Visualizing the Chandrasekhar Limit Using Python

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1 Introduction

" Have you ever wondered what happens to a star when it runs out of fuel ? "

To answer this question let's go to the very beginning of the star's life.

The story begins in the giant clouds of dust and gases. These clouds collapse under their own gravity to form the core. As they collapse the temperature and density of the core increases. Initially the gravitational collapse is counteracted by the action of thermal pressure and radiation pressure, but eventually gravity wins.

As the collapse continues, the temperature in the core rises drastically - reaching around 100 Million Kelvin. At this point the hydrogen fusion in the core starts and two hydrogen fuse together to form a helium atom.

Fun Fact – Stars with mass less than 0.08 Solar Mass never reaches a temperature to fuse H ydrogen fusion and are called as Brown Dwarfs.

The hydrogen fusion reaction in a star can occur via two mechanisms:

1. P–P Chain Reaction

$$\begin{aligned}
\text{Step 1:}\quad & {}^{1}\mathrm{H} + {}^{1}\mathrm{H} \rightarrow {}^{2}\mathrm{H} + e^{+} + \nu_e \\
\text{Step 2:}\quad & {}^{2}\mathrm{H} + {}^{1}\mathrm{H} \rightarrow {}^{3}\mathrm{He} + \gamma \\
\text{Step 3:}\quad & {}^{3}\mathrm{He} + {}^{3}\mathrm{He} \rightarrow {}^{4}\mathrm{He} + 2\,{}^{1}\mathrm{H}
\end{aligned}$$

This generally happens in low-mass stars.

2. CNO Cycle

$$\begin{aligned}
& {}^{12}\mathrm{C} + {}^{1}\mathrm{H} \rightarrow {}^{13}\mathrm{N} + \gamma \\
& {}^{13}\mathrm{N} \rightarrow {}^{13}\mathrm{C} + e^{+} + \nu_e \\
& {}^{13}\mathrm{C} + {}^{1}\mathrm{H} \rightarrow {}^{14}\mathrm{N} + \gamma \\
& {}^{14}\mathrm{N} + {}^{1}\mathrm{H} \rightarrow {}^{15}\mathrm{O} + \gamma \\
& {}^{15}\mathrm{O} \rightarrow {}^{15}\mathrm{N} + e^{+} + \nu_e \\
& {}^{15}\mathrm{N} + {}^{1}\mathrm{H} \rightarrow {}^{12}\mathrm{C} + {}^{4}\mathrm{He}
\end{aligned}$$

This generally occurs in high-mass stars.

At the stage when Hydrogen fuses in the core of a star, the star is said to enter the main sequence of the HR diagram and this timeline is known as the birth of the star.

When the hydrogen fuel in a star vanishes, we are left with a helium core. But hydrogen is still burning or fusing into the outer layers of the star. As the core contracts under gravity, the temperature and pressure rise again. If the core is massive enough, it ignites helium fusion, producing heavier elements like carbon and oxygen. However, this helium-burning phase is short-lived compared to the stable hydrogen-burning stage. Once helium is exhausted, the star begins to lose its outer layers, forming a glowing shell of gas known as a planetary nebula. What remains at the center is a hot and dense stellar remnant called a white dwarf.

Now comes the concept of the **Chandrasekhar Limit** — a critical mass threshold of approximately 1.4 times the mass of our Sun. This is the maximum mass a white dwarf can have to remain stable, supported against gravitational collapse by a quantum mechanical force called electron degeneracy pressure. This pressure arises due to the Pauli exclusion principle, which prevents electrons from occupying the same quantum state.

Fun Fact – The Chandrasekhar Limit is named after Indian-American astrophysicist **Subrahmanyan Chandrasekhar**, who proposed it at the young age of 19!

If the mass of the white dwarf is less than this limit, it will simply continue to cool and fade over billions of years, eventually becoming a cold and dark object known as a black dwarf (although the universe is not yet old enough for any black dwarfs to exist). However, if the core's mass exceeds the Chandrasekhar Limit, the electron degeneracy pressure will no longer be sufficient to counteract gravity.

In this case, the core collapses further. The immense pressure forces electrons and protons to combine into neutrons, forming a new type of stellar remnant called a neutron star. Neutron stars are incredibly dense — a single teaspoon of neutron-star material would weigh billions of tons on Earth.

