Module -3

Math-II

Experiments, Outcomes, Events

An experiment is a process of measurement or observation, in a laboratory, in a factory, on the street, in nature, or wherever; so "experiment" is used in a rather general sense.. A trials a single performance of an experiment. Its result is called an outcome or a sample point. n trials then give a sample of size consisting of sample points. The sample spaces of an experiment is the set of all possible outcomes.

Find the sample spaces in the following experiment.

Q)

Drawing 3 screws from a lot of right-handed and left-handed screws

A)

Let L denote "left-handed screw was drawn". Let R denote "right-handed screw was drawn".

$$S = \{(L, L, L), (L, L, R), (L, R, L), (L, R, R), (R, L, L), (R, L, R), (R, R, L), (R, R, R)\}$$

Q)

Rolling a die until the first *Six* appears

A) Let A= six occurs , N =Six does not occurs ,So S={A ,NA,NNA,NNNA,….}

Q)

In rolling 3 dice, are the events *A: Sum divisible by* 3 and *B: Sum divisible by* 5 mutually exclusive?

In rolling 3 dice the Events are as follows:

A: Sum divisible by 3.
B: Sum divisible by 5.

Determine whether the events are manually Exclusive or not.

Two events are said to be mutually Exclusive if and only if they can't Occur together at same time. For Example while tossing a Coin the Occurance of both head and tail is impossible at one toss.

The Sample Space for the Event will be:
$S = \{(1,1,1), (1,1,2), \ldots, (6,6,6)\}$

So there will be 216 pairs in the Sample space. But out of which One pair will be (5, 5, 5) whose Sum
$5+5+5 = 15$ is divisible by both 3 and 5.
Therefore the event
A: Sum divisible by 3, B: Sum divisble by 5 are
not mutually Exclusive.

Probability

First Definition of Probability

If the sample space S of an experiment consists of finitely many outcomes (points) that are equally likely, then the probability $P(A)$ of an event A is

$$P(A) = \frac{\text{Number of points in } A}{\text{Number of points in } S}. \tag{1}$$

Axioms of probability:

Given a sample space S, with each event A of S (subset of S) there is associated a number $P(A)$, called the **probability** of A, such that the following **axioms of probability** are satisfied.

1. For every A in S,

$$0 \leqq P(A) \leqq 1. \tag{4}$$

2. The entire sample space S has the probability

$$P(S) = 1. \tag{5}$$

3. For mutually exclusive events A and B ($A \cap B = \varnothing$; see Sec. 24.2),

$$P(A \cup B) = P(A) + P(B) \qquad (A \cap B = \varnothing). \tag{6}$$

Complementation Rule

For an event A and its complement A^c in a sample space S,

$$P(A^c) = 1 - P(A). \tag{7}$$

Example:

Five coins are tossed simultaneously. Find the probability of the event A: *At least one head turns up.* Assume that the coins are fair.

Solution. Since each coin can turn up heads or tails, the sample space consists of $2^5 = 32$ outcomes. Since the coins are fair, we may assign the same probability (1/32) to each outcome. Then the event A^c (*No heads turn up*) consists of only 1 outcome. Hence $P(A^c) = 1/32$, and the answer is $P(A) = 1 - P(A^c) = 31/32$. ■

Addition Rule for Arbitrary Events

For events A and B in a sample space,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{9}$$

Example:

In tossing a fair die, what is the probability of getting an odd number or a number less than 4?

Solution. Let A be the event "*Odd number*" and B the event "*Number less than* 4." Then Theorem 3 gives the answer

$$P(A \cup B) = \tfrac{3}{6} + \tfrac{3}{6} - \tfrac{2}{6} = \tfrac{2}{3}$$

because $A \cap B$ = "*Odd number less than* 4" = $\{1, 3\}$.

Conditional Probability:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \qquad [P(A) \neq 0]. \tag{11}$$

Similarly, the *conditional probability of A given B* is

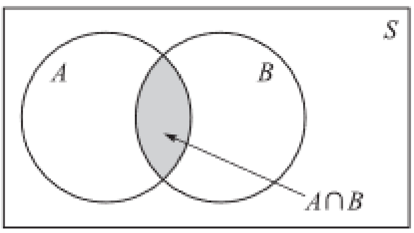
$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad [P(B) \neq 0]. \tag{12}$$

Multiplication Rule

If A and B are events in a sample space S and $P(A) \neq 0$, $P(B) \neq 0$, *then*

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B). \tag{13}$$

$$P(A \cap B) = P(A)\,P(B\,|\,A)$$



Bayes Rule:

Now

$$P(A)P(B|A) = P(B)P(A|B).$$

So

$$P(A|B) = \frac{P(A)\,P(B|A)}{P(B)}$$

This is called Bayes Formula.

This can be written as

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(\text{not } A)P(B|\text{not } A)}$$

Because P(B) = $P(A)P(B|A) + P(\text{not } A)P(B|\text{not } A)$

In general ,

$$P(A1|B) = \frac{P(A1)P(B|A1)}{P(A1)P(B|A1) + P(A2)P(B|A2) + P(A3)P(B|A3) + \ldots\text{etc}}$$

Example:

Example: The Art Competition has entries from three painters: Pam, Pia and Pablo

- Pam put in 15 paintings, 4% of her works have won First Prize.
- Pia put in 5 paintings, 6% of her works have won First Prize.
- Pablo put in 10 paintings, 3% of his works have won First Prize.

