Homework2 (CS-594)

Question1) The most interesting topic that I came across during this course is Expanders in Graph representations. This topic is something I never knew or heard of in the field of graph theory and it made me curious to know the actual applications of Expanders. It is an interesting combination of already known topics like Random walks and geometric group theory. Though I do not completely comprehend Expanders, this course has given me enough introduction and curiosity to explore this topic further.

Question2) An interesting topic that could be introduced in the lectures are Cayley graphs. They have a lot of different applications and many other important classes of graphs like circulant graphs, Hamming graphs and Hypercube graphs are all Cayley graphs, which makes it even more important. Cayley graphs could also cover some important concepts from Group theory alternating group and dihedral group. There was a brief presentation on Cayley graphs during the research presentations in class, but it could have been comprehended better if it were introduced into the course curriculum.

Question 3) Matrix Multiplication Tensor:

For fixed n, the matrix multiplication tensor $T \in \mathbb{C}^{nm \times mp \times pn}$ defined by

$$T_{ij', jk', ki'} = 1_{\{i=i', j=j', k=k'\}}$$

for $i, i', j, j', k, k' \in \{1, \dots, n\}$

and tensor is denoted as $\langle n, m, p \rangle$.

Rank 1 Tensor:

A tensor $T \in \mathbb{C}^{nm \times mp \times np}$ is a rank 1 tensor if it can be expressed as $T = u \otimes v \otimes w$ where $u \in \mathbb{C}^{nm}$, $v \in \mathbb{C}^{mp}$, $w \in \mathbb{C}^{np}$

i.e. $T_{ijk} = u_i v_j w_k$

The matrix multiplication tensor $\langle 2,2,2 \rangle$ can be written as sum of the following Rank 1 tensors:

$T_1 = (1,0,0,1) \otimes (1,0,0,1) \otimes (1,0,0,1)$

$T_2 = (0,0,1,1) \otimes (1,0,0,0) \otimes (0,0,1,-1)$

$T_3 = (1,0,0,0) \otimes (0,1,0,-1) \otimes (0,1,0,1)$

$T_4 = (0,0,0,1) \otimes (-1,0,1,0) \otimes (1,0,1,0)$

$T_5 = (1,1,0,0) \otimes (0,0,0,1) \otimes (-1,1,0,0)$

$T_6 = (-1,0,1,0) \otimes (1,1,0,0) \otimes (0,0,0,1)$

$T_7 = (0,1,0,-1) \otimes (0,0,1,1) \otimes (1,0,0,0)$

$\langle 2,2,2 \rangle$ can be written as linear combination of above 7 tensors. Therefore, the $R(\langle 2,2,2 \rangle) \leq 7$ since we know rank of a tensor is the number of Rank 1 tensors required in linear combination to form the tensor.

Question 4) $R(\langle nn', mm', pp' \rangle) = R(\langle n,m,p \rangle \otimes \langle n', m', p' \rangle)$
$\leq R(\langle n,m,p \rangle) R(\langle n', m', p' \rangle)$.

Deconstructing $\langle nn', mm', pp' \rangle$ into a tensor of tensors.

Let $\langle n,m,p \rangle = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$, $\langle n', m', p' \rangle = \sum_{j=1}^{r'} u_j' \otimes v_j' \otimes w_j'$

Then,

$$\langle nn', mm', pp' \rangle = \sum_{i,j=1}^{r,r'} (u_i \otimes u_j') \otimes (v_i \otimes v_j') \otimes (w_i \otimes w_j')$$

General Bound can be written as

$R(\langle nmp, nmp, nmp \rangle) \leq R(\langle n,m,p \rangle) \cdot R(\langle m,p,n \rangle) R(\langle p,n,m \rangle)$

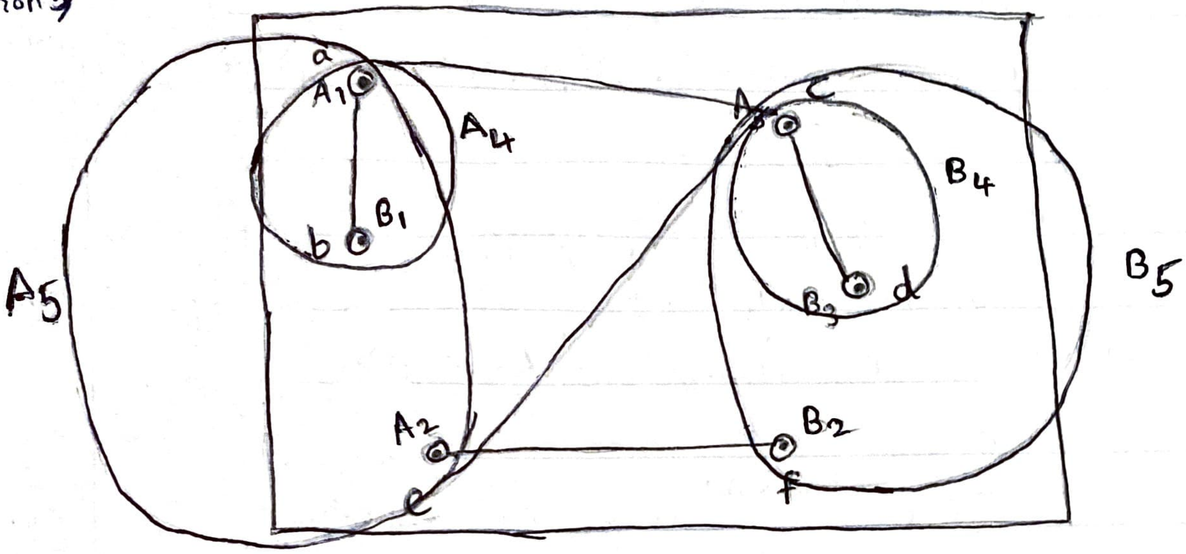
since $R(\langle n,m,p \rangle) = R(\langle mpn \rangle) = R(\langle p,n,m \rangle)$

$R(\langle nmp, nmp, nmp \rangle) \leq R(\langle n,m,p \rangle)^3$

$$\omega \leq \log_{nmp} R(\langle n,m,p \rangle)^3 = 3\log_{nmp} R(\langle n,m,p \rangle)$$

$$\Rightarrow \quad \omega \leq \frac{3\log R(\langle n,m,p \rangle)}{\log(nmp)}$$

Question 5)



For a well-separated pair $\{A, B\}$ the distance between all point pairs in $A \otimes B := \{\{a, b\} \mid a \in A, b \in B, a \neq b\}$ is similar

→ For a set of points P and $s > 0$ an s-well separated pair decomposition (s-WSPD) is a set of pairs $\{\{A_1, B_1\}, \ldots, \{A_m, B_m\}\}$ with

- $A_i, B_i \subset P$ for all i
- $A_i \cap B_i = \emptyset$ for all i
- $\bigcup_{i=1}^{m} A_i \otimes B_i = P \otimes P$
- $\{A_i, B_i\}$ s-well separated for all i (~~radius~~ distance $\geq s \times$ radius)