R-2.5 Let T be a binary tree such that all the external nodes have the same depth. Let D_e be the sum of the depths of all the external nodes of T, and let D_i be the sum of the depths of all the internal nodes of T. Find constants a and b such that

$$D_e + 1 = aD_i + bn,$$

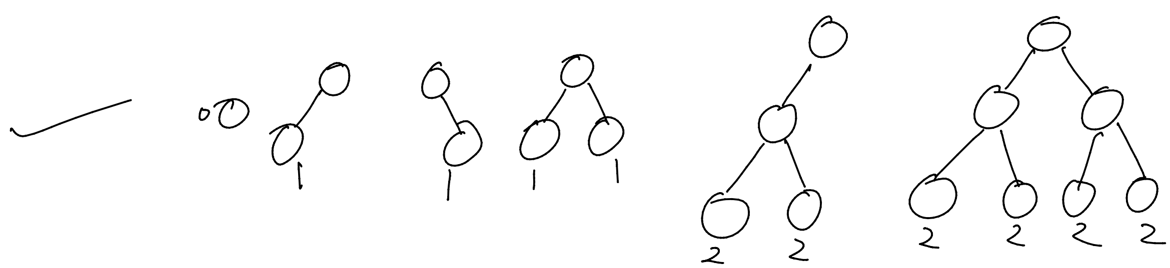
D_e = depth of 1^{st} external node + depth of 2^{nd} ext node ....

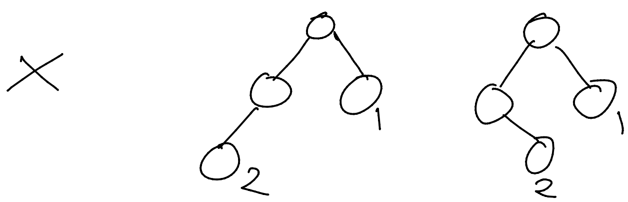
$= \sum_{t=1}^{n} depth(e_t)$

$D_i = \sum_{t=1}^{n} depth(i_t)$

Given: "A binary tree such that all the **external nodes** have the **same depth**"

Lets draw few trees that satisfy these properties





for binary tree with 1 node that satisfies given condition ($1 = n$)

since it doesn't have any child nodes its not an internal node

$D_e = depth(e_1) = 0$

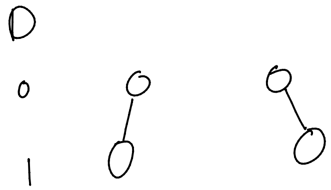
$D_i = 0$ since no internal nodes

$D_e + 1 = a D_i + bn$

$0 + 1 = a(0) + b(1)$

$\Rightarrow b = 1$

for binary tree with 2 node that satisfies given condition
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; \| $
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; n$



$$D_e = depth(e_1) = 1$$

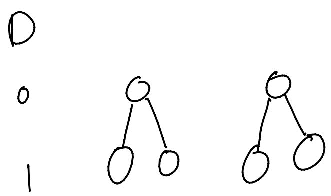
$$D_i = depth(i_1) = 0$$

$$D_e + 1 = a D_i + b n$$

$$1 + 1 = a(0) + b(2)$$

$$2 = b(2) \Rightarrow b = 1$$

for binary tree with 3 node that satisfies given condition
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; \| $
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; n$



$$D_e = depth(e_1) + depth(e_2) = 1 + 1 = 2$$

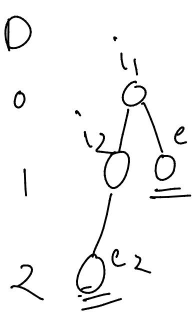
$$D_i = depth(i_1) = 0$$

$$D_e + 1 = a D_i + b n$$

$$2 + 1 = a(0) + b(3)$$

$$3 = b(3) \Rightarrow b = 1$$

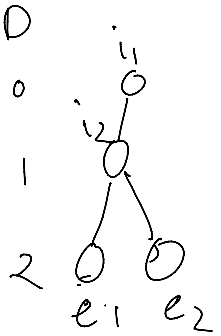
for binary tree with 4 node
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; \| $
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; n$



e_1 & e_2 are at different levels

$\therefore$ violating given condition

for binary tree with 4 node that satisfies given condition
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad n$



