt\right]$

$= \int_0^\infty \dfrac{d}{ds}[e^{-st}f(t)]dt = \int_0^\infty (-t)e^{-st}f(t)dt$

$= -\int_0^\infty e^{-st}[tf(t)]dt = -L[tf(t)]$

(or) $L[tf(t)] = -\dfrac{d}{ds}[F(s)]$

Example: Find $L[te^t]$

Since $L[e^t] = \dfrac{1}{s-1}$, we get

$L[te^t] = \dfrac{d}{ds}\left[\dfrac{1}{s-1}\right] = \dfrac{1}{(s-1)^2}$

III. If $L[f(t)] = F(s)$ then $L\left[\dfrac{f(t)}{t}\right] = \int_s^\infty F(u)du$ provided $\lim_{t\to 0}\dfrac{f(t)}{t}$ exists.

Example: We know that $L[\sin t] = \dfrac{1}{s^2+1}$ and $\lim_{t\to 0}\dfrac{\sin t}{t} = 1$, then prove that $L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{du}{u^2+1} = \tan^{-1}\left(\frac{1}{s}\right)$.

$\because \int_s^\infty \frac{du}{u^2+1} = [\tan^{-1} u]_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$

(using a theorem, namely $\cot^{-1} x + \tan^{-1} x = \frac{\pi}{2}$)

Also using the fact that $\tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right), \cot^{-1} s = \tan^{-1}\left(\frac{1}{s}\right)$.

3.3 Initial value and Final value theorems

1. Initial Value Theorem

If $L[f(t)] = F(s)$ then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$, provided the limits exist.

Proof: We know that

$$L[f'(t)] = sL[f(t)] - f(0) = sF(s) - f(0) \tag{3.2}$$

Taking limits on both sides of (3.2) as $s \to \infty$, we get

$\lim_{s\to\infty} L[f'(t)] = \lim_{s\to\infty}[sF(s) - f(0)]$

$= \lim_{s\to\infty} \int_0^\infty e^{-st} f'(t)dt = 0$

$\therefore \lim_{s\to\infty} sF(s) = f(0) = \lim_{t\to 0} f(t)$

(i.e.) $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

2. Final Value Theorem

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$, when $L[f(t)] = F(s)$ such that the limits exist.

Proof: Again, $L[f'(t)] = sL[f(t)] - f(0) = sF(s) - f(0)$

(i.e) $sF(s) - f(0) = L[f'(t)] = \int_0^\infty e^{-st} f'(t)dt$

$\Rightarrow \lim_{s\to 0}[sF(s) - f(0)] = \lim_{s\to 0} \int_0^\infty e^{-st} f'(t)dt$

$= \int_0^\infty f'(t)dt = [f(t)]_0^\infty$

(i.e.) $\lim_{s\to 0} sF(s) - f(0) = \lim_{t\to\infty} f(t) - f(0)$

(i.e.) $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Convolution of the Functions:

Definition: The convolution of the two functions $f(t)$ and $g(t)$ is denoted as $f(t) * g(t)$ and is defined as

$f(t) * g(t) = \int_0^t f(u).g(t-u)du$

3.4 Convolution Theorem

$L[f(t) * g(t)] = L[f(t)].L[g(t)] = F(s).G(s)$

(i.e.) The Laplace transform of the convolution of any two functions is the product of their Laplace transforms.

Proof: Let $L[f(t) * g(t)] = \int_0^\infty e^{-st}[f(t) * g(t)]dt$, by definition.

$= \int_0^\infty e^{-st}\left[\int_0^t f(u)g(t-u)du\right]dt$

$$L[f(t) * g(t)] = \int_0^\infty \int_0^t e^{-st} f(u)g(t-u)dudt \tag{3.3}$$

By changing the order of integration, we get

$$L[f(t) * g(t)] = \int_0^\infty f(u)\left[\int_0^t e^{-st} g(t-u)dt\right]du \tag{3.4}$$

Let $t - u = v \Rightarrow dt = dv$. When $t = u \Rightarrow v = 0$

When $t = \infty \Rightarrow v = \infty$

$\therefore$ The equation (3.4) becomes,

$= \int_0^\infty f(u)\left[\int_0^\infty e^{-s(u+v)} g(v)dv\right]du$

$= \int_0^\infty f(u)\left[e^{-su}\int_0^\infty e^{-sv} g(v)dv\right]du$

$= \int_0^\infty e^{-su} f(u)du. \int_0^\infty e^{-sv} g(v)dv$

$= \int_0^\infty e^{-st} f(t)dt. \int_0^\infty e^{-st} g(t)dt = L[f(t)].L[g(t)]$

Thus, $L[f(t) * g(t)] = L[f(t)].L[g(t)] = F(s).G(s)$

Example 3.1. *Find* $L[2e^{-3t} + 3t^2 - 4\sin 2t + 2\cos 3t]$

Solution: By the linearity property, we have

$L[2e^{-3t}+3t^2-4\sin 2t+2\cos 3t] = 2L[e^{-3t}]+3L[t^2]-4L[\sin 2t]+2L[\cos 3t]$

$= 2\left(\frac{1}{s+3}\right) + 3\left(\frac{2!}{s^3}\right) - 4\left(\frac{2}{s^2+4}\right) + 2\left(\frac{s}{s^2+9}\right)$

$= \frac{2}{s+3} + \frac{6}{s^3} - \frac{8}{s^2+4} + \frac{2s}{s^2+9}$

Example 3.2. *Find (i)* $L[e^{-2t}\sin 5t]$ *(ii)* $L[e^{3t}\cosh 4t]$

Solution: (i) Let $L[e^{-2t}\sin 5t] = \frac{5}{(s+2)^2+5^2} = \frac{5}{s^2+4s+29}$

(ii) $L[e^{3t}\cosh 4t] = \frac{s-3}{(s-3)^2-4^2} = \frac{s-3}{s^2-6s-7}$

Example 3.3. *Given that* $L\left[\frac{\sin t}{t}\right] = \tan^{-1}\left(\frac{1}{s}\right)$, *find* $L\left[\frac{\sin 4t}{t}\right]$

Solution: By the change of scale property, we have

$L\left[\frac{\sin 4t}{4t}\right] = \frac{1}{4}L\left[\frac{\sin 4t}{t}\right] = \frac{1}{4}\tan^{-1}\left(\frac{1}{(s/4)}\right)$

$= \frac{1}{4}\tan^{-1}\left(\frac{4}{s}\right)$

$\therefore L\left[\frac{\sin 4t}{t}\right] = \tan^{-1}\left(\frac{4}{s}\right)$

Example 3.4. *Find* $L[\sin 3t\cos 4t + \cos^2 2t + 3]$

Solution: $L[\sin 3t \cos 4t + \cos^2 2t + 3]$

$$= L\left[\frac{\sin(3t+4t)+\sin(3t-4t)}{2}\right] + L\left[\frac{1+\cos 4t}{2}\right] + 3L(1)$$

$$= \frac{1}{2}[L(\sin 7t) + L(-\sin t)] + \frac{1}{2}[L(1) + L(\cos 4t)] + 3L(1)$$

$$= \frac{1}{2}\left[\frac{7}{s^2+49} - \frac{1}{s^2+1}\right] + \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right] + \frac{3}{s}$$

Example 3.5. *Find $L[e^t(\cosh t + \sinh t)]$*

Solution: We know that $L[e^{at}f(t)] = F(s-a)$, where $F(s) = L[f(t)]$

(i.e.) $F(s) = L[\cosh t] + L[\sinh t] = \frac{s}{s^2-1} + \frac{1}{s^2-1} = \frac{s+1}{s^2-1}$

$$\therefore L[e^t(\cosh t + \sinh t)] = \frac{(s-1)+1}{(s-1)^2-1} = \frac{s}{s^2-2s} = \frac{1}{s-2}$$

Example 3.6. *Find $L[t^3e^{-4t}]$*

Solution: We know that $L[e^{-at}f(t)] = F(s+a)$,

where $F(s) = L[f(t)]$

Here $L[f(t)] = L[t^3] = \frac{3!}{s^4}$

(i.e.) $F(s) = \frac{6}{s^4}$

$$\therefore L[t^3e^{-4t}] = \frac{6}{(s+4)^4}$$

Example 3.7. *Find i) $L[t\sin 2t]$ ii) $L[t^2\cos 3t]$*

Solution: i) We have $L[f(t)] = -\frac{d}{ds}F(s)$,

where $F(s) = L[f(t)] = L[\sin 2t] = \frac{2}{s^2+4}$

$$\therefore L[t\sin 2t] = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right] = 2.\frac{2s}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2}$$

ii) Now, $L[\cos 3t] = \frac{s}{s^2+9}$

$$\therefore L[t^2\cos 3t] = \frac{d^2}{ds^2}F(s), \text{ where } F(s) = L[\cos 3t] = \frac{s}{s^2+9}$$

$$L[t^2\cos 3t] = \frac{d^2}{ds^2}\left(\frac{s}{s^2+9}\right)$$

$$= \frac{d}{ds}\left(\frac{(s^2+9)(1) - s.2s}{(s^2+9)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{9-s^2}{(s^2+9)^2}\right) = \frac{2s^3-54s}{(s^2+9)^3}$$

Example 3.8. *Find $L[f(t)]$, if $f(t) = \begin{cases} \sin t, \; 0 < t < \pi \\ 0, \; t > \pi \end{cases}$*

Solution: We know that $L[f(t)] = \int_0^\infty e^{-st}f(t)dt$

$= \int_0^{\pi} e^{-st} f(t)dt + \int_{\pi}^{\infty} e^{-st} f(t)dt$

$= \int_0^{\pi} e^{-st} \sin t dt + 0 = \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_0^{\pi}$

$= \frac{e^{-s\pi}}{s^2+1}(0 - \cos\pi) - \frac{(0-1)}{s^2+1}$

$= \frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1} = \frac{1+e^{-s\pi}}{s^2+1}$

Example 3.9. *Find* $L[g(t)]$, *if* $g(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), t > \frac{2\pi}{3} \\ 0, \quad t < \frac{2\pi}{3} \end{cases}$

Solution: By second shifting property, we know if $L[f(t)] = F(s)$ and $g(t) = \begin{cases} f(t-a), t > a \\ 0, t < a \end{cases}$, that $L[g(t)] = e^{-as}F(s)$.

