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Example: min. $\|Ax-b\|_1/(c^Tx+d)$ subject to $\|x\|_\infty \leq 1$. where $A\in\mathbb{R}^{m\times n}$, $b\in\mathbb{R}^m$, $c\in\mathbb{R}^n$. and $d\in\mathbb{R}$. We assume that $d\geq\|c\|_1$, which implies that $c^Tx+d>0$ $\forall$ feasible x. Then

a) show that this is a quasi-convex optimization problem.

b) This problem is equivalent to the convex optimization problem

min. $\|Ay-bt\|_1$ subject to $\|y\|_\infty\leq t$, $c^Ty+dt=1$. —(*)

Sol.- Let $y=\dfrac{x}{c^Tx+d}$, $t=\dfrac{1}{c^Tx+d}$.

$$\|Ay-bt\|_1 = \left\|\frac{Ax}{c^Tx+d}-\frac{b}{c^Tx+d}\right\| = \left\|\frac{Ax-b}{c^Tx+d}\right\|$$

$$= \frac{1}{c^Tx+d}\|Ax-b\|.$$

$$\|y\|_\infty\leq t$$

$$\left\|\frac{x}{c^Tx+d}\right\|_\infty\leq t \Rightarrow \|x\|_\infty\leq t.(c^Tx+d)\Rightarrow \|x\|_\infty\leq t.$$

Suppose y,t are feasible in (*) problem and ~~do~~ if we use the ~~formal~~ transformation $x=y/t$, $(t>0)$

$$\|x\|_\infty\leq 1 \Rightarrow \|y/t\|_\infty\leq 1\Rightarrow \|y\|_\infty\leq t.$$

$$\|A(y/t)-b\|_1/(c^T\tfrac{y}{t}+d) \Leftrightarrow \frac{\|Ay-bt\|_1}{c^Ty+dt} \Rightarrow \|Ay-bt\|_1$$

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Fractional Linear Transformation:

An optimization problem of the form
$\min f_0(x)$ subject to $Gx \leq b$, $Ax = q$
and $f_0(x) = \frac{a^T x + b}{c^T x + d}$ then the problem is frac. linear transf.

Practical Examples.

i) Let there are n products and m raw materials
ii) Let every unit of product j uses a_{ij} unit of material i.
iii) If there are b_i units of material i available
iv) Product J yields ~~to~~ profit c_j dollar per unit and requires an investment of e_j dollar per unit to produce, with f as a fix count. facility wants to maximize return rate of investment.

$\Rightarrow$ No. of products are $x_1, x_2, ---, x_n$.
Then investment will be
$e_1 x_1 + e_2 x_2 + --- + e_n x_n + f$
$= e^T x + f$
The profit $= c_1 x_1 + c_2 x_2 + --- + c_n x_n$
$= c^T x$
return rate of investment $= \frac{c^T x}{e^T x + f}$

$$\max. \frac{c^T x}{e^T x + f}$$

Sub. to $a_i^T x \leq b_i$, $i = 1, 2, ---, n$.
$x_j > 0$, $j = 1, 2, ---, n$.

From (ii) & (iii).
$a_{i1} x_1 + a_{i2} x_2 + --- + a_{in} x_n \leq b_i \Rightarrow a_i^T x \leq b_i$

Generalized linear fractional Programming.

A fractional programming

$$f_0(x) = \max_{i=1,2,\dots,r} \left(\frac{c_i^T x + d_i}{e_i^T x + f_i} \right)$$

with $\operatorname{dom} f_0 = \{x \mid e_i^T x + f_i > 0\}$

Example: Von-neumann growing economy problem. (See from book)

Quadratic Program (Q.P.I.)

An opt. problem of the form

$$\min. (1/2) x^T P x + q^T x + r$$
$$\text{subject to } Gx \preceq h. \quad , \quad P \in S^n_+$$
$$Ax = b.$$

\# Another class of Q.P. is called Quadratically Constrained Quadratic Programming (QCQP) if it is of the form

$$\min (1/2) x^T P_0 x + q_0^T x + r$$
$$\text{Sub. } (1/2) x^T P_i x + q_i^T x + r_i \quad , i = 1, \dots, m. \quad , P_i \in S^n_+$$
$$Ax = b.$$

i) if $P_i \, (i = 1, \dots, m) = 0$ then QCQP $\Rightarrow$ QP.

ii) if $P_i \, (i = 0, 1, \dots, m) = 0$ then QCQP $\Rightarrow$ LP.

Example:

$$\|Ax - b\|_2^2 = (Ax-b)^T (Ax-b) = ((Ax)^T - b^T)(Ax-b).$$
$$= (x^T A^T - b^T)(Ax-b) = x^T A^T A x - x^T A^T b - b^T A x + b^T b.$$
$$= x^T A^T A x - (x^T A^T b + b^T A x) + b^T b.$$
$$= x^T A^T A x - ((b^T A x)^T + (b^T A x)) \cancel{2} + b^T b.$$
$$= x^T A^T A x - 2(b^T A x) + b^T b.$$

Ex.- Distance b/w two polyhedron.

$P_1 = \{x \mid A_1 x \le b_1\}$, $P_2 = \{x \mid A_2 x \le b_2\}$.

