Interactive EM Math

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1 Symbols

For the remainder of this text $\vec{p}$ will be the point that the test charge is placed at and the distance from a point charge is r. Each object has a center $\vec{c}$ and rotation θ, or it's position is defined in a unique way in it's section. Finally, each has either a charge Q, linear charge density λ, or an area charge density σ. The symbol K will be used as Coloumb's constant. Definitionally, voltage is $V = \frac{KQ}{r}$ and electric field is $\vec{E} = \frac{KQ}{r^2}\hat{r}$ where $\hat{r}$ points away from the source. It also follows that $\vec{E} = -\nabla V$.

2 Point charges

2.1 Voltage

The voltage of a point charge is definitionally defined as

$$V = \frac{KQ}{||\vec{p} - \vec{c}||} \tag{1}$$

2.2 Electric Field

The electric field of a point charge is also definitionally defined. The field of a point charge is simply $\vec{E} = \frac{KQ}{r^2}\frac{\vec{r}}{r}$.

$$\frac{KQ}{||\vec{p} - \vec{c}||^3}(\vec{p} - \vec{c}) \tag{2}$$

2.3 Torque

Point charges have no radius, so they have no moment of inertia and also have no torque put upon them.

3 Infinite Plane

3.1 Voltage

The integral of $\frac{KQ}{r}$ over the surface of an infinite plane is infinity, we will use the path integral of integral of $\vec{E} = 2\pi K\sigma$, which is already known. The electric field is obviously constant, so moving a distance d away from the plane will decrease the potential energy by $2\pi K\sigma d$. Adding on a constant accounts for the possible infinity that results from the other method. In reality, this constant could be a reasonable number because all voltage is relative. Anyway, we still have to adjust to fit the confines of the program, including "center" of the plane (which will be a reference for a point on the plane).

From vector math, we know the distance from a plane is $d = \vec{n} \cdot \vec{P_0 P}$, so the distance in our program is $d = \langle -\cos\theta, \sin\theta\rangle \cdot (\vec{p} - \vec{c})$. Therefore, the final formula is shown below

$$V(\vec{p}) = C - 2\pi K\sigma\langle -\cos\theta, \sin\theta\rangle \cdot (\vec{p} - \vec{c}) \tag{3}$$

3.2 Electric Field

The field of an infinite plane is well-known to be $2\pi K\sigma\hat{r}$. In our program, we have to figure out how to determine the $\hat{r}$, but it's obviously either equal to $\vec{n}$ or $-\vec{n}$. We used the sign function with the dot product of the normal and a vector from the plane to the point, which will be positive if they face the same direction and negative if they face the opposite. The final formula is

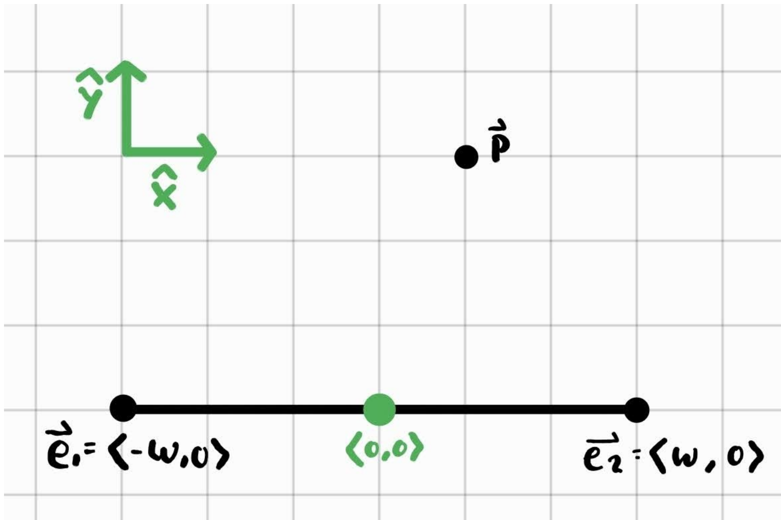
$$\vec{E}(\vec{p}) = 2\pi K\sigma\vec{n}\ \mathrm{sgn}((\vec{p} - \vec{c}) \cdot \vec{n}) \text{ where } \vec{n} = \langle -\sin(\theta), \cos(\theta)\rangle \tag{4}$$

3.3 Torque

Planes are infinite so their moment of inertia is infinity and torque on them has no effect.

