Fluid Dynamics and Solute Dispersion

Anish Ambreth
B.Tech Mathematics and Computing and Engineering Science
es20btech11001@iith.ac.in

Karthik Kurugodu
B.Tech Mathematics and Computing
ma20btech11008@iith.ac.in

***Abstract*—This report explores the dynamics of fluids and the dispersion of solutes within them. It investigates the fundamental principles governing fluid behavior, including turbulence and laminar flow, and analyzes the mechanisms influencing solute dispersion, such as diffusivity and boundary conditions.**

I. Introduction

In the realm of partial differential equations (PDEs), the flow of fluids and dispersion of solute are of interest in the scope of research with immense applications in fields such as hydrology, integration of engineering systems, and modeling in diverse branches of science. Performing the role of the backbone of fluid dynamics is PDEs such as Navier-Stokes equations. Behavior of fluid with movement in both momentum and energy is comprehended by it. This range is from laminar flows to turbulent eddies. Simultaneously, going through solute dispersion among flowing bodies of fluid mediums, modeled by advection-diffusion equations and the related PDE‘s, gives an understanding of the transport process of dissatisfied substances.

Here we investigate the domain of solute transport and fluid flow and its application to partial differential equations(PDEs). By using a brief overview of important concept and mathematical formulations, we plan to reveal in a simple language the fundamental rules that control fluid dynamics and solute motion.

II. Fluid Dynamics

A. Reynolds Transport Theorem

In thermodynamics and solid mechanics we often work with a system (also called a closed system), defined as a quantity of matter of fixed identity. In fluid dynamics, it is more common to work with a control volume (also called an open system), defined as a region in space chosen for study. The size and shape of a system may change during a process, but no mass crosses its boundaries. A control volume, on the other hand, allows mass to flow in or out across its boundaries, which are called the control surface.

The general form of the Reynolds transport theorem can be derived by considering a system with an arbitrary shape and arbitrary interactions, but the derivation is rather involved. To help you grasp the fundamental meaning of the theorem, we derive it first in a straightforward manner using a simple geometry and then generalize the results.

Consider flow from left to right through a diverging (expanding) portion of a flow field as sketched in below figure. The upper and lower bounds of the fluid under consideration are streamlines of the flow, and we assume uniform flow through any cross section between these two streamlines. We choose the control volume to be fixed between sections (1) and (2) of the flow field. Both (1) and (2) are normal to the direction of flow. At some initial time t, the system coincides with the control volume, and thus the system and control volume are identical (the greenish-shaded region in the figure). During time interval Δt, the system moves in the flow direction at uniform speeds V_1 at section (1) and V_2 at section (2). The system at this later time is indicated by the hatched region. The region uncovered by the system during this motion is designated as section I (part of the CV), and the new region covered by the system is designated as section II (not part of the CV). Therefore, at time $t + \Delta t$, the system consists of the same fluid, but it occupies the region CV - I + II. The control volume is fixed in space, and thus it remains as the shaded region marked CV at all times.

Let B represent any extensive property (such as mass, energy, or momentum), and let $b = B/m$ represent the corresponding intensive property. Noting that extensive properties are additive, the extensive property B of the system at times t and $t + \Delta t$ is expressed as

$$B_{sys,t} = B_{CV,t} \quad \text{(System and CV coincide at time t)}$$
$$B_{sys,t+\Delta t} = B_{CV,t+\Delta t} - B_{I,t+\Delta t} + B_{II,t+\Delta t}$$

Subtracting the first equation from the second one and dividing by Δt gives

$$\frac{B_{sys,t+\Delta t} - B_{sys,t}}{\Delta t} = \frac{B_{CV,t+\Delta t} - B_{CV,t}}{\Delta t} - \frac{B_{I,t+\Delta t}}{\Delta t} + \frac{B_{II,t+\Delta t}}{\Delta t}$$

Taking the limit as $\Delta t \to 0$, and using the definition of derivative, we get

$$\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} - \dot{B}_{in} + \dot{B}_{out} \tag{1}$$

or

$$\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} - b_1 \rho_1 V_1 A_1 + b_2 \rho_2 V_2 A_2$$

since

$$B_{I,t+\Delta t} = b_1 m_{I,t+\Delta t} = b_1 \rho_1 \vartheta_{I,t+\Delta t} = b_1 \rho_1 V_1 A_1 \Delta t$$
$$B_{II,t+\Delta t} = b_2 m_{II,t+\Delta t} = b_2 \rho_2 \vartheta_{II,t+\Delta t} = b_2 \rho_2 V_2 A_2 \Delta t$$

where A_1 and A_2 are the cross-sectional areas at locations 1 and 2. Equation (1) states that the time rate of change of the property B of the system is equal to the time rate of change of B of the control volume plus the net flux of B out of the

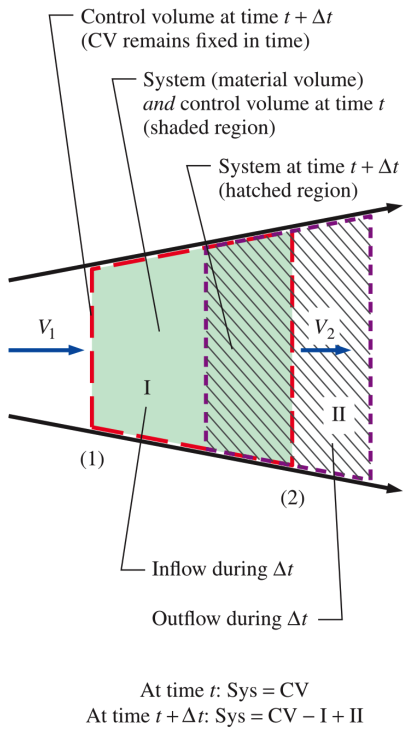

Fig. 1. A moving system (hatched region) and a fixed control volume (shaded region) in a diverging portion of a flow field at times t and $t + \Delta t$. The upper and lower bounds are streamlines of the flow.

control volume by mass crossing the control surface. This is the desired relation since it relates the change of a property of a system to the change of that property for a control volume. Note that equation (1) applies at any instant in time, where it is assumed that the system and the control volume occupy the same space at that particular instant in time.

The influx $\dot{B}_{in}$ and out flux $\dot{B}_{out}$ of the property B in this case are easy to determine since there is only one inlet and one outlet, and the velocities are approximately normal to the surfaces at sections (1) and (2). In general, how- ever, we may have several inlet and outlet ports, and the velocity may not be normal to the control surface at the point of entry. Also, the velocity may not be uniform. To generalize the process, we consider a differential surface area dA on the control surface and denote its unit outer normal by $\vec{n}$. The flow rate of property b through dA is $\rho b \vec{V} \cdot \vec{n} dA$ since the dot product $\vec{V} \cdot \vec{n}$ gives the normal component of the velocity. Then the net rate of outflow through the entire control surface is determined by integration to be

$$\dot{B}_{net} = \dot{B}_{out} - \dot{B}_{in} = \int_{CS} \rho b \vec{V} \cdot \vec{n}\, dA \quad \text{(inflow if negative)}$$

The properties within the control volume may vary with position, in general. In such a case, the total amount of property B within the control volume must be determined by integration:

$$B_{CV} = \int_{CV} \rho b \, d\vartheta$$

Then the equation (1) becomes:

$$\frac{dB_{sys}}{dt} = \frac{d}{dt} \int_{CV} \rho b \, d\vartheta + \int_{CS} \rho b \vec{V} \cdot \vec{n}\, dA \qquad (2)$$

The above equation can also be written as

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\rho b)\, d\vartheta + \int_{CS} \rho b \vec{V} \cdot \vec{n}\, dA$$

These equations are for fixed CV.

For the case when the CV is not fixed

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\rho b)\, d\vartheta + \int_{CS} \rho b \vec{V_r} \cdot \vec{n}\, dA \qquad (3)$$

where, $\vec{V_r}$ is the relative velocity.

$$\vec{V_r} = \vec{V} - \vec{V}_{CS}$$

V_{CS} is the local velocity of the control surface.

During steady flow, the amount of the property B within the control volume remains constant in time, and thus the time derivative in equation (3) becomes zero. Then the Reynolds transport theorem reduces to

$$\frac{dB_{sys}}{dt} = \int_{CS} \rho b \vec{V_r} \cdot \vec{n}\, dA$$

B. Conservation of Mass Principle

The conservation of mass principle for a control volume can be expressed as: The net mass transfer to or from a control volume during a time interval Δt is equal to the net change (increase or decrease) of the total mass within the control volume during Δt. That is,

$$m_{in} - m_{out} = \Delta m_{CV}$$

where Δm_{CV} is the change in the mass of the control volume. It can also be expressed in rate form as

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt}$$

where $\dot{m_{in}}$ and $\dot{m_{out}}$ are the total rates of mass flow into and out of the control volume, and dm_{CV}/dt is the rate of change of mass within the control volume boundaries. Proof for this can be obtained from Reynolds's transport theorem.

Let B = m, this implies b = 1, Equation (1) becomes,

$$\frac{dm_{sys}}{dt} = \frac{dm_{CV}}{dt} - \dot{m}_{in} + \dot{m}_{out}$$

The mass of a closed system is constant, and thus dm_{sys}/dt is zero. We get,

$$\frac{dm_{CV}}{dt} - \dot{m}_{in} + \dot{m}_{out} = 0$$

$$\frac{dm_{CV}}{dt} = \dot{m}_{in} - \dot{m}_{out}$$

A simple rule in selecting a control volume is to make the control surface normal to the flow at all locations where it crosses the fluid flow, whenever possible. This way the dot product $\vec{V} \cdot \vec{n}$ simply becomes the magnitude of the velocity, and the integral $\int_{CV} \rho(\vec{V} \cdot \vec{n})\, dA$ becomes simply $\rho V A$.

