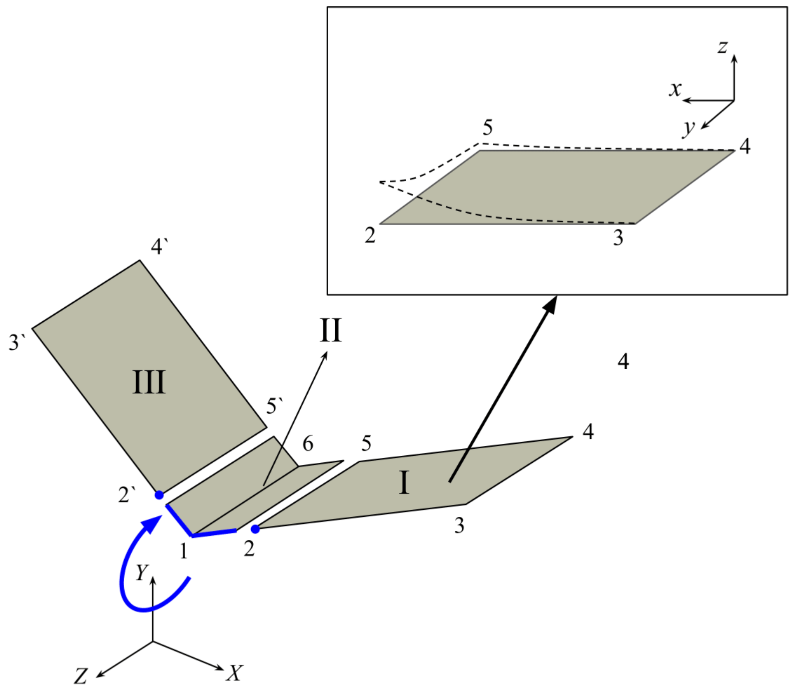
Derivation of stiffness of equivalent torsion spring

The problem of deriving the stiffness of an equivalent torsion spring of a hexagonal honeycomb unit-cell reduces (due to symmetry) to finding the stiffness of the system shown below. Here θ is angle of honeycomb. l, w, and t_w are the length 2-3, width 3-4 and thickness of the plate respectively. d is the width of the attached bilayer (not shown) and is equal to length 2-2'.

In[41]:=



In[1]:=

```
$Assumptions = E2 > 0 && w > 0 && t_w > 0 && l > 0 &&
   {x, y, z, ψ, θ, u, v, w, a, b, c, d, e, f} ∈ Reals;
```

* We assume all the material properties and dimensions are positive and restrict ourselves to the Reals.

In[2]:=

$$u_y = f\, y\, x^3;$$
$$u_z = \left(a + b\, y^2 + c\, y^4\right) \left(x^3\right);$$
$$u_x = u_z \operatorname{Tan}[\theta];$$

* Polynomial approximation of displacement in section I. Here, a, b, c, and e are constants to be found through energy minimization. θ is angle of honecomb.

In[5]:=
```
bEliminate = Simplify[Flatten[Solve[
      {(D[(u_z / Cos[θ]), y] /. {x → l, y → w / 2}) == ψ}, {b}]]];
(*Solving for b as function of tip angle*)
u_x = u_x /. bEliminate; (*Substituting b*)
u_z = u_z /. bEliminate; (*Substituting b*)
u_y = u_y /. bEliminate; (*Substituting b*)
```

In[9]:=
```
ϵ_xx = Simplify[-z D[u_z, {x, 2}] + D[u_x, x]];
ϵ_yy = Simplify[-z D[u_z, {y, 2}] + D[u_y, y]];
ϵ_xy = Simplify[(D[u_x, y] + D[u_y, x]) / 2 - z D[D[u_z, x], y]];
```

* Evaluating the strains and simplifying the equations

In[12]:=
```
σ_xx = E2 (ϵ_xx) / (1);
σ_yy = E2 ( ϵ_yy) / (1);
σ_xy = (E2 / 2) ϵ_xy;
```

* Evaluating the stresses. Here ‘E2’ is used in place of ‘E’ since ‘E’ is reserved in MATHEMATICA.

