Visualizing the Chandrasekhar Limit Using Python

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1 Introduction

" Have you ever wondered what happens to a star when it runs out of fuel ? "

To answer this question let's go to the very beginning of the star's life.

The story begins in the giant clouds of dust and gases. These clouds collapse under their own gravity to form the core. As they collapse the temperature and density of the core increases. Initially the gravitational collapse is counteracted by the action of thermal pressure and radiation pressure, but eventually gravity wins.

As the collapse continues, the temperature in the core rises drastically - reaching around 100 Million Kelvin. At this point the hydrogen fusion in the core starts and two hydrogen fuse together to form a helium atom.

Fun Fact – Stars with mass less than 0.08 Solar Mass never reaches a temperature to fuse H ydrogen fusion and are called as Brown Dwarfs.

The hydrogen fusion reaction in a star can occur via two mechanisms:

1. P–P Chain Reaction

$$\begin{aligned}
\text{Step 1:}\quad & {}^{1}\mathrm{H} + {}^{1}\mathrm{H} \rightarrow {}^{2}\mathrm{H} + e^{+} + \nu_e \\
\text{Step 2:}\quad & {}^{2}\mathrm{H} + {}^{1}\mathrm{H} \rightarrow {}^{3}\mathrm{He} + \gamma \\
\text{Step 3:}\quad & {}^{3}\mathrm{He} + {}^{3}\mathrm{He} \rightarrow {}^{4}\mathrm{He} + 2\,{}^{1}\mathrm{H}
\end{aligned}$$

This generally happens in low-mass stars.

2. CNO Cycle

$$\begin{aligned}
& {}^{12}\mathrm{C} + {}^{1}\mathrm{H} \rightarrow {}^{13}\mathrm{N} + \gamma \\
& {}^{13}\mathrm{N} \rightarrow {}^{13}\mathrm{C} + e^{+} + \nu_e \\
& {}^{13}\mathrm{C} + {}^{1}\mathrm{H} \rightarrow {}^{14}\mathrm{N} + \gamma \\
& {}^{14}\mathrm{N} + {}^{1}\mathrm{H} \rightarrow {}^{15}\mathrm{O} + \gamma \\
& {}^{15}\mathrm{O} \rightarrow {}^{15}\mathrm{N} + e^{+} + \nu_e \\
& {}^{15}\mathrm{N} + {}^{1}\mathrm{H} \rightarrow {}^{12}\mathrm{C} + {}^{4}\mathrm{He}
\end{aligned}$$

This generally occurs in high-mass stars.

At the stage when Hydrogen star to fuse in the core of a star, the star is said to enter the main sequence of the HR diagram and this timeline is known as the birth of the star.

When the Hydrogen fuel in a star vanishes, we are left with a Helium core. But hydrogen is still burning/ fusing into the outer layers of the star.

When a star around the mass of our Sun ($\approx 2 \times 10^{30} Kg$ or $1\, M_{\odot}$) runs out of fuel, it collapses into what we call a White Dwarf. White dwarfs are fascinating objects because they are prime examples of how Quantum Mechanics and Astrophysics go hand in hand. Below, we take a look at some of the physics behind these White Dwarf stars and what keeps them stable.

2 The Physics of Degenerate Matter

First, it is important for us to understand the difference between a quantum gas and a classical or ideal gas. The statistics governing classical gases are Maxwell-Boltzmann statistics or Bose-Einstein statistics, while for a quantum gas it is Fermi-Dirac statistics. We can distinguish between a classical and quantum gas by using the concept of Thermal wavelength (λ_{dB}). For a gas with interparticle spacing d,

$$\text{Thermal de Broglie wavelength:} \quad \lambda_{\mathrm{dB}} = \frac{h}{\sqrt{2\pi m k_B T}} \tag{1}$$

$$\text{Classical gas:} \quad \lambda_{\mathrm{dB}} \ll d \tag{2}$$

$$\text{Quantum gas:} \quad \lambda_{\mathrm{dB}} \gtrsim d \tag{3}$$

A classical gas can be treated as a quantum gas in 2 situations - very high density or very low temperature. In case of white dwarfs the former comes into picture.