What is the chance that Pam will win First Prize?

$$P(Pam|First) = \frac{P(Pam)P(First|Pam)}{P(Pam)P(First|Pam) + P(Pia)P(First|Pia) + P(Pablo)P(First|Pablo)}$$

Put in the values:

$$P(Pam|First) = \frac{(15/30) \times 4\%}{(15/30) \times 4\% + (5/30) \times 6\% + (10/30) \times 3\%}$$

$= 50\%$

Example:

In producing screws, let A mean "screw too slim" and B "screw too short." Let $P(A) = 0.1$ and let the conditional probability that a slim screw is also too short be $P(B|A) = 0.2$. What is the probability that a screw that we pick randomly from the lot produced will be both too slim and too short?

Solution. $P(A \cap B) = P(A)P(B|A) = 0.1 \cdot 0.2 = 0.02 = 2\%$, [redacted]

Sampling. Our next example has to do with randomly drawing objects, *one at a time,* from a given set of objects. This is called **sampling from a population**, and there are two ways of sampling, as follows.

1. In **sampling with replacement,** the object that was drawn at random is placed back to the given set and the set is mixed thoroughly. Then we draw the next object at random.
2. In **sampling without replacement** the object that was drawn is put aside.

A box contains 10 screws, three of which are defective. Two screws are drawn at random. Find the probability that neither of the two screws is defective.

Solution. We consider the events

A: First drawn screw nondefective.

B: Second drawn screw nondefective.

Clearly, $P(A) = \frac{7}{10}$ because 7 of the 10 screws are nondefective and we sample at random, so that each screw has the same probability ($\frac{1}{10}$) of being picked. If we sample with replacement, the situation before the second drawing is the same as at the beginning, and $P(B) = \frac{7}{10}$. The events are independent, and the answer is

$$P(A \cap B) = P(A)P(B) = 0.7 \cdot 0.7 = 0.49 = 49\%.$$

If we sample without replacement, then $P(A) = \frac{7}{10}$, as before. If A has occurred, then there are 9 screws left in the box, 3 of which are defective. Thus $P(B|A) = \frac{6}{9} = \frac{2}{3}$, and Theorem 4 yields the answer

$$P(A \cap B) = \frac{7}{10} \cdot \frac{2}{3} = 47\%.$$

Is it intuitively clear that this value must be smaller than the preceding one?

Q)

In rolling 3 fair dice, what is the probability of obtaining a sum not greater than 16?

A)

Q1 Since each dice can give 6 outcomes (1 to 6).

~~Total~~ The sample space contains $6 \times 6 \times 6 = 216$ outcomes.

Since, the dice are fair, we may assign the same probability ($1/216$) to each outcome.

Event A^c (sum greater than 16) $= \{(6,6,6), (6,6,5), (5,6,6), (6,5,6)\}$

$$P(A^c) = (4)(1/216)$$

$$P(A) = 1 - P(A^c) = 1 - \frac{\cancel{4}\,2}{\cancel{216}\,108} = 1 - \frac{1}{54}$$

$$\boxed{P(A) = \frac{53}{54}}$$

Q)

Three screws are drawn at random from a lot of 100 screws, 10 of which are defective. Find the probability of the event that all 3 screws drawn are nondefective, assuming that we draw (**a**) with replacement, (**b**) without replacement.

A)

Replacement case:

Let

$$A_i = \{\text{on the } i\text{th draw, we have drawn a nondefective screw}\}$$

Notice that

$$P(A_i) = \frac{\text{number of nondefective screws}}{\text{number of screws}} = \frac{90}{100} = \frac{9}{10}$$

since we are drawing with replacement.

We must find

$$P(\text{we have drawn 3 nondefective screws in 3 draws}) = P(A_1 \cap A_2 \cap A_3)$$

These events are independent (we are drawing with replacement), and so

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3) = \frac{9}{10} \cdot \frac{9}{10} \cdot \frac{9}{10} = \frac{729}{1000}$$

Therefore,

$$\boxed{P(\text{we have drawn 3 nondefective screws in 3 draws}) = \frac{729}{1000}}$$

Without replacement case:

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) \tag{1}$$

First of all,

$$P(A_1) = \frac{\text{number of nondefective screws}}{\text{number of screws}} = \frac{90}{100} = \frac{9}{10}$$

Now notice that

$$P(A_2 \mid A_1) = \frac{89}{99}$$

This is because in the first draw we have drawn a nondefective screw, and we do not return it. Therefore, there is a total of 99 screws left, out of which 89 are defective.

Similarly,

$$P(A_3 \mid A_1 \cap A_2) = \frac{88}{98} = \frac{44}{49}$$

since after the first two draws, there are 98 screws left, out of which 88 are nondefective.

Finally, we use (1) to conclude that

$$P(A_1 \cap A_2 \cap A_3) = \frac{9}{10} \cdot \frac{89}{99} \cdot \frac{44}{49} = \frac{35244}{48510} = \frac{178}{245}$$

Thus,

$$\boxed{P(\text{we have drawn 3 nondefective screws in 3 draws}) = \frac{178}{245}}$$

Q)

Under what conditions will it make *practically* no difference whether we sample with or without replacement?

A)

7) when we are drawing small samples from a large set of objects, the replacement or non replacement doesn't matter practically.

