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Stirlings Formula : an asymptotic approximation for factorials

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Abstract

Stirling's formula (or Stirling's approximation) is an approximation for factorials. It is a good approximation, leading to accurate results even for small values of n .Stirling's formula states that as $n \to \infty$, $n! \sim \frac{n^n}{e^n}\sqrt{2.\pi.n}$. That is , $\lim_{n\to\infty} \frac{n!}{\frac{n^n}{e^n}\sqrt{2.\pi.n}} = 1$.Though it is used in different forms especially in its logarithm form in this paper we will keep our focus mainly on focus its classical form.

1 Introduction

The purpose of this paper is to proof the asymptotic estimate for n! of Stirling's formula.Stirling's formula was born in the 1720s from the correspondence between James Stirling and Abraham de Moivre.The formula was first discovered by Abraham de Moivre in the form :

$$n! \sim [\text{ constant }] \cdot n^{n+\frac{1}{2}} e^{-n}$$

De Moivre gave an approximate rational-number expression for the natural logarithm of the constant. Stirling's contributed by showing that the constant is precisely $\sqrt{2\pi}$.Stirling's approximation is widely used in physics in quantum mechanics and thermodynamics for factorial calculations.
There are other different proofs of Stirling's approximation but we will focus in the proof where we use a formula $\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$ which is used for the evaluation of the Gaussian Integral.

2 Proof of Stirling's Formula

We begin our proof at Euler's Integral for n!.We will be using induction and integration by parts to proof Euler's integral for n!.Euler's integral for n! states that:

$$n! = \int_0^\infty x^n.e^{-x}\, dx$$

Proof 1: **Base Case :** $n = 0$ is a direct calculation: $\int_0^\infty e^{-x} dx = -e^{-x}|_0^\infty = 0 - (-1) = 1$.

Induction hypothesis : $n! = \int_0^\infty x^n e^{-x} dx$ for some n

Induction Step : We will show that the hypothesis is also true for n+1

$$\int_0^\infty x^{n+1} e^{-x} dx = \int_0^\infty u dv$$

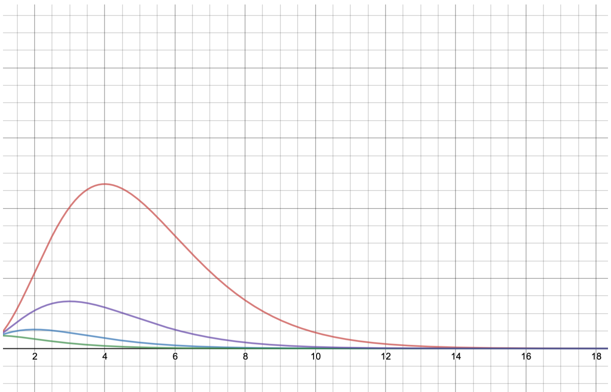

Figure 1

where $u = x^{n+1}$ and $dv = e^{-x}dx$. Then $du = (n+1)x^n dx$ and $v = -e^{-x}$, so

$$\begin{aligned}\int_0^\infty x^{n+1}e^{-x}dx &= uv|_0^\infty - \int_0^\infty vdu \\ &= -\left.\frac{x^{n+1}}{e^x}\right|_0^\infty + \int_0^\infty (n+1)e^{-x}x^n dx \\ &= \lim_{b\to\infty} -\frac{b^{n+1}}{e^b} + 0 + (n+1)\int_0^\infty x^n e^{-x}dx \\ &= (n+1)\int_0^\infty x^n e^{-x}dx\end{aligned}$$

Then by the induction hypothesis $(n+1)\int_0^\infty x^n e^{-x}dx = (n+1).n!$.which closes our induction.