In[15]:=

$$dU = \mathrm{Simplify}\left[\frac{1}{2}\ (\sigma_{xx}\,\epsilon_{xx} + \sigma_{xy}\,\epsilon_{xy} + \sigma_{yy}\,\epsilon_{yy})\right]$$

```
 (* Writing the energy of infinitesimal element*)
```

Out[15]=

$$\frac{1}{16\, l^{6}\, w^{2}} E2\, x^{2} \Big(8\, x^{4} \left(l^{3}\, w \left(f + c \left(w^{2} - 12\, y^{2}\right) z\right) - 2\, z\, \psi\, \mathrm{Cos}[\theta]\right)^{2} + 18 \left(l^{3}\, w \left(2\, a - c\, w^{2}\, y^{2} + 2\, c\, y^{4}\right) + 2\, y^{2}\, \psi\, \mathrm{Cos}[\theta]\right)^{2} (-2\, z + x\, \mathrm{Tan}[\theta])^{2} + x^{2}\, y^{2} \left(-12\, z\, \psi\, \mathrm{Cos}[\theta] + 2\, x\, \psi\, \mathrm{Sin}[\theta] + l^{3}\, w \left(3 \left(f + 2\, c \left(w^{2} - 4\, y^{2}\right) z\right) - c\, x \left(w^{2} - 4\, y^{2}\right) \mathrm{Tan}[\theta]\right)\right)^{2}\Big)$$

In[16]:=
```
U1 = Integrate[dU, {z, -t_w / 2, t_w / 2}];
(*Integrating wrt z*)
U2 = Integrate[U1, {y, 0, w / 2}]; (*Integrating wrt y*)
U3 = Integrate[U2, {x, 0, l}]; (*Integrating wrt x*)
U = Simplify[U3] (*Simplifying the total energy*)
```

Out[19]=

$$\frac{1}{11289600\, l^3\, w} E2 \,\mathrm{Cos}[\theta]\, t_w \Big(7 \big(403200\, a^2 l^6 w^2 - 23520\, a\, c\, l^6 w^6 + 3840\, c^2 l^{10} w^6 + 576\, c^2 l^8 w^8 + 535\, c^2 l^6 w^{10} + 9600\, l^4 \psi^2 + 5040\, l^2 w^2 \psi^2 + 2520\, w^4 \psi^2 + 24\, l^3 w^3 \left(2800\, a - 3\, c\, w^2 \left(56\, l^2 + 45\, w^2\right)\right) \psi\, \mathrm{Cos}[\theta] + 120 \left(80\, l^4 + 42\, l^2 w^2 + 21\, w^4\right) \psi^2\, \mathrm{Cos}[2\,\theta]\big)\, \mathrm{Sec}[\theta]\, t_w^2 + 3\, l^2 w^2 \Big(20\, l^2\, \mathrm{Sec}[\theta]^3 \left(441\, f^2 l^4 w^2 + 8\, c^2 l^6 w^6 + 35\, \psi^2 + l^4 w^2 \left(441\, f^2 - 8\, c^2 l^2 w^4\right) \mathrm{Cos}[2\,\theta] + 28\, c\, l^3 w^3 \psi\, \mathrm{Cos}[3\,\theta] - 35\, \psi^2\, \mathrm{Cos}[4\,\theta] + 245\, f\, l^2 w\, \psi\, \mathrm{Sin}[3\,\theta]\right) + 504\, l^3 w\, (560\, a - 27\, c\, w^4)\, \psi\, \mathrm{Tan}[\theta]^2 - 140\, l^4 w\, \psi\, \mathrm{Sec}[\theta]^2 \left(4\, c\, l\, w^2 - 35\, f\, \mathrm{Tan}[\theta]\right) + 7\, \mathrm{Sec}[\theta] \left(19200\, f^2 l^8 - 560\, c\, f\, l^7 w^4\, \mathrm{Tan}[\theta] + 3 \left(80640\, a^2 l^6 - 4704\, a\, c\, l^6 w^4 + 107\, c^2 l^6 w^8 + 504\, w^2 \psi^2 + 504\, w^2 \psi^2\, \mathrm{Cos}[2\,\theta]\right) \mathrm{Tan}[\theta]^2\right)\Big)\Big)$$

In[26]:=
```
eqn1 = Simplify[D[U, a] == 0];
(*Minimization equation wrt a*)
eqn2 = Simplify[D[U, c] == 0];
(*Minimization equation wrt c*)
eqn3 = Simplify[D[U, f] == 0];
(*Minimization equation wrt e*)
eqn4 = Simplify[D[U, ψ] == M] ;
(*Minimization equation wrt ψ, i.e. dU/dψ=M *)

solution = Simplify[
  Flatten[Solve[{eqn1, eqn2, eqn3, eqn4}, {a, c, f, ψ}]]]
 (*Solving the 4 equations simultaneuosly*)
```

