$-r = (r_1 \cdots r_n)^T$

$$r = A^T \begin{pmatrix} \frac{1}{O_1} & & \\ & \ddots & \\ & & \frac{1}{O_n} \end{pmatrix} r := A^T O^{-1} r$$

$$r_i = \sum \frac{r_j}{O_j} \begin{matrix} \rightarrow \text{pagerank cont)} \\ \rightarrow \text{outdegree} \end{matrix}$$

A:

$$a_{ij} = \begin{cases} 1 & i\text{-}j \text{ node connected} \\ \text{zero} & \text{no connection} \end{cases}$$

$\therefore \sum a_{in} \rightarrow$ row sum of row i

$=$ total # of outdegrees for node i

$= O_i = O_{ii}$

$A \rightarrow A^T$: row sum $\rightarrow$ col sum

$$M = A^T O^{-1}$$

③ colsum M $= \text{colsum}(A^T O^{-1})$

$= \text{colsum } a_{in} \cdot \frac{1}{O_{ii}}$

$= O_{ii} \cdot \frac{1}{O_{ii}}$

$= 1$

$\therefore$ col sum $M = 1$

$$\begin{bmatrix} O_{ii} & & \\ & \ddots & \\ & & O_{mn} \end{bmatrix} [$$

① $A^T O^{-1} r$

$$= \sum_{\substack{(1,1) \\ a_{ij} = 1 \text{ or } 0}}^{(n,n)} a_{ij} \cdot \text{Diag}\left(\frac{1}{O_{ii}}\right)_{i \in [1,n]} \cdot \sum \frac{r_j}{O_j}$$

$$= \begin{bmatrix} \cdots 1 \cdots 0 \cdots 1 \\ \cdots 0 \cdots 1 \cdots 1 \cdot 0 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{o_{ii}} & & \\ & \ddots & \\ & & \frac{1}{o_{nn}} \end{bmatrix} \begin{bmatrix} \sum \frac{r_j}{o_j} \\ \vdots \\ \sum \frac{r_n}{o_n} \end{bmatrix}$$

$$= A^T \cdot \left.\begin{bmatrix} \frac{\sum \frac{r_j}{o_j}}{o_{ii}} \\ \vdots \end{bmatrix}\right\} n$$

$$= \left.\begin{bmatrix} \mathrm{sum}(\mathrm{row}\ i) \cdot \frac{\sum \frac{r_j}{o_j}}{o_{ii}} \\ \vdots \end{bmatrix}\right\} n$$

$$= \left.\begin{bmatrix} \cancel{o_{ii}} \cdot \frac{\sum \frac{r_j}{o_j}}{\cancel{o_{ii}}} \\ \vdots \end{bmatrix}\right\} n$$

$$= \left.\begin{bmatrix} \sum \frac{r_j}{o_j} \\ \vdots \end{bmatrix}\right\} n$$

$$= r$$

③ $\det(M) = \det(M^T)$

$(M - \lambda I) = (M - \lambda I)^T$

$\therefore \det(M^T - \lambda I) = \det((M - \lambda I)^T) = \det(M - \lambda I)$

4 $\rho(M) = \lambda = \|M\| = 1$

$\boxed{1}$ $\because$ M is left stochastic, 2

$\therefore$ col sum $=1$

$\therefore$ $\lambda=1$ is always a valid and max eigenvalue

$\because$ all $m_{ij} \geq 0$,

1 $\therefore \sum_{i\neq j} |m_{ij}| = \sum_{i\neq j} m_{ij}$

convex:

$\therefore$ take $a_{ii} \in [0,1]$, $\sum_{i\neq j} a_{ij} = 1 - a_{ii}$

$\therefore$ 1 is the max & valid eigenvalue

$\therefore$ $\lambda_{max} = 1 = \|M\| = \rho(M)$

2 $M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ & \ddots & & \\ & & & m_{nn} \end{bmatrix}$

Mx ($x \in$ vector (n))

$\|Mx\|_1 = |m_{11}x_1 + m_{12}x_2 \cdots| + | \cdots m_{nn}x_n|$

norm:

$\leq m_{11}|x_1| \cdots m_{nn}|x_n|$

$= \sum |x_n|$ $\because m_{ii} > 0$

$= \|x\|_1$

$\Rightarrow$ $\because \lambda = 1$ is max eigenvalue,

$\therefore \rho(M) = 1$ $\|M\| = 1$

$\boxed{2}$ $\hookrightarrow \lambda$: $Mv = \lambda v$

from ① from ①, r is an eigenvector with $\lambda = 1$

$\therefore$ $Mr = 1 \cdot r$ holds

$\therefore$ $\lambda = 1$ ✓

$\therefore$ $\|M\| = 1 = \lambda = \rho(M)$