Convex Optimization

Chapter 3

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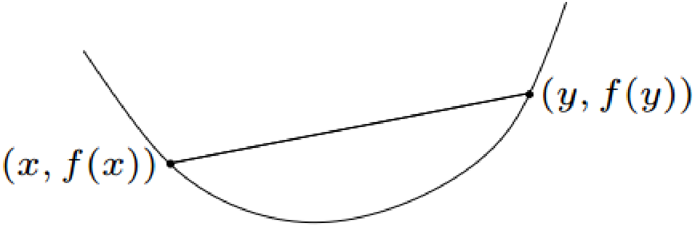
Convex Functions

Let $f : \mathbf{R}^n \to \mathbf{R}$ be a convex function if $\forall x, y \in \text{dom } f$ and $0 \leq \theta \leq 1$ such that

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y).$$

It is also known as the Jensen's inequality.



- if f is concave then $-f$ is convex.

Strictly Convex Functions

Let $f : \mathbf{R}^n \rightarrow \mathbf{R}$ be a convex function if $\forall x, y \in \mathrm{dom}\, f$ and $0 < \theta < 1$ such that

$$f(\theta x + (1 - \theta)y) < \theta f(x) + (1 - \theta)f(y).$$

Extended Convex Functions

Let $\hat{f} : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ be an extended convex function if $\forall x, y \in \operatorname{dom} f$ and $0 \le \theta \le 1$ such that

$$\hat{f}(x) = \begin{cases} f(x) & x \in \operatorname{dom} f \\ \infty & \text{otherwise} \end{cases}$$

and

$$\hat{f}(\theta x + (1-\theta)y) \le \theta \hat{f}(x) + (1-\theta)\hat{f}(y).$$

Example of Convex Functions

convex:

- affine: $ax + b$ on $\mathbf{R}$, for any $a, b \in \mathbf{R}$
- exponential: e^{ax}, for any $a \in \mathbf{R}$
- powers: x^α on $\mathbf{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$
- powers of absolute value: $|x|^p$ on $\mathbf{R}$, for $p \geq 1$
- negative entropy: $x \log x$ on $\mathbf{R}_{++}$

concave:

- affine: $ax + b$ on $\mathbf{R}$, for any $a, b \in \mathbf{R}$
- powers: x^α on $\mathbf{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on $\mathbf{R}_{++}$

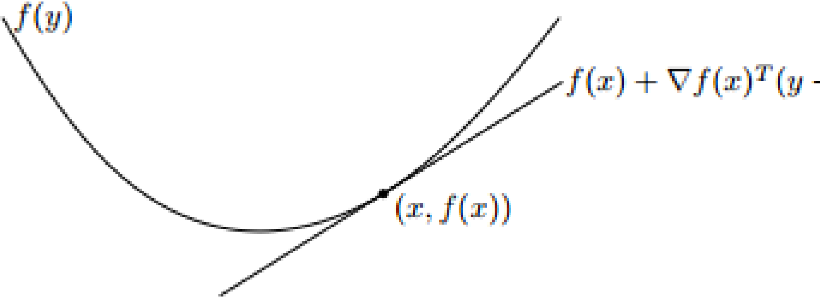
First-order conditions

f is differentiable then f is convex iff dom f is convex and

$$f(y) > f(x) + \nabla f(x)^T (y - x)$$

hold $\forall x, y \in$ dom f and

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_n} \right).$$



Second-order conditions

f is twice differentiable then f is convex if

$$\nabla^2 f(x) \geq 0$$

hold $\forall x \in$ dom f and each element is defined as

$$\nabla^2 f(x)_{ij} = \frac{\partial f(x)}{\partial x_i x_j}.$$

Sublevel Sets

The a-sublevel set of a function $f : R^n \rightarrow R$ is defined as

$$C_a = \{x \in \text{ dom } f \mid f(x) \leq a\}.$$

f is a convex function if $\forall x, y \in C_a$ and $0 \leq \theta \leq 1$ such that $f(x) \leq a$ and $f(y) \leq a$ then

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta)f(y).$$

hold $\forall x,\ y$.

Epigraph

The Ephigraph is defined as

$$\text{epi } f = \{(x,t) | x \in \text{ dom } f, f(x) \leq t\}$$

From *a*-sublevel sets, epigraph is convex.

