1. Given PDF

$$f(x) = \begin{cases} \frac{1}{30} & 0 \le x < 20 \\ \frac{1}{60} & 20 \le x < 40 \\ 0 & \text{otherwise} \end{cases}$$

(a). CDF

$F(y)$ for $0 \le y < 20$

$$\Rightarrow \int_0^y \frac{1}{30}\, dx = \frac{1}{30}\left[x\right]_0^y = \frac{y}{30}$$

$F(y)$ for $20 \le x < 40$

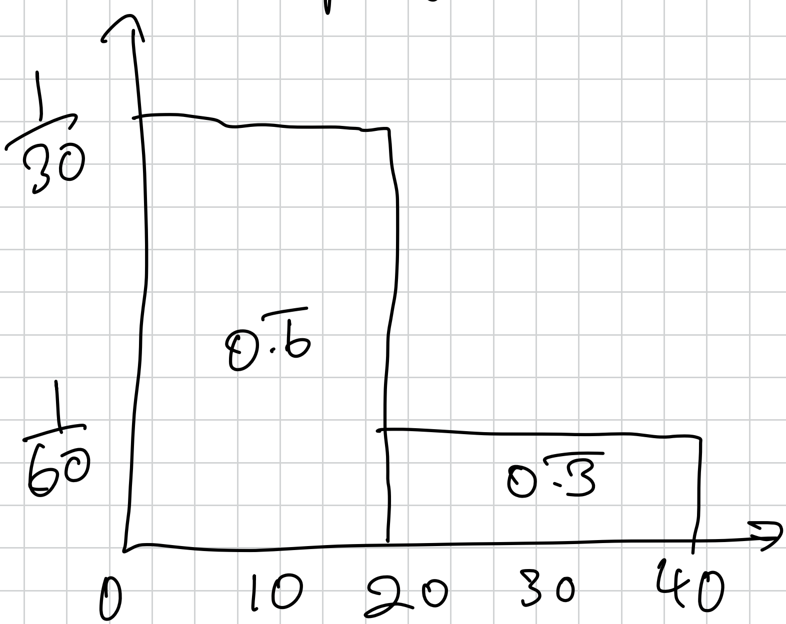
$$= \int_{20}^{y} \frac{1}{60} \cdot dx = \frac{1}{60}\int_{20}^{y} 1 \cdot dx = \frac{1}{60}\left[x\right]_{20}^{y}$$

$$= \frac{1}{60}\left[y - 20\right] = \frac{y-20}{60}$$

$$F(y) = \begin{cases} \dfrac{y}{30} & 0 \leq y < 20 \\[2ex] \dfrac{y-20}{60} & 20 \leq y < 40 \\[2ex] 1 & y \geq 40 \end{cases}$$

(b).

Graph of PDF



To calculate the Expected value we need to calculate the expected values of individual rectangles and add them up.

Left Rectangle area $= 20 \times \frac{1}{30} = 0.66$

Right Rectangle area $= 20 \times \frac{1}{60} = 0.33$

- $2/3^{rd}$ of the time we have a uniform $(0, 20)$ with an average of 10
- $1/3^{rd}$ of the time we have a uniform $(20, 40)$ with an average of 30

Overall expected value =

$$(0.66 \times 10) + (0.33 \times 30) = 6.6 + 9.9$$

$$E(x) = \boxed{16.5}$$

(c)

$F(y) = 0.5$

The left block has most area. So the median (0.5) will lie here.

height $= 1/30$

base $= y$

$\frac{1}{30} \times y = 0.5$

$\boxed{y = 15}$ ; 15 is in b/w 0 & 20

$\therefore$ $y$ is smaller than $E(x)$ because $y = 15$ and $E(x) = 16.5$

2. To verify the PDF of a continuous random variable, we need to check 2 things :

- Always non-negative
- Total area under $f(x)$ is 1

Given :

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 2(x-1) & \text{if } 1 \leq x \leq 2 \\ 0 & \text{if } x > 2 \end{cases}$$

- for $x < 1$, it is 0
- for $x = 1$, $2(1-1) = 0$
- for $x = 2$, $2(2-1) = 2$

  All values of $f(x)$ for $x$ b/w 1 & 2, will be b/w 0 and 2 based on the above 2 points
- for $x > 2$, it is 0

$\therefore$ $f(x)$ is always non-negative

$$\int_1^2 2(x-1)\cdot dx = 2\int_1^2 x-1 \cdot dx$$

$$= 2\left[\int_1^2 x - \int_1^2 1\cdot dx\right] = 2\left[\left[\frac{x^2}{2}\right]_1^2 - \left[x\right]_1^2\right]$$

$$= 2\left[\left[\frac{4}{2} - \frac{1}{2}\right] - \left[2-1\right]\right] = 2\left[2-0.5-1\right]$$

$$= 2\left[0.5\right] = 1$$

$f(x)$ is always non-negative and total area under $f(x)$ is 1. Given PDF is valid.

