Module 1:

Vector Calculus

vector: A physical quantity which has both magnitude and direction.

Eg: velocity,

Dot product and cross product:

If $\vec{A} = a_1 i + a_2 j + a_3 k$ and $\vec{B} = b_1 i + b_2 j + b_3 k$ be two vectors then.

$$\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(or) $\vec{A} \cdot \vec{B} = |\vec{A}| \, |\vec{B}| \cos\theta .$

$$\Rightarrow \cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| \, |\vec{B}|} .$$

Note: If $\vec{A} = a_1 i + a_2 j + a_3 k$ then

$$|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

and unit vector $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|}$

Differentiation of Vector Valued function:

Let $\vec{r} = xi + yj + zk$ be the position vector, where x, y, z are functions of 't'. then its derivative is given by.

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k$$

Velocity & Acceleration:

velocity, $\vec{v} = \frac{d\vec{r}}{dt}$.

Acceleration, $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$.

Operators:

1. ∇ – vector differential operator.

and $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k = \Sigma \frac{\partial}{\partial x} i$

2. ∇^2 – Laplacian operator

and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \Sigma \frac{\partial^2}{\partial x^2}$.

Gradient of a Scalar:

If $\phi(x,y,z) = C$ be a scalar function then gradient of ϕ is defined as.

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

obviously, $\nabla\phi$ is a vector quantity.

Divergence of a vector:

If $\vec{F} = f_1 i + f_2 j + f_3 k$ be a vector then divergence of $\vec{F}$ is defined as.

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

obviously, $\nabla\cdot\vec{F}$ is a scalar quantity.

curl of a vector

If $\vec{F} = f_1 i + f_2 j + f_3 k$ be a vector then curl of a vector $\vec{F}$ is defined as.

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Clearly, $\nabla\times\vec{F}$ is a vector quantity.

Note:

1. vector normal to the surface ϕ is $\nabla\phi$.
& unit vector normal to surface is $\frac{\nabla\phi}{|\nabla\phi|}$

2. Directional derivative of surface ϕ along the director $\vec{D}$ is given by
$\nabla\phi \cdot \hat{n}$ where $\hat{n} = \frac{\vec{D}}{|\vec{D}|}$.

Problems:

1. Find unit vector normal to surface.
$x^3y + 2xz = 4$ at $(2, -2, 3)$.

Sol:- Given $\phi(x, y, z) = x^3y + 2xz$

we have,

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} K.$$

$$\nabla\phi = \frac{\partial}{\partial x}(x^3y + 2xz)i + \frac{\partial}{\partial y}(x^3y + 2xz)j + \frac{\partial}{\partial z}(x^3y + 2xz)K$$

$$\nabla\phi = (3x^2y + 2z)i + (x^3 + 0)j + (0 + 2x)K.$$

at $(2, -2, 3)$

$$\nabla\phi = (-24 + 6)i + 8j + 4K.$$

$\nabla\phi = -18i + 8j + 4K$ is vector normal to surface.

Unit vector normal to surface is

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{-18i+8j+4k}{\sqrt{(-18)^2+8^2+4^2}}$$

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{2[-9i+4j+2k)}{\sqrt{404}} = \frac{2(-9i+4j+2k)}{2\sqrt{101}}$$

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{\sqrt{101}}(-9i+4j+2k)$$

2. Find unit vector normal to the surface $xy^3z^2=4$ at $(-1,-1,2)$.

Sol:- Let $\phi = xy^3z^2$.

we have,

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\nabla\phi = \frac{\partial}{\partial x}(xy^3z^2)i + \frac{\partial}{\partial y}(xy^3z^2)j + \frac{\partial}{\partial z}(xy^3z^2)k.$$

$$\nabla\phi = y^3z^2i + 3xy^2z^2j + 2xy^3zk$$

at $(-1,-1,2)$

$$\nabla\phi = (-1)(4)i + 3(-1)(1)(4)j + 2(-1)(-1)(2)k$$

$\nabla\phi = -4i-12j+4k$ is vector normal to surface.

Then Unit vector normal to surface is

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{-4i-12j+4k}{\sqrt{(-4)^2+(-12)^2+4^2}} = \frac{4(-i-3j+k)}{\sqrt{176}}.$$

$$= \frac{\cancel{4}(-i-3j+k)}{\cancel{4}\sqrt{11}} = \frac{1}{\sqrt{11}}(-i-3j+k)$$

3. Find Unit vector normal to surface

$x^2y - 2xz + 2y^2z^4 = 10$ at $(2, 1, -1)$

Sol:- Let $\phi = x^2y - 2xz + 2y^2z^4$

WKT,

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}K$$

$$\nabla\phi = \frac{\partial}{\partial x}(x^2y - 2xz + 2y^2z^4)i + \frac{\partial}{\partial y}(x^2y - 2xz + 2y^2z^4)j$$
$$+ \frac{\partial}{\partial z}(x^2y - 2xz + 2y^2z^4)K$$

$$\nabla\phi = (2xy - 2z)i + (x^2 + 4yz^4)j + (-2x + 8y^2z^3)K$$

at $(2, 1, -1)$

$$\nabla\phi = \left(2(2)(1) - 2(-1)\right)i + \left[4 + 4(1)(1)\right]j + \left[-2(2) + 8(1)(-1)\right]K$$

$\nabla\phi = 6i + 8j - 12K$ is a vector normal to surface.

Now, unit vector normal to surface is

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{6i + 8j - 12K}{\sqrt{6^2 + 8^2 + (-12)^2}}$$

$$= \frac{2[3i + 4j - 6K]}{\sqrt{244}}$$

$$= \frac{2[3i + 4j - 6K]}{2\sqrt{61}}$$

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{\sqrt{61}}[3i + 4j - 6K]$$

Scanned with CamScanner

4. Find the angle between surfaces.

$x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at $(2,-1,2)$.

Sol: note that angle b/n surfaces is equal to angle b/n vector normal to surfaces.

Let $\phi_1=x^2+y^2+z^2$ and $\phi_2=x^2+y^2-z$.

we have, by the defn of gradient

$$\nabla\phi=\frac{\partial\phi}{\partial x}i+\frac{\partial\phi}{\partial y}j+\frac{\partial\phi}{\partial z}k.$$

$$\nabla\phi_1=\frac{\partial}{\partial x}(x^2+y^2+z^2)i+\frac{\partial}{\partial y}(x^2+y^2+z^2)j+\frac{\partial}{\partial z}(x^2+y^2+z^2)k$$

$$\nabla\phi_1=2x\hat{i}+2y\hat{j}+2z\hat{k}$$

at $(2,-1,2)$

$$\nabla\phi_1=2(2)i+2(-1)j+2(2)k$$

$$\nabla\phi_1=4i-2j+4k.$$

Now,

$$\nabla\phi_2=\frac{\partial}{\partial x}(x^2+y^2-z)i+\frac{\partial}{\partial y}(x^2+y^2-z)j+\frac{\partial}{\partial z}(x^2+y^2-z)k.$$

$$\nabla\phi_2=2xi+2yj-k$$

at $(2,-1,2)$

$$\nabla\phi_2=2(2)i+2(-1)j-k$$

$$\nabla\phi_2=4i-2j-k$$

Angle between the surfaces is

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

$$\cos\theta = \frac{[4i-2j+4K]\cdot[4i-2j-K]}{\sqrt{4^2+(-2)^2+4^2}\,\sqrt{4^2+(-2)^2+(-1)^2}}$$

$$\cos\theta = \frac{4(4)+(-2)(-2)+4(-1)}{\sqrt{36}\,\sqrt{21}}$$

$$\cos\theta = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\boxed{\theta = \cos^{-1}\left[\frac{8}{3\sqrt{21}}\right]}$$

5. Find the angle between normal to surface $xy = z^2$ at $(4,1,2)$ and $(3,3,-3)$

Sol: Let $\phi = xy - z^2$

We have,

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}K$$

$$\nabla\phi = \frac{\partial}{\partial x}(xy-z^2)i + \frac{\partial}{\partial y}(xy-z^2)j + \frac{\partial}{\partial z}(xy-z^2)K$$

$$\nabla\phi = yi + xj - 2zK$$

at $(4,1,2)$

$$\nabla\phi_1 = (1)i + 4j - 2(2)K$$

$$\nabla\phi_1 = i + 4j - 4K$$

at $(3,3,-3)$

$$\nabla\phi_2 = 3i + 3j - 2(-3)K$$

$$\nabla\phi_2 = 3i + 3j + 6K.$$

Angle b/n surfaces is given by

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\ |\nabla\phi_2|}$$

$$\cos\theta = \frac{[i + 4j - 4K] \cdot [3i + 3j + 6K]}{\sqrt{(1)^2 + 4^2 + (-4)^2}\ \sqrt{3^2 + 3^2 + 6^2}}$$

$$\cos\theta = \frac{1(3) + 4(3) - 4(6)}{\sqrt{33}\ \sqrt{54}.}$$

$$\cos\theta = \frac{-9}{9\sqrt{22}} = -\frac{1}{\sqrt{22}}$$

$$\boxed{\theta = \cos^{-1}\left(-\frac{1}{\sqrt{22}}\right)}$$

Note: The directional derivative is maximum along $\nabla\phi$.

6. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ along $2i - j - 2k$.

Sol^n: We have,

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k.$$

$$\nabla\phi = \frac{\partial}{\partial x}(x^2yz + 4xz^2)i + \frac{\partial}{\partial y}(x^2yz + 4xz^2)j + \frac{\partial}{\partial z}(x^2yz + 4xz^2)k.$$

$$\nabla\phi = (2xyz + 4z^2)i + [x^2z + 0]j + [x^2y + 8xz]k$$

at $(1,-2,-1)$

$$\nabla\phi = [2(1)(-2)(-1) + 4(1)]i + [1(-1)]j + [1(-2) + 8(1)(-1)]k$$

$$\nabla\phi = 8i - j - 10k$$

The Directional derivative along $\vec{a} = 2i - j - 2k$ is given by $\nabla\phi \cdot \hat{n}$

Where $\hat{n} = \frac{\vec{a}}{|\vec{a}|} = \frac{2i - j - 2k}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{1}{3}(2i - j - 2k)$

$$\therefore D.D = \nabla\phi \cdot \hat{n} = [8i - j - 10k] \cdot \frac{1}{3}(2i - j - 2k)$$

$$= \frac{1}{3}[8(2) + (-1)(-1) + (-10)(-2)]$$

$$= \frac{37}{3}$$

7. Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ along $2i-3j+6K$.

