assignment1

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1 Assignment 1 - Backpropagation

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1.1 2. Getting Started

[1]:
```python
import numpy as np
from matplotlib import pyplot
```

1.2 3. Read MNIST Data

[2]:
```python
import MNISTtools
help(MNISTtools.load)
help(MNISTtools.show)
```

```
Help on function load in module MNISTtools:

load(dataset='training', path=None)
    Import either the training or testing MNIST data set.
    It returns a pair with the first element being the collection of
    images stacked in columns and the second element being a vector
    of corresponding labels from 0 to 9.

    Arguments:
        dataset (string, optional): either "training" or "testing".
            (default: "training")
        path (string, optional): the path pointing to the MNIST dataset
```

```
        If path=None, it looks succesively for the dataset at:
        '/datasets/MNIST' and './MNIST'. (default: None)

    Example:
        x, lbl = load(dataset="testing", path="/Folder/for/MNIST")

Help on function show in module MNISTtools:

show(image)
    Render a given MNIST image provided as a column vector.

    Arguments:
        image (array): an array of shape (28*28) or (28, 28) representing a
            grey level image of size 28 x 28. Values are expected to be in the
            range [0, 1].

    Example:
        x, lbl = load(dataset="training", path="/datasets/MNIST")
        show(x[:, 0])
```

Question 1

[3]:
```python
# Load the data
xtrain, ltrain = MNISTtools.load(path='./datasets/MNIST')
print(xtrain.shape)
print(ltrain.shape)
```

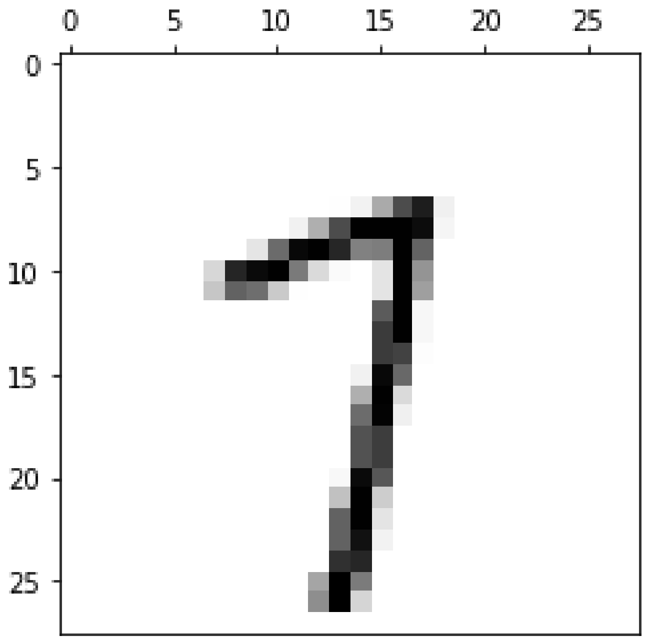
```
(784, 60000)
(60000,)
```

The shape of *xtrain* is $(784, 60000)$ The shape of *ltrain* is $(60000,)$ The size of the training set, i.e., the number of images in the training set is 60000 The feature dimension is 784

Question 2

[4]:
```python
# Displaying the image at index 42
MNISTtools.show(xtrain[:, 42])

# Print its corresponding label
print(ltrain[42])
```



7

The image at the index 42 has been displayed. The corresponding label has been printed and is found to be 7

Question 3

[5]:
```python
# Find the range and type of xtrain
min_x = np.amin(xtrain)
max_x = np.amax(xtrain)

print("Range of xtrain is from ", min_x, " to ", max_x)
print("Data type of xtrain is ", xtrain.dtype)
```

```
Range of xtrain is from  0  to  255
Data type of xtrain is  uint8
```

The range of values for *xtrain* is from 0 to 255 The type of *xtrain* is *uint*8

Question 4

[6]:
```python
def normalize_MNIST_images(x):
    # Convert the uint8 input into float32 for ease of normalization
    fl_x = x.astype(np.float32)

    # Normalize [0 to 255] to [-1 to 1]
    # This means mapping 0 to -1, 255 to 1, and 127.5 to 0
```

```
    ret = 2*(fl_x - 255/2.0) / 255
    return ret
```

[7]:
```
norm_x_train = normalize_MNIST_images(xtrain)
print(norm_x_train.shape)
print("Range of normalized xtrain is", np.amin(norm_x_train), "to", np.
 ↪amax(norm_x_train))
print("Data type of normalized xtrain is", norm_x_train.dtype)
```

```
(784, 60000)
Range of normalized xtrain is -1.0 to 1.0
Data type of normalized xtrain is float32
```

We wrote the function to normalize the training data from $[0 to 255]$ to $[-1 to 1]$ We converted $xtrain$ which was of type $uint8$ into a vector of type $float32$ We then mapped 0 to -1, 255 to 1 by subtracting the mid, which is 127.5 and then dividing by mid, which is 127.5 We then stored the normalized $xtrain$ in the variable $norm\_x\_train$

Question 5

[8]:
```
# Complete the code below
def label2onehot(lbl):
    # Creates a placeholder of size (10, 60000)
    d = np.zeros((lbl.max() + 1, lbl.size))

    # One-hot encode the labels
    d[lbl, np.arange(lbl.size)] = 1
    return d
```

[9]:
```
dtrain = label2onehot(ltrain)
print(dtrain.shape)
print(np.amin(dtrain), np.amax(dtrain))
print("Label at index 42 is", ltrain[42])
print("Corresponding one-hot encodiing is", dtrain[:, 42])
```

```
(10, 60000)
0.0 1.0
Label at index 42 is 7
Corresponding one-hot encodiing is [0. 0. 0. 0. 0. 0. 0. 1. 0. 0.]
```

