R-2.4 Answer the following questions so as to justify Theorem 2.8.

a. Draw a binary tree with height 7 and maximum number of external nodes.
b. What is the minimum number of external nodes for a binary tree with height h? Justify your answer.
c. What is the maximum number of external nodes for a binary tree with height h? Justify your answer.
d. Let T be a binary tree with height h and n nodes. Show that

$$\log(n+1)-1 \le h \le (n-1)/2.$$

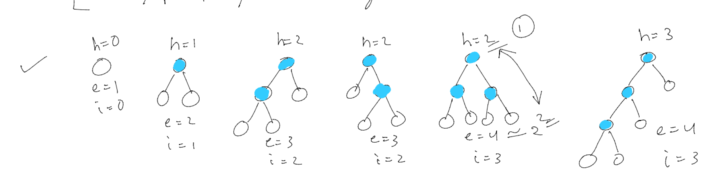
e. For which values of n and h can the above lower and upper bounds on h be attained with equality?

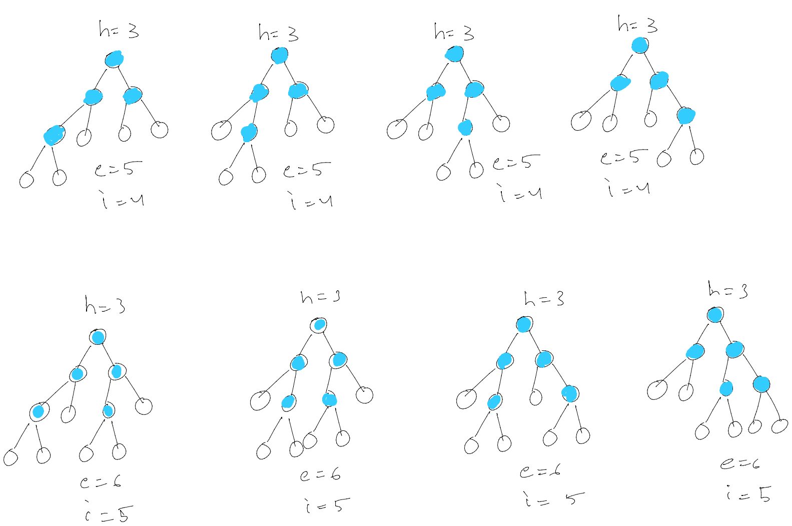
Theorem 2.8: *Let T be a (proper) binary tree with n nodes, and let h denote the height of T. Then T has the following properties:*

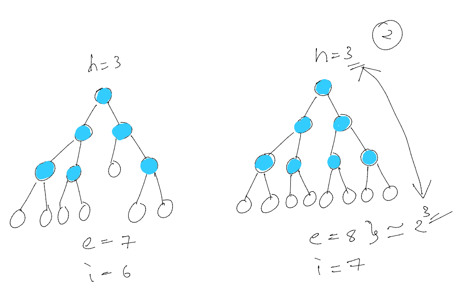
1. *The number of external nodes in T is at least $h+1$ and at most 2^h.*
2. *The number of internal nodes in T is at least h and at most 2^h-1.*
3. *The total number of nodes in T is at least $2h+1$ and at most $2^{h+1}-1$.*
4. *The height of T is at least $\log(n+1)-1$ and at most $(n-1)/2$, that is $\log(n+1)-1 \le h \le (n-1)/2$.*

b) lets draw some proper BTs (since Theorem 2.8 is about them)
c)

[strict/proper/Full Binary tree: each node can have 0 or 2 childrens]



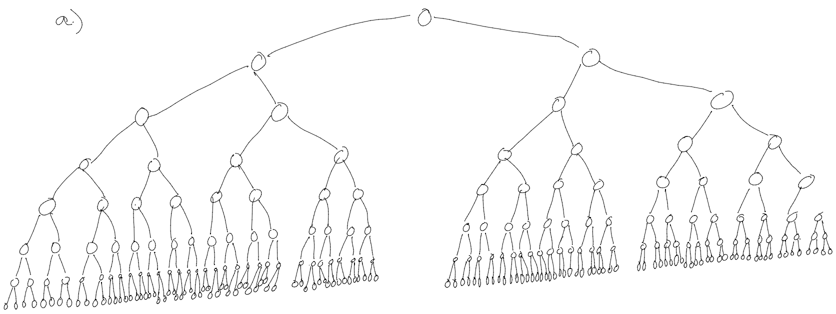




It can be observed that when all levels are fully filled a proper BT is having maximum possible external nodes, having relation to height h as 2^h

$\therefore\ e \le 2^h$ or $2^h \ge e$

a)

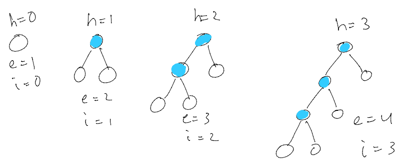


D	H
0	7
1	6
2	5
3	4
4	3
5	2
6	1
7	0

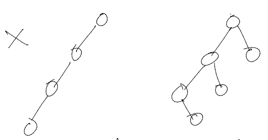
till now in all the examples number of external nodes 'e' has always been 1 more than internal nodes 'i'