(C.) $P(1.5 < X < 1.75)$

$$= \int_{1.5}^{1.75} 2(x-1) \cdot dx = 2\left[\int_{1.5}^{1.75} x - \int_{1.5}^{1.75} 1 \cdot dx\right]$$

$$= 2\left[\left[\frac{x^2}{2}\right]_{1.5}^{1.75} - [x]_{1.5}^{1.75}\right]$$

$$= 2\left[\left[\frac{(1.75)^2}{2} - \frac{(1.5)^2}{2}\right] - [1.75 - 1.5]\right]$$

$$= 2\left[1.53 - 1.125 - 0.25\right]$$

$$= \boxed{0.31}$$

3. given mean $= 120$

$\sigma = 20$

(a) Using pnorm

pnorm(115, 120, 20) = 0.40

pnorm(135, 120, 20) = 0.7734

$P(115 < x < 135)$

$= 0.7734 - 0.40 = 0.37$

(b) $P(X > 160) = 1 -$ pnorm(160, 120, 20)

$= 1 - 0.9772$

$= 0.0227$

Probability of BPM being more than 160 out of 10

$= 1 - P($No musice $>$ than 160 BPM$)$

$= 1 - (0.97725)^{10} = 0.2056$

(c) $\mu = \mu_1 + \mu_2$

$120 + 120 = 240$

Variance $= \sigma_1^2 + \sigma_2^2$

$\sigma_{sum}^2 = 20^2 + 20^2 = 800$

$\sigma_{sum} = 28.284$

$\sigma_{average} = \frac{28.284}{1.414} = 20$

$P(X_1 + X_2 > 320) = 1 - pnorm(320, 240, 28.284)$

$= 0.0023$

4. Given 2 normal independent random variables

$X_1 \sim \text{Normal}(1, 9)$ and $X_2 \sim \text{Normal}(3, 16)$

The mean and variance of $X_1 + X_2$ will equal sum of mean and sum of variance respectively.

(a) $X_1 + X_2$

mean $= 1 + 3 = 4$

variance $= 9 + 16 = 25$

(b) $-X_2$

mean $= -3$

variance $= 16$ ; because $\sigma^2 = -4^2$
$= 16$

(c) $X_1 - X_2$ This will be same as (a) except we have to subtract

mean $= 1 - 3 = -2$

Variance $= \sigma_1^2 - \sigma_2^2 = (-3)^2 + (-4)^2 = 9 + 16 = 25$

Because variance cannot be -ve

(d) $2X_1$

This is same as writing $X_1 + X_1$

$\therefore$, mean $= 2(1) = 2$

variance $= (2 \cdot 3)^2 = 6^2 = 36$

(e) $2X_1 \sim \text{Normal}(2, 36)$

$2X_2 \rightarrow$ mean $= 2(3) = 6$

variance $= (2(4))^2 = 8^2 = 64$

$2X_1 - 2X_2$ mean $= 2 - 6 = -4$

variance $= 2\sigma_1^2 - 2\sigma_2^2$

$= 36 + 64 = 100$

5. Given PDF:

$$f(x) = \begin{cases} 0.1 & 0 \le x < 1 \\ 0.2 & 1 \le x < 2 \\ 0.4 & 2 \le x < 3 \\ 0.3 & 3 \le x < 4 \\ 0 & \text{otherwise} \end{cases}$$

To calculate the CDF we need to calculate the integral in that interval up until $y$.

For $y < 0$, $F(0) = 0$

For $0 \le y < 1 \Rightarrow \int_0^y 0.1 \cdot dx = 0.1 [x]_0^y$

$= 0.1y$

For $1 \le y < 2 \Rightarrow \int_1^y 0.2 dx + \int_0^1 0.1 \cdot dx$

$= 0.2y - 0.2 + 0.1 = 0.2y - 0.1$

For $2 \le y < 3 \Rightarrow \int_0^1 0.1 \cdot dx + \int_1^2 0.2 \cdot dx + \int_2^y 0.4 \cdot dx$

$= 0.1 + 0.2[x]_1^2 + 0.4[x]_2^y$

$= 0.1 + 0.2 + 0.4y - 0.8 = 0.4y - 0.5$

For $3 \le y \le 4 \Rightarrow \int_0^1 0.1\,dx + \int_1^2 0.2\,dx + \int_2^3 0.4 \cdot dx +$

$\int_3^y 0.3\,dx$

$= 0.1 + 0.2 + 0.4[x]_2^3 + 0.3[x]_3^y$

$= 0.3 + 0.4[3-2] + 0.3[y-3]$

$= 0.3 + 0.4 + 0.3y - 0.9 = -0.2 + 0.3y$

CDF

$$F(y) = \begin{cases} 0 & y < 0 \\ 0.1y & 0 \le y < 1 \\ 0.2y - 0.1 & 1 \le y < 2 \\ 0.4y - 0.5 & 2 \le y < 3 \\ 0.3y - 0.2 & 3 \le y < 4 \\ 1 & y \ge 4 \end{cases}$$

