**Cont….from module 3**

NOTE:- Kth moment of $X$ is $E(X^K)$.

It is defined by

$$E(X^K) = \begin{cases} \sum_{\forall x} x^K f(x) & \text{(Discrete)} \\ \int_{-\infty}^{\infty} x^K f(x)\,dx & \text{(cont.)} \end{cases}$$

Kth central moment of $X$ is

$$E[(X-\mu)^K]$$

It is defined by

$$E[(X-\mu)^K] = \begin{cases} \sum_{\forall x} (x-\mu)^K f(x) \\ \int_{-\infty}^{\infty} (x-\mu)^K f(x)\,dx \end{cases}$$

When $K=1$,

$E(X^K) = E(X)$ is known as mean of $X$.

$E[(X-\mu)^K] = E[(X-\mu)^2]$ is known as variance of $X$.

So $\sigma^2 = E[(X-\mu)^2]$

Remember: $E(1)=1$, $E(2)=2$ $\cdots$, $E(c)=c$.

**Q)**

Find the expectation of $g(X) = X^2$, where $X$ is uniformly distributed on the interval $-1 \leqq x \leqq 1$.

**Ans:**

Since $X$ is uniformly distributed over $-1 \leqslant x \leqslant 1$, its density is

$$f(x) = k, \quad -1 \leqslant x \leqslant 1$$

Since it is the density of $X$, the following must hold:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Since

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{1} k\,dx = kx\Big|_{-1}^{1} = 2k,$$

we get

$$k = \frac{1}{2}$$

$$E[X^2] = \int_{-1}^{1} \frac{x^2}{2}\,dx = \frac{x^3}{6}\Big|_{-1}^{1} = \frac{1}{6} - \frac{(-1)}{6} = \frac{1}{3}$$

# Module-4

## Binomial Distribution

It is discrete probability distribution. Probability mass function is given by

$$f(x) = \binom{n}{x} p^x q^{n-x} \qquad (x = 0, 1, \cdots, n)$$

Where x= number of success in n trials.

p is the probability of success and q=1-p is the probability of failure in a trial

How to know a a distribution is binomial distribution ?

If in a distribution ,the outcomes are either or type of events ,like true or false, agree or disagree,right or wrong ,valid or invalid, success or failure etc .. ,then we apply binomail distribution.

Mean of the binomial distibution random variable x is

$$\mu = np$$

Variance of the binomial distibution random variable x is

$$\sigma^2 = npq.$$

**Q)**

Five fair coins are tossed simultaneously. Find the probability function of the random variable $X = \textit{Number of heads}$ and compute the probabilities of obtaining no heads, precisely 1 head, at least 1 head, not more than 4 heads.

**Ans:**

Let $X$ = no. of heads in tossing 5 coins.

$p$ = probability of getting a head $= \frac{1}{2}$

$q$ = " " a tail $= \frac{1}{2}$

$n = 5$

So $f(0) = {}^5C_0\, p^0 q^5 = 0.03125$

$f(1) = {}^5C_1\, p^1 q^4 = 0.15625$

$P(X \geq 1) = 1 - P(X < 1) = 1 - P(X=0)$

$= 1 - f(0) = 1 - 0.03125 = 0.96875$

$P(X \leq 4) = P(X=0) + P(X=1) + \cdots + P(X=4)$

$= f(0) + f(1) + \cdots + f(4)$

$= 0.96875$

or, $P(X \leq 4) = 1 - P(X > 4) = 1 - P(X=5)$

$= 1 - f(5) = 1 - {}^5C_5\, p^5 q^0 = 1 - (0.5)^5$

$= 0.96875$

**Q)**

Let $p = 2\%$ be the probability that a certain type of lightbulb will fail in a 24-hour test. Find the probability that a sign consisting of 15 such bulbs will burn 24 hours with no bulb failures.

**Ans:**

Let $X$ be the number of burnt lightbulbs. Then $X$ is a binomial distribution with parameters

$$n = 15, \quad p = 0.02, \quad q = 1 - p = 0.98$$

Therefore,

$$f(x) = \binom{n}{x} p^x q^{n-x} = \binom{15}{x} 0.02^x 0.98^{15-x}$$

We must find the probability that no lightbulbs have burnt — this probability is

$$P(X = 0) = f(0) = \binom{15}{0} 0.02^0 \cdot 0.98^{15} \approx 0.7386$$

**Q)**

Compute the probability of obtaining at least two *"Six"* in rolling a fair die 4 times.

**Ans:**

Let $X$ = no. of six appears in rolling a fair die 4 times.

$p$ = probability of getting a six, in rolli a die $= \frac{1}{6}$.

$q = 5/6$

$n = 4$

$$P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[ {}^4C_0 \, p^0 q^4 + {}^4C_1 \, p^1 q^3 \right]$$

$$= 1 - \left[ \left(\tfrac{5}{6}\right)^4 + 4\left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right)^3 \right]$$

$$= 0.132$$

**Q)**

② If the probability of hitting a target is 10% and 10 shots are fired independently, what is the probability that the target will be hit at least once?

Ans: Given $n = 10$, $p = 10\% = \frac{10}{100}$

$q = 1 - p = 1 - \frac{10}{100} = \frac{90}{100}$

$\therefore f(x) = \binom{n}{x} p^x q^{n-x} = \binom{10}{x} \left(\frac{10}{100}\right)^x \left(\frac{90}{100}\right)^{10-x}$

P(the target will be hit at least once) =

$1 - P(\text{not hitting the target}) = 1 - P(X=0)$

$= 1 - \binom{10}{0} \left(\frac{10}{100}\right)^0 \left(\frac{90}{100}\right)^{10} = 1 - \left(\frac{90}{100}\right)^{10}$

# Poisson Distribution

It is discrete probability distribution. Probability mass function is given by

$$f(x) = \frac{\mu^x}{x!} e^{-\mu} \qquad (x = 0, 1, \cdots)$$

Where x= number of success in n trials ,( n is a large number)

$\mu$ is the parameter of the poisson distribution. Poisson distribution is the limiting case of binomial distribution.when $p \to 0$ and $n \to \infty$ Poisson dist. becomes binomial dist.

