1. $h(x,y) = xy^2 + x^3y$, point $p = (1,2)$

gradient ∇ of a function representing a surface is the direction of steepest ascent

$$\nabla h(x,y) = \begin{bmatrix} \partial h/\partial x \\ \partial h/\partial y \end{bmatrix} \text{ or } \begin{bmatrix} h_x \\ h_y \end{bmatrix}$$ i.e each co-ordinate is partial derivative of respective axis

$$\frac{\partial h}{\partial x} = y^2 + 3x^2y \;,\; \frac{\partial h}{\partial x}_{(1,2)} = 2^2 + 3(1)^2(2) = 4+6 = 10$$

$$\frac{\partial h}{\partial y} = 2xy + x^3 \;,\; \frac{\partial h}{\partial y}_{(1,2)} = 2(1)(2) + (1)^3 = 4+1 = 5$$

$$\nabla h(1,2) = \begin{bmatrix} 10 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\frac{\nabla h(1,2)}{\|\nabla h(1,2)\|} = \frac{\cancel{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix}}{\cancel{5}\sqrt{2^2+1^2}} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$ is vector/direction

in which if we move we gain height fastest

If we move in direction that is orthogonal to $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ we will remain at same height i.e no change

so the vector orthogonal to $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is $\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\therefore\; v = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

a vector orthogonal to v is $\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ -1 \end{bmatrix} = \frac{-1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\therefore\; w = \frac{-1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2 Find the plane tangent to xyz = α at the point (β,β,β).

$$F(x,y,z) = xyz - \alpha = 0$$

$$F_x = \frac{d}{dx} F(x,y,z) = \frac{d}{dx}(xyz - \alpha)$$

$$= yz\left(\frac{d}{dx}(x) - 0\right)$$

$$= yz \text{ at } (\beta,\beta,\beta) = \beta \times \beta \Rightarrow \beta^2$$

$$F_y = \frac{d}{dy}(xyz - \alpha) = xz \text{ at } (\beta,\beta,\beta) = \beta \times \beta \Rightarrow \beta^2$$

$$F_z = \frac{d}{dz}(xyz - \alpha) = xy \text{ at } (\beta,\beta,\beta) = \beta \times \beta \Rightarrow \beta^2$$

eq of tangent plane:

$$F_x(x_0,y_0,z_0) \times (x - x_0) + F_y(x_0,y_0,z_0) \times (y - y_0) + F_z(x_0,y_0,z_0) \times (z - z_0) = 0$$

$$\beta^2(x - \beta) + \beta^2(y - \beta) + \beta^2(z - \beta) = 0$$

$$\beta^2(x + y + z - 3\beta) = 0$$

$$x + y + z - 3\beta = 0$$

$$x + y + z = 3\beta$$

3 $z(x, y) = \dfrac{f(x-y)}{y}$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \frac{f(x-y)}{y} = \frac{1}{y} \frac{\partial}{\partial x} f(x-y)$$

Chain Rule: $h(x) = f(g(x))$ then $h'(x) = f'(g(x))g'(x)$ $= \dfrac{1}{y} f'(x-y) \dfrac{\partial}{\partial x}(x-y)$

$$= \frac{1}{y} \times f'(x-y) \left(\frac{\partial x}{\partial x} - \frac{\partial y}{\partial x} \right)$$

$$= \frac{f'(x-y)}{y} (1-0)$$

$$= \frac{f'(x-y)}{y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} \frac{f(x-y)}{y} \quad \text{or} \quad \frac{\partial}{\partial y} y^{-1} \cdot f(x-y)$$

Product Rule: $h(x) = f(x)g(x)$ then $h'(x) = f'(x)g(x) + f(x)g'(x)$

Quotient Rule: $h(x) = \frac{f(x)}{g(x)}$ then $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

$$= \frac{-f'(x-y) \cdot y - f(x-y)}{y^2} \quad \text{or} \quad y^{-1} f'(x-y)(0-1) + f(x-y)(-1)y^{-2}$$

$$= \frac{-(y \cdot f'(x-y) + f(x-y))}{y^2}$$

then

$$z + y\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{f(x-y)}{y} + y \cdot \frac{f'(x-y)}{y} + y \cdot \frac{-(y f'(x-y) + f(x-y))}{y^2}$$

$$= \frac{f(x-y)}{y} + f'(x-y) - f'(x-y) - \frac{f(x-y)}{y}$$

$$= 0$$

4.) $g(x,y) = x^2 + 3y^2 \Rightarrow \frac{\partial g}{\partial x} = 2x, \quad \frac{\partial g}{\partial y} = 6y$

$$\nabla g(x,y) = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ 6y \end{bmatrix}$$