The one hot encoding line works as the 1^{st} index is traveresed independently of the 2^{nd} index So, for each image given by the 2^{nd} axis, only the row index given by the value of the label is assigned 1 Thus, 0 maps to $[1, 0, 0, 0, 0, 0, 0, 0, 0, 0]$ and 9 maps to $[0, 0, 0, 0, 0, 0, 0, 0, 0, 1]$ We also checked the label for image 42. The label is 7 and the corresponding one-hot encoding is $[0, 0, 0, 0, 0, 0, 0, 1, 0, 0]$

Question 6

[10]:
```python
def onehot2label(d):
    lbl = d.argmax(axis=0)
    return lbl
```

[11]:
```python
# Checking if this works
lab = dtrain[:, 42]
che = onehot2label(lab)

print("One-hot answer", che, "| Original:", ltrain[42])
assert(che == ltrain[42])
```

```
One-hot answer 7 | Original: 7
```

We have thus checked if our implementation of recovering the label from one-hot encoding is correct The label of the image at index at 42 is 7 The *onehot2label*() function recovers this correctly

1.3 4. Activation Functions

Question 7

[12]:
```python
# Implement the softmax function
def softmax(a):
    # Calculate the max value
    M = np.max(a, axis=0)

    # Subtract for easier exponential calculation
    a_m = a - M

    # Calculate the exponent for each class for each image
    exp_a_m = np.exp(a_m)

    # Calculate the sum for each image
    den = np.sum(exp_a_m, axis=0)

    # Get the probabilities for each class for each image
    g_a = exp_a_m / den
    # print(np.min(g_a), np.max(g_a))
    return g_a
```

1.3.1 Question 8

We need to show that

$$\frac{\partial g(a)_i}{\partial a_i} = g(a)_i(1 - g(a)_i)$$

By definition above, Softmax is

$$y_i = g(a)_i = \frac{exp(a_i)}{\sum_{j=1}^{10} exp(a_j)}$$

So,

$$\frac{\partial g(a)_i}{\partial a_i} = \frac{\partial \left(\frac{exp(a_i)}{\sum_{j=1}^{10} exp(a_j)} \right)}{\partial a_i}$$

Using the division rule of derivatives, we have

$$\frac{\partial g(a)_i}{\partial a_i} = \frac{\sum_{j=1}^{10} exp(a_j) \frac{\partial exp(a_i)}{\partial a_i} - exp(a_i) \frac{\partial \left(\sum_{j=1}^{10} exp(a_j) \right)}{\partial a_i}}{\left(\sum_{j=1}^{10} exp(a_j) \right)^2}$$

Simplifying, we have

$$\frac{\partial g(a)_i}{\partial a_i} = \frac{exp(a_i) * \sum_{j=1}^{10} exp(a_j) - exp(a_i) * exp(a_i)}{\left(\sum_{j=1}^{10} exp(a_j) \right)^2}$$

Taking $\frac{exp(a_i)}{\sum_{j=1}^{10} exp(a_j)}$ outside, we have

$$\frac{\partial g(a)_i}{\partial a_i} = \frac{exp(a_i)}{\sum_{j=1}^{10} exp(a_j)} * \left(\frac{\sum_{j=1}^{10} exp(a_j) - exp(a_i)}{\left(\sum_{j=1}^{10} exp(a_j) \right)} \right)$$

We know that $g(a)_i = \frac{exp(a_i)}{\sum_{j=1}^{10} exp(a_j)}$ Thus, we have

$$\frac{\partial g(a)_i}{\partial a_i} = g(a)_i * (1 - g(a)_i)$$

1.3.2 Question 9

We need to show that

$$\frac{\partial g(a)_i}{\partial a_j} = -g(a)_i * g(a)_j for j \neq i$$

By definition above, Softmax is

$$y_i = g(a)_i = \frac{exp(a_i)}{\sum_{j=1}^{10} exp(a_j)}$$

So,

$$\frac{\partial g(a)_i}{\partial a_j} = \frac{\partial \left(\frac{exp(a_i)}{\sum_{j=1}^{10} exp(a_j)} \right)}{\partial a_j}$$

Taking the term $exp(a_i)$ outside, we have,

$$\frac{\partial g(a)_i}{\partial a_j} = exp(a_i) * \frac{\partial \left(\frac{1}{\sum_{j=1}^{10} exp(a_j)} \right)}{\partial a_j}$$

Using inverse rule of derivatives, we have

$$\frac{\partial g(a)_i}{\partial a_j} = exp(a_i) * \frac{-1 * exp(a_j)}{\left(\sum_{j=1}^{10} exp(a_j)\right)^2}$$

We know that $g(a)_i = \frac{exp(a_i)}{\sum_{j=1}^{10} exp(a_j)}$ Thus, we have

$$\frac{\partial g(a)_i}{\partial a_j} = -1 * g(a)_i * g(a)_j for j \neq i$$

1.3.3 Question 10

1. **We need to prove that the Jacobian of the softmax() function is symmetric** We have the Jacobian listed as follows,

$$\frac{\partial g(a)}{\partial a} = \begin{bmatrix} \frac{\partial g(a)_1}{\partial a_1} & \frac{\partial g(a)_1}{\partial a_2} & \cdots & \frac{\partial g(a)_1}{\partial a_{10}} \\ \frac{\partial g(a)_2}{\partial a_1} & \frac{\partial g(a)_2}{\partial a_2} & \cdots & \frac{\partial g(a)_2}{\partial a_{10}} \\ \vdots & & & \vdots \\ \frac{\partial g(a)_{10}}{\partial a_1} & \frac{\partial g(a)_{10}}{\partial a_2} & \cdots & \frac{\partial g(a)_{10}}{\partial a_{10}} \end{bmatrix}$$

