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Assignment 1

Linear Algebra

1)
$$\langle A, B \rangle = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & & & & \\ a_{22} & & & & \\ \vdots & & & & \\ a_{2n} & & & & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & & & & \\ b_{22} & & & & \\ \vdots & & & & \\ b_{2n} & & & & b_{nn} \end{bmatrix}$$

$$= \{ a_{11} b_{11} + a_{12} b_{12} + a_{13} b_{13} \cdots$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ji} a_{ji}$$

$$B^T A = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & & & & \\ b_{31} & & & & \\ \vdots & & & & \\ b_{n1} & & & & b_{nn} \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} & \cdots & & a_{1n} \\ \vdots & & & & \\ a_{n1} & & & & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{21} & b_{31} & \cdots & b_{n1} \\ b_{12} & b_{22} & & & \\ b_{13} & & & & \\ b_{1n} & & & & b_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ \vdots & a_{22} & & \\ a_{n1} & & & a_{nn} \end{bmatrix}$$

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$$= \begin{bmatrix} b_{11}a_{11} + b_{21}a_{21} + b_{31}a_{31} + \dots & & b_{n1}a_{n1} \\ & b_{12}a_{12} + b_{22}a_{22} + \dots & b_{n2}a_{n2} \\ & & \\ & & b_{1n}a_{1n} + b_{2n}a_{2n} \dots b \end{bmatrix}$$

~~sum of product of~~ elements ~~item~~ summed over
~~Square~~ each ~~row~~ column with ~~with~~ every row.
~~element~~

$$\therefore \text{trace}(B^T A) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ji} a_{ji} = \cancel{\text{trace}}(A \quad \langle A, B \rangle$$

also can be proved more formally by using induction. Let base case be $n = 2$. ~~(AS)~~

$$\begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} + b_{21} + a_{21} & b_{11}a_{12} + b_{21}a \\ b_{12}a_{11} + b_{22}a_{21} & b_{12}a_{12} + b_{22} \end{bmatrix}$$

$$\text{trace}(B^T A) = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$$

$$\left\langle \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22} = \text{trace}(B$$

~~Assum~~ Hence proved.

Assume true until $n = k$

check for $n = k+1$

⋮

solve

3) Calculus & optimization.

1) a) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^3 + 2y$

$$\text{gradient} = \begin{bmatrix} 2x^2 \\ 2 \end{bmatrix}$$

$$\text{hessian} = \begin{bmatrix} 4x & 0 \\ 0 & 0 \end{bmatrix}$$

b) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \sin(x) + y\log(x)$

$$\text{gradient} = \begin{bmatrix} \cos x + \frac{y}{x} \\ \log x \end{bmatrix}$$ ~~Hessian~~ =

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$$\text{Hessian} = \begin{bmatrix} -\sin x \cancel{z} - \dfrac{y}{x^2} & \dfrac{1}{x} \\ \dfrac{1}{x} & 0 \end{bmatrix}$$

5) a) $\min\limits_{x,y} \quad 3x^2 + 2y^2 + x + 4y$

$$\text{gradient}: \begin{bmatrix} 6x+1 \\ 4y+4 \end{bmatrix} = 0$$

$6x+1 = 0$ | $4y = -4$

$x = -1/6$ | $y = -1$

$$\text{min value}: 3\left(\frac{-1}{6}\right)^2 + 2\left(-1\right)^2 + \left(\frac{-1}{6}\right) - 4$$

$$3\left(\frac{1}{36}\right) + 2 - \frac{1}{6} - 4$$

$$\frac{1}{12} + 2 - \frac{1}{6} - 4$$

$$-2 - \frac{1}{12} = \frac{-24}{12} - \frac{1}{12} = \frac{-23}{12}$$

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b) $\nabla f(x,y) = \lambda(\nabla g(x, \cancel{6} y))$

$$\begin{bmatrix} 6x+1 \\ 4y+4 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$6x+1 = 2\lambda x \rightarrow$ ~~x~~ $\cancel{1 = 2x(2\lambda - 6)}$

$4y+4 = 2\lambda y$

$x^2+y^2 = \cancel{0}1$

$6x+1 = 2\lambda x$

$1 = x(\lambda 2 - 6)$

$x = \dfrac{1}{\cancel{6-2\lambda}\ 2\lambda - 6}$

$4y+4 = 2\lambda y$

$4 = y(2\lambda - 4)$

$y = \dfrac{4}{2\lambda - 4}$

$$\left(\frac{1}{2\lambda-6}\right)^2 + \left(\frac{4}{2\lambda-4}\right)^2 = 1$$

$$(2\lambda-4)^2 + 16(2\lambda-6)^2 = (2\lambda-6)^2(2\lambda-4)^2$$

$4\lambda^2 + 16 - \cancel{4}16\lambda + 16(4\lambda^2+36-24\lambda) = \cancel{(2\lambda-6)^2}$

$= (4\lambda^2+16-16\lambda)(4\lambda^2+36-24\lambda)$

$52\lambda^2 + \cancel{576}\ 592 - 368\lambda = 16\lambda^4 + 144\lambda^2 - 96\lambda^3$

$+ 64\lambda^2 + 576 - 384\lambda$

3
36
×16
216
360
576

2
24
×16
144

On input to online multivariable simultaneous eq. solver..

