$\Rightarrow g(\theta x_1 + (1-\theta)x_2) \le \theta g(x_1) + (1-\theta) g(x_2)$.

$\Rightarrow g$ is convex.

Ex.- $\text{dis}(x, S) = \inf \{\|x-y\|, y \in S\}$

$S \to$ set

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Quasi-Convex Function:

A function $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ is called quasi-convex function if domf is convex and

$S_\alpha = \{x \in \text{dom} f \mid f(x) \le \alpha\}$ is convex ~~for~~ $\forall \alpha \in \mathbb{R}$.

Example- $\log(x) : \mathbb{R}$

Quasi-Concave function:

A function $f: \mathbb{R}^n \to \mathbb{R}$ is called quasi-concave if $-f$ is quasi-convex.

eg.- $f: \mathbb{R}^2_+ \subseteq \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = x_1 x_2$.

Then find if f is quasi-concave/convex function

Linear fractional func:

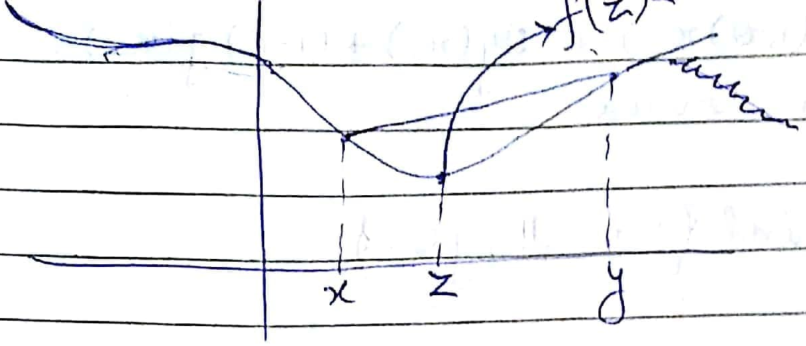
$$f(x) = \frac{a^T x + b}{c^T x + d}, \quad \text{dom} f = \{x \mid c^T x + d > 0\}$$

$$S_\alpha = \left\{x \in \text{dom} f \;\middle|\; \frac{a^T x + b}{c^T x + d} \le \alpha\right\}$$

$\Leftrightarrow \{x \mid a^T x + b \le \alpha(c^T x + d),\ c^T x + d > 0\} \to$ Intersection of 2 half-spaces. Thus, always convex.

Result: A function f is quasi-convex $\Leftrightarrow$ domf is convex and $\forall\, x, y \in \text{dom} f$ and $0 \le \theta \le 1$

$$f(\theta x + (1-\theta) y) \le \max\{f(x), f(y)\}.$$



Result: A continuous function $f: \mathbb{R} \to \mathbb{R}$ is quasi-convex $\Longleftrightarrow$ at least one of the following condⁿ hold

i) f is non-decreasing.

ii) f is non-increasing.

iii) There is a point $c \in \text{dom} f$ such that $t \le c$ and (and $t \in \text{dom} f$), f is non-increasing, and for $t \geq c$ (and $t \in \text{dom} f$) f is non-increasing.

Q.- Let $f: \mathbb{R} \to \mathbb{R}$ be a func. Let $c \in \mathbb{R}$ ~~be~~ be a ~~part~~ point ~~of~~ for $t \le c$, f is decreasing & for $t \geq c$, f is increasing. Prove that f is quasi-convex.

Log Concave Function:

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a positive function i.e. $f(x) > 0$. Then f is said to be log concave if $\log(f(x))$ is concave function.

Similarly, A function f is log convex if $\log f$ is a convex function.

Result: f is log convex $\Longleftrightarrow$ $\frac{1}{f}$ is log concave.

Example: i) $f(x) = a^T x + b$ is log concave on $\{x \mid a^T x + b > 0\}$

ii) $f(x) = e^{ax}$ is log convex / log concave function

Both. Prove it.

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Exercise (1) Prove that product of log concave function is log concave. What about product of log convex function.

(2) Is sum of log concave func. (resp. log convex) is log concave (resp. log convex).

