Module -5

Statistics:

Sample Mean:

$$\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j = \frac{1}{n}(x_1 + x_2 + \cdots + x_n).$$

Sample Variance:

$$s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{x})^2 = \frac{1}{n-1}[(x_1-\bar{x})^2 + \cdots + (x_n - \bar{x})^2],$$

Maximum Likelihood Method

1) Construct MLE function

$$l = f(x_1)f(x_2)\cdots f(x_n).$$

2) Take logarithm of function l
3) Take derivative wrt. Parameter

$$\frac{\partial \ln l}{\partial \theta} = 0,$$

Q)

Find maximum likelihood estimates for $\theta_1 = \mu$ and $\theta_2 = \sigma$ in the case of the normal distribution.

Solution. From (1), Sec. 24.8, and (4) we obtain the likelihood function

$$l = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\sigma}\right)^n e^{-h} \qquad \text{where} \qquad h = \frac{1}{2\sigma^2}\sum_{j=1}^{n}(x_j - \mu)^2.$$

Taking logarithms, we have

$$\ln l = -n \ln \sqrt{2\pi} - n \ln \sigma - h.$$

The first equation in (8) is $\partial(\ln l)/\partial\mu = 0$, written out

$$\frac{\partial \ln l}{\partial \mu} = -\frac{\partial h}{\partial \mu} = \frac{1}{\sigma^2}\sum_{j=1}^{n}(x_j - \mu) = 0. \qquad \text{hence} \qquad \sum_{j=1}^{n} x_j - n\mu = 0.$$

The solution is the desired estimate $\hat{\mu}$ for μ: we find

$$\hat{\mu} = \frac{1}{n}\sum_{j=1}^{n} x_j = \bar{x}.$$

The second equation in (8) is $\partial(\ln l)/\partial\sigma = 0$, written out

$$\frac{\partial \ln l}{\partial \sigma} = -\frac{n}{\sigma} - \frac{\partial h}{\partial \sigma} = -\frac{1}{\sigma} + \frac{1}{\sigma^3}\sum_{j=1}^{n}(x_j - \mu)^2 = 0.$$

Replacing μ by $\hat{\mu}$ and solving for σ^2, we obtain the estimate

$$\tilde{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}(x_j - \bar{x})^2$$

Q)

. Find the maximum likelihood estimate of θ in the density $f(x) = \theta e^{-\theta x}$ if $x \geqq 0$ and $f(x) = 0$ if $x < 0$.

Ans:

Let our density function be

$$f(x) = \begin{cases} \theta e^{-\theta x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

We need to find an estimate for the parameter θ. Since we need to apply maximum likelihood method, we need to define Likelihood function $L : \theta \mapsto \mathbb{R}$,

$$L(\theta) = f(x_1|\theta) \cdots f(x_n|\theta).$$

Let our likelihood function be written as

$$\begin{aligned} L &= \prod_{i=1}^{n} \theta e^{-\theta x_i} \\ &= \theta^n e^{\sum_{i=1}^{n} x_i} \end{aligned}$$

So, by taking the log of both sides, we get:

$$\log L = n \log \theta - \theta \sum_{i=1}^{n} x_i$$

Now we need to calculate the derivation of parameter θ and equate that to zero. Applying the chain rule from calculus, we get:

$$\begin{aligned} \frac{d}{d\theta} \log L = 0 \\ \frac{n}{\theta} - \sum_{i=1}^{n} x_i = 0 \\ \frac{n}{\theta} = \sum_{i=1}^{n} x_i \\ \hat{\theta} = \frac{n}{\sum_{i=1}^{n} x_i} \end{aligned}$$

Hence we got that our maximum likelihood estimator for parameter θ is $\frac{n}{\sum_{i=1}^{n} x_i}$, or in other words

$$\hat{\theta} = \frac{n}{\overline{x}}.$$

Confidence Intervals

Confidence Interval for μ of the Normal Distribution with Known σ^2

Table 25.1 Determination of a Confidence Interval for the Mean μ of a Normal Distribution with Known Variance σ^2

Step 1. Choose a confidence level γ (95%, 99%, or the like).