Here $F(s) = L[f(t)] = L[\cos t] = \frac{s}{s^2+1}$

$\therefore L[g(t)] = e^{\frac{-2\pi s}{3}} \frac{s}{s^2+1}$

Example 3.10. *Find* $L\left[\frac{e^{-t} - e^{-3t}}{t}\right]$

Solution: By Laplace transform of $f(t)/t$, we have that

$L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(u)du$, if $F(s) = L[f(t)]$.

Now $L[f(t)] = L[e^{-t} - e^{-3t}] = L[e^{-t}] - L[e^{-3t}]$

$= \frac{1}{s+1} - \frac{1}{s+3}$

$\therefore L\left[\frac{e^{-t} - e^{-3t}}{t}\right] = \int_s^{\infty}\left(\frac{1}{u+1} - \frac{1}{u+3}\right) du = \ln\left(\frac{s+3}{s+1}\right)$

Example 3.11. *Show that* $L\left[\frac{\cos at - \cos bt}{t}\right]$

Solution: $L[f(t)] = L[\cos at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$\therefore L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^{\infty}\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) du$

$= \frac{1}{2}\ln\left(\frac{s^2+b^2}{s^2+a^2}\right)$

Example 3.12. *Verify the initial value theorem for the function* $3 - 2\cos t$.

Solution: Let $f(t) = 3 - 2\cos t$

$$\lim_{t\to 0} f(t) = 3 - 2(1) = 1 \qquad (3.5)$$

$F(s) = L[f(t)] = 3L(1) - 2L[\cos t] = \frac{3}{s} - \frac{2s}{s^2+1}$

$\therefore \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[3 - 2\frac{s^2}{s^2+1}\right]$

$$= 3 - \lim_{s\to\infty}\frac{1}{1+\frac{1}{s^2}} = 3 - 2(1) = 1 \qquad (3.6)$$

From (3.5) and (3.6), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Thus the initial value theorem is verified.

Example 3.13. *Verify the final value theorem for the function* $1 + e^{-t}(\sin t + \cos t)$

Solution: Let $f(t) = 1 + e^{-t}(\sin t + \cos t)$

$$\lim_{t\to\infty} f(t) = 1 + 0 = 1 \tag{3.7}$$

Now $F(s) = L[f(t)] = L[1] + L[e^{-t}\sin t] + L[e^{-t}\cos t]$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s}{(s+1)^2+1}$$

$$= \frac{1}{s} + \frac{1}{s^2+2s+2} + \frac{s}{s^2+2s+2}$$

$$\therefore \lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{s}{s^2+2s+2} + \frac{s^2}{s^2+2s+2} + 1\right]$$

$$\lim_{s\to 0} sF(s) = 1 + 0 + 0 = 1 \tag{3.8}$$

From (3.7) and (3.8), $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Thus the final value theorem is verified.

Example 3.14. *Use Laplace transform to evaluate* $\int_0^\infty \frac{e^{-t}\sin\sqrt{3}t}{t}dt$

Solution:

$$\int_0^\infty \frac{e^{-t}\sin\sqrt{3}t}{t}dt = \left[\int_0^\infty e^{-st} f(t)dt\right]_{s=1} \quad \text{where } f(t) = \frac{\sin\sqrt{3}t}{t} \tag{3.9}$$

Now, since $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)du$, where

$$F(s) = L[f(t)] = L[\sin\sqrt{3}t] = \frac{\sqrt{3}}{s^2+3}$$

$$\therefore L\left[\frac{\sin\sqrt{3}t}{t}\right] = \int_s^\infty \frac{\sqrt{3}}{u^2+3}du$$

$$= \left[\sqrt{3}\frac{1}{\sqrt{3}}\cot^{-1}\left(\frac{u}{\sqrt{3}}\right)\right]_s^\infty = \cot^{-1}\left(\frac{s}{\sqrt{3}}\right) \tag{3.10}$$

Using (3.10) in (3.9), we get

$$\int_0^\infty \frac{e^{-t}\sin\sqrt{3}t}{t}dt = \left[\cot^{-1}\left(\frac{s}{\sqrt{3}}\right)\right]_{s=1}$$

$$= \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}$$

3.5 Laplace transform of periodic functions

Definition: A function $f(t)$ is said to be periodic, if $f(t+T) = f(t)$ for all t, $(T > 0)$

Theorem 3.1. *If $f(t)$ has period $T > 0$, then*

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$

Proof: We know that $L[f(t)] = \int_0^\infty e^{-st}f(t)dt$

$= \int_0^T e^{-st} f(t)dt + \int_T^{2T} e^{-st} f(t)dt + \int_{2T}^{3T} e^{-st} f(t)dt + \cdots$

In the second integral, let $t = u + T$, in the third integral, let $t = u + 2T$, etc.

Then $L[f(t)] = \int_0^T e^{-su} f(u)du + \int_0^T e^{-s(u+T)} f(u+T)du$

$+ \int_0^T e^{-s(u+2T)} f(u+2T)du + \cdots$

$= (1 + e^{-sT} + e^{-2sT} + \cdots) \int_0^T e^{-su} f(u)du$

$= \dfrac{1}{1-e^{-sT}} \int_0^T e^{-su} f(u)du$

($\because f(u+T) = f(u), f(u+2T) = f(u)$, etc., and using the fact that s_∞ of a G.P. with first term 1 and common ratio e^{-sT} is $\dfrac{1}{1-e^{-sT}}$)

(or) $L[f(t)] = \dfrac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt$

Example 3.15. *Find the Laplace transform of a periodic function $f(t)$, with period 2, given by* $f(t) = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \end{cases}$

Solution: We know that the Laplace transform of a periodic function $f(t)$ with period T is $L[f(t)] = \dfrac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt$

Here $f(t)$ is a periodic function with period 2.

$\therefore L[f(t)] = \dfrac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)dt$

$= \dfrac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} f(t)dt + \int_1^2 e^{-st} f(t)dt\right]$

$= \dfrac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} (1)dt + \int_1^2 e^{-st} (-1)dt\right]$

$= \dfrac{1}{1-e^{-2s}} \left[\left(\dfrac{e^{-st}}{-s}\right)_0^1 - \left(\dfrac{e^{-st}}{-s}\right)_1^2\right]$

$= \dfrac{1}{1-e^{-2s}} \left[\dfrac{e^{-s}}{-s} + \dfrac{1}{s} + \dfrac{e^{-2s}}{s} - \dfrac{e^{-s}}{s}\right]$

$= \dfrac{1}{1-e^{-2s}} \left[\dfrac{1 - 2e^{-s} + e^{-2s}}{s}\right] = \dfrac{1}{s} \cdot \dfrac{(1-e^{-s})^2}{(1+e^{-s})(1-e^{-s})}$

$= \dfrac{1}{s} \cdot \dfrac{1-e^{-s}}{1+e^{-s}} = \dfrac{1}{s} \cdot \dfrac{(1 - e^{-s/2}.e^{-s/2})}{(1 + e^{-s/2}.e^{-s/2})}$

$= \dfrac{1}{s} \cdot \dfrac{(e^{s/2} - e^{-s/2})}{(e^{s/2} + e^{-s/2})}$

$\therefore L[f(t)] = \dfrac{1}{s} \tanh\left(\dfrac{s}{2}\right) \quad \left(\because \tanh\theta = \dfrac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}}\right)$

Example 3.16. *Find the Laplace transform of the full - sine wave rectifier $f(t)$ defined as $f(t) = |\sin\omega t|, t \geq 0$.*

Solution: $f(t) = |\sin\omega t|$ is periodic with period $\dfrac{\pi}{\omega}$.

