07/01/2020

Laplace Transform

Laplace Transform :-

→ Laplace transform always transform a function of t to a function of s.

→ Let, $f(t)$ be a function where $t > 0$.
Let us multiply e^{-st} with the function $f(t)$ & integrate it w.r.t from $t=0$ to $t=\infty$.
If this integration exist, then it will be a function of s which is denoted by $F(s)$, where $F(s)$ is called the Laplace transform of $f(t)$.

So, $$\boxed{L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt}$$

L → Laplacian operator

Inverse Laplace transform :-

According to the definition of Laplace transform, we know that $L\{f(t)\} = F(s)$

$$\Rightarrow \boxed{f(t) = L^{-1}\{F(s)\}}$$

L^{-1} → Inverse Laplace operator

Theorem :-

Laplace transform is linear, that means if $L\{f(t)\}$ & $L\{g(t)\}$ exist, then

$$L\{\alpha f(t) + \beta g(t)\} = \alpha L\{f(t)\} + \beta L\{g(t)\}$$

where α & β are any constants.

It is given that $f(t)$ and $g(t)$ are two function whose Laplace Transform exist.

We have to show that

$$L\{\alpha f(t) + \beta g(t)\} = \alpha L\{f(t)\} + \beta L\{g(t)\}$$

Proof:

$$\text{LHS} = L\{\alpha f(t) + \beta g(t)\}$$

$$= \int_0^\infty e^{-st}\{\alpha f(t) + \beta g(t)\}\,dt$$

$$= \int_0^\infty (e^{-st}\alpha f(t) + e^{-st}\beta g(t))\,dt$$

$$= \int_0^\infty e^{-st}\alpha f(t)\,dt + \int_0^\infty e^{-st}\beta g(t)\,dt$$

$$= \alpha\int_0^\infty e^{-st} f(t)\,dt + \beta\int_0^\infty e^{-st} g(t)\,dt$$

$$= \alpha L\{f(t)\} + \beta L\{g(t)\}$$

$$= \text{RHS}$$

[Proved]

Some basic formulas of Laplace Transform:

i) $f(t) = 1$

$$L\{1\} = \int_0^\infty e^{-st}\cdot 1\cdot dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{-1}{s}\left[e^{-st}\right]_0^\infty = \frac{-1}{s}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{-1}{s}\left[0-1\right] = \frac{1}{s}$$

$$\boxed{L\{1\} = \frac{1}{s}} \qquad \boxed{L^{-1}\left\{\frac{1}{s}\right\} = 1}$$

2) $f(t) = t$

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$$L\{t\} = \int_0^\infty \underset{v}{e^{-st}} \cdot \underset{u}{t} \cdot dt$$

$u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$

$\int u \cdot v\,dx$

$= uv_1 - u'v_{11} + u''v_{111} - u'''v_{IV}$

$$= \cancel{t\int e^{-st}\,dt}$$

$$= \left[t\left(\frac{e^{-st}}{-s}\right) - 1\cdot\left(\frac{e^{-st}}{s^2}\right)\right]_0^\infty$$

$$= \left[t\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{s^2}\right)\right]_0^\infty$$

$$= \left[(0-0) - \left(0 - \frac{1}{s^2}\right)\right]$$

$$= \frac{1}{s^2}$$

$$\boxed{L\{t\} = \frac{1}{s^2}} \qquad \boxed{L^{-1}\left\{\frac{1}{s^2}\right\} = t}$$

3) $f(t) = t^2$

$$L\{t^2\} = \int_0^\infty \underset{v}{e^{-st}} \cdot \underset{u}{t^2} \cdot dt$$

$$= \left[t^2\left(\frac{e^{-st}}{-s}\right) - 2t\left(\frac{e^{-st}}{s^2}\right) + 2\left(\frac{e^{-st}}{-s^3}\right)\right]_0^\infty$$

$$= 0 - \left(0 - 0 + 2\left(\frac{-1}{s^3}\right)\right)$$

$$\boxed{L\{t^2\} = \frac{2}{s^3}} \qquad \boxed{2L^{-1}\left\{\frac{1}{s^3}\right\} = t^2} \qquad \boxed{L^{-1}\left\{\frac{1}{s^3}\right\} = t^2/2}$$

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$$\boxed{L\{t^n\} = \frac{\lfloor n}{s^{n+1}}}, \quad n \in N$$

4) $f(t) = e^{at}$, $s > a$

$$L\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at} \cdot dt$$

$$= \int_0^\infty e^{-st+at} \cdot dt$$

$$= \int_0^\infty e^{-t(s-a)} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{-1}{s-a}\left[e^{-\infty} - e^0\right]$$

$$= \frac{-1}{s-a}\left[0-1\right]$$

$$\boxed{L\{e^{at}\} = \frac{1}{s-a}}, s > a \qquad \boxed{L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}}$$

5) $f(t) = e^{-at}$, $s > a$

$$L\{e^{-at}\} = \int_0^\infty e^{-st} \cdot e^{-at} \cdot dt$$

$$= \int_0^\infty e^{-st-at} dt$$

$$= \int_0^\infty e^{-t(s+a)} dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{-1}{s+a}\left[e^{-\infty} - e^0\right]$$

$$= \frac{-1}{s+a}\left(0-1\right]$$

$$\boxed{L\{e^{-at}\} = \frac{1}{s+a}}$$

$$\boxed{L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}}$$

Note :-

According to Euler's formula, we know that

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{--- (i)}$$

So, $$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$= \cos\theta - i\sin\theta \quad \text{--- (ii)}$$

(i) + (ii)

$$\boxed{\frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta}$$

(i) − (ii)

$$\boxed{\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta}$$

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6) $f(t) = \sin at = \dfrac{e^{iat} - e^{-iat}}{2i}$

$$L\{\sin at\} = L\left\{\frac{e^{iat} - e^{-iat}}{2i}\right\}$$

$$= \frac{1}{2i}\left[L\{e^{iat}\} - L\{e^{-iat}\}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right]$$

$$= \frac{1}{2i}\left[\frac{\cancel{s} + ia - \cancel{s} + ia}{s^2+a^2}\right]$$

$$= \frac{1}{\cancel{2i}}\left[\frac{\cancel{2i}a}{s^2+a^2}\right]$$

$$= \frac{a}{s^2+a^2}$$

$$\boxed{L\{\sin at\} = \frac{a}{s^2+a^2}}$$

$$\boxed{L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at}$$

$$\boxed{L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}}$$

7) $f(t) = \cos at = \dfrac{e^{iat}+e^{-iat}}{2}$

$$L\{\cos at\} = L\left\{\frac{e^{iat}+e^{-iat}}{2}\right\}$$

$$= \frac{1}{2}\left[L\{e^{iat}\} + L\{e^{-iat}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{s}+\cancel{ia}+\cancel{s}-\cancel{ia}}{s^2+a^2}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s}{s^2+a^2}\right]$$

$$= \frac{s}{s^2+a^2}$$

$$\boxed{L\{\cos at\} = \frac{s}{s^2+a^2}} \qquad \boxed{L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at}$$

8) $f(t) = \sinh at = \dfrac{e^{at}-e^{-at}}{2}$

$$L\{\sinh at\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}a}{s^2-a^2}\right]$$

$$= \frac{a}{s^2-a^2}$$

$$\boxed{L\left\{\sinh at\right\} = \frac{a}{s^2-a^2}} \qquad \boxed{L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at}$$

$$\boxed{L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh at}{a}}$$

9) $f(t) = \cosh at = \dfrac{e^{at}+e^{-at}}{2}$

$$L\left\{\cosh at\right\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left[L\left\{e^{at}\right\} + L\left\{e^{-at}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+\cancel{a}+s-\cancel{a}}{s^2-a^2}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s}{s^2-a^2}\right]$$

$$= \frac{s}{s^2-a^2}$$

$$\boxed{L\left\{\cosh at\right\} = \frac{s}{s^2-a^2}} \qquad \boxed{L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at}$$

Q: Find $L\{2t + 5t^3 + e^{-3t} + \sin 3t\}$.

ans: $L\{2t+5t^3+e^{-3t}+\sin 3t\} = 2L\{t\} + 5L\{t^3\} + L\{e^{-3t}\} + L\{\sin 3t\}$

$= 2 \cdot \frac{1}{s^2} + 5\frac{\lfloor 3}{s^4} + \frac{1}{s+3} + \frac{3}{s^2+9}$

$= \frac{2}{s^2} + \frac{30}{s^4} + \frac{1}{s+3} + \frac{3}{s^2+9}$

Q: Find $L\{e^{4t} + 3^t + \cosh 4t\}$

ans: $L\{e^{4t} + 3^t + \cosh 4t\} = L\{e^{4t}\} + L\{3^t\} + L\{\cosh 4t\}$

$= \frac{1}{s-4} + L\{e^{\ln 3^t}\} + \frac{s}{s^2-4^2}$

$= \frac{1}{s-4} + L\{e^{t\ln 3}\} + \frac{s}{s^2-16}$

$= \frac{1}{s-4} + \frac{1}{s-\ln 3} + \frac{s}{s^2-16}$ //

Q: Find $L\{\sin^2 t + \cos^3 t\}$.

ans: $L\{\sin^2 t + \cos^3 t\} = L\{\sin^2 t\} + L\{\cos^3 t\}$

$= L\left\{\frac{1-\cos 2t}{2}\right\} + L\left\{\frac{3\cos t + \cos 3t}{4}\right\}$

$= \frac{1}{2}\left[L\{1\} - L\{\cos 2t\}\right] + \frac{3}{4}L\{\cos t\} + \frac{1}{4}L\{\cos 3t\}$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] + \frac{3}{4}\frac{s}{s^2+1} + \frac{1}{4}\frac{s}{s^2+9}$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] + \frac{3s}{4(s^2+1)} + \frac{s}{4(s^2+9)}$$

Q. Find $L\{\sin(3t+4)\}$.

ans. $L\{\sin(3t+4)\} = L\{\sin 3t \cos 4 + \cos 3t \sin 4\}$

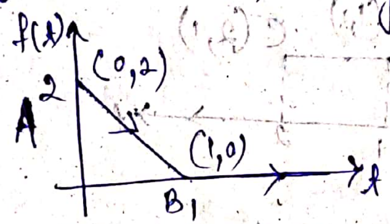
$= L\{\sin 3t \cos 4\} + L\{\cos 3t \sin 4\}$

$= \cos 4 \, L\{\sin 3t\} + \sin 4 \, L\{\cos 3t\}$

$= \cos 4 \dfrac{3}{s^2+9} + \sin 4 \dfrac{s}{s^2+9}$

$= \dfrac{1}{s^2+9}\left(3\cos 4 + s \sin 4\right)$

Q. Find $L\{f(t)\}$, where $f(t)$ is given in the following diagram.



ans. Equation of line AB is $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$f(t) - 2 = \dfrac{0-2}{1-0}(t-0)$

$\Rightarrow f(t) - 2 = -2t$

$\Rightarrow f(t) = -2t + 2, \; 0 < t < 1$

$t > 1, \quad f(t) = 0$

$$f(t) = \begin{cases} 2-2t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}(2-2t)\,dt + \int_0^\infty 0\,dt$$

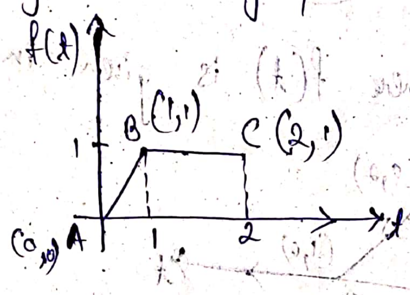
$$= 2-2t\int e^{-st}\,dt + 2\int \frac{e^{-st}}{-s}\,dt - 0$$

$$= \left[(2-2t)\left(\frac{e^{-st}}{-s}\right) + 2\,\frac{e^{-st}}{s^2}\right]_0^1$$

$$= 0 + 2\,\frac{e^{-s}}{s^2} - \left(2\cdot\frac{1}{-s} + 2\cdot\frac{e^0}{s^2}\right)$$

$$= \frac{2e^{-s}}{s^2} + \frac{2}{s} - \frac{2}{s^2}$$

Q2 Find $L\{f(t)\}$ whose graph is given below.



Equation of AB $= \dfrac{y-y_1}{x-x_1} = \dfrac{y_2-y_1}{x_2-x_1}$

$$\Rightarrow \frac{f(t)-0}{t-0} = \frac{1-0}{1-0}$$

$$\Rightarrow f(t) = t \quad , \; 0<t<1$$

$1<t<2, \; f(t)=1$

$t>2, \; f(t)=0$

$$f(t) = \begin{cases} t, & 0<t<1 \\ 1, & 1<t<2 \\ 0, & t>2 \end{cases}$$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st}(t)\, dt + \int_1^2 e^{-st} \cdot 1\, dt + \int_2^\infty 0\, dt$$

$$= \left[t \cdot \frac{e^{-st}}{-s} - 1 \cdot \frac{e^{-st}}{s^2}\right]_0^1 + \left[\frac{e^{-st}}{-s}\right]_1^2$$

$$= \frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} - \left(0 - \frac{1}{s^2}\right) + \frac{e^{-2s}}{-s} - \frac{e^{-s}}{-s}$$

$$= \cancel{-\frac{e^{-s}}{s}} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} - \frac{e^{-2s}}{s} + \cancel{\frac{e^{-s}}{s}}$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s}$$

Q? Find $L^{-1}\left\{\frac{1}{s^2} + \frac{2}{s+3} + \frac{5}{s^2+16}\right\}$.

ans? $L^{-1}\left\{\frac{1}{s^2} + \frac{2}{s+3} + \frac{5}{s^2+16}\right\} = L^{-1}\left\{\frac{1}{s^2}\right\} + 2L^{-1}\left\{\frac{1}{s+3}\right\} + 5L^{-1}\left\{\frac{1}{s^2+16}\right\}$

$$= t + 2e^{-3t} + 5 \cdot \frac{\sin 4t}{4}$$

Q? Find $L^{-1}\left\{\frac{s}{s^2-9} + \frac{s}{s^2+25}\right\}$.

ans?

$$L^{-1}\left\{\frac{s}{s^2-9} + \frac{s}{s^2+25}\right\} = L^{-1}\left\{\frac{s}{s^2-9}\right\} + L^{-1}\left\{\frac{s}{s^2+25}\right\}$$

$$= \cosh 3t + \cos 5t$$

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Q: Find $L^{-1}\left\{\frac{1}{s^3}+\frac{2}{s+3}+\frac{5}{s^2-16}+\frac{4}{s^2+9}\right\}$

ans: $L^{-1}\left\{\frac{1}{s^3}+\frac{2}{s+3}+\frac{5}{s^2-16}+\frac{4}{s^2+9}\right\} = L^{-1}\left\{\frac{1}{s^3}\right\} + 2L^{-1}\left\{\frac{1}{s+3}\right\} + 5L^{-1}\left\{\frac{1}{s^2-16}\right\}$

$$+ 4L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \frac{t^2}{2} + 2e^{-3t} + 5\frac{\sinh 4t}{4} + 4\frac{\sin 3t}{3}$$

Q: Find $L^{-1}\left\{\frac{2s}{s^2+5s+6}\right\}$

ans: $L^{-1}\left\{\frac{2s}{s^2+5s+6}\right\} = L^{-1}\left\{\frac{2s}{(s+3)(s+2)}\right\}$

Now, $\frac{2s}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2}$

$$A = \left.\frac{2s}{s+2}\right]_{s=-3} = \frac{-6}{-1} = 6$$

$$B = \left.\frac{2s}{s+3}\right]_{s=-2} = \frac{-4}{1} = -4$$

$$\frac{2s}{(s+3)(s+2)} = \frac{6}{s+3} - \frac{4}{s+2}$$

$$L^{-1}\left\{\frac{2s}{(s+3)(s+2)}\right\} = 6L^{-1}\left(\frac{1}{s+3}\right) - 4L^{-1}\left(\frac{1}{s+2}\right)$$

$$= 6e^{-3t} - 4\cdot e^{-2t}$$

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Q. Find $L^{-1}\left\{\frac{s^2+3}{s^3-s^2-6s}\right\}$

ans. $L^{-1}\left\{\frac{s^2+3}{s^3-s^2-6s}\right\} = L^{-1}\left\{\frac{s^2+3}{s(s^2-s-6)}\right\}$

