Sampling Theorem

We will show that a band limited signal

$$G(f) = 0 \quad for \ \ |f| > B$$

can be reconstructed exactly from its discrete time samples.

Recall: That a time sampled signal is like taking a snap shot or picture of signal periodically.

Consider the signal $g(t)$ as follows

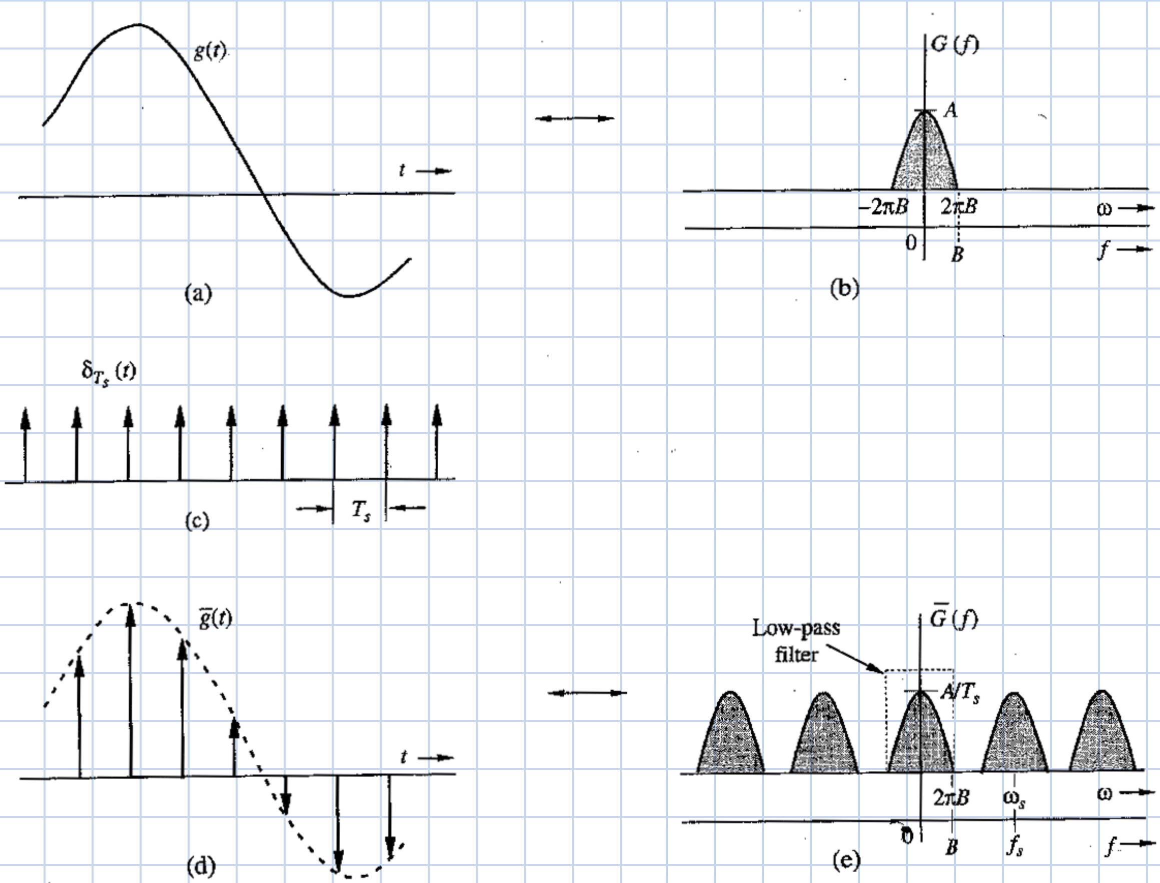


Figure 1: Sampled Signal

Multiplying $g(t)$ by a pulse train $\delta_{T_s}(t)$, which is periodic with period T_s, we get the signal $\bar{g}(t)$ which is a sampled version of $g(t)$. We find that

$$\bar{g}(t) = g(t)\delta_{T_s}(t) = \sum_n g(nT_s)\delta(t - nT_s)$$

Because $\delta_{T_s}(t)$ is periodic we can represent it as a Fourier series

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} D_n \, e^{jn\omega_s t}$$

Where

$$D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta_{T_s}(t)\, e^{-jn\omega_s t} dt \qquad T_0 = T_s \quad \omega_s = 2\pi\frac{1}{T_s}$$

From this we find that $D_n = \frac{1}{T_s}$ and thus that

$$\delta_{T_s}(t) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

so

$$\bar{g}(t) = g(t)\delta_{T_s}(t) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} g(t)e^{jn\omega_s t}$$

Finally taking the Fourier transform $\bar{g}(t)$ yields

$$\mathcal{F}\{\bar{g}(t)\} = \int_{-\infty}^{\infty} \bar{g}(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{T_s}\sum_{n=-\infty}^{\infty} g(t)e^{jn\omega_s t}\, e^{-j\omega t}dt$$