Here $\mu = np$ is the mean of the poisson distribution.

$\sigma^2 = np$ is the variance of the poisson distribution.

**Q)**

Suppose that 4% of steel rods made by a machine are defective, the defectives occurring at random during production. If the rods are packaged 100 per box, what is the Poisson approximation of the probability that a given box will contain $x = 0, 1, \cdots, 5$ defectives?

**Ans:**

$x$ = no. of defectives in a packet of 100.

$p = 4\%$

$n = 100$

Since $p$ is very small compared to $n = 100$,
this is a poisson distribution.

$$\mu = np = 0.04 \times 100 = 4$$

$$f(0) = \frac{e^{-4} \cdot 4^0}{0!} = e^{-4}$$

$$f(1) = \frac{e^{-4} \cdot 4^1}{1!} = 4e^{-4}$$

$$f(2) = \frac{e^{-4} \cdot 4^2}{2!} = 8e^{-4}$$

$\vdots$

$$f(5) = \frac{e^{-4} \cdot 4^5}{5!}$$

**Q)**

Let $X$ be the number of cars per minute passing a certain point of some road between 8 A.M. and 10 A.M. on a Sunday. Assume that $X$ has a Poisson distribution with mean 5. Find the probability of observing 4 or fewer cars during any given minute.

**Ans:**

The probability function of the Poisson distribution with mean $\mu$ is given by

$$f(x) = \frac{\mu^x}{x!}e^{-\mu}, \quad x = 0, 1, 2, \ldots$$

Here the mean is $\mu = 5$, so

$$f(x) = \frac{5^x}{x!}e^{-5}, \quad x = 0, 1, 2, \ldots$$

Now, the probability of observing 4 cars or fewer is

$$P(X \leqslant 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$

(Recall that $f(x) = P(X = X)$)

$$= f(0) + f(1) + f(2) + f(3) + f(4)$$

$$= \frac{5^0}{0!}e^{-5} + \frac{5^1}{1!}e^{-5} + \frac{5^2}{2!}e^{-5} + \frac{5^3}{3!}e^{-5} + \frac{5^4}{4!}e^{-5}$$

$$= \frac{523}{8}e^{-5} \approx 0.4405$$

**Q)**

Suppose that a telephone switchboard of some company on the average handles 300 calls per hour, and that the board can make at most 10 connections per minute. Using the Poisson distribution, estimate the probability that the board will be overtaxed during a given minute.

**Ans**

Let $X$ be the number of calls per minute. The average number of calls per hour is 300, so the mean of $X$ (the average number of calls per minute) is

$$\mu = \frac{300}{60} = 5$$

The probability function of the Poisson distribution with mean $\mu$ is given by

$$f(x) = \frac{\mu^x}{x!}e^{-\mu}, \quad x = 0, 1, 2, \ldots$$

Thus, here

$$f(x) = \frac{5^x}{x!}e^{-5}, \quad x = 0, 1, 2, \ldots$$

Here we must find

$$P(X > 10)$$

The complementary event is $X \leqslant 10$, so

$$P(X > 10) = 1 - P(X \leqslant 10) = 1 - F(10),$$

where $F$ is the distribution function of $X$, thus $P(X \leqslant x) = F(x)$. From the table we see that $F(10) = 0.9863$, so

$$P(X > 10) = 1 - 0.9863 = 0.0137$$

**Q)**

Suppose that a certain type of magnetic tape contains, on the average, 2 defects per 100 meters. What is the probability that a roll of tape 300 meters long will contain **(a)** $x$ defects, **(b)** no defects?

**Ans:**

We can model this using the Poisson distribution. Let $X$ be the number of defects on a tape which is 300 meters long. Since there are, on average, 2 defects per 100 meters, there are, on average, $3 \cdot 2 = 6$ defects per 300 meters. This means that the mean of $X$, which we will denote by $\mu$, is $\mu = 6$. The probability function of the Poisson distribution with mean $\mu$ is

$$f(x) = \frac{\mu^x}{x!}e^{-\mu}, \quad x = 0, 1, 2, \ldots$$

Here $\mu = 6$, so

$$f(x) = \frac{6^x}{x!}e^{-6}, \quad x = 0, 1, 2, \ldots$$

**(a)**

By definition of the probability function,

$$P(X = x) = f(x) = \frac{6^x}{x!}e^{-6}, \quad x = 0, 1, 2, \ldots$$

**(b)**

Similarly,

$$P(X = 0) = f(0) = \frac{6^0}{0!}e^{-6} = e^{-6} \approx 0.0025$$

**Q)**

If a ticket office can serve at most 4 customers per minute and the average number of customers is 120 per hour, what is the probability that during a given minute customers will have to wait? (Use the Poisson

**Ans:**

Average number of costomers : 120/hour = $\frac{120}{60}$ /min = 2/min

$\Rightarrow \mu = 2$

P( castomer has to wait) = P(more than 4 customers/min)

The distribution can be assumed to be Poisson distribution with $\mu = 2$

$\Rightarrow f(x) = \frac{e^{-4} 2^x}{x!}$

P(more than 4 /min) = 1 - P(less than or equal to 4)

$= 1 - (f(0) + f(1) + f(2) + f(3) + f(4))$

$= 1 - e^{-2}\left(\frac{4^0}{0!} + \frac{4^1}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!}\right)$

or

from table

$= 1 - (0.1353 + 0.2707 + 0.2707 + 0.1804 + 0.0902)$

$= 1 - 0.9473 = 0.0527$

---

**Q)**

Suppose that in the production of 60-ohm radio resistors, nondefective items are those that have a resistance between 58 and 62 ohms and the probability of a resistor's being defective is 0.1%. The resistors are sold in lots of 200, with the guarantee that all resistors are nondefective. What is the probability that a given lot will violate this guarantee? (Use the Poisson distribution.)

