EE49001: Control and Electronic System Design

Assignment-2: Inverted Pendulum, Part:1

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1. A pendulum consisting of a mass-less rod of length $l = 0.36m$ and a bob of mass $m = 0.26kg$ is attached to a cart of mass $M = 2.4kg$ moving on a rail. Let the position of the cart on the rail be denoted by x and the angle of the rod with respect to the upward vertical by θ. The cart is propelled by a horizontal force of u along the rail.

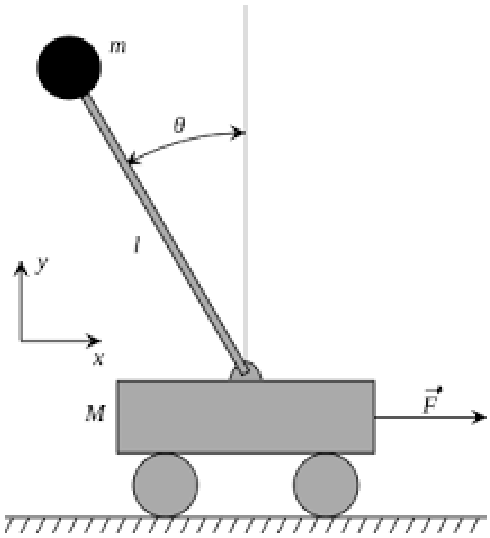

Fig: Diagram for Inverted Pendulum Setup

For the following system, we have 4 states: $z = [x, \dot{x}, \theta, \dot{\theta}]$, where:

$\dot{x} = v \equiv$ Translational velocity of the cart along x −direction

$\dot{\theta} = \omega$ Angular velocity of the rod, $F = u \equiv$ Horizontal force on the cart

1.1. Represent the nonlinear dynamics of the plant in the form $\dot{z} = f(z, u)$ where z denotes the states and u denotes the input

We can, using Newton's Laws of Motion, say that:

$$\dot{x} = v$$

$$\dot{\theta} =$$

$$(m + M)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = u$$

$$ml^2\ddot{\theta} = mglsin\theta - ml\ddot{x}cos$$

Considering the state space, $z = \begin{bmatrix} x \\ v \\ \theta \\ \omega \end{bmatrix}$

$$\dot{x} = f_1 = v$$

$$\dot{v} = f_2 = \frac{u + ml\omega^2\sin\theta - mg\sin\theta\cos\theta}{m\sin^2\theta + M}$$

$$\dot{\theta} = f_3 = \omega$$

$$\dot{\omega} = f_4 = \frac{g(m + M)\sin\theta - u\cos\theta - ml\omega^2\sin\theta\cos\theta}{(m\sin^2\theta + M)l}$$

1.2. Determine the equilibrium points of the above dynamical system.

The system is at equilibrium, means $\sum \vec{F}_{system} = 0$ and $\sum \vec{\tau}_{system} = 0$. It means, $\ddot{x} = \ddot{\theta} = 0$. Putting these values in the non-linear dynamics' equations, we get: $\theta \in \{0, \pi\}$. For equilibrium we have $\dot{z} = 0$. That is: $\dot{x} = \ddot{x} = \dot{\theta} = \ddot{\theta} = 0$.

1.3. Linearize the above nonlinear dynamics around the unstable equilibrium point which has $\theta = 0$, and obtain the linearized dynamics of the form: $\dot{z} = Az + Bu$ for suitable matrices A and B. Determine the dimensions of A and B.

For Linearization of the given state equations in the neighbourhood of $\theta = 0$ i.e. $\theta = \Delta\theta$ where $\Delta\theta \to 0\,°$, we can assume the small angle approximations to be valid. Therefore, we have $sin\theta \approx \theta$ and $cos\theta \approx 1$.