But the story doesn't end there. If the remnant's mass is even greater — usually more than 2 to 3 solar masses — not even neutron degeneracy pressure can resist the collapse. The core then becomes a black hole, a region of spacetime with gravity so intense that nothing, not even light, can escape its pull.

2 The Physics of Degenerate Matter

First, it is important for us to understand the difference between a quantum gas and a classical or ideal gas. The statistics governing classical gases are Maxwell-Boltzmann statistics or Bose-Einstein statistics, while for a quantum gas it is Fermi-Dirac statistics. We can distinguish between a classical and quantum gas by using the concept of Thermal wavelength (λ_{dB}). For a gas with interparticle spacing d,

$$\text{Thermal de Broglie wavelength:} \quad \lambda_{\mathrm{dB}} = \frac{h}{\sqrt{2\pi m k_B T}} \tag{1}$$

$$\text{Classical gas:} \quad \lambda_{\mathrm{dB}} \ll d \tag{2}$$

$$\text{Quantum gas:} \quad \lambda_{\mathrm{dB}} \gtrsim d \tag{3}$$

A classical gas can be treated as a quantum gas in 2 situations - very high density or very low temperature. In case of white dwarfs the former comes into picture.

2.1 Fermi-Dirac Statistics

The distribution function for Fermi–Dirac statistics is given by:

$$f(E) = \frac{1}{1 + \exp\left(\frac{E - E_F}{k_B T}\right)} \tag{4}$$

where E_F is the Fermi energy of the system.

The Fermi energy is defined as the highest occupied energy level of the system at absolute zero temperature (0 K). The figure below illustrates how the energy states are occupied for a gas of fermions governed by Fermi–Dirac statistics:

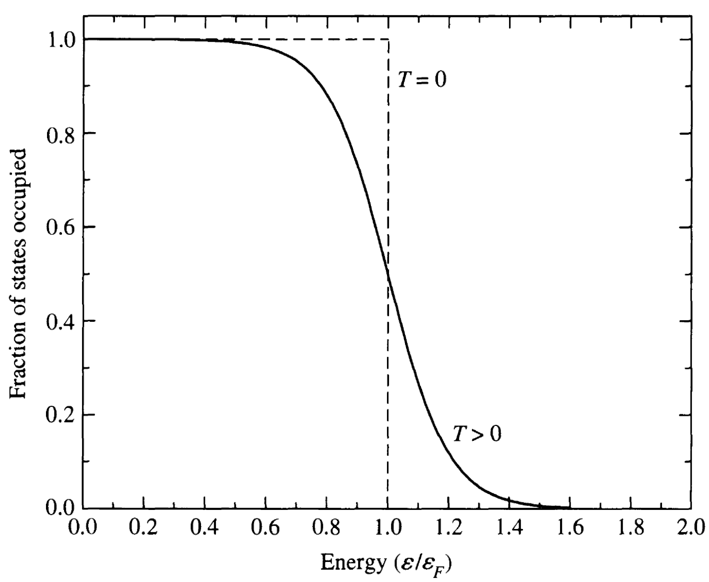

Figure 1: Fraction of energy states ϵ occupied by fermions, as a function of energy. At $T = 0$, all fermions occupy states with $\epsilon < \epsilon_F$.

To derive an expression for the fermi energy of a system, we can use an approach similar to the one used to calculate Pressure in KTG by assuming the quantum gas to occupy the volume of a cube of side L and writing out the de-Broglie wavelengths and momentum of the particles in this situation. After doing this we get the fermi energy to be

$$\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} \tag{5}$$

here, n represents the number of electrons per unit volume of the gas. To express the same in terms of the density of the gas, we can substitue for n as

$$n = \frac{Z}{A}\frac{\rho}{m_H} \tag{6}$$

which gives fermi energy to be,

$$\epsilon_F = \frac{\hbar^2}{2m_e}\left[3\pi^2\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{2/3} \tag{7}$$

This is a very important expression and will be used in the future to derive an expression for degeneracy pressure.

2.2 Electron Degeneracy Pressure

Electron degeneracy pressure arises from 2 of the most fundamental principles in Quantum Mechanics:
1. Pauli's exclusion principle - states that no 2 identical fermions can occupy the same quantum state
2. Heisenberg's uncertainity principle - which says that the product of uncertainity in momentum and position of a quantum mehcanical particle must be of the order of $\hbar$.