C. Cauchy Momentum Equation

In the above Reynolds's equation, let's consider $B = m\vec{v}$. We get,

$$\frac{d(m\vec{v})}{dt} = \int_{CV} \frac{\partial}{\partial t}(\rho\vec{v})\; dV + \int_{CS} \rho\vec{v}(\vec{v}\cdot\vec{n})\; dA$$

We know from Newton's second law,

$$\frac{d(m\vec{v})}{dt} = \sum \vec{F}$$

Therefore,

$$\sum \vec{F} = \int_{CV} \frac{\partial}{\partial t}(\rho\vec{v})\; dV + \int_{CS} \rho\vec{v}(\vec{v}\cdot\vec{n})\; dA$$

$$\sum \vec{F}_{body} + \sum \vec{F}_{surface} = \int_{CV} \frac{\partial}{\partial t}(\rho\vec{v})\; dV + \int_{CS} \rho\vec{v}(\vec{v}\cdot\vec{n})\; dA \tag{4}$$

Now let's consider an infinitesimal control volume, we consider a box-shaped control volume, at the center of the box, we define the density as ρ and the velocity components as u, v, and w. We also define the stress tensor as σ_{ij} at the centre of the box. For simplicity, we consider the x-component of equation (4), obtained by setting $\sum \vec{F}$ equal to it's x - component, $\sum F_x$ and $\vec{v}$ to it's x - component, u.

$$\sum F_x = \sum \vec{F}_{x,body} + \sum \vec{F}_{x,surface}$$

$$= \int_{CV} \frac{\partial}{\partial t}(\rho u)\; dV + \sum_{out} \dot{m}u - \sum_{in} \dot{m}u \tag{5}$$

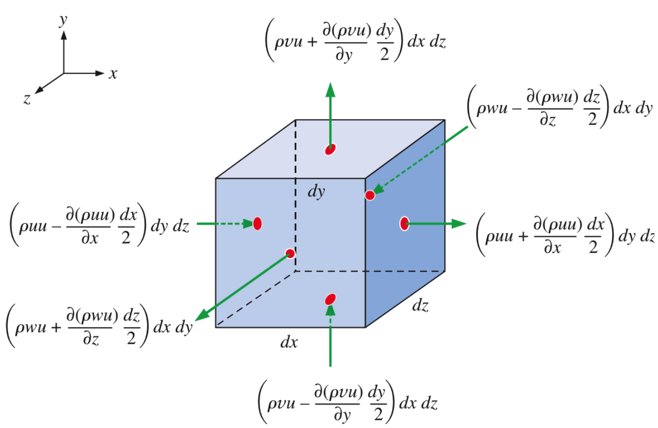

Fig. 2. Inflow and outflow of the x-component of linear momentum through each face of an infinitesimal control volume; the red dots indicate the center of each face.

As the control volume is very small,

$$\int_{CV} \frac{d}{dt}(\rho u)u\; dV \cong \frac{d}{dt}(\rho u)\; dx\,dy\,dz$$

Since the volume of the differential element is $dx\; dy\; dz$. We apply first-order truncated Taylor series expansions at locations away from the center of the control volume to approximate the inflow and outflow of momentum in the x-direction.

The above figure shows these momentum fluxes at the center point of each of the six faces of the infinitesimal control volume. Only the normal velocity component at each face needs to be considered, since the tangential velocity components contribute no mass flow out of (or into) the face, and hence no momentum flow through the face either.

By summing all the outflows and subtracting all the inflows shown in the figure, we obtain an approximation,

$$\sum_{out} \dot{m}u - \sum_{in} \dot{m}u \cong \left(\frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho vu) + \frac{\partial}{\partial z}(\rho wu)\right) dx\,dy\,dz$$

Next, we sum all the forces acting on our infinitesimal control volume in the x-direction. We need to consider both body forces and surface forces. Gravity force (weight) is the only body force we take into account. For the general case in which the coordinate system may not be aligned with the z-axis (or with any coordinate axis for that matter), the gravity vector is written as

$$\vec{g} = g_x\vec{i} + g_y\vec{j} + g_z\vec{k}$$

Thus, in the x-direction, the body force on the control volume is

$$\sum \vec{F}_{x,body} = \sum \vec{F}_{x,gravity} \cong \rho g_x\, dx\,dy\,dz$$

Next we consider the net surface force in the x-direction. Recall that stress tensor σ_{ij} has dimensions of force per unit area. Thus, to obtain a force, we must multiply each stress component by the surface area of the face on which it acts. We need to consider only those components that point in the x (or $-x$) direction. (The other components of the stress tensor, although they may be nonzero, do not contribute to a net force in the x-direction.) Using truncated Taylor series expansions, we sketch all the surface forces that contribute to a net x-component of surface force acting on our differential fluid element

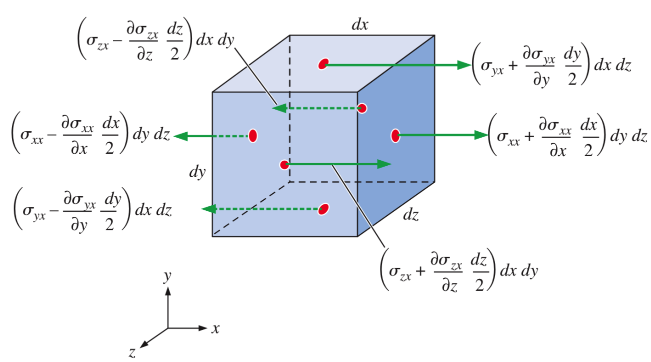

Fig. 3. Sketch illustrating the surface forces acting in the x-direction due to the appropriate stress tensor component on each face of the differential control volume; the red dots indicate the center of each face.

Summing all the surface forces illustrated in the above figure, we obtain an approximation for the net surface force acting on the differential fluid element in the x-direction

$$\sum \vec{F}_{x,surface} \cong \left(\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx}) \right) dx\, dy\, dz$$

We now substitute these equations into equation (5) , noting that the volume of the differential element of fluid, $dx\ dy\ dz$, appears in all terms and can be eliminated. After some rearrangement we obtain the differential form of the x-momentum equation,

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho vu) + \frac{\partial}{\partial z}(\rho wu)$$
$$= \rho g_x + \frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})$$

In similar fashion, we generate differential forms of the y and z momentum equations,

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho vv) + \frac{\partial}{\partial z}(\rho wv)$$
$$= \rho g_y + \frac{\partial}{\partial x}(\sigma_{xy}) + \frac{\partial}{\partial y}(\sigma_{yy}) + \frac{\partial}{\partial z}(\sigma_{zy})$$

and

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho uw) + \frac{\partial}{\partial y}(\rho vw) + \frac{\partial}{\partial z}(\rho ww)$$
$$= \rho g_z + \frac{\partial}{\partial x}(\sigma_{xz}) + \frac{\partial}{\partial y}(\sigma_{yz}) + \frac{\partial}{\partial z}(\sigma_{zz})$$

Finally, we combine these equation into one vector equation,

$$\frac{\partial}{\partial t}(\rho \vec{V}) + \vec{\nabla} \cdot (\rho \vec{V}\vec{V}) = \rho \vec{g} + \vec{\nabla} \cdot \sigma_{ij}$$

Applying the product rule to the first term on the left side of the equation, we get

$$\frac{\partial}{\partial t}(\rho \vec{V}) = \rho \frac{\partial \vec{V}}{\partial t} + \vec{V} \frac{\partial \rho}{\partial t}$$

We can also write,

$$\vec{\nabla} \cdot (\rho \vec{V}\vec{V}) = \vec{V}\vec{\nabla} \cdot (\rho \vec{V}) + \rho(\vec{V} \cdot \vec{\nabla})\vec{V}$$

Thus we have eliminated the second-order tensor represented by $\vec{V}\vec{V}$.
After some rearrangements and substitution we get,

$$\rho \frac{\partial \vec{V}}{\partial t} + \vec{V} \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) \right] + \rho(\vec{V} \cdot \vec{\nabla})\vec{V} = \rho \vec{g} + \vec{\nabla} \cdot \sigma_{ij}$$

But the expression in square brackets in this equation is identically zero by the continuity equation, and by combining the remaining two terms on the left side, we write

$$\rho \left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V} \right] = \rho \frac{D\vec{V}}{Dt} = \rho \vec{g} + \vec{\nabla} \cdot \sigma_{ij}$$

In Cartesian coordinates, the three components of Cauchy's equation are

$$\rho \frac{Du}{Dt} = \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{dy} + \frac{\partial \sigma_{zx}}{\partial z}$$
$$\rho \frac{Dv}{Dt} = \rho g_y + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}$$
$$\rho \frac{Dw}{Dt} = \rho g_z + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}$$

D. The Angular Momentum Equation

The linear momentum equation is useful for determining the relationship between the linear momentum of flow streams and the resultant forces. Many engineering problems involve the moment of the linear momentum of flow streams, and the rotational effects caused by them. Such problems are best analyzed by the angular momentum equation, also called the moment of momentum equation. An important class of fluid devices, called turbo machines, which include centrifugal pumps, turbines, and fans, is analyzed by the angular momentum equation.

