Harmonic functions.

The function 'u' is said to be harmonic in a domain D, if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ {laplace eqn $\nabla^2\phi = 0$}.

ll^{ly} the function 'V' is said to be harmonic in a domain 'D' if $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$.

Problems.

1. Show that $u = x^2 - y^2$ is harmonic.

Soln:- Given, $u = x^2 - y^2$.

$$\frac{\partial u}{\partial x} = 2x. \qquad \frac{\partial u}{\partial y} = -2y.$$

$$\frac{\partial^2 u}{\partial x^2} = 2. \qquad \frac{\partial^2 u}{\partial y^2} = -2.$$

Consider, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$.

$\therefore$ The given fun is harmonic.

2. S.T $V = 2xy$ & $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic

Soln:- (i) Given, $V = 2xy$.

$$V_x \; \frac{\partial V}{\partial x} = 2y, \qquad \frac{\partial V}{\partial y} = 2x$$

$$\left\langle \frac{\partial^2 V}{\partial x^2} = 0 \right\rangle \qquad \left\langle \frac{\partial^2 V}{\partial y^2} = 0 \right\rangle.$$

Consider, $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 + 0 = 0$.

$\therefore$ The fun $V = 2xy$ is harmonic.

Given, $u = \frac{1}{2}\log(x^2+y^2)$.

$$\frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}(2x) \quad , \quad \frac{\partial u}{\partial y} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}(2y).$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}. \qquad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}.$$

$$\left\langle \frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \right\rangle \qquad \left\langle \frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \right\rangle$$

Consider,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0.$$

$\therefore$ The given fun^n is harmonic.

Theorem 3:- P.T the real & imaginary parts of analytic fun^n are harmonic.

(or)

P.T the real & imaginary parts of analytic fun^n satisfies laplace Eq^n.

Sol:- let $f(z) = u+iv$ be the analytic fun^n.

Then, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

D.B S.p.w.r.t 'x'. DB S.p.w.r.t 'y'.

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right). \qquad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right).$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{---(1)}. \qquad -\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{---(2)}.$$

From (1) & (2), $\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$\therefore$ u is harmonic.

i.e., Real part of analytic fun^n is harmonic.

Consider,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \& \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

DBS w.r.t 'y'. DBS w.r.t 'x'.

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right). \qquad \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right).$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}. \qquad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \quad \text{---} (4).$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \quad \text{---} (3).$$

From (3) & (4).

$$\frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2} \implies \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

$\therefore$ V is harmonic.

i.e., Imaginary part of analytic fun^n is harmonic