Now, $\frac{s^2+3}{s(s^2-3s+2s-6)} = \frac{s^2+3}{s(s-3)(s+2)}$

$$\frac{s^2+3}{s(s-3)(s+2)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s+2}$$

$$A = \left.\frac{s^2+3}{(s-3)(s+2)}\right]_{s=0} = \frac{3}{-6} = \frac{1}{-2}$$

$$B = \left.\frac{s^2+3}{s(s+2)}\right]_{s=3} = \frac{12}{3(5)} = \frac{\cancel{12}^{4}}{\cancel{15}_{5}} = 4/5$$

$$\cancel{C = \frac{s^2+3}{s(s+2)}}$$

$$C = \left.\frac{s^2+3}{s(s-3)}\right]_{s=-2} = \frac{7}{+10}$$

$$\frac{s^2+3}{s(s-3)(s+2)} = \frac{-1}{2s} + \frac{4}{5(s-3)} + \frac{7}{10(s+2)}$$

$$L^{-1}\left\{\frac{s^2+3}{s(s-3)(s+2)}\right\} = \frac{-1}{2}L^{-1}\left\{\frac{1}{s}\right\} + \frac{4}{5}L^{-1}\left\{\frac{1}{s-3}\right\} + \frac{7}{10}L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{-1}{2}.1 + \frac{4}{5}e^{3t} + \frac{7}{10}e^{-2t}$$

Q. Q2 find $L^{-1}\left\{\frac{2s+3}{(s-1)(s^2+9)}\right\}$.

a ans: $L^{-1}\left\{\frac{2s+3}{(s-1)(s^2+9)}\right\} = \frac{A}{s-1} + \frac{Bs+C}{s^2+9}$

$$A = \left.\frac{2s+3}{s^2+9}\right]_{s=1} = \frac{5}{10} = \frac{1}{2}$$

$$\frac{2s+3}{\cancel{(s-1)(s^2+9)}} = \frac{\frac{1}{2}(s^2+9) + (Bs+C)(s-1)}{\cancel{(s-1)(s^2+9)}}$$

$$2s+3 = \frac{1}{2}(s^2+9) + B(s^2-s) + C(s-1)$$

s^2, $0 = \frac{1}{2} + B$

$\boxed{B = -\frac{1}{2}}$

s, $2 = -B + C$

$C = 2 + B = 2 - \frac{1}{2} = \frac{3}{2}$

$$\frac{2s+3}{(s-1)(s^2+9)} = \frac{1}{2(s-1)} + \frac{-\frac{1}{2}s + \frac{3}{2}}{s^2+9}$$

$$L^{-1}\left\{\frac{2s+3}{(s-1)(s^2+9)}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{-\frac{1}{2}s+\frac{3}{2}}{s^2+3^2}\right\}$$

$$= \frac{1}{2}e^t + L^{-1}\left\{\frac{-\frac{1}{2}s}{s^2+3^2}\right\} + L^{-1}\left\{\frac{\frac{3}{2}}{s^2+3^2}\right\}$$

$$= \frac{e^t}{2} - \frac{1}{2}\cos 3t + \frac{\cancel{3}}{2}\cdot\frac{\sin 3t}{\cancel{3}}$$

$$= \frac{e^t}{2} - \frac{\cos 3t}{2} + \frac{\sin 3t}{2} \quad //$$

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Q> Find $L^{-1}\left\{\frac{s+2}{(s^2+9)(s^2+7)}\right\}$.

ans> $L^{-1}\left\{\frac{s+2}{(s^2+9)(s^2+7)}\right\} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+7}$

$\cancel{A = \frac{s+2}{s^2+7}}\Big|_{s^2=-9} = \cancel{\frac{-7}{88}}$

$\cancel{B = \frac{s+2}{}}$

$$\frac{s+2}{\cancel{(s^2+9)(s^2+7)}} = \frac{(As+B)(s^2+7)+(Cs+D)(s^2+9)}{\cancel{(s^2+9)(s^2+7)}}$$

$$s+2 = A(s^3+7s)+B(s^2+7)+C(s^3+9s)+D(s^2+9)$$

$s^3,\ 0 = A+C$ $\qquad$ $s^2,\ 0 = B+D$ $\qquad$ $s,\ 1 = 7A+9C$

$A = -C$ $\qquad$ $B = -D$ $\qquad$ $1 = 7A-9A$

$-2A = 1$

$\boxed{C = 1/2}$ $\qquad$ $\boxed{A = -1/2}$

$2 = 7B+9D$

$= 7B - 9B$ $\qquad$ $\boxed{D = 1}$

$\cancel{2} = -\cancel{2}B$

$\boxed{B = -1}$

$$\frac{s+2}{(s^2+9)(s^2+7)} = \frac{-\frac{1}{2}s - 1}{s^2+9} + \frac{\frac{1}{2}s+1}{s^2+7}$$

$$L^{-1}\left\{\frac{s+2}{(s^2+9)(s^2+7)}\right\} = -\frac{1}{2}L^{-1}\left\{\frac{s}{s^2+9}\right\} - 1\,L^{-1}\left\{\frac{1}{s^2+3^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{s}{s^2+(\sqrt{7})^2}\right\}$$

$$+ L^{-1}\left\{\frac{1}{s^2+(\sqrt{7})^2}\right\}$$

$$= -\frac{1}{2}\cos 3t - \frac{\sin 3t}{3} + \frac{1}{2}\cos\sqrt{7}t + \frac{\sin\sqrt{7}t}{\sqrt{7}} \ //$$

First shifting theorem / shifting on s-axis :-

→ If $L\{f(t)\} = F(s)$, then

$$\boxed{L\{e^{at} f(t)\} = F(s-a)}$$

It is given that $F(s) = L\{f(t)\}$

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \quad \text{—— (1)}$$

We have to show that $L\{e^{at} f(t)\} = F(s-a)$

Proof :- RHS

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt \quad (\text{By } (1))$$

$$\boxed{L\{x\} = \int_0^\infty e^{-st} x\, dt}$$

$$= \int_0^\infty e^{-st} \cdot e^{at} f(t)\, dt$$

$$\cancel{= \int_0^\infty e^{at} f(t)} = L\{e^{at} f(t)\}$$

$$= LHS$$

[Proved]

→ If $L^{-1}\{F(s)\} = f(t)$, then

$$L^{-1}\{F(s-a)\} = e^{at} f(t)$$

Q:- Find $L\{t \cdot e^{5t}\}$.

ans:- Here $a = 5$,

$f(t) = t$

$F(s) = L\{t\} = \frac{1}{s^2}$

We know that, $L\{e^{at} f(t)\} = F(s-a)$

$$\rightarrow L\{e^{5t} \cdot t\} = F(s-a) = \frac{1}{(s-5)^2}$$

Q. Find $L\{(t+t^2-1)e^{3t}\}$.

ans. Here $a = 3$

$f(t) = t^2 + t - 1$

$F(s) = L\{t^2+t-1\}$

$= \frac{1}{s^2} + \frac{2}{s^3} - \frac{1}{s}$

We know that, $L\{e^{at} f(t)\} = F(s-a)$

$\Rightarrow L\{e^{at}(t+t^2-1)\} = F(s-3)$

$= \frac{1}{(s-3)^2} + \frac{2}{(s-3)^3} - \frac{1}{(s-3)}$ //

Q. Find $L\{\sin 4t \cdot e^{-7t}\}$.

ans. Here $a = -7$

$f(t) = \sin 4t$

$F(s) = L\{\sin 4t\}$

$= \frac{4}{s^2+16}$

We know that, $L\{e^{at} f(t)\} = F(s-a)$

$\Rightarrow L\{e^{-7t}(\sin 4t)\} = F(s+7)$

$= \frac{4}{(s+7)^2+16} = \frac{4}{s^2+49+14s+16}$

$= \frac{4}{s^2+14s+65}$ //

Q: Find $L\{\cosh 3t . e^{5t}\}$.

ans: Here $a = 5$

$f(t) = \cosh 3t$

$F(s) = L\{\cosh 3t\} = \frac{s}{s^2 - 9}$

We know that, $L\{e^{at} f(t)\} = F(s-a)$

$L\{e^{5t} \cosh 3t\} = F(s-5)$

$= \frac{s-5}{(s-5)^2 - 9}$

$= \frac{s-5}{s^2+25-10s-9} = \frac{s-5}{s^2-10s+16}$

Q: Find $L^{-1}\left\{ \frac{1}{(s-4)^3} \right\}$.

ans: Here $a = 4$

$F(s-4) = \frac{1}{(s-4)^3}$

$F(s) = \frac{1}{s^3}$

$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2}$

We know that $L^{-1}\{F(s-a)\} = e^{at} f(t)$

$\Rightarrow L^{-1}\{F(s-4)\} = e^{4t} f(t)$

$\Rightarrow L^{-1}\left\{ \frac{1}{(s-4)^3} \right\} = e^{4t} . \frac{t^2}{2}$

1) Find $L^{-1}\left\{\frac{1}{(s-2)^2+9}\right\}$ by 1st shifting theorem.

ans: Here $a=2$

$$f(s-2)=\frac{1}{(s-2)^2+9}$$

$$f(s) = \frac{1}{s^2+3^2}$$

$$f(t) = L^{-1}\left\{\frac{1}{s^2+3^2}\right\} = \frac{\sin 3t}{3}$$

We know that $L^{-1}\{F(s)\} = f(t)$

$$L^{-1}\{F(s-a)\} = e^{at} f(t)$$

$$\Rightarrow L^{-1}\{F(s-2)\} = e^{2t} f(t)$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(s-2)^2+9}\right\} = e^{2t} \cdot \frac{\sin 3t}{3}$$

2) Find $L^{-1}\left\{\frac{s}{(s-3)^2-16}\right\}$.

ans: Here $a=3$

$$F(s-3) = \frac{s}{(s-3)^2-4^2} = \frac{s-3+3}{(s-3)^2-4^2}$$

$$F(s) = \frac{s+3}{s^2-4^2}$$

$$f(t) = L^{-1}\left\{\frac{s}{s^2-4^2} + \frac{3}{s^2-4^2}\right\}$$

$$= \cosh 4t + 3\,\frac{\sinh 4t}{4}$$

$$L^{-1}\{F(s-3)\} = e^{3t} f(t)$$

$$\Rightarrow L^{-1}\left\{\frac{s}{(s-3)^2-16}\right\} = e^{3t}\left(\cosh 4t + \frac{3}{4}\sinh 4t\right)$$

3) Find $L^{-1}\left\{\frac{1}{s^2+3s-1}\right\}$

ans) $L^{-1}\left\{\frac{1}{s^2+\frac{3}{2}s\cdot 2+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2-1}\right\}$

$$F(s)=\frac{1}{\left(s+\frac{3}{2}\right)^2-\frac{13}{4}}$$

Here $a=-3/2$

$$F(s+3/2)=\frac{1}{(s+3/2)^2-\frac{13}{4}}$$

$$F(s)=\frac{1}{s^2-\left(\frac{\sqrt{13}}{2}\right)^2}$$

$$f(t)=L^{-1}\left\{\frac{1}{s^2-\left(\frac{\sqrt{13}}{2}\right)^2}\right\}=\sinh \sqrt{13}/2$$

$$f(t)=\frac{2}{\sqrt{13}}\sinh\frac{\sqrt{13}}{2}$$

$$L^{-1}\{F(s+3/2)\}=e^{-3/2\,t}f(t)$$

$$L^{-1}\left\{\frac{1}{\left(s+\frac{3}{2}\right)^2-\frac{13}{4}}\right\}=e^{-3/2\,t}\frac{2}{\sqrt{13}}\sinh\frac{\sqrt{13}}{2}t$$

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4) Find $L^{-1}\left\{\frac{3}{s^2-s+7}\right\}$

ans:

$$L^{-1}\left\{\frac{3}{s^2-2\cdot s\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+7}\right\}$$

$$=L^{-1}\left\{\frac{3}{\left(s-\frac{1}{2}\right)^2+\frac{27}{4}}\right\}$$

$$a=\frac{1}{2}$$

$$F\left(s-\frac{1}{2}\right)=\frac{3}{\left(s-\frac{1}{2}\right)^2+\frac{27}{4}}$$

$$F(s)=\frac{3}{s^2+\left(\frac{\sqrt{27}}{2}\right)^2}$$

$$f(t)=3L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{27}}{2}\right)^2}\right\}$$

$$=3\cdot\frac{2}{\sqrt{27}}\sin\frac{\sqrt{27}}{2}t$$

$$=\frac{2}{\sqrt{3}}\sin\frac{3\sqrt{3}}{2}t$$

$$L^{-1}\left\{F\left(s-\frac{1}{2}\right)\right\}=e^{t/2}f(t)$$

$$\Rightarrow L^{-1}\left\{\frac{3}{\left(s-\frac{1}{2}\right)^2+\frac{27}{4}}\right\}=e^{t/2}\cdot\frac{2}{\sqrt{3}}\sin\frac{3\sqrt{3}}{2}t$$

Laplace transform of derivatives and integration

Theorem 1: If $L\{f(t)\} = F(s)$, then

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

~~LHS~~ $\cancel{L\{f'(t)\} = \int_0^\infty e^{-st} f(t)\,dt}$

It is given that $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

We have to show that $L\{f'(t)\} = sL\{f(t)\} - f(0)$

LHS: $L\{f'(t)\} = \int_0^\infty e^{-st}.f'(t)\,dt$

$$= e^{-st}.f(t)\Big]_0^\infty - \int_0^\infty e^{-st}.(-s).f(t)\,dt$$

$$= 0 - 1.f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + sL\{f(t)\}$$

$$= RHS$$

[Proved]

Theorem 2: If $L\{f(t)\} = F(s)$, then

$$L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f'(0)$$

It is given that $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

We have to show that ~~L{~~

$$L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f'(0)$$

LHS $L\{f''(t)\} = \int_0^\infty \underset{1}{e^{-st}}.\underset{2}{f''(t)}\,dt$

4) Find $L^{-1}\left\{\frac{3}{s^2-s+7}\right\}$

ans:

$$L^{-1}\left\{\frac{3}{s^2-2\cdot s\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+7}\right\}$$

$$= L^{-1}\left\{\frac{3}{\left(s-\frac{1}{2}\right)^2+\frac{27}{4}}\right\}$$

$$a = \frac{1}{2}$$

$$F\left(s-\frac{1}{2}\right) = \frac{3}{\left(s-\frac{1}{2}\right)^2+\frac{27}{4}}$$

$$F(s) = \frac{3}{s^2+\left(\frac{\sqrt{27}}{2}\right)^2}$$

$$f(t) = 3L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{27}}{2}\right)^2}\right\}$$

$$= 3\cdot\frac{2}{\sqrt{27}}\sin\frac{\sqrt{27}}{2}t$$

$$= \frac{2}{\sqrt{3}}\sin\frac{3\sqrt{3}}{2}t$$

$$L^{-1}\left\{F\left(s-\frac{1}{2}\right)\right\} = e^{t/2}f(t)$$

$$\Rightarrow L^{-1}\left\{\frac{3}{\left(s-\frac{1}{2}\right)^2+\frac{27}{4}}\right\} = e^{t/2}\cdot\frac{2}{\sqrt{3}}\sin\frac{3\sqrt{3}}{2}t$$