$\therefore L[f(t)] = \dfrac{1}{1-e^{-\pi s/\omega}} \int_0^{\frac{\pi}{\omega}} e^{-st} |\sin\omega t| dt$

$$= \frac{1}{1-e^{-\pi s/\omega}} \int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t dt \; (\because \sin \omega t > 0 \text{ in } [0, \pi/\omega])$$

$$= \frac{1}{1-e^{-\pi s/\omega}} \left[\frac{e^{-st}}{s^2+\omega^2}(-s \sin \omega t - \omega \cos \omega t) \right]_0^{\pi/\omega}$$

$$= \frac{1}{(s^2+\omega^2)(1-e^{-\pi s/\omega})} [\omega e^{-\pi s/\omega} + \omega]$$

$$= \frac{\omega}{s^2+\omega^2} \left[\frac{1+e^{-\pi s/\omega}}{1-e^{-\pi s/\omega}} \right] = \frac{\omega}{s^2+\omega^2} \left[\frac{1+e^{-\pi s/2\omega} e^{-\pi s/2\omega}}{1-e^{-\pi s/\omega} e^{-\pi s/2\omega}} \right]$$

$$= \frac{\omega}{s^2+\omega^2} \left[\frac{e^{\pi s/2\omega} + e^{-\pi s/2\omega}}{e^{\pi s/2\omega} - e^{-\pi s/2\omega}} \right]$$

$$= \frac{\omega}{s^2+\omega^2} \coth\left(\frac{\pi s}{2\omega}\right) \quad \left[\because \coth \theta = \frac{e^\theta + e^{-\theta}}{e^\theta - e^{-\theta}} \right]$$

Example 3.17. *Find the Laplace transform of a function $f(t)$ given by $f(t) = \begin{cases} t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$ and that $f(t+2\pi) = f(t)$.*

Solution: Given $f(t) = \begin{cases} t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$ such that $f(t)$ is a periodic function with period 2π.

$$\therefore L[f(t)] = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} f(t) dt + \int_\pi^{2\pi} 0 dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[t\left(\frac{e^{-st}}{s}\right)_0^\pi - (1)\left(\frac{e^{-st}}{s^2}\right)_0^\pi \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{-\pi e^{-s\pi}}{s} - \frac{(e^{-s\pi}-1)}{s^2} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{1-e^{-s\pi}}{s^2} - \frac{\pi e^{-s\pi}}{s} \right]$$

Example 3.18. *Find the Laplace transform of the periodic function $f(t) = \begin{cases} t, & 0 < t < 1 \\ 2-t, & 1 < t < 2 \end{cases}$ where $f(t+2) = f(t)$ for all t where t is positive.*

Solution: The given function is a periodic function with period 2.

$$\therefore L[f(t)] = \frac{\int_0^2 e^{-st} f(t) dt}{1-e^{-2s}} = \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} t dt + \int_1^2 e^{-st}(2-t) dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\left(\frac{e^{-st} t}{s} - 1.\frac{e^{-st}}{s^2} \right)_0^1 + \left(\frac{e^{-st}(2-t)}{s} - (-1)\frac{e^{-st}}{s^2} \right)_1^2 \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{t}{s} + \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{1-2e^{-s}+e^{-2s}}{s^2} \right] = \frac{(1-e^{-s})^2}{(1+e^{-s})(1-e^{-s})s^2}$$

$$= \frac{1-e^{-s}}{(1+e^{-s})s^2} = \frac{1}{s^2} \cdot \frac{1-e^{-s}}{1+e^{-s}} = \frac{1}{s^2} \frac{e^{s/2}-e^{-s/2}}{e^{s/2}+e^{-s/2}}$$

$$= \frac{1}{s^2} \tanh\left(\frac{s}{2}\right)$$

3.6 Inverse Laplace Transform

If $L[f(t)] = F(s)$, then $f(t)$ is called Inverse Laplace Transform of $F(s)$. This we write as $f(t) = L^{-1}[F(s)]$. Here the symbol L^{-1} is called the Inverse Laplace transform operator.

Example:

We know that $L[e^{2t}] = \dfrac{1}{s-2}$.

Then we can write this as $L^{-1}\left[\dfrac{1}{s-2}\right] = e^{2t}$.

Remark: For uniqueness of the inverse Laplace transform, the sufficient conditions for the existence of $L[f(t)]$ should hold.

From the table of Laplace transform of some elementary functions, it follows easily the inverse Laplace transforms of $F(s)$, as given in the following table:

S.No.	$F(s)$	$L^{-1}F(s) = f(t)$
1	$\frac{1}{s}$	1
2	$\frac{1}{s^2}$	t
3	$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$
4	$\frac{1}{s-a}$	e^{at}
5	$\frac{1}{s+a}$	e^{-at}
6	$\frac{a}{s^2+a^2}$	$\sin at$
7	$\frac{s}{s^2+a^2}$	$\cos at$
9	$\frac{a}{s^2-a^2}$	$\sinh at$
10	$\frac{s}{s^2-a^2}$	$\cosh at$

As in the case of Laplace transform, we now write some important properties of Inverse Laplace transforms.

Property 3.1. ***Linearity Property:***

$$L^{-1}[aF(s) + bG(s)] = aL^{-1}F(s) + bL^{-1}G(s) = af(t) + bg(t)$$

Example:

$$L^{-1}\left[\frac{3}{s-1} - \frac{2s}{s^2+9}\right] = 3L^{-1}\left[\frac{1}{s-1}\right] - 2L^{-1}\left[\frac{s}{s^2+9}\right]$$

$$= 3e^t - 2\cos 3t$$

Property 3.2. ***First shifting property:***

If $L^{-1}F(s) = f(t)$, then $L^{-1}F(s-a) = e^{at}f(t)$

Example:

$$L^{-1}\left[\frac{s-2}{s^2-4s+13}\right] = L^{-1}\left[\frac{s-2}{(s-2)^2+9}\right]$$

$$= L^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right] = e^{2t}\cos 3t$$

Property 3.3. ***Second shifting property:***

If $L^{-1}F(s) = f(t)$, then $L^{-1}[e^{-as}F(s)] = \begin{cases} f(t-a), \ t > a \\ 0, \ t < a \end{cases}$

Example:

$$L^{-1}\left[\frac{e^{-\pi s/6}}{s^2+1}\right] = \begin{cases} \sin\left(t - \frac{\pi}{6}\right), t > \frac{\pi}{6} \\ 0, \quad t < \frac{\pi}{6} \end{cases}$$

Property 3.4. ***Change of Scale Property:***

If $L^{-1}F(s) = f(t)$, then $L^{-1}[F(as)] = \frac{1}{a}f\left(\frac{t}{a}\right)$

Example:

$$L^{-1}\left[\frac{3s}{(3s)^2+16}\right] = \frac{1}{3}\cos\left(\frac{4t}{3}\right)$$

Property 3.5. ***Inverse Laplace transform of derivatives:***

If $L^{-1}F(s) = f(t)$, then

$$L^{-1}[F^{(n)}(s)] = L^{-1}\left[\frac{d^n}{ds^n}F(s)\right] = (-1)^n t^n f(t)$$

Example:

$$L^{-1}\left[\frac{-4s}{(s^2+4)^2}\right] = -t\sin 2t \begin{cases} \because L^{-1}\left[\frac{2}{(s^2+4)}\right] = \sin 2t \\ \text{and } \frac{d}{ds}\left[\frac{2}{s^2+4}\right] = \frac{-4s}{(s^2+4)^2} \end{cases}$$

Property 3.6. ***Inverse Laplace transform of integral:***

If $L^{-1}F(s) = f(t)$, then $L^{-1}\left[\int_s^\infty F(u)du\right] = \frac{f(t)}{t}$

Property 3.7. ***Multiplication by s:***

If $L^{-1}F(s) = f(t)$ and $f(0) = 0$, then $L^{-1}[sF(s)] = f'(t)$.

Example:

$L^{-1}\left[\frac{s}{s^2+1}\right] = \frac{d}{dt}(\sin t) = \cos t$, since $L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$ and $\sin 0 = 0$

Property 3.8. ***Division by s:***

If $L^{-1}F(s) = f(t)$, then $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(u)du = \int_0^t L^{-1}F(s)dt$

Example:

Since $L^{-1}\left[\frac{4}{(s^2+16)}\right] = \sin 4t$, we have

$$L^{-1}\left[\frac{4}{s(s^2+16)}\right] = \int_0^t \sin 4u du = \frac{1}{4}(1-\cos 4t)$$

3.7 Convolution Theorem

If $L^{-1}F(s) = f(t)$ and $L^{-1}G(s) = g(t)$, then

$L^{-1}[F(s).G(s)] = \int_0^t f(u)g(t-u)du = f * g = L^{-1}[F(s)] * L^{-1}[G(s)]$, $f * g$ is called the convolution of $f(t)$ and $g(t)$.

Note: $f * g = g * f$, (i.e.) the convolution is abelian.

Example 3.19. *Find* $L^{-1}\left[\frac{1}{s^2+9}\right]$.

Solution: Since $L[\sin 3t] = \dfrac{3}{s^2+9}$

$$\frac{1}{3}L[\sin 3t] = \frac{1}{s^2+9}$$

$$\therefore L^{-1}\left[\frac{1}{s^2+9}\right] = \frac{\sin 3t}{3}$$

Example 3.20. *Find* $L^{-1}\left[\frac{1}{s^2-3}\right]$.

