Math 133 # Taylor Series **Stewart §11.10**

Series representation of a function. The main purpose of series is to write a given complicated quantity as an infinite sum of simple terms; and since the terms get smaller and smaller, we can approximate the original quantity by taking only the first few terms of the series. In this section, we finally develop the tool that lets us do this in most cases: a way to write any reasonable function as an explicit power series. This will allow us to compute outputs of the function by plugging into the series.

Our functions must behave decently near the center point of the desired power series. We say $f(x)$ is *analytic* at $x = a$ if it is possible to write $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for some coefficients c_n, with positive radius of convergence. In practice, any formula involving standard functions and operations defines an analytic function, provided the formula gives real number values in a small interval around $x = a$. For example $\frac{1}{x-a}$ is *not* analytic at $x = a$, because it gives $\pm\infty$ at $x = a$; and $\sqrt{x-a}$ is *not* analytic at $x = a$ because for x slightly smaller than a, it gives the square root of a negative number.*

> *Taylor Series Theorem:* Let $f(x)$ be a function which is analytic at $x = a$. Then we can write $f(x)$ as the following power series, called the *Taylor series* of $f(x)$ at $x = a$:
>
> $$f(x) \;=\; f(a)+f'(a)\,(x-a)+\frac{f''(a)}{2!}\,(x-a)^2+\frac{f'''(a)}{3!}\,(x-a)^3+\frac{f''''(a)}{4!}\,(x-a)^4+\cdots,$$
>
> valid for x within a radius of convergence $|x-a| < R$ with $R > 0$, or convergent for all x.
>
> If we write the nth derivative of $f(x)$ as $f^{(n)}(x)$, this becomes:
>
> $$f(x) \;=\; \sum_{n=0}^{\infty} c_n(x-a)^n \qquad \text{with coefficients} \qquad c_n = \frac{f^{(n)}(a)}{n!}\,.$$

WARNING: The coefficients are *constants* with no x, so $c_1 = f'(a)$, NOT $f'(x)$.

Proof. Given that $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for *some* c_n, we will derive the desired formula for these coefficients. Since $f(a) = \sum_{n=0}^{\infty} c_n(a-a)^n = c_0 + c_1(0) + c_2(0^2) + \cdots$, we get $c_0 = f(a)$. Next, by the Theorem in §11.9, we have $f'(x) = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$, so $f'(a) = c_1 + 2c_2(0) + 3c_3(0^2) + \cdots$, and $c_1 = f'(a)$. Next, $f''(x) = \sum_{n=0}^{\infty} n(n-1)c_n(x-a)^{n-2}$, so $f''(a) = (2)(1)c_2$ and $c_2 = \frac{1}{2}f''(a)$. Continuing, we get:

$$f^{(N)}(x) \;=\; \sum_{n=1}^{\infty} n(n-1)\cdots(n-N+1)\,c_n\,(x-a)^{n-N}\,.$$

Notes by Peter Magyar `magyar@math.msu.edu`

*The function $\sqrt[3]{x-a}$ is also not analytic near $x = a$, ever though it gives real number values. The problem is that it has a vertical tangent at $x = a$, so it is not differentiable.

The terms for $n=0,1,\ldots,N{-}1$ are all zero because of the factors $n(n{-}1)\cdots(n{-}N{+}1)$, so the first non-zero term is for $n=N$. Plugging in $x=a$ gives: $f^{(N)}(a) = N(N{-}1)\cdots(1)c_N$, and $c_N = \frac{1}{N!}f^{(N)}(a)$ as desired. Q.E.D.

