<p style="text-align:center">Research Assignment</p>

Fuzzy Arithmetic on Gaussian Fuzzy Number

<p style="text-align:center">Course : BITS F343 Fuzzy Logic and Applications
Instructor: Dr. Chandra Shekhar</p>

Anish Kulkarni	P V S Tarak Shree Vallabha	Dr. Chandra Shekhar
2020A7PS0975P	2020A7PS1513P	csbits.fuzzy@gmail.com

<p style="text-align:center">December 10, 2022</p>

Abstract

Arithmetic operations like addition, subtraction, multiplication, division on the Gaussian Fuzzy Number are discussed in the document. Exponentiation and Logarithm are also discussed briefly.

1 Introduction

Fuzzy numbers are a special type of fuzzy set. There exist different types of fuzzy numbers like, triangular fuzzy number, trapezoidal fuzzy number, pentagonal fuzzy number, L-R fuzzy numbers etc. One of these types if the Gaussian Fuzzy Number. The document begins with an introduction of the preliminaries that we require to understand the Gaussian Fuzzy Number and any operations performed on it.

2 Preliminaries

2.1 Fuzzy Set

Consider a non-empty Crisp set X. A fuzzy set $\tilde{A}$ on set X is defined as follows:

$$\tilde{A} = \{\, (x, \mu_{\tilde{A}}(x)) \,:\, x \in X,\, \mu_{\tilde{A}} : X \to [0,1] \,\}, \tag{1}$$

where $\mu_{\tilde{A}}$ is called the membership function of the $\tilde{A}$ and set X is called the universe of discourse of $\tilde{A}$.

2.2 Normal Fuzzy Set

Consider a fuzzy set $\tilde{A}$ with universe of discourse X. $\tilde{A}$ is said to be a normal fuzzy set if there exist at least one $x \in X$ such that $\mu_{\tilde{A}} = 1$.

2.3 Support of a Fuzzy Set

Consider a fuzzy set $\tilde{A}$ with universe of discourse X. The support of $\tilde{A}$ denoted by $Supp(\tilde{A})$ is a crisp set given by :

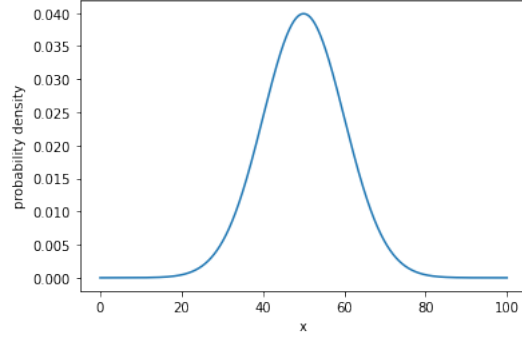$$Supp(\tilde{A}) = \{\, x \,:\, x \in X,\, \mu_{\tilde{A}}(x) > 0 \,\} \tag{2}$$

Figure 1: Gaussian distribution with $\mu = 50$ and $\sigma = 10$.

2.4 α-cut a Fuzzy set

For a fuzzy set $\tilde{A}$ defined on X, the α-cut of $\tilde{A}$ for $\alpha \in [0, 1]$, denoted by $\tilde{A}_\alpha$ is a crisp set defined as:

$$\tilde{A}_\alpha = \{\, x \,:\, x \in X,\, \mu_{\tilde{A}}(x) \geq \alpha \,\} \tag{3}$$

2.5 Fuzzy Number

A fuzzy set $\tilde{A}$ defined on $\mathbb{R}$ is said to be a fuzzy number if it has the following properties:

1. $\tilde{A}$ must be a normal fuzzy set.

2. $\tilde{A}_\alpha$ must be closed for every $\alpha \in [0, 1]$.

3. The support of $\tilde{A}$, $Supp(\tilde{A})$ must be bounded.

2.6 Gaussian Distribution

Gaussian Distribution, also known as Normal Distribution is a continuous probability distribution for a real-valued random variable. The probability density function of a Gaussian Distribution is given by:

$$\mathcal{N}(\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} \tag{4}$$

where $\mu \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}^+$ represent the mean and variance of the Gaussian probability distribution on $x \in \mathbb{R}$. Gaussian distribution is sometimes informally called a bell-shaped curve. Figure 1 shows a Gaussian distribution with $\mu = 50$ and $\sigma = 10$.

2.7 Gaussian Fuzzy Number

The membership function of a Gaussian fuzzy number $\tilde{A}$ on $\mathbb{R}$ is given by:

$$\mu_{\tilde{A}}(x) = e^{\left[\frac{-(x-m)^2}{2\sigma^2}\right]} \forall x \in \mathbb{R},\ \sigma > 0,\ m \in \mathbb{R} \tag{5}$$

Figure 2 shows Matlab plot for a Gaussian Fuzzy number with $\mu = 20$ and $\sigma = 5$.

