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Dynamical Systems and Discrete Markov Chains

Undergraduate Seminar

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Abstract

A dynamical system is a system whose state is uniquely specified by a set of variables and whose behavior is described by predefined rules.I will start with dynamical systems,stochastic processes and all the way go to Discrete State Markov Chains.The paper is structured in such a way so that you can get the whole idea and relate the idea of dynamical systems to Markov Processes.

1 Introduction

Dynamical systems theory is the very foundation of almost any kind of complex systems with a set of predefined protocols. A Markov chain is a stochastic model describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event.Predicting traffic flows, communications networks, genetic issues, and queues are examples where Markov chains can be used to model performance.The sketch of the paper is as follows:

- Dynamical Systems: This section is dedicated to the understanding of dynamical systems,Phase space and different types of dynamical systems.
- Markov chains: This section is dedicated to understanding of Markov chains,IIDs,different examples of Markov chains and existence of Markov chains.
- Discrete State Markov Process: This section is dedicated to understanding of how we connect the concept of dynamical systems and Markov chain.It mainly discusses continuous time discrete state Markov chains and an example of the same.

2 Dynamical Systems

Dynamical systems theory describes general patterns found in the solutions of systems of nonlinear differential equations representing the change of process in time.Geometric and analytic study of simple examples have led to tremendous insights into universal aspects of nonlinear dynamics.Diverse areas like fluid flows,chemical reactions,laser dynamics,cardiac rhythms have confirmed the ubiquity of these dynamical patterns.

Definition 1 : (Dynamical system) Given a phase space $\mathcal{B} \subset \mathbb{R}^{N_z}$ embedded into an N_z -dimensional Euclidean space, a dynamical system is a map $\psi : \mathcal{B} \to \mathcal{B}$. Given an initial condition $z^0 \in \mathcal{B}$, a dynamical system defines a solution sequence or trajectory $\{z^n\}_{n\geq 0}$ via the iteration

$$z^{n+1} = \psi\left(z^n\right) \tag{1}$$

Definition 2 : (Phase Space) In dynamical system theory, a phase space is a space in which all possible states of a system are represented,where each state is a unique point in the phase space.The concept of phase space was developed in the 19th century by Ludwig Boltzmann,Henry Poincaré and Josiah Willard Gibbs.

Example 1 : (Phase Space) For mechanical systems, the phase space is the space which consists of all possible values of position and momentum variables.

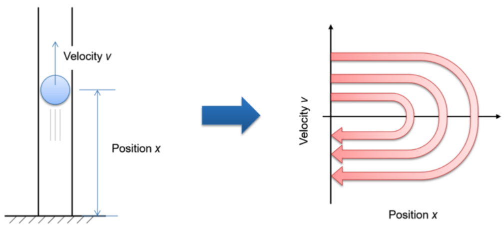

Figure 1: A ball is thrown upward in a vertical tube (left) and a illustration of its phase space (right). The dynamic behavior of the ball can be visualize as a static trajectory in the phase space (red arrows).

Definition 3 : (Autonomous and Non-Autonomous Dynamical System) If the map ψ(from equation 1) depends on the iteration index n, then the system is called a non autonomous dynamical system.Otherwise it is called a autonomous dynamical system.

Dynamical systems can be described over either discrete time steps or continuous time line.The general mathematical formulations are as follows :

Definition 4 : (Discrete-time dynamical system)

$$x_t = F\left(x_{t-1}, t\right)$$

This type of model is called a difference equation, a recurrence equation, or an iterative *map*.

Definition 5 : (Continuous-time dynamical system)

$$\frac{dx}{dt} = F(x, t)$$

This type of model is called a differential equation.