Eg: There are 10,000 screws with 10 defective screws in it

So if we are drawing one screw at a time, the probability of a non defective screw drawn three times is

(a) with replacement $\frac{9990}{10000} \times \frac{9990}{10000} \times \frac{9990}{10000} = 0.9970029$

(b) without replacement

$$= \frac{9990}{10,000} \times \frac{9989}{9999} \times \frac{9988}{9998} = .9970026$$

Both are approximately equal.

Q)

If a certain kind of tire has a life exceeding 40,000 miles with probability 0.90, what is the probability that a set of these tires on a car will last longer than 40,000 miles?

A)

A set of tires consists of 4 tires. Let

$$A_i = \{\text{the } i\text{th tire lasted longer than } 40{,}000 \text{ miles}\}$$

The set of tires lasted longer than 40,000 miles if every of the 4 tires lasted longer than 40,000 miles. Thus, we need to find

$$P(A_1 \cap A_2 \cap A_3 \cap A_4)$$

These events are independent, so

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1)P(A_2)P(A_3)P(A_4)$$

Now we will use that

$$P(A_i) = 0.9,$$

which is stated in the problem. Therefore,

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = 0.9^4 = 0.6561$$

Thus,

$$\boxed{P(\text{the set of tires lasted longer than } 40{,}000 \text{ miles}) = 0.6561}$$

Q)

A pressure control apparatus contains 3 electronic tubes. The apparatus will not work unless all tubes are operative. If the probability of failure of each tube during some interval of time is 0.04, what is the corresponding probability of failure of the apparatus?

A)

Let F denote the event of failure of the instrument, and let W denote the event that the instrument has worked during the whole time period.

Let F_i denote the event of failure of the ith tube, and let W_i denote the event that the ith tube has worked during the whole time period. Then $W_i = F_i^c$, so, using [redacted] (**Complementation Rule**), we get

$$P(W_i) = P(F_i^c) = 1 - P(F_i) = 1 - 0.04 = 0.96$$

Now, $W = W_1 \cap W_2 \cap W_3$ (all 3 tubes must work for the instrument to work), and since W_1, W_2, W_3 are independent, we have

$$P(W) = P(W_1 \cap W_2 \cap W_3) = P(W_1)P(W_2)P(W_3) = 0.96^3$$

Finally, $F = W^c$, so, using [redacted] (**Complementation Rule**) again, we get

$$P(F) = P(W^c) = 1 - P(W) = 1 - 0.96^3$$

Therefore,

$$\boxed{P(F) = 1 - 0.96^3 = 0.115264}$$

Permutations and Combinations

Permutations

(a) ***Different things.*** *The number of permutations of n different things taken all at a time is*

(1) $$n! = 1 \cdot 2 \cdot 3 \cdots n \qquad \text{(read "}n\text{ factorial")}.$$

(b) ***Classes of equal things.*** *If n given things can be divided into c classes of alike things differing from class to class, then the number of permutations of these things taken all at a time is*

(2) $$\frac{n!}{n_1!\,n_2!\cdots n_c!} \qquad (n_1 + n_2 + \cdots + n_c = n)$$

Where n_j is the number of things in the jth class.

Example:

If a box contains 6 red and 4 blue balls, the probability of drawing first the red and then the blue balls is

$$P = 6!4!/10! = 1/210 \approx 0.5\%.$$

Combinations

The number of different combinations of n different things taken, k at a time, without repetitions, is

(4a) $$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdots k},$$

and the number of those combinations with repetitions is

(4b) $$\binom{n+k-1}{k}.$$

Example:

The number of samples of five lightbulbs that can be selected from a lot of 500 bulbs is [redacted]

$$\binom{500}{5} = \frac{500!}{5!495!} = \frac{500 \cdot 499 \cdot 498 \cdot 497 \cdot 496}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 255{,}244{,}687{,}600.$$

Q)

In how many different ways can we select a committee consisting of 3 engineers, 2 physicists, and 2 computer scientists from 10 engineers, 5 physicists, and 6 computer scientists? [redacted]

A)

We choose a group of 3 engineers from 10 of them and the order in which we choose them does not matter. So, there are

$$\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120$$

ways to choose them. Similarly, there are

$$\binom{5}{2} = \frac{5 \cdot 4}{1 \cdot 2} = 10$$

ways to choose physicists and

$$\binom{6}{2} = \frac{6 \cdot 5}{1 \cdot 2} = 15$$

ways to choose computer scientists.

Total number is the product of these three numbers (the choice a group of one "type" of people is independent of the choice of other groups):

$$120 \cdot 10 \cdot 15 = 18,000$$

Random Variables. Probability Distributions

Random Variable

A random variable X is a function defined on the sample space S of an experiment. Its values are real numbers. For every number a the probability

$$P(X = a)$$

A *Random Variable* is a **set of possible values** from a random experiment.

Notations:

The probability that X takes on the value x, P(X=x).

may denote P(X=x) by p(x) or $p_X(x)$.

$P_X(i)=P_i$

Or,

$f(x) = P(X = x)$ and $F(x) = P(X \leqq x)$

Here f(x) is called probability function and F(x) is called cumulative probability distribution.

Note:

$$P(a < X \leqq b) = F(b) - F(a) =$$

Explanation: P(a<x<=b)= P(x<=b) – p(x<=a) = F(b)- F(a)

Example: Coin Flips

When a coin is flipped 3 times, the sample space will be

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}.

If *X* is the number of heads, then *X* is a random variable whose probability distribution is as follows:

Possible Events	*x*	*P*(*x*)
TTT	0	1/8
HTT, THT, TTH	1	3/8
HHT, HTH, THH	2	3/8
HHH	3	1/8
Total		1

Another Example:

Example: Two dice are tossed.