Step 2. Determine the corresponding c:

γ	0.90	0.95	0.99	0.999
c	1.645	1.960	2.576	3.291

Step 3. Compute the mean $\bar{x}$ of the sample $x_1, \cdots, x_n$.

Step 4. Compute $k = c\sigma/\sqrt{n}$. The confidence interval for μ is

$$\text{CONF}_{\gamma}\,\{\bar{x} - k \leqq \mu \leqq \bar{x} + k\}. \tag{3}$$

Confidence Interval for μ of the Normal Distribution with Known σ^2

Determine a 95% confidence interval for the mean of a normal distribution with variance $\sigma^2 = 9$, using a sample of $n = 100$ values with mean $\bar{x} = 5$.

Solution. ***Step 1.*** $\gamma = 0.95$ is required. ***Step 2.*** The corresponding c equals 1.960; see Table 25.1. ***Step 3.*** $\bar{x} = 5$ is given. ***Step 4.*** We need $k = 1.960 \cdot 3/\sqrt{100} = 0.588$. Hence $\bar{x} - k = 4.412$, $\bar{x} + k = 5.588$ and the confidence interval is $\text{CONF}_{0.95}\,\{4.412 \leqq \mu \leqq 5.588\}$.

Q)

Find a 95% confidence interval for the mean of a normal population with standard deviation 4.00 from the sample 39, 51, 49, 43, 57, 59. Does that interval get longer or shorter if we take $\gamma = 0.99$ instead of 0.95? By what factor?

Ans:

Lets first assume that $\gamma = 0.95$. Using the statistical tables, we can see that corresponding $c = 1.960$ for that particular γ.

$$\begin{aligned}\overline{X} &= \frac{1}{6}(39 + 51 + 49 + 43 + 57 + 59) \\ &= 49.67\end{aligned}$$

The corresponding κ will be

$$\begin{aligned}\kappa &= \frac{c\sigma}{\sqrt{n}} \\ &= \frac{1.96 \cdot 4}{\sqrt{6}} \\ &= 3.2\end{aligned}$$

Let us observe the case $\gamma = 0.99$. From the statistical tables, we can see that corresponding $c = 2.576$. From that, we can calculate corresponding κ:

$$\kappa = \frac{c\sigma}{\sqrt{n}} = \frac{2.576 \cdot 4}{\sqrt{6}} = 2.45$$

Hence, the confidence interval is

$$(\overline{x} - \kappa \leq \mu \leq \overline{x} + \kappa) = (47.22 \leq \mu \leq 52.18)$$

From our results, we can see that the confidence interval is smaller the bigger confidence level is. The interval gets shorter by factor

$$\frac{52.87 - 46.47}{52.12 - 47 - 22} = 1.3$$

Q)

Determine a 95% confidence interval for the mean μ of a normal population with variance $\sigma^2 = 16$, using a sample of size 200 with mean 74.81.

Ans:

Our sample size is 200, so $n = 200$. Mean is $\mu = 74.81$
Let $\gamma = 0.95$. Using the statistical tables, we can see that corresponding $c = 1.960$ for that particular confidence level. We also have that variance $\sigma^2 = 16$, hence standard deviation is $\sigma = 4$. This is a normal population.

Using the formula for κ value, we get

$$\begin{aligned}\kappa &= \frac{c\sigma}{\sqrt{n}} \\ &= \frac{1.96 \cdot 4}{\sqrt{200}} \\ &= \frac{7.84}{14.14} \\ &= 0.55\end{aligned}$$

Finally, the 95% confidence interval for the mean μ is

$$\begin{aligned}I &= (\overline{x} - \kappa \le \mu \le \overline{x} + \kappa) \\ &= (74.81 - 0.55 \le \mu \le 74.81 + 0.55) \\ &= (74.26 \le \mu \le 75.36)\end{aligned}$$

Hence, our interval is [74.26, 75.36].

Sample Size Needed for a Confidence Interval of Prescribed Length

How large must n be in Example 1 if we want to obtain a 95% confidence interval of length $L = 0.4$?

Solution. The interval (3) has the length $L = 2k = 2c\sigma/\sqrt{n}$. Solving for n, we obtain

$$n = (2c\sigma/L)^2.$$

In the present case the answer is $n = (2 \cdot 1.960 \cdot 3/0.4)^2 \approx 870$.