2.8 L-R Representation of a Fuzzy number

In the L-R representation of a fuzzy number, $\tilde{P}$ is represented by :

$$\tilde{P} \,=\, (m, \alpha, \beta)_{L,R}$$

Here, m represents the modal value of the membership function, α and β represent the left-spread and the right-spread of the membership curve respectively, and L and R represent the left and right shape
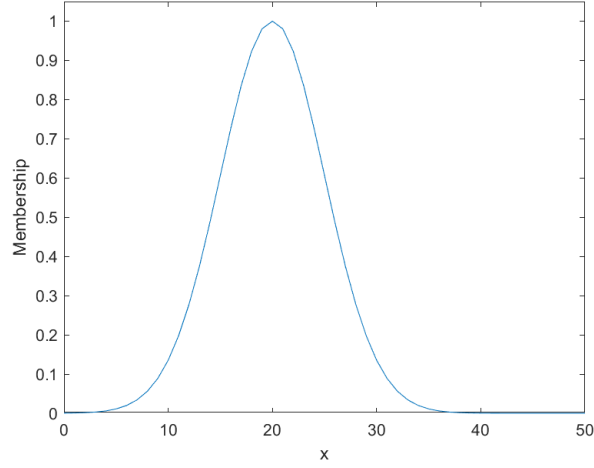
Figure 2: Gaussian Fuzzy Number with $\mu = 20$ and $\sigma = 5$

functions of the membership curve of $\tilde{P}$. The membership function of an L-R fuzzy number, $\tilde{P}$ is given by:

$$\tilde{P} = (m, \alpha, \beta)_{L,R} = \begin{cases} L\left(\frac{m-x}{\alpha}\right) & \text{if } x \leq m \\ R\left(\frac{x-m}{\beta}\right) & \text{if } x > m \end{cases} \tag{6}$$

The α-cut for an L-R fuzzy number $\tilde{P} = (m, p, q)_{L,R}$ is given by:

$$\tilde{P}_\alpha = \left[m - pL^{-1}(\alpha), m + qR^{-1}(\alpha)\right] \tag{7}$$

For a Gaussian fuzzy number $\tilde{A}$ with mean m and standard deviation σ,

$$L(u) = R(u) = e^{\frac{-u^2}{2}} , \ \tilde{A} = (m, \sigma, \sigma)_{L,R} \tag{8}$$

2.9 Tangent and Secant Approximations

To simplify calculation of result during multiplication, reciprocal, and division of L-R fuzzy numbers, two approximations were proposed, namely *Tangent Approximation* and *Secant Approximation*. Consider two fuzzy numbers, $\tilde{A} = (m_1, p_1, q_1)_{L,R}$ and $\tilde{B} = (m_2, p_2, q_2)_{L,R}$. The α-cuts of $\tilde{A}$ and $\tilde{B}$ $\alpha \in (0, 1]$ using equation (7) for are:

$$\tilde{A}_\alpha = \left[m_1 - p_1 L^{-1}(\alpha), m_1 + q_1 R^{-1}(\alpha)\right]$$

$$\tilde{B}_\alpha = \left[m_2 - p_2 L^{-1}(\alpha), m_2 + q_2 R^{-1}(\alpha)\right]$$

2.9.1 Tangent Approximation

If p_1 and q_1 are close to m_1, and p_2 and q_2 are close to m_2, we can consider $\mu_{\tilde{A}}$, $\mu_{\tilde{B}}$ to be very close to 1. Then, the quadratic term, $\left[L^{-1}(\alpha)\right]^2$ can be ignored during calculation for approximation.
The tangent approximation for the reciprocal of a $\tilde{A}$ is given by:

$$\left(\tilde{A}^{-1}\right)_t = \left(\frac{1}{m_1}, \frac{p_2}{m_1^2}, \frac{p_1}{m_1^2}\right)_{R,L} \approx \tilde{A}^{-1}, \quad 0 \notin Supp(\tilde{A}) \tag{9}$$

2.9.2 Secant approximation

If p_1, and q_1 are not negligible to m_1, and p_2 and q_2 are not negligible to m_2, the approximate result is estimated by approximating the quadratic term, $\left[L^{-1}(\alpha)\right]^2$ to the linear term $\left[L^{-1}(\alpha)\right]$. The secant approximation for the reciprocal of a $\tilde{A}$ is given by:

$$\left(\tilde{A}^{-1}\right)_s = \left(\frac{1}{m_1}, \frac{p_2}{m_1(m_1 + p_2)}, \frac{p_1}{m_1(m_1 - p_1)}\right)_{R,L} \approx \tilde{A}^{-1}, \quad 0 \notin Supp(\tilde{A}) \tag{10}$$

3

3 Discussion

This section looks at performing basic arithmetic operations on Gaussian Fuzzy Numbers. α-cut method is used to perform arithmetic operations on Gaussian fuzzy numbers and calculate the membership function of the resulting fuzzy number.

3.1 Calculating α-cut for Gaussian Fuzzy Number

Consider a Gaussian fuzzy number with mean μ and standard deviation σ. The membership function is given by:

$$\mu_{\tilde{A}}(x) = e^{\left[\frac{-(x-m)^2}{2\sigma^2}\right]} \forall x \in \mathbb{R}, \ \sigma > 0, \ m \in \mathbb{R} \tag{11}$$

The α-cut $\tilde{A}^\alpha$ can be calculated as follows:

$$\tilde{A}_\alpha = \{x \in \mathbb{R} : \mu(x) \geq \alpha\} \forall \alpha \in [0,1] \implies e^{\left[\frac{-(x-m)^2}{2\sigma^2}\right]} \geq \alpha \tag{12}$$

By solving for x, we obtain the α-cut as:

$$\tilde{A}_\alpha = [m - \sigma\sqrt{-2\ln\alpha}, \ m + \sigma\sqrt{-2\ln\alpha}] \tag{13}$$

3.2 Addition of Gaussian Fuzzy Numbers

Consider two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ defined on $\mathbb{R}$, with membership functions given by:

$$\mu_{\tilde{A}}(x) = e^{\left[\frac{-(x-m_{\tilde{A}})^2}{2\sigma_{\tilde{A}}^2}\right]} ; \ \mu_{\tilde{B}}(x) = e^{\left[\frac{-(x-m_{\tilde{B}})^2}{2\sigma_{\tilde{B}}^2}\right]}$$

