COMPLEX VARIABLES

Some basic concepts

complex numbers

The numbers of the form where x and y are real no. and i is imaginary

Here x is called the real part of the complex no. & y is called the imaginary part of complex no.

conjugate of complex no.

The conjugate of complex no. is denoted by $\bar{z}$ and is defined by $\bar{z} = x - iy$.

NOTE:

1) If $z = x+iy$ then

$|z| = \sqrt{x^2+y^2}$

$\text{amp}(z) = \tan^{-1}\left(\frac{y}{x}\right)$

2) $\sin ix = i \sinh x$, $\cos ix = \cosh x$.

Analytic function or regular function or Holomorphic function

The function $f(z)$ is said to be analytic in a region R if $f'(z)$ exist at all points of R.

Theorem:-1 : The necessary condition for $f(z) = u+iv$ to be analytic in a domain D is u and v satisfies.

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Cauchy riemann equation

i.e, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ or

Derive cauchy's rieman equation in cartesian form

Proof

given $f(z) = u + iv$ is analytic in a region R.

$\Rightarrow f'(z)$ exists at all points in a region R.

Let $\boxed{f(z) = u + iv} \rightarrow (1)$.

$f(z+\Delta z) = u + \Delta u + i(v + \Delta v)$

$\boxed{f(z+\Delta z) = u + \Delta u + iv + i\Delta v} \rightarrow (2)$.

(2) − (1)

$\Rightarrow f(z+\Delta z) - f(z) = (u + \Delta u + iv + i\Delta v) - (u + iv)$

$f(z+\Delta z) - f(z) = \cancel{u} + \Delta u + \cancel{iv} + i\Delta v - \cancel{u} - \cancel{iv}$

$f(z+\Delta z) - f(z) = \Delta u + i\Delta v.$

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$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta z}$$

$\lim_{\Delta z \to 0}$ on both sides

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \left(\frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}\right)$$

$$f'(z) = \lim_{\Delta x + i\Delta y \to 0} \left[\frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}\right]$$

Case-1.) As $\Delta x \to 0$

$$f'(z) = \lim_{i\Delta y \to 0} \left[\frac{\Delta u + i\Delta v}{i\Delta y}\right]$$

$$= \lim_{i\Delta y \to 0} \left[\frac{\Delta u}{i\Delta y} + \frac{i\Delta v}{i\Delta y} \right] \qquad \frac{1}{i} = -i$$

$$= \lim_{i\Delta y \to 0} \left[-i\frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta y} \right]$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y} \longrightarrow (3)$$

case-2.)

$\Delta y \to 0.$

$$f'(z) = \lim_{\Delta x \to 0} \left[\frac{\Delta u + i\Delta v}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x} \right]$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \longrightarrow (4)$$

comparing (3) and (4).

$$-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

equating real and imaginary part

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \quad \text{and} \quad -\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x}$$

$\therefore$ cauchy riieman equation is satisfied.

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Cauchy Rieman equation in polar form.

or

The necessary condition for the function $f(z) = u + iv$ is analytic at any point $z = re^{i\theta}$

or. $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ & $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$

Proof

Let $f(z) = u + iv$

put $z = re^{i\theta}$

$f(re^{i\theta}) = u + iv \rightarrow (*)$

Diff. both sides w.r.t partially w.r.t r.

$f'(re^{i\theta})\, e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \rightarrow (1)$

Diff. (*) b.s. partially w.r.t θ

$re^{i\theta}\, i f'(re^{i\theta}) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$

$ir\left[e^{i\theta} f'(re^{i\theta})\right] = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$

$ir\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$ [from (1)]

$ii = -1$ $\quad ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$

equating the real and imaginary part

$-r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta}$ $\qquad r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}$

$$\boxed{\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}} \qquad \boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$$

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polar $z = re^{i\theta}$
cartesian $z = x+iy$ } $w = u+iv = f(z)$ $w = f(z) = u+iv$.

cartesian $f'(z) = u_x + iv_x$

polar → $f'(z) = \frac{1}{e^{i\theta}}[u_r + iv_r]$ DATE: PAGE:

1) show that $w = z+e^z$ is analytic and hence find $\frac{dw}{dz}$ $[f'(z)]$

Soln

$w = z+e^z$

$u+iv = (x+iy) + e^{x+iy}$.

$w = f(z)$
$u+iv$

$u+iv = x+iy + e^x e^{iy}$

$= x+iy + e^x[\cos y + i\sin y]$

$= x+iy + e^x\cos y + ie^x\sin y$.

$u+iv = (x+e^x\cos y) + i(y+e^x\sin y)$

$u = x+e^x\cos y$ $v = y+e^x\sin y$

$\frac{\partial u}{\partial x} = 1+e^x\cos y$ $\frac{\partial v}{\partial x} = e^x\sin y$

$\frac{\partial u}{\partial y} = -e^x\sin y$ $\frac{\partial v}{\partial y} = 1+e^x\cos y$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ Cauchy Rieman are satisfied

$\therefore$ The given function is analytic

To find $\frac{dw}{dz}$ $[f'(z)]$

$w = z+e^z$

$\frac{dw}{dz} = 1+e^z$

WKT,

$\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= 1+e^x\cos y + ie^x\sin y$.

