7.1 The Fourier Integral Representation

To help us understand and appreciate the topics of this section, let us recall from Section 2.3 the Fourier series representation theorem. Given a $2p$-periodic function f, we have

$$(1) \qquad f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right),$$

where

$$a_0 = \frac{1}{2p} \int_{-p}^{p} f(t)\, dt,$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(t) \cos \frac{n\pi}{p} t\, dt, \quad b_n = \frac{1}{p} \int_{-p}^{p} f(t) \sin \frac{n\pi}{p} t\, dt.$$

Now suppose that f is defined on the entire real line but is not periodic. Can we represent f by a Fourier series? It turns out that we no longer have a Fourier series representation, but a Fourier integral representation. The answer is given by the following important theorem.

THEOREM 1 FOURIER INTEGRAL REPRESENTATION

Suppose that f is piecewise smooth on every finite interval and that $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$. Then f has the following **Fourier integral representation**

$$(2) \qquad f(x) = \int_0^{\infty} [A(\omega) \cos \omega x + B(\omega) \sin \omega x]\, d\omega \quad (-\infty < x < \infty),$$

where, for all $\omega \geq 0$,

$$(3) \qquad A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t\, dt; \qquad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t\, dt.$$

The integral in (2) converges to $f(x)$ if f is continuous at x and to $\frac{f(x+)+f(x-)}{2}$ otherwise.

Refer to Section 2.2 for the definition of *piecewise smooth.*

Note the similarity between the Fourier integral (2) and the Fourier series (1). The sum in (1) is replaced by an integral in (2), and the integrals from $-p$ to p that define the Fourier coefficients are replaced by integrals from $-\infty$ to ∞ in (3). Also, in (3), the "Fourier coefficients" are computed over a continuous range $\omega \geq 0$, whereas the Fourier coefficients of a periodic function are computed over a discrete range of values $n = 0, 1, 2, \ldots$.

As with Fourier series, for Theorem 1 to hold we imposed sufficient conditions on f, including the condition

$$\int_{-\infty}^{\infty} |\, f(x)\, |\, dx < \infty,$$

which is expressed by saying that f is **integrable on the entire real line.** This condition also ensures the existence of the improper integrals defining $A(\omega)$ and $B(\omega)$ in (3).

We will omit the proof of Theorem 1, which involves ideas similar to those in the proof of the Fourier series representation theorem (see [1]). The fact that the series in (1) is changed into an integral in the Fourier series representation as the period tends to infinity can be motivated as follows.

Suppose that f is an integrable function on the real line. Restrict f to a finite interval $(-p, p)$. Take the part of f that is inside $(-p, p)$ and extend it periodically outside this interval. The periodic extension agrees with f on $(-p, p)$ and has a Fourier series as in (1), which represents $f(x)$ for x in $(-p, p)$. The question now is, What happens to this representation as $p \to \infty$? To answer this question, let us investigate the Fourier coefficients as $p \to \infty$. Since f is integrable, it follows that $a_0 \to 0$ as $p \to \infty$. Also, we can draw a connection between a_n and b_n and $A(\omega)$ and $B(\omega)$ as follows. The integrability of f implies that the integrals in (3) can be approximated by merely integrating over the (large) finite interval $(-p, p)$. The difference is just the tail ends of the integrals, which can be made arbitrarily small. Thus, for large p, we can write

$$a_n \approx \frac{1}{p}\int_{-\infty}^{\infty} f(t)\cos\frac{n\pi}{p}t\,dt = A(\omega_n)\,\Delta\omega \qquad \text{(by (3))},$$

where $\omega_n = (n\pi)/p$ and $\Delta\omega = \frac{\pi}{p}$. Similarly, $b_n \approx B(\omega_n)\,\Delta\omega$. Plugging these values into (1), we see that for very large p, we have

$$f(x) \approx \sum_{n=1}^{\infty} \left(A(\omega_n)\cos\omega_n x + B(\omega_n)\sin\omega_n x\right)\Delta\omega. \tag{4}$$

We have conveniently used a notation that suggests that the sum in (4) is a Riemann sum. This sum samples the integrand of (2) at equally spaced points ω_n with a partition size $\Delta\omega$ being precisely the distance between two consecutive ω_n. The fact that n goes to infinity in (4) indicates that this Riemann sum is not over a finite interval but (regardless of p) spans the entire nonnegative ω-axis. As $p \to \infty$, $\Delta\omega \to 0$ and this Riemann sum converges to the integral in ω, given by (2).

EXAMPLE 1 A Fourier integral representation

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Solution From (3),

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt = \frac{1}{\pi}\int_{-1}^{1}\cos\omega t\,dt = \left[\frac{\sin\omega t}{\pi\omega}\right]_{-1}^{1} = \frac{2\sin\omega}{\pi\omega}.$$

(Strictly speaking, we should treat the case $\omega = 0$ separately. However, as you can check, the formula that we obtained for $A(\omega)$ is valid in the limit as $\omega \to 0$.) Since $f(x)$ is even, $B(\omega) = 0$. For $|x| \neq 1$ the function is continuous and Theorem 1 gives

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin\omega \cos\omega x}{\omega}\, d\omega\,.$$

For $x = \pm 1$, points of discontinuity of f, Theorem 1 yields the value 1/2 for the last integral. Thus we have the Fourier integral representation of f

$$\frac{2}{\pi} \int_0^\infty \frac{\sin\omega \cos\omega x}{\omega}\, d\omega = \begin{cases} 1 & \text{if } |x| < 1, \\ 1/2 & \text{if } |x| = 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

■

In Example 1 we have used the evenness of f to infer that $B(\omega) = 0$ for all ω. Similarly, if f is odd, then $A(\omega) = 0$ for all ω. These observations simplify the computation of the Fourier integral representations of even and odd functions.

Theorem 1 can be used to evaluate interesting improper integrals. For example, setting $x = 0$ in the integral representation of Example 1 yields the important integral

$$\int_0^\infty \frac{\sin\omega}{\omega}\, d\omega = \frac{\pi}{2}\,, \tag{5}$$

known as the **Dirichlet integral**, after the German mathematician Peter Gustave Lejeune Dirichlet (1805–1859).

EXAMPLE 2 Computing integrals via the Fourier integral

Show that

$$\frac{2}{\pi} \int_0^\infty \frac{\cos\frac{\pi\omega}{2}}{1-\omega^2} \cos\omega x\, d\omega = \begin{cases} \cos x & \text{if } |x| < \pi/2, \\ 0 & \text{if } |x| > \pi/2. \end{cases}$$

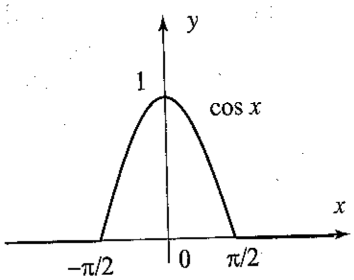

Figure 1 $f(x)$ in Example 2.

Solution Let $f(x)$ denote the function defined on the right side of this equality, as shown in Figure 1. It is even and vanishes outside the interval $[-\pi/2, \pi/2]$. Thus $B(\omega) = 0$ for all $\omega \geq 0$, and

$$\begin{aligned} A(\omega) &= \frac{2}{\pi} \int_0^{\pi/2} \cos x \cos\omega x\, dx && \text{(by evenness)} \\ &= \frac{1}{\pi} \int_0^{\pi/2} [\cos(1+\omega)x + \cos(1-\omega)x]\, dx && \text{(Trig identity)} \\ &= \frac{1}{\pi}\left[\frac{\sin[(1+\omega)\pi/2]}{1+\omega} + \frac{\sin[(1-\omega)\pi/2]}{1-\omega}\right] && \text{(for } \omega \neq 1) \\ &= \frac{1}{\pi}\left[\frac{\cos(\omega\pi/2)}{1+\omega} + \frac{\cos(\omega\pi/2)}{1-\omega}\right] = \frac{2\cos(\omega\pi/2)}{\pi(1-\omega^2)}\,. \end{aligned}$$

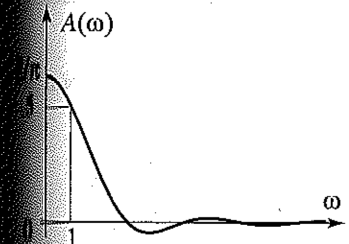

Figure 2 $A(\omega)$ is continuous.

