Gravitation Wave Identification

Using Metropolis Hastings Algorithm

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Problem Statement

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Given a time series strain data with added noise with the structure of a gravitational wave as given below

$$h(t) = \alpha e^t [1 - \tanh\{2(t - \beta)\}] \sin(\gamma t)$$

α, β, γ are parameters that signify the physical properties of the given wave. There value ranges are

$$0 < \alpha < 2$$
$$1 < \beta < 10$$
$$1 < \gamma < 20$$

We need to determine the parameter values using a Metropolis Hastings Random walk algorithm in the 3 dimensional space.

Understanding Wave Parameters

Let us visualize how the parameters α, β, and γ influence the waveform

1. α controls the amplitude of the signal
2. β shifts the signal in time
3. γ controls the oscillation frequency

Animation Paramater effects

Methodology

Random Walks

1. Initialization: We start with initial parameter values at the midpoints of the given ranges so

$$\alpha = 1, \beta = 5, \gamma = 10$$

2. Random Walk:

For each iteration we propose a new set of parameters using

$$\theta_{\text{new}} \sim N(\theta_{\text{initial}}, \sigma^2) \quad \text{where} \ \ \sigma = [0.005, 0.081, 0.2]$$

The new value is discarded or chosen based on an Acceptance Probability defined as

$$A(\theta_{\text{new}}, \theta_{\text{initial}}) = \min\left(1, \frac{\text{Posterior}(\theta_{\text{new}})}{\text{Posterior}(\theta_{\text{initial}})}\right)$$

The Posterior function is defined as the following

$$\text{Posterior}(\theta_{\text{i}}) = P(\theta_{\text{i}} \mid \text{data}) \propto P(\text{data} \mid \theta_{\text{i}})P(\theta_{\text{i}})$$

Due to the assumptions taken, $P(\theta)$ has no effect on the acceptance ratio and our algorithm as a whole.

$$P(\theta) := \begin{cases} \text{constant if } \theta \in \theta_{\text{constraint}} \\ 0 \text{ everywhere else} \end{cases}$$

The only significant metric to consider is now the Likelihood Function $L(\theta)$ which we define as

$$L(\theta) = \exp\left(-\frac{1}{2N} \sum_{i} \frac{\left(\mathrm{y}_{\mathrm{i}} - \mathrm{f}(\theta_{\mathrm{i}})\right)^2}{\sigma_{\mathrm{i}}^2}\right)$$

Heteroscedastic Gaussians and Likelihood Choice

1. Gaussian Noise Assumption: We assume the observed data can be written as

$$y_i = f(t_i; \theta) + \varepsilon_i$$

where each ε_i is sampled from a Gaussian distribution.

1. Heteroscedasticity: The noise variance depends on the data index i, so each point has its own standard deviation

$$\sigma_i \propto y_i$$

This gives us a Heteroscedastic Gaussian noise model.

1. Resulting Likelihood: By substituting

$$\varepsilon_i = y_i - f(\theta)_i$$

into the Gaussian PDF, the likelihood becomes

$$P(\varepsilon_i) = \exp\left(-\frac{1}{\sqrt{2\pi\sigma^2}}\frac{\varepsilon_i}{\sigma_{\mathrm{i}}^2}\right)$$

1. Why the $\frac{1}{N}$ Scaling?

- Large N will yield bad scores to good parameters.
- Softens the curvature of posterior geography.
- Improves stability and mixing without changing relative acceptance ratios.

1. Why Gaussian? Gaussian noise implies:

- Smooth deviations
- No strong outliers
- Residuals cluster around zero

leading to the exponential–squared–error form.

1. Why Not L1 or Student–t?

- Laplace / L1: better when data has sharp spikes or extreme outliers.
- Student–t: heavy tails, suited for bursty or glitchy noise. Very useful for practical implementations.

Hello

Results

Parameter Values

Parameter	α (alpha)	β (beta)	γ (gamma)
Median Value	1.38	3.91	10.00
95% Credibility Interval	0.92 - 1.91	3.63 - 4.19	9.92 - 10.08
Effective Sample Size	25.0	61.1	1800.0
MC Standard Error	0.049	0.019	0.001

The MCMC Algorithm ran with Acceptance Ratio of 0.222.