Preserve Convexity Operations: Nonnegative weighted sums

For $i = 1, \ldots, n$ such that $w_i \geq 0$ and f_i is a convex function. Then the function

$$f(x) = \sum_{i=1}^{n} w_i f_i(x).$$

is convex.

Proof: Let $x_1, x_2 \in \text{dom } g$, $0 \leq \theta \leq 1$, and $\hat{\theta} = 1 - \theta$. We have

$$\begin{aligned} g(\theta x_1 + \hat{\theta} x_2) &= \sum_{i=1}^{n} w_i f_i(\theta x_1 + \hat{\theta} x_2) \leq \sum_{i=1}^{n} w_i \left[\theta f_i(\theta x_1) + \hat{\theta} f_i(x_2) \right] \\ &= \sum_{i=1}^{n} w_i \theta f_i(\theta x_1) + \sum_{i=1}^{n} w_i \hat{\theta} f_i(x_2) \\ &= \theta g(x_1) + \hat{\theta} g(x_2). \end{aligned}$$

Preserve Convexity Operations: Composition with an affine mapping

Let $x \in \operatorname{dom} g$ the affine mapping function is defined as

$$g(x) = f(Ax + b).$$

If f is convex function then g is a convex function..
Proof: Let $x_1, x_2 \in \operatorname{dom} g$ and $Ax + b \in \operatorname{dom} f$, $0 \leq \theta \leq 1$, and $\hat{\theta} = 1 - \theta$. We have

$$\begin{aligned}
g(\theta x_1 + \hat{\theta} x_2) &= f(A(\theta x_1 + \hat{\theta} x_2) + b) \\
&= f(A\theta x_1 + A\hat{\theta} x_2 + b) \\
&= f(\theta(Ax_1 + b_1) + \hat{\theta}(Ax_2 + b_2)) \\
&\leq \theta f(Ax_1 + b_1) + \hat{\theta} f(Ax_2 + b_2) \\
&= g(x_1) + g(x_2),
\end{aligned}$$

where $b = \theta b_1 + \hat{\theta} b_2$.

Preserve Convexity Operations: Pointwise maximum and supremum

The perspective function: Let $x \in \operatorname{dom} f$ and f_1, f_2 be convex functions such as

$$f(x) = \max\{f_1(x), f_2(x)\}$$

then f is a convex function.
Proof: Let $x_1, x_2 \in \operatorname{dom} f$, $0 \leq \theta \leq 1$, and $\hat{\theta} = 1 - \theta$. We have

$$\begin{aligned} f(\theta x_1 + \hat{\theta} x_2) &= \max(f_1(\theta x_1 + \hat{\theta} x_2), f_2(\theta x_1 + \hat{\theta} x_2)) \\ &= \max(\theta f_1(x_1) + \hat{\theta} f(x_2), \theta f_2(x_1) + \hat{\theta} f_2(x_2)) \\ &= \max(\theta f_1(x_1), \theta f_2(x_1)) + \max(\hat{\theta} f_1(x_2) + \hat{\theta} f_2(x_2)) \\ &= \theta f(x_1) + \hat{\theta} f(x_2). \end{aligned}$$

Preserve Convexity Operations: Minimization

If f is convex in (x, y), and C is a convex nonempty set then the function

$$h(x) = \inf_{y \in C} f(x, y)$$

is a convex function. The domain of g is

$$\text{dom } g = \{x | (x, y) \in \text{dom } f \text{ for some } y \in C\}.$$

Preserve Convexity Operations: Minimization

Proof: Let $x_1, x_2 \in$ dom h, $y_1, y_2 \in C$, and $0 \leq \theta \leq 1$, and $\hat{\theta} = 1 - \theta$. We have

$$\begin{aligned} h(\theta x_1 + \hat{\theta} x_2) &= \inf_{y \in C} f(\theta x_1 + \hat{\theta} x_2, y) \\ &\leq f(\theta x_1 + \hat{\theta} x_2, \theta y_1 + \hat{\theta} y_2) \\ &\leq \theta f(x_1, y_1) + \hat{\theta} f(x_2, y_2) \\ &\leq \theta g(x_1) + \hat{\theta} g(x_2) + \epsilon \end{aligned}$$

where $\epsilon > 0$.