From question 9, we know that

$$\frac{\partial g(a)_i}{\partial a_j} = -1 * g(a)_i * g(a)_j for j \neq i$$

From question 8, we have

$$\frac{\partial g(a)_i}{\partial a_i} = g(a)_i * (1 - g(a)_i)$$

So,

$$\frac{\partial g(a)}{\partial a} = \begin{bmatrix} g(a)_1(1-g(a)_1) & -g(a)_1 g(a)_2 & \cdots & -g(a)_1 g(a)_{10} \\ -g(a)_2 g(a)_1 & g(a)_2(1-g(a)_2) & \cdots & -g(a)_2 g(a)_{10} \\ \vdots & & & \vdots \\ -g(a)_{10} g(a)_1 & -g(a)_{10} g(a)_2 & \cdots & g(a)_{10}(1-g(a)_{10}) \end{bmatrix}$$

Taking transpose of the Jacobian, we have

$$\left[\frac{\partial g(a)}{\partial a}\right]^T = \begin{bmatrix} g(a)_1(1-g(a)_1) & -g(a)_2 g(a)_1 & \cdots & -g(a)_{10} g(a)_1 \\ -g(a)_1 g(a)_2 & g(a)_2(1-g(a)_2) & \cdots & -g(a)_{10} g(a)_2 \\ \vdots & & & \vdots \\ -g(a)_1 g(a)_{10} & -g(a)_2 g(a)_{10} & \cdots & g(a)_{10}(1-g(a)_{10}) \end{bmatrix}$$

Thus, we have proved that

$$\left[\frac{\partial g(a)}{\partial a}\right]^T = \frac{\partial g(a)}{\partial a}$$

2. **Now, we need to prove** $\delta = g(a) \otimes e - < g(a), e > g(a)$

$$\text{We know that} \quad \delta = \left[\frac{\partial g(a)}{\partial a}\right]^T \times e$$

LHS

$$\delta = \left[\frac{\partial g(a)}{\partial a}\right]^T \times e$$

From above, we know that

$$\left[\frac{\partial g(a)}{\partial a}\right]^T = \frac{\partial g(a)}{\partial a}$$

So,

$$\delta = \frac{\partial g(a)}{\partial a} \times e$$

$$\delta = \begin{bmatrix} g(a)_1(1-g(a)_1) & -g(a)_1 g(a)_2 & \cdots & -g(a)_1 g(a)_{10} \\ -g(a)_2 g(a)_1 & g(a)_2(1-g(a)_2) & \cdots & -g(a)_2 g(a)_{10} \\ \vdots & & & \vdots \\ -g(a)_{10} g(a)_1 & -g(a)_{10} g(a)_2 & \cdots & g(a)_{10}(1-g(a)_{10}) \end{bmatrix} \times e$$

$$\text{where} \quad e = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{10} \end{pmatrix}$$

$$\text{So,} \quad \delta = \begin{bmatrix} g(a)_1(1-g(a)_1) & -g(a)_1 g(a)_2 & \cdots & -g(a)_1 g(a)_{10} \\ -g(a)_2 g(a)_1 & g(a)_2(1-g(a)_2) & \cdots & -g(a)_2 g(a)_{10} \\ \vdots & & & \vdots \\ -g(a)_{10} g(a)_1 & -g(a)_{10} g(a)_2 & \cdots & g(a)_{10}(1-g(a)_{10}) \end{bmatrix} \times \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{10} \end{pmatrix}$$

$$\implies \delta = \begin{bmatrix} g(a)_1(1-g(a)_1)e_1 + -g(a)_1 g(a)_2 e_2 + \cdots + -g(a)_1 g(a)_{10} e_{10} \\ -g(a)_2 g(a)_1 e_1 + g(a)_2(1-g(a)_2)e_2 + \cdots + -g(a)_2 g(a)_{10} e_{10} \\ \vdots \\ -g(a)_{10} g(a)_1 e_1 + -g(a)_{10} g(a)_2 e_2 + \cdots + g(a)_{10}(1-g(a)_{10})e_{10} \end{bmatrix}$$

$$\text{So,} \quad \delta = \begin{bmatrix} g(a)_1 \left[e_1 - g(a)_1 e_1 + -g(a)_2 e_2 + \cdots + -g(a)_{10} e_{10}\right] \\ g(a)_2 \left[e_2 - g(a)_1 e_1 + -g(a)_2 e_2 + \cdots + -g(a)_{10} e_{10}\right] \\ \vdots \\ g(a)_{10} \left[e_{10} - g(a)_1 e_1 + -g(a)_2 e_2 + \cdots + -g(a)_{10} e_{10}\right] \end{bmatrix}$$

$$\text{So,} \quad \delta = \begin{bmatrix} g(a)_1 \left[e_1 - \sum_{i=1}^{10} g(a)_i e_i\right] \\ g(a)_2 \left[e_2 - \sum_{i=1}^{10} g(a)_i e_i\right] \\ \vdots \\ g(a)_{10} \left[e_{10} - \sum_{i=1}^{10} g(a)_i e_i\right] \end{bmatrix}$$

RHS

$$g(a) \otimes e - < g(a), e > g(a)$$

Individually, these terms are as follows

$$g(a) \otimes e = \begin{bmatrix} g(a)_1.e_i \\ g(a)_2.e_2 \\ \vdots \\ g(a)_{10}.e_{10} \end{bmatrix}$$

$$< g(a), e >= \left(\sum_{i=1}^{10} g(a)_i e_i \right)$$

Plugging these in RHS, we get

$$g(a) \otimes e - < g(a), e > g(a) = \begin{bmatrix} g(a)_1.e_1 \\ g(a)_2.e_2 \\ \vdots \\ g(a)_{10}.e_{10} \end{bmatrix} - \left(\sum_{i=1}^{10} g(a)_i e_i \right) \begin{bmatrix} g(a)_1 \\ g(a)_2 \\ \vdots \\ g(a)_{10} \end{bmatrix}$$