The previous equation can be rewritten as

$$\mathcal{F}\{\bar{g}(t)\} = \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} g(t)e^{jn\omega_s t}e^{-j\omega t}dt$$

Using the phase shift property of the Fourier transform we find that

$$\mathcal{F}\{\bar{g}(t)\} = \bar{G}(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} G(f - nf_s)$$

Where $f_s = \frac{1}{T_s}$

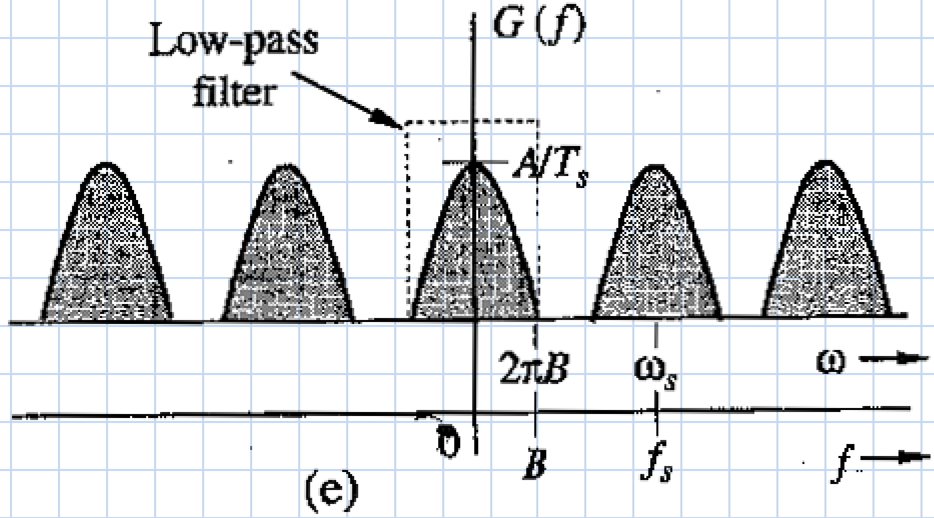


Figure 2: Spectrum of a Sampled Signal $g(t)$

The interesting result of this exercise is that we find the sampled signal $\bar{g}(t)$ has a spectrum that contains copies of the transform $G(f)$ centered at every $n\omega_s$. To recover this signal $g(t)$ in entirety we simply have to place a low-pass filter of bandwidth B centered at $0Hz$. Notice that we could also place a band-pass filter of bandwidth B around any $n\omega_s$ to get a modulated version of the signal where the carrier would be $n\omega_s$.

What we also need to take note of is that if f_s is less than $2B$ the signal spectrum will have the copies overlapping or aliasing each other. In order to have no overlap we must have $f_s > 2B$ or $T_s < \frac{1}{2B}$ which is referred to as the Nyquist rate.

For a look at an aliased spectrum take a look at Figure 3.

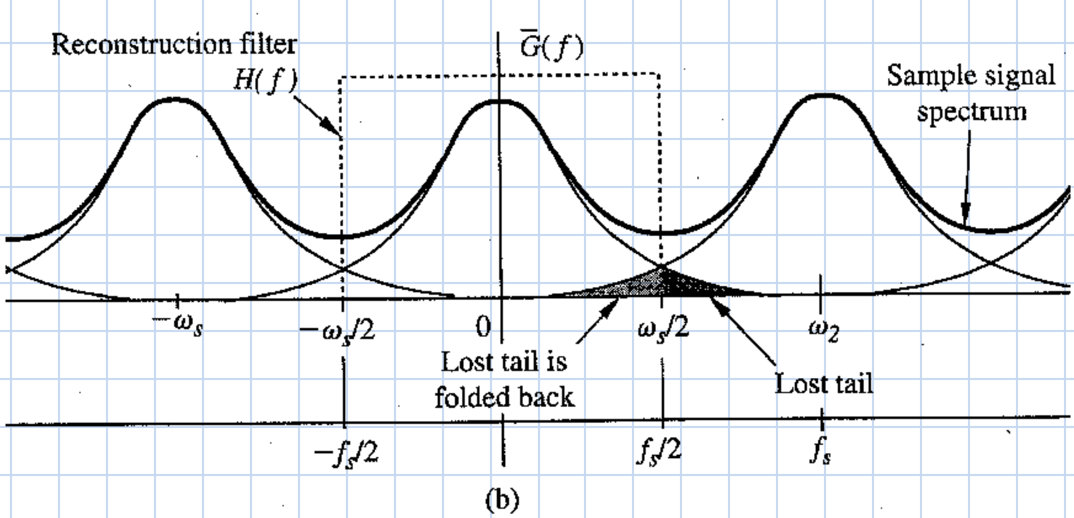
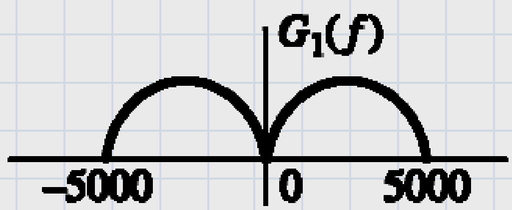


Figure 3: Aliasing Due to Under Sampling $f_s < 2B$

Example 1: Sampling Theorem

Question: Given the following signal find the NYquist interval and the sampling rate.