$= 1+e^x(\cos y + i\sin y)$

$\frac{dw}{dz} = 1+e^x e^{iy} = 1+e^{x+iy} = \boxed{1+e^z = \frac{dw}{dz}}$

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Solⁿ $f(z) = \log z$

$u + iv = \log(x+iy)$

$= \log (r\cos\theta + ir\sin\theta)$

$= \log r(\cos\theta + i\sin\theta)$

$= \log r e^{i\theta}$

$= \log r + \log e^{i\theta}$

$= \log r + i\theta \log e.$

$u + iv = \log r + i\theta$

put $x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$x^2+y^2 = r^2$

$r = \sqrt{x^2+y^2}$ $\quad \theta = \tan^{-1}(y/x)$

$u = \log r$ $\quad v = \theta$

$u = \frac{1}{2}\log(x^2+y^2)$

$u_x = \frac{1}{2}\frac{1}{x^2+y^2} 2x$

$u_x = \frac{x}{x^2+y^2}$

$u_y = \frac{1}{2}\frac{1}{x^2+y^2}(2y)$

$u_y = \frac{y}{x^2+y^2}$

$v = \tan^{-1}(y/x)$

$v_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} y\left(\frac{-1}{x^2}\right)$

$v_x = \frac{y}{1+\frac{y^2}{x^2}}\left(\frac{-1}{x^2}\right)$

$= \frac{y}{\frac{x^2+y^2}{\cancel{x^2}}}\left(\frac{-1}{x^2}\right)$

$v_x = \frac{-y}{x^2+y^2}$

$v_y = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)$

$v_y = \frac{1}{\frac{x^2+y^2}{x^2}}\left(\frac{1}{x}\right)$

$v_y = \frac{x}{x^2+y^2}$

$\therefore u_x = -v_y$ and

$u_y = -v_x.$

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(a) polar form

$w = \log z$

$u+iv = \log re^{i\theta}$

$u+iv = \log r + \log e^{i\theta}$

$u+iv = \log r + i\theta \log e$

$u+iv = \log r + i\theta$

Equating real and imaginary part.

$u = \log r \qquad v = \theta$

$\frac{\partial u}{\partial r} = u_r = \frac{1}{r} \qquad v_r = 0$

$u_\theta = 0 \qquad v_\theta = 1$

$u_r = \frac{1}{r} v_\theta \qquad v_r = -\frac{1}{r} u_\theta$

$\therefore$ cauchy's reiman eqn are satisfied, hence the given ~~method~~ function is analytic.

To find $f'(z)$

WKT $f'(z) = \frac{1}{e^{i\theta}} [u_r + i v_r]$

$= \frac{1}{e^{i\theta}} \left[\frac{1}{r} + i\cancel{0}\right]$

$= \frac{1}{e^{i\theta}} \left[\frac{1}{r}\right]$

$= \frac{1}{re^{i\theta}} = \frac{1}{z}$

Show that z^n is analytic and hence find its derivative.

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z's cartesian
$z^{pow} > 1 \to$ polar

$f(z) = z^n$

$u+iv = (re^{i\theta})^n$

$u+iv = r^n (e^{i\theta})^n$

$= r^n e^{i\theta n}$

$= r^n [\cos n\theta + i \sin n\theta]$

$u+iv = r^n \cos n\theta + i r^n \sin n\theta$

equating real and imaginary parts.

$u = r^n \cos n\theta$ $\quad\quad$ $v = r^n \sin n\theta$

$u_r = n r^{n-1} \cos n\theta$ $\quad\quad$ $v_r = n r^{n-1} \sin n\theta$

$u_\theta = -n \sin n\theta \, r^n$ $\quad\quad$ $v_\theta = \cos n\theta \times n r^n$

$u_r = \frac{1}{r} v_\theta$ $\quad$ & $v_r = -\frac{1}{r} u_\theta$

To find $f'(z)$

WKT, $f'(z) = \frac{1}{e^{i\theta}} [u_r + i v_r]$

$= \frac{1}{e^{i\theta}} [n r^{n-1} \cos n\theta + i \, n r^{n-1} \sin n\theta]$

$= \frac{n r^{n-1}}{e^{i\theta}} [\cos n\theta + i \sin n\theta]$

$= \frac{1}{e^{i\theta}} n r^{n-1} e^{in\theta}$

$= n r^{n-1} e^{in\theta - i\theta}$

$= n r^{n-1} e^{i\theta(n-1)}$

$= n (r e^{i\theta})^{n-1}$

$f'(z) = n z^{n-1}$

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Construction of analytic function by the real or imaginary part [Milne's Thompson method]

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Now substituting $r = z$ and $\theta = 0$ to obtain $f'(z)$ as a function of z and then integrating w.r.t z to get $f(z)$

1.) Find the analytic function whose real part is $e^x \sin y$.

sol: $f(z) = u + iv$ be the required analytic function.

$u = e^x \sin y$

$u_x = e^x \sin y$

$u_y = e^x \cos y$

WKT, $f'(z) = U_x + i V_x$ — Cauchy: $U_x = V_y$, $-U_y = V_x$

$= U_x + i(-U_y)$ $(\therefore V_x = -U_y)$

$f'(z) = e^x \sin y - ie^x \cos y$

By Milne's thompson method, put $x = z$ and $y = 0$.

$f'(z) = 0 - e^z (1)(i)$

$f'(z) = -ie^z$

$\int$ BS w.r.t z

$\int f'(z)\, dz = \int -ie^z\, dz$

$\boxed{f(z) = -ie^z}$

2) Find the analytic function whose real part is sinx coshy.

sol: $f(z) = u + iv$ be the required analytic function.

$u = \sin x \cosh y \Rightarrow u_x = \cos x \cosh y$

$u_y = \sin x \sinh y$

$q_x = \cos x \sinh y$.