Thus, we have

$$g(a) \otimes e - < g(a), e > g(a) = \begin{bmatrix} g(a)_1.e_1 \\ g(a)_2.e_2 \\ \vdots \\ g(a)_{10}.e_{10} \end{bmatrix} - \begin{bmatrix} g(a)_1 \sum_{i=1}^{10} g(a)_i e_i \\ g(a)_2 \sum_{i=1}^{10} g(a)_i e_i \\ \vdots \\ g(a)_{10} \sum_{i=1}^{10} g(a)_i e_i \end{bmatrix}$$

So, finally we have

$$g(a) \otimes e - < g(a), e > g(a) = \begin{bmatrix} g(a)_1 \left[e_1 - \sum_{i=1}^{10} g(a)_i e_i \right] \\ g(a)_2 \left[e_2 - \sum_{i=1}^{10} g(a)_i e_i \right] \\ \vdots \\ g(a)_{10} \left[e_{10} - \sum_{i=1}^{10} g(a)_i e_i \right] \end{bmatrix}$$

We notice that LHS = RHS Hence, we have proved that

$$\delta = g(a) \otimes e - < g(a), e > g(a)$$

[13]:
```
# Implementation of the gradient of the softmax function
# The directional derivative of the softmax function is as follows:-
# delta = elementwise product (g(a) and e) - <g(a),e> g(a)
def softmaxp(a, e):
    # Calculate g(a)
    g_a = softmax(a)

    # Calculate term 1
    t1 = g_a * e
```

```
    # Calculate the directional derivative
    delta = t1 - np.sum(t1, axis=0)*g_a
    return delta
```

Question 11

[14]:
```
# Check if softmaxp is correct
# finite difference step
eps = 1e-6

# random inputs
a = np.random.randn(10, 200)

# random directions
e = np.random.randn(10, 200)

# testing part
diff = softmaxp(a, e)

# From the definition of a derivative, we have
diff_approx = (softmax(a + eps*e) - softmax(a)) / eps

# Calculate the relative error of these 2 approaches
rel_error = np.abs(diff - diff_approx).mean() / np.abs(diff_approx).mean()

# print the relative error
print(rel_error, 'should be smaller than 1e-6')
```

```
4.882725480374909e-07 should be smaller than 1e-6
```

We have implemented the code to compute the directional derivative of g at point a in the direction of e using the $softmaxp(a, e)$ function We tested the implementation of our code by comparing with the fundamental definition of directional derivative, where,

$$\delta = \frac{\partial g(a)}{\partial a} \times e = \lim_{\varepsilon \to 0} \frac{g(a + \varepsilon e) - g(a)}{\varepsilon}$$

We verified that our implementation of $softmaxp()$ is correct and that the relative error is smaller that $1e - 6$

1.3.4 Question 12

Computing the derivative of relu() We know that for ReLU, $g(a)_i = max(a_i, 0)$ Since for ReLU the activation is element-wise, i.e., $g(a)_i = g(a_i)$, the Jacobian is diagonal Hence,

$$\delta = \frac{\partial g(a)}{\partial a} \otimes e$$

$$\implies \delta = g'(a) \otimes e$$

[15]:
```
# Compute the ReLU(a) = max(ai, 0)
def relu(a):
    # Create a copy of the array a
    g_a = np.copy(a)

    # Set those values less than 0 to 0
    g_a[a < 0] = 0
    return g_a

def relup(a, e):
    # Relup is the directional derivative of ReLU(a) in the direction of e
    # Taking the Jacobian for ReLU and then deriving, we have found that the␣
 ↪derivative is as given:-
    # It is the element-wise product of gradient of relu and the vector e
    # Create a copy of the array a
    del_a = np.copy(a)

    # Set the values less than 0 to 0
    del_a[a < 0] = 0

    # Set the values greater than 0 to 1
    del_a[a > 0] = 1

    # Compute delta as the element-wise product of the gradient of relu and the␣
 ↪vector e
    delta = del_a * e
    return delta
```

We have implemented the relu function and its directional derivative now We used the Jacobian to derive the relation of relup to vector operations We shall now test $reulp()$

[16]:
```
# Check if relup is correct
# finite difference step
eps = 1e-6

# random inputs
a = np.random.randn(10, 200)

# random directions
e = np.random.randn(10, 200)

# testing part
diff = relup(a, e)

# From the definition of a derivative, we have
```

```
diff_approx = (relu(a + eps*e) - relu(a)) / eps

# Calculate the relative error of these 2 approaches
rel_error = np.abs(diff - diff_approx).mean() / np.abs(diff_approx).mean()

# print the relative error
print(rel_error, 'should be smaller than 1e-6')
```

```
4.3060646935183765e-11 should be smaller than 1e-6
```

We have implemented the code to compute the directional derivative of g at point a in the direction of e using the $relup(a, e)$ function We tested the implementation of our code by comparing with the fundamental definition of directional derivative, where,

$$\delta = \frac{\partial g(a)}{\partial a} \times e = \lim_{\varepsilon \to 0} \frac{g(a + \varepsilon e) - g(a)}{\varepsilon}$$

We verified that our implementation of $relup()$ is correct and that the relative error is smaller that $1e - 6$

1.4 5. Backpropagation

Question 13

[17]:
```
# define and initialize our shallow network
def init_shallow(Ni, Nh, No):
    """
    Ni - dimension of the input layer. Ni = 784
    Nh - dimension of the hidden layer. Nh = 64
    No - dimension of the output layer. No = 10
    """
    # Create the bias vector for the 1st layer
    # We are using He initialization method
    b1 = np.random.randn(Nh, 1) / np.sqrt((Ni + 1.) / 2.)
    # Create the synaptic weights between the input and the hidden neurons
    W1 = np.random.randn(Nh, Ni) / np.sqrt((Ni + 1.) / 2.)

    # Create the bias vector for the 2nd layer
    # We are using Xavier initialization method
    b2 = np.random.randn(No, 1) / np.sqrt((Nh + 1.))
    # Create the synaptic weights between the hidden and the output neurons
    W2 = np.random.randn(No, Nh) / np.sqrt((Nh + 1.))
    return W1, b1, W2, b2

# Initialize our shallow network
Ni = norm_x_train.shape[0]
Nh = 64
No = dtrain.shape[0]
netinit = init_shallow(Ni, Nh, No)
```

