

AI1103 - Challenging Problem 17

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Download latex-tikz codes from

<https://github.com/Anirudh-Srinivasan-CS20/AI1103/blob/main/Challenging-Problem-17//Challenging-Problem-17.tex>

QUESTION

Let X and Y be independent and identically distributed random variables such that $\Pr(X = 0) = \Pr(X = 1) = \frac{1}{2}$. Let $Z = X + Y$ and $W = |X - Y|$. Then which statement is not correct?

- A) X and W are independent.
- B) Y and W are independent.
- C) Z and W are uncorrelated.
- D) Z and W are independent.

SOLUTION

Since, X and Y are i.i.d random variables

$$\Pr(X = 0) = \Pr(Y = 0) = \frac{1}{2} \quad (0.0.1)$$

$$\Pr(X = 1) = \Pr(Y = 1) = \frac{1}{2} \quad (0.0.2)$$

X	Y	Z	W	ZW	Pr(X, Y)
0	0	0	0	0	$\frac{1}{4}$
0	1	1	1	1	$\frac{1}{4}$
1	0	1	1	1	$\frac{1}{4}$
1	1	2	0	0	$\frac{1}{4}$

Table 4: This table shows probability associated with each value that the random variable Z, W and ZW can take when $X = x, Y = y, Z = x+y, W = |x - y|$ and $ZW = (x+y)|x - y|$.

\Rightarrow For Z , we have:

$$\Pr(Z = 0) = \frac{1}{4} \quad (0.0.3)$$

$$\Pr(Z = 1) = \frac{1}{2} \quad (0.0.4)$$

$$\Pr(Z = 2) = \frac{1}{4} \quad (0.0.5)$$

$$E(Z) = \sum_{i=0}^2 \Pr(Z = i) \times z_i \quad (0.0.6)$$

$$= \frac{1}{4} \times 0 + \frac{1}{2} \times 1 + \frac{1}{4} \times 2 \quad (0.0.7)$$

$$= 1 \quad (0.0.8)$$

\Rightarrow For W , we have:

$$\Pr(W = 0) = \frac{1}{2} \quad (0.0.9)$$

$$\Pr(W = 1) = \frac{1}{2} \quad (0.0.10)$$

$$E(W) = \sum_{i=0}^1 \Pr(W = i) \times w_i \quad (0.0.11)$$

$$= \frac{1}{2} \times 0 + \frac{1}{2} \times 1 \quad (0.0.12)$$

$$= \frac{1}{2} \quad (0.0.13)$$

\Rightarrow For ZW , we have:

$$\Pr(ZW = 0) = \frac{1}{2} \quad (0.0.14)$$

$$\Pr(ZW = 1) = \frac{1}{2} \quad (0.0.15)$$

$$E(ZW) = \sum_{i=0}^1 \Pr(WZ = i) \times (wz)_i \quad (0.0.16)$$

$$= \frac{1}{2} \times 0 + \frac{1}{2} \times 1 \quad (0.0.17)$$

$$= \frac{1}{2} \quad (0.0.18)$$

$$E(ZW) = \frac{1}{2} \quad (0.0.19)$$

$$E(Z) \times E(W) = 1 \times \frac{1}{2} \quad (0.0.20)$$

$$E(ZW) = E(Z) \times E(W) \quad (0.0.21)$$

\Rightarrow Z and W are uncorrelated.

From the above table:

$$\Pr(X = 0|W = 0) = \frac{1}{2} = \Pr(X = 0) \quad (0.0.22)$$

$$\Pr(X = 0|W = 1) = \frac{1}{2} = \Pr(X = 0) \quad (0.0.23)$$

$$\Pr(X = 1|W = 0) = \frac{1}{2} = \Pr(X = 1) \quad (0.0.24)$$

$$\Pr(X = 1|W = 1) = \frac{1}{2} = \Pr(X = 1) \quad (0.0.25)$$

\implies X and W are independent.

As the distribution of X and Y are identical, similarly we get:

\implies Y and W are independent.

$$\Pr(Z = 0|W = 0) = \frac{1}{2} \neq \Pr(Z = 0) \quad (0.0.26)$$

\implies Z and W are dependent.

Answer: Option (D)