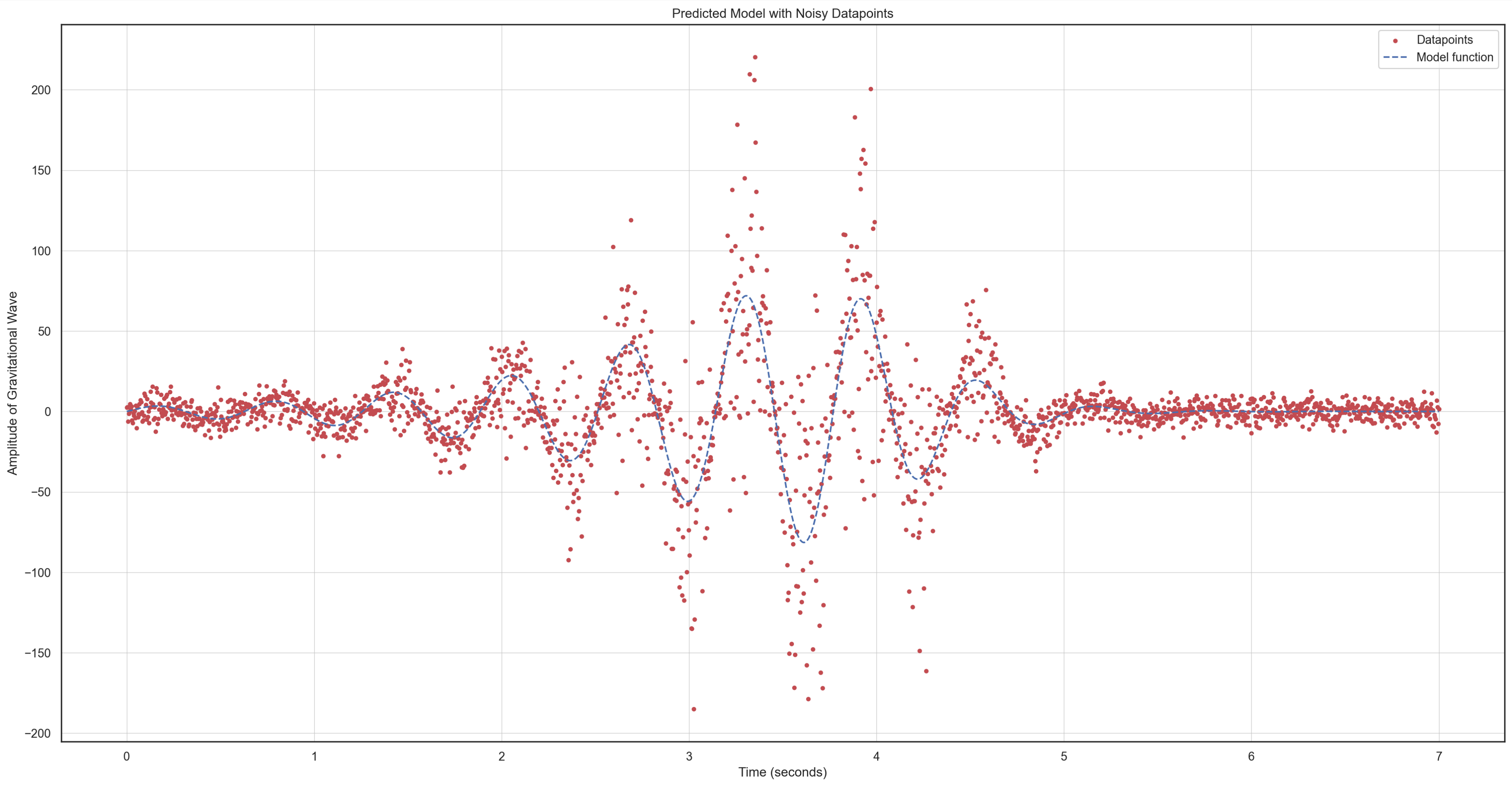
The Global Signal to Noise Ratio was 0.97, with Local SNR of 0.99