Laplace transform of derivatives and integration

Theorem 1: If $L\{f(t)\} = F(s)$, then

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

~~LHS: $L\{f'(t)\} = \int_0^\infty e^{-st} f(t)\,dt$~~

It is given that $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

We have to show that $L\{f'(t)\} = sL\{f(t)\} - f(0)$

LHS: $L\{f'(t)\} = \int_0^\infty e^{-st} \cdot f'(t)\,dt$

$$= e^{-st} \cdot f(t)\Big]_0^\infty - \int_0^\infty e^{-st} \cdot (-s) \cdot f(t)\,dt$$

$$= 0 - 1 \cdot f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + sL\{f(t)\}$$

$$= RHS$$

[Proved]

Theorem 2: If $L\{f(t)\} = F(s)$, then

$$L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f'(0)$$

It is given that $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

We have to show that

$$L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f'(0)$$

LHS $L\{f''(t)\} = \int_0^\infty \underset{1}{e^{-st}} \cdot \underset{2}{f''(t)}\,dt$

$$= e^{-st} \cdot f'(t) \Big]_0^{\infty} - \int_0^{\infty} e^{-st} \cdot (-s) \cdot f'(t) dt$$

$$= 0 - 1 \cdot f'(0) + s \int_0^{\infty} e^{-st} f'(t) dt$$

$$= -f'(0) + sL\{f'(t)\}$$

$$= -f'(0) + s\{sL\{f(t)\} - f(0)\}$$

We know that

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

$$L\{f''(t)\} = s^2 L\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$$

$$= RHS$$

[Proved]

<u>Note:</u> Continuing in similare manner ; we get

$$L\{f'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - s f'(0) - f''(0)$$

$$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) + \dots - f^{n-1}(0)$$

1) Find $L\{\sin t\}$ using laplace transforem of derivatives.

ans: $f(t) = \sin t$

$f(0) = 0$

$f'(t) = \cos t$

$f'(0) = 1$

$f''(t) = -\sin t$

$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$$

$\Rightarrow L\{-\sin t\} = s^2 L\{\sin t\} - s.0 - 1$

$\Rightarrow -L\{\sin t\} - s^2 L\{\sin t\} = -1$

$\Rightarrow \cancel{-}L\{\sin t)(1+s^2) = \cancel{-}1$

$\Rightarrow L\{\sin t\} = \frac{1}{s^2+1}$ //

2) Find $L\{\cosh 2t\}$ using laplace transform of derivatives.

ans: $f(t) = \cosh 2t$

$f(0) = 1$

$f'(t) = 2\sinh 2t$

$f'(0) = 0$

$f''(t) = 4\cosh 2t$

$L\{f''(t)\} = s^2 L\{f(t)\} - s.f(0) - f'(0)$

$\Rightarrow L\{4\cosh 2t\} = s^2 L\{\cosh 2t\} - s.1 - 0\}$

$\Rightarrow L\{4\cosh 2t\} - s^2 L\{\cosh 2t\} = -s$

$\Rightarrow L\{\cosh 2t\}(4 - s^2) = -s$

$\Rightarrow L\{\cosh 2t\} = \frac{-s}{4-s^2}$,

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23) $L^{-1}\left\{\frac{s}{L^2 s^2 + n^2\pi^2}\right\}$

$= L^{-1}\left\{\frac{s}{L^2\left(s^2 + \frac{n^2\pi^2}{L^2}\right)}\right\}$

$= \frac{1}{L^2} L^{-1}\left\{\frac{s}{s^2 + \left(\frac{n\pi}{L}\right)^2}\right\}$

Solutions of ordinary differential equations using Laplace transform :-

To find the solution of any initial value problem, we have to proceed as per the following three steps:-

Step 1:- Let, the equation be $ay'' + by' + cy = f(x)$, where a, b, c are constants. $y(0) = \alpha$

Let y be the solution such that $y'(0) = \beta$

$$L\{y\} = Y$$

Find $L\{y'\}$ and $L\{y''\}$

$$L\{y'\} = sY - y(0)$$

$$= sY - \alpha$$

$$L\{y''\} = s^2 L\{y\} - sy'(0) - y'(0)$$

$$= s^2 Y - s.\beta - \alpha$$

Step 2:- Taking laplace transform of both sides of the given problem and substituting the value of $L\{y'\}$ and $L\{y''\}$, we get

$$Y = F(s)$$

Step-3:- Taking L^{-1} on both sides of $Y = F(s)$, we get the reqd. solⁿ of the given differential equation.

Q. Solve $y'' - 5y' + 6y = 0$ where $y(0) = 0$ & $y'(0) = 3$, by using laplace transform.

ans. Let y be the solution of the given eqn, such

$$L\{y\} = Y$$

$$L\{y'\} = sL\{y\} - y(0)$$

$$= s.Y - 0$$

$$= sY$$

$$L\{y''\} = s^2L\{y\} - sy(0) - y'(0)$$

$$= s^2Y - s.0 - 3$$

$$= s^2Y - 3$$

Taking Laplace transform of both sides of the given problem, we get

$$L\{y''\} - 5L\{y'\} + 6L\{y\} = L\{0\}$$

$$\Rightarrow s^2Y - 3 - 5sY - 6Y = 0$$

$$\Rightarrow s^2Y - 5sY + 6Y = 3$$

$$\Rightarrow Y(s^2 - 5s + 6) = 3$$

$$\Rightarrow Y = \frac{3}{s^2 - 3s - 2s + 6}$$

$$\Rightarrow Y = \frac{3}{(s-3)(s-2)} = \frac{A}{s-3} + \frac{B}{s-2}$$

$$A = \left.\frac{3}{s-2}\right]_{s=3} = \frac{3}{1} = 1$$

$$B = \left.\frac{3}{s-3}\right]_{s=2} = \frac{3}{-1} = -3$$

$$\Rightarrow Y = \frac{3}{s-3} - \frac{3}{s-2}$$

$$\Rightarrow L\{y\} = \frac{3}{s-3} - \frac{3}{s-2}$$

$$\Rightarrow y = 3L^{-1}\left\{\frac{1}{s-3}\right\} - 3L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$\Rightarrow y = 3e^{3t} - 3e^{2t}$$

Q: Solve $y'' + 7y' + 10y = 0$ where $y(0) = 1$ & $y'(0) = 2$, by using laplace transform.

ans: Let y be the solution of the given eqn, such that $L\{y\} = Y$

$$\Rightarrow L\{y'\} = sL\{y\} - y(0)$$

$$= sY - 1$$

$$L\{y''\} = s^2L\{y\} - sy(0) - y'(0)$$

$$= s^2Y - s.1 - 2$$

$$= s^2Y - s - 2$$

Taking laplace transform of both sides of the given problem, we get.

$$L\{y''\} + 7L\{y'\} + 10L\{y\} = L\{0\}$$

$$\Rightarrow s^2Y - s - 2 + 7(sY - 1) + 10Y = 0$$

$$\Rightarrow Y(s^2 + 7s) - s - 2 - 7 + 10Y = 0$$

$\Rightarrow Y(s^2+7s) - s - 9 + 10Y = 0$

$\Rightarrow Y(s^2+7s+10) = s+9$

$$\Rightarrow Y = \frac{s+9}{s^2+5s+2s+10}$$

$$\Rightarrow Y = \frac{s+9}{(s+5)(s+2)} = \frac{A}{s+2} + \frac{B}{s+5}$$

$$A = \left.\frac{s+9}{s+5}\right]_{s=-2} = \frac{7}{3}$$

$$B = \left.\frac{s+9}{s+2}\right]_{s=-5} = \frac{-4}{3}$$

$$\Rightarrow Y = \frac{7}{3(s+2)} - \frac{4}{3(s+5)}$$

$$\Rightarrow L^{-1}\{Y\} = \frac{7}{3} L^{-1}\left\{\frac{1}{s+2}\right\} - \frac{4}{3} L^{-1}\left\{\frac{1}{s+5}\right\}$$

$$= \frac{7}{3} e^{-2t} - \frac{4}{3} e^{-5t}$$

4.

Q7 Solve $y'' - 4y' + 3y = 6t - 8$, where $y(0) = y'(0) = 0$ using laplace transform.

ans Let, y be the solution of the given eqn, such that

$L\{y\} = Y$

$L\{y'\} = sL\{y\} - y(0)$

$= s \cdot Y - 0$

$= sY$

$L\{y''\} = s^2 L\{y\} - s\cdot y(0) - y'(0)$

$= s^2 y - s\cdot 0 - 0$

$= s^2 y$

Taking laplace transform of both sides of the given problem we get

$L\{y''\} - 4L\{y'\} + 3L\{y\} = 6t - 8$

$\Rightarrow s^2 y - 4sy + 3y = 6L\{t\} - 8L\{1\}$

$\Rightarrow y(s^2 - 4s + 3) = \frac{6}{s^2} - \frac{8}{s}$

$\Rightarrow y((s-3)(s-1)) = \frac{6}{s^2} - \frac{8}{s}$

$\Rightarrow y((s-3)(s-1)) = \frac{6-8s}{s^2}$

$\Rightarrow y = \frac{6-8s}{s^2(s-3)(s-1)} = \frac{A}{s-3} + \frac{B}{s-1} + \frac{Cs+D}{s^2}$

$A = \left.\frac{6-8s}{s^2(s-1)}\right]_{s=3} = \frac{6-24}{9(2)} = \frac{-18}{18} = -1$

$B = \left.\frac{6-8s}{s^2(s-3)}\right]_{s=1} = \frac{6-8}{-2} = \frac{-2}{-2} = 1$

$\frac{6-8s}{\cancel{s^2(s-1)(s-3)}} = \frac{As^2(s-1) + Bs^2(s-3) + (Cs+D)(s-1)(s-3)}{\cancel{s^2(s-1)(s-3)}}$

$\Rightarrow 6 - 8s = -s^2(s-1) + s^2(s-3) + Cs(s^2-4s+3) + D(s^2-4s+3)$

$0 = -1 + 1 + C$

$\boxed{C = 0}$

s^2

$0 = 1 - 3 - 4C + D$

$0 = -2 - 0 + D$

$\boxed{D = 2}$

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$$Y = \frac{-1}{S-3} + \frac{1}{S-1} + \frac{0.S+2}{S^2}$$

$$\Rightarrow L\{y\} = \frac{-1}{S-3} + \frac{1}{S-1} + \frac{2}{S^2}$$

$$\Rightarrow y = -L^{-1}\left\{\frac{1}{S-3}\right\} + L^{-1}\left\{\frac{1}{S-1}\right\} + 2L^{-1}\left\{\frac{1}{S^2}\right\}$$

$$\Rightarrow y = -e^{3t} + e^{t} + 2t$$

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Laplace transform of integration:

Note: If $F(x) = \int_{\alpha=(x)}^{\beta} f(t)\,dt$

$$F'(x) = f(\beta)\frac{d}{dt}(\beta) - f(\alpha)\frac{d}{dx}(\alpha)$$

Theorem: If $L\{f(t)\} = F(s)$

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(S)}{S}$$

To prove this, let us take $g(t) = \int_0^t f(t)\,dt$ — ①

$$\Rightarrow g(t) = \int_0^t f(\alpha)\,d\alpha$$

$(\because$ DI is independent on variable)

$$\Rightarrow g(0) = \int_0^0 f(\alpha)\,d\alpha$$

$$\boxed{g(0) = 0} \quad \text{—— ②}$$

Differentiating both the sides w.r.t t, we get

$$g'(t) = f(t)\frac{d}{dt}(t) - f(0)\frac{d}{dt}(0)$$

$\Rightarrow g'(t) = f(t)$

Taking laplace on both sides, we get

$$L\{g'(t)\} = L\{f(t)\}$$

$$sL\{g(t)\} - g(0) = F(s)$$

$$\Rightarrow sL\{g(t)\} - 0 = F(s)$$

$$\Rightarrow L\{g(t)\} = \frac{F(s)}{s}$$

$$\Rightarrow \boxed{L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}}$$

[Proved]

If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt$

Q: Find $L^{-1}\left\{\frac{1}{s(s-2)}\right\}$

ans: Here $F(s) = \frac{1}{s-2}$

$$f(t) = L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$$

$$L^{-1}\left\{\frac{1}{s(s-2)}\right\} = \int_0^t e^{2t}\,dt$$

$$= \left[\frac{e^{2t}}{2}\right]_0^t$$

$$= \frac{1}{2}\left[e^{2t} - 1\right]$$

$$= \frac{e^{2t}-1}{2}$$

Q: Find $L^{-1}\left\{\dfrac{1}{s^3+5s^2}\right\}$

ans: $L^{-1}\left\{\dfrac{1}{s^3+5s^2}\right\} = L^{-1}\left\{\dfrac{1}{s^2(s+5)}\right\}$

Here $F(s) = \dfrac{1}{s+5}$

$f(t) = e^{-5t}$

We know that $L^{-1}\left\{\dfrac{F(s)}{s}\right\} = \int_0^t f(t)\,dt$

$$\therefore L^{-1}\left\{\frac{1}{s(s+5)}\right\} = \int_0^t e^{-5t}\,dt$$

$$= \left[\frac{e^{-5t}}{-5}\right]_0^t$$

$$= \frac{-1}{5}\left(e^{-5t}-1\right)$$

$$= \frac{1-e^{-5t}}{5}$$

$$L^{-1}\left\{\frac{1}{s^2(s+5)}\right\} = \int_0^t \left(\frac{1-e^{-5t}}{5}\right) dt$$

$$\cancel{\frac{1}{5}\int_0^t dt}$$

$$= \frac{1}{5}\left[t + \frac{e^{-5t}}{5} - \left(0+\frac{1}{5}\right)\right]$$

$$= \frac{1}{5}\left[t + \frac{e^{-5t}}{5} - \frac{1}{5}\right]$$

Q: Find $L^{-1}\left\{\dfrac{1}{s(s^2-36)}\right\}$.

ans: $L^{-1}\left\{\dfrac{1}{s(s^2-36)}\right\} = L^{-1}\left\{\dfrac{1}{s(s^2-6^2)}\right\}$

Here $F(s) = \frac{1}{s^2-6^2}$

$f(t) = \frac{\sinh 6t}{6}$

We know that $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)dt$

$\Rightarrow L^{-1}\left\{\frac{1}{s(s^2-36)}\right\} = \int_0^t \frac{\sinh 6t}{6} dt$

$= \frac{1}{6}\left(\frac{\cosh 6t}{6}\right)_0^t$

$= \frac{1}{36}\left(\cosh 6t - 1\right)$

Q7 Find $L^{-1}\left\{\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)\right\}$

$L^{-1}\left\{\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)\right\} = L^{-1}\left\{\frac{1}{s^2}\left(\frac{s}{s+1} - \frac{1}{s+1}\right)\right\}$

$= L^{-1}\left\{\left(\frac{1}{s(s+1)} - \frac{1}{s^2(s+1)}\right)\right\}$ ——— (1)

Here $F(s) = \frac{1}{s+1}$

$f(t) = e^{-t}$

$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)dt$

$L^{-1}\left\{\frac{1}{s(s+1)}\right\} = \int_0^t e^{-t}dt$

$= \left(\frac{e^{-t}}{-1}\right)_0^t$

~~Find $L^{-1}\{$~~

$$= -(e^{-t} - e^{0})$$

$$= 1 - e^{-t}$$

$$L^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = \int_0^t (1-e^{-t})\,dt$$

$$= \left(t + e^{-t}\right)_0^t$$

$$= t + e^{-t} - 1$$

Q2. $$L^{-1}\left\{\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)\right\}$$

$$= L^{-1}\left\{\frac{1}{s(s+1)}\right\} - L^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$$

$$= 1 - e^{-t} - (t + e^{-t} - 1)$$

$$= 1 - e^{-t} - t - e^{-t} + 1$$

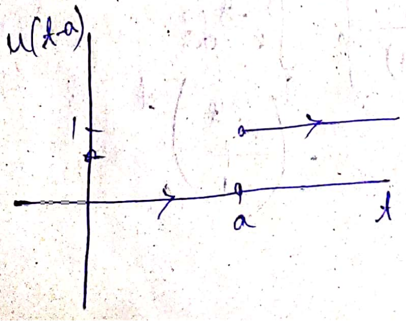
$$= 2 - 2e^{-t} - t$$

Unit step function and Dirac's Delta function:

If $a > 0$, then the unit step function of a denoted by $u(t-a)$ is defined as

$$u(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a \end{cases}$$

Graph of unit step function



Laplace transform of unit step function:

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\,dt$$

$$= \int_0^a 0\,dt + \int_a^\infty e^{-st} \cdot 1\,dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_a^\infty = \frac{-1}{s}\left(e^{-st}\right)_a^\infty$$

$$= \frac{-1}{s}\left(0 - e^{-as}\right)$$

$$\boxed{L\{u(t-a)\} = \frac{e^{-as}}{s}}$$

Ans

$$\boxed{L^{-1}\left\{\frac{e^{-as}}{s}\right\} = u(t-a)}$$

Second shifting theorem or shifting on t-axis:

If $L\{f(t)\} = F(s)$, then

$$L\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$$

$$L\{e^{at} f(t)\} = F(s-a)$$

Q: Find $L\{(t-1)u(t-1)\}$.

ans: Here $a = 1$

$f(t-1) = t-1$

$f(t) = t$

$F(s) = L\{t\} = \frac{1}{s^2}$

We know that

$$L\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$$

$$\Rightarrow L\{f(t-a_1)\,u(t-1)\} = e^{-s}F(s)$$

$$\Rightarrow L\{(t-1)\,u(t-1)\} = e^{-s}\cdot\frac{1}{s^2}$$ //

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Proof We know that $L\{f(t)\} = F(s)$

$$L\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

<u>RHS</u> $e^{-as}F(s) = e^{-as}L\{f(t)\}$

$$= e^{-as}\int_0^\infty e^{-st}f(t)\,dt$$

$$= \int_0^\infty e^{-st}e^{-as}f(t)\,dt$$

$$= \int_0^\infty e^{-st-as}\,f(t)\,dt$$

$$= \int_0^\infty e^{-s(t+a)}f(t)\,dt$$

Let, $t+a = x$

$dt = dx$

When $t=0,\ x=a$

$t=\infty,\ x=\infty$

Substituting

$$= \int_a^\infty e^{-sx}f(x-a)\,dx$$

$$= \int_a^\infty e^{-st}f(t-a)\,dt$$

$$= \int_a^\infty e^{-st}f(t-a)\,\underbrace{u(t-a)}_{\text{unit step function}}\,dt$$

$$= \underbrace{\int_0^a e^{-st} f(t-a)u(t-a)dt}_{0} + \int_a^\infty e^{-st} f(t-a)u(t-a)dt$$

$$= \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$$

$$= L\{f(t-a)u(t-a)\}$$

$= RHS.$

[Proved]

Formula:

If $L^{-1}\{F(s)\} = f(t)$

$$\boxed{L^{-1}\{e^{-as}F(s)\} = \{f(t-a)u(t-a)\}}$$

Q: Find $L\{(t-1)^3 u(t-1)\}$.

ans: Here, $a=1$

$f(t-1) = (t-1)^3$

$f(t) = t^3$

$F(s) = L\{t^3\}$

$$= \frac{\lfloor 3}{s^4} = \frac{6}{s^4}$$

We know that $L\{f(t-a)u(t-a)\} = e^{-as}F(s)$

$\Rightarrow L\{f(t-1)u(t-1)\} = e^{-s}F(s)$

$\Rightarrow L\{(t-1)^3 . u(t-1)\} = e^{-s} . \frac{6}{s^4}$

//

Q: Find $L\{\sin(t-\pi)u(t-\pi)\}$.

ans: Here, $a = \pi$

$f(t-\pi) = (t-\pi)\sin$

$f(t) = \sin t$

$F(s) = L\{\sin t\}$

$= \frac{1}{s^2+1}$

We know that $L\{f(t-a)u(t-a)\} = e^{-as}F(s)$

$L\{f(t-\pi)u(t-\pi)\} = e^{-\pi s}F(s)$

$\Rightarrow L\{\sin(t-\pi)u(t-\pi)\} = e^{-\pi s}\frac{1}{s^2+1}$

Q: Find $L\{t\,u(t-5)\}$.

ans: Here, $a = 5$

$f(t-5) = t-5+5$

$\Rightarrow f(t) = t+5$

$\Rightarrow F(s) = L\{t\} + 5L\{1\}$

$= \frac{1}{s^2} + \frac{5}{s}$

We know that $L\{f(t-a)u(t-a)\} = e^{-as}F(s)$

$L\{f(t-5)u(t-5)\} = e^{-5s}F(s)$

$\Rightarrow L\{t\cdot u(t-5)\} = e^{-5s}\left(\frac{1}{s^2}+\frac{5}{s}\right)$ //

Q: Find $L\{t^2 \cdot u(t-3)\}$.

ans: Here, $a = 3$

$f(t-3) = (t-3+3)^2$

$f(t) = (t+3)^2$

$= t^2 + 9 + 6t$

$F(s) = \frac{2}{s^3} + \frac{9}{s} + \frac{6}{s^2}$

We know that $L\{f(t-a)u(t-a)\} = e^{-as}F(s)$

$\Rightarrow L\{f(t-3)\,u(t-3)\} = e^{-3s}F(s)$

$\Rightarrow L\{t^2 \cdot u(t-3)\} = e^{-3s}\left(\frac{2}{s^3} + \frac{9}{s} + \frac{6}{s^2}\right)$ //

Q: Find $L\{\sin t\, u(t-\pi)\}$.

ans: Here, $a = \pi$

$f(t-\pi) = \sin t$

$= \sin((t-\pi)+\pi)$

$= -\sin(t-\pi)$

$f(t) = -\sin t$

$F(s) = \frac{-1}{s^2+1}$

We know that $L\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$

$\Rightarrow L\{f(t-\pi)u(t-\pi)\} = e^{-\pi s}F(s)$

$\Rightarrow L\{\sin t\, u(t-\pi)\} = e^{-\pi s} \cdot \left(\frac{-1}{s^2+1}\right)$ //

Q: Find $L\{(t-1)^2 u(t+3)\}$.

Here, $a = -3$

$f(t+3) = (t-1)^2$

$= (t+3-4)^2$

$f(t) = (t-4)^2$

$= t^2 + 16 - 8t$

$F(s) = \frac{2}{s^3} + \frac{16}{s} - \frac{8}{s^2}$

We know that $L\{f(t-a)\,u(t-a)\} = e^{-as} F(s)$

$\Rightarrow L\{f(t+3)\,u(t+3)\} = e^{3s} F(s)$

$\Rightarrow L\{(t-1)^2 u(t+3)\} = e^{3s}\left(\frac{2}{s^3} + \frac{16}{s} - \frac{8}{s^2}\right)$ //

Representation of discontinuous function in terms of unit step function:

Sometimes, a ~~func~~ discontinuous function is given, but we can represent this function ~~as the addition of~~ in terms of unit step function.

Let us consider the discontinuous function

$$f(t) = \begin{cases} A, & 0 \le t \le b \\ B, & b < t < c \\ D, & t \geq c \end{cases}$$

$f(t) = A\{u(t-0) - u(t-b)\} + B\{u(t-b) - u(t-c)\} + D\,u(t-c)$ //

Q. Find $L\{f(t)\}$ where $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$ ① 23/01/20

ans- As $f(t)$ is a discontinuous function,

$$f(t) = t^2[u(t-0) - u(t-1)]$$

$$= t^2u(t-0) - t^2u(t-1)$$

$$L\{f(t)\} = L\{t^2u(t-0)\} - L\{t^2u(t-1)\} \quad \text{—— ①}$$

$a = 0$

$f(t-0) = t^2$

$F(s) = \frac{2}{s^3}$

$a = 1$

$f(t-1) = t^2 = (t-1+1)^2$

$f(t) = (t+1)^2$

$= t^2 + 1 + 2t$

$F(s) = \frac{2}{s^3} + \frac{1}{s} + \frac{2}{s^2}$

$$L\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$$

when $a = 0$

$$L\{f(t-0)\,u(t-0)\} = e^{-0\cdot s}F(s)$$

$$L\{t^2u(t-0)\} = \frac{2}{s^3}$$

& $a = 1$

$$L\{f(t-1)\,u(t-1)\} = e^{-s}F(s)$$

$$\Rightarrow L\{t^2u(t-1)\} = e^{-s}\left(\frac{2}{s^3} + \frac{1}{s} + \frac{2}{s^2}\right)$$

$$L\{f(t)\} = \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{1}{s} + \frac{2}{s^2}\right)$$

Q. Find $L\{f(t)\}$ where $f(t) = \begin{cases} 2, & 0 \le t \le \pi \\ 0, & \pi < t < 2\pi \\ \sin t, & t > 2\pi \end{cases}$

ans: As $f(t)$ is a discontinuous function,

$\cancel{f(t) = L\{2u(t-0)}$

$$f(t) = 2[u(t-0) - u(t-\pi)] + \sin t\, u(t-2\pi)$$

$$= 2u(t-0) - 2u(t-\pi) + \sin t\, u(t-2\pi)$$

$$L\{f(t)\} = 2L\{u(t-0)\} - 2L\{u(t-\pi)\} + L\{\sin t\, u(t-2\pi)\}$$

$$\Rightarrow L\{f(t)\} = 2\frac{1}{s} - 2\frac{e^{-\pi s}}{s} + L\{\sin t\, u(t-2\pi)\} \quad \text{—— (1)}$$

$$a = 2\pi$$

$$f(t-2\pi) = \sin t$$

$$= \sin(t - 2\pi + 2\pi)$$

$$f(t) = \sin(t+2\pi) = \sin t$$

$$F(s) = L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

$$\Rightarrow L\{f(t-2\pi)\cdot u(t-2\pi)\} = e^{-2\pi s}F(s)$$

$$\Rightarrow L\{\sin t\, u(t-2\pi)\} = e^{-2\pi s}\frac{1}{s^2+1}$$

Substituting in eq^n (1)

$$L\{f(t)\} = \frac{2}{s} - \frac{2e^{-\pi s}}{s} + e^{-2\pi s}\left(\frac{1}{s^2+1}\right)$$

//

Q7 Find $L^{-1}\{e^{-2s} \cdot \frac{1}{s^2}\}$. (3)

ans Here $a = 2$.

$$F(s) = \frac{1}{s^2}$$

$$f(t) = L^{-1}\{\frac{1}{s^2}\} = t$$

We know that $L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$

Putting $a = 2$

$$L^{-1}\{e^{-2s} \frac{1}{s^2}\} = f(t-2)u(t-2).$$
$$= (t-2)u(t-2)$$

Q7 Find $L^{-1}\{\frac{e^{-3s}}{s^2+25}\}$

Here $a = 3$

$$F(s) = \frac{1}{s^2+5^2}$$

$$f(t) = \frac{\sin 5t}{5}$$

We know that $L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$

Putting $a = 3$

$$L^{-1}\{\frac{e^{-3s}}{s^2+5^2}\} = f(t-3)u(t-3)$$

$$L^{-1}\{\frac{e^{-3s}}{s^2+5^2}\} = \frac{\sin 5(t-3)}{5} u(t-3)$$

Q7 Find $L^{-1}\{\frac{e^{-2\pi s}}{s^2+2s+2}\}$

Here $a = 2\pi$

$$F(s) = \frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1^2}$$

$$f(t) = \sin t \, e^{-t}$$ (Using L^{-1} of 1st shifting theorem)

We know that $L^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)\}$ (4)

$$L^{-1}\left\{\frac{e^{-2\pi s}}{s^2+2s+2}\right\} = e^{-(t-2\pi)} \cdot \sin(t-2\pi)\,u(t-2\pi)$$

$$= e^{2\pi-t}\sin(-(2\pi-t))\,u(t-2\pi)$$

$$= e^{2\pi-t}(-\sin(2\pi-t)\,u(t-2\pi))$$

$$= e^{2\pi-t}\sin t\;u(t-2\pi)$$

Q. Find $L^{-1}\left\{\frac{e^{3s}}{s^2+5s+7}\right\}$

ans Here $a=-3$

$$F(s) = \frac{1}{s^2+5s+7} = \frac{1}{(s+5/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2\sin\frac{\sqrt{3}}{2}t}{\sqrt{3}} \cdot e^{-5/2t}$$

We know that $L^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$

$$L^{-1}\left\{\frac{e^{3s}}{s^2+5s+7}\right\} = f(t+3)\,u(t+3)$$

$$\cancel{= \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t \cdot e^{-5/2t}}$$

$$= \frac{2}{\sqrt{3}}e^{-5/2t}\sin\frac{\sqrt{3}}{2}(t+3)\,u(t+3)$$

Q. Find $L^{-1}\left\{e^{-2s}\left(\frac{1}{(s-4)^3}+\frac{1}{(s+5)^2}+\frac{1}{s}\right)\right\}$

Here $a=2$

$$F(s) = \frac{1}{(s-4)^3} + \frac{1}{(s+5)^2} + \frac{1}{s}$$

$$f(t) = \frac{t^2}{2} e^{4t} + t e^{-5t} + 1$$

(5)

We know that $L^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$

$$L^{-1}\left\{ e^{-2s}\left(\frac{1}{(s-4)^3} + \frac{1}{(s+5)^2} + \frac{1}{s}\right)\right\} = f(t-2)\,u(t-2)$$

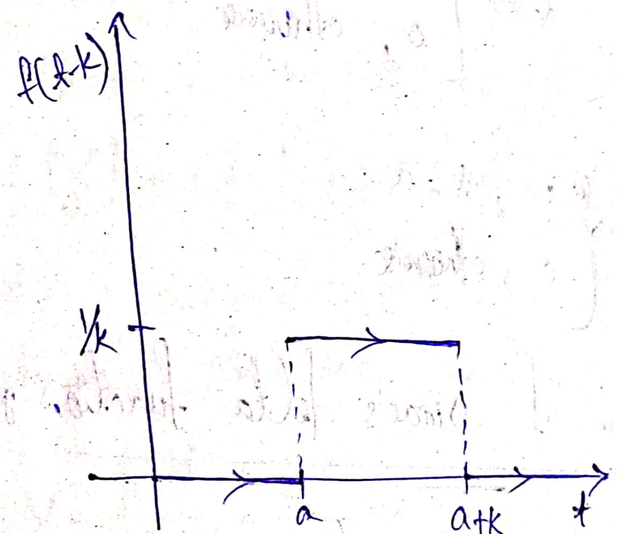
$$= \left[\frac{(t-2)^2}{2} e^{4(t-2)} + (t-2)e^{-5(t-2)} + 1\right] u(t-2)$$

Dirac's Delta function :

The impulse of a force for any time interval k is denoted by $f(t-k)$ and it is defined as

$$f(t-k) = \begin{cases} 1/k, & a \le t \le a+k \\ 0 & \text{otherwise} \end{cases}$$

Graph of impulse function / $f(t-k)$:



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Laplace transform of impulse of a force :

We know that $f(t-k) = \begin{cases} 1/k, & a \le t \le a+k \\ 0, & \text{otherwise} \end{cases}$

$\Rightarrow f(t-k) = \frac{1}{k}\left[u(t-k) - u(t-(a+k))\right]$

$$L\{f(t-k)\} = \frac{1}{k}\left[L(u(t-a)) - L\{u(t-(a+k))\}\right]$$

$$= 1/k\left[\frac{e^{-as}}{s} - \frac{e^{-(a+k)s}}{s}\right]$$

$$= \frac{1}{ks}\left[e^{-as} - e^{-as}.e^{-ks}\right]$$

$$L\{f(t-k)\} = \frac{e^{-as}}{ks}\left[1 - e^{-ks}\right]$$

* Dirac's delta function is the limiting value of $f(t-k)$ when k approaches to zero.

i.e., $\delta(t-a) = \lim_{k\to 0} f(t-k)$

$$= \lim_{k\to 0}\begin{cases} 1/k, & a \le t \le a+k \\ 0, & \text{otherwise}\end{cases}$$

$$\Rightarrow \delta(t-a) = \begin{cases} \infty, & t = a \\ 0, & \text{otherwise}\end{cases}$$