The case $\omega = 1$ should be treated separately. It yields $A(1) = 1/2$ (check it!). As Figure 2 shows, the graph of $A(\omega)$ is continuous at $\omega = 1$. In fact, you can check that $\lim_{\omega\to 1} A(\omega) = A(1) = 1/2$. Now using (2), we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos(\omega\pi/2)}{1-\omega^2}\cos\omega x\, d\omega\,.$$

Replacing $f(x)$ by its formula, we get the desired identity. ■

Partial Fourier Integrals and the Gibbs Phenomenon

In analogy with the partial sums of Fourier series, we define the partial Fourier integral of f by

(6) $$S_\nu(x) = \int_0^\nu [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, d\omega \quad (\text{for } \nu > 0),$$

where $A(\omega)$ and $B(\omega)$ are given by (3). With this notation, Theorem 1 states

$$\lim_{\nu\to\infty} S_\nu(x) = \frac{f(x+)+f(x-)}{2}\,.$$

Like Fourier series, near a point of discontinuity the Fourier integral exhibits a Gibbs phenomenon. To illustrate this phenomenon, we introduce the **sine integral function**

(7) $$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt \qquad (-\infty < x < \infty).$$

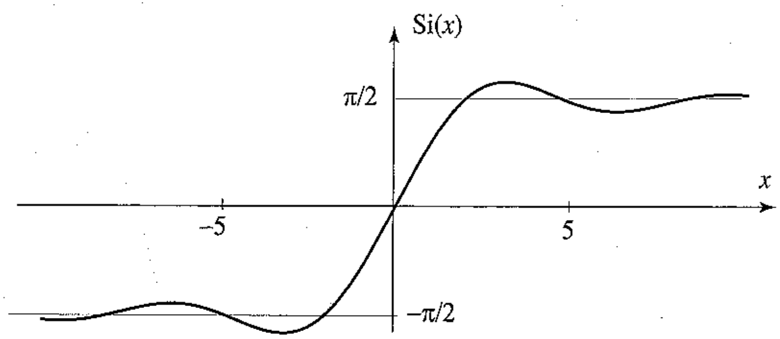

Figure 3 Graph of $\mathrm{Si}(x)$. Even though there is no expression of $\mathrm{Si}(x)$ in terms of elementary functions, you can still compute its numerical values using a power series expansion (see Exercise 23(c)).

Because of its frequent occurrence, the function $\mathrm{Si}(x)$ is tabulated and is available as a standard function in most computer systems. See Figure 3 for its graph. From (5), it follows that

(8) $$\lim_{x\to\infty}\mathrm{Si}(x) = \int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}\,.$$

EXAMPLE 3 Gibbs phenomenon for partial Fourier integrals

(a) Show that the partial Fourier integral of the function in Example 1 can be written as

$$S_\nu(x) = \frac{1}{\pi}[\mathrm{Si}(\nu(1+x)) + \mathrm{Si}(\nu(1-x))].$$

(b) To illustrate the representation of the function by its Fourier integral, plot several partial Fourier integrals and discuss their behavior near the points $x = \pm 1$.

Solution We have from Example 1 and (6)

$$\begin{aligned}
S_\nu(x) &= \frac{2}{\pi}\int_0^\nu \frac{\sin\omega\cos\omega x}{\omega}\,d\omega \\
&= \frac{1}{\pi}\int_0^\nu \frac{\sin\omega(1+x)}{\omega}\,d\omega + \frac{1}{\pi}\int_0^\nu \frac{\sin\omega(1-x)}{\omega}\,d\omega \\
&\quad \text{(Let } u=\omega(1+x) \text{ in the first integral, and } u=\omega(1-x) \text{ in the second.)} \\
&= \frac{1}{\pi}\left[\int_0^{\nu(1+x)} \frac{\sin u}{u}\,du + \int_0^{\nu(1-x)} \frac{\sin u}{u}\,du\right] \\
&= \frac{1}{\pi}[\mathrm{Si}(\nu(1+x)) + \mathrm{Si}(\nu(1-x))], \qquad \text{by (7).}
\end{aligned}$$

(b) In Figure 4 we have plotted the graphs of $S_\nu(x)$, for $\nu = 1, 4, 7, 10$, using the sine integral function and the formula given by (a). Observe how the partial integrals approximate the function in a way reminiscent of the approximation of a periodic function by its Fourier series. In particular, note the Gibbs phenomenon at the points of discontinuity of f where the partial integrals overshoot their limiting values. ■

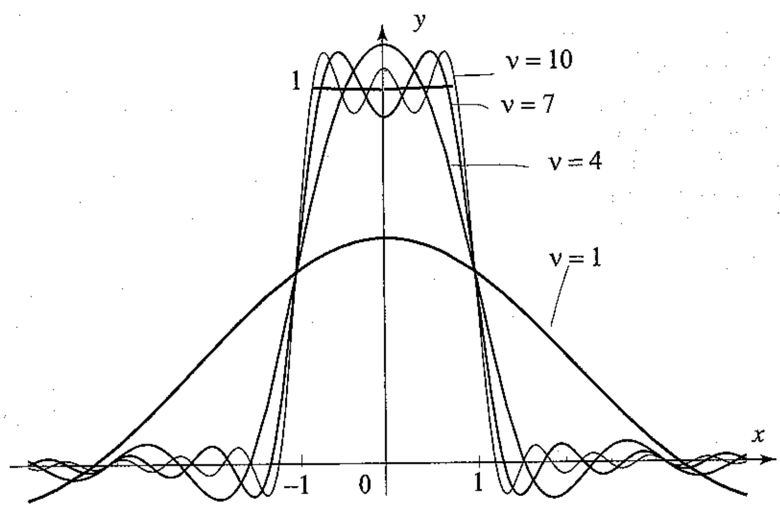

Figure 4 Approximation by partial Fourier integrals and Gibbs phenomenon.

Exercises 7.1

In Exercises 1–12, *find the Fourier integral representation of the given function.*

1.

$$f(x)=\begin{cases} 1 & \text{if } -a<x<a,\\ 0 & \text{otherwise,}\end{cases}$$

where $a>0$.

2.

$$f(x)=\begin{cases} -1 & \text{if } -1<x<0,\\ 1 & \text{if } 0<x<1,\\ 0 & \text{otherwise.}\end{cases}$$

3.

$$f(x)=\begin{cases} 1-\cos x & \text{if } -\pi/2<x<\pi/2,\\ 0 & \text{otherwise.}\end{cases}$$

4.

$$f(x)=\begin{cases} 1-|x| & \text{if } -1<x<1,\\ 0 & \text{otherwise.}\end{cases}$$

5.

$$f(x)=e^{-|x|}.$$

6.

$$f(x)=\begin{cases} 1-x^2 & \text{if } -1<x<1,\\ 0 & \text{otherwise.}\end{cases}$$

7.

$$f(x)=\begin{cases} 1 & \text{if } 0<x<1,\\ 0 & \text{otherwise.}\end{cases}$$

8.

$$f(x)=\begin{cases} 0 & \text{if } -1<x<1,\\ 1 & \text{if } 1<|x|<2,\\ 0 & \text{otherwise.}\end{cases}$$

9.

$$f(x)=\begin{cases} x & \text{if } -1<x<1,\\ 2-x & \text{if } 1<x<2,\\ -2-x & \text{if } -2<x<-1,\\ 0 & \text{otherwise.}\end{cases}$$

10.

$$f(x)=\begin{cases} x & \text{if } -1<x<1,\\ 0 & \text{otherwise.}\end{cases}$$

11.

$$f(x)=\begin{cases} \sin x & \text{if } 0<x<\pi,\\ 0 & \text{otherwise.}\end{cases}$$

12.

$$f(x)=\begin{cases} e^{-x} & \text{if } x>1,\\ 0 & \text{otherwise.}\end{cases}$$

13. (a) Use Example 1 to show that

$$\int_0^\infty \frac{\sin\omega\cos\omega}{\omega}\,d\omega=\frac{\pi}{4}.$$

(b) Use integration by parts and (a) to obtain

$$\int_0^\infty \frac{\sin^2\omega}{\omega^2}\,d\omega=\frac{\pi}{2}.$$

14. Use the identity $\sin^2\omega+\cos^2\omega=1$ and Exercise 13(b) to obtain

$$\int_0^\infty \frac{\sin^4\omega}{\omega^2}\,d\omega=\frac{\pi}{4}\,.$$

[Hint: $\sin^2\omega=\sin^4\omega+\cos^2\omega\sin^2\omega=\sin^4\omega+\frac{1}{4}\sin^2 2\omega$.]