Measurement Metrics for Metropolis Hastings

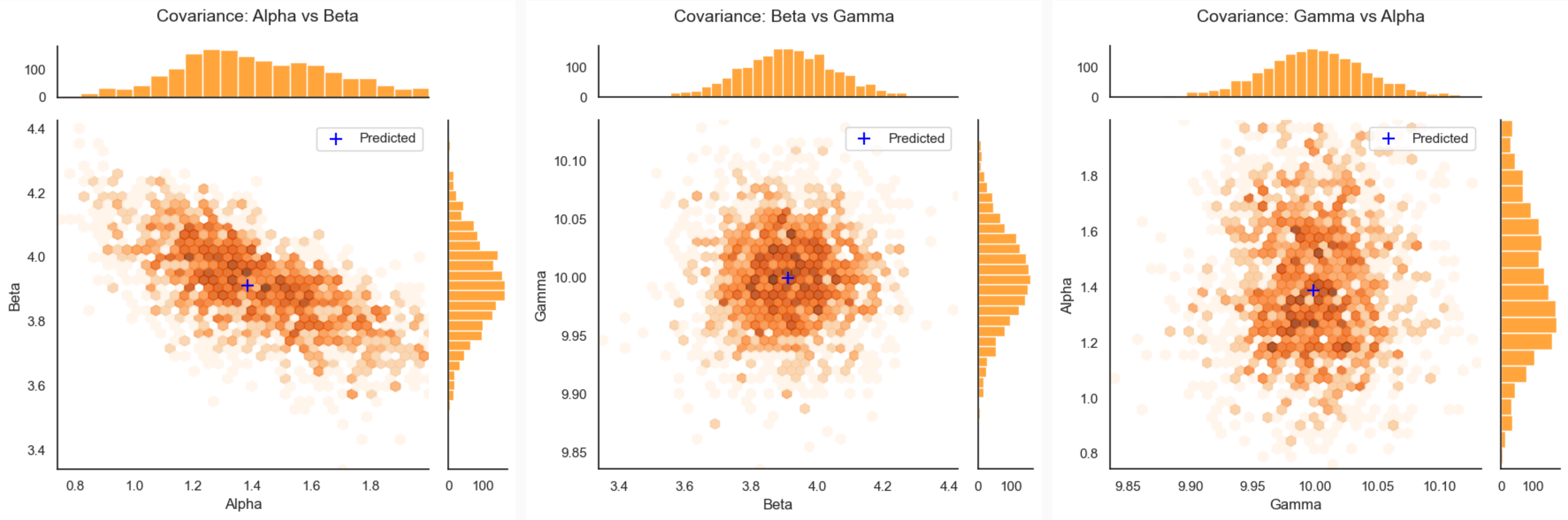
Talk about signal to noise ratio, ESS, AFC, MC Std Err

Autocorrelation measures how much each MCMC sample depends on its predecessors.