Answer: The bandwidth of this signal is 5 kHz and therefore the sampling rate 10 kHz.

Signal Reconstruction

We will now show that a signal $g(t)$ can in fact be reconstructed in its entirety simply by a well-placed low-pass filter.

For a band limited signal with bandwidth B we find the transfer function of the filter for reconstruction to be

$$H(f) = T_s \Pi\left(\frac{f}{2B}\right)$$

Note: the addition of the scaling factor T_s.

In the time domain we find the transfer function to be

$$h(t) = 2BT_s sinc(2\pi Bt)$$

For arguments sake we will say that $2BT_s$ is equal to 1 so that

$$h(t) = sinc(2\pi Bt)$$

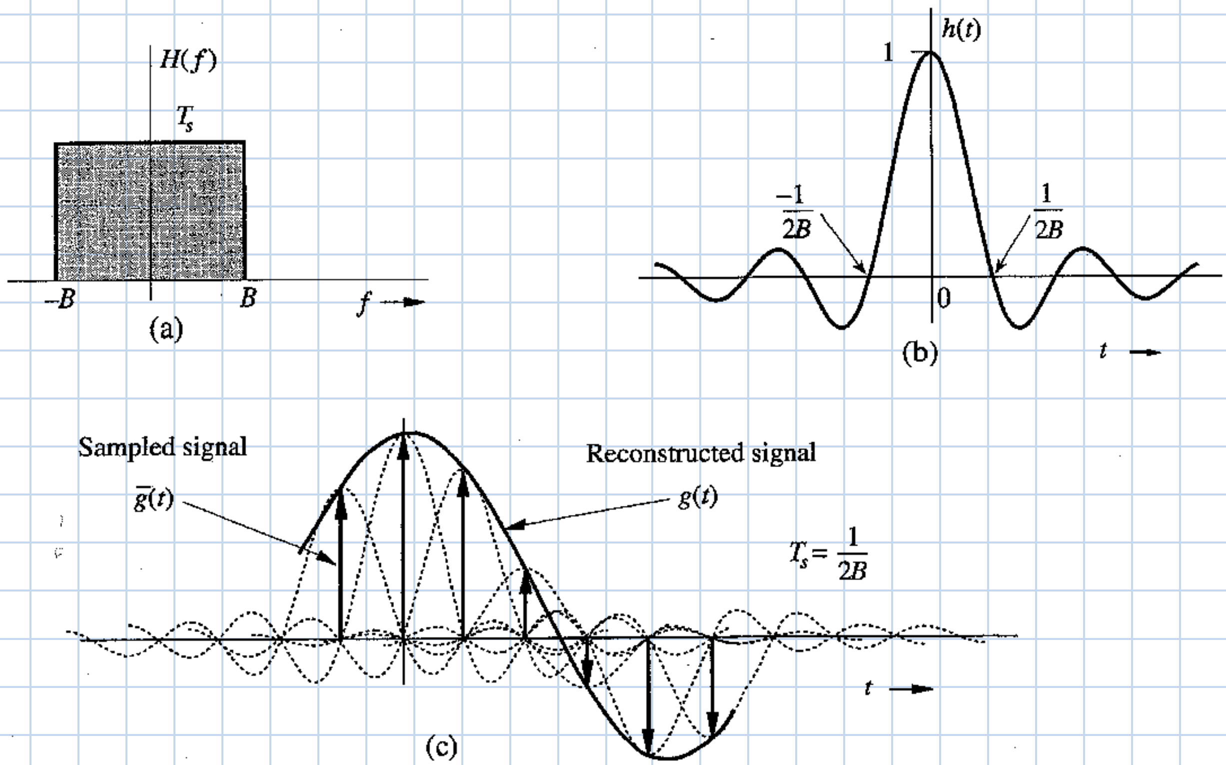


Figure 4: Reconstructing Sampled Signal

In Figure 4 we can see that at all Nyquist sampling instants $t = \left(\pm {}^{n}/_{2B}\right)$, $h(t) = 0$ except at $t = 0$. When $\bar{g}(t)$ is inputted to the transfer function of our filter each sample, being an impulse, generates a sinc function of height equal to the strength of the sample at that instant in time seen in Figure 4.c. So

$$g(t) = \bar{g}(t) * h(t) = \int_{-\infty}^{\infty} \bar{g}(\tau)h(t-\tau)\tau t$$

$$= \int_{-\infty}^{\infty} \sum_{n} g(nT_s)\delta(\tau - nT_s)\, h(t-\tau)d\tau$$

$$= \sum_{n} g(nT_s) \int_{-\infty}^{\infty} \delta(\tau - nT_s)\, h(t-\tau)d\tau$$

$$= \sum_{n} g(nT_s)\, h(t - nT_s) = \sum_{n} g(nT_s)\, sinc\big(2\pi B(t - nT_s)\big)$$

Recall: The integral of a delta function is $\int_{-\infty}^{\infty} \delta(t-a)x(t)dt = x(a)$

We also find that

$$g(t) = \sum_n g(nT_s)\, sinc\big(2\pi B(t-nT_s)\big)$$
$$= \sum_n g(nT_s)\, sinc\big(2\pi B(t-n\pi)\big)$$

What we showed above is that the reconstruction of the signal $g(t)$ from $\bar{g}(t)$ is an **interpolation formula** which yields $g(t)$ as a weight sum of all the samples.

