IITI SOC

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1. Introduction

Stars are huge spheres of hot plasma which go through nuclear fusion in their centers. During their lifetimes, they have a balance between collapsing gravitationally and the outward pressure from heat. When stars run out of nuclear fuel, their ultimate destiny depends on what mass they started out with. For low to intermediate-mass stars (up to around 8 solar masses), the final stage is a white dwarf — a small stellar remnant stabilized by electron degeneracy pressure. These stars no longer burn fusion but resist collapsing because of quantum mechanical forces. As the remnant contracts and cools, its internal behavior is controlled no longer by thermal pressure, but by the physics of degenerate matter. Knowledge of the boundary beyond which this degeneracy fails — the Chandrasekhar limit — is the key to understanding why white dwarfs cannot be over a specific mass without collapsing into neutron stars or black holes.

2. Physics of Degenerate Matter

White dwarfs are under the jurisdiction of quantum, rather than classical, physics. Whereas classical gases conform to Maxwell-Boltzmann statistics, quantum gases of fermions follow Fermi-Dirac statistics. This crossover happens when the thermal de Broglie wavelength,

$$\lambda_{dB} = \frac{h}{\sqrt{2\pi m k_B T}},$$

approaches or surpasses interparticle spacing d. In white dwarfs, high density makes this unavoidable, leading to degenerate matter.

2.1 Fermi-Dirac Statistics

Fermions fill energy states according to the Fermi-Dirac distribution:

$$f(E) = \frac{1}{1 + \exp\left(\frac{E - E_F}{k_B T}\right)}$$

The states below the Fermi energy E_F are filled at $T = 0$. The Fermi energy is:

$$\epsilon_F = \frac{\hbar^2}{2m_e}\left(3\pi^2 \frac{Z}{A}\frac{\rho}{m_H}\right)^{2/3}$$

where ρ is mass density and m_H is the hydrogen mass.

2.2 Electron Degeneracy Pressure

Electron degeneracy pressure is a result of the Pauli exclusion principle and Heisenberg uncertainty, which in combination push electrons into higher momentum states during compression.

Estimate the momentum of compressed electrons:

$$p \sim \hbar \left(\frac{Z}{A}\frac{\rho}{m_H}\right)^{1/3}$$

Substituting into the pressure formula $P = \frac{1}{3} n_e p v$, we get:

$$P = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}\left(\frac{Z}{A}\frac{\rho}{m_H}\right)^{5/3}$$

This is the equation of state for a non-relativistic degenerate electron gas, with polytropic index $n = 1.5$.

3. The Chandrasekhar Limit

White dwarfs are stellar remnants supported against gravitational collapse by the degeneracy pressure of electrons. As stars exhaust their nuclear fuel, those with masses less than about eight times the mass of the Sun end their lives as white dwarfs. The final destiny of such a star will hinge on whether the internal pressure can resist gravitational forces — a state controlled by the physics of degenerate matter.

Degenerate Electron Pressure

Inside a white dwarf, electrons are compressed so tightly that quantum mechanical effects become dominant. Depending upon whether the electrons are traveling much more slowly or nearer to the speed of light, the pressure assumes various forms.

Non-relativistic regime (slow electrons):

$$P = K_1 \rho^{5/3}$$

where K_1 is a constant in terms of fundamental physical constants, ρ is the mass density, and this equation gives how pressure depends on density when electron velocities are low.

Relativistic regime (ultra-fast electrons):

$$P = K_2 \rho^{4/3}$$

When the mass of the white dwarf grows, the electrons are very relativistic, and this softer pressure dependence cannot sustain growing gravity above a certain mass limit.

Hydrostatic Equilibrium

A white dwarf needs to achieve hydrostatic equilibrium in order to be structurally stable. This state is characterized by the hydrostatic equilibrium equation:

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

where $M(r)$ is the mass enclosed within radius r, $\rho(r)$ is the density profile, and G is the gravitational constant.

Polytropic Equation of State

Degenerate matter can be represented by a polytropic equation of state:

$$P = K\rho^{1+\frac{1}{n}}$$

where n is the polytropic index. For white dwarfs:

- $n = \frac{3}{2}$ corresponds to non-relativistic degeneracy.
- $n = 3$ is the ultra-relativistic case.