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$u_x = v_y$
$-u_y = v_x$

wkt, $f'(z) = u_x + i v_x$
$f'(z) = u_x - i u_y$
$f'(z) = \cos x \cosh y - i \sin x \sinh y$

By Milne's thompson method, put $x = z$ and $y = 0$.
$f'(z) = \cos z$
$f'(z) = \cos z$
$\int f'(z)\, dz = \int \cos z\, dz$
$\boxed{f(z) = \sin z}$

3.) Find the analytic $f(z)$ function whose real part is ~~$x \sin x \cosh y$~~ $x \sin x \cosh y - y \cos x \sinh y$

Sol^n ~~Let $f(z) = x \sin x \cosh y - y \cos x \sinh y$ be the required~~
Let $f(z) = u + iv$ be the required analytic function

$u = x \sin x \cosh y - y \cos x \sinh y$
$u_x = \cosh y\,[x \cos x + \sin x] + y \sinh y \sin x$
$u_y = x \sin x \sinh y - \cos x\,[y \cosh y + \sinh y]$
wkt
$f'(z) = u_x + i v_x$
$f'(z) = u_x - i u_y$
$f'(z) = \cosh y\,[x \cos x + \sin x] + y \sinh y \sin x - i\,(x \sin x \sinh y - \cos x\,(y \cosh y + \sinh y))$

By Milne's thompson method, put $x = z$ and $y = 0$.
$f'(z) = 1\,[z \cos z + \sin z] + 0 - i\,[0]$
$f'(z) = z \cos z + \sin z$
$f(z) = \int (z \cos z + \sin z)\, dz$
$f(z) = [z\,(\sin z) - (-\cos z) - \cos z]$

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$f(z) = z \sin z$ is the required analytic function.

4) Find the analytic function whose real part is $\log(\sqrt{x^2+y^2})$

Sol^n Let $f(z) = u + iv \Rightarrow$ required analytic function

$u = \log(x^2+y^2)^{1/2} = 1/2 \log(x^2+y^2)$

$u_x = \frac{1}{2} \frac{1}{x^2+y^2}(2x)$

$u_x = \frac{x}{x^2+y^2}$

$u_y = \frac{1}{2} \frac{2y}{x^2+y^2} \Rightarrow u_y = \frac{y}{x^2+y^2}$

WKT, $f'(z) = u_x + i v_x$

$f'(z) = u_x - i u_y$

$f'(z) = \frac{x}{x^2+y^2} - i\left(\frac{y}{x^2+y^2}\right)$

By Milne' Thompson method, put $x = z$ $y = 0$.

$f'(z) = \frac{z}{z^2+0} = \frac{1}{z}$

$f(z) = \int \frac{1}{z} dz$

$f(z) = \log z \Rightarrow$ required analytic function.

QP. 5) Find the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$

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Sol: Let $f(z) = u + iv$ be the required analytic function

$u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$u_x = \dfrac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$

$u_y = \dfrac{-\sin 2x}{(\cosh 2y - \cos 2x)^2}(2\sinh 2y)$

WKT, $f'(z) = u_x + iv_x$ $\qquad u_x = v_y$

$f'(z) = u_x - iu_y$ $\qquad -v_x = u_y \Rightarrow -u_y = +v_x$

$f'(z) = \dfrac{(\cosh 2y - \cos 2x)(2\cos 2x) - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2} -$

$i\left[\dfrac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}\right]$

By using Milne Thompson method put $x = z,\ y = 0$.

$f'(z) = \dfrac{(1 - \cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1 - \cos 2z)^2} - i[0]$.

$f'(z) = \dfrac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$

$= \dfrac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1 - \cos 2z)^2}$

$= 2(\cos 2z - 1) / (1 - \cos 2z)^2$

$$= -2 \frac{\cancel{(1-\cos 2z)}}{(1-\cos 2z)^{\cancel{2}}}$$

$$= \frac{-2}{(1-\cos 2z)} = \frac{-\cancel{2}}{\cancel{2}\sin^2 z}$$

$$f''(z) = \frac{-1}{\sin^2 z} = -\operatorname{cosec}^2 z$$

Int B.S w.r.t x.

$$\boxed{f(z) = +\cot z}$$

6) Find the analytic function whose imaginary part is $e^x \sin y$ and hence find its real part.

Sol: Let $f(z) = u + iv$ be the required analytic function.

$v = e^x \sin y$

$v_x = e^x \sin y$

$v_y = e^x \cos y$

WKT,

$f'(z) = u_x + i v_x$

$= v_y + i v_x$

$= e^x \cos y + i e^x \sin y$

By Milne Thompson method. put $x = z$, $y = 0$.

$f'(z) = e^z$

$\int f'(z) = \int e^z dz$

$\boxed{f(z) = e^z} \Rightarrow$ required analytic function.

$u + iv = e^{x+iy}$.

$u + iv = e^x e^{iy}$.

$u + iv = e^x [\cos y + i \sin y]$

$u+iv = e^x\cos y + ie^x\sin y$

$\boxed{u = e^x\cos y}$

7.) Find the analytic function whose imaginary part is $e^x(x\sin y + y\cos y)$

Sol^n Let $f(z) = u+iv \Rightarrow$ required analytic function.

$V = \mathrm{Im}[e^x(x\sin y + y\cos y)]$ $v = e^x(x\sin y + y\cos y)$

$v_x = e^x(\sin y) + (x\sin y + y\cos y)e^x$

$v_y = e^x[x\cos y - y\sin y + \cos y]$

We,

$f'(z) = u_x + iv_x$

$= v_y + iv_x$

$= e^x[x\cos y - y\sin y + \cos y] + ie^x[\sin y + x\sin y + y\cos y]$

By Milne Thompson method put $x = z$, $y = 0$.

