Bilinear Transformation (BLT)

A Transformation of the form $w = \dfrac{az+b}{cz+d}$ — (1) where a, b, c and d are real real (or) complex constants such that $ad-bc \neq 0$ and $\dfrac{dw}{dz} \neq 0$ is called Bilinear or Mobius Transformation.

i.e the transformation is conformal.

If $ad-bc=0$ every point of the z-plane is a critical point. the Inverse mapping of eqn (1) is $z = \dfrac{-dw+b}{cw-a}$ is also bilinear transformation.

Properties:

1) A bilinear transformation maps circles to circles

2) A bilinear transformation preserves cross-ratio of four points. i.e the points z_1, z_2, z_3, z_4 of the z-plane map on to the points w_1, w_2, w_3, w_4 of the w-plane respectively then

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

Note:

1) If any one of point say $z_1 \to \infty$, the quotient of those two difference which contain z_1 is replaced by 1.

2) Invariant points: If z maps in to itself in the w-plane i.e ($w=z$) then eqn (1) becomes, $z = \dfrac{az+b}{cz+d}$ (or)

$(z^2+(d-a)-b=0$

The roots of this eqn are defined as the Invariant (or) fixed points of the bilinear transformation.

Problems:

1. Find BLT maps $0, -i, -1$ of z-plane on to the points $i, 1, 0$ of w-plane respectively.

Soln: Let $w = \dfrac{az+b}{cz+d}$ be the required BLT

Let the points $z_1 = 0$, $z_2 = -i$, $z_3 = -1$ and $z_4 = z$ map on to the points $w_1 = i$, $w_2 = 1$, $w_3 = 0$ and $w_4 = w$.

$w = \dfrac{az+b}{cz+d}$

Since the cross ratio unchanged under a BLT we have

$z=0, w=i$: $i = \dfrac{b}{d}$ $\quad b - di = 0$

$z=-i, w=1$: $1 = \dfrac{-ai+b}{-ci+d}$ $\quad -ai+b+ci-d=0$

$z=-1, w=0$: $0 = \dfrac{-a+b}{-c+d}$ $\quad -a+b=0$

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

$$\frac{(i-1)(0-w)}{(1-0)(w-i)} = \frac{(0+i)(-1-z)}{(-i+1)(z-0)}$$

$$\frac{-w(i-1)}{w-i} = \frac{i(-1-z)}{z(-i+1)}$$

$$\frac{-iw+w}{w-i} = \frac{-i-iz}{-iz+z}$$

$$(-iw+w)(-iz+z) = (-i-iz)(w-i)$$

$$i^2wz - iwz - iwz + wz = -iw - iwz + i^2 + i^2 z$$

$$-\cancel{wz} - iwz + \cancel{wz} = -iw - 1 - z$$

$$iw - iwz = -(z+1)$$

$$iw(1-z) = -(z+1)$$

$$w = \frac{-(z+1)}{i(1-z)} = \frac{z+1}{i(z-1)}$$

$w = \dfrac{z+1}{i(z-1)}$ is the required BLT.

Find the BLT which maps the points $1, i, -1$ on the points $2, i, -2$ respectively. Also find the Invariant points of the points of the transformation.

Soln: Let $w = \frac{az+b}{cz+b}$

Let the points $z_1 = 1, z_2 = i, z_3 = -1$ and $z_4 = z$ maps on to the points $w_1 = 2, w_2 = i, w_3 = -2$ and $w_4 = w$

Since the cross-ratio unchanged a bilinear transformation we have,

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

$$\frac{(2-i)(-2-w)}{(i+2)(w-2)} = \frac{(1-i)(-1-z)}{(i+1)(z-1)}$$

$$(2-i)(i+1)(-2-w)(z-1) = (1-i)(i+2)(-1-z)(w-2)$$

$$(2i - i^2 + 2 - i)(-2z - wz + 2 + w) = (i - i^2 + 2 - 2i)(-w - zw + 2 + 2z)$$

$$(3+i)(-2z - wz + 2 + w) = (3-i)(-w - zw + 2 + 2z)$$

$$-6z - \cancel{2iz} - 3wz - iwz + \cancel{6} + 2i + 3w + \cancel{iw} = -3w + \cancel{iw} - 3wz + iwz + 6 - 2i + 6z - \cancel{2iz}$$