Out[26]=

$$
\begin{aligned}
\{a \to\ & \Big(525\,\text{M}\,w^2 \left(9\,l^2\,\text{Sin}[\theta]^2 + 5\,\text{Cos}[\theta]^2\,t_w^2\right) \\
& \Big(-3\,l^2 \left(32\,256\,000\,l^6 + 3\,000\,960\,l^4\,w^2 - 656\,509\,l^2\,w^4 - 70\,560\,w^6\right) \text{Sin}[\theta]^2\,t_w^2 - \\
& 140 \left(384\,000\,l^6 + 37\,440\,l^4\,w^2 - 4901\,l^2\,w^4 - 420\,w^6\right) \text{Cos}[\theta]^2\,t_w^4 + \\
& 27\,l^4\,w^2 \left(28\,800\,l^4 + 48\,965\,l^2\,w^2 + 7056\,w^4\right) \text{Sin}[\theta]^2\,\text{Tan}[\theta]^2\Big)\Big) \Big/ \\
& \Big(\text{E2}\,t_w \Big(45\,l^4 \left(2\,903\,040\,000\,l^{10} + 2\,858\,976\,000\,l^8\,w^2 + 1\,821\,083\,520\,l^6\,w^4 + \right. \\
& \left. 288\,250\,417\,l^4\,w^6 + 15\,648\,780\,l^2\,w^8 + 635\,040\,w^{10}\right) \text{Cos}[\theta]\,\text{Sin}[\theta]^4\,t_w^4 + \\
& 25\,l^2 \left(5\,806\,080\,000\,l^{10} + 4\,940\,352\,000\,l^8\,w^2 + 2\,124\,365\,040\,l^6\,w^4 + \right. \\
& \left. 289\,303\,959\,l^4\,w^6 + 13\,496\,980\,l^2\,w^8 + 423\,360\,w^{10}\right) \text{Cos}[\theta]^3\,\text{Sin}[\theta]^2\,t_w^6 + 3500 \\
& \left(11\,520\,000\,l^{10} + 9\,288\,000\,l^8\,w^2 + 3\,212\,760\,l^6\,w^4 + 388\,521\,l^4\,w^6 + 16\,430\,l^2\,w^8 + 420\,w^{10}\right) \\
& \text{Cos}[\theta]^5\,t_w^8 + 405\,l^6\,w^2 \left(51\,840\,000\,l^8 + 101\,506\,800\,l^6\,w^2 + \right. \\
& \left. 20\,043\,625\,l^4\,w^4 + 1\,473\,612\,l^2\,w^6 + 84\,672\,w^8\right) \text{Sin}[\theta]^5\,t_w^2\,\text{Tan}[\theta] + \\
& 729\,l^8\,w^4 \left(240\,000\,l^6 + 672\,875\,l^4\,w^2 + 215\,180\,l^2\,w^4 + 21\,168\,w^6\right) \text{Sin}[\theta]^5\,\text{Tan}[\theta]^3\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
c \to\ & \Big(882\,000\,\text{M} \left(9\,l^2\,\text{Sin}[\theta]^2 + 5\,\text{Cos}[\theta]^2\,t_w^2\right) \Big(l^2 \left(410\,880\,l^4 + 278\,203\,l^2\,w^2 + 30\,240\,w^4\right) \\
& \text{Sin}[\theta]^2\,t_w^2 + 20 \left(10\,080\,l^4 + 4523\,l^2\,w^2 + 420\,w^4\right) \text{Cos}[\theta]^2\,t_w^4 + \\
& 9\,l^4 \left(9600\,l^4 + 23\,075\,l^2\,w^2 + 3024\,w^4\right) \text{Sin}[\theta]^2\,\text{Tan}[\theta]^2\Big)\Big) \Big/ \\
& \Big(\text{E2}\,t_w \Big(45\,l^4 \left(2\,903\,040\,000\,l^{10} + 2\,858\,976\,000\,l^8\,w^2 + 1\,821\,083\,520\,l^6\,w^4 + \right. \\
& \left. 288\,250\,417\,l^4\,w^6 + 15\,648\,780\,l^2\,w^8 + 635\,040\,w^{10}\right) \text{Cos}[\theta]\,\text{Sin}[\theta]^4\,t_w^4 + \\
& 25\,l^2 \left(5\,806\,080\,000\,l^{10} + 4\,940\,352\,000\,l^8\,w^2 + 2\,124\,365\,040\,l^6\,w^4 + \right. \\
& \left. 289\,303\,959\,l^4\,w^6 + 13\,496\,980\,l^2\,w^8 + 423\,360\,w^{10}\right) \text{Cos}[\theta]^3\,\text{Sin}[\theta]^2\,t_w^6 + 3500 \\
& \left(11\,520\,000\,l^{10} + 9\,288\,000\,l^8\,w^2 + 3\,212\,760\,l^6\,w^4 + 388\,521\,l^4\,w^6 + 16\,430\,l^2\,w^8 + 420\,w^{10}\right) \\
