1. Monic Legendre polynomials on $[-1,1]$ are defined as follows:

$$q_0(x) = 1 \tag{1}$$
$$q_1(x) = x \tag{2}$$

$q_n(x)$ is a monic polynomial of degree n such that $\displaystyle\int_{-1}^{1} q_n(x)q_m(x)dx = 0$ for all $m \neq n$.

- Show that the Legendre polynomials satisfy the recurrence

$$q_{n+1} = xq_n - \left(\frac{n^2}{4n^2-1}\right) q_{n-1}$$

- Prove that if $p(x)$ is a monic polynomial of degree n minimizing $\|p(x)\|_2^2 = \displaystyle\int_{-1}^{1} p^2(x)dx$, then $p(x) = q_n(x)$. Hence, conclude that Legendre nodes (roots of the Legendre polynomial) minimize $\displaystyle\int_{-1}^{1}\left(\prod_{k=0}^{n}(x-x_k)\right)^2 dx$.
- Consider the Runge function $f(x) = \dfrac{1}{1+25x^2}$. Interpolate the Runge function using the roots of the Legendre polynomial of degree n. Plot the error as a function of $n \in \{3,4,5,6,\ldots,21\}$.

2. The Chebyshev polynomials of the first kind are defined as:

$$T_n(x) = \cos(n \arccos(x))$$

- Show that the Chebyshev polynomials of the first kind satisfy the recurrence:

$$T_{n+1} = 2xT_n - T_{n-1}$$

with $T_0 = 1$ and $T_1 = x$.
- Show that $T_n(x)$ is a polynomial of degree n with leading coefficient as 2^{n-1} for $n \geq 1$.
- All zeros of $T_{n+1}(x)$ are in the interval $[-1,1]$ and given by $x_k = \cos\left(\dfrac{2k+1}{2n+2}\pi\right)$, where $k \in \{0,1,2,\ldots,n\}$.
- Conclude that $T_n(x)$ alternates between ± 1 exactly $n+1$ times.
- Show that $\left|\displaystyle\prod_{k=0}^{n}(x-x_k)\right| \leq \dfrac{1}{2^n}, \quad \forall x \in [-1,1]$.
- For any choice of nodes $\{y_k\}_{k=0}^{n}$, consider the polynomial $P_{n+1}(x) = \displaystyle\prod_{k=0}^{n}(x-y_k)$ and look at $F(x) = P_{n+1}(x) - \dfrac{T_{n+1}(x)}{2^n}$. If $|P_{n+1}(x)| \leq \dfrac{1}{2^n}$, show that $F(x)$ alternates in sign $n+2$ on $[-1,1]$. Hence, conclude that $F(x)$ has to be identically zero and therefore conclude that Chebyshev nodes minimizes $\max_{x\in[-1,1]}\left|\displaystyle\prod_{k=0}^{n}(x-x_k)\right|$.