$f'(z) = e^z[z(1) - 0 + 1) + i[0]$

$f'(z) = e^z[z+1]$

$f'(z) = ze^z + e^z$

$\int f'(z) = \int(\underset{u}{z}\underset{v}{e^z} + e^z)\,dz$

$f(z) = ze^z - \cancel{e^z} + \cancel{e^z}$ mt

$\boxed{f(z) = ze^z}$ $\Rightarrow$ required analytic function.

$f(z) = u+iv$, $z \Rightarrow x+iy$

8.) Find the analytic function whose imaginary part is $\left(r - \frac{1}{r}\right)\sin\theta$ $r \neq 0$. MQP

Sol^n Let $f(z) = u+iv \Rightarrow$ required analytic function

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$v = \left(r - \frac{1}{r}\right) \sin\theta$

$v_r = \left(1 + \frac{r}{r^2}\right) \sin\theta$

$v_\theta = \left(r - \frac{1}{r}\right) \cos\theta$

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We, $f'(z) = \frac{1}{e^{i\theta}} (u_r + i v_r)$ $\quad u_r = \frac{1}{r} v_\theta$

$= \frac{1}{e^{i\theta}} \left[\frac{1}{r} v_\theta + i v_r\right]$ $\quad v_r = -\frac{1}{r} u_\theta$

$f'(z) = \frac{1}{e^{i\theta}} \left[\frac{1}{r}\left(r - \frac{1}{r}\right)\cos\theta + i\left(1 + \frac{1}{r^2}\right)\sin\theta\right]$

By using Milne thompson method put $r = z$, $\theta = 0$

$f'(z) \; \frac{1}{1}\left[\frac{1}{z}\left(z - \frac{1}{z}\right) + i\,(0)\right]$

$f'(z) = \frac{1}{z}\left(z - \frac{1}{z}\right) = 1 - \frac{1}{z^2}$

$\int f'(z) = \int \left(1 - \frac{1}{z^2}\right) dz$

$\boxed{f(z) = z + \frac{1}{z}}$ => required analytic function.

9.) ~~Find~~ construct an the analytic function whose real part is $r^2 \cos 2\theta$ and hence find its imaginary part.

Sol^n Let $f(z) = u + iv$ => required analytic function.

$u = r^2 \cos 2\theta$

$u_r = 2r \cos 2\theta$ $\quad u_\theta = -2r^2 \sin 2\theta$

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$u_r = \frac{1}{r} v_\theta$

WKT, $f'(z) = \frac{1}{e^{i\theta}}\left[u_r + i v_r\right]$

$v_r = -\frac{1}{r} u_\theta$.

$= \frac{1}{e^{i\theta}}\left[u_r - \frac{i u_\theta}{r}\right]$

$f'(z) = \frac{1}{e^{i\theta}}\left[2r\cos 2\theta + i\frac{1}{r}2r^2\sin 2\theta\right]$

By Milne Thompson method put $r = z$ $\theta = 0$

$f'(z) = 1\left[2z\right]$.

$\int f'(z) = \int 2z \, dz$

$\boxed{f(z) = z^2}$ $\Rightarrow$ required analytic function

$u + iv = (re^{i\theta})^2 \quad r^2 e^{i2\theta}$ $(z = re^{i\theta})$

$\cancel{u + iv = r^2(\cos\theta + i\sin\theta)^2}$

$u + iv = r^2(\cos 2\theta + i\sin 2\theta)$.

$u + iv = r^2\cos 2\theta + r^2 i\sin 2\theta$.

$\boxed{v = r^2\sin 2\theta}$ $\Rightarrow$ Imaginary part.

(c) Find the analytic function $f(z) = u + iv$ given $u - v = e^x[\cos y - \sin y]$

Sol^n Let $f(z) = u + iv$ be the required analytic function

Given $u - v = e^x[\cos y - \sin y] \longrightarrow (*)$

Diff $(*)$ w.r.t x.

$u_x - v_x = e^x[\cos y - \sin y] \rightarrow (1)$

Diff $(*)$ w.r.t y.

$u_y - v_y = e^x[-\sin y - \cos y] \rightarrow$

$\Rightarrow -v_x - u_x = e^x[-\sin y - \cos y] \rightarrow (2)$ $\qquad u_x = v_y$

$\qquad u_y = -v_x$

(1) + (2)

$\Rightarrow u_x - v_x - v_x - u_x = e^x(\cos y - \sin y) + e^x(-\sin y - \cos y)$

$-2v_x = e^x[\cos y - \sin y - \sin y - \cos y]$

$-2v_x = -2e^x \sin y$

$\boxed{v_x = e^x \sin y}$

(1) − (2) ⇒

$(u_x - v_x) - (-v_x - u_x) = e^x(\cos y - \sin y) - e^x(-\sin y - \cos y)$

$u_x - \cancel{v_x} + \cancel{v_x} + u_x = e^x[\cos y - \cancel{\sin y} + \cancel{\sin y} + \cos y]$

$2u_x = 2e^x \cos y$

$\boxed{u_x = e^x \cos y}$

WKT $f'(z) = u_x + iv_x$

$= e^x \cos y + i\, e^x \sin y$

By using Milne thompson method put $x = z, y = 0$.

$f'(z) = e^z$

Int w.r.t z

$\int f'(z)\, dz = \int e^z\, dz$

$\boxed{f(z) = e^z}$

11.) Find the analytic function $f(z)$ given that the sum of real and imaginary part $x^3 - y^3 + 3xy(x-y)$

$u+v$

Sol: Let $f(z) = u + iv$ be the required analytic function.

given $u + v = x^3 - y^3 + 3xy(x-y) \rightarrow (*)$

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$u_x + iv_x = 3x^2 + 6xy - 3y^2 \rightarrow (1)$ $\quad x^3 - y^3 + 3x^2y - 3xy^2 \rightarrow (*)$

$u_y + iv_y = -3y^2 + 3x^2 - 6xy \rightarrow (2)$ $\quad u_x = v_y$

$-v_x + iu_x = -3y^2 + 3x^2 - 6xy \rightarrow (2)$ $\quad u_y = -v_x$

(1) + (2) $\Rightarrow$

$u_x + \cancel{v_x} - \cancel{v_x} + u_x = 3x^2 + \cancel{6xy} - 3y^2 - 3y^2 + 3x^2 - \cancel{6xy}$.