The Chandrasekhar Mass Limit

Once electrons are relativistic, degeneracy pressure is no longer able to maintain sufficient pressure against gravity. A theoretical mass limit for stable white dwarfs, the Chandrasekhar limit, results:

$$M_{Ch} = \frac{5.83}{\mu_e^2} M_\odot$$

where μ_e is the mean molecular weight per electron. For ordinary carbon-oxygen white dwarfs, $\mu_e \approx 2$, so that:

$$M_{Ch} \approx 1.44 M_\odot$$

Any white dwarf with this mass approaching will collapse into a black hole or neutron star since any stable electron degeneracy pressure cannot halt the implosion.

4. Python-Based Visualization of the Chandrasekhar Limit

4.1 Libraries and Tools Used

We developed a Python-based simulation to assist in visualizing and understanding white dwarf physics and the Chandrasekhar limit. The implementation is based on NumPy for array-based numerical computation, Matplotlib for generating interactive plots, and SciPy for numerical optimization procedures that identify equilibrium states. These packages enable us to construct a completely interactive framework wherein users can adjust input parameters using sliders and observe how the system responds.

4.2 Physical Modeling

The model takes a simplified but realistic model of a white dwarf. It computes the total energy as the sum of energy due to gravitational binding and energy due to electron degeneracy pressure. The degeneracy pressure is computed using three cases: the non-relativistic case, applicable to low densities; the ultra-relativistic case, applicable to extremely high densities; and a general relativistic formula that smoothly interpolates between the two. For each group of stellar parameters, such as composition and mass, the code determines the radius that will minimize total energy, showing that there is a stable white dwarf configuration. If there is no such radius, then the star is unstable and collapses.

4.3 Interactive Parameters

There are sliders in the user interface for mean molecular weight per electron (μ_e), mass (in solar masses), and a density scale factor on the pressure-density plots. These are linked directly to the physical equations. Altering any of the sliders updates all visual outputs instantaneously. This layout allows users to navigate the stability landscape of white dwarfs. They can observe how slight changes in major parameters lead to huge differences in stellar behavior.

4.4 Visual Plots and Interpretations

The plot of mass vs radius reveals the inverse relationship between the mass and the radius of a white dwarf. With increasing mass, the white dwarf's radius decreases due to greater gravitational compression. This is the case up to the Chandrasekhar limit is achieved. The radius then abruptly falls to zero, indicating no stable arrangement can exist; the star would implode under its own weight. This graph illustrates the shocking yet critical feature of white dwarf structure: more massive white dwarfs are actually smaller.

The energy vs radius plot shows the total energy of the white dwarf system vs. radius for constant mass. For low masses, there is a definite minimum in the energy curve, indicating a stable equilibrium radius. But as mass builds up, the minimum becomes less pronounced and finally vanishes completely, suggesting that no radius provides a stable configuration. This disappearance of the energy minimum for high masses graphically marks the onset of gravitational instability and collapse.

In the degeneracy pressure vs density graph, we contrast three degeneracy pressure models across a very broad range of electron number density: the non-relativistic regime, in which pressure rises rapidly with density; the ultra-relativistic regime, in which pressure increases more slowly; and the general relativistic model, which interpolated smoothly between the two. With increasing density, electrons become more relativistic, and the response of pressure gets weaker and ceases to be capable of resisting gravity. This reduction in the equation of state is the primary reason why there is a Chandrasekhar limit.

The minimum energy vs mass graph demonstrates how the smallest possible total energy, which symbolizes the most stable arrangement, varies as we vary the white dwarf's mass. For masses smaller than the Chandrasekhar limit, the system has a minimum negative energy, showing a stable star. When we approach the critical mass, this minimum increases and eventually disappears. This plot offers quantitative confirmation of the theoretical Chandrasekhar limit and shows how stability disappears above some mass threshold.

The mass-radius stability heatmap offers a glimpse of stability for a broad range of mass and radius values. Stable configurations where energy minima lie are represented by cooler colors. Unstable regions with no equilibrium are represented by warmer colors. A marker shows the current parameter values chosen by the user. This plot graphically displays the phase space of white dwarf stability and enables users to immediately check whether a specific mass-radius pair is within the stable region.

4.5 Simulation Output and Conclusions

When executed, the simulation outputs a summary of the current setup, including the calculated Chandrasekhar mass for the value of μ_e, the energy of the system, and the radius at energy minimization. It also reports whether the configuration is stable and issues a warning if no equilibrium was found. Through this interactive tool, users can observe how two relativistic effects cause a softening in degeneracy pressure, and how this leads to the loss of white dwarf stability above the Chandrasekhar limit. The visualizations make this otherwise abstract transition vividly clear, offering both qualitative and quantitative insights into one of astrophysics' most important thresholds.