Gravitation Wave Identification

Using Metropolis Hastings Algorithm

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Problem Statement

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Given a time series strain data with added noise with the structure of a gravitational wave as given below

$$h(t) = \alpha e^{t}[1 - \tanh\{2(t - \beta)\}]\sin(\gamma t)$$

α, β, γ are parameters that signify the physical properties of the given wave. There value ranges are

$$0 < \alpha < 2$$
$$1 < \beta < 10$$
$$1 < \gamma < 20$$

We need to determine the parameter values using a Metropolis Hastings Random walk algorithm in the 3 dimensional space.

Understanding Wave Parameters

Let us visualize how the parameters α, β, and γ influence the waveform

1. α controls the amplitude of the signal
2. β shifts the signal in time
3. γ controls the oscillation frequency

Animation Paramater effects

Methodology

Random Walks

1. **Initialization:** We start with initial parameter values at the midpoints of the given ranges so

$$\alpha = 1, \beta = 5, \gamma = 10$$

2. **Random Walk:**

For each iteration we propose a new set of parameters using

$$\theta_{\text{new}} \sim N\left(\theta_{\text{initial}}, \sigma^2\right) \quad \text{where} \;\; \sigma = [0.005, 0.081, 0.2]$$

The new value is discarded or chosen based on an Acceptance Probability defined as

$$A(\theta_{\mathrm{new}}, \theta_{\mathrm{initial}}) = \min\left(1, \frac{\mathrm{Posterior}(\theta_{\mathrm{new}})}{\mathrm{Posterior}(\theta_{\mathrm{initial}})}\right)$$

The Posterior function is defined as the following

$$\mathrm{Posterior}(\theta_{\mathrm{i}}) = P(\theta_{\mathrm{i}} \mid \mathrm{data}) \propto P(\mathrm{data} \mid \theta_{\mathrm{i}})P(\theta_{\mathrm{i}})$$

Due to the assumptions taken, $P(\theta)$ has no effect on the acceptance ratio and our algorithm as a whole.

$$P(\theta) \coloneqq \begin{cases} \text{const if } \theta \in \theta_{\text{constraint}} \\ 0 \text{ everywhere else} \end{cases}$$

The only significant metric to consider is now the Likelihood Function $L(\theta)$ which we define as

$$L(\theta) = \exp\left(-\frac{1}{2N} \sum_{i} \frac{\left(\mathrm{y_i} - \mathrm{f}(\theta_\mathrm{i})\right)^2}{\sigma_\mathrm{i}^2}\right)$$

Results

Numerical Analysis

Parameter Values

Parameter	α (alpha)	β (beta)	γ (gamma)
Median Value	1.38	3.91	10.00
95% Credibility Interval	0.92 - 1.91	3.63 - 4.19	9.92 - 10.08
Effective Sample Size	25.0	61.1	1800.0
MC Standard Error	0.049	0.019	0.001

The MCMC Algorithm ran with **Acceptance Ratio** of 0.222.

The Global **Signal to Noise Ratio** was 0.97, with Local SNR of 0.99

Measurement Metrics for Metropolis–Hastings

- **Signal-to-Noise Ratio (SNR):**

Measures how strongly the true signal stands out from the noise. A Global SNR of 0.97 indicates an extremely noisy signal.

- **Effective Sample Size (ESS):**

MCMC samples are correlated, so the **true** number of independent samples is smaller. ESS quantifies this:

$$\mathrm{ESS} = \frac{N_{\mathrm{samples}}}{1 + 2\sum_{k=1}^{\infty} \rho_{\mathrm{k}}}$$

where ρ_{k} is the autocorrelation at lag k.