Preserve Convexity Operations: Composition

Let $x \in \text{dom } h$ be a composition function

$$h(x) = g(f(x)).$$

1. g is convex & non-decreasing and f is convex, then h is convex.
2. g is convex & non-increasing and f is concave, then h is convex.
3. g is concave & non-increasing and f is convex, then h is concave.
4. g is concave & non-decreasing and f is concave, then h is concave.

Proof: Let $x_1, x_2 \in \text{dom } h$, $0 \leq \theta \leq 1$, and $\hat{\theta} = 1 - \theta$

$$h(\theta x_1 + \hat{\theta} x_2) = g(h(\theta x_1 + \hat{\theta} x_2)) = g(\theta h(x_1) + \hat{\theta} h(x_2))$$
$$= \theta g(h(x_1)) + \hat{\theta} g(h(x_2)).$$

Preserve Convexity Operations: Perspective

The perspective function: Let $x, t \in \text{dom } g$

$$g(x,t) = t \times f(x/t).$$

The perspective function g is convex if f is convex.
Proof: Let $x_1, x_2, t_1, t_2 \in \text{dom } g$, $0 \leq \theta \leq 1$, and $\hat{\theta} = 1 - \theta$. We have

$$\begin{aligned}
&g(\theta x_1 + \hat{\theta} x_2, \theta t_1 + \hat{\theta} t_2) = \\
&= (\theta t_1 + \hat{\theta} t_2) \times f\left(\frac{\theta x_1 + \hat{\theta} x_2}{\theta t_1 + \hat{\theta} t_2}\right) \\
&= (\theta t_1 + \hat{\theta} t_2) \times f\left(\frac{\theta \frac{t_1}{t1} x_1 + \hat{\theta} \frac{t_2}{t2} x_2}{(\theta t_1 + \hat{\theta} t_2)}\right) \\
&\leq (\theta t_1 + \hat{\theta} t_2) \left[\frac{\theta_1 t_1}{\theta t_1 + \hat{\theta} t_2} f\left(\theta x_1 / t_1\right) + \frac{\theta_2 t_2}{\theta t_1 + \hat{\theta} t_2} f\left(\hat{\theta} x_2 / t_2\right)\right] \\
&= t_1 g(x_1, t) + t_2 g(x_2, t_2).
\end{aligned}$$

Practical Methods to Establish Convexity of a Function

- Verify definition (often simplified by restricting to a line).
- Twice differentiable functions, show

$$\nabla^2 f(x) \geq 0.$$

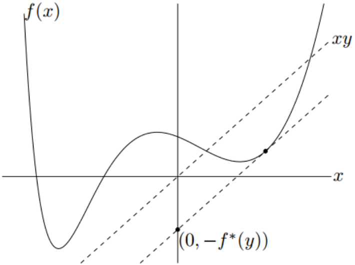
- Show that f is obtained from simple convex functions by operations that preserve convexity.

The Conjugate Function

Let $f : R^n \to R$. The conjugate function of f is $f^* : R^n \to R$ is

$$f^*(x) = \sup_{x \in \text{ dom } f} (y^t x - f(x))$$

f^* is convex (even if f is not convex)



- Negative logarithm

$$f(x) = -\log(x)$$

The deriviate of $f(x)$ w.r.t x is

$$\frac{\partial}{\partial x} yx - f(x) = y + \frac{1}{x} = 0 \implies x^* = -\frac{1}{y.}$$

We have

$$yx^* - f(x^*) = -\frac{y}{y} + \log\left(-\frac{1}{y}\right) = -1 - \log(-y).$$

$$f^*(y) = \sup_{x>0}(xy + \log(x)) = \begin{cases} -1 - log(-y) & y < 0 \\ \infty & \text{otherwise} \end{cases}$$

- Strictly convex quadratic

$$f(x) = (1/2)x^T Qx$$

with $Q \in S^n_{++}$. The deriviate of $f(x)$ w.r.t x is

$$\frac{\partial}{\partial x} y^T x - f(x) = y^T - x^T Q = 0 \implies x^* = Q^{-1}y.$$

Plug x^* into x, we get

$$y^T x^* + f(x^*) = y^T Q^{-1} y - \frac{1}{2} y^T Q^{-1} y = \frac{1}{2} y^T Q^{-1} y.$$

Thus, we have

$$f^*(y) = \sup_x (y^t x - (1/2)x^t Q) = 1/2 y^T Q^{-1} y$$

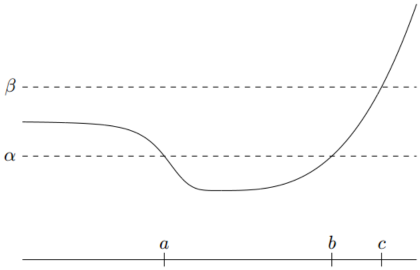
Quasiconvex Functions

A function $f : R^n \to R$ is quasiconvex if dom f is convex and the sublevel sets

$$S_a = \{x \in \text{dom } f \mid f(x) \leq a\}$$

are convex for all a.