Solⁿ: we have

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} K$$

$$\therefore \nabla\phi = \frac{\partial}{\partial x}[4xz^3 - 3x^2y^2z]i + \frac{\partial}{\partial y}(4xz^3 - 3x^2y^2z]j + \frac{\partial}{\partial z}[4xz^3 - 3x^2y^2z]K.$$

$$\nabla\phi = [4z^3 - 6xy^2z]i + [0 - 6x^2yz]j + [12xz^2 - 3x^2y^2]K.$$

at $(2,-1,2)$

$$\nabla\phi = [4(8) - 6(2)(1)(2)]i + [-6(4)(-1)(2)]j + [12(2)(4) - 3(4)(1)]K$$

$$\underline{\nabla\phi} = 8i + 48j + 84K$$

The directional derivative along the direction $\vec{d} = 2i - 3j + 6K$ is $\nabla\phi \cdot \hat{n}$.

Where $\hat{n} = \frac{\vec{d}}{|\vec{d}|} = \frac{2i-3j+6K}{\sqrt{2^2+(-3)^2+6^2}} = \frac{1}{7}(2i-3j+6K)$

$$\therefore D \cdot D = \nabla\phi \cdot \hat{n} = [8i+48j+84K] \cdot \frac{1}{7}(2i-3j+6K)$$

$$= \frac{1}{7}\{8(2) + 48(-3) + 84(6)\}$$

$$= \frac{376}{7}.$$

8. In which direction the directional derivative of x^2yz^3 is maximum at $(2,1,-1)$ and find magnitude of this maximum.

Sol:- WKT, the directional derivative is maximum along $\nabla\phi$.

$$\therefore \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\therefore \nabla\phi = \frac{\partial}{\partial x}(x^2yz^3)i + \frac{\partial}{\partial y}(x^2yz^3)j + \frac{\partial}{\partial z}(x^2yz^3)K.$$

$$\nabla\phi = 2xyz^3 i + x^2z^3 j + 3x^2yz^2 K$$

at $(2,1,-1)$

$$\nabla\phi = 2(2)(1)(-1)i + 4(-1)j + 3(4)(1)(1)K$$

$$\underline{\nabla\phi = -4i - 4j + 12K}$$

$\therefore$ Its magnitude is

$$|\nabla\phi| = \sqrt{(-4)^2 + (-4)^2 + 12^2} = \sqrt{176}$$

9. If the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(-1,1,2)$ has a maximum magnitude of 32 units in the direction parallel to y-axis, find a, b, c.

Sol:- Max. directional derivative is along $\nabla\phi$ and in the direction parallel to y-axis the magnitude is given by 32 units.

$\nabla\phi \cdot j = 32$ units at $(-1,1,2)$

Next,
$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} K$$

$$\nabla\phi = \frac{\partial}{\partial x}(axy^2 + byz + cz^2x^3) i + \frac{\partial}{\partial y}(axy^2 + byz + cz^2x^3) j$$
$$+ \frac{\partial}{\partial z}(axy^2 + byz + cz^2x^3) K$$

$$\nabla\phi = [ay^2 + 0 + 3cz^2x^2] i + [2axy + bz + 0] j$$
$$+ [0 + by + 2czx^3] K$$

at $(-1,1,2)$
$$\nabla\phi = [a(1) + 3c(4)(1)] i + [2a(-1)(1) + 2b] j$$
$$+ [b(1) + 2c(2)(-1)] K$$

$$\nabla\phi = (a + 12c) i + (-2a + 2b) j + (b - 4c) K$$

Now,
$$\nabla\phi \cdot j = [(a + 12c) i + (-2a + 2b) j + (b - 4c) K] \cdot j$$

$$\nabla\phi \cdot j = (-2a + 2b)$$

$$\therefore \quad -2a + 2b = 32 \quad \text{——— (1)}$$

Since max. magnitude in the dir^n $\parallel^{lel}$ to y-axis

$\therefore$ $a + 12c = 0$ —— (2) and $b - 4c = 0$ —— (3)

on solving (1), (2), and (3), we get

$a = -12$, $b = 4$, $c = 1$

10. If $\vec{A} = 2x^2 i - 3yz j + xz^2 K$ and $\phi = 2z - x^3 y$.
compute $\vec{A} \cdot \nabla\phi$ and $\vec{A} \times \nabla\phi$ at $(1, -1, 1)$

Sol: We have

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} K$$

$$\nabla\phi = \frac{\partial}{\partial x}(2z - x^3 y) i + \frac{\partial}{\partial y}(2z - x^3 y) j + \frac{\partial}{\partial z}(2z - x^3 y) K$$

$$\nabla\phi = (0 - 3x^2 y) i + (0 - x^3) j + (2 - 0) K$$

$$\nabla\phi = -3x^2 y i - x^3 j + 2K$$

at $(1, -1, 1)$

$$\nabla\phi = -3(1)(-1) i - (1) j + 2K$$

$$\nabla\phi = 3i - j + 2K$$

and $\vec{A} = 2(1) i - 3(-1)(1) j + (1)(1) K$

$$\vec{A} = 2i + 3j + K$$

Now,

$$\vec{A} \cdot \nabla\phi = [2i + 3j + K] \cdot (3i - j + 2K)$$

$$= 2(3) + 3(-1) + 1(2)$$

$$\underline{\underline{\vec{A} \cdot \nabla\phi = 5}}$$

next, $\vec{A} \times \nabla\phi = \begin{vmatrix} i & j & K \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{vmatrix}$

$$= i[6 + 1] - j(4 - 3) + K[-2 - 9]$$

$$\vec{A} \times \nabla\phi = 7i - j - 11K$$

11. Find $div\vec{F}$ and $curl\vec{F}$ where

$\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$.

Sol^n:- Let $\phi = x^3+y^3+z^3-3xyz$

then

$$\vec{F} = \nabla\phi$$

$$\text{i.e., } \vec{F} = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} K$$

$$\vec{F} = [3x^2-3yz]i + [3y^2-3xz)j + [3z^2-3xy]K$$

We have

$$div\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}[3x^2-3yz] + \frac{\partial}{\partial y}[3y^2-3xz) + \frac{\partial}{\partial z}[3z^2-3xy]$$

$$div\vec{F} = 6x-0+6y-0+6z-0$$

$$div\vec{F} = 6[x+y+z]$$

next

$$curl\vec{F} = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_2 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= i\left\{\frac{\partial}{\partial y}[3z^2-3xy] - \frac{\partial}{\partial z}[3y^2-2xz]\right\}$$
$$-j\left\{\frac{\partial}{\partial x}[3z^2-3xy) - \frac{\partial}{\partial z}[3x^2-3yz]\right\}$$
$$+K\left\{\frac{\partial}{\partial x}[3y^2-3xz] - \frac{\partial}{\partial y}[3x^2-3yz]\right\}$$

$$= i\{-3x-(-3x)\} - j\{-3y-(-3y)\} + k\{-3z-(-3z)\}$$

$$= 0i + 0j + 0k$$

$$\text{curl}\,\vec{F} = \vec{0}$$

12. If $\vec{F} = (x+y+1)i + j - (x+y)k$ Show that $\vec{F}\cdot\text{curl}\,\vec{F} = 0$

Sol:- We have

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$\text{curl}\,\vec{F} = i\left[\frac{\partial}{\partial y}(-x-y) - \frac{\partial}{\partial z}(1)\right] - j\left[\frac{\partial}{\partial x}(-x-y) - \frac{\partial}{\partial z}(x+y+1)\right] + k\left[\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(x+y+1)\right]$$

$$\text{curl}\,\vec{F} = i[-1-0] - j[-1-0] + k[0-1]$$

$$\text{curl}\,\vec{F} = -i + j - k$$

Now

$$\vec{F}\cdot\text{curl}\,\vec{F} = [(x+y+1)i + j - (x+y)k]\cdot[-i+j-k]$$

$$\vec{F}\cdot\text{curl}\,\vec{F} = (x+y+1)(-1) + (1)(1) + (-x-y)(-1)$$

$$= -x-y-1+1+x+y$$

$$\vec{F}\cdot\text{curl}\,\vec{F} = 0$$

13. ~~Find div F and curl F~~

If $\vec{F} = (3x^2y - z)i + (xz^3 + y^4)j - 2x^3z^2K$

find grad(div $\vec{F}$) at $(2,-1,0)$.

Sol:- we have

$$\nabla\cdot\vec{F} = \text{div}\,\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\text{div}\vec{F} = \frac{\partial}{\partial x}(3x^2y - z) + \frac{\partial}{\partial y}(xz^3 + y^4) + \frac{\partial}{\partial z}(-2x^3z^2)$$

$$\text{div}\vec{F} = 6xy - 0 + 0 + 4y^3 - 4x^3z$$

$$\text{div}\vec{F} = 6xy + 4y^3 - 4x^3z = \phi \text{ (say)}$$

Next

$$\text{grad}(\text{div}\vec{F}) = \text{grad}(\phi)$$

$$= \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}K$$

$$= \frac{\partial}{\partial x}(6xy + 4y^3 - 4x^3z)i + \frac{\partial}{\partial y}(6xy + 4y^3 - 4x^3z)j$$

$$+ \frac{\partial}{\partial z}[6xy + 4y^3 - 4x^3z)K$$

$$\text{grad}[\text{div}\vec{F}] = (6y - 12x^2z)i + (6x + 12y^2 - 0)j$$

$$+ (-4x^3)K$$

at $(2,-1,0)$

$$\text{grad}(\text{div}\vec{F}) = [6(-1) - 0]i + [6(2) + 12(1)]j - 4(8)K$$

$$= 2[-3i + 12j - 16K]$$

Scanned with CamScanner

14. Find curl (curl $\vec{A}$) given that

$$\vec{A} = xy\hat{i} + y^2z\hat{j} + z^2y\hat{k}$$

Sol: we have

$$\text{curl}\,\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{curl}\,\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2z & z^2y \end{vmatrix}$$

$$\text{curl}\,\vec{A} = \hat{i}\left\{\frac{\partial}{\partial y}(z^2y) - \frac{\partial}{\partial z}(y^2z)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(z^2y) - \frac{\partial}{\partial z}(xy)\right\}$$
$$+ \hat{k}\left\{\frac{\partial}{\partial x}(y^2z) - \frac{\partial}{\partial y}(xy)\right\}$$

$$\text{curl}\,\vec{A} = (z^2-y^2)\hat{i} + 0\hat{j} + [-x]\hat{k}$$

Now,

$$\text{curl}\,[\text{curl}\,\vec{A}] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2-y^2 & 0 & -x \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(-x) - \frac{\partial}{\partial z}(0)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial z}(z^2-y^2)\right\}$$
$$+ \hat{k}\left\{\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(z^2-y^2)\right\}$$

$$= 0\hat{i} - \hat{j}\,(-1-2z) + \hat{k}\,[2y]$$

$$\underline{\text{curl}\,[\text{curl}\,\vec{A}] = (1+2z)\hat{j} + 2y\hat{k}}$$

Solenoidal and Irrotational Vector fields

* If $\operatorname{div}\vec{F}=0$, then $\vec{F}$ is called Solenoidal vector field.
* If $\operatorname{curl}\vec{F}=0$, then $\vec{F}$ is called Irrotational (or) conservative (or) potential field.