The α-cuts for $\tilde{A}$ and $\tilde{B}$ are:

$$\tilde{A}_\alpha = \left[m_{\tilde{A}} - \sigma_{\tilde{A}}\sqrt{-2\ln\alpha}, \ m_{\tilde{A}} + \sigma_{\tilde{A}}\sqrt{-2\ln\alpha}\right]$$

and,

$$\tilde{B}_\alpha = \left[m_{\tilde{B}} - \sigma_{\tilde{B}}\sqrt{-2\ln\alpha}, \ m_{\tilde{B}} + \sigma_{\tilde{B}}\sqrt{-2\ln\alpha}\right]$$

respectively, for $\alpha \in (0,1]$. Upon interval addition of α-cuts of $\tilde{A}$ and $\tilde{B}$, we get:

$$(\tilde{A}\oplus\tilde{B})_\alpha = \tilde{A}_\alpha + \tilde{B}_\alpha = \left[(m_{\tilde{A}} + m_{\tilde{B}}) - (\sigma_{\tilde{A}} + \sigma_{\tilde{B}})\sqrt{-2\ln\alpha}, \ (m_{\tilde{A}} + m_{\tilde{B}}) + (\sigma_{\tilde{A}} + \sigma_{\tilde{B}})\sqrt{-2\ln\alpha}\right]; \quad \alpha \in [0,1] \tag{14}$$

By comparing (14) to *alpha*-cut of a Gaussian fuzzy number given by (13), we get:

$$\sigma_{\tilde{A}\oplus\tilde{B}} = \sigma_{\tilde{A}} + \sigma_{\tilde{B}}, \quad m_{\tilde{A}\oplus\tilde{B}} = m_{\tilde{A}} + m_{\tilde{B}}, \quad \mu_{\tilde{A}\oplus\tilde{B}}(x) = e^{\left[\frac{-(x-(m_{\tilde{A}}+m_{\tilde{B}}))^2}{2(\sigma_{\tilde{A}}+\sigma_{\tilde{B}})^2}\right]}; \forall x \in \mathbb{R} \tag{15}$$

Example : Let $\tilde{A}$ and $\tilde{B}$ be two Gaussian fuzzy numbers defined on $\mathbb{R}$ with $\sigma_{\tilde{A}} = 5$, $m_{\tilde{A}} = 20$; and $\sigma_{\tilde{B}} = 4$, $m_{\tilde{B}} = 25$ respectively. We get membership functions of $\tilde{A}$ and $\tilde{B}$ as $\mu_{\tilde{A}}(x) = e^{\left(\frac{-(x-20)^2}{2(5)^2}\right)}$ and $\mu_{\tilde{B}}(x) = e^{\left(\frac{-(x-25)^2}{2(4)^2}\right)}$. On addition of $\tilde{A}$ and $\tilde{B}$, we get $\mu_{\tilde{A}\oplus\tilde{B}}(x) = e^{\left(\frac{-(x-45)^2}{2(9)^2}\right)}$. Figure 3 shows MATLAB plots of $\tilde{A}$, $\tilde{B}$ and $\tilde{A} \oplus \tilde{B}$.

3.3 Subtraction of Gaussian Fuzzy Numbers

Consider two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ defined on $\mathbb{R}$, with membership functions given by:

$$\mu_{\tilde{A}}(x) = e^{\left[\frac{-(x-m_{\tilde{A}})^2}{2\sigma_{\tilde{A}}^2}\right]} ; \ \mu_{\tilde{B}}(x) = e^{\left[\frac{-(x-m_{\tilde{B}})^2}{2\sigma_{\tilde{B}}^2}\right]}$$
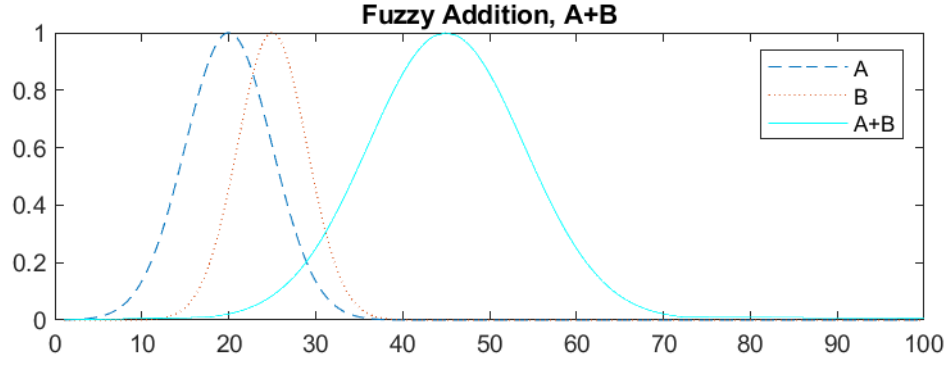
Figure 3: Graphs of $\mu_{\tilde{A}}(x) = e^{\left(\frac{-(x-20)^2}{2(5)^2}\right)}$, $\mu_{\tilde{B}}(x) = e^{\left(\frac{-(x-25)^2}{2(4)^2}\right)}$, and $\mu_{\tilde{A}\oplus\tilde{B}}(x) = e^{\left(\frac{-(x-45)^2}{2(9)^2}\right)}$