(3) Show that if $g: \mathbb{R}^n \to \mathbb{R}$ is quasi-~~convex~~ convex and $h: \mathbb{R} \to \mathbb{R}$ is non-decreasing then $f = h \circ g$ is also quasiconvex.

(4) The composition of a quasiconvex func. with an affine func. yields a quasi-convex func.

Optimization Problem:

1. Maximize/minimize $f_0(x)$, here $f_0(x)$ is called objective func.
   Constraints: (Inequality Const.): $f_i(x) \leq 0$ for $i = 1, 2, \ldots, m$
   (equality const.): $h_i(x) = 0$, $i = 1, 2, \ldots, p$.

i) $x \in \mathbb{R}^n$ is called optimization variable.
ii) $f_0: \mathbb{R}^n \to \mathbb{R}$ is called objective func.
iii) $f_i: \mathbb{R}^n \to \mathbb{R}$ is called inequality constraint.
iv) $h_i: \mathbb{R}^n \to \mathbb{R}$ " " equality ".

$x = (x_1, x_2, \ldots, x_n)$, each x_i is called decision variable.

Domain of optimization $\Rightarrow$ $(\cap D(f_i)) \cap (\cap \mathrm{Dom}(h_i)) = D$
i.e. if $\exists$ $x \in D$. Then x is called feasible point and D is called feasible region. Also the problem is

called feasible, otherwise infeasible.

$$p^* = \begin{cases} \inf\{f_0(x)\} \\ \infty \text{ , if the problem is infeasible.} \\ -\infty \text{ , if problem is unbounded below.} \end{cases}$$

$$p^* = \begin{cases} \sup\{f_0(x)\} \\ \infty \text{ ; unbounded above} \\ -\infty \text{ ; infeasible} \end{cases} \quad \text{for maxz.}$$

-/3/18.

Diet Mix Problem:

A kitchen manager at (A) hospital has to decide the food mix for the patients. Dietary instructions are that each patient must get at least

i) 1 gram of protein.

ii) 1 gram of fat.

iii) 3 grams of carbohydrate.

Additional inst. are that in no ~~case~~ case the carbohydrate content should exceed 6 grams per patient.

The availability of protein, fat & carbohydrates in gram per kg of chicken, rice and bread is as given below:

	Protein	Fat	Carbohydrate	Price/kg
Chicken	10	2	0	30
Rice	2	1	15	5
Bread	2	0	10	4

Formulate a suitable optimization problem for the diet mix problem assuming 100 patients ~~are~~ on that day in order to minimize ~~the~~ the cost.

Sol.- Let x kg chicken, y kg rice, z kg bread.

Cost of chicken = 30 rs.

" " rice = 5 rs.

" " bread = 4 rs.

$\min(30x + 5y + 4z)$ subject to

$f_1 = 10x + 2y + 2z \geq 100$

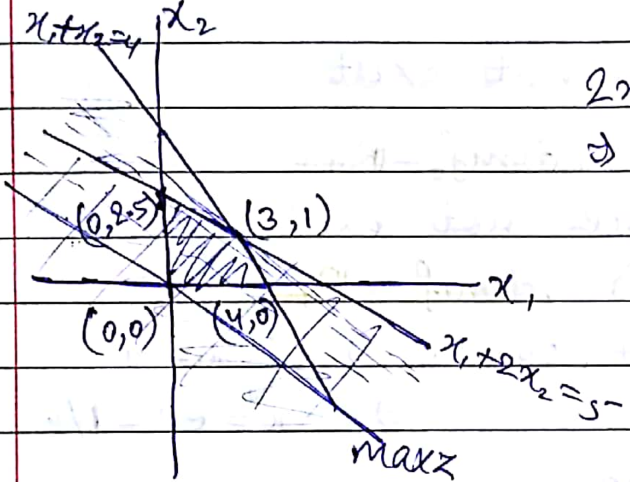
$f_2 = 2x + y + 0.z \geq 100$

$600 \geq \underbrace{0.x + 15y + 10z}_{f_3} \geq 100$, $x \geq 0, y \geq 0, z \geq 0$.