Note: If $\vec{F}$ is a Irrotational vector then there exist a scalar function ϕ such that $\vec{F}=\nabla\phi$. ϕ is known as potential function.

Problems

1. Show that $\vec{F}=\dfrac{x\hat{i}+y\hat{j}}{x^2+y^2}$ is both Solenoidal and Irrotational.

Sol:- Given $\vec{F}=\dfrac{x}{x^2+y^2}\hat{i}+\dfrac{y}{x^2+y^2}\hat{j}$

Now $\operatorname{div}\vec{F}=\dfrac{\partial f_1}{\partial x}+\dfrac{\partial f_2}{\partial y}+\dfrac{\partial f_3}{\partial z}$

$$\operatorname{div}\vec{F}=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)+0$$

$$=\frac{(x^2+y^2)(1)-x(2x+0)}{(x^2+y^2)^2}+\frac{(x^2+y^2)(1)-y(0+2y)}{(x^2+y^2)^2}$$

$$=\frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$\boxed{\operatorname{div}\vec{F}=0}$ $\Rightarrow$ F is Solenoidal we have

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$$

$$\operatorname{curl}\vec{F} = i\left\{\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{y}{x^2+y^2}\right)\right\} - j\left\{\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right)\right\} + k\left\{\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right\}$$

$$\operatorname{curl}\vec{F} = i[0-0] - j[0-0] + k\left\{\frac{(x^2+y^2)(0) - y(2x)}{(x^2+y^2)^2} - \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2}\right\}$$

$$\operatorname{curl}\vec{F} = 0i + 0j + k\left\{\frac{-2xy+2xy}{(x^2+y^2)^2}\right\}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\boxed{\operatorname{curl}\vec{F} = \vec{0}}$$

$\therefore$ $\vec{F}$ is Irrotational.

2. Show that $\vec{F} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$ is Irrotational. Also find Scalar function ϕ such that $\vec{F} = \nabla\phi$.

Sol:- we have

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$\text{curl}\,\vec{F} = i\left\{\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right\} - j\left\{\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right\} + k\left\{\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right\}$$

$$\text{curl}\,\vec{F} = i[1-1] - j[1-1] + k[1-1]$$

$$\text{curl}\,\vec{F} = 0i + 0j + 0k$$

$$\text{curl}\,\vec{F} = \vec{0}$$

$\therefore \vec{F}$ is Irrotational.

To find ϕ: we have $\nabla\phi = \vec{F}$

ie., $$\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$$

on Comparing, we can write

$$\frac{\partial \phi}{\partial x} = y+z, \quad \frac{\partial \phi}{\partial y} = z+x, \quad \frac{\partial \phi}{\partial z} = x+y$$

Consider $\frac{\partial\phi}{\partial x} = y+z$

Int w.r.t 'x'

$$\phi = \int (y+z)\,dx + f_1(y,z)$$

$$\phi = xy + xz + f_1(y,z) \quad \text{—} ①$$

now, $\frac{\partial\phi}{\partial y} = z+x$

Int w.r.t y

$$\phi = \int (z+x)\,dy + f_2(x,z)$$

$$\phi = yz + xy + f_2(x,z) \quad \text{—} ②$$

next, $\frac{\partial\phi}{\partial z} = x+y$

Int w.r.t 'z'

$$\phi = \int (x+y)\,dz + f_3(x,y)$$

$$\phi = xz + yz + f_3(x,y) \quad \text{—} ③.$$

we assume that

$f_1(y,z) = yz$

$f_2(x,z) = xz$

$f_3(x,y) = xy$

$\therefore$ $\boxed{\phi = xy + xz + yz}$ is required potential function

3: Find the values of a, b, c if

$$\vec{F} = (x+y+az)i + (bx+2y-z)j + (x+cy+2z)K$$

such that $\text{curl}\,\vec{F} = 0$, then find scalar function ϕ such that $\vec{F} = \nabla\phi$.

Sol:- Given $\text{curl}\,\vec{F} = \vec{0}$.

ie.,
$$\begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+az & bx+2y-z & x+cy+2z \end{vmatrix} = \vec{0}$$

$$\Rightarrow i\left\{ \frac{\partial}{\partial y}(x+cy+2z) - \frac{\partial}{\partial z}(bx+2y-z)\right\}$$
$$-j\left\{ \frac{\partial}{\partial x}(x+cy+2z) - \frac{\partial}{\partial z}(x+y+az)\right\}$$
$$+K\left\{ \frac{\partial}{\partial x}(bx+2y-z) - \frac{\partial}{\partial y}(x+y+az)\right\} = \vec{0}$$

$$\Rightarrow i\{c+1) - j[1-a) + K[b-1) = 0i + 0j + 0K$$

on comparing, we can write

$$c+1=0, \quad 1-a=0, \quad b-1=0$$

$$\boxed{c=-1}, \quad \boxed{a=1}, \quad \boxed{b=1}$$

To find ϕ: we have $\nabla\phi = \vec{F}$

ie.,
$$\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}K = (x+y+z)i + (x+2y-z)j + (x-y+2z)K$$

on comparing we can write

$$\frac{\partial \phi}{\partial x} = x + y + z, \quad \frac{\partial \phi}{\partial y} = x + 2y - z, \quad \frac{\partial \phi}{\partial z} = x - y + 2z$$

Firstly consider, $\frac{\partial \phi}{\partial x} = x + y + z$

Int w.r.t 'x'

$$\phi = \int (x + y + z)\, dx + f_1(y, z)$$

$$\phi = \frac{x^2}{2} + xy + xz + f_1(y, z) \quad \text{—} ①$$

now, $\frac{\partial \phi}{\partial y} = x + 2y - z$

Int w.r.t 'y'

$$\phi = \int (x + 2y - z)\, dy + f_2(x, z)$$

$$\phi = xy + \cancel{2}\cdot\frac{y^2}{\cancel{2}} - yz + f_2(x, z) \quad \text{—} ②$$

next, $\frac{\partial \phi}{\partial z} = x - y + 2z$

Int w.r.t 'z'

$$\phi = \int (x - y + 2z)\, dz + f_3(x, y)$$

$$\phi = xz - yz + \cancel{2}z^2/\cancel{2} + f_3(x, y) \quad \text{—} ③$$

we assume that

$$f_1(y, z) = y^2 - yz + z^2$$

$$f_2(x, z) = x^2/2 + xz + z^2$$

$$f_3(x, y) = x^2/2 + xy + y^2$$

Thus

$$\boxed{\phi = \frac{x^2}{2} + xy + xz + y^2 - yz + z^2}$$

4. Find a, b such that

$\vec{F} = (axy + z^3)i + (3x^2 - z)j + (bxz^2 - y)K$

is irrotational. Also find scalar function ϕ such that $\vec{F} = \nabla\phi$.

Sol:- Given $\vec{F}$ is Irrotational.

i.e., curl $\vec{F} = \vec{0}$

$$\begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy+z^3 & 3x^2-z & bxz^2-y \end{vmatrix} = \vec{0}$$

$$\Rightarrow i\left\{\frac{\partial}{\partial y}(bxz^2-y) - \frac{\partial}{\partial z}(3x^2-z)\right\}$$
$$- j\left\{\frac{\partial}{\partial x}(bxz^2-y) - \frac{\partial}{\partial z}(axy+z^3)\right\}$$
$$+ K\left\{\frac{\partial}{\partial x}(3x^2-z) - \frac{\partial}{\partial y}(axy+z^3)\right\} = \vec{0}$$

$$\Rightarrow i[-1+1] - j[bz^2 - 3z^2] + K[6x - ax]$$
$$= 0i + 0j + 0K$$

on Comparing, we can write

$bz^2 - 3z^2 = 0$ $\qquad$ $6x - ax = 0$

$z^2(b-3) = 0$ $\qquad$ $x(6-a) = 0.$

$\Rightarrow b-3 = 0$ $\qquad$ $6 - a = 0$

$\boxed{b=3}$ $\qquad$ $\boxed{a=6}$

To find ϕ: we have $\nabla\phi = \vec{F}$

$$\Rightarrow \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k = (6xy + z^3) i + (3x^2 - z) j + (3xz^2 - y) k.$$

on Comparing, we can write

$$\frac{\partial\phi}{\partial x} = 6xy + z^3.$$

Int w.r.t x.

$$\phi = \int (6xy + z^3) dx + f_1(y, z)$$

$$\phi = 6\frac{x^2}{2} y + xz^3 + f_1(y, z). \quad \text{——} ①$$

now,

$$\frac{\partial\phi}{\partial y} = 3x^2 - z.$$

Int w.r.t 'y'

$$\phi = \int (3x^2 - z) dy + f_2(x, z)$$

$$\phi = 3x^2 y - yz + f_2(x, z) \quad \text{——} ②$$

next $\frac{\partial\phi}{\partial z} = 3xz^2 - y.$

Int w.r.t 'z'

$$\phi = \int (3xz^2 - y) dz + f_3(x, y)$$

$$\phi = 3x\frac{z^3}{3} - yz + f_3(x, y) \quad \text{——} ③$$

we assume that

$f_1(y, z) = -yz$ $\quad f_2(x, z) = xz^3$

and $f_3(x, y) = 3x^2 y$

Thus

$$\boxed{\phi = 3x^2 y + xz^3 - yz.}$$

5. Show that $\vec{F} = (2xy^2 + yz)i + (2x^2y + xz + 2yz^2)j + (2y^2z + xy)k$ is Irrotational. Hence find its scalar function.

Sol:- W.K.T

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2+yz & 2x^2y+xz+2yz^2 & 2y^2z+xy \end{vmatrix}$$

$$= i\{(4yz+x) - (0+x+4yz)\}$$
$$- j\{(0+y) - (0+y)\}$$
$$+ k\{(4xy+z+0) - (4xy+z)\}$$

$$= 0i + 0j + 0k$$

$$\text{curl } \vec{F} = \vec{0}$$

$\therefore \vec{F}$ is Irrotational.