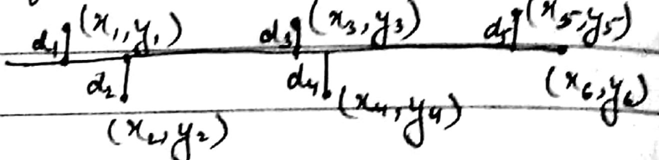
$\Rightarrow$ distance $(P_1, P_2) = \inf\{\|x_1 - x_2\| \mid x_1 \in P_1, x_2 \in P_2\}$.

This can be solved by Q.P. $\min \|x_1 - x_2\|_2^2$.

Sub to $A_1 x \le b_1$, $A_2 x \le b_2$.

Q.- Method of Least Square.

$\min \left(\sum_{i=1} d_i^2\right)$



$\Rightarrow$ $y = aX + b$.

then $d_1 = (ax_1 + b - y_1)$

$d_2 = (ax_2 + b - y_2)$

$d_3 = (ax_3 + b - y_3)$.

$d_1^2 + d_2^2 + \dots + d_5^2$

$\min \sum_{i=1}^{6} d_i^2 = \min \sum_{i=1}^{6} \|ax_i + b - y_i\|$

Sub. to $l \le x_i \le m$

Geometric Programming Problem:

Monomial: A function $f: \mathbb{R}^n \to \mathbb{R}$ with $\operatorname{dom} f = \mathbb{R}^n_{++}$ defined by

$f(x_1, x_2, \dots, x_n) = c\, x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$.

where $c > 0$ and $a_i \in \mathbb{R}$. is called monomial.

Polynomial: A sum of monomial i.e. of the form

$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \dots x_n^{a_{nk}}$

where $c_k > 0$ and $a_{ik} \in \mathbb{R}$ is called polynomial.

Exercise : i) Show that polynomials are closed under addition/multiplication and non-negative scaling.
ii) Show that monomials are closed under multiplication & division.

Geometric Programming

An optimization problem of the form:

$$\left.\begin{aligned} \min f_0(x) \quad \text{sub. to} \quad & f_i(x) \leq 1, \; i=1,\dots,m. \\ & h_i(x) = 1, \; i=1,\dots,p. \end{aligned}\right\} \text{---} (*)$$

where $f_0, f_1, \dots, f_m$ are polynomials.
and $h_1, h_2, \dots, h_p$ are monomials.

Transformation under which a geometric program can be converted to convex optimization

Let $y_i = \log x_i \Leftrightarrow x_i = e^{y_i}$

then

$$\begin{aligned} \text{monomial} \quad f(x) &= c x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}. \\ &= f(x_1, x_2, \dots, x_n) \\ &= f(e^{y_1}, e^{y_2}, \dots, e^{y_n}) \\ &= c e^{y_1 a_1} \cdot e^{y_2 a_2} \dots e^{y_n a_n}. \end{aligned}$$

$$\Rightarrow f(x) = c e^{y_1 a_1 + a_2 y_2 + \dots + a_n y_n}. \cancel{= e^{a^T y + b}}.$$

$$= e^{a^T y + b}, \quad b = \log c.$$

Similarly, a polynomial can be written as

$$f(x) = \sum_{K=1}^{K} e^{a_K^T y + b_K}$$

$$(*) \Rightarrow \min. \sum_{K=1}^{K_0} e^{a_{0K}^T y + b_{0K}}.$$

$$\text{Sub. to} \quad \sum_{K=1}^{K_i} e^{a_{iK}^T y + b_{iK}} \leq 1, \; i=1,\dots,m.$$

$$e^{g_i^T y + h_i} = 1, \quad i=1,\dots,p.$$

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Now transform the objective & affine func by taking log.

$$\min.\ \tilde{f}_0(y) = \log\left(\sum_{k=1}^{K_0} e^{a_{0k}^T y + b_{0k}}\right)$$

$$\text{sub. to}\quad \tilde{f}_i(y) = \log\left(\sum_{k=1}^{K_i} e^{a_{ik}^T + b_{ik}}\right) \le 0.$$

$$\tilde{h}_i(y) = g_i^T y + h_i = 0$$

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Duality Optimization Problem :

$$(*) - \begin{cases} \min.\ f_0(x) & \\ \text{Sub. to}\quad f_i(x) \le 0 & , i=1,\dots,m. \\ \qquad\quad h_i(x) = 0 & , i=1,\dots,p. \end{cases} \Rightarrow \text{primal}$$

$$D = \left(\bigcap_{i=0}^{m} \operatorname{dom} f_i\right) \cap \left(\bigcap_{i=1}^{p} \operatorname{dom} h_i\right) \subseteq \mathbb{R}^n.$$

p^* is optimal value

Lagrangian of $(*)$ is defined as

$$L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \longrightarrow \mathbb{R}.$$

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i + \sum_{i=1}^{p} \nu_i h_i(x)$$

$$= f_0 + \lambda^T F + \nu^T H.$$

Here λ's are called ~~lang~~ lagrange multipli corresponding ~~f_i's~~ ~~f_i vs h_i~~ f_i.
ν_i's are called lagrange multipli corresponding to h_i.