Practical Signal Reconstruction

Let's face it there is no such thing as an ideal sampling function i.e. a delta function. In the real world we are limited to more practical signals for instance $p(t)$ shown in Figure 5.

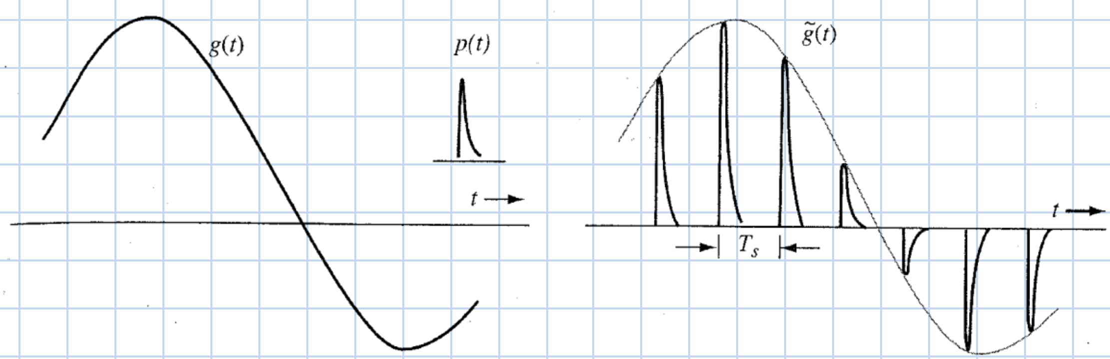


Figure 5: Practical Sampling

For practicality $p(t)$ should be easy to generate. For $p(t)$ let analyze the accuracy of the reconstructed signal. Let's say that

$$\tilde{g}(t) = \sum_n g(nT_s)p(t-nT_s)$$

We can rewrite $\tilde{g}(t)$ as follows

$$\tilde{g}(t) = \sum_n g(nT_s)p(t) * \delta(t-nT_s)$$

After some manipulation we find that

$$\tilde{g}(t) = p(t) * \sum_n g(nT_s)\delta(t - nT_s)$$

The Fourier transform yields

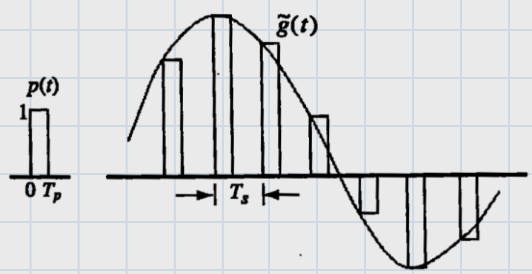
$$\tilde{G}(f) = P(f)\bar{G}(f) = P(f)\frac{1}{T_s}\sum_{n=-\infty}^{\infty} G\left(f - nf_s\right)$$

This is an interesting result because what it says is that rather than having to walk through a barrage of mathematically equations to determine how to reconstruct $g(t)$, we can simply find a transfer function that will essentially undue the results of sampling a signal with a non-ideal sampling function in this case $p(t)$. We will refer to this transfer function as an equalizer. The end result of apply an equalizer is that afterwards the signal simply becomes $\bar{g}(t)$.

Recall: We showed earlier that $g(t)$ can be extracted by a simple, well placed, low-pass filter.

Example 2: Non-Ideal Sampling

Let's consider a simple interpolating pulse generator that generates short (zero-order hold) pulses.



We have

$$p(t) = \Pi\left(\frac{t - 0.5T_p}{T_p}\right)$$

For reconstruction we first have

$$\tilde{g}(t) = \sum_n g(nT_s)\Pi\left(\frac{t - 0.5T_p}{T_p}\right)$$

The transfer function of $p(t)$ is

$$P(f) = T_p sinc(\pi f T_p)e^{-j\pi f T_p}$$

As a result the equalizer must satisfy

$$E(f) = \begin{cases} T_s/P(f) & |f| \le B \\ Flexible & B < |f| < (1/T_s - B) \\ 0 & |f| \ge (1/T_s - B) \end{cases}$$

