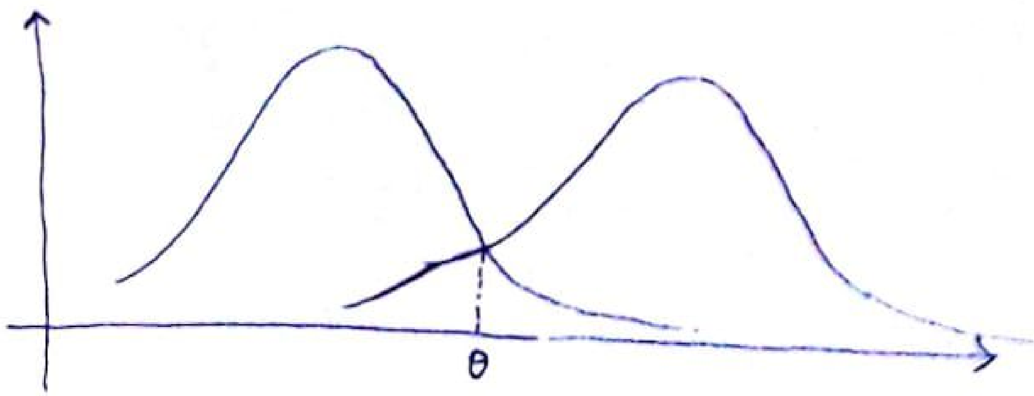
SMAI HW - 6 Problem 2



Let us assume the prior probabilities for background and foreground be $P(Bg)$, $P(Fg)$ respectively.

Let $I_{x,y}$ be the intensity value of cell (x, y).

If $I_{x,y} > \theta$, $I_{x,y}$ is a background pixel.

Else $I_{x,y}$ is a foreground pixel.

$$\cancel{P(I_{x,y} \mid \text{background}) =}$$

$$P(\text{background} \mid I_{x,y}) = P(I_{x,y} \mid \text{background})\, P(\text{background})$$

$$= P(Bg) \times \frac{1}{\sqrt{2\pi}\,\sigma_1} \exp\left(- \frac{(I_{x,y} - \mu_1)^2}{2\sigma_1^2} \right)$$

Similarly,

$$P(\text{foreground} \mid I_{x,y}) = P(I_{x,y} \mid \text{foreground})\, P(\text{foreground})$$

$$= P(Fg) \times \frac{1}{\sqrt{2\pi}\,\sigma_2} \exp\left(- \frac{(I_{x,y} - \mu_2)^2}{2\sigma_2^2} \right)$$

Let θ = value of intensity for which probability of density of background = probability of density of foreground.

$$\Rightarrow \frac{P(Bg)}{\cancel{\sqrt{2\pi}}\,\sigma_1} \exp\left(-\frac{(\theta-\mu_1)^2}{2\sigma_1^2}\right)$$

$$= \frac{P(Fg)}{\cancel{\sqrt{2\pi}}\,\sigma_2} \exp\left(-\frac{(\theta-\mu_2)^2}{2\sigma_2^2}\right).$$

$$\Rightarrow \log\left(\frac{P(Bg)}{\sigma_1}\right) + \log\left(\exp\left(-\frac{(\theta-\mu_1)^2}{2\sigma_1^2}\right)\right)$$

$$= \log\left(\frac{P(Fg)}{\sigma_2}\right) + \log\left(\exp\left(-\frac{(\theta-\mu_2)^2}{2\sigma_2^2}\right)\right)$$

$$\Rightarrow \log(P(Bg)) - \log(\sigma_1) - \frac{(\theta-\mu_1)^2}{2\sigma_1^2}$$

$$= \log(P(Fg)) - \log(\sigma_2) - \frac{(\theta-\mu_2)^2}{2\sigma_2^2}.$$

$\Rightarrow$ a) $\cancel{\log(P(Bg))} - \cancel{\log(\sigma_1)} - \frac{(\theta-\mu_1)^2}{2\sigma_1^2}$

$= \cancel{\log(P(Fg))} - \cancel{\log(\sigma_2)} - \frac{(\theta-\mu_2)^2}{2\sigma_2^2}$ [$\because$ Given, $P(Bg) = P(Fg)$ & $\sigma_1 = \sigma_2$]

$$\Rightarrow \frac{(\theta - u_1)^2}{\cancel{2}\cancel{\sigma_1^2}} = \frac{(\theta - u_2)^2}{\cancel{2}\cancel{\sigma_2^2}}.$$

$$\Rightarrow \theta - u_1 = \pm(\theta - u_2).$$

[$\theta - u_1 = \theta - u_2$ is in valid because $\Rightarrow u_1 = u_2$]

$$\Rightarrow \theta - u_1 = -\theta + u_2.$$

$$\Rightarrow 2\theta = u_1 + u_2.$$

$$\boxed{\theta = \frac{u_1 + u_2}{2}.}$$

$\Rightarrow$ Putting $\theta = \frac{u_1 + u_2}{2}$.

$$\log(P(By)) - \log \sigma_1 - \frac{\left(\frac{(u_1+u_2)}{2} - u_1\right)^2}{2\sigma_1^2}$$

$$= \log(P(Fy)) - \log \sigma_2 - \frac{\left(\frac{(u_1+u_2)}{2} - u_2\right)^2}{2\sigma_2^2}.$$

$$\Rightarrow \log\left(\frac{P(By)}{P(Fy)}\right) - \log\left(\frac{\sigma_1}{\sigma_2}\right)$$

$$= \frac{\left(\frac{u_2 - u_1}{2}\right)^2}{2\sigma_1^2} - \frac{\left(\frac{u_1 - u_2}{2}\right)^2}{2\sigma_2^2}$$

$$\Rightarrow \log\left(\frac{P(By)}{P(Fy)}\right) - \log\left(\frac{\sigma_1}{\sigma_2}\right)$$

$$= \frac{(u_2-u_1)^2}{8\sigma_1^2} - \frac{(u_2-u_1)^2}{8\sigma_2^2}$$

$$= \frac{(u_2-u_1)^2}{8}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right).$$

$$\Rightarrow \log\left(\frac{P(By)}{P(Fy)}\right) - \log\left(\frac{\sigma_1}{\sigma_2}\right) = \frac{(u_2-u_1)^2}{8}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right)$$

c) $$\log(4 \times P(Fy)) - \cancel{\log(\sigma_1)} - \frac{(\theta-100)^2}{2\sigma_1^2}$$

$$= \log(P(Fy)) - \cancel{\log(\sigma_2)} - \frac{(\theta-200)^2}{2\sigma_2^2}$$

$$\Rightarrow \log(4) + \cancel{\log(P(Fy))} - \frac{(\theta-100)^2}{2\sigma_1^2}$$

$$= \cancel{\log(P(Fy))} - \frac{(\theta-200)^2}{2\sigma_2^2}$$

$$\Rightarrow \log(4) = \frac{(\theta^2 - 200\theta + 10000)}{2\sigma_1^2} - \frac{(\theta^2 - 400\theta + 40000)}{2\sigma_2^2}$$

$$\Rightarrow 2\sigma_1^2 \log(4) = 200\theta - 30000$$

$$\Rightarrow \theta = \frac{2\log(4)\,\sigma_1^2 + 30000}{200} = \frac{\overset{2.773}{\cancel{1.204}}\,\sigma_1^2 + 30000}{200}.$$