$D_e = 2 + 2 = 4$

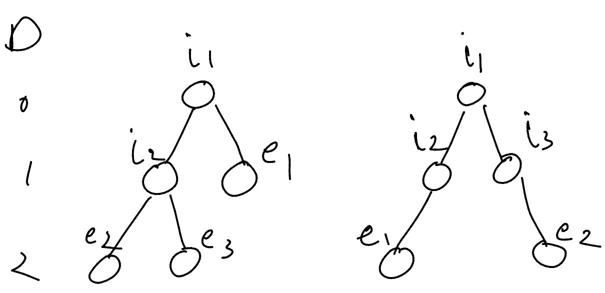
$D_i = 1 + 0 = 1$

$D_e + 1 = a D_i + b n$

$4 + 1 = a(1) + b(4)$

from previous examples we have seen one possible value for b i.e $b=1$ if it is true in all cases then $a=1$ fits perfectly in above equation

for binary tree with 5 node that satisfies given condition
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| $
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad n$

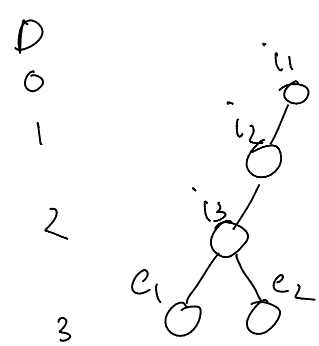


✗ ✓

$D_e = 2+2 = 4$, $D_i = 0+1+1 = 2$

$D_e + 1 = a D_i + b n$

$4 + 1 = a(2) + b(5)$



$D_e = 3+3 = 6$

$D_i = 0+1+2 = 3$

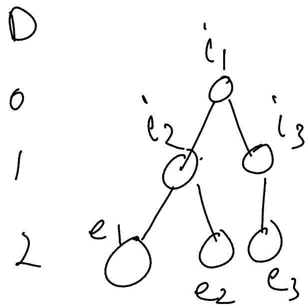
$6+1 = a(3) + b(5)$

though trees are as per given condition there is no one single pair of (a,b) that satisfies both the equations & previous equations

$$2a + 5b = 5$$

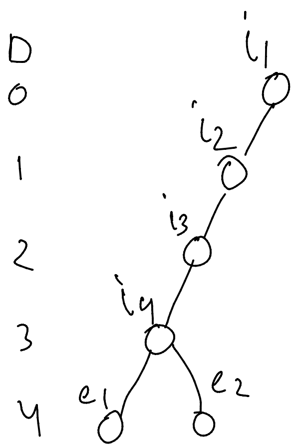
$$3a + 5b = 7$$

for binary tree with $\underset{n}{\underset{\|}{6}}$ node that satisfies given condition



$$6 + 1 = a(2) + b(6)$$

$$7 = 2a + 6b$$



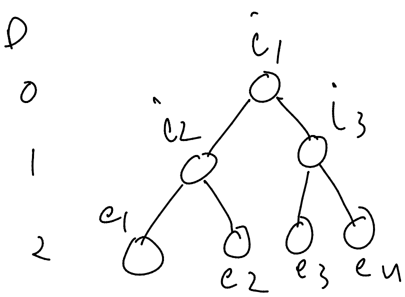
$$8+1 = a(6) + b(6)$$

$$9 = 6a + 6b$$

$$3 = 2a + 2b$$

can't conclude on a & b

for binary tree with $\underset{n}{\underset{\|}{7}}$ node that satisfies given condition

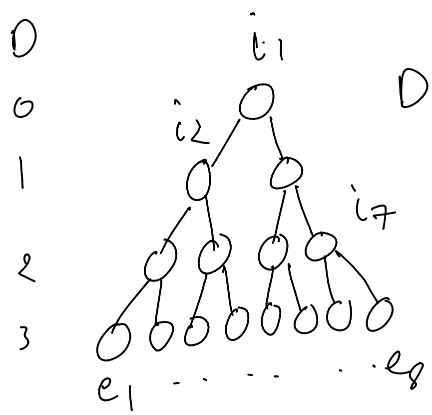


$$8+1 = a(2) + b(7)$$

$$9 = 2a + 7b$$

with 7 nodes the only way for all external nodes / leafs are at the same level is when all levels are full (i.e perfect binary tree)

so let's check perfect BT's

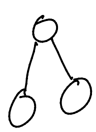


$$D(e_1) + \dots D(e_8) + 1 = a(D(i_1) + \dots D(i_7)) + b(n)$$

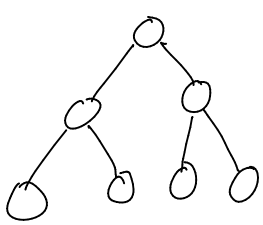
$$(3 \times 8) + 1 = a(2 \times 4 + 1 \times 2) + b(15)$$

$$24 + 1 = a(8 + 2) + b(15)$$

$$25 = 10a + 15b$$



$$3 = a(0) + b(3)$$



$$9 = 2a + 7b$$

for all perfect binary trees of height 'h' there exists a single pair of constants $(a, b) = (1, 1)$ that satisfies $D_e + 1 = a D_i + b(n)$

and $n = 2^h - 1$