**Ans:**

Let $X$ be the number of defective resistors. Since $p$ is small and $n$ is big, we can use Poisson distribution. The mean is

$$\mu = n \cdot p = 200 \cdot 0.1\% = 200 \cdot 0.001 = 0.2$$

The probability function of the Poisson distribution with mean $\mu$ is

$$f(x) = \frac{\mu^x}{x!}e^{-\mu}$$

Therefore, in this problem,

$$f(x) = \frac{0.2^x}{x!}e^{-0.2}$$

We must find $P(X \geqslant 1)$ (the probability that the guarantee is violated is the same as the probability of having at least 1 defective resistor).

The complementary event of $X \geqslant 1$ is $X < 1$, so we compute

$$P(X < 1) = P(X = 0) = f(0) = \frac{0.2^0}{0!}e^{-0.2} = e^{-0.2} \approx 0.8187$$

Therefore,

$$P(X \geqslant 1) = 1 - P(X < 1) = 1 - e^{-0.2} \approx 0.1813$$

**Q)**

**Rutherford–Geiger experiments.** In 1910, E. Rutherford and H. Geiger showed experimentally that the number of alpha particles emitted per second in a radioactive process is a random variable $X$ having a Poisson distribution. If $X$ has mean 0.5, what is the probability of observing two or more particles during any given second?

**Ans:**

The probability function of the Poisson distribution with mean $\mu$ is given by

$$f(x) = \frac{\mu^x}{x!}e^{-\mu}, \quad x = 0, 1, 2, \ldots$$

Here the mean is $\mu = 0.5$, so

$$f(x) = \frac{0.5^x}{x!}e^{-0.5}, \quad x = 0, 1, 2, \ldots$$

Now, the probability of observing less than 2 particles is

$$\begin{aligned} P(X < 2) &= P(X \leqslant 1) \\ &= P(X = 0) + P(X = 1) \qquad \text{(Recall that } f(x) = P(X = X)\text{)} \\ &= f(0) + f(1) \\ &= \frac{0.5^0}{0!}e^{-0.5} + \frac{0.5^1}{1!}e^{-0.5} \\ &= 1.5e^{-0.5} \approx 0.9098 \end{aligned}$$

Therefore, since $X < 2$ and $X \geqslant 2$ are complementary events,

$$P(X \geqslant 2) = 1 - P(X < 2) = 1 - 1.5e^{-0.5} \approx 0.0902$$

**Q)**

Q. Suppose that 3% of bolts made by a machine are defective, the defective occuring at random during production. If the bolts are packaged 50 per box, what is the poisson approximation of the probability that a given box will contain $x$ defectives?

Sol$^n$. Given $n = 50$, $p = 3\%$

$\therefore$ Mean $= \mu = np = 50 \times \frac{3}{100} = 1.5$

$$P(\text{box contains } x \text{ defectives}) = \frac{e^{-1.5}(1.5)^x}{x!}$$

# Normal Distribution

The **normal distribution** or *Gauss distribution* is defined as the distribution with the density

**(1)** $$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \qquad (\sigma > 0)$$

f(x) has following features.

1. $\mu$ is the mean and $\sigma$ the standard deviation.
2. $1/(\sigma\sqrt{2\pi})$ is a constant factor that makes the area under the curve of $f(x)$ from $-\infty$ to $\infty$ equal to 1, as it must be by (10), Sec. 24.5.
3. The curve of $f(x)$ is symmetric with respect to $x = \mu$ because the exponent is quadratic. Hence for $\mu = 0$ it is symmetric with respect to the $y$-axis $x = 0$ (Fig. 519, *"bell-shaped curves"*).
4. The exponential function in (1) goes to zero very fast—the faster the smaller the standard deviation $\sigma$ is, as it should be (Fig. 519).

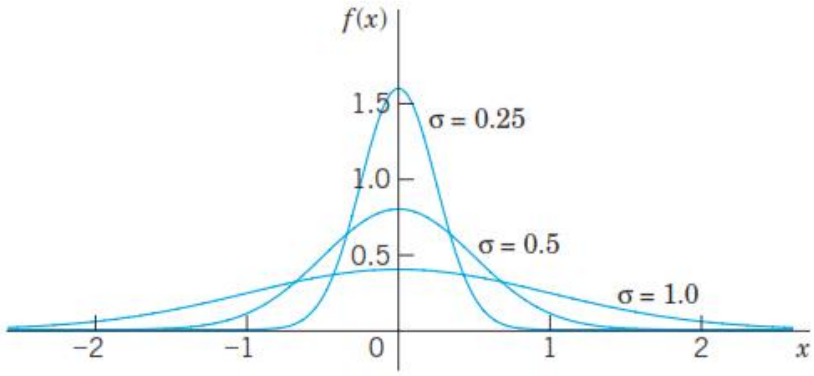


**Fig. 519.** Density (1) of the normal distribution with $\mu = 0$ for various values of $\sigma$

Cummulative distribution function is

**(2)** $$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[-\frac{1}{2}\left(\frac{v-\mu}{\sigma}\right)^2\right] dv.$$

Here we needed $x$ as the upper limit of integration and wrote $v$ (instead of $x$) in the integrand.

