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RESEARCH ARTICLE

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An observation on the existence of stable generalized complex structures on ruled surfaces

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Abstract
We point out that any stable generalized complex structure on a sphere bundle over a closed surface of genus at least two must be of constant type.

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1 | INTRODUCTION

The canonical line bundle $K_{\mathcal{J}} \subset \wedge^{\bullet} T^*_{\mathbb{C}} M$ of a stable generalized complex structure $\mathcal{J}$ on a smooth n-manifold M is generated pointwise by the complex differential form

$$\rho = e^{B+i\omega} \wedge \Omega, \tag{1.1}$$

where B and ω are real two forms and the complex form on the right side of (1.1) decomposes into $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ for $(\theta_1, \dots, \theta_k)$ a basis for $L \cap T^*_{\mathbb{C}}$ and L is the $+i$-eigenbundle of $\mathcal{J}$ ([9, 10] and Cavalcanti–Gualtieri [4] for background and further details). A fundamental invariant of a stable generalized complex structure $(M, \mathcal{J})$ is its type and its defined by

$$\mathrm{Type}(\mathcal{J}) = \deg(\Omega) = k,$$

see [10, Section 3.1]. In dimension 4, the possible constant types for a given $(M, \mathcal{J})$ are $k \in \{0, 1, 2\}$. The value $k = 0$ corresponds to a stable generalized complex structure induced by a symplectic structure, while $k = 2$ corresponds to one induced by a complex structure. Stable generalized complex structures of type $k = 1$ have been studied by Bailey–Cavalcanti–Gualtieri [1] and Chen–Nie [6].

Moreover, there is now a myriad of examples of stable generalized complex structures in dimension 4 whose type jumps from $k = 0$ to $k = 2$ [2, 3, 5, 8, 10, 16, 17]. As Cavalcanti–Gualtieri have shown, for these kinds of stable generalized complex structures, the underlying smooth four-manifold need not support a symplectic nor a complex structure [2, 3]. The type change occurs along a smoothly embedded torus $T \subset M$, which is a path-connected component of the type change locus of $\mathcal{J}$. All known examples of such stable generalized complex structures in the literature share the common trait that the underlying closed four-manifold has nonnegative Euler characteristic. This raises the following question.

Question 1. Does there exist a closed four-manifold of negative Euler characteristic that admits a stable generalized complex structure with nonempty-type change locus?

The main result of this short note puts Question 1 into perspective by exhibiting four-manifolds of negative Euler characteristic that solely admit stable generalized complex structures of constant type, a behavior that was not previously known to occur; cf. Remark 7.

Theorem A. *The type change locus of a stable generalized complex structure on a sphere bundle over a closed surface of genus strictly greater than one is empty.*

In particular, any stable generalized complex structure on such a four-manifold arises from a Kähler structure.

This note aims to make well-known classification theorems and results addressing the minimal genus function of a symplectic four-manifold (see [7, Chapter 2]) more widespread within the generalized complex geometry community. The main ingredient in the proof of Theorem A is Li–Li's solution to the minimal genus problem of smooth embeddings in sphere bundles over surfaces in [12] (see Theorems 4 and 5). The other ingredients in the proof are a pair of generalized complex cut-and-paste constructions of Cavalcanti–Gualtieri [2, 3], the symplectic Thom conjecture due to Oszváth–Szabó [15, Theorem 1.1], and Liu's classification of irrational ruled symplectic four-manifolds [14] (see Theorem 3).

2 | BACKGROUND RESULTS

Key background results that are used in the proof of Theorem A are collected in this section with the purpose of making the short note as self-contained as possible.

2.1 | A relation between the symplectic Kodaira dimension and a stable generalized complex surgery

Cavalcanti–Gualtieri introduced in [2, Section 3] and [3, Section 4] a four-dimensional cut-and-paste operation that has been a fruitful source of examples of stable generalized complex four-manifolds; cf. Goto-Hayano [8]. Their main results can be rephrased into the following theorem.