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix} - \alpha \nabla f\left(\begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix}\right)$$

(i) $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$ & $\alpha = 0.1$ and $1 <= k <= 6$

for $k=1$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \alpha \nabla g\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 3 \\ 8 \end{bmatrix} - 0.1 \begin{bmatrix} 2(3) \\ 6(8) \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 8 \end{bmatrix} - 0.1 \begin{bmatrix} 6 \\ 48 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 8 \end{bmatrix} - \begin{bmatrix} 0.6 \\ 4.8 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4 \\ 3.2 \end{bmatrix}$$

$k=2$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.4 \\ 3.2 \end{bmatrix} - 0.1 \times \nabla g\left(\begin{bmatrix} 2.4 \\ 3.2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2.4 \\ 3.2 \end{bmatrix} - 0.1 \times \begin{bmatrix} 2(2.4) \\ 6(3.2) \end{bmatrix}$$

$$= \begin{bmatrix} 2.4 \\ 3.2 \end{bmatrix} - 0.1 \times \begin{bmatrix} 4.8 \\ 19.2 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4 \\ 3.2 \end{bmatrix} - \begin{bmatrix} 0.48 \\ 1.92 \end{bmatrix} = \begin{bmatrix} 1.92 \\ 1.28 \end{bmatrix}$$

k=3

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1.92 \\ 1.28 \end{bmatrix} - 0.1 \begin{bmatrix} 2(1.92) \\ 6(1.28) \end{bmatrix}$$

$$= \begin{bmatrix} 1.92 \\ 1.28 \end{bmatrix} - 0.1 \begin{bmatrix} 3.84 \\ 7.68 \end{bmatrix}$$

$$= \begin{bmatrix} 1.92 \\ 1.28 \end{bmatrix} - \begin{bmatrix} 0.384 \\ 0.768 \end{bmatrix} = \begin{bmatrix} 1.536 \\ 0.512 \end{bmatrix}$$

k=4

$$\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1.536 \\ 0.512 \end{bmatrix} - 0.1 \begin{bmatrix} 2(1.536) \\ 6(0.512) \end{bmatrix}$$

$$= \begin{bmatrix} 1.536 \\ 0.512 \end{bmatrix} - 0.1 \begin{bmatrix} 3.072 \\ 3.072 \end{bmatrix}$$

$$= \begin{bmatrix} 1.2288 \\ 0.2048 \end{bmatrix}$$

k=5

$$\begin{bmatrix} x_5 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1.2288 \\ 0.2048 \end{bmatrix} - 0.1 \begin{bmatrix} 2(1.2288) \\ 6(0.2048) \end{bmatrix}$$

$$= \begin{bmatrix} 1.2288 \\ 0.2048 \end{bmatrix} - \begin{bmatrix} 0.24576 \\ 0.12288 \end{bmatrix}$$

$$= \begin{bmatrix} 0.98304 \\ 0.08192 \end{bmatrix}$$

k=6

$$\begin{bmatrix} x_6 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0.98304 \\ 0.08192 \end{bmatrix} - 0.1 \begin{bmatrix} 2(0.98304) \\ 6(0.08192) \end{bmatrix}$$

$$= \begin{bmatrix} 0.98304 \\ 0.08192 \end{bmatrix} - \begin{bmatrix} 0.196608 \\ 0.049152 \end{bmatrix}$$

$$= \begin{bmatrix} 0.786432 \\ 0.032768 \end{bmatrix}$$

$\therefore$ minimum for $g(x,y) = x^2 + 3y^2$ for $1 <= k <= 6$, $\alpha = 0.1$

and start point $\begin{bmatrix} 3 \\ 8 \end{bmatrix}$ is $\begin{bmatrix} 0.786432 \\ 0.032768 \end{bmatrix}$

(ii) start point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\alpha = 0.1$

$k=1$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0.1 \begin{bmatrix} 2(1) \\ 6(0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0.2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0 \end{bmatrix}$$

$k=2$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0 \end{bmatrix} - 0.1 \begin{bmatrix} 2(0.8) \\ 6(0) \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 \\ 0 \end{bmatrix} - \begin{bmatrix} 0.16 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.64 \\ 0 \end{bmatrix}$$