Solution: Since $L\left[\frac{\sqrt{3}}{s^2-(\sqrt{3})^2}\right] = \sinh\sqrt{3}t$

$$\frac{1}{\sqrt{3}}L[\sinh\sqrt{3}t] = \frac{1}{s^2-3}$$

$$L^{-1}\left[\frac{1}{s^2-3}\right] = \frac{\sinh\sqrt{3}t}{\sqrt{3}}$$

Example 3.21. *Find* $L^{-1}\left[\frac{4}{2s-4}+\frac{4+2s}{16s^2-4}\right]$

Solution:

$$L^{-1}\left[\frac{4}{2s-4}+\frac{4+2s}{16s^2-4}\right] = L^{-1}\left[\frac{4}{2s-4}\right] + L^{-1}\left[\frac{4}{16s^2-4}\right] + 2L^{-1}\left[\frac{s}{16s^2-4}\right]$$

$$= L^{-1}\left[\frac{2}{s-2}\right] + L^{-1}\left[\frac{4}{16\left(s^2-\frac{4}{16}\right)}\right] + 2L^{-1}\left[\frac{s}{16\left(s^2-\frac{4}{16}\right)}\right]$$

$$= 2L^{-1}\left[\frac{1}{s-2}\right] + L^{-1}\left[\frac{1/4}{s^2-(1/4)}\right] + \frac{2}{16}L^{-1}\left[\frac{s}{s^2-(1/4)}\right]$$

$$= 2L^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{2}L^{-1}\left[\frac{1/2}{s^2-(1/2)^2}\right] + \frac{1}{8}L^{-1}\left[\frac{s}{s^2-(1/2)^2}\right]$$

$$= 2e^{2t} + \frac{1}{2}\sinh\frac{1}{2}t + \frac{1}{8}\cosh\frac{1}{2}t$$

Example 3.22. *Find* $L^{-1}\left[\dfrac{3s+7}{s^2-2s-3}\right]$

Solution: $L^{-1}\left[\dfrac{3s+7}{s^2-2s-3}\right] = L^{-1}\left[\dfrac{3s+7}{(s-1)^2-4}\right]$

$$= L^{-1}\left[\frac{3(s-1)+10}{(s-1)^2-4}\right] = 3L^{-1}\left[\frac{s-1}{(s-1)^2-2^2}\right] + 5L^{-1}\left[\frac{2}{(s-1)^2-\ldots}\right]$$

$$= 3e^t\cosh 2t + 5e^t\sinh 2t = e^t[3\cosh 2t + 5\sinh 2t]$$

Example 3.23. *Using Laplace transform of the derivatives, find* $L(t\cos at)$.

Solution: We know that

$$L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0) \tag{3.11}$$

Let $f(t) = t\cos at \Rightarrow f(0) = 0$

Then $f'(t) = -at\sin at + \cos at \quad \therefore f'(0) = 1$

Also $f''(t) = -a[at\cos at + \sin at] - a\sin at$

Equation (3.11) becomes

$$L(-a[at\cos at + \sin at + a\sin at]) = s^2L[t\cos at] - 0 - 1$$

$$L[-a^2t\cos at] - L[2a\sin at] = s^2L[t\cos at] - 1$$

$$1 - L[2a\sin at] = (s^2+a^2)L[t\cos at]$$

$$\Rightarrow (s^2+a^2)L[t\cos at] = 1 - 2aL[\sin at] = 1 - \frac{2a^2}{s^2+a^2}$$

$$= \frac{s^2+a^2-2a^2}{s^2+a^2} = \frac{s^2-a^2}{s^2+a^2}$$

$$\text{(iii) } L(t\cos at) = \frac{s^2-a^2}{(s^2+a^2)^2}$$

Example 3.24. *Find* $L^{-1}\left[\dfrac{s+1}{s^2+s+1}\right]$

Solution: $L^{-1}\left[\dfrac{s+1}{s^2+s+1}\right] = L^{-1}\left[\dfrac{(s+1/2)+1/2}{(s+1/2)^2+3/4}\right]$

$$= L^{-1}\left[\frac{(s+1/2)}{(s+1/2)^2+(\sqrt{3}/2)^2}\right] + \frac{1}{\sqrt{3}}L^{-1}\left[\frac{\sqrt{3}/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right]$$

$$= e^{-t/2}\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}}{2}t$$

$$= \frac{e^{-t/2}}{\sqrt{3}}\left[\sqrt{3}\cos\frac{\sqrt{3}}{2}t + \sin\frac{\sqrt{3}}{2}t\right]$$

Example 3.25. *Find* $L^{-1}\left[\dfrac{s}{(s+2)^2}\right]$

Solution: $L^{-1}\left[\dfrac{s}{(s+2)^2}\right] = L^{-1}\left[s.\dfrac{1}{(s+2)^2}\right]$

$$= \frac{d}{dt}L^{-1}\left[\frac{1}{(s+2)^2}\right] = \frac{d}{dt}[e^{-2t}L^{-1}(1/s^2)]$$

$$= \frac{d}{dt}(e^{-2t}t) = e^{-2t}(1) + t.e^{-2t}(-2) = e^{-2t}(1-2t)$$

Example 3.26. *Find* $L^{-1}\left[\dfrac{(s+1)^2}{(s^2+2s+5)^2}\right]$

Solution:

$L^{-1}\left[\dfrac{(s+1)^2}{(s^2+2s+5)^2}\right] = e^{-t}L^{-1}\left[\dfrac{s^2}{(s^2+2^2)^2}\right]$, by the first shifting property.

$$= e^{-t}L^{-1}\left[s.\frac{s}{(s^2+2^2)^2}\right] = e^{-t}\frac{d}{dt}\left[L^{-1}\frac{s}{(s^2+2^2)^2}\right]$$

$$= e^{-t}\frac{d}{dt}\left[\frac{t}{4}\sin 2t\right] \quad \left(\because L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}\sin at,\right.$$

provided $\left. f(0)=0\right)$

$$= e^{-t}\left[\frac{1}{4}(\sin 2t + 2t\cos 2t)\right] = \frac{1}{4}e^{-t}(\sin 2t + 2t\cos 2t)$$

Example 3.27. *Given that* $L^{-1}\left[\dfrac{s}{(s^2-a^2)^2}\right] = \dfrac{1}{2a}\sinh at$, *find* $L^{-1}\left[\dfrac{s^2}{(s^2-a^2)^2}\right]$

Solution: $L^{-1}\left[\dfrac{s^2}{(s^2-a^2)^2}\right] = L^{-1}\left[s.\dfrac{s}{(s^2-a^2)^2}\right]$

$$= \frac{d}{dt}\left[\frac{s}{(s^2-a^2)^2}\right] = \frac{d}{dt}\left[\frac{t}{2a}\sinh at\right]$$

$$= \frac{1}{2a}[t\cosh at.a + \sinh at] = \frac{1}{2a}[\sinh at + at\cosh at]$$

Example 3.28. *Find* $L^{-1}\left[\dfrac{1}{s(s^2+2s+2)}\right]$

Solution:

$$L^{-1}\left[\frac{1}{s(s^2+2s+2)}\right] = L^{-1}\left[\frac{1}{s}F(s)\right] \text{ where } F(s) = \frac{1}{s^2+2s+2}$$

$$\therefore \int_0^t L^{-1}[F(s)]dt = \int_0^t L^{-1}\left[\frac{1}{s^2+2s+2}\right]dt \tag{3.12}$$

Now, $L^{-1}\left[\dfrac{1}{s^2+2s+2}\right] = L^{-1}\left[\dfrac{1}{(s+1)^2+1^2}\right]$

$$= e^{-t}L^{-1}\left[\frac{1}{s^2+1^2}\right] = e^{-t}\sin t \tag{3.13}$$

Using (3.13) in (3.12), we get

$$L^{-1}\left[\frac{1}{s^2+2s+2}\right] = \textstyle\int_0^t e^{-t}\sin t dt$$

$$= \frac{e^{-t}}{1+1}[(1-\sin t+\cos t)] = \frac{e^{-t}}{2}[1-\sin t+\cos t]$$

Example 3.29. *Find* $L^{-1}\left[\dfrac{1}{s(s+2)^3}\right]$

Solution:

$$L^{-1}\left[\frac{1}{s(s+2)^3}\right] = \int_0^t L^{-1}F(s)dt, \text{ where } F(s) = \frac{1}{(s+2)^3} \tag{3.14}$$

$$L^{-1}F(s) = e^{-2t}L^{-1}\left[\frac{1}{s^3}\right] = e^{-2t}\frac{t^2}{2} \tag{3.15}$$

Using (3.15) in (3.14), we get

$$L^{-1}\left[\frac{1}{s(s+2)^3}\right] = \int_0^t e^{-2t}\frac{t^2}{2}dt = \frac{1}{2}\int_0^t t^2e^{-2t}dt$$

$$= \frac{1}{2}\left[t^2\left(\frac{e^{-2t}}{2}\right) - 2t\left(\frac{e^{-2t}}{4}\right) + 2\left(\frac{e^{-2t}}{-8}\right)\right]_0^t$$

$$= \frac{1}{2}\left[e^{-2t}\left(-\frac{t^2}{2} - \frac{t}{2} - \frac{1}{4}\right) + \frac{1}{4}\right]$$

$$= \frac{1}{8}[e^{-2t}(-2t^2 - 2t - 1) + 1] = \frac{1}{8}[1 - e^{-2t}(2t^2 + 2t + 1)]$$

Example 3.30. *Evaluate each of the following, using convolution theorem:*

(i) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$, *(ii)* $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

Solution:

(i) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ can be written as $L^{-1}\left[\frac{s}{(s^2+a^2).(s^2+a^2)}\right]$

$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right].L^{-1}\left[\frac{s}{(s^2+a^2)}\right] = f(t)*g(t)$, by convolution theorem

$= \int_0^t f(u)g(t-u)du$, [where $f(t) = L^{-1}\left[\frac{s}{(s^2+a^2)}\right] = \cos at$,

