1D Advection Equation

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Contents

1 Description

Consider the 1D advection equation,

$$u_t + au_x = 0$$

defined on the domain $[a, b]$

With periodic boundary conditions, that is,

$$u(x+1,t) = u(x,t)$$

2 Discretization

Partition the domain into N points, such that

$$x_i = i\Delta x$$

where, Δx is the partition size defined as,

$$\Delta x = \frac{b-a}{N-1}$$

The time is partitioned so that,

$$t_n = n\Delta t$$

Where Δt is the time interval

Let U_i^n be the approximation to the function u, that is,

$$U_i^n \approx u(x_i, t_n)$$

By the finite difference method the time derivative is approximated using a forward difference scheme.

$$\left.\frac{\partial}{\partial t} u(x_i, t)\right|_{t=t_n} \Rightarrow \frac{U_i^{n+1} - U_i^n}{\Delta t}$$

The space derivative can be approximated in three different schemes:

2.1 Forward Difference

2.1.1 Description

$$\left.\frac{\partial}{\partial x} u(x, t_n)\right|_{x=x_i} \Rightarrow \frac{U_{i+1}^{n+1} - U_i^n}{\Delta x}$$

Thus we get,

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + a\frac{U_{i+1}^{n+1} - U_i^n}{\Delta x} = 0$$

Define,

$$\sigma = \frac{a\Delta t}{\Delta x}$$

Rearranging we get,

$$U_i^{n+1} = (1+\sigma)\, U_i^n - \sigma U_{i+1}^n$$

where,

$$i = 0, 1, \ldots, N-2 \qquad n = 0, 1, 2, \ldots$$

With the periodic boundary condition we define,

$$U_{N-1}^{n+1} = (1+\sigma)\, U_{N-1}^n - \sigma U_1^n$$

2.1.2 Stability

Using Fourier Analysis we get, the solution

$$U_i^n = \beta^n e^{ilx_i}$$

Substituting in the scheme we get,

$$\beta^{n+1} e^{ilx_i} = (1+\sigma)\, \beta^n e^{ilx_i} - \sigma \beta^n e^{ilx_{i+1}}$$

Thus,

$$\beta = 1 + \sigma - \sigma e^{il\Delta x}$$

$$\beta = 1 + \sigma - \sigma\left(cos(l\Delta x) + isin(\Delta x)\right)$$

$$|\beta| = 1 + 2\sigma\,(\sigma+1)\,(1 - cos(l\Delta x))$$

- If $a < 0$

  $$\sigma < 0$$

  For a stable solution we need $|\beta| < 1$

  $$1 + 2\sigma\,(\sigma+1)\,(1 - cos(l\Delta x)) < 1$$

  $$2\sigma\,(\sigma+1)\,(1 - cos(l\Delta x)) < 0$$

  Since $\sigma < 0$ and $1 - cos(l\Delta x) > 0$

  $$\sigma + 1 > 0 \Rightarrow \sigma > -1$$

  Thus if $a < 0$ the forward scheme is stable iff $\sigma > -1$

- If $a > 0$

$$\sigma > 0$$

For a stable solution we need $|\beta| < 1$

$$1 + 2\sigma\left(\sigma + 1\right)\left(1 - cos(l\Delta x)\right) < 1$$

$$2\sigma\left(\sigma + 1\right)\left(1 - cos(l\Delta x)\right) < 0$$

Since $\sigma > 0$ and $1 - cos(l\Delta x) > 0$

$$\sigma + 1 < 0 \Rightarrow \sigma < -1$$

This is not possible

Thus if $a > 0$ the forward scheme is unconditionally unstable.

2.2 Backward Difference

2.2.1 Description

$$\frac{\partial}{\partial x}u(x, t_n)\bigg|_{x=x_i} \Rightarrow \frac{U_i^{n+1} - U_{i-1}^n}{\Delta x}$$

Thus we get,

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + a\frac{U_i^{n+1} - U_{i-1}^n}{\Delta x} = 0$$

Define,

$$\sigma = \frac{a\Delta t}{\Delta x}$$

Rearranging we get,

$$U_i^{n+1} = \left(1 - \sigma\right)U_i^n + \sigma U_{i-1}^n$$

where,

$$i = 1, 2, \ldots, N - 1 \qquad n = 0, 1, 2, \ldots$$

With the periodic boundary condition we define,

$$U_0^{n+1} = \left(1 - \sigma\right)U_0^n + \sigma U_{N-2}^n$$

2.2.2 Stability

Using Fourier Analysis we get, the solution

$$U_i^n = \beta^n e^{ilx_i}$$

Substituting in the scheme we get,

$$\beta^{n+1} e^{ilx_i} = (1-\sigma)\,\beta^n e^{ilx_i} + \sigma \beta^n e^{ilx_{i-1}}$$