We defined the network architecture and parameters and initialized them in the snippet above We used He initialization for the input neurons to hidden neurons connections, and we used Xavier initialization for the hidden neurons to output neurons connections

Question 14

[18]:
```
# define the forward_prop function to propagate the activations through the␣
↪network
def forwardprop_shallow(x, net):
    W1 = net[0]
    b1 = net[1]
    W2 = net[2]
    b2 = net[3]

    # Input to hidden neurons
    a1 = W1.dot(x) + b1
    h1 = relu(a1)

    # Hidden to output neurons
    a2 = W2.dot(h1) + b2
    y = softmax(a2)
    return y

# Calculate the initial output for the random initializations
yinit = forwardprop_shallow(norm_x_train, netinit)
```

[19]:
```
print(norm_x_train.shape)
print(yinit.shape)

print(np.min(norm_x_train), np.max(norm_x_train))
print(np.min(yinit), np.max(yinit))
```

```
(784, 60000)
(10, 60000)
-1.0 1.0
0.0018577635031450167 0.8057231387657491
```

We have implemented the function to propagate forward through the network We subsequently calculated the initial output for our initialization of the network with random parameter values

Question 15

[20]:
```
# Function to compute the cross-entropy loss
def eval_loss(y, d):
    # Calculates the log of the predicted probabilities
    log_y = np.log(y)

    # Element-wise multiplication with d
```

```
    mult = d*log_y

    # Take the negative to get cross-entropy
    mult = -1 * mult

    # calculate the sum over all probabilities and sum over all the input vectors
    sum_pro = np.sum(mult)

    # Calculate the average of the cross-entropy
    ret = np.mean(mult)
    return ret

# Check the evaluation loss for the initial predictions
print(eval_loss(yinit, dtrain), 'should be around .26')
```

```
0.2638596794329442 should be around .26
```

We have thus implemented the function to calculate the loss We have verified that the initial loss is around .26

Question 16

[21]:
```
# Function to calculate the percentage
def eval_perfs(y, lbl):
    # Convert the given probabilities to corresponding label
    pred_lbl = onehot2label(y)

    # Compare the groundtruth with the predicted label and identify␣
↪Misclassified samples
    comps = [pred_lbl != lbl]
    nums = np.sum(comps)
    ret = (nums * 1.0 / lbl.shape[0]) * 100.0
    return ret

# Print the percentage of "mis-classified" samples for the initial predicted␣
↪probabilities
# and the groundtruth labels
print(eval_perfs(yinit, ltrain), "% of the images are misclassified")
```

```
91.175 % of the images are misclassified
```

We implemented the function to calculate the percentage of mis-classified samples We picked the index with the maximum value(probability) for each column using the $y.argmax(axis = 0)$ function This index is basically the predicted class of the given image(column) We then compared the predicted label with the actual label, calculated the number of misclassified images, and then divided by the total number of images to get the percentage of mis-classification

1.4.1 Question 17

We need to show that

$$(\nabla_y E)_i = -\frac{d_i}{y_i}$$

E is cross-entropy loss, and is given by

$$E = -\sum_{i=1}^{10} d_i log(y_i)$$

Differentiating with respect to y_i and taking $-d_i$ as common, we have

$$(\nabla_y E)_i = -d_i * \frac{\partial\left(\sum_{i=1}^{10} log(y_i)\right)}{\partial y_i}$$

Thus, we have

$$(\nabla_y E)_i = -d_i * \left(\frac{1}{y_i}\right)$$

Hence, proved that

$$(\nabla_y E)_i = -\frac{d_i}{y_i}$$

[22]:
```
# Function to perform backpropagation in the network
def update_shallow(x, d, net, gamma=.05):
    W1 = net[0]
    b1 = net[1]
    W2 = net[2]
    b2 = net[3]

    Ni = W1.shape[1]
    Nh = W1.shape[0]
    No = W2.shape[0]

    # Normalize the gamma by the training dataset size
    gamma = gamma / x.shape[1]

    ## Backprop begins!
    # Forward prop through the network using current parameters
    # This calculates the predicted probabilities for each class
    # working dim - 64*60000 - h1; 10*60000 - y
    a1 = W1.dot(x) + b1
    h1 = relu(a1)

    # Hidden to output neurons
    a2 = W2.dot(h1) + b2
    y = softmax(a2)

    # Calculate the loss
```

```
# e = eval_loss(y_pred, dtrain)

## Backprop through output neurons to hidden neurons
# Calculate the gradient of the error for output neurons
# working dim - 10*60000
# DEBUG
#print(d.shape, y_pred.shape)
#print(np.min(d), np.max(d))
#print(np.min(y_pred), np.max(y_pred))
e2 = -1.0 * d / y

# calculate derivative of softmax() activation
# working dim - 10*60000
#delta2 = softmaxp(y_pred, e2)
#delta2 = softmaxp(y, e2)
delta2 = softmaxp(a2, e2)

# Calculate the derivative of E wrt W2
# working dim - 10*60000 * 60000*64(h1.T)
grad_w2_e = delta2.dot(h1.T)

# Calculate the derivative of E wrt b2
# working dim - 10*60000 * 60000*1 = 10*1
grad_b2_e = delta2.dot(np.ones((delta2.shape[1], 1)))

# Calculate the gradient of the error for the hidden neurons
# Working dim - 64*60000
# 64*10(W2 is 10*64) * 10*60000
e1 = W2.T.dot(delta2)

# Calculate the derivative of the relu() activation
# working dim - 64*60000
#delta1 = relup(h1, e1)
delta1 = relup(a1, e1)

# Calculate the derivative of E wrt W1
# working dim - 64*60000 * 60000*784(x.T) (h0 = x)
grad_w1_e = delta1.dot(x.T)

# Calculate the derivative of E wrt b1
# working dim - 64*60000 * 60000*1 = 64*1
grad_b1_e = delta1.dot(np.ones((delta1.shape[1], 1)))

## UPDATE the parameters
W2 = W2 - gamma * grad_w2_e
W1 = W1 - gamma * grad_w1_e
b2 = b2 - gamma * grad_b2_e
```

```
    b1 = b1 - gamma * grad_b1_e

    # return the updated parameters
    return W1, b1, W2, b2
```