$x = 0.416$

$y = 0.909$

$\lambda\ \cancel{z} = 4.2$

is one of the solns. others are

$x = 1.8i - 0.625$	$x = -1.8i - 0.625$
$y = 0.55i + 2.3$	$y = 2.3 - 0.55i$
$\lambda\ \cancel{z} = 2.91 - 0.247i$	$\lambda = 0.247i + 2.91$

$x = \cancel{0.41}\quad -0.165$

$y = \cancel{0.910}\quad -0.98$

$z = \cancel{4.2}\quad -0.27$

2) $h_\theta(x) = g(\theta^T x) = g(\theta^T x) = \dfrac{1}{1 + \exp(-\theta^T x)}$

$J(\theta) =$

$$\frac{1}{m}\sum_{i=1}^{m}\left[-y^{(i)}\log\left(\frac{1}{1+e^{\theta^T x^{(i)}}}\right) - (1-y^{(i)})\log\left(1 - \frac{1}{1+e^{-\theta^T x^{(i)}}}\right)\right.$$

$$= \frac{1}{m}\sum_{i=1}^{m}\left[-y^{(i)}\log\left(\frac{1}{1+e^{\theta^T x^{(i)}}}\right) - (1-y^{(i)})\log\left(\frac{e^{-\theta^T x^{(i)}}}{1+e^{-\theta^T x^{(i)}}}\right)\right]$$

Diff wrt θ.

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$$= \frac{1}{m}\sum_{i=1}^{m}\Bigg[-y^i\left(1+e^{-\theta^T x^{(i)}}\right)(-x^{(i)}) \cancel{(1-y^{(i)})}$$

$$- (1-y^{(i)})\left(\frac{1}{e^{-\theta^T x^i}}\right)(-x^{(i)})$$

$$+ (1-y^{(i)})\left(1+e^{-\theta^T x^{(i)}}\right)(-x^{(i)})$$

$$\Bigg]$$

4) $E = \|Ax - b\|^2$

$Ax = b \rightarrow$ $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$

$\therefore$, $E = (C-0)^2 + (C+D-8)^2 + (C+3D-8)^2 + (C+4D-20)^2$

$$\frac{\partial E}{\partial C} = 0 = 2C + 2(C+D-8) + 2(C+3D-8) + 2(C+4D-20)$$

$$= \cancel{2C} + 8C + 16D - 72 = 0$$

$$4C + 8D = 36$$

$$C + 2D = 9 \quad (1)$$

$$\frac{\partial E}{\partial D} = 2C + 2(C+D-8) + 6(C+3D-8) + 8(C+4D-20)$$

$$= 16C + 52D - 224 = 0$$

$$16C + 52D = 224$$

$18 \overline{)224}$
18
44

$$A^T A x = A^T b$$

$$AA^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 5 \\ 1 & 4 & 10 & 13 \\ 1 & 5 & 13 & 17 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

$$4C + 8D = 36$$

$$8C + 26D = 112$$

which is same as diff eq obtained

4. Rigid Body Transformations

2) One at a time as the no. of mults & adds will be lower as we are doing matrix × vector product at each step rather than matrix × matrix product that is done in the other ~~m~~ method.

3) $^{B}p = {}^{B}R_{A}\,{}^{A}p$

a) $^{A}p = R_x(30^\circ)\,R_z(20^\circ)\,{}^{B}p$

$$^{B}p = \underbrace{R_z^{-1}(20^\circ)\,R_x^{-1}(30^\circ)}_{^{B}R_{A}}\,{}^{A}p$$

$$^{B}R_{A} = \begin{bmatrix} \cos 20 & \sin 20 & 0 \\ -\sin 20 & \cos 20 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30 & \sin 30 \\ 0 & -\sin 30 & \cos 30 \end{bmatrix}$$

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$$^{B}R_{A} = \begin{bmatrix} \cos 20 & \sin 20 \cos 30 & \sin 20 \sin 30 \\ -\sin 20 & \cos 20 \cos 30 & \cos 20 \sin 30 \\ 0 & -\sin 30 & \cos 30 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4081 & 0.1408 & -0.9020 \\ -0.9129 & 0.0629 & -0.4032 \\ 0 & 0.9880 & 0.1543 \end{bmatrix}$$

b) $\cancel{^{B}_{A}}\ {}^{A}P = \cancel{R_y(20^\circ)}\ R_z(20^\circ) R_x(20^\circ)\ {}^{B}P$

$$^{B}P = \underbrace{R_y^{-1}(20^\circ) R_z^{-1}(20^\circ)}_{^{B}R_{A}}\ {}^{A}P$$

compute similarly!

1) $^{A}P\cancel{^{A}} = R_{z(30)} R_{x(45^\circ)}\ {}^{B}P\cancel{^{B}}$

current axis rotation.

given rotation matrix $= R = R_{z(30^\circ)}\ R_{x(45^\circ)}$

computed on matlab!

$$= \begin{bmatrix} 0.1543 & 0.5190 & -0.8407 \\ -0.9880 & 0.0810 & -0.1313 \\ 0 & 0.8509 & 0.5253 \end{bmatrix}$$

4) a) b) substitute required value in ~~written~~ code for c).