Multi-Grid Optimisation

M.Tech. project report

In

Thermal Engineering

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1. Introduction

One of the most powerful acceleration schemes for the convergence of iterative methods in solving elliptic problems is the multigrid algorithm. The method is based on the realization that different components of solution converge to the exact solution at different rates and hence should be treated differently. Suppose the residual or the error vector in the solution is represented as linear combination of a set of basis vectors which when plotted on the grid would range from smooth to rapidly varying (just like low and high frequency sines and cosines). It turns out that smooth component of the residual converges very slowly to zero and the rough part converges quickly. The multigrid algorithm takes advantage of this to substantially reduce the overall effort required to obtain a converged solution.

2. Methodology

The basic dual-grid multigrid algorithm is summarized below:

1. Perform a few iterations on the original equation, $A\phi = b$, on the fine grid with the mesh spacing *h*. Let the resulting solution be denoted by ψ. Calculate the residual $r = b - A\psi$ on the same grid.
2. Transfer the residual to a coarse grid(restriction) of mesh spacing 2*h*, and on this grid iterate on the error equation $A\varepsilon = r$, with the initial guess $\varepsilon_0 = 0$
3. Interpolate (prolongation) the resulting ε to the fine grid. Make a correction on the previous ψ by adding it to ε, i.e. $\psi_{new} = \psi + \varepsilon$. Using ψ_{new} as the initial guess the initial problem is reiterated.
4. This process is repeated until the solution converges to desired convergence criteria.
5. For a multigrid with more than 2 steps the steps 1-2 are repeated until the last step and then the step 3 is applied with the error equation.

3 1-D Multi-Grid Validation

Equation being used for validation:

$$\frac{d^2u}{dx^2} = \frac{1}{2}[\sin \pi x + \sin 16\pi x]$$

With B.C.s $u_0 = u_N = 0$

The discretized equation used to evaluate the PDE is:

$$u_j^{n+1} = \frac{1}{2}[u_{j+1}^n + u_{j-1}^{n+1}] - \frac{h^2}{4}[\sin(\pi jh) + \sin(16\pi jh)]$$

Restriction:

The residual $r = A\phi$ is mapped onto the coarser grid. This process of mapping the residual on coarser grid is called restriction. A simple intuitive restriction would be to retain only the residuals at the remaining grid points. Practice shows Laplacian smoothing leads to much faster convergence. Thus, residual at coarser grid is calculated as:

$$r_j^{2h} = \frac{1}{4}[r_{2j-1}^h + 2r_{2j}^h + r_{2j+1}^h] \forall j = 1 - N/2 - 1$$

At each grid level Gauss-Seidel iteration is performed on the residual equation

$$A\varepsilon = r$$

Prolongation:

The new errors calculated at all the grid levels is interpolated to finer grid sizes. This process of interpolating errors on finer grid is called prolongation

Error is interpolated on the finer grid with following equations which follows common intuition:

$$\epsilon_{2j}^{h} = \epsilon_{j}^{2h} \forall j = 0 - N$$

$$\epsilon_{2j+1}^{h} = \frac{\epsilon_{j}^{2h} + \varepsilon_{j+1}^{2h}}{2} \forall j = 0, N-1$$

At each grid level Gauss-Seidel iteration (or any other solver can be used) is performed again on the residual equation.

For a finest grid size of 65 nodes and application of full V-cycle we obtain the following figure of maximum residual ***r*** v/s V-cycle number. We find that it takes 5 full V-cycle for the maximum absolute residual to fall below 10^{-3}. The residual falls below 10^{-12} after 15 V-cycles. The results are in conformance with the ones provided.

