Q! $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Finding the eigen values we find those values of λ for which $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 4 & 5-\lambda & 6 \\ 7 & 8 & 9-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)\begin{vmatrix} 5-\lambda & 6 \\ 8 & 9-\lambda \end{vmatrix} - (2)\begin{vmatrix} 4 & 6 \\ 7 & 9-\lambda \end{vmatrix} + (3)\begin{vmatrix} 4 & 5-\lambda \\ 7 & 8 \end{vmatrix}$$

$$= (1-\lambda)((5-\lambda)(9-\lambda) - (6)(8)) - (2)((4)(9-\lambda) - (6)(7)) + (3)((4)(8) - (7)(5-\lambda))$$

$$= (1-\lambda)(45 - 5\lambda - 9\lambda + \lambda^2 - 48) - (2)(36 - 4\lambda - 42) + (3)(32 - 35 + 7\lambda)$$

$$= (1-\lambda)(\lambda^2 - 14\lambda - 3) + (-2)(-4\lambda - 6) + (3)(7\lambda - 3)$$

$$= \lambda^2 - 14\lambda - 3 - \lambda^3 + 14\lambda^2 + 3\lambda + 8\lambda + 12 + 21\lambda - 9$$

$$= -\lambda^3 + 15\lambda^2 + 18\lambda$$

Now we solve for $\det(A - \lambda I) = 0$

$$\lambda^3 - 15\lambda^2 - 18\lambda = 0$$

$\boxed{\lambda = 0 \text{ is a root}}$

$$\therefore (\lambda)(\lambda^2 - 15\lambda - 18) = 0$$

$$\lambda = \frac{-(-15) \pm \sqrt{225 - 4(1)(-18)}}{2(1)} = \frac{15 \pm \sqrt{297}}{2}$$

Eigen values

$\boxed{\lambda = 0}$

$\boxed{\lambda = \frac{15 + 3\sqrt{33}}{2}}$

$\boxed{\lambda = \frac{15 - 3\sqrt{33}}{2}}$

Q2 vector X – q dimensional   Y – p dimensional

$$\underset{p\times 1}{y} = \underset{p\times q}{A}\,\underset{q\times 1}{x}$$

① A has to be $p\times q$ for diimensionality reduction

② Euclidean distance – For two ~~matrices~~ vectors p and q,
the euclidean distance $= (p-q)^T(p-q)$

Given, euclidean distance between y_1 and y_2 is same as x_1 and x_2.

$$(y_1-y_2)^T(y_1-y_2) = (x_1-x_2)^T(x_1-x_2)$$

$$y_1 = Ax_1 \qquad y_2 = Ax_2$$

$$(A(x_1-x_2))^T(A(x_1-x_2)) = (x_1-x_2)^T(x_1-x_2)$$

$$((x_1-x_2)^T A^T)(A(x_1-x_2)) = (x_1-x_2)^T(x_1-x_2)$$

$$\cancel{(x_1-x_2)^T}\, A^T A\, \cancel{(x_1-x_2)} = \cancel{(x_1-x_2)^T}\,\cancel{(x_1-x_2)}$$

$$A^T A = I$$

Only possible when $A^T = A^{-1}$

③ (a) $q=2$ $p=2$ Let $y = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$y = Ax$$

$$\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x_1 + 2x_2 = 1/2$$
$$x_3 + 2x_4 = 1$$

one possible solution is $A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$

$$[\;] = [\;][\;] \qquad [\;] = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

2x1   2x4 4x1   1x1   1x2 2x1

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad 1+4+9+16 \quad 4+6+6+4$$

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(b) $q = 2$ and $p = 1$ Let $y = 3$ and $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$y = Ax \qquad 3 = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x_1 + 2x_2 = 3$$

One possible solution is $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$

(c) $q = 4$ $p = 2$ Let $y = \begin{bmatrix} 30 \\ 20 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$y = Ax$ One possible $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$

Q3   Line $l \Rightarrow \quad w_1 x_1 + w_2 x_2 + w_3 = 0$

~~$[\mu_1, \mu_2]$~~

(1) $D_1 \Rightarrow \quad \{ x_i = [x_1^i, x_2^i]^T \}$

$\mu \Rightarrow \quad [\mu_1, \mu_2]$

$$A' = [x_1' - \mu_1, x_2' - \mu_2][x_1' - \mu_1, x_2' - \mu_2]^T$$

Let us define $x = x_1^i - \mu_1 \;\forall i$, $y = x_2^i - \mu_2 \;\forall i$

$A' = [x, y][x, y]^T$ Also, here $y = mx + c$ line equation

$$A' = \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix} = \begin{bmatrix} x^2 & mx^2 + cx \\ mx^2 + cx & m^2x^2 + 2mx + c^2 \end{bmatrix}$$

Converting to it's row echelon form using $R_2 \rightarrow R_2 - \left(m + \frac{c}{x}\right) R_1$

$$A' = \begin{bmatrix} x^2 & mx^2 + cx \\ 0 & 0 \end{bmatrix}$$

The number of eigen values is the number of non zero rows in the row echelon form. Here it is 1.