The Random Variable is **X** = "The sum of the scores on the two dice".

Let's make a table of all possible values:

		1st Die					
		1	**2**	**3**	**4**	**5**	**6**
2nd Die	**1**	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

There are **6 × 6 = 36** possible outcomes, and the Sample Space (which is the sum of the scores on the two dice) is {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

Let's count how often each value occurs, and work out the probabilities:

- 2 occurs just once, so P(X = 2) = 1/36
- 3 occurs twice, so P(X = 3) = 2/36 = 1/18
- 4 occurs three times, so P(X = 4) = 3/36 = 1/12
- 5 occurs four times, so P(X = 5) = 4/36 = 1/9
- 6 occurs five times, so P(X = 6) = 5/36
- 7 occurs six times, so P(X = 7) = 6/36 = 1/6
- 8 occurs five times, so P(X = 8) = 5/36
- 9 occurs four times, so P(X = 9) = 4/36 = 1/9
- 10 occurs three times, so P(X = 10) = 3/36 = 1/12
- 11 occurs twice, so P(X = 11) = 2/36 = 1/18
- 12 occurs just once, so P(X = 12) = 1/36

Example (continued) What is the probability that the sum of the scores is 5, 6, 7 or 8?

In other words: What is **P(5 ≤ X ≤ 8)**?

$$
\begin{aligned}
P(5 \le X \le 8) &= P(X=5) + P(X=6) + P(X=7) + P(X=8) \\
&= (4+5+6+5)/36 \\
&= 20/36 \\
&= 5/9
\end{aligned}
$$

Example: Find F(6)?

F(6)=P(x<=6)=P(x=6)+P(x=5)+….+P(x=0)=5/36+4/36+3/36+2/36+1/36=15/36

Example: Find P(2<x<= 6) = F(6)- F(2) =15/35-1/36=14/36

Random Variables are two types:

⌘ Discrete Random Variables:

a random variable that can assume only a countable number of values. The value of a discrete random variable comes from counting.

⌘ Continuous Random Variable:

random variables that can assume any value on a continuum. Measurement is required to determine the value for a continuous random variable.

- A discrete random variable is one which may take on only a countable number of distinct values such as 0, 1, 2, 3, 4,….
- Discrete random variables are usually (but not necessarily) counts.
- **Examples:**
  - number of children in a family
  - the Friday night attendance at a cinema
  - the number of patients a doctor sees in one day
  - the number of defective light bulbs in a box of ten
  - the number of “heads” flipped in 3 trials

- A continuous random variable is one which takes an **infinite** number of possible values.
- Continuous random variables are usually measurements.
- Examples:
  - height
  - weight
  - the amount of sugar in an orange
  - the time required to run a mile.

Probability Distribution

A mathematical description of Probabilities for a random variable.

Properties of Discrete probability distribution:

. Each probability $P(x)$ must be between 0 and 1: $0 \leq P(x) \leq 1$.

. The sum of all the probabilities is 1: $\Sigma P(x) = 1$.

Here P(x) or f(x) is called probability mass function

Properties of Continuous probability distribution:

Continuous random variables are ***simpler than discrete ones*** with respect to intervals.

i) $\boldsymbol{F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx}$ **,so** $\boldsymbol{f(x) = F'(x)}$

ii) $\boldsymbol{P(a < x < b) = P(a < x \leq b) = P(a \leq x < b)}$

$\boldsymbol{= P(a \leq x \leq b) = \int_{a}^{b} f(x)dx \,.\, where\ a, b\ are\ any\ real\ numbers}$

iii) $\boldsymbol{\int_{-\infty}^{\infty} f(x)dx = 1}$ **(very important information)**

Here f(x) is called probability density function.

Question 1:

Find and

Graph the probability function $f(x) = kx^2$ ($x = 1, 2, 3, 4, 5$; k suitable) and the distribution function.

Ans:

mass

For this to be a [redacted] function, we must have

$$\sum_{x} f(x) = 1$$

(the sum is taken for all values of x such that $f(x) > 0$, since we know that there are only finitely many of them).
So,

$$1 = \sum_{i=1}^{5} f(i) = \sum_{i=1}^{5} ki^2 = k + 4k + 9k + 16k + 25k = 55k$$

Therefore,

$$\boxed{k = \frac{1}{55}}$$

So, the [redacted] function is

$$f(x) = \frac{x^2}{55}, \quad x = 1, 2, 3, 4, 5$$

Distribution Function means cumulative distribution function.

$$F(x) = P(X \le x) = \sum_{i=0}^{x} P(X = i) = \sum_{i=0}^{x} f(i)$$

$$F(1) = f(1) = \frac{1}{55}$$

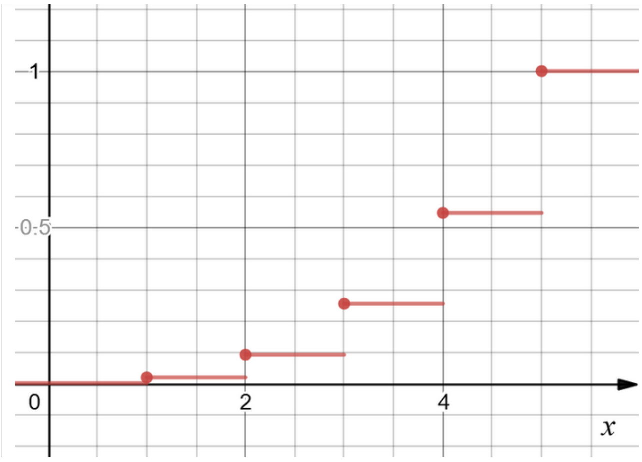
$$F(2) = f(1) + f(2) = \frac{1}{55} + \frac{4}{55} = \frac{5}{55}$$