Laplace transform of Dirac's delta function:

We know that, $\delta(t-a) = \lim_{k\to 0} f(t-k)$

$$L\{\delta(t-a)\} = L\left\{\lim_{k\to 0} f(t-k)\right\}$$

$$= \lim_{k\to 0} L\{f(t-k)\}$$

$$= \lim_{k \to 0} \frac{e^{-as}}{ks} (1 - e^{-ks})$$

$$= e^{-as} \lim_{k \to 0} \frac{-(e^{-ks} - 1)}{ks}$$

$$= e^{-as} \lim_{k \to 0} \frac{e^{-ks} - 1}{-ks}$$

$$= e^{-as} \cdot 1$$

$$\boxed{L\{\delta(t-a)\} = e^{-as}}$$

Q: Solve $y'' + y = \delta(t-\pi) - \delta(t-2\pi)$ where $y(0) = 0$ and $y'(0) = 1$.

ans: Let Y be the solution.

$$L\{y''\} = s^2 Y - s \cdot 0 - 1$$
$$= s^2 Y - 1$$

Taking laplace of both sides, we get

$$L\{y''\} + L\{y\} = L\{\delta(t-\pi)\} - L\{\delta(t-2\pi)\}$$

$$\Rightarrow s^2 Y - 1 + Y = e^{-\pi s} - e^{-2\pi s}$$

$$\Rightarrow s^2 Y + Y = e^{-\pi s} - e^{-2\pi s} + 1$$

$$\Rightarrow Y = \frac{e^{-\pi s}}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1} + \frac{1}{s^2+1}$$

$$\Rightarrow L\{y\} = \frac{e^{-\pi s}}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1} + \frac{1}{s^2+1}$$

$$y = L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}_{a=\pi} - L^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\}_{a=2\pi} + L^{-1}\left\{\frac{1}{s^2+1}\right\} \quad \text{—} (1)$$

$$F(s) = \frac{1}{s^2+1}$$

$$f(t) = \sin t$$

$$L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

$$\therefore L^{-1}\left\{e^{-\pi s}\frac{1}{s^2+1}\right\} = f(t-\pi)\,u(t-\pi)$$

$$= \sin(t-\pi)\,u(t-\pi)$$

$$= \sin(-(\pi-t))\,u(t-\pi)$$

$$= -\sin(\pi-t)\,u(t-\pi)$$

$$= -\sin t\,u(t-\pi)$$

$$L^{-1}\left\{e^{-2\pi s}\frac{1}{s^2+1}\right\} = f(t-2\pi)\,u(t-2\pi)$$

$$= \sin(t-2\pi)\,u(t-2\pi)$$

$$= \sin(-(2\pi-t))\,u(t-2\pi)$$

$$= -\sin(2\pi-t)\,u(t-2\pi)$$

$$= \sin t\,u(t-2\pi)$$

Substituting these in eqn ①, we get

$$y = -\sin t\,u(t-\pi) - \sin t\,u(t-2\pi) + \sin t$$

Application of Laplace transform in RLC Circuits

The voltage drop across resistance, inductance and Capacitance are RI, $L\frac{dI}{dt}$, $\frac{1}{C}\int I\,dt$.

Q: Find the current in LC circuit where L = 1 Henry, C = 1 Farad, 0 initial current and charge, and

$$v(t) = \begin{cases} 1, & 0 < t < a \\ 0, & \text{otherwise.} \end{cases}$$

The voltage drop eqⁿ for LC circuit is

ans: $$L\frac{di}{dt} + \frac{1}{C}\int i\,dt = v(t)$$

$$\frac{di}{dt} + \int i\,dt = u(t-0) - u(t-a)$$

Taking laplace transform of both sides

$$L\left\{\frac{di}{dt}\right\} + L\left\{\int i(t)\,dt\right\} = L\{u(t-0)\} - L\{u(t-a)\}$$

$$sL\{i\} - i(0) + \frac{L(i)}{s} = \frac{1}{s} - \frac{e^{-as}}{s}$$

$$sI + \frac{I}{s} = \frac{1}{s} - \frac{e^{-as}}{s}$$

$$\Rightarrow I\left(s + \frac{1}{s}\right) = \frac{1}{s} - \frac{e^{-as}}{s}$$

$$\Rightarrow I\frac{(s^2+1)}{s} = \frac{1-e^{-as}}{s}$$

$$\Rightarrow I = \frac{1}{s^2+1} - \frac{e^{-as}}{s^2+1}$$

$$\Rightarrow L\{i\} = \frac{1}{s^2+1} - \frac{e^{-as}}{s^2+1}$$

$$\Rightarrow i = L^{-1}\left\{\frac{1}{s^2+1}\right\} - L^{-1}\left\{\frac{e^{-as}}{s^2+1}\right\}$$

$$i = \sin t - L^{-1}\left\{\frac{e^{-as}}{s^2+1}\right\} \quad \text{--- (1)}$$

$a = a$

$$F(s) = \frac{1}{s^2+1}$$

$$f(t) = \sin t$$

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$$L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

$$L^{-1}\left\{e^{-as}\frac{1}{s^2+1}\right\} = f(t-a)\,u(t-a)$$

$$= \sin(t-a)\,u(t-a)$$

Substituting these in eq^n (1)

$$\boxed{i = \sin t - \sin(t-a)\,u(t-a)}$$

Derivative and Integration of Laplace transform:

Theorem:

If $L\{f(t)\} = F(s)$, then

$$\boxed{L\{t\,f(t)\} = -\frac{d}{ds}(F(s))}$$

RHS $= -\frac{d}{ds}(F(s))$

$$= -\frac{d}{ds}\left(L\{f(t)\}\right)$$

$$= -\frac{d}{ds}\left(\int_0^\infty e^{-st}f(t)\,dt\right)$$

$$= -\int_0^\infty \frac{d}{ds}(e^{-st})\,f(t)\,dt$$

$$= -\int_0^\infty e^{-st}(-t)\,f(t)\,dt$$

$$= \int_0^\infty e^{-st}\,t\,f(t)\,dt$$

$= LHS$

[Proved]

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Formula :

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If $L^{-1}\{F(s)\} = f(t)$

$L^{-1}\left\{\frac{d}{ds}F(s)\right\} = -t f(t)$

1) Find $L\{t \sin t\}$

ans: Here, $f(t) = \sin t$

$F(s) = L\{\sin t\}$

$= \frac{1}{s^2+1}$

We know that $L\{t f(t)\} = \frac{-d}{ds}(F(s))$

$\Rightarrow L\{t \sin t\} = \frac{-d}{ds}\left(\frac{1}{s^2+1}\right)$

$= \frac{-d}{ds}(s^2+1)^{-1}$

$= -(-1)(s^2+1)^{-2} \cdot 2s$

$= \frac{2s}{(s^2+1)^2}$

2) Find $L\{t \cosh 5t\}$

ans: Here, $f(t) = \cosh 5t$

$F(s) = L\{\cosh 5t\}$

$= \frac{s}{s^2-5^2}$

We know that $L\{t f(t)\} = \frac{-d}{ds}(F(s))$

$= \frac{-d}{ds}\left(\frac{s}{s^2-5^2}\right)$

$$= \left(\frac{1(s^2-25)-s\cdot 2s}{(s^2-25)^2}\right)$$

$$= \left(\frac{-s^2-25}{(s^2-25)^2}\right)$$

$$= \frac{s^2+25}{(s^2-25)^2}$$

Note:

If $L\{f(t)\} = F(s)$, then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(F(s))$

3) Find $L\{t^3 \cdot e^{5t}\}$.

ans: Here $f(t) = e^{5t}$

$$F(s) = \frac{1}{s-5}$$

$$L\{t\, e^{5t}\} = \frac{-d}{ds}\left(\frac{1}{s-5}\right)$$

$$= -(-1)\frac{1}{(s-5)^2}$$

$$= \frac{1}{(s-5)^2} = (s-5)^{-2}$$

$$L\{t^2 e^{5t}\} = \frac{-d}{ds}(s-5)^{-2}$$

$$= -(-2)(s-5)^{-3}$$

$$= 2(s-5)^{-3}$$

$$L\{t^3 e^{5t}\} = \frac{-d}{ds}\left[2(s-5)^{-3}\right]$$

$$= -2(-3)(s-5)^{-4} = \frac{6}{(s-5)^4}$$

4) Find $L\{t^2 \sinh t\}$

ans: Here $f(t) = \sinh t$

$$F(s) = \frac{1}{s^2 - 1}$$

$$L\{t \sinh t\} = -\frac{d}{ds}\left(\frac{1}{s^2-1}\right)$$

$$= -\frac{d}{ds}(s^2-1)^{-1}$$

$$= (-1)(-1)(s^2-1)^{-2}\cdot 2s$$

$$= \frac{2s}{(s^2-1)^2}$$

$$L\{t^2 \sinh t\} = -\frac{d}{ds}\left(\frac{2s}{(s^2-1)^2}\right)$$

$$= -\left[\frac{(s^2-1)^2\cdot 2 - 2s\cdot 2(s^2-1)2s}{(s^2-1)^4}\right]$$

$$= \frac{-2\cancel{(s^2-1)}\{s^2-1-4s^2\}}{(s^2-1)^{\cancel{4}\,3}}$$

$$= -2\left(\frac{-3s^2-1}{(s^2-1)^3}\right)$$

$$= \frac{6s^2+2}{(s^2-1)^3}$$

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Integration of Laplace transform :

If $L\{f(t)\} = F(s)$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

~~To prove~~ It is given that $L\{f(t)\} = F(s)$

We have to show that $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

$$RHS = \int_s^\infty F(s)\,ds$$

$$= \int_s^\infty L\{f(t)\}\,ds$$

$$= \int_s^\infty \left(\int_0^\infty e^{-st} f(t)\,dt\right) ds$$

$$= \int_s^\infty \int_0^\infty e^{-st} f(t)\,dt\,ds$$

$$= \int_0^\infty \left(\int_s^\infty e^{-st}\,ds\right) f(t)\,dt$$

$$= \int_0^\infty \left(\frac{e^{-st}}{-t}\right)_0^\infty f(t)\,dt$$

$$= \int_0^\infty -\frac{1}{t}\left(e^{-\infty} - e^{-st}\right) f(t)\,dt$$

$$= \int_0^\infty -\frac{1}{t}\left(0 - e^{-st}\right) f(t)\,dt$$

$$= \int_0^\infty e^{-st}\,\frac{f(t)}{t}\,dt$$

$$= L\left\{\frac{f(t)}{t}\right\} = LHS$$

[Proved]

1) Find $L\left\{\frac{\sin t}{t}\right\}$.

ans: Here, $f(t) = \sin t$

$F(s) = \frac{1}{s^2+1}$

We know that $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \left[\tan^{-1} s\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}s$$

$$= \pi/2 - \tan^{-1}s$$

$$= \cot^{-1}s$$

If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\int_s^\infty F(s)\,ds\right\} = \frac{f(t)}{t}$

Notes

→ If $F(s)$ be any logarithmic function or inverse trigonometric function, then to find its inverse laplace, let us apply the inverse formula of derivative of laplace transform.

→ To find the inverse laplace of $F(s)$ where $F(s)$ is a rational function in which the power of the denominator is at least 2, then in this case, we apply inverse formula of integration of laplace transform.

1) Find $L^{-1}\{\log(s^2+25)\}$.

Here $F(s) = \log(s^2+25)$

~~We know that~~

Let, $L^{-1}\{F(s)\} = f(t)$

$\Rightarrow L^{-1}\left\{\frac{d}{ds}(F(s))\right\} = -t\,f(t)$

$\Rightarrow L^{-1}\left\{\frac{d}{ds}\left(\log(s^2+25)\right)\right\} = -t\,f(t)$

$\Rightarrow L^{-1}\left\{\frac{2s}{s^2+25}\right\} = -t\,f(t)$

$\Rightarrow 2\cos 5t = -t\,f(t)$

$\Rightarrow f(t) = \frac{-2\cos 5t}{t}$

$L^{-1}\{\log(s^2+25)\} = \frac{-2\cos 5t}{t}$ //

2) Find $L^{-1}\left\{\log\left(\frac{s+5}{s-4}\right)\right\}$

Here $F(s) = \log\left(\frac{s+5}{s-4}\right)$

Let, $L^{-1}\{F(s)\} = f(t)$

$\Rightarrow L^{-1}\left\{\frac{d}{ds}(F(s))\right\} = -t\,f(t)$

$\Rightarrow L^{-1}\left\{\frac{d}{ds}\left(\log\left(\frac{s+5}{s-4}\right)\right)\right\} = -t\,f(t)$

$L^{-1}\left\{\frac{d}{ds}\left(\log(s+5) - \log(s-4)\right)\right\} = -t\,f(t)$

$\Rightarrow L^{-1}\left\{\frac{1}{s+5} - \frac{1}{s-4}\right\} = -t\,f(t)$

$\Rightarrow e^{-5t} - e^{-4t} = -t f(t)$

$$f(t) = \frac{e^{-5t} - e^{-4t}}{-t}$$

$$L^{-1}\left\{\log\left(\frac{s+5}{s-4}\right)\right\} = \frac{e^{-5t} - e^{-4t}}{-t}$$

3) Find $L^{-1}\left\{\tan^{-1}\frac{s}{3}\right\}$

Here $F(s) = \tan^{-1}\frac{s}{3}$

Let, $L^{-1}\{F(s)\} = f(t)$

$\Rightarrow L^{-1}\left\{\frac{d}{ds}(F(s))\right\} = -t f(t)$

$\Rightarrow L^{-1}\left\{\frac{d}{ds}\left(\tan^{-1}\frac{s}{3}\right)\right\} = -t f(t)$

$\Rightarrow L^{-1}\left\{\frac{1}{1+\frac{s^2}{9}} \frac{d}{ds}\left(\frac{s}{3}\right)\right\} = -t f(t)$

$\Rightarrow L^{-1}\left\{\frac{\cancel{9}\,3}{s^2+9} \times \frac{1}{\cancel{3}}\right\} = -t f(t)$

$\Rightarrow L^{-1}\left\{\frac{3}{s^2+3^2}\right\} = -t f(t)$

$\Rightarrow \sin 3t = -t f(t)$

$\Rightarrow f(t) = \frac{-\sin 3t}{t}$

$L^{-1}\left\{\tan^{-1}\frac{s}{3}\right\} = \tan^{-1}\frac{s}{3}$ //

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Q. Find $L^{-1}\left\{\frac{2s+3}{(s^2+3s+1)^2}\right\}$

Let, $L^{-1}\left\{\frac{2s+3}{(s^2+3s+1)^2}\right\} = f(t)$

$\Rightarrow L^{-1}\left\{\int_s^\infty \frac{2s+3}{(s^2+3s+1)^2}\,ds\right\} = \frac{F(t)}{t}$

Let, $s^2+3s+1 = x$

$\Rightarrow (2s+3)\,ds = dx$

when $s = s$, $x = s^2+3s+1$

$s = \infty$, $x = \infty$

Now,

$$L^{-1}\left\{\int_{s^2+3s+1}^{\infty} \frac{dx}{x^2}\right\} = \frac{F(t)}{t}$$

$$\Rightarrow L^{-1}\left\{\int_{s^2+3s+1}^{\infty} x^{-2}\,dx\right\} = \frac{F(t)}{t}$$

$$\Rightarrow L^{-1}\left\{-\left(\frac{1}{x}\right)_{s^2+3s+1}^{\infty}\right\} = \frac{F(t)}{t}$$

$$\Rightarrow -L^{-1}\left\{0 - \frac{1}{s^2+3s+1}\right\} = \frac{F(t)}{t}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(s+3/2)^2 - \left(\frac{\sqrt{5}}{2}\right)^2}\right\} = \frac{F(t)}{t}$$

$$\Rightarrow \frac{\sinh \frac{\sqrt{5}}{2}t}{\frac{\sqrt{5}}{2}}\, e^{-3/2\, t} = \frac{F(t)}{t}$$