$$-6z - iwz + 2i + 3w = -3w + iwz - 2i + 6z$$

$$-iwz + 3w + 3w - iwz = -2i + 6z + 6z - 2i$$

$$-2iwz + 6w = -4i + 12z$$

$$2w(-iz+3) = 2(-2i + 6z)$$

$$w = \frac{6z - 2i}{-iz + 3}$$ is the required BLT

To find Invariant points

Consider, $w = \frac{6z - 2i}{-iz + 3}$

put $w = z$ we get,

$$z = \frac{6z - 2i}{-iz + 3}$$

$$z(-iz+3) = 6z - 2i$$

$$-iz^2 + 3z - 6z + 2i = 0$$

$$-iz^2 - 3z + 2i = 0$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $a = -i, b = -3, c = 2i$

$$= \frac{3 \pm \sqrt{9 - 4(-i)(2i)}}{2(-i)} = \frac{3 \pm 1}{-2i}$$

$\frac{3+1}{-2i}$, $\frac{4}{-2i}$, $\frac{2}{-i}$, $2i$ | $\frac{3-1}{-2i}$, $\frac{2}{-2i}$, $\frac{1}{-i}$, i

$\therefore z = i, 2i$ are the Invariant points

3) Find the BLT that maps the points $z=-1, i, 1$ onto the points $w=1, i, -1$ respectively.

Soln: Let $w = \frac{az+b}{bz+d}$ be the required BLT

Let the points $z_1=-1$, $z_2=i$, $z_3=1$ and $z_4=z$ maps on to the points $w_1=1$, $w_2=i$, $w_3=-1$ and $w_4=w$

Since the cross ratio unchanged a bilinear transformation we have,

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

$$\frac{(1-i)(-1-w)}{(i+1)(w-1)} = \frac{(-1-i)(1-z)}{(i-1)(z+1)}$$

$$(1-i)(i-1)(-1-w)(z+1) = (-1-i)(i+1)(1-z)(w-1)$$

$$(i-\underset{(+1)}{i^2}-\cancel{1}+i)(-z-wz-1-w) = (-i-\underset{(+1)}{i^2}-\cancel{1}-i)(w-wz-1+z)$$

$$\cancel{2i}(-z-wz-1-w) = -\cancel{2i}(w-wz-1+z)$$

$$-\cancel{z}-wz-1-\cancel{w} = -\cancel{w}+wz+1-\cancel{z}$$

$$-wz-1 = wz+1$$

$$-2wz = 2$$

$$-wz = 1$$

$i, 1, -1$ & $1, 0, \infty$.

$w = -\frac{1}{z}$ is the required BLT.

4) Find BLT which maps the points $z=1, i, -1$ on to the points $w=0, 1, \infty$.

Soln: Let $w = \frac{az+b}{cz+d}$ be the required BLT

Let the points $z_1=1$, $z_2=i$, $z_3=-i$ and $z_4=z$ maps on to the points $w_1=0$, $w_2=1$, $w_3=\infty$ and $w_4=w$

Find BLT which maps the points $z = 1, i, -1$ on to the points $w = 0, 1, \infty$.

Soln: Let $w = \frac{az+b}{cz+d}$ be the required BLT

Let the points $z_1 = 1$, $z_2 = i$, $z_3 = -i$ and $z_4 = z$ maps onto the points

$w_1 = 0$, $w_2 = 1$, $w_3 = \infty$ and $w_4 = w$

Since the cross ratio unchanged a BLT we have

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

$$\frac{(0-1)(\infty - w)}{(1-\infty)(w-0)} = \frac{(1-i)(-1-z)}{(i+1)(z-1)}$$

$$\frac{(-1)(\cancel{\infty})}{(\cancel{\infty})(w)} = \frac{(1-i)\cancel{-}(1+z)}{(1+i)(z-1)}$$

$$\frac{1}{w} = \frac{(1-i)(1+z)}{(1+i)(z+1)}$$

$$w = \frac{(1+i)(z-1)}{(1-i)(z+1)} \times \frac{1+i}{1+i}$$

$$= \frac{(1+i)^2(z-1)}{(1-i^2)(z+1)} = \frac{1-i^2+2i}{2} \times \frac{z-1}{z+1}$$

$w = i\left(\frac{z-1}{z+1}\right)$ is the required BLT.

