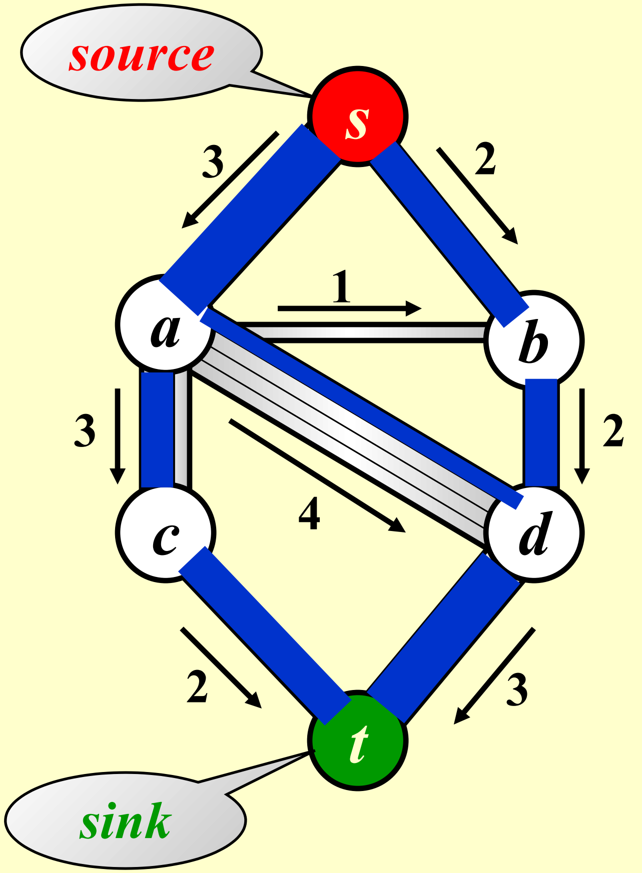
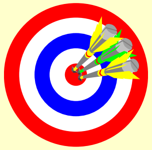
§4 Network Flow Problems

〖 Example 〗 Consider the following network of pipes:

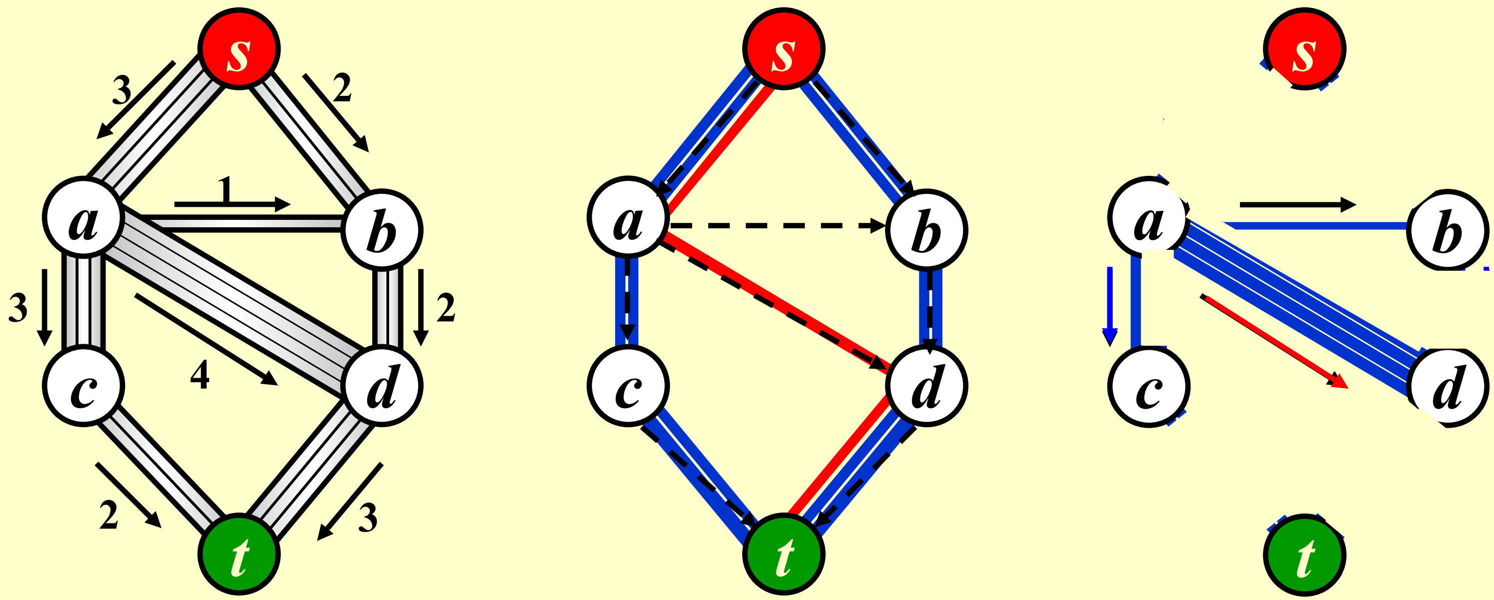


Note: Total coming in (v) $\equiv$ Total going out (v) where $v \notin \{ s, t \}$



Determine the maximum amount of flow that can pass from *s* to *t*.

1. A Simple Algorithm



G | Flow G_f | Residual G_r

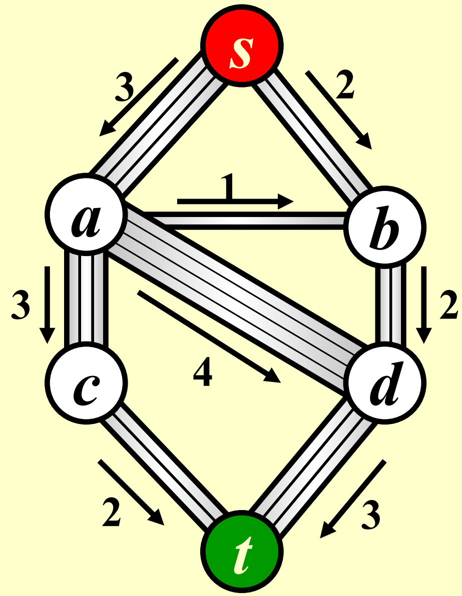
Step 1: Find any path $s \rightarrow t$ in G_r ;

Step 2: Take the minimum edge on this path as the amount of fl
ow and add to G_f ;

augmenting path

Step 3: Update G_r and remove the 0 flow edges;

Step 4: If (there is a path $s \rightarrow t$ in G_r)
Goto **Step 1**;
Else
End.

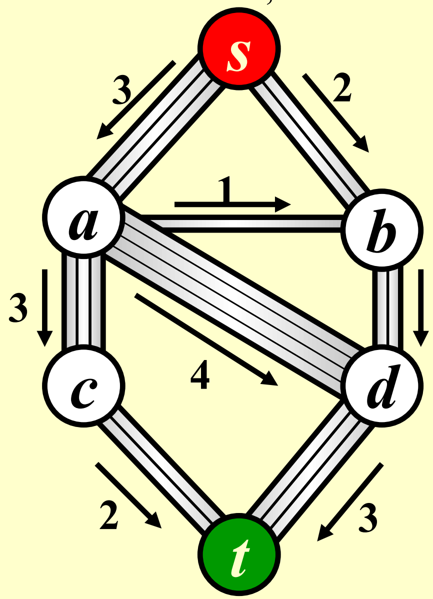

s
3
2
a
1
b
3
2
4
c
d
2
3
t

G

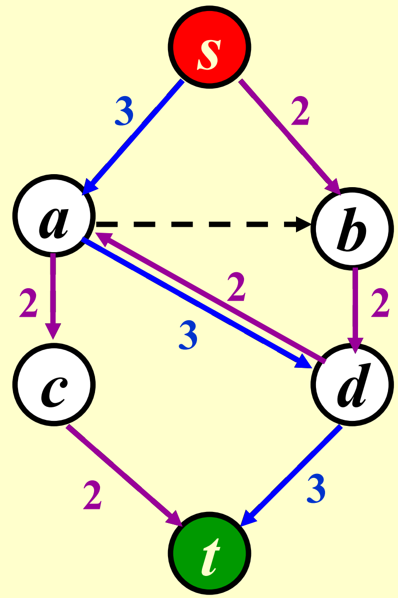

Uh-oh…
Seems we cannot be
greedy at this point.

