1D Advection Equation

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1 Description

Consider the 1D advection equation,

$$u_t + au_x = 0$$

defined on the domain [a, b]

With periodic boundary conditions, that is,

$$u(x+1,t) = u(x,t)$$

2 Discretization

Partition the domain into N points, such that

$$x_i = i\Delta x$$

where, Δx is the partition size defined as,

$$\Delta x = \frac{b-a}{N-1}$$

The time is partitioned so that,

$$t_n = n\Delta t$$

Where Δt is the time interval

Let U_i^n be the approximation to the function u, that is,

$$U_i^n \approx u(x_i, t_n)$$

By the finite difference method the time derivative is approximated using a forward difference scheme.

$$\left. \frac{\partial}{\partial t} u(x_i, t) \right|_{t=t_n} \Rightarrow \frac{U_i^{n+1} - U_i^n}{\Delta t}$$

The space derivative can be approximated in three different schemes:

2.1 Forward Difference

2.1.1 Description

$$\left. \frac{\partial}{\partial x} u(x, t_n) \right|_{x=x_i} \Rightarrow \frac{U_{i+1}^{n+1} - U_i^n}{\Delta x}$$

Thus we get,

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + a \frac{U_{i+1}^{n+1} - U_i^n}{\Delta x} = 0$$

Define,

$$\sigma = \frac{a\Delta t}{\Delta x}$$

Rearranging we get,

$$U_i^{n+1} = (1+\sigma) U_i^n - \sigma U_{i+1}^n$$

where,

$$i = 0, 1, \dots, N - 2$$
 $n = 0, 1, 2, \dots$

With the periodic boundary condition we define,

$$U_{N-1}^{n+1} = (1+\sigma) U_{N-1}^{n} - \sigma U_{1}^{n}$$

2.1.2 Stability

Using Fourier Analysis we get, the solution

$$U_i^n = \beta^n e^{ilx_i}$$

Substituting in the scheme we get,

$$\beta^{n+1}e^{ilx_i} = (1+\sigma)\beta^n e^{ilx_i} - \sigma\beta^n e^{ilx_{i+1}}$$

Thus,

$$\beta = 1 + \sigma - \sigma e^{il\Delta x}$$

$$\beta = 1 + \sigma - \sigma \left(\cos(l\Delta x) + i\sin(\Delta x)\right)$$

$$|\beta| = 1 + 2\sigma \left(\sigma + 1\right) \left(1 - \cos(l\Delta x)\right)$$

• If a < 0

$$\sigma < 0$$

For a stable solution we need $|\beta| < 1$

$$1 + 2\sigma \left(\sigma + 1\right) \left(1 - \cos(l\Delta x)\right) < 1$$

$$2\sigma \left(\sigma + 1\right)\left(1 - \cos(l\Delta x)\right) < 0$$

Since $\sigma < 0$ and $1 - cos(l\Delta x) > 0$

$$\sigma + 1 > 0 \Rightarrow \sigma > -1$$

Thus if a < 0 the forward scheme is stable iff $\sigma > -1$

• If a > 0

$$\sigma > 0$$

For a stable solution we need $|\beta| < 1$

$$1 + 2\sigma (\sigma + 1) (1 - \cos(l\Delta x)) < 1$$

$$2\sigma (\sigma + 1) (1 - \cos(l\Delta x)) < 0$$

Since $\sigma > 0$ and $1 - cos(l\Delta x) > 0$

$$\sigma+1<0\Rightarrow\sigma<-1$$

This is not possible

Thus if a > 0 the forward scheme is unconditionally unstable.

2.2 Backward Difference

2.2.1 Description

$$\left. \frac{\partial}{\partial x} u(x, t_n) \right|_{x=x_i} \Rightarrow \frac{U_i^{n+1} - U_{i-1}^n}{\Delta x}$$

Thus we get,

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + a \frac{U_i^{n+1} - U_{i-1}^n}{\Delta x} = 0$$

Define,

$$\sigma = \frac{a\Delta t}{\Delta x}$$

Rearranging we get,

$$U_i^{n+1} = (1 - \sigma) U_i^n + \sigma U_{i-1}^n$$

where,

$$i = 1, 2, \dots, N - 1$$
 $n = 0, 1, 2, \dots$

With the periodic boundary condition we define,

$$U_0^{n+1} = (1-\sigma) U_0^n + \sigma U_{N-2}^n$$

2.2.2 Stability

Using Fourier Analysis we get, the solution

$$U_i^n = \beta^n e^{ilx_i}$$

Substituting in the scheme we get,

$$\beta^{n+1}e^{ilx_i} = (1 - \sigma)\beta^n e^{ilx_i} + \sigma\beta^n e^{ilx_{i-1}}$$

