CHAPTER 2

ALGORITHM ANALYSIS

【Definition】 An algorithm is a finite set of instructions that, if followed, accomplishes a particular task. In addition, all algorithms must satisfy the following criteria:

(1) Input There are zero or more quantities that are externally supplied.

(2) Output At least one quantity is produced.

(3) Definiteness Each instruction is clear and unambiguous.

(4) Finiteness If we trace out the instructions of an algorithm, then for all cases, the algorithm terminates after finite number of steps.

(5) Effectiveness Every instruction must be basic enough to be carried out, in principle, by a person using only pencil and paper. It is not enough that each operation be definite as in(3); it also must be feasible.

Note: A **program** is written in some programming language, and does not have to be finite (*e.g. an operation system*).

An **algorithm** can be described by human languages, flow charts, some programming languages, or pseudo-code.

〖Example〗 **Selection Sort**: Sort a set of n 1 integers in increasing order.

From those integers that are currently unsorted, find the smallest and place it next in the sorted list.

Where and how are they stored?

Where?

```
for ( i = 0; i < n; i++) {
    Examine list[i] to list[n−1] and suppose that the smallest
      integer is at list[min];
  Interchange list[i] and list[min];
}
```

Algorithm in pseudo-code

Sort = Find the smallest integer + Interchange it with list[i].

§1 What to Analyze

➢ **Machine & compiler-dependent run times.**

➢ **Time & space complexities : machine & compiler-independent.**

- **Assumptions:**

① **instructions are executed sequentially**

② **each instruction is simple, and takes exactly one time unit**

③ **integer size is fixed and we have infinite memory**

- **Typically the following two functions are analyzed:**

$T_{avg}(N)$ & $T_{worst}(N)$ -- the average and worst case time complexities, respectively, as functions of input size N.

If there is more than one input, these functions may have more than one argument.

〖 Example 〗 Matrix addition

```
void  add ( int  a[ ][ MAX_SIZE ],
            int  b[ ][ MAX_SIZE ],
            int  c[ ][ MAX_SIZE ],
            int  rows,  int  cols )
{
   int  i,  j ;
   for ( i = 0; i < rows; i++ )      /* rows + 1 */
       for ( j = 0; j < cols; j++ )     /* rows(cols+1) */
           c[ i ][ j ] = a[ i ][ j ] + b[ i ][ j ];    /* rows · cols */
}
```

A: Exchange rows and cols.

$T(\text{rows}, \text{cols}) = 2\ \text{rows} \cdot \text{cols} + 2\text{rows} + 1$

〖 Example 〗 Iterative function for summing a list of numbers

```
float  sum ( float  list[ ],  int  n )
{  /* add a list of numbers */
   float  tempsum = 0;   /* count = 1 */
   int  i ;
   for ( i = 0; i < n; i++ )
           /* count ++ */
      tempsum  += list [ i ] ;   /* count ++ */
    /* count ++ for last execution of for */
   return  tempsum;    /* count ++ */
}
```