Table 25.2 Determination of a Confidence Interval for the Mean μ of a Normal Distribution with Unknown Variance σ^2

Step 1. Choose a confidence level γ (95%, 99%, or the like).

Step 2. Determine the solution c of the equation

$$F(c) = \tfrac{1}{2}(1 + \gamma) \tag{9}$$

from the table of the t-distribution with $n - 1$ degrees of freedom (Table A9 in App. 5; [redacted] n = sample size).

Step 3. Compute the mean $\bar{x}$ and the variance s^2 of the sample $x_1, \cdots, x_n$.

Step 4. Compute $k = cs/\sqrt{n}$. The confidence interval is

$$\mathrm{CONF}_\gamma \{\bar{x} - k \leqq \mu \leqq \bar{x} + k\}. \tag{10}$$

Confidence Interval for μ of the Normal Distribution with Unknown σ^2

Five independent measurements of the point of inflammation (flash point) of Diesel oil (D-2) gave the values (in °F) 144 147 146 142 144. Assuming normality, determine a 99% confidence interval for the mean.

Solution. *Step 1.* $\gamma = 0.99$ is required.

Step 2. $F(c) = \frac{1}{2}(1 + \gamma) = 0.995$, and Table A9 in App. 5 with $n - 1 = 4$ d.f. gives $c = 4.60$.

Step 3. $\bar{x} = 144.6, \quad s^2 = 3.8.$

Step 4. $k = \sqrt{3.8} \cdot 4.60/\sqrt{5} = 4.01$. The confidence interval is $\mathrm{CONF}_{0.99}\{140.5 \leqq \mu \leqq 148.7\}$.

If the variance σ^2 were known and equal to the sample variance s^2, thus $\sigma^2 = 3.8$, then Table 25.1 would give $k = c\sigma/\sqrt{n} = 2.576\sqrt{3.8}/\sqrt{5} = 2.25$ and $\mathrm{CONF}_{0.99}\{142.35 \leqq \mu \leqq 146.85\}$. [redacted]

Q)

[redacted] Find a 99% confidence interval for the mean of a normal population from the sample:

- [redacted] Copper content (%) of brass 66, 66, 65, 64, 66, 67, 64, 65, 63, 64

Ans:

Since our sample is $66, 66, 65, 64, 66, 67, 64, 65, 63, 64$, from there we can conclude that $n = 10$.
Since this is a normal population, by using the statistical tables, we can see that for the 99% confidence level, and at $n-1=9$ degrees of freedom the value of corresponding c is 3.25.

The formula for mean is

$$\overline{X} = \frac{1}{n}(x_1 + \cdots + x_n)$$

and by using it we can calculate the mean of the given sample:

$$\begin{aligned}\overline{x} &= \frac{1}{n}\sum_{i=1}^{10} x_i \\ &= \frac{1}{10}(66+66+65+64+67+64+65+63+64) \\ &= 65\end{aligned}$$

Using the formula for variance, we can also calculate that:

$$\begin{aligned}\sigma^2 &= \frac{1}{n-1}\sum_{i=1}^{10}(x_i - \overline{x})^2 \\ &= \frac{1}{9}(1+4+0+1+4+1+1+0+1+1) \\ &= \frac{14}{9} \\ &= 1.55\end{aligned}$$

Standard deviation is $\sqrt{\sigma^2} = \sigma = \sqrt{1.55} = 1.25$.

Now we need to calculate the corresponding κ value:

$$\begin{aligned}\kappa &= \frac{c\sigma}{n} \\ &= \frac{4.0625}{\sqrt{10}} \\ &= 1.29\end{aligned}$$

Finally, we can obtain the confidence interval for the mean μ of the normal distribution. It will be

$$\begin{aligned}I &= (65 - 1.29 \leq \mu \leq 65 + 1.29) \\ &= (63.71 \leq \mu \leq 66.29)\end{aligned}$$

Table 25.3 Determination of a Confidence Interval for the Variance σ^2 of a Normal Distribution, Whose Mean Need Not Be Known

Step 1. Choose a confidence level γ (95%, 99%, or the like).