- **Monte Carlo Standard Error (MCSE):**

Estimates the uncertainty in the posterior mean **due to sampling noise**. It is defined as

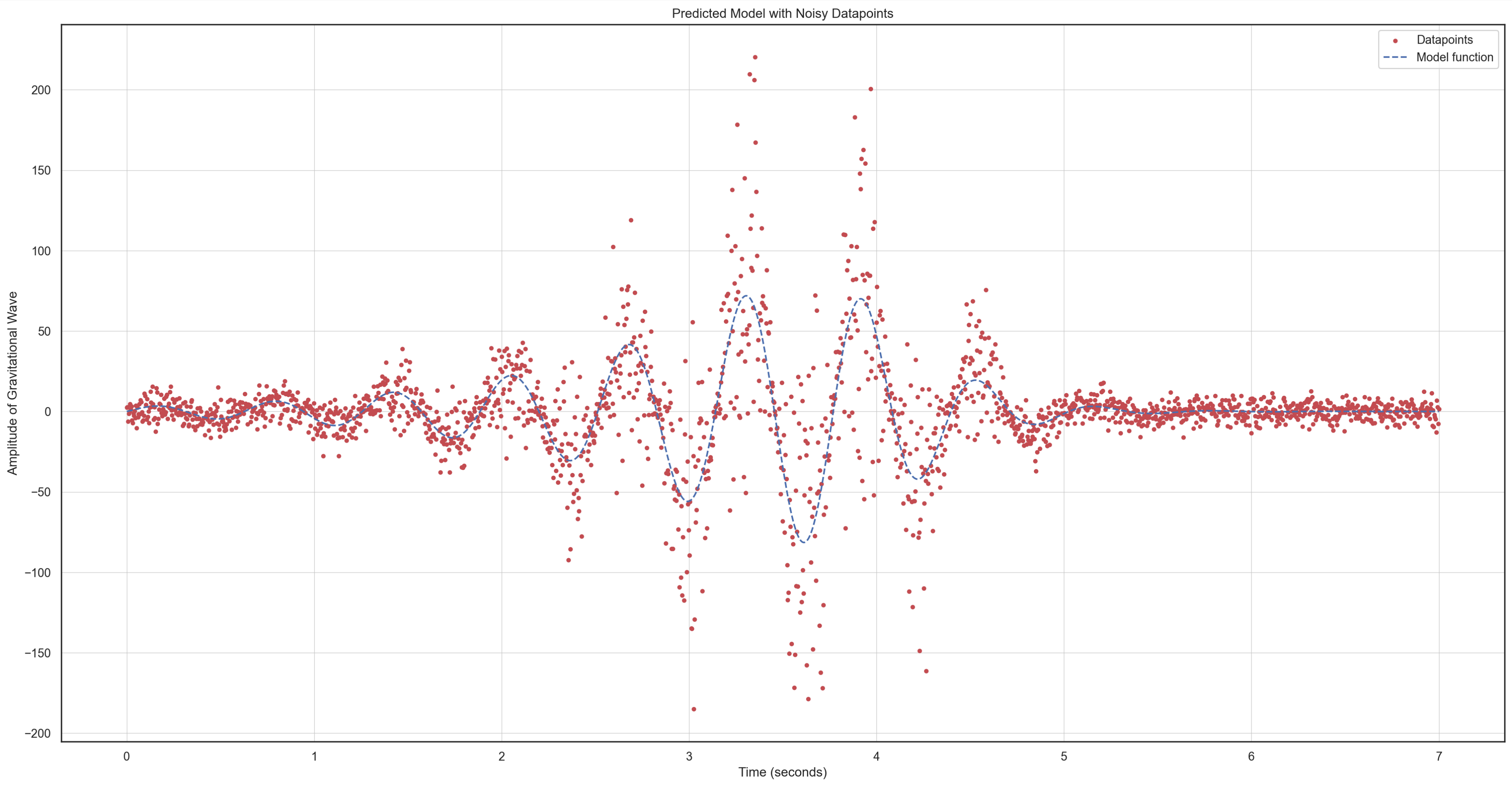
$$\text{MCSE} = \frac{\sigma}{\sqrt{\text{ESS}}}$$

Low MCSE indicates that the chain produced enough effective samples for reliable estimation of parameter statistics.