$\Rightarrow \frac{2t}{\sqrt{5}} \sinh \frac{\sqrt{5}}{2} + e^{3/2 t} = F(t)$

Q. Find $L^{-1}\left\{ \frac{s}{(s^2+9)^2} \right\}$

Let, $L^{-1}\left\{ \frac{s}{(s^2+9)^2} \right\} = F(t)$

$\Rightarrow L^{-1}\left\{ \int_s^\infty \frac{s}{(s^2+9)^2} ds \right\} = \frac{F(t)}{2}$

Let, $s^2+9 = x$

$\Rightarrow s.ds = \frac{dx}{2}$

When $s=s, x = s^2+9$

$s=\infty, x=\infty$

Now,

$L^{-1}\left\{ \int_{s^2+9}^\infty \frac{dx}{2x^2} \right\} = \frac{F(t)}{t}$

$\Rightarrow \frac{1}{2} L^{-1}\left\{ \left(\frac{-1}{x}\right)_{s^2+9}^\infty \right\} = \frac{F(t)}{t}$

$\Rightarrow \frac{1}{2} L^{-1}\left\{ 0 - \frac{1}{s^2+9} \right) = \frac{F(t)}{t}$

$\Rightarrow \frac{1}{2} \frac{\sin 3t}{3} = \frac{F(t)}{t}$

Q. Find $L^{-1}\left\{ \frac{s^2+1}{(s^2-25)^2} \right\}$.

Ans: $L^{-1}\left\{ \frac{s^2-25+26}{(s^2-25)^2} \right\}$

$= L^{-1}\left\{ \frac{s^2-25}{(s^2-25)^2} + \frac{26}{(s^2-25)^2} \right\}$

$= L^{-1}\left\{ \frac{1}{s^2-25} \right\} + 26 L^{-1}\left\{ \frac{1}{(s-5)^2 (s+5)^2} \right\}$

$= \frac{\sinh 5t}{5} + 26 L^{-1}\left\{ \frac{1}{(s-5)^2 (s+5)^2} \right\}$ —— (1)

$\frac{1}{(s-5)^2 (s+5)^2} = \frac{A}{s-5} + \frac{B}{(s-5)^2} + \frac{C}{s+5} + \frac{D}{(s+5)^2}$

$B = \left. \frac{1}{(s+5)^2} \right]_{s=5} = \frac{1}{100}$ $\quad D = \left. \frac{1}{(s-5)^2} \right]_{s=-5} = \frac{1}{100}$

$\frac{1}{\cancel{(s-5)^2} \cancel{(s+5)^2}} = \frac{A(s-5)(s+5)^2 + B(s+5)^2 + C(s+5)(s-5)^2 + D(s-5)^2}{\cancel{(s-5)^2} \cancel{(s+5)^2}}$

$$1 = A(s-5)(s^2+25+105) + B(s^2+25s+105) + C(s+5)(s^2+25-105) + D(s^2+25-105)$$

$s^3, 0 = A + C$

$A = -C$

Const $1 = -125A + 25B + 125C + 250$

$\Rightarrow 1 = 250C + \frac{50}{100}$

$\Rightarrow 1 - \frac{1}{2} = 250C$

$\frac{1}{2} = 250C$, $C = \frac{1}{500}$, $A = \frac{-1}{500}$

$$\frac{1}{(s-5)^2(s+5)^2} = \frac{\frac{-1}{500}}{s-5} + \frac{\frac{1}{100}}{(s-5)^2} + \frac{\frac{1}{500}}{s+5} + \frac{\frac{1}{100}}{(s+5)^2}$$

$$L^{-1}\left\{\frac{1}{(s-5)^2(s+5)^2}\right\} = \frac{-1}{100}e^{5t} + \frac{1}{100} \cdot te^{5t} + \frac{1}{500}e^{-5t} + \frac{1}{100}te^{-5t}$$

Convolution:

Let $f(t)$ & $g(t)$ be two functions such that

$L\{f(t)\} = F(s)$ and $L\{g(t)\} = G(s)$

Let, $H(s) = F(s).G(s)$

If $L^{-1}\{H(s)\}$ exists, then it will be a function of t assuming that $L^{-1}\{H(s)\} = h(t)$

This $h(t)$ is called the convolution of $f(t)$. This $h(t)$ is called the convolution of $f(t)$ & $g(t)$. And symbolically it is written as:

$$L^{-1}\{H(s)\} = h(t) = (f * g)(t) = \int_0^t F(\tau) g(t-\tau) d\tau$$

Q: Find $1 * t$.

ans: Here $f(t) = 1$, $g(t) = t$ $\quad F(\tau) = 1$

$g(t-\tau) = t - \tau$

But $f*g = \int_0^t f(\tau)g(t-\tau)d\tau$

$$1*t = \int_0^t 1.(t-\tau)d\tau$$

$$= t(\tau)_0^t - \left(\frac{\tau^2}{2}\right)_0^t$$

$$= t(t-0) - \frac{1}{2}(t^2-0)$$

$$= t^2 - \frac{t^2}{2} = \frac{t^2}{2}$$

Q. Find $t * e^{2t}$.

$f(t) = t$, $g(t) = e^{2t}$

$f(\tau) = \tau$, $g(t-\tau) = e^{2t-2\tau}$

$$f*g = \int_0^t \tau . e^{2t-2\tau} d\tau$$

$$= \int_0^t \tau . e^{2t} . e^{-2\tau} d\tau$$

$$= e^{2t} \int_0^t \tau . e^{-2\tau} d\tau$$

$$= e^{2t} \left[\tau \int e^{-2\tau} d\tau - \int \left[\frac{d}{d\tau}(\tau) \int e^{-2\tau} . d\tau \right] d\tau \right]$$

or

$$= e^{2t} \left[\tau . \frac{e^{-2\tau}}{-2} - 1 . \frac{e^{-2\tau}}{4} \right]_0^t$$

$$= e^{2t} \left[-\frac{t}{2} e^{-2t} - \frac{e^{-2t}}{4} - \left(0 - \frac{1}{4}\right) \right]$$

$$= e^{2t} \left[-\frac{t}{2} e^{-2t} - \frac{e^{-2t}}{4} + \frac{1}{4} \right]$$

$$= -\frac{t}{2} - \frac{1}{4} + \frac{e^{2t}}{4}$$

Q. $\sin t * \cos t$. ~~Find it~~ Find it.

$f(t) = \sin t$ $\qquad f(\tau) = \sin \tau$

$g(t) = \cos t$ $\qquad g(t-\tau) = \cos(t-\tau)$

$$\sin t * \cos t = \int_0^t \cos(t-\tau)\sin\tau \, d\tau$$

$$= \int_0^t \frac{\sin(\tau+(t-\tau)) + \sin(\tau-t+\tau)}{2} d\tau$$

$$= \frac{1}{2}\int_0^t (\sin t + \sin(2\tau - t)) d\tau$$

$$= \frac{1}{2} \sin t \int_0^t d\tau + \frac{1}{2}\int_0^t \sin(2\tau - t) d\tau$$

$$= \frac{\sin t}{2} (\tau)_0^t - \frac{1}{2}\left(\frac{\cos(2\tau - t)}{2}\right)_0^t$$

$$= \frac{t \sin t}{2} - \frac{1}{4}(\cos t - \cos(-t))$$

$$= \frac{t\sin t}{2} - \left(\frac{1}{4}\times 0\right)$$

$$= \frac{t \sin t}{2},$$

Q. Find $e^{2t} * e^{3t}$.

$f(t) = e^{2t}$, $g(t) = e^{3t}$

$f(\tau) = e^{2\tau}$, $g(t-\tau) = e^{3(t-\tau)}$

$$f * g = \int_0^t e^{2\tau} \cdot e^{3(t-\tau)} d\tau$$

$$= \int_0^t e^{2\tau} \cdot e^{3t} \cdot e^{-3\tau} d\tau$$

$$= e^{3t} \int_0^t e^{-\tau} d\tau$$

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Q7 Find $u(t-3) * e^{-2t}$.

ans: $f(t) = u(t-3)$ $\qquad g(t) = e^{-2t}$

$f(\tau) = u(\tau-3)$ $\qquad g(t-\tau) = e^{-2(t-2)}$

$$f * g = \int_0^t f(\tau) g(t-\tau) d\tau$$

$$u(\tau-3) = \begin{cases} 1, & \tau > 3 \\ 0, & \tau < 3 \end{cases}$$

$$u(t-3) * e^{-2t} = \int_0^t u(\tau-3) \cdot e^{-2(t-\tau)} d\tau$$

$$= \int_0^3 u(\tau-3) e^{-2(t-\tau)} d\tau + \int_3^t u(\tau-3) e^{-2(t-\tau)} d\tau$$

$$= 0 + \int_3^t 1 . e^{2t-2\tau} d\tau$$

$$= \int_3^t e^{-2t} . e^{+2\tau} d\tau .$$

$$= e^{-2t} \left(\frac{2^{2\tau}}{2} \right)_3^t$$

$$= \frac{e^{-2t}}{2} (2^t - e^6)$$

$$= \frac{1}{2} - \frac{e^{6-2t}}{2}$$

Q2 Using convolution, find $L^{-1}\left\{ \frac{6}{s(s+3)} \right\}$.

ans2 Here, $H(s) = \dfrac{6}{s(s+3)}$

$$\Rightarrow F(s)\, G(s) = \frac{6}{s} \cdot \frac{1}{s+3}$$

$F(s) = \dfrac{6}{s}$ $\qquad G(s) = \dfrac{1}{s+3}$

$\Rightarrow f(t) = L^{-1}\left\{\frac{6}{s}\right\} = 6$, $g(t) = L^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$

$f(\tau) = 6$ $\quad g(t-\tau) = e^{-3(t-\tau)}$

$L^{-1}\{H(s)\} = h(t) = f * g$

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_0^t 6 \cdot e^{-3(t-\tau)}\, d\tau$$

$$= 6 \int_0^t e^{-3(t-\tau)}\, d\tau$$

$$= 6 \int_0^t e^{-3t} \cdot e^{3\tau}\, d\tau$$

$$= 6e^{-3t}\left(\frac{e^{3\tau}}{3}\right)_0^t$$

$$= 2e^{-3t}\left(e^{3t} - 1\right)$$

$$= 2 - 2e^{-3t} \text{ //}$$

Q7. Using convolution, find $L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}$.

ans: Here, $H(s) = \frac{s}{(s^2+1)(s^2+4)}$

$\Rightarrow F(s)\,G(s) = \frac{1}{s^2+1^2} \times \frac{s}{s^2+2^2}$

$F(s) = \frac{1}{s^2+1^2}$ $\quad G(s) = \frac{s}{s^2+2^2}$

$f(t) = \sin t$ $\quad g(t) = \cos 2t$

$f(\tau) = \sin\tau$ $\quad g(t-\tau) = \cos(2t - 2\tau)$

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$$L^{-1}\{H(s)\} = h(t) = f * g$$

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_0^t \sin\tau \cos(2t-2\tau)\, d\tau$$

$$= \int_0^t \frac{\sin(\tau+2t-2\tau) + \sin(\tau-2t+2\tau)}{2}\, d\tau$$

$$= \frac{1}{2}\int_0^t \sin(2t-\tau) + \sin(3\tau-2t)\, d\tau$$

$$= \frac{1}{2}\left[\frac{\cos(2t-\tau)}{-1} - \frac{\cos(3\tau-2t)}{3}\right]_0^t$$

$$= \frac{1}{2}\left[\cos t - \frac{\cos t}{3} - \left(\cos 2t - \frac{\cos(-2t)}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{2\cos t}{3} - \frac{2\cos 2t}{3}\right]$$

$$= \frac{1}{3}(\cos t - \cos 2t)$$ //

Q. Find $L^{-1}\left\{\frac{1}{s^2(s-3)}\right\}$

ans: Here, $H(s) = \frac{1}{s^2(s-3)}$

$$F(s)\, G(s) = \frac{1}{s^2} \cdot \frac{1}{s-3}$$

$F(s) = \frac{1}{s^2}$ $\quad$ $G(s) = \frac{1}{s-3}$

$f(t) = t$ $\quad$ $g(t) = e^{3t}$

$f(\tau) = \tau$ $\quad$ $g(t-\tau) = e^{3(t-\tau)}$

$$L^{-1}\{H(s)\} = h(t) = f * g$$

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$$f*g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_0^t \tau\, e^{3(t-\tau)}\, d\tau$$

$$= \int_0^t \tau\, e^{3t} . e^{-3\tau}\, d\tau$$

$$= e^{3t} \int_0^t \tau . e^{-3\tau}\, d\tau$$

$$= e^{3t} \left[\tau . \frac{e^{-3\tau}}{-3} - 1 . \frac{e^{-3\tau}}{9} \right]_0^t$$

$$= e^{3t} \left(-t\, \frac{e^{-3t}}{3} - \frac{e^{-3t}}{9} - \left(0 - \frac{1}{9}\right) \right)$$

$$= e^{3t} \left(-\frac{t e^{-3t}}{3} - \frac{e^{-3t}}{9} + \frac{1}{9} \right)$$

$$= \frac{-t}{3} - \frac{1}{9} + \frac{e^{3t}}{9}$$

$$= \frac{e^{3t} - 1 - 3t}{9} \quad //$$

Q. Applying convolution solve $y'' + y = t$; $y(0) = y'(0) = 0$.

ans Let y be the solⁿ such that $L\{y\} = Y$

$$L\{y''\} = s^2 L\{y\} - s\, y(0) - y'(0)$$

$$= s^2 Y - s.0 - 0$$

$$= s^2 Y$$

Taking laplace on both sides of the given problem, we get

$$L\{y''\} + L\{y\} = L\{t\}$$

$$\Rightarrow s^2 Y + Y = \frac{1}{s^2}$$

$$\Rightarrow (s^2+1)\, Y = \frac{1}{s^2}$$

$\Rightarrow Y = \frac{1}{s^2(s^2+1)}$

$\Rightarrow L\{y\} = \frac{1}{s^2(s^2+1)}$

$\Rightarrow y = L^{-1}\left\{ \frac{1}{s^2(s^2+1)} \right\}$

Let, $H(s) = \frac{1}{s^2(s^2+1)}$

$F(s)\, G(s) = \frac{1}{s^2} \cdot \frac{1}{s^2+1}$

$F(s) = \frac{1}{s^2}$ $\qquad G(s) = \frac{1}{s^2+1}$

$f(t) = t$ $\qquad g(t) = \cancel{\cos t}\ \sin t$

$f(\tau) = \tau$ $\qquad g(t-\tau) = \sin(t-\tau)$

$$L^{-1}\{H(s)\} = h(t) = f * g$$

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_0^t \tau \sin(t-\tau)\, d\tau$$

$$= \left[\tau \left(- \frac{\cos(t-\tau)}{-1} \right) - 1 \cdot \frac{\sin(t-\tau)}{-1} \right]_0^t$$

$$= t\cos 0 + \sin 0 - (0 + \sin(t-0))$$

$$= t + 0 - \sin t$$

$y = t - \sin t$ is the reqd. solⁿ.

