## 3D Rotation

- 3-D Rotation around coordinate axes counter-clockwise

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}, R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

### Rodrigues Formula

For a vector $\mathbf{x}$ and a unit vector $\mathbf{k}$, the rotation of $\mathbf{x}$ by an angle $\theta$ about $\mathbf{k}$ is given by

$$\mathbf{x_{rot}} = \mathbf{x}\cos(\theta) + \sin(\theta)(\mathbf{k} \times \mathbf{x}) + (\mathbf{1} - \cos\theta)(\mathbf{k} \cdot \mathbf{x})\mathbf{k}$$

- Axis-Angle to $\mathbf{R}$: $\mathbf{R} = \mathbf{I} + (\sin\theta)\mathbf{K} + (1 - \cos\theta)K^2$, where

$$K = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

- $\mathbf{R}$ to Axis-Angle: $\theta = \cos^{-1}(\frac{Tr(R)-1}{2})$, $\mathbf{k} = \frac{1}{2\sin\theta}\begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$

## Camera parameters

- Homogeneous coordinates: $\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix}^T \rightarrow \mathbf{x}' = \begin{bmatrix} x \cdot z & y \cdot z & z \end{bmatrix}^T, z \neq 0$
- Perspective Projection Transformation: $\mathbf{x}' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

### Intrinsic Properties

- Principal Axis: Line from the camera centre perpendicular to the image plane.
- Normalized Camera Coordinate: Camera centre at the origin ($C$), $x$ and $y$ axes are aligned with the image axes and image plane ($P_z = f$); units in m/cm/ft etc.
- Principal point: point where the principal axis intersects the image plane (z=f)
- Pixel coordinate frame: Origin (0,0) is in the corner of an image; units are in pixels.
- Skew: Not necessary that image plane axis are perpendicular

$$K = \begin{bmatrix} 1 & s & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} m_x & & \\ & m_y & \\ & & 1 \end{bmatrix} \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} = \begin{bmatrix} \alpha_x & s & \beta_x \\ & \alpha_y & \beta_y \\ & & 1 \end{bmatrix}$$

$s$ is the skew parameter. Skew parameter is non-zero, only if x and y axes are non-orthogonal, i.e., pixels are not rectangular. $m_x$ and $m_y$ are the scaling factors in x and y directions. $f$ is the focal length, and $p_x$ and $p_y$ are the principal point coordinates.

### Steps to convert world view to pixels

1. World to Camera Coordinate Transformation: $\mathbf{X_{cam}} = \mathbf{R}(\mathbf{X} - \mathbf{C})$ [Extrinsic parameters]
2. Perspective Projection: camera 3D to to image plane 2D
3. Scaling and shifting: image plane 2D to pixel 2D [Intrinsic parameters]

$$\mathbf{X}' = \mathbf{K}[\mathbf{R}(\mathbf{X} - \mathbf{C})] = \mathbf{K}[\mathbf{R}|\mathbf{t}]\mathbf{X}$$

Here, $\mathbf{K}$ is the intrinsic matrix, $\mathbf{R}$ is the rotation matrix, $\mathbf{t}$ is the translation vector, and $\mathbf{X}$ is the 3D point in the world coordinate system.

### Degrees of Freedom

- Intrinsic: Focal length (2-dof) $f_x$ and $f_y$, Principal point (2-dof) $p_x, p_y$, Skew factor (1-dof) $s$
- Extrinsic: Rotation (3-dof): $R$, Translation (3-dof): $t$
- Total degrees of freedom: (3 + 3) + (2 + 2 + 1) = 11

# Projective Geometry

## Basics

- Planes passing through origin and $\perp$ to vector $\mathbf{n}$: $\mathbf{n} \cdot \mathbf{x} = 0$ i.e. $ax_1 + bx_2 + cx_3 = 0$
- Vector $\parallel$ to intersection of 2 planes $(a, b, c)$ and $(a', b', c')$: $\mathbf{n}'' = \mathbf{n} \times \mathbf{n}'$
- Planes passing through two points $\mathbf{x}$ and $\mathbf{x}'$: $\mathbf{n} = \mathbf{x} \times \mathbf{x}'$
- To each point $m$ of the plane $P$ we can associate a single ray $\mathbf{x} = (x_1, x_2, x_3)$
- To each line $l$ of the plane $P$ we can associate a single point $\mathbf{l} = (l_1, l_2, l_3)$

We can go in reverse as well - If we have a line $ax + by + c = 0$, then the point $\mathbf{l} = (a, b, c)$ is the corresponding plane in 3D, and if $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in l$, then $\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}^T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$ - Point of intersection of 2 lines $x = l_1 \times l_2$ - Line at infinity: $l_\infty = (0, 0, 1)^T$, Point at infinity is always of the form: $\mathbf{x}_\infty = (a, b, 0)^T$ - For a line $l = \begin{bmatrix} a & b & c \end{bmatrix}^T$, the point at infinity is $\mathbf{x}_\infty = (b, -a, 0)^T$