56 Find the BLT that maps the points $1, i, -1$ respectively on to the points $i, 0, -i$ under this transformation. Hence find

(i) the image of $|z| < 1$

Soln: Let $w = \frac{az+b}{cz+d}$ be the required BLT

Let the points $z_1 = 1$, $z_2 = i$, $z_3 = -1$ and $z_4 = z$ maps on to the points

$w_1 = i$, $w_2 = 0$, $w_3 = -i$ and $w_4 = w$

Since the cross ratio unchanged a BLT we have,

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

$$\frac{(i-0)(-i-w)}{(0+i)(w-i)} = \frac{(1-i)(-1-z)}{(i+1)(z-1)}$$

$$\frac{\cancel{i}(-i-w)}{\cancel{i}(w-i)} = \frac{(1-i)(-1-z)}{(1+i)(z-1)}$$

$$\frac{w+i}{w-i} = \frac{(1-i)(1+z)}{(1+i)(z-1)}$$

$$(1+i)(w+i)(z-1) = (1-i)(w-i)(z+1)$$

$$(1+i)(wz+iz-w-i) = (1-i)(wz-iz+w-i)$$

$$\cancel{wz}+iz-w-\cancel{i}+iwz-z-\cancel{iw}+1 = \cancel{wz}-iz+w-\cancel{i}-iwz+z-\cancel{iw}-1$$

$$iwz+iwz-w-w = -iz-1-iz-1$$

$$2iwz-2w = -2iz-2$$

$$iwz-w = -iz-1$$

$$w(iz-1) = -(iz+1)$$

$$w = \frac{-(iz+1)}{iz-1} = \frac{1+iz}{1-iz}$$ is the required BLT

To find image $|z|<1$

consider, $w = \dfrac{1+iz}{1-iz}$

$$w(1-iz) = 1+iz$$

$$w-iwz = 1+iz$$

$$iz+iwz = w-1$$

$$iz(1+w) = w-1$$

$$z = \frac{w-1}{i(1+w)} \qquad \boxed{\frac{1}{i} = -i}$$

$$z = \frac{i(1-w)}{1+w}$$

∴, consider

$$|z|<1$$

$$|i|\left|\frac{1-w}{1+w}\right|<1 \qquad |i|=1$$

$$|1-w|<|1+w|$$

$$|1-u-iv|<|1+u+iv|$$

$$(1-u)^2+\cancel{v^2}<(1+u)^2+\cancel{v^2}$$

$$1+u^2-2u<1+u^2+2u$$

$$-2u<2u$$

$$0<4u$$

$$4u>0$$

$$u>0$$

∴ $u>0$ is the image of $|z|<1$.

The Invariant points are obtained by taking $w=z$

we have, $w=\dfrac{1+iz}{1-iz}$

$$z=\frac{1+iz}{1-iz}$$

$$z(1-iz)=1+iz$$

$$z-iz^2=1+iz$$

$$iz^2+iz-z+1=0$$

$$iz^2+(i-1)z+1=0$$

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$a=i,\ b=i-1,\ c=1$

$$=\frac{-(i-1)\pm\sqrt{(i-1)^2-4(i)(1)}}{2(i)}$$

$$=\frac{(1-i)\pm\sqrt{-6i}}{2i}$$ the invariant points.

6) Find the Bilinear transformation which send the points $z=0,1,\infty$ in to the points $w=-5,-1,3$ respectively. What are the Inverse points of this transformation.

Soln: Let $w=\dfrac{az+b}{cz+d}$ be the required BLT

Let the points $z_1=0,\ z_2=1,\ z_3=\infty$ and $z_4=z$ maps on to the points $w_1=-5,\ w_2=-1,\ w_3=3$ and $w_4=w$

Since the cross ratio unchanged a bilinear transformation we have,

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

$$\frac{(-5+1)(-3+w)}{(-1-3)(w+5)} = \frac{(0-1)(\infty-z)}{(1-\infty)(z-0)}$$

$$\frac{(-4)(3-w)}{(-1-3)(w+5)} = \frac{(-1)(\infty)}{\infty(z)}$$

$$z(3-w) = -(w+5)$$

$$3z - wz = -w-5$$

$$3z+5 = wz - w$$

$$w(z-1) = 3z+5$$

$w = \frac{3z+5}{z-1}$ is the required BLT

Invariant points:

consider $w = \frac{3z+5}{z-1}$

put $w=z$ we get

$$z = \frac{3z+5}{z-1}$$

$$z^2-z-3z-5=0$$

$$z^2-4z-5=0$$

$z=-1,5$ are the Invariant points

7. Find the Bilinear transformation which maps $z_1=-1, z_2=0, z_3=1$ onto $w_1=0$, $w_2=i$, $w_3=3i$.

Soln: Let $w = \frac{az+b}{cz+d}$ be the required BLT

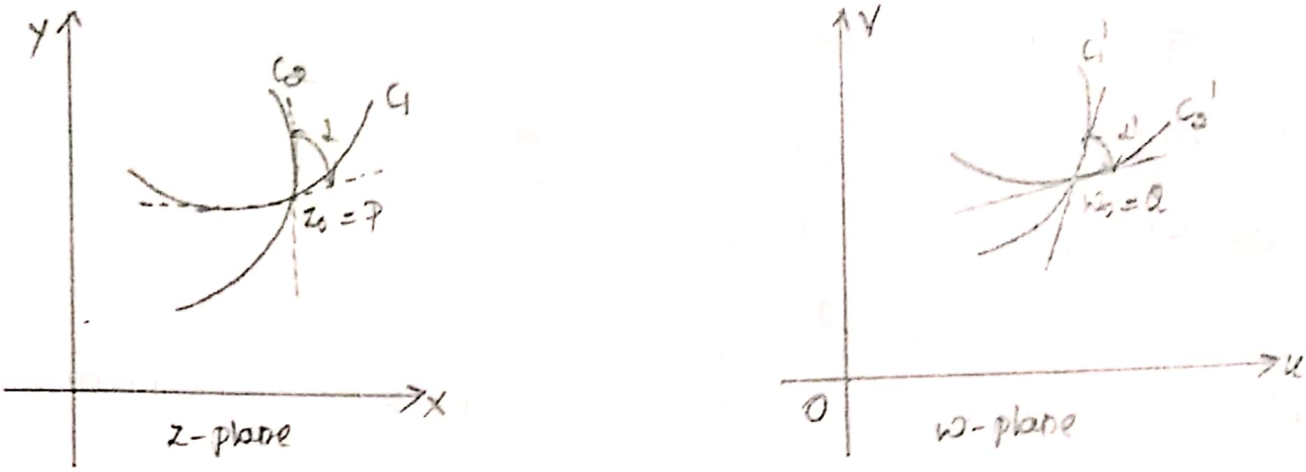
Let the points $z_1=-1$, $z_2=0$, $z_3=1$ and $z_4=z$ maps onto the points $w_1=0$, $w_2=i$, $w_3=3i$, and $w_4=w$

So $$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

$$\frac{(0-i)(3i-w)}{(i-3i)(w-0)} = \frac{(-1-0)(1-z)}{(0-1)(z+1)}$$

$$\frac{(-i)(3i-w)}{(-2i)(w)} = \frac{(-1)(1-z)}{(-1)(z+1)}$$

Conformal Transformation (Mapping)



Let C_1 and C_2 be the two curves in the z-plane intersecting at a point $P = z_0$. Let $w = f(z)$ transforms C_1 and C_2 to C_1' and C_2' intersecting at a point $Q = w_0$ in w-plane.

The mapping $w = f(z)$ is said to be conformal if angle b/w C_1 and C_2 at z_0 = angle between C_1' and C_2' at w_0

The mapping $w = f(z)$ is conformal at a point z if $f'(z) \neq 0$

(OR) If $f(z)$ is analytic and $f'(z) \neq 0$ in a region R of the z-plane, then the mapping $w = f(z)$ is conformal at all the points of R.

Discussion of conformal transformation

Transformation of $w = e^z$

Let $w = f(z) = e^z$

$\Rightarrow f'(z) = e^z$

since $f'(z) \neq 0$ for all $z \neq 0$. $f(z)$ is conformal at every point

Let $z = x+iy$ and $w = u+iv$

Now, $w = e^z$

$u+iv = e^{x+iy} = e^x e^{iy}$

$u+iv = e^x(\cos y + i \sin y)$

$u+iv = e^x\cos y + i\, e^x \sin y$

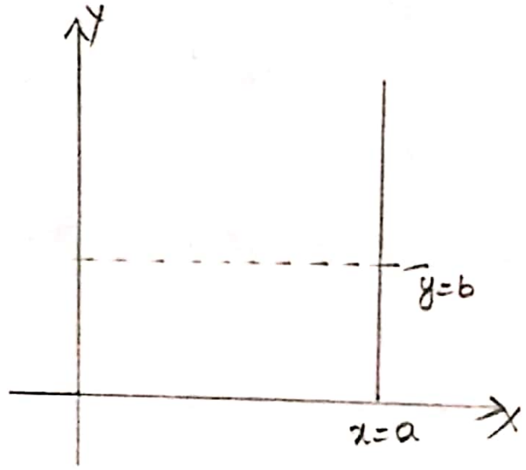
$\Rightarrow u = e^x\cos y \qquad v = e^x \sin y$

