Analogy between **Signals** and **Vectors**

As we know, a **vector space V**, like : **"Euclidian Space"** is a set of **vectors, whose elements, the vectors, satisfy certain conditions.**

- The **sum** of any two vectors belonging to V is another vector which also belongs to V.
- There exists in V, a vector, called the **zero vector**, denoted by O), which is such that A + O = A, for any vector A belonging to V.
- For any vector A belonging to V, there exists another vector B, also belonging to V and such that **A + B = the zero vector O**.
- If any vector, say A belonging to V is **multiplied by a** scalar (a real number or complex number), the **resultant vector also belongs to V.**

F1 F2 F3 F4 F5 F6 F7

by its own magnitude.

Linearly Independent Vectors

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The **vectors i, j and k** are said to be **linearly independent** in the sense that **none of them can be expressed as a linear combination of the rest.**

Example : Show that the vectors : X=(1,6,5) , Y=(1,1,0) and Z=(7,5,2) belonging to the **Euclidian space** are **linearly independent**.
Solution :

- if X, Y, and Z are **linearly independent** THEN (see the definition)
- if X, Y, and Z are **NOT linearly independent** THEN : **det(A) = 0**.

Basis set

Basis set

A **set of vectors** belonging to a **vector space V** (like the Euclidian space) which are **linearly independent** and which **span** (or generate) **that space**, are said to be **forming a basis set for that space.**

Thus, **i, j and k form a basis set of vectors for the Euclidian space** since they **are linearly independent** and also **span the Euclidian space** in the sense that **any vector belonging to that space can be expressed as a linear combination of these three vectors.**

Note :

A **basis set** for a **vector space** is **not unique**, since there can be any number of basis sets (For example, rotation of the X, Y, Z axes creates a **new set of basis vectors**)

Orthogonality of vectors

- Two vectors are said to be orthogonal to each other if their dot product is zero
- A set of vectors are said to be an orthogonal set of vectors if any two distinct vectors from the set are orthogonal to each other.

- The vectors i, j and k are not only linearly independent, they are also mutually orthogonal.
- If, in addition, each one of the vectors of the set has a unit magnitude, the set is said to be an orthonormal set.

Dimension

Dimension

- The **number of vectors** in **any basis set for a given space will be the same** and this **number is called the dimension of that space**.
- It represents the **minimum number of linearly independent vectors** required to **generate that vector space**.

An n-dimensional vector space $(n > 3)$ in which each vector will be a point in an n-dimensional space and can be represented by a column vector having the n-coordinates of that point as the entries. Thus, if a is a vector in an n-dimensional vector space.

$$\boldsymbol{\alpha} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdot \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ \cdot \end{bmatrix} + \ldots + x_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \end{bmatrix}$$

Example :

Given two vectors : $X = (0, 2, 1)$ and $Y = (1, -2, 1)$:

1. Find the magnitude of each vector.
2. Check whether they are orthogonal.
3. If they are not orthogonal, find the angle between them.

Component of a Vector along Another Vector

Component of a Vector **along** Another Vector

- Consider **n non-zero orthogonal vectors**, $\alpha_1, \alpha_2, \ldots \alpha_n$. consider the following **linear combination of these n vectors :**

$$\beta = x_1.\alpha_1 + x_2.\alpha_2 + \cdots + +x_k.\alpha_k + \cdots + x_n.\alpha_n$$

where $x_1, x_2, \ldots, x_n$ are some **real numbers** and β is the **resultant vector.**

- Taking the **dot product** of both sides with the vector α_k, we get :

$$\beta.\alpha_k = x_1.(\alpha_1.\alpha_k) + x_2.(\alpha_2.\alpha_k) + \cdots + +x_k.(\alpha_k.\alpha_k) + \cdots + x_n.(\alpha_n.\alpha_k)$$

- Since $\alpha_i, i = 1, 2, 3, \ldots, n$ are **orthogonal vectors** all the products on the **RHS** are **zero**, except $(\alpha_k.\alpha_k)$,which we know, is equal to $|\alpha_k|^2$.

$$\beta.\alpha_k = x_k.|\alpha_k|^2$$

Hence,

$$x_k = \frac{\beta.\alpha_k}{|\alpha_k|^2} \quad \text{is coordinate of} \quad \beta \quad \textbf{along} \quad \alpha_k$$

and the component of β **along** α_k is $x_k.\alpha_k = \frac{\beta.\alpha_k}{|\alpha_k|^2}.\alpha_k$

Component of a Vector along Another Vector

Example :

For the two vectors X =(0,2,1) and Y= (1,-2,1), find the **component of Y along X**.