$k=3$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.64 \\ 0 \end{bmatrix} - 0.1 \begin{bmatrix} 2(0.64) \\ 6(0) \end{bmatrix}$$

$$= \begin{bmatrix} 0.64 \\ 0 \end{bmatrix} - \begin{bmatrix} 0.128 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.512 \\ 0 \end{bmatrix}$$

$k=4$

$$\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0.512 \\ 0 \end{bmatrix} - 0.1 \begin{bmatrix} 2(0.512) \\ 6(0) \end{bmatrix} = \begin{bmatrix} 0.4096 \\ 0 \end{bmatrix}$$

$k=5$

$$\begin{bmatrix} x_5 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0.4096 \\ 0 \end{bmatrix} - 0.1 \begin{bmatrix} 2(0.4096) \\ 6(0) \end{bmatrix} = \begin{bmatrix} 0.32768 \\ 0 \end{bmatrix}$$

$k=6$

$$\begin{bmatrix} x_6 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0.32768 \\ 0 \end{bmatrix} - 0.1 \begin{bmatrix} 2(0.32768) \\ 6(0) \end{bmatrix} = \begin{bmatrix} 0.26214 \\ 0 \end{bmatrix}$$

$\therefore$ minimum for $g(x,y) = x^2 + 3y^2$ for $1 <= k <= 6$, $\alpha = 0.1$ and start point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} 0.26214 \\ 0 \end{bmatrix}$

for start point $\begin{bmatrix} 3 \\ 8 \end{bmatrix}$ at $k=11$ $\begin{bmatrix} x_{11} \\ y_{11} \end{bmatrix} = \begin{bmatrix} 0.25769 \\ 0.00053 \end{bmatrix}$

both starting points seems to converge around $\begin{bmatrix} 0.26 \\ 0 \end{bmatrix}$

for (3,8) it took 11 steps but for (1,0) just 6 steps.

5. function $y^4 - 62y^2 - \alpha y + 9$ is maximum at $y = 1$

a function f is maximum when $\frac{df}{dy} = 0$

$$\frac{df}{dy} = 4y^3 - 124y - \alpha = 0$$

at $y = 1 \Rightarrow 4(1)^3 - 124(1) - \alpha = 0$

$$4 - 124 - \alpha = 0$$

$$\alpha = 120$$

6. Compute the derivative $\frac{df}{dx}$ of the following function

$$f(x) = \sin(\sqrt{x} + \ln(x)) + \cos(\sqrt{x} + \ln(x)) + (\sqrt{x} + \ln(x))$$

illustrating back propagation method.

$$\left[\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}, \quad \frac{d}{dx}\ln(x) = \frac{1}{x}, \quad \frac{d}{dx}\sin(x) = \cos(x), \quad \frac{d}{dx}\cos(x) = -\sin(x) \right]$$

$$\frac{d\,f(x)}{dx} = \frac{d}{dx}(\sin(\sqrt{x} + \ln(x))) + \frac{d}{dx}(\cos(\sqrt{x} + \ln(x))) + \frac{d}{dx}(\sqrt{x} + \ln(x))$$

$$= \cos(\sqrt{x} + \ln(x)) \times \frac{d}{dx}(\sqrt{x} + \ln(x)) - \sin(\sqrt{x} + \ln(x)) \times \frac{d}{dx}(\sqrt{x} + \ln(x))$$

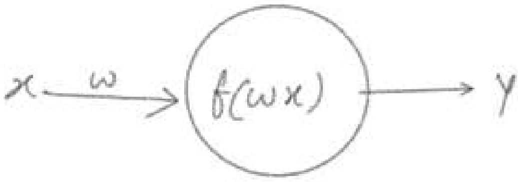
$$+ \frac{d}{dx}(\sqrt{x} + \ln(x))$$

$$= (\cos(\sqrt{x} + \ln(x)) - \sin(\sqrt{x} + \ln(x)) + 1) \times \left(\frac{1}{2\sqrt{x}} + \frac{1}{x} \right)$$

$$= (\cos(\sqrt{x} + \ln(x)) - \sin(\sqrt{x} + \ln(x)) + 1) \times \left(\frac{\sqrt{x}}{2x} + \frac{2}{2x} \right)$$

$$= \frac{(\cos(\sqrt{x} + \ln(x)) - \sin(\sqrt{x} + \ln(x)) + 1)(\sqrt{x} + 2)}{2x}$$

