

# AI1103 - Challenging Problem 5

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<https://github.com/Anirudh-Srinivasan-CS20/AI1103/blob/main/Challenging-Problem-5//Challenging-Problem-5.tex>

## QUESTION

Suppose  $X_1, X_2, X_3$  and  $X_4$  are independent and identically distributed random variables, having density function  $f$ . Then,

- A)  $\Pr(X_4 > \text{Max}(X_1, X_2) > X_3) = \frac{1}{6}$
- B)  $\Pr(X_4 > \text{Max}(X_1, X_2) > X_3) = \frac{1}{8}$
- C)  $\Pr(X_4 > X_3 > \text{Max}(X_1, X_2)) = \frac{1}{12}$
- D)  $\Pr(X_4 > X_3 > \text{Max}(X_1, X_2)) = \frac{1}{6}$

## SOLUTION

Given,  $X_1, X_2, X_3, X_4$  are i.i.d random variables.

**Lemma 0.1.** Every i.i.d sequence of random variables is exchangeable. Any value of a finite sequence is as likely as any permutation of those values. The joint probability distribution is invariant under the symmetric group.

*Proof.*

$$f_{X_1, X_2, X_3, \dots, X_n}(x) = f_{X_1}(x) \times f_{X_2}(x) \times \dots \times f_{X_n}(x) \quad (0.0.1)$$

As  $X_i$ s are i.i.d random variables, their joint probability density function is the product of their marginal probability density functions and as multiplication is commutative, it is exchangeable.  $\square$

**Definition 1** (Symmetric Group). It is the group of permutation on a set with  $n$  elements and has  $n!$  elements. Order of a symmetric group represents the number of elements in it.

**Lemma 0.2.** If  $n$  elements of a set are random, then probability of each element ' $E_i$ ' of the symmetric group ' $S$ ' is  $\frac{1}{n!}$ .

*Proof.* As the  $n$  values are completely random, there will be no bias for a particular arrangement and

hence all the elements of the symmetric group are equally likely.

$$O(S) = n! \quad (0.0.2)$$

where:  $O(S)$  denotes the order of the symmetric group.

$$\implies \Pr(E_i) = \frac{1}{n!} \forall E_i \in S \quad (0.0.3)$$

$\square$

Hence, any permutation of  $X_1, X_2, X_3, X_4$  is equally likely. As there are 4 random values that the random variables represent, they can be arranged in  $4!$  ways. By (0.0.3), we have:

$$\Pr(X_1 > X_2 > X_3 > X_4) = \Pr(X_1 > X_2 > X_4 > X_3) \quad (0.0.4)$$

$$= \dots \quad (0.0.5)$$

$$= \frac{1}{24} \quad (0.0.6)$$

## 1) Options A and B:

$$\Pr(X_4 > \text{Max}(X_1, X_2) > X_3) =$$

$$\Pr(X_4 > \text{Max}(X_1, X_2) > X_3 | X_1 > X_2) +$$

$$\Pr(X_4 > \text{Max}(X_1, X_2) > X_3 | X_2 > X_1) \quad (0.0.7)$$

Clearly, by definition:

$$\text{Max}(X_1, X_2) = \begin{cases} X_1, & \text{if } X_1 > X_2 \\ X_2, & \text{if } X_2 > X_1 \end{cases} \quad (0.0.8)$$

$$\implies \Pr(X_4 > \text{Max}(X_1, X_2) > X_3) =$$

$$\Pr(X_4 > X_1 > X_3 | X_1 > X_2) +$$

$$\Pr(X_4 > X_2 > X_3 | X_2 > X_1) \quad (0.0.9)$$

$$\Pr(X_4 > X_1 > X_3 | X_1 > X_2) =$$

$$\Pr(X_4 > X_1 > X_2 > X_3)$$

$$+ \Pr(X_4 > X_1 > X_3 > X_2) = \frac{1}{12} \quad (0.0.10)$$

As  $Max(X_1, X_2)$  being  $X_1$  or  $X_2$  is equally likely,

$$\Pr(X_4 > X_2 > X_3 | X_2 > X_1) = \frac{1}{12} \quad (0.0.11)$$

$$\Pr(X_4 > Max(X_1, X_2) > X_3) = 2 \times \frac{1}{12} = \frac{1}{6} \quad (0.0.12)$$

## 2) Options C and D:

$$\begin{aligned} \Pr(X_4 > X_3 > Max(X_1, X_2)) &= \\ \Pr(X_4 > X_3 > Max(X_1, X_2) | X_1 > X_2) &+ \\ \Pr(X_4 > X_3 > Max(X_1, X_2) | X_2 > X_1) & \quad (0.0.13) \end{aligned}$$

$$\begin{aligned} \Pr(X_4 > X_3 > Max(X_1, X_2)) &= \\ \Pr(X_4 > X_3 > X_1 > X_2) &+ \\ \Pr(X_4 > X_3 > X_2 > X_1) & \quad (0.0.14) \end{aligned}$$

$$\Pr(X_4 > X_3 > Max(X_1, X_2)) = 2 \times \frac{1}{24} = \frac{1}{12} \quad (0.0.15)$$

Answer: Option (A) and (C)