The α-cuts for $\tilde{A}$ and $\tilde{B}$ are:

$$\tilde{A}_\alpha = \left[m_{\tilde{A}} - \sigma_{\tilde{A}}\sqrt{-2\ln\alpha},\ m_{\tilde{A}} + \sigma_{\tilde{A}}\sqrt{-2\ln\alpha}\right]$$

and,

$$\tilde{B}_\alpha = \left[m_{\tilde{B}} - \sigma_{\tilde{B}}\sqrt{-2\ln\alpha},\ m_{\tilde{B}} + \sigma_{\tilde{B}}\sqrt{-2\ln\alpha}\right]$$

respectively, for $\alpha \in (0,1]$. Upon interval subtraction of α-cuts of $\tilde{A}$ and $\tilde{B}$, we get:

$$(\tilde{A}\ominus\tilde{B})_\alpha = \tilde{A}_\alpha - \tilde{B}_\alpha = \left[(m_{\tilde{A}} - m_{\tilde{B}}) - (\sigma_{\tilde{A}} + \sigma_{\tilde{B}})\sqrt{-2\ln\alpha},\ (m_{\tilde{A}} - m_{\tilde{B}}) + (\sigma_{\tilde{A}} + \sigma_{\tilde{B}})\sqrt{-2\ln\alpha}\right]; \quad \alpha \in [0,1] \tag{16}$$

By comparing (16) to *alpha*-cut of a Gaussian fuzzy number given by (13), we get:

$$\sigma_{\tilde{A}\ominus\tilde{B}} = \sigma_{\tilde{A}} - \sigma_{\tilde{B}}, \quad m_{\tilde{A}\ominus\tilde{B}} = m_{\tilde{A}} - m_{\tilde{B}}, \quad \mu_{\tilde{A}\ominus\tilde{B}}(x) = e^{\left[\frac{-(x-(m_{\tilde{A}} - m_{\tilde{B}}))^2}{2(\sigma_{\tilde{A}} + \sigma_{\tilde{B}})^2}\right]}; \forall x \in \mathbb{R} \tag{17}$$

Example : Let $\tilde{A}$ and $\tilde{B}$ be two Gaussian fuzzy numbers defined on $\mathbb{R}$ with $\sigma_{\tilde{A}} = 5$, $m_{\tilde{A}} = 20$; and $\sigma_{\tilde{B}} = 4$, $m_{\tilde{B}} = 25$ respectively. We get membership functions of $\tilde{A}$ and $\tilde{B}$ as $\mu_{\tilde{A}}(x) = e^{\left(\frac{-(x-20)^2}{2(5)^2}\right)}$ and $\mu_{\tilde{B}}(x) = e^{\left(\frac{-(x-25)^2}{2(4)^2}\right)}$. On addition of $\tilde{A}$ and $\tilde{B}$, we get $\mu_{\tilde{A}\ominus\tilde{B}}(x) = e^{\left(\frac{-(x+5)^2}{2(9)^2}\right)}$. Figure 4 shows MATLAB plots of $\tilde{A}$, $\tilde{B}$ and $\tilde{A} \ominus \tilde{B}$.
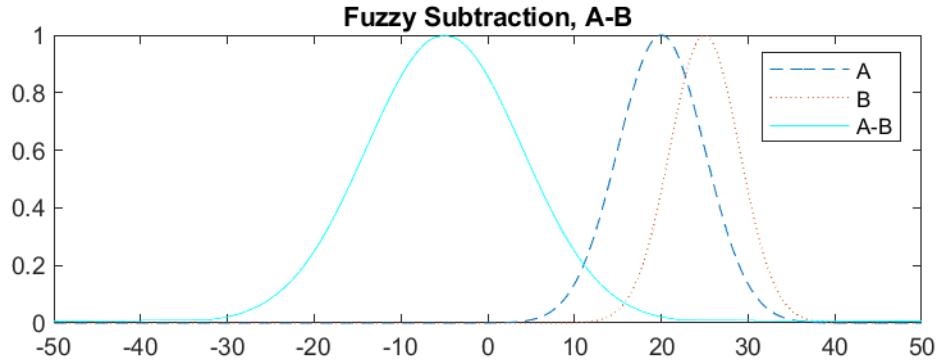

Figure 4: Graphs of $\mu_{\tilde{A}}(x) = e^{\left(\frac{-(x-20)^2}{2(5)^2}\right)}$, $\mu_{\tilde{B}}(x) = e^{\left(\frac{-(x-25)^2}{2(4)^2}\right)}$, and $\mu_{\tilde{A}\ominus\tilde{B}}(x) = e^{\left(\frac{-(x+5)^2}{2(9)^2}\right)}$

3.4 Multiplication of Gaussian Fuzzy Numbers

Consider two Gaussian fuzzy numbers $\tilde{A} = (m_{\tilde{A}}, \sigma_{\tilde{A}}, \sigma_{\tilde{A}})_{L,R}$ and $\tilde{B} = (m_{\tilde{B}}, \sigma_{\tilde{B}}, \sigma_{\tilde{B}})_{L,R}$, defined on $\mathbb{R}$ with $L(u) = R(u) = e^{\frac{-u^2}{2}} \implies L^{-1}(u) = R^{-1}(u) = \sqrt{-2\ln x}$. The membership functions of $\tilde{A}$ and $\tilde{B}$ given by:

$$\mu_{\tilde{A}}(x) = e^{\left[\frac{-(x - m_{\tilde{A}})^2}{2\sigma_{\tilde{A}}^2}\right]} \; ; \; \mu_{\tilde{B}}(x) = e^{\left[\frac{-(x - m_{\tilde{B}})^2}{2\sigma_{\tilde{B}}^2}\right]} \forall x \in \mathbb{R}$$