4 Finite Line

For the finite line, it also has the property w which is half of the width. We will start by assuming that the line's center is $\vec{0}$ and the rotation is zero, so the line will be along the x-axis.



4.1 Voltage

$$
\begin{aligned}
V &= \int_{-w}^{w} \frac{K\lambda dx}{\sqrt{(p_x - x)^2 + p_y^2}} \\
&\text{let } x = p_x - x \\
&dx = -dx \\
&= -K\lambda \int_{p_x+w}^{p_x-w} \frac{dx}{\sqrt{x^2+y^2}} \\
&= K\lambda \int_{p_x-w}^{p_x+w} \frac{dx}{\sqrt{x^2+y^2}} \\
&= K\lambda \ln\left(\sqrt{x^2+y^2}+x\right)|_{p_x-w}^{p_x+w} \\
&= K\lambda \ln\left|\frac{\sqrt{(p_x+w)^2+p_y^2}+p_x+w}{\sqrt{(p_x-w)^2+p_y^2}+p_x-w}\right|
\end{aligned}
$$

Define $\vec{e_1} = \vec{c} - w\langle\cos(\theta), \sin(\theta)\rangle$ and $\vec{e_2} = \vec{c} + w\langle\cos(\theta), \sin(\theta)\rangle$ to be the end-points of the line.

$$V = K\lambda \ln\left(\frac{\|\vec{p} - \vec{e_1}\| + p_x + w}{\|\vec{p} - \vec{e_2}\| + p_x - w}\right)$$

In order to adjust for the angle and center of the line, we will replace p_x with the linear transformation $\vec{p} \cdot \langle\cos(\theta), sin(\theta)\rangle$ and rename it as g.

$$V(\vec{p}) = K\lambda \ln\left(\frac{\|\vec{p} - \vec{e_1}\| + g + w}{\|\vec{p} - \vec{e_2}\| + g - w}\right) \tag{5}$$

4.2 Electric Field

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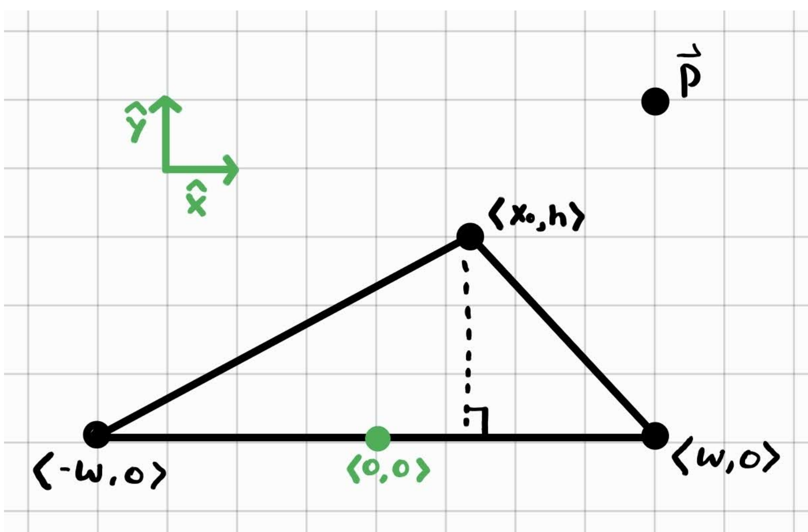
4.3 Torque

The moment of inertia of a finite line about the center is well known to be $\frac{1}{12}ML^2$.