From the Heisenberg uncertainity principle we can get a crude approximation of the momentum of the particle in one coordinate direction as //

$$p_x \approx \Delta p_x \approx \frac{\hbar}{\Delta x} \approx \hbar {n_e}^{1/3} \tag{8}$$

So the total momentum p is just

$$p = \sqrt{3}p_x = \sqrt{3}\hbar\left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{1/3} \tag{9}$$

From here we can calculate velocity and plug into the equation of pressure //

$$P = \frac{1}{3}n_e p v \tag{10}$$

On doing so we get the final expression for degenracy pressure to be,

$$P = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}\left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{5/3} \tag{11}$$

We see that equation (11) is an equation of state between pressure and density with an adiabatic index of 5/3 and a corresponding polytropic index of 1.5. Hence we can solve the stellar-structure equations like Lane-Emden equation with the help of this expression.

2.3 The Chandrasekhar Limit

2.3.1 Non-Relativistic case

To get an idea of the mass-radius relationship in the non-relativistic case, we can use the polytropic relation derived above. From equation (11), we can write:

$$P \propto \rho^{5/3} \tag{12}$$

Since pressure is force per unit area, we can approximate:

$$P \propto \frac{M^2}{R^4} \tag{13}$$

Plugging this into equation (12), along with density written in terms of mass and radius, we get:

$$MR^3 = \text{constant} \quad \text{or} \quad MV_{\text{wd}} = \text{constant} \tag{14}$$

So, the mass of a non-relativistic white dwarf is inversely proportional to its volume. As a result, more massive white dwarfs are smaller in size. This implies that continuously adding mass would eventually cause the radius to approach zero. However, at densities around $10^9\,\mathrm{kg\,m^{-3}}$, nearly all electrons in the white dwarf attain speeds comparable to the speed of light. In this regime, the non-relativistic polytropic relation derived earlier is no longer valid, and a relativistic treatment becomes necessary.

2.3.2 Relativistic case

Above, we have used the polytropic equation between pressure and density, which was derived without any relativistic effects. However, in reality, it is special relativity that gives rise to the Chandrasekhar limit. Let us now derive a relation between pressure and density while incorporating relativistic effects.

From quantum mechanics, we know that the density of states in momentum space is given by

$$g(p)\,dp = \frac{8\pi p^2}{h^3}\,dp \tag{15}$$

Since the Fermi temperature in white dwarfs is typically of the order of $10^9\,\mathrm{K}$, while the actual temperature is much lower (around $10^7\,\mathrm{K}$), we can safely assume that the electron gas is completely degenerate. Therefore, we use the $T = 0$ approximation, where all momentum states are filled up to the Fermi momentum p_F, and none beyond. This justifies the limits of integration from 0 to p_F.

The degeneracy pressure is given by

$$P_{\text{deg}} = \frac{1}{3} \int_0^{p_F} v(p)\, g(p)\, p\, dp \tag{16}$$

where $v(p)$ is the velocity of electrons as a function of momentum. In the ultra-relativistic regime, this can be approximated as $v(p) \approx c$. Substituting $g(p)$ and $v(p)$ into the above expression, we get

$$P_{\text{deg}} = \frac{8\pi c}{3h^3} \int_0^{p_F} p^3\, dp \quad \Rightarrow \quad P_{\text{deg}} \propto p_F^4 \tag{17}$$

We clearly see that the degeneracy pressure is proportional to the fourth power of the Fermi momentum. We can relate number density and momentum as:

$$n_e = \int_0^{p_F} g(p)\, dp = \frac{8\pi}{h^3} \int_0^{p_F} p^2\, dp \quad \Rightarrow \quad n_e \propto p_F^3 \tag{18}$$

Assuming constant composition, $n_e \propto \rho$, hence $p_F \propto \rho^{1/3}$. Substituting into the expression for pressure gives:

$$P \propto \rho^{4/3} \tag{19}$$

Now we can repeat the analysis from the non-relativistic case to understand the mass-radius relationship. Plugging this polytropic relation into the equilibrium condition (e.g., Equation 13), we get:

$$\rho^{4/3} \propto \frac{M^2}{R^4} \quad \Rightarrow \quad \frac{M^{4/3}}{R^4} \propto \frac{M^2}{R^4} \tag{20}$$

But the radius R cancels out completely! This implies that in the ultra-relativistic case, the mass is independent of radius. That is, even as the radius tends to zero, the white dwarf mass asymptotically approaches a finite value — the Chandrasekhar limit ($1.44\ M_\odot$).

References

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