Thus, we have written the function to perform one backpropagation update for our shallow network. We have also proved that

$$(\nabla_y E)_i = -\frac{d_i}{y_i}$$

We then used $softmaxp()$ and $relup()$ to calculate the gradients We implemented the backpropagation layer-wise from the output neurons to the hidden neurons. We coded the backpropagation as given:-

$$W_k^{t+1} = W_k^t - \gamma \nabla_{w_k} E^t$$
$$b_k^{t+1} = b_k^t - \gamma \nabla_{b_k} E^t$$
$$where \quad \nabla_{w_k} E = \delta_k h_{k-1}^T$$
$$and \quad \nabla_{b_k} E = \delta_k 1_N$$
$$and \quad \delta_k = \left[\frac{\partial g_k(a_k)}{\partial a_k}\right]^T \times e_k$$
$$where \quad e_k = \begin{cases} \nabla_y E & \text{if k is an output layer} \\ W_{k+1}^T \delta_{k+1} & \text{otherwise} \end{cases}$$

We finally return the network parameters $W_1, b_1, W_2 and b_2$ after one iteration of backpropagation to the caller

Question 18

[23]:
```
# To compute backprop_shallow
def backprop_shallow(x, d, net, T, gamma=.05):
    # Get the label given the one-hot encoding
    lbl = onehot2label(d)

    # Compute and display the loss and performance measure initially
    y = forwardprop_shallow(x, net)
    tr_loss = eval_loss(y, d)
    print("Initial loss is:", tr_loss)
    tr_perf = eval_perfs(y, lbl)
    print(tr_perf, "% of images are misclassified initially\n")

    for t in range(T):
        # update the parameters using the update_shallow() function
        net = update_shallow(x, d, net, gamma)

        # Compute and display the loss and performance measure for every 10␣
↪iterations
        if t%10 == 9:
            y = forwardprop_shallow(x, net)
```

```
            tr_loss = eval_loss(y, d)
            print("Training loss after iteration", t+1, "is:", tr_loss)
            tr_perf = eval_perfs(y, lbl)
            print(tr_perf, "% of images are misclassified after iteration",␣
↪t+1,"\n")
    return net
```

[24]:
```
# Train the net for 2 iterations initially. The output is the final parameters␣
↪after training
# Now training net for 100 iterations
nettrain = backprop_shallow(norm_x_train, dtrain, netinit, 100)
```

```
Initial loss is: 0.2638596794329442
91.175 % of images are misclassified initially

Training loss after iteration 10 is: 0.16191209907405543
42.126666666666665 % of images are misclassified after iteration 10

Training loss after iteration 20 is: 0.11597192658503634
28.148333333333333 % of images are misclassified after iteration 20

Training loss after iteration 30 is: 0.09096238435774094
22.475 % of images are misclassified after iteration 30

Training loss after iteration 40 is: 0.08444967562032382
26.521666666666665 % of images are misclassified after iteration 40

Training loss after iteration 50 is: 0.07062631389995717
20.724999999999998 % of images are misclassified after iteration 50

Training loss after iteration 60 is: 0.061937629673339566
16.713333333333333 % of images are misclassified after iteration 60

Training loss after iteration 70 is: 0.05701452523889525
15.273333333333333 % of images are misclassified after iteration 70

Training loss after iteration 80 is: 0.05346541739673409
14.456666666666667 % of images are misclassified after iteration 80

Training loss after iteration 90 is: 0.05070839570076598
13.766666666666666 % of images are misclassified after iteration 90

Training loss after iteration 100 is: 0.048489191601058614
13.243333333333332 % of images are misclassified after iteration 100
```

Wrote the code for *backprop_shallow*() to train the network This function performs T updates of the network by calling one instance of the backpropagation function *update_shallow*() each time

We evaluate the loss initially and after each iteration by calling the $eval_loss()$ function and then display it Similarly, we evaluate the percentage of images mis-classified initially and after each iteration by calling the $eval_perfs()$ function and then display it This function finally returns the completely trained parameters W_1, b_1, W_2, b_2 after T iterations and stores it in *nettrain*

2 iterations

We tried running the code with 2 iterations initially. 2 iterations worked in reducing the loss from 0.255 to 0.219 It also reduced the percentage of misclassified images from 91.41% to 80.23% So we moved onto testing the function with 5 iterations

5 iterations

5 iterations worked in reducing the loss from 0.264 to 0.183 It also reduced the percentage of misclassified images from 89.82% to 49.83% So, we moved onto testing the function with 20 iterations

20 iterations

20 iterations worked in reducing the loss from 0.270 to 0.127 It also reduced the percentage of misclassified images from 91.122% to 34.36% We finally run the network with 100 iterations

100 iterations

100 iterations worked in reducing the loss from 0.264 to 0.048 It also reduced the percentage of misclassified images from 91.175% to 13.243% This result is also consistent with the value given in the question. We indeed reached about 13% of training errors with $T = 100$ iterations

