AI1103 - Assignment 4

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Download latex-tikz codes from

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https://github.com/Anirudh−Srinivasan−CS20/
    AI1103/blob/main/Assignment−4/Assignment
    −4.tex
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Question

Let $X_1, X_2, \dots$ be independent random variables with X_n being uniformly distributed between -n and 3n, n=1,2,.... Let $S_N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \frac{X_n}{n}$ for N=1,2,... and let F_N be the distribution function of S_N. Also, let ϕ denote the distribution function of a standard normal variable. Which of the following is/are true?

A) $\lim_{N\to\infty} F_N(0) \le \phi(0)$
B) $\lim_{N\to\infty} F_N(0) \ge \phi(0)$
C) $\lim_{N\to\infty} F_N(1) \le \phi(1)$
D) $\lim_{N\to\infty} F_N(1) \ge \phi(1)$

Solution

Given, $X_1, X_2, \dots$ are independent random variables with $X_i \sim \mathcal{U}(-i, 3i)$. Let us define:

$$Y_i = \frac{X_i}{i} \ \forall \text{ i} \tag{0.0.1}$$

$$\implies Y_i \sim \mathcal{U}(-1, 3) \tag{0.0.2}$$

Now, $Y_1, Y_2, \dots$ are i.i.d (independent and identically distributed) uniform random variables.

Definition 1. *The probability density function of a uniformly distributed continuous random variable in the interval [a,b] is*

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & \text{for } a \le x \le b \\ 0, & \text{otherwise} \end{cases} \tag{0.0.3}$$

Lemma 0.1. *For a uniformly distributed continuous random variable in the interval [a,b], the mean and variance are given by:*

$$\mu = \frac{a+b}{2} \tag{0.0.4}$$

$$\sigma^2 = \frac{(b-a)^2}{12} \tag{0.0.5}$$

Proof.

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x)\, dx \tag{0.0.6}$$

$$\mu = \int_{-\infty}^{a} 0x\, dx + \int_{a}^{b} \frac{1}{b-a} x\, dx + \int_{b}^{+\infty} 0x\, dx \tag{0.0.7}$$

$$= 0 + \frac{1}{b-a}\left(\frac{x^2}{2}\right)\Bigg|_a^b + 0 \tag{0.0.8}$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2} \tag{0.0.9}$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx \tag{0.0.10}$$

$$E(X^2) = \int_{-\infty}^{a} 0x^2\, dx + \int_{a}^{b} \frac{1}{b-a} x^2\, dx + \int_{b}^{+\infty} 0x^2\, dx \tag{0.0.11}$$

$$= 0 + \frac{1}{b-a}\left(\frac{x^3}{3}\right)\Bigg|_a^b + 0 \tag{0.0.12}$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{a^2+ab+b^2}{3} \tag{0.0.13}$$

$$\sigma^2 = E(X^2) - [E(X)]^2 \tag{0.0.14}$$

$$= \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 \tag{0.0.15}$$

$$= \frac{(b-a)^2}{12} \tag{0.0.16}$$

□

Using (0.1), we get:

$$\mu = \frac{-1+3}{2} = 1 \tag{0.0.17}$$

$$\sigma^2 = \frac{[3-(-1)]^2}{12} = \frac{16}{12} = \frac{4}{3} \tag{0.0.18}$$

And, we have:

$$S_N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \frac{X_n}{n} \tag{0.0.19}$$

$$= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} Y_n \tag{0.0.20}$$

Theorem 0.1 (Central Limit theorem)**.** *Let $X_1, X_2, \ldots, X_n$ be i.i.d random variables with finite μ and σ^2. If Z_n is defined as*

$$Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \tag{0.0.21}$$

Then,

$$\lim_{n\to\infty} Z_n \sim \mathcal{N}(\mu, \sigma^2) \tag{0.0.22}$$

By (0.1), we can conclude:

$$\lim_{N\to\infty} S_N \sim \mathcal{N}(\mu, \sigma^2) \tag{0.0.23}$$

$$\implies \lim_{N\to\infty} F_N(x) = \phi\left(\frac{x-\mu}{\sigma}\right) \tag{0.0.24}$$

$$= \phi\left(\frac{\sqrt{3}(x-1)}{2}\right) \tag{0.0.25}$$

Substituting x=1, we get:

$$\lim_{N\to\infty} F_N(1) = \phi\left(\frac{\sqrt{3}(1-1)}{2}\right) \tag{0.0.26}$$

$$= \phi(0) \tag{0.0.27}$$

The distribution function F_N is non-decreasing. So,

$$\lim_{N\to\infty} F_N(0) \leq \lim_{N\to\infty} F_N(1) = \phi(0) \tag{0.0.28}$$

$$\lim_{N\to\infty} F_N(0) \leq \phi(0) \tag{0.0.29}$$

The distribution function ϕ is non-decreasing. So,

$$\lim_{N\to\infty} F_N(1) = \phi(0) \leq \phi(1) \tag{0.0.30}$$

$$\lim_{N\to\infty} F_N(1) \leq \phi(1) \tag{0.0.31}$$

Answer: Option (A) and (C)