Challenging Problem - 3

Anirudh Srinivasan

IIT Hyderabad

July 2, 2021

Problem Definition

Question

Let X_1, X_2, X_3, X_4, X_5 be i.i.d random variables having a continuous distribution function. Given: $X_1 = max(X_1, X_2, X_3, X_4, X_5)$. Then, the value of p $= \Pr(X_1 > X_2 > X_3 > X_4 > X_5)$ is:

1. $\frac{1}{4}$
2. $\frac{1}{5}$
3. $\frac{1}{4!}$
4. $\frac{1}{5!}$

Key Concepts and Definitions

Lemma

Every i.i.d sequence of random variables is exchangeable. Any value of a finite sequence is as likely as any permutation of those values. The joint probability distribution is invariant under the symmetric group.

Proof

$$f_{X_1,X_2,X_3,\ldots,X_n}(x) = f_{X_1}(x) \times f_{X_2}(x) \times \ldots f_{X_n}(x) \tag{1}$$

As X_is are i.i.d random variables, their joint probability density function is the product of their marginal probability density functions and as multiplication is commutative, it is exchangeable.

Key Concepts and Definitions Contd.

Symmetric group

It is the group of permutation on a set with n elements and has $n!$ elements. Order of a symmetric group represents the number of elements in it.

Lemma

If n elements of a set are random, then probability of each element 'E_i' of the symmetric group 'S' is $\frac{1}{n!}$.

Key Concepts and Definitions Contd.

Proof

As the n values are completely random, there will be no bias for a particular arrangement and hence all the elements of the symmetric group are equally likely.

$$O(S) = n! \tag{2}$$

where: O(S) denotes the order of the symmetric group.

$$\implies \Pr(E_i) = \frac{1}{n!} \forall E_i \in S \tag{3}$$

Solution

Given, X_1, X_2, X_3, X_4, X_5 are i.i.d random variables. Clearly, in the conditional world where: $X_1 = max(X_1, X_2, X_3, X_4, X_5)$, we have:

$$p = \Pr\left(X_2 > X_3 > X_4 > X_5\right) \tag{4}$$

As any permutation of X_2, X_3, X_4, X_5 is equally likely. As there are 4 random values that the random variables represent, they can be arranged in 4! ways. By (3), we get:

$$\begin{aligned}\Pr\left(X_2 > X_3 > X_4 > X_5\right) &= \frac{1}{4!} \quad (5)\\ &= \frac{1}{24} \quad (6)\end{aligned}$$