$g(t) = L^{-1}\left[\frac{1}{(s^2+a^2)}\right] = \frac{1}{a}\sin at]$

$$= \int_0^t \cos au \frac{1}{a}\sin a(t-u)du$$

$$= \frac{1}{a}\int_0^t \cos au[\sin at\cos au - \cos at\sin au]du$$

$$= \frac{1}{a}\left[\int_0^t \sin at\cos^2 au du - \int_0^t \cos at\sin au\cos au du\right]$$

$$= \frac{1}{a}\left[\sin at\int_0^t\left(\frac{1+\cos 2au}{2}\right)du - \cos at\int_0^t\frac{\sin 2au}{2}du\right]$$

$$= \frac{1}{a}\left[\sin at\left(\frac{1}{2}u + \frac{\sin 2au}{4a}\right)_0^t - \frac{\cos at}{2}\left(\frac{-\cos 2au}{2a}\right)_0^t\right]$$

$$= \frac{1}{a}\left[\sin at\left(\frac{t}{2} + \frac{\sin 2at}{4a}\right) + \frac{\cos at}{2}\left(\frac{\cos 2at - 1}{2a}\right)\right]$$

$$= \frac{1}{a}\left[\sin at\left(\frac{t}{2} + \frac{\sin at\cos at}{2a}\right) - \cos at\left(\frac{1-\cos 2at}{4a}\right)\right]$$

$$= \frac{1}{a}\left[\sin at\left(\frac{t}{2} + \frac{\sin at\cos at}{2a}\right) - \cos at\left(\frac{2\sin^2 at}{2.2a}\right)\right]$$

$$= \frac{1}{a}\left[\frac{t}{2}\sin at + \frac{\sin^2 at\cos at}{2a} - \frac{\sin at\cos at}{2a}\right] = \frac{t\sin at}{2a}$$

(ii) $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ can be written as

$$L^{-1}\left[\frac{s}{(s^2+a^2)}.\frac{s}{(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left(\frac{s}{(s^2+b^2)}\right) = f * g = \cos at * \cos bt$$

$$= \int_0^t \cos au \cos b(t-u)du$$

$$= \int_0^t \frac{\cos(au+bt-bu)+\cos(au-bt-bu)}{2}du$$

$$= \frac{1}{2}\left[\int_0^t \cos\{(a-b)u+u\}du + \int_0^t \cos\cos\{(a-b)u-bt\}du\right]$$

$$= \frac{1}{2}\left[\left(\frac{\sin(bt+(a-b)u)}{a-b}\right)_0^t + \left(\frac{\sin(-bt+(a+b)u)}{a+b}\right)_0^t\right]$$

$$= \frac{1}{2}\left[\left(\frac{\sin(bt+at-bt)}{a-b}\right) + \left(\frac{\sin(-bt+at+bt)}{a+b}\right) - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at - 2b\sin bt}{a^2-b^2}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}$$

Example 3.31. *Using partial fractions, find* $L^{-1}\left[\dfrac{3s+16}{s^2-s-16}\right]$

Solution: $\dfrac{3s+16}{s^2-s-16} = \dfrac{3s+16}{(s-3)(s+2)} = \dfrac{A}{s-3} + \dfrac{B}{s+2}$

$\Rightarrow 3s+16 = A(s+2) + B(s-3)$

$s = 3 \Rightarrow 25 = 5A \quad$ (i.e.) $A = 5$

$s = -2 \Rightarrow 10 = -5B \quad$ (i.e.) $B = -2$

$$L^{-1}\left[\frac{3s+16}{s^2-s-16}\right] = L^{-1}\left[\frac{5}{s-3}\right] + L^{-1}\left[\frac{-2}{s+2}\right] = 5e^{3t} - 2e^{-2t}$$

Example 3.32. *Find* $L^{-1}\left[\dfrac{11s^2-2s+5}{(s-2)(2s-1)(s+1)}\right]$

Solution: $\dfrac{11s^2-2s+5}{(s-2)(2s-1)(s+1)} = \dfrac{A}{s-2} + \dfrac{B}{2s-1} + \dfrac{C}{s+1}$

$\Rightarrow 11s^2-2s+5 = A(2s-1)(s+1)+B(s-2)(s+1)+C(s-2)(2s-1)$

$s = 2 \Rightarrow 45 = 9A \Rightarrow A = 5$

$s = -1 \Rightarrow 18 = 9C \Rightarrow C = 2$

$s = 1/2 \Rightarrow \dfrac{27}{4} = -\dfrac{9}{4}B \Rightarrow B = -3$

$$L^{-1}\left[\frac{11s^2-2s+5}{(s-2)(2s-1)(s+1)}\right] = L^{-1}\left[\frac{5}{s-2}\right] + L^{-1}\left[\frac{-3}{2s-1}\right] + L^{-1}\left[\frac{2}{s+1}\right]$$

$$= 5L^{-1}\left[\frac{1}{s-2}\right] - 3L^{-1}\left[\frac{1}{2s-1}\right] + 2L^{-1}\left[\frac{1}{s+1}\right]$$

$$= 5L^{-1}\left[\frac{1}{s-2}\right] - \frac{3}{2}L^{-1}\left[\frac{1}{s-(1/2)}\right] + 2L^{-1}\left[\frac{1}{s+1}\right]$$

$$= 5e^{2t} - \frac{3}{2}e^{\frac{1}{2}t} + 2e^{-t}$$

Example 3.33. *Find* $L^{-1}\left[\dfrac{s^2-2s+3}{(s-1)^2(s+1)}\right]$

Solution: $\dfrac{s^2-2s+3}{(s-1)^2(s+1)} = \dfrac{A}{(s-1)^2} + \dfrac{B}{s-1} + \dfrac{C}{s+1}$

$\Rightarrow s^2-2s+3 = A(s+1) + B(s-1)s+1) + C(s-1)^2$

$s=-1 \Rightarrow 4C = 6 \Rightarrow C = 3/2$

$s=1 \Rightarrow 2A = 2 \Rightarrow A = 1$

$s=0 \Rightarrow A - B + C = 3 \Rightarrow B = -1/2$

$\therefore L^{-1}\left[\frac{s^2-2s+3}{(s-1)^2(s+1)}\right] = L^{-1}\left[\frac{1}{(s-1)^2}\right] - L^{-1}\left[\frac{1/2}{s-1}\right] + L^{-1}\left[\frac{3/2}{s+1}\right]$

$= e^tL^{-1}\left[\dfrac{1}{s^2}\right] - \dfrac{1}{2}L^{-1}\left[\dfrac{1}{s-1}\right] + \dfrac{3}{2}L^{-1}\left[\dfrac{1}{s+1}\right]$

$= te^t - \dfrac{1}{2}e^t + \dfrac{3}{2}e^{-t} = \dfrac{1}{2}e^t(2t-1) + \dfrac{3}{2}e^{-t}$

Example 3.34. *Find* $L^{-1}\left[\dfrac{27-2s}{(s+4)(s^2+9)}\right]$

Solution: $\dfrac{27-2s}{(s+4)(s^2+9)} = \dfrac{A}{s+4} + \dfrac{Bs+C}{s^2+9}$

$\Rightarrow 27-2s = A(s^2+9) + (Bs+C)(s+4)$

Comparing like co-efficients of s^2, s and constant terms, on both

sides of the above equation, we get

$$A + B = 0 \quad (3.16)$$

$$4B + C = -2 \quad (3.17)$$

$$9A + 4C = 27 \quad (3.18)$$

From (3.16), $B = -A$

(i.e.) (3.17) becomes $-4A + C = -2$

Equation (3.18) is $9A + 4C = 27$

Solving these two equations, we get $-25A = -35$

$\therefore A = 7/5$ and $B = -7/5$

From equation (3.17), $C = -2 + 28/5 = 18/5$

$\therefore L^{-1}\left[\dfrac{s^2-2s+3}{(s-1)^2(s+1)}\right] = \dfrac{7}{5}L^{-1}\left[\dfrac{1}{s+4}\right] - \dfrac{7}{5}L^{-1}\left[\dfrac{s}{s^2+9}\right] + \dfrac{18}{5}L^{-1}\left[\dfrac{s}{s^2+9}\right]$

$= \dfrac{7}{5}e^{-4t} - \dfrac{7}{5}\cos 3t + \dfrac{18}{5}\sin 3t$

Example 3.35. *Evaluate* $L^{-1}\left[\dfrac{s^2-3}{(s+2)(s-3)(s^2+2s+5)}\right]$

Solution: $\dfrac{s^2-3}{(s+2)(s-3)(s^2+2s+5)} = \dfrac{As+B}{s^2-s-6} + \dfrac{Cs+D}{s^2+2s+5}$

$\Rightarrow s^2-3 = (As+B)(s^2+2s+5) + (Cs+D)(s^2-s-6) \quad (3.19)$

Comparing the co-efficients of s^3, s^2, s and constant terms on both sides of above equation, we get

$$A + C = 20 \tag{3.20}$$

$$2A + B - C + D = 1 \tag{3.21}$$

$$5A + 2B - 6C - D = 0 \tag{3.22}$$

$$5B - 6D = -3 \tag{3.23}$$

$$\text{Equation (3.21)} + \text{(3.22) gives } 7A + 3B - 7C = 1 \tag{3.24}$$

$$\text{Equation (3.22)} \times 6 - \text{(3.23)} \Rightarrow 30A + 7B - 36C = 3 \tag{3.25}$$

$$\text{Equation (3.24)} \times 7 - \text{(3.25)} \times 3 \text{ gives } -41A + 59C = -2 \tag{3.26}$$

$$\text{Equation (3.19)} \times 59 \text{ gives } 59A + 59C = 0 \tag{3.27}$$

Equation (3.26) – (3.27) $\Rightarrow -100A = -2 \Rightarrow A = 1/50$

From (3.26), $59C = -2 + \dfrac{41}{50} = -\dfrac{59}{50} \Rightarrow C = -1/50$

Equation (3.24), $\dfrac{7}{50} + 3B + \dfrac{7}{50} = 1 \Rightarrow 3B = 1 - \dfrac{14}{50} = \dfrac{36}{50}$

