MATHEMATICS

SECTION A

January 27, 2024

1. Find the integrating factor of the differential equation $x\dfrac{dy}{dx} - 2y = 2x^2$.

2. Find $\dfrac{dy}{dx}$, if $xy^2 - x^2 = 4$.

3. If A is a square matrix of order 3 with $|A| = 4$, then write the value of $|-2A|$.

4. If a line has the direction ratios $-18, 12, -4$, then what are its direction cosines?

5. Find the cartesian equation of the line which passes through the point $(-2, 4, -5)$ and is parallel to the line $\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$.

6. Let * be a binary operation on $R - \{-1\}$ defined by $a * b = \frac{a}{b+1}$, for all $a, b \in R - \{-1\}$.
   show that * is neither commutative nor associative in $R - \{-1\}$.

7. If $A = \begin{pmatrix} -3 & 6 \\ -2 & 4 \end{pmatrix}$, then show that $A^3 = A$.

8. Form the differential equation representing the family of curves $y^2 = m\left(a^2 - x^2\right)$ by eliminating the arbitrary constants $'m'$ and $'a'$.

9. Find:
$$\int \frac{\sin x - \cos x}{\sqrt{1 + \sin 2x}}\,dx, 0 < x < \frac{\pi}{2}$$

10. Find:
$$\int \frac{\sin(x - a)}{\sin(x + a)}\,dx$$

11. Find:
$$\int (\log x)^2\,dx$$

12. Mother, father and son line up at random for a family photo. If A and B are two events given by A = son on one end, B = Father in the middle, find $P\left(\frac{B}{A}\right)$.

13. Let X be a random variable which assumes values x_1, x_2, x_3, x_4 such that
$$2P(X = x_1) = 3P(X = x_2) = P(X = x_3) = 5P(X = x_4).$$
Find the probability distribution of X.

14. A coin is tossed 5 times. Find the probability of getting
    (i) at least 4 heads, and

(ii) at most 4 heads.

15. Find a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$, where $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.

16. Show that the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$, $-2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\hat{i} - 3\hat{j} + 5\hat{k}$ are coplanar.

17. Show that the relation R on the set Z of all integers, given by $R = \{(a, b) : 2 \text{ divides } (a - b)\}$ is an equivalence relation.

18. If $f(x) = \frac{4x+3}{6x-4}$, $x \neq \frac{2}{3}$, show that $fof(x) = x$ for all $x \neq \frac{2}{3}$. Also, find the inverse of f.

19. If $\sin y = x \sin(a + y)$, prove that

$$\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$$

20. If $(\sin x)^y = x + y$, find $\dfrac{dy}{dx}$.

21. If $\sin^{-1}\left(\frac{3}{x}\right) + \sin^{-1}\left(\frac{4}{x}\right) = \frac{\pi}{2}$, then find the value of x.

22. Using properties of determinants, prove that

$$\begin{vmatrix} a^2 + 1 & ab & ac \\ ab & b^2 + 1 & bc \\ ac & bc & c^2 + 1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$$

23. If $y = \left(\cot^{-1} x\right)^2$, show that $\left(x^2 + 1\right)^2 \frac{d^2y}{dx^2} + 2x\left(x^2 + 1\right)\frac{dy}{dx} = 2$.

24. Find the equations of the tangent and normal to the curve $y = \frac{x-7}{(x-2)(x-3)}$ at the point where it cuts the x-axis.

25. Find the local maxima and local minima, if any, of the following function. Also find the local maximum and the local minimum values, as the case may be :

$$f(x) = \sin x + \frac{1}{2}\cos 2x, \ 0 \leq x \leq \frac{\pi}{2}$$

26. If $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, find A^2 and show that $A^2 = A^{-1}$.

27. Using matrix method, solve the following system of equations:

$$2x - 3y + 5z = 13$$
$$3x + 2y - 4z = -2$$
$$x + y - 2z = -2$$