E969205: Signal Processing System Design
Indian Institute of Technology, Kharagpur

Operations on 1D Signals using DFT, DCT and DST Algorithms

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1. Objective

The objective of this experiment is to implement the Discrete Fourier Transform (DFT), Discrete Cosine Transform (DCT), and Discrete Sine Transform (DST) algorithms for one-dimensional (1D) signals, and to reconstruct these signals using their respective inverse transforms: Inverse DFT, Inverse DCT, and Inverse DST. Additionally, the experiment aims to evaluate how the reconstruction quality of these signals is affected by retaining only a subset of their coefficients and to analyze the impact of coefficient retention on reconstruction accuracy by measuring the Mean Squared Error (MSE) between the original and reconstructed signals.

Notations: During the course of this entire report, a uniform representation of notations is followed as follows:

- $x[n]$ represents the signal strength of n^{th} sample of a discrete signal
- $x(t)$ represents the strength of a continuous signal at time instant t
- $X(k)$ is the k^{th} transform coefficient
- $X(\omega)$ represents transform coefficient corresponding to frequency ω
- N represents total number of samples in a signal

2. Algorithms for Feature Extraction from Signals

2.1. Discrete Fourier Transform—The DFT transforms a discrete 1D signal $x[n]$ into its frequency components:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, \ldots, N-1$$

This provides the spectrum of the signal. The magnitude spectrum $|X[k]|$ represents the amplitude of each frequency component, while the phase spectrum $\arg(X[k])$ represents the phase shift.

2.2. Discrete Sine Transform—The DST transforms a discrete 1D signal $x[n]$ using sine functions:

$$X[k] = \sum_{n=1}^{N} x[n] \sin\left(\frac{\pi nk}{N+1}\right), \quad k = 0, 1, \ldots, N-1$$

It is useful in spectral analysis. For DST, the spectrum is purely real, hence it only has a magnitude spectrum, $|X[k]|$, without a phase component.

2.3. Discrete Cosine Transform—The DCT transforms a discrete 1D signal $x[n]$ using cosine functions:

$$X[k] = \sqrt{\frac{2}{N}} \cdot c_{(k)} \sum_{n=0}^{N-1} x[n] \cos\left(\frac{\pi}{N} k \left(n + \frac{1}{2}\right)\right), \quad k = 0, 1, \ldots, N-1$$

$$c_{(k)} = \begin{cases} \frac{1}{\sqrt{2}} & \text{k = 0} \\ 1 & \text{k = 1, 2, ..., N-1} \end{cases}$$

It is widely used in image and signal compression. Like DST, the DCT has a purely real spectrum, so it only has a magnitude spectrum, $|X[k]|$, and no phase component.

3. Inverse Transforms

3.1. Inverse Discrete Fourier Transform (IDFT)—The formula for the Inverse Discrete Fourier Transform (IDFT) is given by:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

3.2. Inverse Discrete Cosine Transform (IDCT)—The formula for the Inverse Discrete Cosine Transform (IDCT) is given by:

$$x[n] = \frac{1}{2} c_0 + \sum_{k=1}^{N-1} c_k \cos\left(\frac{\pi}{N} k \left(n + \frac{1}{2}\right)\right)$$

3.3. Inverse Discrete Sine Transform (IDST)—The formula for the Inverse Discrete Sine Transform (IDST) is given by:

$$x[n] = \sum_{k=1}^{N} X[k] \sin\left(\frac{\pi}{N+1} kn\right)$$

4. Operations on Periodic Signal

In this section, our focus will be on operations with a periodic signal. For all the following cases in this section, we'll be applying operations on the given signal:

$$x(t) = \sin(\omega t) + \frac{1}{3} \cos(3\omega t)$$

The sampling frequency F is chosen keeping the Nyquist Criterion under consideration. For all the following expeiments we consider, $\omega = 150Hz$ and sampling frequency $F_s = 1500Hz$. The given signal is passed through the DFT, DCT and DST functions made on Python and the output response is plotted.