Case (i): Consider a straight line parallel to Y-axis

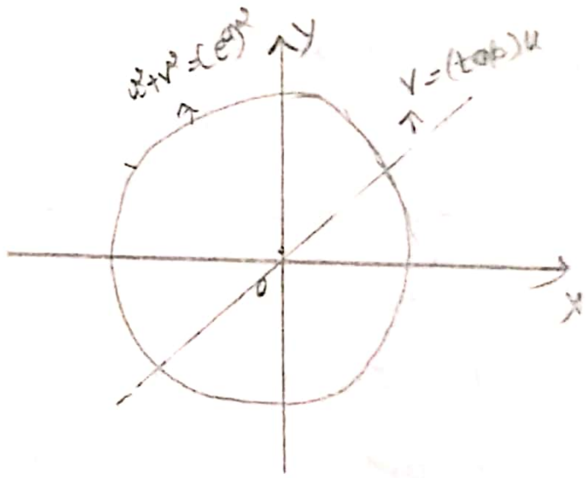
Eqn of straight line is $x = a$

Now, $u = e^a\cos y$ and $v = e^a \sin y$

$u^2+v^2 = e^{2a} = (e^a)^2$ is the eqn of circle with centre at origin and radius e^a.



(Z- plane)



(W- plane)

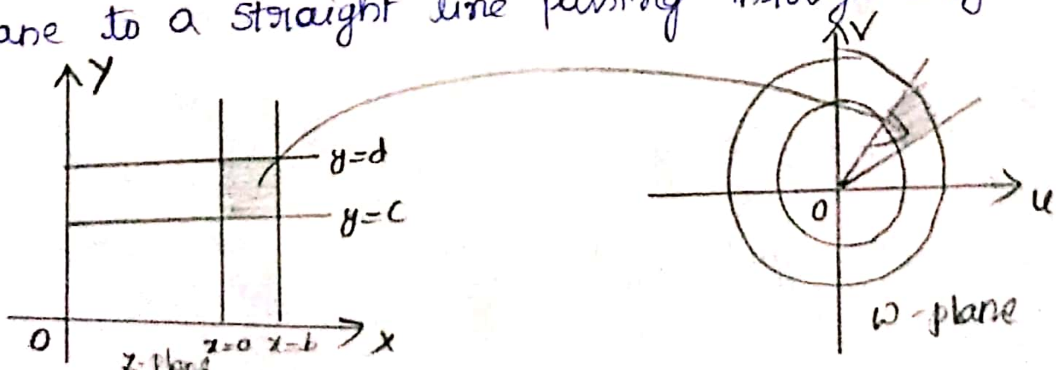
Case (ii): Consider a straight line parallel to X-axis

Eqn of straight line parallel to X-axis is $y=b$

$u = e^x\cos b$ and $v = e^x \sin b$

$\Rightarrow \frac{v}{u} = \frac{e^x \sin b}{e^x \cos b} \Rightarrow v = (\tan b)\, u$ is the eqn of straight line passing through the origin and having slope $(\tan b)$ in w-plane

$\therefore$ $w = e^z$ transforms the straight line parallel to X-axis in Z-plane to a straight line passing through origin in w-plane.



Discuss the Conformal Transformation $w = z^2$

Consider $w = z^2$

$\Rightarrow f(z) = z^2$ and $f'(z) = 2z$

Since $f'(z) \neq 0$ for all $z \neq 0$, $f(z)$ is conformal at all the point except at $z = 0$.

Now, $w = z^2$

$u + iv = (x+iy)^2$

$u + iv = (x^2 - y^2) + i(2xy)$

$\Rightarrow u = x^2 - y^2$ and $v = 2xy$ —— $(*)$

Case(i): Consider an equation of straight line $x = a$ parallel to y-axis in z-plane. Where a is any real constant.