Now, $A = \frac{1}{N} \sum_{i=1}^{N} [x_i - \mu][x_i - \mu]^T$

$$= \begin{bmatrix} \Sigma x_i^2 & m\Sigma x_i^2 + c\,\Sigma x_i^2 \\ m\Sigma x_i^2 + c\Sigma x & m^2 \Sigma x_i^2 + c^2 + 2cm\Sigma x \end{bmatrix}$$

If we subtract the mean from every point, the point corresponding to the mean itself is at $(0,0)$

$\therefore \quad c = 0$

$$A = \begin{bmatrix} \Sigma x_i^2 & m\Sigma x_i^2 \\ m\Sigma x_i^2 & m^2 \Sigma x_i^2 \end{bmatrix} \qquad R_2 \rightarrow R_2 - mR_1$$

$$A = \begin{bmatrix} \Sigma x_i^2 & m\Sigma x_i^2 \\ 0 & 0 \end{bmatrix} \qquad \text{No. of non-zero eigen values} = 1$$

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(2) line l $w_1 x_1 + w_2 x_2 + w_3 = 0$

The slope of this line is $m = -\frac{w_1}{w_2}$

$\therefore$ The slope of the line perpendicular to this is $m = \frac{w_2}{w_1}$

The equation of the line is $y = \frac{w_2}{w_1} x + c$

$D_2 = \{x_j = [x_1^j, x_2^j]^T\}$

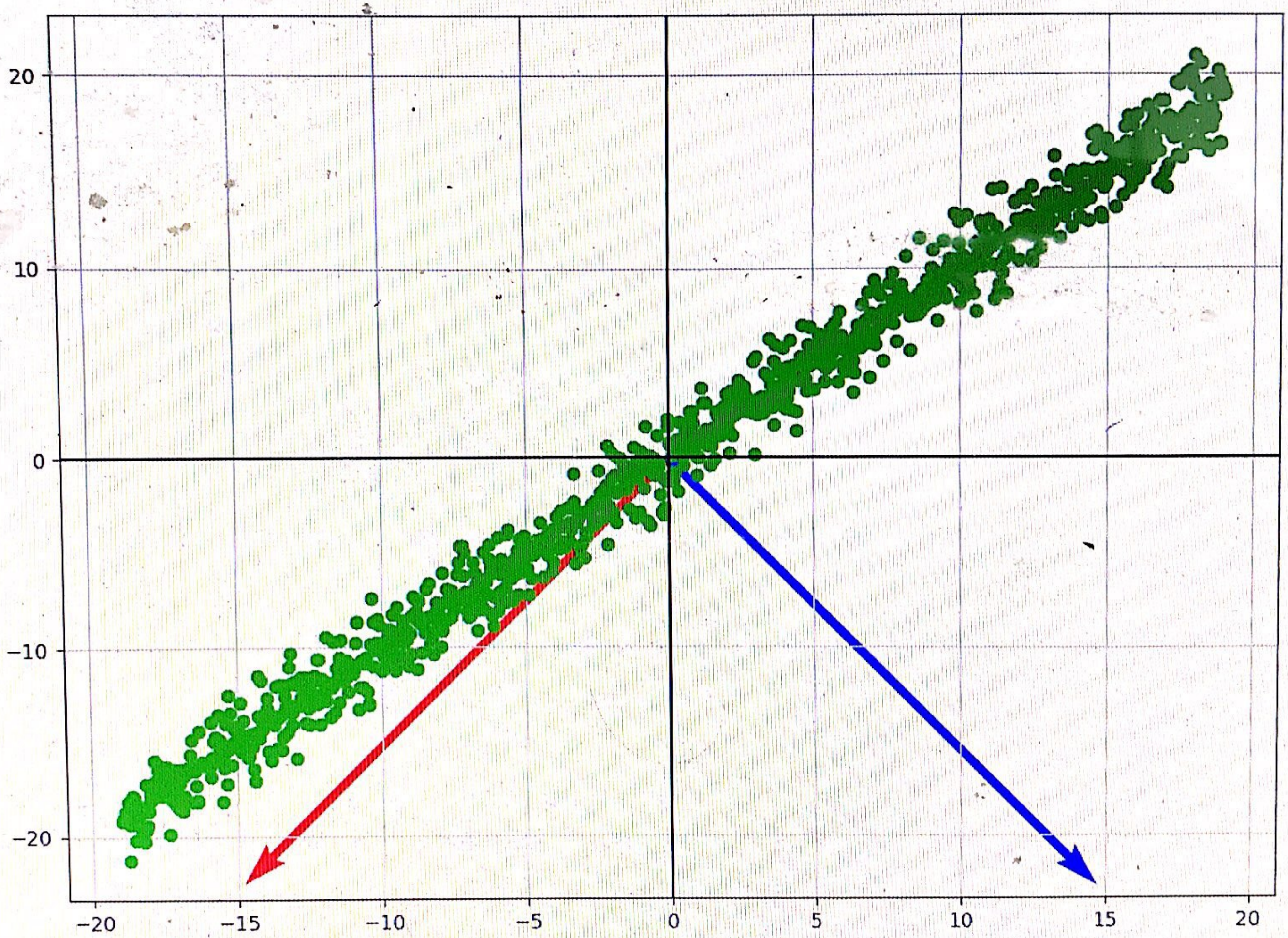
$\mu = [\mu_1, \mu_2]^T$

μ_1, μ_2 is a point on this line, so $c = w_1 \mu_2 - w_2 \mu_1$

$$y = \frac{w_2}{w_1} x + w_1 \mu_2 - w_2 \mu_1$$

This is the same equation as the previous two lines. so the number of eigen values is also the same.

(3) The points around the main line. The points will be random, so the co-variance matrix will have two eigen values and ~~their~~ second very small compared to the first one as the points will lies close to the 1D line. It will only be large along one dimension. The eigen vector along the larger eigen value will be along the line as the points are scattered randomly, and the other eigen vector will be perpendicular to the lines (as points are scattered randomly.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