DSO530 Week1 Technical Notes: Review of Probability II

Random Variable (r.v.)

… is a variable whose value depends on the outcome of an experiment (e.g. coin tossing)

- *Random variable* assigns a value (typically a number) to each possible outcome in the sample space S = {all possible outcomes}

Experiment: Toss a coin 3 times. X = # of "heads".

Outcome	TTT	TTH	THT	HTT	HHT	HTH	THH	HHH
Value of X	0	1	1	1	2	2	2	3

Random Variables

Discrete random variable: Possible values can be listed or counted

- e.g. the number of defective units in a batch of 20

Continuous random variable: May assume any numerical value in one or more intervals

- e.g. the waiting time for a credit card authorization

Discrete Random Variable

- **(probability)** **<u>distribution</u>** of a discrete random variable is a table, graph, or formula that gives the probability associated with each possible value
- probabilities of all possible values must sum to 1

e.g. X = number of heads in 3 tosses

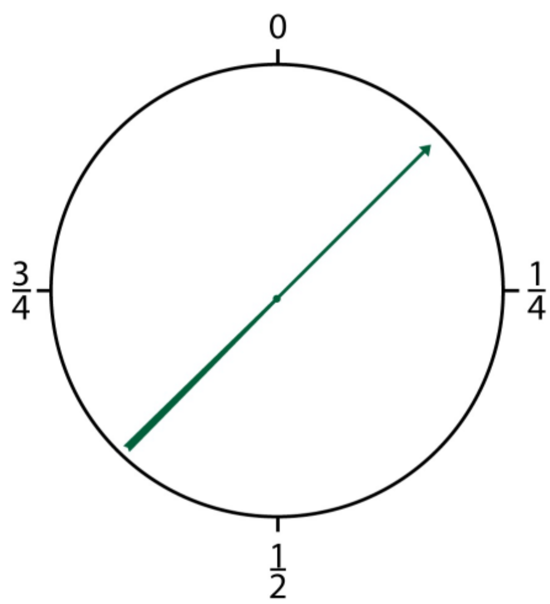
Value	0	1	2	3
Probability	1/8	3/8	3/8	1/8

Example (continuous random variable)

A spinner turns freely on its axis and slowly comes to a stop

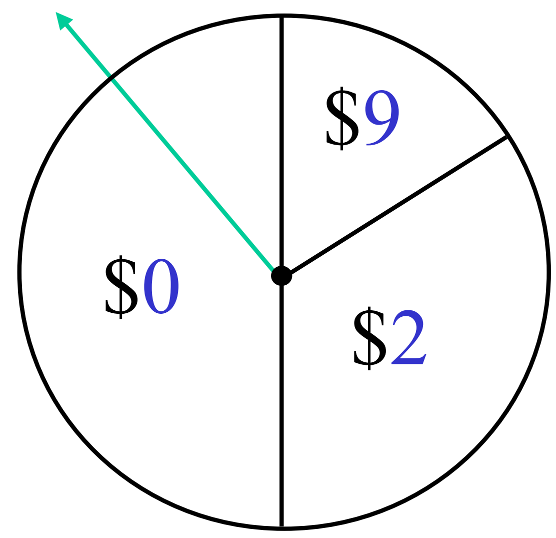
random variable X: location of the pointer
can be anywhere on a circle that is marked from 0 to 1

(does not favor any part of the circle)



Another r.v. defined on the same experiment

E.g. Y = payoff in spinner game

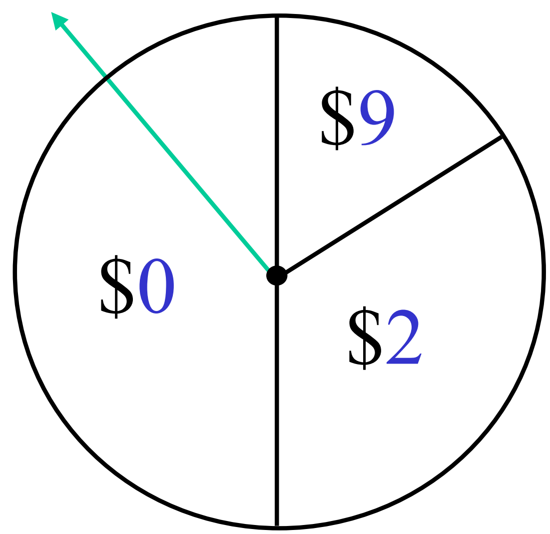


Distribution of Y:

$$Y = \begin{cases} 0 & \text{with prob } 0.5 \\ 2 & \text{with prob } 0.3 \\ 9 & \text{with prob } 0.2 \end{cases}$$

Expected Value (mean) of random variable

E.g. Y = payoff in spinner game



Distribution of Y:

$$Y = \begin{cases} 0 & \text{with prob } 0.5 \\ 2 & \text{with prob } 0.3 \\ 9 & \text{with prob } 0.2 \end{cases}$$

notation: expected value (mean) of $Y = E(Y) = \mu$

$E(Y) = 0(0.5) + 2(0.3) + 9(0.2) = 2.4$ dollars

Why is "mean" defined this way?