For the corresponding **standardized normal distribution** with mean 0 and standard deviation 1 we denote $F(x)$ by $\Phi(z)$. Then we simply have from (2)

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du. \tag{3}$$

*The distribution function $F(x)$ of the normal distribution with any $\mu$ and $\sigma$ [[redacted]] is related to the standardized distribution function $\Phi(z)$ in (3) by the formula*

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right). \tag{4}$$

**Normal Probabilities for Intervals**

*The probability that a normal random variable $X$ with mean $\mu$ and standard deviation $\sigma$ assume any value in an interval $a < x \leqq b$ is*

$$P(a < X \leqq b) = F(b) - F(a) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right). \tag{5}$$

NOTE:

$$(6) \qquad \begin{array}{ll} \textbf{(a)} & P(\mu - \sigma < X \leqq \mu + \sigma) \approx 68\% \\ \textbf{(b)} & P(\mu - 2\sigma < X \leqq \mu + 2\sigma) \approx 95.5\% \\ \textbf{(c)} & P(\mu - 3\sigma < X \leqq \mu + 3\sigma) \approx 99.7\%. \end{array}$$

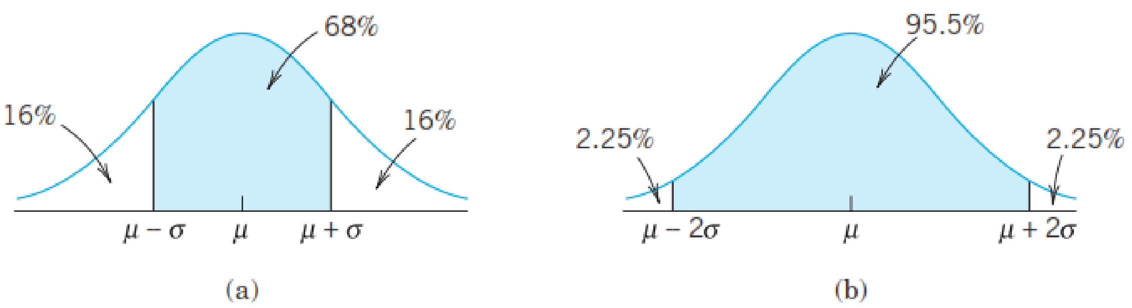


**Fig. 521.** Illustration of formula (6)

Similarly,

(7)

(a) $$P(\mu - 1.96\sigma < X \leqq \mu + 1.96\sigma) = 95\%$$

(b) $$P(\mu - 2.58\sigma < X \leqq \mu + 2.58\sigma) = 99\%$$

(c) $$P(\mu - 3.29\sigma < X \leqq \mu + 3.29\sigma) = 99.9\%.$$

**Table A7 Normal Distribution**

Values of the distribution function $\Phi(z)$ [see (3), Sec. 24.8]. $\Phi(-z) = 1 - \Phi(z)$