Now eqn $(*)$ becomes

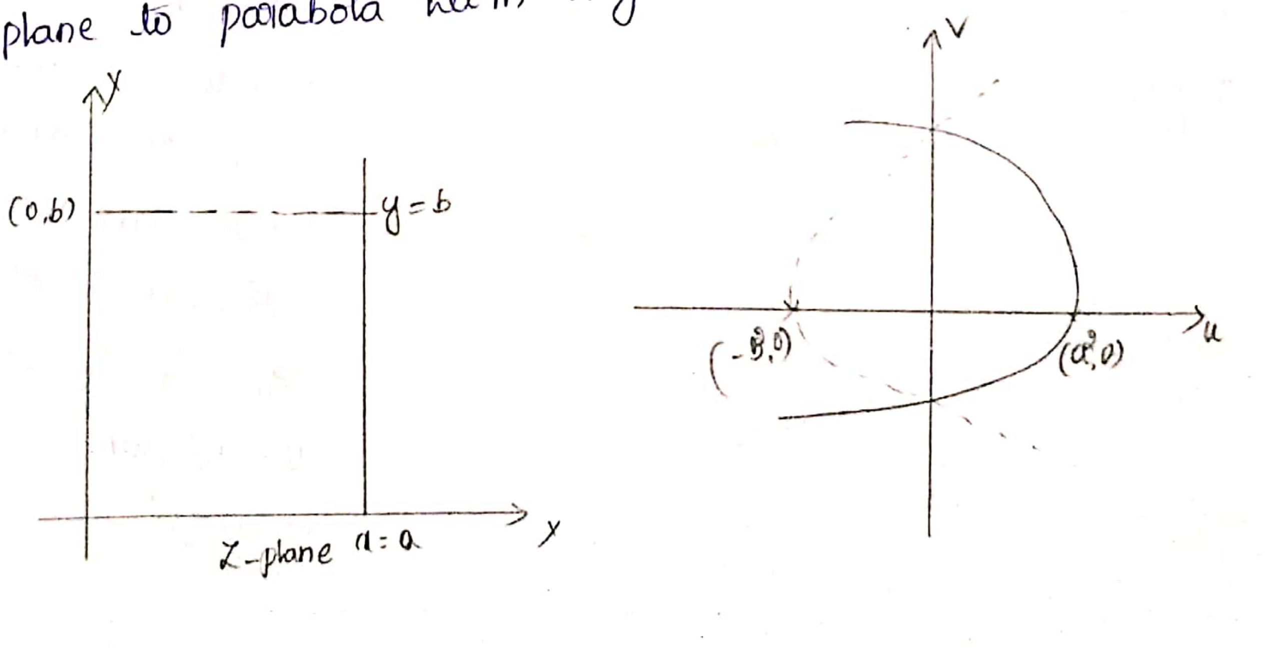
$u = a^2 - y^2$ and $v = 2ay$

$y^2 = a^2 - u$ and $v^2 = 4a^2y^2$

$= 4a^2(a^2 - u)$

$v^2 = -4a^2(u - a^2)$ which is the eqn of parabola with $(a^2, 0)$ as vertex and focus at the origin.

$\therefore w = z^2$ transforms a straight line parallel to y-axis in z-plane to parabola with negative u-axis as its axis.



case(ii) : Consider an eqn of straight line $y=b$ parallel parallel to x-axis in z-plane where b is any real constant. Now eqn (*) becomes,

$u = x^2 - b^2$ and $v = 2xb$

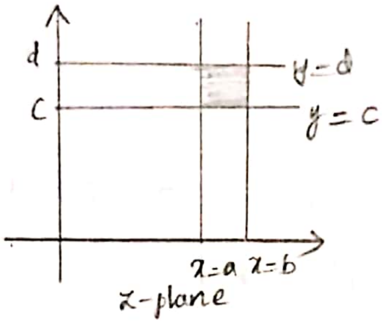
$x^2 = u + b^2$

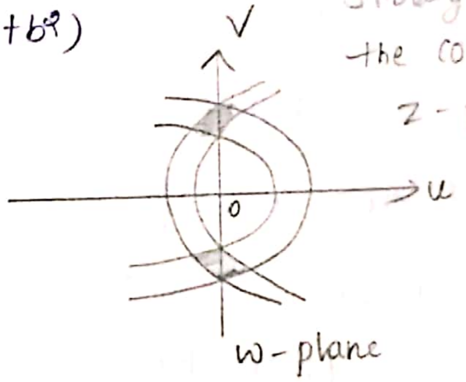
$v^2 = 4x^2 b^2$

$v^2 = 4(u+b^2) b^2$

$v^2 = 4b^2(u+b^2)$

Hence from these two cases we conclude that the straight line parallel to the coordinate axes in the z-plane map on to parabolas in the w-plane





The parabola with $(-b^2, 0)$ as vertex and positive u-axis as its axis.