2. A Solution – allow the algorithm to undo its decisions

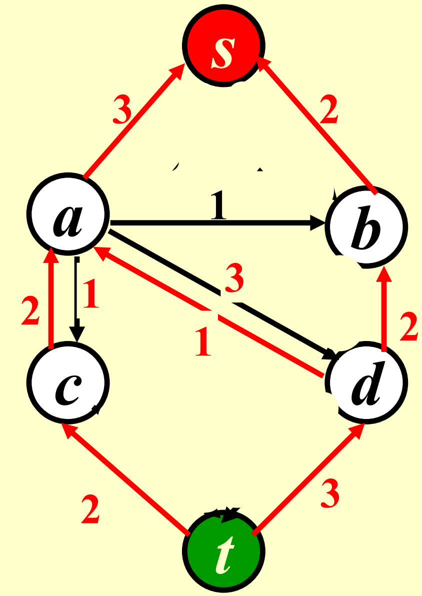
For each edge (*v*, *w*) with flow $f_{v,\,w}$ in G_f , add an edge (*w*, *v*) wit
h flow $f_{v,\,w}$ in G_r .



G



Flow G_f



Residual G_r

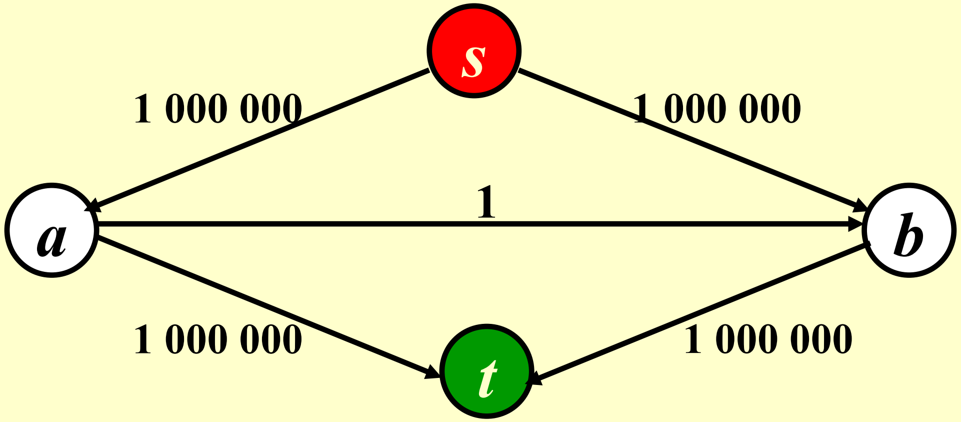
〖 **Proposition** 〗 If the edge capabilities are rational numbers, this algorithm always terminate with a maximum flow.

Note: The algorithm works for *G* with *cycles* as well.

3. Analysis (If the capacities are all integers)

☞ An augmenting path can be found by an unweighted shortest p ath algorithm.

$T = O(\ f \cdot |E|\ )$ where f is the maximum flow.



☞ Always choose the augmenting path that allows the largest incr ease in flow. /* modify Dijkstra's algorithm */

$$T = T_{augmentation} * T_{find\ a\ path}$$
$$= O(\ |E| \log cap_{max}\ ) * O(\ |E| \log |V|\ )$$
$$= O(\ |E|^2 \log |V|\ ) \text{ if } cap_{max} \text{ is a small integer.}$$

☞ **Always choose the augmenting path that has the least number of edges.**

$$T = T_{augmentation} * T_{find\ a\ path}$$
$$= O(\,|E|\,) * O(\,|E| \cdot |V|\,) \quad \text{/* unweighted shortest path algorithm */}$$
$$= O(\,|E|^2\,|V|\,)$$