The α-cuts for $\tilde{A}$ and $\tilde{B}$ are:

$$\tilde{A}_\alpha = \left[m_{\tilde{A}} - \sigma_{\tilde{A}}\sqrt{-2\ln\alpha}, \; m_{\tilde{A}} + \sigma_{\tilde{A}}\sqrt{-2\ln\alpha}\right]$$

and,

$$\tilde{B}_\alpha = \left[m_{\tilde{B}} - \sigma_{\tilde{B}}\sqrt{-2\ln\alpha}, \; m_{\tilde{B}} + \sigma_{\tilde{B}}\sqrt{-2\ln\alpha}\right]$$

respectively, for $\alpha \in [0, 1]$.

$$\text{Let} \quad a = m_{\tilde{A}} - \sigma_{\tilde{A}}\sqrt{-2\ln\alpha},$$
$$b = m_{\tilde{A}} + \sigma_{\tilde{A}}\sqrt{-2\ln\alpha},$$
$$c = m_{\tilde{B}} - \sigma_{\tilde{B}}\sqrt{-2\ln\alpha},$$
$$d = m_{\tilde{B}} + \sigma_{\tilde{B}}\sqrt{-2\ln\alpha}.$$

Upon interval multiplication of α-cuts, we get:

$$(\tilde{A} \otimes \tilde{B})_\alpha = \tilde{A}_\alpha \times \tilde{B}_\alpha = [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}] \tag{18}$$

If we try to calculate the exact value of the membership function using equation (18), the resulting fuzzy number may not be a Gaussian fuzzy number. The exact interval of the α-cut in equation (18) depends upon the signs of a, b, c, d, which give rise to various cases. Let us consider the case where $\tilde{A} > 0$, (i.e., $m_{\tilde{A}} > 0$, $m_{\tilde{A}} - \sigma_{\tilde{A}} > 0$, $m_{\tilde{A}} + \sigma_{\tilde{A}} > 0$, so, $\mu_{\tilde{A}} \approx 0 \; \forall x \leq 0$) and similarly, $\tilde{B} > 0$. From equation (18), we get:

$$(\tilde{A} \otimes \tilde{B})_\alpha = \tilde{A}_\alpha \times \tilde{B}_\alpha = \left[\left(m_{\tilde{A}} - \sigma_{\tilde{A}}\sqrt{-2\ln\alpha}\right)\left(m_{\tilde{B}} - \sigma_{\tilde{B}}\sqrt{-2\ln\alpha}\right), \; \left(m_{\tilde{A}} + \sigma_{\tilde{A}}\sqrt{-2\ln\alpha}\right)\left(m_{\tilde{B}} + \sigma_{\tilde{B}}\sqrt{-2\ln\alpha}\right)\right] \tag{19}$$

$$\alpha \in [0, 1]$$

On simplification and using tangent approximation on equation (19), we get:

$$\tilde{A} \otimes \tilde{B} \approx (m_{\tilde{A}}m_{\tilde{B}}, \; m_{\tilde{A}}\sigma_{\tilde{B}} + m_{\tilde{B}}\sigma_{\tilde{A}}, \; m_{\tilde{A}}\sigma_{\tilde{B}} + m_{\tilde{B}}\sigma_{\tilde{A}})_{L,R} \quad ; L(u) = R(u) = e^{\frac{-u^2}{2}} \forall u \in \mathbb{R} \tag{20}$$

Similarly, on simplification and secant approximation, we get:

$$\tilde{A} \otimes \tilde{B} \approx (m_{\tilde{A}}m_{\tilde{B}}, \; m_{\tilde{A}}\sigma_{\tilde{B}} + m_{\tilde{B}}\sigma_{\tilde{A}} + \sigma_{\tilde{A}}\sigma_{\tilde{B}}, \; m_{\tilde{A}}\sigma_{\tilde{B}} + m_{\tilde{B}}\sigma_{\tilde{A}} + \sigma_{\tilde{A}}\sigma_{\tilde{B}})_{L,R} \quad ; L(u) = R(u) = e^{\frac{-u^2}{2}} \tag{21}$$

Figure (5) shows the multiplication of two Gaussian fuzzy numbers $\tilde{A} = (5, 1, 1)_{L,R}$ and $\tilde{B} = (5, 2, 2)_{L,R}$ with $L(u) = R(u) = e^{\frac{-u^2}{2}} \; \forall u \in \mathbb{R}$.

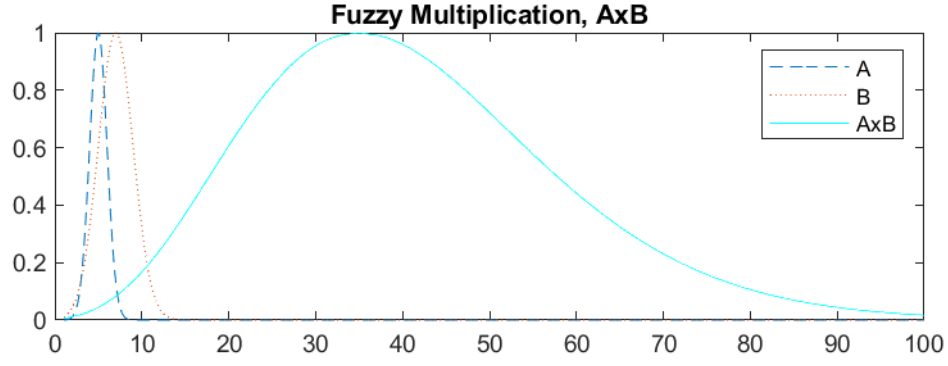Figure 5: Graphs of $\mu_{\tilde{A}}(x) = e^{\left(\frac{-(x-5)^2}{2(1)^2}\right)}$, $\mu_{\tilde{B}}(x) = e^{\left(\frac{-(x-7)^2}{2(2)^2}\right)}$, and $\mu_{\tilde{A} \otimes \tilde{B}}(x)$