We plot the magnitude and phase spectrum in case of a Discrete Fourier Transform and only the magnitude spectrume in case of Discrete Sine and Cosine Transforms. As DST and DCT are real-valued transforms, calculating phase for them makes no sense. Later on, we use the custom inverse functions in attempt to regain the original signal from their transformed versions.

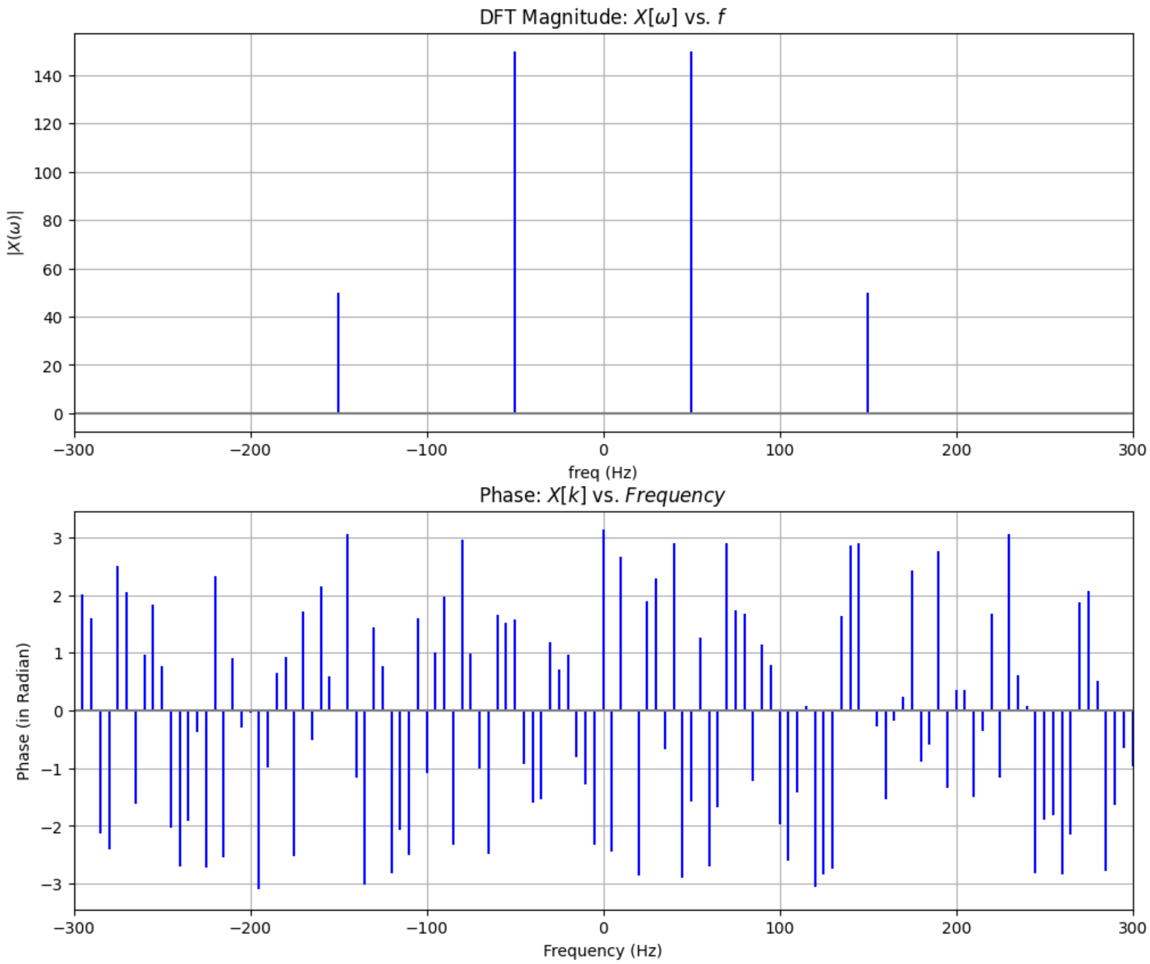