| $z$ | $\Phi(z)$ | $z$ | $\Phi(z)$ | $z$ | $\Phi(z)$ | $z$ | $\Phi(z)$ | $z$ | $\Phi(z)$ | $z$ | $\Phi(z)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | **0.** | | **0.** | | **0.** | | **0.** | | **0.** | | **0.** |
| 0.01 | 5040 | 0.51 | 6950 | 1.01 | 8438 | 1.51 | 9345 | 2.01 | 9778 | 2.51 | 9940 |
| 0.02 | 5080 | 0.52 | 6985 | 1.02 | 8461 | 1.52 | 9357 | 2.02 | 9783 | 2.52 | 9941 |
| 0.03 | 5120 | 0.53 | 7019 | 1.03 | 8485 | 1.53 | 9370 | 2.03 | 9788 | 2.53 | 9943 |
| 0.04 | 5160 | 0.54 | 7054 | 1.04 | 8508 | 1.54 | 9382 | 2.04 | 9793 | 2.54 | 9945 |
| 0.05 | 5199 | 0.55 | 7088 | 1.05 | 8531 | 1.55 | 9394 | 2.05 | 9798 | 2.55 | 9946 |
| 0.06 | 5239 | 0.56 | 7123 | 1.06 | 8554 | 1.56 | 9406 | 2.06 | 9803 | 2.56 | 9948 |
| 0.07 | 5279 | 0.57 | 7157 | 1.07 | 8577 | 1.57 | 9418 | 2.07 | 9808 | 2.57 | 9949 |
| 0.08 | 5319 | 0.58 | 7190 | 1.08 | 8599 | 1.58 | 9429 | 2.08 | 9812 | 2.58 | 9951 |
| 0.09 | 5359 | 0.59 | 7224 | 1.09 | 8621 | 1.59 | 9441 | 2.09 | 9817 | 2.59 | 9952 |
| 0.10 | 5398 | 0.60 | 7257 | 1.10 | 8643 | 1.60 | 9452 | 2.10 | 9821 | 2.60 | 9953 |
| 0.11 | 5438 | 0.61 | 7291 | 1.11 | 8665 | 1.61 | 9463 | 2.11 | 9826 | 2.61 | 9955 |
| 0.12 | 5478 | 0.62 | 7324 | 1.12 | 8686 | 1.62 | 9474 | 2.12 | 9830 | 2.62 | 9956 |
| 0.13 | 5517 | 0.63 | 7357 | 1.13 | 8708 | 1.63 | 9484 | 2.13 | 9834 | 2.63 | 9957 |
| 0.14 | 5557 | 0.64 | 7389 | 1.14 | 8729 | 1.64 | 9495 | 2.14 | 9838 | 2.64 | 9959 |
| 0.15 | 5596 | 0.65 | 7422 | 1.15 | 8749 | 1.65 | 9505 | 2.15 | 9842 | 2.65 | 9960 |
| 0.16 | 5636 | 0.66 | 7454 | 1.16 | 8770 | 1.66 | 9515 | 2.16 | 9846 | 2.66 | 9961 |
| 0.17 | 5675 | 0.67 | 7486 | 1.17 | 8790 | 1.67 | 9525 | 2.17 | 9850 | 2.67 | 9962 |
| 0.18 | 5714 | 0.68 | 7517 | 1.18 | 8810 | 1.68 | 9535 | 2.18 | 9854 | 2.68 | 9963 |
| 0.19 | 5753 | 0.69 | 7549 | 1.19 | 8830 | 1.69 | 9545 | 2.19 | 9857 | 2.69 | 9964 |
| 0.20 | 5793 | 0.70 | 7580 | 1.20 | 8849 | 1.70 | 9554 | 2.20 | 9861 | 2.70 | 9965 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.21 | 5832 | 0.71 | 7611 | 1.21 | 8869 | 1.71 | 9564 | 2.21 | 9864 | 2.71 | 9966 |
| 0.22 | 5871 | 0.72 | 7642 | 1.22 | 8888 | 1.72 | 9573 | 2.22 | 9868 | 2.72 | 9967 |
| 0.23 | 5910 | 0.73 | 7673 | 1.23 | 8907 | 1.73 | 9582 | 2.23 | 9871 | 2.73 | 9968 |
| 0.24 | 5948 | 0.74 | 7704 | 1.24 | 8925 | 1.74 | 9591 | 2.24 | 9875 | 2.74 | 9969 |
| 0.25 | 5987 | 0.75 | 7734 | 1.25 | 8944 | 1.75 | 9599 | 2.25 | 9878 | 2.75 | 9970 |
| 0.26 | 6026 | 0.76 | 7764 | 1.26 | 8962 | 1.76 | 9608 | 2.26 | 9881 | 2.76 | 9971 |
| 0.27 | 6064 | 0.77 | 7794 | 1.27 | 8980 | 1.77 | 9616 | 2.27 | 9884 | 2.77 | 9972 |
| 0.28 | 6103 | 0.78 | 7823 | 1.28 | 8997 | 1.78 | 9625 | 2.28 | 9887 | 2.78 | 9973 |
| 0.29 | 6141 | 0.79 | 7852 | 1.29 | 9015 | 1.79 | 9633 | 2.29 | 9890 | 2.79 | 9974 |
| 0.30 | 6179 | 0.80 | 7881 | 1.30 | 9032 | 1.80 | 9641 | 2.30 | 9893 | 2.80 | 9974 |
| 0.31 | 6217 | 0.81 | 7910 | 1.31 | 9049 | 1.81 | 9649 | 2.31 | 9896 | 2.81 | 9975 |
| 0.32 | 6255 | 0.82 | 7939 | 1.32 | 9066 | 1.82 | 9656 | 2.32 | 9898 | 2.82 | 9976 |
| 0.33 | 6293 | 0.83 | 7967 | 1.33 | 9082 | 1.83 | 9664 | 2.33 | 9901 | 2.83 | 9977 |
| 0.34 | 6331 | 0.84 | 7995 | 1.34 | 9099 | 1.84 | 9671 | 2.34 | 9904 | 2.84 | 9977 |
| 0.35 | 6368 | 0.85 | 8023 | 1.35 | 9115 | 1.85 | 9678 | 2.35 | 9906 | 2.85 | 9978 |
| 0.36 | 6406 | 0.86 | 8051 | 1.36 | 9131 | 1.86 | 9686 | 2.36 | 9909 | 2.86 | 9979 |
| 0.37 | 6443 | 0.87 | 8078 | 1.37 | 9147 | 1.87 | 9693 | 2.37 | 9911 | 2.87 | 9979 |
| 0.38 | 6480 | 0.88 | 8106 | 1.38 | 9162 | 1.88 | 9699 | 2.38 | 9913 | 2.88 | 9980 |
| 0.39 | 6517 | 0.89 | 8133 | 1.39 | 9177 | 1.89 | 9706 | 2.39 | 9916 | 2.89 | 9981 |
| 0.40 | 6554 | 0.90 | 8159 | 1.40 | 9192 | 1.90 | 9713 | 2.40 | 9918 | 2.90 | 9981 |
| 0.41 | 6591 | 0.91 | 8186 | 1.41 | 9207 | 1.91 | 9719 | 2.41 | 9920 | 2.91 | 9982 |
| 0.42 | 6628 | 0.92 | 8212 | 1.42 | 9222 | 1.92 | 9726 | 2.42 | 9922 | 2.92 | 9982 |
| 0.43 | 6664 | 0.93 | 8238 | 1.43 | 9236 | 1.93 | 9732 | 2.43 | 9925 | 2.93 | 9983 |
| 0.44 | 6700 | 0.94 | 8264 | 1.44 | 9251 | 1.94 | 9738 | 2.44 | 9927 | 2.94 | 9984 |
| 0.45 | 6736 | 0.95 | 8289 | 1.45 | 9265 | 1.95 | 9744 | 2.45 | 9929 | 2.95 | 9984 |
| 0.46 | 6772 | 0.96 | 8315 | 1.46 | 9279 | 1.96 | 9750 | 2.46 | 9931 | 2.96 | 9985 |
| 0.47 | 6808 | 0.97 | 8340 | 1.47 | 9292 | 1.97 | 9756 | 2.47 | 9932 | 2.97 | 9985 |
| 0.48 | 6844 | 0.98 | 8365 | 1.48 | 9306 | 1.98 | 9761 | 2.48 | 9934 | 2.98 | 9986 |
| 0.49 | 6879 | 0.99 | 8389 | 1.49 | 9319 | 1.99 | 9767 | 2.49 | 9936 | 2.99 | 9986 |
| 0.50 | 6915 | 1.00 | 8413 | 1.50 | 9332 | 2.00 | 9772 | 2.50 | 9938 | 3.00 | 9987 |