In Exercises 15–18, establish the given identity.

15. $\displaystyle\int_0^\infty \frac{\sin\omega\cos 2\omega}{\omega}\,d\omega = 0.$ [Hint: Example 1.]

16. $\displaystyle\int_0^\infty \frac{1-\cos t}{t^2}\,dt = \frac{\pi}{2}.$ [Hint: Integrate by parts and use Example 1.]

17.
$$\int_0^\infty \frac{\cos x\omega + \omega\sin x\omega}{1+\omega^2}\,d\omega = \begin{cases} 0 & \text{if } x<0,\\ \pi/2 & \text{if } x=0,\\ \pi e^{-x} & \text{if } x>0.\end{cases}$$

18.
$$\frac{2}{\pi}\int_0^\infty \frac{\sin\pi\omega}{1-\omega^2}\sin\omega x\,d\omega = \begin{cases} \sin x & \text{if } |x|\le\pi,\\ 0 & \text{if } |x|>\pi.\end{cases}$$

19. Let $a>0$. Derive the formula
$$e^{-a\,x} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{a^2+\omega^2}\,d\omega \quad (x\ge 0).$$
[Hint: Fourier integral representation of $e^{-a|x|}$.]

20. Define the **signum function**, $\operatorname{sgn}(x)$, by
$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x>0,\\ 0 & \text{if } x=0,\\ -1 & \text{if } x<0.\end{cases}$$
Establish the identity
$$\frac{2}{\pi}\int_0^\infty \frac{\sin xt}{t}\,dt = \operatorname{sgn}(x).$$
[Hint: Make a change of variables $\omega = xt$.]

In Exercises 21–22, express the given function using the signum function and its translates.

21. $f(x)$ as in Example 1.

22. $f(x) = \begin{cases} 2 & \text{if } -1<x<1,\\ 1 & \text{if } 1<|x|<2,\\ 0 & \text{otherwise.}\end{cases}$

Project Problem: The sine integral. Do Exercises 23 and 24 and any one of Exercises 25–28.

23. Properties of the sine integral Use (7) to establish the following properties.
(a) $\operatorname{Si}(x)$ is an odd function.
(b) $\lim_{x\to-\infty}\operatorname{Si}(x) = -\frac{\pi}{2}$.
(c) $\operatorname{Si}(x) = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!}$. [Hint: $\displaystyle\frac{\sin t}{t} = \sum_{n=0}^\infty (-1)^n \frac{t^{2n}}{(2n+1)!}$.]

7.2 The Fourier Transform

We will use the complex exponential function to write the Fourier integral representation of Section 7.1 in complex form. This new representation features an important pair of transforms: the Fourier transform and its inverse Fourier transform. As you will see in this section, the concept of transform pairs provides a convenient way to state the fundamental operational properties of the Fourier transform, which are very useful in solving boundary value problems.

Consider a continuous piecewise smooth integrable function f. Starting with the Fourier integral representation, we have

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\left(\cos\omega t\cos\omega x + \sin\omega t\sin\omega x\right)\,dt\,d\omega \\ &\qquad\qquad (\cos(a-b) = \cos a\cos b + \sin a\sin b) \\ &= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\omega(x-t)\,dt\,d\omega \\ &\qquad\qquad (\cos u = \tfrac{1}{2}(e^{iu}+e^{-iu})) \\ &= \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(t)(e^{i\omega(x-t)}+e^{-i\omega(x-t)})\,dt\,d\omega \\ &= \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(t)e^{i\omega(x-t)}\,dt\,d\omega + \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(t)e^{-i\omega(x-t)}\,dt\,d\omega \end{aligned}$$

If we change ω to $-\omega$ in the second term and adjust the limits on ω from $-\infty$ to 0, we obtain, after adding the two integrals,

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)e^{i\omega(x-t)}\,dt\,d\omega \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{i\omega x}\overbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(t)e^{-i\omega t}\,dt}^{\hat{f}(\omega)}\,d\omega. \end{aligned}$$

This is the **complex form of the Fourier integral representation**, which features the following transform pair:

FOURIER TRANSFORM

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{-i\omega x}dx \qquad (-\infty < \omega < \infty) \tag{1}$$

and

INVERSE FOURIER TRANSFORM

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{i\omega x}\widehat{f}(\omega)\,d\omega \qquad (-\infty < x < \infty). \tag{2}$$

There are other conventions for the Fourier transform. For example, we could choose $\widehat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$, and then the inverse Fourier transform becomes $f(x) = \int_{-\infty}^{\infty} e^{i\omega x}\widehat{f}(\omega)\,d\omega$. In the definition of $\widehat{f}$ we have used x as a variable of integration, instead of t. The symbols $\mathcal{F}(f)(\omega)$ and $\mathcal{F}^{-1}(f)(x)$ are also used to denote the Fourier transform and its inverse, respectively. Sometimes, to be more specific, we will write $\mathcal{F}(f(x))(\omega)$ instead of $\mathcal{F}(f)(\omega)$. According to Theorem 1 of Section 7.1, if f is not continuous at x, the left side of (2) is to be replaced by $(f(x+) + f(x-))/2$. The integral for the inverse Fourier transform may not exist as a two-sided improper integral; in general, this integral should be computed as a Cauchy principal value: $f(x) = \frac{1}{\sqrt{2\pi}}\lim_{a\to\infty}\int_{-a}^{a} e^{i\omega x}\widehat{f}(\omega)\,d\omega$ (see [1], Section 11.1).

Putting $\omega = 0$ in (1), we find that

$$\boxed{\widehat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,dx\,.}$$

Thus the value of the Fourier transform at $\omega = 0$ is equal to the signed area between the graph of $f(x)$ and the x-axis, multiplied by a factor of $1/\sqrt{2\pi}$.

EXAMPLE 1 A Fourier transform

(a) Find the Fourier transform of the function in Figure 1, given by

$$f(x) = \begin{cases} 1 & \text{if } |x| < a, \\ 0 & \text{if } |x| > a. \end{cases}$$

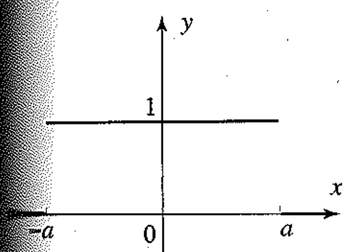

Figure 1 Graph of f in Example 1.

What is $\widehat{f}(0)$? (b) Express f as an inverse Fourier transform.

Solution For $\omega \neq 0$ we have

$$\begin{aligned} \widehat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-i\omega x}\,dx \\ &= \frac{-1}{\sqrt{2\pi}i\omega}e^{-i\omega x}\Big|_{-a}^{a} = \sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega}. \end{aligned}$$

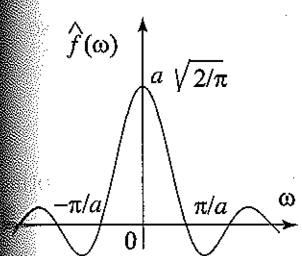

Figure 2 Graph of $\widehat{f}$ in Example 1.

For $\omega = 0$ we have $\widehat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} dx = a\sqrt{2/\pi}$. Since

$$\lim_{\omega\to 0}\widehat{f}(\omega) = \lim_{\omega\to 0}\sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega} = a\sqrt{\frac{2}{\pi}} = \widehat{f}(0),$$

it follows that $\widehat{f}(\omega)$ is continuous at 0 (Figure 2), and we may write

$$\widehat{f}(\omega) = \sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega} \qquad \text{for all } \omega.$$

(b) To express f as an inverse Fourier transform, we use (2) and get

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \sqrt{\frac{2}{\pi}} \frac{\sin a\omega}{\omega}\, d\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\omega x} \frac{\sin a\omega}{\omega}\, d\omega \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} (\cos \omega x - i \sin \omega x) \frac{\sin a\omega}{\omega}\, d\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos \omega x \sin a\omega}{\omega}\, d\omega, \end{aligned}$$

because $\sin \omega x \, \frac{\sin a\omega}{\omega}$ is an odd function of ω and so its integral is zero.