Hence, we can write:

$$ml^2\ddot{\theta} = mgl\,\theta - ml\ddot{x}$$

$$(m + M)\ddot{x} + ml\ddot{\theta} = u$$

For small θ, assuming $\dot{\theta}^2 \approx 0$

These are the linearized dynamics equations in the neighbourhood of $\theta = 0°$

The 4 nonlinear equations can be linearised in the form $\dot{z} = Az + Bu$, where the matrices are defined as follows:

$$[A_{ij}] = \left[\frac{\partial f_i}{\partial z_j}\right]_{z_0} \text{ and} [B_i] = \left[\frac{\partial f_i}{\partial u}\right]_{z_0}$$

z_0 is the unstable equilibrium point, $z_0 = 0$

Dimensionally, $A \equiv 4 \times 4$ and $B \equiv 4 \times 1$

z_0 is the unstable equilibrium point $z_0 = 0$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{(m+M)g}{Ml} & 0 \end{bmatrix} \backslash\backslash \text{ and } B = \begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ -\dfrac{1}{Ml} \end{bmatrix}$$

1.4. Obtain the transfer functions $\frac{\Theta(s)}{U(s)}$ and $\frac{X(s)}{U(s)}$ assuming θ to be close to zero.

Taking Laplace Transform we get

$$(m + M)s^2X + mls^2\Theta = U\text{\bigmmls}^2\Theta - mg\Theta + ms^2X = 0$$

Where,

$$X = X(s) = \mathcal{L}\{x(t)\} \| 1\Theta = \Theta(s) = \mathcal{L}\{\theta(t)\}\text{\bigmU} = U(s) = \mathcal{L}\{u(t)\}$$

On solving and simplifying we get,

$$\frac{X(s)}{U(s)} = \frac{\frac{1}{M}\left(s^2 - \frac{g}{l}\right)}{s^2\left(s^2 - \dfrac{(M+m)g}{Ml}\right)}$$

$$\frac{\Theta(s)}{U(s)} = -\frac{\dfrac{1}{Ml}}{s^2 - \dfrac{(M+m)g}{Ml}}$$

Numerical Values of transfer function:

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sys1 =

  0.4167 s^2 - 12.41
  ------------------
   s^4 - 30.17 s^2
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Fig: Transfer function $\frac{X(s)}{U(s)}$

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sys2 =

     -0.15 s^2
  ---------------
  s^4 - 30.17 s^2
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Fig: Transfer function $\frac{\Theta(s)}{U(s)}$

1.5. The pendulum is to be balanced in the inverted position with the cart being within the rail by applying a suitable u. Explain if the pendulum can be balanced using only θ or only x feedback. In other words, determine if the system is controllable when the input u is equal to either θ or x.

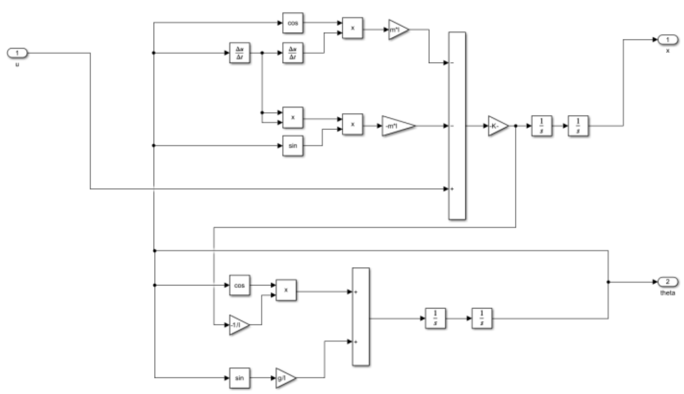
For $G(s) = \frac{\Theta(s)}{U(s)}$: There is a pole-zero cancellation for the pole at $s = 0$

$H(s) = \frac{X(s)}{U(s)}$: Though, theoretically, there is no pole-zero cancellation, but pole and zero are very close. Therefore, for all practical purposes, pole-zero cancellation can be considered for the system.