Quantization

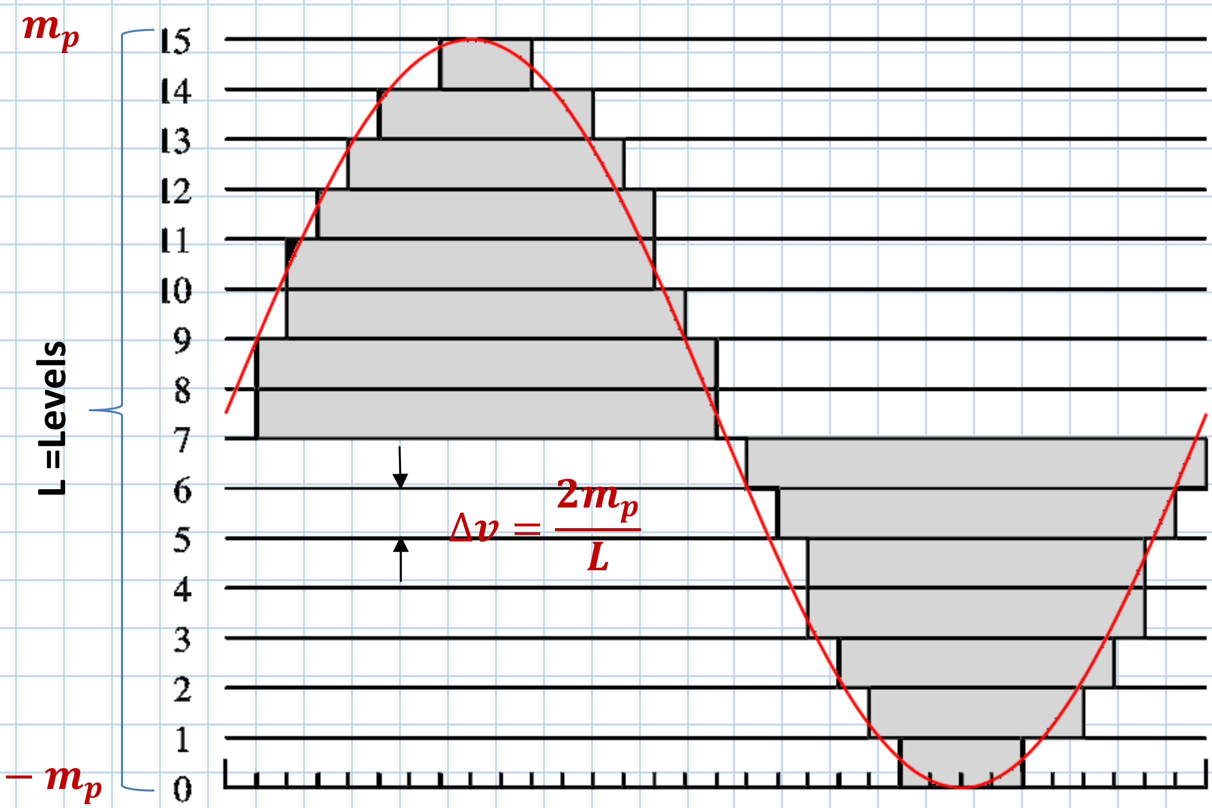


Figure 6: Quantization

What it means to quantize a signal is to limit the possible values that a signal can take on. For instants, a discrete signal can have any range of amplitudes at a discrete time interval. Well in order to represent such a signal digitally, let's say, we would have to have an infinite amount of bits to do this. Since no computer exist that can handle an infinite bit data sample we quantize the data to fit within a range that we are comfortable with and fits our requirements for the system.

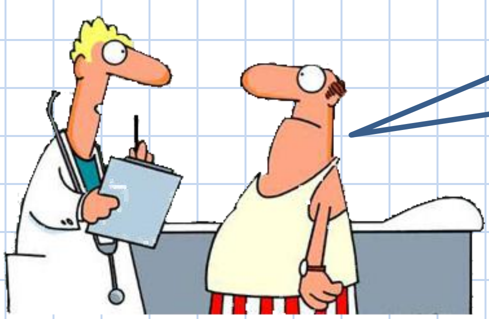
Example 3: Quantization

We have a radio receiver that will never receive a signal greater than $1V_p$ (p = peak refer to Figure 6). Well at this point we can say to ourselves we do not need to have a signal greater that $1V_p$ so all signals will only take on values between $\pm 1V$. Our accuracy or resolution of the signal depends on L. For larger values of L we will have move voltage resolution. Let's say we later find out that our voltage accuracy is $0.25V$ we can use an L value of 8 where

$$\mathbf{L} = \frac{\mathbf{2m_p}}{\boldsymbol{\Delta}\mathbf{v}} = \frac{\mathbf{2(1}\boldsymbol{V_p}\mathbf{)}}{\mathbf{0.25}} = \mathbf{8}$$

If we decided to represent this signal digitally we could say we need $2^3 = 8$ or 3 bits and now any incoming signal will take on values between -1 and 1 in $0.25V$ steps. Another way to say it is that we can only detect a $0.25V$ change in the incoming signals voltage. For an 8 bit binary number we have

Binary	000	001	010	011	100	101	110	111
Decimal	0	1	2	3	4	5	6	7
x0.25V	0	0.25	0.50	0.75	1	1.25	1.50	1.75
-1V	-1.00	-0.75	-0.50	-0.25	0.00	0.25	0.50	0.75



It turns out that we will not change the signal per se, rather we will increase the noise level of the signal.