2.1 Fermi-Dirac Statistics

The distribution function for Fermi–Dirac statistics is given by:

$$f(E) = \frac{1}{1 + \exp\left(\frac{E - E_F}{k_B T}\right)} \tag{4}$$

where E_F is the Fermi energy of the system.

The Fermi energy is defined as the highest occupied energy level of the system at absolute zero temperature (0 K). The figure below illustrates how the energy states are occupied for a gas of fermions governed by Fermi–Dirac statistics:

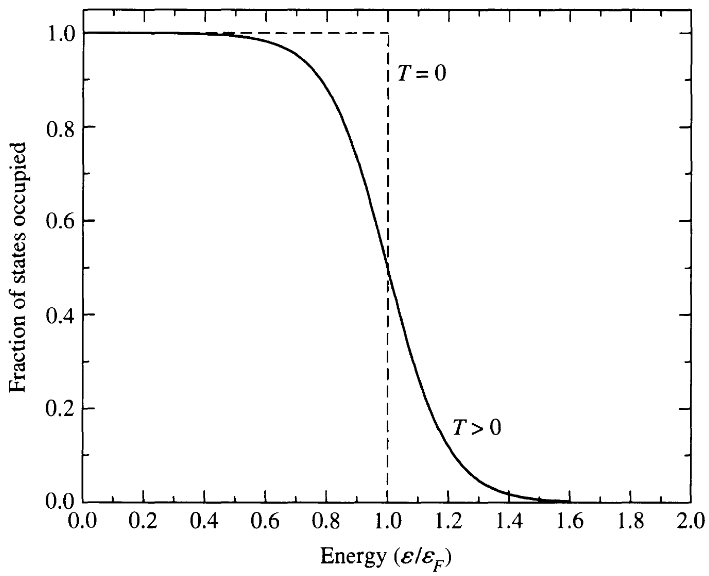

Figure 1: Fraction of energy states ϵ occupied by fermions, as a function of energy. At $T = 0$, all fermions occupy states with $\epsilon < \epsilon_F$.

To derive an expression for the fermi energy of a system, we can use an approach similar to the one used to calculate Pressure in KTG by assuming the quantum gas to occupy the volume of a cube of side L and writing out the de-Broglie wavelengths and momentum of the particles in this situation. After doing this we get the fermi energy to be

$$\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} \tag{5}$$

here, n represents the number of electrons per unit volume of the gas. To express the same in terms of the density of the gas, we can substitue for n as

$$n = \frac{Z}{A}\frac{\rho}{m_H} \tag{6}$$

which gives fermi energy to be,

$$\epsilon_F = \frac{\hbar^2}{2m_e}\left[3\pi^2\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{2/3} \tag{7}$$

This is a very important expression and will be used in the future to derive an expression for degeneracy pressure.

2.2 Electron Degeneracy Pressure

Electron degeneracy pressure arises from 2 of the most fundamental principles in Quantum Mechanics:
1. Pauli's exclusion principle - states that no 2 identical fermions can occupy the same quantum state
2. Heisenberg's uncertainity principle - which says that the product of uncertainity in momentum and position of a quantum mehcanical particle must be of the order of $\hbar$.

From the Heisenberg uncertainity principle we can get a crude approximation of the momentum of the particle in one coordinate direction as //

$$p_x \approx \Delta p_x \approx \frac{\hbar}{\Delta x} \approx \hbar {n_e}^{1/3} \tag{8}$$

So the total momentum p is just

$$p = \sqrt{3} p_x = \sqrt{3} \hbar \left[\left(\frac{Z}{A} \right) \frac{\rho}{m_H} \right]^{1/3} \tag{9}$$

From here we can calculate velocity and plug into the equation of pressure //

$$P = \frac{1}{3} n_e p v \tag{10}$$

On doing so we get the final expression for degenracy pressure to be,

$$P = \frac{(3\pi^2)^{2/3}}{5} \frac{\hbar^2}{m_e} \left[\left(\frac{Z}{A} \right) \frac{\rho}{m_H} \right]^{5/3} \tag{11}$$

We see that equation (11) is an equation of state between pressure and density with an adiabatic index of 5/3 and a corresponding polytropic index of 1.5. Hence we can solve the stellar-structure equations like Lane-Emden equation with the help of this expression.