Not surprisingly, when $a = 1$, this representation coincides with the integral representation we found in Example 1 of Section 7.1. ∎

The Fourier transform in Example 1 is continuous on the entire real line even though the function has jump discontinuities at $x = \pm a$. In fact, it can be shown that the Fourier transform of an integrable function is *always continuous.*

In our next example we will use the absolute value of complex numbers. Let us recall that if $z = a + ib$, then $|z| = \sqrt{a^2 + b^2}$. In particular, if $z = e^{-ix\omega}$, then

$$\left| e^{-ix\omega} \right| = |\cos(x\omega) - i \sin(x\omega)| = \sqrt{\cos^2(x\omega) + \sin^2(x\omega)} = 1\,.$$

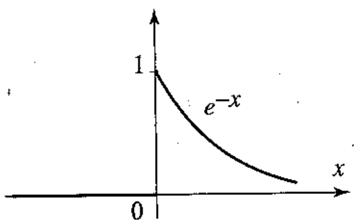

Figure 3 Graph of f in Example 2.

EXAMPLE 2 Computing Fourier transforms

Find the Fourier transform of the function in Figure 3,

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0\,. \end{cases}$$

Solution We have

$$\begin{aligned} \widehat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x} e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x(1+i\omega)}\, dx \\ &= \frac{-1}{\sqrt{2\pi}(1 + i\omega)} e^{-i\omega x} e^{-x} \Big|_0^{\infty}\,. \end{aligned}$$

Since $\left|e^{-ix\omega}\right| = 1$, it follows that $\lim_{x\to\infty} \left|e^{-x(1+i\omega)}\right| = \lim_{x\to\infty} e^{-x} = 0$, and so

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}(1 + i\omega)} = \frac{1 - i\omega}{\sqrt{2\pi}(1 + \omega^2)}\,.$$

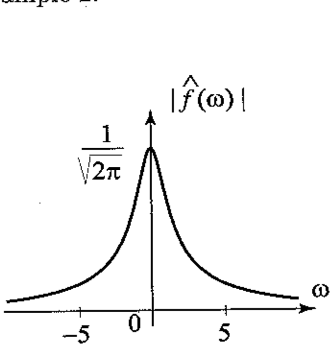

Figure 4 Graph of $|\widehat{f}|$ in Example 2.

Figure 4 shows the graph of the absolute value of $\widehat{f}$. Here again, it is worth noting that $\widehat{f}$ and $|\,\widehat{f}\,|$ are both continuous even though f is not. ∎

Example 2 illustrates a noteworthy fact that the Fourier transform may be complex-valued even though the function is real-valued. Also, the Fourier transform is continuous but not integrable, not even as an improper two-sided integral. In this case, the integral in (2) for the inverse Fourier transform should be computed as a Cauchy principal value. Indeed, you can check that, at $x = 0$, we do have $1/2 = \big(f(0+) + f(0-)\big)/2$, and the Cauchy principal value of the inverse Fourier transform yields

$$\begin{aligned}\lim_{a\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \widehat{f}(\omega)\, d\omega &= \lim_{a\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \frac{1 - i\omega}{\sqrt{2\pi}(1+\omega^2)}\, d\omega \\ &= \frac{1}{2\pi} \lim_{a\to\infty} \Big[\int_{-a}^{a} \frac{1}{1+\omega^2}\, d\omega - i \overbrace{\int_{-a}^{a} \frac{\omega}{1+\omega^2}\, d\omega}^{=0} \Big] = \frac{1}{2}.\end{aligned}$$

As defined by (1), the Fourier transform takes a function f and produces a new function $\widehat{f}$, and the inverse transform recovers the original function f from $\widehat{f}$. This process makes of transform pairs a powerful tool in solving partial differential equations. As we will see in the following sections, the idea is to "Fourier transform" a given equation into one that may be easier to solve. After solving the transformed equation involving $\widehat{f}$, we recover the solution of the original problem with the inverse transform. To assist us in handling the transformed equations, we develop the operational properties of the Fourier transform.

Operational Properties

We shall investigate the behavior of the Fourier transform in connection with the common operations on functions: linear combination, translation, dilation, differentiation, multiplication by polynomials, and convolution.

THEOREM 1 LINEARITY

The Fourier transform is a linear operation; that is, for any integrable functions f and g and any real numbers a and b,

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g).$$

Proof Exercise 18. ■

THEOREM 2 FOURIER TRANSFORMS OF DERIVATIVES

(i) Suppose $f(x)$ is piecewise smooth, $f(x)$ and $f'(x)$ are integrable, and $f(x) \to 0$ as $|x| \to \infty$, then

$$\mathcal{F}(f') = i\omega\, \mathcal{F}(f).$$

(ii) If in addition $f''(x)$ is integrable, and $f'(x)$ is piecewise smooth and $\to 0$ as $|x| \to \infty$, then

$$\mathcal{F}(f'') = i\omega\mathcal{F}(f') = -\omega^2\mathcal{F}(f).$$

(iii) In general, if f and $f^{(k)}(x)$ $(k = 1, 2, \ldots, n-1)$ are piecewise smooth and tend to 0 as $|x| \to \infty$, and f and its derivatives of order up to n are integrable, then

$$\mathcal{F}(f^{(n)}) = (i\omega)^n\mathcal{F}(f).$$

Proof Parts (ii) and (iii) are obtained by repeated applications of (i). To prove (i), we use the definition of $\mathcal{F}(f')$ and integrate by parts. To simplify the proof, we suppose further that f is smooth. Then

$$\begin{aligned}
\mathcal{F}(f')(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)e^{-i\omega x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left[f(x)\, e^{-i\omega x}\Big|_{-\infty}^{\infty} - (-i\omega) \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx \right] \\
&= 0 + i\omega\mathcal{F}(f) \qquad (\text{since } f(x) \to 0 \text{ as } |x| \to \infty, \text{ and } |e^{\pm i\omega x}| = 1).
\end{aligned}$$

THEOREM 3 DERIVATIVES OF FOURIER TRANSFORMS

(i) Suppose $f(x)$ and $xf(x)$ are integrable; then

$$\mathcal{F}(xf(x))(\omega) = i\left[\widehat{f}\right]'(\omega) = i\frac{d}{d\omega}\mathcal{F}(f)(\omega).$$

(ii) In general, if $f(x)$ and $x^n f(x)$ are integrable, then

$$\mathcal{F}(x^n f(x)) = i^n \left[\widehat{f}\right]^{(n)}(\omega).$$

Sketch of Proof Part (ii) follows from (i). To motivate (i) we will assume that we can differentiate under the integral. Then

$$\begin{aligned}
\left[\widehat{f}\right]'(\omega) &= \frac{d}{d\omega}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\frac{d}{d\omega}e^{-i\omega x}\, dx \\
&= -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xf(x)e^{-i\omega x}\, dx = -i\mathcal{F}(xf(x))(\omega),
\end{aligned}$$

and (i) follows upon multiplying both sides by i. This proof is valid if for example f is smooth and vanishes outside a finite interval. For an arbitrary function f, we approximate f by functions that are smooth and vanish outside a finite interval. The details are beyond the level of this book and will be omitted.

Convolution of Functions

We expand our list of operational properties by introducing the convolution of two functions f and g by

CONVOLUTION

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)g(t)\, dt. \tag{3}$$

(The factor $\frac{1}{\sqrt{2\pi}}$ is merely for convenience. If we drop it from the definition of the convolution, it will reappear in its Fourier transform.) The convolution of f and g is a binary operation, which combines translation, multiplication of functions, and integration. Its effect on the functions f and g is difficult to explain directly, as the following examples illustrate. (It does have a simple description in terms of the Fourier transform, as we will see in Theorem 4.) Let us first observe that convolution is a commutative operation; that is, $f * g(x) = g * f(x)$. This follows by making a change of variables ($t \leftrightarrow t - x$) in (3) (Exercise 55).

EXAMPLE 3 Convolution with the cosine

Suppose that f is integrable and even ($f(-x) = f(x)$ for all x) and let $g(x) = \cos ax$. Show that, for all real numbers a: $f * g(x) = \cos(ax)\,\widehat{f}(a)$.

