Problem Set 5

1. Given :

PDF

$$f(x) = \begin{cases} 0 & x < 0 \\ x & x \in (0,1) \\ (3-x)/4 & x \in (1,3) \\ 0 & x > 3 \end{cases}$$

Before we find the population median, let's find the CDF.

For $y < 0$, $F(y) = 0$

For $y \in (0,1)$, $\int_0^y x \cdot dx = \left[\frac{x^2}{2}\right]_0^y = \frac{y^2}{2}$

For $y \in (1,3)$, $\int_0^1 x \cdot dx + \int_1^y \frac{3-x}{4} \cdot dx \Rightarrow \frac{1}{2} + \frac{1}{4}\int_1^y 3 - x \cdot dx$

$\Rightarrow \frac{1}{2} + \frac{1}{4}\left[3[x]_1^y - \left[\frac{x^2}{2}\right]_1^y\right] \Rightarrow \frac{1}{2} + \frac{1}{4}\left[3y - 3 - \frac{y^2}{2} + \frac{1}{2}\right]$

$\Rightarrow \frac{1}{2} + \frac{1}{4}\left[-2.5 + 3y - \frac{y^2}{2}\right]$

$$\Rightarrow \frac{1}{2} + \frac{1}{4}\left[-2.5 + 3y - \frac{y^2}{2}\right]$$

$$\Rightarrow \frac{1}{2} + \frac{-2.5}{4} + \frac{3y}{4} - \frac{y^2}{8} \Rightarrow \frac{-y^2}{8} + \frac{3y}{4} - \frac{2.5}{4} + \frac{2}{4}$$

$$\Rightarrow \frac{-y^2}{8} + \frac{3y}{4} - \frac{0.5}{4}$$

For $y \geq 3$, $F(y) = 1$

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{y^2}{2} & 0 \leq y < 1 \\ \frac{-y^2}{8} + \frac{3y}{4} - \frac{0.5}{4} & 1 \leq y < 3 \\ 1 & y > 3 \end{cases}$$

(a) To find the population median, let's check if it's there in the interval $\boxed{0 \leq y < 1}$

$$\frac{y^2}{2} = 0.5$$

$$\Rightarrow y^2 = 1$$

$$\Rightarrow y = 1$$

The only issue here is 1 is not in $0 \leq y < 1$

$\boxed{\text{Let's try } 1 \leq y < 3}$

$$\frac{-y^2}{8} + \frac{3y}{4} - \frac{0.5}{4} = 0.5$$

$$\frac{-y^2 + 6y - 1}{8} = 0.5$$

$$-y^2 + 6y - 1 = 4$$

$$\Rightarrow y^2 - 6y + 5 = 0$$

$$\Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1$; $b = -6$; $c = 5$

$$\Rightarrow \frac{+6 \pm \sqrt{36 - 4(1)(5)}}{2(1)}$$

$$\Rightarrow \frac{+6 \pm \sqrt{36 - 20}}{2}$$

$$\Rightarrow \frac{+6 \pm \sqrt{16}}{2} = \frac{6 \pm 4}{2}$$

$\frac{10}{2} = 5$ (or) $\frac{2}{2} = 1$

$\boxed{\therefore \text{Only 1 is b/w } 1 \leq y < 3}$

(a) $\boxed{q_2 = 1}$

(b).

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx = \int_{0}^{1} x \cdot x \cdot dx + \int_{1}^{3} \frac{3-x}{4} \cdot dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + 3\left[x\right]_1^3 - \left[\frac{x^2}{2}\right]_1^3 = \frac{1}{3} + 3[2] - \left[\frac{9}{2} - \frac{1}{2}\right]$$

$$= \frac{1}{3} + 6 - \frac{8}{2} = \frac{1}{3} + 6 - 4 = \boxed{2.33}$$

$$\boxed{\therefore E(x) > q_2}$$

(c) $P(0.5 < X < 1.5) = P(1.5) - P(0.5)$

$$= \left[\frac{-y^2}{8} + \frac{3y}{4} - \frac{0.5}{4}\right] - \left[\frac{y^2}{2}\right] = \frac{(-1.5)^2}{8} + \frac{4.5}{4} - \frac{0.5}{4} - \frac{0.5^2}{4}$$