$$F(3) = f(1) + f(2) + f(3) = \frac{1}{55} + \frac{4}{55} + \frac{9}{55} = \frac{14}{55}$$

$$F(4) = f(1) + f(2) + f(3) + f(4) = \frac{1}{55} + \frac{4}{55} + \frac{9}{55} + \frac{16}{55} = \frac{30}{55}$$

$$F(5) = f(1) + f(2) + f(3) + f(4) + f(5) = \frac{1}{55} + \frac{4}{55} + \frac{9}{55} + \frac{16}{55} + \frac{25}{55} = \frac{55}{55} = 1$$

Its graph always a step function. See below



Question 2)

Graph the density function $f(x) = kx^2$ ($0 \leqq x \leqq 5$; k suitable) and the distribution function.

Ans:

We must have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Since $f(x) = 0$ when $x < 0$ or $x > 5$,

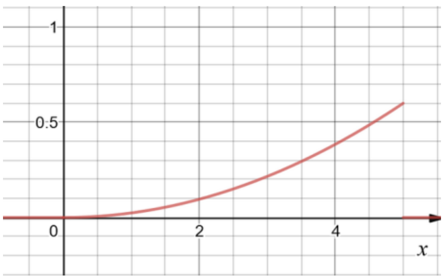
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^5 f(x)\,dx = \int_0^5 kx^2\,dx = \frac{kx^3}{3}\bigg|_0^5 = \frac{5^3k}{3} - \frac{0^3k}{3} = \frac{125}{3}k$$

Thus,

$$\frac{125}{3}k = 1 \implies \boxed{k = \frac{3}{125}}$$

This means that

$$f(x) = \begin{cases} \frac{3}{125}x^2, & 0 \leqslant x \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$$



The distribution function is given by

$$F(x) = \int_{-\infty}^{x} f(v)\, dv$$

If $x < 0$, then $f(v) = 0$ on $v \leqslant x$, so

$$F(x) = \int_{-\infty}^{x} 0\, dv = 0$$

If $0 \leqslant x \leqslant 5$, then

$$F(x) = \underbrace{\int_{-\infty}^{0} \underbrace{f(v)}_{=0}\, dv}_{=0} + \int_{0}^{x} f(v)\, dv = \int_{0}^{x} \frac{3}{125} v^2\, dv = \frac{1}{125} v^3 \bigg|_{0}^{x} = \frac{1}{125} x^3$$

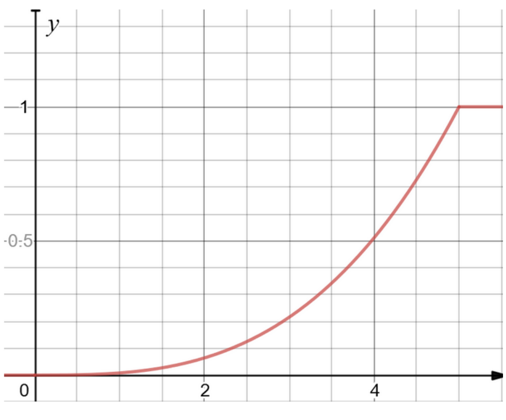
If $x > 5$, then

$$F(x) = \underbrace{\int_{-\infty}^{0} \underbrace{f(v)}_{=0}\, dv}_{=0} + \underbrace{\int_{0}^{5} f(v)\, dv}_{=1} + \underbrace{\int_{5}^{x} \underbrace{f(v)}_{=0}\, dv}_{=0} = 1$$

Therefore,

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{1}{125} x^3 & 0 \leqslant x \leqslant 5 \\ 1 & x > 5 \end{cases}$$

Graph this function:



Question 3)

If density function

is $f(x) = k = \text{const}$ if $-2 \leqq x \leqq 2$ and 0 else-
e. Find $P(0 \leqq X \leqq 2)$.

Ans:

We must have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Since $f(x) = 0$ when $x < -2$ or $x > 2$,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-2}^{2} f(x)\,dx = \int_{-2}^{2} k\,dx = kx\Big|_{-2}^{2} = 4k$$

Thus,

$$4k = 1 \implies \boxed{k = \frac{1}{4}}$$

This means that

$$f(x) = \begin{cases} \frac{1}{4}, & -2 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

The distribution function is given by

$$F(x) = \int_{-\infty}^{x} f(v)\,dv$$

If $x < -2$, then $f(v) = 0$ on $v \leqslant x$, so

$$F(x) = \int_{-\infty}^{x} 0\;dv = 0$$

If $-2 \leqslant x \leqslant 2$, then

$$F(x) = \underbrace{\int_{-\infty}^{-2} \underbrace{f(v)}_{=0}\;dv}_{=0} + \int_{-2}^{x} f(v)\;dv = \int_{-2}^{x} \frac{1}{4}\,dv = \frac{1}{4}v\Big|_{-2}^{x} = \frac{1}{4}(x+2) = \frac{x}{4} + \frac{1}{2}$$

If $x > 2$, then

$$F(x) = \underbrace{\int_{-\infty}^{-2} \underbrace{f(v)}_{=0}\;dv}_{=0} + \underbrace{\int_{-2}^{2} f(v)\,dv}_{=1} + \underbrace{\int_{2}^{x} \underbrace{f(v)}_{=0}\;dv}_{=0} = 1$$

Therefore,

$$F(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{4}x + \frac{1}{2} & -2 \leqslant x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

Question 4)

[redacted] when $f(-2) = f(2) = \frac{1}{8}$, $f(-1) = f(1) = \frac{3}{8}$. Can f have further positive values?