$2u_x = 6x^2 - 6y^2$

$\boxed{u_x = 3x^2 - 3y^2}$

(1) - (2) $\Rightarrow$

$(u_x + v_x) - (-v_x + u_x) = 3x^2 + 6xy - 3y^2 - (-3y^2 + 3x^2 - 6xy)$

$\cancel{u_x} + v_x + v_x - \cancel{u_x} = \cancel{3x^2} + 6xy - \cancel{3y^2} + \cancel{3y^2} - \cancel{3x^2} + 6xy$

$2v_x = 12xy$

$\boxed{v_x = 6xy}$

WKT $\quad f'(z) = u_x + iv_x$

$= (3x^2 - 3y^2) + i(6xy)$

By Milne Thompson method put $x = z \quad y = 0$.

$f'(z) = 3z^2$

$\int f'(z) = \int 3z^2 \, dz$

$\boxed{f(z) = z^3} \Rightarrow$ required analytic function.

12) If $f(z) = u + iv$ is analytic and given that $u + v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$ where $r \neq 0$ then determine the analytic function $f(z)$.

Sol^n let $f(z) = u + iv \Rightarrow$ required analytic function.

given, $u + v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta) \rightarrow (*)$

$u_r + v_r = \frac{-2}{r^3}(\cos 2\theta - \sin 2\theta) \rightarrow (1)$

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Diff w.r.t θ partially. $u_r = \frac{1}{r} v_\theta$

$u_\theta + v_\theta = \frac{1}{r^2}(-2\sin 2\theta - 2\cos 2\theta)$ $v_r = -\frac{1}{r} u_\theta$

$-r v_r + r u_r = \frac{1}{r^2}(-2\sin 2\theta - 2\cos 2\theta)$

$\div r$

$-v_r + u_r = \frac{1}{r^3}(-2\sin 2\theta - 2\cos 2\theta) \rightarrow (2)$

$(1) + (2) \Rightarrow u_r + v_r + (-v_r + u_r) = \frac{-2}{r^3}(\cos 2\theta - \sin 2\theta) + \frac{1}{r^3}(-2\sin 2\theta - 2\cos 2\theta)$

(x)

$2u_r = \frac{-2}{r^3}(\cos 2\theta - \sin 2\theta + \sin 2\theta + 2\cos 2\theta)$

$\cancel{2} u_r = -\frac{\cancel{2}}{r^3}(-2\sin 2\theta)$

$$\boxed{u_r = \frac{+2\sin 2\theta}{r^3}}$$

✓ $u_r = -\frac{1}{r^3}(2\cos 2\theta)$

$(1) - (2) \Rightarrow$

$(u_r + v_r) - (-v_r + u_r) = \frac{-2}{r^3}(\cos 2\theta - \sin 2\theta) - \frac{1}{r^3}[-2\sin 2\theta - 2\cos 2\theta]$

$2v_r = \frac{-2}{r^3}[\cos 2\theta - \sin 2\theta - \sin 2\theta - \cos 2\theta]$

$v_r = -\frac{1}{r^3}[-2\sin 2\theta]$

$$\boxed{v_r = \frac{2\sin 2\theta}{r^3}}$$

WKT, $f'(z) = [u_r + i v_r]\frac{1}{e^{i\theta}}$

$= \frac{1}{e^{i\theta}}\left[-\frac{1}{r^3}[2\cos 2\theta] + \frac{2}{r^3}\sin 2\theta\right]$

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By Milne's Thompson method, $x = z$, $y = 0$.

$$f'(z) = -\frac{1}{z^3}(2)$$

$$\int f'(z) = -2\int \frac{1}{z^3}\, dz = -2\int z^{-3}\, dz$$

$$f(z) = \cancel{-2}\frac{z^{-2}}{\cancel{(-2)}}$$

$$\boxed{f(z) = \frac{1}{z^2}}$$ ⇒ required Analytic function

Complex Integration / complex line Integral

* Cauchy's Theorem / Cauchy Integral theorem

If $f(z)$ is analytic everywhere inside & on a simple closed curve c then $\int_c f(z)\,dz = 0$.

Proof

let $f(z) = u + iv$ be the analytic function.

Int B.S w.r.t z.

$$\int_c f(z)\,dz = \int_c (u + iv)\,dz$$

$$\int_c f(z)\,dz = \int_c (u + iv)(dx + i\,dy)$$

$$\int_c f(z)\,dz = \int_c u\,dx + iu\,dy + iv\,dx - v\,dy$$

$$\int_c f(z)\,dz = \int_c (u\,dx - v\,dy) + i(u\,dy + v\,dx)$$

$$\int_c f(z)\,dz = \int_c \overbrace{u\,dx - v\,dy}^{M\,dx + N\,dy} + i\int_c u\,dy + v\,dx \rightarrow (1)$$

using green's theorem $\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$\int_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

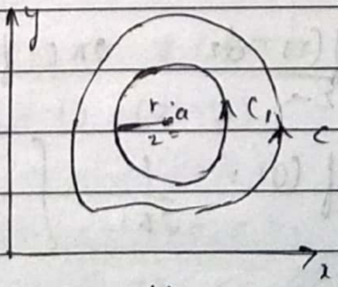
$$\int_C f(z)\,dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$\boxed{\int_C f(z)\,dz = 0}$$ Hence proved.