(b) Median

$$F(q_2) = 0.5$$

Let's try the interval $2 \le y < 3$

$$0.4y - 0.5 = 0.5$$

$$y = \frac{1}{0.4} = 2.5$$

And 2.5 lies in b/w 2 & 3

$$\boxed{\text{The mean is } > 2}$$

(c) $$E(x) = \int_0^\infty x \cdot f(x) \cdot dx$$

$$= \int_0^1 0.1 \cdot x \cdot dx + \int_1^2 0.2 \cdot x \cdot dx + \int_2^3 0.4 \cdot x \cdot dx + \int_3^4 0.3 \cdot x \cdot dx$$

$$= 0.1\left[\frac{x^2}{2}\right]_0^1 + 0.2\left[\frac{x^2}{2}\right]_1^2 + 0.4\left[\frac{x^2}{2}\right]_2^3 + 0.3\left[\frac{x^2}{2}\right]_3^4$$

$$= 0.1\left[\frac{1}{2}\right] + 0.2\left[2 - 0.5\right] + 0.4\left[4.5 - 2\right] + 0.3\left[8 - 4.5\right]$$

$$= \frac{0.1}{2} + 0.3 + 1 + 1.05 = \boxed{2.4}$$

$\therefore$ Expected value is greater than 2.

6. given

$g: R \to R$

$$g(x) = \begin{cases} 0 & x < 0 \\ x & x \in [0,1] \\ 1 & x \in [1,2] \\ 3-x & x \in [2,3] \\ 0 & x > 3 \end{cases}$$

(a) What is C such that f is a PDF?

$$\int_{-\infty}^{\infty} f(x) \cdot dx = 1$$

$$c \left[ \int_0^1 x \cdot dx + \int_1^2 1 \cdot dx + \int_2^3 (3-x)\, dx \right] = 1$$

$$c \left[ \left[\frac{x^2}{2}\right]_0^1 + [x]_1^2 + \left[ 3[x]_2^3 - \left[\frac{x^2}{2}\right]_2^3 \right] \right] = 1$$

$$\Rightarrow c\left(\left(\frac{1}{2}\right)^2 + 2 + \left(3 - \frac{3^2}{2} - \frac{2^2}{2}\right)\right) = 1$$

$$\Rightarrow c\left(\frac{1}{2} + 1 + \frac{1}{2}\right) = 1$$

$$2c = 1 \Rightarrow \boxed{c = 0.5}$$

(b) $P(1.5 < X < 2.5)$

$$\Rightarrow \int_{1.5}^{2.0} 1 \times 0.5 \cdot dx + \int_{2.0}^{2.5} 0.5 \times (3 - x) \cdot dx$$

$$= 0.5\left[x\right]_{1.5}^{2} + 1.5\left[x\right]_{2}^{2.5} - 0.5\left[\frac{x^2}{2}\right]_{2}^{2.5}$$

$$= 0.5\left[2 - 1.5\right] + 1.5\left[2.5 - 2\right] - 0.5\left[\frac{2.5^2}{2} - \frac{4}{2}\right]$$

$$= 0.4375$$

(c) $E(x) = \int x \cdot f(x) \cdot dx$

$$\Rightarrow \int_0^1 0.5x^2 + \int_1^2 0.5x \cdot dx + \int_2^3 0.5x(3-x)dx$$

$$E(x) = 0.5\left[\frac{x^3}{3}\right]_0^1 + 0.5\left[\frac{x^2}{2}\right]_1^2 + 0.5\left[3 \cdot \left[\frac{x^2}{2}\right]_2^3 - \left[\frac{x^3}{3}\right]_2^3\right]$$

(d) $F(1) = ?$

(e) $F(2) = F(1) + \int_1^2 0.5\,dx$

$$F(2) = 0.25 + 0.5(1)$$

$$F(2) = 0.75$$

$$\int_2^c 0.5 \times (3-x)\,dx = 0.5\left[\int_2^c 3\,dx - \int_2^c x\,dx\right]$$

$$= 0.5\left[3[x]_2^c - \left[\frac{x^2}{2}\right]_2^c\right] = 0.9 - 0.75$$

$$= 0.5\left[3c - 6 - \left[\frac{c^2}{2} - \frac{4}{2}\right]\right] = 0.15$$

$$= 0.5\left[3c - 6 - \frac{c^2}{2} + 2\right] = 0.5\left[3c - \frac{c^2}{2} - 4\right]$$

$\times 2$

$$-2\left[3c - \frac{c^2}{2} - 4\right] = -0.3$$

$$+c^2 - 6c + 8 = -0.3$$