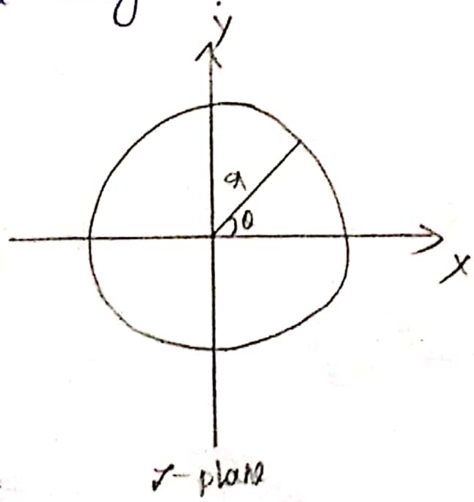
case(iii): Consider a circle with centre origin and radius r in z-plane.

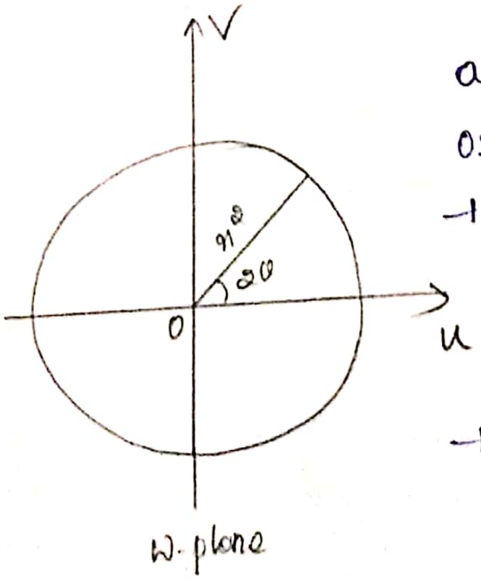
i.e $|z| = r$, $z = re^{i\theta}$

$w = z^2 = (re^{i\theta})^2 = r^2 e^{2i\theta}$

$w = Re^{i\phi}$ where $R = r^2$ and $\phi = 2\theta$ which is also a circle in the w-plane having radius r^2 and subtending an angle 2θ at origin

Hence we conclude that a circle with centre at origin and radius r in the z-plane maps on to a circle at origin and radius r^2 in the w-plane.





Transformation of $w = z + \frac{1}{z}$

Let $w = z + \frac{1}{z}$ where $z \neq 0$ — (*)

$\Rightarrow f'(z) = 1 - \frac{1}{z^2}$

Clearly $f(z)$ is conformal for all the values of z except $z = \pm 1$

Let $z = re^{i\theta}$

Now eqn (*) becomes

$$u + iv = re^{i\theta} + \frac{1}{re^{i\theta}}$$

$$= re^{i\theta} + \frac{1}{r} e^{-i\theta}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Case(i): Let $r = b$ be a circle with centre at the origin and radius b in the z-plane

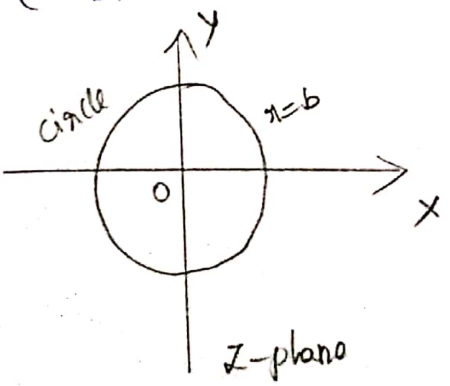
Now, $u = \frac{u}{\left(b+\frac{1}{b}\right)} = \cos\theta$ and $\frac{v}{\left(b-\frac{1}{b}\right)} = \sin\theta$