3 Markov Chains

Definition 6 : (Markov Chain) Let $\mathbb{S} = \{a_j : j = 1, 2, \ldots, K\}, K \leq \infty$ be a finite or countable set. Let $\boldsymbol{P} = ((p_{ij}))_{K\times K}$ be a stochastic matrix, i.e., $p_{ij} \geq 0$, for every i, $\sum_{j=1}^{K} p_{ij} = 1$ and $\mu = \{\mu_j : 1 \leq j \leq K\}$ be a probability distribution, i.e. $\mu_j \geq 0$ for all j and $\sum_{j=1}^{K} \mu_j = 1$

A sequence $\{X_n\}_{n=0}^{\infty}$ of $\mathbb{S}$-valued random variables on some probability space $(\Omega, \mathcal{F}, P)$ is called a Markov chain with stationary transition probabilities $\boldsymbol{P} = ((p_{ij}))$, initial distribution μ, and state space $\mathbb{S}$ if

(i) $X_0 \sim \mu$, i.e., $P(X_0 = a_j) = \mu_j$ for all j, and

(ii) $P\left(X_{n+1} = a_j \mid X_n = a_i, X_{n-1} = a_{i_{n-1}}, \ldots, X_0 = a_{i_n}\right) = p_{ij}$ for all $a_i, a_j, a_{i_{n-1}}, \ldots, a_{i_0} \in \mathbb{S}$ and $n = 0, 1, 2, \ldots$

i.e., the sequence is memoryless. Given X_n, X_{n+1} is independent of $\{X_j : j \leq n-1\}$. More generally, given the present (X_n), the past $(\{X_j : j \leq\ n-1\}$) and the future $(\{X_j : j > n\})$ are stochastically independent.

Definition 7 : (IID or Independent and Identically Distributed random variables) A collection of random variables where each random variable has the same probability distribution as the others and all are mutually independent are called Independent and Identically Distributed random variables.This property is usually abbreviated as i.i.d. or **iid** or IID.

Markov Chain Examples

Example 2 : (IID sequence) Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of iid $\mathbb{S}$-valued random variables with distribution $\mu = \{\mu_j\}$. Then $\{X_n\}_{n=0}^{\infty}$ is a Markov chain with initial distribution μ and transition probabilities given by $p_{ij} = \mu_j$ for all i, i.e., all rows of P are identical.The converse is also true.The converse can be proven as follows i.e., if all rows of P are identical, then $\{X_n\}_{n=1}^{\infty}$ are iid and independent of X_0.

Example 3 : (Birth and death chains) Let $\mathbb{S} = \mathbb{Z}_+$. Define P by

$$\begin{aligned} p_{i,i+1} = \alpha_i, \quad p_{i,i-1} = \beta_i = 1 - \alpha_i, \text{ for } i \geq 1 \\ p_{0,1} = \alpha_{0,} \qquad p_{0,0} = \beta_0 = 1 - \alpha_0 \end{aligned}$$

The population increases at rate α_i and decreases at rate $1 - \alpha_i$.

Example 4 : (Iterated function systems) Let $G := \{h_i : h_i : \mathbb{S} \to \mathbb{S}, i = 1, 2, \ldots, L\}, L \leq \infty$. Let $\mu = \{p_i\}_{i=1}^{L}$ be a probability distribution. Let $\{f_n\}_{n=1}^{\infty}$ be iid, such that $P(f_n = h_i) = p_i, 1 \leq i \leq L$. Let X_0 be a $\mathbb{S}$-valued random variable independent of $\{f_n\}_{n=1}^{\infty}$. Then, the iid random iteration scheme

$$\begin{aligned} X_1 = f_1(X_0) \\ X_2 = f_2(X_1) \end{aligned}$$

is a Markov chain with transition probability matrix

$$p_{ij} = P(f_1(i) = j) = \sum_{r=1}^{L} p_r I(h_r(i) = j)$$

Remark 1 : Any discrete state space Markov chain can be generated in this way. (see section 3.2)

3.1 Existence of Markov Chain(Kolmogorov's approach)

Definition 8 : (Stochastic process) A stochastic process with index set A is a family $\{X_\alpha : \alpha \in A\}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$.