Solution From the definition and the fact that $f * g = g * f$, we have

$$\begin{aligned} f * g(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cos[a(x-t)]\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \big[\cos(ax)\cos(at) + \sin(ax)\sin(at) \big]\, dt. \end{aligned}$$

Since f is even, the product $f(t)\sin at$ is odd; hence $\int_{-\infty}^{\infty} f(t) \sin at\, dt = 0$, and so

$$\begin{aligned} f * g(x) &= \cos(ax) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cos(at)\, dt \\ &= \cos(ax) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \big[\cos(at) - i \sin(at) \big]\, dt \\ &= \cos(ax) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iat}\, dt = \cos(ax) \widehat{f}(a). \end{aligned}$$

■

EXAMPLE 4 Convolution as an average

For $n = 1, 2, \ldots$, let $g_n(x) = n\sqrt{\pi/2}$ if $|x| < 1/n$ and $g_n(x) = 0$ otherwise. Suppose that f is continuous on $(-\infty, \infty)$ and let F denote an antiderivative of f. Show that

$$f * g_n(x) = \frac{F(x + 1/n) - F(x - 1/n)}{2/n}.$$

Solution From the definition, we have

$$\begin{aligned} f * g_n(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t) g_n(t)\, dt = \frac{n}{2} \int_{-1/n}^{1/n} f(x-t)\, dt \\ &= \frac{n}{2} \int_{x-1/n}^{x+1/n} f(t)\, dt = \frac{n}{2} \big(F(x + 1/n) - F(x - 1/n) \big), \end{aligned}$$

where we have used the change of variables $t \leftrightarrow x - t$ in the second integral. Thus the desired result follows. ■

We mention some noteworthy properties of the convolution in Example 4. The interval of integration, $(x - 1/n,\, x + 1/n)$, in the expression $f * g_n(x) = \frac{n}{2} \int_{x-1/n}^{x+1/n} f(t)\, dt$ is centered at x and has length $2/n$. Thus the convolution $f * g_n(x)$ is the average of the function f over the interval $(x - 1/n,\, x + 1/n)$. As the length of the interval shrinks to 0 (that is, as $n \to \infty$), we expect this average to converge to $f(x)$. In other terms, we expect $\lim_{n\to\infty} f * g_n(x) =$ $f(x)$. This is indeed the case, since we have

$$\begin{aligned}
\lim_{n\to\infty} f * g_n(x) &= \lim_{n\to\infty} \frac{n}{2}\big(F(x+1/n) - F(x-1/n)\big) \\
&= \frac{1}{2} \lim_{1/n\to 0} \left[\frac{F(x+1/n) - F(x)}{1/n} + \frac{F(x+(-1/n)) - F(x)}{(-1/n)}\right] \\
&= F'(x),
\end{aligned}$$

by definition of the derivative. But $F'(x) = f(x)$, by the fundamental theorem of calculus; and so $\lim_{n\to\infty} f * g_n(x) = f(x)$.

The fact that the convolution of f with a sequence of functions converges to f will be at the heart of solutions of important boundary value problems such as the Dirichlet problem in the upper half-plane and the heat equation of the real line. In each one of these problems, the sequence of functions, or kernels, will be different but it will share properties similar to the following properties of the sequence (g_n) in Example 4:

- $g_n(x) \geq 0$ for all x. The area under the graph of $g_n(x)$ and above the x-axis is equal to $\sqrt{2\pi}$, for all $n \geq 1$. That is, $\int_{-\infty}^{\infty} g_n(x)\,dx = \sqrt{2\pi}$. Moreover, the area is more and more concentrated around 0. That is, $\lim_{n\to\infty} \int_a^b g_n(x)\,dx = 0$ if 0 is not in $[a, b]$.
- On the Fourier transform side, we have $\lim_{n\to\infty} \widehat{g}_n(\omega) = 1$ for all ω (Exercise 8).

Convolutions can be tedious to compute directly from definition (3). One way to avoid a direct computation is to use the following important property of convolutions and the Fourier transform. The process is illustrated in Example 5 below.

THEOREM 4 FOURIER TRANSFORMS OF CONVOLUTIONS

Suppose that f and g are integrable; then

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g).$$

Theorem 4 is expressed by saying that the Fourier transform takes convolutions into products.

Proof Using (3) and (1), and then interchanging the order of integration, we get

$$\begin{aligned}
\mathcal{F}(f * g)(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t) e^{-i\omega x}\,dx\, g(t)\,dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i\omega u}\,du\, e^{-i\omega t} g(t)\,dt \\
&\qquad\qquad (u = x - t,\ du = dx) \\
&= \mathcal{F}(f)(\omega)\mathcal{F}(g)(\omega).
\end{aligned}$$

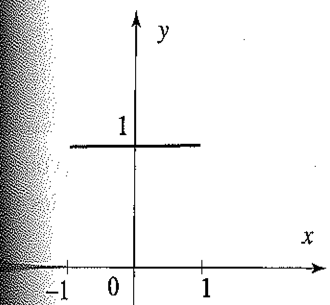

Figure 5 Graph of f.

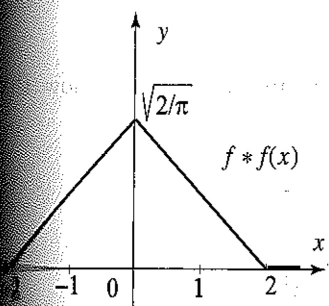

Figure 6 Graph of $f * f$.

EXAMPLE 5 Fourier transform of a convolution

Consider the function $f(x) = 1$ if $|x| < 1$ and 0 otherwise. The graph of this function is shown in Figure 5. From Example 1, we have

$$\widehat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega}.$$

We want to compute $f * f$, the convolution of f with itself. Instead of computing directly from (3), we will use Theorem 4 as follows. We have

$$\mathcal{F}(f * f)(\omega) = \widehat{f}(\omega)^2 = \frac{2}{\pi} \frac{\sin^2\omega}{\omega^2}.$$

Using the inverse Fourier transform, with the help of the table of Fourier transforms in Appendix B, we find

$$f * f(x) = \mathcal{F}^{-1}\left(\frac{2}{\pi}\frac{\sin^2\omega}{\omega^2}\right) = \begin{cases} \sqrt{\frac{2}{\pi}}\left(1 - \frac{|x|}{2}\right) & \text{if } |x| < 2, \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

The graph of $f * f(x)$ is shown in Figure 6. Note that $f * f$ is continuous even though f is not. ■

The rest of this section is devoted to studying the **Gaussian function** $f(x) = e^{-x^2}$ and its Fourier transform. This function plays a key role in the solution of the heat equation on the line (Section 7.4). We need the famous improper integral

$$(4) \qquad \boxed{I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}\,.}$$

We have computed this integral in Section 4.7, Exercise 35, in connection with the gamma function. Let us give here a more direct proof. We square the integral, use polar coordinates ($r^2 = x^2 + y^2$, $dx\,dy = r\,dr\,d\theta$), and get

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy \\ &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta = \int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_0^{\infty}\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} d\theta = \pi, \end{aligned}$$

and (4) follows upon taking square roots.

THEOREM 5
TRANSFORM OF THE GAUSSIAN

Let $a > 0$. We have

$$\mathcal{F}\left(e^{-\frac{ax^2}{2}}\right)(\omega) = \frac{1}{\sqrt{a}}\, e^{-\frac{\omega^2}{2a}}.$$

Proof We give an indirect proof based on the operational properties of the Fourier transform. Let $f(x) = e^{-\frac{ax^2}{2}}$. A simple verification shows that f satisfies the first order linear differential equation

$$f'(x) + axf(x) = 0\,.$$

Taking Fourier transforms and using Theorems 1, 2, and 3, we get

$$\omega\widehat{f}(\omega) + a\,\frac{d}{d\omega}[\widehat{f}](\omega) = 0\,.$$

Thus $\widehat{f}$ satisfies a similar first order linear ordinary differential equation. Solving this equation in $\widehat{f}$, we find

$$\widehat{f}(\omega) = A\, e^{-\frac{\omega^2}{2a}},$$

where A is an arbitrary constant. To complete the proof, we must show that $A = \frac{1}{\sqrt{a}}$. We have

$$\begin{aligned} A &= \widehat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}}\,dx \\ &= \frac{1}{\sqrt{a\pi}}\int_{-\infty}^{\infty} e^{-u^2}\,du \qquad \left(u = \sqrt{\frac{a}{2}}x,\ \sqrt{\frac{2}{a}}\,du = dx\right) \\ &= \frac{1}{\sqrt{a}} \qquad \text{(by (4))}. \end{aligned}$$

Replacing a by $2a$ in Theorem 5 yields

$$\boxed{\mathcal{F}\left(e^{-ax^2}\right)(\omega) = \frac{1}{\sqrt{2a}}\, e^{-\frac{\omega^2}{4a}} \qquad (a > 0)\,.} \tag{5}$$

Taking $a = 1$ in Theorem 5 gives

$$\boxed{\mathcal{F}\left(e^{-\frac{x^2}{2}}\right)(\omega) = e^{-\frac{\omega^2}{2}}\,.}$$

This remarkable identity states that $e^{-\frac{x^2}{2}}$ is its own Fourier transform. Is this the only function with this property? See Exercise 59 for an answer.