Step 2. Determine solutions c_1 and c_2 of the equations

$$F(c_1) = \tfrac{1}{2}(1-\gamma), \qquad F(c_2) = \tfrac{1}{2}(1+\gamma) \tag{15}$$

from the table of the chi-square distribution with $n-1$ degrees of freedom (Table A10 in App. 5; or use a CAS; n = sample size).

Step 3. Compute $(n-1)s^2$, where s^2 is the variance of the sample $x_1, \cdots, x_n$.

Step 4. Compute $k_1 = (n-1)s^2/c_1$ and $k_2 = (n-1)s^2/c_2$. The confidence interval is

$$\text{CONF}_\gamma \{k_2 \leqq \sigma^2 \leqq k_1\}. \tag{16}$$

Confidence Interval for the Variance of the Normal Distribution

Determine a 95% confidence interval (16) for the variance, using Table 25.3 and a sample (tensile strength of sheet steel in kg/mm^2, rounded to integer values)

89 84 87 81 89 86 91 90 78 89 87 99 83 89.

Solution. ***Step 1.*** $\gamma = 0.95$ is required.

Step 2. For $n - 1 = 13$ we find

$$c_1 = 5.01 \quad \text{and} \quad c_2 = 24.74.$$

Step 3. $13s^2 = 326.9$.

Step 4. $13s^2/c_1 = 65.25$, $13s^2/c_2 = 13.21$.

The confidence interval is

$$\text{CONF}_{0.95}\{13.21 \leqq \sigma^2 \leqq 65.25\}.$$

Q)

Find a 95% confidence interval for the variance of a normal population from the sample:

[redacted] Length of 20 bolts with sample mean 20.2 cm and sample variance 0.04 cm^2

Ans:

Since we have 20 bolts in our sample, $n = 20$ and from that we have $n-1 = 19$ degrees of freedom. That gives corresponding $c_1 = 8.91$ and $c_2 = 32.852$.

We're also given that sample mean $\overline{X} = 20.2$ and sample variance $s^2 = 0.04$. The last thing that we need to do is computing corresponging k_1 and k_2 values to find the said interval. We do that the following way:

$$\begin{aligned} k_1 &= \frac{(n-1)s^2}{c_1} \\ &= \frac{19 \cdot 0.04}{8.91} \\ &= \frac{0.76}{8.94} \\ &= 0.84 \end{aligned}$$

Similarly, we calculate k_2:

$$\begin{aligned} k_2 &= \frac{(n-1)s^2}{c_2} \\ &= \frac{19 \cdot 0.04}{32.852} \\ &= \frac{0.76}{32.852} \\ &= 0.023 \end{aligned}$$

Finally, we have that the 95% confidence interval for the variance is

$$\begin{aligned} I &= (k_2 \leq \sigma^2 \leq k_1) \\ &= [0.023,\ 0.85] \end{aligned}$$

Q)

Find a 95% confidence interval for the variance of a normal population from the sample:

Carbon monoxide emission (grams per mile) of a certain type of passenger car (cruising at 55 mph): 17.3, 17.8, 18.0, 17.7, 18.2, 17.4, 17.6, 18.1

Ans:

From the given sample, we can calculate its mean and variance.
The formula for mean is

$$\overline{X} = \frac{1}{n}(x_1 + \cdots + x_n),$$

for a random sample $x_1, \ldots, x_n$.
Using that formula, we can get this:

$$\begin{aligned}\overline{X} &= \frac{1}{8}(17.3 + 17.8 + 18.0 + 17.7 + 18.2 + 17.4 + 17.6 + 18.1) \\ &= \frac{142.1}{8} \\ &= 17.76\end{aligned}$$

Similarly, using the formula for variance, we get:

$$\begin{aligned}S^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 \\ &= \frac{1}{7}(0.214 + 0.016 + 0.056 + 0.0036 + 0.1936 + 0.1296 + 0.0256 + 0.1156) \\ &= 0.106\end{aligned}$$

The values of c_1 and c_2 corresponding to $n - 1 = 7$ degrees of freedom are 16.013 and 1.7.