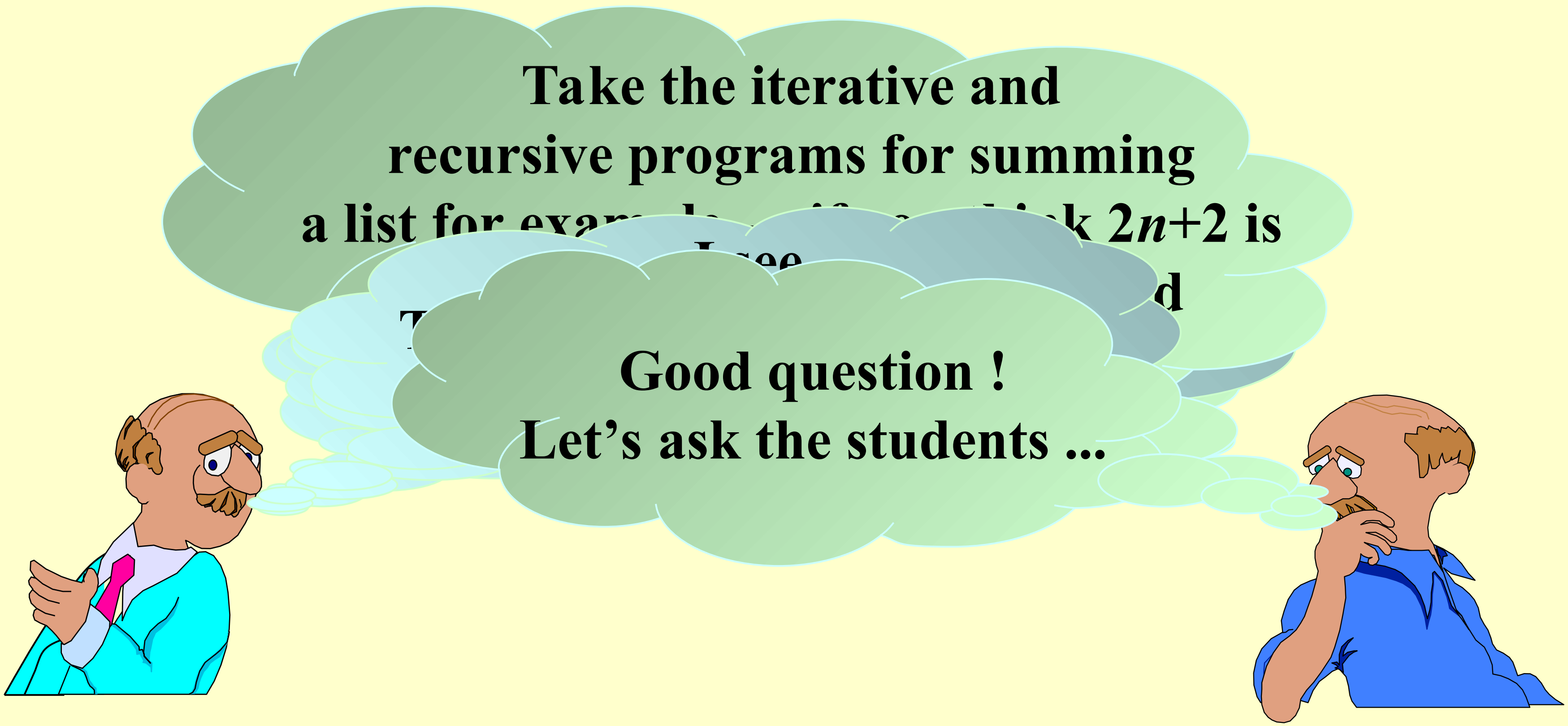
$$T_{sum}(n) = 2n + 3$$

〖 Example 〗 Recursive function for summing a list of numbers

```
float  rsum ( float  list[ ],  int  n )
{  /* add a list of numbers */
   if ( n )   /* count ++ */
      return rsum(list, n–1) + list[n – 1];
       /* count ++ */
   return   0;   /* count ++ */
}
```

$$T_{rsum}(n) = 2n + 2$$

But it takes more time to compute each step.