Lagrangian Dual function / Dual function:

$$g : \mathbb{R}^m \times \mathbb{R}^p \longrightarrow \mathbb{R}$$

$$g(\lambda, \nu) = \inf_{\cancel{p^*}} L(x, \lambda, \nu) \quad , x \in D.$$

Claim: g is concave function.

$$g(\theta(\lambda_1, \nu_1) + (1-\theta)(\lambda_2, \nu_2)) \geq \theta g(\lambda_1, \nu_1) + (1-\theta) g(\lambda_2, \nu_2)$$

$$\cancel{f(x) \geq \inf_{x \in A} (f(x)) \; x \in A}$$

$$\cancel{g(x) \geq \inf_{x \in A} (g(x))}$$

$$\left.\begin{aligned} f(x) &\geq \inf_{x \in A} (f(x)) \\ g(x) &\geq \inf_{x \in A} (g(x)) \end{aligned}\right\} f(x) + g(x) \geq \inf(f(x)) + \cancel{g(x)} \inf(g(x)). \quad \forall x \in A.$$

$$\Rightarrow \inf_{x \in A} (f(x) + g(x)) \geq \inf_{x \in A} (f(x)) + \inf_{x \in A} (g(x))$$

Exercise Claim: $g(\lambda, \nu)$ is a ~~convex~~ concave function.

$$g(\theta(\lambda_1, \nu_1) + (1-\theta)(\lambda_2, \nu_2))$$

$$= g(\theta\lambda_1 + (1-\theta)\lambda_2, \theta\nu_1 + (1-\theta)\nu_2)$$

Result: ~~for~~ for $\lambda \succeq 0$ and for any ν $g(\lambda, \nu) \leq p^*$

Proof: Let $\bar{x}$ be a feasible point of (*) i.e.
$f_i(\bar{x}) \leq 0$ and $h_i(\bar{x}) = 0$.

Then

$$\sum_{i=1}^{m} \lambda_i f_i(\bar{x}) + \sum_{j=1}^{p} \nu_j h_j(x) \leq 0.$$

$$g(\lambda, \nu) = \inf L(x, \lambda, \nu) \leq L(\bar{x}, \lambda, \nu) = f_0(\bar{x}) + \sum_{i=1}^{m} \lambda_i f_i(\bar{x}) + \sum_{i=1}^{p} \nu_i h_i(\bar{x}) \leq f(\bar{x})$$

$\forall \bar{x}$ feasible.

$$\Rightarrow g(\lambda, \nu) \leq f(\bar{x}) \Rightarrow \boxed{g(\lambda, \nu) \leq p^*}$$

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This $g(\lambda, \nu)$ gives a lower bound of p^* for $\lambda \succeq 0$.

Therefore, consider another optimization problem $\max g(\lambda, \nu)$ sub to $\lambda \succeq 0$ $\Big\}$ (**)

which is called as dual problem of primal(*).

Exercise: (**) is convex optimization problem.

$$\boxed{\begin{array}{l} \max g(\lambda, \nu) \text{ sub to } \lambda \succeq 0 \\ \text{st } g(\lambda, \nu) \le p^* \end{array} \Big\} \text{ dual of primal}(*) \\ \hookrightarrow \text{lagrangian dual}}$$

Exercise: $\min x^T x$ sub. to $Ax = b$.

$$l(x, \nu) = x^T x + \nu^T (Ax - b)$$

$$g(\nu) = \inf_{x \in D} l(x, \nu)$$

$$\nabla_x l(x, \nu) = 2x + \nu^T A = 2x + A^T \nu = 0$$

$$\Rightarrow x = -\frac{1}{2} A^T \nu$$

Then $g(\nu) = l\left(-\frac{1}{2} A^T \nu, \nu\right)$

$$= -\frac{1}{2} (A^T \nu)^T (A^T \nu) + \nu^T (A (A^T \nu) - b)$$

$$\Rightarrow \boxed{g(\nu) = -\frac{1}{4} \nu^T A A^T \nu - b^T \nu}$$

Exercise: find lagrangian dual $g(\lambda, \nu)$ for

$\min c^T x$ sub. $Ax = b$, $x \succeq 0$

$$g(\lambda, \nu) = \begin{cases} -b^T \nu & A^T \nu - \lambda + c = 0 \\ -\infty & \text{o.w.} \end{cases}$$

$$\text{dom } g(\lambda, \nu) = \{(\lambda, \nu) \mid g(\lambda, \nu) > -\infty\}$$

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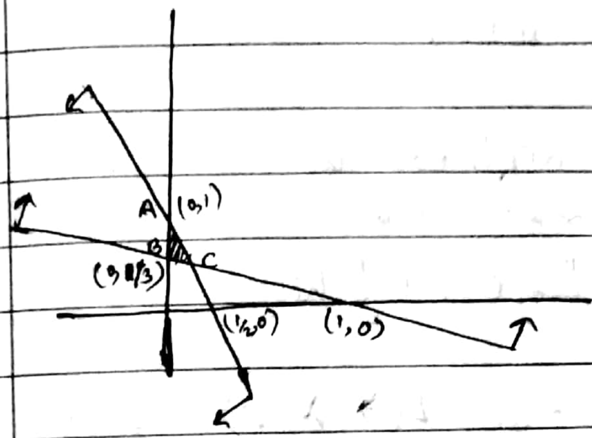
Q.- min. (x_1+x_2) sub. to $2x_1+x_2 \le 1$ $\left(\tfrac{1}{2},0\right),(0,1)$, $x_1+3x_2 \ge 1$ $(1,0),\left(0,\tfrac{1}{3}\right)$