<u>To find ϕ:</u> we have $\nabla\phi = \vec{F}$

on simplifying

$$\phi = x^2y^2 + y^2z^2 + xyz.$$

Scanned with CamScanner

6. Find the value of constant "a" such that $\vec{F} = (axy - z^3)i + ((a-2)x^2)j + (1-a)xz^2 k$ is irrotational and Hence find a scalar function such that $\vec{F} = \nabla\phi$.

Sol: Given $\vec{F}$ is Irrotational

ie., $\text{curl}\,\vec{F} = \vec{0}$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = \vec{0}$$

$$\Rightarrow i\left\{\frac{\partial}{\partial y}((1-a)xz^2) - \frac{\partial}{\partial z}((a-2)x^2)\right\}$$
$$- j\left\{\frac{\partial}{\partial x}((1-a)xz^2) - \frac{\partial}{\partial z}(axy - z^3)\right\}$$
$$+ k\left\{\frac{\partial}{\partial x}((a-2)x^2) - \frac{\partial}{\partial y}(axy - z^3)\right\} = \vec{0}$$

$$\Rightarrow i[0 + 0] - j\{(1-a)z^2 - (0 - 3z^2)\}$$
$$+ k[2(a-2)x - ax] = \vec{0}$$

$$\Rightarrow 0i - j[(4-a)z^2] + k((a-4)x) = 0i + 0j + 0k$$

On comparing, we can write

$(4-a)z^2 = 0$ and $(a-4)x = 0$

$4 - a = 0$ $\quad$ $a - 4 = 0$

$\boxed{a = 4}$ $\quad$ $a = 4$

To find ϕ: we have $\nabla\phi = \vec{F}$

i.e., $\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = (4xy - z^3) i + 2x^2 j - 3xz^2 k.$

on comparing, we can write

$\frac{\partial \phi}{\partial x} = 4xy - z^3$

Int w.r.t 'x'

$\phi = \int (4xy - z^3)\, dx + f_1(y, z)$

$\phi = 4\frac{x^2}{2} y - xz^3 + f_1(y, z)$ —— ①

$\frac{\partial \phi}{\partial y} = 2x^2$

Int w.r.t 'y'

$\phi = \int (2x^2)\, dy + f_2(x, z)$

$\phi = 2x^2 y + f_2(x, z)$ —— ②

& $\frac{\partial \phi}{\partial z} = -3xz^2$

Int w.r.t 'z'

$\phi = \int (-3xz^2)\, dz + f_3(x, y)$

$\phi = -\cancel{3}x \cdot \frac{z^3}{\cancel{3}} + f_3(x, y)$ —— ③.

we shall assume that

$f_1(y, z) = 0$ $f_2(x, z) = -xz^3$

& $f_3(x, y) = 2x^2 y$

Thus $\boxed{\phi = 2x^2 y - xz^3}$

If $\vec{F} = \nabla(xy^3z^2)$. find $\operatorname{div}\vec{F}$ and $\operatorname{curl}\vec{F}$ at the point $(1,-1,1)$.

Sol:- Let $\phi = xy^3z^2$. then

we have $\vec{F} = \nabla\phi$.

$$\vec{F} = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\vec{F} = \frac{\partial}{\partial x}(xy^3z^2)i + \frac{\partial}{\partial y}(xy^3z^2)j + \frac{\partial}{\partial z}(xy^3z^2)k$$

$$\vec{F} = y^3z^2 i + 3xy^2z^2 j + 2xy^3z k$$

W.K.T

$$\operatorname{div}\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(y^3z^2) + \frac{\partial}{\partial y}(3xy^2z^2) + \frac{\partial}{\partial z}(2xy^3z)$$

$$\operatorname{div}\vec{F} = 0 + 6xyz^2 + 2xy^3$$

at $(1,-1,1)$

$$\operatorname{div}\vec{F} = 6(1)(-1)(1) + 2(1)(-1) = \underline{\underline{-8}}$$

Now

$$\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$$

$$= i[6xy^2z - 6xy^2z] - j[2y^3z - 2y^3z] + k[3y^2z^2 - 3y^2z^2]$$

$$= 0i + 0j + 0k$$

$$\underline{\underline{\operatorname{curl}\vec{F} = \vec{0}}}$$

Scanned with CamScanner

Vector Integration

The line integral of vector $\vec{A}(x,y,z)$ along C is defined to be sum of. Scalar product of $\vec{A}$ and $d\vec{r}$ is. represented by $\int_C \vec{A}\cdot dr$.

The total work done by force is given by $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F}$ is force acted upon by a particle in displacing it along the curve C.

Note that $\vec{F}$ is irrotational if $\oint_C \vec{F}\cdot d\vec{r} = 0$.

Problems

1. If $\vec{F} = xyi + yzj + zxK$, Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where 'c' is curve represented by $x=t$, $y=t^2$, $z=t^3$, and $-1\le t\le 1$.

Sol:- Since $\vec{r} = xi + yj + zK$

$\therefore d\vec{r} = dxi + dyj + dzK$

$\therefore \vec{F}\cdot d\vec{r} = (xyi + yzj + zxK)\cdot(dxi + dyj + dzK)$

$\vec{F}\cdot d\vec{r} = xy\,dx + yz\,dy + zx\,dz$.

given $x = t$, $y = t^2$, $z = t^3$

$\therefore dx = dt$, $dy = 2t\,dt$, $dz = 3t^2 dt$.

Hence
$$\vec{F}\cdot d\vec{r} = t(t^2)\,dt + t^2(t^3)\,2t\,dt + t^3(t)\,3t^2 dt$$
$$\vec{F}\cdot d\vec{r} = t^3 dt + 2t^6 dt + 3t^6 dt.$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{t=-1}^{1} (t^3 + 5t^6)\,dt$$
$$= \left[\frac{t^4}{4} + 5\cdot\frac{t^7}{7}\right]_{-1}^{1}$$
$$= \frac{1}{4}(1-1) + \frac{5}{7}\left[1-(-1)\right]$$

$$\int_C \vec{F}\cdot d\vec{r} = \frac{10}{7}$$

2. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = xyi + (x^2+y^2)j$ along (i) the path of straight line from $(0,0)$ to $(1,0)$ and then $(1,1)$.

(ii) the st. line joining the origin & $(1,2)$.

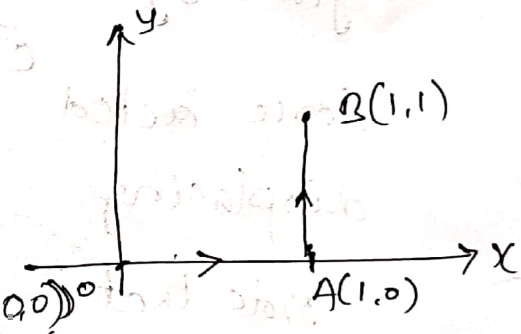
Sol:.. Since $d\vec{r} = dx\,i + dy\,j + dz\,K$.

$\therefore \vec{F}\cdot d\vec{r} = [xyi + (x^2+y^2)j]\cdot(dx\,i + dy\,j + dz\,k)$

$\vec{F}\cdot d\vec{r} = xy\,dx + (x^2+y^2)\,dy$.

(i). C is st. line from $(0,0)$ to $(1,0)$ & then $(1,1)$

$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r}$ $\longrightarrow$ ①



Along OA: $y=0 \Rightarrow dy = 0$

& x varies from 0 to 1.

$\therefore \int_{OA} \vec{F}\cdot d\vec{r} = \int_{OA} x(0)\,dx + (x^2+0^2)\,0 = 0$

Along AB: $x=1 \Rightarrow dx = 0$

$\therefore$ y varies from 0 to 1

$\therefore \int_{AB} \vec{F}\cdot d\vec{r} = \int_{AB} 1(y)\,0 + (1^2+y^2)\,dy = \int_{y=0}^{1} (1+y^2)\,dy$

$= \left(y + y^3/3\right)_0^1 = [1+\frac{1}{3}] - [0+0] = 4/3$

Thus ① $\Rightarrow$

$\int_C \vec{F}\cdot d\vec{r} = 0 + 4/3 = \frac{4}{3}$

(ii). C is the straight joining the points (0,0) and (1,2)

we have Eq^n of st. line joining (x_1, y_1) and (x_2, y_2) is

$$\frac{(y-y_1)}{(x-x_1)} = \frac{(y_2-y_1)}{(x_2-x_1)}$$

$$\therefore \frac{(y-0)}{(x-0)} = \frac{(2-0)}{(1-0)} \Rightarrow \frac{y}{x} = 2$$

$\therefore y = 2x$. then $dy = 2dx$.

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C xy\,dx + (x^2+y^2)\,dy$$

$$= \int_{x=0}^{1} x(2x)\,dx + (x^2+4x^2)\,2dx.$$

$$= \int_{x=0}^{1} (2x^2 + x^2 + 4x^2)\,2dx$$

$$= 2\int_{x=0}^{1} 7x^2\,dx$$

$$= 2 \times 7 \cdot \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{14}{3}[1-0]$$

$$\int_C \vec{F} \cdot d\vec{r} = \frac{14}{3}$$

3. If $\vec{F} = (3x^2+6y)i - 14yzj + 20xz^2K$, Evaluate $\int_C \vec{F}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve given by $x=t$, $y=t^2$, $z=t^3$.

Sol:- Since $d\vec{r} = dx\,i + dy\,j + dz\,K$.

$\therefore \vec{F}\cdot d\vec{r} = [(3x^2+6y)i - 14yzj + 20xz^2k] \cdot [dx\,i + dy\,j + dz\,k]$

$\vec{F}\cdot d\vec{r} = (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz.$

Given $x=t$, $y=t^2$, $z=t^3$

$dx = dt$ $\quad dy = 2t\,dt$ $\quad dz = 3t^2dt$.

$\therefore \vec{F}\cdot d\vec{r} = (3t^2+6t^2)dt - 14(t^2)(t^3)\,2t\,dt + 20\,t(t^3)^2(3t^2dt)$

$\vec{F}\cdot d\vec{r} = 9t^2dt - 28\,t^6dt + 60\,t^9dt$.

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)\,dt$$

$$= \left\{9\frac{t^3}{3} - 28\cdot\frac{t^7}{7} + 60\frac{t^{10}}{10}\right\}_{t=0}^{1}$$

$$= \left[\frac{9}{3} - \frac{28}{7} + \frac{60}{10}\right] - [0 - 0 + 0]$$

$$= 3 - 4 + 6.$$

$$\int_C \vec{F}\cdot d\vec{r} = 5$$

4. If $\vec{F} = x^2 i + xy j$ Evaluate $\int_C \vec{F} \cdot d\vec{r}$ from $(0,0)$ to $(1,1)$ along.

(i) the line $y = x$

(ii) the parabola $y = \sqrt{x}$.