X	Y
1.1	1.9



here our activation function is

$$f(x) = \sin(\sqrt{x} + \ln(x)) + \cos(\sqrt{x} + \ln(x)) + (\sqrt{x} + \ln(x))$$

and its derivative is

$$\frac{d}{dx}(\sin(\sqrt{x} + \log(x)) + \cos(\sqrt{x} + \log(x)) + (\sqrt{x} + \log(x))) = \frac{(\sqrt{x} + 2)(-\sin(\sqrt{x} + \log(x)) + \cos(\sqrt{x} + \log(x)) + 1)}{2x}$$

lets initialize weights w_1 & w_2 in range of $(0,1)$

$w_1 = 4$ and learning rate $\alpha = 0.1$

loss function $E(out) = \frac{1}{2}(target - out)^2$

$$\frac{dE}{d\,out} = -(target - out)$$

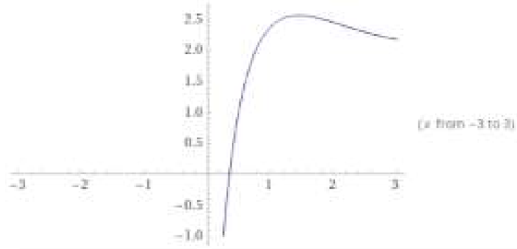
for $x = 1.1$, $y = 1.9$

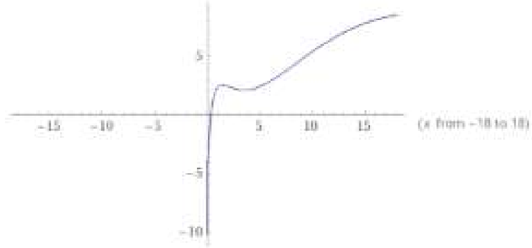
$w \times x = 4 \times 1.1$ (net_1)

$f(4 \times 1.1) = 2.24$ (out_1)

plot	$\sin(\sqrt{x} + \log(x)) + \cos(\sqrt{x} + \log(x)) + (\sqrt{x} + \log(x))$

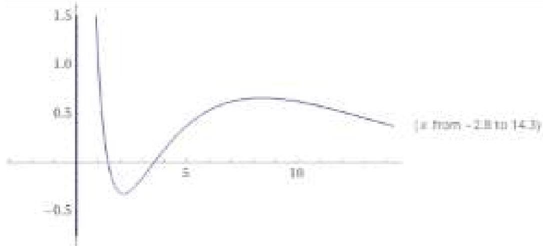
Plots

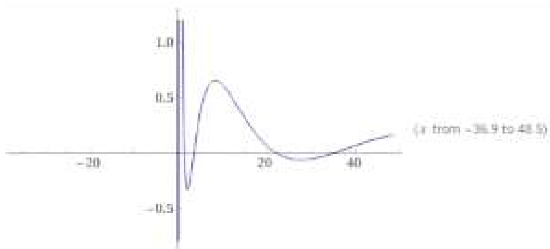




plot	$\frac{(\sqrt{x} + 2)(-\sin(\sqrt{x} + \log(x)) + \cos(\sqrt{x} + \log(x)) + 1)}{2x}$

Plots





$$\frac{dE}{d\,out} = -(1.9 - 2.24)$$
$$= 0.34$$

$$\frac{d\,out}{d\,net} = \frac{d}{dx}\left(\sin(\sqrt{x} + \log(x)) + \cos(\sqrt{x} + \log(x)) + (\sqrt{x} + \log(x))\right) = \frac{(\sqrt{x} + 2)\left(-\sin(\sqrt{x} + \log(x)) + \cos(\sqrt{x} + \log(x)) + 1\right)}{2x}$$

$$\frac{d\,out(0.34)}{d\,net} = 8.94$$

$$\frac{d\,net}{dw} = \frac{d}{dw}(w \times out_1) = out_1 = 2.24$$

$$\frac{dE}{d\,out} \times \frac{d\,out}{d\,net} \times \frac{d\,net}{dw} = \frac{dE}{dw} \Rightarrow 0.34 \times 8.94 \times 2.24 = 6.8$$

$$w_{new} = w_{old} - \alpha \times \frac{dE}{dw} = 4 - 0.1 \times 6.8 = 3.32$$

to check new output

$$f(w_{new} \times X) = f(3.32 \times 1.1) = 2.14$$

actual Y	Y_{old}	Y_{new}
1.9	2.24	2.14

we can see after updating weight using backpropagation new output is closer to the actual expected output

