Conditional Probability

Snacks and Stats

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1 Axioms of Probability

For a sample space Ω containing all possible outcomes, class $\mathcal{F}$ of events, and a function P that assigns probability to each event in $\mathcal{F}$:

- $0 \leq P(A) \leq 1 \rightarrow$ the probability of an event A is between 0 and 1
- $P(\Omega) = 1 \rightarrow$ the probability of any of the possible outcomes occurring is 1
- $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \rightarrow$ the probability of either an event A_1 or an event A_2 occurring is the probability of A_1 plus the probability of A_2 minus the probability of both A_1 and A_2 occurring.

2 Conditional Probability

If two events A_1 and A_2 are **independent**, then $P(A_1 \cap A_2) = P(A_1)P(A_2)$. If the events are **not independent**, then

$$P(A_1 \cap A_2) = P(A_1)P(A_2|A_1) \tag{1}$$

Equation 1 states that the probability of both A_1 and A_2 occurring is equal to the probability of A_1 occurring, times the probability of A_2 occurring, *given that A_1 has already occurred.*

2.1 Example: Cancer screening

Assume that 1% of a population has cancer. There is a test that has an 80% success rate at detecting that cancer. The test also has a false-positive rate of 9.6%. If a patient tests positive for cancer, what is the probability that they have cancer?

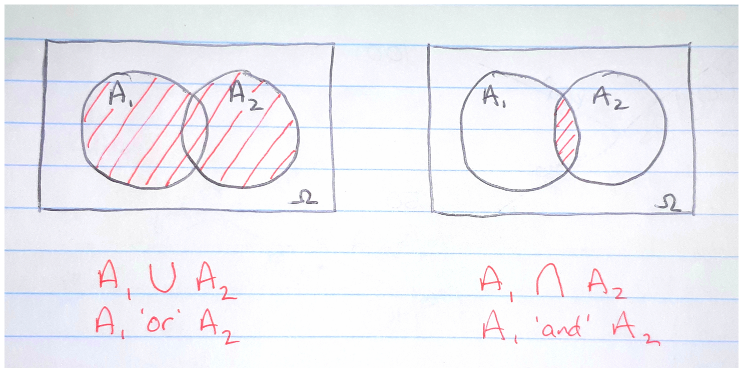

Figure 1: Illustration of the notation $A_1 \cup A_2$ and $A_1 \cap A_2$ for two events A_1 and A_2 that are independent and not mutually exclusive.

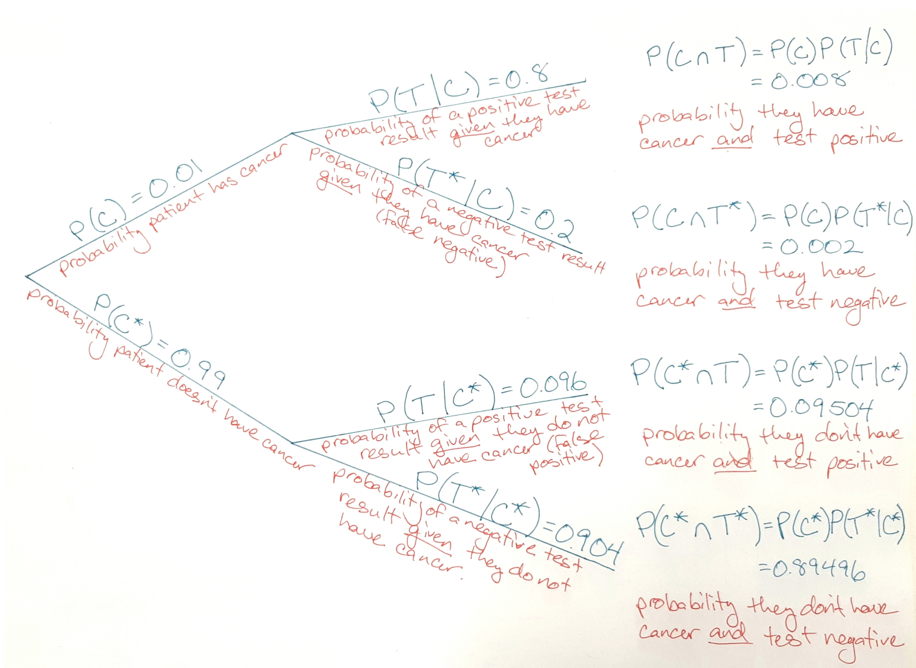

Figure 2: Tree diagram illustrating the 'Cancer Screening' example.

3 Bayes' Theorem

This example leads us directly to Bayes' Theorem!

The probability that a patient has cancer, given that they received a positive test result is $P(C|T)$. This is found by first determining the probability that they have cancer and gets a positive test result:

$$\begin{aligned} P(C \cap T) &= P(C)P(T|C) \\ &= 0.01 \times 0.80 \\ &= 0.008 \end{aligned}$$

and comparing that to the total probability of getting a positive test result $P(T)$:

$$\begin{aligned} P(T) &= P(C)P(T|C) + P(C^*)P(T|C^*) \\ &= 0.01 \times 0.80 + 0.99 \times 0.096 \\ &= 0.103 \end{aligned}$$

Therefore the probability that a patient has cancer, given that they received a positive test result is:

$$\begin{aligned} P(C|T) &= \frac{P(C)P(T|C)}{P(T)} \\ &= \frac{0.008}{0.103} \\ &= 0.078 = 7.8\% \end{aligned}$$

This is Bayes' Theorem! To use the terminology of Bayesian inference, we'd say that the *a priori* probability that a patient has cancer is 1%. Given the observation of the positive test result, we then revise this to find that the *a posteriori* probability that the patient has cancer is 7.8%.

3.1 Example: Coin Toss

Suppose you have 5 coins: 4 are 'fair' having both heads and tails, while one is 'unfair' having only heads. You choose one of the 5 coins at random, flip it 3 times and observe 3 heads. What is the probability that you chose one of the fair coins?

3.2 Example: Monty Hall

You're on a game show called the 'Monty Hall Show'. The host, Monty Hall, presents you with three doors. Behind two of the doors is a goat, but behind one of the doors is a shiny new car! You choose a door, and before you open it, Monty opens one of the other doors to reveal a goat. Monty then gives you a choice: either stick with the choice you made, or choose the other door. What should you do?

3.3 Example: Two Envelopes

You are presented with two identical sealed envelopes and told that one contains \$50 while the other contains \$100. You choose one at random. You're then given the opportunity to switch, along with the argument that if you switch you are still guaranteed \$50, but have a 50% chance of getting \$100, so you should switch! Is that true?

3.4 Example: Two Daughters

If a family has two children, at least one of whom is a daughter, what is the probability that both of them are daughters? If a family has two children, the elder of which is a daughter, what is the probability that both of them are daughters? Are these the same?

3.5 Example: Random Draw

You are presented with a bag that contains 12 balls of three colours: red, blue and yellow. You know that 4 are red and the rest are either blue or yellow in unknown proportion. If you draw a single ball at random, what is the probability that it is blue?