: Line integral

Cauchy's integral formula

If $f(z)$ is analytic inside and on a simple closed curve C, and if a is any point within C then $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\,dz$



Since a is a point within C, we shall enclose it by a circle C_1 with $z = a$ as a centre and r as a radius such that C_1 lies entirely within C. The function $\frac{f(z)}{z-a}$ is analytic inside and on the boundary of the anular region b/w C and C_1. Now as the consequence of cauchy's theorem we have

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz \quad \text{--- (1)}.$$

The eqn of C_1 [circle with centre a and radius r] can be written as $|z-a| = re^{i\theta}$

$z - a = re^{i\theta} \Rightarrow z = a + re^{i\theta}$

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$dz = ire^{i\theta} d\theta \qquad 0 \le \theta \le 2\pi$

using these results in RHS part of (1)

we get $\int_c \frac{f(z)}{z-a} dz = \int_{\theta=0}^{2\pi} f\frac{(a+re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta$

$= i \int_{\theta=0}^{2\pi} f(a+re^{i\theta}) d\theta$

This is true for any r is positive, however small.

Hence $r \to 0$

$\int_c \frac{f(z)}{z-a} dz = i \int_{\theta=0}^{2\pi} f(a) d\theta$

$= i f(a) \int_{\theta=0}^{2\pi} d\theta$

$= i f(a) [\theta]_{\theta=0}^{2\pi}$

$= i f(a)(2\pi)$

$\int_c \frac{f(z)}{z-a} dz = 2\pi i f(a)$

$\therefore f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a} dz$

NOTE: Generalised cauchy's integral formula

If $f(z)$ is analytic inside and on a simple closed curve c $f^n(a) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}} dz$

Problems on cauchy's integral formula

1.) Evaluate $\int_c \frac{z^2-z+1}{z-1}$ c: i) $|z| = 1$

$|z| = 1/2$

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i) $f(z) = z^2 - z + 1$ $|z|$

By cauchy's integral formula $|1|$

$\therefore \int_C \frac{z^2 - z + 1}{z - 1} = 2\pi i f(1)$ $z = 1 \Rightarrow$ lies in C.

$f(z) = z^2 - z + 1$

$= 2\pi i(1)$ $= 1 - 1 + 1$

$= 2\pi i$ $= 1$

ii) $f(z) = \underline{z^2 - z + 1}$

By cauchy's integral formula $|z|$

$\int_C \frac{z^2 - z + 1}{z - 1} dz = 0.$ $|1|$

$= 0$ $1 > 1/2$

lies outside C

2) Evaluate by using cauchy's integral formula

$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is the circle $C: |z| = 3$

both are I

Soln
$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz + \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$

but the soln are 1 - in 2 - out

$\int_C \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)}}{(z-1)} dz \quad \int_C \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)}}{(z-2)} dz$

$f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{z - 2}, \quad g(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{z - 1}$

By cauchy's integral formula we have.

$2\pi i f(1) + 2\pi i g(2)$ $f(1) = \frac{\sin \pi + \cos \pi}{1 - 2} = \frac{-1}{-1} = 1$

$2\pi i [f(1) + g(2)]$

$2\pi i [1 + 1]$ $g(2) = \frac{\sin 4\pi + \cos 4\pi}{1} = \frac{1}{1} = 1$

$\cos$ odd $\pi = -1$
$\cos$ even $\pi = 1$

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$4\pi i$

3.) Evaluate $\int_c \frac{e^{2z}}{(z+1)(z+2)} dz$ $c: |z| = 3.$

Sol^n $\int_c \frac{e^{2z}}{(z+1)(z+2)} dz + \int_{c_1} \frac{e^{2z}}{(z+1)(z+2)} dz$

$\int_c \frac{\frac{e^{2z}}{(z+2)}}{(z+1)} dz + \int_c \frac{\frac{e^{2z}}{z+1}}{z+2} dz$

$f(z) = \frac{e^{2z}}{z+2}$, $g(z) = \frac{e^{2z}}{z+1}$

By cauchy's integral formula

$2\pi i \, f(-1) + 2\pi i \, g(-2)$

$2\pi i \, [f(1) + g(2)]$

$2\pi i \, (e^{-2} - e^{-4})$

$f(-1) = \frac{e^{-2}}{-1+2} = e^{-2}$

$g(-2) = \frac{e^{-4}}{-1} = -e^{-4}$

4) Evaluate $\int_c \frac{e^{tz}}{z^2+1} dz.$ $c: |z| = 3.$

$\int_c \frac{e^{tz}}{(z+i)(z-i)} dz$

z^2+1

$z^2 - i^2 = (z+i)(z-i)$

$\int_c \frac{\frac{e^{tz}}{z+i}}{(z-i)} dz + \int_c \frac{e^{tz}}{\left(\frac{z-i}{z+i}\right)} dz$

5) $\int_c \frac{3z^2+7z+1}{z+1}\, dz \quad c : |z| = 1/2$

sol: Let $f(z) = \frac{3z^2+7z+1}{z+1}$ lies outside the circle

$\therefore$ By cauchy integral formula

$\int_c f(z) = 0.$

6.) $\int_c \frac{2z+1}{z^2+z}\, dz \quad c: |z| = 3/2$

sol: $\int_c \frac{2z+1}{z(z+1)}$

$z=0 \quad I \quad 0$

$\int_c \frac{\frac{2z+1}{z+1}}{z}\, dz$

$f(z) = \frac{2z+1}{z+1}$ is analytic everywhere inside and on c.