$x_1 \ge 0$, $x_2 \ge 0$.



$$f_0(x_1+x_2) = x_1+x_2$$

$$\nabla f_0 = \begin{pmatrix}\partial f/\partial x_1 \\ \partial f/\partial x_2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \neq 0$$

$$\frac{\partial f}{\partial x_1} \neq 0, \quad \frac{\partial f}{\partial x_2} \neq 0$$

Boundary of D $\Rightarrow$ ~~ACA~~ $AC \cup BC \cup AB$.

for AB, $f_0(0,x_2) = x_2$.

$$\frac{\partial f_0}{\partial x_2} = 1 \neq 0.$$

$\because f_0'(x_2) \neq 0$ in dom. f_0. It will attain its min. at boundary i.e. (A, B). Then

$f_0(x_1,x_2)/_A = 1$, $f_0(x_1,x_2)/_B = 1/3$. ✓

$x_1 + \frac{1-x_1}{3} = \frac{2}{3}x_1 + 1/3$

for BC, $f_0(x_1,x_2) = \cancel{x_1+3x_2} \Rightarrow \nabla f_0 = \begin{pmatrix}2/3\\0\end{pmatrix} \neq 0$

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$f_0(x_1,x_2)/_C = \frac{2}{3}\times\frac{2}{5} + \frac{1}{3} = \frac{3}{5}$, $C = (2/5, 1/5)$

3.33
0.33
0.6
$x_1 + 3x_2 = 1$
$6x_1 + 3x_2 = 3$
$-5x_1 = -2$
$x_1 = 2/5$
$\frac{2}{5} + 3x_2 = 1$
$\frac{5-2}{5} = \frac{3}{5} = 3x_2$
$x_2 = 1/5$.

for AC, $f_0(x_1,x_2) = \cancel{2x_1+x_2}$ $x_1 + 1 - 2x_1 = 1 - x_1$

$$\nabla f_0 = \begin{pmatrix}-1\\0\end{pmatrix} \neq 0.$$

Q.- min. $x_1^2 + 9x_2^2$ sub. to $2x_1+x_2 \ge 1$, $x_1+3x_2 \ge 1$.
$x_1, x_2 \ge 0$

Standard form of Optimization Problem: is
min. $f_0(x)$ subject to $f_i(x) \le 0$, $i=1,\dots,m$.
$h_i(x)=0$, $i=1,\dots,p$.

$$D = \left(\bigcap_{i=1}^{m} \mathrm{dom} f_i\right) \cap \left(\bigcap_{i=1}^{p} \mathrm{dom} h_i\right)$$

Then lagrangian is defined as

$$L(x,\lambda,\nu) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}.$$

$$L(x,\lambda,\nu) = f_0(x) + \sum_{i=1}^{m} f_i(x) + \sum_{i=1}^{p} h_i(x).$$

Lagrange dual $g(\lambda,\nu) = \inf L(x,\lambda,\nu)$, $x \in D$.
$g(\lambda,\nu) \le p^* \rightarrow$ primal optimal.
for $\lambda \geq 0$.

$$\left.\begin{array}{l} \max g(\lambda,\nu) \\ \text{Sub. to } \lambda \geq 0 \end{array}\right\} \text{———} (**)$$

This is convex optimization problem.

Ex.- min. $(c^T x)$ sub. to $Ax=b$, $x \geq 0 \Leftrightarrow -x \le 0$.

Sol.- $L(x,\lambda,\nu) = c^T x - \lambda^T x + \nu^T(Ax-b)$.

Then $g(\lambda,\nu) = \inf_x L(x,\lambda,\nu)$.

$$= \inf_x \left(c^T x - \lambda^T x + \nu^T(Ax-b)\right)$$

$$g(\lambda,\nu) = -b^T\nu + \inf_x (c-\lambda+A^T\nu)^T x$$

i.e.

$$\left[\begin{array}{l} \max. -b^T\nu \\ \text{Sub. to } (c-\lambda+A^T\nu)^T = 0 \\ \qquad \lambda \geq 0 \end{array}\right]$$

Q. min. $c^T x$ sub. to $Ax \le b$.

$$L(x,\lambda,\nu) = c^T x - \lambda^T(Ax-b)$$

$$\left[\begin{array}{l} \max. -b^T\lambda \\ \text{Sub. to } (c-A^T\lambda)^T = 0 \end{array}\right]$$

Primal Optimization Problem: (p^*).
dual problem $\underbrace{\max.\ g(\lambda, \nu),\ \lambda \succeq 0}_{\to d^*}$.

$p^* \geq d^*$.
This is called weak duality.
Value $p^* - d^*$ is called optimal duality gap.
If $p^* = d^*$ then the condition is called strong duality.

1) Convex opt. problem, then we can have strong duality
+
Slater's Condition: There exits
$x \in \text{relint}(D)$ s.t. $f_i(x) < 0$, $i = 1, \dots, m$.
$h_i(x) = 0$, $i = 1, \dots, p$.