Question 19

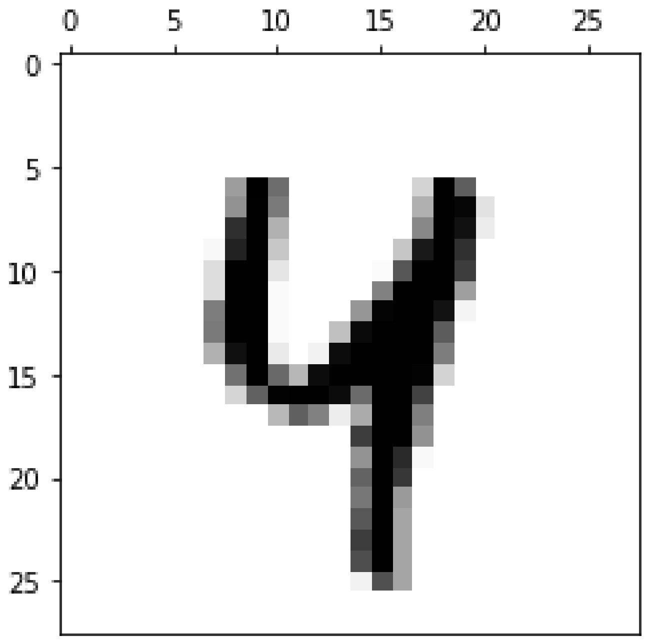
[25]:
```
# Load the testing data
xtest, ltest = MNISTtools.load(dataset='testing', path='./datasets/MNIST')
print(xtest.shape)
print(ltest.shape)
```

```
(784, 10000)
(10000,)
```

The size of the testing set of images is found to be $(784, 10000)$, that is, it contains 10000 images

[26]:
```
# Displaying the image at index 42
MNISTtools.show(xtest[:, 42])

# Print its corresponding label
print(ltest[42])
```



4

[27]:
```
# Find the range and type of xtest
min_te_x = np.amin(xtest)
max_te_x = np.amax(xtest)

print("Range of xtest is from ", min_te_x, " to ", max_te_x)
print("Data type of xtest is ", xtest.dtype)
```

```
Range of xtest is from  0  to  255
Data type of xtest is  uint8
```

[28]:
```
# Normalize the test images
norm_x_test = normalize_MNIST_images(xtest)
print(norm_x_test.shape)
print("Range of normalized xtest is", np.amin(norm_x_test), "to", np.
 ↪amax(norm_x_test))
print("Data type of normalized xtest is", norm_x_test.dtype)
```

```
(784, 10000)
Range of normalized xtest is -1.0 to 1.0
Data type of normalized xtest is float32
```

[29]:
```
dtest = label2onehot(ltest)
print(dtest.shape)
print(np.amin(dtest), np.amax(dtest))
```

```
print("Label at index 42 is", ltest[42])
print("Corresponding one-hot encodiing is", dtest[:, 42])
```

```
(10, 10000)
0.0 1.0
Label at index 42 is 4
Corresponding one-hot encodiing is [0. 0. 0. 0. 1. 0. 0. 0. 0. 0.]
```

[30]:
```
# Compute and display the loss and performance measure on the test set
y_test = forwardprop_shallow(norm_x_test, nettrain)
te_loss = eval_loss(y_test, dtest)
print("Test loss is:", te_loss)
te_lbl = onehot2label(dtest)
te_perf = eval_perfs(y_test, te_lbl)
print(te_perf, "% of images are misclassified in the test set\n")
```

```
Test loss is: 0.04600023010648992
12.24 % of images are misclassified in the test set
```

Thus, we have loaded the testing dataset

2 iterations testing vs training

We tested the performance of the network parameters that were trained for 2 iterations and we got a loss of 0.218 on this test set Training loss after 2 iterations was: 0.219 79.28% of the images are mis-classified from the test set after the training 80.23% of images are misclassified in the training set after 2 iterations

5 iterations testing vs training

We tested the performance of the network parameters that were trained for 5 iterations and we got a loss of 0.181 on this test set Training loss after 5 iterations was: 0.183 48.14% of the images are mis-classified from the test set after the training 49.83% of images are misclassified in the training set after 5 iterations

20 iterations testing vs training

We tested the performance of the network parameters that were trained for 20 iterations and we got a loss of 0.124 on this test set Training loss after 20 iterations was: 0.127 33.42% of the images are mis-classified from the test set after the training 34.36% of images are misclassified in the training set after 20 iterations

100 iterations testing vs training

We tested the performance of the network parameters that were trained for 100 iterations and we got a loss of 0.046 on this test set Training loss after 100 iterations was: 0.048 12.24% of the images are mis-classified from the test set after the training 13.243% of images are misclassified in the training set after 100 iterations

Question 20

[31]:
```
# Backprop using minibatch
def backprop_minibatch_shallow(x, d, net, T, B=100, gamma=0.05):
```

```
    # Get the number of images
    N = x.shape[1]

    # Calculate the number of batches
    NB = int((N+B-1)/B)

    # Convert one-hot encoded data to a label
    lbl = onehot2label(d)

    # Compute and display the loss and performance measure initially
    y = forwardprop_shallow(x, net)
    tr_mini_loss = eval_loss(y, d)
    print("Initial minibatch loss is:", tr_mini_loss)
    tr_mini_perf = eval_perfs(y, lbl)
    print(tr_mini_perf, "% of images are misclassified initially using minibatch␣
↪method\n")

    # For every iteration(epoch)
    for t in range(T):
        # shuffle the indices to access the data
        shuffled_indices = np.random.permutation(range(N))

        # For each minibatch
        for l in range(NB):
            # get the shuffled indices for a given minibatch
            minibatch_indices = shuffled_indices[B*l:min(B*(l+1), N)]
            # Backprop through the minibatch and update the parameters of the␣
↪network
            net = update_shallow(x[:, minibatch_indices], d[:,␣
↪minibatch_indices], net, gamma)

        y = forwardprop_shallow(x, net)
        tr_mini_loss = eval_loss(y, d)
        print("Training loss using minibatches after epoch", t+1, "is:",␣
↪tr_mini_loss)
        tr_mini_perf = eval_perfs(y, lbl)
        print(tr_mini_perf, "% of images are misclassified using minibatches␣
↪after epoch", t+1,"\n")
    return net
```