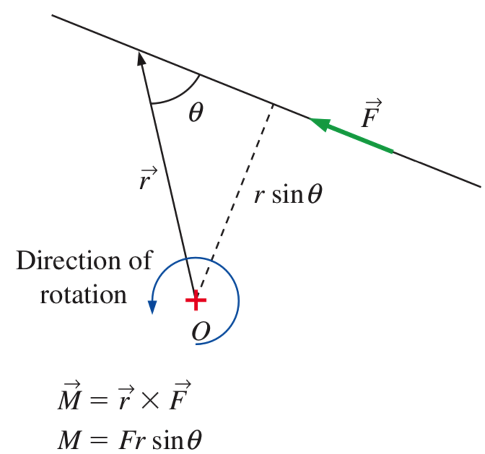

Fig. 4. The moment of force $\vec{F}$ about a point O is the vector product of the position vector $\vec{r}$ and $\vec{F}$

The moment of a force $\vec{F}$ about a point O is the vector (or cross) product

$$\vec{M} = \vec{r} \times \vec{F}$$

where $\vec{r}$ is the position vector from point O to any point on the line of action of $\vec{F}$.
The vector product of $\vec{r}$ and the momentum vector $m\vec{v}$ gives the moment of momentum, also called the angular momentum, about a point O as

$$\vec{H} = \vec{r} \times m\vec{v}$$

Therefore, $\vec{r} \times \vec{V}$ represents the angular momentum per unit mass, and the angular momentum of a differential mass $\partial m = \rho dV$ is $\partial \vec{H} = (\vec{r} \times \vec{v})\rho\, dV$.

Then the angular momentum of a system is determined by integration to be

$$\vec{H}_{sys} = \int_{sys} (\vec{r} \times \vec{v})\rho \, dV$$

The rate of change of the moment of momentum is

$$\frac{d\vec{H}_{sys}}{dt} = \frac{d}{dt}\int_{sys} (\vec{r} \times \vec{v})\rho \, dV$$

We also know,

$$\vec{M} = \vec{r} \times \vec{F}$$

$$\vec{H} = \vec{r} \times m\vec{v}$$

Therefore,

$$\sum \vec{M} = \frac{d\vec{H}_{sys}}{dt}$$

Now by using Reynolds's Theorem, Let B=$\vec{H}$, this implies b=$\vec{r} \times \vec{v}$.
Substituting the values in equation (3) we get,

$$\frac{d\vec{H}_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho(\vec{r} \times \vec{v}) \, dV + \int_{CS} \rho(\vec{r} \times \vec{v})(\vec{v}_r \cdot \vec{n}) \, dA$$

$$\sum \vec{M} = \frac{d}{dt}\int_{CV} \rho(\vec{r} \times \vec{v}) \, dV + \int_{CS} \rho(\vec{r} \times \vec{v})(\vec{v}_r \cdot \vec{n}) \, dA \quad (6)$$

Equation (6) can be stated as,
The sum of all external moments acting on a CV=
The time rate of change of the angular momentum of the contents of the CV + The net flow rate of angular momentum out of the control surface by the mass flow
Note: For fixed CV $\vec{v}_r = \vec{v}$
Let's take one of the cases where $\vec{r}$ is either constant along the inlet or outlet (as in radial flow turbo machines) or is large compared to the diameter of the inlet or outlet pipe (as in rotating lawn sprinklers).In such cases, the average value of $\vec{r}$ is used throughout the cross-sectional area of the inlet or outlet. Then, an approximate form of the angular momentum equation in terms of average properties at inlets and outlets becomes,

$$\sum \vec{M} = \frac{d}{dt}\int_{CV} \rho(\vec{r} \times \vec{v}) \, dV + \sum_{out}(\vec{r} \times \dot{m}\vec{v}) - \sum_{in}(\vec{r} \times \dot{m}\vec{v}) \quad (7)$$

For steady flow,

$$\sum \vec{M} = \sum_{out}(\vec{r} \times \dot{m}\vec{v}) - \sum_{in}(\vec{r} \times \dot{m}\vec{v})$$

The above equation states that the net torque acting on the control volume during steady flow is equal to the difference between the outgoing and incoming angular momentum flow rates

E. Bernoulli Equation

We know from newton's second law,

$$\sum F_{sys} = m \, a_{sys}$$

$$a = \frac{dV}{dt}$$

$$dV = \frac{dV}{ds}ds + \frac{dV}{dt}dt$$

If the flow is steady $\frac{dV}{dt} = 0$, then

$$a = \frac{ds}{dt} \cdot \frac{dV}{ds} = V \cdot \frac{dV}{ds}$$

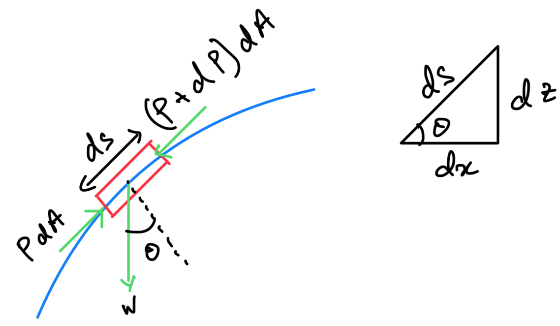

Fig. 5.

The above figure shows an element in the fluid, where P is the pressure and w is the weight of the element and ds being the length of the element and dA being the area of the element.

Using the newton's second law,

$$P \, dA - (P + dP) \, dA - W \, sin\theta = m \cdot V \cdot \frac{dV}{ds} \quad (8)$$

$$W = mg = \rho \, dA \, ds \cdot g = \rho g \, dA \, ds$$

$$sin\theta = \frac{dZ}{ds}$$

Equation (8) becomes,

$$-dP \, dA - \rho g \, dA \, dZ = \rho V \, dA \, dV$$

$$-dP - \rho g \, dZ = \rho V \, dV$$

$$\frac{dP}{\rho} + g \, dZ + V \, dV = 0$$

$$\frac{dP}{\rho} + g \, dZ + \frac{1}{2} \, d(V^2) = 0 \quad (9)$$

$$\frac{dP}{\rho} = \text{flow energy}$$

$$g \, dZ = \text{Potential energy}$$

$$\frac{1}{2} \, d(V^2) = \text{Kinetic energy}$$

This is the case for steady flow, for the case of unsteady flow.

$$\frac{dP}{\rho} + \frac{dV}{dt}ds + g\,dZ + \frac{1}{2}\,d(V^2) = 0$$

Equation (9) after integration becomes,

$$P + \rho g Z + \frac{\rho V^2}{2} = 0$$

$$P = \text{Static Pressure}$$
$$\rho g Z = \text{Hydrostatic Pressure}$$
$$\frac{\rho V^2}{2} = \text{Dynamic Pressure}$$

Bernoulli Equation states that the total pressure along a streamline is constant.

F. Viscosity, Newtonian and Non-Newtonian Fluids

When two solid bodies in contact move relative to each other, a friction force develops at the contact surface in the direction opposite to motion. To move a table on the floor, for example, we have to apply a force to the table in the horizontal direction large enough to overcome the friction force. The magnitude of the force needed to move the table depends on the friction coefficient between the table legs and the floor.
The situation is similar when a fluid moves relative to a solid or when two fluids move relative to each other. We move with relative ease in air, but not so in water. Moving in oil would be even more difficult, as can be observed by the slower downward motion of a glass ball dropped in a tube filled with oil. It appears that there is a property that represents the internal resistance of a fluid to motion or the "fluidity," and that property is the viscosity. The force a flowing fluid exerts on a body in the flow direction is called the drag force, and the magnitude of this force depends, in part, on viscosity.
To obtain a relation for viscosity, consider a fluid layer between two very large parallel plates (or equivalently, two parallel plates immersed in a large body of a fluid) separated by a distance l. Now a constant parallel force F is applied to the upper plate while the lower plate is held fixed. After the initial transients, it is observed that the upper plate moves continuously under the influence of this force at a constant speed V. The fluid in contact with the upper plate sticks to the plate surface and moves with it at the same speed, and the shear stress t acting on this fluid layer is

$$\tau = \frac{F}{A}$$

where A is the contact area between the plate and the fluid. Note that the fluid layer deforms continuously under the influence of shear stress.
The fluid in contact with the lower plate assumes the velocity of that plate, which is zero (because of the no-slip condition). In steady laminar flow, the fluid velocity between the plates varies linearly between 0 and V, and thus the velocity profile and the velocity gradient are

$$u(y) = \frac{y}{l}V \quad \text{and} \quad \frac{du}{dy} = \frac{V}{l}$$

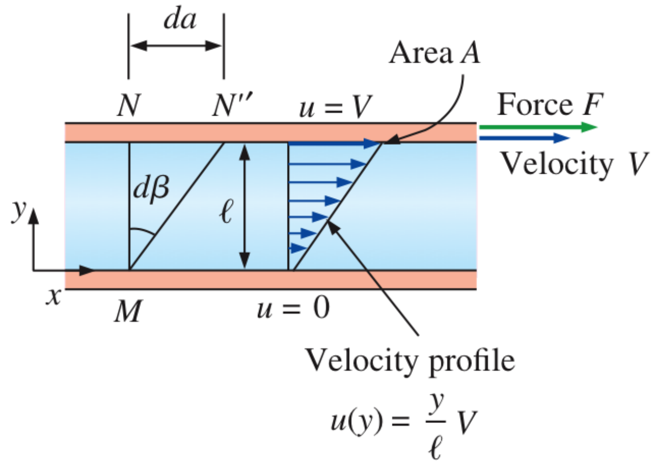

Fig. 6.

where y is the vertical distance from the lower plate.
During a differential time interval dt, the sides of fluid particles along a vertical line MN rotate through a differential angle $d\beta$ while the upper plate moves a differential distance $da = V dt$. The angular displacement or deformation (or shear strain) can be expressed as

$$d\beta \approx \tan d\beta = \frac{da}{l} = \frac{V\ dt}{l} = \frac{du}{dy}dt$$

Rearranging, the rate of deformation under the influence of shear stress τ becomes

$$\frac{d\beta}{dt} = \frac{du}{dy}$$

Thus we conclude that the rate of deformation of a fluid element is equivalent to the velocity gradient du/dy. Further, it can be verified experimentally that for most fluids the rate of deformation (and thus the velocity gradient) is directly proportional to the shear stress τ,

$$\tau \propto \frac{d\beta}{dt} \quad \text{or} \quad \tau \propto \frac{du}{dy}$$