Stochastic Processes Examples

Example 5 : Let $\Omega = [0,1], \mathcal{F} = \mathcal{B}([0,1]), P =$ the Lebesgue measure on $[0,1]$. Let $A_1 = \{1,2,3,\ldots\}, A_2 = [0,T], 0 < T < \infty$. For $\omega \in \Omega, n \in A_1, t \in A_2,$ let

$X_n(\omega) = \sin 2\pi n\omega$

$Y_t(\omega) = \sin 2\pi t\omega$

$V_{n,t}(\omega) = X_n^2(\omega) + Y_t^2(\omega)$

Then $\{X_n : n \in A_1\}, \{V_{n,t} : (n,t) \in A_1 \times A_2\}, \{Y_t : t \in A_2\}$ are all stochastic processes.

Definition 9 : (Family of Finite Dimensional Distributions) The family $\left\{\mu_{(\alpha_1,\alpha_2,\ldots,\alpha_k)}(\cdot) \equiv P\left((X_{\alpha_1},\ldots,X_{\alpha_k}) \in\right) : (\alpha_1,\alpha_2,\ldots,\alpha_k) \in A^k, 1 \le k < \infty\right\}$ of probability distributions is called the family of finite dimensional distributions (fdds) associated with the stochastic process $\{X_\alpha : \alpha \in A\}$.

This family of finite dimensional distributions satisfies the following consistency conditions: For any $(\alpha_1,\alpha_2,\ldots,\alpha_k) \in A^k, 2 \le k < \infty$, and any $B_1, B_2, \ldots, B_k$ in $\mathcal{B}(\mathbb{R})$

C1: $\mu_{(\alpha_1,\alpha_2\ldots\ldots\alpha_k)}\left(B_1 \times \cdots \times B_{k-1} \times \mathbb{P}\right) = \mu_{(\alpha_1,\alpha_2\ldots\ldots\alpha_{k-1})}\left(B_1 \times \cdots \times B_{k-1}\right)$

C2: For any permutation $(i_1, i_2, \ldots, i_k)$ of $(1,2,\ldots,k)$,

$\mu_{\left(\alpha_i,\alpha_{1_2},\ldots,\alpha_{i_k}\right)}\left(B_{i_1} \times B_{i_2} \times \cdots \times B_{i_k}\right) = \mu_{(\alpha_1,\ldots,\alpha_k)}\left(B_1 \times B_2 \times \cdots \times B_k\right)$

Kolmogorov's Consistency Theorem

Let A be a nonempty set. Let $Q_A \equiv \left\{\nu_{(\alpha_1,\alpha_2,\ldots,\alpha_k)} : (\alpha_1,\alpha_2,\ldots,\alpha_k) \in A^k, 1 \le k < \infty\right\}$ be a family of probability distributions such that for each $(\alpha_1,\alpha_2,\ldots,\alpha_k) \in A^k, 1 \le k < \infty,$

(i) $\nu_{(\alpha_1,\alpha_2,\ldots,\alpha_k)}$ is a probability distribution on $\left(\mathbb{R}^k, \mathcal{B}\left(\mathbb{R}^k\right)\right)$

(ii) C1 and C2 hold, i.e., for all $B_1, B_2, \ldots, B_k \in \mathcal{B}(\mathrm{R}), 2 \le k < \infty$

$$\begin{aligned}&\nu_{(\alpha_1,\alpha_2,\ldots,\alpha_k)}\left(B_1 \times B_2 \times \cdots \times B_{k-1} \times \mathbb{R}\right)\\&= \nu_{(\alpha_1,\alpha_2,\ldots,\alpha_{k-1})}\left(B_1 \times B_2 \times \cdots \times B_{k-1}\right)\end{aligned}$$

and for any permutation $(i_1, i_2, \ldots, i_k)$ of $(1, 2, \ldots, k)$,

$$\mu_{\left(\alpha_{i_1}, \alpha_{12}, \ldots, \alpha_{i_k}\right)}\left(B_{i_1} \times B_{i_2} \times \cdots \times B_{i_k}\right)$$
$$= \mu_{(\alpha_1, \alpha_2, \ldots, \alpha_k)}\left(B_1 \times B_2 \times \cdots \times B_k\right)$$

Then, there exists a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $X_A \equiv \{X_\alpha : \alpha \in A\}$ on $(\Omega, \mathcal{F}, P)$ such that Q_A is the family of finite dimensional distributions associated with X_A.

