Indian Institute of Information Technology Allahabad
Mid Semester Examination, March 2017
Tentative Marking Scheme

Date of Examination (Meeting): 05.03.2017 (1st meeting)

Program Code & Semester: B.Tech. (IT) & Dual Degree B.Tech.-M.Tech. IV Semester
Paper Title: Convex Optimization, Paper Code: SMAT430C
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Max Marks: 35 Duration: 2 hours

Attempt each question on a new page, and attempt all the parts of a question at the same place. Numbers indicated on the right in [ ] are full marks of that particular problem.

Notations: $\mathbb{N}$: Set of natural numbers, $\mathbb{Z}$: Set of integers, K_1 and K_2 are Cones, inf is the infimum, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \succeq 0\}$, $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$.

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1. Prove or disprove the following statements.

   (a) Let $A, B \subset \mathbb{R}$ such that $A \subseteq B$. Then $\inf B \leq \inf A$. [2]

   Solution. Let $a = \inf A,\ b = \inf B$.
   $\implies b \leq x \quad \forall\ x \in B$.
   $\implies b \leq x \quad \forall\ x \in A$. [1]
   $\implies b \leq a$. [1]

   (b) The set $\{x \in \mathbb{R} : -\frac{1}{2}x^2 + x + 1 \leq 0\}$ is convex. [2]

   Solution. The function $f(x) = -\frac{1}{2}x^2 + x + 1$ is concave, and has two distinct roots, say x_1 and x_2. [1]
   Let $x_1 < x_2$, then $\{x \in \mathbb{R} : -\frac{1}{2}x^2 + x + 1 \leq 0\} = (-\infty, x_1) \cup (x_2, \infty)$ which is not convex. [1]

   (c) The average value of a continuous and convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ on any line segment is less than or equal to the average of its values at the endpoints of the segment. [3]

   Solution. As f is convex, we have for $0 \leq \lambda \leq 1$, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$. [1]
   Integrating both sides from 0 to 1,
   $$\int_0^1 f(\lambda x + (1-\lambda)y) \leq \int_0^1 [\lambda f(x) + (1-\lambda)f(y)]d\lambda = \frac{f(x)+f(y)}{2}.$$ [2]

   (d) The cone $\mathbb{R}^n_+$ is self-dual. [3]

   Solution. We need to prove $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$.
   For $y \succeq 0$, $x^T y \geq 0\ \ \forall\ x \succeq 0 \implies \mathbb{R}^n_+ \subseteq (\mathbb{R}^n_+)^*$. [1]
   Now, let $x^T y \geq 0\ \ \forall\ x \succeq 0$. Choose $x = e_i,\ i = 1, 2, \ldots, n$, where e_i denotes the vector with a 1 in the i^{th} coordinate and 0's elsewhere. [1]
   $\implies y_i \geq 0,\ i = 1, 2, \ldots, n \implies y \succeq 0 \implies (\mathbb{R}^n_+)^* \subseteq \mathbb{R}^n_+$. [1]

   (e) $K_1 \subseteq K_2 \implies K_1^* \subseteq K_2^*$. [2]

   Solution. $\mathbb{R}^2_+ \subset \mathbb{R}^2$ but $(\mathbb{R}^2_+)^* = \mathbb{R}^2_+ \nsubseteq (\mathbb{R}^2)^* = \{0\}$. [2]

2. Let $f : \mathbb{R} \times \mathbb{R}_{++} \longrightarrow \mathbb{R}$ be defined as $f(x,y) = \dfrac{x^2}{y}$. Determine whether f is convex or not. Is f quasiconvex. [4]

   Solution. $\bigtriangledown^2 f(x,y) = \begin{pmatrix} 2/y & -2x/y^2 \\ -2x/y^2 & 2x^2/y^3 \end{pmatrix}$. [1]

   $\bigtriangledown^2 f(x,y)$ is positive semidefinite, i.e., $\bigtriangledown^2 f(x,y) \succeq 0$, because determinant is zero, $2/y > 0$, and $2x^2/y^3 \geq 0$. [1]
   Therefore, f is convex, [1]
   Because f is convex, it is also quasiconvex. [1]

3. Show that the conjugate of a function is always a convex function. Derive the conjugate of the exponential function on $\mathbb{R}$. [6]

   Solution. Let $f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$. The conjugate of f is the function $f^* : B \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, defined as $f^*(y) = \sup\limits_{x \in A} (y^T x - f(x))$, where $B = \{y \in \mathbb{R}^n : \sup_{x\in A} (y^T x - f(x)) < \infty\}$. [1]

   For each x, $y^T x - f(x)$ is an affine function of y, and hence convex. Now, pointwise supremum of convex functions is convex. Hence, f is convex. [1]

   Let $f(x) = e^x$.

   $xy - e^x$ is unbounded if $y < 0$. [1]

   For $y > 0$, $xy - e^x$ attains maximum at $x = \log y$, [1]

   so we have $f^*(y) = y \log y - y$. [1]

   For $y = 0$, $f^*(y) = \sup (-e^x) = 0$. [1]

4. Find minimum and minimal element(s) of the set $\{x \in \mathbb{R}^2 : ||x||_\infty \leq 1\}$. [3]

   Solution. $||x||_\infty = \max\{|x_1|, |x_2|\}$. [1]

   The minimum element is $(-1, -1)$ because all other points of the above set lie above and to the right of $(-1, -1)$. [1]

   If a set has minimum element, then it is unique, and it's also the minimal element. Hence, minimal element is $(-1, -1)$. [1]

5. Find the supremum and infimum of the set $\{\frac{m}{|m|+n} : n \in \mathbb{N},\ m \in \mathbb{Z}\}$. [2]

   Solution. $\sup = 1$, $\inf = -1$. [2]

6. Prove that a function is convex if and only if its epigraph is a convex set. [8]

   Solution. ($\implies$) $\text{epi}f = \{(x, t) : f(x) \leq t\}$. [1]

   Let $(x, t),\ (y, s) \in \text{epi}f$. Hence, $f(x) \leq t,\ f(y) \leq s$. [1]

   As f is convex, for $0 \leq \lambda \leq 1$, $f(\lambda x + (1-\lambda)y)) \leq \lambda f(x) + (1-\lambda))f(y) \leq \lambda t + (1-\lambda))s$. [1]

   This implies that $(\lambda x + (1-\lambda)y), \lambda t + (1-\lambda)s) = \lambda(x, t) + (1-\lambda)(y, s) \in \text{epi}f$. [1]

   Hence, epigraph of f is a convex set.

   ($\Longleftarrow$) $(x, f(x)),\ (y, f(y)) \in \text{epi}f$. [1]

   As $\text{epi}f$ is convex, for $0 \leq \lambda \leq 1$,

   $\lambda(x, f(x)) + (1-\lambda)(y, f(y)) = (\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda)f(y)) \in \text{epi}f$. [1]

   Therefore, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y))$. [1]

   Hence, f is a convex function. [1]