Previously we demonstrated the interpolation formula

$$g(t) = \sum_k g(kT_s) sinc(2\pi Bt - k\pi)$$

If we consider the $m(kT_s)$ as to be the k^{th} sample of the message signal $m(t)$ the interpolation is as follows

Sampling Theorem Notes

$$m(t) = \sum_k m(kT_s) sinc(2\pi Bt - k\pi)$$

And for the quantized signal let $\widehat{m}(kT_s)$ be the k^{th} sample of the signal $\widehat{m}(t)$ then

$$q(t) = \sum_k \widehat{m}(kT_s) sinc(2\pi Bt - k\pi)$$

The distortion component $q(t)$ is then

$$q(t) = \sum_k [\widehat{m}(kT_s) - m(kT_s)] sinc(2\pi Bt - k\pi)$$
$$= \sum_k q(kT_s) sinc(2\pi Bt - k\pi)$$

The signal $q(t)$ is an undesired and acts as noise which leads to the term quantization noise.

To calculate the power of the quantization noise we do the following

$$\widetilde{\widetilde{q^2(t)}} = \lim_{T\to\infty} \frac{1}{T} \int_T q^2(t) dt$$
$$= \lim_{T\to\infty} \frac{1}{T} \int_T \left[\sum_k q(kT_s) sinc(2\pi Bt - k\pi) \right]^2 dt$$

Where $\widetilde{\widetilde{q^2(t)}}$ denotes the mean square power. Since the a since function $sinc(2\pi Bt - n\pi)$ and $sinc(2\pi Bt - m\pi)$ are orthogonal ever where but at $n = m$ we find that

$$\widetilde{\widetilde{q^2(t)}} = \lim_{T\to\infty} \frac{1}{T} \int_T \sum_k q^2(kT_s) sinc^2(2\pi Bt - k\pi)\, dt$$
$$= \lim_{T\to\infty} \frac{1}{T} \sum_k q^2(kT_s) \int_T sinc^2(2\pi Bt - k\pi) dt$$

From the orthogonality relationship it follows that

$$\widetilde{\widetilde{q^2(t)}} = \lim_{T\to\infty} \frac{1}{2BT} \sum_k q^2(kT_s)$$

Assuming that the error is equally likely be anywhere between $\left(-\frac{\Delta v}{2}, \frac{\Delta v}{2}\right)$ we will find that the quantization noise is

$$N_q = \widetilde{\widetilde{q^2(t)}} = \frac{{m_p}^2}{3L^2}$$

Since the power of the message signal

$$S_0 = \widetilde{\widetilde{m^2(t)}}$$

It is clear that the Signal to Quantization Noise Ratio (SQNR) is

$$\frac{S_0}{N_q} = 3L^2 \frac{\widetilde{m^2(t)}}{{m_p}^2}$$

Quantization will be in addition to whatever other noise is present in the system.

Example 4: Power of Quantized Signal

Question: If an audio signal has average power of 0.1W and a peak voltage of 1V. What is the resulting SQNR if L = 50,000 levels.

Answer: When you see average power think mean square power or $\widetilde{\mathrm{m}^2(\mathrm{t})}$ unless otherwise noted. With this we have

$$\mathbf{\frac{S_0}{N_q} = 3L^2 \frac{\widetilde{m^2(t)}}{{m_p}^2} = 3(50,000)^2 \frac{0.1}{(1)^2} = 88.7506\mathit{dB}}$$

Where a the decibel value is calculated as

$$\mathbf{\frac{S_0}{N_q} dB = 10 log\left(3L^2 \frac{\widetilde{m^2(t)}}{{m_p}^2}\right)}$$

Question: How many bits would be required to represent one sample from this audio signal?

Answer:

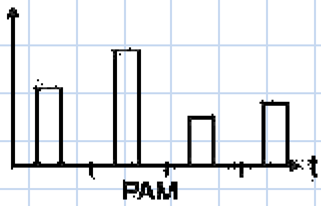
$$\boldsymbol{log_b(x) = \frac{log(x)}{log(b)} = \frac{log(50000)}{log(2)} = 15.61}$$

Well there is no such thing as 15.61 bits so we must round up to 16 bits in order to represent the largest possible quantization of the signal.

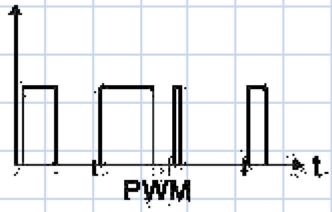
Some Applications of the Sampling Theorem

Sampling theorem is important because it allows a continuous signal to be sampled and then transmitted as a discrete number rather than a continuous time signal. Processing a continuous signal is then equivalent to processing a discrete signal. Discrete values can be transmitted by multiple variations of a pulse train.