Slater's Theorem:
Strong duality holds if slater's condition holds & the problem is convex.

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Optimization Problem:
$\min f_0(x)$ sub. $f_i(x) \leq 0$, $h_i(x) = 0$.
$L(x, \lambda, \nu) = (f_0(x) + \sum \lambda_i f_i(x) + \sum \nu_i h_i)$
$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu)$.

min. $f_0(x)$ sub. $x = 0$.
$L(x, \nu) = f_0(x) + \nu^T x$.
$g(\nu) = \inf (f_0(x) + \nu^T x)$
$= -\sup(-f_0(x) - \nu^T x)$.
$= -\sup((-\nu^T)x - f_0(x))$.
$\boxed{g(\nu) = -f^*(-\nu)}$

$\max g(\lambda, \nu)$ d^*.
Sub. $\lambda \succeq 0$

opt. prob $\min f_0(x)$
Sub. $f_i(x) \le 0$
$h_i(x) = 0$ $\Big\} p^*$

$p^* \ge d^*$.

Weak Duality

Exercise: find the relation b/w conjugate and dual for the fallowing $\min. f_0(x)$ Sub. $Ax \preceq b$, $cx = d$ i.e. under what condition $d^* = p^*$ (Strong duality)

$$f(x) = \left(\frac{-x^2}{1+x^2}\right)$$

$$\inf(f(x)) = -\sup(-f(x)).$$

Slatus Theorem:

If $\exists$ $x^* \in \text{relD}$ s.t. $f_i(x^*) < 0$ and $h_i(x^*) = 0$ then strong duality will hold when the problem is convex optimization.

Convex Opt. + Slatus Theorem = Strong Duality.

Suppose strong duality holds i.e. x^* and (λ^*, ν^*) are pair of optimal points for primal and dual

$$f_0(x^*) = g(\lambda^*, \nu^*) = \inf_x L(x, \lambda^*, \nu^*)$$

$$\le \left(f_0(x^*) + \sum \lambda_i^* f_i(x) + \sum \nu_i h_i(x)\right)$$

$\Rightarrow f_0(x^*) \le f_0(x^*) + \sum \lambda_i^* f_i(x^*)$

$\Rightarrow \sum \lambda_i^* f_i(x^*) = 0 \Rightarrow \boxed{\lambda_i^* f_i(x^*) = 0 \quad \forall\, i = 1, \dots, m}$

Complementary Condition

$\lambda_i^* f_i(x^*) = 0 \iff f_i(x^*) \le 0 \Rightarrow \lambda_i^* = 0$

$\lambda^* \ge 0 \Rightarrow f_i(x^*) = 0$

$\nabla_x L(x, \lambda, \nu) = 0$

$\nabla f(x^*) + \sum \lambda_i \nabla f_i(x^*) + \sum \nu_i h_i(x^*) = 0$ —— (5)

$f_0(x^*) \le 0$ —— (1)

$h_i(x^*) = 0$ —— (2)

$\lambda^* f_i(x^*) = 0$ —— (3)

$\lambda_i^* \ge 0$ —— (4)

$\nu_i h_i = 0$ —— (6)

KKT Conditions

KKT conditions are necessary conditions. In particular if the opt. problem is convex opt. prob. then K.K.T. conditions are also sufficient.

Q.- $\max. z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

sub. to $x_1 + x_2 \le 2$, $2x_1 + 3x_2 \le 12$

Sol.- $L(x, \lambda, \nu) = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2 + \lambda_1(x_1 + x_2 - 2) + \lambda_2(2x_1 + 3x_2 - 12)$

$-2x_1 + 4 + \lambda_1 + 2\lambda_2 = 0$ —— (1) $\quad \partial f / \partial x_1$

$-2x_2 + 6 + \lambda_1 + 3\lambda_2 = 0$ —— (2) $\quad \partial f / \partial x_2$

$-2x_3 = 0$ —— (3)

$x_1 + x_2 \le 2$ —— (4) $\quad 2x_1 + 3x_2 \le 12$ —— (5)

$\lambda_1(x_1 + x_2 - 2) = 0$ —— (6) $\quad \lambda_2(2x_1 + 3x_2 - 12) = 0$ —— (7)

$\lambda_1, \lambda_2 \ge 0$ —— (8), (9)

Case 1: $\lambda_1 = 0$, $\lambda_2 = 0$ —— X (4) fails.

Case 2: $\lambda_1 \ne 0$, $\lambda_2 \ne 0$ X

$2x_1 + 3x_2 - 12$
$2x_1 + 2x_2 - 4$ $\Rightarrow x_2 - 8 = 0 \quad x_2 = 8$ $\quad x_1 = -6$

$\lambda \,|\, -16 - 2\lambda_2 + 3\lambda_2 = 10 \quad \lambda_2 = 26 \quad \lambda_1 = -16 - 52 < 0$

Case 3: $\lambda_1 = 0$, $\lambda_2 \ne 0$ X.