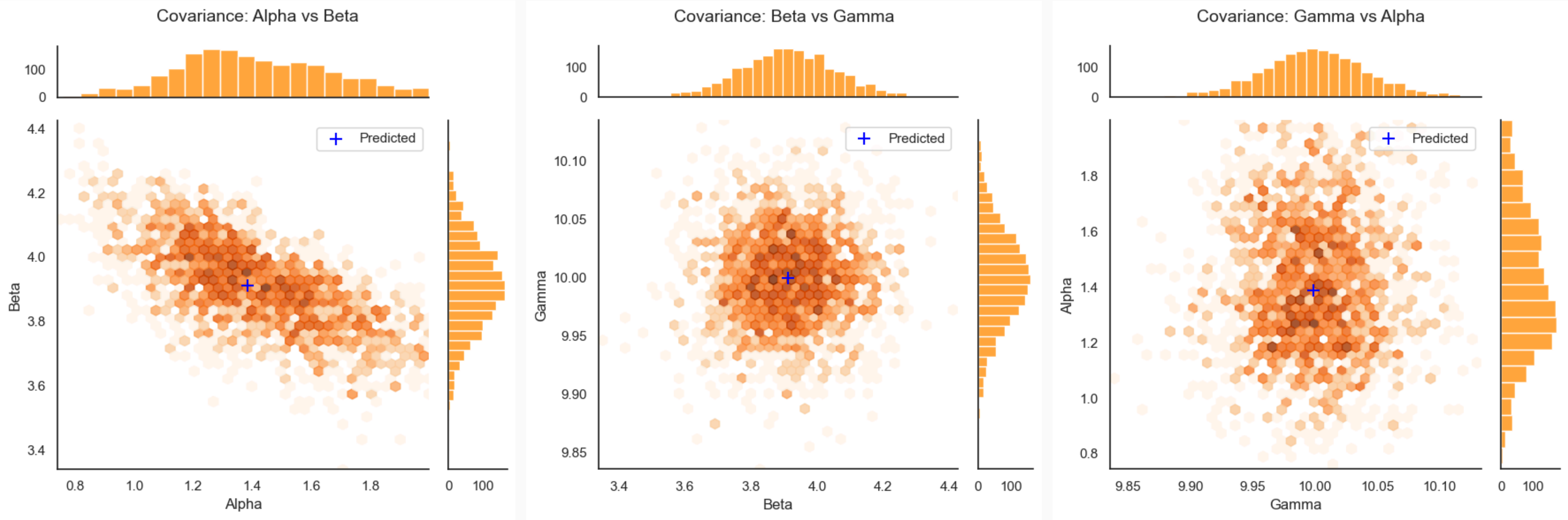
- **Autocorrelation:**

Measures how strongly each MCMC sample depends on earlier samples. High autocorrelation means slow exploration and fewer effectively independent samples, directly reducing ESS and increasing MCSE.

