AI1103 - Challenging Problem 5

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Download latex-tikz codes from

https://github.com/Anirudh−Srinivasan−CS20/AI1103/blob/main/Challenging−Problem−5/CP−5.tex

Question

Suppose X_1,X_2,X_3 and X_4 are independent and identically distributed random variables, having density function f. Then,

A) $\Pr(X_4 > Max(X_1, X_2) > X_3) = \frac{1}{6}$
B) $\Pr(X_4 > Max(X_1, X_2) > X_3) = \frac{1}{8}$
C) $\Pr(X_4 > X_3 > Max(X_1, X_2)) = \frac{1}{12}$
D) $\Pr(X_4 > X_3 > Max(X_1, X_2)) = \frac{1}{6}$

Solution

Given, X_1, X_2, X_3, X_4 are i.i.d random variables. As every i.i.d sequence of random variables is exchangeable. Any value of a finite sequence is as likely as any permutation of those values. The joint probability distribution is invariant under the symmetric group.
Hence, any permutation of X_1, X_2, X_3, X_4 is equally likely. As there are 4 random values that the random variables represent, they can be arranged in 4! ways.

$$\Pr(X_1 > X_2 > X_3 > X_4) = \Pr(X_1 > X_2 > X_4 > X_3) \tag{0.0.1}$$
$$= \dots \tag{0.0.2}$$
$$= \frac{1}{24} \tag{0.0.3}$$

Options A and B

Without loss of generality, let $Max(X_1, X_2) = X_1$

$$\Pr(X_4 > Max(X_1, X_2) > X_3) = \Pr(X_4 > X_1 > X_3) \tag{0.0.4}$$

$$\Pr(X_4 > X_1 > X_3) = \Pr(X_4 > X_1 > X_2 > X_3) + \Pr(X_4 > X_1 > X_3 > X_2) = \frac{1}{12} \tag{0.0.5}$$

As $Max(X_1, X_2)$ being X_1 or X_2 is equally likely,

$$\Pr(X_4 > X_2 > X_3) = \frac{1}{12} \tag{0.0.6}$$
$$\Pr(X_4 > Max(X_1, X_2) > X_3) = 2 \times \frac{1}{12} = \frac{1}{6} \tag{0.0.7}$$

Options C and D

Without loss of generality, let $Max(X_1, X_2) = X_1$

$$\Pr(X_4 > X_3 > Max(X_1, X_2)) = \Pr(X_4 > X_3 > X_1 > X_2) \tag{0.0.8}$$
$$= \frac{1}{24} \tag{0.0.9}$$

As $Max(X_1, X_2)$ being X_1 or X_2 is equally likely,

$$\Pr(X_4 > X_3 > X_2 > X_1) = \frac{1}{24} \tag{0.0.10}$$
$$\Pr(X_4 > X_3 > Max(X_1, X_2)) = 2 \times \frac{1}{24} = \frac{1}{12} \tag{0.0.11}$$

Answer: Option (A) and (C)