Once we have a power series for $f(x)$ with known coefficients $c_n = \frac{f^{(n)}(a)}{n!}$, we can approximate $f(x)$ by taking a finite partial sum of the series up to some cutoff term N. This partial sum is called a *Taylor polynomial*, denoted $T_N(x)$:

$$f(x) \;\approx\; T_N(x) \;=\; \sum_{n=0}^{N} c_n(x{-}a)^n = f(a) + f'(a)(x{-}a) + \cdots + \frac{f^{(N)}(a)}{N!}(x{-}a)^N\,.$$

Note that $T_1(x) = f(a) + f'(a)(x-a)$ is just the linear approximation near $x=a$, whose graph is the tangent line (Calculus I §2.9). We can improve this approximation of $f(x)$ in two ways:

- Take more terms, increasing N.
- Take the center a close to x, giving small $(x{-}a)$ and tiny $(x{-}a)^n$.

A Taylor series centered at $a=0$ is specially named a *Maclaurin series.*

Example: sine function. To find Taylor series for a function $f(x)$, we must determine $f^{(n)}(a)$. This is easiest for a function which satisfies a simple differential equation relating the derivatives to the original function. For example, $f(x)=\sin(x)$ satisfies $f''(x) = -f(x)$, so coefficients of the Maclaurin series (center $a=0$) are:

n	0	1	2	3	4	5	6	7
$f^{(n)}(x)$	$\sin(x)$	$\cos(x)$	$-\sin(x)$	$-\cos(x)$	$\sin(x)$	$\cos(x)$	$-\sin(x)$	$-\cos(x)$
$f^{(n)}(0)$	0	1	0	-1	0	1	0	-1
$c_n = \frac{f^{(n)}(0)}{n!}$	0	1	0	$-\frac{1}{3!}$	0	$\frac{1}{5!}$	0	$-\frac{1}{7!}$

That is:

$$\sin(x) \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \;=\; \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n{+}1)!}\,.$$

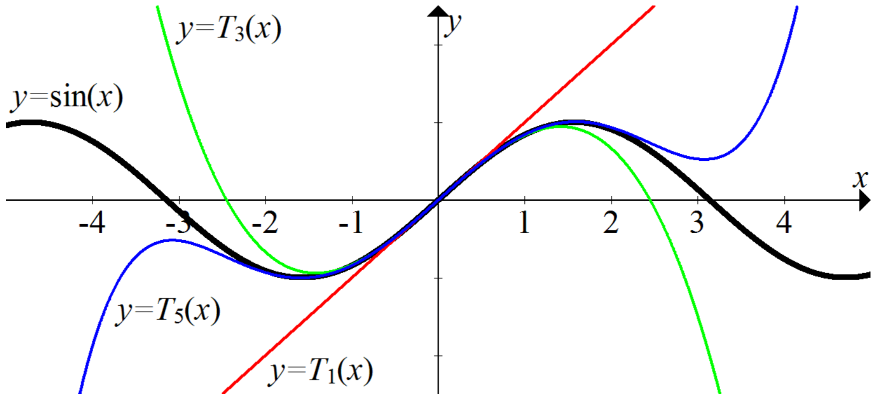
To find the domain of convergence, we apply the Ratio Test (11.6/I):

$$L \;=\; \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \;=\; \lim_{n\to\infty}\left|\frac{x^{2(n+1)+1}}{(2(n{+}1){+}1)!} \Bigg/ \frac{x^{2n+1}}{(2n{+}1)!}\right|$$

$$=\; \lim_{n\to\infty}\frac{|x|^{2n+3}}{|x|^{2n+1}}\cdot\frac{(2n{+}1)!}{(2n{+}3)!} \;=\; \lim_{n\to\infty}\frac{|x|^2}{(2n{+}2)(2n{+}3)} \;=\; 0$$

for any fixed $x \neq 0$. Since $L = 0 < 1$ regardless of x, the series converges for all x.

This formula for $\sin(x)$ astonishes us because the right side is a simple algebraic series having no apparent relation to trigonometry. We can try to understand and check the series by graphically comparing $\sin(x)$ with its first few Taylor polynomial approximations:



- The Taylor polynomial $T_1(x) = x$ (in red) is just the linear approximation or tangent line of $y = \sin(x)$ at the center point $x = 0$. The curve and line are close (to within a couple of decimal places) near the point of tangency and up to about $|x| \le 0.5$. Once they veer apart, the approximation is useless.