Wrote the code for *backprop_minibatch_shallow*() to train the network In minibatch backpropagation, we divided the dataset into a number of mini-batches, each sized 100 images We thus updated the parameters of our network TN/B times We first calculate the number of batches to train for We then shuffle the entire dataset, and for each minibatch, we update the parameters of the network using $update_shallow()$ function We evaluate the loss initially and after each epoch by calling the *eval_loss*() function and then display it Similarly, we evaluate the percentage of images mis-classified initially and after each epoch by calling the *eval_perfs*() function and then display it This function finally returns the completely trained parameters W_1, b_1, W_2, b_2 after T epochs and stores it in *netminibatch*

Question 21

[32]:
```
# Train the network for a few epochs
print(np.min(norm_x_train), np.max(norm_x_train))
netminibatch = backprop_minibatch_shallow(norm_x_train, dtrain, netinit, 5,␣
 ↪B=100)
```

```
-1.0 1.0
Initial minibatch loss is: 0.2638596794329442
91.175 % of images are misclassified initially using minibatch method

Training loss using minibatches after epoch 1 is: 0.03001518878674801
8.746666666666666 % of images are misclassified using minibatches after epoch 1

Training loss using minibatches after epoch 2 is: 0.024412366297609372
7.1499999999999995 % of images are misclassified using minibatches after epoch 2

Training loss using minibatches after epoch 3 is: 0.020629920459641838
6.0183333333333335 % of images are misclassified using minibatches after epoch 3

Training loss using minibatches after epoch 4 is: 0.01799265925320068
5.258333333333334 % of images are misclassified using minibatches after epoch 4

Training loss using minibatches after epoch 5 is: 0.01602968399180054
4.575 % of images are misclassified using minibatches after epoch 5
```

2 epochs minibatch

We tried running the code with 2 epochs initially. 2 epochs using minibatches worked in reducing the loss from 0.255 to 0.025 Using minibatches also reduced the percentage of misclassified images from 91.41% to 7.145%

Without Minibatch - 2 iterations

This is different from running with 2 iterations over the whole training set 2 iterations worked in reducing the loss from 0.255 to 0.219 It also reduced the percentage of misclassified images from 91.41% to 80.23%

5 epochs minibatch

We tried running the code with 5 epochs after this. 5 epochs using minibatches worked in reducing the loss from 0.264 to 0.015 Using minibatches also reduced the percentage of misclassified images from 89.82% to 4.6%

Without Minibatch - 5 iterations

This is different from running with 5 iterations over the whole training set 5 iterations worked in reducing the loss from 0.264 to 0.183 It also reduced the percentage of misclassified images from 89.82% to 49.83%

[33]:
```
# Compute and display the loss and performance measure on the test set
y_mini_test = forwardprop_shallow(norm_x_test, netminibatch)
te_mini_loss = eval_loss(y_mini_test, dtest)
print("Test loss after minibatch gradient descent is:", te_mini_loss)
te_mini_perf = eval_perfs(y_mini_test, te_lbl)
print(te_mini_perf, "% of images are misclassified in the test set after␣
 ↪minibatch gradient descent\n")
```

```
Test loss after minibatch gradient descent is: 0.016790431647965853
4.9799999999999995 % of images are misclassified in the test set after minibatch
gradient descent
```

2 epochs minibatch - Test vs. Train

We tested the performance of the network parameters that were trained for 2 epochs and we got a loss of 0.025 on this test set Training loss after 2 epochs was: 0.025 6.99% of the images are mis-classified from the test set after the minibatch training 7.145% of images are misclassified in the training set after 5 epochs

2 iterations without minibatch testing set vs. 2 epochs minibatch testing set

Testing loss after training without minibatches for 2 iterations was : 0.218 79.28% of the images are mis-classified from the test set without minibatch training

5 epochs minibatch - Test vs. Train

We tested the performance of the network parameters that were trained for 5 epochs and we got a loss of 0.016 on this test set Training loss after 5 epochs was: 0.015 4.82% of the images are mis-classified from the test set after the minibatch training 4.6% of images are misclassified in the training set after 5 epochs

5 iterations without minibatch testing set vs. 5 epochs minibatch testing set

Testing loss after training without minibatches for 5 iterations was : 0.181 48.14% of the images are mis-classified from the test set without minibatch training

1.4.2 Observations

After training for 100 iterations over the entire dataset, our network had a training loss of 0.048 and testing set loss of 0.046. This is higher compared to training the network parameters with minibatches for just 5 epochs, where the training loss was 0.015 and the testing set loss was 0.016 The percentage of mis-classified samples follows the same trend. After training for 100 iterations over the entire dataset, our network had mis-classification percentage as 13.243% on the training set and 12.24% on the testing set. This again is a higher error percentage for mis-classified images compared to training the network parameters on random minibatches for just 5 epochs With just 5 epochs of training on minibatches, our network had mis-classification percentage as 4.6% on the training set and 4.82% on the testing set, which is better than the results we obtained after training the network parameters for 100 iterations on the entire dataset

1.4.3 Inferences

Comparing the performance of the network using minbatches vs. not using it, we conclude that training using minibatch gradient descent gives improved performance compared to training on the entire dataset. One possible reason could be that taking random minibatches introduces the network to a large number of sets of images, thus leading to more network parameter updates. This is also contributed by the fact that this small minibatch manages to capture the distribution of the entire dataset, thus leading to minimal information loss that helps to update the parameters successfully.

1.5 Conclusion

Thus, we have learnt about shallow networks, and implemented a simple shallow feedforward network to classify MNIST handwritten images. We trained the whole network over multiple iterations, as well as trained it using minibatches and compared their performance.

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