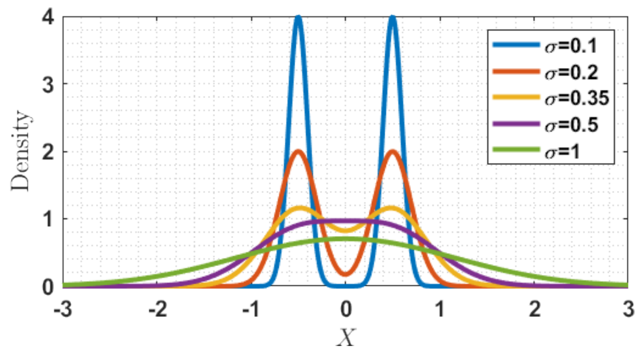
Homework 7

Spring 2024
(Due: Mar 1, 2024 Friday)

Name: ______________________ Email: ______________________

Homework is due at 11:59pm (midnight) Eastern Time. Please print this homework, write your solution, and scan the solution. Submit your homework through Gradescope. No late homework will be accepted.

Exercise 1. Separating two Gaussians
Suppose you have two Gaussians located at μ and $-\mu$ respectively, i.e., Gaussian$(x;\mu,\sigma^2)$ and Gaussian$(x;-\mu,\sigma^2)$. We know that if μ is large enough, then the two Gaussians will be separated. We also know that if σ is small enough, the two Gaussians will be separated. Fix μ. What is the largest σ you can have before the two Gaussian become merged together? By merge, I mean that you will no longer be able to see the two peaks. The figure below is an example for $\mu = 0.5$. As you can see, when $\sigma = 0.5$, the two Gaussians cannot be distinguished. Use paper and pencil to answer this question, and use Python/MATLAB to verify your answer.



Exercise 2.
Define a zero-mean Gaussian function as

$$\phi(x; 0, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}. \tag{1}$$

We want to approximate the Gaussian function using a boxcar function

$$\varphi(x; W) = \begin{cases} \frac{1}{W}, & -\frac{W}{2} \leq x \leq \frac{W}{2}, \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

Find the best W that can achieve this. Hint: Go to Wikipedia and learn the concept of KL-divergence.

Practice Me! Do not Hand in.

Exercise 3. LEON GARCIA 4.70
A Gamma distribution has a PDF

$$f_X(x) = \frac{\lambda(\lambda x)^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)}, \qquad x > 0,$$

where $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$, for $z > 0$ is the Gamma function.

(a) Find $\mathbb{E}[X]$.

(b) Find $\mathbb{E}[X^2]$ and $\text{Var}[X]$.

(c) If $\alpha = 1$, what is the resulting distribution?

(d) If $\lambda = 1/2$, $\alpha = k/2$, find $\mathbb{E}[X]$ and $\text{Var}[X]$. This is called the *chi-square* distribution.

(e) If $\alpha = m$, find $\mathbb{E}[X]$ and $\text{Var}[X]$. This is called the *m-Erlang* distribution.

Hint: You may use the facts that

$$\begin{aligned} \Gamma(1/2) &= \sqrt{\pi}, \\ \Gamma(z+1) &= z\Gamma(z), \text{ for } z > 0 \\ \Gamma(m+1) &= m! \text{ for} m \in \mathbb{N}^+ \end{aligned}$$

Exercise 4.
A random variable X has a PDF
$$f_X(x) = ae^x e^{-ae^x}, 0 \le x < \infty.$$
Let $Y = e^X$. Find $f_Y(y)$, for $-\infty < y < \infty$.

Exercise 5.
The random variable X has the PDF
$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$
Let Y be a new random variable
$$Y = \begin{cases} 0, & X < 0, \\ \sqrt{X}, & 0 \le X \le 1, \\ 1, & X > 1. \end{cases}$$

Exercise 6.
Compute $\mathbb{E}[Y]$ and $\mathbb{E}[Y^2]$ for the following random variables:

(a) $Y = A\cos(\omega t + \theta)$, where $A \sim \mathcal{N}(\mu,\, \sigma^2)$.

(b) $Y = a\cos(\omega t + \Theta)$, where $\Theta \sim \text{Uniform}(0,\, 2\pi)$.

(c) $Y = a\cos(\omega T + \theta)$, where $T \sim \text{Uniform}\left(-\frac{\pi}{\omega}, \frac{\pi}{\omega}\right)$.

Exercise 7.
Let X be a Poisson random variable with PMF

$$p_X(k) = \frac{\lambda^k}{k!}e^{-\lambda}, \quad k = 0, 1, \ldots$$

Let $Y = \cos(\pi X)$. Find $\text{Var}[Y]$.

Exercise 8.

A random variable X has the PDF

$$f_X(x) = ce^{-2x}, \qquad x \geq 0.$$

Let $g(\cdot)$ be the function:

$$g(x) = \begin{cases} x, & 0 \leq x < 1, \\ 1, & x \geq 1. \end{cases}$$

(a) Determine c.

(b) Find and sketch the CDF of $Y = g(X)$.

(c) Calculate $\mathbb{E}[Y]$ without computing the PDF.

Exercise 9.
Consider a transmission system which sends a binary signal $\{+1, -1\}$ with equal probability $\mathbb{P}[X = +1] = \mathbb{P}[X = -1] = \frac{1}{2}$. Let N be a noise random variable with PDF

$$f_N(x) = \begin{cases} \alpha^2 x e^{-\alpha x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Let $Y = X + N$ be the received signal. The receiver implements a simple detection rule as follows:

- Say $X = +1$ if $Y \geq \ell$, where ℓ is a parameter that you need to determine;
- Say $X = -1$ if $Y < \ell$.

Suppose that $\alpha = \ln \sqrt{3}$.

(a) Show that the probability of detection error is

$$P_e = C\left(1 - \int_A^B \alpha^2 x e^{-\alpha x} dx\right).$$

What are A, B, C?

(b) Determine ℓ that minimizes the probability of error.

Exercise 10.
Let X be a continuous random variable with PDF

$$f_X(x) = 2xe^{-x^2}, \qquad x \geq 0. \tag{3}$$

Consider a function

$$g(X) = 1 - e^{-X^2}, \qquad X \geq 0.$$

If $Y = g(X)$, find the PDF of Y. Specify the range for which $f_Y(y)$ is zero.

Exercise 11.
Let X be a continuous random variable with PDF

$$f_X(x) = \frac{1}{2}e^{-|x|}, \qquad -\infty < x < \infty. \tag{4}$$

Consider a function

$$g(X) = e^{-X}, \qquad -\infty < x < \infty.$$

If $Y = g(X)$, find the PDF of Y. Specify the range for which $f_Y(y)$ is zero.

Exercise 12.
Let X be a continuous random variable with PDF

$$f_X(x) = (1+x)^{-2}, \qquad x \geq 0. \tag{5}$$

Consider a function

$$g(X) = \frac{X}{1+X}, \qquad \text{for all } X.$$

If $Y = g(X)$, find the PDF of Y. Specify the range for which $f_Y(y)$ is zero.

Exercise 13.
Let X be a continuous random variable with PDF

$$f_X(x) = \frac{1}{x^2}, \qquad x \geq 1. \tag{6}$$

Consider a function

$$g(X) = \frac{X}{1+X}, \qquad \text{for all } X.$$

If $Y = g(X)$, find the PDF of Y. Specify the range for which $f_Y(y)$ is zero.