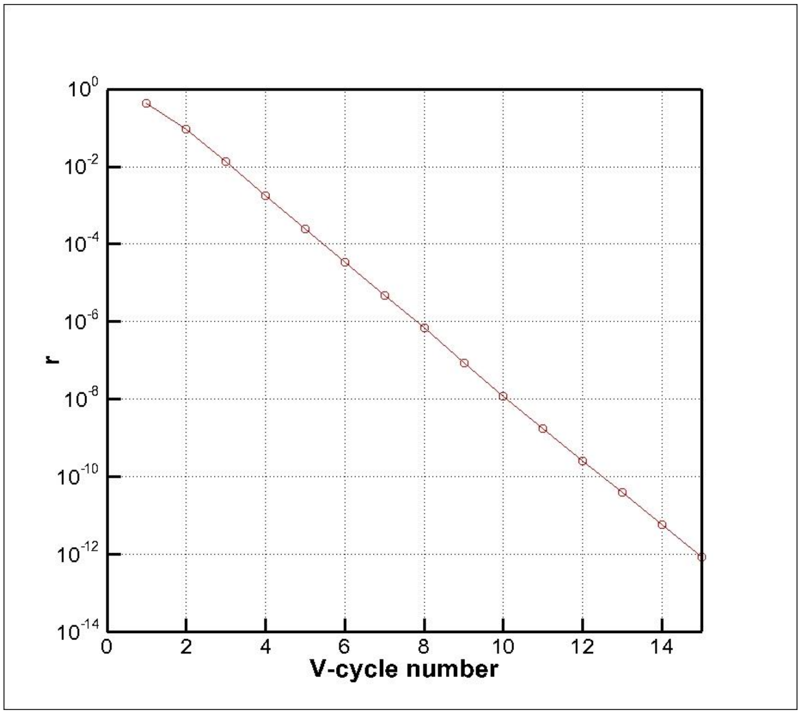

Fig 1. Maximum residual v/s V cycle number

4 2-D Multi-Grid Validation

For 2-D validation the Poisson equation has been taken into consideration of the form:

$$\frac{\partial^2 \mathrm{T}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{T}}{\partial \mathrm{y}^2} = \mathrm{S}$$

For the validation problem $S = -2(2 - x^2 - y^2)$

Here, we have the analytical solution for the given source term being:

$$T = (x^2 - 1)(y^2 - 1)$$

The restriction step for 2-D is

$$r_{i,j}^{2h} = \frac{1}{16}[r_{2i-1,2j-1}^{h} + r_{2i+1,2j-1}^{h} + r_{2i+1,2j+1}^{h} + r_{2i-1,2j+1}^{h} + 2(r_{2i,2j-1}^{h} + r_{2i,2j+1}^{h} + r_{2i-1,2j}^{h} + r_{2i+1,2j}^{h}) + 4r_{2i,2j}^{h}]$$

The prolongation of error is done according to following algebra

$$\epsilon_{2i,2j}^{h} = \epsilon_{i,j}^{2h}$$

$$\epsilon_{2i+1,2j}^{h} = \frac{1}{2}[\epsilon_{i,j}^{2h} + \epsilon_{i+1,j}^{2h}]$$

$$\epsilon_{2i,2j+1}^{h} = \frac{1}{2}[\epsilon_{i,j}^{2h} + \epsilon_{i,j+1}^{2h}]$$

$$\epsilon_{2i+1,2j+1}^{h} = \frac{1}{4}[\epsilon_{i,j}^{2h} + \epsilon_{i+1,j}^{2h} + \epsilon_{i,j+1}^{2h} + \epsilon_{i+1,j+1}^{2h}]$$