By Cauchy integral formula

$\int_c \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$

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$$\int_c \frac{\frac{2z+1}{z+1}}{(z-0)} dz = 2\pi i\, f(0) \qquad f(0) = \frac{1}{1}$$

$$= 2\pi i$$

7) $\int_c \frac{z}{z^2-3z+2} dz \qquad c: |z-2| = 1/2$

consider $\int_c \frac{z}{z^2-3z+2} = \int_c \frac{z}{(z-2)(z-1)} dz$

$|z-a|$

$|2-2|$

$0 < 1/2$ in.

$|1-2|$

$1 > 1/2$ = out

then $f(z) = \int_c \frac{\frac{z}{z-1}}{z-2} dz$ is analytic everywhere inside and on c.

out

$\therefore$ By cauchy integral over c.

$f(z) = \frac{z}{z-1}$

By $\int_c \frac{f(z)}{z-a} dz = 2\pi i\, f(a)$

$$\int_c \frac{\frac{z}{z-1}}{z-2} dz$$

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Poles

Consider the expansion of a Laurent series about the point $z=a$ in the form $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_n (z-a)^{-n}$

then 1st term is called analytic part of $f(z)$ and 2nd part term is called as principle part of $f(z)$.

If

is a pole of order m. In particular the pole of order 1 is called the simple pole.

Residue

The co-efficient of $\frac{1}{z-a}$ i.e, a_{-1} in the expansion of $f(z)$ is called the residue of $f(z)$ at the pole $z=a$

Formula for the residue at the pole.

If $z=a$ is a pole of order m then the residue of $f(z)$ at $z=a$ is denoted by resi $R[m,a]$ and is defined by - $R[m,a] = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)]$

In particular for a simple pole i.e, when $m=1$ we have $R[1,a] = \lim_{z\to a} [(z-a) f(z)]$

Cauchy's Residue theorem

If $f(z)$ is analytic inside and on the boundary of simple closed curve C except for a finite no. of poles $a, b, c, \ldots$ then $\int_c f(z)\, dz = 2\pi i \sum R_i$

where $\sum R_i$ is sum of residues

1) Find the residues of $f(z) = \frac{z}{(z+1)(z-2)^2}$ i) $z=-1$ ii) $z=2$

or Evaluate $\int_c \frac{z}{(z+1)(z-2)^2}\,dz$ where c is $|z|=-1$, $z=2$

Sol^n $z=-1$ is a pole of order 1.

$a=-1$ $m=1$

$z=2$ is a pole of order 2

$a=2$ $m=2$

case 1.) $z=-1$ is a pole of order 1

$a=-1$ $m=1$

$R[1,a] = \lim_{z\to a} (z-a)\, f(z)$

$= \lim_{z\to -1} (z+1)\, f(z)$

$= \lim_{z\to -1} (z+1)\, \frac{z}{(z+1)(z-2)^2}$

$= -\frac{1}{9} = R_1$

case 2.) $a=2$ $m=2$

$R[m,a] = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$

$R[2,2] = \lim_{z\to 2} \frac{d}{dz} \left[(z-2)^2 \frac{z}{(z+1)(z-2)^2} \right]$

$= \lim_{z\to 2} \left[\frac{(z+1)(1) - z(1)}{(z+1)^2} \right]$

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$= \lim_{z \to 2} \left[\frac{1}{(z+1)^2} \right] = \frac{1}{9} = R_2$

$\therefore \int_C f(z) = 2\pi i \Sigma R$

$= 2\pi i \left[R_1 + R_2 \right]$

$= 2\pi i \left[-\frac{1}{9} + \frac{1}{9} \right]$

$= 0$

2) $\int_C \frac{2z+1}{z^2 - z - 2} \, dz \Rightarrow$ sol^n $\int_C \frac{2z+1}{(z-2)(z+1)}$

Sol^n Here. $z = 2$, order: $z = -1$ are simple poles.

case 1) $z = 2$ is a pole of order 1

$R(1, a) = \lim_{z \to a} (z-a) f(z)$

$= \lim_{z \to 2} (z-2) \left(\frac{2z+1}{(z-2)(z+1)} \right)$

$= \frac{4+1}{3} = \frac{5}{3} = R_1$

case 2) $z = -1$ is a pole of order 1

$R(1, a) = \lim_{z \to a} (z-a) f(z)$

$= \lim_{z \to -1} (z+1) \left\{ \frac{2z+1}{(z-2)(z+1)} \right\}$

$= \frac{-1}{-3} = \frac{1}{3} = R_2$

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$\therefore$ By Cauchy's theorm $\int_c f(z)\, dz = 2\pi i\, \Sigma R$

$$\int_c \frac{2z+1}{(z-2)(z+1)}\, dz = 2\pi i \left(\frac{5}{3} + \frac{1}{3}\right)$$

$$= 4\pi i.$$

3) Evaluate $\int_c \frac{e^{2z}}{(z+1)(z-2)}\, dz \qquad |z| = 3$

Sol: Here $z = -1$, $z = 2$ are the poles of order 1

case (i) $z=-1$ is a pole of order 1 $\quad a = -1 \quad m = 1$

$$R[1, a] = \lim_{z\to a} (z-a) f(z)$$

$$= \lim_{z\to -1} (z+\cancel{1}) \frac{e^{2z}}{\cancel{(z+1)}(z-2)}$$

$$R_1 = \frac{e^{-2}}{-3}$$

case (ii) $z = 2 \quad a = 2, \; m = 1$

$$R[1, 2] = \lim_{z\to 2} (z-\cancel{2}) \frac{e^{2z}}{(z+1)(z-\cancel{2})}$$

$$= \lim_{z\to 2} \frac{e^{2z}}{(z+1)}$$

$$R_2 = \frac{e^4}{3}$$

$$\int_c \frac{e^{2z}}{(z+1)(z-2)} = 2\pi i\, \Sigma R$$

$$= 2\pi i \left(\frac{e^{-2}}{-3} + \frac{e^4}{3}\right)$$

Bilinear transformation

The transformation of the form $w = \frac{az+b}{cz+d}$ → ①

where a, b, c and d are real or constants, such that $ad-bc \neq 0$ and $\frac{dw}{dz} \neq 0$ is called bilinear transformation.