Finally, the corresponding k_1 and k_2 values are

$$\begin{aligned}k_1 &= \frac{(n-1)\sigma^2}{c_1} = \frac{7 \cdot 1.106}{16.013} \\ &= 0.0463\end{aligned}$$

Similarly,

$$\begin{aligned}k_2 &= \frac{(n-1)\sigma^2}{c_2} = \frac{7 \cdot 1.106}{1.7} \\ &= 0.4364\end{aligned}$$

We can finally obtain our confidence interval for the variance of this population, which is:

$$\begin{aligned}I &= (k_1 \leq \sigma^2 \leq k_2) \\ &= [0.046,\ 0.437]\end{aligned}$$

Table A9 ***t*-Distribution**

Values of z for given values of the distribution function $F(z)$ (see (8) in Sec. 25.3). Example: For 9 degrees of freedom, $z = 1.83$ when $F(z) = 0.95$.

$F(z)$	Number of Degrees of Freedom									
	1	2	3	4	5	6	7	8	9	10
0.5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.6	0.32	0.29	0.28	0.27	0.27	0.26	0.26	0.26	0.26	0.26
0.7	0.73	0.62	0.58	0.57	0.56	0.55	0.55	0.55	0.54	0.54
0.8	1.38	1.06	0.98	0.94	0.92	0.91	0.90	0.89	0.88	0.88
0.9	3.08	1.89	1.64	1.53	1.48	1.44	1.41	1.40	1.38	1.37
0.95	6.31	2.92	2.35	2.13	2.02	1.94	1.89	1.86	1.83	1.81
0.975	12.7	4.30	3.18	2.78	2.57	2.45	2.36	2.31	2.26	2.23
0.99	31.8	6.96	4.54	3.75	3.36	3.14	3.00	2.90	2.82	2.76
0.995	63.7	9.92	5.84	4.60	4.03	3.71	3.50	3.36	3.25	3.17
0.999	318.3	22.3	10.2	7.17	5.89	5.21	4.79	4.50	4.30	4.14

$F(z)$	Number of Degrees of Freedom									
	11	12	13	14	15	16	17	18	19	20
0.5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.6	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26
0.7	0.54	0.54	0.54	0.54	0.54	0.54	0.53	0.53	0.53	0.53
0.8	0.88	0.87	0.87	0.87	0.87	0.86	0.86	0.86	0.86	0.86
0.9	1.36	1.36	1.35	1.35	1.34	1.34	1.33	1.33	1.33	1.33
0.95	1.80	1.78	1.77	1.76	1.75	1.75	1.74	1.73	1.73	1.72
0.975	2.20	2.18	2.16	2.14	2.13	2.12	2.11	2.10	2.09	2.09
0.99	2.72	2.68	2.65	2.62	2.60	2.58	2.57	2.55	2.54	2.53
0.995	3.11	3.05	3.01	2.98	2.95	2.92	2.90	2.88	2.86	2.85
0.999	4.02	3.93	3.85	3.79	3.73	3.69	3.65	3.61	3.58	3.55

$F(z)$	Number of Degrees of Freedom									
	22	24	26	28	30	40	50	100	200	∞
0.5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.6	0.26	0.26	0.26	0.26	0.26	0.26	0.25	0.25	0.25	0.25
0.7	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.53	0.52
0.8	0.86	0.86	0.86	0.85	0.85	0.85	0.85	0.85	0.84	0.84
0.9	1.32	1.32	1.31	1.31	1.31	1.30	1.30	1.29	1.29	1.28
0.95	1.72	1.71	1.71	1.70	1.70	1.68	1.68	1.66	1.65	1.65
0.975	2.07	2.06	2.06	2.05	2.04	2.02	2.01	1.98	1.97	1.96
0.99	2.51	2.49	2.48	2.47	2.46	2.42	2.40	2.36	2.35	2.33
0.995	2.82	2.80	2.78	2.76	2.75	2.70	2.68	2.63	2.60	2.58
0.999	3.50	3.47	3.43	3.41	3.39	3.31	3.26	3.17	3.13	3.09

Table A10 Chi-square Distribution

Values of x for given values of the distribution function $F(z)$ (see Sec. 25.3 before (17)). Example: For 3 degrees of freedom, $z = 11.34$ when $F(z) = 0.99$.