$\therefore \quad e = i+1 \leq 2^h$

$\Rightarrow i \leq 2^h - 1$

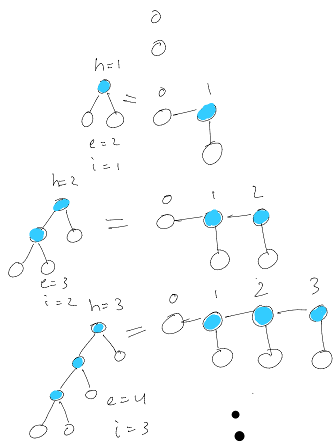


These are the only cases we see minimum no of external and internal nodes against a height h. i.e when the "at each level only 1 node is having 2 children, except leafs" at last level there are 2 nodes to satisfy Proper BT condition.



skewed BT

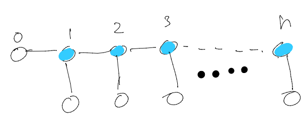
last node has 1 child



if we shift perspective we can observe that

for each internal node there is 1 external node

for each external node there is 1 internal node, except for 1



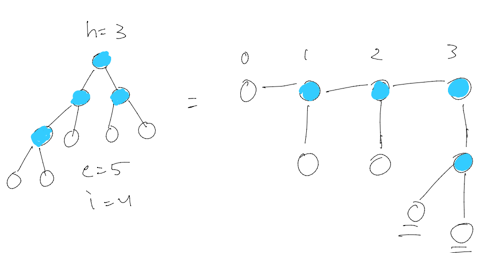
$i \geq h$

$e = i + \underline{1}$

$e \geq h+1$

how is $i \geq h$ isn't it $i = h$ in above diagram

$i = h$ only when each level has only 2 nodes or node, except leafs



when at some level both nodes starts having children then $i > h$

since i can be exactly h and greater than h

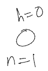
$\therefore i \geq h$

since $e = i + 1 \Rightarrow e \geq h + 1$

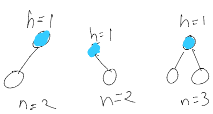
finally $h + 1 \leq e \leq 2^h$

$h \leq i \leq 2^h - 1$

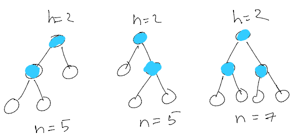
d)



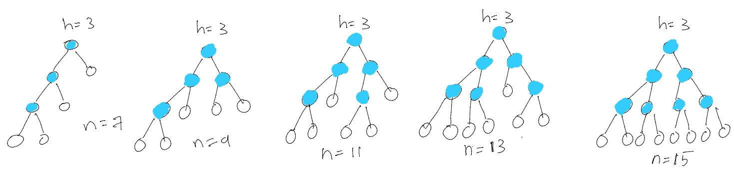
$\rightarrow h = n - 1$ & $h < n$



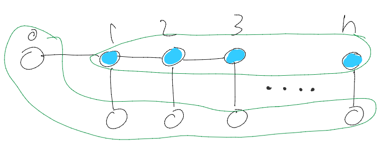
when,
all levels are not full $h = n - 1$
all levels are full $h = \frac{n-1}{2}$



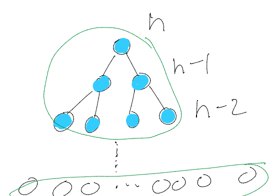
when,
all levels are not full $h = \frac{n-1}{2}$
all levels are full $h = \frac{n-1}{3}$



we can observe no. of nodes 'n' is minimum only when each level has only 2 nodes or node, except leafs



minimum nodes



maximum nodes

no. of internal + external nodes = total nodes

we know,

$h + 1 \leq e \leq 2^h$

$h \leq i \leq 2^h - 1$

$i + e = n$

$h + h + 1 \leq n \Rightarrow 2h + 1 \leq n$

$\Rightarrow h \leq \frac{(n-1)}{2}$

$n \leq 2^h + 2^h - 1 \Rightarrow n \leq 2 \times 2^h - 1$

$\Rightarrow 2^{h+1} \geq (n+1)$

we need expression in terms of 'h' so apply log that will bring 'h' from exponent to multiple

$$\log_2(2^{h+1}) \geq \log_2(n+1)$$

$$(h+1)\log_2 2 \geq \log_2(n+1)$$

$$h+1 \geq \log(n+1)$$

$$h \geq \log(n+1)-1$$

finally

$$\log(n+1)-1 \leq h \leq \frac{n-1}{2}$$

e) n h

0 0

$$\log(0+1)-1 \leq 0 \leq \frac{0-1}{2}$$

$$-1 \neq 0 \neq -0.5$$

1 0

$$\log(1+1)-1 \leq 0 \leq \frac{1-1}{2}$$

$$\log(2)-1 \leq 0 \leq \frac{0}{2}$$

$$1-1 \leq 0 = 0$$

$$0 = 0 = 0$$

found 1 pair of $n=1$, $h=0$ for which above expression is true