Theorem 1 (Cavalcanti–Gualtieri [2, 3]). *Let $(\widehat{M}, \mathcal{J})$ be a stable generalized complex four-manifold whose type change locus has path-connected components $\{\widehat{T}_1, \dots, \widehat{T}_n\}$ for some $n \in \mathbb{N}$. Suppose that each of these tori has self-intersection zero. There is a symplectic four-manifold (M, ω) with Euler* *characteristic and signature given by*

$$\chi(\widehat{M}) = \chi(M) and \sigma(\widehat{M}) = \sigma(M),$$

as well as symplectically embedded tori $T_1, \dots, T_n \subset (M, \omega)$ *of self-intersection zero.*

The second Stiefel–Whitney class of the four-manifolds of Theorem 1 need not coincide. Indeed, the setting of the example considered by Cavalcanti–Gualtieri [2, Example 4.2] is $\widehat{M} = 3\mathbb{CP}^2 \# 19\overline{\mathbb{CP}^2}$ and M is a K3 surface.

Before discussing a relation between the symplectic Kodaira dimension of M and the characteristic numbers of $\widehat{M}$, first recall the definition of the former. Let (M, ω) be a minimal symplectic four-manifold and let K_ω be its canonical class. The symbol K_ω also denotes the first Chern class of any almost-complex structure on M that is compatible with the symplectic structure ω.

Definition 2. The symplectic Kodaira dimension. Li [13, Definition 2.2]. The symplectic Kodaira dimension $\mathrm{Kod}(M, \omega)$ of a minimal symplectic four-manifold (M, ω) is defined by

$$\mathrm{Kod}(M,\omega) = \begin{cases} -\infty & \text{if } K_\omega \cdot [\omega] < 0 \text{ or } K_\omega \cdot K_\omega < 0, \\ 0 & \text{if } K_\omega \cdot [\omega] = 0 \text{ and } K_\omega \cdot K_\omega = 0, \\ 1 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } K_\omega \cdot K_\omega = 0, \text{ or} \\ 2 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } K_\omega \cdot K_\omega > 0. \end{cases}$$

The symplectic Kodaira dimension of a nonminimal symplectic four-manifold is defined as the symplectic Kodaira dimension of any of its minimal models.

A minimal symplectic four-manifold (M, ω) with $\pi_1(M)$ of exponential growth has $\mathrm{Kod}(M, \omega) = -\infty$ if and only if M is irrationally ruled by a result of Liu [14]. All such four-manifolds admit a Kähler structure and are diffeomorphic to either the product $S^2 \times \Sigma_g$ or the nontrivial bundle $S^2 \widetilde{\times} \Sigma_g$ in the absence of smoothly embedded spheres of self-intersection minus one; see [13, Section 3] for a guide to the literature on the beautiful characterizations of these four-manifolds. For our purposes, we state Liu's classification as the following result.

Theorem 3 (Liu [14, Main Theorem A, Theorem B]). *Let* (M, ω) *be a symplectic four-manifold with* $\mathrm{Kod}(M, \omega) = -\infty$ *and whose fundamental group has exponential growth. Then, M is diffeomorphic to* $M_g \# k\overline{\mathbb{CP}^2}$ *where* M_g *is a sphere bundle over a surface of genus at least two and* $k \in \mathbb{Z}_{\geqslant 0}$.

The Euler characteristic of a minimal symplectic four-manifold (M, ω) with $\mathrm{Kod}(M, \omega) = -\infty$ is negative if and only if its fundamental group has exponential growth. The precise diffeomorphism type of the four-manifold of Theorem 3 is determined from its intersection form over $\mathbb{Z}$ [7, Definition 1.2.1] and/or its Euler characteristic, signature, and second Stiefel–Whitney class.

2.2 | The minimal genus of symplectic surfaces in irrational ruled symplectic four-manifolds

The main ingredient in the proof of Theorem A is work of B.-H. Li and T.-J. Li [12, Theorems 1 and 2] that solves the minimal genus problem for smooth embeddings of surfaces in irrational ruled symplectic four-manifolds. We now recall their statements in order to make this note as self-contained as possible.