~~Ma~~

Q.- $\max z = 2x_1 + 4x_2$ subject to $x_1 + 2x_2 \leq 5$

$x_1 + x_2 \leq 4$, $x_1, x_2 \geq 0$.

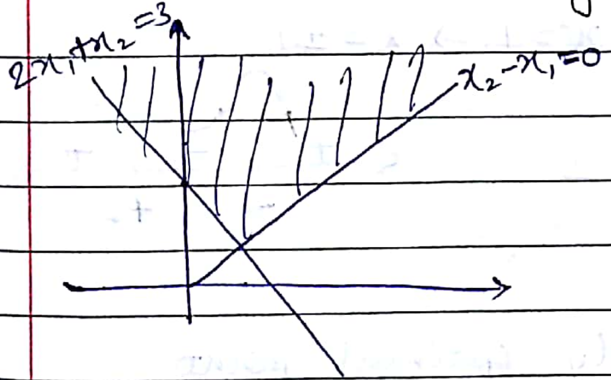


$2x_1 + 4x_2 = 0$

$\Rightarrow \frac{x_1}{x_2} = -2$. $(-2, 1)$, $(0,0)$.

Sol. $\Rightarrow$ $\boxed{(3,1)}$

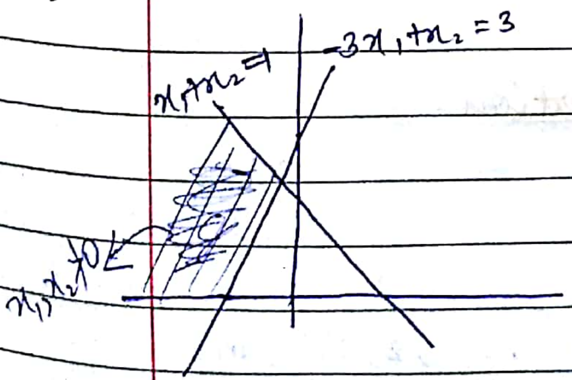
Q.- $\max z = 6x_1 + x_2$. subject to $2x_1 + x_2 \geq 3$, $x_2 - x_1 \geq 0$, $x_1, x_2 \geq 0$



~~Not feasible~~. (Unbounded)

$p^* = +\infty$.

Q.- $\max z = x_1 + x_2$, s.t. $x_1 + x_2 \leq 1$, $-3x_1 + x_2 \geq 3$, $x_1, x_2 \geq 0$



Infeasible problem.

$p^* = -\infty$.

Optimal and locally optimal points:

i) x is feasible if $x \in \text{dom} f_0$ and it satisfies all the constraints.

ii) A feasible x is optimal if $f_0(x) = p^*$,

iii) x is locally optimal if there exists $R > 0$ s.t. x is optimal for (P) for $\|z - x\|_2 \le R$.

Example 1: $\min. f_0(x) = \frac{1}{x}$ $\quad \text{dom} f_0 = \mathbb{R}_{++}$.

Sol.- $\quad p^* = 0$, x_0 does not exist.

Ex 2- $\min. (f_0(x) = -\log x)$, $\text{dom} f_0 = \mathbb{R}_{++}$.

Sol.: $\quad p^* = -\infty$ & x_0 does not exist.

Ex.3- $\min. (f_0(x) = x \log x)$, $\text{dom} f_0 = \mathbb{R}_{++}$

Sol. $\quad \cancel{p^* =}$ $f_0'(x) = 1 + \log x = 0 \Rightarrow \log x = -1$

$\Rightarrow x = e^{-1} = 1/e$.

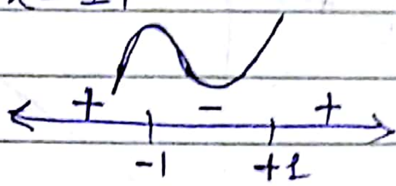
$p^* = \frac{-1}{e}$, $x_0 = 1/e$.

Ex.4- $\min. (f_0(x) = x^3 - 3x)$. $\text{dom.} = \mathbb{R}$.

