AI1103 - Assignment 4

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Download latex-tikz codes from

https://github.com/Anirudh−Srinivasan−CS20/AI1103/blob/main/Assignment−4/Assignment−4.tex

Question

Let $X_1, X_2, \ldots$ be independent random variables with X_n being uniformly distributed between -n and 3n, n=1,2,.... Let $S_N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \frac{X_n}{n}$ for N=1,2,...and let F_N be the distribution function of S_N. Also, let ϕ denote the distribution function of a standard normal variable. Which of the following is/are true?

A) $\lim_{N\to\infty} F_N(0) \le \phi(0)$
B) $\lim_{N\to\infty} F_N(0) \ge \phi(0)$
C) $\lim_{N\to\infty} F_N(1) \le \phi(1)$
D) $\lim_{N\to\infty} F_N(1) \ge \phi(1)$

Solution

Given, $X_1, X_2, \ldots$ are independent random variables with $X_i \sim \mathcal{U}(-i, 3i)$. Let us define:

$$Y_i = \frac{X_i}{i} \ \forall \text{ i} \tag{0.0.1}$$

$$\implies Y_i \sim \mathcal{U}(-1, 3) \tag{0.0.2}$$

Now, $Y_1, Y_2, \ldots$ are i.i.d (independent and identically distributed) random variables with:

$$\mu = \frac{-1+3}{2} = 1 \tag{0.0.3}$$

$$\sigma^2 = \frac{[3-(-1)]^2}{12} = \frac{16}{12} = \frac{4}{3} \tag{0.0.4}$$

And, we have:

$$S_N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \frac{X_n}{n} \tag{0.0.5}$$

$$= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} Y_n \tag{0.0.6}$$

By Central Limit Theorem, we can conclude:

$$\lim_{N\to\infty} S_N \sim \mathcal{N}(\mu, \sigma^2) \tag{0.0.7}$$

$$\implies \lim_{N\to\infty} F_N(x) = \phi\left(\frac{x-\mu}{\sigma}\right) \tag{0.0.8}$$

$$= \phi\left(\frac{\sqrt{3}(x-1)}{2}\right) \tag{0.0.9}$$

Substituting x=1, we get:

$$\lim_{N\to\infty} F_N(1) = \phi\left(\frac{\sqrt{3}(1-1)}{2}\right) \tag{0.0.10}$$

$$= \phi(0) \tag{0.0.11}$$

The distribution function F_N is non-decreasing. So,

$$\lim_{N\to\infty} F_N(0) \le \lim_{N\to\infty} F_N(1) = \phi(0) \tag{0.0.12}$$

$$\lim_{N\to\infty} F_N(0) \le \phi(0) \tag{0.0.13}$$

The distribution function ϕ is non-decreasing. So,

$$\lim_{N\to\infty} F_N(1) = \phi(0) \le \phi(1) \tag{0.0.14}$$

$$\lim_{N\to\infty} F_N(1) \le \phi(1) \tag{0.0.15}$$

Answer: Option (A) and (C)