Fig. 1. Discrete Fourier Transform of input signal $x(t)$

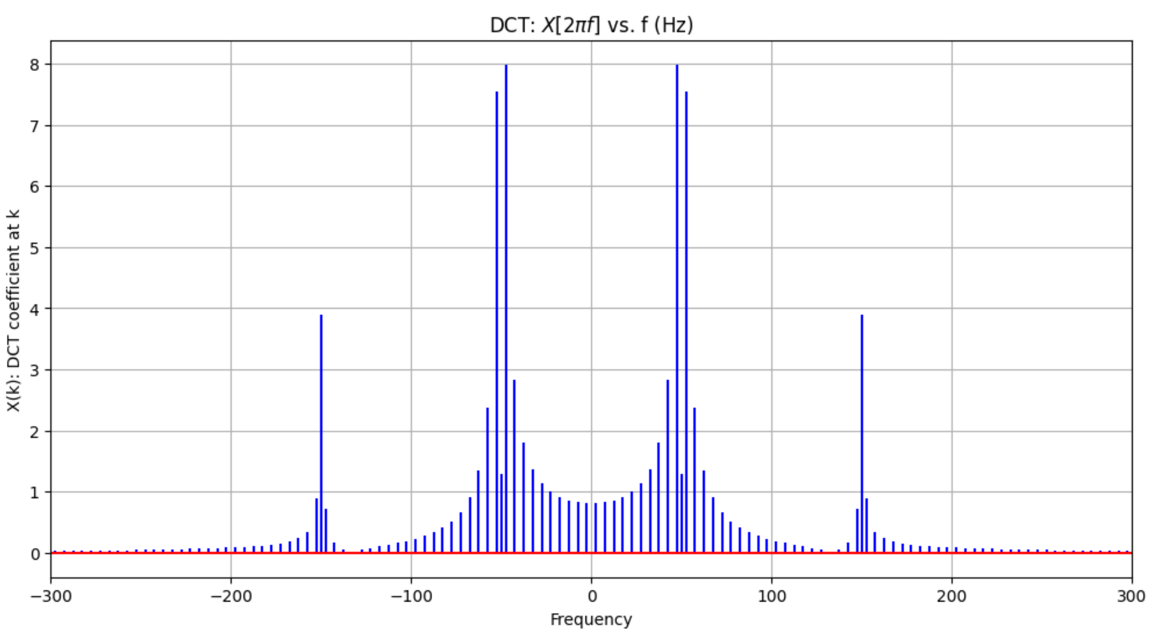

Fig. 2. Discrete Cosine Transform of Input Signal

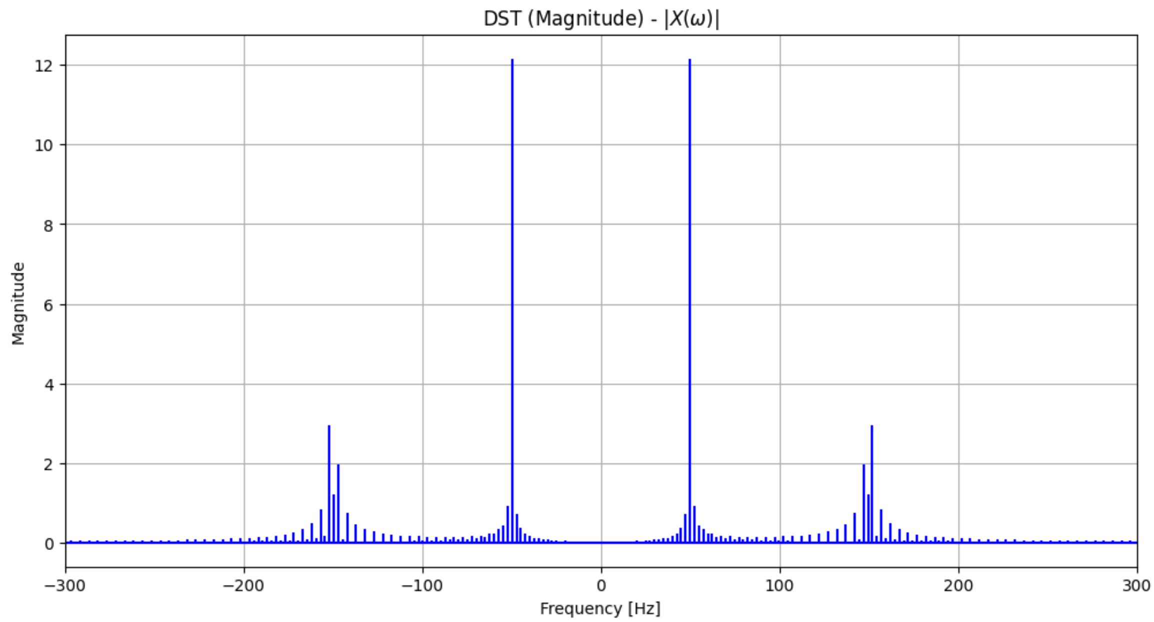

Fig. 3. Discrete Sine Transform of Input Signal

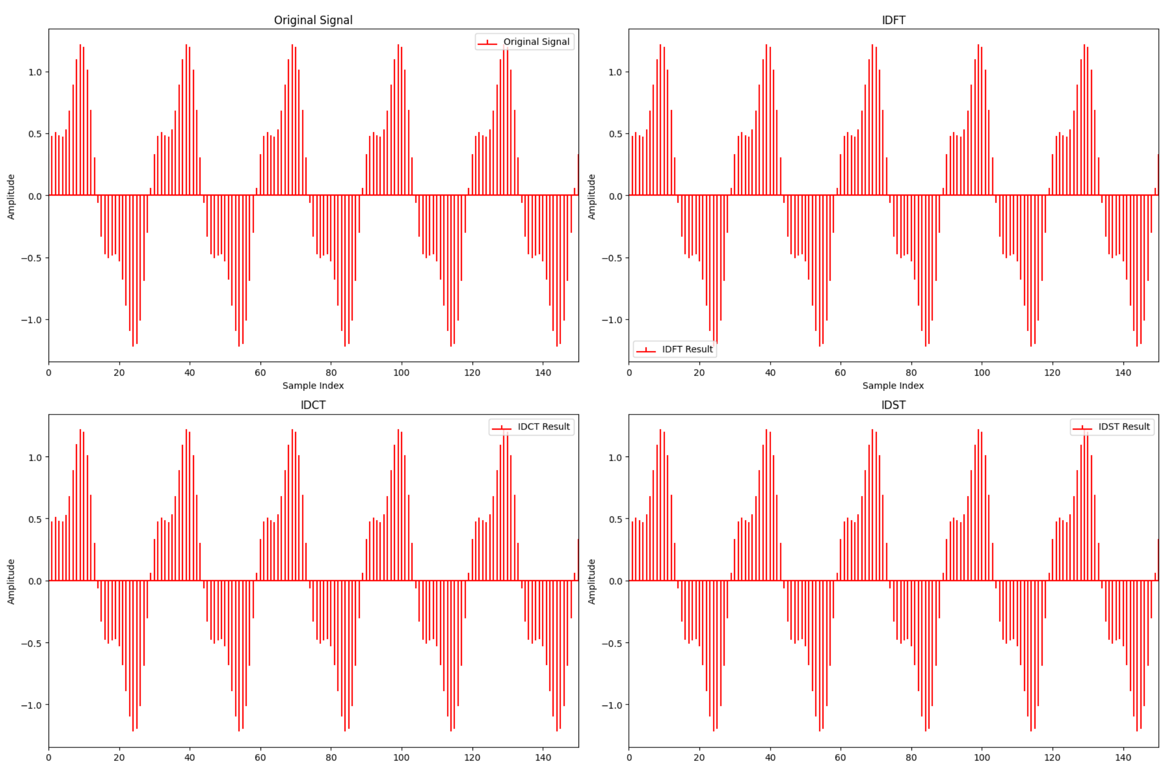

Fig. 4. Reconstruction of Original Signal using Inverse Transforms

5. Operations on Aperiodic Signal

In this section, our focus will be on operations with a periodic signal. For all the following cases in this section, we'll be applying operations on the given signal:

$$x[n] = \frac{1}{2^n}u[n]$$

To capture the behaviour of the signal, we use a sample length of 100. Using a higher sample length doesn't help us capture the behaviour of signal well because as $n \to \infty, x \to 0$. The given signal is passed through the DFT, DCT and DST functions made on Python and the output response is plotted.