Fluids for which the rate of deformation is linearly proportional to the shear stress are called Newtonian fluids . Most common fluids such as water, air, gasoline, and oils are Newtonian fluids. Blood and liquid plastics are examples of non-Newtonian fluids.
In one-dimensional shear flow of Newtonian fluids, shear stress can be expressed by the linear relationship,

$$\tau = \mu\frac{du}{dy}$$

where the constant of proportionality μ is called the coefficient of viscosity or the dynamic (or absolute) viscosity of the fluid, whose unit is kg/m·s, or equivalently, N·s/m^2 (or Pa·s where Pa is the pressure unit pascal). A common viscosity unit is poise, which is equivalent to 0.1 Pa·s (or centipoise, which

is one-hundredth of a poise). The viscosity of water at 20°C is 1.002 centipoise, and thus the unit centipoise serves as a useful reference.
A plot of shear stress versus the rate of deformation (velocity gradient) for a Newtonian fluid is a straight line whose slope is the viscosity of the fluid, as shown below,

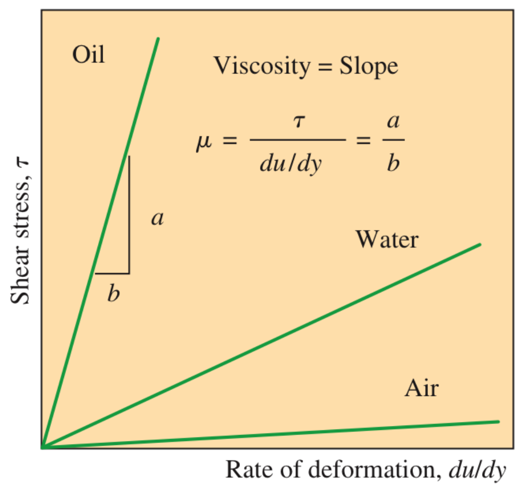

Fig. 7. The rate of deformation (velocity gradient) of a Newtonian fluid is proportional to shear stress, and the constant of proportionality is the viscosity.

The above equation can also be written as,

$$\tau = \mu\dot{\gamma}$$

where, $\dot{\gamma}$ is the strain rate, which is nothing but $\frac{d\beta}{dt}$.
Note that viscosity is independent of the rate of deformation for Newtonian fluids. Since the rate of deformation is proportional to the strain rate, it reveals that viscosity is actually a coefficient in a stress–strain relationship.
For non-Newtonian fluids, the relationship between shear stress and rate of deformation is not linear, as shown below. The slope of the curve on the τ versus du/dy chart is referred to as the apparent viscosity of the fluid. Fluids for which the apparent viscosity increases with the rate of deformation (such as solutions with suspended starch or sand) are referred to as dilatant or shear thickening fluids, and those that exhibit the opposite behavior (the fluid becoming less viscous as it is sheared harder, such as some paints, polymer solutions, and fluids with suspended particles) are referred to as pseudo plastic or shear thinning fluids. Some materials such as toothpaste can resist a finite shear stress and thus behave as a solid, but deform continuously when the shear stress exceeds the yield stress and behave as a fluid. Such materials are referred to as Bingham plastics.
In case of dilatant, μ increases with force, but in case of pseudo plastic, μ decreases with force.
The general form for the equation can be written as,

$$\tau = K\left[\frac{du}{dy}\right]^n$$

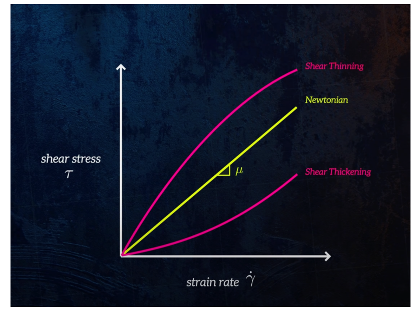

Fig. 8.

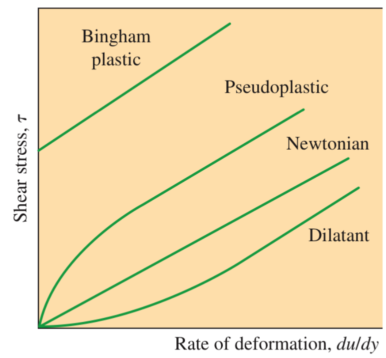

Fig. 9. Variation of shear stress with the rate of deformation for Newtonian and non-Newtonian fluids (the slope of a curve at a point is the apparent viscosity of the fluid at that point).

n<1 : pseudo plastic; n=1 & K=μ : Newtonian; n>1 : Dilatant
For the above-mentioned fluids, the shear stress will disappear when the shear rate approaches zero. Some other fluids will start to deform only when the shear stress exceeds a certain critical level. These fluids are called plastic fluids (e.g., toothpaste or even blood). One needs to overcome a given shear force before the fluid flows. This critical value of the corresponding shear stress is called the yield stress, τ_0. For such fluids, the shear stress can be generally defined as,

$$\tau^{\frac{1}{m}} = \tau_0^{\frac{1}{m}} + K\left[\frac{du}{dy}\right]^{\frac{1}{m}}$$

It is called Bingham model when m=1, and Casson model when m=2.

G. Navier Stokes Equation

When a fluid is at rest, the only stress acting at any surface of any fluid element is the local hydrostatic pressure P, which always acts inward and normal to the surface. Thus, regardless of the orientation of the coordinate axes, for a fluid at rest the stress tensor reduces to

$$\textit{Fluid at rest: } \sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}$$

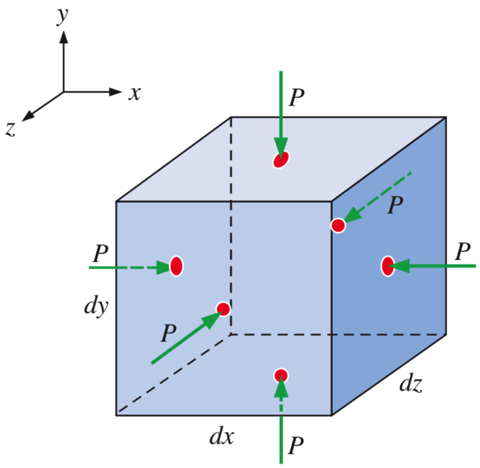

Fig. 10. For fluids at rest, the only stress on a fluid element is the hydrostatic pressure, which always acts inward and normal to any surface.

In case of moving fluids.

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

where we have introduced a new tensor, τ_{ij}, called the viscous stress tensor or the deviatoric stress tensor.
Viscous stress tensor for an incompressible Newtonian fluid with constant properties:

$$\tau_{ij} = 2\mu\epsilon_{ij}$$

where ϵ_{ij} is the strain rate tensor.
In Cartesian coordinates, the nine components of the viscous stress tensor are listed, only six of which are independent due to symmetry:

$$\tau_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \begin{pmatrix} 2\mu\frac{\partial u}{\partial x} & \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & 2\mu\frac{\partial v}{\partial y} & \mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) & \mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) & 2\mu\frac{\partial w}{\partial z} \end{pmatrix}$$

Therefore, in cartesian coordinates the stress tensor (σ_{ij}) becomes,

$$\begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} 2\mu\frac{\partial u}{\partial x} & \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & 2\mu\frac{\partial v}{\partial y} & \mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) & \mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) & 2\mu\frac{\partial w}{\partial z} \end{pmatrix}$$

Now we substitute this into the three Cartesian components of Cauchy's equation. Let's consider the x-component first.

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho g_x + 2\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) + \mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)$$

After re-arranging and using this,

$$\mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right) = \mu\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z}\right)$$

We get,

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

The term in parentheses is zero because of the continuity equation for incompressible flow.
We also recognize the last three terms as the Laplacian of velocity component u in Cartesian coordinates. Thus, we write the x-component of the momentum equation as,

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho g_x + \mu\nabla^2 u$$

Similarly, the y and z-components of the momentum equation reduce to,

$$\rho\frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \rho g_y + \mu\nabla^2 v$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \rho g_z + \mu\nabla^2 w$$

Finally, we combine the three components into one vector equation, the result is the Navier–Stokes equation for incompressible flow with constant viscosity.

$$\rho\frac{D\vec{V}}{Dt} = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V} \tag{10}$$

X-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]$$

Y-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right]$$

Z-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right]$$

The Navier–Stokes equation are expanded in cylindrical coordinates (r, u, z) and (u_r, u_θ, u_z):
The r-component:

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)$$
$$=-\frac{\partial P}{\partial r}+\rho g_r$$
$$+\mu\left[\frac{1}{r}\frac{\partial}{dr}\left(r\frac{\partial u_r}{\partial r}\right)-\frac{u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right]$$

The θ-component:

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}-\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)$$
$$=-\frac{1}{r}\frac{\partial P}{\partial r}+\rho g_\theta$$
$$+\mu\left[\frac{1}{r}\frac{\partial}{dr}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right]$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)$$
$$=-\frac{\partial P}{\partial z}+\rho g_z+\mu\left[\frac{1}{r}\frac{\partial}{dr}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]$$

H. Laminar and Turbulent flows

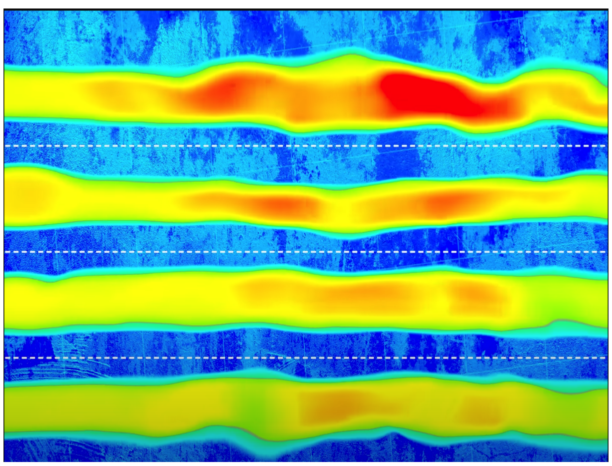

Fig. 11. Laminar Flow

Eddys in turbulent flow creates mixing of fliud layers.