$\Rightarrow B = 12/50$

From (3.25), $\dfrac{60}{50} - 6D = -3 \Rightarrow -6D = -3 - \dfrac{60}{50} = -\dfrac{210}{50}$

$\Rightarrow D = 35/50$

$$L^{-1}\left[\frac{s^2-3}{(s+2)(s-3)(s^2+2s+5)}\right]$$

$$= L^{-1}\left[\frac{\frac{1}{50}s + \frac{12}{50}}{s^2 - s - 6}\right] + L^{-1}\left[\frac{-\frac{1}{50}s + \frac{35}{50}}{s^2 + 2s + 5}\right] \qquad \rightarrow (I)$$

Consider,

$$L^{-1}\left[\frac{-\frac{1}{50}s + \frac{35}{50}}{s^2 + 2s + 5}\right] = -\frac{1}{50}L^{-1}\left[\frac{s+1}{(s+1)^2 + 2^2} + \frac{35}{50}\frac{1}{(s+1)^2 + 2^2}\right]$$

$$= -\frac{1}{50}e^{-t}\cos 2t + \frac{1}{50}L^{-1}\left[\frac{1}{(s+1)^2 + 2^2}\right] + \frac{35}{50}L^{-1}\left[\frac{1}{(s+1)^2 + 2^2}\right]$$

$$= -\frac{1}{50}e^{-t}\cos 2t + \frac{1}{100}e^{-t}\sin 2t + \frac{35}{100}e^{-t}\sin 2t \qquad \rightarrow (i)$$

Now consider $L^{-1}\left[\dfrac{\frac{1}{50}s + \frac{12}{50}}{s^2 - s - 6}\right] = L^{-1}\left[\dfrac{\frac{1}{50}s + \frac{12}{50}}{(s+2)(s-3)}\right]$

$$\Rightarrow \frac{\frac{1}{50}s + \frac{12}{50}}{(s+2)(s-3)} = \frac{A}{s+2} + \frac{B}{s-3}$$

$\Rightarrow A = -\dfrac{1}{25}$ and $B = \dfrac{3}{50}$

$$L^{-1}\left[\frac{\frac{1}{50}s + \frac{12}{50}}{s^2 - s - 6}\right] = -\frac{1}{25}\left[\frac{1}{s+2}\right] + \frac{3}{50}\left[\frac{1}{s-3}\right]$$

$$L^{-1}\left[\frac{-\frac{1}{50}s+\frac{35}{50}}{s^2+2s+5}\right]=-\frac{1}{25}e^{-2t}+\frac{3}{50}e^{3t} \qquad \to (ii)$$

Using (i) and (ii) in (I), we get

$$L^{-1}\left[\frac{s^2-3}{(s+2)(s-3)(s^2+2s+5)}\right]=\frac{3}{50}e^{3t}-\frac{1}{25}e^{-2t}-\frac{1}{50}e^{-t}\cos 2t+\frac{9}{25}e^{-t}\sin 2t$$

3.8 Application of Laplace Transform to Solve Differential Equations

Laplace transform is useful in solving differential (both ordinary and partial) equations. We confine ourselves to apply this method to solve only ordinary linear differential equations with constant co-efficients.

The following steps are to be followed while solving the linear differential equations.

Step 1: Take Laplace transforms of both sides of the given differential equation.

Step 2: Use the formula of Laplace transform of the derivatives and substitute the initial conditions given.

Step 3: Finally, take Inverse Laplace transforms to get the required solution.

Solved Problems:

Using Laplace transform, solve the following differential equations:

Example 3.36. $y''-3y'-4y=2e^{-t}, y(0)=1, y'(0)=1.$

Solution:

Given that $y''-3y'-4y=2e^{-t}$ such that $y(0)=y'(0)=1$ (3.28)

Taking Laplace transforms of both sides of equation (3.28),

we get $L[y''-3y'-4y]=2L[e^{-t}]$

(i.e.) $L[y'']-3L[y']-4L[y]=2L[e^{-t}]$

$$[s^2\overline{y}(s)-sy(0)-y'(0)]-3[s\overline{y}(s)-y(0)]-4\overline{y}(s)=2.\frac{1}{s+1}$$

$$(s^2-3s-4)\overline{y}(s)-s-1+3=\frac{2}{s+1}\ (\because y(0)=y'(0)=1)$$

$$(i.e.)\ (s^2-3s-4)\overline{y}(s)=\frac{2}{s+1}+s-2$$

$$\overline{y}(s)=\frac{2}{(s+1)(s^2-3s-4)}+\frac{s-2}{(s^2-3s-4)}$$

$$=2\left[\frac{1}{(s+1)^2(s-4)}\right]+\frac{s-2}{(s^2-3s-4)}$$

$$\overline{y}(s)=2\left[\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{s-4}\right]+\left[\frac{D}{s+1}+\frac{E}{s-4}\right]$$

$$-2\left[\frac{F}{s+1}+\frac{G}{s-4}\right]$$

$$=2\left[\frac{-1/25}{s+1}-\frac{1/5}{(s+1)^2}+\frac{1/25}{s-4}\right]+\left[\frac{1/5}{s+1}+\frac{4/5}{s-4}\right]$$

$$-2\left[\frac{-1/5}{s+1}+\frac{1/5}{s-4}\right]$$ (by usual partial fraction method, the vlaues of A, B, C, D, E, F and G, can be found)

Taking the Inverse Laplace transform of both sides, we get

$$y(t)=-\frac{2}{25}e^{-t}-\frac{2}{5}te^{-t}+\frac{2}{25}e^{4t}+\frac{1}{5}e^{-t}+\frac{4}{5}e^{4t}+\frac{2}{5}e^{-t}-\frac{2}{5}e^{4t}$$

$$=\frac{13}{25}e^{-t}+\frac{12}{15}e^{4t}-\frac{2}{5}te^{-t}$$

$$=\frac{1}{25}(13e^{-t}-10te^{-t}+12e^{4t})$$

Example 3.37. *Solve:* $(D^2+4D+13)y=e^{-t}\sin t, y=0$ *and* $Dy=0$ *at* $t=0$, *where* $D\equiv\frac{d}{dt}$.

Solution: Given $(D^2+4D+13)y=e^{-t}\sin t$

Taking Laplace transform on both sides, we get

$$(s^2+4s+13)\overline{y}(s)=L[e^{-t}\sin t]=\frac{1}{(s+1)^2+1^2}=\frac{1}{s^2+2s+2}$$

$$\therefore \overline{y}(s)=\frac{1}{(s^2+2s+2)(s^2+4s+13)}$$

$$=\frac{As+B}{s^2+2s+2}+\frac{Cs+D}{s^2+4s+13}$$

By the usual partial fraction method, the constants can be found as $A=-2/85, B=7/85, C=2/85$ and $D=-3/85$.

$$\therefore \overline{y}(s)=\frac{1}{85}\left[\frac{-2s+7}{s^2+2s+2}+\frac{2s-3}{s^2+4s+13}\right]$$

$$=\frac{1}{85}\left[\frac{-2(s+1)+9}{(s+1)^2+1^2}+\frac{2(s+2)-7}{(s+2)^2+3^2}\right]$$

Taking Inverse Laplace transforms on both sides, we get

$$L^{-1}[\overline{y}(s)]=\frac{1}{85}\left[L^{-1}\left(\frac{-2(s+1)}{(s+1)^2+1^2}\right)+9L^{-1}\left(\frac{1}{(s+1)^2+1^2}\right)\right.$$

$$\left.+2L^{-1}\left(\frac{s+2}{(s+2)^2+3^2}\right)-\frac{7}{3}L^{-1}\left(\frac{3}{(s+2)^2+3^2}\right)\right]$$

$$\therefore y(t)=\frac{1}{85}\left[e^{-t}(-2\cos t+9\sin t)+e^{-2t}\left(2\cos 3t-\frac{7}{3}\sin 3t\right)\right]$$

Example 3.38. *Solve:* $x''+4x'+3x=10\sin t, x(0)=x'(0)=0$.

Solution: Given $x''+4x'+3x=10\sin t, x(0)=x'(0)=0$

Taking Laplace transforms on both sides and using initial conditions, we get

$$(s^2+4s+3)\overline{x}(s)=\frac{10}{s^2+1}$$

$$\therefore \overline{x}(s) = \frac{10}{(s^2+1)(s^2+4s+3)}$$

$$= 10\left[\frac{As+B}{s^2+1} + \frac{C}{s+1} + \frac{D}{s+3}\right] \text{ (using partial fractions)}$$

$$= 10\left[\frac{(-1/5)s+1/10}{s^2+1} + \frac{1/4}{s+1} + \frac{(-1/20)}{s+3}\right]$$

Taking Inverse Laplace transforms on both sides, we get

$$L^{-1}[\overline{x}(s)] = \frac{5}{2}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right]$$

$$-2L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} - 2\cos t + \sin t$$

Example 3.39. *Solve:* $y'' + 2y' - 3y = \sin t$, *given* $y(0) = y'(0) = 0$.