	Grey	Red	Glazed	Operating Cost
UNIT 1	3000	2000	300	400
UNIT 2	2000	5000	1500	600
	18000	34000	9000	

Objective is to minimize operating cost of unit I per x_1 days & of unit II per x_2 days

from this we can formulate Linear programming problem (LPP) as

$$\min_{x \in \mathbb{R}^2} \begin{bmatrix} 400 \\ 600 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{subject to} \begin{bmatrix} 3000 & 2000 \\ 2000 & 5000 \\ 300 & 1500 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 18000 \\ 34000 \\ 9000 \end{bmatrix}$$

To derive dual linear program using Lagrange duality,

$$\text{lagrangian } L(x,\lambda) = \begin{bmatrix} 400 \\ 600 \end{bmatrix}^T x + \lambda^T \left(\begin{bmatrix} 3000 & 2000 \\ 2000 & 5000 \\ 300 & 1500 \end{bmatrix} \cdot x - \begin{bmatrix} 18000 \\ 34000 \\ 9000 \end{bmatrix} \right)$$

$$L(x,\lambda) = \left(\begin{bmatrix} 400 \\ 600 \end{bmatrix}^T + \lambda^T \begin{bmatrix} 3000 & 2000 \\ 2000 & 5000 \\ 300 & 1500 \end{bmatrix} \right) x - \lambda^T \begin{bmatrix} 18000 \\ 34000 \\ 9000 \end{bmatrix}$$

differentiate L w.r.t x and set to 0

$$\frac{d}{dx}(L(x,\lambda)) = \begin{bmatrix} 400 \\ 600 \end{bmatrix}^T + \lambda^T \begin{bmatrix} 3000 & 2000 \\ 2000 & 5000 \\ 300 & 1500 \end{bmatrix} = 0$$

by substituting above equation into lagrangian we get

dual lagrangian $D(\lambda) = -\lambda^T \begin{bmatrix} 18000 \\ 34060 \\ 9000 \end{bmatrix}$

Thus our dual linear program or dual optimization problem is

$$\max_{\lambda \in R^m} -\lambda^T \begin{bmatrix} 18000 \\ 34000 \\ 9000 \end{bmatrix}$$

$$\text{subject to} \quad \begin{bmatrix} 400 \\ 600 \end{bmatrix}^T + \lambda^T \begin{bmatrix} 3000 & 2000 \\ 2000 & 5000 \\ 300 & 1500 \end{bmatrix} = 0$$

$$\lambda \geq 0$$

8. Quadratic approximation of $f(x,y) = x^3y^2 + x + y^3$ at $(x_0, y_0) = (1,1)$

In general quadratic approximation of $f(x,y)$ is given by

$$f(x_0, y_0) + \frac{d}{dx}(f(x_0,y_0))(x-x_0) + \frac{d}{dy}(f(x_0,y_0)).(y-y_0)$$

$$+ \frac{d^2}{dx^2}(f(x_0,y_0))(x-x_0)^2 + \frac{d^2}{dxdy}(f(x_0,y_0))(x-x_0)(y-y_0) + \frac{d^2}{dy^2}(f(x_0,y_0))(y-y_0)^2$$

finding terms in above equation

$$f(1,1) = (1)^3(1)^2 + 1 + (1)^3 = 3$$

$$\frac{df}{dx} = 3(xy)^2 + 1 \quad \& \quad \frac{d}{dx}(f(1,1)) = 3(1)^2 + 1 = 4$$

$$\frac{df}{dy} = 2x^3y + 3y^2 \quad \& \quad \frac{df}{dy}(1,1) = 2(1)^3(1) + 3(1)^2 = 5$$

$$\frac{d^2f}{dx^2} = 3y^2(2x) = 6xy^2 \quad \& \quad \frac{d^2f(1,1)}{dx^2} = 6(1)(1)^2 = 6$$

$$\frac{d^2f}{dxdy} = 3x^2(2y) = 6x^2y \quad \& \quad \frac{d^2f(1,1)}{dxdy} = 6(1)^2(1) = 6$$

$$\frac{d^2f}{dy^2} = 2x^3 + 6y \quad \& \quad \frac{d^2f(1,1)}{dy^2} = 2(1)^3 + 6(1) = 8$$

putting all terms together we get,

$$3 + 4(x-1) + 5(y-1) + 6(x-1)^2 + 6(x-1)(y-1) + 8(y-1)^2$$

$$= 3 + 4x - 4 + 5y - 5 + 6x^2 - 12x + 6 + 6xy - 6x - 6y + 6 + 8y^2 - 16y + 8$$

$$= 6x^2 + 8y^2 + 6xy - 14x - 17y + 14$$

$$\therefore f(x,y) = x^3y^2 + x + y^3 \simeq 6x^2 + 8y^2 + 6xy - 14x - 17y + 14$$

