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On the minimal period of integer tilings

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Abstract

If a finite set A tiles the integers by translations, it also admits a tiling whose period M has the same prime factors as $|A|$. We prove that the minimal period of such a tiling is bounded by $\exp(c(\log D)^2/\log\log D)$, where D is the diameter of A. In the converse direction, given $\epsilon > 0$, we construct tilings whose minimal period has the same prime factors as $|A|$ and is bounded from below by $D^{3/2-\epsilon}$. We also discuss the relationship between minimal tiling period estimates and the Coven–Meyerowitz conjecture.

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1 | INTRODUCTION

A finite set $A \subset \mathbb{Z}$ tiles the integers by translations if there exists a covering of $\mathbb{Z}$ by pairwise disjoint translates of A. More formally, there exists a translation set $T \subset \mathbb{Z}$ such that every integer $n \in \mathbb{Z}$ has a unique representation $n = a + t$ with $a \in A$ and $t \in T$.

Newman [13] proved that any tiling of $\mathbb{Z}$ by a finite set A must be periodic, so that $T = B \oplus M\mathbb{Z}$ for some $M \in \mathbb{N}$ and $B \subset \{0, 1, \dots, M-1\}$. His pigeonholing argument shows that the least period of the tiling must satisfy $M \leqslant 2^D$, where

$$D := \mathrm{diam}(A) = \max(A) - \min(A)$$

is the diameter of A. This bound was subsequently improved by Kolountzakis [6], Ruzsa [23, Appendix] and Biró [1]. The best estimate to date is due to Biró, who proved the following: for any $\epsilon > 0$ there exists a number $D(\epsilon)$ such that if $A \oplus T = \mathbb{Z}$ is a tiling and $D = \mathrm{diam}(A) \geqslant D(\epsilon)$, then the least period of T is at most $\exp(D^{1/3+\epsilon})$.

In the other direction, Kolountzakis [6] constructed tilings with the least period M bounded from below by cD^2 for some absolute constant c. Steinberger [19] gave an improved construction with

$$M \geqslant \exp((\log D)^2/4\log\log D);$$

in particular, there exist tilings whose least period is superpolynomial in D.

In the sequel, we will always assume that $A \subset \mathbb{Z}$ is finite and non-empty, and we will always use D to denote the diameter of A. In order to avoid trivial cases, we will also assume that

$$D \geqslant 2. \tag{1}$$

If $D = 0$ (so that A is a singleton) or $D = 1$ (with $A = \{n, n+1\}$ for some $n \in \mathbb{Z}$), $\mathbb{Z}$ can only be tiled by translates of A in one way, with period $M = D + 1$.

Definition 1. Assume that A tiles $\mathbb{Z}$ by translation. We define the *minimal tiling period of A* to be the smallest number $\mathcal{P}(A)$ such that there exists a $\mathcal{P}(A)$-periodic set $T \subset \mathbb{Z}$ satisfying $A \oplus T = \mathbb{Z}$.

If $A \oplus T$ is a tiling and M is the least period of T, we clearly have $\mathcal{P}(A) \leqslant M$. The inequality may be strict, since there may exist a different tiling of $\mathbb{Z}$ whose least period is smaller. For example, we have the following reduction, due to Coven and Meyerowitz [3] and based on the dilation theorem of Tijdeman [22].

Lemma 1 [3, Lemma 2.3]. *Assume that A tiles $\mathbb{Z}$ with period M, and suppose that $M = mM'$, where M' has the same prime factors as $|A|$ and m is relatively prime to $|A|$. Then A also tiles $\mathbb{Z}$ with period M'.*

The tilings constructed in [6, 19] have long periods M, but in both cases we have $M = m|A|$ with m relatively prime to A. It follows that A also admits the tiling $A \oplus |A|\mathbb{Z} = \mathbb{Z}$, so that the minimal tiling period for A is $\mathcal{P}(A) = |A| \leqslant D + 1$.

Our first result is an upper bound on $\mathcal{P}(A)$ in terms of D.

Theorem 1. *Assume that $A \oplus T = \mathbb{Z}$, and that the least period M of T has the same prime factors as $|A|$. (By Lemma 1, if A tiles $\mathbb{Z}$, then such a tiling always exists.) Assume further that (1) holds. Then*

$$\mathcal{P}(A) \leqslant M \leqslant \exp(c(\log D)^2/\log\log D)$$

for some absolute constant $c > 0$.