3.5 Reciprocal of Gaussian Fuzzy Number

Consider a Gaussian fuzzy number $\tilde{A} = (m, \sigma, \sigma)_{L,R}$ on $\mathbb{R}$ with $L(u) = R(u) = e^{\frac{-u^2}{2}} \implies L^{-1}(u) = R^{-1}(u) = \sqrt{-2 \ln x}$. For a Gaussian Fuzzy number, the value of membership is not 0 at $x = 0$. So, reciprocal is not defined for a Gaussian fuzzy number. But, if we assume that $m > 0$, $\sigma + m > 0'$ $m - \sigma > 0$ or $m < 0$, $\sigma + m < 0$, $m - \sigma < 0$, the Gaussian fuzzy number approximately lies entirely on one side of $x = 0$. In such a case, $\mu_{\tilde{A}}(0) \approx 0$. So, we can neglect the membership function at $x = 0$. To calculate the approximate value of the reciprocal of Gaussian Fuzzy Number $\tilde{A}$, we use tangent and secant approximation. Using tangent approximation, we get:

$$\left(\tilde{A}^{-1}\right)_t = \left(\frac{1}{m}, \frac{\sigma}{m^2}, \frac{\sigma}{m^2}\right)_{R,L} \approx \tilde{A}^{-1} \tag{22}$$

Using secant approximation, we get:

$$\left(\tilde{A}^{-1}\right)_s = \left(\frac{1}{m}, \frac{\sigma}{m(m+\sigma)}, \frac{\sigma}{m(m-\sigma)}\right)_{R,L} \approx \tilde{A}^{-1} \tag{23}$$

3.6 Division of Gaussian Fuzzy Numbers

Consider two Gaussian fuzzy numbers $\tilde{A} = (m_{\tilde{A}}, \sigma_{\tilde{A}}, \sigma_{\tilde{A}})_{L,R}$ and $\tilde{B} = (m_{\tilde{B}}, \sigma_{\tilde{B}}, \sigma_{\tilde{B}})_{L,R}$, defined on $\mathbb{R}$ with $L(u) = R(u) = e^{\frac{-u^2}{2}} \implies L^{-1}(u) = R^{-1}(u) = \sqrt{-2 \ln x}$. The membership functions of $\tilde{A}$ and $\tilde{B}$ given by:

$$\mu_{\tilde{A}}(x) = e^{\left[\frac{-(x-m_{\tilde{A}})^2}{2\sigma_{\tilde{A}}^2}\right]} \, ; \, \mu_{\tilde{B}}(x) = e^{\left[\frac{-(x-m_{\tilde{B}})^2}{2\sigma_{\tilde{B}}^2}\right]}$$

The α-cuts for $\tilde{A}$ and $\tilde{B}$ are:

$$\tilde{A}_\alpha = \left[m_{\tilde{A}} - \sigma_{\tilde{A}}\sqrt{-2\ln\alpha}, \, m_{\tilde{A}} + \sigma_{\tilde{A}}\sqrt{-2\ln\alpha}\right]$$

and,

$$\tilde{B}_\alpha = \left[m_{\tilde{B}} - \sigma_{\tilde{B}}\sqrt{-2\ln\alpha}, \, m_{\tilde{B}} + \sigma_{\tilde{B}}\sqrt{-2\ln\alpha}\right]$$

respectively, for $\alpha \in [0, 1]$.

$$\text{Let} \quad a = m_{\tilde{A}} - \sigma_{\tilde{A}}\sqrt{-2\ln\alpha},$$
$$b = m_{\tilde{A}} + \sigma_{\tilde{A}}\sqrt{-2\ln\alpha},$$
$$c = m_{\tilde{B}} - \sigma_{\tilde{B}}\sqrt{-2\ln\alpha},$$
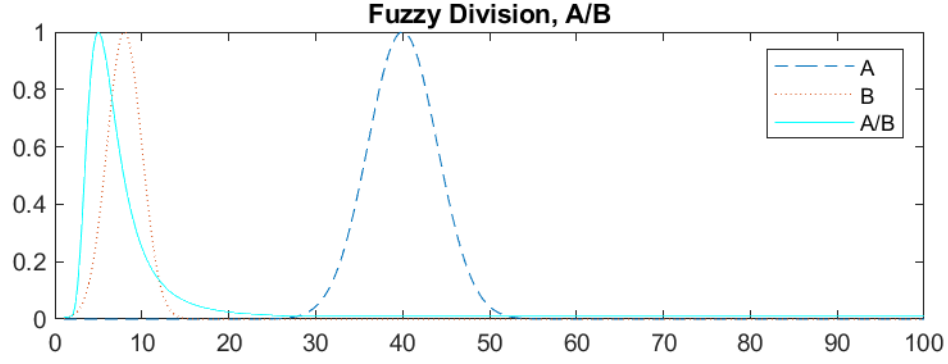$$d = m_{\tilde{B}} + \sigma_{\tilde{B}}\sqrt{-2\ln\alpha}.$$

7

Figure 6: Graphs of $\mu_{\tilde{A}}(x) = e^{\left(\frac{-(x-40)^2}{2(4)^2}\right)}$, $\mu_{\tilde{B}}(x) = e^{\left(\frac{-(x-8)^2}{2(2)^2}\right)}$, and $\mu_{\tilde{A} \oslash \tilde{B}}(x)$