**Table A8** **Normal Distribution**

Values of $z$ for given values of $\Phi(z)$ [see (3), Sec. 24.8] and $D(z) = \Phi(z) - \Phi(-z)$
Example: $z = 0.279$ if $\Phi(z) = 61\%$; $z = 0.860$ if $D(z) = 61\%$.

| % | $z(\Phi)$ | $z(D)$ | % | $z(\Phi)$ | $z(D)$ | % | $z(\Phi)$ | $z(D)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | −2.326 | 0.013 | 41 | −0.228 | 0.539 | 81 | 0.878 | 1.311 |
| 2 | −2.054 | 0.025 | 42 | −0.202 | 0.553 | 82 | 0.915 | 1.341 |
| 3 | −1.881 | 0.038 | 43 | −0.176 | 0.568 | 83 | 0.954 | 1.372 |
| 4 | −1.751 | 0.050 | 44 | −0.151 | 0.583 | 84 | 0.994 | 1.405 |
| 5 | −1.645 | 0.063 | 45 | −0.126 | 0.598 | 85 | 1.036 | 1.440 |
| 6 | −1.555 | 0.075 | 46 | −0.100 | 0.613 | 86 | 1.080 | 1.476 |
| 7 | −1.476 | 0.088 | 47 | −0.075 | 0.628 | 87 | 1.126 | 1.514 |
| 8 | −1.405 | 0.100 | 48 | −0.050 | 0.643 | 88 | 1.175 | 1.555 |
| 9 | −1.341 | 0.113 | 49 | −0.025 | 0.659 | 89 | 1.227 | 1.598 |
| 10 | −1.282 | 0.126 | 50 | 0.000 | 0.674 | 90 | 1.282 | 1.645 |
| 11 | −1.227 | 0.138 | 51 | 0.025 | 0.690 | 91 | 1.341 | 1.695 |
| 12 | −1.175 | 0.151 | 52 | 0.050 | 0.706 | 92 | 1.405 | 1.751 |
| 13 | −1.126 | 0.164 | 53 | 0.075 | 0.722 | 93 | 1.476 | 1.812 |
| 14 | −1.080 | 0.176 | 54 | 0.100 | 0.739 | 94 | 1.555 | 1.881 |
| 15 | −1.036 | 0.189 | 55 | 0.126 | 0.755 | 95 | 1.645 | 1.960 |
| 16 | −0.994 | 0.202 | 56 | 0.151 | 0.772 | 96 | 1.751 | 2.054 |
| 17 | −0.954 | 0.215 | 57 | 0.176 | 0.789 | 97 | 1.881 | 2.170 |
| 18 | −0.915 | 0.228 | 58 | 0.202 | 0.806 | 97.5 | 1.960 | 2.241 |
| 19 | −0.878 | 0.240 | 59 | 0.228 | 0.824 | 98 | 2.054 | 2.326 |
| 20 | −0.842 | 0.253 | 60 | 0.253 | 0.842 | 99 | 2.326 | 2.576 |
| 21 | −0.806 | 0.266 | 61 | 0.279 | 0.860 | 99.1 | 2.366 | 2.612 |
| 22 | −0.772 | 0.279 | 62 | 0.305 | 0.878 | 99.2 | 2.409 | 2.652 |
| 23 | −0.739 | 0.292 | 63 | 0.332 | 0.896 | 99.3 | 2.457 | 2.697 |
| 24 | −0.706 | 0.305 | 64 | 0.358 | 0.915 | 99.4 | 2.512 | 2.748 |
| 25 | −0.674 | 0.319 | 65 | 0.385 | 0.935 | 99.5 | 2.576 | 2.807 |
| 26 | −0.643 | 0.332 | 66 | 0.412 | 0.954 | 99.6 | 2.652 | 2.878 |
| 27 | −0.613 | 0.345 | 67 | 0.440 | 0.974 | 99.7 | 2.748 | 2.968 |
| 28 | −0.583 | 0.358 | 68 | 0.468 | 0.994 | 99.8 | 2.878 | 3.090 |
| 29 | −0.553 | 0.372 | 69 | 0.496 | 1.015 | 99.9 | 3.090 | 3.291 |
| 30 | −0.524 | 0.385 | 70 | 0.524 | 1.036 | | | |
| 31 | −0.496 | 0.399 | 71 | 0.553 | 1.058 | 99.91 | 3.121 | 3.320 |
| 32 | −0.468 | 0.412 | 72 | 0.583 | 1.080 | 99.92 | 3.156 | 3.353 |
| 33 | −0.440 | 0.426 | 73 | 0.613 | 1.103 | 99.93 | 3.195 | 3.390 |
| 34 | −0.412 | 0.440 | 74 | 0.643 | 1.126 | 99.94 | 3.239 | 3.432 |
| 35 | −0.385 | 0.454 | 75 | 0.674 | 1.150 | 99.95 | 3.291 | 3.481 |
| 36 | −0.358 | 0.468 | 76 | 0.706 | 1.175 | 99.96 | 3.353 | 3.540 |
| 37 | −0.332 | 0.482 | 77 | 0.739 | 1.200 | 99.97 | 3.432 | 3.615 |
| 38 | −0.305 | 0.496 | 78 | 0.772 | 1.227 | 99.98 | 3.540 | 3.719 |
| 39 | −0.279 | 0.510 | 79 | 0.806 | 1.254 | 99.99 | 3.719 | 3.891 |
| 40 | −0.253 | 0.524 | 80 | 0.842 | 1.282 | | | |

Q)