One might ask whether the condition on the prime factors of M prevents the tiling from having long periods altogether. We provide a partial answer in Theorem 2. Our lower bound on the tiling period is worse than that in [6] and especially in [19], but, unlike in [6, 19], M does not have any new prime factors that are not already present in $|A|$. For a discussion of the formal similarity between our upper bound in Theorem 1 and Steinberger's lower bound in [19], see the remark at the end of Section 2.

Theorem 2. *For any $0 < \beta < 3/2$, there exists a tiling $A \oplus T = \mathbb{Z}$ with least period M such that $M \geqslant D^\beta$ and M has the same prime factors as $|A|$.*

We do not have examples of sets that tile *only* with long periods. The set A constructed in the proof of Theorem 2 also tiles with period $|A|$, as do those in [6, 19]. Intuitively, a 'simple' tile A allows significant freedom in the choice of the tiling complement, ranging from simple to more complicated. (For an analogue of this in metric geometry, one may think of cube tilings in high-dimensional spaces.)

We further note that if a tile A with $|A| \geqslant 2$ satisfies the Coven–Meyerowitz conditions (T1) and (T2) [3], it admits a tiling with period at most $2D$. Thus any tile A with $|A| \geqslant 2$ and $\mathcal{P}(A) > 2D$ would also have to provide a counterexample to the Coven–Meyerowitz conjecture. We discuss this connection in Section 4.

2 | PROOF OF THEOREM 1

We will need the polynomial formulation of integer tilings. Assume that $A \oplus T = \mathbb{Z}$ and that T is M-periodic, so that $T = B \oplus M\mathbb{Z}$ for some finite set $B \subset \mathbb{Z}$. Then $A \oplus B$ mod M is a factorization of the cyclic group $\mathbb{Z}_M$. We write this as $A \oplus B = \mathbb{Z}_M$. We also assume that (1) holds.

By translational invariance, we may assume that $A, B \subset \{0, 1, \dots\}$ and that $0 \in A \cap B$. The *mask polynomials* of A and B are

$$A(X) = \sum_{a \in A} X^a, \; B(X) = \sum_{b \in B} X^b.$$

Then $A \oplus B = \mathbb{Z}_M$ means that

$$A(X)B(X) = 1 + X + \cdots + X^{M-1} \quad \text{mod } (X^M - 1). \tag{2}$$

We may rephrase this in terms of cyclotomic polynomials as follows. Recall that the sth cyclotomic polynomial Φ_s for $s \in \mathbb{N}$ is the minimal polynomial of $e^{2\pi i/s}$. Alternatively, Φ_s may be defined inductively via the identity

$$X^N - 1 = \prod_{s|N} \Phi_s(X).$$

Then (2) is equivalent to

$$|A||B| = M \text{ and } \Phi_s(X) \mid A(X)B(X) \text{ for all } s|M, \; s \neq 1. \tag{3}$$

Since Φ_s are irreducible, each $\Phi_s(X)$ with $s|M$ must divide at least one of $A(X)$ and $B(X)$.

Lemma 2. *Assume that $A \oplus T = \mathbb{Z}$ is a tiling with the least period M, and let $M = p_1^{n_1} \cdots p_d^{n_d}$ be the prime factorization of M. Then for every $j \in \{1, \dots, d\}$ there exists $s|M$ such that $\Phi_s|A$ and $p_j^{n_j}|s$.*

Proof. We argue by contradiction. Write $T = B \oplus M\mathbb{Z}$ for some $B \subset \{0, 1, \dots, M-1\}$, so that $A \oplus B = \mathbb{Z}_M$. Suppose that there is a $j \in \{1, \dots, d\}$ such that

$$\text{if } s|M \text{ and } p_j^{n_j}|s, \text{ then } \Phi_s \nmid A.$$

It follows that

$$P(X) := \frac{X^M - 1}{X^{M/p_j} - 1} = \prod_{s|M,\, s\nmid \frac{M}{p_j}} \Phi_s(X)$$

divides $B(X)$. Hence

$$B(X) = P(X)B_0(X) = \left(1 + X^{M/p_j} + X^{2M/p_j} + \cdots + X^{(p_j-1)M/p_j}\right)B_0(X)$$

for some polynomial $B_0 \in \mathbb{Z}[X]$. If $B_0(X)$ had degree M/p_j or higher, then $B(X)$ would have degree M or higher, contradicting the assumption that $B \subset \{0, 1, \dots, M-1\}$. It follows that $B_0(X)$ is the mask polynomial of a set $B_0 \subset \{0, 1, \dots, (M/p_j) - 1\}$, and that B is (M/p_j)-periodic, contradicting the minimality of M. □

