1. PROOF OF THEOREM 1.3

Proof. When $m \geqslant 9.6446 \times 10^{35}$, by Theorem 9.3 the result is immediate. So assume that $m < 9.6446 \times 10^{35}$. If m is an icosahedral number, then the theorem follows trivially. Otherwise, consider the following steps.

Step 1. Let n be the integer such that $g(n) < m < g(n+1)$. If $m \in (g(n), g(n) + 2000)$, then let $m_1 = m - g(n-1)$. Otherwise, let $m_1 = m - g(n)$.

Step 2. Taking $m - m_1$ to be the first term in the sum. Reduce to problem to writing m_1 as a sum of one less icosahedral numbers.

Step 3. Repeat Steps 1 and 2 with m_1 until we reduce the problem to writing the numbers as a sum of 8 icosahedral numbers.

We will apply the above steps to some positive integer $m < 9.6446 \times 10^{35}$. If $m \in [g(n) + 2000, g(n+1))$ for some $n \in \mathbb{N}$, then let $m_1 = m - g(n)$. We deduce that

$$m_1 < g(n+1) - g(n) = \frac{15}{2}n^2 + \frac{5}{2}n + 1. \tag{1.1}$$

Since $m < 9.6446 \times 10^{35}$, we have

$$g(n) = \frac{5}{2}n^3 - \frac{5}{2}n^2 + n < 9.6446 \times 10^{35},$$

which implies that $n < 7.3 \times 10^{11}$. Substituting this into (1.1), we conclude that $m_1 < 4 \times 10^{24}$. When $m \in (g(n), g(n) + 2000)$, we have

$$m_1 < g(n) + 2000 - g(n-1) = \frac{15}{2}n^2 - \frac{25}{2}n + 2006. \tag{1.2}$$

Again since $m < 9.6446 \times 10^{35}$, we conclude that $m_1 < 4 \times 10^{24}$ as well. Therefore, in both cases, it suffices to write $2000 \leqslant m_1 \leqslant 4 \times 10^{24}$ as a sum of 14 icosahedral numbers.

Iterating the above six more times, the original problem is reduced to writing $2000 \leqslant m_7 \leqslant 6860$ as a sum of 8 icosahedral numbers. This was confirmed true by using our Python algorithm (see Python algorithm.py file). For positive integers less than 2000, we checked using the same algorithm that each of them can be written as a sum of at most 15 icosahedral numbers. This completes the proof of the theorem. □

Remark 1.1. *While using the algorithm to check positive integers less than* 2000*, we found that the numbers that cannot be written as a sum of at most 13 icosahedral numbers, but can be written as a sum of 14 icosahedral numbers are* $47, 83, 94$ *and* 119*. Further, the only number that cannot be written as a sum of at most 14 icosahedral numbers is* 95*. These five numbers are the only counterexamples to the original Pollock's conjecture for icosahedral numbers.*

2. PROOF OF THEOREM 1.4

Proof. The proof follows the same steps as that of Theorem 1.7. We have

$$g(n+1) - g(n) = \frac{27}{2}n^2 + \frac{9}{2}n + 1 \tag{2.1}$$

and

$$g(n) + 2000 - g(n-1) = \frac{27}{2}n^2 - \frac{45}{2}n + 2010 \tag{2.2}$$

After repeating the Steps 1-3 as in the proof of Theorem 1.7 eight times, it suffices to write any positive integer m such that $2000 \leqslant m < 5290$ as a sum of at most 14 dodecahedral numbers. This is confirmed true using our algorithm. For positive integers less than 2000, we checked using the same algorithm that each of them can be written as a sum of at most 22 dodecahedral numbers. This completes the proof of the theorem. □

Remark 2.1. *While using the algorithm to check whether all positive integers less than* 2000 *can be written as a sum of at most 22 icosahedral numbers, we found out that the only number that cannot be written as a sum of at most 21 dodecahedral numbers is* 79*. This is the only counterexample to the original Pollock's conjecture.*