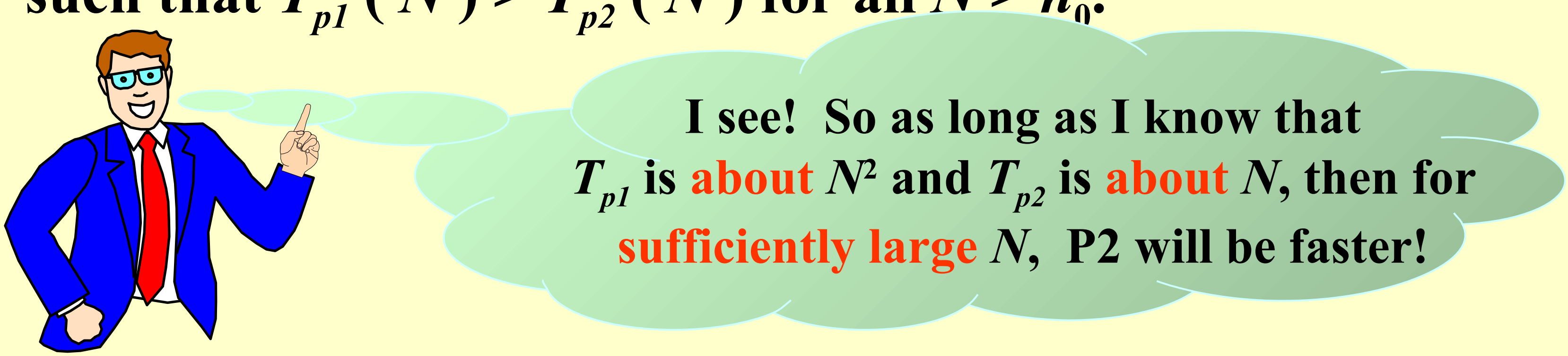

Take the iterative and
recursive programs for summing
Good question !
Let's ask the students ...

§2 Asymptotic Notation (O, Ω, Θ, o)

The point of counting the steps is to predict the growth in run time as the N change, and thereby compare the time complexities of two programs. So what we really want to know is the asymptotic behavior of T_p.

Suppose $T_{p1}(N) = c_1N^2 + c_2N$ and $T_{p2}(N) = c_3N$. Which one is faster?

No matter what c_1, c_2, and c_3 are, there will be an n_0 such that $T_{p1}(N) > T_{p2}(N)$ for all $N > n_0$.



【 Definition 】 $T(N) = O(f(N))$ if there are positive constants c and n_0 such that $T(N) \le c \cdot f(N)$ for all N n_0.

【 Definition 】 $T(N) = \Omega(g(N))$ if there are positive constants c and n_0 such that $T(N)$ $c \cdot g(N)$ for all N n_0.

【 Definition 】 $T(N) = \Theta(h(N))$ if and only if $T(N) = O(h(N))$ and $T(N) = \Omega(h(N))$.

【 Definition 】 $T(N) = o(p(N))$ if $T(N) = O(p(N))$ and $T(N) \ne \Theta(p(N))$.

Note:

- $2N + 3 = O(N) = O(N^{k1}) = O(2^N) = \cdots$ We shall always take the **smallest** $f(N)$.
- $2^N + N^2 = \Omega(2^N) = \Omega(N^2) = \Omega(N) = \Omega(1) = \cdots$ We shall always take the **largest** $g(N)$.

Rules of Asymptotic Notation

☞ If $T_1(N) = O(f(N))$ and $T_2(N) = O(g(N))$, then

(a) $T_1(N) + T_2(N) = \max(O(f(N)), O(g(N)))$,

(b) $T_1(N) * T_2(N) = O(f(N) * g(N))$.