Theorem 5 can be used to compute some interesting integrals.

60. Give an example of two functions f and g such that neither one vanishes identically on any interval but $f * g$ is identically 0. [Hint: Think on the Fourier transform side. Let $\widehat{f}$ be a tent function supported over the interval $(-2, 2)$, as in Figure 6. Define $\widehat{g}$ by translating $\widehat{f}$ to the left (or right) by 4 units.]

7.3 The Fourier Transform Method

The partial differential equations that we encountered in Chapters 3–5 were in most cases defined on finite regions (for example, vibrating strings of finite length, heat transfer in a finite rod, vibrating rectangular membrane). Their solutions depended in an essential way on the boundary conditions. When modeling problems over regions that extend very far in at least one direction, we can often idealize the situation to that of a problem having infinite extent in one or more directions, where any boundary conditions that would have applied on the far-away boundaries are discarded in favor of simple boundedness conditions on the solution as the appropriate variable is sent to infinity. Such problems are mathematically modeled by differential equations defined on infinite regions. For one-dimensional problems we distinguish two types of infinite regions: infinite intervals extending from $-\infty$ to ∞ and semi-infinite intervals extending from one point (usually the origin) to infinity (usually $+\infty$).

In this section we develop the Fourier transform method and apply it to solve the wave and the heat equations on the real line. Appropriate tools for solving problems on a semi-infinite interval are developed and used in later sections. They include the cosine and sine transforms and the Laplace and Hankel transforms.

Transforms of Partial Derivatives

Throughout this section we will suppose that $u(x, t)$ is a function of two variables x and t, where $-\infty < x < \infty$ and $t > 0$. Because of the presence of two variables, care is needed in identifying the variable with respect to which the Fourier transform is computed. For example, for fixed t, the function $u(x, t)$ becomes a function of the spatial variable x, and as such, we can take its Fourier transform with respect to the x variable. We denote this transform by $\widehat{u}(\omega, t)$. Thus

FOURIER TRANSFORM IN THE x VARIABLE

$$\boxed{(1) \qquad \mathcal{F}\big(u(x,t)\big)(\omega) = \widehat{u}(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-i\omega x}\, dx\,.}$$

To illustrate the use of this notation we compute some very useful transforms. We will assume that the function $u(x, t)$, as a function of x, has sufficient properties that enable us to use freely the operational properties of the Fourier transform from Section 7.2.

FOURIER TRANSFORM AND PARTIAL DERIVATIVES

Given $u(x,t)$ with $-\infty < x < \infty$ and $t > 0$, we have

(2) $$\mathcal{F}\left(\tfrac{\partial}{\partial t}\,u(x,t)\right)(\omega) = \tfrac{d}{dt}\,\widehat{u}(\omega,t);$$

(3) $$\mathcal{F}\left(\tfrac{\partial^n}{\partial t^n}\,u(x,t)\right)(\omega) = \tfrac{d^n}{dt^n}\,\widehat{u}(\omega,t), \quad n = 1, 2, \ldots;$$

(4) $$\mathcal{F}\left(\tfrac{\partial}{\partial x}\,u(x,t)\right)(\omega) = i\omega\widehat{u}(\omega,t);$$

(5) $$\mathcal{F}\left(\tfrac{\partial^n}{\partial x^n}\,u(x,t)\right)(\omega) = (i\omega)^n\widehat{u}(\omega,t), \quad n = 1, 2, \ldots.$$

Note that on the right sides of (2) and (3) we have used ordinary derivatives in t instead of partial derivatives. The reason is to emphasize the crucial fact that, in applying the Fourier transform method, we will transform a partial differential equation in $u(x,t)$ into an ordinary differential equation in $\widehat{u}(\omega,t)$, where t is the variable. This will become more apparent in the examples.

The last two identities are consequences of (1) and Theorem 2 of Section [illegible]. To prove (2) we start with the right side and differentiate under the integral sign with respect to t:

$$\frac{d}{dt}\,\widehat{u}(\omega,t) = \frac{1}{\sqrt{2\pi}}\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\,u(x,t)e^{-i\omega x}\,dx.$$

The last expression is the Fourier transform of $\frac{\partial}{\partial t}\,u(x,t)$ as a function of x, and (2) follows. Repeated differentiation under the integral sign with respect to t yields (3).

The Fourier Transform Method

The use of the Fourier transform to solve partial differential equations is best described by examples. We start with the wave equation.

EXAMPLE 1 The wave equation for an infinite string
Solve the boundary value problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2} && (-\infty < x < \infty, t > 0),\\ u(x,0) &= f(x) && \text{(initial displacement)},\\ \frac{\partial}{\partial t}u(x,0) &= g(x) && \text{(initial velocity)}. \end{aligned}$$

Assume that both f and g have Fourier transforms, and give the answer as an inverse Fourier transform.

Solution We fix t and take the Fourier transform of both sides of the differential equation and the initial conditions. Using (3) and (5) with $n = 2$, we get

(6) $$\frac{d^2}{dt^2}\widehat{u}(\omega,t) = -c^2\omega^2\widehat{u}(\omega,t),$$

(7) $$\widehat{u}(\omega,0) = \widehat{f}(\omega),$$

(8) $$\frac{d}{dt}\widehat{u}(\omega,0) = \widehat{g}(\omega).$$

It is clear that (6) is an ordinary differential equation in $\widehat{u}(\omega, t)$, where t is the variable. Let us write (6) in the standard form

$$\frac{d^2}{dt^2}\,\widehat{u}(\omega, t) + c^2\omega^2\widehat{u}(\omega, t) = 0\,.$$

The general solution of this equation is

$$\widehat{u}(\omega, t) = A(\omega)\cos c\omega t + B(\omega)\sin c\omega t,$$

See Appendix A.2 for the solution of second order linear differential equations (Case III).

where $A(\omega)$ and $B(\omega)$ are constant in t. (You should note that while A and B are constant in t, they can depend on ω, which explains writing $A(\omega)$ and $B(\omega)$.) We determine $A(\omega)$ and $B(\omega)$ from the initial conditions (7) and (8) as follows:

$$\begin{aligned} \widehat{u}(\omega, 0) = A(\omega) &= \widehat{f}(\omega), \\ \frac{d}{dt}\widehat{u}(\omega, 0) &= c\omega B(\omega) = \widehat{g}(\omega)\,. \end{aligned}$$

So

$$\widehat{u}(\omega, t) = \widehat{f}(\omega)\cos c\omega t + \frac{1}{c\omega}\,\widehat{g}(\omega)\sin c\omega t\,.$$

To obtain the solution we use the inverse Fourier transform ((2), Section 7.2) and get

$$u(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\widehat{f}(\omega)\cos c\omega t + \frac{1}{c\omega}\,\widehat{g}(\omega)\sin c\omega t\right]e^{i\omega x}d\omega\,. \tag{9}$$

■

We summarize the Fourier transform method as follows.

Step 1: Fourier transform the given boundary value problem in $u(x, t)$ and get an ordinary differential equation in $\widehat{u}(\omega, t)$ in the variable t.
Step 2: Solve the ordinary differential equation and find $\widehat{u}(\omega, t)$.
Step 3: Inverse Fourier transform $\widehat{u}(\omega, t)$ to get $u(x, t)$.

This simple method is successful in treating a variety of partial differential equations, but it has its limitations, since we have to assume that the functions in the problem and its solutions have Fourier transforms. Nevertheless, the method offers us opportunities beyond these limitations, as we now illustrate. Consider formula (9), which gives you the solution of the wave boundary value problem in the form of an inverse Fourier transform. To simplify the presentation, take $g = 0$. Then $\widehat{g} = 0$ and (9) becomes

$$u(x,\, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\widehat{f}(\omega)\cos c\omega t\right]e^{i\omega x}d\omega.$$

Thus, u is the inverse Fourier transform of $\widehat{f}(\omega)\cos c\omega t$. Appealing to Exercise 20(b), Section 7.2, we find that $u(x,\, t) = \big(f(x - ct) + f(x + ct)\big)/2$, which we recognize as d'Alembert's form of the solution of the wave equation from Section 3.4. (For the general case, see Exercise 21). In this form, we can verify the solution directly by using the equations in the boundary value problem. Thus, even though the Fourier transform method requires the existence of the Fourier transform of f, the solution that we derived applies in much more general situations.