- -f is quasiconcave.
- f is quasilinear then every level set $\{x|f(x) = a\}$ is convex. Any monotonically increasing (decreasing) f is quasilinear.



Quasiconvex Proof

If f is convex then f is quasiconvex.
Proof: Let f be a convex function such as

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-x)f(y)$$

and the set $S_a = \{x \in \text{dom } f | f(x) \leq a\}$ are convex for all a. Suppose that $0 \leq \theta \leq 1$ and $x, y \in S_a$ implies that $f(x) \leq a$ and $f(y) \leq a$. We have

$$\begin{aligned} f(\theta x + (1-\theta)y) &\leq \theta f(x) + (1-\theta)f(y) \\ &\leq \theta a + (1-\theta)a = a. \end{aligned}$$

Thus, $\theta a + (1-\theta)a \in S_a$ and f is quasiconvex.

Quasiconcave Proof

If f is convex then f is quasiconcave.
Proof: Let f be a concave function such as

$$f(\theta x + (1-\theta)y) \geq \theta f(x) + (1-x)f(y)$$

and the set $S_a = \{x \in \text{dom } f | f(x) \geq a\}$ are convex for all a.
Suppose that $0 \leq \theta \leq 1$ and $x, y \in S_a$ implies that $f(x) \geq a$ and $f(y) \geq a$. We have

$$\begin{aligned} f(\theta x + (1-\theta)y) &\geq \theta f(x) + (1-\theta)f(y) \\ &\geq \theta a + (1-\theta)a = a. \end{aligned}$$

Thus, $\theta a + (1-\theta)a \in S_a$ and f is quasiconcave.

Quasiconvex and Quasiconcave Function Example

Let $f : R \to R$ and dom $f \in R_{++}$ defined as

$$f(x) = \log(x)$$

The 2nd deriviate is

$$f''(x) = -\frac{1}{x^2} < 0$$

The function f is concave implies that it is quasiconcave (proof next page).

Quasiconvex and Quasiconcave Function Example

The function f is quasiconcave if the set

$$s_a = \{x \in R_{++} | \log(x) \geq a\}$$

is convex for all a. Let $x, y \in S_a$ and $0 \leq \theta \leq 1$, implies that $\log(x) \geq a$ and $\log(y) \geq a$. Then we have we have

$$\begin{aligned}\log(\theta x + (1-\theta)y) \geq & \ \theta \log(x) + (1-\theta)\log(y) \\ & \geq \theta a + (1-\theta)a = a.\end{aligned}$$

It implies that

$$\log(\theta x + (1-\theta)y) \geq a.$$

Thus, we have $\theta x + \ (1-\theta)y \in S_a$ and $\log(x)$ and the function is quasiconcave.

Quasiconvex and Quasiconcave Function Example

The function f is quasiconvex if the set

$$s_a = \{x \in R_{++} | \log(x) \leq a\}$$

is convex for all a. Let $x, y \in S_a$ and $0 \leq \theta \leq 1$, then we have $\log(x) \leq a \implies x \leq \exp(a)$ so as $y \leq \exp(a)$.

$$\theta x + (1-\theta)y \leq \ \theta \exp(a) + (1-\theta)\exp(a) = \exp(a).$$

Take the log of both side and we get

$$\log(\theta x + (1-\theta)y) \leq a.$$

Thus, we have $\theta x + \ (1-\theta)y \in S_a$ and $\log(x)$ is quasiconvex. The above showed that the function $\log(x)$ is quasilinear.

Quasiconvex and Quasiconcave Function Example

Let $f : R^2 \to R$ and dom $f \in R_+$ defined as

$$f(x_1, x_2) = x_1 x_2$$

The Hessian Matrix is

$$\nabla f''(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The function f is not convex nor concave. But f is quasiconcave and quasiconvex.