Now we plot the graph for $y = x^n e^{-x}$for $1 \leq n \leq 4$ in figure 1. Now by considering the graph for $y = x^n.e^{-x}$ for $x \geq 0$ using calculus we get that $y = x^n.e^{-x}$ attains a maxima at x = n and has inflection points at $x = n + \sqrt{n}$ and $x = n - \sqrt{n}$. For large values of n these graphs look similar to bell curves in **probability theory**.So in order to further our proof we will use a substitution inspired from probability theory which is analogous to our situation. In probability theory the density function of a normal random variable X with mean μ and standard deviation σ has its maximum at μ and inflection points at $\mu + \sigma$ and $\mu - \sigma$,and the random variable $Z = \frac{X-\mu}{\sigma}$ is then the normal with mean 0 and standard deviation 1.

We consider the analogy $\mu \leftrightarrow n$ and $\sigma \leftrightarrow \sqrt{n}$ make the substitution :

$$t = (x-n)/\sqrt{n}$$

Equation 1 :

$$\begin{aligned}n! &= \int_0^\infty x^n e^{-x}dx \\ &= \int_{-\sqrt{n}}^\infty (n+\sqrt{n}t)^n e^{-(n+\sqrt{n}t)}\sqrt{n}dt \\ &= \frac{n^n\sqrt{n}}{e^n}\int_{-\sqrt{n}}^\infty \left(1+\frac{t}{\sqrt{n}}\right)^n e^{-\sqrt{n}t}dt\end{aligned}$$

As we can see the terms extracted out of the integral are exactly what appears in String's formula.So now only concern ourselves with the integral

$$\int_{-\sqrt{n}}^{\infty}(1+\frac{t}{\sqrt{n}})^{n}.e^{-\sqrt{n}t}\,dt \qquad \textbf{(I1)}$$

. So from here we will use an idea from another formula from Probability theory used in the evaluation of Gaussian Integral:

$$\int_{-\infty}^{\infty}e^{-x^2/2}\,dx=\sqrt{2\pi} \qquad \textbf{(I2)}$$

Therefore we have to show that

$$\int_{-\sqrt{n}}^{\infty}(1+\frac{t}{\sqrt{n}})^{n}.e^{-\sqrt{n}t}\,dt \to \int_{-\infty}^{\infty}e^{-t^2/2}\,dt=\sqrt{2\pi}$$

Now we will convert our integral **I1** in such a way that it is analogous to the above formula in **I2** so we get $\sqrt{2\pi}$.
To write the integral **I1** over the whole real line , we set :

$$\textbf{Function 1 :}\quad f_n(t)=\begin{cases}0, & \text{for } t\leq -\sqrt{n}\\ (1+t/\sqrt{n})^n.e^{-\sqrt{n}t} & \text{for } t\geq -\sqrt{n}\end{cases}$$

Hence $n!=\frac{n^n.\sqrt{n}}{e^n}\int_{-\infty}^{\infty}f_n(t)\,dt$. By plotting $f_n(t)$ the graph suggests that as

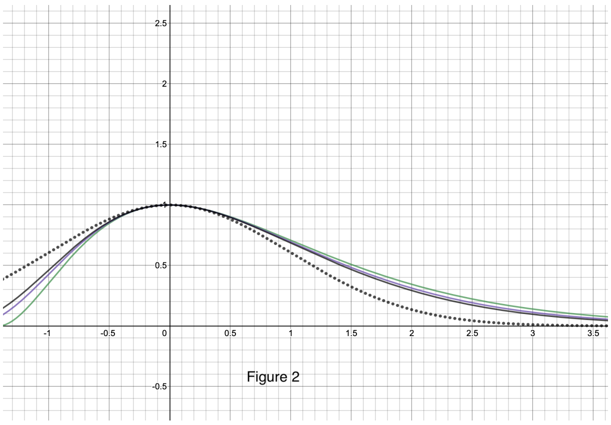

Figure 2

$n\to\infty$,$f_n(t)\to e^{-t^2/2}$.The dotted lines in figure 2 represents $e^{-t^2/2}$.
Proposition 1 : for each $t\in R$, $f_n(t)\to e^{-t^2/2}$as $n\to\infty$