In our next example we use the Fourier transform method to solve the heat equation on the real line. The example models the transfer of heat in a very long rod extending in both directions on the x-axis.

EXAMPLE 2 The heat equation for an infinite rod
Solve the boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= c^2 \frac{\partial^2 u}{\partial x^2} && (-\infty < x < \infty,\ t > 0), \\ u(x,0) &= f(x) && \text{(initial temperature distribution).} \end{aligned}$$

Assume that f has a Fourier transform, and give your answer in the form of an inverse Fourier transform.

Solution Fourier transforming the boundary value problem, we get

$$\begin{aligned} \frac{d}{dt}\widehat{u}(\omega,t) &= -c^2\omega^2\widehat{u}(\omega,t), \\ \widehat{u}(\omega,0) &= \widehat{f}(\omega). \end{aligned}$$

The general solution of the first order differential equation in t is

$$\widehat{u}(\omega,t) = A(\omega)e^{-c^2\omega^2 t},$$

Once again, since we are dealing with a family of differential equations (one for each ω), we must use a different constant for each equation, which explains the use of the notation $A(\omega)$.

where $A(\omega)$ is a constant that depends on ω. Setting $t = 0$ and using the transformed initial condition, we get

$$\widehat{u}(\omega,0) = A(\omega) = \widehat{f}(\omega).$$

Hence

$$\widehat{u}(\omega,t) = \widehat{f}(\omega)e^{-c^2\omega^2 t}.$$

Taking inverse Fourier transforms, we get the solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{-c^2\omega^2 t}e^{i\omega x}\,d\omega. \tag{10}$$

Formula (10) gives you the solution of the heat boundary value problem in the form of an integral involving the Fourier transform of the initial heat distribution. In the next section, we will compute this integral and give the answer as a convolution of f and a fixed function known as the heat or Gauss's kernel. As in the case of the wave equation, you will see that the convolution form of the solution may be applied even if f does not have a Fourier transform.

Our next two examples illustrate the use of the Fourier transform method in solving problems with mixed and higher-order derivatives.

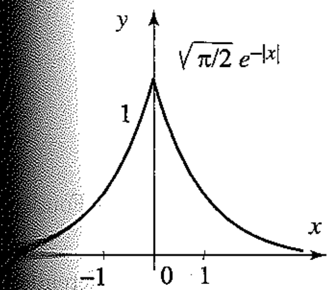

Figure 1 Graph of $\sqrt{\frac{\pi}{2}}e^{-|x|}$.

EXAMPLE 3 The Fourier transform method with mixed derivatives
Solve the boundary value problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t \partial x} &= \frac{\partial^2 u}{\partial x^2} \quad (-\infty < x < \infty,\ t > 0), \\ u(x,0) &= \sqrt{\frac{\pi}{2}} e^{-|x|}. \end{aligned}$$

Solution The function in the initial condition is shown in Figure 1. Combining (2) and (4), we obtain

$$\mathcal{F}\left(\frac{\partial^2 u}{\partial t \partial x}\right) = \mathcal{F}\left(\frac{\partial}{\partial t}\frac{\partial u}{\partial x}\right) = \frac{d}{dt}\mathcal{F}\left(\frac{\partial u}{\partial x}\right) = i\omega \frac{d\widehat{u}}{dt}(\omega, t).$$

Fourier transforming the differential equation and using (5), we get

$$i\omega\, \frac{d\widehat{u}}{dt}(\omega,t) = -\omega^2 \widehat{u}(\omega,t) \quad \text{or} \quad \frac{d\widehat{u}}{dt}(\omega,t) = i\omega \widehat{u}(\omega,t).$$

Solving this first order ordinary differential equation, we find

$$\widehat{u}(\omega,t) = A(\omega)e^{i\omega t}.$$

The initial condition implies that

$$\widehat{u}(\omega,0) = \mathcal{F}\left(\sqrt{\frac{\pi}{2}} e^{-|x|}\right) = \frac{1}{1+\omega^2},$$

and so $A(\omega) = \frac{1}{1+\omega^2}$. Hence

$$\widehat{u}(\omega,t) = \frac{e^{i\omega t}}{1+\omega^2},$$

and the solution u is obtained by taking inverse Fourier transforms. In this case, we can determine u explicitly by using the shifting property of the Fourier transform (Exercise 19, Section 7.2). We first note that the inverse Fourier transform of $\frac{1}{1+\omega^2}$ is $\sqrt{\frac{\pi}{2}}e^{-|x|}$ (use the table of Fourier transforms). Hence by the shifting property,

$$u(x,t) = \mathcal{F}^{-1}\left(\frac{e^{i\omega t}}{1+\omega^2}\right) = \sqrt{\frac{\pi}{2}} e^{-|x+t|}.$$

The solution is illustrated in Figure 2. ■

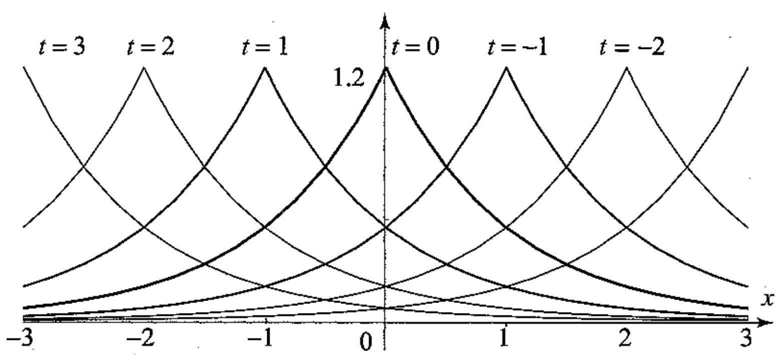

Figure 2 Graphs of $u(x,t)$ in Example 3 at various values of

Our next example deals with a fourth order differential equation that arises in modeling the transverse vibrations of an elastic beam. (See Section 6.5.)

EXAMPLE 4 Transverse vibrations of an elastic beam
Solve the boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4} \qquad (-\infty < x < \infty,\ t > 0),$$
$$u(x,\,0) = f(x), \quad u_t(x,\,0) = g(x)\,,$$

where $c > 0$ is a constant. Give your answer in the form of an inverse Fourier transform.

Solution Fix t and take the Fourier transform of both sides of the differential equation and the initial conditions. Using (3) with $n = 2$, we obtain $\mathcal{F}(u_{tt}) = \frac{d^2}{d\omega^2}\widehat{u}(\omega,\,t)$. Using (5) with $n = 4$ and remembering that $i^4 = 1$, we find $\mathcal{F}(u_{xxxx}) = \omega^4\widehat{u}(\omega,\,t)$. Thus,

$$\tfrac{d^2\widehat{u}}{dt^2}(\omega,t) + c^2\omega^4\widehat{u}(\omega,t) = 0 \quad (-\infty < \omega < \infty),$$
$$\widehat{u}(\omega,\,0) = \widehat{f}(\omega), \quad \tfrac{d}{d\omega}\widehat{u}(\omega,\,0) = \widehat{g}(\omega)\,.$$

The second order differential equation in t is similar to the one in Example 1. Its general solution is

See Appendix A.2 for the solution of second order linear differential equations (Case III).

$$\widehat{u}(\omega,t) = A(\omega)\cos(c\omega^2 t) + B(\omega)\sin(c\omega^2 t),$$

where $A(\omega)$ and $B(\omega)$ are constant in t. We determine $A(\omega)$ and $B(\omega)$ from the transformed initial conditions as follows:

$$\begin{aligned} \widehat{u}(\omega,\,0) = A(\omega) &= \widehat{f}(\omega), \\ \frac{d}{dt}\widehat{u}(\omega,\,0) &= c\omega^2 B(\omega) = \widehat{g}(\omega)\,. \end{aligned}$$

So

$$\widehat{u}(\omega,\,t) = \widehat{f}(\omega)\cos(c\omega^2 t) + \frac{1}{c\omega^2}\,\widehat{g}(\omega)\sin(c\omega^2 t)\,.$$

Applying the inverse Fourier transform ((2), Section 7.2), we get

$$(11) \qquad u(x,\,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big[\widehat{f}(\omega)\cos(c\omega^2 t) + \frac{1}{c\omega^2}\,\widehat{g}(\omega)\sin(c\omega^2 t)\Big]e^{i\omega x}\,d\omega\,.$$

Unlike Example 1, inverting the Fourier transform here is a much more challenging task (see Exercise 26).