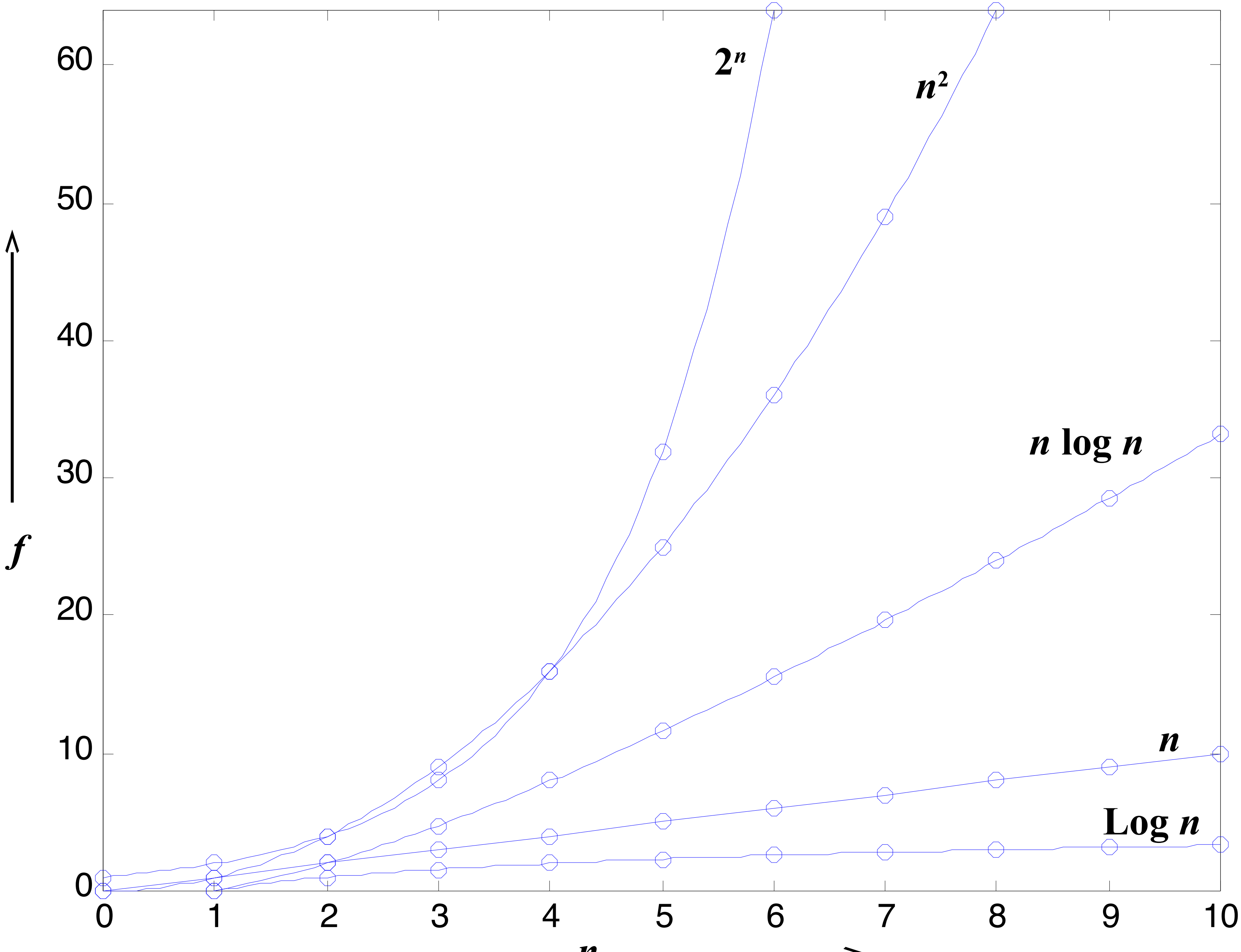
☞ If $T(N)$ is a polynomial of degree k, then $T(N) = \Theta(N^k)$.

☞ $\log^k N = O(N)$ for any constant k. This tells us that logarithms grow very slowly.

Note: When compare the complexities of two programs asymptotically, make sure that N is sufficiently large.

For example, suppose that $T_{p1}(N) = 10^6 N$ and $T_{p2}(N) = N^2$. Although it seems that $\Theta(N^2)$ grows faster than $\Theta(N)$, but if $N < 10^6$, P2 is still faster than P1.