Upon interval division of α-cuts, we get:

$$(\tilde{A} \oslash \tilde{B})_\alpha = \tilde{A}_\alpha / \tilde{B}_\alpha = [\min\{a/c, a/d, b/c, b/d\}, \max\{a/c, a/d, b/c, b/d\}]; \quad \alpha \in [0, 1] \qquad (24)$$

Similar to what was seen while calculating the reciprocal of a Gaussian Fuzzy number, division of two Gaussian Fuzzy numbers is not defined, since the value of the membership function of the denominator at $x = 0$ is not zero. But, if we assume that for the Gaussian fuzzy number $\tilde{B}$ in the denominator, $m_{\tilde{B}} > 0$, $\sigma_{\tilde{B}} + m_{\tilde{B}} > 0'$ $m_{\tilde{B}} - \sigma_{\tilde{B}} > 0$ or $m_{\tilde{B}} < 0$, $\sigma_{\tilde{B}} + m_{\tilde{B}} < 0$, $m_{\tilde{B}} - \sigma_{\tilde{B}} < 0$, the Gaussian fuzzy number approximately lies entirely on one side of $x = 0$. In such a case, $\mu_{\tilde{B}}(0) \approx 0$. So, we can neglect the value of the membership function of the denominator $\tilde{B}$ at $x = 0$. Still, calculating the exact curve of equation (24) would be complicated. So we use use tangent and secant approximation and multiply $\tilde{A}$ with the reciprocal $1/\tilde{B}$ to get the approximate result of division.

$$\tilde{A} \oslash \tilde{B} = \tilde{A} \times 1/\tilde{B} \qquad (25)$$

Figure (6) shows the division of two Gaussian fuzzy numbers $\tilde{A} = (40, 4, 4)_{L,R}$ by $\tilde{B} = (8, 2, 2)_{L,R}$ with $L(u) = R(u) = e^{\frac{-u^2}{2}} \ \forall u \in \mathbb{R}$.

3.7 Exponent of a Gaussian Fuzzy Number

Consider Gaussian fuzzy number $\tilde{A}$ with mean m and standard deviation $\sigma_{\tilde{A}}$.
α-cut of $\tilde{A}$ is $\tilde{A}_\alpha = \left[m_{\tilde{A}} - \sigma_{\tilde{A}}\sqrt{-2\ln\alpha}, \ m_{\tilde{A}} + \sigma_{\tilde{A}}\sqrt{-2\ln\alpha}\right]$.
To calculate $\exp(\tilde{A})$ or $e^{(\tilde{A})}$ we first take the exponent of the α-cut.

$$(exp(\tilde{A}))_\alpha = exp(\tilde{A}_\alpha) = \left[exp(m_{\tilde{A}} - \sigma_{\tilde{A}}\sqrt{-2\ln\alpha}), exp(m_{\tilde{A}} + \sigma_{\tilde{A}}\sqrt{-2\ln\alpha})\right] \quad \alpha \in [0, 1]$$

The next step is to fuzzify the obtained interval. Let

$$x = exp(m_{\tilde{A}} - \sigma_{\tilde{A}}\sqrt{-2\ln\alpha})$$

$$\ln x = m_{\tilde{A}} - \sigma_{\tilde{A}}\sqrt{-2\ln\alpha}$$

$$\ln\alpha = \frac{-1}{2}\left(\frac{m_{\tilde{A}} - \ln x}{\sigma_{\tilde{A}}}\right)^2$$

$$\alpha = exp\left(\frac{-1}{2}\left(\frac{m_{\tilde{A}} - \ln x}{\sigma_{\tilde{A}}}\right)^2\right) \quad ; \quad 0 < x \le exp(m_{\tilde{A}})$$

Similarly, for the right interval value, we get,

$$\alpha = exp\left(\frac{-1}{2}\left(\frac{-m_{\tilde{A}} + \ln x}{\sigma_{\tilde{A}}}\right)^2\right) \quad ; \quad x \ge exp(m_{\tilde{A}})$$

8

Therefore,

$$e^{\tilde{A}} = \begin{cases} exp\left(\frac{-1}{2}\left(\frac{m_{\tilde{A}} - \ln x}{\sigma_{\tilde{A}}} \right)^2 \right) & \text{if } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Example : Let $\tilde{A}$ be a Gaussian fuzzy number defined on $\mathbb{R}$ with $\sigma_{\tilde{A}} = 1$, $m_{\tilde{A}} = 5$. We get membership function of $\tilde{A}$ as $\mu_{\tilde{A}}(x) = e^{\left(\frac{-(x-5)^2}{2(1)^2} \right)}$. On exponentiation of $\tilde{A}$, we get $\mu_{e^{\tilde{A}}} = exp\left(\frac{-1}{2}\left(\frac{5 - \ln x}{1} \right)^2 \right)$; $x > 0$. Figure 7 shows plots of $\tilde{A}$ and $e^{\tilde{A}}$.
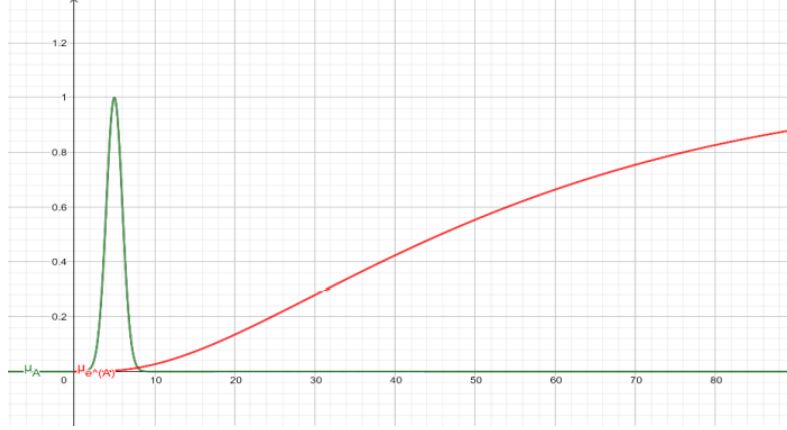