Sol.- $f_0'(x) = 3x^2 - 3 = 0 \Rightarrow x^2 = 1. \Rightarrow x = \pm 1$

$f_0(1) = 1 - 3 = -2.$

$f_0(-1) = -1 + 3 = 2.$



$p^* = -\infty$.

however, ~~where~~ $x = 1$ is locally optimal point.

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Standard form of Optimization:

Problem

$$\min f_0(x)$$

Subject to $\quad f_i(x) \le 0$, $i = 1, 2, \ldots, m$

$\qquad h_i(x) = 0$, $i = 1, 2, \ldots, p$.

~~$D = \bigcap_{i=1}^{m} f_i$~~ $D = \left(\bigcap_{i=1}^{m} \operatorname{dom} f_i\right) \cap \left(\bigcap_{i=1}^{p} \operatorname{dom} h_i\right)$ is also

called implicit constraints of the problem. And h_i & f_i $(i=1,\ldots,m)$ are called explicit ~~por~~ constraints.

Def^n: A problem is called unconstraint if it has no explicit constraints.

Ex.- $\min f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x)$

$$b_i - a_i^T x > 0$$

i.e. $b_i > a_i^T x$

Dom: $\{x \mid b_i > a_i^T x\}$

Def^n: Feasibility Problem:

Convex Optimization Problem:

An optimization problem of the form

$\min f_0(x)$ subject to $f_i(x) \le 0$, $i=1,\ldots,m$

$$Ax = B \Leftrightarrow \{a_i^T x = b_i \quad, i=1,\ldots,p\}$$

is called convex optimization problem if all f_i are convex function and equality constraints are affine function.

Feasibility reason of a convex problem is convex set.

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Ex- i) $\min f_0(x) = x_1^2 + x_2^2$ subject to $\frac{x_1}{(1+x_2^2)} \le 0$

$h_i(x) = (x_1+x_2)^2 = 0 \longrightarrow$ Not an affine func

$\therefore$ Not Convex Opt. Prob.

$D = \{(x_1, x_2) \mid x_1 = -x_2 \,,\, x_1 \le 0\}$

ii) $\min \; x_1^2 + x_2^2$ subject to $x_1 \le 0$, $x_1 + x_2 = 0$

$D = \{(x_1, x_2) \mid x_1 = -x_2 \le 0\}$

Result:

For a convex optimization problem, when we minimize a convex function over a convex set then every local minima is a global minima

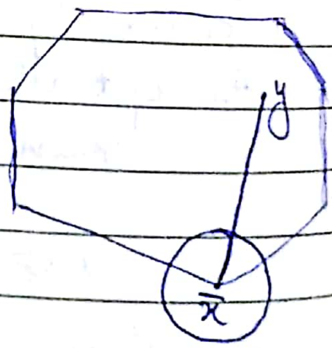
Proof: Let $\bar{x}$ be a local minima of a set D.

Then $\exists \; \delta > 0$ s.t. $\forall \; x \in B_\delta(\bar{x}) \cap D$

we have $f(x) \geq f(\bar{x})$

Let y be a point in D and ~~to~~ join it to $\bar{x}$

$z \in [\bar{x}, y]$



$z = \lambda y + (1-\lambda)\bar{x}$ for $\lambda \in [0\;1]$

$\lambda = 0 \;,\; z = \bar{x}$

$\lambda = 1 \;,\; z = y$

then $\exists \; \mathring{\lambda} \in (0\;1)$ s.t. $\forall \; \lambda \in [0, \mathring{\lambda}]$

$z_\lambda = \lambda y + (1-\lambda)\bar{x} \in B_\delta(\bar{x}) \cap C$.

Now, $f(z_\lambda) \geq f(\bar{x})$

$\Rightarrow f(\lambda y + (1-\lambda)\bar{x}) \geq f(\bar{x})$

$\Rightarrow f(\bar{x}) \leq f(\lambda y + (1-\lambda)\bar{x}) \leq \lambda f(y) + (1-\lambda) f(\bar{x})$

$\Longleftrightarrow f(\bar{x}) \leq \lambda f(y) + f(\bar{x}) - \lambda f(\bar{x})$

$\lambda f(\bar{x}) \leq \lambda f(y)$

$\Longleftrightarrow f(\bar{x}) \leq f(y)$.

