AI1103 - Challenging Problem 3

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Download latex-tikz codes from

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https://github.com/Anirudh−Srinivasan−CS20/
    AI1103/blob/main/Challenging−Problem−3//
    Challenging−Problem−3.tex
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Question

Let X_1, X_2, X_3, X_4, X_5 be i.i.d random variables having a continuous distribution function. Given: $X_1 = max(X_1, X_2, X_3, X_4, X_5)$. Then, the value of p = $\Pr(X_1 > X_2 > X_3 > X_4 > X_5)$ is:

A) $\frac{1}{4}$
B) $\frac{1}{5}$
C) $\frac{1}{4!}$
D) $\frac{1}{5!}$

Solution

Given, X_1, X_2, X_3, X_4, X_5 are i.i.d random variables.

Lemma 0.1. *Every i.i.d sequence of random variables is exchangeable. Any value of a finite sequence is as likely as any permutation of those values. The joint probability distribution is invariant under the symmetric group.*

Proof.

$$f_{X_1,X_2,X_3,\ldots,X_n}(x) = f_{X_1}(x) \times f_{X_2}(x) \times \ldots f_{X_n}(x) \quad (0.0.1)$$

As X_is are i.i.d random variables, their joint probability density function is the product of their marginal probability density functions and as multiplication is commutative, it is exchangeable. □

Definition 1 (Symmetric Group)**.** *It is the group of permutation on a set with n elements and has n! elements. Order of a symmetric group represents the number of elements in it.*

Lemma 0.2. *If n elements of a set are random, then probability of each element 'E$_i$' of the symmetric group 'S' is $\frac{1}{n!}$.*

Proof. As the n values are completely random, there will be no bias for a particular arrangement and hence all the elements of the symmetric group are equally likely.

$$O(S) = n! \quad (0.0.2)$$

where: O(S) denotes the order of the symmetric group.

$$\implies \Pr(E_i) = \frac{1}{n!} \forall E_i \in S \quad (0.0.3)$$

□

Clearly, in the conditional world where: $X_1 = max(X_1, X_2, X_3, X_4, X_5)$, we have:

$$p = \Pr(X_2 > X_3 > X_4 > X_5) \quad (0.0.4)$$

As any permutation of X_2, X_3, X_4, X_5 is equally likely. As there are 4 random values that the random variables represent, they can be arranged in 4! ways. By (0.0.3), we get:

$$\Pr(X_2 > X_3 > X_4 > X_5) = \frac{1}{4!} \quad (0.0.5)$$

$$= \frac{1}{24} \quad (0.0.6)$$

Answer: Option (C)