9. find cube root of 1002 using Quadratic approximation
i.e Taylor polynomial upto degree 2

$1000 \approx 1002$, 1000 & 1002 are very near so

$\sqrt[3]{1000} \approx \sqrt[3]{1002}$ their cube roots will also be near

we know that $\sqrt[3]{1000} = 10$

so lets approximate $f(x) = \sqrt[3]{x}$ at $a = 1000$

taylor polynomial $T(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$

$f(1000) = \sqrt[3]{1000} = 10$

$f'(x) = \frac{d}{dx} x^{1/3} = \frac{1}{3} x^{\frac{1}{3}-1} = \frac{1}{3x^{2/3}}$, $f'(1000) = \frac{1}{3(1000)^{2/3}} = \frac{1}{300}$

$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left(\frac{1}{3x^{2/3}}\right) = \frac{-2}{9x^{5/3}}$, $f''(1000) = \frac{-2}{9(1000)^{5/3}} = \frac{-1}{450000}$

$$T(x) = 10 + \frac{(x-1000)}{3 \times 10^2} - \frac{(x-1000)^2}{9 \times 10^5}$$

to test how close our approximation is

$$T(1002) = 10 + \frac{(1002-1000)}{3 \times 10^2} - \frac{(1002-1000)^2}{9 \times 10^5}$$

$$= 10 + \frac{2}{3 \times 10^2} - \frac{4}{9 \times 10^5}$$

$$= 10 + 0.00666666\bar{6} - 0.00000444\bar{4}$$

$$= 10.0066622\bar{2}$$

$$\therefore \sqrt[3]{1002} \simeq 10.00666222\bar{2}$$

using technology i.e a calculator the exact value of $\sqrt[3]{1002} = 10.0066622271539$

by looking at approximated value & exact value we can tell we are close upto 8th decimal place.

Surface: $F(x,y,z)=0$ Point of tangency: (x_0,y_0,z_0)
Equation of Tangent Plane:
$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$

differentiate $g(x,y)$ w.r.t x

$$\frac{\partial}{\partial x}\left((6y^2-5)^2(x^2+y^2-1)^2\right)=4x(5-6y^2)^2(x^2+y^2-1)$$

differentiate $g(x,y)$ w.r.t y

$$\frac{\partial}{\partial y}\left((6y^2-5)^2(x^2+y^2-1)^2\right)=4y(6y^2-5)(x^2+y^2-1)(6x^2+12y^2-11)$$

$$g_x(1,-1) = 4(1)(5-6(-1)^2)^2(1^2+(-1)^2-1) = 4(-1)^2(1) = 4$$

$$g_x(-1,1) = 4(-1)(5-6(1)^2)^2((-1)^2+1^2-1) = -4$$

$$g_x(1,1) = 4(1)(5-6(1)^2)^2(1^2+1^2-1) = 4$$

$$g_y(1,1) = 4(1)(6(1)^2-1)(1^2+1^2-1)(6(1)^2+12(1)^2-11)$$

$$= 4(1)(6+12-11) = 4(18-11) = 4(7) = 28$$

$$g_y(1,-1) = 4(-1)(1)(18-11) = -28$$

$$g_y(-1,1) = 4(1)(18-11) = 28$$

$$\nabla f(1,-1,1) = \begin{bmatrix} 4+28 \\ -4-28 \\ 4+28 \end{bmatrix} = \begin{bmatrix} 32 \\ -32 \\ 32 \end{bmatrix} = \begin{bmatrix} \nabla f_x \\ \nabla f_y \\ \nabla f_z \end{bmatrix}$$

by substituting partial gradients into tangent plane equation we get,

$$32(x-x_0) - 32(y-y_0) + 32(z-z_0) = 0$$

given $(x_0, y_0, z_0) = (1,-1,1)$

$$x-1-y+(-1)+z-1 = 0$$

$$x-y+z = 3$$