Ans:

Since

$$\sum_x f(x) = f(-2) + f(-1) + f(1) + f(2) = 1,$$

f cannot have further positive values (otherwise, we would have $\sum_x f(x) > 1$). Also, $\sum_x f(x)$ makes sense since $f(x) > 0$ only for finitely many x. [redacted]

Question 5)

[redacted] the distribution function $F(x) = 1 - e^{-3x}$ if $x > 0$, $F(x) = 0$ if $x \leqq 0$, and the density $f(x)$. Find x such that $F(x) = 0.9$.

Ans:

Now we must find $f(x)$. Recall that

$$f(x) = F'(x)$$

Also,

$$\frac{d}{dx}(1 - e^{-3x}) = 3e^{-3x},$$

so

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0 & x \leqslant 0 \end{cases}$$

Now

$$\begin{aligned} 0.9 = F(x) = 1 - e^{-3x} &\implies 1 - e^{-3x} = 0.9 \\ &\implies e^{-3x} = 0.1 \\ &\implies -3x = \ln 0.1 \\ &\implies \boxed{x = \frac{\ln 0.1}{-3} \approx 0.7675} \end{aligned}$$

Question 6)

Let X [millimeters] be the thickness of washers. Assume that X has the density $f(x) = kx$ if $0.9 < x < 1.1$ and 0 otherwise. Find k. What is the probability that a washer will have thickness between 0.95 mm and 1.05 mm?

Ans:

We must have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Since $f(x) = 0$ when $x \leqslant 0.9$ or $x \geqslant 1.1$, the integral can be simplified:

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{0.9}^{1.1} f(x)\,dx \overset{(*)}{=} \int_{0.9}^{1.1} kx\,dx = \left.\frac{kx^2}{2}\right|_{0.9}^{1.1} = \frac{k}{2}(1.1^2 - 0.9^2) = 0.2k$$

In $(*)$ we used that $f(x) = kx$ on $0.9 < x < 1.1$. Therefore,

$$0.2k = 1 \implies \boxed{k = 5}$$

Therefore, the density is given by

$$f(x) = \begin{cases} 5x & 0.9 < x < 1.1 \\ 0 & \text{otherwise} \end{cases}$$

Now we need to find

$$P(0.95 < X < 1.05) = P(0.95 < X \leqslant 1.05)$$

(the equality holds because X is continuous). Furthermore, [redacted] in the book,

$$P(0.95 < X \leqslant 1.05) = \int_{0.95}^{1.05} 5x\,dx = \left.\frac{5x^2}{2}\right|_{0.95}^{1.05} = \frac{5}{2}(1.05^2 - 0.95^2) = 0.5$$

Question 7)

If the diameter X of axles has the density $f(x) = k$ if $119.9 \leqq x \leqq 120.1$ and 0 otherwise, how many defectives will a lot of 500 axles approximately contain if defectives are axles slimmer than 119.91 or thicker than 120.09?

Ans:

We must have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Since $f(x) = 0$ when $x < 119.9$ or $x > 120.1$, the integral can be simplified:

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{119.9}^{120.1} f(x)\,dx \overset{(*)}{=} \int_{119.9}^{120.1} k\,dx = kx\Big|_{119.9}^{120.1} = k(120.1 - 119.9) = 0.2k$$

In $(*)$ we used that $f(x) = k$ on $119.9 < x < 120.1$. Therefore,

$$0.2k = 1 \implies \boxed{k = 5}$$

Therefore, the density is given by

$$f(x) = \begin{cases} 5 & 119.9 < x < 120.1 \\ 0 & \text{otherwise} \end{cases}$$

Now we will find $P(X < 119.91 \text{ or } X > 120.09)$. First of all,

$$P(X < 119.91) = P(X \leqslant 119.91)$$

since X is continuous, and

$$P(X \leqslant 119.91) = \int_{-\infty}^{119.91} f(x)\,dx = \int_{119.9}^{119.91} 5\,dx = 5x\Big|_{119.9}^{119.91} = 0.05$$

Furthermore, $X > 120.09$ is a complementary event of $X \leqslant 120.09$, so

$$P(X > 120.09) = 1 - P(X \leqslant 120.09)$$

Now,

$$P(X \leqslant 120.09) = \int_{-\infty}^{120.09} f(x)\,dx = \int_{119.9}^{120.09} 5\,dx = 5x\Big|_{119.9}^{120.09} = 0.95$$

Thus,

$$P(X > 120.09) = 1 - 0.95 = 0.05$$

Finally, $X < 119.91$ and $X > 120.09$ are mutually exclusive, so

$$P(X < 119.91 \text{ or } X > 120.09) = \underbrace{P(X < 119.91)}_{=P(X \leqslant 119.91)} + P(120.09) = 0.1$$

So, among 500 axles, there are approximately

$$500 \cdot P(X < 119.91 \text{ or } X > 120.09) = 500 \cdot 0.1 = 50$$

defectives.