Since, y was arbitrary. Here $\bar{x}$ is globally minimum.

Result:

Let $f : C \to \mathbb{R}$ be a convex func. defined over a convex set $C \subseteq \mathbb{R}^n$. Then the set of optimal solution of the problem $\min\{f(x) \mid x \in C\}$ which is denoted by X^*, is convex. In addition, if f is strictly convex on C, then $\exists$ at most one optimal solution of the problem.

Proof:

$X^* = \{$set of optimal solution of problem$\}$

Let optimal value is f^*

If $X^* = \phi$, then nothing to prove.

Suppose $X^* = \phi$. Let $x \neq y$

$x, y \in X^*$, $f(x) = f(y) = f^*$

Then to show that

$\lambda x + (1-\lambda) y \in X^*$, for $\lambda \in [0\ 1]$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$
$$\leq \lambda f^* + (1-\lambda) f^*$$
$$\leq f^*$$

$$\Rightarrow \cancel{f(\lambda x + (1-} \ f(\lambda x + (1-\lambda) y) = f^*$$

Suppose $x \neq y$, $x, y \in X^*$

$\frac{1}{2}x + \frac{1}{2}y \in X^*$

$$f\left(\tfrac{1}{2}x + \tfrac{1}{2}y\right) < \tfrac{1}{2}f(x) + \tfrac{1}{2}f(y) \quad \text{(for strict case).}$$

$$\Rightarrow f\left(\tfrac{1}{2}x + \tfrac{1}{2}y\right) < f^*$$

a contradiction.

Equivalent Problems:

Two optimization problems are said to be equivalent if solution of one can be obtained from other & vice-versa.

Example: $\min f_0(x) = x_1^2 + x_2^2$

subject to $f_1(x) = \dfrac{x_1}{(1+x_2^2)} \leq 0$, $h(x) = x_1 + x_2 = 0$.

$\min.\ x_1^2 + x_2^2$ subject to $x_1 \leq 0$, $x_1 + x_2 = 0$.

feasible in $D = \{(x_1, x_2) \mid x_1 = -x_2 \leq 0\}$

1.) Eliminating equality Constraints:

$\min f_0(x)$

subject to $f_i(x) \leq 0$, $i = 1, 2, \ldots, m$

$AX = b$.

is equivalent to

$\min_{(\text{over } z)} f_0(FZ + x_0)$

subject to $f_i(FZ + x_0) \leq 0$

where F & x_0 are s.t.

$AX = b \iff x = FZ + x_0$

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Optimality Criteria for Convex Optimization

Problem: $\min f_0(x)$ subject to $f_i(x) \leq 0$, $i = 1, \ldots, m$
$Ax = b$.

Result: Suppose that f_0 is a convex optimization problem s.t. f_0 is differentiable. Let X denote the feasible set. Then x is optimal $\iff$

$\nabla f_0(x)^T (y - x) \geq 0 \quad \forall\ y \in X$.

Proof: Let $\nabla f_0(x)^T (y - x) \geq 0 \quad \forall\ y \in X$.

To show that x is optimal point i.e.

$$f_0(y) \geq f_0(x) \quad \forall\, y \in X.$$

Since f_0 is a convex function and differentiable $\forall\, y \in X$ we have

$$f_0(y) \geq f_0(x) + \nabla f_0(x)^T (y-x)$$

$$\Rightarrow \quad f_0(y) - f_0(x) \geq 0$$

$$\Rightarrow \quad f_0(y) \geq f_0(x) \quad \forall\, y \in X$$

$\Rightarrow$ This x is optimal point.

