AI1103 - Challenging Problem 5

Anirudh Srinivasan
CS20BTECH11059

Download latex-tikz codes from

https://github.com/Anirudh−Srinivasan−CS20/AI1103/blob/main/Challenging−Problem−5//Challenging−Problem−5.tex

Question

Suppose X_1,X_2,X_3 and X_4 are independent and identically distributed random variables, having density function f. Then,

A) $\Pr(X_4 > Max(X_1, X_2) > X_3) = \frac{1}{6}$
B) $\Pr(X_4 > Max(X_1, X_2) > X_3) = \frac{1}{8}$
C) $\Pr(X_4 > X_3 > Max(X_1, X_2)) = \frac{1}{12}$
D) $\Pr(X_4 > X_3 > Max(X_1, X_2)) = \frac{1}{6}$

Solution

Given, X_1, X_2, X_3, X_4 are i.i.d random variables.

Lemma 0.1. *Every i.i.d sequence of random variables is exchangeable. Any value of a finite sequence is as likely as any permutation of those values. The joint probability distribution is invariant under the symmetric group.*

Proof.

$$f_{X_1,X_2,X_3,\dots,X_n}(x) = f_{X_1}(x) \times f_{X_2}(x) \times \dots f_{X_n}(x) \quad (0.0.1)$$

As X_is are i.i.d random variables, their joint probability density function is the product of their marginal probability density functions and as multiplication is commutative, it is exchangeable. □

Definition 1 (Symmetric Group)**.** *It is the group of permutation on a set with n elements and has $n!$ elements. Order of a symmetric group represents the number of elements in it.*

Lemma 0.2. *If n elements of a set are random, then probability of each element 'E_i' of the symmetric group 'S' is $\frac{1}{n!}$.*

Proof. As the n values are completely random, there will be no bias for a particular arrangement and hence all the elements of the symmetric group are equally likely.

$$O(S) = n! \quad (0.0.2)$$

where: O(S) denotes the order of the symmetric group.

$$\implies \Pr(E_i) = \frac{1}{n!} \forall E_i \in S \quad (0.0.3)$$

□

Hence, any permutation of X_1, X_2, X_3, X_4 is equally likely. As there are 4 random values that the random variables represent, they can be arranged in 4! ways. By (0.0.3), we have:

$$\Pr(X_1 > X_2 > X_3 > X_4) = \Pr(X_1 > X_2 > X_4 > X_3) \quad (0.0.4)$$

$$= \dots \quad (0.0.5)$$

$$= \frac{1}{24} \quad (0.0.6)$$

1) **Options A and B:**

$$\Pr(X_4 > Max(X_1, X_2) > X_3) = \Pr(X_4 > Max(X_1, X_2) > X_3 | X_1 > X2) + \Pr(X_4 > Max(X_1, X_2) > X_3 | X_2 > X_1) \quad (0.0.7)$$

Clearly, by definition:

$$Max(X_1, X_2) = \begin{cases} X_1, & \text{if } X_1 > X_2 \\ X_2, & \text{if } X_2 > X_1 \end{cases} \quad (0.0.8)$$

$$\implies \Pr(X_4 > Max(X_1, X_2) > X_3) = \Pr(X_4 > X_1 > X_3 | X_1 > X_2) + \Pr(X_4 > X_2 > X_3 | X_2 > X_1) \quad (0.0.9)$$

$$\Pr(X_4 > X_1 > X_3 | X_1 > X_2) = \frac{\Pr((X_4 > X_1 > X_3) \cap (X_1 > X_2))}{\Pr(X_1 > X_2)} \quad (0.0.10)$$

The condition $X_1 > X_2$ on $X_4 > X_1 > X_3$

restricts the possibility of $X_2 > X_4$ and only allows $X_4 > X_2$ (as $(X_4 > X_1) \cap (X_1 > X_2) \implies X_4 > X_2$) . Hence,

$$\Pr((X_4 > X_1 > X_3) \cap (X_1 > X_2)) = \Pr(X_4 > X_2) \times \Pr(X_4 > X_1 > X_3) \times \Pr(X_1 > X_2) \tag{0.0.11}$$

$$\Pr((X_4 > X_1 > X_3) \cap (X_1 > X_2)) = \frac{1}{2!} \times \frac{1}{3!} \times \frac{1}{2!} = \frac{1}{2} \times \frac{1}{6} \times \frac{1}{2} = \frac{1}{24} \tag{0.0.12}$$

$$\Pr(X_1 > X_2) = \frac{1}{2!} = \frac{1}{2} \tag{0.0.13}$$

Substituting (0.0.12) and (0.0.13) in (0.0.10):

$$\Pr(X_4 > X_1 > X_3 | X_1 > X_2) = \frac{\frac{1}{24}}{\frac{1}{2}} = \frac{1}{12} \tag{0.0.14}$$