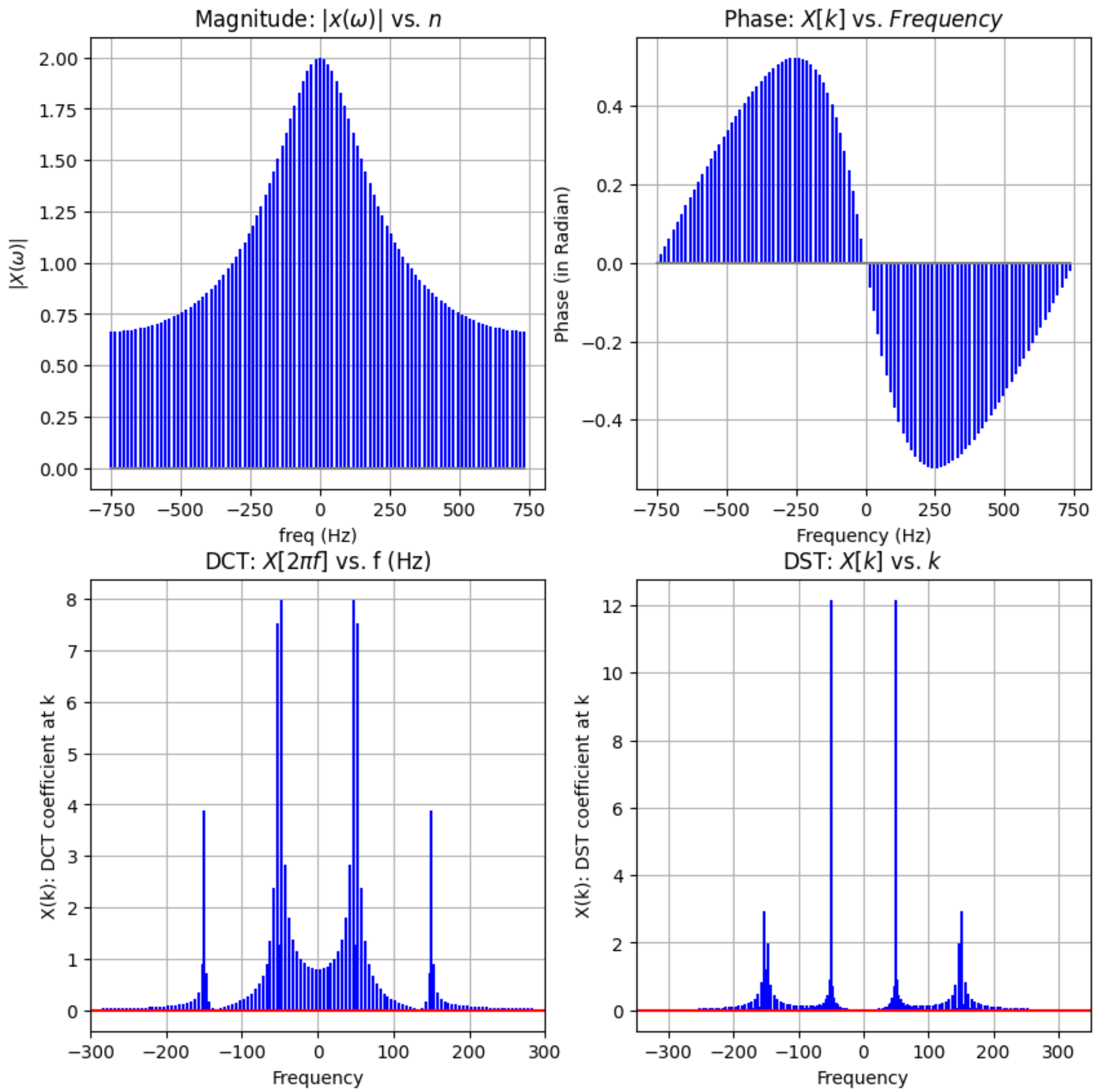

Fig. 5. DFT, DCT and DST applications on signal S_2

Following the transformation, we attempt to reconstruct the original signal from their transformed counterparts. We perfrom the reconstruction steps by retaining top $p\%$ of the coefficients in the corresponding transform (DFT, DCT and DST). The reconstruction is then performed and we calculate mean squared error between the reconstructed signal and original signal. The motive is to obtain a sweet-spot between the compression factor and the signal quality.

$$x[n] \xrightarrow{\text{DFT/DCT/DST}} X(\omega) \xrightarrow{\text{retain top } p\% \text{ coefficients}} \hat{X}(\omega) \xrightarrow{\text{IDFT/IDCT/IDST}} \hat{x}[n]$$

We want to find minimum value of p such that we have minimum value of $\|x[n]-\hat{x}[n]\|_2$. Thus we vary the percentage of retained coefficients in the transform and attempt to reconstruct the signal from those coefficients. In the table below, Mean Squared Error (MSE) values corresponding to different retention percentages and different techniques has been listed.

Table 1. Mean Squared Errors for Different Retention Percentages

	45%	**75%**	**90%**
DFT-IDFT	0.0032	0.0012	0.0004
DCT-IDCT	0.0003	0.0000	0.0000
DST-IDST	0.0020	0.0002	0.0000

Based on the different values of the Mean Squared Error, we conclude that Discrete Fourier Transform is the best compression technique that can increase the compression factor with the minimum reconstruction error.