Converse,

Let x is an optimal point and suppose if possible $\exists\, y \in X$ s.t. $\nabla f_0(x)^T (y-x) \leq 0$

Since, $x, y \in X$ and X is a convex set $\Rightarrow$

$$z(t) = ty + (1-t)x \in X \text{ for } t \in [0\ \ 1]$$

$$\lim_{t \to 0} \frac{f_0(z(t)) - f_0(x)}{t} = \lim_{t \to 0} \frac{(f_0(z(t)) - f_0(x))(z(t)-x)}{(z(t)-x)t}$$

$$= \lim_{t \to 0} \frac{(f_0(z(t)) - f_0(x)) . \cancel{t}(y-x)}{t(y-x)\cancel{t}}$$

$$= \lim_{t \to 0} \frac{f_0(z(t)) - f_0(x)}{t(y-x)} . (y-x)$$

$$= \nabla f_0(x)^T . (y-x) < 0$$

for small ~~to~~ $t > 0$

$$f_0(z(t)) - f_0(x) < 0 \Rightarrow f_0(z(t)) < f_0(x)$$

which is a contradiction.

Problem: Prove that $x^* = (1, 1/2, -1)$ is optimal for the optimization problem $\min . \frac{1}{2} x^T P x + q^T x + r$ where subject to $-1 \leq x_i \leq 1$, $i = 1, 2, 3$.

$$P = \begin{pmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{pmatrix}, \quad q = \begin{pmatrix} -12 \\ -14.5 \\ 13 \end{pmatrix}, \quad r = 1.$$

Sol.- To show that x^* is optimal it is supposed to show that

$$\left\{\begin{array}{l}\nabla f_0(x^*)^T(y-x^*) \geq 0 \quad \forall\, y \in x.\\ f: \mathbb{R}^n \longrightarrow \mathbb{R} \text{ and } f \text{ is diff.}\end{array}\right.$$

$\downarrow$

To use this, we first have to show that the problem is ~~an op~~ convex opt. problem.

$\nabla f_0'(x)$ $\leftarrow$

$= P \to$ +ve definite.

$\Rightarrow$ convex.

$$\nabla f_{/x^*} = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

$$f_0 : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$f_0(x) = \frac{1}{2}(x_1 \ x_2 \ x_3)\left(\quad P \quad \right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + (-12 \ -14.5 \ 13)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + 1$$

$$= \frac{1}{2}(13x_1+12x_2-2x_3 \quad 12x_1+17x_2+6x_3 \quad -2x_1+6x_2+12x_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$-12x_1-14.5x_2+13x_3+1$$

$$= \frac{13x_1^2}{2}+6x_2x_1-x_3x_1+6x_1x_2+\frac{17}{2}x_2^2+3x_3x_2-\cancel{x_1x_3}+3x_2\cancel{x_3}+6x_3^2$$

$$-12x_1 \cancel{+} -14.5x_2+13x_3+1$$

$$= \cancel{\frac{13}{2}x_1^2+\frac{17}{2}x_2^2+6x_3^2-7x_1-5.5x_2+15x_3+1}$$

$$= \frac{13}{2}x_1^2+\frac{17}{2}x_2^2+6x_3^2+12x_1x_2-2x_1x_3+6x_2x_3-12x_1$$
$$-14.5x_2+13x_3+1$$

$$\nabla f_0 \quad \frac{\partial f}{\partial x_1} = 13x_1+12x_2-2x_3-12$$
$$= 13(1)+12(+1/2)-2(-1)-12 = 9 > 0$$

Similarly find $\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} > 0$.

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Condition for Unconstraint Problem i.e. $\min f_0(x)$

Then condition is x_0 is optimum point if

$$\nabla f_0(x) = 0.$$

Proof: Let x is optimum. This means

$$\nabla f_0(x)^T (y - x) \geq 0 \quad \forall y \in X.$$

$\because$ f_0 is diff. on X. So for y sufficiently close to x are feasible.

$$y = x - t \nabla f_0(x) \quad , t \in \mathbb{R}.$$

for t small, all possible y is feasible.

$$\nabla f_0(x)^T (-t \nabla f_0(x)) \geq 0$$

$$= -t \|\nabla f_0(x)\|_2^2 \geq 0 \qquad \because t > 0$$

$$\Rightarrow -\|\nabla f_0(x)\|_2^2 \geq 0 \Rightarrow \|\nabla f_0(x)\|_2^2 \leq 0 \Leftrightarrow \|\nabla f_0(x)\|_2^2 = 0$$

$$\Leftrightarrow \boxed{\nabla f_0(x) = 0}$$

Convex Optimization Problem with equality Constraints:

i.e. $\min f_0(x)$ subject to $Ax = b$.