or Mobius transformation i.e., the transformation is conformal. If $ab-bc=0$ then every point of the z plane is a critical point. The inverse maping of eqⁿ ① is $z = \frac{-dw+b}{cw-a}$ is also a bialinear transformation.

Properties

1) Bilinear transformation maps circles to circles.

2) Bilinear transformation preserves cross ratio of 4 points i.e., the points z_1, z_2, z_3, z_4 of the z plane maps on to the points w_1, w_2, w_3, w_4 of the w plane respectively then,

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

NOTE :-

1) If any one of the point say $z \to \infty$, the quotient of those two difference which contain 'z_1' is replaced by 1.

2) The invariant points of 'z' maps into itself is

the roots of this eq^n are defined as the invariant point or fixed points of a bilinear transformation

1.) Find the Bilinear Transformation maps $0, -i, -1$ on to the points $i, 1, 0$ of w plane respectively.

sol^n Let $w = \dfrac{az+b}{cz+d}$ be the required Bilinear transformation

Let the points be $z_1 = 0,\ z_2 = -i,\ z_3 = -1,\ z_4 = z$

$w_1 = i \quad w_2 = 1 \quad w_3 = 0 \quad w_4 = w$

Since the cross ratio remains unaltered under a bilinear transformation

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

$$\frac{(i-1)(0-w)}{(i-w)(0-1)} = \frac{(0+i)(-1-z)}{(0-z)(-1+i)}$$

$$\frac{(1-i)\,w}{(w-i)} = \frac{+i\,(1+z)}{(-z)(1-i)}$$

$$(1-i)^2(-wz) = i(1+z)(w-i)$$

$$\cancel{1} + \cancel{i^2} - 2i(-wz) = (i+z)(w-i)$$

$$2wzi = wi + 1 + zw - zi$$

$$(2wz\cancel{i}) = \cancel{i}(w + 1 + zw - z)$$

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$(2wz = w + 1 + 2w - 2)$

$2wz = i\,[w - i + wz - iz]$

$2wz = w - i + wz - iz$

$2wz - w - wz = -i - iz$

$wz - w = -i - iz$

$w(z-1) = i(1+z)$

$w = \dfrac{-i(1+z)}{z+1}.$

2) Find BLT which maps $z: -1, i, 1$ $\quad w = 1, i, -1$

sol: Let $w = \dfrac{az+b}{cz+d}$ be the required bilinear transformation

Let $z_1 = -1$, $z_2 = i$, $z_3 = 1$, $z_4 = z$

$w_1 = 1$, $w_2 = i$, $w_3 = -1$, $w_4 = w$

Since the cross ratio remain unaltered in BLT we have,

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

$$\frac{(1-i)(-1-w)}{(1-w)(-1-i)} = \frac{(-1-i)(1-z)}{(-1-z)(1-i)}$$

$\Rightarrow (1-i)(-1-w)(-1-z)(1-i) = (-1-i)(1-z)(1-w)(-1-i)$

~~$(1-i)^2(-1-w)(1-z) = (-1-i)^2(1-z)(1-w)$~~

$(1-i)^2(-1-w)(-1-z) = (-1-i)^2(1-z)(1-w) \Rightarrow$

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$$\frac{(1-i)}{(1+1)}\frac{(1+w)}{(1-w)} = \frac{(1+i)}{(1-i)}\frac{(1-z)}{(1+z)}$$

$$(1-i)^2(1+w)(1+z) = (1+i)^2(1-w)(1-z)$$

$$(\cancel{1}-\cancel{1}-\cancel{2i})(1+z+w+wz) = (\cancel{1}+\cancel{i^2}+\cancel{2i})(1-z-w+zw)$$

$$-1-\cancel{z}-\cancel{w}-wz = 1-\cancel{z}-\cancel{w}+zw$$

$$-1-wz-zw = 1$$

$$-1-2wz = 1$$

$$-\cancel{2}wz = \cancel{2}$$

$$\boxed{w = \frac{-1}{2}}$$ Required BLT.

3) Find the BLT which maps the points $z = 1, i, -1$ on to the points $0, 1, \infty$.

Soln $w = \dfrac{az+b}{cz+d}$ be the required BLT

w $z_1 = 1$ $z_2 = i$ $z_3 = -1$ $z_4 = z$

$w_1 = 0$ $w_2 = 1$ $w_3 = \infty$ $w_4 = w$

Since the cross ratio remain unaltered under a BLT we have,

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

$$\frac{(0-1)(\infty-w)}{(0-w)(\infty-1)} = \frac{(1-i)(-1-z)}{(1-z)(-1-i)}$$

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$$\frac{1}{w} = \frac{(1-i)(z+1)}{(1-z)(1+i)}$$

$$w = \frac{zi-i}{z}$$

$(1-z)(1+i) = w(1-i)(z+1)$

$1+i-z-zi = w-wi\ (z+1)$

$1+i-z-zi = wz-wiz+w-wi$

$1+i-z-zi = w$

wT

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