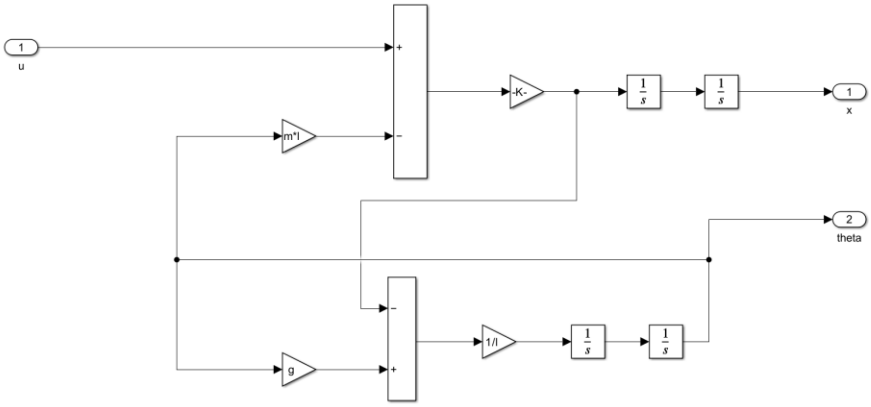
Though it is theoretically possible to control the system by only providing feedback of x, it is not practically possible.

Hence, the system is not controllable by providing feedback of only θ or x

1.6. Create a model of the nonlinear dynamical system in SIMULINK



1.7. Create a model of the linearized system in SIMULINK.



1.8. Plots for x and θ

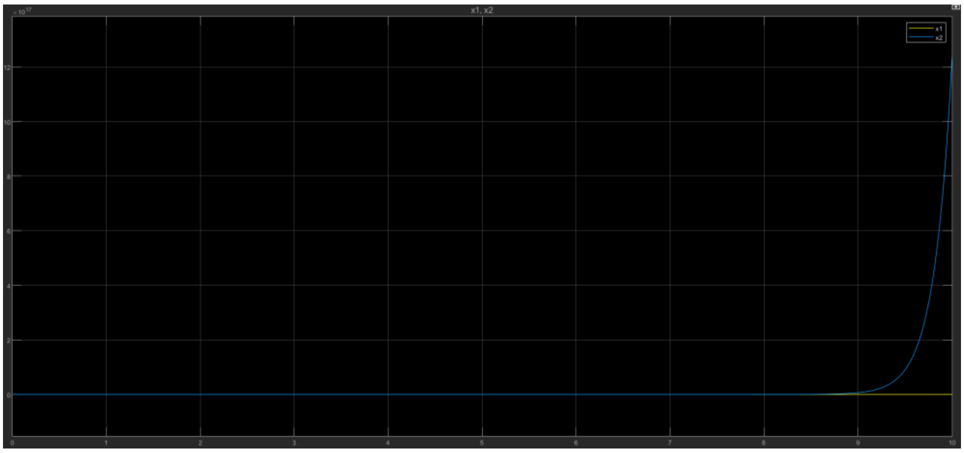


Fig. Comparison between x_1 (non-linear) and x_2 (linear)

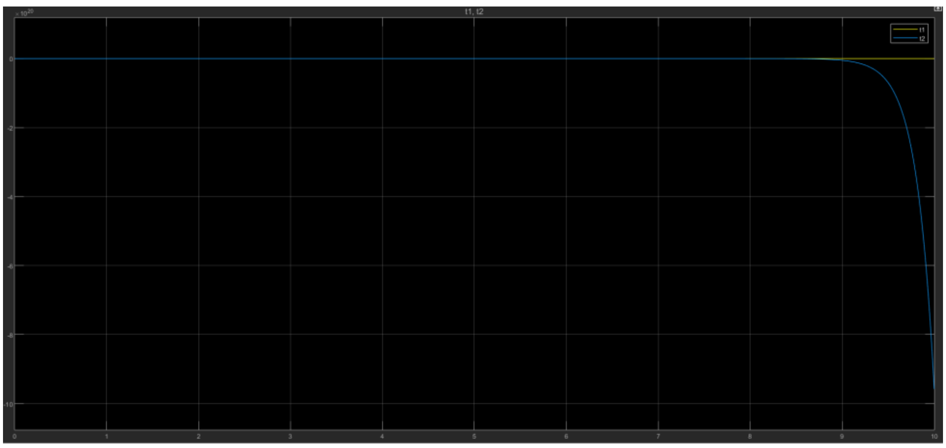


Fig. Comparison between θ_1 (non-linear) and θ_2 (linear)