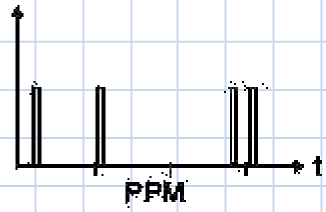
- Pulse Amplitude Modulation (PAM)



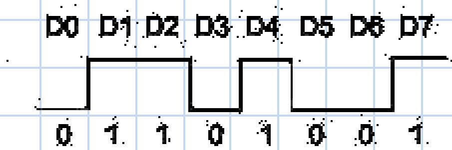
- Pulse Width Modulation (PWM)



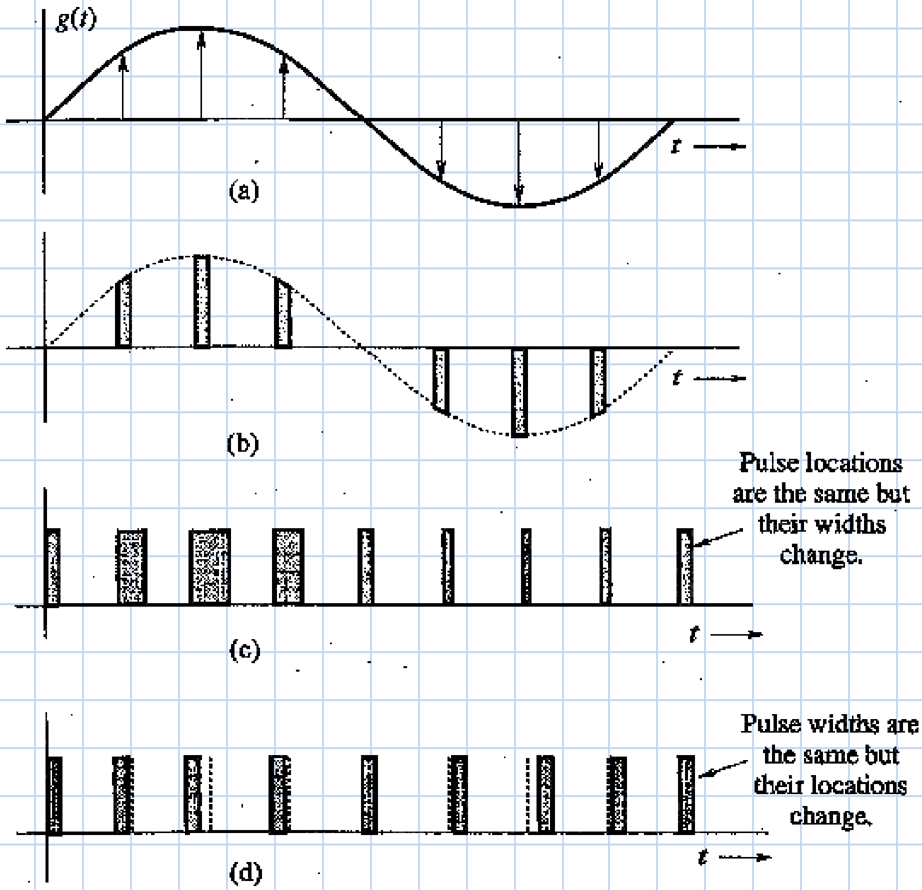
- Pulse Position Modulation (PPM)



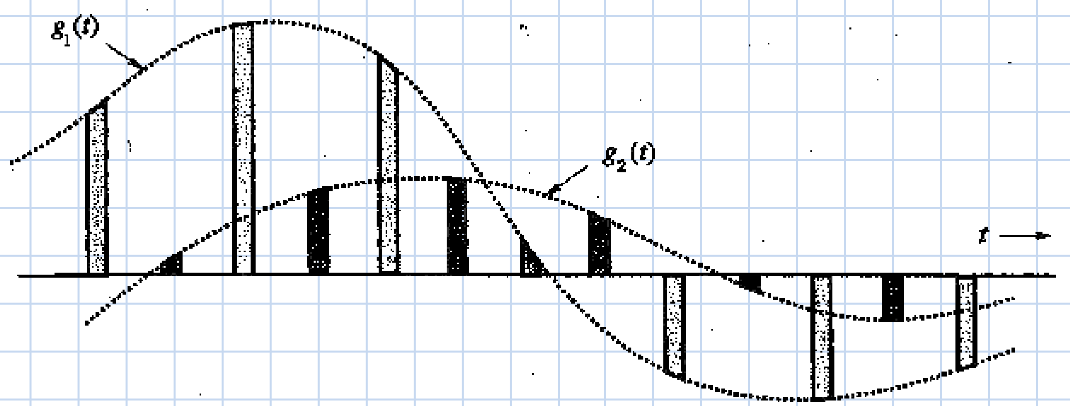
- Pulse Code Modulation (PCM)