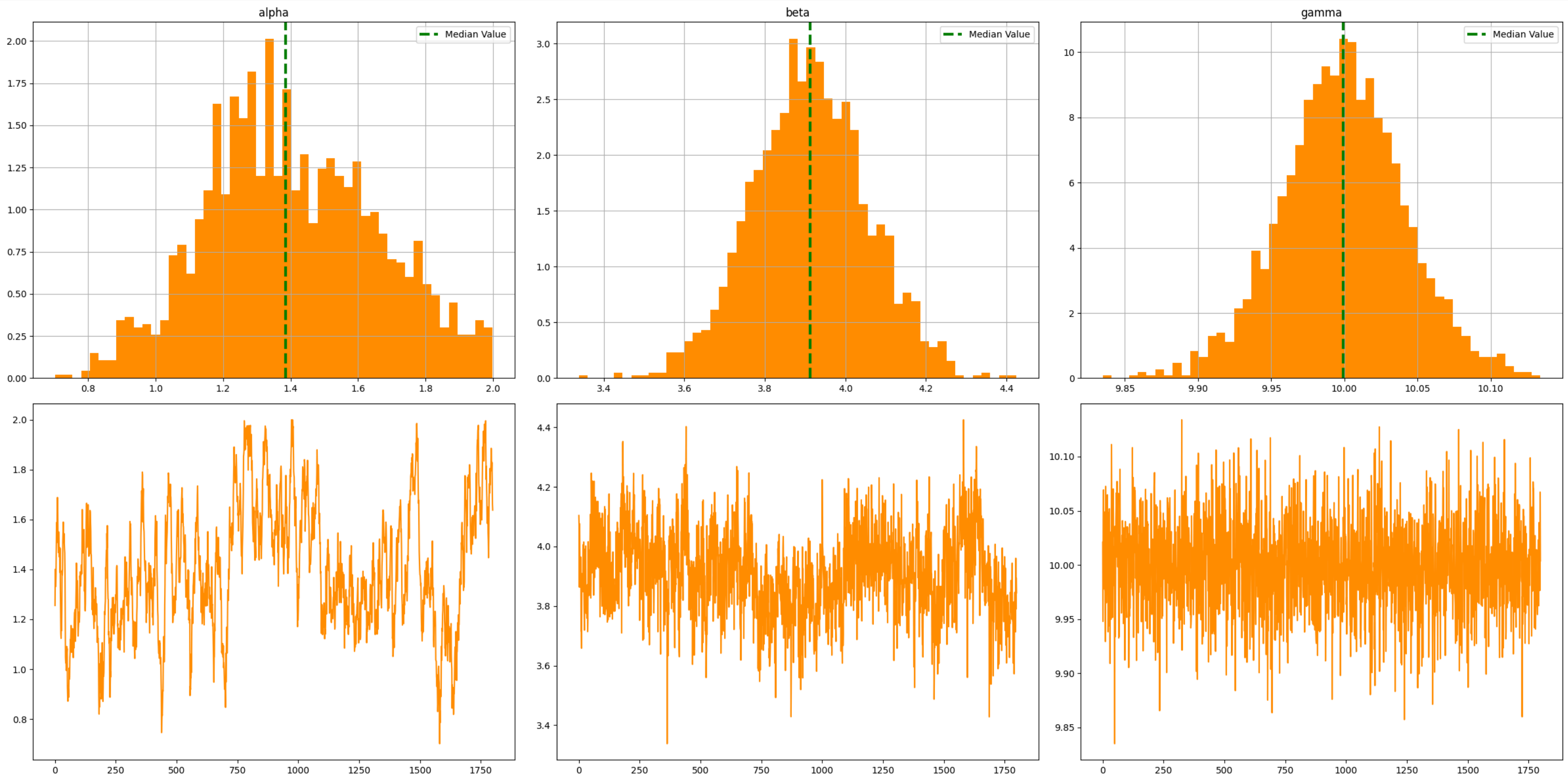
Prediction vs Data



Covariance Scatter Plots

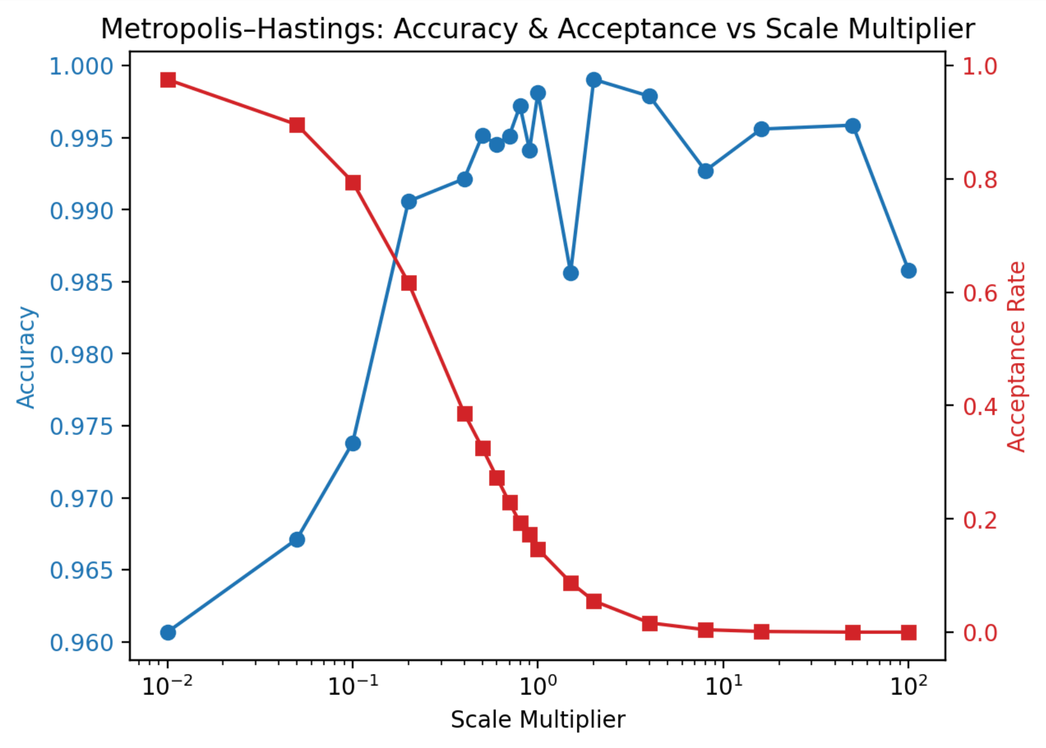


Histograms and Trace Plots



Optimization

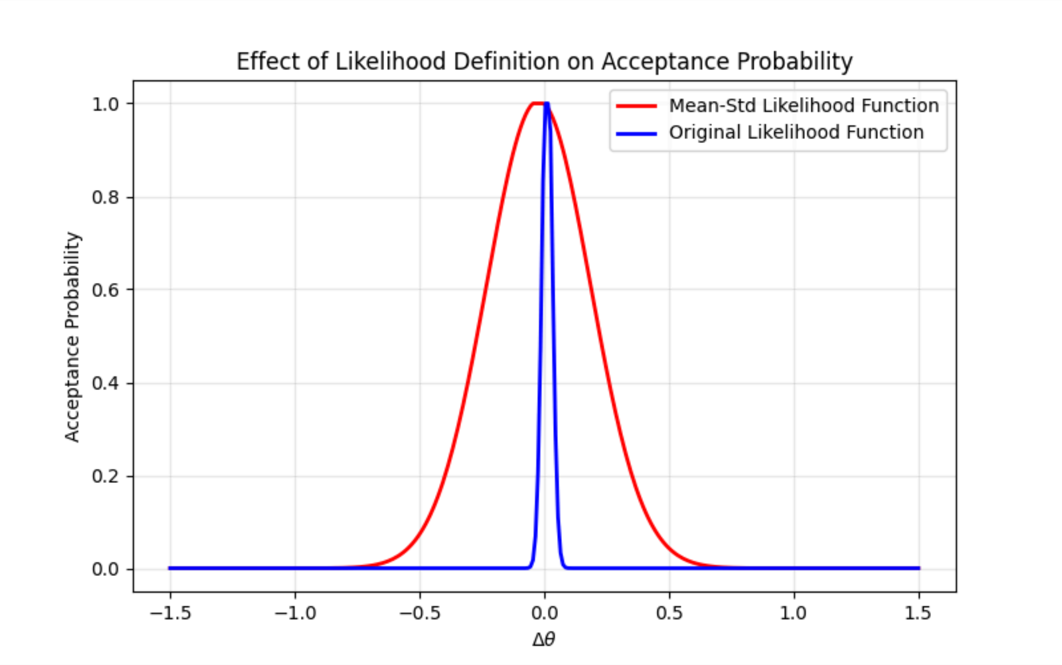
Scale Selection



- Small Scales ($< 10^{-1}$)
  - Chain barely moves ⟶ strong autocorrelation ⟶ low accuracy.
- Chosen Scales (α: 0.005, β: 0.1, γ: 0.2)
  - (Roberts & Rosenthal, 2004) predicts optimal acceptance ≈ 0.234 for high-dimensional targets.
  - Accuracy peaks — this is the optimal region for efficient sampling.
- Large Scales (> 5)
  - Acceptance rate falls ⟶ chain stagnates ⟶ accuracy degrades.