& \text{Cos}[\theta]^5\,t_w^8 + 405\,l^6\,w^2 \left(51\,840\,000\,l^8 + 101\,506\,800\,l^6\,w^2 + \right. \\
& \left. 20\,043\,625\,l^4\,w^4 + 1\,473\,612\,l^2\,w^6 + 84\,672\,w^8\right) \text{Sin}[\theta]^5\,t_w^2\,\text{Tan}[\theta] + \\
& 729\,l^8\,w^4 \left(240\,000\,l^6 + 672\,875\,l^4\,w^2 + 215\,180\,l^2\,w^4 + 21\,168\,w^6\right) \text{Sin}[\theta]^5\,\text{Tan}[\theta]^3\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
f \to\ & -\Big(\Big(1\,176\,000\,l\,\text{M}\,w^2 \left(9\,l^2\,\text{Sin}[\theta]^2 + 5\,\text{Cos}[\theta]^2\,t_w^2\right) \text{Tan}[\theta] \\
& \Big(18\,l^2 \left(2100\,l^4 + 107\,l^2\,w^2 + 70\,w^4\right) \text{Sin}[\theta]^2\,t_w^2 + \\
& 35 \left(600\,l^4 + 27\,l^2\,w^2 + 10\,w^4\right) \text{Cos}[\theta]^2\,t_w^4 + 81\,l^4\,w^2 \left(5\,l^2 + 14\,w^2\right) \text{Sin}[\theta]^2\,\text{Tan}[\theta]^2\Big)\Big) \Big/ \\
& \Big(\text{E2}\,t_w \Big(45\,l^4 \left(2\,903\,040\,000\,l^{10} + 2\,858\,976\,000\,l^8\,w^2 + 1\,821\,083\,520\,l^6\,w^4 + \right. \\
& \left. 288\,250\,417\,l^4\,w^6 + 15\,648\,780\,l^2\,w^8 + 635\,040\,w^{10}\right) \text{Cos}[\theta]\,\text{Sin}[\theta]^4\,t_w^4 + \\
& 25\,l^2 \left(5\,806\,080\,000\,l^{10} + 4\,940\,352\,000\,l^8\,w^2 + 2\,124\,365\,040\,l^6\,w^4 + \right. \\
& \left. 289\,303\,959\,l^4\,w^6 + 13\,496\,980\,l^2\,w^8 + 423\,360\,w^{10}\right) \text{Cos}[\theta]^3\,\text{Sin}[\theta]^2\,t_w^6 + \\
& 3500 \left(11\,520\,000\,l^{10} + 9\,288\,000\,l^8\,w^2 + 3\,212\,760\,l^6\,w^4 + 388\,521\,l^4\,w^6 + \right. \\
& \left. 16\,430\,l^2\,w^8 + 420\,w^{10}\right) \text{Cos}[\theta]^5\,t_w^8 + 405\,l^6\,w^2 \left(51\,840\,000\,l^8 + 101\,506\,800\,l^6\,w^2 + \right. \\
& \left. 20\,043\,625\,l^4\,w^4 + 1\,473\,612\,l^2\,w^6 + 84\,672\,w^8\right) \text{Sin}[\theta]^5\,t_w^2\,\text{Tan}[\theta] + \\
& 729\,l^8\,w^4 \left(240\,000\,l^6 + 672\,875\,l^4\,w^2 + 215\,180\,l^2\,w^4 + 21\,168\,w^6\right) \text{Sin}[\theta]^5\,\text{Tan}[\theta]^3\Big)\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\psi \to\ & \Big(25\,200\,l^3\,\text{M}\,w\,\text{Sec}[\theta]^2 \Big(840 \left(3200\,l^6 + 900\,l^4\,w^2 + 223\,l^2\,w^4 + 21\,w^6\right) t_w^2 + \\
& l^2\,w^2 \left(96\,000\,l^4 + 245\,945\,l^2\,w^2 + 31\,752\,w^4\right) \text{Tan}[\theta]^2\Big)\Big) \Big/ \\
& \Big(\text{E2}\,t_w \Big(140 \left(11\,520\,000\,l^{10} + 9\,288\,000\,l^8\,w^2 + 3\,212\,760\,l^6\,w^4 + \right. \\
& \left. 388\,521\,l^4\,w^6 + 16\,430\,l^2\,w^8 + 420\,w^{10}\right) t_w^4 + \\
& 5\,l^2\,w^2 \left(51\,840\,000\,l^8 + 101\,026\,800\,l^6\,w^2 + 18\,697\,875\,l^4\,w^4 + 1\,043\,252\,l^2\,w^6 + 42\,336\,w^8\right) \\
& t_w^2\,\text{Tan}[\theta]^2 + 9\,l^4\,w^4 \left(240\,000\,l^6 + 672\,875\,l^4\,w^2 + 215\,180\,l^2\,w^4 + 21\,168\,w^6\right) \text{Tan}[\theta]^4\Big)\Big)\}
\end{aligned}
$$