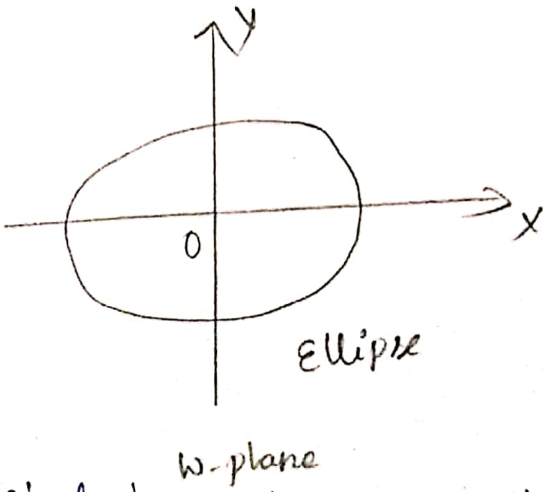
Squaring and adding we get,

$$\frac{u^2}{\left(b+\frac{1}{b}\right)^2} + \frac{v^2}{\left(b-\frac{1}{b}\right)^2} = \cos^2\theta + \sin^2\theta$$

$$\frac{u^2}{\left(b+\frac{1}{b}\right)^2} + \frac{v^2}{\left(b-\frac{1}{b}\right)^2} = 1$$

which is eqn of ellipse with centre at the origin in w-plane.





$\therefore w = z + \frac{1}{z}$ transforms a circle in z-plane to an ellipse in w-plane

Case(ii): Let $\theta = c$ be any line in z-plane

$$u = \left(r+\frac{1}{r}\right)\cos c \quad \text{and} \quad v = \left(r-\frac{1}{r}\right)\sin c$$

Squaring,

$$\cancel{u^2+v^2}\ \frac{u^2}{\cos^2 c} = \left(r+\frac{1}{r}\right)^2 \quad \text{and} \quad \frac{v^2}{\sin^2 c} = \left(r-\frac{1}{r}\right)^2$$

$$\frac{u^2}{\cos^2 c} - \frac{v^2}{\sin^2 c} = 4$$

$\div 4$ we get

$$\frac{u^2}{(2\cos c)^2} - \frac{v^2}{(2\sin c)^2} = 1$$ is the eqn of hyperbola in w-plane

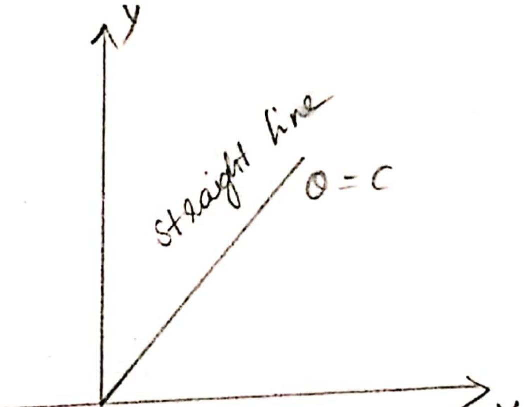
$$\frac{u^2}{\cos^2 c} = r^2+\frac{1}{r^2}+2$$

$$\frac{v^2}{\sin^2 c} = r^2+\frac{1}{r^2}-2$$

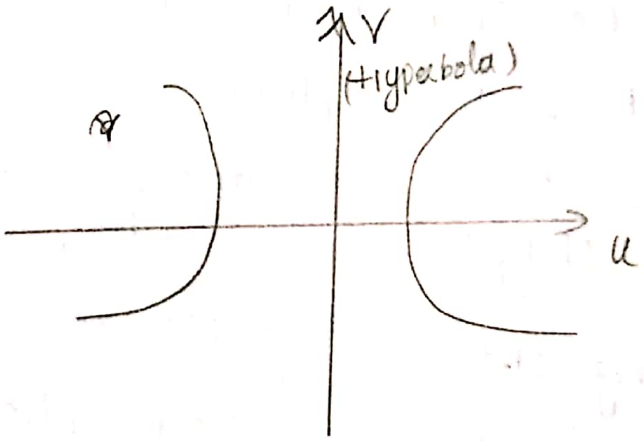
$$\frac{u^2}{\cos^2 c} - \frac{v^2}{\sin^2 c} = 4$$

$$\frac{u^2}{(2\cos c)^2} - \frac{v}{(2\sin c)^2} = 1.$$

$\therefore\ w = z+\frac{1}{z}$ transforms a straight line passing through origin in z-plane to a hyperbola in w-plane.



Z-plane



W-plane