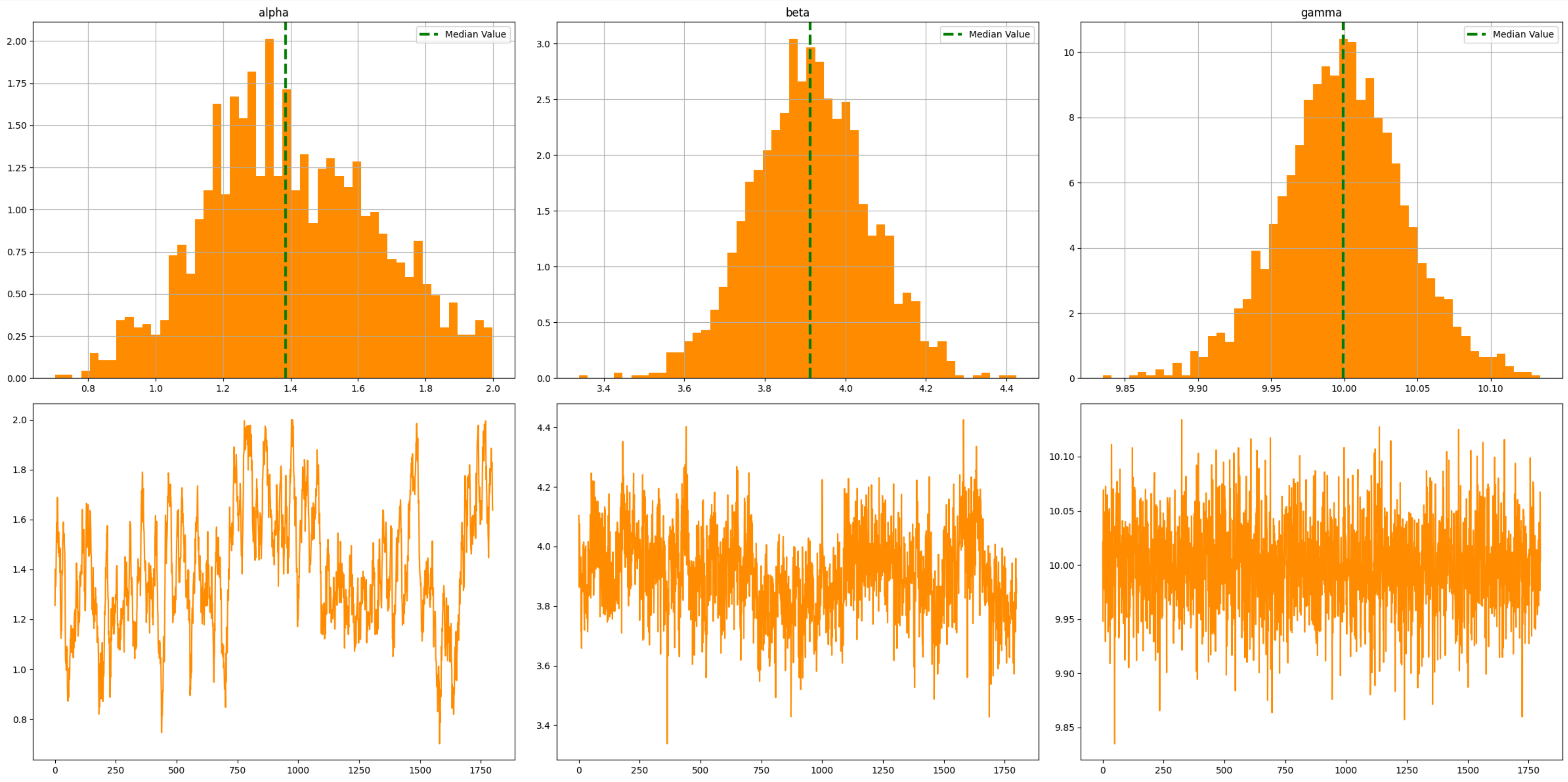
Prediction vs Data



Covariance Scatter Plots

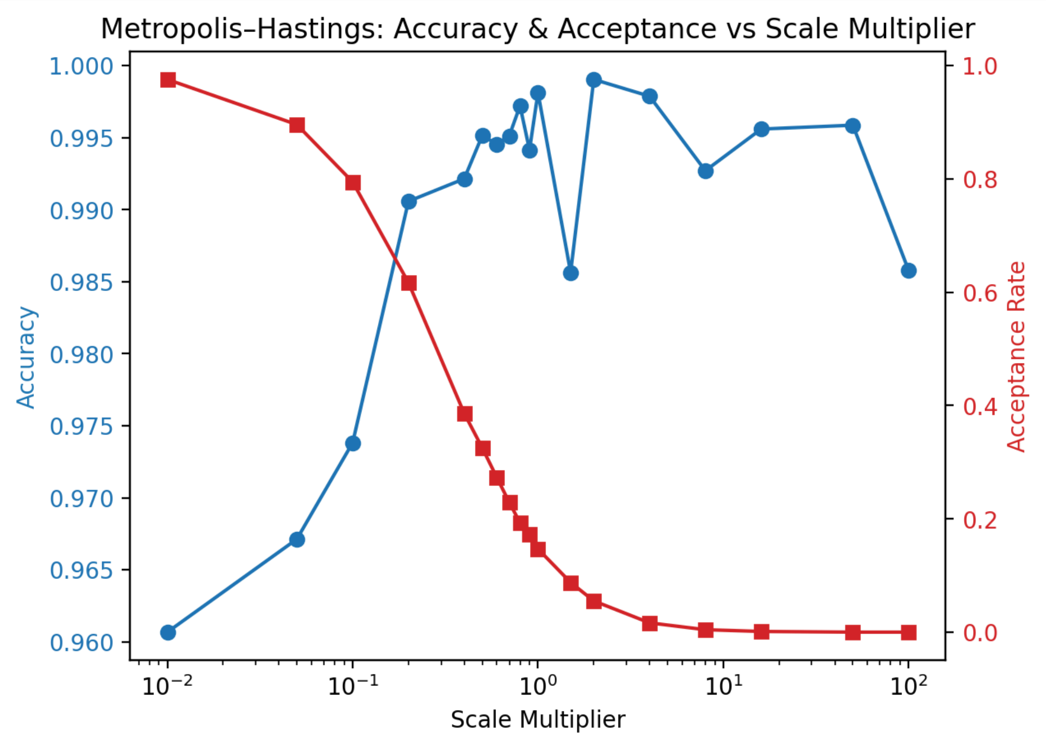


Histograms and Trace Plots



Optimization

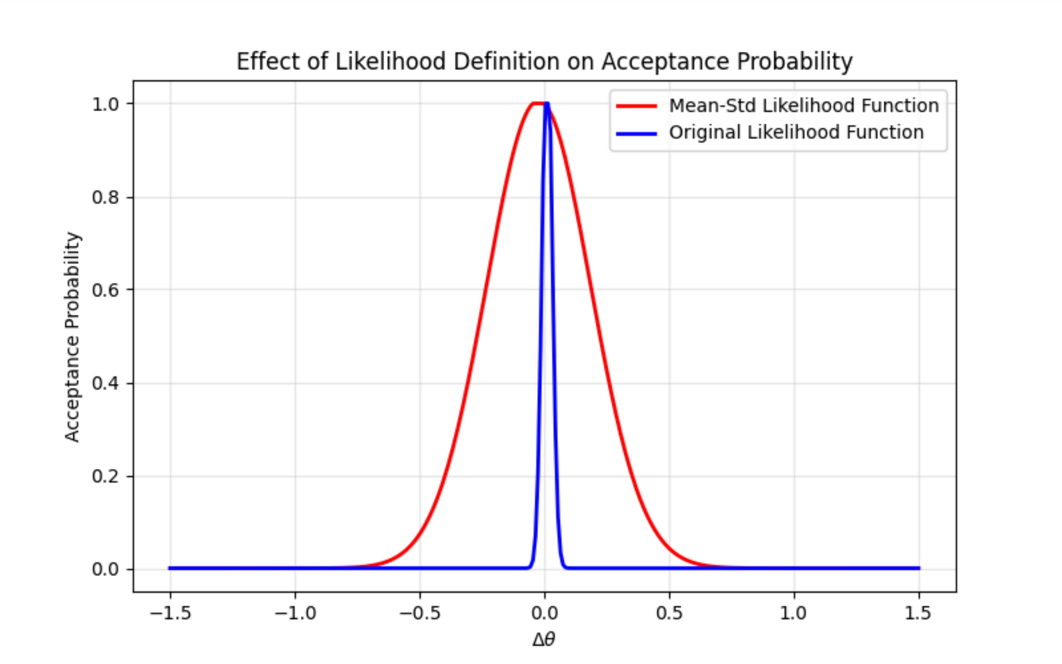
Scale Selection



- **Small Scales** ($< 10^{-1}$)
  - Chain barely moves → strong autocorrelation → **low accuracy.**
- **Chosen Scales** (α: 0.005, β: 0.1, γ: 0.2)
  - (Roberts & Rosenthal, 2004) predicts optimal acceptance ≈ **0.234** for high-dimensional targets.
  - Accuracy peaks — this is the optimal region for efficient sampling.
- **Large Scales** (> 5)
  - Acceptance rate falls → chain stagnates → **accuracy degrades.**

Likelihood Function

-> The original likelihood makes even tiny parameter changes look catastrophically unlikely, driving acceptance to near zero. The new version fixes this by taking a mean instead of summation, keeping acceptance stable.




```python
def likelihood_new(y_data, y_prior):
    y_err = 0.1 * np.std(y_data)
    Y = np.mean((y_data-y_prior)**2)/y_err**2
    return -0.5 * Y

def likelihood_original(y_datay, y_priory):
    y_err = 0.1 * (y_data + y_prior) + 1e-6
    Y = np.sum(((y_data - y_prior)/y_err) ** 2 )
    return -0.5 * Y
```