Aliter:
The event $(X_4 > X_1 > X_3 | X_1 > X_2)$ can be decomposed into its constituent sub-events and hence we have:

$$\Pr(X_4 > X_1 > X_3 | X_1 > X_2) = \Pr(X_4 > X_1 > X_2 > X_3) + \Pr(X_4 > X_1 > X_3 > X_2) = \frac{1}{12} \tag{0.0.15}$$

As $Max(X_1, X_2)$ being X_1 or X_2 is equally likely,

$$\Pr(X_4 > X_2 > X_3 | X_2 > X_1) = \frac{1}{12} \tag{0.0.16}$$

$$\Pr(X_4 > Max(X_1, X_2) > X_3) = 2 \times \frac{1}{12} = \frac{1}{6} \tag{0.0.17}$$

2) **Options C and D:**

$$\Pr(X_4 > X_3 > Max(X_1, X_2)) = \Pr(X_4 > X_3 > Max(X_1, X_2) | X_1 > X_2) + \Pr(X_4 > X_3 > Max(X_1, X_2) | X_2 > X_1) \tag{0.0.18}$$

$$\Pr(X_4 > X_3 > Max(X_1, X_2)) = \Pr(X_4 > X_3 > X_1 | X_1 > X_2) + \Pr(X_4 > X_3 > X_2 | X_2 > X_1) \tag{0.0.19}$$

$$\Pr(X_4 > X_3 > X_1 | X_1 > X_2) = \frac{\Pr((X_4 > X_3 > X_1) \cap (X_1 > X_2))}{\Pr(X_1 > X_2)} \tag{0.0.20}$$

The condition $X_1 > X_2$ on $X_4 > X_3 > X_1$ restricts the possibility of $X_2 > X_4$ and $X_2 > X_3$ and only allows $X_4 > X_2$ and $X_3 > X_2$ (as $(X_4 > X_1) \cap (X_1 > X_2) \implies X_4 > X_2$ and $(X_3 > X_1) \cap (X_1 > X_2) \implies X_3 > X_2$) . Hence,

$$\Pr((X_4 > X_3 > X_1) \cap (X_1 > X_2)) = \Pr(X_3 > X_2) \times \Pr(X_4 > X_2) \times \Pr(X_4 > X_1 > X_3) \times \Pr(X_1 > X_2) \tag{0.0.21}$$

$$\Pr((X_4 > X_3 > X_1) \cap (X_1 > X_2)) = \frac{1}{2!} \times \frac{1}{2!} \times \frac{1}{3!} \times \frac{1}{2!} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{6} \times \frac{1}{2} = \frac{1}{48} \tag{0.0.22}$$

$$\Pr(X_1 > X_2) = \frac{1}{2!} = \frac{1}{2} \tag{0.0.23}$$

Substituting (0.0.22) and (0.0.23) in (0.0.20):

$$\Pr(X_4 > X_3 > X_1 | X_1 > X_2) = \frac{\frac{1}{48}}{\frac{1}{2}} = \frac{1}{24} \tag{0.0.24}$$

Aliter:
The event $(X_4 > X_3 > X_1 | X_1 > X_2)$ can be decomposed into its constituent sub-events and hence we have:

$$\Pr(X_4 > X_3 > X_1 | X_1 > X_2) = \Pr(X_4 > X_3 > X_1 > X_2) = \frac{1}{24} \tag{0.0.25}$$

As $Max(X_1, X_2)$ being X_1 or X_2 is equally likely,

$$\Pr(X_4 > X_3 > X_2 | X_2 > X_1) = \frac{1}{24} \tag{0.0.26}$$

$$\Pr(X_4 > X_3 > Max(X_1, X_2)) = 2 \times \frac{1}{24} = \frac{1}{12} \tag{0.0.27}$$

Answer: Option (A) and (C)