- The next Taylor polynomial $T_3(x) = x - \frac{x^3}{3!} = x - \frac{1}{6}x^3$ (in green) matches $y = \sin(x)$ in its first three derivatives at $x = 0$, and stays close to the original curve up to about $|x| \le 1.5$.

- The next $T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ is even closer to $f(x)$ for even larger x. Taking enough terms in the Taylor series will give a good approximation for *any* x, since the series converges everywhere.

PROBLEM: Compute $\sin(10°)$. A geometric method would be to construct a right triangle with a $10°$ angle, and measure the opposite side divided by the hypotenuse; but this would only work for a couple of decimal places of accuracy. Of course, a calculator can produce many decimal places, but how does it know? Taylor series!

As always when doing calculus on trig functions, we must first convert to radians (see end of §2.5): $10° = \frac{2\pi}{360}(10) = \frac{\pi}{18}$. Here $|x| = \frac{\pi}{18} \approx \frac{1}{6}$ is small, so the Maclaurin series centered at 0 should converge very quickly, giving high-accuracy approximations. That is:

$$\sin(\tfrac{\pi}{18}) \;\approx\; T_3(\tfrac{\pi}{18}) \;=\; \tfrac{\pi}{18} - \tfrac{1}{6}(\tfrac{\pi}{18})^3 \;\approx\; \underline{0.17364}68\,.$$

It turns out this is correct to 5 decimal places (underlined), using only two non-zero terms of the Taylor series and a good estimate for π. We could verify this by taking more terms and seeing that these 5 digits do not change; or by the Remainder Estimates below.

Example: square roots. We compute $\sqrt{2}$ to 5 decimal places.[†] First, we must consider $\sqrt{2}$ to be an output of the function $f(x) = \sqrt{x}$ at $x = 2$. Next, we must choose the center a for its Taylor series.

- $a = 0$ does not work because $\sqrt{x}$ is not analytic at $x = 0$. Indeed, if there were a convergent Taylor series $\sqrt{x} = c_0 + c_1 x + c_2 x^2 + \cdots$, we could plug in $x = -0.1$ to get: $\sqrt{-0.1} = c_0 + c_1(-0.1) + c_2(-0.1)^2 + \cdots$, a real value for the square root of a negative number!

- $a = 1$ is too far from $x = 2$: it turns out $|x{-}a| = |2{-}1| = 1$ is beyond the radius of convergence of the Taylor series.

- $a = 2$ is useless, since writing the Taylor series requires us to know $f^{(n)}(2)$, including $f(2) = \sqrt{2}$, the same number we are trying to compute.

- A useful choice of a requires: $a > 0$ so that the Taylor series exists; a is close to $x = 2$, making $|x{-}a|$ small so the series converges quickly; and $f(a) = \sqrt{a}$ is easy to compute so we can find the coefficients. A value satisfying all three conditions is: $a = \frac{9}{4}$.

Now we have:

n	0	1	2	3	4
$f^{(n)}(x)$	$x^{1/2}$	$\frac{1}{2}\, x^{-1/2}$	$-\frac{1\cdot 1}{2\cdot 2}\, x^{-3/2}$	$\frac{1\cdot 1\cdot 3}{2\cdot 2\cdot 2}\, x^{-5/2}$	$-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 2\cdot 2\cdot 2}\, x^{-7/2}$
$f^{(n)}(\frac{9}{4})$	$\frac{3}{2}$	$\frac{1}{3}$	$-\frac{2}{27}$	$\frac{4}{81}$	$-\frac{40}{729}$
$c_n = \frac{f^{(n)}(\frac{9}{4})}{n!}$	$\frac{3}{2}$	$\frac{1}{3}$	$-\frac{1}{27}$	$\frac{2}{243}$	$-\frac{5}{2187}$