Recall that the degree of the cyclotomic polynomial Φ_s is equal to $\varphi(s)$, the Euler totient function. By Lemma 2, for each $j \in \{1, \dots, d\}$ there is an s_j such that $p_j^{n_j} \mid s_j$ and $\Phi_{s_j}|A$, so that

$$D = \deg A(X) \geqslant \deg \Phi_{s_j} = \varphi(s_j) \geqslant \varphi(p_j^{n_j}) = (1 - p_j^{-1})p_j^{n_j} \geqslant p_j^{n_j}/2.$$

Taking the product over $j \in \{1, \dots, d\}$ leads to an upper bound $M \leqslant (2D)^d$.

On the other hand, since $|A|$ and M have the same prime factors, we have

$$D \geqslant |A| - 1 \geqslant p_1 \cdots p_d - 1 \geqslant (d+1)! - 1,$$

where we crudely lower bounded the size of the jth prime number by $j + 1$. This readily implies that

$$d \leqslant c\frac{\log D}{\log\log D} \tag{4}$$

for some absolute constant $c > 0$. Plugging this into the upper bound on M finishes the proof.

Remark. Our upper bound in Theorem 1, and Steinberger's lower bound in [19], have the same form because they are both based on the same estimate for prime numbers. Specifically, our proof above uses that if $p_1, \dots, p_d$ are primes such that $p_1 \cdots p_d \leqslant D$, then d obeys the bound (4). We then combine this with $M \geqslant (2D)^d$ to get our conclusion. Steinberger optimizes the estimate (4) and constructs the set A so that $|A| = p_1 \cdots p_d$ and $|A| \leqslant D$, but then the period of the tiling he constructs is divisible by additional d primes $q_1, \dots, q_d$ of size about D; thus $M \geqslant D^d$ in his construction. This leads to a lower bound of the same general form as our upper bound in Theorem 1, but we were not able to replicate his construction without introducing the additional large primes not appearing in $|A|$.

3 | PROOF OF THEOREM 2

3.1 | Preliminaries

Let p_1, p_2, p_3 be large primes such that $p_1 < p_2 < p_3 < 2p_1$. Let $N = p_1p_2p_3$ and $M = (p_1p_2p_3)^n$ for some large n to be determined later. We will need an explicit version of the Chinese remainder theorem, as follows.

Lemma 3. *Let $M_j := M/p_j^n$ for $j = 1, 2, 3$. Then the mapping $\pi : \mathbb{Z}_{p_1^n} \times \mathbb{Z}_{p_2^n} \times \mathbb{Z}_{p_3^n} \to \mathbb{Z}_M$, defined by*

$$\pi(x_1, x_2, x_3) = \sum_{j=1}^{3} x_j M_j, \tag{5}$$

is an isomorphism.

Proof. To see that π is well defined, suppose that $x_j \equiv x_j' \mod p_j^n$ for $j = 1, 2, 3$; then $\pi(x_1, x_2, x_3) - \pi(x_1', x_2', x_3') = \sum (x_j - x_j')M_j$ is divisible by $p_j^n M_j = M$.

It remains to prove that π is one to one. Indeed, suppose $\pi(x_1, x_2, x_3) \equiv \pi(y_1, y_2, y_3) \mod M$, so that p_j^n divides $\sum_{i=1}^{3} (x_i - y_i)M_i$ for each $j = 1, 2, 3$. Since $p_j^n | M_i$ for each $i \neq j$, it follows that $p_j^n | (x_j - y_j)M_j$. Since $p_j \nmid M_j$, this is only possible when $x_j = y_j$. This has to hold for each $j \in \{1, 2, 3\}$, and the conclusion follows. □

Lemma 3 allows us to visualize $\mathbb{Z}_M$ as a three-dimensional lattice, p_j^n-periodic in the jth direction. To each $x \in \mathbb{Z}_M$, we assign coordinates (x_1, x_2, x_3) so that $x = \sum x_j M_j$ in $\mathbb{Z}_M$, and this representation is unique for each x. We furthermore have

$$(x, p_j^n) = (x_j, p_j^n); \tag{6}$$

this follows since $p_j^n | M_i$ for $i \neq j$.