Thus,

$$\beta = 1 - \sigma + \sigma e^{-il\Delta x}$$

$$\beta = 1 - \sigma + \sigma cos(l\Delta x) - i\sigma sin(l\Delta x)$$

$$|\beta| = 1 + 2\sigma\,(\sigma - 1)\,(1 - cos(l\Delta x))$$

- If $a < 0$

  $$\sigma < 0$$

  For a stable solution we need $|\beta| < 1$

  $$1 + 2\sigma\,(\sigma - 1)\,(1 - cos(l\Delta x)) < 1$$

  $$2\sigma\,(\sigma - 1)\,(1 - cos(l\Delta x)) < 0$$

  Since $\sigma < 0$ and $1 - cos(l\Delta x) > 0$

  $$\sigma - 1 > 0 \Rightarrow \sigma > 1$$

  This is not possible

  Thus if $a < 0$ the backward scheme is unconditionally unstable.

- If $a > 0$

  $$\sigma > 0$$

  For a stable solution we need $|\beta| < 1$

  $$1 + 2\sigma\,(\sigma - 1)\,(1 - cos(l\Delta x)) < 1$$

  $$2\sigma\,(\sigma - 1)\,(1 - cos(l\Delta x)) < 0$$

  Since $\sigma > 0$ and $1 - cos(l\Delta x) > 0$

  $$\sigma - 1 < 0 \Rightarrow \sigma < 1$$

  Thus if $a > 0$ the backward scheme is stable iff $\sigma < 1$.

2.3 Central Difference

2.3.1 Description

$$\frac{\partial}{\partial x}u(x,t_n)\bigg|_{x=x_i} \Rightarrow \frac{U_{i+1}^n - U_{i-1}^n}{\Delta x}$$

Thus we get,

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + a\frac{U_{i+1}^n - U_{i-1}^n}{\Delta x} = 0$$

Define,

$$\sigma = \frac{a\Delta t}{\Delta x}$$

Rearranging we get,

$$U_i^{n+1} = U_i^n - \frac{\sigma}{2}U_{i+1}^n + \frac{\sigma}{2}U_{i-1}^n$$

where,

$$i = 1, 2, \dots, N-2 \qquad n = 0, 1, 2, \dots$$

With the periodic boundary condition we define,

$$U_0^{n+1} = U_0^n - \frac{\sigma}{2}U_1^n + \frac{\sigma}{2}U_{N-2}^n$$

$$U_{N-1}^{n+1} = U_{N-1}^n - \frac{\sigma}{2}U_1^n + \frac{\sigma}{2}U_{N-2}^n$$

2.3.2 Stability

Using Fourier Analysis we get, the solution

$$U_i^n = \beta^n e^{ilx_i}$$

Substituting in the scheme we get,

$$\beta^{n+1}e^{ilx_i} = \beta^n e^{ilx_i} - \frac{\sigma}{2}\beta^n e^{ilx_{i+1}} + \frac{\sigma}{2}\beta^n e^{ilx_{i-1}}$$

Thus,

$$\beta = 1 - \sigma/2e^{il\Delta x} + \sigma/2e^{-il\Delta x}$$
$$|\beta| = 1 + \sigma^2\sin^2(l\Delta x)$$

Clearly, $|\beta| > 1$. Hence, the central difference scheme is unconditionally unstable.

But β is close to one if $\sigma < 1$, meaning the solution slowly grows in amplitude for $\sigma < 1$

2.4 Lax Wendroff Scheme

2.4.1 Description

Consider the Taylor Series Expansion of $u(x_i, t_{n+1})$,

$$u(x_i, t_{n+1}) = u(x_i, t_n) + \Delta t \cdot u_t(x_i, t_n) + \frac{\Delta t^2}{2} \cdot u_{tt}(x_i, t_n) + O(\Delta t^2)$$

From the advection equation we have

$$u_t = -a \cdot u_x$$

Thus,

$$u_{tt} = -a \cdot u_{xt}$$

$$u_{tt} = a^2 \cdot u_{xx}$$

Substituting we have,

$$u(x_i, t_{n+1}) = u(x_i, t_n) - a\Delta t \cdot u_x(x_i, t_n) + a^2 \frac{\Delta t^2}{2} \cdot u_{xx}(x_i, t_n) + O(\Delta t^2)$$

Replacing with the approximation and using central difference for the derivatives the scheme becomes,

$$U_i^{n+1} = U_i^n - \frac{a\Delta t}{2\Delta x}(U_{i+1}^n - U_{i-1}^n) + \frac{a^2 \Delta t^2}{2\Delta x^2}(U_{i-1}^n - 2U_i^n + U_{i+1}^n)$$