Hence:

$$\begin{aligned}\sqrt{x} \;&=\; \tfrac{3}{2} + \tfrac{1}{3}(x{-}\tfrac{9}{4}) - \tfrac{1}{27}(x{-}\tfrac{9}{4})^2 + \tfrac{2}{243}(x{-}\tfrac{9}{4})^3 - \tfrac{5}{2187}(x{-}\tfrac{9}{4})^4 + \cdots \\ &=\; \tfrac{3}{2} + \tfrac{1}{3}(x{-}\tfrac{9}{4}) + \sum_{n=2}^{\infty}(-1)^{n-1}\,\tfrac{(2n-3)!!}{n!}\,\tfrac{2^{n-1}}{3^{2n-1}}\,(x{-}\tfrac{9}{4})^n\,,\end{aligned}$$

where we use the odd factorial notation $(2n{-}3)!! = (1)(3)(5)\cdots(2n{-}3)$. For $x = 2$, we have $x{-}\frac{9}{4} = -\frac{1}{4}$, so:

$$\begin{aligned}\sqrt{2} \;&=\; \tfrac{3}{2} + \tfrac{1}{3}(-\tfrac{1}{4}) - \tfrac{1}{27}(-\tfrac{1}{4})^2 + \tfrac{2}{243}(-\tfrac{1}{4})^3 - \tfrac{5}{2187}(-\tfrac{1}{4})^4 + \cdots\,, \\ &\approx\; \tfrac{3}{2} - \tfrac{1}{3}\,\tfrac{1}{4} - \tfrac{1}{27}\,\tfrac{1}{4^2} - \tfrac{2}{243}\,\tfrac{1}{4^3} - \tfrac{5}{2187}\,\tfrac{1}{4^4} \;\approx\; \underline{1.41421}43\,,\end{aligned}$$

which is correct to 5 decimal places (underlined).

[†]We saw another very good algorithm for this in Calculus I §3.8: Newton's Method, in which we found approximate solutions to equations like $x^2 - 2 = 0$ by repeatedly taking a linear approximation to $f(x) = x^2 - 2$.

Common Taylor series

- $\dfrac{1}{1-x} \;=\; \displaystyle\sum_{n=0}^{\infty} x^n$ for $|x| < 1$ (*Geometric Series*).

- $\ln(1+x) \;=\; \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$ for $|x| < 1$.

- $(1+x)^p \;=\; \displaystyle\sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!}\,x^n$ for $|x| < 1$ (*Binomial Series*).

- $\exp(x) \;=\; \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

- $\sin(x) \;=\; \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x.

- $\cos(x) \;=\; \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ for all x.

Bounding the remainder to determine accuracy. For a function with Taylor series $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, we define the *remainder term* as the difference between a function and its Taylor polynomial approximation:

$$R_N(x) \;=\; f(x) - T_N(x) \;=\; \sum_{n=N+1}^{\infty} c_n (x-a)^n.$$

That is, $f(x) = T_N(x) + R_N(x)$, so that $R_N(x)$ is the error in the approximation $f(x) \approx T_N(x)$.

> *Lagrange Remainder Formula:* For any Taylor polynomial approximation $f(x) = T_N(x) + R_N(x)$, the remainder term is equal to:
>
> $$R_N(x) \;=\; \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$
>
> for some point c between a and x.

This allows an *a priori* estimate of the error, provided we can find an upper bound for the derivative: if $|f^{(N+1)}(t)| \le M$ for all $t \in [a, x]$ or $[x, a]$, then:

$$|R_N(x)| \;\le\; \max_{t\in[a,x]} \left| \frac{f^{(N+1)}(t)}{(N+1)!}(x-a)^{N+1} \right| \;\le\; \frac{M}{(N+1)!}\,|x-a|^{N+1},$$

since we can apply the bound $|f^{(N+1)}(t)| \le M$ to $t = c$ in the Lagrange Remainder Formula. This generalizes the error estimate for the linear approximation (Calculus I §2.9 end and §3.2 end). Note the similarity of the error expression $\frac{1}{(N+1)!} f^{(N+1)}(t)\,(x-a)^{N+1}$ to the next term in the Taylor series, $\frac{1}{(N+1)!} f^{(N+1)}(a)\,(x-a)^{N+1}$. We give a partial proof at the end of this section.