For a set $A \subset \mathbb{Z}_M$, we define its divisor set

$$\mathrm{Div}(A) = \{(a - a', M) : \ a, a' \in A\}. \tag{7}$$

Theorem 3 (Sands [17]). *Let $A, B \subset \mathbb{Z}_M$. Then $A \oplus B = \mathbb{Z}_M$ if and only if*

$$|A|\,|B| = M \text{ and } \mathrm{Div}(A) \cap \mathrm{Div}(B) = \{M\}. \tag{8}$$

We will need only the fact that (8) implies tiling. This has a short proof as follows. Suppose that we have $a + b \equiv a' + b' \mod M$ for some $a, a' \in A$ and $b, b' \in B$. Then $a - a' \equiv b' - b \mod M$, so that $(a - a', M) \in \mathrm{Div}(A) \cap \mathrm{Div}(M)$. If (8) holds, we must have $a = a'$ and $b - b'$. Thus (8) implies that all sums $a + b$ with $a \in A, b \in B$ are distinct. Together with the condition $|A|\,|B| = M$, this implies $A \oplus B = \mathbb{Z}_M$.

3.2 | Construction of A and B

We now begin our construction. Define

$$A = \left\{ \sum a_i M_i \ : \ a_i \in \{0, 1, 2, \dots, p_i - 1\} \text{ for } i = 1, 2, 3 \right\}, \tag{9}$$

$$B^* = \left\{ \sum b_i M_i \ : \ b_i \in \{0, p_i, 2p_i, \dots, (p_i^{n-1} - 1)p_i\} \text{ for } i = 1, 2, 3 \right\}. \tag{10}$$

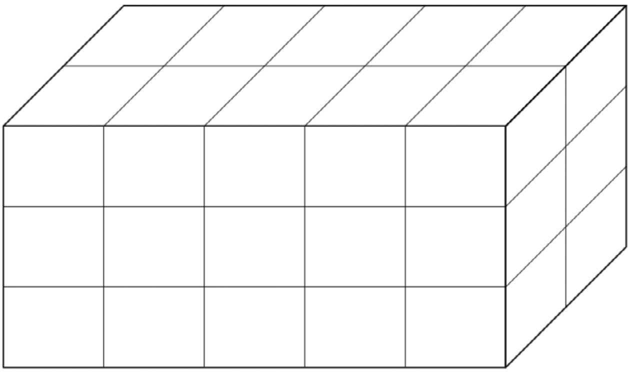

FIGURE 1 Illustration of the tiling $A \oplus B^* = \mathbb{Z}_M$ in the special case $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and $n = 2$. Three coordinate axes represents the groups $\mathbb{Z}_{p_i^2}$ and the set A is a $2 \times 3 \times 5$ axis aligned box.

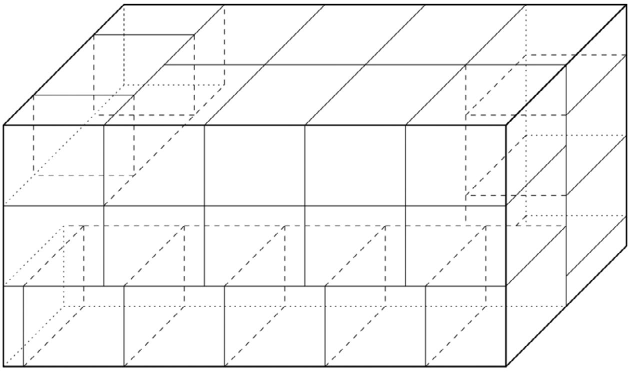

FIGURE 2 To make the tiling non-periodic, we choose three non-overlapping columns in different directions and shift the cubes in those columns.

By Lemma 3, A and B^* are sets with $|A| = p_1 p_2 p_3 = N$ and $|B^*| = M/N$. Hence $|A|\,|B^*| = M$. We also have

$$\mathrm{Div}(A) = \{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ : \ \alpha_i \in \{0, n\} \text{ for } i = 1, 2, 3\}, \tag{11}$$

Indeed, let $a, a' \in A$ with $a \neq a'$. Then, with the obvious notation, we have $a - a' = \sum (a_j - a_j')M_j$, with each $a_j - a_j'$ either equal to 0 or not divisible by p_j.

