Queue Simulation Results

Discrete-Event Simulation Output

50,000 Customers per Test Case

Restaurant Wait Time Modeling Project

1 Simulation Configuration

- **Simulation type:** Discrete-event simulation
- **Customers per test:** 50,000 (5,000 warmup discarded)
- **Random seed:** 42 (reproducible)
- **Service discipline:** First-Come-First-Served (FCFS)

2 Phase 1: M/M/1 and M/M/k Validation

2.1 Test 1: M/M/1 Queue (Moderate Load)

Parameters: $\lambda = 0.8$, $\mu = 1.0$, $k = 1$, $\rho = 0.80$

Analytical Results:

Metric	Value
ρ	0.800000
$E[L]$	4.000000
$E[W]$	5.000000
$E[W_q]$	4.000000
$E[L_q]$	3.200000

Simulation Results:

Metric	Value
num_customers	45,001
avg_wait_in_queue	4.322459
avg_time_in_system	5.328370
avg_queue_length	3.393343
avg_system_length	4.199331
simulation_time	62,362.21
prob_wait	0.809782

Comparison:

Metric	Simulation	Analytical	Error %
$E[W_q]$	4.322459	4.000000	8.06%
$E[W]$	5.328370	5.000000	6.57%
$E[L]$	4.199331	4.000000	4.98%

Verdict: Good agreement. 5-8% error is typical for 45,000 samples.

2.2 Test 2: M/M/1 Queue (High Load)

Parameters: $\lambda = 0.95$, $\mu = 1.0$, $k = 1$, $\rho = 0.95$

Analytical Results:

Metric	Value
ρ	0.950000
$E[L]$	19.000000
$E[W]$	20.000000
$E[W_q]$	19.000000
$E[L_q]$	18.050000

Simulation Results:

Metric	Value
num_customers	45,001
avg_wait_in_queue	24.866331
avg_time_in_system	25.872242
avg_queue_length	22.556367
avg_system_length	23.512980
simulation_time	52,542.87
prob_wait	0.959068

Comparison:

Metric	Simulation	Analytical	Error %
$E[W_q]$	24.866331	19.000000	30.88%
$E[W]$	25.872242	20.000000	29.36%
$E[L]$	23.512980	19.000000	23.75%

Verdict: Expected deviation. Near $\rho = 1$, variance explodes: $\mathrm{Var}[L] = \rho/(1-\rho)^2 = 380$. Would need 500,000+ samples for tight convergence.

2.3 Test 3: M/M/3 Queue (Multi-Server)

Parameters: $\lambda = 2.0$, $\mu = 1.0$, $k = 3$, $\rho = 0.6667$

Analytical Results:

Metric	Value
ρ	0.666667
π_0	0.111111
$P(\text{wait})$	0.444444
$E[L_q]$	0.888889
$E[L]$	2.888889
$E[W_q]$	0.444444
$E[W]$	1.444444

Simulation Results:

Metric	Value
num_customers	45,001
avg_wait_in_queue	0.477445
avg_time_in_system	1.483353
avg_queue_length	0.937559
avg_system_length	2.952623
simulation_time	24,944.38
prob_wait	0.457545

Comparison:

Metric	Simulation	Analytical	Error %
$E[W_q]$	0.477445	0.444444	7.43%
$E[W]$	1.483353	1.444444	2.69%
$E[L]$	2.952623	2.888889	2.21%
$P(\text{wait})$	0.457545	0.444444	2.95%

Verdict: Excellent agreement. 2-7% error validates Erlang-C formula.

2.4 Test 4: M/M/5 Queue (Multi-Server)

Parameters: $\lambda = 4.0$, $\mu = 1.0$, $k = 5$, $\rho = 0.80$

Analytical Results:

Metric	Value
ρ	0.800000
π_0	0.012987
$P(\text{wait})$	0.554113
$E[L_q]$	2.216450
$E[L]$	6.216450
$E[W_q]$	0.554113
$E[W]$	1.554113

Simulation Results:

Metric	Value
num_customers	45,001
avg_wait_in_queue	0.603261
avg_time_in_system	1.609160
avg_queue_length	2.375725
avg_system_length	6.405527
simulation_time	12,473.22
prob_wait	0.574965

Comparison:

Metric	Simulation	Analytical	Error %
$E[W_q]$	0.603261	0.554113	8.87%
$E[W]$	1.609160	1.554113	3.54%
$E[L]$	6.405527	6.216450	3.04%
$P(\text{wait})$	0.574965	0.554113	3.76%

Verdict: Good agreement. 3-9% error confirms M/M/k implementation.