5 Triangle

For this section, we will split the "shape" class into several triangles. We will make several assumptions about the positioning of the triangle to make the initial integration easier, and we can apply those linear transformations later. To start, we will assume the hypotenuse of the triangle is along the x-axis, it's center is at the origin, and the other tip has a positive y co-ordinate. We can assume these because every triangle can be translated and rotated into this state.

$$\text{Let } w = \text{Half the width of the triangle}$$
$$\text{Let } h = \text{The height of the triangle}$$
$$\text{Let } x_0 = \text{The x coordinate of the tip relative to the center of the hypotenuse}$$



5.1 Voltage

Let's start with the following integral. We will do dx slices first so it doesn't have to be split into two integrals.

$$V(\vec{p}) = K\sigma \int_0^h \int_{\frac{y}{h}(x_0-w)+w}^{\frac{y}{h}(x_0+w)-w} \frac{dxdy}{\sqrt{(x-p_x)^2+(y-p_y)^2}}$$

$$V(\vec{p}) = K\sigma \int_{-p_y}^{h-p_y} \int_{\frac{y+p_y}{h}(x_0-w+w)-p_x}^{\frac{y+p_y}{h}(x_0+w)-w-p_x} \frac{dxdy}{\sqrt{x^2+y^2}}$$

$$\text{Given } \int \frac{dx}{\sqrt{x^2+y^2}} = \operatorname{sgn}(y)\sinh^{-1}\left(\frac{x}{y}\right)$$

$$\text{Let } a_1 = \frac{x_0-w}{h}$$

$$\text{Let } b_1 = \frac{p_y(x_0-w)}{h} + w - p_x$$

$$\text{Let } a_2 = \frac{x_0+w}{h}$$

$$\text{Let } b_2 = \frac{p_y(x_0+w)}{h} - w - p_x$$

$$V(\vec{p}) = K\sigma \int_{-p_y}^{h-p_y} \operatorname{sgn}(y)\left[\sinh^{-1}\left(a_1+\frac{b_1}{y}\right) - \sinh^{-1}\left(a_2+\frac{b_2}{y}\right)\right] dy$$

The following integral is done courtesy of integral-calculator.com.

$$g(y,a,b) = \frac{a\sqrt{\left(a+\frac{b}{y}\right)^2+1}-a}{a+\frac{b}{y}}$$

$$AD(y,a,b) = \int \sinh^{-1}\left(a+\frac{b}{y}\right) dx$$

$$= y\sinh^{-1}\left(a+\frac{b}{y}\right) + \frac{b}{\sqrt{a^2+1}} \ln\left(\left|\frac{g(y,a,b)+\sqrt{a^2+1}+1}{g(y,a,b)-\sqrt{a^2+1}+1}\right|\right)$$

Since there is a discontinuity, we will consider two cases. First, the two limits could be on the same side of zero, or the limits could cross zero. In the first case, the answer is simply:

$$V(\vec{p}) = K\sigma \operatorname{sgn}(-p_y)\left[AD(y,a_1,b_1) - AD(y,a_2,b_2)\right]_{y=-p_y}^{y=h-p_y}$$

If the limits are on opposite sides of zero, then we have to take the improper integral of both sides.

$$\text{Let } AD_0^+ = \lim_{y\to 0^+} AD(y,a,b) = \frac{b}{\sqrt{a^2+1}} \ln\left(\left|\frac{a+\sqrt{a^2+1}+1}{a-\sqrt{a^2+1}+1}\right|\right)$$
$$\text{Let } AD_0^- = \lim_{y\to 0^-} AD(y,a,b) = \frac{b}{\sqrt{a^2+1}} \ln\left(\left|\frac{-a+\sqrt{a^2+1}+1}{-a-\sqrt{a^2+1}+1}\right|\right)$$
$$\text{Let } AD_0 = -AD_0^+ - AD_0^- = \frac{-b}{\sqrt{a^2+1}} \ln\left(\left|\frac{2\sqrt{a^2+1}+2}{-2\sqrt{a^2+1}+2}\right|\right)$$
$$= \frac{b}{\sqrt{a^2+1}} \ln\left(\frac{\sqrt{a^2+1}+a^2}{\sqrt{a^2+1}+1}\right)$$
$$V = K\sigma\left[AD(h-p_y,a_1,b_1) + AD_0(a_1,b_1) + AD(-p_y,a_1,b_1)\right]$$
$$- K\sigma\left[AD(h-p_y,a_2,b_2) + AD_0(a_2,b_2) + AD(-p_y,a_2,b_2)\right]$$

5.2 Electric Field

Since the calculations for a charged triangle are really complicated, we will split the x and y directions separately.