In[27]:=

```
τ_i = Simplify[(M / (ψ /. solution))]
 (*Stiffness of section I -
  moment (M) by tip angle (found as ψ above)*)
```

Out[27]=

$$\Big(E2\,\mathrm{Cos}[\theta]^2\,t_w\,\big(140\,(11\,520\,000\,l^{10}+9\,288\,000\,l^8 w^2+3\,212\,760\,l^6 w^4+388\,521\,l^4 w^6+16\,430\,l^2 w^8+420\,w^{10})\,t_w^4+5\,l^2 w^2\,(51\,840\,000\,l^8+101\,026\,800\,l^6 w^2+18\,697\,875\,l^4 w^4+1\,043\,252\,l^2 w^6+42\,336\,w^8)\,t_w^2\,\mathrm{Tan}[\theta]^2+9\,l^4 w^4\,(240\,000\,l^6+672\,875\,l^4 w^2+215\,180\,l^2 w^4+21\,168\,w^6)\,\mathrm{Tan}[\theta]^4\big)\Big)\Big/\Big(25\,200\,l^3 w\,\big(840\,(3200\,l^6+900\,l^4 w^2+223\,l^2 w^4+21\,w^6)\,t_w^2+l^2 w^2\,(96\,000\,l^4+245\,945\,l^2 w^2+31\,752\,w^4)\,\mathrm{Tan}[\theta]^2\big)\Big)$$

In[28]:=

```
τ_ii = E2 I2 / (w / 2); (*Stiffness of section II. 'I2'
 is used in place of 'I' since it is reserved*)
I2 = ((d / 2) Tan[θ] * t_w^3 / 12 + ((d / 2) Tan[θ])^3 * t_w / 12) +
   ((d / 2) Tan[θ] * t_w^3 / 12 - ((d / 2) Tan[θ])^3 * t_w / 12) * Cos[2 θ];
(* Area moment of inertia of section II found
 by rotating that of a rectangular section by
 θ and invoking symmetry to double the value*)
 τ_ii = Simplify[E2 I2 / (w / 2)] (*Substituting*)
```

Out[29]=

$$\frac{d\,E2\,\mathrm{Sin}[\theta]\,t_w\,\left(4\,\mathrm{Cos}[\theta]\,t_w^2+d^2\,\mathrm{Sin}[\theta]\,\mathrm{Tan}[\theta]^3\right)}{24\,w}$$

In[30]:=

```
τ = Simplify[τ_ii + 2 τ_i]
 (*Total τ. Since τ_i=τ_iii by symmetry*)
```

Out[30]=

$$\frac{1}{12\,600\,w}E2\,t_w\,\Big(525\,d\,\mathrm{Sin}[\theta]\,\left(4\,\mathrm{Cos}[\theta]\,t_w^2+d^2\,\mathrm{Sin}[\theta]\,\mathrm{Tan}[\theta]^3\right)+\Big(\mathrm{Cos}[\theta]^2\,\big(140\,(11\,520\,000\,l^{10}+9\,288\,000\,l^8 w^2+3\,212\,760\,l^6 w^4+388\,521\,l^4 w^6+16\,430\,l^2 w^8+420\,w^{10})\,t_w^4+5\,l^2 w^2\,(51\,840\,000\,l^8+101\,026\,800\,l^6 w^2+18\,697\,875\,l^4 w^4+1\,043\,252\,l^2 w^6+42\,336\,w^8)\,t_w^2\,\mathrm{Tan}[\theta]^2+9\,l^4 w^4\,(240\,000\,l^6+672\,875\,l^4 w^2+215\,180\,l^2 w^4+21\,168\,w^6)\,\mathrm{Tan}[\theta]^4\big)\Big)\Big/\Big(840\,(3200\,l^9+900\,l^7 w^2+223\,l^5 w^4+21\,l^3 w^6)\,t_w^2+l^5 w^2\,(96\,000\,l^4+245\,945\,l^2 w^2+31\,752\,w^4)\,\mathrm{Tan}[\theta]^2\Big)\Big)$$