Let the feasible set is non-empty. Otherwise the problem will be infeasible.

The optimality criteria

$$\nabla f_0(x)^T (y - x) \geq 0 \quad \forall y \in X.$$

then $Ay = b$.

Since x is feasible, therefore, every $y \in X$ can be written as $y = x + v$ where ~~$v \in$~~ $v \in N(A)$

$$\Rightarrow \nabla f_0(x)^T . v \geq 0 \quad \forall \quad v \in N(A)$$

$$\Rightarrow \nabla f_0(x)^T v = 0 \quad \Rightarrow \nabla f_0(x) \perp N(A)$$

$$N(A)^\perp = R(A^T) \Rightarrow \nabla f_0(x) \in R(A^T) \quad \text{i.e.}$$

$$\exists \cancel{v \in R^p} \; v \in R^p \text{ s.t. } \boxed{\nabla f_0(x) + A^T v = 0}$$

~~or $v \in \mathrm{dom}(A^T)$~~

Proof: If $x \in X$ is optimum then

$$\nabla f_0(x)^T (y-x) \geq 0 \quad \forall\, y \in X \qquad \text{--- } (*)$$

$$y = x + v \quad \text{where } v \in N(A)$$

$$(*) \Rightarrow \nabla f_0(x)^T v \geq 0 \quad , \forall\, v \in N(A)$$

$$\Rightarrow \nabla f_0(x)^T v = 0 \quad \forall\, v \in N(A)$$

$$\Rightarrow \nabla f_0(x)^T \in N(A)^{\perp}$$

$$\Rightarrow \nabla f_0(x)^T \in R(A^T)$$

$$\Rightarrow \exists\, \lambda \in \mathrm{dom}(A^T) \text{ s.t.}$$

$$\Rightarrow \boxed{A^T\lambda = \nabla f_0(x)^T \Leftrightarrow \nabla f_0(x)^T - A^T\lambda = 0}$$

Convex Optimization Problem over a positive orthant i.e.

$$\min f_0(x)$$
$$\text{subject to } x \geq 0$$

If $x \geq 0$ is an optimal point then

$$\nabla f_0(x)^T (y-x) \geq 0 \quad \forall\, y \geq 0$$

$$\nabla f_0(x)^T y \geq 0 \quad \forall\, y \in X \text{ i.e. } y \geq 0 \qquad \{\text{for } x=0\}$$

This term is bounded below under

$$\nabla f_0(x)^T \geq 0$$

$$\Rightarrow \nabla f_0(x)^T y - \nabla f_0(x)^T x \geq 0$$

$$\Leftrightarrow -\nabla f_0(x)^T \geq 0$$

$$\Leftrightarrow \nabla f_0(x)^T \leq 0 \Leftrightarrow \boxed{\nabla f_0(x)^T = 0}$$

$(x_1, x_2, \dots, x_n) \quad (y_1, y_2, \dots, y_n)$

$\nabla f_0(x)^T x = 0.$

$\Leftrightarrow \quad \sum_{i=1}^{n} (\nabla f_0(x))_i \cdot x_i = 0.$

$\Leftrightarrow \quad \nabla f_0(x_i) . x_i = 0 \quad \forall\ i = 1, \dots, n.$

$$\boxed{\begin{array}{c} x \geq 0 \text{ if } \nabla f_0(x_i).x_i = 0 \\ \text{and } \nabla f_0(x)^T \geq 0 \end{array}}$$

Exercise. In convex optimization problem $\min f_0(x)$ s.to $Ax = b$. If x is feasible point then any feasible y can be expressed as $y = x + v$, $v \in N(A)$.