$F(z)$	Number of Degrees of Freedom									
	1	2	3	4	5	6	7	8	9	10
0.005	0.00	0.01	0.07	0.21	0.41	0.68	0.99	1.34	1.73	2.16
0.01	0.00	0.02	0.11	0.30	0.55	0.87	1.24	1.65	2.09	2.56
0.025	0.00	0.05	0.22	0.48	0.83	1.24	1.69	2.18	2.70	3.25
0.05	0.00	0.10	0.35	0.71	1.15	1.64	2.17	2.73	3.33	3.94
0.95	3.84	5.99	7.81	9.49	11.07	12.59	14.07	15.51	16.92	18.31
0.975	5.02	7.38	9.35	11.14	12.83	14.45	16.01	17.53	19.02	20.48
0.99	6.63	9.21	11.34	13.28	15.09	16.81	18.48	20.09	21.67	23.21
0.995	7.88	10.60	12.84	14.86	16.75	18.55	20.28	21.95	23.59	25.19

$F(z)$	Number of Degrees of Freedom									
	11	12	13	14	15	16	17	18	19	20
0.005	2.60	3.07	3.57	4.07	4.60	5.14	5.70	6.26	6.84	7.43
0.01	3.05	3.57	4.11	4.66	5.23	5.81	6.41	7.01	7.63	8.26
0.025	3.82	4.40	5.01	5.63	6.26	6.91	7.56	8.23	8.91	9.59
0.05	4.57	5.23	5.89	6.57	7.26	7.96	8.67	9.39	10.12	10.85
0.95	19.68	21.03	22.36	23.68	25.00	26.30	27.59	28.87	30.14	31.41
0.975	21.92	23.34	24.74	26.12	27.49	28.85	30.19	31.53	32.85	34.17
0.99	24.72	26.22	27.69	29.14	30.58	32.00	33.41	34.81	36.19	37.57
0.995	26.76	28.30	29.82	31.32	32.80	34.27	35.72	37.16	38.58	40.00

$F(z)$	Number of Degrees of Freedom									
	21	22	23	24	25	26	27	28	29	30
0.005	8.0	8.6	9.3	9.9	10.5	11.2	11.8	12.5	13.1	13.8
0.01	8.9	9.5	10.2	10.9	11.5	12.2	12.9	13.6	14.3	15.0
0.025	10.3	11.0	11.7	12.4	13.1	13.8	14.6	15.3	16.0	16.8
0.05	11.6	12.3	13.1	13.8	14.6	15.4	16.2	16.9	17.7	18.5
0.95	32.7	33.9	35.2	36.4	37.7	38.9	40.1	41.3	42.6	43.8
0.975	35.5	36.8	38.1	39.4	40.6	41.9	43.2	44.5	45.7	47.0
0.99	38.9	40.3	41.6	43.0	44.3	45.6	47.0	48.3	49.6	50.9
0.995	41.4	42.8	44.2	45.6	46.9	48.3	49.6	51.0	52.3	53.7

$F(z)$	Number of Degrees of Freedom							
	40	50	60	70	80	90	100	> 100 (Approximation)
0.005	20.7	28.0	35.5	43.3	51.2	59.2	67.3	$\frac{1}{2}(h - 2.58)^2$
0.01	22.2	29.7	37.5	45.4	53.5	61.8	70.1	$\frac{1}{2}(h - 2.33)^2$
0.025	24.4	32.4	40.5	48.8	57.2	65.6	74.2	$\frac{1}{2}(h - 1.96)^2$
0.05	26.5	34.8	43.2	51.7	60.4	69.1	77.9	$\frac{1}{2}(h - 1.64)^2$
0.95	55.8	67.5	79.1	90.5	101.9	113.1	124.3	$\frac{1}{2}(h + 1.64)^2$
0.975	59.3	71.4	83.3	95.0	106.6	118.1	129.6	$\frac{1}{2}(h + 1.96)^2$
0.99	63.7	76.2	88.4	100.4	112.3	124.1	135.8	$\frac{1}{2}(h + 2.33)^2$
0.995	66.8	79.5	92.0	104.2	116.3	128.3	140.2	$\frac{1}{2}(h + 2.58)^2$