Thus,

$$\beta = 1 - \sigma + \sigma e^{-il\Delta x}$$

$$\beta = 1 - \sigma + \sigma \cos(l\Delta x) - i\sigma \sin(l\Delta x)$$

$$|\beta| = 1 + 2\sigma (\sigma - 1) (1 - \cos(l\Delta x))$$

• If a < 0

For a stable solution we need $|\beta| < 1$

$$1 + 2\sigma \left(\sigma - 1\right)\left(1 - \cos(l\Delta x)\right) < 1$$

$$2\sigma (\sigma - 1) (1 - \cos(l\Delta x)) < 0$$

Since $\sigma < 0$ and $1 - cos(l\Delta x) > 0$

$$\sigma - 1 > 0 \Rightarrow \sigma > 1$$

This is not possible

Thus if a < 0 the backward scheme is unconditionally unstable.

• If a > 0

$$\sigma > 0$$

For a stable solution we need $|\beta| < 1$

$$1 + 2\sigma \left(\sigma - 1\right)\left(1 - \cos(l\Delta x)\right) < 1$$

$$2\sigma \left(\sigma - 1\right)\left(1 - \cos(l\Delta x)\right) < 0$$

Since $\sigma > 0$ and $1 - cos(l\Delta x) > 0$

$$\sigma - 1 < 0 \Rightarrow \sigma < 1$$

Thus if a > 0 the backward scheme is stable iff $\sigma < 1$.

2.3 Central Difference

2.3.1 Description

$$\left. \frac{\partial}{\partial x} u(x, t_n) \right|_{x=x} \Rightarrow \frac{U_{i+1}^{n+1} - U_{i-1}^n}{\Delta x}$$

Thus we get,

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + a \frac{U_{i+1}^{n+1} - U_{i-1}^n}{\Delta x} = 0$$

Define,

$$\sigma = \frac{a\Delta t}{\Delta x}$$

Rearranging we get,

$$U_i^{n+1} = U_i^n - \frac{\sigma}{2}U_{i+1}^n + \frac{\sigma}{2}U_{i-1}^n$$

where,

$$i = 1, 2, \dots, N - 2$$
 $n = 0, 1, 2, \dots$

With the periodic boundary condition we define,

$$U_0^{n+1} = U_0^n - \frac{\sigma}{2}U_1^n + \frac{\sigma}{2}U_{N-2}^n$$

$$U_{N-1}^{n+1} = U_{N-1}^n - \frac{\sigma}{2}U_1^n + \frac{\sigma}{2}U_{N-2}^n$$

2.3.2 Stability

Using Fourier Analysis we get, the solution

$$U_i^n = \beta^n e^{ilx_i}$$

Substituting in the scheme we get,

$$\beta^{n+1}e^{ilx_i} = \beta^n e^{ilx_i} - \frac{\sigma}{2}\beta^n e^{ilx_{i+1}} + \frac{\sigma}{2}\beta^n e^{ilx_{i-1}}$$

Thus,

$$\beta = 1 - \sigma/2e^{il\Delta x} + \sigma/2e^{-il\Delta x}$$
$$|\beta| = 1 + \sigma^2 \sin^2(l\Delta x)$$

Clearly, $|\beta| > 1$. Hence, the central difference scheme is unconditionally unstable.

But β is close to one if $\sigma < 1$, meaning the solution slowly grows in amplitude for $\sigma < 1$

3 Implementation

#include <iostream>
#include <vector>

3.1 Forward Difference

// This method returns the current state of the solution after n time steps
vector<double> nsol(const int n, const double sigma, const vector<double> &u)
{

```
int size = u.size();
vector<double> u_prev(size,0.0);
u_prev = u;
```

```
vector<double> u_next(size,0.0);
    for(unsigned int k=0;k<n;k++)</pre>
    {
        u_next[size-1] = (1+sigma)*u_prev[size-1] - sigma*u_prev[1];
        for(unsigned int i=0;i<size-1;i++)</pre>
            u_next[i] = (1+sigma)*u_prev[i] - sigma*u_prev[i+1];
        u_prev = u_next;
    return u_next;
}
3.2
     Backward Difference
// This method returns the current state of the solution after n time steps
vector<double> nsol(const int n, const double sigma, const vector<double> &u)
{
    int size = u.size();
    vector<double> u_prev(size,0.0);
    u_prev = u;
    vector<double> u_next(size,0.0);
    for(unsigned int k=0;k<n;k++)</pre>
    {
        u_next[0] = (1-sigma)*u_prev[0] + sigma*u_prev[size-2];
        for(unsigned int i=1;i<size;i++)</pre>
            u_next[i] = (1-sigma)*u_prev[i] + sigma*u_prev[i-1];
        u_prev = u_next;
    }
    return u_next;
}
     Central Difference
3.3
// This method returns the current state of the solution after n time steps
vector<double> nsol(const int n, const double sigma, const vector<double> &u)
{
    int size = u.size();
```