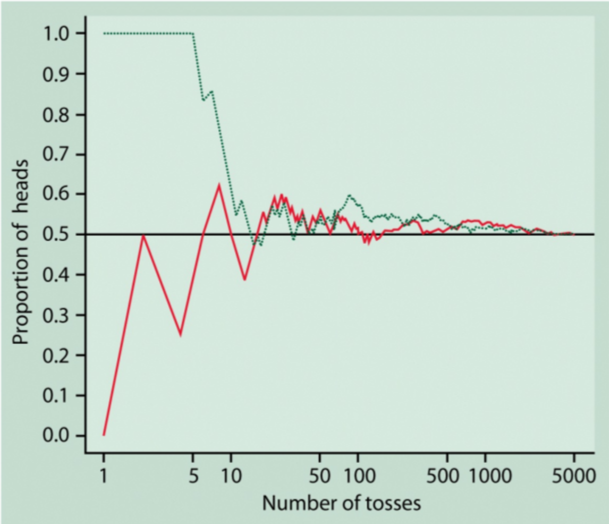
"long run frequency" interpretation of probability:

Suppose P(an event) = p (e.g. 0.5). Define

$$F_n = \frac{\#\text{ times this event occurs in } n \text{ independent trials}}{n}$$

As n increases, this fraction will approach p :

$$F_n \to p$$



Why is “mean” defined this way?

Imagine playing the game n times (n large).

Total payoff:

$$x_1 + x_2 + \cdots + x_n = 0(\#\text{ of }0\text{'s}) + 2(\#\text{ of }2\text{'s}) + 9(\#\text{ of }9\text{'s})$$

Average payoff:

$$\bar{x}_n = 0\underbrace{\left(\frac{\#\text{ of }0\text{'s}}{n}\right)}_{\downarrow\ 0.5} + 2\underbrace{\left(\frac{\#\text{ of }2\text{'s}}{n}\right)}_{\downarrow\ 0.3} + 9\underbrace{\left(\frac{\#\text{ of }9\text{'s}}{n}\right)}_{\downarrow\ 0.2}$$

i.e. $\bar{x}_n \to 0(0.5) + 2(0.3) + 9(0.2) = E(X)$

Law of large numbers

As we do many independent repetitions of the experiment, drawing more and more numbers from the same distribution, the mean of our sample will approach the mean of the distribution more and more closely:

$$\bar{X}_n \to E(X) \qquad \text{as } n \text{ increases}$$

General formula for discrete r.v.'s

If X has distribution

value of X	v_1	v_2	…	v_k
probability	p_1	p_2	…	p_k

• Then $E(X) = \mu = v_1 p_1 + \ldots + v_k p_k$

Example

- Suppose a day trader buys one share of IBM
- Let X represent the change in price of IBM
- She pays \$100 today, and the price tomorrow can be either \$105, \$100, or \$95

Stock Price	Change x	Probability $P(X = x)$
Increases	\$5	0.11
Stays same	0	0.80
Decreases	−\$5	0.09

Question

Suppose you buy one share of IBM today ($100). How much are you expected to earn tomorrow?

A. -5
B. 0
C. 0.1
D. 1

Mean of X

Stock Price	Change x	Probability $P(X = x)$
Increases	\$5	0.11
Stays same	0	0.80
Decreases	$-\$5$	0.09

$$\begin{aligned}\mu &= -5\,p(-5) + 0\,p(0) + 5\,p(5) \\ &= -5(0.09) + 0(0.80) + 5(0.11) \\ &= \$.10\end{aligned}$$

SD and Variance of random variable

Notation: $SD(X) = \sigma$ $Var(X) = \sigma^2$

Definition: $Var(X) = E\ (X - \mu)^2$

Expected value and SD: properties

Adding or Subtracting a Constant (c)

- Changes the expected value by a fixed amount:

$$E(X \pm c) = E(X) \pm c$$

- Does not change the standard deviation (SD):

$$SD(X \pm c) = SD(X)$$

Expected value and SD: properties

Multiplying by a Constant (c)

- $$E(cX) = c\, E(X)$$
- $$SD(cX) = |c|\, SD(X)$$

Question?

McDonald’s has a monthly return with mean 0.53% and SD 7.6%, and Disney has a monthly return with mean 0.61% and SD 8.3%. As an investor, how would you choose between McDonald's and Disney stocks?

A. McDonald’s
B. Disney
C. Neither as both are risky

The Sharpe Ratio

- Popular in finance for comparing investments: the higher the Sharpe ratio, the better the investment
- Is the ratio of an investment's net expected gain to its standard deviation

$$Sharpe(X) = \frac{\mu - r_f}{\sigma}$$

- μ and σ are the mean and SD of the return on the investment
- r_f stands for the return on a risk-free investment (e.g. interest rate on a savings account)

Example

- Summary of monthly returns in 2000-2006:

Company	Random Variable	Mean	SD
Disney	D	0.61%	8.3%
McDonald's	M	0.53%	7.6%

- Suppose the risk free rate is 0.4% per month
- Sharpe(D) = (0.61-0.4)/8.3 = 0.0253
- Sharpe(M) = (0.53-0.4)/7.6 = 0.0171
- Disney is preferred to McDonald's

Density Curves

- Density curves: A curve that
  - 1) lies above the x-axis,
  - 2) has total area 1 under the curve (and above the x-axis).
- Using density curves to describe continuous probability distributions (think about normal distribution)