AI1103 - Challenging Problem 17

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Download latex-tikz codes from

https://github.com/Anirudh−Srinivasan−CS20/AI1103/blob/main/Challenging−Problem−17//Challenging−Problem−17.tex

Question

Let X and Y be independent and identically distributed random variables such that $\Pr(X = 0) = \Pr(X = 1) = \frac{1}{2}$. Let $Z = X + Y$ and $W = |X - Y|$. Then which statement is not correct?

A) X and W are independent.
B) Y and W are independent.
C) Z and W are uncorrelated.
D) Z and W are independent.

Solution

Since, X and Y are i.i.d random variables

$$\Pr(X = 0) = \Pr(Y = 0) = \frac{1}{2} \tag{0.0.1}$$

$$\Pr(X = 1) = \Pr(Y = 1) = \frac{1}{2} \tag{0.0.2}$$

X	Y	Z	W	ZW	$\Pr(X, Y)$
0	0	0	0	0	$\frac{1}{4}$
0	1	1	1	1	$\frac{1}{4}$
1	0	1	1	1	$\frac{1}{4}$
1	1	2	0	0	$\frac{1}{4}$

Table 4: This table shows probability associated with each value that the random variable Z,W and ZW can take when X = x, Y = y, Z = x+y, W = $|x - y|$ and ZW = (x+y)$|x - y|$.

$\implies$ For Z, we have:

$$\Pr(Z = 0) = \frac{1}{4} \tag{0.0.3}$$

$$\Pr(Z = 1) = \frac{1}{2} \tag{0.0.4}$$

$$\Pr(Z = 2) = \frac{1}{4} \tag{0.0.5}$$

$$E(Z) = \sum_{i=0}^{2} Pr(Z = i) \times z_i \tag{0.0.6}$$

$$= \frac{1}{4} \times 0 + \frac{1}{2} \times 1 + \frac{1}{4} \times 2 \tag{0.0.7}$$

$$= 1 \tag{0.0.8}$$

$\implies$ For W, we have:

$$\Pr(W = 0) = \frac{1}{2} \tag{0.0.9}$$

$$\Pr(W = 1) = \frac{1}{2} \tag{0.0.10}$$

$$E(W) = \sum_{i=0}^{1} Pr(W = i) \times w_i \tag{0.0.11}$$

$$= \frac{1}{2} \times 0 + \frac{1}{2} \times 1 \tag{0.0.12}$$

$$= \frac{1}{2} \tag{0.0.13}$$

$\implies$ For ZW, we have:

$$\Pr(ZW = 0) = \frac{1}{2} \tag{0.0.14}$$

$$\Pr(ZW = 1) = \frac{1}{2} \tag{0.0.15}$$

$$E(ZW) = \sum_{i=0}^{1} Pr(WZ = i) \times (wz)_i \tag{0.0.16}$$

$$= \frac{1}{2} \times 0 + \frac{1}{2} \times 1 \tag{0.0.17}$$

$$= \frac{1}{2} \tag{0.0.18}$$

$$E(ZW) = \frac{1}{2} \tag{0.0.19}$$

$$E(Z) \times E(W) = 1 \times \frac{1}{2} \tag{0.0.20}$$

$$E(ZW) = E(Z) \times E(W) \tag{0.0.21}$$

$$cov(ZW) = E(ZW) - E(Z) \times E(W) = 0 \tag{0.0.22}$$

The correlation coefficient between Z and W is given by:

$$\rho_{Z,W} = \frac{cov(ZW)}{\sigma_z \times \sigma_w} \tag{0.0.23}$$

$$= 0 \tag{0.0.24}$$

$\implies$ Z and W are uncorrelated.

From the above table:

$$\Pr(X = 0|W = 0) = \frac{1}{2} = \Pr(X = 0) \tag{0.0.25}$$

$$\Pr(X = 0|W = 1) = \frac{1}{2} = \Pr(X = 0) \tag{0.0.26}$$

$$\Pr(X = 1|W = 0) = \frac{1}{2} = \Pr(X = 1) \tag{0.0.27}$$

$$\Pr(X = 1|W = 1) = \frac{1}{2} = \Pr(X = 1) \tag{0.0.28}$$

$\implies$ X and W are independent.

As the distribution of X and Y are identical, similarly we get:

$\implies$ Y and W are independent.

$$\Pr(Z = 0|W = 0) = \frac{1}{2} \neq \Pr(Z = 0) \tag{0.0.29}$$

$\implies$ Z and W are dependent.

Answer: Option (D)