Likelihood Function

-> The original likelihood makes even tiny parameter changes look catastrophically unlikely, driving acceptance to near zero. The new version fixes this by taking a mean instead of summation, keeping acceptance stable.




```
def likelihood_new(y_data, y_prior):
    y_err = 0.1 * np.std(y_data)
    Y = np.mean((y_data-y_prior)**2)/y_err**2
    return -0.5 * Y

def likelihood_original(y_datay, y_priory):
    y_err = 0.1 * (y_data + y_prior) + 1e-6
    Y = np.sum(((y_data - y_prior)/y_err) ** 2 )
    return -0.5 * Y
```

Data Generation for Better Inference

System Overview

The data generation pipeline accepts an analytical gravitational wave model function and produces a CSV file containing noisy observations at discrete time points. This system serves two primary purposes:

- Algorithm Validation: Generate data with known ground truth parameters to verify MCMC recovery accuracy
- Sensitivity Analysis: Test algorithm performance under varying noise conditions and proposal scales
- Robustness Check : Validate algorithm stability under heteroscedastic noise to assess reliability across signal regimes.

Noise Formula

The noise standard deviation for each data point is calculated as:

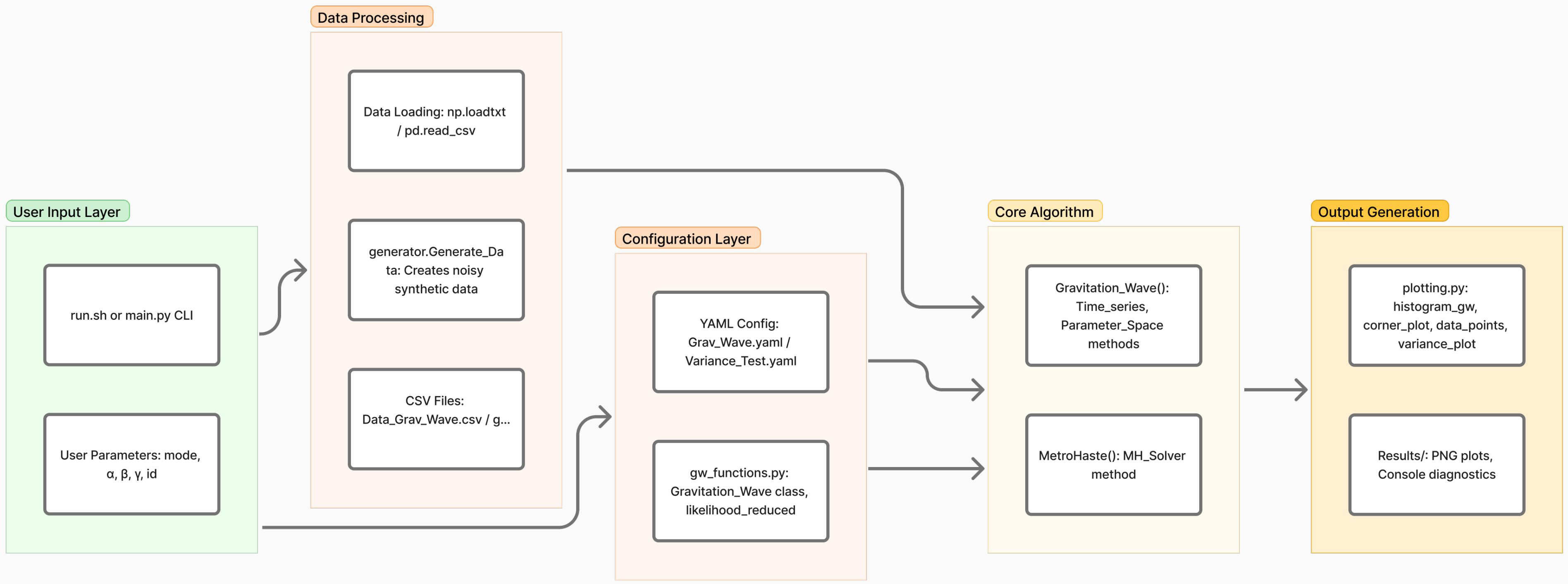
```
f_error = noise x f_points + 0.1 x std(f_points)
f_noisy = f_points +
        np.random.normal(0, np.abs(f_error), num)
```

This creates heteroscedastic noise where measurement error grows with signal amplitude.

Noise Characteristics

Component	Purpose	Typical Magnitude
Proportional	Scales with signal amplitude	Dominant near peaks
Baseline	Constant floor noise	4–8 for typical signals

Code Structure



Thank You