I. Reynolds Number

The transition from laminar to turbulent flow depends on the geometry, surface roughness, flow velocity, surface temperature, and type of fluid, among other things. After exhaustive experiments in the 1880s, Osborne Reynolds discovered that the flow regime depends mainly on the ratio of inertial forces to viscous forces in the fluid. This ratio is called the Reynolds number and is expressed for internal flow in a circular pipe as

$$Re=\frac{\rho V_{avg} D}{\mu} \tag{11}$$

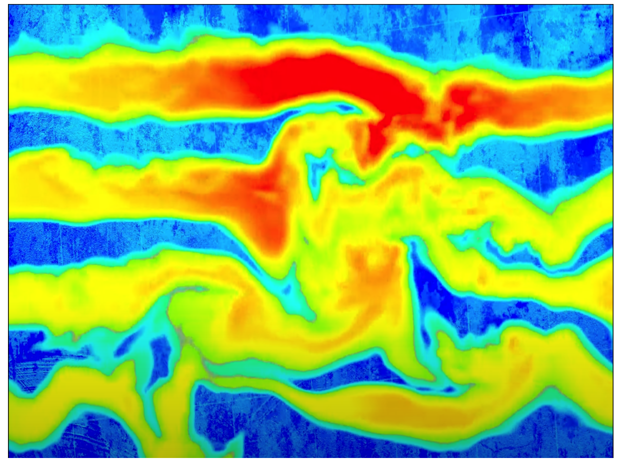

Fig. 12. Turbulent Flow

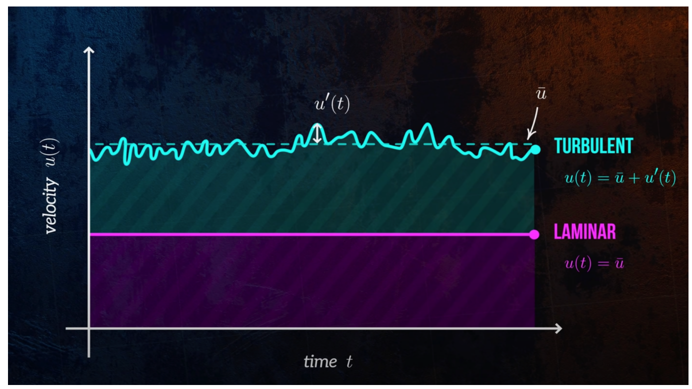

Fig. 13.

where, V_{avg} = average flow velocity, D= characteristic length of the geometry.
At large Reynolds numbers, the inertial forces, which are proportional to the fluid density and the square of the fluid velocity, are large relative to the viscous forces, and thus the viscous forces cannot prevent the random and rapid fluctuations of the fluid. At small or moderate Reynolds numbers, however, the viscous forces are large enough to suppress these fluctuations and to keep the fluid "in line." Thus the flow is turbulent in the first case and laminar in the second.
The Reynolds number at which the flow becomes turbulent is called the critical Reynolds number, Re_{cr}. The value of the critical Reynolds number is different for different geometries and flow conditions. For internal flow in a circular pipe, the generally accepted value of the critical Reynolds number is $Re_{cr}=2300$
It certainly is desirable to have precise values of Reynolds numbers for laminar, transitional, and turbulent flows, but this is not the case in practice. It turns out that the transition from laminar to turbulent flow also depends on the degree of disturbance of the flow by surface roughness, pipe vibrations, and fluctuations in the upstream flow. Under most

practical conditions, the flow in a circular pipe is laminar for $Re \leq 2300$, turbulent for $Re \geq 4000$ and transitional in between.

Let us take ΔP as $P_{in} - P_{out}$.

$$Laminar\ \Delta P < Turbulent\ \Delta P$$

Darcy-Weisbach Equation

$$\frac{\Delta P}{L} = f \cdot \frac{\rho}{2} \cdot \frac{u_{avg}^2}{D}$$

where f is the friction factor.
In case of laminar flow,

$$f = \frac{64}{Re}$$

For turbulent flow we need to use Colebrook equation and solving it iteratively. But this equation is difficult to use, so instead we use moody diagram.

$$\frac{1}{\sqrt{f}} = -2\log\left(\frac{\epsilon}{3.7D} + \frac{2.51}{Re\sqrt{f}}\right)$$

where $\frac{\epsilon}{D}$ is relative roughness.

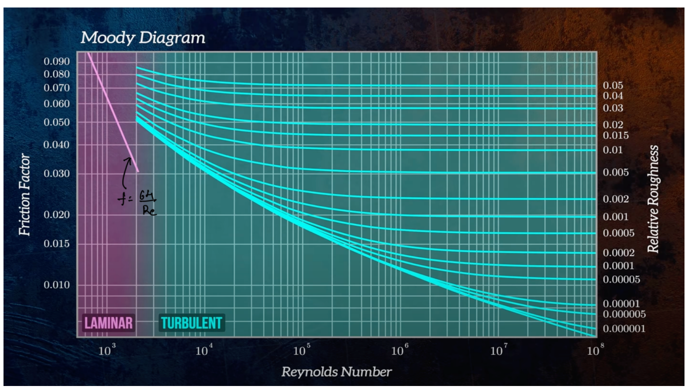

Fig. 14. Moody diagram

There exists a very thin layer where viscous force dominate because of no slip condition, due to this there is a laminar flow. This layer is called Laminar Sublayer. The thickness of this layer decreases when Re is increases.
If the roughness of the surface is contained entirely in the laminar sublayer then the surface is called hydraulically smooth.
This roughness has no effect on the turbulent flow.

J. Flows

Poiseuille and Couette Flow are two kinds of flow, which are two exact solutions to the continuity and Navier-Stokes equation.
Poiseuille Flow: Laminar flow between fixed parallel plates.
Consider the plates are very wide and long, this implies,

$$v = w = 0$$

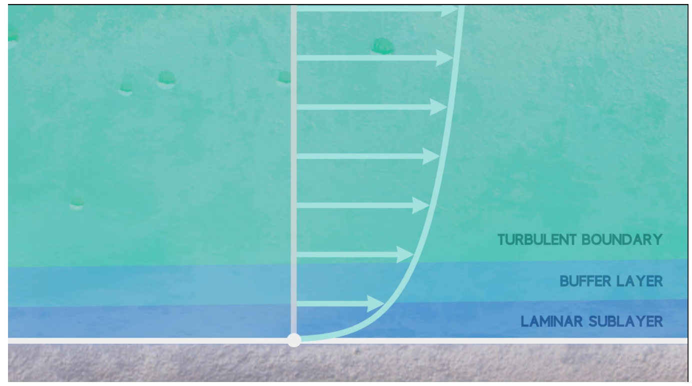

Fig. 15.

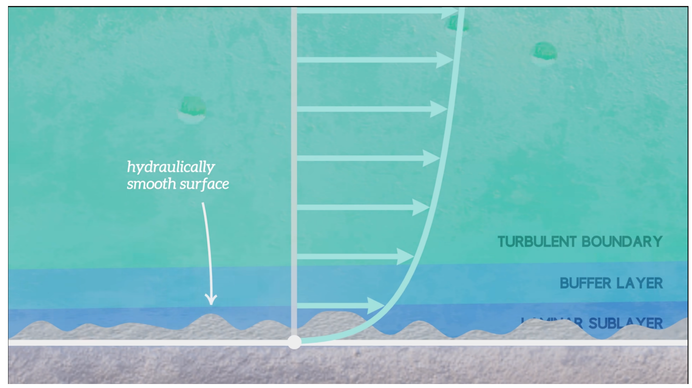

Fig. 16.

We know the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

This implies,

$$\frac{\partial u}{\partial x} = 0$$

This flow is called fully developed flow.

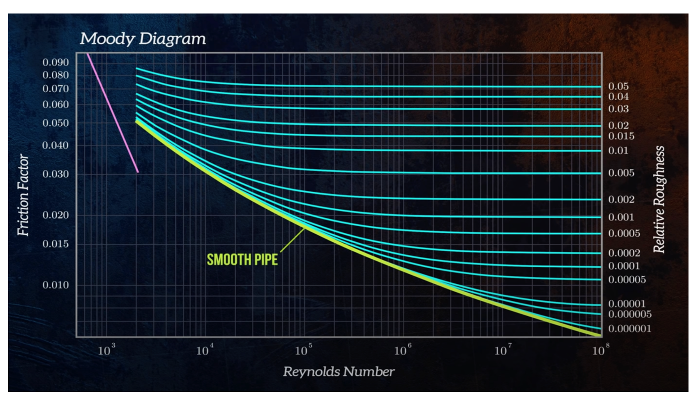

Fig. 17.