EXAMPLE: We previously computed $\sin(\frac{\pi}{18}) = T_3(\frac{\pi}{18}) + R_3(\frac{\pi}{18})$, centered at $a = 0$. We have the upper bound:

$$f^{(N+1)}(t) \;=\; \sin^{(4)}(t) \;=\; \sin(t) \;\leq\; M \;=\; 1 \quad \text{for} \quad t \in [0, \tfrac{\pi}{18}]\,.$$

Thus, the error term is at most:

$$\left|R_3(\tfrac{\pi}{18})\right| \;\leq\; \tfrac{M}{(N+1)!}\,|x{-}a|^{N+1} \;=\; \tfrac{1}{4!}(\tfrac{\pi}{18})^4 \;\approx\; 4\times 10^{-5}\,.$$

Approximation to n decimal places means with error smaller than 0.5×10^{-n}, so our approximation is accurate to at least 4 places (though it is actually 5 places).

EXAMPLE: We previously computed $\sqrt{2} = T_4(2) + R_4(2)$, centered at $a = \frac{9}{4}$. We have the upper bound:

$$|f^{(N+1)}(t)| \;=\; |\tfrac{d^5}{dt^5}(t^{1/2})| \;=\; \tfrac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 2\cdot 2\cdot 2\cdot 2}\, t^{-9/2} \;\leq\; \tfrac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 2\cdot 2\cdot 2\cdot 2}\, 2^{-9/2} \;\leq\; M \;=\; \tfrac{1}{5}$$

for $t \in [2, \frac{9}{4}]$: we plug in the left endpoint $t = 2$ since $t^{-9/2}$ is a decreasing function. Thus, the error term is at most:

$$|R_4(2)| \;\leq\; \tfrac{M}{(N+1)!}\,|x{-}a|^{N+1} \;=\; \tfrac{1/5}{5!}\,\left|2{-}\tfrac{9}{4}\right|^5 \;\approx\; 2\times 10^{-6}\,.$$

Our approximation is accurate to at least 5 decimal places.

EXAMPLE: The function $f(x) = e^{-1/x^2}$ is not analytic at $x = 0$, since $1/x^2$ is undefined at that point. However, we can easily check that $\lim_{x\to 0} f(x) = 0$, so $x = 0$ is a removable discontinuity (§1.8); and in fact $\lim_{x\to 0} f^{(n)}(x) = 0$. Thus the Taylor Series Theorem would give $c_n = \frac{1}{n!}f^{(n)}(0) = 0$, but this would give the trivial Taylor series $f(x) \overset{??}{=} 0 + 0x + 0x^2 + \cdots$, which is clearly nonsense. This is because no matter how small $|x| \neq 0$, the remainder $R_N(x)$ does not go to zero as $N{\to}\infty$: the numerator $f^{(N+1)}(c)$ overwhelms $\frac{1}{(N+1)!}\,|x|^{N+1}$.

Proof of Remainder Bound. Write the Taylor polynomial at $x = a$ as: $T_a(x) = f(a) + f'(a)\,(x{-}a) + \cdots + \frac{f^{(n)}(a)}{n!}(x{-}a)^N$. Now fix a constant value of x and let a vary, replacing a with the variable t. Then the remainder function is:

$$\epsilon(t) \;=\; f(x) - T_t(x) \;=\; f(x) - f(t) - f'(t)(x{-}t) - \frac{f''(t)}{2!}(x{-}t)^2 - \cdots - \frac{f^{(N)}(t)}{N!}(x{-}t)^N.$$

