Convex Optimization: SMAT430C
Practice Problem Set-I

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1. From Boyd's Book, Chapter 2 (Convex Set), Exercises: 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.12, 2.14, 2.16.

2. If C is an affine set and $x_0 \in C$, then show that the set $V = C - x_0 = \{x - x_0 \,|\, x \in C\}$ is a subspace.

3. Determine Interior (A), Closure(A) and Boundary(A) for the following:

   (a) $A = (0, 1]$
   (b) $A = (-1, 1) \cup \{2\}$
   (c) $A = \{(x, 0) \in \mathbb{R}^2 \,|\, x \in \mathbb{R}\}$
   (d) $A = \mathbb{Q}$
   (e) $A = \mathbb{Q}^c$
   (f) $A = \{(x, y) \in \mathbb{R}^2 \,|\, x > 0, y \neq 0\}$
   (g) $A = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$
   (h) $A = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \neq 1\}$
   (i) $A = \mathbb{R} \setminus \mathbb{Z}$

4. Draw figures of the unit ball $B[x_c, r]$ for

   (a) $(\mathbb{R}^2, ||\,.||_2)$ (the standard Euclidean norm)
   (b) $(\mathbb{R}^2, ||\,.||_1)$ (the l_1 norm)
   (c) $(\mathbb{R}^2, ||\,.||_\infty)$ (the l_∞ norm)

5. Show that a norm ball is convex set.

6. Show that $B[x_c, r] = x_c + rB[0, 1]$, $r > 0$.

7. Prove that an arbitrary intersection of convex sets is convex set. Is the result true for arbitrary union? Justify.

8. Express the positive semidefinite cone S_n^+ as the intersection of half spaces.

9. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called affine if it is of the form $f(x) = Ax + b$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Show that the image of a convex set (resp. the inverse image of a convex set) under affine function is again convex.

10. Show that the sum of two convex sets S_1 and S_2, defined as $S_1 + S_2 = \{x_1 + x_2 \,|\, x_1 \in S_1, x_2 \in S_2\}$ is convex.

11. Show that the Cartesian product of two convex sets S_1 and S_2, defined as $S_1 \times S_2 = \{(x_1, x_2) \,|\, x_1 \in S_1, x_2 \in S_2\}$ is convex.

12. Prove that a positive definite matrix can not have a zero or a negative number on its diagonal.

13. The function $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$, defined by $P(x_1, \ldots, x_n, t) = (\frac{x_1}{t}, \ldots, \frac{x_n}{t})$ for $t > 0$, is called a perspective function. Show that image of a convex set (resp. inverse image of a convex set) under the perspective function is convex.

14. The function $f : \mathbb{R}^n \to \mathbb{R}^m$, defined by $f(x) = \frac{(Ax+b)}{(c^Tx+d)}$, where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ with $\mathbf{dom}\ f = \{x : c^T x + d > 0\}$, is called a linear-fractional function. Show that image of a convex set (resp. inverse image of a convex set) under the perspective function is convex.

15. Determine whether the following systems of linear equations is consistent. In case of consistent, find the dimension of their solution set.

$$\begin{aligned} x_1 - 2x_2 + x_3 + 2x_4 &= 1 \\ x_1 + x_2 - x_3 + x_4 &= 2 \\ x_1 + 7x_2 - 5x_3 - x_4 &= 3 \end{aligned}$$

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + 2x_3 &= 1 \\ x_1 - 3x_2 + 4x_3 &= 2 \end{aligned}$$

$$\begin{aligned} 2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 &= -2 \\ x_1 - 2x_2 - 4x_3 + 3x_4 + x_5 &= -2 \\ 2x_1 - 4x_3 + 2x_4 + x_5 &= 3 \\ x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 &= -7 \end{aligned}$$

16. Determine whether the following matrices are positive definite, positive semi-definite, negative definite, negative semi-definite or indefinite.

$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \\ -1 & -1 & -3 \end{bmatrix} \quad \begin{bmatrix} -2 & 4 & -1 \\ 4 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & -2 \\ -1 & -2 & -4 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 3 \\ 3 & 3 & 9 \end{bmatrix} \quad \begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -5 \end{bmatrix}$$

17. For what value(s) of b, the following matrix is positive definite.

$$\begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$$

18. Let S^2_+ denote the set of all 2×2 positive semidefinite matrices. Then show that

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in S^2_+ \quad \Leftrightarrow x \geq 0,\ z \geq 0,\ xz \geq y^2$$