## Reading Entries from Table A7

If $X$ is standardized normal (so that $\mu = 0, \sigma = 1$), then

$$P(X \leqq 2.44) = 0.9927 \approx 99\tfrac{1}{4}\%$$

$$P(X \leqq -1.16) = 1 - \Phi(1.16) = 1 - 0.8770 = 0.1230 = 12.3\%$$

$$P(X \geqq 1) = 1 - P(X \leqq 1) = 1 - 0.8413 = 0.1587) \text{ by (7), Sec. 24.3}$$

$$P(1.0 \leqq X \leqq 1.8) = \Phi(1.8) - \Phi(1.0) = 0.9641 - 0.8413 = 0.1228.$$

Q)

## Probabilities for Given Intervals, Table A7

Let $X$ be normal with mean 0.8 and variance 4 (so that $\sigma = 2$). Then by (4) and (5)

$$P(X \leqq 2.44) = F(2.44) = \Phi\left(\frac{2.44 - 0.80}{2}\right) = \Phi(0.82) = 0.7939 \approx 80\%$$

or, if you like it better, (similarly in the other cases)

$$P(X \leqq 2.44) = P\left(\frac{X - 0.80}{2} \leqq \frac{2.44 - 0.80}{2}\right) = P(Z \leqq 0.82) = 0.7939$$

$$P(X \geqq 1) = 1 - P(X \leqq 1) = 1 - \Phi\left(\frac{1 - 0.8}{2}\right) = 1 - 0.5398 = 0.4602$$

$$P(1.0 \leqq X \leqq 1.8) = \Phi(0.5) - \Phi(0.1) = 0.6915 - 0.5398 = 0.1517.$$

Q)

## Unknown Values $c$ for Given Probabilities, Table A8

Let $X$ be normal with mean 5 and variance 0.04 (hence standard deviation 0.2). Find $c$ or $k$ corresponding to the given probability

$$P(X \leqq c) = 95\%, \qquad \Phi\left(\frac{c - 5}{0.2}\right) = 95\%, \qquad \frac{c - 5}{0.2} = 1.645, \qquad c = 5.329$$

$$P(5 - k \leqq X \leqq 5 + k) = 90\%, \qquad 5 + k = 5.329 \qquad \text{(as before; why?)}$$

$$P(X \geqq c) = 1\%, \qquad \text{thus } P(X \leqq c) = 99\%, \qquad \frac{c - 5}{0.2} = 2.326, \qquad c = 5.465.$$

**Q)**

In a production of iron rods let the diameter $X$ be normally distributed with mean 2 in. and standard deviation 0.008 in.

**(a)** What percentage of defectives can we expect if we set the tolerance limits at $2 \pm 0.02$ in.?

**(b)** How should we set the tolerance limits to allow for 4% defectives?

**Ans:**

**(a)** $1\frac{1}{4}\%$ because from (5) and Table A7 we obtain for the complementary event the probability

$$\begin{aligned} P(1.98 \leqq X \leqq 2.02) &= \Phi\left(\frac{2.02 - 2.00}{0.008}\right) - \Phi\left(\frac{1.98 - 2.00}{0.008}\right) \\ &= \Phi(2.5) - \Phi(-2.5) \\ &= 0.9938 - (1 - 0.9938) \\ &= 0.9876 \\ &= 98\tfrac{3}{4}\%. \end{aligned}$$

**(b)** $2 \pm 0.0164$ because, for the complementary event, we have

$$0.96 = P(2 - c \leqq X \leqq 2 + c)$$

or

$$0.98 = P(X \leqq 2 + c)$$

so that Table A8 gives

$$0.98 = \Phi\left(\frac{2 + c - 2}{0.008}\right),$$

$$\frac{2 + c - 2}{0.008} = 2.054, \qquad c = 0.0164.$$

## Normal Approximation of the Binomial Distribution

$$P(a \leqq X \leqq b) = \sum_{x=a}^{b} \binom{n}{x} p^x q^{n-x} \sim \Phi(\beta) - \Phi(\alpha),$$

$$\alpha = \frac{a - np - 0.5}{\sqrt{npq}}, \qquad \beta = \frac{b - np + 0.5}{\sqrt{npq}}.$$

**Q)**

If the lifetime $X$ of a certain kind of automobile battery is normally distributed with a mean of 5 years and a standard deviation of 1 year, and the manufacturer wishes to guarantee the battery for 4 years, what percentage of the batteries will he have to replace under the guarantee?

**Ans:**

Here

$$\mu = 5 \quad \text{and} \quad \sigma^2 = 1 \Longrightarrow \sigma = 1$$

The percentage of batteries which will need to be replaced is the same as $P(X < 4) = P(X \leqslant 4)$ (the equality follows from the fact that the normal distribution is a continuous distribution). Furthermore, we can use [redacted]

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right) = \Phi(x - 5)$$

Therefore,

$$P(X \leqslant 4) = F(4) = \Phi(-1) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587 = 15.87\%$$

(we also used that $\Phi(-z) = 1 - \Phi(z)$).

**Q)**

A manufacturer knows from experience that the resistance of resistors he produces is normal with mean

$\mu = 150\ \Omega$ and standard deviation $\sigma = 5\ \Omega$. What percentage of the resistors will have resistance between $148\ \Omega$ and $152\ \Omega$? Between $140\ \Omega$ and $160\ \Omega$?

**Ans:**

$$\mu = 150\,\Omega \qquad \sigma = 5\,\Omega$$

$$P(148 \leq X \leq 152) = \Phi\left(\frac{152-150}{5}\right) - \Phi\left(\frac{148-150}{5}\right)$$

$$= \Phi(+0.4) - \Phi(-0.4)$$

$$= 0.6554 - 0.3446$$

$$= \underline{\underline{0.3108}} \quad (\text{ie } 31.08\%)$$

$$P(140 \leq X \leq 160) = \Phi\left(\frac{160-150}{5}\right) - \Phi\left(\frac{140-150}{5}\right)$$

$$= \Phi(2) - \Phi(-2)$$

$$= 0.9775 - 0.02275$$

$$= 0.9547 \quad (\text{ie } 95.47\%)$$

**Q)**

The breaking strength $X$ [kg] of a certain type of plastic block is normally distributed with a mean of 1500 kg and a standard deviation of 50 kg. What is the maximum load such that we can expect no more than 5% of the blocks to break?

