1. In class, in the proof of the Euler-Maclaurin formula, we left the following as an exercise.

$$\int_a^b \frac{B_{2m+2}(\{t\})}{(2m+2)!} g^{(2)}(t)dt = \int_a^b \frac{B_{2m}(\{t\})}{(2m)!} g(t)dt + \frac{b_{2m+2}}{(2m+2)!}\left(g'(b) - g'(a)\right)$$

   Use integration by parts to prove the above. Note that $B_k(x)$ are the Bernoulli polynomials and $\{t\}$ is the fractional part of t.

2. In class, we obtained that

$$n! \sim C\sqrt{n}\left(\frac{n}{e}\right)^n, \quad (\spadesuit)$$

   The notation $f(n) \sim g(n)$ means that $\lim_{n\to\infty} \dfrac{f(n)}{g(n)} = 1$. In this exercise, we will obtain C by following the sequence of steps.

   - Show that

$$I_n = \int_0^{\pi/2} \sin^n(x)dx = \begin{cases} \dfrac{\pi}{2^{2k+1}}\dbinom{2k}{k} & \text{if } n = 2k \\ \dfrac{2^{2k}}{2k+1}\dfrac{1}{\dbinom{2k}{k}} & \text{if } n = 2k+1 \end{cases}$$

   - Show that

$$\lim_{n\to\infty} \frac{I_{2n-1}}{I_{2n+1}} = 1$$

   - Prove that I_n is a monotone decreasing sequence, i.e.,

$$I_{2n+1} < I_{2n} < I_{2n-1}$$

   - Conclude that

$$\lim_{n\to\infty} \frac{I_{2n}}{I_{2n+1}} = 1$$

   - Hence, conclude that

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{n\pi}}$$

   - Expand the central binomial coefficient and derive that $C = \sqrt{2\pi}$ using $(\spadesuit)$.

3. Let $I = \displaystyle\int_0^1 \exp\left(x^2\right)dx$. The exact value of the integral accurate upto 16 digits is 1.46265174590718161. Compute the integral using the following methods by sub-dividing $[0,1]$ into $N \in \{2,4,8,16,32,64,128,256,512,1024\}$ panels. Plot the decay of the absolute error as a function of N on a log-log plot.

   - Midpoint rule
   - Trapezoidal rule
   - Trapezoidal rule with end point correction using the first derivative
   - Trapezoidal rule with end point correction using the first and third derivatives

   Repeat the same using Gauss Legendre quadrature with $N \in \{2,3,4,\ldots,51\}$ and plot the absolute error as a function of N on a log-log plot. For an accuracy of 10^{-12}, report the number of nodes required by

   - Midpoint rule
   - Trapezoidal rule
   - Trapezoidal rule with end point correction using the first derivative
   - Trapezoidal rule with end point correction using the first and third derivatives
   - Gauss Legendre quadrature

4. Evaluate

$$I = \int_0^2 \frac{e^{-x}}{\sqrt{x}}dx$$

   using the following methods

- Use the midpoint rule to avoid the singularity of the integrand at $x = 0$
- Make a change of variable $x = t^2$ and again use the midpoint rule to evaluate the resulting integral

Compare the accuracy of the two methods by plotting the error as a function of N (number of panels) on a log-log scale, where $N \in \{2, 4, 8, 16, 32, 64, 128, 256, 512, 1024\}$.

5. Prove that the error due to Hermite interpolation as discussed in class is given by

$$f(x) - p_{2n+1}(x) = \frac{f^{2n+2}(\xi_x)}{(2n+2)!} \prod_{j=0}^{n} (x - x_j)^2$$

for some ξ_x in the interpolating interval. (**HINT**: The proof is extremely similar to the proof we gave for Lagrange interpolation)

6. **Error estimate of Gaussian quadrature**: Hence, prove that

$$\int_{-1}^{1} f(x)dx - \sum_{i=0}^{n} w_i f(x_i) = \frac{f^{(2n)}(\xi)}{(2n)!} \int_{-1}^{1} q_n^2(x)dx$$

for some $\xi \in (-1, 1)$ and q_n is the monic Legendre polynomial of degree n.