Equivalent Reformulation:

Problem: $\begin{cases} \min f_0(x) \\ \text{sub. to } f_i(x) \leq 0, \quad i = 1, \dots, m. \\ Ax = b. \end{cases}$ —(*)

i) Eliminating or introducing equality constraints.

(*) is equivalent to

$\min_{z}. \ f_0(Fz + x_0)$

sub. to $f_i(Fz + x_0) \leq 0, \quad i = 1, \dots, m.$

where F & x_0 are such that

$Ax = b \Leftrightarrow x = Fz + x_0.$

ii) Introducing slack variable.

(*) is equivalent to

$\min_{x, s}. \ f_0(x)$

sub. to $f_i(x) + s_i = 0 \quad \forall\ i = 1, \dots, m$

$Ax = b, \quad s_i \geq 0, \quad i = 1, \dots, m.$

Here s_i are slack variables

iii) Epigraph form of (*)

or (*) is also equivalent to

$$\min_{x,t} \quad t$$

Sub. to $f_0(x) - t \leq 0$

$f_i(x) \leq 0$, $i = 1, \ldots, m$.

$Ax = b$.

Quasi-Convex Optimization Problem.

$\min.\ f_0(x)$

s.t. $f_i(x) \leq 0$, $Ax = b$.

where $f_0 : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a quasi convex func.

and $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ are convex func.

Example. $\min \left(\dfrac{f_0(x)}{c^T x + d} \right)$ Sub. to $f_i(x) \leq 0$, $i = 1, \ldots, m$.

$Ax = b$.

where $f_0, f_1, \ldots, f_m$ are convex function and domain of objective function is $\{x \in \operatorname{dom} f_0 \mid c^T x + d > 0\}$

Then

a) Show that the problem is a quasi-convex problem.

b) This is equivalent to $\min g_0(y,t)$ sub. to $g_i(y,t) \leq 0$, $Ay = bt$, $c^T y + dt = 1$

where g_i is the perspective of f_i. The variable $y \in \mathbb{R}^n$ & $t \in \mathbb{R}$ and show that the equivalent problem is convex.

$f_0(x)$ is convex thus all α-sublevel sets are convex. So, given func. will be quasi-convex.

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Example: min. $\|Ax-b\|_1/(c^Tx+d)$ subject to $\|x\|_\infty \le 1$.
where $A\in\mathbb{R}^{m\times n}$, $b\in\mathbb{R}^m$, $c\in\mathbb{R}^n$. and $d\in\mathbb{R}$. We assume that $d\ge\|c\|_1$, which implies that $c^Tx+d>0$ $\forall$ feasible x. Then

a) show that this is a quasi-convex optimization problem.

b) This problem is equivalent to the convex optimization problem

min. $\|Ay-bt\|_1$ subject to $\|y\|_\infty \le t$, $c^Ty+dt=1$. $(*)$

Sol.- Let $y=\dfrac{x}{c^Tx+d}$, $t=\dfrac{1}{c^Tx+d}$.

$$\|Ay-bt\|_1 = \left\|\frac{Ax}{c^Tx+d}-\frac{b}{c^Tx+d}\right\| = \left\|\frac{Ax-b}{c^Tx+d}\right\|$$
$$= \frac{1}{c^Tx+d}\|Ax-b\|.$$

$\|y\|_\infty \le t$

$\left\|\dfrac{x}{c^Tx+d}\right\|_\infty \le t \Rightarrow \|x\|_\infty \le t.(c^Tx+d) \Rightarrow \|x\|_\infty \le t.$

Suppose y,t are feasible in $(*)$ problem and ~~do~~ if we use the ~~formal~~ transformation $x=y/t$, $(t>0)$

$\|x\|_\infty \le 1 \Rightarrow \|y/t\|_\infty \le 1 \Rightarrow \|y\|_\infty \le t.$

$\|A(y/t)-b\|_1/(c^T\frac{y}{t}+d) \Leftrightarrow \dfrac{\|Ay-bt\|_1}{c^Ty+dt} \Rightarrow \|Ay-bt\|_1$