Case 4: $\lambda_1 \ne 0$, $\lambda_2 = 0$ X

$x_1 + x_2 = 2$ $\quad 4 + x_2 \le 12 \Rightarrow x_2 \le 8$

$x_1 = 2 - 8$ $\quad x_1 = -6 \Rightarrow x_1 \ge -6$

~~$\lambda \ne 0$~~

$-2x_1 + 2x_2 - 2 = 0$. $\quad -2x_1 + 4 - 2x_1 - 2 = 0$. $\quad -4x_1 = -2 \quad x_1 = 1/2$.

$-1 + 4 + \lambda_1 = 0 \Rightarrow \lambda_1 = -3$

We are doing it for $-z$.

$z = x_1^2 + x_2^2 + x_3^2 - 4x_1 - 6x_2$

Q. min $f_0(x)$, sub $Ax \le b$, $cx = d$.

$g(\lambda, \nu) = f^*$

$L(x, \lambda, \nu) = f_0(x) + \lambda^T(Ax - b) + \nu^T(cx - d)$

$g(\lambda, \nu) = \inf(f_0(x) + \lambda^T(Ax - b) + \nu^T(cx - d))$

$= -b^T\lambda - d^T\nu + \inf(f_0(x) + \lambda^T Ax + \nu^T cx)$

$= -b^T\lambda - d^T\nu - \sup(-f_0(x) - ($

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for an optimization problem

$$\left.\begin{array}{l} \min f_0(x) \\ \text{sub. to } \; f_i(x) \le 0 \;, i = 1, \dots, m \\ \qquad\quad\; h_i(x) = 0 \;, i = 1, \dots, p \end{array}\right\} p^*$$

$p^* \ge d^*$.

If any pair x^* and (λ^*, ν^*) are optimal of primal and dual respectively with $p^* = d^*$ then

$$\left.\begin{array}{l} \max g(\lambda, \nu) \\ \text{sub. to } \lambda \succeq 0 \end{array}\right\} d^*$$

then x^* and (λ^*, ν^*) will satisfy KKT ~~(Karush~~

$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$

$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu)$

$$\left.\begin{array}{ll} \nabla f_0(x^*) + \sum \lambda_i \nabla f_i(x^*) + \sum \nu_i \nabla h_i(x^*) = 0 & \\ f_i(x^*) \le 0 & , \lambda^* \succeq 0 \\ h_i(x^*) = 0 & , \lambda_i^* f_i(x^*) = 0. \end{array}\right\} \text{KKT cond}^n$$

Q.- $\min. -\sum_{i=1}^{n} \log(\alpha_i + x_i)$, $i = 1, \dots, n$

Sub. to $x \succeq 0$, $1^T x = 1$ where $\alpha_i > 0$

Allocating power to a set of n communication channels. The variables x_i ~~for~~ represents the transmitter power allocated to the i^{th} channel and $\log(\alpha_i + x_i)$ gives the capacity or communication rate of channel. So the problem is to allocate a total power of one to the channels in order to maximise the total communication rate.

Sol.- $x^* \succeq 0$, $1^T x^* = 1$, $\lambda^* \succeq 0$, $\lambda_i^* x_i^* = 0$, $i = 1, \dots, n.$ $\}$ $(*)$.

$$\frac{-1}{(\alpha_i + x_i^*)} - \lambda_i^* + \nu^* = 0, \quad i = 1, \dots, n.$$

$$x^* \succeq 0, \quad 1^T x^* = 1.$$

$$x_i^* \left(\nu^* - \frac{1}{(\alpha_i + x_i^*)} \right) = 0. \quad \text{—— } (3).$$

$$\nu^* \geq \frac{1}{(\alpha_i + x_i^*)}$$

If $\nu^* < \frac{1}{\alpha_i} \Rightarrow \alpha_i < \frac{1}{\nu^*}$

$$\alpha_i + x_i^* \geq \frac{1}{\nu^*} > \alpha_i$$

$$\Rightarrow x_i^* > 0.$$

By (3), $\nu^* = \frac{1}{(\alpha_i + x_i^*)} \Rightarrow x_i^* = \frac{1}{\nu^*} - \alpha_i$

then $x_i^* = \frac{1}{\nu^*} - \alpha_i$ if $\nu^* < \frac{1}{\alpha_i}$.

If $\nu_i^* \geq \frac{1}{\alpha_i}$, then $x_i^* > 0$ is not possible

because if it is there

$\nu^* \geq \frac{1}{\alpha_i} > \frac{1}{(\alpha_i + x_i^*)} \Rightarrow \nu^* > \frac{1}{(\alpha_i + x_i^*)} \Rightarrow x_i^* = 0$. A contradiction

Thus, $x_i^* = \begin{cases} \frac{1}{\nu^*} - \alpha_i & \text{if } \nu^* < \frac{1}{\alpha_i} \\ 0 & \text{if } \nu^* \geq \frac{1}{\alpha_i} \end{cases}$

~~Def^n: Subgradient of a function $f: \mathbb{R}^n \to \mathbb{R}$.~~
~~A vector~~

Subgradient of a function $f: \mathbb{R}^n \to \mathbb{R}$

A vector $g \in \mathbb{R}^n$ is a subgradient of the func. ~~of~~ f at $x \in \text{dom} f$ if for all $y \in \text{dom} f$, we have.

$$\boxed{f(y) \geq f(x) + g^T(y-x)}$$

Recall that if f is convex and differentiable then $\Rightarrow$ $f(y) \geq f(x) + (\nabla f(x))^T (y-x)$. $\forall\, x, y \in \text{dom} f$.