Sol:- Since $d\vec{r} = dx\, i + dy\, j + dz\, k$

$\therefore \vec{F} \cdot d\vec{r} = (x^2 i + xy j) \cdot (dx\, i + dy\, j + dz\, k)$

$\vec{F} \cdot d\vec{r} = x^2 dx + xy\, dy$

(i) along the line $y = x$

$\therefore dy = dx$.

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C x^2 dx + xy\, dy$$

$$= \int_0^1 x^2 dx + x(x)\, dx.$$

$$= \int_{x=0}^{1} (2x^2)\, dx = 2 \cdot \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3}\left[1^3 - 0^3\right]$$

$$\int_C \vec{F} \cdot d\vec{r} = 2/3$$

(ii) Along the parabola $y = \sqrt{x}$ (or) $y^2 = x$.

$\therefore 2y\, dy = dx$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C x^2 dx + xy\, dy$$

$$= \int_0^1 (y^2)^2 2y\, dy + (y^2)\, y\, dy.$$

$$= \int_0^1 (2y^5 + y^3)\, dy = \left[2\frac{y^6}{6} + \frac{y^4}{4}\right]_0^1$$

$$\int_C \vec{F} \cdot d\vec{r} = \left(\frac{2}{6} + \frac{1}{4}\right) - (0+0) = \frac{7}{12}$$

5. Find the total work done by force represented by $\vec{F} = 3xyi - yj + 2zxK$ in moving a particle round the circle $x^2+y^2=4$.

Sol:- we have, total work done is given by

$$W = \int_C \vec{F} \cdot d\vec{r}$$

Since $d\vec{r} = dx\,i + dy\,j + dz\,K$.

$\therefore \vec{F} \cdot d\vec{r} = 3xy\,dx - y\,dy + 2zx\,dz$.

For a circle $x^2+y^2=4$, its parametric Eq^n are $x = 2\cos\theta$, $y = 2\sin\theta$, $z=0$

$\Rightarrow dx = -2\sin\theta\,d\theta$, $dy = 2\cos\theta\,d\theta$.

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C 3xy\,dx - y\,dy + 2zx\,dz$$

$$= \int_{\theta=0}^{2\pi} 3(2\cos\theta)(2\sin\theta)(-2\sin\theta)d\theta - 2\sin\theta\, 2\cos\theta\, d\theta + 0$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \underbrace{(-24)\sin^2\theta\cos\theta\, d\theta}_{I_1} - 2.\sin 2\theta\, d\theta \quad -(1)$$

To find I_1: take $t = \sin\theta \Rightarrow dt = \cos\theta\, d\theta$.

$$\therefore I_1 = -24 \int t^2 dt = -\cancel{24}^{8} \frac{t^3}{\cancel{3}} = -8.\sin^3\theta$$

$$\int_C \vec{F} \cdot d\vec{r} = \left(-8\sin^3\theta\right)_0^{2\pi} - \cancel{2}.\frac{(-\cos 2\theta)}{\cancel{2}}\Big|_0^{2\pi} = [0-0] + [1-1]$$

$= 0$//

Scanned with CamScanner

Green's theorem in a plane:

Statement: If R is a closed region of the xy-plane bounded by a simple closed curve C and if M and N are two continuous function of x, y having continuous first order partial derivatives in region R then

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

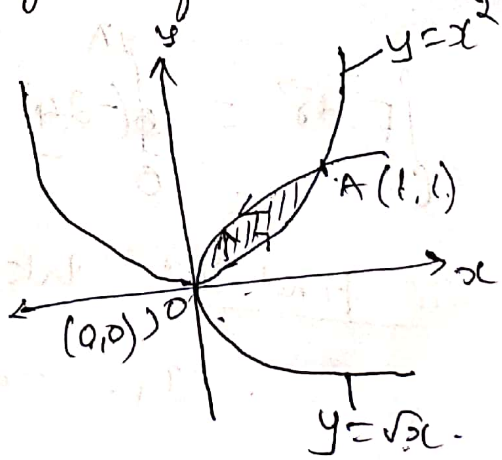
problems

1. Verify Green's theorem in a plane $\int_C (3x^2-8y^2)dx + (4y-6xy)dy$, where C is the boundary of region enclosed by $y=\sqrt{x}$ and $y=x^2$.

Sol:- Here $M = 3x^2-8y^2$ & $N = 4y-6xy$

$\frac{\partial M}{\partial y} = 0-16y$ $\quad$ $\frac{\partial N}{\partial x} = 0-6y.$

'C' is the boundary of region of enclosed by $y=\sqrt{x}$ & $y=x^2$.



To find pt. of Intersection:

we have $y=\sqrt{x}$, $y=x^2$

$\sqrt{x} = x^2$

$\Rightarrow x = x^4 \Rightarrow x^4 - x = 0$

$x(x^3-1) = 0$

$x=0,\ x=1$

$\therefore$ then $y=0, \quad y=1$

$\therefore$ pt. of Intersections are $(0,0)$ & $(1,1)$

By Green's theorem, we have

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \rightarrow (1).$$

$$LHS = \int_C M\,dx + N\,dy = \int_{OA} (M\,dx + N\,dy) + \int_{AO} (M\,dx + N\,dy)$$

$$= I_1 + I_2 \text{ (say)}.$$

Along OA: $y = x^2 \Rightarrow dy = 2x\,dx$.

and x varies from '0' to '1'

$$\therefore I_1 = \int_{OA} M\,dx + N\,dy = \int_{OA} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_{x=0}^{1} (3x^2 - 8x^4)\,dx + (4x^2 - 6x \cdot x^2)\,2x\,dx.$$

$$= \int_{x=0}^{1} (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx$$

$$= \int_{x=0}^{1} (3x^2 - 20x^4 + 8x^3)\,dx$$

$$= \left[\cancel{3} \cdot \frac{x^3}{\cancel{3}} - \overset{4}{\cancel{20}}\frac{x^5}{\cancel{5}} + \overset{2}{\cancel{8}} \cdot \frac{x^4}{\cancel{4}}\right]_0^1$$

$$= [1 - 4 + 2] - [0 + 0 + 0]$$

$$I_1 = -1.$$

<u>Along AO</u>: $y=\sqrt{x}$ (or) $x=y^2 \Rightarrow dx = 2y\,dy$.

& y varies from to 1 to 0

$$\therefore I_2 = \int_{AO} M\,dx + N\,dy = \int_{AO} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_{y=1}^{0} (3y^4 - 8y^2)\,2y\,dy + (4y - 6\cdot y^2 \cdot y)\,dy.$$

$$= \int_{y=1}^{0} (6y^5 - 16y^3 + 4y - 6y^3)\,dy$$

$$= \int_{y=1}^{0} (6y^5 - 22y^3 + 4y)\,dy$$

$$= \left[\cancel{6}\cdot\frac{y^6}{\cancel{6}} - \overset{11}{\cancel{22}}\cdot\frac{y^4}{\cancel{4}} + \cancel{4}\,\frac{y^2}{\cancel{2}}\right]_{y=1}^{0}$$

$$= [0 - 0 + 0] - \left[1 - \frac{11}{2} + 2\right]$$

$$= -\left[-5/2\right]$$

$$I_2 = 5/2 \qquad (6)$$

$\therefore LHS = -1 + 5/2 = 3/2.$

Now consider

$$RHS = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

$$= \iint_R [-6y - (-16y)]\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} 10y \, dy\, dx.$$

$$= \int_{x=0}^{1} \cancel{10}^{5} \cdot \left[\frac{y^2}{\cancel{2}}\right]_{y=x^2}^{\sqrt{x}} dx$$

$$= 5 \int_{x=0}^{1} (x - x^4)\, dx$$

$$= 5 \cdot \left(\frac{x^2}{2} - \frac{x^5}{5}\right]_{x=0}^{1}$$

$$= 5\left\{\left[\frac{1}{2} - \frac{1}{5}\right] - [0 - 0]\right\}$$

$$= 5\left[\frac{3}{10}\right]$$

$RHS = 3/2$

$\therefore LHS = RHS$

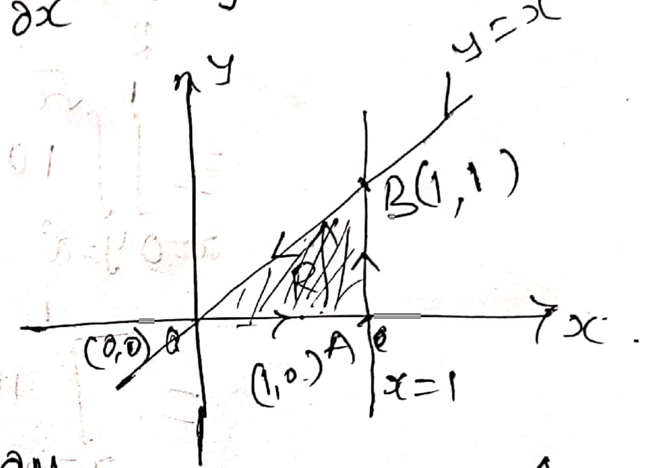
Hence Green's theorem is verified.

Scanned with CamScanner

⑨ Evaluate $\int_C (xy - x^2)\,dx + x^2y\,dy$ where C is the enclosed curve formed by $y=0$, $x=1$, and $y=x$ by Green's theorem

Sol:- let $M = xy - x^2$ & $N = x^2y$

$$\frac{\partial M}{\partial y} = x - 0 \qquad \frac{\partial N}{\partial x} = 2xy$$



By Green's theorem

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{x} (2xy - x)\,dy\,dx.$$

$$= \int_{x=0}^{1} \left[\cancel{2}x.\frac{y^2}{\cancel{2}} - xy\right]_{y=0}^{x} dx$$

$$= \int_{x=0}^{1} \{[x^3 - x^2] - [0-0]\}\,dx$$

$$= \left\{\frac{x^4}{4} - \frac{x^3}{3}\right\}_{x=0}^{1}$$

$$= \left[\frac{1}{4} - \frac{1}{3}\right] - [0-0]$$

$$= -1/12$$

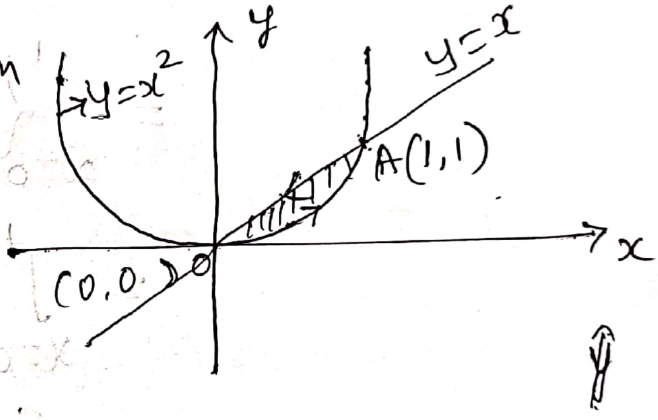
3. Verify Green's theorem for $\int_C (xy + y^2)\,dx + x^2\,dy$ where 'C' the enclosed curve of the region bounded by $y = x$ and $y = x^2$.