Data Generation for Better Inference

System Overview

The data generation pipeline accepts an analytical gravitational wave model function and produces a CSV file containing noisy observations at discrete time points. This system serves two primary purposes:

- Algorithm Validation: Generate data with known ground truth parameters to verify MCMC recovery accuracy
- Sensitivity Analysis: Test algorithm performance under varying noise conditions and proposal scales
- Robustness Check : Validate algorithm stability under heteroscedastic noise to assess reliability across signal regimes.

Noise Formula

The noise standard deviation for each data point is calculated as:

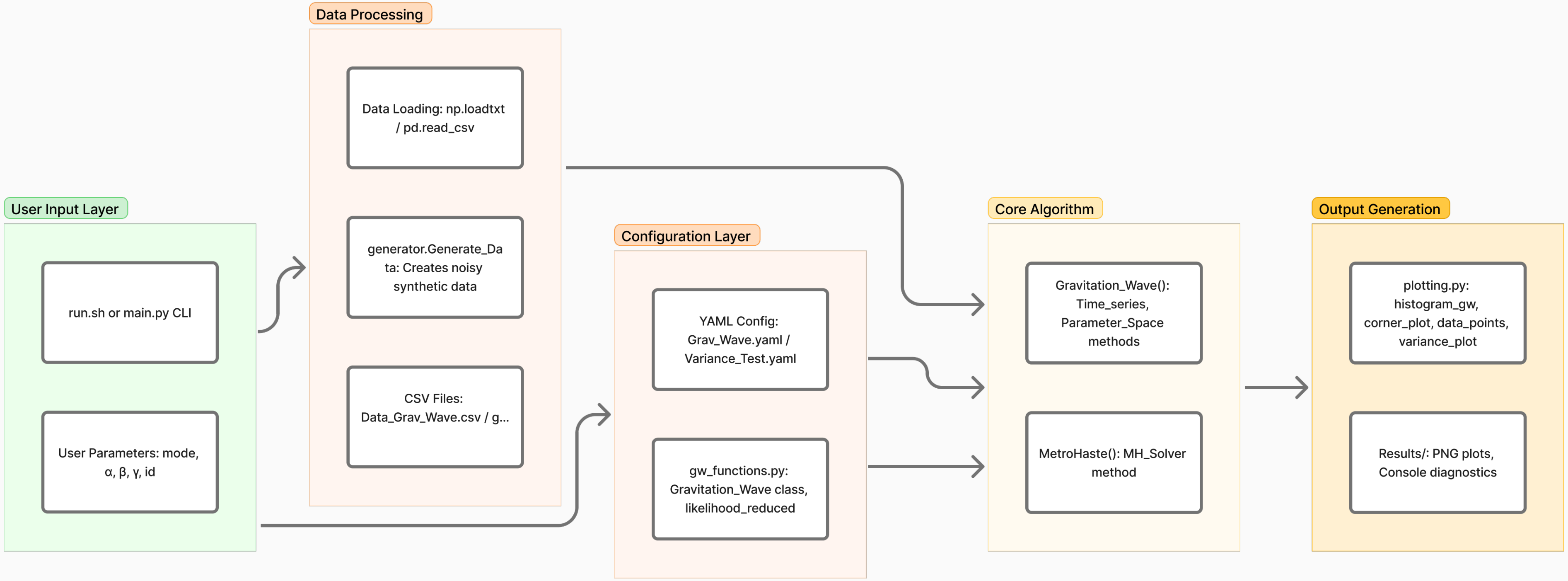
```
f_error = noise x f_points + 0.1 x std(f_points)
f_noisy = f_points +
            np.random.normal(0, np.abs(f_error), num)
```

This creates heteroscedastic noise where measurement error grows with signal amplitude.

Noise Characteristics

Component	Purpose	Typical Magnitude
Proportional	Scales with signal amplitude	Dominant near peaks
Baseline	Constant floor noise	4–8 for typical signals

Code Structure



Thank You

Any Questions?