Proof 2: We will show that for each $t\in R$, $f_n(t)\to e^{-t^2/2}$as $n\to\infty$.
To prove the above statement we will use a result from R. Michel, "The (n + 1)-th Proof of Stirling's Formula," Amer. Math. Monthly 115 (2008), 844–845. that is **if** $|x|\leq 1/2$ **then** $\log(1+x)=x-x^2/2+O(|x|^3/3)$ **.** We begin our proof from **Function 1** and by taking logarithm on both sides we get

$$\log(f_n(t))=n\log(1+t/\sqrt{n})-\sqrt{n}t.$$

For $n > 4t^2, |t/\sqrt{n}| < 1/2$. Then by using the mentioned result $|x| \leq 1/2,^2$

$$\log\left(f_n(t)\right) = n\left(\frac{t}{\sqrt{n}} - \frac{(t/\sqrt{n})^2}{2} + O\left((t/\sqrt{n})^3\right)\right) - \sqrt{n}t = -\frac{t^2}{2} + O\left(t^3/\sqrt{n}\right)$$

As $n \to \infty$, the O -term tends to 0, so the limit is $-t^2/2$

To deduce from the results of **proof 2** that $\int_{-\infty}^{\infty} f_n(t)\,dt \to \int_{-\infty}^{\infty} e^{-t^2/2}\,dt$ we will use dominated convergence theorem where we will determine a positive integrable function on $\mathbb{R}$ dominating $|f_n| = f_n$ for all n.Now by observing the plot in Figure 2 one such function will be :

$$\textbf{Function 2 :} \quad g(t) := \begin{cases} e^{-}t^2/2, & \text{for } t < 0 \\ f_1(t) & \text{for } t \geq 0 \end{cases}$$

So now we just have to show that $0 \leq f_n(t) \leq g(t)$ .Which can be done by taking cases and comparing them.

Case 1 : We will check that $\log f_n(t) \leq -t^2/2$ for $-\sqrt{n} < t \leq 0$ We take the difference

$$\textbf{Equation 2 :} \quad \log f_n(t) + \frac{t^2}{2} = n\log\left(1 + \frac{t}{\sqrt{n}}\right) - \sqrt{n}t + \frac{t^2}{2}$$

for $-\sqrt{n} < t \leq 0$ is increasing .Also we can see that at $t = 0$ which implies it is negative for $-\sqrt{n} < t < 0$. The derivative of **Equation 2** is

$$\frac{n}{1 + t/\sqrt{n}}\frac{1}{\sqrt{n}} - \sqrt{n} + t = \frac{t^2}{t + \sqrt{n}}$$

which is positive for $-\sqrt{n} < t < 0$.

Case 2 : We will check $\log f_n(t) \leq \log(1 + t) - t$ for $t \geq 0$.

For $n = 1$, since $\log f_1(t) = \log(1 + t) - t$ for $t \geq 0$. Hence we take $n > 1$. Again we take the difference

$$\textbf{Equation 3 :} \quad \log(1+t) - t - \log f_n(t) = \log(1+t) - t - n\log\left(1 + \frac{t}{\sqrt{n}}\right) + \sqrt{n}t$$

for $t \geq 0$ is increasing,and it vanishes at $t = 0$ which implies it is positive for $t > 0$. Now by taking the derivative of **Equation 3**

$$\frac{1}{1+t} - 1 - \frac{n}{1 + t/\sqrt{n}}\frac{1}{\sqrt{n}} + \sqrt{n} = \frac{(\sqrt{n} - 1)t^2}{(t+1)(t + \sqrt{n})}$$

we see it is positive when $t > 0$ and $n \geq 2$. Now we have successfully shown that

$$\int_{-\sqrt{n}}^{\infty}(1 + \frac{t}{\sqrt{n}})^n.e^{-\sqrt{n}t}\,dt \to \int_{-\infty}^{\infty} e^{-t^2/2}\,dt = \sqrt{2\pi}$$

Our proof of Stirling's formula is now complete.

3 References

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