Sol:- Let $M = xy + y^2$, & $N = x^2$

$\therefore \frac{\partial M}{\partial y} = x + 2y \qquad \frac{\partial N}{\partial x} = 2x.$

Here c is the enclosed curve of the region bounded by $y = x$ & $y = x^2$

To find pt of Intersection we have



$y = x$ & $y = x^2$

$\Rightarrow x = x^2$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0 \quad x = 1 \quad \Rightarrow y = 0$ & $y = 1$

$\therefore (0,0), (1,1)$ are the point of Intersection

By Green's theorem, we have

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Consider

$$LHS = \int_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AO} M\,dx + N\,dy$$

$$LHS = I_1 + I_2 \text{ (say)}$$

Along OA: $y = x^2 \Rightarrow dy = 2x\,dx$, $0 \le x \le 1$.

$\therefore I_1 = \int_{OA} M\,dx + N\,dy = \int_{OA} (xy + y^2)\,dx + x^2\,dy$

$= \int_{x=0}^{1} (x x^2 + x^4)\,dx + x^2\, 2x\,dx.$

$= \int_{x=0}^{1} (3x^3 + x^4)\,dx$

$= \left[3 \cdot \frac{x^4}{4} + \frac{x^5}{5}\right]_{x=0}^{1}$

$= \left[\frac{3}{4} + \frac{1}{5}\right] - [0 + 0]$

$I_1 = \frac{19}{20}$

Along AO: $y = x \Rightarrow dy = dx$,

& x varies from '1' to '0'.

$\therefore I_2 = \int_{AO} M\,dx + N\,dy = \int_{AO} (xy + y^2)\,dx + x^2\,dy$

$= \int_{x=1}^{0} (x \cdot x + x^2)\,dx + x^2\,dx$

$= \int_{x=1}^{0} 3x^2\,dx.$

$= \cancel{3}\left[\frac{x^3}{\cancel{3}}\right]_{x=1}^{0}$

$= 0 - 1$

$I_2 = -1$

$\therefore LHS = I_1 + I_2 = \frac{19}{20} - 1 = \frac{-1}{20}$

Next consider

$$RHS = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_R [2x - (x+2y)]\, dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (x - 2y)\, dy\, dx$$

$$= \int_{x=0}^{1} \left[x \cdot y - \cancel{2}\frac{y^2}{\cancel{2}}\right]_{y=x^2}^{x} dx$$

$$= \int_{x=0}^{1} \{[x^2 - x^2] - [x^3 - x^4]\}\, dx$$

$$= \int_{x=0}^{1} (x^4 - x^3)\, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_{x=0}^{1}$$

$$= \left[\frac{1}{5} - \frac{1}{4}\right] - [0 - 0]$$

$$RHS = -\frac{1}{20}$$

$\therefore$ LHS = RHS

Hence Green's theorem is Verified.

4. Find the Area between parabolas $y^2=4x$ and $x^2=4y$ with the help of Green's theorem in a plane.

Sol:- WKT, Area $= \iint_R dx\,dy$.

Taking $N = \frac{x}{2}$, $M = -\frac{y}{2}$

$\therefore \frac{\partial N}{\partial x} = \frac{1}{2}$ $\quad \frac{\partial M}{\partial y} = -\frac{1}{2}$

and $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$

$\therefore$ By Green's theorem, we have

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

$$\int_C -\frac{y}{2} dx + \frac{x}{2} dy = \iint_R (1)\, dx\,dy.$$

$$\Rightarrow \text{Area} = \frac{1}{2}\int_C x\,dy - y\,dx \quad \text{—— (1)}.$$

To find pt of Intersection

we have

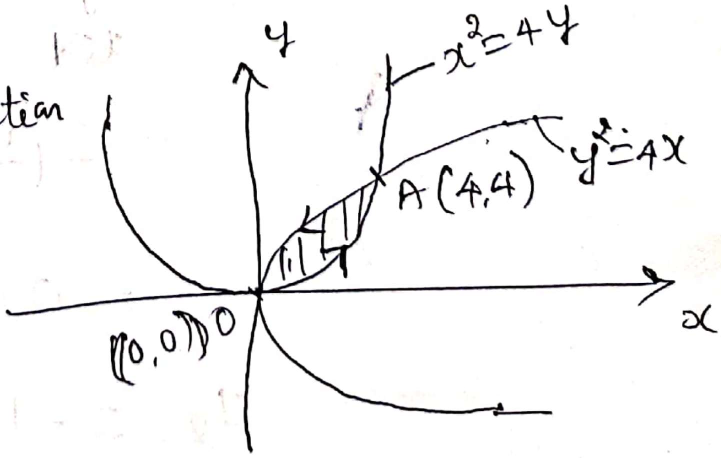
$y^2 = 4x$ & $x^2 = 4y$

$\left(\frac{x^2}{4}\right)^2 = 4x$

$\Rightarrow x^4 = 64x$

$x^4 - 64x = 0$

$x(x^3 - 64) = 0$.



$x = 0, \quad x^3 = 64$

$x_1 = 0 \quad x = 4$

$\therefore y = 0$ and $y = 4$

$\therefore (0,0)$ & (4.4) are the point of Intersection

① $\Rightarrow$

$$\text{Area} = \frac{1}{2}\left\{ \int_{OA} x\,dy - y\,dx + \int_{AO} x\,dy - y\,dx \right\}$$

$$\text{Area} = \frac{1}{2}\left[I_1 + I_2 \right] \quad \ldots \text{(Say)} \quad — ②$$

Along OA: $x^2 = 4y \Rightarrow y = x^2/4 \Rightarrow dy = \frac{2x\,dx}{4_2}$

and x varies from 0 to 4

$$\therefore I_1 = \int_{OA} x\,dy - y\,dx = \int_{OA} x \cdot \frac{x\,dx}{2} - \frac{x^2}{4}\,dx.$$

$$I_1 = \int_{x=0}^{4} \left(\frac{x^2}{4}\right) dx = \left[\frac{x^3}{12}\right]_0^4 = \left[\frac{4^3}{12} - 0\right]$$

$I_1 = 64/12.$

Along AO: $y^2 = 4x \Rightarrow x = \frac{y^2}{4} \Rightarrow dx = \frac{2y\,dy}{4}$

& y varies from 4 to 0.

$$\therefore I_2 = \int_{AO} x\,dy - y\,dx = \int_{y=4}^{0} \frac{y^2}{4}\,dy - y \cdot \frac{y\,dy}{2}$$

$$I_2 = \int_{y=4}^{0} \left(\frac{-y^2}{4}\right) dy = -\left(\frac{y^3}{12}\right)_4^0 = -\left\{0 - \frac{64}{12}\right\}$$

$I_2 = 64/12$

Thus $\text{Area} = \frac{1}{2}\left\{\frac{64}{12} + \frac{64}{12}\right\} = \frac{16}{3}$

Stoke's theorem:

Statement: If S is a surface bounded by simple closed curve C and if $\vec{F}$ is any continuously differentiable vector function then

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds$$

problems

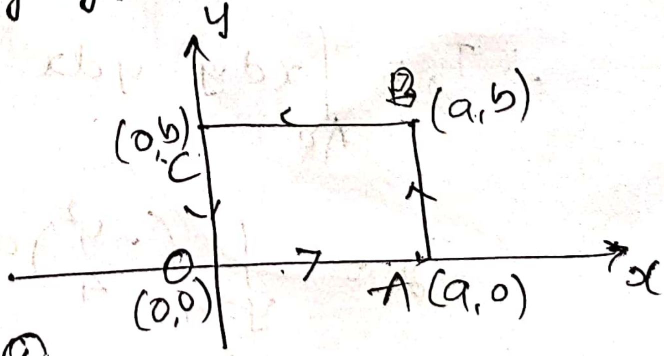
1. verify Stokes theorem for the vector $\vec{F} = (x^2+y^2)i - 2xyj$ taken round the rectangle bounded by $x=0$, $x=a$ and $y=0$, $y=b$.

Sol:- By stoke's theorem, we have

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds$$

$$\therefore \vec{F}\cdot d\vec{r} = \left[(x^2+y^2)\,i - 2xyj\right]\cdot\left[dx\,i + dy\,j + dz\,k\right]$$

$$\vec{F}\cdot d\vec{r} = (x^2+y^2)dx - 2xydy.$$



$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO} \longrightarrow (1)$$

Along OA: $y=0 \Rightarrow dy=0$
& x varies from 0 to a

$$\therefore \int_{OA} \vec{F}\cdot d\vec{r} = \int_{OA} (x^2+y^2)\,dx - 2xy\,dy$$

$$= \int_0^a (x^2+0)\,dx - 0 = \int_0^a x^2\,dx$$

$$\int_{OA} \vec{F}\cdot d\vec{r} = \left(\frac{x^3}{3}\right)_0^a = \left[\frac{a^3}{3} - \frac{0}{3}\right] = a^3/3$$

Along AB: $x=a \Rightarrow dx=0$ & y varies from '0' to 'b'

$$\therefore \int_{AB} \vec{F}\cdot d\vec{r} = \int_{AB} (x^2+y^2)\,dx - 2xy\,dy = \int_{AB} 0 - 2a.y\,dy$$

$$= -2a\int_{y=0}^{b} y\,dy = -\cancel{2}a\left[\frac{y^2}{\cancel{2}}\right]_{y=0}^{b}$$

$$\int_{AB} \vec{F}\cdot d\vec{r} = -a\left[b^2 - 0^2\right] = -ab^2$$

Along BC: $y=b \Rightarrow dy=0$ & x varies from 'a' to '0'

$$\therefore \int_{BC} \vec{F}\cdot d\vec{r} = \int_{BC} (x^2+y^2)\,dx - 2xy\,dy = \int_{BC}^{0} (x^2+b^2)\,dx - 0.$$

$$= \left[\frac{x^3}{3} + b^2x\right]_a^0 = [0+0] - \left[\frac{a^3}{3} + ab^2\right]$$

$$\int_{BC} \vec{F}\cdot d\vec{r} = -\frac{a^3}{3} - ab^2$$

Along CO: $x=0 \Rightarrow dx=0$

and y varies from 'b' to 'o'

$$\therefore \int_{CO} \vec{F}\cdot d\vec{r} = \int_{CO} (x^2+y^2)dx - 2xy\,dy = \int_{CO} 0 - 0. = 0$$