Solution : (X , Y) = –3 and $|X| = 5$, so the **component** of Y along X is :

$$\frac{Y.X}{|X|^2}.X = -\frac{3}{5}.(0,2,1) = (0,-\frac{6}{5},-\frac{3}{5}) = (0,-1.2,-0.6)$$

Signal Spaces

- In **digital communications**, in general, **one of a set of M** ($M \geq 2$) **possible signals**, $s_i(t), i = 1, 2, 3, \ldots, M$,**is transmitted every T sec**.
- **These M signals are known a priori to the receiver**. What the receiver **does not know,** however, is, **which one of the M signals has been transmitted during a given T sec period**
- The **job of the receiver** is then to **correctly identify, during each T sec period, the transmitted signal**, in the presence of noise.
- These **M signals** are **continuous-time real-valued signals** having a **finite energy over a T sec period**, i.e., the time period for which one of the M signals is transmitted.

- Let us now consider the set S of all possible continuous-time signals having a **finite energy over a period of T sec.**
- These **signals** can easily be shown to satisfy all the four important conditions (vectors)
  1. Any **two signals**, if added, will still give us a signal that is again continuous-time and having a finite energy over the period [0, T]
  2. There exists a **zero signal** in S. (over the [0, T] and so is a continuous-time signal with zero energy over that interval)
  3. For energy signal, s(t), over [0, T], there exists in S, another signal −**s(t)** over the same interval, and the sum of these two yields the zero signal
  4. if any $s(t)$ belonging to S is **multiplied by a real number** (or a complex number), the resulting signal is also a continuous-time signal with some finite energy over [0, T], and thus belong to S.
- **Thus, the set S of all continuous-time signals with finite energy, defined over [0, T]**, forms what we may call as a **"signal space"**, which is analogous to a " vector space". It is **formed by a set of signals satisfying certain (four) conditions**

Linear Independence of Signals

Now that the **analogy between vectors and signals** has been **clearly established**, we can extend the concept like "**linear independence**", "**basis set**", "**dimension**", "**orthogonality**" associated with vectors to the signals too.

1-Linear Independence of Signals

A **set of signals** is said to be a **linearly independent set of signals**, provided **none of them can be expressed as a linear combination of the rest.**

2-Basis set for a signal space

A **basis set B** for a **signal space S**, is a **set of linearly independent signals** which **span (i.e., generate) the signal space S.**

=**the signals in a basis set should not only be linearly independent** but **should also be able to generate any signal belonging to S through a** linear combination **of some or all of them (basis signals).**

3-Dimension of a signal space

The **dimension** of a **signal space** S is the **number of basis signals** in any **basis set** for S.

Orthogonality of Signals

- **For vectors : two distinct vectors A and B are orthogonal** if their **dot product is zero**.
- For **two distinct signals** $s_1(t)$ and $s_2(t)$ are **orthogonal** to each other over $[t_1 t_2]$ if **their inner product over that interval is zero** :

$$\boxed{< s_1(t), s_2(t) >= \int_{t_1}^{t_2} s_1(t).s_2^*(t).dt = 0} \tag{6}$$

Norm or Length of a Signal

$$< s_i(t), s_i(t) >= \|s_i(t)\|^2 = \int_{t_1}^{t_2} s_i(t).s_i^*(t)dt = \int_{t_1}^{t_2} |s_i(t)|^2 dt = E_s$$

E_s : Energy of the signal $s_i(t)$ over the interval t_1 to t_2.
Norm or length of the finite energy signal $s_i(t)$ is $\sqrt{E_s}$

Distance between Two Signals

Definition

The **distance between two signals** $s_i(t)$ and $s_j(t)$, is defined as the **norm of their difference** :

$$\text{distance} = \quad \|s_i(t) - s_j(t)\|$$

$$< (s_i(t) - s_j(t)), (s_i(t) - s_j(t)) >^{\frac{1}{2}} = \left[\int_{t_1}^{t_2} |[s_i(t) - s_j(t)]|^2\right]^{\frac{1}{2}}$$

$$d_{s_i \text{ to } s_j} = \sqrt{E_{s_k(t)}}$$

where $s_k(t) \triangleq s_i(t) - s_j(t)$

- A set S of non-zero signals $s_i(t), i = 1, 2, 3, \cdots$ is said to be **orthogonal set** over $[t_1 t_2]$ if :

$$\int_{t_1}^{t_2} s_i(t).s_j^*(t)dt = \begin{cases} 0 & \text{for } i \neq j \\ \text{constant k}, & \text{for } i = j \end{cases}$$

  The inner product for any two distinct signals must be ZERO.

- In case , $k = 1$, then the signals are said to form an **orthonormal set since every signal in the set has a norm = 1.**

Any signal s(t) can be normalized so as to have a unit norm by divising the signal by its own norm :

$$\| \frac{s(t)}{\|s(t)\|} \|$$

Example :

Show that the signals $x_n(t) = A.\cos(nw_0 t), n = 0, 1, 2, \ldots,$ where $w_0 = \frac{2\pi}{T}$, form a **set of orthogonal functions** over $[0, T]$.