Q. Solve $y''+4y = r(t)$, $y(0)=1$, $y'(0)=0$, $r(t)= \begin{cases} 1, & 0<t<1 \\ 0, & \text{otherwise} \end{cases}$

Ans The given eq^n can be written as

$$y''+4y = \begin{cases} 1, & 0<t<1 \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow y''+4y = 1\{u(t-0)-u(t-1)\}$

$\Rightarrow y''+4y = u(t-0)-u(t-1) \longrightarrow (1)$

Let y be the sol^n such that $L\{y\}=Y$

$$L\{y''\} = s^2L\{y\} - s\,y(0) - y'(0)$$
$$= s^2Y - s.1 - 0$$
$$= s^2Y - s$$

Taking laplace on both the sides of the given problem, we get

$L\{y''\}+4L\{y\} = L\{u(t-0)\} - L\{u(t-1)\}$

$\Rightarrow s^2Y - s + 4Y = \frac{1}{s} - \frac{e^{-s}}{s}$

$\Rightarrow s^2Y + 4Y = s + \frac{1}{s} - \frac{e^{-s}}{s}$

$\Rightarrow Y(s^2+4) = s + \frac{1}{s} - \frac{e^{-s}}{s}$

$\Rightarrow Y = \frac{s}{s^2+4} + \frac{1}{s(s^2+4)} - \frac{e^{-s}}{s(s^2+4)}$

$\Rightarrow y = L^{-1}\left\{\frac{s}{s^2+4}\right\} + L^{-1}\left\{\frac{1}{s(s^2+2^2)}\right\} - L^{-1}\left\{\frac{e^{-s}}{s(s^2+4)}\right\}$

$= \cos 2t + L^{-1}\left\{\frac{1}{s(s^2+2^2)}\right\} - L^{-1}\left\{\frac{e^{-s}}{s(s^2+4)}\right\} \longrightarrow (2)$

$$\left\{ \frac{1}{s(s^2+2^2)} \right\} = H(s)$$

$$H(s) = F(s)\, G(s) = \frac{1}{s} \cdot \frac{1}{s^2+4}$$

$$F(s) = \frac{1}{s} \qquad G(s) = \frac{1}{s^2+4}$$

$$f(t) = 1 \qquad g(t) = \frac{\sin 2t}{2}$$

$$f(\tau) = 1 \qquad g(t-\tau) = \frac{\sin 2(t-\tau)}{2}$$

$$L^{-1}\left\{ \frac{1}{s(s^2+2^2)} \right\} = \int_0^t \frac{\sin 2(t-\tau)}{2}\, d\tau$$

$$= \frac{1}{2}\left(\frac{-\cos(2t-2\tau)}{-2} \right)_0^t$$

$$= \frac{1}{4}\left(\cos 0 - \cos 2t \right)$$

$$= \frac{1-\cos 2t}{4}$$

$$L^{-1}\left\{ e^{-as} F(s) \right\} = f(t-a)\, u(t-a)$$

$$\Rightarrow L^{-1}\left\{ e^{-1s} \frac{1}{s(s^2+2^2)} \right\} = f(t-1)\, u(t-1)$$

$$= \frac{1-\cos 2(t-1)}{4}\, u(t-1)$$

Putting these in eq^n (2)

$$y = \cos 2t + \frac{1-\cos 2t}{4} - \frac{1-\cos 2(t-1)}{4}\, u(t-1)$$

Integral Equation :

06/02/20020

The equation in which some integral terms are present is called the integral equation.

e.g.:- $y(t) = 1 + \int_0^t y(\tau)\, g(t-\tau)\, d\tau$

Solution of integral equations :

The solution of integral equation can be determined by using the help of convolution.

1) Using convolution, solve the integral equation

$$y = 2t - 4 \int_0^t \underbrace{y(\tau)\,(t-\tau)}\, d\tau$$

$f(\tau) = y(\tau)$
$g(t-\tau) = t-\tau$
$f(t) = y(t)$
$g(t) = t$

ans. The eq^n can be written as

$$y = 2t - 4\, y * t$$

Taking laplace on both sides, we get

$$L\{y\} = 2L\{t\} - 4L\{y * t\}$$

$$Y = 2\frac{1}{s^2} - 4\, L\{y\}\, L\{t\} \quad (\because L\{f * g\} = L\{f\}L\{g\})$$

$$Y = \frac{2}{s^2} - \frac{4Y}{s^2}$$

$$Y + \frac{4Y}{s^2} = \frac{2}{s^2}$$

$$\Rightarrow Y\left(\frac{s^2+4}{\cancel{s^2}}\right) = \frac{2}{\cancel{s^2}}$$

$$\Rightarrow Y = \frac{2}{s^2+2^2}$$

$$\Rightarrow L\{y\} = L^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \sin 2t$$

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2) Solve the integral equation

$$y(t) = \sin 2t + \int_0^t y(\tau)\sin 2(t-\tau)\,d\tau$$

ans: $y(t) = \sin 2t + y(t) * \sin 2t$

Taking laplace on both the sides

$$L\{y\} = L\{\sin 2t\} + L\{y * \sin 2t\}$$

$$\Rightarrow Y = \frac{2}{s^2+4} + L\{y\}.L\{\sin 2t\}$$

$$\Rightarrow Y = \frac{2}{s^2+4} + Y.\frac{2}{s^2+4}$$

$$\Rightarrow Y - \frac{2Y}{s^2+4} = \frac{2}{s^2+4}$$

$$\Rightarrow \frac{Ys^2+4Y-2Y}{\cancel{s^2+4}} = \frac{2}{\cancel{s^2+4}}$$

$$\Rightarrow Y(s^2+2) = 2$$

$$\Rightarrow Y = \frac{2}{s^2+2}$$

$$\Rightarrow L\{y\} = L^{-1}\left\{\frac{2}{s^2+(\sqrt{2})^2}\right\}$$

$$\Rightarrow L\{y\} = \frac{2\sin\sqrt{2}t}{\sqrt{2}}$$

$$\Rightarrow L\{y\} = \sqrt{2}\sin\sqrt{2}t$$

3) Solve the integral equation

$$y(t) = 1 - \sinh t + \int_0^t (1+\tau)\, y(t-\tau)\, d\tau$$

ans- The eqⁿ can be written as

$$y(t) = 1 - \sinh t + (1+t) * y(t)$$

Taking laplace on both the sides

$$L\{y\} = L\{1\} - L\{\sinh t\} + L\{(1+t) * (y(t))\}$$

$$\Rightarrow Y = \frac{1}{s} - \frac{1}{s^2-1} + L\{(1+t)\} \cdot L\{y(t)\}$$

$$\Rightarrow Y = \frac{1}{s} - \frac{1}{s^2-1} + L\{(1+t)\} \cdot L\{y(t)\}$$

$$\Rightarrow Y = \frac{1}{s} - \frac{1}{s^2-1} + \left(\frac{1}{s} + \frac{1}{s^2}\right) Y$$

$$\Rightarrow Y - \left(\frac{1}{s} + \frac{1}{s^2}\right) Y = \frac{1}{s} - \frac{1}{s^2-1}$$

$$\Rightarrow Y\left(1 - \frac{1}{s} + \frac{1}{s^2}\right) = \frac{1}{s} - \frac{1}{s^2-1}$$

$$\Rightarrow Y\left(1 - \frac{1}{s} - \frac{1}{s^2}\right) = \frac{s^2-1-s}{s(s^2-1)}$$

$$\Rightarrow Y\left(\frac{\cancel{s^2-s-1}}{s^{\cancel{2}}}\right) = \frac{\cancel{s^2-s-1}}{\cancel{s}(s^2-1)}$$

$$\Rightarrow Y = \frac{s}{s^2-1^2}$$

$$L\{y\} = \frac{s}{s^2-1^2}$$

$$y = L^{-1}\left\{\frac{s}{s^2-1^2}\right\}$$

$$y = \cosh t$$

Properties of convolution

1) It is commutative that means $f * g = g * f$
2) It is associative that means $(f * g) * h = f * (g * h)$
3) It is distributive that means $f * (g + h) = f * g + f * h$

1) LHS

$$f * g = \int_0^t f(\tau) g(t-\tau) d\tau$$

Let $t - \tau = x$

$$0 - d\tau = dx$$

$$d\tau = -dx$$

when $\tau = 0, \; x = t$

$\tau = t, \; x = 0$

so $$f * g = \int_t^0 f(t-x)\, g(x)\, (-dx)$$

$$= \int_0^t g(x)\, f(t-x)\, dx$$

$$= \int_0^t g(\tau)\, f(t-\tau)\, d\tau$$ (x is independent variable)

$$= g * f$$

Q: Find $L^{-1}\left\{\frac{s-2}{s-3}\right\}$.

ans: $$L^{-1}\left\{\frac{s-2}{s-3}\right\} = L^{-1}\left\{\frac{s-3+1}{s-3}\right\}$$

$$= L^{-1}\{1\} + L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= \delta(t) + e^{3t}$$

Deteremination of inverse laplace by using partial fraction :-

Fraction : If $f(x)$ & $g(x)$ be any two functions, then $\frac{f(x)}{g(x)}$ is called a fraction.

Types of fractions :

There are two types of fractions : i) Propere fraction
ii) Impropere fraction

Propere fraction :

→ The fraction in which the degree of denominatore is greatere than the degree of numeratore is called a propere fraction.

e.g : $\frac{x+2}{x^2+1}$

Impropere fraction :

→ The fraction in which either the degree of numeratore and denominatore are equal or the degree of denominatore is less than the degree of numerator, then it is called an impropere fraction.

e.g :- $\frac{x+1}{x-1}$, $\frac{x^2+2}{x+3}$

Partial fraction :

→ It is a process of splitting or breaking each factore of the denominatore of a propere fraction into separate terms.

e.g :- $\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$

Some basic formulas of partial fraction :

1) Partial fraction of non-repeated linear factors.

$$\frac{f(x)}{(x+\alpha)(x+\beta)(x+\gamma)} = \frac{A}{x+\alpha} + \frac{B}{x+\beta} + \frac{C}{x+\gamma},$$

where A, B, C are constants & can be determined.

2) Partial fraction of repeated linear factors.

$$\frac{f(x)}{(x+\alpha)^n} = \frac{A_1}{x+\alpha} + \frac{A_2}{(x+\alpha)^2} + \cdots + \frac{A_n}{(x+\alpha)^n}$$

where $A_1, A_2, \ldots A_n$ are constants.

3) Partial fraction of non-repeated quadratic factors.

$$\frac{f(x)}{(x^2+\alpha)(x^2+\beta)} = \frac{Ax+B}{x^2+\alpha} + \frac{Cx+D}{x^2+\beta}$$

where A, B, C, D are constants.

4) Partial fraction of repeated quadratic factors.

$$\frac{f(x)}{(x^2+\alpha)^n} = \frac{A_1x+B_1}{x^2+\alpha} + \frac{A_2x+B_2}{(x^2+\alpha)^2} + \cdots + \frac{A_nx+B_n}{(x^2+\alpha)^n},$$

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where $A_1, B_1, A_2, B_2 \ldots$ are constants.

1) Using partial fraction find $L^{-1}\left\{\frac{1}{(s+2)(3s-5)}\right\}$.

ans: $\frac{1}{(s+2)(3s-5)} = \frac{A}{s+2} + \frac{B}{3s-5}$

$$A = \left.\frac{1}{3s-5}\right]_{s=-2} = \frac{1}{-11} = \frac{-1}{11}$$

$$B = \left.\frac{1}{s+2}\right]_{s=5/3} = \frac{1}{5/3+2} = \frac{1}{\frac{5+6}{3}} = \frac{3}{11}$$

Substituting the values of A & B we get

$$\frac{1}{(s+2)(3s-5)} = \frac{\frac{-1}{11}}{s+2} + \frac{\frac{3}{11}}{3s-5}$$

$$L^{-1}\left\{\frac{1}{(s+2)(3s-5)}\right\} = \frac{-1}{11} L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{3}{11} L^{-1}\left\{\frac{1}{3s-5}\right\}$$

$$= \frac{-1}{11} e^{-2t} + \frac{\cancel{3}}{11} L^{-1}\left\{\frac{1}{\cancel{3}(s-5/3)}\right\}$$

$$= \frac{-1}{11} e^{-2t} + \frac{1}{11} e^{5/3\, t}$$

//

2) Find $L^{-1}\left\{\frac{2s^3}{s^4-81}\right\}$.

$$\frac{2s^3}{s^4-81} = \frac{2s^3}{(s^2)^2-(9)^2} = \frac{2s^3}{(s^2-9)(s^2+9)} = \frac{2s^3}{(s-3)(s+3)(s^2+9)}$$

$$\frac{2s^3}{(s-3)(s+3)(s^2+9)} = \frac{A}{s-3} + \frac{B}{s+3} + \frac{Cs+D}{s^2+9} \quad \text{--- (1)}$$

$$A = \left.\frac{2s^3}{(s+3)(s^2+9)}\right]_{s=3} = \frac{\cancel{2}\times \cancel{27}^{3}}{\cancel{6}_{\cancel{2}}\times \cancel{18}_{2}} = \frac{1}{2}$$

$$B = \left.\frac{2s^3}{(s-3)(s^2+9)}\right]_{s=-3} = \frac{\cancel{2}\times(\cancel{-27})^{3}}{\cancel{-6}_{\cancel{2}}\times \cancel{18}_{2}} = \frac{1}{2}$$

eq^n (1) can be simplified as

$$\frac{2s^3}{\cancel{(s-3)(s+3)(s^2+9)}} = \frac{A(s+3)(s^2+9) + B(s-3)(s^2+9) + (Cs+D)(s^2-9)}{\cancel{(s-3)(s+3)(s^2+9)}}$$

$$2s^3 = A(s^3+9s+3s^2+27) + B(s^3+9s-3s^2-27) + C(s^3-9s) + D(s^2-9)$$

s^3: $2 = A+B+C$ $\quad$ s^2: $0 = 3A-3B+D$

$\Rightarrow 2 = 1+C$ $\quad$ $0 = \frac{3}{2} - \frac{3}{2} + D$

$\Rightarrow \boxed{C=1}$ $\quad$ $\boxed{D=0}$

Substituting A, B, C, D in eq^n (i) we get;

$$\frac{2s^3}{(s+3)(s-3)(s^2+9)} = \frac{1/2}{s-3} + \frac{1/2}{s+3} + \frac{s}{s^2+9}$$

$$L^{-1}\left\{\frac{2s^3}{(s+3)(s-3)(s^2+9)}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s-3}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{s+3}\right\} + L^{-1}\left\{\frac{1}{s^2+3^2}\right\}$$

$$= \frac{1}{2}e^{3t} + \frac{1}{2}e^{-3t} + \cos 3t$$

3) Prove that $L^{-1}\left\{\frac{1}{s^4+4a^4}\right\} = \frac{1}{4a^3}(\cosh at \sin at - \sinh at \cos at)$.

ans → To prove:-

$$\left\{\frac{1}{s^4+4a^4}\right\} = \frac{1}{4a^3} L\{\cosh at \sin at - \sinh at \cos at\}$$

RHS

$$\frac{1}{4a^3} L\{\cosh at \sin at - \sinh at \cos at\}$$

$$= \frac{1}{4a^3}\left[L\left\{\frac{e^{at}+e^{-at}}{2}\sin at\right\} - L\left\{\frac{e^{at}-e^{-at}}{2}\cos at\right\}\right]$$

$$= \frac{1}{8a^3}\left[L\{e^{at}\sin at\} + L\{e^{-at}\sin at\} - L\{e^{at}\cos at\} + L\{e^{-at}\cos at\}\right]$$

$$= \frac{1}{8a^3}\left[\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2} - \frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{8a^3}\left\{ \frac{a}{s^2+2a^2-2as} + \frac{a}{s^2+2a^2+2as} - \frac{s-a}{s^2+2a^2-2as} + \frac{s+a}{s^2+2a^2+2as} \right\}$$

$$= \frac{1}{8a^3}\left\{ \frac{2a-s}{s^2+2a^2-2as} + \frac{2a+s}{s^2+2a^2+2as} \right\}$$

$$= \frac{1}{8a^3}\left\{ \frac{(2a-s)(s^2+2a^2+2as)+(2a+s)(s^2+2a^2-2as)}{(s^2+2a^2-2as)(s^2+2a^2+2as)} \right\}$$

$$\cancel{= \frac{1}{8a^3}\left\{ 2as^2+4a^3+4a^2s-s^3 \right.}$$

$$= \frac{1}{8a^3}\left\{ \frac{2a(s^2+2a^2+\cancel{2as}+s^2+2a^2-\cancel{2as})+s(\cancel{s^2}+\cancel{2a^2}-2as-\cancel{s^2}-\cancel{2a^2}-2as)}{(s^2-2a^2)^2-4a^2s^2} \right\}$$

$$= \frac{1}{8a^3}\left\{ \frac{\cancel{4as^2}+8a^3-\cancel{4as^2}}{s^4+4a^4+\cancel{4a^2s^2}-\cancel{4a^2s^2}} \right\}$$

$$= \frac{1}{\cancel{8a^3}} \cdot \frac{\cancel{8a^3}}{s^4+4a^4}$$

$$= \frac{1}{s^4+4a^4} = LHS$$

[proved]

4) Prove that $L^{-1}\left\{ \frac{s}{s^4+4a^4} \right\} = \frac{1}{2a^2}(\sinh at \sin at)$

To prove $\frac{s}{s^4+4a^4} = \frac{1}{2a^2} L(\sinh at \sin at)$

RHS

$\frac{1}{2a^2}(\sinh at \sin at)$

$$= \frac{1}{2a^2}\left[L\left\{ \frac{e^{at}-e^{-at}}{2} \sin at \right\} \right]$$

$$= \frac{1}{4a^2}\left[L\{e^{at}\sin at\} - L\{e^{-at}\sin at\} \right]$$

$$= \frac{1}{4a^2}\left[\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{4a^2}\left[\frac{a\left((s+a)^2+a^2\right) - a\left((s-a)^2+a^2\right)}{\left((s-a)^2+a^2\right)\left((s+a)^2+a^2\right)}\right]$$

$$= \frac{1}{4a^{\cancel{2}}}\left[\frac{\cancel{s^2}+\cancel{2a^2}+2as-(\cancel{s^2}+\cancel{2a^2}-2as)}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right]$$

$$= \frac{1}{\cancel{4a}}\left[\frac{\cancel{4a}s}{(s^2+2a^2)^2-4a^2s^2}\right]$$

$$= \frac{s}{s^4+4a^4+\cancel{4a^2s^2}-\cancel{4a^2s^2}}$$

$$= \frac{s}{s^4+4a^4} = LHS$$

[proved]

Periodic Functions :

→ A function $f(x)$ is said to be a periodic function with period T, if $f(x+T) = f(x)$.