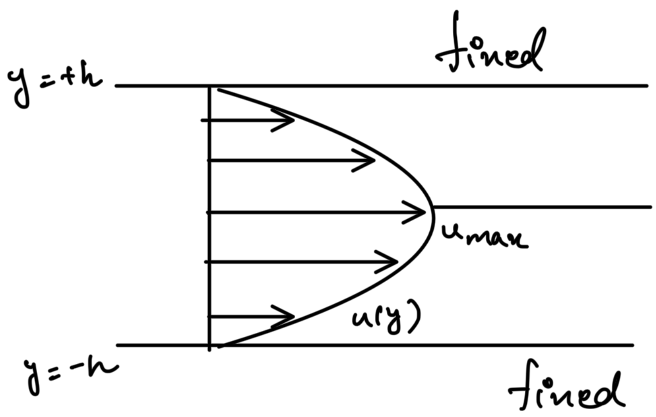

Fig. 18. Poiseuille Flow

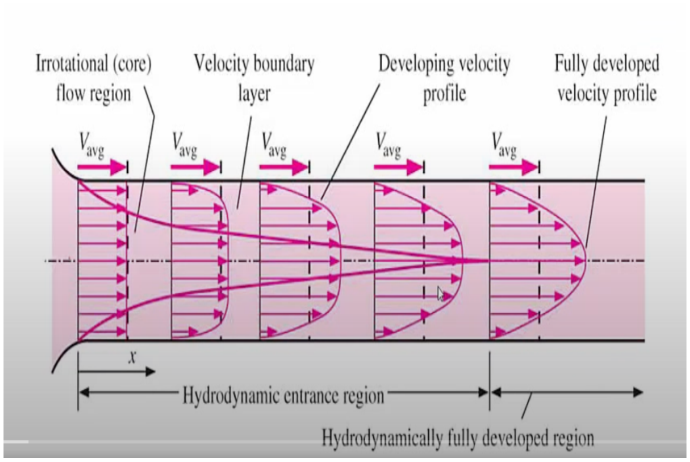

Fig. 19. Poiseuille Flow

Now we use Navier-Stokes equation in x-direction,

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)$$
$$=-\frac{\partial P}{\partial x}+\rho g_x+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]$$

Since it is a steady flow,

$$\frac{\partial u}{\partial t}=0$$

After substituting the values, we get,

$$\mu\frac{d^2u}{dy^2}=\frac{dP}{dx}$$

From "separation of variables" method: Two equal quantities, one varies with x only (p(x)), one varies with y only (u(y)). The can be true is only if they equal a constant. Otherwise they would be independent!

$\frac{dP}{dx}$ is the pressure gradient supplied by the pump to overcome the viscous shear stress at the wall, i.e. the skin friction drag. The shear force at the walls will not change with x since the flow is fully developed, $\frac{du}{dx}=0$. So, the pressure gradient will be constant, i.e not change with x.

Therefore,

$$\mu\frac{d^2u}{dy^2}=\frac{dP}{dx}=\text{const.}$$

Integrating the above equation with dy,

$$\frac{du}{dy}=\frac{1}{\mu}\frac{dP}{dx}y+c_1$$

$$c_1=0 \text{ (using } \frac{du}{dy} \text{ at y=0 is 0.)}$$

Again integrating,

$$u=\frac{1}{2\mu}\frac{dP}{dx}y^2+c_2$$

At $y=\pm h$, $u=0$, this implies,

$$c_2=\frac{-1}{2\mu}\frac{dP}{dx}h^2$$

After substituting the value of c_2 we get,

$$u=\frac{1}{2\mu}\frac{dP}{dx}y^2-\frac{-1}{2\mu}\frac{dP}{dx}h^2$$

$$u=\frac{1}{2\mu}\frac{dP}{dx}\left(y^2-h^2\right)$$

$$\frac{u}{u_{max}}=\left(1-\frac{y^2}{h^2}\right)$$

Couette Flow : Flow between parallel plates with upper plate moving

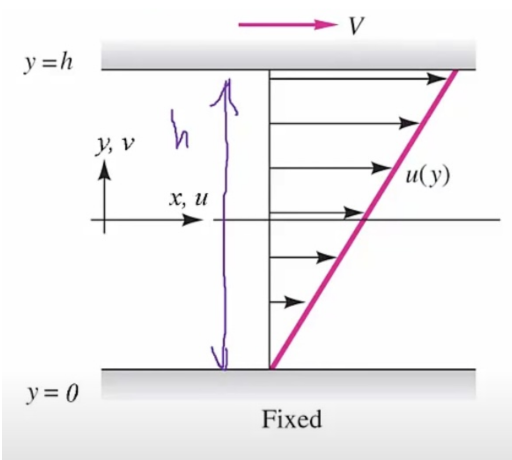

Fig. 20. Couette Flow

Steady Laminar incompressible flow between parallel plates at a distance h apart.

Flow is driven by upper plate moving at velocity V. Therefore, no pressure gradient, i.e $\frac{dP}{dx}=0$.

Consider the plates are very wide and long, this implies,

$$v=w=0$$

Similar to Poiseuille flow, using Navier stokes equation we get,

$$\frac{d^2u}{dy^2} = 0$$

$$\frac{du}{dy} = c_1$$

$$u = c_1 y + c_2$$

At y = 0, u = 0 and y = h, u = V, from this we get the equation as,

$$u = \frac{y}{h} V$$

Pulsatile Flow : Consider the flow in a circular vessel. Assuming the only non-zero component of the velocity vector is in the axial direction and is denoted by u. Using cylindrical polar coordinates, and taking the problem to be axisymmetric (i.e, $u = u(r, x, t)$, $P = P(r, x, t)$), the continuity equation reduces to,

$$\frac{\partial u}{\partial x} = 0$$

The Navier-Stokes equations becomes,

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} + \frac{\mu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right)$$

$$0 = -\frac{\partial P}{\partial r}$$

The above equations implies that,

$$u = u(r, t) \text{ and } p = p(x, t)$$

Using this on the above equation, we get,

$$\frac{\mu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) - \rho \frac{\partial u}{\partial t} = \frac{\partial P}{\partial x}$$

Observing that the left-hand side of the above equation is a function of (r, t) only and the right-hand side a function of (x, t) only, we conclude that both sides must be a function of t only.

Consider now a pulsatile sinusoidal flow with the pressure gradient and the axial velocity as

$$\frac{\partial p}{\partial x} = -P e^{iwt}$$

$$u(r, t) = U(r) e^{iwt}$$

where P is a constant, and $U(r)$ is the velocity profile across the tube of radius a . We assume that the flow is identical at each section along the tube so that a travelling wave solution can be neglected. It is clear that when w = 0 (the steady case), the flow becomes that of Poiseuille flow discussed earlier.

It is clear that the real part gives the velocity for the pressure gradient $Pcoswt$ and the imaginary part gives the velocity for the pressure gradient $Psinwt$. Upon substituting we get,

$$\frac{d^2U}{dr^2} + \frac{1}{r} \frac{dU}{dr} - \frac{iw\rho}{\mu} U = -\frac{P}{\mu}$$

The solution of this equation involving Bessel function of complex argument.

$$U(r) \approx A J_0 \left(i \sqrt{(iw\rho/\mu)} r \right) + B Y_0 \left(i \sqrt{(iw\rho/\mu)} r \right) + \frac{P}{w\rho i}$$

As U must be finite on the axis (ie., at r = 0), and since $Y_0(0)$ is not finite, then B has to be zero. Also, because of the no-slip condition $U(r)|_{r=a}$ = 0, we have

$$A J_0 \left(i \sqrt{(iw\rho/\mu)} a \right) + \frac{P}{w\rho i} = 0$$

Let us introduce a non-dimensional parameter called Womersley parameter(α)

$$\alpha = a \sqrt{\frac{w\rho}{\mu}} \text{ or } \alpha = a \sqrt{\frac{w}{v}}$$

where $v = \mu/\rho$ is the kinematic viscosity.

$$A = -\frac{P}{w\rho i} \frac{1}{J_0(i^{3/2}\alpha)}$$

After substituting this value in the above equation we get,

$$U(r) = -\frac{iP}{w\rho i} \left(1 - \frac{J_0(i^{3/2}\alpha r/a)}{J_0(i^{3/2}\alpha)} \right)$$

Hence, the final result for the velocity of pulsatile flow in a cylindrical tube of radius a is,

$$u(r, t) = -\frac{iP}{w\rho i} \left(1 - \frac{J_0(i^{3/2}\alpha r/a)}{J_0(i^{3/2}\alpha)} \right) e^{iwt}$$

III. Diffusion

In this model, the capillary blood vessel is represented by a straight circular cylindrical tube and the tissue is represented by a stationary concentric cylindrical tube as shown in the figure below.

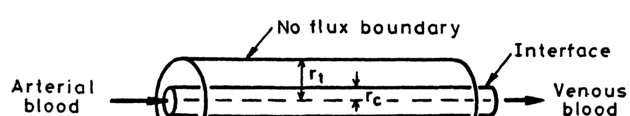

Fig. 21. Krogh tissue cylinder

A large number of identical tubes are closely packed to represent the tissue of an organ.

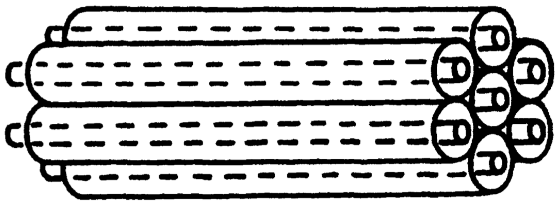

Fig. 22. Closely packed identical tubes

The model is used to study the transport of molecular oxygen from the blood plasma to the skeletal muscle tissues or lungs or brain, across the capillary walls. The model will give an indication of the effect of the velocity of blood flow in capillaries, axial and radial diffusion of oxygen in blood, oxygen diffusivity in tissues, and permeability of capillary walls.
The model hypothesizes that, in a given portion of tissue, all capillaries and surrounding tissues are of equal diameter and are homogeneously dispersed in the tissue. In the capillary vessel, oxygen transport takes place both by convection and by diffusion. In the tissue there is diffusion of oxygen, as well as consumption of oxygen by the tissue cells.
Since diffusion is the main mechanism, we will first introduce the basic equations of the process of diffusion.Let $c(x,t)$ be the concentration of a solute or the amount of solute per unit volume at the point x at time t. Due to the concentration gradient, there is a flow of solute given by the flux vector or the current density vector $\mathbf{J}$. This vector consists of two parts: diffusive $\mathbf{J}_{diff}$ and convective $\mathbf{J}_{conv}$. The former obeys Fick's law, according to which the flux is proportional to the concentration gradient. The latter is proportional to the product of velocity and concentration. Hence, we have

$$\mathbf{J}_{diff} = -D\nabla c$$

$$\mathbf{J}_{conv} = cq$$

where $\mathbf{q}$ (with components u, v, $+--w$) is the flow velocity vector and D is the coefficient of diffusion. Combining these two current densities we have a total flux vector $\mathbf{J} = (\mathbf{J}_x, \mathbf{J}_y, \mathbf{J}_z)$ given by,

$$J_x = -D_x\frac{\partial c}{\partial x}+uc, \quad J_y = -D_y\frac{\partial c}{\partial y}+vc, \quad J_z = -D_z\frac{\partial c}{\partial z}+wc \tag{12}$$

where D_x, D_y, D_z are the coefficient of diffusion in the x, y, z direction.
Let's consider a very small cubical volume of the solution

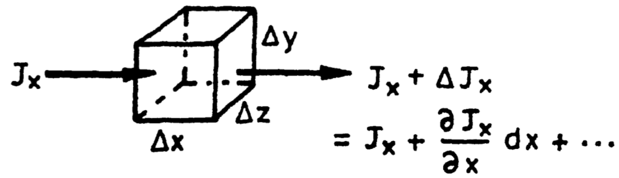

Fig. 23. Conservation of solute transport into and out of a small volume element.