Solution: Given $y'' + 2y' - 3y = \sin t$, given $y(0) = y'(0) = 0$

Taking Laplace transforms on both sides, we get

$$s^2L[y(t) - sy(0) - y'(0) + 2(sL(y(t)) - y(0)) - 3L[y(t)]] = L[\sin t]$$

$$\text{(i.e.) } (s^2+2s-3)\overline{y}(s) = \frac{1}{s^2+1} \quad (\because y(0) = y'(0) = 0)$$

$$\therefore \overline{y}(s) = \frac{1}{s^2+1}\cdot\frac{1}{s^2+2s-3}$$

$$\therefore \overline{y}(s) = \frac{1}{(s-1)(s+3)(s^2+1)} \tag{3.29}$$

Consider

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1} \tag{3.30}$$

Using usual procedure, we can find the constants A, B, C and D as $A = 1/8, B = -1/40, C = -1/10$ and $D = -1/5$

(i.e.) The equation (3.30) becomes,

$$\therefore \overline{y}(s) = \frac{1/8}{(s-1)} - \frac{1/40}{s+3} - \frac{(1/10)s+1/5}{s^2+1}$$

Taking Inverse Laplace transforms on both sides, we get

$$L^{-1}[\overline{y}(s)] = \frac{1}{8}L^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right]$$

$$-\frac{1}{10}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{5}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$\text{(i.e.) } y(t) = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t$$

Example 3.40. *Solve:* $(D^2+4)y = \cos 2t, y(0) = 3, y'(0) = 4$

Solution: Given $(D^2+4)y = \cos 2t, y(0) = 3, y'(0) = 4$

Taking Laplace transforms on both sides, we get

$$L[y''(t)] + 4L[y(t)] = L[\cos 2t]$$

$$\Rightarrow \{s^2 L[y(t)] - s(3) - 4\} + L[y(t)] = \frac{s}{s^2+4}$$

$$\text{(i.e.) } \overline{y}(s)(s^2+4) = \frac{s}{s^2+4} + 3s + 4$$

$$\text{(or) } \overline{y}(s) = \frac{s}{(s^2+4)^2} + \frac{3s+4}{s^2+4}$$

$$L^{-1}[\overline{y}(s)] = L^{-1}\left[\frac{s}{(s^2+4)^2}\right] + 3L^{-1}\left[\frac{s}{s^2+4}\right] + \frac{4}{2}L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$= \frac{t}{2\times 2}\sin 2t + 3\cos 2t + 2\sin 2t$$

$$= \frac{t}{4}\sin 2t + 3\cos 2t + 2\sin 2t$$

Example 3.41. *Solve:* $x'' - 2x' + x = e^{-t}$*, given that* $x(0) = 2, x'(0) = 1$.

Solution: Given $x'' - 2x' + x = e^{-t}$

Taking Laplace transforms on both sides, we get

$$L[x''] - 2L[x'] + L[x] = L[e^{-t}]$$

$$\{s^2\overline{x}(s) - s(2) - 1\} - 2\{s\overline{x}(s) - 2\} + \overline{x}(s) = \frac{1}{s+1}$$

$$\text{(i.e.) } \overline{x}(s)(s^2 - 2s + 1) = \frac{1}{s+1} + 2s - 3$$

$$\overline{x}(s) = \frac{1}{(s+1)(s^2-2s+1)} + \frac{2s-3}{s^2-2s+1} \qquad (3.31)$$

$$\text{Consider } \frac{1}{(s+1)(s^2-2s+1)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2} \qquad (3.32)$$

It can be found that $A = 1/4, B = -1/4, C = 1/2$.

Using the values of A, B, C (from (3.32)) in equation (3.31), we get

$$\overline{x}(s) = \frac{1}{4(s+1)} - \frac{1}{4(s-1)} + \frac{1}{2(s-1)^2} + \frac{2s}{(s-1)^2} - \frac{3}{(s-1)^2}$$

Taking Inverse Laplace transforms on both sides, we get

$$L^{-1}[\overline{x}(s)] = \frac{1}{4}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{4}L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{2s-3+1/2}{(s-1)^2}\right]$$

$$= \frac{1}{4}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{4}L^{-1}\left[\frac{1}{s-1}\right] + 2L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{5}{2}L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$\text{(i.e.) } x(t) = \frac{1}{4}e^{-t} - \frac{1}{4}e^{t} + 2(e^t + te^t) - \frac{5}{2}te^t$$

$$\left\{\because L^{-1}\left[\frac{1}{(s-1)^2}\right] = e^t + te^t\right\}$$

$$x(t) = e^t\left(2 - \frac{1}{4}\right) + \frac{1}{4}e^{-t} + te^t\left(2 - \frac{5}{2}\right)$$

$$= \frac{7}{4}e^t + \frac{1}{4}e^{-t} - \frac{1}{2}te^t = \frac{1}{4}(7e^t + e^{-t} - 2te^t)$$

Example 3.42. *Solve:* $(D^2 - 4D + 8)y = e^{2t}$, *given* $y(0) = 2, y'(0) = -2$.

Solution: Given $(D^2 - 4D + 8)y = e^{2t}$

Taking Laplace transforms on both sides, we get

$$\{s^2\overline{y}(s) - s(2) + 2\} - 4\{s\overline{y}(s) - 2\} + 8\overline{y}(s) = \frac{1}{s-2}$$

$$\overline{y}(s)[s^2 - 4s + 8] = \frac{1}{s-2} + 2s - 10$$

$$\therefore \overline{y}(s) = \frac{1}{(s-2)(s^2-4s+8)} + \frac{2s-10}{s^2-4s+8}$$

By the partial fraction method, the above equation can be written as

$$\overline{y}(s) = \frac{1/4}{s-2} + \frac{\frac{7}{4}s - \frac{19}{2}}{s^2-4s+8}$$

$$= \frac{1}{4}.\frac{1}{s-2} + \frac{\frac{7}{4}(s-2) - 6}{s^2-4s+8}$$

Taking Inverse Laplace transforms on both sides, we get

$$L^{-1}[\overline{y}(s)] = \frac{1}{4}L^{-1}\left[\frac{1}{s-2}\right] + \frac{7}{4}L^{-1}\left[\frac{s-2}{(s-2)^2+4}\right]$$

$$-\frac{6}{2}L^{-1}\left[\frac{2}{(s-2)^2+4}\right]$$

$$\therefore y(t) = \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}\cos 2t - 3e^{2t}\sin 2t$$

$$= \frac{1}{4}e^{2t}[1 + 7\cos 2t - 12\sin 2t]$$

EXERCISE

1. Define Laplace transforms.
2. State the conditions for the existence of Laplace transform of a function.
3. Define unit step function.
4. State first shifting property and second shifting property of Laplace transforms.
5. Find $L[\cosh at \cos at]$ $\left[\text{Ans:} \dfrac{s^3}{s^4+4a^4}\right]$
6. Find $L[2\cos 4t - 3\sin 4t]$ $\left[\text{Ans:} \dfrac{2s-12}{s^2+16}\right]$
7. Find $L[t^2 \cos at]$ $\left[\text{Ans:} \dfrac{2s^3-6a^2s}{(s^2+a^2)^3}\right]$
8. Find $L[5 - 3t - 2e^{-t}]$ $\left[\text{Ans:} \dfrac{3s^2+2s-3}{s^2(s+1)}\right]$
9. Find $L[\sin 2t \sin 3t]$ $\left[\text{Ans:} \dfrac{12s}{(s^2+1)(s^2+25)}\right]$

10. Find $L[e^{-2t}\cos t]$ $\left[\text{Ans:}\dfrac{s+2}{s^2+4s+5}\right]$

11. Evaluate $L[e^{-t}(3\sinh 2t - 5\cosh 2t)]$ $\left[\text{Ans:}\dfrac{1-5s}{s^2+2s+3}\right]$

12. Find $L[f(t)]$ if $f(t) = \begin{cases} (t-1)^2,\ t>1 \\ 0,\ 0<t<1 \end{cases}$

$\left[\text{Ans:}\dfrac{2e^{-s}}{s^3}\right]$

13. Show that $\int_0^\infty te^{-3t}\sin t dt = \dfrac{3}{50}$, by using Laplace transform.

14. Evaluate $\int_0^\infty \dfrac{\cos 6t - \cos 4t}{t} dt$ using Laplace transform

$\left[\text{Ans: } \ln\dfrac{3}{2}\right]$

15. Find the Laplace transform of the following periodic functions:

(i) $f(t) = \begin{cases} t,\ 0<t<\pi \\ 0,\ \pi<t<2\pi \end{cases}$ given that $f(t+2\pi) = f(t)$

(ii) $f(t) = \begin{cases} \sin t,\ 0<t<\pi \\ 0,\ \pi<t<2\pi \end{cases}$

$Ans:(i)\dfrac{1}{1-e^{-2\pi s}}\left[\dfrac{1}{s^2}(1-e^{-\pi s}) - \dfrac{\pi}{s}e^{-\pi s}\right]$