It is clear from (10) that $N | b - b'$ for each $b, b' \in B^*$. This and (11) imply that (8) holds, so that $A \oplus B^* = \mathbb{Z}_M$ and M has the same prime factors as $|A|$. However, this tiling does not meet the conditions of Theorem 2 because B^* is obviously $M_i p_i$-periodic for each i. We invite the reader to verify that A is in fact a complete residue system mod N (the argument is similar to the proof of Lemma 3), and that $B^* = N\mathbb{Z}_M$. In the Chinese Remainder Theorem geometric representation, we may interpret A as a discrete rectangular box, and $A \oplus B^*$ is a lattice tiling by translates of that box (see Figure 1).

To construct a non-periodic set tiling complement B, we perform a 'column shift' in each direction (see Figure 2). Constructions of this type go back to the work of Szabó [21]. For $i = 1, 2, 3$, let

$$z^{(i)} := (p_i^{n-1} - 1)M_i p_i, \;\; F_i := \{0, M_i p_i, 2M_i p_i, \ldots, (p_i^{n-1} - 1)M_i p_i\}.$$

Define

$$\tilde{B}_1 := z^{(3)} + F_1, \;\; \tilde{B}_2 := z^{(1)} + F_2, \;\; \tilde{B}_3 := z^{(2)} + F_3,$$

$$B_1 := z^{(3)} + M_1 + F_1, \;\; B_2 := z^{(1)} + M_2 + F_2, \;\; B_3 := z^{(2)} + M_3 + F_3,$$

Let

$$B_0 := B^* \setminus (\tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3) \text{ and } B := B_0 \cup B_1 \cup B_2 \cup B_3.$$

3.3 | Proof that B is a set and that $A \oplus B = \mathbb{Z}_M$

The sets $\tilde{B}_1, \tilde{B}_2, \tilde{B}_3$ are disjoint and contained in B^*, hence B_0 is a set. Let

$$\begin{aligned}\mathcal{D} :=\{p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} \;:\; & \beta_i \in \{0, 1, \ldots, n\} \text{ for } i = 1, 2, 3, \text{ and} \\ & 1 \leqslant \beta_i \leqslant n - 1 \text{ for at least one } i.\}\end{aligned}$$

We will prove that

(i) if $b, b' \in B_j$ for some $j \in \{0, 1, 2, 3\}$, and if $b \neq b'$, then $(b - b', M) \in \mathcal{D}$;
(ii) if $b \in B_i$ and $b' \in B_j$ for some $i, j \in \{0, 1, 2, 3\}$ with $i \neq j$, then $(b - b', M) \in \mathcal{D}$.

This will prove that the four sets B_0, B_1, B_2, B_3 are disjoint, since $(b - b', M) \in \mathcal{D}$ in (ii) implies in that $b - b' \neq 0$. In particular, B is a set with $|B| = |B^*| = M/N$. It will also prove that $\mathrm{Div}(B) \subset \mathcal{D} \cup \{M\}$. Since $\mathcal{D}$ is disjoint from $\mathrm{Div}(A)$ by (11), it follows by (8) that $A \oplus B = \mathbb{Z}_M$.

We start with (i). Let $b, b' \in B_0$ with $b = \sum b_i M_i$ and $b' = \sum b_i' M_i$. If $b \neq b'$, then $b_i \neq b_i'$ for at least one i. Since p_i divides both b_i and b_i', we have

$$p_i | (b_i - b_i') \text{ but } p_i^n \nmid (b_i - b_i'), \tag{12}$$

as claimed. Similarly, if $b, b' \in B_i$ for some $i \in \{1, 2, 3\}$, and if $b \neq b'$, then (12) holds for that i.

We now prove (ii). Suppose first that $b \in B_0$ and $b' \in B_1$, with $b = \sum b_i M_i$ and $b' = \sum b_i' M_i$. Then

$$b_1' = 1 + k p_1 M_1, \;\; b_2' = 0, \;\; b_3' = (p_3^{n-1} - 1) M_3 p_3 \tag{13}$$

for some $k \in \{0, 1, \ldots, p_1^{n-1} - 1\}$. Since $b \in B^*$ but $b \notin \tilde{B}_0$, we cannot have both $b_2 = b_2'$ and $b_3 = b_3'$. Therefore $b_i \neq b_i'$ for at least one $i \in \{2, 3\}$, and for that i we have (12). The cases when $b' \in B_2$ and $b' \in B_3$ are similar.