Using conservation principle, which states that, within a given infinitesimal volume element of a solution in which solute currents exist, whatever the cause of the currents may be, the rate at which matter accumulates or disappears within the region is equal to the net flux across the surface bounding the infinitesimal region. This is stated by the following expression:

$$\begin{gathered}\left\{J_x dydz - \left(J_x + \frac{\partial J_x}{\partial x}dx\right)dydx\right\}\\ +\\ \left\{J_y dxdz - \left(J_y + \frac{\partial J_y}{\partial y}dy\right)dxdz\right\}\\ +\\ \left\{J_z dxdy - \left(J_z + \frac{\partial J_z}{\partial z}dz\right)dxdy\right\}\\ = \left(\frac{\partial c}{\partial t}\right)dxdydz\end{gathered}$$

The LHS part of the equation is the net diffusive flux of solute across the surface bounding of the small volume element and the RHS is the increment per unit time of the concentration of the infinitesimal volume element. Here, we assumed that due to chemical reaction in the volume element, the metabolite is consumed at a rate g moles per unit volume per unit time.
Using this the above equation becomes,

$$\frac{\partial c}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0$$

Using equation 12 and the above equation we get,

$$\begin{gathered}\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D_x\frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_y\frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(D_z\frac{\partial c}{\partial z}\right)\\ -c\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - u\frac{\partial c}{\partial x} - v\frac{\partial c}{\partial y} - w\frac{\partial c}{\partial z}\end{gathered}$$

The above equation is the basic equation for diffusion.
We can write the above equation in terms of cylindrical coordinates as

$$\frac{\partial c}{\partial t} + v(r,t)\nabla c = D\nabla^2 c$$

since, ∇v = 0. for in compressible flow.

Suppose the fluid in the tube is in fully developed time-dependent parallel laminar flow described by the axial velocity $u = u(t, r)$.

$$\frac{\partial c}{\partial t} = D\left(\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial z^2}\right), \quad 0 < r < r_c$$

which can be written as,

$$\frac{\partial C}{\partial t} = D\left(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial x^2}\right), \quad 0 < r < r_c$$

Boundary Conditions:
(i)The initial condition

$$C(0, x, r) = C_0\psi_1(x)Y_1(r)$$

The distribution is assumed to be a separable function of x and r.
(ii) Due to the symmetry of the capillary flow about the axis, the radial component of the concentration gradient vanishes, ie.,

$$\frac{\partial C}{\partial r} = 0 \quad r = 0$$
$$C(t, x, 0) = finite$$

(iii)

$$-D\frac{\partial C}{\partial r}(t, x, R) = K_s C(t, x, R)$$

(iv) Since the amount of solute in the system is finite,

$$C(t, \infty, r) = \frac{\partial C}{\partial x}(t, \infty, r) = 0$$

Using some dimensionless quantities,

$$\tau = \frac{Dt}{R^2}, \quad X = \frac{Dx}{R^2 u_0}, \quad y = \frac{r}{R}, \quad U(\tau, y) = \frac{u(t, r)}{u_0}$$
$$Pe = \frac{Ru_0}{D}, \quad \theta = \frac{C}{C_0} \quad \text{and} \quad \beta = \frac{k_s R}{D}$$

where, u_0 is a reference velocity and C_0 is a reference concentration, the diffusion equation and the boundary condition becomes,

$$\frac{\partial \theta}{\partial \tau} + U(\tau, y)\frac{\partial \theta}{\partial X} = \frac{1}{y}\frac{\partial}{\partial y} y\frac{\partial \theta}{\partial y} + \frac{1}{(Pe)^2}\frac{\partial^2 \theta}{\partial X^2} \tag{13}$$

$$\theta(\tau, X, 0) = \text{ finite}$$
$$\theta(0, X, y) = \psi(X)Y(y),$$
$$\frac{\partial \theta}{\partial y}(\tau, X, 1) = -\beta\theta(\tau, X, 1),$$
$$\theta(\tau, \infty, y) = \frac{\partial \theta}{\partial X}(\tau, \infty, y) = 0$$

The solution for the above equation is,

$$\theta = \sum_{k=0}^{\infty} f_k(\tau, y)\frac{\partial^k \theta_m}{\partial X^k}$$

where the dimensionless area-average concentration θ_m is defined as

$$\theta_m = 2\int_0^1 \theta y \ \mathrm{d}y$$

Multiplying equation (13) by $2y$ and integrated with respect to y from y = 0 to y = 1, and then using the value of θ_m,

$$\frac{\partial \theta_\mathrm{m}}{\partial \tau} = \frac{1}{(Pe)^2}\frac{\partial^2 \theta_\mathrm{m}}{\partial X^2} + 2\frac{\partial \theta}{\partial y}(\tau, X, 1) - 2\frac{\partial}{\partial X}\int_0^1 U(\tau, y)\theta y \ \mathrm{d}y$$

Substitute the value,

$$\theta = \sum_{k=0}^{\infty} f_k(\tau, y)\frac{\partial^k \theta_\mathrm{m}}{\partial X^k}$$

in the above equation, we get

$$\frac{\partial \theta_\mathrm{m}}{\partial \tau} = \sum_{i=0}^{\infty} K_i(\tau)\frac{\partial^i \theta_\mathrm{m}}{\partial X^i}$$

where,

$$K_i(\tau) = \frac{\delta_{i2}}{(Pe)^2} + 2\frac{\partial f_i}{\partial y}(\tau, 1) - 2\int_0^1 f_{i-1}(\tau, y)U(\tau, y)y \ \mathrm{d}y \tag{14}$$

$f_{-1} \equiv 0$ and δ_{i2} is the Kronecker delta defined by

$$\left.\begin{aligned} \delta_{ij} &= 1, \quad i = j \\ &= 0, \quad i \neq j \end{aligned}\right\}$$

Now substitute the value,

$$\theta = \sum_{k=0}^{\infty} f_k(\tau, y)\frac{\partial^k \theta_\mathrm{m}}{\partial X^k}$$

in equation (13) we get,

$$\sum_{i=0}^{\infty}\frac{\partial f_i(\tau, y}{\partial \tau}\frac{\partial^i \theta_m}{\partial X^i} + \sum_{i=0}^{\infty} f_i(\tau, y)\sum_{k=0}^{\infty} K_k\frac{\partial^{k+i}\theta_m}{\partial X^{k+i}} +$$
$$\sum_{i=0}^{\infty} U(\tau, y)f_{i-1}(\tau, y)\frac{\partial^i \theta_m}{\partial X^i} = \sum_{i=0}^{\infty}\frac{1}{y}\frac{\partial}{\partial y} y\frac{\partial f_i(\tau, y)}{\partial y}\frac{\partial^i \theta_m}{\partial X^i} +$$
$$\sum_{i=0}^{\infty}\frac{1}{Pe^2} f_{i-2}(\tau, y)\frac{\partial^i \theta_m}{\partial X^i}$$

Equating the coefficients of $\frac{\partial^i \theta_m}{\partial X^i}$, we get

$$\frac{\partial f_i}{\partial \tau} = \frac{1}{y}\frac{\partial}{\partial y} y\frac{\partial f_i}{\partial y} - U(\tau, y)f_{i-1} + \frac{1}{(Pe)^2} f_{i-2} - \sum_{k=0}^{i} K_k f_{i-k} \tag{15}$$

$f_{-1} \equiv 0$ and $f_{-2} \equiv 0$.
The initial and boundary conditions on θ_m and f_k can be obtained from the initial conditions equations

$$\theta_\mathrm{m}(0, X) = 2\int_0^1 \theta(0, X, y)y \ \mathrm{d}y = 2\psi(X)\int_0^1 Y(y)y \ \mathrm{d}y$$

Let's now assume,

$$f_k(0, y) = 0$$

$$\theta = f_0\theta_m$$
$$\theta(0, X, y) = f_0(0, y)\theta_m(0, X)$$

this gives

$$f_0(0,y) = \frac{\theta(0,X,y)}{\theta_m(0,X)} = \frac{Y(y)}{2\int_0^1 Y(y)y \ \mathrm{d}y} \tag{16}$$

Boundary conditions:

1)

$$\frac{\partial f_k}{\partial y} \text{ at y=0}$$
$$f_k(\tau,y) = \text{finite}$$

2)

$$\frac{\partial f_k}{\partial y}(\tau,y) = -\beta f_k(\tau,y)$$

3)

$$\theta_m(\tau,\infty) = \frac{\partial \theta_m}{\partial X}(\tau,\infty) = 0$$