Remembering that x is a constant with respect to the variable t, we find the derivative:

$$\begin{aligned}\epsilon'(t) \;=\; 0 - f'(t) - f'(t)(-1) - f''(t)(x{-}t) - \frac{f''(t)(-2)(x{-}t)}{2!} - \frac{f'''(t)(x{-}t)^2}{2!} \\ - \cdots - \frac{f^{(N)}(t)\,(-N)(x{-}t)^{N-1}}{N!} - \frac{f^{(N+1)}(t)\,(x{-}t)^N}{N!}.\end{aligned}$$

Every term cancels except the last, so $\epsilon'(t) = -\frac{f^{(N+1)}(t)\,(x-t)^N}{N!}$. Now we apply the Mean Value Theorem (§3.2) to $\epsilon(t)$ on the interval $t \in [a, x]$, which says: $\frac{\epsilon(x)-\epsilon(a)}{x-a} \;=\; \epsilon'(t)$ for some $t \in (a, x)$. Considering $\epsilon(x) = f(x) - f(x) = 0$, we have:

$$f(x) - T_a(x) \;=\; \epsilon(a) \;=\; -\epsilon'(t)\,(x{-}a) \;=\; \frac{f^{(N+1)}(t)}{N!}(x{-}t)^N(x{-}a).$$

Assuming $|f^{N+1)}(t)| < M$ for $t \in (a, x)$, and using $|x{-}t| < |x{-}a|$, we get a slightly weaker remainder bound than above:

$$|R_N(x)| \;=\; |f(x) - T_a(x)| \;<\; \frac{M}{N!}\,|x{-}a|^{N+1}$$

Extra Topic: Irrationality of e. In §11.2, we saw that repeating decimals represent rational numbers (fractions), and every fraction can be written by long division as a repeating decimal. Thus, the non-repeating infinite decimals are the real numbers which cannot be written as fractions: they are *irrational*. However, it is diffficult to prove that any given number (such as π or $\sqrt{2}$) is irrational.

We can use our series to prove the irrationality of the constant $e = 2.7182818284590\cdots$. To prove the negative proposition that e is not equal to any possible fraction $\frac{a}{b}$, we use the *method of contradiction*: that is, we assume that there *were* some fraction with $e = \frac{a}{b}$, and use this to deduce an impossible statement, which will show that the original assumption $e = \frac{a}{b}$ is also impossible.

Thus, using the Taylor series definition for e, we assume:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \overset{\text{def}}{=} e \overset{?}{=} \frac{a}{b}\,.$$

It is easy to show $2 < e < 3$, so e cannot be a whole number, and has denominator $b > 1$. We multiply by $b!$ to clear denominators up to the $\frac{1}{b!}$ term:

$$b! + \frac{b!}{1!} + \frac{b!}{2!} + \cdots + \frac{b!}{b!} + b!R_b \;=\; b!\,e \;=\; b!\,\frac{a}{b} \;=\; (b{-}1)!\,a\,,$$

where $b!R_b = \sum_{n \geq b+1} \frac{b!}{n!}$. The terms $b!, \frac{b!}{1!}, \frac{b!}{2!}, \ldots, \frac{b!}{b!}$ on the left are whole numbers, and $(b{-}1)!\,a$ on the right is a whole number, so the remainder $b!R_b$ must also be a whole number. But it must also be very small, as we can see from a simple geometric series estimate:[‡]

$$\begin{aligned} b!R_b &= \tfrac{1}{b+1} + \tfrac{1}{(b+1)(b+2)} + \tfrac{1}{(b+1)(b+2)(b+3)} + \cdots \\ &< \tfrac{1}{b+1} + \tfrac{1}{(b+1)^2} + \tfrac{1}{(b+1)^3} + \cdots \\ &= \tfrac{1}{b+1}\,\tfrac{1}{1-\frac{1}{b+1}} \;=\; \tfrac{1}{b+1}\,\tfrac{b+1}{b+1-1} \;=\; \tfrac{1}{b} \;<\; 1. \end{aligned}$$

Thus, the same positive number $b!R_b$ is both a whole number and less than 1, which is impossible. Thus the original assumption $e = \frac{a}{b}$ is also impossible.

[‡]We do not need the powerful Lagrange remainder formula here.