$$= +\frac{2.25}{8} + \frac{9}{8} - \frac{1}{8} - \frac{0.25}{4}$$

$$= +\frac{2.25}{8} + \frac{8}{8} - \frac{0.5}{8}$$

$$= \frac{1.75}{8} = \boxed{0.218}$$

(d) iqr(x)

$$iqr(x) = q_3 - q_1$$

$$q_3 = 0.75$$

$$\frac{-y^2}{8} + \frac{3y}{4} - \frac{0.5}{4} = 0.75$$

$$-y^2 + 6y - 1 = 6$$

$$y^2 - 6y + 7 = 0$$

$$\Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1;\ b = -6;\ c = 7$$

$$q_1 = 0.25$$

$$\int_0^{q_1} x \cdot dx = 0.25$$

$$\left[\frac{x^2}{2}\right]_0^{q_1} = 0.25$$

$${q_1}^2 = 0.5$$

$${q_1}^2 = \sqrt{0.5} = \sqrt{\frac{1}{2}} = \frac{1}{1.414}$$

$$q_1 = \boxed{0.707}$$

$$\frac{+6 \pm \sqrt{36-4(1)(7)}}{2}$$

$$\Rightarrow \frac{6 \pm \sqrt{8}}{2}$$

$$\Rightarrow \frac{6 \pm 2\sqrt{2}}{2}$$

$$\Rightarrow 3 \pm \sqrt{2}$$

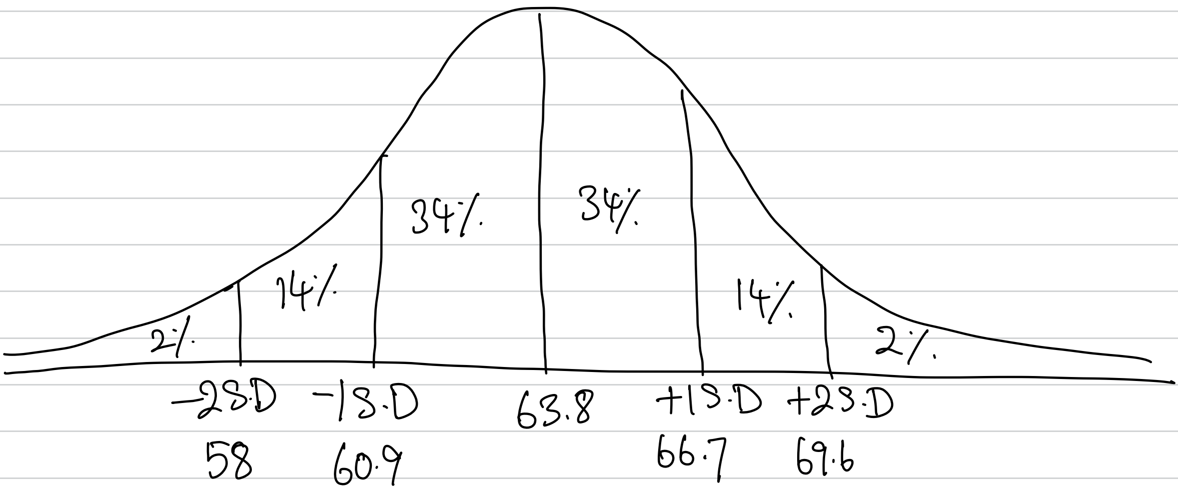
$x = 4.41$ or $x = 1.58$

$x \neq 4.41$ because it's not in the $1 \leq y < 3$ interval

$$IQR = q_3 - q_1 = 1.58 - 0.707 = \boxed{0.873}$$

2. We are given a normal$(63.8, 2.9^2)$

$\sigma = 2.9$



(a)

First we can calculate the percentile rank of 65.5

By using R, we can calculate the percentile rank of 65.5.

