INDIAN INSTITUTE OF INFORMATION TECHNOLOGY, LUCKNOW

Quiz - II Marking Scheme: April, 2018
Date of Examination (Session): 12.04.2018

Program Code & Semester: B.Tech. (IT) 4th semester
Paper Title: Convex Optimization, Paper Code: SMAT430C

Max Marks: 20
Duration: 1 hour

---

1. Let f be a linear-fractional function. Prove or disprove that f is quasilinear. [4]

   Solution: A function $f : \mathbb{R}^n \to \mathbb{R}$ is called quasiconvex if its domain and all its sublevel sets $S_\alpha = \{x \in \mathbf{dom} f | f(x) \leq \alpha\}$ for $\alpha \in \mathbb{R}$, are convex. A function is quasiconcave if every superlevel set $\{x \in \mathbf{dom} f | f(x) \geq \alpha\}$is convex. A function that is both quasiconvex and quasiconcave is called quasilinear. [1]

   A linear fractional function is defined by $f(x) = \frac{a^T x+b}{c^T x+d}$, with $\mathbf{dom} f = \{x | c^T x + d > 0\}$[1]

   Its α-sublevel set is $S_\alpha = \{x | (c^T x + d) > 0, \frac{(a^T x+b)}{(c^T x+d)} \leq \alpha\}$
   $= \{x | (c^T x + d) > 0, (a^T x + b) \leq \alpha(c^T x + d)\}$, which is convex, since it is the intersection of an open halfspace and a closed halfspace. [1]

   Similarly its α-superlevel set is $= \{x | (c^T x + d) > 0, \frac{(a^T x+b)}{(c^T x+d)} \geq \alpha\}$
   $= \{x | (c^T x + d) > 0, (a^T x + b) \geq \alpha(c^T x + d)\}$, which is convex, since it is the intersection of an open halfspace and a closed halfspace. [1]

   As the function is both quasiconvex and quasiconcave it is quasilinear.

2. Prove that a function is convex if and only if its epigraph is a convex set. [8]

   Solution:

   Consider the function f to be convex, we need tol prove that its epigraph is a convex set:

$epif = \{(x,t) : f(x) \leq t\}$.[1]

Let $(x,t), (y,s) \in epif$. Hence, $f(x) \leq t, f(y) \leq s$.[1]

As f is convex, for $0 \leq \lambda \leq 1, f(\lambda x + (1-\lambda)y)) \leq \lambda f(x) + (1-\lambda))f(y) \leq \lambda t + (1-\lambda))s$.[1]

This implies that $(\lambda x + (1-\lambda)y), \lambda t + (1-\lambda)s) = \lambda(x,t) + (1-\lambda)(y,s) \in epif$.[1]

Hence, epigraph of f is a convex set.

Conversely, consider epi f is a convex set, we need to prove that f is a convex function:

$(x, f(x)), (y, f(y)) \in epif$.[1]

As $epif$ is convex, for $0 \leq \lambda \leq 1, \lambda(x, f(x)) + (1-\lambda)(y, f(y))$
$= (\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda)f(y)) \in epif$.[1]

Therefore, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y))$.[1]

Hence, f is a convex function. [1]

3. Show that if f is convex in (x,y) and C is a convex non-empty set, then the function $g(x) = inf_{y \in C} f(x,y)$ is convex in x. [4]

Solution:

If f is convex in (x,y), and C is a convex nonempty set, then the function $g(x) = \inf_{y \in C} f(x,y)$ is convex in x, provided $g(x) > -\infty$ for all x. The domain of g is $domg = \{x | (x,y) \in domf forsomey \in C\}$.

We prove this by verifying Jensens inequality for $x_1, x_2 \in domg$. Let $\epsilon > 0$. Then there are $y_1, y_2 \in C$ such that $f(x_i, y_i) \leq g(x_i) + \epsilon$ for $i = 1, 2$.[1]

Now let $\theta \in [0,1]$. We have

$g(\theta x_1 + (1-\theta)x_2) = \inf_{y \in C} f(\theta x_1 + (1-\theta)x_2, y)$[1]

$\leq f(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2)$[1]

$\leq \theta f(x_1, y_1) + (1-\theta)f(x_2, y_2)$[1]

$\leq \theta g(x_1) + (1-\theta)g(x_2) + \epsilon$[1]

Since this holds for any $\epsilon > 0$, we have
$g(\theta x_1 + (1-\theta)x_2) \leq \theta g(x_1) + (1-\theta)g(x_2)$.[1]

4. In composition of two functions h and g, the requirement that monotonicity hold for the extended - value extension $\tilde{h}$, and not just the function h, cannot be removed from the composition rules. Give reason. [4]

Solution:

The requirement that monotonicity hold for the extended-value extension $\tilde{h}$ and not just the function h, cannot be removed from vector composition rules. This is evident from the following example:

Consider the function, $g(x) = x^2$, with $domg = \mathbb{R}$, and $h(x) = 0$, with $domh = [1, 2]$.[1]

Here g is convex, and h is convex and nondecreasing. [1]

But the function $f = h \circ g$, given by $f(x) = 0, domf = [\sqrt{2}, 1] \bigcup [1, \sqrt{2}]$, is not convex, since its domain is not convex.[1]

Here, of course, the function $\tilde{h}$ is not nondecreasing for any $x, y \in \mathbb{R}$ with $x < y$, as we do not have $\tilde{h}(x) \leq \tilde{h}(y)$[1] .