Quasiconvex and Quasiconcave Function Example

Proof: $S_a = \{x \in R^2_+ | x_1 x_2 \geq a\}$, $x, y \in S_a$, $0 \leq \theta \leq 1$ $\hat{\theta} = 1 - \theta$.

$$
\begin{aligned}
(\theta x_1 + \hat{\theta} x_2)(\theta y_1 + \hat{\theta} y_2) &= \theta^2 x_1 y_1 + \hat{\theta}^2 x_2 y_2 + \theta\hat{\theta}(x_1 y_2 + x_2 y_1) \\
&= \theta^2 x_1 y_1 + \hat{\theta}^2 x_2 y_2 + \theta\hat{\theta}\left(\frac{x_1}{x_2} x_2 y_2 + \frac{x_2}{x_1} x_1 y_1\right) \\
&\geq \theta^2 a + \hat{\theta}^2 a + \theta\hat{\theta}\left(\frac{x_1}{x_2} a + \frac{x_2}{x_1} a\right) \\
&= a\left[\theta^2 + \hat{\theta}^2 + \theta\hat{\theta}\left(\frac{x_1}{x_2} + \frac{x_2}{x_1} + 2 - 2\sqrt{\frac{x_1}{x_2}}\sqrt{\frac{x_2}{x_1}}\right)\right. \\
&= a\left[\theta^2 + \hat{\theta}^2 + \theta\hat{\theta}\left\{\left(\sqrt{\frac{x_1}{x_2}} - \sqrt{\frac{x_2}{x_1}}\right)^2 + 2\right\}\right] \\
&\geq a\left[\theta^2 + \hat{\theta}^2 + 2\theta\hat{\theta}\right] \\
&\geq a(\theta^2 + \theta^2 - 2\theta + 1 - 2\theta^2 + 2\theta) = a.
\end{aligned}
$$

Thus, we have $\theta x + (1 - \theta) y \in S_a$ and S_a is a convex for all a.
The function xy is quasiconcave.

Quasiconvex and Quasiconcave Function Example

Proof: Let $S_a = \{x \in R_+^2 | x_1 x_2 \leq a\}$, $x, y \in S_a$, $0 \leq \theta \leq 1$, and $\hat{\theta} = 1 - \theta$.

$$\begin{aligned}
(\theta x_1 + \hat{\theta} x_2)(\theta y_1 + \hat{\theta} y_2) &= \theta^2 x_1 y_1 + \hat{\theta}^2 x_2 y_2 + \theta\hat{\theta}(x_1 y_2 + x_2 y_1) \\
&\leq \theta^2 x_1 y_1 + \hat{\theta}^2 x_2 y_2 \\
&\leq \theta x_1 y_1 + \hat{\theta} x_2 y_2 \\
&\leq \theta a + (1 - \theta) a = a.
\end{aligned}$$

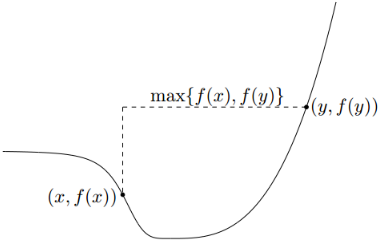
Thus, we have $\theta x + (1 - \theta) y \in S_a$ and S_a is a convex for all a. The function xy is quasiconvex. Since f is both quasiconvex and quasiconcave, f is quasilinear.

Quasiconvex Functions

A function f is quasiconvex iff the dom f is convex and for any $x, y \in \operatorname{dom} f$ and $0 \leq \theta \leq 1$ such that

$$f(\theta x + (1-\theta)y) \leq \max\{f(x), f(y)\}.$$

It is known as the Jensens inequality for quasiconvex functions.



The value of the function on a segment does not exceed the maximum of its values at the endpoints.

Quasiconvex Functions

A function f is quasiconvex, that is $S_a = \{x | f(x) \leq a\}$ is convex for all $a \in R \implies f(\theta x + (1-\theta)y) \leq \max\{f(x), f(y)\}$.
Proof: WLOG let $\max(f(x), f(y)) = f(x) = a$. Let $x, y \in S_a$ such that

$$f(x) \leq a \text{ and } f(y) \leq a$$

and let $0 \leq \theta \leq 1$. Since S_a is convex, we have $\theta x + (1-\theta)y \in S_a$, thus we have

$$f(\theta x + (1-\theta)y) \leq a = \max(f(x), f(y)).$$

Thus, f is quasiconvex.

f$(\theta x + (1-\theta)y) \leq \max(f(x), f(y)) \implies S_a = \{x | f(x) \leq a\}$ is convex for al $a \in R$.