Assume now that $b \in B_3$ and $b' \in B_1$, with $b = \sum b_i M_i$ and $b' = \sum b_i' M_i$. Then (13) holds, and

$$b_1 = 0, \;\; b_2 = (p_2^{n-1} - 1) M_2 p_2, \;\; b_3 = 1 + \ell p_3 M_3$$

for some $\ell \in \{0, 1, \ldots, p_1^{n-1} - 1\}$. Thus (12) holds with $i = 2$. The remaining cases are identical up to a permutation of indices.

3.4 | Proof that B has no periods smaller than M

It suffices to prove that B is not M/p_i-periodic for $i = 1, 2, 3$. Fix i, and observe that $z^{(i)} = (p_i^{n-1} - 1)M_i p_i \notin B$ since it was removed together with one of the sets $\tilde{B}_j$. However, $z^{(i)} + M/p_i = M_i(p_i^{n-1} - p_i) \in B$.

3.5 | Proof that M is large relative to D

Given $0 < \beta < 3/2$, we need to verify that $M \geqslant D^\beta$ for an appropriate choice of the parameters of the construction. We have

$$\operatorname{diam}(A) = \frac{(p_1 - 1)M}{p_1^n} + \frac{(p_2 - 1)M}{p_2^n} + \frac{(p_3 - 1)M}{p_3^n} \leqslant \frac{3M}{p_1^{n-1}}.$$

Choose a small $\epsilon > 0$ to be fixed later, and let

$$\alpha = \frac{(3 - \epsilon)n}{2n + 1}.$$

Note that $0 < \alpha < 3/2$. Then

$$\begin{aligned} M = p_1^n p_2^n p_3^n &= p_1^{\epsilon n} p_1^\alpha p_1^{n-\alpha-n\epsilon} p_2^n p_3^n \\ &\geqslant p_1^{\epsilon n} 2^{-(n-\alpha-n\epsilon)} p_1^\alpha p_2^{(3n-\alpha-n\epsilon)/2} p_3^{(3n-\alpha-n\epsilon)/2} \\ &= p_1^{\epsilon n} 2^{-(n-\alpha-n\epsilon)} 3^{-\alpha} (3 p_1 p_2^n p_3^n)^\alpha, \end{aligned}$$

since we chose α so that $n\alpha = (3n - \alpha - n\epsilon)/2$.

Given β with $0 < \beta < 3/2$, we may choose $\epsilon > 0$ small enough and n large enough so that $\beta < \alpha < 3/2$. We fix these choices. Next,

$$\frac{p_1^{\epsilon n}}{2^{n-\alpha-n\epsilon} 3^\alpha} = \left(\frac{2}{3}\right)^\alpha \frac{p_1^{\epsilon n}}{2^{n-n\epsilon}} \geqslant \left(\frac{2}{3}\right)^{3/2} \left(\frac{p_1^\epsilon}{2}\right)^n.$$

If we choose p_1 large enough so that $(\frac{p_1^\epsilon}{2})^n > (\frac{3}{2})^{3/2}$, we may conclude that

$$M \geqslant (3 p_1 p_2^n p_3^n)^\beta \geqslant (\operatorname{diam}(A))^\beta$$

as claimed.

4 | THE MINIMAL TILING PERIOD AND THE COVEN–MEYEROWITZ CONJECTURE

Coven and Meyerowitz [3] proved the following theorem.

Theorem 4. *Let S_A be the set of prime powers p^α such that $\Phi_{p^\alpha}(X)$ divides $A(X)$. Consider the following conditions;*

(T1) $A(1) = \prod_{s \in S_A} \Phi_s(1)$;
(T2) if $s_1, \dots, s_k \in S_A$ *are powers of different primes, then* $\Phi_{s_1 \dots s_k}(X)$ *divides* $A(X)$.

Then

(i) *if A satisfies (T1), (T2), then A tiles* $\mathbb{Z}$;
(ii) *if A tiles* $\mathbb{Z}$ *then (T1) holds;*
(iii) *if A tiles* $\mathbb{Z}$ *and* $|A|$ *has at most two distinct prime factors, then (T2) holds.*

The conjecture that (T2) holds for all finite integer tiles has become known as the *Coven–Meyerowitz conjecture*. It remains open in general; see [7–10] for recent progress.

