CMSC 351 Spring 2022 Exam 2

Version Tue/Thu

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1. Mark each of the following as true or false. Write the word TRUE or FALSE in the box to the right. No justification is required. [10 pts]

(a)	MergeSort is stable.	TRUE
(b)	A list with three elements can be sorted with just two comparisons.	FALSE
(c)	The Master Theorem applies to $T(n) = T(n/2) + f(n)$ with $f(n) = \mathcal{O}(n)$.	FALSE
(d)	In a max heap the smallest element must be a leaf.	TRUE
(e)	If a list is $[5, 8, 3, 7, 10, 11]$ after the first iteration of QuickSort, the pivot value could have been 7.	FALSE

2. Suppose a list A with n elements contains integers between 0 and $2^{(n^2)}-1$ inclusive. Using base 2 to represent these numbers and RadixSort + CountingSort, calculate the Θ time complexity. [15 pts]

Solution:

If we use base $b = 2$ then we'll need $d = n^2$ digits to represent all of 0 through $2^{(n^2)} - 1$. Then the underlying CountingSort has $k = \max = 1$ and is therefore $\Theta(n^2 + 1)$ and so the resulting RadixSort will be $\Theta(n^2(n + 1))$.

3. Apply the Master Theorem to solve the following recurrence relation: [15 pts]

$$T(n) = 2\,T(n/4) + \sqrt{n} + 5$$

Solution:

We have $\log_b a = \log_4 2 = \frac{1}{2}$ and $\sqrt{n} + 5 = \Theta(n^{1/2})$ so $c = 1/2$ and so $\log_b a = c$.

The Master Theorem then tells us that the result is $\Theta(n^{1/2} \lg n)$.

4. Apply the Master Theorem to solve the following recurrence relation: [15 pts]

$$T(n) = 5\,T(n/4) + \lg n$$

Solution:

We have $\log_b a = \log_4 5 > 1$ and $\lg n = \mathcal{O}(n^1)$ so $c = 1$ and so $\log_b a > c$.

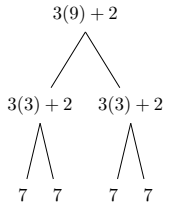
The Master Theorem then tells us that the result is $\Theta(n^{\log_4 5})$.

5. Consider the recurrence relation:

$$T(n) = 2T(n/3) + 3n + 2 \quad \text{with} \quad T(1) = 7$$

(a) Draw the complete recursion tree for $T(9)$ with its values filled in. [10 pts]

Solution:



(b) From (a) what is the value of $T(9)$? [5 pts]

Solution:

The total time would be the sum of all the values, so:

$$29 + 11 + 11 + 7 + 7 + 7 + 7 = 79$$

6. Here is the pseudocode for the partition function from QuickSort in case you need it: [10 pts]

```
function partition(A,L,R)
    pivotvalue = A[R]
    t = L
    for i = L to R-1
        if A[i] <= pivotvalue
            A[t] <-> A[i]
            t = t + 1
        end
    end
    A[t] <-> A[R]
    return t
end
```

Consider the list:

`[5, 8, 3, 4, 10, 7]`

If `partition(A,0,6)` is run on this list, show the result after each swap which actually exchanges different elements.

Solution:

We have:

After the first swap: `[5, 3, 8, 4, 10, 7]`

After the second swap: `[5, 3, 4, 8, 10, 7]`

After the third swap: `[5, 3, 4, 7, 10, 8]`

7. Suppose you have a heap which has N layers and is full on every layer, thus it contains $n = 2^N - 1$ nodes. You select a node uniformly at random and run `maxheapify` on it. If a swap takes 2 seconds and nothing else takes any time at all, what is the expected value for the time it takes `maxheapify` to run in the worst case? You can leave your answer as a summation including n but it should not include N. [20 pts]

Solution:

In a worst case `maxheapify` will need to swap all the way to the leaves.

- There is 1 node in layer 1 and so there is a $\frac{1}{n}$ probability of choosing such an node and in the worst case this will require $N-1$ swaps, all the way to the bottom.
- There are 2 nodes in layer 2 and so there is a $\frac{2}{n}$ probability of choosing such a node and in the worse case this will require $N-2$ swaps, all the way to the bottom.
- There are 4 nodes in layer 3 and so there is a $\frac{4}{n}$ probability of choosing such a node and in the worse case this will require $N-3$ swaps, all the way to the bottom.
- And so on ...

In general for $1 \leq k \leq N$ there are 2^{k-1} nodes in layer k and so there is a $\frac{2^{k-1}}{n}$ probability of choosing such a node and in the worst case this will require $N-k$ swaps and hence take time $2(N-k)$.

All together then the expected time will be:

$$\sum_{k=1}^{N} \frac{2^{k-1}}{n} (2(N-k)) = \sum_{k=1}^{\lg(n+1)} \frac{2^k}{n} (\lg(n+1) - k)$$

Alternate Solution:

There is a $\frac{1}{n}$ probability of choosing any particular node $1 \leq i \leq n$. We know that node i is in layer $1 + \lfloor \lg i \rfloor$ and that the maximum layer is $N = \lg(n+1)$. Consequently in a worse case there will be swaps all the way down, meaning $\lg(n+1) - (1 + \lfloor \lg i \rfloor)$ swaps. This means it will take $2\left(\lg(n+1) - (1 + \lfloor \lg i \rfloor)\right)$ seconds for node i and hence an expected value of:

$$\sum_{i=1}^{n} \frac{1}{n} \cdot 2\left(\lg(n+1) - (1 + \lfloor \lg i \rfloor)\right)$$

Alternate Solution Note:

Alternately in this case $\lg(n+1)$ may be replaced by $1 + \lfloor \lg n \rfloor$ if the student used that fact that node n is in layer $1 + \lfloor \lg n \rfloor$. Then it would simplify to:

$$\sum_{i=1}^{n} \frac{1}{n} \cdot 2\left(\lfloor \lg n \rfloor - \lfloor \lg i \rfloor\right)$$