So far we have used the Fourier transform method to solve partial differential equations with constant coefficients. As our next example illustrates, the method is also useful in solving problems with nonconstant coefficients.

EXAMPLE 5 An equation with nonconstant coefficients

Solve

$$\begin{aligned} t\,\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} &= 0 \qquad (-\infty < x < \infty,\ t > 0), \\ u(x,0) &= f(x). \end{aligned}$$

Simplify your answer as much as possible. What are appropriate assumptions on f?

Solution The new feature in this example is the presence of the term $t\,\frac{\partial u}{\partial x}$ in the equation. Since we will use the Fourier transform with respect to x, the variable t will be treated as a constant. Thus

$$\mathcal{F}\left(t\,\frac{\partial u}{\partial x}\right) = t\mathcal{F}\left(\frac{\partial u}{\partial x}\right) = i\omega t\widehat{u}(\omega, t).$$

Going back to our problem, we use the Fourier transform and get

$$\begin{gathered} i\omega t\,\widehat{u}(\omega, t) + \tfrac{d}{dt}\widehat{u}(\omega, t) = 0, \\ \widehat{u}(\omega, 0) = \widehat{f}(\omega). \end{gathered}$$

At this point we have to assume that f has a Fourier transform. Solving the first order differential equation in t yields

$$\widehat{u}(\omega, t) = A(\omega)e^{-i\frac{t^2}{2}\omega},$$

where the arbitrary constant, $A(\omega)$, is allowed to be a function of ω. Putting $t = 0$ implies

$$\widehat{u}(\omega, t) = \widehat{f}(\omega)e^{-i\frac{t^2}{2}\omega}.$$

To determine u we use a property of the Fourier transform as we did in Example 3. This property says that translating a function by a constant a corresponds to multiplying its Fourier transform by $e^{ia\omega}$ (see Exercise 19, Section 7.2). Thus

$$u(x, t) = f\left(x - \frac{t^2}{2}\right).$$

Let us check our answer. Assuming that f is differentiable, we have $u_x(x,t) = f'(x - \frac{t^2}{2})$; and, by the chain rule, $u_t(x,t) = -tf'(x - \frac{t^2}{2})$. Thus $tu_x + u_t = 0$, verifying the equation. At $t = 0$, we get $u(x, 0) = f(x)$, as desired. The solution that we derived requires only that f be differentiable. ■

Exercises 7.3

In Exercises 1–6, determine the solution of the given wave or heat problem. Write your answer in the form of an inverse Fourier transform. Take the variables in the ranges $-\infty < x < \infty$, $t > 0$.

1. $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, \quad u(x,0) = \dfrac{1}{1+x^2}, \quad \dfrac{\partial u}{\partial t}(x,0) = 0.$

2. $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, \quad u(x,0) = \begin{cases} \cos x & \text{if } -\frac{\pi}{2} \le x \le \frac{\pi}{2}, \\ 0 & \text{otherwise,} \end{cases} \quad \dfrac{\partial u}{\partial t}(x,0) = 0.$

3. $\dfrac{\partial u}{\partial t} = \dfrac{1}{4}\dfrac{\partial^2 u}{\partial x^2}, \quad u(x,0) = e^{-x^2}$. [Hint: Theorem 5, Section 7.2.]

4. $\dfrac{\partial u}{\partial t} = \dfrac{1}{100}\dfrac{\partial^2 u}{\partial x^2}, \quad u(x,0) = \begin{cases} 100 & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$

5. $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}, \quad u(x,0) = \sqrt{\dfrac{2}{\pi}}\dfrac{\sin x}{x}, \quad \dfrac{\partial u}{\partial t}(x,0) = 0.$

6. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \quad u(x,0) = \begin{cases} 1 - \frac{|x|}{2} & \text{if } -2 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$

In Exercises 7–20, solve the given problem. Assume that the functions in each problem have Fourier transforms. Take $-\infty < x < \infty$, *and* $t > 0$.

7. $\dfrac{\partial u}{\partial x} + 3\dfrac{\partial u}{\partial t} = 0, \quad u(x,0) = f(x).$

8. $a\dfrac{\partial u}{\partial x} + b\dfrac{\partial u}{\partial t} = 0, \quad u(x,0) = f(x).$

9. $t^2\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial t} = 0, \quad u(x,0) = f(x).$

10. $a(t)\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial t} = 0, \quad u(x,0) = f(x).$

11. $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial t}, \quad u(x,0) = f(x).$

12. $\dfrac{\partial u}{\partial t} + \sin t\dfrac{\partial u}{\partial x} = 0, \quad u(x,0) = \sin x.$

13. $\dfrac{\partial u}{\partial t} = t\dfrac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x).$

14. $\dfrac{\partial u}{\partial t} = a(t)\dfrac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x)$, where $a(t) > 0$.

15. $\dfrac{\partial^2 u}{\partial t^2} + 2\dfrac{\partial u}{\partial t} = -u, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x).$

16. $\dfrac{\partial u}{\partial t} = e^{-t}\dfrac{\partial^2 u}{\partial x^2}, \quad u(x,0) = e^{-|x|}.$

17. $\dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^4 u}{\partial x^4} = 0, \quad u(x,0) = \begin{cases} .1 & \text{if } |x| < 1, \\ 0 & \text{otherwise,} \end{cases} \quad u_t(x,0) = 0.$

18. $\dfrac{\partial u}{\partial t} = t\,\dfrac{\partial^4 u}{\partial x^4}, \quad u(x,0) = f(x)\,.$

19. $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^3 u}{\partial t \partial x^2}, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x)\,.$

20. $\dfrac{\partial^2 u}{\partial t^2} - 4\,\dfrac{\partial^3 u}{\partial t \partial x^2} + 3\dfrac{\partial^4 u}{\partial x^4} = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x)\,.$

21. D'Alembert's solution of the wave equation.
(a) Verify that

$$u(x,\, t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$$

is a solution of the boundary value problem of Example 1.
(b) Derive d'Alembert's solution from (9) and Exercises 19 and 20 in Section 7.2.

22. (a) Use D'Alembert's solution to describe the vibration of a very long string with $c = 1$, $f(x) = \cos x$ for $|x| \leq \frac{\pi}{2}$ and 0 otherwise, and $g(x) = 0$.
(b) Draw the shape of the string at $t = 0, \pi/4, \pi/2, \pi$.

Project Problem: Do Exercises 23 and 24 to solve a **heat problem with convection.**

23. Assume that $f(x)$ has a Fourier transform. Solve the boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= c^2\,\frac{\partial^2 u}{\partial x^2} + k\,\frac{\partial u}{\partial x}, \qquad -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x)\,. \end{aligned}$$

This problem models heat transfer in a long heated bar that is exchanging heat with the surrounding medium. This phenomenon is called **convection**, and k is a positive constant called the **coefficient of convection**. (See Exercise 10, Section 7.4, for the convolution form of the solution.)

24. Specialize Exercise 23 to the case $c = 1,\ k = .5,\ f(x) = e^{-x^2}$.

25. Solve the boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= c^2\,\frac{\partial^3 u}{\partial x^3}, \qquad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= f(x). \end{aligned}$$

The equation is known as the **linearized Korteweg–de Vries equation.**

26. Project Problem: Transverse vibrations of a beam. In this problem, we outline a method for computing the inverse Fourier transform in Example 4 when $g = 0$. You will need the Fourier transform of e^{ix^2}, which is given by

$$\begin{aligned} (12) \qquad & \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \big(\cos x^2 + i \sin x^2\big) e^{-i\omega x}\, dx \\ & = \frac{1}{2}\Big(\cos\frac{\omega^2}{4} + \sin\frac{\omega^2}{4}\Big) + \frac{i}{2}\Big(\cos\frac{\omega^2}{4} - \sin\frac{\omega^2}{4}\Big). \end{aligned}$$