The proof of Theorem 4(i) in [3] is by explicit construction. Assuming that A satisfies (T1) and (T2). Under that assumption, Coven and Meyerowitz construct an explicit tiling of the integers by A with period

$$M := \operatorname{lcm}(S_A) = p_1^{n_1} \cdots p_k^{n_k}.$$

We fix this value of M for the rest of this section.

Assume that $\min(A) = 0$ and $\max(A) \geqslant 1$ (the case $|A| = 1$ being trivial). We claim that if A satisfies (T2), then

$$D = \operatorname{diam}(A) \geqslant M/2.$$

Indeed, since $\Phi_M | A$ by (T2), we have $A(e^{2\pi i/M}) = 0$. This would not be possible with $D < M/2$, since then we would have $\operatorname{Im}(e^{2\pi i a/M}) \geqslant 0$ for all $a \in A$, with a strict inequality at least once. This implies that if A tiles the integers and satisfies (T2), then it also admits a tiling with period $M \leqslant 2D$.

Coven and Meyerowitz [3, Remarks after Lemma 2.1, p. 167] also make the stronger claim without proof that for any set A of non-negative integers we have $M \leqslant pD/(p-1)$, or equivalently,

$$\operatorname{diam}(A) \geqslant \frac{p-1}{p} M, \tag{14}$$

where p is the smallest prime factor of $|A|$. This would imply that if A tiles the integers and satisfies (T2), then it also tiles with period at most $pD/(p-1)$. However, we could not reproduce their proof and do not believe their claim to be true in general.

The details are as follows. For general sets of integers (that do not necessarily tile the integers or satisfy (T2)), an easy counterexample is provided by the set A with the mask polynomial

$$A(X) = \Phi_{p^2}(X)\Phi_{q^2}(X) = (1 + X^p + \cdots + X^{(p-1)p})(1 + X^q + \cdots + X^{(q-1)q}),$$

where p, q are large primes such that $p < q < 2p$. Then $M = \operatorname{lcm}(S_A) = p^2q^2$, but $D = (p-1)p + (q-1)q$ which is much smaller than $\frac{p-1}{p}M$.

The more interesting question is whether (14) holds if we assume that A satisfies (T1) and (T2). This is in fact true if $|A|$ has at most two distinct prime factors. Indeed, in that case M also has at most two distinct prime factors, say p and q. As above, (T2) implies that $\Phi_M | A$. By the 2-prime case of the structure theorem for vanishing sums of roots of unity [4] (see also [11, Theorem 3.3 and Corollary 3.4]), we have

$$A(X) = P(X)\left(1 + X^{M/p} + \cdots + X^{(p-1)M/p}\right) + Q(X)\left(1 + X^{M/q} + \cdots + X^{(q-1)M/q}\right)$$

mod $(X^M - 1)$, for some polynomials $P(X)$ and $Q(X)$ with non-negative integer coefficients. In other words, A mod M is a union of 'fibers' — cosets of subgroups of $\mathbb{Z}_M$ of order p and q. Since the diameter of each such coset is either $(p-1)M/p$ or $(q-1)M/q$, the inequality (14) follows. If $|A|$ is a prime power, the same proof applies with the terms involving q removed.

The above argument no longer works when $|A|$ (therefore M) has three or more distinct prime factors. We still have $\Phi_M | A$, but the corresponding structure theory in this case [4, 15, 16, 18] is more complicated, and there are many examples of sets A such that $\Phi_M | A$ but A does not contain any fibers (see, for example, [14] for a long list of examples). Instead of just the fact that $\Phi_M | A$, we could try to use the full strength of the assumption that A tiles and satisfies (T2); however, there are also examples of sets A that tile the integers, satisfy (T2), satisfy $\Phi_M | A$, and do not contain any fibers [2].

A deeper theorem due to Steinberger [20] states that (14) holds when $2/p_1 > 1/p_2 + \cdots + 1/p_d$, where the primes dividing M are ordered so that $p_1 < \cdots < p_d$. In particular, this is always true when $d = 3$, since there are only two terms on the right side and $p_1 < p_2 < p_3$. Steinberger does not believe that (14) should hold in general, and neither do we.

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No new data were generated or analyzed during this study.

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