Code:

```
pnorm(65.5, mean = 63.8, sd = 2.9) * 100
```

≈ 72

Which means there 100-72 = 28% of woman who are taller than 65.5 inches.

(b). To calculate the IQR, we can use R again.

Code:

```
qnorm(0.75,63.8,2.9) - qnorm(0.25,63.8,2.9)
```

We get q_3 as 65.75 and q_1 as 61.84

$$q_3 - q_1 = \boxed{3.91}$$

(c). The shortest 2.5% of women are shorter than ____

→ We can find this by using qnorm because it gives us the value at a given percentile

Code: `qnorm(0.025,63.8,2.9)` $\simeq \boxed{58.1}$

The tallest 2.5% women are taller than ____

Code: `qnorm(0.975,63.8,2.9)` $\simeq \boxed{69.5}$

3.(a) Please refer to the uploaded .R file also

Code:

```
X <- rnorm(10000)
Y <- X^2
P <- sum(Y>1/1000)
```

We get $P(Y>1)$ as $\boxed{0.31}$

(b) 0.9 quantile of y will be $P(Y<y) = 0.9$

$Y = X^2$

$P(X^2 < y) = 0.9$

$P(-\sqrt{y} < X < \sqrt{y}) = 0.9$

We can rewrite this as $2P(0 < X < \sqrt{y}) = 0.9$

$P(0 < X < \sqrt{y}) = 0.45$

$P(X < \sqrt{y}) - P(X < 0) = 0.45$

$$P(X < \sqrt{y}) - 0.5 = 0.45$$

$$P(X < \sqrt{y}) = 0.95$$

Now we can finally compute the value of y by using

qnorm.

Code:

```
(qnorm(0.95))^2
```

$\simeq 2.7$

$\therefore$ The 0.9 quantile of $Y \simeq 2.7$

4. Given:

$$f(x) = \begin{cases} 0.3 & 0 \leq x < 1 \\ 0.7 & 1 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Before we find a and b values, let us calculate the CDF for the given PDF.

For $y < 0$, $F(y) = 0$ [given in PDF]

For $0 \leq y < 1$, $\Rightarrow \int_0^y 0.3 \, dx = 0.3 [x]_0^y = 0.3y$

For $1 \leq y < 2$, $\Rightarrow \int_0^1 0.3 \, dx + \int_1^y 0.7 \, dx \Rightarrow 0.3[x]_0^1 + 0.7[x]_1^y$

$\Rightarrow 0.3 + 0.7[y-1] \Rightarrow 0.3 + 0.7y - 0.7 \Rightarrow 0.7y - 0.4$

$$F(y) = \begin{cases} 0 & y < 0 \\ 0.3y & 0 \leq y < 1 \\ 0.7y - 0.4 & 1 \leq y < 2 \\ 1 & y > 2 \end{cases}$$

(a).

Now, if we want to find a value 'a' that minimizes $E|X-a|$, we can find the median and that can be used to minimize $E|X-a|$.

Let's check if the median is present in the interval $1 \leq y < 2$

$$0.7y - 0.4 = 0.5$$

$$0.7y = 0.9$$

$$y = \frac{0.9}{0.7} = 1.28$$

1.28 does lie in the interval $1 \leq y < 2$. So, $a = 1.28$

(b). Now for $E[(X-b)^2]$, the value b that will minimize our equation will be the expected value.

$$E(x) = \int_0^1 0.3x\,dx + \int_1^2 0.7x\,dx = 0.3\left[\frac{x^2}{2}\right]_0^1 + 0.7\left[\frac{x^2}{2}\right]_1^2$$

$$= 0.3\,\frac{1}{2} + 0.7\left[\frac{4}{2} - \frac{1}{2}\right] = \frac{0.3}{2} + 0.7\left[\frac{3}{2}\right]$$

$$= 0.15 + 1.05 = 1.2$$

$\therefore$ The value b that minimizes $E[(X-b)^2]$ will be 1.2