5.2.1 Field in the $\hat{x}$ direction

5.2.2 Field in the $\hat{y}$ direction

$$\text{Let } f = h - a$$
$$\text{Let } n = m\left(p - hm\right) - a$$
$$\text{Let } j = \sqrt{n\frac{m\left(p\left(mp-2a\right)+h^2m\right)+a^2}{a-m\left(p-hm\right)}}$$
$$\text{Let } k = \sqrt{m^2+1}\sqrt{\left(m^2+1\right)u^2 - 2m\left(mp+f\right)u + m^2p^2 + 2fmp + h^2 - 2ah + a^2}$$
$$\text{Let } l = mp + h - a$$
$$p = p_x \quad a = p_y$$
$$V = \frac{\ln\left(\left|k+\left(m^2+1\right)u - m^2p + fm\right|\right)}{m\sqrt{m^2+1}}$$
$$+ \frac{2\left(mp-a\right)\arctan\left(\frac{nk+m\left(m^2+1\right)lu-n|l|-m^4p^2-2fm^3p-f^2m^2}{\sqrt{m^2+1}\,j\left(\left(m^2+1\right)u-m^2p-fm\right)}\right)}{mj}$$

5.3 Moment of inertia

For ease of calculation, we will choose a simple origin (in this case, the center of the hypotenuse) to do the integration around, then use the parallel axis theorem to move that axis. In order to continue with the common theme, I have decided to put the origin at

6 Conductors

For this entire program, we are working in 2d, so there is no obvious way to have conductors. We could either have sheets of conductor that

6.1 Setup

The goal of a conductor in this program is to have a constant voltage along the edge and a constant voltage on the inside of a conductor. We made a generic class that only takes in the location of points of charge and test points where the voltage should be measured. In order to not mess with the dimensionality, the point charges are actually line charges that extend a couple of meters in either direction because an infinitely thin conductor has no effect, and a plate conductor facing the viewer would not look as cool.

6.2 Linear Algebra

The following is the system of linear equations that we have to solve, or at least get the closest possible answers to:

$$\begin{array}{ccccccccc}
Q_1 & + & Q_2 & + & Q_3 & +\ldots+ & Q_n & = Q_{net} \\
\frac{KQ_1}{r_{11}} & + & \frac{KQ_2}{r_{12}} & + & \frac{KQ_3}{r_{13}} & +\ldots+ & \frac{KQ_n}{r_{1n}} & = C - V_1 \\
\frac{KQ_1}{r_{21}} & + & \frac{KQ_2}{r_{22}} & + & \frac{KQ_3}{r_{23}} & +\ldots+ & \frac{KQ_n}{r_{2n}} & = C - V_2 \\
\frac{KQ_1}{r_{31}} & + & \frac{KQ_2}{r_{32}} & + & \frac{KQ_3}{r_{13}} & +\ldots+ & \frac{KQ_n}{r_{3n}} & = C - V_3 \\
\vdots & & \vdots & & \vdots & \ddots & & \vdots \\
\frac{KQ_1}{r_{m1}} & + & \frac{KQ_2}{r_{m2}} & + & \frac{KQ_3}{r_{m3}} & +\ldots+ & \frac{KQ_n}{r_{mn}} & = C - V_m
\end{array}$$

Where Q_i is the ith charge point, r_ij is the distance between the ith test point and the jth charge point, C is the voltage of the conductor, and V_j is the external voltage at the jth test point. If we subtract C from both sides, this leads to a

$m+1$ by $n+1$ matrix equation of the form

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 0 \\ \frac{K}{r_{11}} & \frac{K}{r_{12}} & \frac{K}{r_{13}} & \dots & \frac{K}{r_{1n}} & 1 \\ \frac{K}{r_{21}} & \frac{K}{r_{22}} & \frac{K}{r_{23}} & \dots & \frac{K}{r_{2n}} & 1 \\ \frac{K}{r_{31}} & \frac{K}{r_{32}} & \frac{K}{r_{33}} & \dots & \frac{K}{r_{3n}} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{K}{r_{m1}} & \frac{K}{r_{m2}} & \frac{K}{r_{m3}} & \dots & \frac{K}{r_{mn}} & 1 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ \vdots \\ Q_n \\ -C \end{bmatrix} = \begin{bmatrix} Q_{net} \\ -V_1 \\ -V_2 \\ -V_3 \\ \vdots \\ -V_n \end{bmatrix}$$

Which we will reduce to write as $A\vec{Q} = -\vec{V}$. Solving this linear system of equations is impossible to do exactly because there might be more test points than charge points, however, we can solve for the least squares solution which is governed by the equation $\vec{V} = (A^T A)^{-1} A^T \vec{Q}$. Since the matrix part is constant for each shape of object, it can be calculated once at the object construction and the charge at each point can be computed with a simple vector transformation.