Theorem 4 (Li-Li, [12, Theorem 1]). *Let a and b be nonnegative integers, and consider the second homology classes*

$$x = [\{pt\} \times \Sigma_g] and y = [S^2 \times \{pt\}].$$

The minimal genus g_ξ of $\xi = ax + by \in H_2(S^2 \times \Sigma_g)$ is given by

$$g_\xi = \begin{cases} (a-1)(g-1+b)+g & (b+g)a \neq 0 \\ 0 & \textit{otherwise.} \end{cases}$$

The two orientable sphere bundles over a closed surface Σ_g are distinguished by their second Stiefel–Whitney class. The corresponding statement in the case where the second Stiefel–Whitney class does not vanish is as follows.

Theorem 5 (Li-Li [12, Theorem 2]). *Let $S^2 \widetilde{\times} \Sigma_g$ be the nontrivial sphere bundle over a surface of genus $g > 0$. Let x be the homology class of its section and y the homology class of a fiber. The minimal genus g_ξ of $\xi = ax + by \in H_2(S^2 \widetilde{\times} \Sigma_g)$ is given by*

$$g_\xi = \begin{cases} (a'-1)(g-1+\frac{1}{2}|a'+2b'|)+g & a \neq 0 \\ 0 & \textit{otherwise,} \end{cases}$$

where $a' = |a|$ and $b' = \frac{a}{|a|}b$ when $a \neq 0$.

The following consequence of Theorems 4 and 5 was pointed out by the referee.

Corollary 6. *Let M be a sphere bundle over a surface of genus strictly greater than one. If a homology class $\xi \in H_2(M)$ is represented by a torus T, then ξ is a multiple of the homology class of the fiber of M. In particular, $\xi^2 = 0$ and T is not a minimum genus representative of ξ.*

3 | PROOF OF THEOREM A

There are stable generalized complex structures of constant type $k \in \{0, 2\}$ on both bundles $S^2 \times \Sigma_g$ and $S^2 \widetilde{\times} \Sigma_g$ that come from their structure as irrational ruled complex surfaces. They can also be equipped with stable generalized complex structures of constant type $k = 1$ by considering the sphere as the complex projective line and the volume form on the closed surface of genus g.

To see that neither of these four-manifolds admits a stable generalized complex structure with at least one connected component in its type change locus, we proceed by contradiction. Let $\widehat{M}$ be a sphere bundle over a closed surface of genus at least two and suppose that there is a stable generalized complex structure $(\widehat{M}, \mathcal{J})$ whose type change locus has at least one connected component. The following two scenarios must be considered.

(1) There is at least one path-connected component $\widehat{T} \subset \widehat{M}$ of the type change locus of $\mathcal{J}$ with nonzero self-intersection.

(2) Every path-connected component $\widehat{T}$ of the type change locus of $\mathcal{J}$ has self-intersection zero.

The first scenario is immediately ruled out by Corollary 6. In order to rule out the second scenario, we apply Theorem 1 to $(\widehat{M}, \mathcal{J})$ to obtain a symplectic four-manifold (M, ω) with signature $\sigma(M) = \sigma(\widehat{M}) = 0$ and Euler characteristic $\chi(M) = \chi(\widehat{M}) < 0$. These values imply that (M, ω) is a minimal symplectic four-manifold whose fundamental group has exponential growth. Theorem 1 implies that there is a symplectic torus $T \subset M$ of self-intersection zero, where M is either $S^2 \times \Sigma_g$ or $S^2 \widetilde{\times} \Sigma_g$ by Liu's classification of irrational ruled symplectic four-manifolds in Theorem 3; cf. [7, Theorem 10.1.19]. Such symplectic surface would be genus minimizing by the proof of the symplectic Thom conjecture of Oszváth–Szabó [15, Theorem 1.1]. However, this contradicts Corollary 6. We conclude that the second scenario is not possible.

Remark 7. It is well known that the number of path-connected components of the type change locus of a stable generalized complex structure on $S^2 \times S^2$ and $S^2 \times T^2$ can be chosen to be arbitrarily large [8, 16, 17].

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JOURNAL INFORMATION

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