Assignment 4: Deadline: 13/02/2024, 4:55pm

1. Consider the linear dynamical system in state-space form

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$
$$y(t) = Cx(t) + Du(t), \tag{2}$$

where $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The vector $x \in \mathbb{R}^4$ denotes the state variables, $u \in \mathbb{R}^2$ denotes the inputs, and $y \in \mathbb{R}^2$ denotes the outputs.

   Create a model in Simulink with four integrator blocks for the states and use matrix multiplication option in the gain blocks, and mux/demux blocks to convert scalar signals into vectors.

2. Excite the system with step inputs for both the input channels and plot the all the states in one figure and both the outputs in another figure.

3. Determine the transfer function between every pair of inputs and outputs. There would be four such transfer functions. Verify if the poles of the transfer functions coincide with the eigenvalues of matrix A. Determine if the above system is stable.

> The state equation (1) or the pair (A, B) is said to be *controllable* if for any initial state $x(0) = x_0$ and any final state x_1, there exists an input that transfers x_0 to x_1 in a finite time. Otherwise (1) or the pair (A, B) is said to be *uncontrollable*. Let $x \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^{n_u}$. A necessary and sufficient condition for controllability of the pair (A, B) is that the following matrix
>
> $$\mathcal{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n_x-1}B] \in \mathbb{R}^{n_x \times n_x n_u}$$
>
> has rank n_x.

4. Verify that the system (1) is controllable for the matrices specified above.

A consequence of the system being controllable is that a linear state-feedback controller of the form

$$u(t) = -Kx(t)$$

can be used to control the system by placing the poles of the closed-loop system $A - BK$ at any desired location under mild conditions. The matrix $K \in \mathbb{R}^{n_u \times n_x}$ can be computed using MATLAB command "$\texttt{K} = \texttt{place}(\texttt{A}, \texttt{B}, \texttt{p})$" where p is the vector of desired pole locations of the closed loop system.

5. Find K such that the eigenvalues of $A-BK$ are at $(-100, -200, -300, -400)$. Implement the above designed state-feedback controller in Simulink and plot the trajectory of the states. Let the initial state be $x_0 = [10, 20, -10, -20]$.

6. Repeat the above for desired eigenvalue/pole locations at $(-10, -20, -30, -40)$ and $(-1, -2, -3, -4)$ for the same initial conditions. Generate four figures, one for each state, and in each figure, show the trajectory of that state under the three different controller gains. Describe and explain the difference in trajectories.

The state equation (1) with output equation (2) or the pair (A, C) is said to be *observable* if for any initial state $x(0) = x_0$, there exists a finite time t_1 such that knowledge of input $u(t)$ and output $y(t)$ over $t \in [0, t_1]$ suffices to uniquely determine the initial state x_0. Otherwise (A, C) is said to be *unobservable*. A necessary and sufficient condition for observability of the pair (A, C) is that the following matrix

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n_x - 1} \end{pmatrix} \in \mathbb{R}^{n_x n_y \times n_x}$$

has rank n_x; here n_y denotes the dimension of y.

7. The above scheme is applicable when the all the states are being measured so that the control signal can be computed. If instead, we have access to output y rather than state x, we need to first estimate the states. State

estimation is possible if the pair (A, C) is observable. Determine if the system with matrices specified above is observable.

8. It can be shown that a pair (A, C) is observable if and only if the pair $(A^\top, C^\top)$ is controllable. Using this fact, find a gain matrix $L \in \mathbb{R}^{n_y \times n_x}$ such that the eigenvalyes of $A - LC$ are located at $p = (-10, -20, -30, -40)$.

9. Implement a Luenberger observer with the above observer gain L. In particular, create a replica of the original system in SIMULINK with four states denoted by $\widehat{x}$ which evolves as

$$\dot{\widehat{x}} = A\widehat{x} + Bu + L(y - C\widehat{x} - Du).$$

Assume that the initial state of this new auxiliary system $\widehat{x}(0) = 0$. Apply the same input $u(t)$ to both the original system and the new system. Compare the trajectories of $x(t)$ and $\widehat{x}(t)$.

10. Let the estimation error be defined as $e(t) := x(t) - \widehat{x}(t)$. Derive the dynamics of $e(t)$, i.e., derive $\dot{e}(t)$ in terms of $e(t)$ and determine conditions under which the error will converge to 0.

11. Instead of using $u(t) = -Kx(t)$, apply $u(t) = -K\widehat{x}(t)$ for K obtained by setting the eigenvalues of $A - BK$ at $p = (-10, -20, -30, -40)$ and discuss/explain how the response differs for the following three choice of eigenvalues of $A - LC$:

   - $p_1 = (-10, -20, -30, -40)$
   - $p_2 = (-1, -2, -3, -4)$
   - $p_3 = (-100, -200, -300, -400)$

   Discuss how one should choose the eigenvalues of the observer for a given set of eigenvalues of $A - BK$. The above scheme is called *observer based state feedback* design and only uses information about the output of the plant to both estimate and control the states.

12. Derive the complete closed-loop dynamics of the system with state $z(t) = [x(t) \quad e(t)]^\top$ and determine conditions under which it is stable. Show that K and L can be designed independently.