ECE/CS 559 - Fall 2017 - Midterm Part #1.

Full Name: **ID Number:**

Q1 (25 pts). Let $f(x) = x^2$.

(a) **(3 pts)** Let $x^\star$ be the global minimizer of f, i.e. $x^\star = \min_{x\in\mathbb{R}} f(x)$, where $\mathbb{R}$ is the set of real numbers. Find $x^\star$.

(b) **(5 pts)** Recall that the gradient descent equations are given by $x_{n+1} = x_n - \eta f'(x_n)$, $n \in \{0, 1, 2, \ldots\}$, where $\eta > 0$ is the learning parameter, and $f'(x)$ is the derivative of $f(x)$ with respect to x. Find x_n for any $n \geq 1$ given $x_0 = 2$ and $\eta = \frac{1}{4}$.

(c) **(11 pts)** Describe the $n \to \infty$ asymptotic behavior of x_n and $f(x_n)$ for every possible initial condition $x_0 \in \mathbb{R}$ and learning parameter $\eta > 0$. For example, your answer should be able to describe where x_n and $f(x_n)$ go as $n \to \infty$ given initial conditions $x_0 = -2444$ and $\eta = 120$ (or any other x_0 and η that will be given to you).

(d) **(6 pts)** Let $k(y,z) = y^2 + z^4$. Let $g(y,z) \triangleq \begin{bmatrix} \frac{\partial k}{\partial y} \\ \frac{\partial k}{\partial z} \end{bmatrix}$ and $H(y,z) \triangleq \begin{bmatrix} \frac{\partial^2 k}{\partial y^2} & \frac{\partial^2 k}{\partial y \partial z} \\ \frac{\partial^2 k}{\partial z \partial y} & \frac{\partial^2 k}{\partial z^2} \end{bmatrix}$ denote the gradient and the Hessian of k, respectively. Recall that the update equations for Newton's method are given by $\left[\begin{smallmatrix} y_{n+1} \\ z_{n+1} \end{smallmatrix}\right] = \left[\begin{smallmatrix} y_n \\ z_n \end{smallmatrix}\right] - \eta \left(H(y_n, z_n)\right)^{-1} g(y_n, z_n)$, $n \in \{0, 1, 2, \ldots\}$. Given $y_0 = z_0 = \eta = 1$, calculate y_n, z_n for every $n \geq 1$.

Q2 (25 pts). Consider a neuron with $n \geq 1$ inputs $x_1, \ldots, x_n$, and output $y = \theta(w_0 + w_1 x_1 + \cdots + w_n x_n)$, where $w_0, w_1, \ldots, w_n$ are the neuron bias and weights, and the activation function is given by $\theta(x) = 1$ if $x \in [0,1]$, and $\theta(x) = 0$ if $x \notin [0,1]$. Note that the activation function is different than the functions that we have encountered throughout the lectures.

(a) **(9 pts):** Let $n = 1$. Does there exist w_0, w_1 such that $y = 1 - x_1$ for $x_1 \in \{0, 1\}$? In other words, can a single neuron with activation function θ implement the `NOT` gate? If your answer is "Yes," find specific w_0, w_1 such that the neuron implements the `NOT` gate. If your answer is "No," prove that no choice for w_0, w_1 can result in a neuron that implements the `NOT` gate.

(b) **(8 pts):** Let $n = 2$. Does there exist w_0, w_1, w_2 such that $y = x_1 x_2$ for $x_1, x_2 \in \{0, 1\}$? In other words, can a single neuron with activation function θ implement the `AND` gate? Justify your answer as in (a).

(c) **(8 pts):** Let $n = 2$. Does there exist w_0, w_1, w_2 such that $y = ((x_1 + x_2) \mod 2)$ for $x_1, x_2 \in \{0, 1\}$? In other words, can a single neuron with activation function θ implement the `XOR` gate? Justify your answer as in (a).