Kolmogorov's approach

Let $\Omega = \mathrm{S}^{\mathbb{Z}_+} = \{\omega : \omega = \{x_n\}_{n=0}^{\infty}, x_n \in \mathbb{S} \text{ for all } n\}$ be the set of all sequences $\{x_n\}_{n=0}^{\infty}$ with values in $\mathbb{S}$. Let $\mathcal{F}_0$ consist of all finite dimensional subsets of Ω of the form

$$A = \{\omega : \omega = \{x_n\}_{n=0}^{\infty}, x_j = a_j, 0 \leq j \leq m\}$$

where $m < \infty$ and $a_j \in \mathbb{S}$ for all $j = 0, 1, \ldots, m$. Let $\mathcal{F}$ be the σ -algebra generated by $\mathcal{F}_0$. Fixing μ and $\boldsymbol{P}$. For A as shown above.

$$\lambda_{\mu, \boldsymbol{P}}(A) := \mu_{a_0} p_{a_0 a_1} p_{a_1 a_2} \cdots p_{a_{m-1} a_m}$$

Then it can be shown, using Kolmogorov's consistency theorem that $\lambda_{\mu, \boldsymbol{P}}$ can be extended to be a probability measure on $\mathcal{F}$. Let $X_n(\omega) = x_n$, if $\omega = \{x_n\}_{n=0}^{\infty}$, be the coordinate projection. Then $\{X_n\}_{n=0}^{\infty}$ will be a sequence of $\mathbb{S}$ -valued random variables on $(\Omega, \mathcal{F}, \lambda_{\mu, \boldsymbol{P}})$, such that it is a **Markov chain** with initial distribution μ and transition probability $\boldsymbol{P}$. A typical element $\omega = \{x_n\}_{n=0}^{\infty}$ of Ω is called a sample path or a sample trajectory.

3.2 Existence of Markov Chain (IIIDRM approach or iteration of iid random maps)

Let $\boldsymbol{P} = ((p_{ij}))_{K \times K}$ be a stochastic matrix. Let $f : \mathbb{S} \times [0, 1] \to \mathbb{S}$ be

$$f(a_i, u) = \begin{cases} a_1 & \text{if} \quad 0 \leq u < p_{i1} \\ a_2 & \text{if} \quad p_{i1} \leq u < p_{i1} + p_{i2} \\ \vdots & \\ a_j & \text{if} \quad p_{i1} + p_{i2} + \cdots + p_{i(j-1)} \leq u < p_{i1} + p_{i2} \ldots + p_{ij} \\ \vdots & \\ a_K & \text{if} \quad p_{i1} + p_{i2} + \cdots + p_{i(K-1)} \leq u < 1 \end{cases}$$

Let $U_1, U_2, \ldots$ be iid Uniform [0,1] random variables.Let $f_n(\cdot) := f(\cdot, U_n)$. Then for each n, f_n maps $\mathbb{S}$ to $\mathbb{S}$. Also $\{f_n\}_{n=1}^{\infty}$ are iid. Let X_0 be independent of $\{U_i\}_{i=1}^{\infty}$ and $X_0 \sim \mu$. Then the sequence $\{X_n\}_{n=0}^{\infty}$ defincd by

$$X_{n+1} = f_{n+1}(X_n) = f(X_n, U_{n+1})$$

is a Markov chain with initial distribution μ and transition probability $\boldsymbol{P}$. The underlying probability space on which X_0 and $\{U_i\}_{i=1}^{\infty}$ are defined can be taken to be the Lebesgue space $([0,1], \mathcal{B}([0,1]), m)$, where m is the Lebesgue measure.

4 Discrete State Markov Process

This section will discuss continuous time Markov chains over a discrete state space.These are very useful in many areas of applications such as queuing theory and mathematical finance.

note: Now you will see how a system or a dynamical system relates with markov chain.