Proof: WLOG Let $\max(f(x), f(y)) = f(x) = a$. Let $x, y \in S_a$ and $0 \leq \theta \leq 1$. We have $f(x) \leq a$ and $f(y) \leq a$. Since

$$f(\theta x + (1-\theta)y) \leq \max(f(x), f(y)) = a,$$

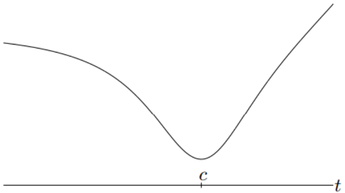
we have $f(\theta x + (1-\theta)y) \leq a$. This implies that $\theta x + (1-\theta)y \in S_a$ and S_a is convex.

Quasiconcave can be proved in a similar fashion.

Quasiconvex Functions

The function $f : \mathbb{R} \to \mathbb{R}$ is quasiconvex, if and only if at least one of the following conditions holds

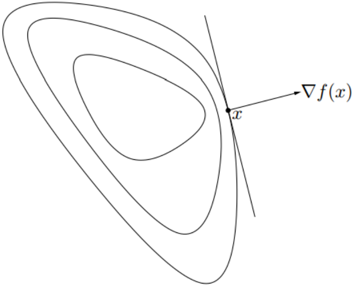
- f is nondecreasing
- f is nonincreasing
- There exist a point $c \in \text{dom } f$ such that for $t \leq c$ (and $t \in \text{dom } f$), f is nonincreasing, and for $t \geq c$ (and $t \in \text{dom } f$), f is nondecreasing.



Differentiable Quasiconvex Functions

First-order conditions:
Suppose $f : R^n \to R$ is differentiable. Then f is quasiconvex iff dom f is convex and for all $x, y \in$ dom f

$$f(y) \leq f(x) \Rightarrow \nabla f(x)^t (y - x) \leq 0. \qquad (1)$$



Differentiable Quasiconvex Functions

Proof: We assume that $f(y) \leq f(x) \Rightarrow f'(x)^t(y-x) \leq 0$ is true. Let the function $f : \mathbb{R}^n \to \mathbb{R}$, $x, y \in \text{dom } f$ such that $y > x$ and $f(y) \leq f(x)$. Let $z \in [x, y]$ and $f(z) \leq f(x)$.

Suppose for contradiction that $f(z) \geq f(x)$, since f is differentiable, we have $f'(z) < 0$. This implies that

$$f(x) \leq f(z) \implies f'(z)(x-z) \leq 0.$$

Since $(x-z) \leq 0$ and $f'(z) < 0$, which contradict the (1).

Differentiable Quasiconvex Functions

Need to show the other direction as well.

Proof: We assume that $f(y) \leq f(x) \Rightarrow f'(x)^t(y-x) \leq 0$ is true. Let the function $f : \mathbb{R}^n \to \mathbb{R}$, $x, y \in \operatorname{dom} f$ such that $y < x$ and $f(y) \leq f(x)$. Let $z \in [y, x]$ and $f(z) \leq f(x)$.

Suppose for contradiction that $f(z) \geq f(x)$, since f is differentiable, we have $f'(z) > 0$. This implies that

$$f(x) \leq f(z) \implies f'(z)(x-z) \leq 0.$$

Since $(x-z) \geq 0$ and $f'(z) > 0$, which contradict the (1). Thus, we have

$$f(y) \leq f(x) \Rightarrow \nabla f(x)^t(y-x) \leq 0.$$

Second-order conditions:
If f is quasiconvex, then for all $x \in \text{dom } f$, and all $y \in R^n$, we have

$$y^t \nabla f(x) = 0 \Rightarrow y^t \nabla^2 f(x) y \geq 0.$$

Preserve Quasiconvex Operations

- Scalar: If $f(x)$ is quasiconvex and $w > 0$, then $g(x) = wf(x)$ is also quasiconvex.