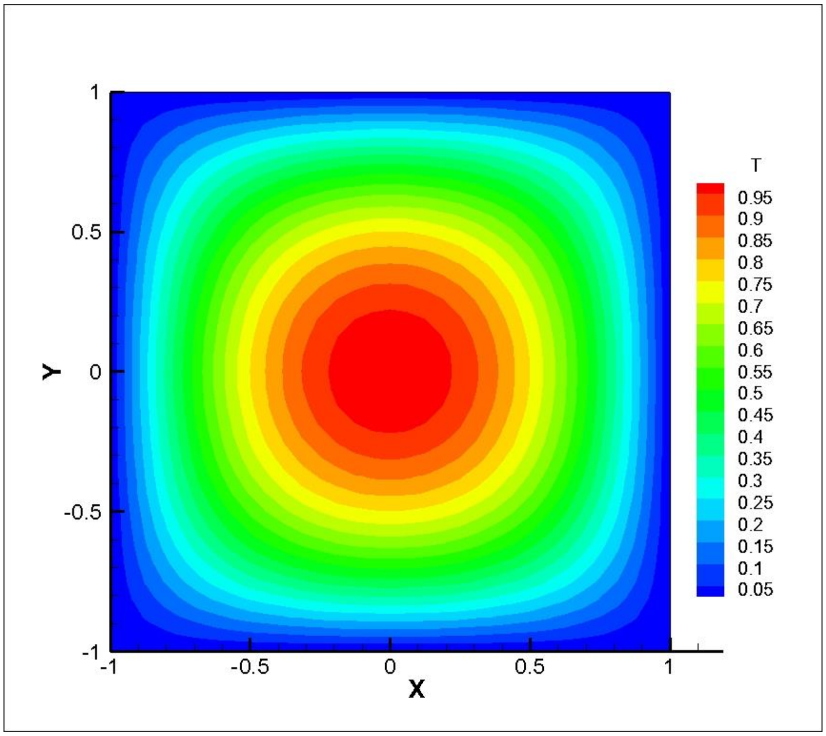

Fig 2: Iso-thermal contour numerical

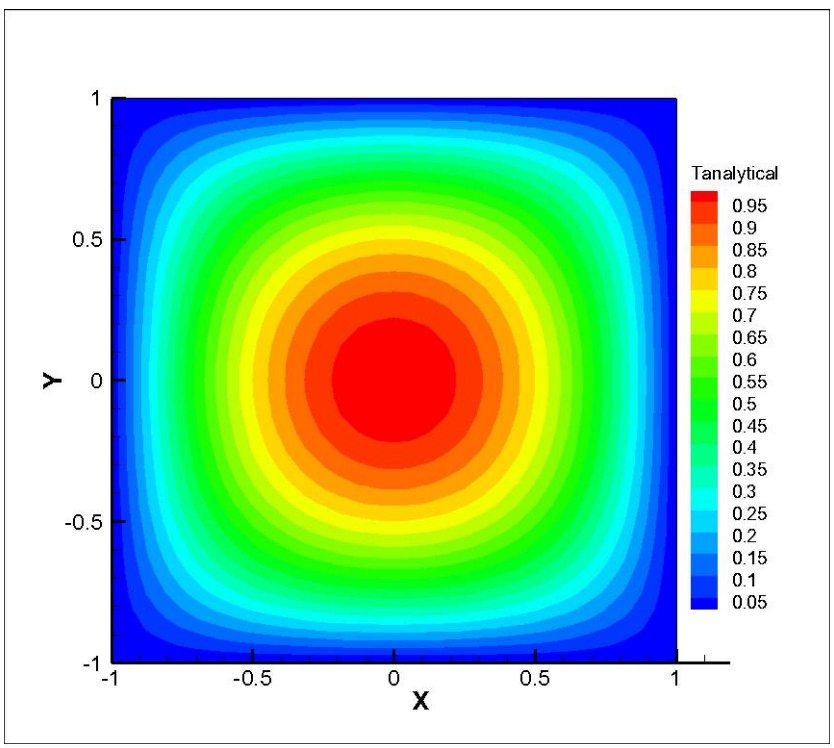

Fig 3: Iso-thermal contour analytical

The iso-thermal contours for both analytical and numerical show strong conformance with each other.

For the solution obtained by Gauss-Siedel iterative solver alone takes 339 iterations to achieve 0.01% accuracy. Where as the multigrid acceleration takes 4 complete V-cycles with 3 iterations at each node of the V-cycle amounting to 24 Gauss-Siedel iterations. This is a huge reduction in the work done because of the application of multigrid acceleration.

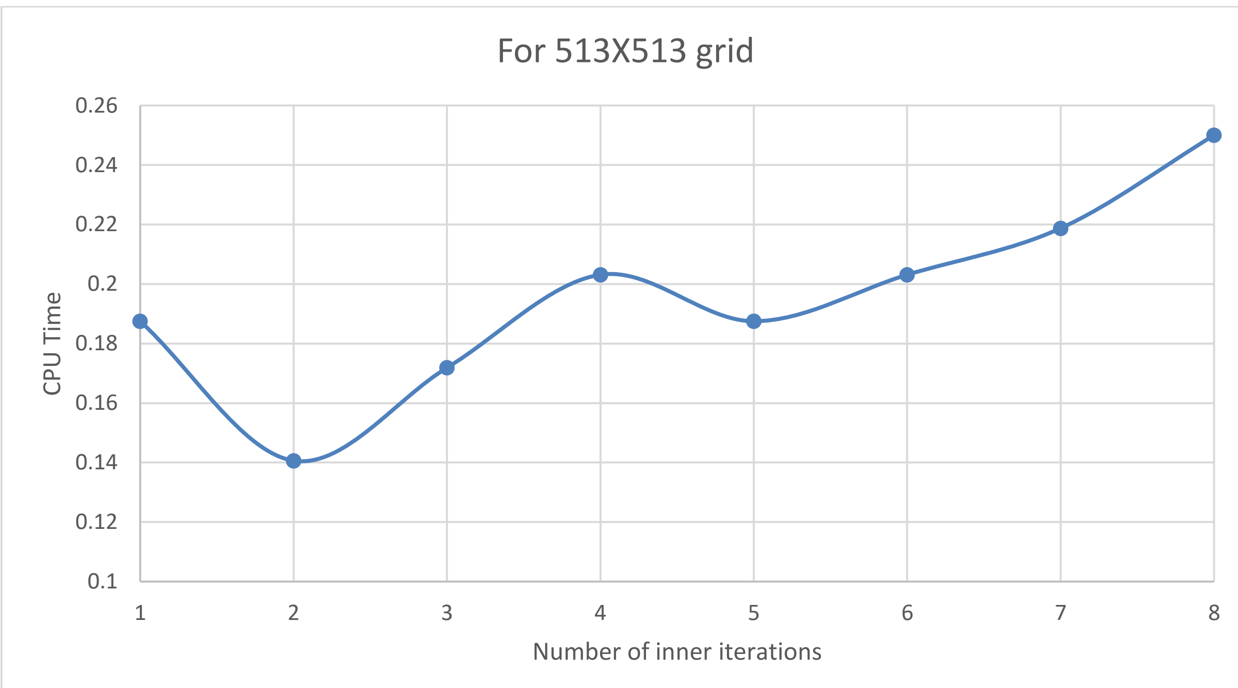

Fig 4: CPU time v/s number of inner iterations at each node of the V-cycle

5 2-D Multi-Grid for a complex source term

Numerical solution of Poisson equation with the source term as:

$$S = -2(1-6x^2)y^2(1-y^2) + (1-6y^2)x^2(1-x^2)$$

The analytical solution for the Poisson equation is given as

$$T(x, y) = x^2 y^2 (1-x^2)(y^2 - 1)$$

The iso=thermal contours of both analytical solution and multigrid preconditioned Gauss-Siedel solver are plotted and compared. Both give nearly identical plots and only difference arise at few decimal places. The plots of numerical and analytical are presented in Figure 5 and 6 respectively.

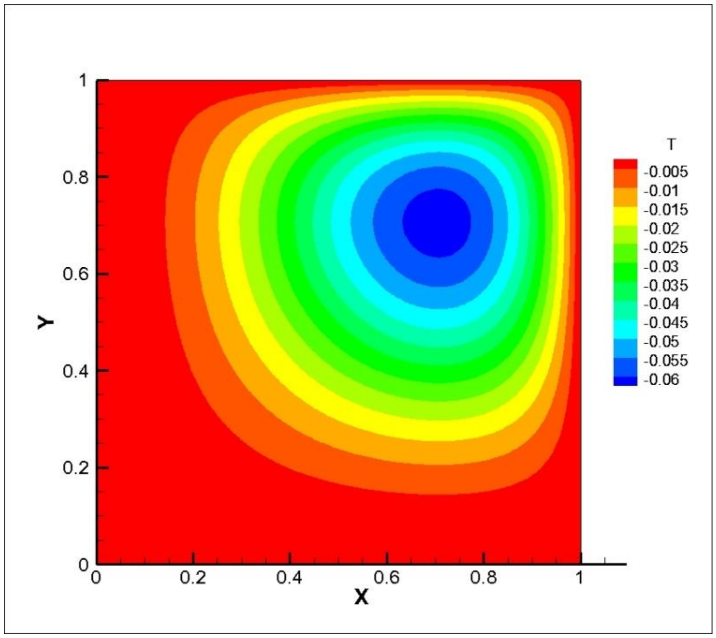

Fig 5 Iso-thermal plot for numerical solution

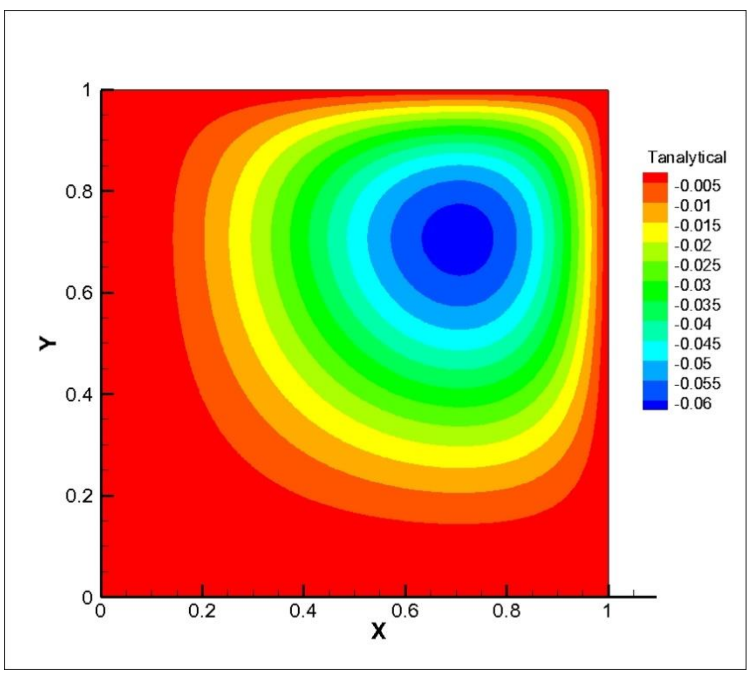

Fig 6: Iso-thermal contour plot for analytical solution

The true potential of multigrid preconditioning/acceleration is evident from the following table comparing Gauss-Siedel iterative solver and Gauss-Seidell with multigrid preconditioning.

	Only Gauss-Seidel	GS with Multigrid
Number of grid points	257*257	257*257
Convergence value of residual	1E-10	1E-10
CPU time	255.5468(s)	0.328125(s)[For 1 inner iterations]