Figure 7: Graphs of $\mu_{\tilde{A}}(x) = e^{\left(\frac{-(x-5)^2}{2(1)^2} \right)}$, and $\mu_{ln(\tilde{A})} = exp\left(\frac{-1}{2}\left(\frac{5 - ln(x)}{1} \right)^2 \right)$

3.8   Natural Logarithm of Gaussian Fuzzy number

Consider Gaussian fuzzy number $\tilde{A}$ with mean m and standard deviation $\sigma_{\tilde{A}}$.
α-cut of $\tilde{A}$ is $\tilde{A}_\alpha = \left[m_{\tilde{A}} - \sigma_{\tilde{A}}\sqrt{-2\ln\alpha},\ m_{\tilde{A}} + \sigma_{\tilde{A}}\sqrt{-2\ln\alpha} \right]$.
To calculate $\ln(\tilde{A})$ we first take the natural logarithm of the α-cut.

$$(ln(\tilde{A}))_\alpha = ln(\tilde{A}_\alpha) = [ln(m_{\tilde{A}} - \sigma_{\tilde{A}}\sqrt{-2\ln\alpha}), ln(m_{\tilde{A}} + \sigma_{\tilde{A}}\sqrt{-2\ln\alpha})] \quad \alpha \in [0,1]$$

The next step is to fuzzify the obtained interval. Let

$$x = ln(m_{\tilde{A}} - \sigma_{\tilde{A}}\sqrt{-2\ln\alpha})$$

$$e^x = m_{\tilde{A}} - \sigma_{\tilde{A}}\sqrt{-2\ln\alpha}$$

$$\ln\alpha = \frac{-1}{2}\left(\frac{m_{\tilde{A}} - e^x}{\sigma_{\tilde{A}}} \right)^2$$

$$\alpha = exp\left(\frac{-1}{2}\left(\frac{m_{\tilde{A}} - e^x}{\sigma_{\tilde{A}}} \right)^2 \right) \quad ; \quad \forall x \in \mathbb{R}$$

We get,

$$ln(\tilde{A}) = \left\{ exp\left(\frac{-1}{2}\left(\frac{m_{\tilde{A}} - e^x}{\sigma_{\tilde{A}}} \right)^2 \right) \quad \forall x \in \mathbb{R} \right.$$

Example : Let $\tilde{A}$ be a Gaussian fuzzy number defined on $\mathbb{R}$ with $\sigma_{\tilde{A}} = 1$, $m_{\tilde{A}} = 10$. We get membership function of $\tilde{A}$ as $\mu_{\tilde{A}}(x) = e^{\left(\frac{-(x-10)^2}{2(1)^2} \right)}$. On taking logarithm of $\tilde{A}$, we get $\mu_{ln(\tilde{A})} = exp\left(\frac{-1}{2}\left(\frac{10 - e^x}{1} \right)^2 \right) \quad \forall x \in \mathbb{R}$. Figure 8 shows plots of $\tilde{A}$ and $ln(\tilde{A})$.
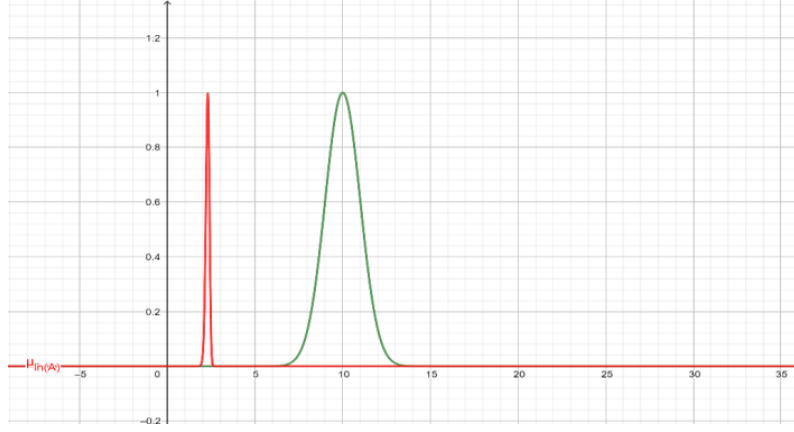
Figure 8: Graphs of $\mu_{\tilde{A}}(x) = e^{\left(\frac{-(x-10)^2}{2(1)^2}\right)}$, and $\mu_{ln(\tilde{A})} = exp\left(\frac{-1}{2}\left(\frac{10-e^x}{1}\right)^2\right)$

4 Conclusion

In this paper, we have discussed the definiton of the Gaussian Fuzzy Number and also shown its representation using the L-R Fuzzy Number. We have used the α-cut method to evalutate various arithmetic operations on the Gaussian fuzzy number.

References

[JZW16] Fan Yang Jian Zhou and Ke Wang. Fuzzy arithmetic on lr fuzzy numbers with applications to fuzzy programming. *Journal of Intelligent Fuzzy Systems*, pages 71–87, 2016.

[KY] George J. Klir and Bo Yuan. Fuzzy sets and fuzzy logic - theory and applications. pages 97–108.

[Zim] H. J. Zimmerman. Fuzzy set theory-and its applications. pages 59–68.