Thus, if f is a convex function which is differentiable then its subgradient at x is nothing but its gradient at x.

Ex.- $f(x) = |x|$

$$\delta f(x) = \begin{cases} 1 & x > 0 \\ & x = 0 \\ -1 & x < 0 \end{cases}$$

Def^n. A function f is called subdifferentiable at x if $\exists$ at least one subgradient at x.

Moreover, the set of subgradients of f at the point x is called subdifferentiable of f at x, and is denoted as $\delta f(x)$.

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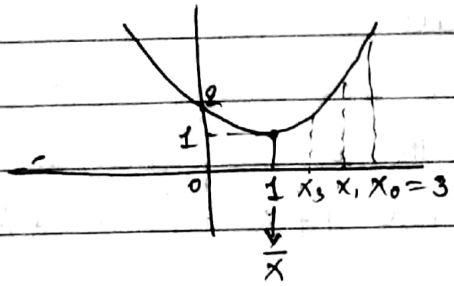
Optimization Problem without Constraints:

i.e. max/min $f_0(x)$.

$f_0(x) = x^2 - 2x + 2$.

min. $f_0(x)$. $\quad f'(x_0) = 2x - 2 = 0 \Rightarrow x = 1$.



In this case, we know $f'(1) = 0$.

Gradient desent Algorithm:

$$X_{i+1} = X_i - \alpha f'(X_i)$$

where X_0 is initial gradient.

$\alpha = 0.2$ Let $X_0 = 3$ and $\alpha = 0.2$. Then

$X_1 = 3 - 0.2(4) = 3 - 0.8 = 2.2$

$X_2 = 2.2 - \cancel{0.2(2.2)}\ 0.2(2.4) = 2.2 - 0.48 = 1.72$

$$(X_i) \longrightarrow \bar{X}$$

If $\alpha = 1$, $\quad X_1 = 3 - 4 = -1$

$X_2 = -1 - 1(-4) = 3$.

$X_3 = -1$, $X_4 = 3$.

eg.- $f(x) = e^{\frac{x^TAx}{2\sigma^2}}$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^n \times \mathbb{R}^n$.

for $n = 2$, let $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

It takes 50 iterations to obtain its soln: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Subgradient desent Algorithm:

$$X_{i+1} = X_i - \alpha v_i$$

where $v_i \in \partial f_i(x_i)$ i.e. v_i is subgradient at x_i i.e. v_i is a member of subdifferential at x_i.

Note that $\{f(x_i)\}$ is not necessarily monotonic decreasing.

Define $v_i = \min\{f(x_0), f(x_1), \dots, f(x_i)\}$
Now obvious question: Under what condition $\{v_k\} \longrightarrow$ optimal value.
To discuss these, we made following assumptions:
<u>Assumption 1</u>: $f_0 : \mathbb{R}^n \to \mathbb{R}$ is lipschitz contineous i.e. $\exists\ l \geq 0$ s.t. $|f(x) - f(u)| \leq l\|x-u\|$.

<u>Assumption 2</u>: $\min. f_0(x)$, $x \in \mathbb{R}^2$ has an optimal solution.

<u>Lemma 1</u>: Let $l \geq 0$ be a lipschitz constant of f. Then $\|v\| \leq l$ where $v \in \delta f(x)\ \forall x \in \mathbb{R}^n$.

<u>Proof</u>:

Since $v \in \delta f(x)$, we have
$$\langle v, h\rangle \leq f(x+h) - f(x) \quad \forall h \in \mathbb{R}^n.$$

Then by Lipschitz continuity of f we have:
$$\langle v, h\rangle \leq f(x+h) - f(x) \leq l\|h\| \quad \forall h \in \mathbb{R}^n.$$
Let $h = v$ then,
$$\langle v, v\rangle \leq l\|v\| \Rightarrow \|v\| \leq l.$$

<u>Proposition 1</u>:
Let l be a lipschitz constant of f. consider the sequence $\{x_k\}$ from subgradient algorithm.
Then $\|x_{k+1} - x\|^2 \leq \|x_k - x\|^2 - 2\alpha_k(f(x_k) - f(x)) + \alpha_k^2 l^2$
$\forall x \in \mathbb{R}, k \in \mathbb{N}$.
α_k is ~~slope~~ step size and l is lipschitz constant.

Proof:

$$\|x_{k+1}-x\|^2 = \|x_k - \alpha_k v_k - x\|^2 = \|(x_k - x) - \alpha_k v_k\|^2.$$
$$= \|x_k - x\|^2 - 2\alpha_k \langle v_k, x_k - x\rangle + \alpha_k^2 \|v_k\|^2.$$

Note that $v_k \in \delta f(x_k)$ and f is lipschitz continuous, therefore by previous lemma $\|v_k\| \leq L$.

In addition,

$\langle v_k, x - x_k\rangle \leq f(x) - f(x_k)$ (by defⁿ of subgradient)

$\Rightarrow f(x_k) - f(x) \leq \langle v_k, x_k - x\rangle$

Therefore,

$$\|x_{k+1}-x\|^2 \leq \|x_k - x\|^2 - 2\alpha_k(f(x_k)-f(x)) + \alpha k^2 L^2$$

— proof.