6.3 3d Correction

Since 2d conductors do not actually exist, we are treating the conductor as if it is extruded a couple of units in the z direction. This would mean that each point charge would behave similar to a line charge, so we used the voltage and field caused by a line charge that was mentioned earlier. However, this approximation did not come out well because the charge density tapers off near the ends. We will assume for simplicity that the charge density changes at a linear rate. Instead of having a Q for each line, we now have λ_1 and λ_2 for the center and the tip respectively. Since the entire simulation is symmetrical across the z-axis, we will just mirror every thing by multiplying or dividing by 2. We will also assume that the conductor has a variable height h. Let's start with the voltage on the $z = 0$ plane.

$$\begin{aligned} V &= \frac{Q}{r} \\ V &= \int_0^h \frac{(\lambda_1 + \frac{z}{h}(\lambda_2 - \lambda_1))dz}{\sqrt{z^2 + r^2}} \\ &= \lambda_1 \int_0^h \frac{1}{\sqrt{z^2 + r^2}} + (\lambda_2 - \lambda_1) \int_0^h \frac{z}{\sqrt{z^2 + r^2}} \\ &= \lambda_1 \ln\left(\frac{h + \sqrt{h^2 + r^2}}{r}\right) + (\lambda_2 - \lambda_1)(\sqrt{h^2 + r^2} - r) \end{aligned}$$

We also need to add another test point at a different location outside of the flat plane. So we'll add a test point next to the position of λ_2, which will also require us to account for the mirroring because the mirrored half is further away. So the density will vary from λ_2 to λ_1 over the distance 0 to h, and then it will vary from λ_1 to λ_2 over the interval h to $2h$. We will define $d_h = \sqrt{h^2+r^2}$, $d_{2h} = \sqrt{(2h)^2+r^2}$, and $\Delta\lambda = \lambda_2 - \lambda_1$.

$$V = \int_0^h \frac{\lambda_2 - \frac{z}{h}\Delta\lambda dz}{\sqrt{z^2+r^2}} + \int_h^{2h} \frac{\frac{z-h}{h}\lambda_2 + \frac{2h-z}{h}\lambda_1}{\sqrt{z^2+r^2}} dz$$
$$V = \lambda_2 \ln\left(\frac{h+d_h}{r}\right) - \Delta\lambda(d_h - r) + \frac{\Delta\lambda}{h}(d_{2h} - d_h) + (\lambda_1 - \Delta\lambda)\ln\left(\frac{2h+d_{2h}}{h+d_h}\right)$$

Now we can replace the $\frac{KQ}{r}$ in the matrix with these formulas. λ_1 and λ_2 will be treated as "charge" points, and the test points will be doubled in the z direction.

7 Calculating Inertia

The parallel axis theorem states that $I = I_C + Md^2$. Where I is the moment of inertia about an arbitrary point, I_C is the moment of inertia about the center, M is the mass of the object, and d is the distance between the arbitrary point and the center of mass of the object.

8 Algorithms

include here the algorithm for getting force between each element

9 Magnetic Field

For moving charges, the formula for the electric field is

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2} \tag{6}$$

Where $\mu_0 = 1.257 \times 10^{-6} \frac{\text{N}}{\text{A}^2}$. For moving charges, the strength of this force is weak because μ_0 is a small number. We will not implement the magnetic field in the program because it is only significant for currents. For two moving charges, the ratio between magnetic force and electric force is $\frac{F_{\vec{B}}}{F_{\vec{E}}} = \frac{v^2}{c^2}$, so for $v \ll c$, the magnetic force is nothing compared to the electric force.