Hence on substituting, Eq ① becomes

$$\int_C \vec{F}\cdot d\vec{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 + 0$$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = -2ab^2}$$

Next, curl $\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$

$$\text{curl}\vec{F} = i[0-0] - j[0-0] + k[-2y-2y]$$

$$\text{curl}\vec{F} = 0i + 0j - 4yk$$

and $\hat{n}ds = d\vec{s} = dydz\,i + dzdx\,j + dxdy\,k$.

$$\therefore \text{curl}\vec{F}\cdot\hat{n}ds = -4y\,dxdy$$

NOW RHS $= \iint_S \text{curl}\vec{F}\cdot\hat{n}ds = \int_{x=0}^{a}\int_{y=0}^{b} -4y\,dydx.$

$$= -4\int_{x=0}^{a}\left[\frac{y^2}{2}\right]_0^b dx = -2\int_{x=0}^{a}(b^2-0^2)dx$$

$$= -2b^2[x]_0^a = -2b^2[a-0]$$

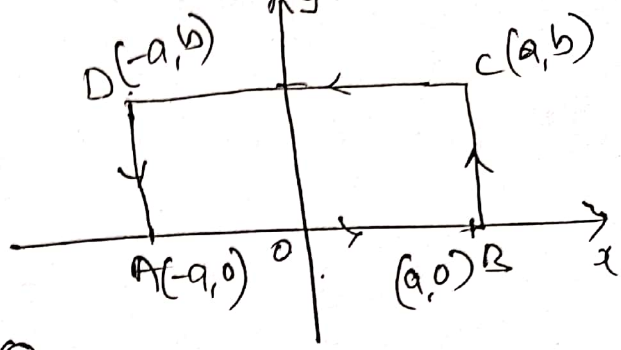
$$\boxed{\iint_S \text{curl}\vec{F}\cdot\hat{n}ds = -2ab^2}$$

Hence Stokes is Verified.

2. Verify Stoke's theorem for the vector $\vec{F} = (x^2+y^2)i - 2xyj$ taken round the rectangle bounded by $x = \pm a,\ y=0,\ y=b$.

Sol: $\because d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\vec{F}\cdot d\vec{r} = (x^2+y^2)dx - 2xydy$$



$$\int_C \vec{F}\cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CO} + \int_{DA} \rightarrow (1)$$

Along AB: $y=0 \Rightarrow dy=0$ & x varies from '$-a$' to 'a'

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{AB} (x^2+y^2)dx - 2xydy = \int_{x=-a}^{a} (x^2+0)dx - 0$$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{a^3}{3} - \left(-\frac{a^3}{3}\right) = 2a^3/3.$$

Along BC: $x=a \Rightarrow dx=0$

and y varies from 0 to b.

$$\therefore \int_{BC} \vec{F}\cdot d\vec{r} = \int_{BC} (x^2+y^2)\,dx - 2xy\,dy$$

$$= \int_{y=0}^{b} 0 - 2ay\,dy = -2a\int_0^b y\,dy$$

$$= -\cancel{2}a\left[\frac{y^2}{\cancel{2}}\right]_0^b = -a\left[b^2 - 0^2\right]$$

$$\int_{BC} \vec{F}\cdot d\vec{r} = -ab^2$$

Along CD: $y=b$, $dy=0$ & x varies from a to $-a$

$$\int_{CD} \vec{F}\cdot d\vec{r} = \int_a^{-a} (x^2+b^2)\,dx = \left(\frac{x^3}{3}+b^2x\right)_a^{-a}$$

$$= \left(-\frac{a^3}{3} - ab^2\right) - \left[a^3/3 + ab^2\right]$$

$$\int_{CD} \vec{F}\cdot d\vec{r} = -2a^3/3 - 2ab^2$$

Along DA: $x=-a$, $dx=0$ & y varies from b to 0

$$\int_{DA} \vec{F}\cdot d\vec{r} = \int_b^0 0 - 2(-a)y\,dy = 2a\, y^2/2\Big|_b^0$$

$$= a(0-b^2) = -ab^2$$

Thus
$$LHS = \int_C \vec{F}\cdot d\vec{r} = 2a^3/3 - ab^2 - 2a^3/3 - 2ab^2 - ab^2$$

$$\underline{LHS = -4ab^2}$$

Next, $$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & K \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$\text{curl}\,\vec{F} = -4yK$$

$$\therefore \text{curl}\,\vec{F}\cdot\hat{n}\,ds = -4y\,dx\,dy$$

now $$RHS = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \int_{x=-a}^{a}\int_{y=0}^{b} -4y\,dy\,dx$$

$$= -4\int_{x=-a}^{a} \left(\frac{y^2}{2}\right)\Big|_0^b dx = -2\int_{x=-a}^{a} (b^2-0^2)\,dx.$$

$$= -2b^2(x)\Big|_{-a}^{a} = -2b^2(a+a) = .$$

$$\boxed{RHS = -4ab^2}$$

Hence Stoke's theorem Verified.

3. verify Stokes's theorem for the vector $\vec{F} = (2x-y)i - yz^2 j - y^2 z k$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$, c is its boundary.

Sol:- $\vec{F}\cdot d\vec{r} = [(2x-y)i - yz^2 j - y^2 z k]\cdot[dx\,i + dy\,j + dz\,k]$

$$\vec{F}\cdot d\vec{r} = (2x-y)\,dx - yz^2\,dy - y^2 z\,dz$$

Here c is the circle

i.e., $x^2+y^2=1$ $(\because z=0)$.

$$\therefore \int_c \vec{F}\cdot d\vec{r} = \int_c (2x-y)\,dx - 0 - 0$$

take $x=\cos\theta$, $y=\sin\theta$.

$dx = -\sin\theta\,d\theta$ and $0 \le \theta \le 2\pi$

$$\therefore \int_c \vec{F}\cdot d\vec{r} = \int_{\theta=0}^{2\pi} [2\cos\theta - \sin\theta](-\sin\theta\,d\theta)$$

$$= \int_{\theta=0}^{2\pi} [-2\sin\theta\cos\theta + \sin^2\theta]\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[-\sin 2\theta + \frac{1-\cos 2\theta}{2}\right] d\theta.$$

$$= \left[\frac{\cos 2\theta}{2} + \frac{1}{2}\theta - \frac{1}{2}\cdot\frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$LHS = \int_C \vec{F}\cdot d\vec{r} = \left[\frac{\cos 4\pi}{2} + \frac{2\pi}{2} - \frac{\sin 4\pi}{4}\right] - \left[\frac{1}{2} + 0 - 0\right]$$

$$= \left(\frac{1}{2} + \pi - 0\right) - \frac{1}{2}$$

$\because \sin 0 = 0$

$\cos 0 = 1$

$\sin n\pi = 0$

$$LHS = \int_C \vec{F}\cdot d\vec{r} = \underline{\underline{\pi}}$$

Next, $$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$\text{curl}\,\vec{F} = i\left[-2yz - (-2yz)\right] - j\left[0-0\right] + k\left[0-(-1)\right]$$

$$\text{curl}\,\vec{F} = 0i + 0j + 1k$$

WKT $\hat{n}\,ds = d\vec{s} = dy\,dz\,i + dz\,dx\,j + dx\,dy\,k.$

$\therefore \text{curl}\,\vec{F}\cdot\hat{n}\,ds = dx\,dy.$

now $$RHS = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$= \iint_S dx\,dy$$

RHS. = Area of the curve [circle].

$$RHS = \underline{\underline{\pi}}$$

$\therefore$ Stoke's theorem is verified.

4. Verify Stoke's theorem for $\vec{F} = yi + zj + xk$ where S is the upper half of the sphere $x^2+y^2+z^2=1$ and C is the boundary.

Sol. $\vec{F}\cdot d\vec{r} = (yi + zj + xk)\cdot(dx\,i + dy\,j + dz\,k)$

$\vec{F}\cdot d\vec{r} = y\,dx + z\,dy + x\,dz.$

Here C is the circle, $x^2+y^2=1$ $(\because z=0)$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C y\,dx + 0 + 0$$

take $x = \cos\theta$, $y = \sin\theta$

$dx = -\sin\theta\,d\theta$ and $0 \le \theta \le 2\pi$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{\theta=0}^{2\pi} \sin\theta(-\sin\theta\,d\theta)$$

$$= -\int_{\theta=0}^{2\pi} \sin^2\theta\,d\theta = -\int_{\theta=0}^{2\pi} \frac{(1-\cos 2\theta)}{2}\,d\theta$$

$$= -\frac{1}{2}\int_{\theta=0}^{2\pi} (1-\cos 2\theta)\,d\theta$$

$$= -\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_{\theta=0}^{2\pi}$$

$$= -\frac{1}{2}\left\{\left[2\pi - \frac{\sin 4\pi}{2}\right] - \left[0 - \frac{\sin 0}{2}\right]\right\}$$

$$= -\frac{1}{2}\left[2\pi - 0 - 0 + 0\right]$$

LHS $= \int \vec{F}\cdot d\vec{r} = -\pi$

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$\text{curl}\,\vec{F} = i[0-1] - j[1-0] + k[0-1]$$

$$\text{curl}\,\vec{F} = -i - j - k$$

WKT $\hat{n}\,ds = d\vec{s} = dy\,dz\,i + dz\,dx\,j + dx\,dy\,k$

$(\because z$

$\therefore \text{curl}\,\vec{F} \cdot \hat{n}\,ds = 0 + 0 + (-1)\,dx\,dy$

$\text{curl}\,\vec{F} \cdot \hat{n}\,ds = -dx\,dy$

NOW,

$$RHS = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\,ds$$

$$= -\iint_S dx\,dy$$

RHS = − Area of curve (circle)

$$= -\pi(1)^2$$

$$RHS = -\pi$$

Thus Stoke's theorm is Verified

Gauss - Divergence theorem :

Statement: If V is the volume bounded by a surface S and $\vec{F}$ is continuously differentiable vector function then

$$\iiint_V div\vec{F}\, dV = \iint_S \vec{F}\cdot\hat{n}\, ds$$

Problems:

1. If $\vec{F} = 2xyi + yz^2j + xzk$ and S is the rectangular parallelopiped bounded by $x=0,\ y=0,\ z=0,\ x=2,\ y=1,\ z=3$. Find the flux across S.