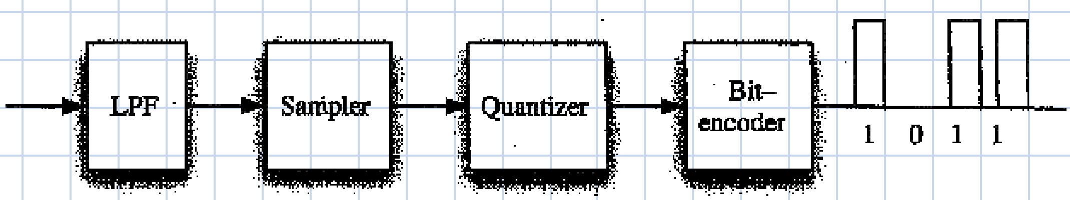
Although the method of creating and representing a quantized signal via a manipulated pulse train may vary the data being transmitted does not. See the image below.



g(t)
t
(a)
t
(b)
Pulse locations are the same but their widths change.
t
(c)
Pulse widths are the same but their locations change.
t
(d)



$g_1(t)$
$g_2(t)$
t



LPF
Sampler
Quantizer
Bit–encoder
1 0 1 1

Time Division Multiplexing

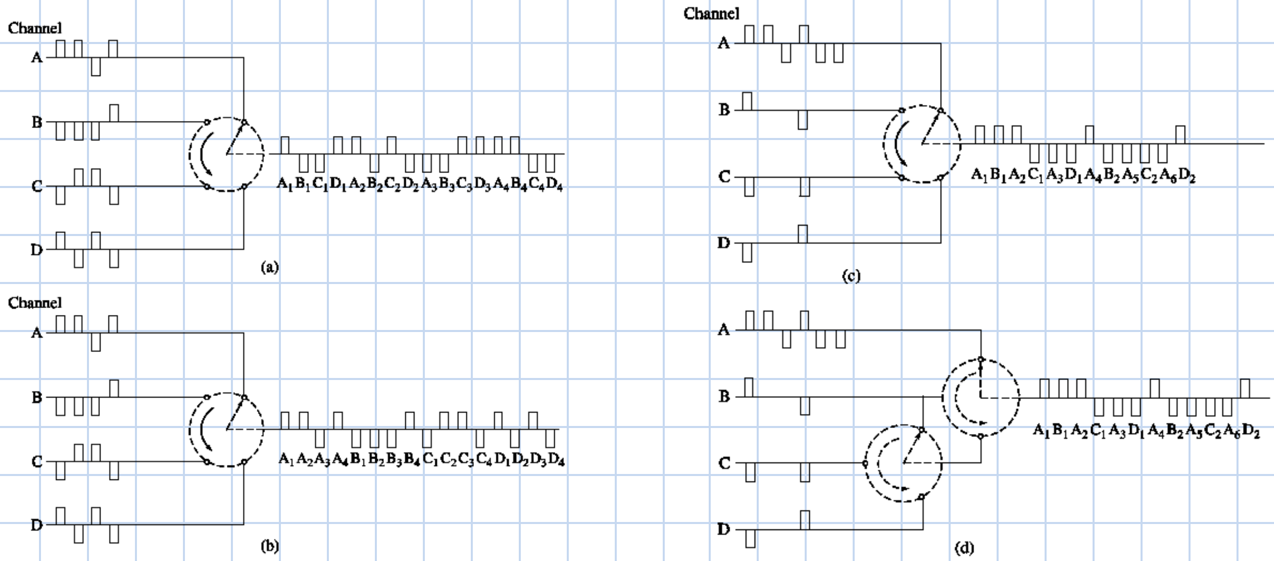


Figure 7: Time Division Multiplexing

In Figure 7 shows various forms of time multiplexing.

- Figure 7.a: Bit interleaving
- Figure 7.b: Word or byte interleaving
- Figure 7.c: Interleaving a of channels with different bit rates
- Figure 7.d: Alternate scheme for channels with different bit rates.

Time division multiplexing allows multiple channels or user to share the same high speed data link. This is an important concept especially in the world of wireless communications because bandwidth is a limited resource.

Example 5: Data Transmission and Time Multiplexing

Question: A wireless communication provider would like to time divide voice data from as many users as possible in order to transmit it over a new wireless link that has a bandwidth of 20MHz. In order to minimize data errors, engineers decide that they must sample data on the link at 20% above the Nyquist rate. If we can only transmit 2bits/s/Hz

(a) What is the maximum sampling rate that can be achieved for this new wireless channel?
(b) How many bits/s can be transmitted on this channel?
(c) If each user will be transmitting data at 500Kbs/s how many users can this channel serve?

Answer: The Nyquist sampling criteria says that we must sample at $T_s < \frac{1}{2B}$. In this case we have a channel that has $2B = 20\text{MHz}$ so $B = 10\text{MHz}$, but we must also account for the 20% over sampling that engineering has advised us of thus $2B = 20(1 - 0.20)\text{MHz}$ which means that our max bandwidth is really only $B = 8\text{MHz}$. Since we can only transmit 2bits/s/Hz we find the maximum bitrate of the channel is $\frac{8MHz}{2bits/sHz} = 4Mbits/s$. If each user transmits 500Kbs/s we find that the channel can service 8 users.