Mean and Variance of a Distribution

The **mean** *(also called the* **expected value***) of a discrete random variable X is the number*

$$\mu = E(X) = \Sigma x\, P(x)$$

The mean of a random variable may be interpreted as the average of the values assumed by the random variable in repeated trials of the experiment.

For Continuous random variable X,

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

Example:

Find the mean of the discrete random variable *X* whose probability distribution is

x	-2	1	2	3.5
$P(x)$	0.21	0.34	0.24	0.21

Solution:

The formula in the definition gives

$$\begin{aligned}\mu &= \Sigma x\, P(x) \\ &= (-2)\cdot 0.21 + (1)\cdot 0.34 + (2)\cdot 0.24 + (3.5)\cdot 0.21 = 1.135\end{aligned}$$

The **variance**, σ^2, *of a discrete random variable X is the number*

$$\sigma^2 = \Sigma (x-\mu)^2\, P(x)$$

For continuous random variable X,

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

standardized random variable *Z corresponding to X,*

(a) *If a random variable X has mean μ and variance σ^2, then the random variable*

$$Z = \frac{X - \mu}{\sigma}$$ *has the mean 0 and the variance 1.*

Example:

The random variable X = *Number of heads in a single toss of a fair coin* has the possible values $X = 0$ and $X = 1$ with probabilities $P(X = 0) = \frac{1}{2}$ and $P(X = 1) = \frac{1}{2}$. From (la) we thus obtain the mean $\mu = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$, and (2a) yields the variance

$$\sigma^2 = (0 - \tfrac{1}{2})^2 \cdot \tfrac{1}{2} + (1 - \tfrac{1}{2})^2 \cdot \tfrac{1}{2} = \tfrac{1}{4}.$$

Question 1)

Find the mean and variance of the random variable X with probability function or density $f(x)$.

$f(x) = kx$ ($0 \leqq x \leqq 2$, k suitable)

Ans:

① $f(x) = kx \quad 0 \leq x \leq 2$

as $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{\infty} kx\,dx = 1 \Rightarrow \int_0^2 kx\,dx = 1$

$\Rightarrow \frac{kx^2}{2}\Big|_0^2 = 1 \Rightarrow k \times \frac{4}{2} = 1 \Rightarrow k = 0.5$

Hence $f(x) = 0.5x$

mean $= \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^2 x(0.5x)\,dx = \int_0^2 0.5x^2\,dx = 0.5\frac{x^3}{3}\Big|_0^2$

$= 0.5 \times \frac{2^3}{3} = \frac{4}{3}$

Question 2)

Find the mean and variance of the random variable X with probability function or density $f(x)$.

X = Number a fair die turns up

Ans:

② x = number a fair die turns up

we know $P(X) = \frac{1}{6}$

Thus $f(x) = \frac{1}{6}$

Mean $\mu = \sum_{i=1}^{6} x_i f(x) = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$

$= \frac{21}{6} = 3.5$

variance, $\sigma^2 = \sum_{i=1}^{6} (x_i - \mu)^2 f(x)$

$= (1-3.5)^2 \frac{1}{6} + (2-3.5)^2 \times\frac{1}{6} + (3-3.5)^2 \times\frac{1}{6} + (4-3.5)^2 \times\frac{1}{6} + (5-3.5)^2 \times\frac{1}{6}$

$+ (6-3.5)^2 \times\frac{1}{6}$

$= 2.917$

Question 3)

Find the mean and variance of the random variable X with probability function or density $f(x)$.

$f(x) = 4e^{-4x}$ $(x \geqq 0)$

Ans:

The mean is given by

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

So,

$$\begin{aligned}
\mu &= \int_0^{\infty} 4xe^{-4x}\, dx \\
&= \left\{ \begin{array}{cc} u = x & du = dx \\ dv = 4e^{-4x}dx & v = \frac{4e^{-4x}}{-4} = -e^{-4x} \end{array} \right\} \\
&= -xe^{-4x}\Big|_0^{\infty} + \int_0^{\infty} e^{-4x}\, dx \\
&= \frac{e^{-4x}}{-4}\Big|_0^{\infty} \\
&= \frac{1}{4}
\end{aligned}$$

The variance is given by

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

Thus,

$$\begin{aligned}
\sigma^2 &= \int_0^{\infty} \left(x - \frac{1}{4}\right)^2 f(x)\, dx \\
&= \int_0^{\infty} \left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) 4e^{-4x}\, dx \\
&= 4\underbrace{\int_0^{\infty} x^2 e^{-4x}\, dx}_{(*)} - 2\underbrace{\int_0^{\infty} xe^{-4x}\, dx}_{=\mu/4} + \frac{1}{4}\underbrace{\int_0^{\infty} e^{-4x}\, dx}_{(**)} \\
&= \frac{1}{8} - \frac{1}{8} + \frac{1}{16} \\
&= \frac{1}{16}
\end{aligned}$$

Question 4)

Find the mean and variance of the random variable X with probability function or density $f(x)$.