Now define,

$$\sigma = \frac{a\Delta t}{\Delta x}$$

Thus,

$$U_i^{n+1} = U_i^n - \frac{\sigma}{2}(U_{i+1}^n - U_{i-1}^n) + \frac{\sigma^2}{2}(U_{i-1}^n - 2U_i^n + U_{i+1}^n)$$

$$U_i^{n+1} = \frac{\sigma(\sigma+1)}{2} U_{i-1}^n + \left(1 - \sigma^2\right) U_i^n + \frac{\sigma(\sigma-1)}{2} U_{i+1}^n$$

where,

$$i = 1, 2, \ldots, N-2 \qquad n = 0, 1, 2, \ldots$$

With the periodic boundary condition we define,

$$U_0^{n+1} = \frac{\sigma(\sigma+1)}{2} U_{N-2}^n + \left(1 - \sigma^2\right) U_0^n + \frac{\sigma(\sigma-1)}{2} U_1^n$$

$$U_{N-1}^{n+1} = \frac{\sigma(\sigma+1)}{2} U_{N-2}^n + \left(1 - \sigma^2\right) U_{N-1}^n + \frac{\sigma(\sigma-1)}{2} U_1^n$$

2.4.2 Stability

Using Fourier Analysis we get, the solution

$$U_i^n = \beta^n e^{ilx_i}$$

Substituting in the scheme we get,

$$\beta^{n+1} e^{il\Delta x} = \frac{\sigma(\sigma+1)}{2}\beta^n e^{ilx_{i-1}} + \left(1-\sigma^2\right)\beta^n e^{ilx_i} + \frac{\sigma(\sigma-1)}{2}\beta^n e^{ilx_{i+1}}$$

Thus,

$$\beta = \frac{\sigma(\sigma+1)}{2} e^{-il\Delta x} + \left(1-\sigma^2\right) + \frac{\sigma(\sigma-1)}{2} e^{il\Delta x}$$

$$\beta = \frac{\sigma}{2}\left(\sigma(e^{il\Delta x} + e^{-il\Delta x}) - (e^{il\Delta x} - e^{-il\Delta x})\right) + 1 - \sigma^2$$

$$\beta = \sigma^2 cos(l\Delta x) - \sigma sin(l\Delta x) + 1 - \sigma^2$$

$$|\beta| = \sigma^2 cos(l\Delta x) - \sigma sin(l\Delta x) + 1 - \sigma^2$$

For $|\beta| < 1$,

$$|\sigma| < 1$$

Thus the scheme is stable for $|\sigma| < 1$.

3 Implementation

```
#include <iostream>
#include <vector>
```

3.1 Forward Difference

```
// This method returns the current state of the solution after n time steps
vector<double> nsol(const int n, const double sigma, const vector<double> &u)
{
    int size = u.size();
    vector<double> u_prev(size,0.0);
    u_prev = u;
    vector<double> u_next(size,0.0);
    for(unsigned int k=0;k<n;k++)
    {
        u_next[size-1] = (1+sigma)*u_prev[size-1] - sigma*u_prev[1];
        for(unsigned int i=0;i<size-1;i++)
        {
            u_next[i] = (1+sigma)*u_prev[i] - sigma*u_prev[i+1];
        }
        u_prev = u_next;
    }
    return u_next;
}
```

3.2 Backward Difference

```
// This method returns the current state of the solution after n time steps
vector<double> nsol(const int n, const double sigma, const vector<double> &u)
{
    int size = u.size();
    vector<double> u_prev(size,0.0);
    u_prev = u;
    vector<double> u_next(size,0.0);
    for(unsigned int k=0;k<n;k++)
    {
        u_next[0] = (1-sigma)*u_prev[0] + sigma*u_prev[size-2];
        for(unsigned int i=1;i<size;i++)
        {
            u_next[i] = (1-sigma)*u_prev[i] + sigma*u_prev[i-1];
        }
        u_prev = u_next;
    }
    return u_next;
}
```

3.3 Central Difference

```
// This method returns the current state of the solution after n time steps
vector<double> nsol(const int n, const double sigma, const vector<double> &u)
{
    int size = u.size();
    vector<double> u_prev(size,0.0);
    u_prev = u;
    vector<double> u_next(size,0.0);
    for(unsigned int k=0;k<n;k++)
    {
        u_next[size-1] = u_prev[size-1] - (sigma/2)*u_prev[1] + (sigma/2)*u_prev[size-
        u_next[0] = u_prev[0] - (sigma/2)*u_prev[1] + (sigma/2)*u_prev[size-2];
        for(unsigned int i=1;i<size-1;i++)
        {
            u_next[i] = u_prev[i] - (sigma/2)*u_prev[i+1] + (sigma/2)*u_prev[i-1];
        }
        u_prev = u_next;
    }
    return u_next;
}
```

3.4 Lax Wendroff Scheme