Sol: Flux across $S = \iint_S \vec{F}\cdot\hat{n}\, ds$

By divergence theorem, WKT

$$\text{Flux} = \iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V div\vec{F}\, dv.$$

$$\text{Flux} = \iiint_V div\vec{F}\, dv \quad \text{———} (1).$$

Since $\vec{F} = 2xyi + yz^2j + xzk$

WKT, $div\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Scanned with CamScanner

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(xz)$$

$$\operatorname{div}\vec{F} = 2y + z^2 + x.$$

$$\therefore \text{flux} = \int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{3} (2y + z^2 + x)\, dz\, dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{1} \left(2yz + \frac{z^3}{3} + xz\right)\Big|_{z=0}^{3} dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{1} \left\{\left[6y + \frac{3^3}{3} + 3x\right] - (0 + 0 + 0)\right\} dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{1} (6y + 9 + 3x)\, dy\, dx$$

$$= \int_{x=0}^{2} \left[6 \cdot \frac{y^2}{2} + 9y + 3xy\right]_{y=0}^{1} dx.$$

$$= \int_{x=0}^{2} \left\{[3 + 9 + 3x] - [0 + 0 + 0]\right\} dx.$$

$$= \int_{x=0}^{2} (12 + 3x)\, dx$$

$$= \left[12x + \frac{3x^2}{2}\right]_{x=0}^{2}$$

$$= \left[24 + \frac{3(4)}{2}\right] - [0 + 0]$$

$$\text{flux} = \underline{\underline{30}}$$

2. Evaluate $\iint_S (ax\,i + by\,j + cz\,k) \cdot \hat{n}\,ds$ where S is surface of the sphere $x^2+y^2+z^2=1$

Sol:- let $\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$

From divergence theorem, we have

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv. \quad \text{---} (1).$$

WKT,
$$\operatorname{div}\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)$$

$$\operatorname{div}\vec{F} = a + b + c$$

Thus Eqn (1) becomes

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (a+b+c)\,dv.$$

$$= (a+b+c)\underbrace{\iiint dv}_{V}.$$

$= (a+b+c)$ volume of the sphere.

$$= (a+b+c)\left[\frac{4}{3}\cdot\pi\cdot(1)^3\right] \qquad \because r=1$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = (a+b+c)\cdot\frac{4\pi}{3}$$

3. Verify divergence theorem for the vector function $\vec{F} = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$. taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

Sol:- By divergence theorem, we have

$$\iiint_V div\vec{F}\, dv = \iint_S \vec{F} \cdot \hat{n}\, ds$$

Now, $LHS = \iiint_V div\vec{F}\, dv$

But $div\vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$

$div\vec{F} = 2x + 2y + 2z$

$$\therefore LHS = 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c}(x+y+z)\, dz\, dy\, dx.$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\left[xz + yz + \frac{z^2}{2}\right]_{z=0}^{c} dy\, dx.$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\left\{\left[cx + cy + \frac{c^2}{2}\right] - (0+0+0)\right\} dy\, dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\left[cx + cy + c^2/2\right] dy\, dx$$

$$= 2\int_{x=0}^{a}\left[cxy + c.\frac{y^2}{2} + \frac{c^2}{2}y\right]_{y=0}^{b} dx.$$

$$= 2\int_{x=0}^{a}\left\{\left[bcx+\frac{b^2c}{2}+\frac{bc^2}{2}\right]-\left[0+0+0\right]\right\}dx$$

$$= 2\int_{x=0}^{a}\left(bcx+\frac{b^2c}{2}+\frac{bc^2}{2}\right)dx$$

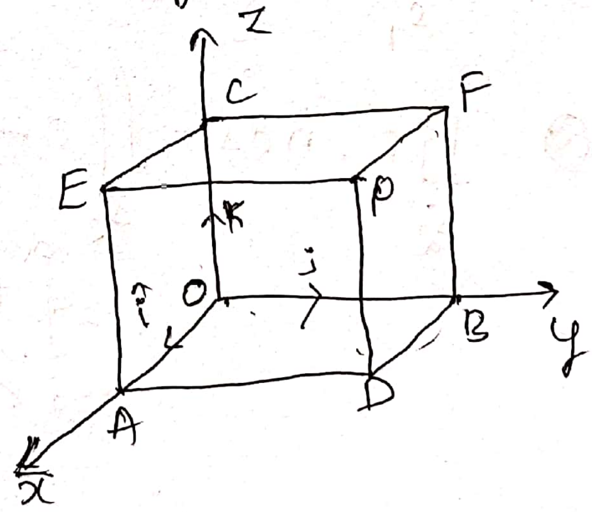
$$= 2\left[bc\cdot\frac{x^2}{2}+\frac{b^2c}{2}x+\frac{bc^2}{2}x\right]_{x=0}^{a}$$

$$= 2\left\{\left[\frac{a^2bc}{2}+\frac{ab^2c}{2}+\frac{abc^2}{2}\right]-\left[0+0+0\right]\right\}$$

$$= \frac{2}{2}\left[a^2bc+ab^2c+abc^2\right]$$

$$LHS = \underline{\underline{abc\,[a+b+c]}}$$

Next, $RHS = \iint_S \vec{F}\cdot\hat{n}\,ds$ is to be determined over 6 faces of the rectangular parallelopiped.



S_1: OADB

S_2: OCEA $\quad$ S_3: OBFC

S_4: CEPF $\quad$ S_5: BFPD.

S_6: ADPE.

The unit outward normals to these faces ($\hat{n}$) are respectively, $-\hat{k}, -\hat{j}, -\hat{i}, \hat{k}, \hat{j}, \hat{i}$.

we have

$$\vec{F}\cdot\hat{n}\,ds = \left[(x^2-yz)i + (y^2-zx)j + (z^2-xy)k\right]\cdot \left[dy\,dz\,i + dz\,dx\,j + dx\,dy\,k\right]$$

① $$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} 0 + 0 + (+xy)\,dx\,dy \qquad (z=0,\ \hat{n}=-k)$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} xy\,dy\,dx.$$

$$= \int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_{y=0}^{b} dx = \int_{x=0}^{a} x\left(\frac{b^2}{2}-0\right)dx.$$

$$= \left[\frac{b^2}{2}\frac{x^2}{2}\right]_0^a = \frac{b^2}{2}\left[\frac{a^2}{2}-0\right]$$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = a^2b^2/4$$

② $$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint_{S_2} 0 + (-zx)\,dz\,dx + 0 \qquad (y=0,\ \hat{n}=-j)$$

$$= \int_{x=0}^{a}\int_{z=0}^{c} zx\,dz\,dx$$

$$= \int_{x=0}^{a}\left(\frac{z^2}{2}x\right)_{z=0}^{c} dx = \int_{x=0}^{a} x\left(\frac{c^2}{2}-0\right)dx$$

$$= c^2/2\,\left(x^2/2\right)\Big|_0^a = \frac{c^2}{2}\left[\frac{a^2}{2}-0\right]$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = a^2c^2/4.$$

Scanned with CamScanner

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \iint_{S_3} (0-yz)(-dy\,dz) + 0 + 0 \qquad (x=0,\ \hat{n}=-\hat{i})$$

$$= \int_{y=0}^{b}\int_{z=0}^{c} yz\,dz\,dy = \int_{y=0}^{b} y\cdot\frac{z^2}{2}\Big|_{z=0}^{c} dy$$

$$= \int_{y=0}^{b} y\cdot\left(\frac{c^2}{2}-0\right)dy =$$

$$= \frac{c^2}{2}\left(\frac{y^2}{2}\right)\Big|_{y=0}^{b} = \frac{c^2}{2}\left(\frac{b^2}{2}-0\right)$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = b^2c^2/4$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \iint_{x=0\ y=0}^{a\ \ b} (c^2-xy)\,dx\,dy \qquad (z=c,\ \hat{n}=\hat{k})$$

$$= \int_{x=0}^{a}\left[c^2y - x\,y^2/2\right]_{y=0}^{b} dx$$

$$= \int_{x=0}^{a}\left\{\left(c^2b - x\,b^2/2\right) - (0-0)\right\}dx$$

$$= \int_{x=0}^{a}\left(bc^2 - \frac{b^2}{2}x\right)dx$$

$$= \left(bc^2x - \frac{b^2}{2}\,x^2/2\right)\Big|_{x=0}^{a}$$

$$= \left(bc^2a - \frac{b^2}{2}\,a^2/2\right) - (0-0)$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = abc^2 - a^2b^2/4$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{a}\int_{z=0}^{c} 0 + (b^2 - zx)\,dz\,dx + 0 \qquad (y=b,\ \hat{n}=\hat{j})$$

$$= \int_{x=0}^{a} \left[b^2 z - x\,z^2/2\right]_{z=0}^{c} dx$$

$$= \int_{x=0}^{a} \left\{\left[b^2 c - x\,c^2/2\right] - (0-0)\right\} dx$$

$$= \left(b^2 c x - \frac{c^2}{2}\cdot x^2/2\right]_{x=0}^{a}$$

$$= \left[b^2 ca - \frac{c^2}{2}\,a^2/2\right) - (0-0)$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = ab^2c - a^2c^2/4$$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_{z=0}^{c}\int_{y=0}^{b} (a^2 - yz)\,dy\,dz + 0 + 0 \qquad (x=a,\ \hat{n}=\hat{i})$$

$$= \int_{z=0}^{c} \left[a^2 y - z\,y^2/2\right]_{y=0}^{b} dz = \int_{z=0}^{c} \left\{\left[a^2 b - z\frac{b^2}{2}\right] - (0-0)\right\} dz$$

$$= \left[a^2 b z - \frac{b^2}{2}\cdot\frac{z^2}{2}\right]_{z=0}^{c} = \left[a^2bc - \frac{b^2}{2}\frac{c^2}{2}\right] - [0-0]$$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = a^2bc - \frac{b^2c^2}{4}$$

Thus $RHS = \iint_S \vec{F}\cdot\hat{n}\,ds = \cancel{\frac{a^2b^2}{4}} + \cancel{\frac{a^2c^2}{4}} + \cancel{\frac{b^2c^2}{4}} + abc^2 - \cancel{\frac{a^2b^2}{4}}$

$$+ab^2c - \cancel{\frac{a^2c^2}{4}} + a^2bc - \cancel{\frac{b^2c^2}{4}}$$

$$= abc(a+b+c)$$

$RHS = LHS$

Thus divergence theorem is Verified.