$f(x) = k(1 - x^2)$ if $-1 \leqq x \leqq 1$ and 0 otherwise

Ans:

We must first find k. Since f is a density, the following must hold:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Since

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= k\int_{-1}^{1}(1-x^2)\,dx \\ &= k\left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1} \\ &= k\left(1-\frac{1}{3}-(-1)+\frac{(-1)^3}{3}\right) \\ &= k\left(2-\frac{2}{3}\right) \\ &= \frac{4k}{3},\end{aligned}$$

we conclude that

$$k = \frac{3}{4}$$

The mean is

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx$$

Therefore,

$$\begin{aligned}\mu &= \int_{-1}^{1} \frac{3}{4}x(1-x^2)\,dx \\ &= \frac{3}{4}\int_{-1}^{1}(x-x^3)\,dx \\ &= \frac{3}{4}\left(\frac{x^2}{2}-\frac{x^4}{4}\right)\Big|_{-1}^{1} \\ &= 0\end{aligned}$$

The variance is

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

Therefore,

$$\begin{aligned}
\sigma^2 &= \int_{-1}^{1} \frac{3}{4} x^2(1-x^2)\, dx \\
&= \frac{3}{4}\int_{-1}^{1} (x^2 - x^4)\, dx \\
&= \frac{3}{4}\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_{-1}^{1} \\
&= \frac{3}{4}\left(\frac{1}{3} - \frac{1}{5} - \frac{(-1)^3}{3} + \frac{(-1)^5}{5}\right) \\
&= \frac{3}{4}\left(\frac{2}{3} - \frac{2}{5}\right) \\
&= \frac{3}{4} \cdot \frac{4}{15} \\
&= \frac{1}{5}
\end{aligned}$$

Question 5)

What is the expected daily profit if a store sells X air conditioners per day with probability $f(10) = 0.1$, $f(11) = 0.3$, $f(12) = 0.4$, $f(13) = 0.2$ and the profit per conditioner is \$55?

Ans:

The mean number of soled air conditioners is

$$\begin{aligned}
\mu &= \sum_j x_j f(x_j) \\
&= 10f(10) + 11f(11) + 12f(12) + 13f(13) \\
&= 10 \cdot 0.1 + 11 \cdot 0.3 + 12 \cdot 0.4 + 13 \cdot 0.2 \\
&= 11.7
\end{aligned}$$

So, the expected profit is

$$\text{mean number of soled air conditioners} \cdot \text{profit per conditioner} = 11.7 \cdot 55 = 643.5$$

Question 6)

What is the mean life of a lightbulb whose life X [hours] has the density $f(x) = 0.001e^{-0.001x}$ $(x \geqq 0)$?

Ans:

$f(x) = 0.001\, e^{-0.001x} \quad x \geq 0$

$$\text{mean} = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x f(x)\,dx = \int x \times 0.001 e^{-0.001x}\,dx$$

$$= 0.001 \int_0^{\infty} x e^{-0.001x}\,dx = 0.001 \left(x \left(\frac{e^{-0.001x}}{-0.001} \right) - 1 \left(\frac{e^{-0.001x}}{(0.001)^2} \right) \right) \Big|_0^{\infty}$$

$$= 0.001 \left(0 - \left(-\frac{1}{(0.001)^2} \right) \right) = \frac{1}{0.001} = 1000$$

So, mean life = 1000 hours

Question 7)

A small filling station is supplied with gasoline every **Saturday** afternoon. Assume that its volume X of sales in ten thousands of gallons has the probability density $f(x) = 6x(1 - x)$ if $0 \leqq x \leqq 1$ and 0 otherwise. Determine the mean, the variance, and the standardized variable.

Ans:

$f(x) = 6x(1-x) \quad 0 \leq x \leq 1$ and 0 otherwise

$$\text{mean } \mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x \cdot 6x(1-x)\,dx = 6 \int_0^1 (6x^2 - 6x^3)\,dx$$

$$= 6 \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1 = 6 \left(\frac{1}{3} - \frac{1}{4} \right) = 6 \left(\frac{1}{12} \right) = \frac{3}{6} = \frac{1}{2}$$

$$\text{Variance } \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_0^1 (x-0.5)^2 6x(1-x)\,dx = 6 \int_0^1 (x-0.5)^2 x(1-x)\,dx$$

$$= 6 \int_0^1 (x^2 + 0.25 - x)(x - x^2)\,dx$$

$$\sigma^2 = 6 \int_0^1 (x^3 + 0.25x - x^2 - x^4 - 0.25x^2 + x^4)\,dx = \frac{1}{20} \Rightarrow \sigma = \frac{1}{\sqrt{20}}$$

$$\text{Standardized variable} = \frac{X - \mu}{\sigma} = \frac{X - 0.5}{\frac{1}{\sqrt{20}}}$$

Question 8)

What capacity must the tank in Prob. 7 above have in order that the probability that the tank will be emptied in a given week be 5%.?

Ans:

Here we must find x such that

$$P(X > x) = 0.05$$

Explanation: if the volume of the container is x, then we must find x such that the probability that the demand for gas will be greater than x equals $5\% = 0.05$.

Since

$$P(X > x) = \int_x^1 f(v)\,dv,$$

we get

$$P(X > x) = \int_x^1 6v(1-v)\,dv = 6\int_x^1 (-v^2+v)\,dv = -2v^3+3v^2\Big|_x^1 = 1+2x^3-3x^2$$

Thus, we must find x such that

$$1 + 2x^3 - 3x^2 = 0.05 \Longrightarrow 2x^3 - 3x^2 + 0.95 = 0$$

Also, notice that we must have $0 \leqslant x \leqslant 1$. Therefore, using calculator, we see that only 1 solution of the above equation makes sense:

$$\boxed{x \approx 0.865}$$