→ Here T is called the period of the function $f(x)$.

Note :

A function has infinite no. of periods if T is a period of $f(x)$, then nT, $n \in N$ is also another period of the function $f(x)$.

Fundamental period / Primitive period :

→ A function has infinite no. of periods but the smallest period is called the fundamental / primitive period.

e.g.: $\sin x$ is a periodic function with period 2π, the other periods of $\sin x$ are $4\pi, 6\pi, 8\pi, 10\pi, \dots,$ but 2π is the fundamental period.

Laplace transform of a periodic function :-

If $f(t)$ be a periodic function with period T, then

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\, dt$$

~~It is given that $f(t)$ is.~~

So, $f(t+nT) = f(t), \quad n \in N$ —— (1)

LHS

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

$$= \int_0^T e^{-st} f(t)\, dt + \int_T^{2T} e^{-st} f(t)\, dt + \int_{2T}^{3T} e^{-st} f(t)\, dt \dots$$

$t - T = x$
$dt = dx$
$t = T \quad x = 0$
$t = 2T, x = T$

$t - 2T = y$
$dt = dy$
$t = 2T \quad y = 0$
$t = 3T \quad y = T$

$$= \int_0^T e^{-st} f(t)\, dt + \int_0^T e^{-s(x+T)} f(x+T) dx + \int_0^T e^{-s(y+2T)} f(y+2T)\, dy + \dots$$

$$= \int_0^T e^{-st} f(t) dt + \int_0^T e^{-s(t+T)} f(t+T) dt + \int_0^T e^{-s(t+2T)} f(t+2T)\, dt + \dots$$

($\because$ D.I are independent on variable)

$$= \int_0^T e^{-st} f(t)\, dt + e^{-sT} \int_0^T e^{-st} f(t+T)\, dt + e^{-2sT} \int_0^T e^{-st} F(t+2T) dt + \dots$$

$$= \int_0^T e^{-st} f(t) dt + e^{-st} \int_0^T e^{-st} f(t) dt + e^{-2sT} \int_0^T e^{-st} \cdot f(t)\, dt + \dots$$

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$$= \int_0^T e^{-st} f(t)\,dt \left[1 + e^{-sT} + (e^{-sT})^2 + (e^{-sT})^3 + \dots\right]$$

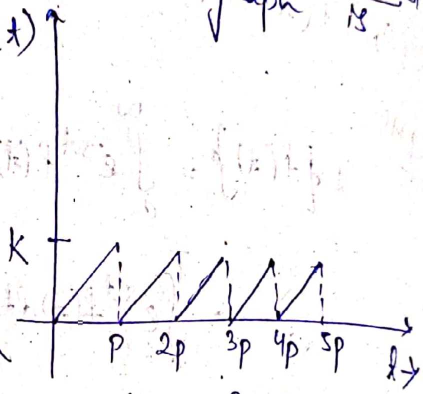
$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$= RHS$

[Proved]

[11/02/20]

1) Find $L\{f(t)\}$ with period P whose graph is given below.



ans Here $f(t)$ is a periodic function with period P

In the 1st interval the graph is a straight line with points $(0,0)$ & (p,k).

Eq^n of line is $y - 0 = \frac{k}{p}(x-0)$

$$f(t) = \begin{cases} \frac{kt}{p}, & 0 \le t \le P \\ 0, & \text{otherwise} \end{cases}$$

$$L\{f(t)\} = \frac{1}{1-e^{-sP}} \int_0^P e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-sP}} \int_0^P e^{-st} \frac{kt}{p}\,dt$$

$$= \frac{k}{p(1-e^{-sP})} \int_0^P t\, e^{-st}\,dt$$

$$= \frac{k}{p(1-e^{-sP})} \left[t \cdot \frac{e^{-st}}{-s} - 1 \cdot \frac{e^{-st}}{s^2}\right]_0^P$$

$$= \frac{K}{P(1-e^{-SP})}\left[\frac{-P}{S}e^{-SP} - \frac{e^{-SP}}{S^2} - \left(0 - \frac{1}{S^2}\right)\right]$$

$$= \frac{K}{P(1-e^{-SP})}\left[\frac{-P}{S}e^{-SP} - \frac{e^{-SP}}{S^2} + \frac{1}{S^2}\right]$$

Solution of system of differential equations :

→ System of differential eq^n means the no. of differential eq^n is more than 1.

(OR)

→ When more than one differential equations are given, then it is called a system of differential equation.

e.g. : $y_1' + 2y_2 + 2 = 0$

$y_1 + 5y_2' = \sin x$

1) Solve the system of differential eq^n.

$y_1' = -y_1 + y_2$ ——(1), $y_1(0) = 1$

$y_2' = -y_1 - y_2$ ——(2), $y_2(0) = 0$

ans: Let, y_1 & y_2 are the sol^n such that $L\{y_1\} = Y_1$ and $L\{y_2\} = Y_2$.

Now, $L\{y_1'\} = SL\{y_1\} - y_1(0)$

$= SY_1 - 1$

$L\{y_2'\} = SL\{y_2\} - y_2(0)$

$= SY_2 - 0$

$= SY_2$

Taking laplace on both sides of eq^n (1)

$L\{y_1'\} = L\{-y_1 + y_2\}$

$\Rightarrow SY_1 - 1 = -L\{y_1\} + L\{y_2\}$

$\Rightarrow sY_1 - 1 = -Y_1 + Y_2$

$\Rightarrow sY_1 + Y_1 = 1 + Y_2$

$\Rightarrow Y_1(s+1) = 1 + Y_2$

$\Rightarrow Y_1(s+1) - Y_2 = 1$ —— (1)

Taking laplace on both eq sides of 2nd eqn.

$L\{y_2'\} = -L\{y_1\} - L\{y_2\}$

$\Rightarrow sY_2 = -Y_1 - Y_2$

$\Rightarrow sY_2 + Y_2 + Y_1 = 0$

$\Rightarrow Y_2(s+1) + Y_1 = 0$ —— (2)

$(1) \times (s+1) \Rightarrow Y_1(s+1)^2 - \cancel{Y_2(s+1)} = s+1$

$Y_1 + \cancel{Y_2(s+1)} = 0$

$Y_1(1 + (s+1)^2) = s+1$

$\Rightarrow Y_1 = \dfrac{s+1}{(s+1)^2 + 1^2}$

$\Rightarrow L\{y_1\} = \dfrac{s+1}{(s+1)^2 + 1^2}$

$\Rightarrow y_1 = L^{-1}\left\{\dfrac{s+1}{(s+1)^2 + 1^2}\right\}$

$\Rightarrow \boxed{y_1 = \cos t e^{-t}}$

$Y_2 = Y_1(s+1) - 1$

$= \dfrac{(s+1)(s+1)}{(s+1)^2 + 1} - 1$

$$= \frac{(s+1)^2 - (s+1)^2 - 1}{(s+1)^2 + 1}$$

$$Y_2 = \frac{-1}{(s+1)^2+1}$$

$$L\{y_2\} = -\frac{1}{(s+1)^2+1^2}$$

$$y_2 = L^{-1}\left\{ \frac{1}{(s+1)^2+1^2} \right\}$$

$$\Rightarrow y_2 = -\sin t \, e^{-t}$$

$$\Rightarrow y_2 = -e^{-t}\sin t$$

Hence the solutions are ~~cot~~ $\cos t e^{-t}$ & $-e^{-t}\sin t$.

2) Q Solve the system of differential equation

$$y_1' + y_2 = 2\cos t \quad , y_1(0) = 0$$

$$y_1 + y_2' = 0 \quad , y_2(0) = 1 .$$

ans Let y_1 & y_2 are the solⁿ such that $L\{y_1\} = Y_1$ and $L\{y_2\} = Y_2$

Now, $L\{y_1'\} = sL\{y_1\} - y_1(0)$

$$= sY_1 - 0$$

$$= sY_1$$

$$L\{y_2'\} = sL\{y_2\} - y_2(0)$$

$$= sY_2 - 1$$

Taking laplace on both sides of eqⁿ (1)

$$L\{y_1'\} + L\{y_2\} = 2L\{\cos t\}$$

$$\Rightarrow sY_1 + Y_2 = 2L\{\cos t\}$$

$\Rightarrow sY_1 + Y_2 = 2\left\{\frac{s}{s^2+1^2}\right\}$

$\Rightarrow sY_1 + Y_2 = \frac{2s}{s^2+1^2}$ —— 1

Taking laplace on both sides of 2nd eq^n

$L\{y_1\} + L\{y_2'\} = 0$

$\Rightarrow Y_1 + sY_2 - 1 = 0$

$\Rightarrow sY_2 + Y_1 = 1$ —— 2

$1 \times s \quad s^2Y_1 + \cancel{sY_2} = \frac{2s^2}{s^2+1}$

$\quad Y_1 + \cancel{sY_2} = 1$

$\quad (-) \quad (-) \quad (-)$

$Y_1(s^2-1) = \frac{2s^2}{s^2+1} - 1$

$\Rightarrow Y_1\cancel{(s^2-1)} = \frac{\cancel{2s^2 - s^2 - 1}}{(s^2+1)}$

$\Rightarrow Y_1 = \frac{1}{s^2+1}$

$\Rightarrow L\{y_1\} = \frac{1}{s^2+1}$

$\Rightarrow y_1 = \sin t$

$sY_2 + \frac{1}{s^2+1} = 1$

$\Rightarrow sY_2 = 1 - \frac{1}{s^2+1}$

$\Rightarrow sY_2 = \frac{s^2+1-1}{s^2+1}$

$\Rightarrow Y_2 = \frac{s}{s^2+1}$

$\Rightarrow L\{y_2\} = \frac{s}{s^2+1}$

$\Rightarrow y_2 = \cos t$

Hence, the solns are $\sin t$ and $\cos t$.

3) Solve the system of eqns

$$2y_1' - y_2' - y_3' = 0 \qquad y_1(0) = 0$$
$$y_1' + y_2' = 4t+2 \qquad y_2(0) = 0$$
$$y_2' + y_3 = t^2+2 \qquad y_3(0) = 0$$

Let y_1, y_2 & y_3 are the solns such that $L\{y_1\} = Y_1$, $L\{y_2\} = Y_2$ & $L\{y_3\} = Y_3$.

$L\{y_1'\} = sL\{y_1\} - y_1(0)$

$= sY_1$

$L\{y_2'\} = sL\{y_2\} - y_2(0)$

$= sY_2$

$L\{y_3'\} = sL\{y_3\} - y_3(0)$

$= sY_3$

Taking laplace on both sides of 1st eqn,

$2L\{y_1'\} - L\{y_2'\} - L\{y_3'\} = 0$

$\Rightarrow 2sY_1 - sY_2 - sY_3 = 0$

$\Rightarrow 2Y_1 - Y_2 - Y_3 = 0$ —— (1)

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Taking laplace on both sides of eq^n (2)

$$sY_1 + sY_2 = \frac{4}{s^2} + \frac{2}{s} \quad \text{——— (2)}$$

Taking laplace on both sides of eq^n (3)

$$sY_2 + Y_3 = \frac{2}{s^3} + \frac{2}{s} \quad \text{——— (3)}$$

$$Y_1 + Y_2 = \frac{4}{s^3} + \frac{2}{s^2}$$

$$\Rightarrow Y_1 = \frac{4}{s^3} + \frac{2}{s^2} - Y_2$$

$$sY_2 + Y_3 = \frac{2}{s^3} + \frac{2}{s}$$

$$\Rightarrow Y_3 = \frac{2}{s^3} + \frac{2}{s} - sY_2$$

Substituting these in eq^n (1), we get

$$2\left(\frac{4}{s^3} + \frac{2}{s^2} - Y_2\right) - Y_2 - \frac{2}{s^3} - \frac{2}{s} + sY_2 = 0$$

$$\Rightarrow \frac{8}{s^3} + \frac{4}{s^2} - 2Y_2 - Y_2 - \frac{2}{s^3} - \frac{2}{s} + sY_2 = 0$$

$$\Rightarrow \frac{6}{s^3} + \frac{2}{s^2} = Y_2(3-s)$$

$$\Rightarrow Y_2 = \frac{6}{s^3(s-3)} - \frac{2}{s^2(s-3)}$$

$$\Rightarrow y_2 = 6L^{-1}\left\{\frac{1}{s^3(s-3)}\right\} - 2L^{-1}\left\{\frac{1}{s^2(s-3)}\right\}$$

$$F(s) = \frac{1}{s-3}$$

$$f(t) = e^{3t}$$

$$L^{-1}\left\{\frac{1}{s(s-3)}\right\} = \int_0^t e^{3t}\,dt$$

$$= \left(\frac{e^{3t}}{3}\right)_0^t$$

$$= \frac{e^{3t}-1}{3}$$

$$L^{-1}\left\{\frac{1}{s^2(s-3)}\right\} = \int_0^t \frac{e^{3t}-1}{3}\,dt$$

$$= \frac{1}{3}\left[\frac{e^{3t}}{3} - t\right]_0^t$$

$$= \frac{1}{3}\left(\frac{e^{3t}}{3} - t - \frac{1}{3}\right)$$

$$= \frac{e^{3t}-3t-1}{9}$$

$$L^{-1}\left\{\frac{1}{s^3(s-3)}\right\} = \int_0^t \frac{e^{3t}-3t-1}{9}\,dt$$

$$= \frac{1}{9}\left(\frac{e^{3t}}{3} - \frac{3t^2}{2} - t\right)_0^t$$

$$= \frac{1}{9}\left(\frac{e^{3t}}{3} - \frac{3t^2}{2} - t - \frac{1}{3}\right)$$

$$y_2 = \frac{6}{9}\left(\frac{e^{3t}}{3} - \frac{3t^2}{2} - t - \frac{1}{3}\right) - \frac{2}{9}\left(e^{3t} - 3t - 1\right)$$