Chapter 2 solutions

2.1 A set of vectors are linearly dependent if $\sum_i a_i |v\rangle_i = 0$, where a_i are complex numbers not all zero, and $|v\rangle_i$ are the vectors. Otherwise they are linearly independent. For the given set of vectors we can see that $(1,-1) + (1,2) - (2,1) = 0$. Thus they are linearly dependent.

2.2 Here we have a linear operator A such that $A|0\rangle = |1\rangle$, and $A|1\rangle = |0\rangle$. From equation 2.12 of Nielsen & Chuang we see that the matrix representation of a linear operator $A : V \to W$ is given by

$$A|v_j\rangle = \sum_i A_{ij} |w_i\rangle \,, \tag{1}$$

where $\{|v_j\rangle\}$ is a basis for V, $\{|w\rangle_i\}$ is a basis for W, and the A_{ij} are the elements of the matrix representing the operator A. Here we have $W = V$. Then we can write

$$\begin{aligned} A|0\rangle &= |1\rangle = 0\,|0\rangle + 1\,|1\rangle \,, \\ A|1\rangle &= |0\rangle = 1\,|0\rangle + 0\,|1\rangle \,. \end{aligned} \tag{2}$$

Thus we have $A_{00} = 0, A_{01} = 1, A_{10} = 1, A_{11} = 0$, which means that the matrix representation of A with respect to the input basis $\{|0\rangle, |1\rangle\}$, and output basis $\{|0\rangle, |1\rangle\}$ is

$$A \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3}$$

We can have a different matrix representation of the operator A, by choosing a different input and output bases. We will use the basis $\{|+\rangle, |-\rangle\}$ as both the input and output bases. Here

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle). \tag{4}$$

Then the action of A on this basis is given by

$$A|\pm\rangle = \frac{1}{\sqrt{2}}(A|0\rangle \pm A|1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle \pm |0\rangle) = \pm|\pm\rangle \,. \tag{5}$$

Thus A converts $|+\rangle$ to $|+\rangle$, and $|-\rangle$ to $-|-\rangle$. Again using equation 2.12 from Nielsen & Chuang, we can write this as

$$\begin{aligned} A|+\rangle &= |+\rangle = 1\,|+\rangle + 0\,|-\rangle \,, \\ A|-\rangle &= -|-\rangle = 0\,|+\rangle - 1\,|-\rangle \,. \end{aligned} \tag{6}$$

Thus we have $A_{++} = -A_{--} = 1$, and $A_{+-} = A_{-+} = 0$. Thus the matrix representing A with respect to this choice of input and output bases is

$$A \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{7}$$

The operator A is nothing but the operator corresponding to the Pauli matrix X.

2.3 A is a linear operator from V to W, and B is a linear operator from W to X, where V, W, X are vector spaces. Then from equation 2.12 of Nielsen & Chuang we have:

$$A\left|v_i\right\rangle = \sum_j A_{ji}\left|w_j\right\rangle, \tag{8}$$

$$B\left|w_j\right\rangle = \sum_k B_{kj}\left|x_k\right\rangle, \tag{9}$$

where $\{\left|v_i\right\rangle\}$, $\{\left|w_j\right\rangle\}$, and $\{\left|x_k\right\rangle\}$ are the bases for V, W, X, respectively, and A_{ji}, and B_{kj} are the elements of the matrices representing A, B, respectively, in these bases. Then we can write the linear transformation $BA : V \to X$ as

$$\begin{aligned} BA\left|v_j\right\rangle &= B(A\left|v_j\right\rangle) \\ &= B\left(\sum_{ji} A_{ji}\left|w_j\right\rangle\right) \\ &= \sum_j A_{ji}(B\left|w_j\right\rangle) \\ &= \sum_j A_{ji}\sum_k B_{kj}\left|x_k\right\rangle \\ &= \sum_{kj} B_{kj}A_{ji}\left|x_k\right\rangle \\ &= \sum_k (BA)_{ki}\left|x_k\right\rangle, \end{aligned} \tag{10}$$

where in the third line we used the linearity of B, and in the last line we used the definition of matrix multiplication. The complex numbers $(BA)_{ki}$ are the elements of the matrix corresponding to the operator BA. Thus the matrix representation of BA is the product of the matrix representations of B, and A.

2.4 The identity operator is defined as $I : V \to V$, such that $I\left|v\right\rangle = \left|v\right\rangle$, where $\left|v\right\rangle$ is a vector in the vector space V. Let $\{\left|v_i\right\rangle\}$ be a basis for V. Then

$$I\left|v_i\right\rangle = \left|v_i\right\rangle = 0\sum_{j<i}\left|v_j\right\rangle + 1\left|v_i\right\rangle + 0\sum_{k>i}\left|v_k\right\rangle. \tag{11}$$

This is possible because a basis is always linearly independent. Thus $I_{ij} = 1$ if $i = j$ and $I_{ij} = 0$ if $i \neq j$. Thus the matrix corresponding to the identity operator is

$$I \equiv \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}, \tag{12}$$

where the missing elements are all 0.

2.5 A function $(\cdot,\cdot) : V \times V \to \mathbb{C}$ is an inner product if it satisfies the requirements that:

1. $(\cdot,\cdot)$ is linear in the second argument,
2. $(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^*$,
3. $(|v\rangle, |v\rangle) \geq 0$ with equality if and only if $|v\rangle = 0$.

This is the definition of inner product given in Nielsen and Chuang. The claim is the function $(\cdot,\cdot) : \mathbb{C}^n \to \mathbb{C}$ defined by

$$((y_1, \ldots, y_n), (z_1, \ldots, z_n)) \equiv \sum_i y_i^* z_i = [y_1^* \ldots y_n^*] \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \tag{13}$$

is an inner product. To verify this claim we need to check if it satisfies all the conditions of being an inner product.

Linearity in the second argument. Let the second argument be $\sum_i \lambda_i (z_{i1}, \ldots, z_{in})$. Then from the definition we have

$$\begin{aligned} ((y_1, \ldots, y_n), \sum_i \lambda_i (z_{i1}, \ldots, z_{in})) &= \sum_j y_j^* \left(\sum_i \lambda_i z_{ij} \right) \\ &= \sum_{ij} \lambda_i y_j^* z_{ij} \\ &= \sum_i \lambda_i \left(\sum_j y_j^* z_{ij} \right) \\ &= \sum_i \lambda_i ((y_1, \ldots, y_n), (z_{i1}, \ldots, z_{in})). \end{aligned} \tag{14}$$

Conjugate symmetry. We can check this by using the symmetry and linearity of complex number conjugation:

$$\begin{aligned} ((y_1, \ldots, y_n), (z_1, \ldots, z_n)) &= \sum_i y_i^* z_i \\ &= \sum_i (z_i^* y_i)^* \\ &= \left(\sum_i z_i^* y_i \right)^* \\ &= ((z_1, \ldots, z_n), (y_1, \ldots, y_n)). \end{aligned} \tag{15}$$

Positive semi-definite. This is happens because $z^* z = |z|^2$:

$$((y_1, \ldots, y_n), (y_1, \ldots, y_n)) = \sum_i y_i^* y_i = \sum_i |y_i|^2, \tag{16}$$

where $|y_i|^2 \geq 0$ with equality if and only if $y_i = 0$.