Consider a physical system that can be in one of a finite or countable number of states $\{0,1,2,\ldots,K\}, K \leq \infty$. Assume that the system evolves in continuous time in the following manner. In each state the system stays a random length of time that is exponentially distributed and then jumps to a new state with a probability distribution that depends only on the current state and not on the past history. Thus, if the state of the system at the time of the n th transition is denoted by $y_n, n=0,1,2,\ldots$, then $\{y_n\}_{n\geq 0}$ is a Markov chain with state space $\mathbb{S} \equiv \{0,1,2,\ldots,K\}, K \leq \infty$ and some transition probability matrix $\boldsymbol{P} \equiv ((p_{ij}))$. If $y_n = i_n$, then the system stays in i_n a random length of time L_n, called the sojourn time, such that conditional on $\{y_n = i_n\}_{n\geq 0}, \{L_n\}_{n\geq 0}$ are independent exponential random variables with L_n having a mean λ_i^{-1}. Now set the state of the system $X(t)$ at time $t \geq 0$ by

$$X(t)=\begin{cases} y_0 & 0 \leq t < L_0 \\ y_1 & L_0 \leq t < L_0 + L_1 \\ \vdots & \\ y_n & L_\nu + L_1 + \cdots + L_{n-1} \leq t < L_0 + L_1 + \cdots + L_n \end{cases}$$

Then $\{X(t) : t \geq 0\}$ is called a continuous time Markov chain (CTMC) with state space $\overline{\mathbb{S}}$, jump probabilities $\boldsymbol{P} \equiv ((p_{ij}))$, waiting time parameters $\{\lambda_i : i \in \mathbb{S}\}$, and embedded Markov chain $\{y_n\}_{n\geq 0}$. To make sure that there are only finite number of transitions in finite time, i.e.,

$$\sum_{i=0}^{\infty} L_i = \infty$$

one needs to impose the nonexplosion condition

$$\sum_{i=0}^{\infty} \frac{1}{\lambda_{y_n}} = \infty$$

A sufficient condition for this is that $\lambda_i < \infty$ for all $i \in S$ and $\{y_n\}_{n\geq 0}$ is an irreducible recurrent Markov chain.

It can be verified using the "memorylessness" property that $\{X(t) : t \geq 0\}$ has the **Markov property**, i.e., for any $0 \leq t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_r < \infty$ and

$$\begin{aligned} & P\left(X\left(t_r\right)=i_r \mid X\left(t_j\right)=i_j, 0 \leq j \leq r-1\right) \\ & =P\left(X\left(t_r\right)=i_r \mid X\left(t_{r-1}\right)=i_{r-1}\right) \end{aligned}$$

Compound Poisson Processes

This is an example of a continuous time discrete state Markov chain.

Let $\{L_i\}_{i\geq 0}$ and $\{\xi_i\}_{i\geq 1}$ be two independent sequences of random variables such that $\{L_i\}_i \geq 0$ are iid exponential with mean α^{-1} and $\{\xi_i\}_{i\geq 1}$ are iid integer valued random variables with distribution $\{p_j\}$. Let $X(t) = k$ if $L_0 + \cdots + I_k \leq t < I_0 + \cdots + I_{k+1}$. Let

$$X(t) = \begin{cases} 0 & 0 \leq t < L_0 \\ 1 & L_0 \leq t < L_0 + L_1 \\ \vdots & \\ k & I_{10} + \cdots + I_{k-1} \leq t < I_0 + \cdots + I_k, \\ \vdots & \end{cases}$$

Let

$$Y(t) = \sum_{i=1}^{X(t)} \xi_i, \quad t \geq 0$$

Then $\{Y(t) : t \geq 0\}$ is a continuous time Markov chain with state space $S \equiv \{0, \pm 1, \pm 2, \ldots\}$, jump probabilities $p_{ij} = P(\xi_1 = j - i) = p_{j-i}$. It is also a Levy process i.e it has stationary and independent increments.It is called a compound Poisson process with jump rate α and jump distribution $\{p_j\}$. If $p_1 \equiv 1$ this reduces to the Poisson process case.

5 References

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