Indian Institute of Information Technology Allahabad
Convex Optimization (SMAT430C)
Tentaive Marking Scheme (Mid-semester Exam, 2018)

---

1. Prove or disprove the following statements.

   (a) The conjugate function f^* of a function f is convex, whether f is convex or not. [2]
   Solution. True. $f^*(y) = \sup\limits_{x \in \mathbf{dom} f} (y^T x - f(x))$ [1]
   $\because$ f^* is the pointwise supremum of family of convex functions (affine functions) of y [1]
   $\Rightarrow$ f^* is also a convex function.

   (b) If all the sublevel sets of a function f is convex, then f is convex. [2]
   Solution. Not true in general.
   Consider a non-convex function f given by $f(x) = \log x$, with **dom** $f = \mathbb{R}_{++}$. [1]
   Then $S_\alpha(f) = \{x : f(x) \leq \alpha\} = \{x : \log x \leq \alpha\} = \{x : x \leq e^\alpha\}$
   $\because$ S_α is an interval for each $\alpha \in \mathbb{R}$, S_α is a convex set, $\forall\ \alpha \in \mathbb{R}$ [1]

   (c) Let $h(x) = x^{3/2}$ with **dom** $h = \mathbb{R}_+$. Then $\tilde{h}$ is not increasing (nondecreasing). [2]
   Solution. True.
   $\because$ $h(x) = x^{3/2}$ is a convex function, $\tilde{h} = \begin{cases} h(x) & x \in \mathbf{dom}\, h \\ \infty & \text{otherwise} \end{cases}$ [1]
   $\because$ $\tilde{h}(-1) = \infty$ and $\tilde{h}(1) = 1$, $\tilde{h}$ is not increasing (nondecreasing). [1]

   (d) If f and g are convex functions, then their composition $f \circ g$ is also convex. [3]
   Solution. Not true in general.
   Consider convex functions, $g(x) = x^2$, with **dom** $g = \mathbb{R}$, and $f(x) = 0$, with **dom** $f = [1, 2]$. [1+1]
   Then $(f \circ g)(x) = 0$, with **dom** $f \circ g = [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$, is not convex, since its domain is not convex. [1]

2. Find the supremum and infimum of the set$\{x + \frac{1}{x} : x > 0\}$ [2]
   Solution. The set is not bounded above. inf = 2. [1+1]

3. Find the conjugate function of $f(x) = \begin{cases} x \log x & x > 0 \\ 0 & x = 0 \end{cases}$ [4]
   Solution. $f(y) = xy - x \log x$ is bounded above on $\mathbb{R}_+$ for all y, hence **dom** $f^* = \mathbb{R}$ [1+1]
   f^* attains its maximum at $x = e^{y-1}$ and $f^*(y) = e^{y-1}$ [1+1]

4. Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex and bounded above. Show that f is constant (Hint: consider $g(t) = f((1-t)x+ty)$. [7]
   Solution. Suppose f is not a constant, then there exist x, y with $f(x) < f(y)$. [1]
   Define $g(t) = f(x + t(y - x))$.
   As f is convex and bounded above, g is also convex and bounded above. [1]
   Moreover, $g(0) < g(1)$ [1]
   By convexity of g (for $x_1 = 0$, $x_2 = t$, $\theta_1 = 1 - \frac{1}{t}$ and $\theta_2 = \frac{1}{t}$) [1]
   we have $g(1) \leq (1 - \frac{1}{t})g(0) + \frac{1}{t}g(t)$ [1]
   $\Rightarrow$ $g(t) \geq g(0) + t(g(1) - g(0))$ [1]
   $\because$ $g(1) - g(0) > 0$, $g(t) \to \infty$ as $t \to \infty$, a contradiction as g is bounded above. [1]

5. Show that a function f is convex if and only if its epigraph is a convex set.

Solution. ($\Longrightarrow$) epi$f = \{(x, t) : f(x) \leq t\}$. [1]

Let $(x, t),\ (y, s) \in$ epi$f$. Hence, $f(x) \leq t,\ f(y) \leq s$. [1]

As f is convex, for $0 \leq \lambda \leq 1$, $f(\lambda x + (1 - \lambda)y)) \leq \lambda f(x) + (1 - \lambda))f(y) \leq \lambda t + (1 - \lambda))s$. [1]

This implies that $(\lambda x + (1 - \lambda)y), \lambda t + (1 - \lambda)s) = \lambda(x, t) + (1 - \lambda)(y, s) \in$ epi$f$. [1]

Hence, epigraph of f is a convex set.

($\Longleftarrow$) $(x, f(x)),\ (y, f(y)) \in$ epi$f$. [1]

As epi$f$ is convex, for $0 \leq \lambda \leq 1$,

$\lambda(x, f(x)) + (1 - \lambda)(y, f(y)) = (\lambda x + (1 - \lambda)y, \lambda f(x) + (1 - \lambda)f(y)) \in$ epi$f$. [1]

Therefore, $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y))$. [1]

Hence, f is a convex function. [1]