* $v_k = \min\{f(x_1), f(x_2), \ldots, f(x_k)\}$

$\bar{v} = \inf(f(x))$.

Proposition 2:

Let $L \geq 0$ be a lipschitz constant of f and let S be the set of optimal solutions of the 1 problem then –

$$0 \leq v_k - \bar{v} \leq d(x, s).$$

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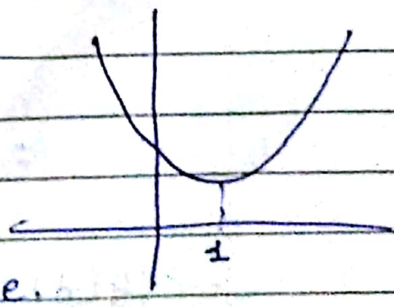
Gradient Desent Method –

$f(x) = x^2 - 2x + 2 = (x-1)^2 + 1.$

min. $f(x) = (x-1)^2 + 1$

$f'(x) = 2(x-1) = 0 \Rightarrow x = 1.$



i e $\{0, 1, \ldots\}$.

$x_{i+1} = x_i - \alpha f'(x_i).$

where α is step size.

$x_1 = x_0 - \alpha f'(x_0)$

let $\alpha = 0.2$, $x_0 = 3$, $x_1 = 2.2$, $x_2 = 1.7$ $\{x_i\} \to 1.$

Example: min. (x^2+1) sub. to $(x-2)(x-4) \leq 0$.

i) Is it a convex optimization problem.

Yes. Because $f''(x) = 2 > 0$.

ii) What is the feasible set?

$\overset{+}{\longleftarrow} \; 2 \; \overset{-}{} \; 4 \; \overset{+}{\longrightarrow}$ ~~(2, 4)~~ $[2, 4]$

iii) What is optimal point & optimum value?

$x^* = 2$, $p^* = 5$.

iv) Is Slater's condition satisfied?

Yes. (Strict inequality + opt. pt.)

Strong duality holds.

v) $g(\lambda)$:

$L(x, \lambda) = (x^2+1) + \lambda(x^2 - 6x + 8)$

$g(\lambda) = \inf L(x, \lambda)$.

$L(x, \lambda) = x^2(\lambda+1) - 6x\lambda + 1 + 8\lambda$

$L'(x, \lambda) = 2x(\lambda+1) - 6\lambda = 0$

$\Rightarrow x = \frac{6\lambda}{2(\lambda+1)} = \frac{3\lambda}{\lambda+1} \rightarrow$ min value

$L''(x, \lambda) = 2(\lambda+1) > 0$ for $\lambda > -1$.

$$g(\lambda) = \inf\left((x^2+1) + \lambda(x^2 - 6x + 8)\right)$$

$$= \left(\frac{3\lambda}{\lambda+1}\right)^2 (\lambda+1) - 6\lambda\left(\frac{3\lambda}{\lambda+1}\right) + 8\lambda + 1$$

$$= \frac{9\lambda^2}{\lambda+1} - \frac{18\lambda^2}{\lambda+1} + 8\lambda + 1$$

$$= \frac{-9\lambda^2 + 8\lambda^2 + 8\lambda + \lambda + 1}{\lambda+1}$$

$$= \frac{-\lambda^2 + 9\lambda + 1}{\lambda+1} \quad \text{for } \lambda > -1.$$

$$\therefore g(\lambda) = \begin{cases} \frac{-\lambda^2 + 9\lambda + 1}{\lambda+1}, & \lambda > -1 \\ -\infty, & \text{o.w.} \end{cases} \Rightarrow \text{dual.}$$

vi) Dual Problem.

$$\max. \left(\frac{-9\lambda^2}{\lambda+1} + (8\lambda+1) \right) \quad \text{sub. to } \lambda \geq 0.$$

$$\frac{(\lambda+1)(-18\lambda) + 9\lambda^2(1)}{(\lambda+1)^2} + 8 = 0.$$

$\Rightarrow -18\lambda^2 - 18\lambda + 9\lambda^2 + 8\lambda^2 + 8 + 16\lambda = 0.$

$\Rightarrow \cancel{-9\lambda} \quad -\lambda^2 - 2\lambda + 8 = 0.$

$\Rightarrow \lambda^2 + 2\lambda - 8 = 0 \Rightarrow (\lambda+1)^2 = 9 \Rightarrow \lambda + 1 = \pm 3.$

$\Rightarrow \lambda = 2 \text{ or } -4.$

-36 +17

+16 +33

$$\boxed{\lambda = 2}$$

Q.2- $\min. \ x_1^2 + x_2^2$

sub. to $(x_1-1)^2 + (x_2-1)^2 \leq 1$

$(x_1-1)^2 + (x_2+1)^2 \leq 1.$

i) find the feasible set. $(1,0)$.

ii) Write KKT conditions.

$$L(x,\lambda) = x_1^2 + x_2^2 + \lambda_1\left((x_1-1)^2 + (x_2-1)^2 - 1\right) + \lambda_2\left((x_1-1)^2 + (x_2+1)^2 - 1\right).$$