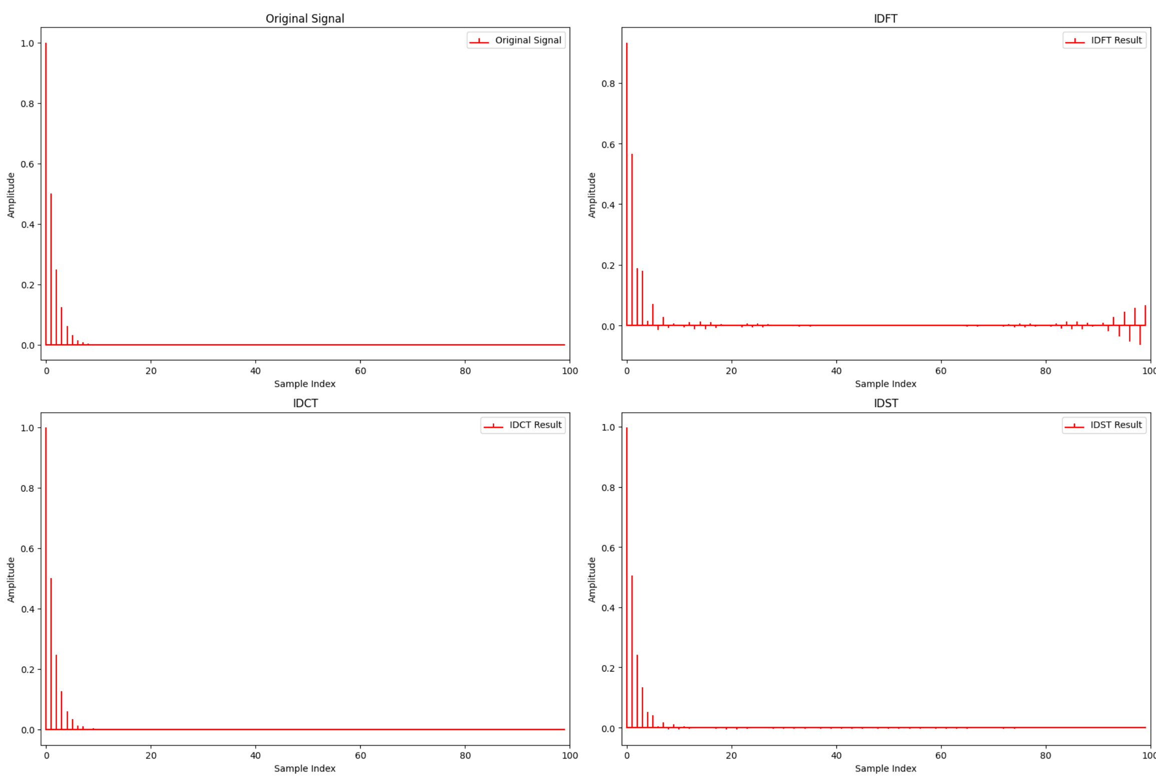

Fig. 6. Signal Reconstruction by retaining 90% coefficients

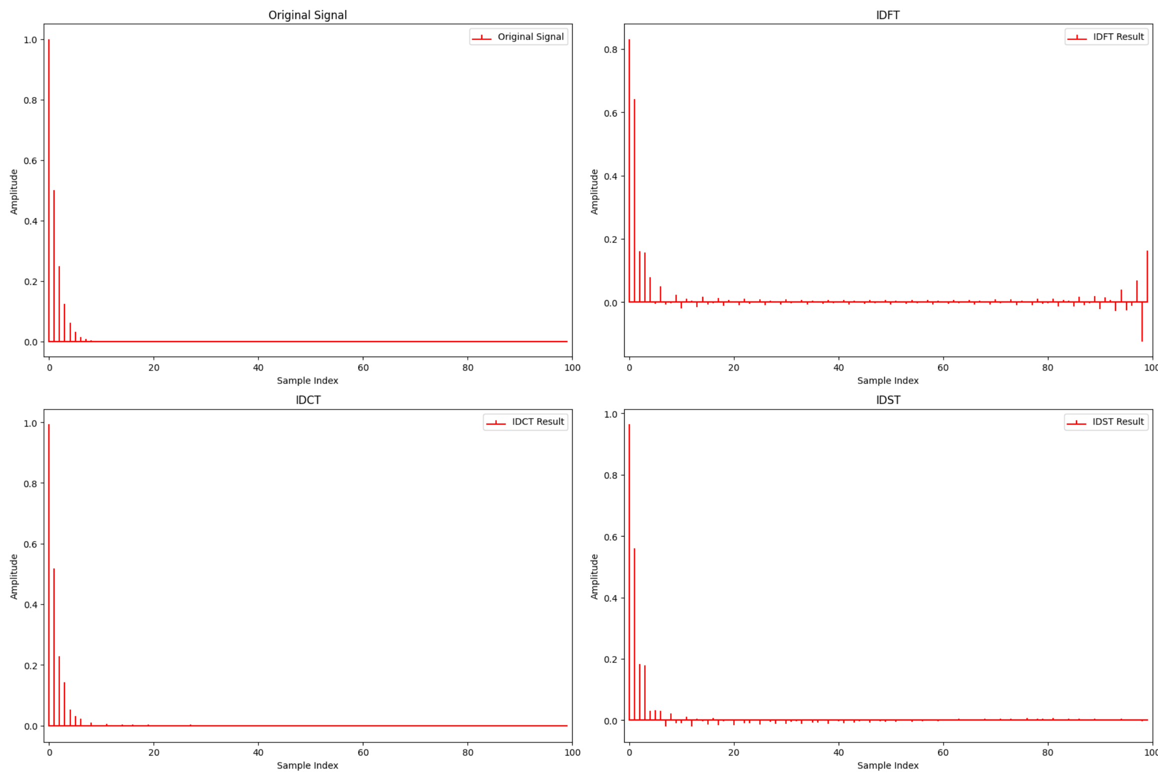

