Indian Institute of Information Technology Allahabad
Convex Optimization (SMAT430C)
Tentaive Marking Scheme (Quiz-I)

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1. Let $D = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 \leq 1,\ z = 0\}$.

   (a) Find the convex hull, affine hull and conic hull of D. [3]
   Solution. The convex hull of D, **conv** $D = D$. [1]
   The affine hull of D, **aff** $D = \{(x, y, z) \in \mathbb{R}^3 \,|\, z = 0\}$. [1]
   The conic hull of D is $\{(x, y, z) \in \mathbb{R}^3 \,|\, z = 0\}$. [1]

   (b) Find the interior (D°), boundary (∂D) and closure ($\overline{D}$) of D. [3]
   Solution. $D^\circ = \emptyset$, [1]
   $\partial D = D$, [1]
   $\overline{D} = D \cup \partial D = D$. [1]

   (c) Find the relative interior and relative boundary of D. [2]
   Solution. The relative interior of D, **relint** $D = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 < 1,\ z = 0\}$, [1]
   The relative boundary of D is $\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1,\ z = 0\}$. [1]

   (d) Is the set D open? Justify your answer. [1]
   Solution. $\because D \neq D^\circ$, D is not open. [1]

   (e) Is the set D closed? Justify your answer. [1]
   Solution. $\because D = \overline{D}$, D is closed. [1]

2. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an affine function. Show that the inverse image of a convex set under f is convex. [4]
   Solution. Let S be a convex subset of $\mathbb{R}^m$. Then $f^{-1}(S) = \{x \in \mathbb{R}^n \,|\, f(x) \in S\}$ [1]
   Given any $x_1, x_2 \in f^{-1}(S)$, we have $f(x_1), f(x_2) \in S$.
   $\because S$ is convex, therefore for $\theta \in [0, 1]$, $\theta f(x_1) + (1 - \theta)f(x_2) \in S$. [1]
   $\because f$ is an affine function, $f(x) = Ax + b$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. [1]
   Therefore,
   $\theta f(x_1) + (1 - \theta)f(x_2) = \theta(Ax_1 + b) + (1 - \theta)(Ax_2 + b) = A(\theta x_1 + (1 - \theta)x_2) + b = f(x) \in S$, [1]
   where $x = \theta x_1 + (1 - \theta)x_2$.
   $\Rightarrow x \in f^{-1}(S)$,
   $\Rightarrow f^{-1}(S)$ is convex.

3. Let $A = \begin{pmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{pmatrix}$. For what values of b, A is a positive definite matrix. [3]
   Solution. A matrix A is positive definite if and only if all leading principle minors are positive.
   Clearly $\det(2) > 0$ and $\det \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} > 0$. [1]
   For A to be positive definite, $\det A > 0 \Longrightarrow -b^2 + b + 2 > 0$. [1]
   $\Longrightarrow (b + 1)(b - 2) < 0$.
   Hence $b \in (-1, 2)$. [1]

4. Let A be an $m \times n$ matrix. Find the dual cone of $\{Ax \,|\, x \succeq 0\}$ [3]

Solution. Let $K = \{Ax \mid x \succeq 0\}$.

$$\begin{aligned}
y \in K^* &\iff z^T y \geq 0, \forall z \in K && [1] \\
&\iff (Ax)^T y \geq 0, \forall x \succeq 0 && [1] \\
&\iff x^T A^T y \geq 0, \forall x \succeq 0 \\
&\iff A^T y \succeq 0. && [1]
\end{aligned}$$

Therefore, $K^* = \{y : A^T y \succeq 0\}$.