3 Phase 2: Assumption Failure Analysis

3.1 M/M/1 with Bursty Arrivals

Parameters: $\lambda = 0.8$, $\mu = 1.0$. All scenarios calibrated to same effective arrival rate.

M/M/1 Analytical (Poisson assumption):

$E[W]$	5.0000
$E[W_q]$	4.0000
$E[L]$	4.0000

Results Comparison:

Arrival Type	$E[W_q]$	$E[W]$	$E[L]$	vs M/M/1
M/M/1 Theory	4.0000	5.0000	4.0000	baseline
Poisson (sim)	4.3225	5.3284	4.1993	+6.6%
Batch ($\mu_B = 2$)	8.6223	9.6218	7.9088	**+92.4%**
Hawkes	7.9380	8.9384	6.9977	**+78.8%**

Key Insights:

- Batch arrivals increase wait time by ∼92% vs Poisson prediction
- Hawkes (self-exciting) increases wait time by ∼79% vs Poisson prediction
- M/M/1 formulas **underestimate** congestion when arrivals are bursty

3.2 M/M/4 with Bursty Arrivals

Parameters: $\lambda = 3.2$, $\mu = 1.0$, $k = 4$, $\rho = 0.80$

M/M/4 Analytical (Poisson assumption):

$E[W]$	1.7455
$E[W_q]$	0.7455
$P(\text{wait})$	0.5964

Results Comparison:

Arrival Type	$E[W_q]$	$E[W]$	$P(\text{wait})$	vs Theory
M/M/k Theory	0.7455	1.7455	0.5964	baseline
Poisson (sim)	0.8123	1.8182	0.6158	+4.2%
Batch ($\mu_B = 2$)	1.7725	2.7720	0.7524	**+58.8%**
Hawkes	1.5077	2.5050	0.6752	**+43.5%**

4 Phase 3: M/M/1 vs M/M/k Burstiness Comparison

Question: Does M/M/k handle burstiness better than M/M/1?
Both systems at $\rho = 0.8$ utilization.

System	Batch Penalty	Hawkes Penalty
M/M/1	+92.4%	+78.8%
M/M/4	+58.8%	+43.5%

Conclusion:

- M/M/1 batch penalty: +92% — M/M/4 batch penalty: +59%
- M/M/1 Hawkes penalty: +79% — M/M/4 Hawkes penalty: +44%
- Multiple servers reduce both absolute wait times and relative burstiness penalty
- This is because k servers can absorb short bursts before queue builds

5 Summary

5.1 Validation Tests

Test	System	Utilization	Error Range	Status
Test 1	M/M/1	$\rho = 0.80$	5-8%	Pass
Test 2	M/M/1	$\rho = 0.95$	24-31%	Expected
Test 3	M/M/3	$\rho = 0.67$	2-7%	Pass
Test 4	M/M/5	$\rho = 0.80$	3-9%	Pass

5.2 Assumption Failure Impact

Arrival Type	M/M/1 Penalty	M/M/4 Penalty	Reduction
Batch ($\mu_B = 2$)	+92.4%	+58.8%	36% better
Hawkes ($\alpha = 0.4$)	+78.8%	+43.5%	45% better

5.3 Key Takeaways

1. M/M/1 and M/M/k formulas validated — simulation matches theory within expected variance
2. High utilization ($\rho \to 1$) causes variance explosion — larger samples needed
3. Bursty arrivals increase wait times by 60-90% compared to Poisson predictions
4. Multiple servers partially mitigate burstiness penalty (reduce by 35-45%)
5. Poisson assumption matters — real-world arrival patterns are often bursty