**Ans:**

Let $c$ be the required load. Then we want to have

$$P(X \leqslant c) = 5\%$$

(the load can be greater than the breaking strength $X$ of the block only in 5% of the cases).

Also, we have

$$\mu = 1500$$

and

$$\sigma = 50$$

Using [redacted],

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-1500}{50}\right)$$

Therefore,

$$P(X \leqslant c) = F(c) = \Phi\left(\frac{c-1500}{50}\right)$$

So, we want to find $c$ such that

$$P(X \leqslant c) = 5\% \Longrightarrow \Phi\left(\frac{c-1500}{50}\right) = 5\%$$

Using the table, we get

$$\frac{c-1500}{50} = -1.645 \Longrightarrow c - 1500 = -82.25 \Longrightarrow \boxed{c = 1417.75}$$

**Q)**

If the mathematics scores of the SAT college entrance exams are normal with mean 480 and standard deviation 100 (these are about the actual values over the past years) and if some college sets 500 as the minimum score for new students, what percent of students would not reach that score?

**Ans:**

$\mu = 480 \qquad \sigma = 100$

$$P(x < 500) = \Phi\left(\frac{500-480}{100}\right)$$
$$= \Phi(0.2)$$
$$= 0.5793$$
$$= \underline{57.93\%}$$

**Q)**

If the monthly machine repair and maintenance cost $X$ in a certain factory is known to be normal with mean \$12,000 and standard deviation \$2000, what is the probability that the repair cost for the next month will exceed the budgeted amount of \$15,000?

**Ans:**

$\mu = \$12000 \qquad \sigma = \$2000$

$$P(\text{Budget} > \$15000) = 1 - \Phi\left(\frac{15000-12000}{2000}\right)$$
$$= 1 - \Phi(1.5)$$
$$= \underline{\underline{0.0668}} \qquad (\approx 6.7\%)$$

**Q)**

If the resistance $X$ of certain wires in an electrical network is normal with mean 0.01 $\Omega$ and standard deviation 0.001 $\Omega$, how many of 1000 wires will meet the specification that they have resistance between 0.009 and 0.011 $\Omega$?

**Ans:**

$\mu = 0.01\,\Omega \qquad \sigma = 0.001\,\Omega$

$$P(0.009 \leq X \leq 0.011) = \Phi\left(\frac{0.011 - 0.01}{0.001}\right) - \Phi\left(\frac{0.009 - 0.1}{0.001}\right)$$

$$= \Phi(1) - \Phi(-1)$$

$$= 0.8413 - 0.1586$$

$$= \underline{\underline{0.6827}}$$

$\therefore$ No. of wires in a set of 1000, that will have resistance in the above range $= 1000 \times 0.6827$

$$\approx \underline{\underline{683}}$$

**Q)**

Let $X$ be normal with mean 50 and variance 9. Determine $c$ such that $P(X < c) = 5\%$, $P(X > c) = 1\%$, $P(50 - c < X < 50 + c) = 50\%$.

**Ans:**

Here

$$\mu = 50$$

and

$$\sigma^2 = 9 \implies \sigma = 3$$

$\underline{P(X < c) = 5\%}$

Since normal distribution is a continuous distribution,

$$P(X < c) = P(X \leqslant c)$$

So, we must find $c$ such that

$$P(X \leqslant c) = 5\%$$

Using [redacted],

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-50}{3}\right)$$

Therefore,

$$P(X \leqslant c) = 5\% \implies F(c) = 5\% \implies \Phi\left(\frac{c-50}{3}\right) = 5\%$$

From the table,

$$\frac{c-50}{3} = -1.645 \implies c - 50 = -4.935 \implies \boxed{c = 45.065}$$

$\underline{P(X > c) = 1\%}$

Since $X > c$ and $X \leqslant c$ are complementary events,

$$P(X \leqslant c) = 1 - P(X > c)$$

Thus, now we must find $c$ such that

$$P(X \leqslant c) = 1 - 1\% = 100\% - 1\% = 99\%$$

So, similarly to the first part,

$$\Phi\left(\frac{c-50}{3}\right) = 99\%,$$

from which we get

$$\frac{c-50}{3} = 2.326 \implies c - 50 = 6.978 \implies \boxed{c = 56.978}$$

$P(50 - c < X < 50 + c) = 50\%$

Since the normal distribution is a continuous distribution,

$$P(50 - c < X < 50 + c) = P(50 - c < X \leqslant 50 + c)$$

Now we can use [redacted].

$$P(a < X \leqslant b) = \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right)$$

Thus,

$$\begin{aligned} P(50 - c < X \leqslant 50 + c) &= \Phi\left(\frac{50 + c - 50}{3}\right) - \Phi\left(\frac{50 - c - 50}{3}\right) \\ &= \Phi\left(\frac{c}{3}\right) - \Phi\left(-\frac{c}{3}\right) \\ &= 2\Phi\left(\frac{c}{3}\right) - 1, \end{aligned}$$

where we used the equality $\Phi(-z) = 1 - \Phi(z)$ in the last equation. So, we must find $c$ such that

$$P(50 - c < X \leqslant 50 + c) = 50\% \Longrightarrow 2\Phi\left(\frac{c}{3}\right) - 1 = 50\% = 0.5$$

Thus,

$$2\Phi\left(\frac{c}{3}\right) = 1.5 \Longrightarrow \Phi\left(\frac{c}{3}\right) = 0.75 = 75\%$$

Using the table, we find

$$\frac{c}{3} = 0.674 \Longrightarrow \boxed{c = 2.022}$$