Fig. 7. Signal Reconstruction by retaining 75% coefficients

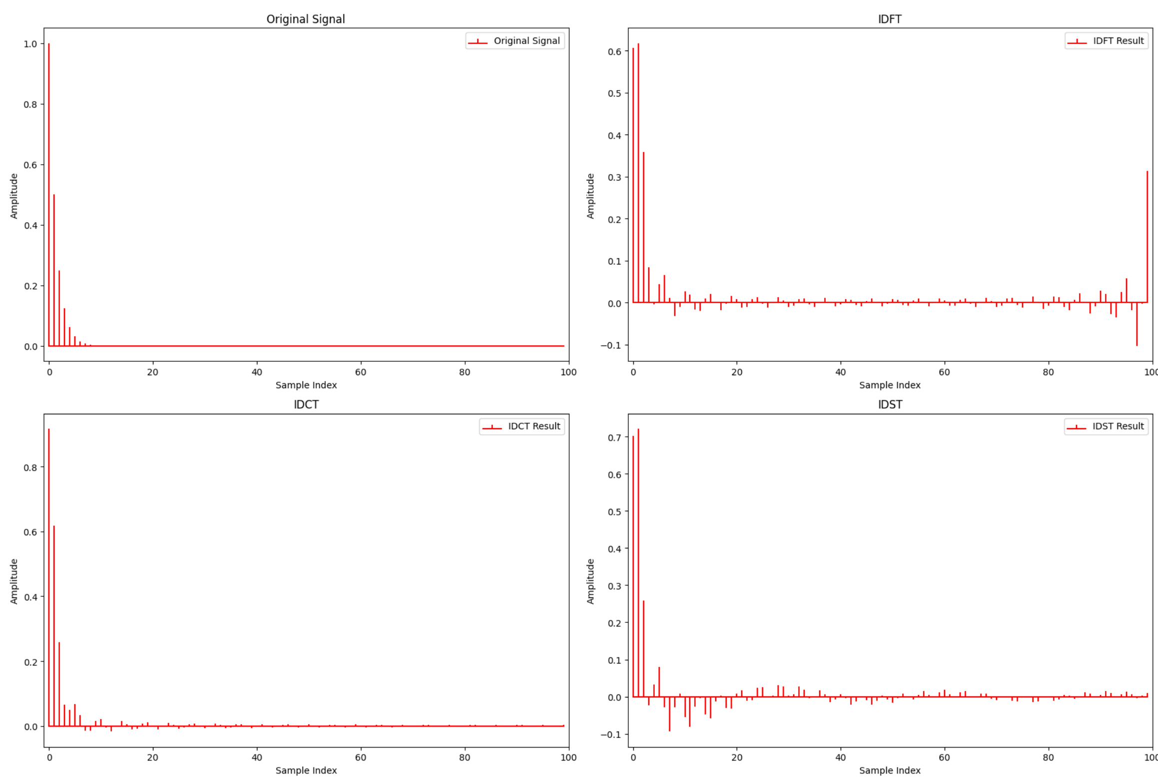

Fig. 8. Signal Reconstruction by retaining 45% coefficients