  Proof: Let $S_a = \{x | g(x) \leq a\}$, $x_1, x_2 \in S_a$, and $0 \leq \theta \leq 1$.

$$g(\theta x + (1-\theta)y) = wf(\theta x + (1-\theta)y) \leq w\theta f(x) + w(1-\theta)f(y)$$
$$\leq \theta a + (1-\theta)a = a.$$

Preserve Quasiconvex Operations

- Pointwise supremum: if f_1 and f_2 are quasiconvex then

$$g(x) = \sup_{i \in \{1,2\}} (f_i(x))$$

  is quasiconvex.

Proof: Let $S_a = \{x | g(x) \leq a\}$, $x_1, x_2 \in S_a$, and $0 \leq \theta \leq 1$.

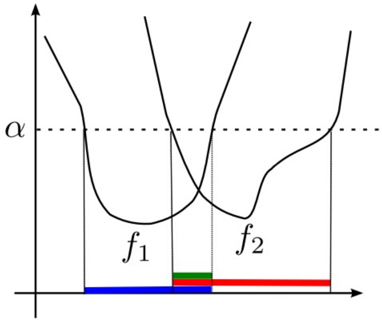
$$\begin{aligned} g(\theta x + (1-\theta)y) &= \sup_{i \in \{1,2\}} (f_i(\theta x + (1-\theta)y)) \\ &\leq \sup_{i \in \{1,2\}} (\theta f_i(x) + (1-\theta) f_i(y)) \\ &\leq \sup_{i \in \{1,2\}} (\theta a + (1-\theta)a) = a \end{aligned}$$

Preserve Quasiconvex Operations

- Pointwise supremum: if f_1 and f_2 are quasiconvex then

$$g(x) = \sup_{i \in \{1,2\}} (f_i(x))$$

is quasiconvex.



Preserve Quasiconvex Operations

- Composition
$$g(x) = h(f(x))$$
is quasiconvex if $f : R^n \to R$ is quasiconvex and $h : R \to R$ is nondecreasing.

Proof: Let $S_a = \{x | g(x) \leq a\}$, $x_1, x_2 \in S_a$, and $0 \leq \theta \leq 1$.

$$\begin{aligned} g(f(\theta x + (1-\theta)y)) &= h(f(\theta x + (1-\theta)y)) \\ &\leq h(\theta f(x) + (1-\theta)f(y)) \\ &\leq \theta h(f(x)) + (1-\theta)h(f(y)) \leq a. \end{aligned}$$

Thus, S_a is convex and $g(x)$ is quasiconvex.

Preserve Quasiconvex Operations

- Minimization If $f(x, y)$ is quasiconvex jointly in x and y and C is a convex set, then the function

$$g(x) = \inf_{y \in C} f(x, y)$$

  is quasiconvex.

Show $S_a = \{x | g(x) \leq a\}$ is convex. Let $x_1, x_2 \in S_a$, $y_1, y_2 \in C$ and $\epsilon > 0$ such that $f(x_1, y_1) \leq a + \epsilon$ and $f(x_2, y_2) \leq a + \epsilon$. Since f is quasiconvex, we have

$$f(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2) \leq a + \epsilon$$

Thus, $g(\theta x_1 + (1-\theta)x_2) \leq a$ and S_a is convex.

Log-concave and Log-convex Functions

A log-concave function $f : R^n \to R$ if $f(x) > 0$ for all $x \in \text{dom } f$ and $\log f(x)$ is concave. For $x, y \in \text{dom } f$ and $0 \leq \theta \leq 1$ then

$$f(\theta x + (1-\theta)y) = f(x)^\theta f(y)^{1-\theta}.$$

Log-concave and Log-convex Functions

Twice differentiable log-convex/concave functions

Suppose f is twice differentiable, with $\mathbf{dom}\, f$ convex, so

$$\nabla^2 \log f(x) = \frac{1}{f(x)} \nabla^2 f(x) - \frac{1}{f(x)^2} \nabla f(x) \nabla f(x)^T.$$

We conclude that f is log-convex if and only if for all $x \in \mathbf{dom}\, f$,

$$f(x) \nabla^2 f(x) \succeq \nabla f(x) \nabla f(x)^T,$$

and log-concave if and only if for all $x \in \mathbf{dom}\, f$,

$$f(x) \nabla^2 f(x) \preceq \nabla f(x) \nabla f(x)^T.$$