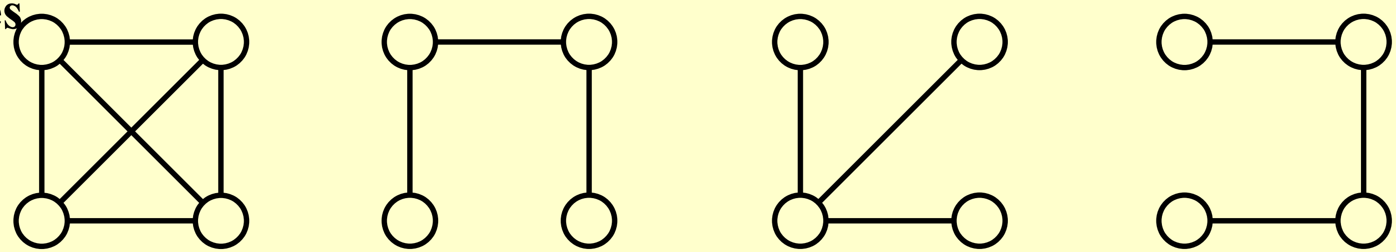
Note:

- **If every $v \notin \{\, s, t \,\}$ has either a single incoming edge of capacity 1 or a single outgoing edge of capacity 1, then time bound is reduced to $O(\,|E|\,|V|^{1/2}\,)$.**
- **The *min-cost flow* problem is to find, among all maximum flows, the one flow of minimum cost provided that each edge has a cost per unit of flow.**

§5 Minimum Spanning Tree

【Definition】 A *spanning tree* of a graph G is a tree which consists of V(G) and a subset of E(G)

〖Example〗 A complete graph and three of its spanning trees



Note:

- The minimum spanning tree is a *tree* since it is acyclic -- the n umber of edges is |V| – 1.
- It is *minimum* for the total cost of edges is minimized.
- It is *spanning* because it covers every vertex.
- A minimum spanning tree exists iff G is *connected*.
- Adding a non-tree edge to a spanning tree, we obtain a *cycle*.

Greedy Method Make the best decision for each stage, under the following constrains :

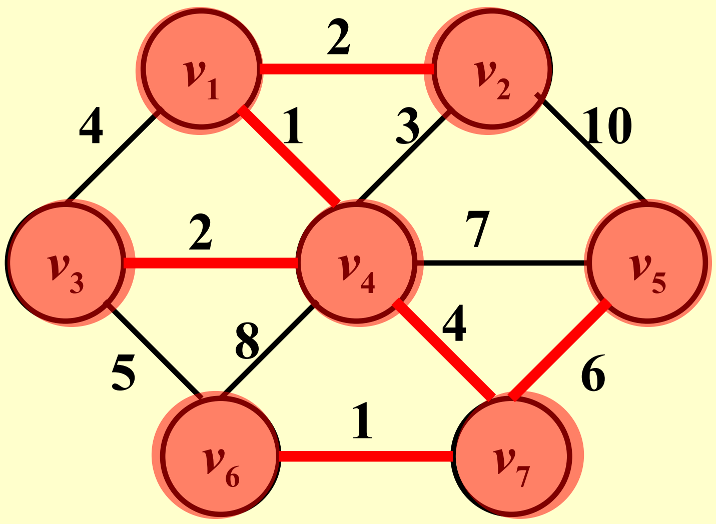
(1) we must use only edges within the graph;

(2) we must use exactly |V| –1 edges;

(3) we may not use edges that would produce a cycle.

1. *Prim*'s Algorithm – grow a tree

/* very similar to Dijkstra's algorithm */



2. *Kruskal*'s Algorithm – maintain a forest

$T = O(\,|E| \log |E|\,)$

```
void Kruskal ( Graph G )
{   T = { } ;
    while  ( T contains less than |V| –1 edges
                 && E is not empty ) {
        choose a least cost edge (v, w) from E ;   /* DeleteMin */
        delete (v, w) from E ;
        if  ( (v, w) does not create a cycle in T )
            add (v, w) to T ;      /* Union / Find */
        else
            discard (v, w) ;
    }
    if  ( T contains fewer than |V| –1 edges )
        Error ( "No spanning tree" ) ;
}
```

A more detailed pseudocode is given by Figure 9.58 on p.321