We know that $\theta_m = 2\int_0^1 \theta y \ \mathrm{d}y$

$$\theta_m = 2\int_0^1 \sum_{k=0}^{\infty} f_k \frac{\partial^k \theta_m}{\partial X^k} y \ \mathrm{d}y$$

$$\theta_m = 2\sum_{k=0}^{\infty} \int_0^1 f_k \frac{\partial^k \theta_m}{\partial X^k} y \ \mathrm{d}y$$

$$\theta_m = 2\int_0^1 f_0 \ y \ \theta_m \ \mathrm{d}y + 2\sum_{k=1}^{\infty} \int_0^1 f_k \frac{\partial^k \theta_m}{\partial X^k} y \ \mathrm{d}y$$

now comparing the LHS and RHS, we get,

$$\frac{\delta_{k0}}{2} = \int_0^1 f_0 \ y \ \mathrm{d}y$$

$$\int_0^1 f_k \frac{\partial^k \theta_m}{\partial X^k} y \ \mathrm{d}y = 0$$

where δ_{k0} is the Kronecker delta function.
The function f_0 and the exchange coefficient K_0 are independent of the velocity field and can be solved for immediately. It should be pointed out here that a simultaneous solution c=has to be obtained for these two quantities since K_0, is

$$K_0(\tau) = 2\frac{\partial f_0}{\partial y}(\tau,1)$$

$$\frac{\partial f_0}{\partial \tau} = \frac{1}{y}\frac{\partial}{\partial y} y \frac{\partial f_0}{\partial y} - f_0 K_0$$

Using the boundary conditions and equation (16) we get the solution to be

$$f_0(\tau,y) = \exp\left\{-\int_0^{\tau} K_0(\eta)\mathrm{d}\eta\right\} g_0(\tau,y) \tag{17}$$

where,

$$\frac{\partial g_0}{\partial \tau} = \frac{1}{y}\frac{\partial}{\partial y} y \frac{\partial g_0}{\partial y}$$

along with,

$$g_0(0,y) = f_0(0,y) = Y(y)/2\int_0^1 yY(y)\mathrm{d}y,$$
$$\frac{\partial g_0}{\partial y}(\tau,1) = -\beta g_0(\tau,1),$$
$$g_0(\tau,0) = \text{ finite.}$$

The Solution for these equations is,

$$g_0(\tau,y) = \sum_{n=0}^{\infty} A_n J_0(\mu_n y) \exp\left\{-\mu_n^2 \tau\right\} \tag{18}$$

Here if we observe carefully we could have done this earlier with f_0. But $\frac{\partial f_0}{\partial t}$ has another term which is $-f_0 k_0$. Hence we reduced to this form.
Here $\mu_n J_1(\mu_n) = \beta J_0(\mu_n)$ $\forall$ n $\in$ W.
Where J_1 and J_0 are Bessel's functions.
We know that

$$\frac{1}{2} = \int_0^1 f_0 \ y \ \mathrm{d}y$$

Substituting the value of f_0 from equation (17), we get

$$\frac{1}{2} = \int_0^1 \exp\left\{-\int_0^{\tau} K_0(\eta)\mathrm{d}\eta\right\} g_0(\tau,y) \ y \ \mathrm{d}y$$

$$\exp\left\{-\int_0^{\tau} K_0(\eta)\mathrm{d}\eta\right\} = \frac{1}{2\int_0^1 g_0(\tau,y) \ y \ \mathrm{d}y}$$

$$f_0(\tau,y) = \exp\left\{-\int_0^{\tau} K_0(\eta)\mathrm{d}\eta\right\} g_0(\tau,y) = \frac{g_0(\tau,y)}{2\int_0^1 g_0(\tau,y) \ y \ \mathrm{d}y}$$

Now substituting the value of $g_0(\tau,y)$ in the above equation from equation(17) and simplifying we get,

$$f_0(\tau,y) = \frac{\sum_{n=0}^{\infty} A_n J_0(\mu_n y) \exp\left\{-\mu_n^2 \tau\right\}}{2\sum_{n=0}^{\infty} \left(\frac{A_n}{\mu_n}\right) J_1(\mu_n) \exp\left\{-\mu_n^2 \tau\right\}}$$

We know that $K_0(\tau) = 2\frac{\partial f_0}{\partial y}(\tau,1)$ from this we get,

$$K_0(\tau) = -\frac{\sum_{n=0}^{\infty} A_n \mu_n J_1(\mu_n) \exp\left\{-\mu_n^2 \tau\right\}}{\sum_{n=0}^{\infty} \left(\frac{A_n}{\mu_n}\right) J_1(\mu_n) \exp\left\{-\mu_n^2 \tau\right\}}$$

As $\tau \to \infty$ we get the asymptotic representations of f_0 and K_0,

$$f_0(\infty,y) = \frac{\mu_0}{2J_1(\mu_0)} J_0(\mu_0 y)$$

$$K_0(\infty) = -\mu_0^2$$

For large value of τ, for steady flow,

$$u(t,r) = u(r) = u_0(1 - \frac{r^2}{R^2})$$

and the reference velocity u_0 is therefore the maximum velocity of flow which occurs at the centre line. In dimensionless form, the above equation becomes,

$$U(\tau,y) = U(y) = 1 - y^2$$

Using this value in equation (15) we get,

$$\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}y\frac{\mathrm{d}f_k}{\mathrm{d}y}+\mu_0^2 f_k=\left(1-y^2\right)f_{k-1}-\frac{1}{(Pe)^2}f_{k-2}+\sum_{i=1}^{k}K_i f_{k-i} \tag{19}$$

The boundary conditions,

1)
$$\int_0^1 f_k y\ \mathrm{d}y=0$$

2)
$$f_k(0)=\text{finite}$$

3)
$$\frac{\partial f_k}{\partial y}(1)=-\beta f_k(1)$$

From equation (14) and after using the value of $U(\tau,y)=1-y^2$ we get,

$$K_k=\frac{\delta_{k2}}{(Pe)^2}+2f_k'(1)-2\int_0^1 y\left(1-y^2\right)f_{k-1}(y)\mathrm{d}y$$

For solving K_k we need to know $f_k'(1)$. For solving $f_k'(1)$ we need to know K_k. A contradiction.
Consider a non-homogeneous differential equation

$$\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}y\frac{\mathrm{d}\phi}{\mathrm{d}y}+\mu^2\phi=0$$

$$\phi(0)=\text{finite},\quad \phi'(1)=-\beta\phi(1)$$

Clearly μ_0 is an eigen value and $\phi_0=J_0(\mu_0,y)$ is the corresponding eigen function. From equation (15),

$$K_i f_0=\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}y\frac{\mathrm{d}f_i}{\mathrm{d}y}+\mu_0^2 f_i-\left(1-y^2\right)f_{i-1}+\frac{1}{(Pe)^2}f_{i-2}-\sum_{k=1}^{i-1}K_k f_{i-k}$$

Multiply $yJ_0(\mu_0 y)$ on both side and integrate from 0 to 1, we get

$$K_k=\frac{1}{\int_0^1 yf_0(y)J_0\left(\mu_0 y\right)\mathrm{d}y}\times$$
$$\left[\int_0^1 yJ_0\left(\mu_0 y\right)\left\{\frac{1}{(Pe)^2}f_{k-2}(y)-\left(1-y^2\right)f_{k-1}(y)-\sum_{i=1}^{k-1}K_i f_{k-i}(y)\right\}\mathrm{d}y\right]$$

Now we have the values of K_i's, with this we can solve equation (19) and using Hildebrand method we get,

$$f_k=\sum_{j=0}^{\infty}B_{j,k}J_0(\mu_i y)$$

where,

$$\begin{aligned}B_{j,k}=&\frac{1}{\mu_j^2-\mu_0^2}\left[\frac{1}{(Pe)^2}B_{j,k-2}-B_{j,k-1}-\sum_{i=1}^{k}K_i B_{j,k-i}\right.\\&\left.+\frac{2\mu_j^2}{\left(\beta^2+\mu_j^2\right)J_0^2\left(\mu_j\right)}\sum_{l=0}^{\infty}B_{l,k-1}I(j,l)\right]\end{aligned} \tag{20}$$

$B_{j,-1}\equiv 0$ and $B_{j,0}\equiv 0$.
The function $I_{j,l}$ is an integral,

$$\begin{aligned}I(j,l)=I(l,j)&=\int_0^1 y^3 J_0\left(\mu_j y\right)J_0\left(\mu_l y\right)\mathrm{d}y\\&=\frac{2\left(\beta^2+\mu_j^2+\mu_l^2\right)}{\left(\mu_j^2-\mu_l^2\right)^2}J_0\left(\mu_j\right)J_0\left(\mu_l\right)\quad (j\neq l),\\&=\frac{\left(\mu_j^2+\beta\right)^2+\beta^2\left(\mu_j^2-3\right)}{6\mu_j^4}J_0^2\left(\mu_j\right)\quad (j=l).\end{aligned}$$

We plot the graph using the above equations for K_0.

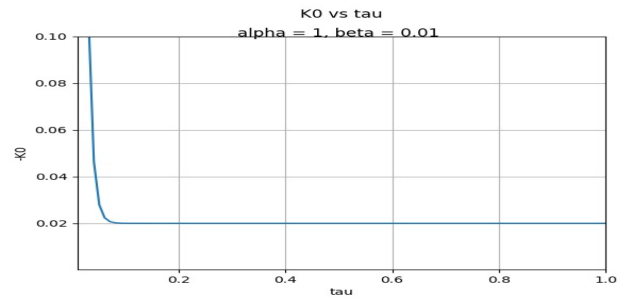

Fig. 24. K_0 vs τ

Please find the code for the above graph here