Input size n							
Time	**Name**	**1**	**2**	**4**	**8**	**16**	**32**
1	constant	1	1	1	1	1	1
$\log n$	logarithmic	0	1	2	3	4	5
n	linear	1	2	4	8	16	32
$n \log n$	log linear	0	2	8	24	64	160
n^2	quadratic	1	4	16	64	256	1024
n^3	cubic	1	8	64	512	4096	32768
2^n	exponential	2	4	16	256	65536	4294967296
$n!$	factorial	1	2	24	40326	2092278988000	26313×10^{33}



2^n
n^2
$n \log n$
n
Log n
60
50
40
30
20
10
0
0
1
2
3
4
5
6
7
8
9
10
f
n

	Time for $f(n)$ instructions on a 10^9 instr/sec computer						
n	$f(n)=n$	$n\log_2 n$	n^2	n^3	n^4	n^{10}	2^n
10	.01μs	.03μs	.1μs	1μs	10μs	10sec	1μs
20	.02μs	.09μs	.4μs	8μs	160μs	2.84hr	1ms
30	.03μs	.15μs	.9μs	27μs	810μs	6.83d	1sec
40	.04μs	.21μs	1.6μs	64μs	2.56ms	121.36d	18.3min
50	.05μs	.28μs	2.5μs	125μs	6.25ms	3.1yr	13d
100	.10μs	.66μs	10μs	1ms	100ms	3171yr	$4*10^{13}$yr
1,000	1.00μs	9.96μs	1ms	1sec	16.67min	$3.17*10^{13}$yr	$32*10^{283}$yr
10,000	10μs	130.03μs	100ms	16.67min	115.7d	$3.17*10^{23}$yr	
100,000	100μs	1.66ms	10sec	11.57d	3171yr	$3.17*10^{33}$yr	
1,000,000	1.0ms	19.92ms	16.67min	31.71yr	$3.17*10^7$yr	$3.17*10^{43}$yr	

μs = microsecond = 10^{-6} seconds
ms = millisecond = 10^{-3} seconds
sec = seconds

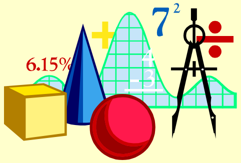
min = minutes
hr = hours
d = days

yr = years

〖 Example 〗 Matrix addition

```
void  add ( int  a[ ][ MAX_SIZE ],
            int  b[ ][ MAX_SIZE ],
            int  c[ ][ MAX_SIZE ],
            int  rows,  int  cols )
{
    int  i,  j ;
    for ( i = 0; i < rows; i++ )   /* Θ (rows) */
        for ( j = 0; j < cols; j++ )   /* Θ (rows · cols ) */
            c[ i ][ j ] = a[ i ][ j ] + b[ i ][ j ];   /* Θ (rows · cols ) */
}
```

$T(\text{rows}, \text{cols}) = \Theta(\text{rows} \cdot \text{cols})$



General Rules

- **FOR LOOPS:** The running time of a for loop is at most the running time of the statements inside the for loop (including tests) times the number of iterations.
- **NESTED FOR LOOPS:** The total running time of a statement inside a group of nested loops is the running time of the statements multiplied by the product of the sizes of all the for loops.
- **CONSECUTIVE STATEMENTS:** These just add (which means that the maximum is the one that counts).
- **IF / ELSE:** For the fragment

```
if ( Condition )  S1;
else  S2;
```

the running time is never more than the running time of the test plus the larger of the running time of S1 and S2.

☞ RECURSIONS:

〖 Example 〗 Fibonacci number:

$\mathrm{Fib}(0) = \mathrm{Fib}(1) = 1,\ \mathrm{Fib}(n) = \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2)$

```
long int  Fib ( int  N )                /* T ( N ) */
{
        if  ( N <= 1 )                  /* O( 1 ) */
            return  1;                  /* O( 1 ) */
        else
            return  Fib( N – 1 ) + Fib( N – 2 );
}           /*O(1)*/   /*T(N –1)*/    /*T(N –2)*/
```

Q: Why is it so bad?

$T(N) = T(N-1) + T(N-2) + 2 \quad \mathrm{Fib}(N)$

Proof by induction

$\left(\frac{3}{2}\right)^N \le \mathrm{Fib}(N) \le \left(\frac{5}{3}\right)^N \Rightarrow$ $T(N)$ grows exponentially