(ii) $\dfrac{1}{(s^2+1)(1-e^{-\pi s})}$

16. Find the Laplace transform of the following functions:

(i) $\dfrac{1-\cos t}{t}$ (ii) $\dfrac{\sin 3t \sin t}{t}$

$\left[\text{Ans:}(i)\dfrac{1}{2}\log\left(\dfrac{s^2+1}{s^2}\right)\ (ii)\dfrac{1}{4}\log\left(\dfrac{s^2+16}{s^2+4}\right)\right]$

17. Using the Laplace transform of the derivatives find the following:

(i) $L[t\cos at]$ (ii) $L[t\sinh at]$ (iii) $L(\sin^2 t)$

$\left[\text{Ans:}(i)\left(\dfrac{s^2-a^2}{(s^2+a^2)^2}\right)\ (ii)\left(\dfrac{2as}{(s^2-a^2)^2}\right)\ (iii)\left(\dfrac{2}{s(s^2+4)}\right)\right]$

18. Evaluate $L\left[\int_0^t te^{-t}dt\right]$ $\left[\text{Ans:}\dfrac{s}{s(s+1)^2}\right]$

19. Evaluate $L\left[\int_0^t \dfrac{1-e^{-t}}{t}dt\right]$ $\left[\text{Ans:}\dfrac{1}{s}\log\left(\dfrac{s-1}{s}\right)\right]$

20. Using convolution theorem, evaluate the following:

$\int_0^t e^{-u}\sin(t-u)du$

21. Verify the initial value theorem for $t+\sin 3t$.

22. Verify the final value theorem for $1 + e^{-t}(\sin t + \cos t)$

23. Find the Inverse Laplace transform of the following:

(i) $\dfrac{s}{s^2-16}$ (ii) $\dfrac{s}{s^2+2}$

[Ans: (i) $\cosh 4t$ (ii) $\cos\sqrt{2}t$]

24. Find (i) $L^{-1}\left[\dfrac{6s-4}{s^2-4s+20}\right]$ (ii) $L^{-1}\left[\dfrac{3s+2}{4s^2+12s+9}\right]$

$$\left[\text{Ans:}(i)2e^{2t}(3\cos 4t+\sin 4t)\ (ii)\frac{3}{4}e^{-3t/2}-\frac{5}{8}te^{-3t/2}\right]$$

25. Find $L^{-1}\left[\dfrac{e^{-3s}}{s^2-2s+5}\right]$

$$\left[\text{Ans: }\begin{cases}\frac{1}{2}e^{(t-3)}\sin 2(t-3), t>3\\ 0, \quad t<3\end{cases}\right]$$

26. Find $L^{-1}\left[\tan^{-1}\dfrac{2}{s^2}\right]$ $\left[\text{Ans:}\dfrac{2\sin t\sinh t}{t}\right]$

27. Find $L^{-1}\left[\dfrac{s+2}{s^2(s+3)}\right]$ $\left[\text{Ans:}\dfrac{2}{3}t+\dfrac{1}{9}-\dfrac{1}{9}e^{-3t}\right]$

28. Using convolution theorem, find the Inverse Laplace transforms of the following:

(i) $\dfrac{1}{(s+1)(s^2+1)}$ (ii) $\dfrac{1}{s(s^2+1)}$ (iii) $\dfrac{1}{s^2(s^2+25)}$

(iv) $\dfrac{s^2}{(s^2+4)^2}$

Ans: (i) $\dfrac{1}{2}(\sin t-\cos t+e^{-t})$ (ii) $1-\cos t$

(iii) $\dfrac{1}{5}(t\cos 4t+\sin 4t-\sin 5t)$ (iv) $\dfrac{1}{2}t\cos 2t+\dfrac{1}{4}\sin 2t$

29. Using partial fractions, find the following:

(i) $L^{-1}\left[\dfrac{s-1}{(s+3)(s^2-2s+2)}\right]$

(ii) $L^{-1}\left[\dfrac{s^2-2s+3}{(s-1)^2(s+1)}\right]$ (iii) $L^{-1}\left[\dfrac{3s+1}{(s-1)(s^2+1)}\right]$

Ans: (i) $\dfrac{1}{5}e^{-t}(4\cos t-3\sin t)-\dfrac{4}{5}e^{-3t}$ (ii) $\dfrac{1}{2}(2t-1)e^t+\dfrac{3}{2}e^{-t}$

(iii) $2e^t-2\cos t+\sin t$

30. Using Laplace transforms solve the following differential equations:

(i) $(D^2+D)y=t^2+2t$, given $y(0)=4, y'(0)=-2$

(ii) $y''-3y'+2y=e^{2t}$, given $y(0)=-3, y'(0)=5$

$$\left[\text{Ans:}(i)\frac{t^3}{3}+4e^{-t}+2\ (ii)te^{2t}-10e^t+7e^{2t}\right]$$

31. Solve $y''+y'-2y=3\cos 3t-11\sin 3t$, given that $y(0)=0$

and $y'(0) = 6$. [Ans: $\sin 3t - e^{-2t} + e^t$]

32. Solve: $x'' - 2x' + x = t^2e^t, x(0) = 2, x'(0) = 3$.

[Ans: $x = (t^4/12 + t + 2)e^t$]

33. Find $L[f(t)]$ if $f(t) = \begin{cases} \sin t, \text{for } 0 < t < \pi \\ 0, \text{ for } t > \pi \end{cases}$

$\left[\text{Ans:} \frac{1}{s-2}(1 - e^{-(s-2)}\right]$

34. Find the Inverse Laplace transform of $\frac{14s + 10}{49s^2 + 28s + 13}$

$\left[\text{Ans:} \frac{2}{7}e^{(2/7)t}\left(\cos\frac{3}{7}t + \sin\frac{3}{7}t\right)\right]$

35. If $L[f(t)] = F(s)$, prove that $L\left[f\left(\frac{t}{a}\right)\right] = aF(as)$.

36. State the initial value theorem in Laplace transforms.

37. State the final value theorem in Laplace transforms.

38. Find $L\left[e^{2t}\left(\cosh 2t + \frac{1}{2}\sinh 2t\right)\right]$

39. Find $L\left[\frac{1+2t}{\sqrt{t}}\right]$ $\left[\text{Ans:}\sqrt{\frac{\pi}{s}}\left(1 + \frac{1}{s}\right)\right]$

40. State the formula for the Laplace transform of a periodic function and hence find the Laplace transform of $f(t) = t$, in $0 < t < 1$.

41. If $L[f(t)] = \frac{1}{(s-2)^2}$, then find $\lim_{t\to 0} f(t)$ [Ans: $-\sin 3$]

42. Find $L^{-1}\left[\frac{s}{s^2 + 4s + 5}\right]$ [Ans: $e^{-2t}(\cos t - 2\sin t)$]

43. State the convolution theorem for Laplace transforms.

44. Prove that $f(t) * g(t) = g(t) * f(t)$

45. Verify initial value theorem for the function $1 + e^{-2t}$

46. Find (i) $L\left[e^t \int_0^t t\cos t dt\right]$ (ii) $L^{-1}\left[s\log\left(\frac{s^2 + a^2}{s^2 + b^2}\right)\right]$

Ans: (i) $\frac{s^2 + 2s}{(s+1)(s^2 + 2s + 2)}$

(ii) $\frac{1}{t}\left[2(a\sin at - b\sin bt) - \frac{2(\cos bt - \cos at)}{t}\right]$

47. Find the Laplace transform of full sine wave rectifier given below:

$f(t) = \begin{cases} E\sin\omega t, 0 < t < \frac{\pi}{\omega} \\ 0, \quad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$ with $f\left(t + \frac{2\pi}{\omega}\right) = f(t)$

$\left[\text{Ans: } L[f(t)] = \frac{\omega}{(1 - e^{-s\pi\omega}(s^2 + \omega^2)}\right]$

48. Prove that Laplace transform of the triangular wave of period 2π defined by $f(t) = \begin{cases} t, & 0 \le t \le \pi \\ 2\pi - t, & \pi \le t \le 2\pi \end{cases}$

is $\dfrac{1}{s^2} \tanh\left(\dfrac{\pi s}{2}\right)$

49. Find the Laplace transform of $f(t) = \begin{cases} t, & 0 < t \le 2 \\ 4 - t, & 2 \le t < 4 \end{cases}$ and satisfy $f(t+4) = f(t)$.

50. Given that $L^{-1}\left[\dfrac{s^2+4}{(s^2-4)^2}\right] = t\cosh^2 t$, find $L^{-1}\left[\dfrac{s^2+1}{(s^2-1)^2}\right]$

[Ans: $t\cosh t$]

4 ANALYTIC FUNCTIONS

Introduction

Let $z = x + iy$ be a complex variable where x and y are real variables. If for every z, there exists one or more values of w, then w can be represented as a function of z.

(i.e.) $w = f(z) = u(x,y) + iv(x,y)$ is a function of the complex variable $z = x + iy$.

Example:

Let $w = z^2$, here for every z there exist a value of w.

Now $z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy = u + iv$

$f(z) = u + iv$ and $u = x^2 - y^2$ and $v = 2xy$

Limit of a function

Let $f(z)$ be a function defined in a set D and z_0 be a limit point of D. Then A is said to be limit of $f(z)$ at z_0, if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - A| < \epsilon$ for all z in D other than z_0 with $|z - z_0| < \delta$. It is denoted by