Session 13: Introduction to Linear Programming (Solutions Only)

Warm Up: Optimal Team Work

Bob and Alice work together as a husband and wife team to make specialized crafts which they sell on Etsy.com. After months of experimentation, they figured out that there are two kinds of products they can make which have high demands. Alice can make an ornamental bracelet by herself in one hour, which makes a profit of 100 dollars per unit. They can also team up to make a wooden toy house, which sells for 300 dollars per unit, but each requires two hours of work by Alice and three hours of work by Bob. Suppose that they can each work six hours per day. What is the maximum profit per day they can make together as a family and how would they achieve it?

a) If X is the number of units of the ornamental bracelet the family makes per day and Y the number of units of the wooden toy house, what combinations of (X, Y) are possible? Enumerate as many as you can by hand.

b) Plot all the feasible (X, Y) combinations in a two-dimensional graph (with X being on the horizontal axis and Y on the vertical axis). What is the optimal combination and the associated profit?

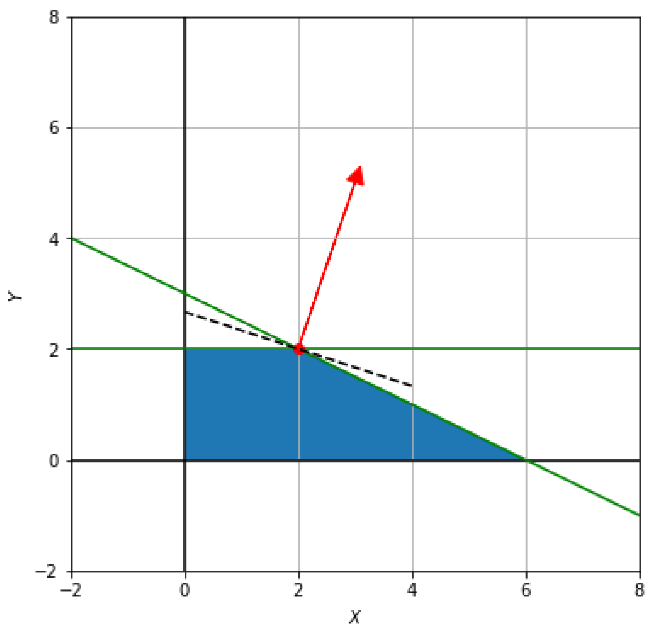
c) How would the answer change if each unit of the ornamental bracelet makes a profit of 200 dollars (instead of 100)?

d) How would the answer change if Alice works an additional hour per day? (Profit of ornamental bracelet is still 100 dollars.)

This problem can be modeled using the following linear program (LP):

$$
\begin{array}{lr}
\text{Maximize:} & 100X + 300Y \\
\text{subject to:} & \\
\text{(Alice)} & X + 2Y \leq 6 \\
\text{(Bob)} & 3Y \leq 6 \\
\text{(Non-negativity)} & X, Y \geq 0
\end{array}
$$

The following plot illustrates the geometry of the above LP.

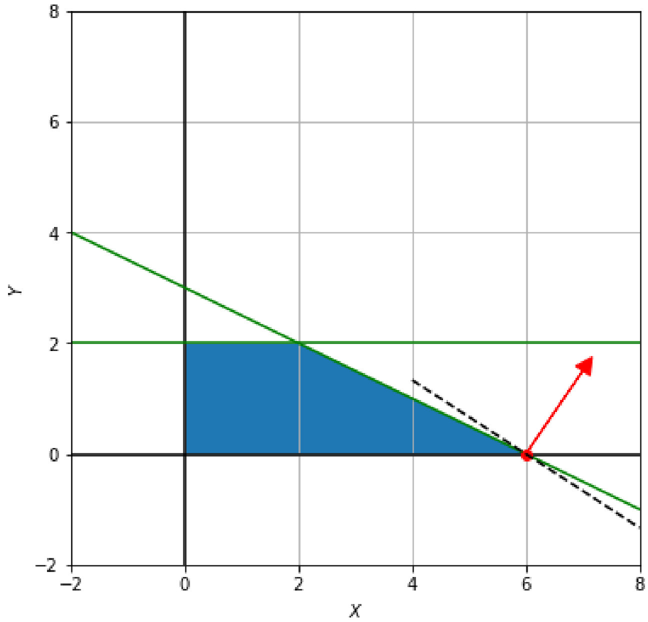


As can be seen, the optimal combination is $(X, Y) = (2, 2)$.

For part c), the LP objective becomes

$$\text{Maximize:} \qquad 200X + 300Y$$

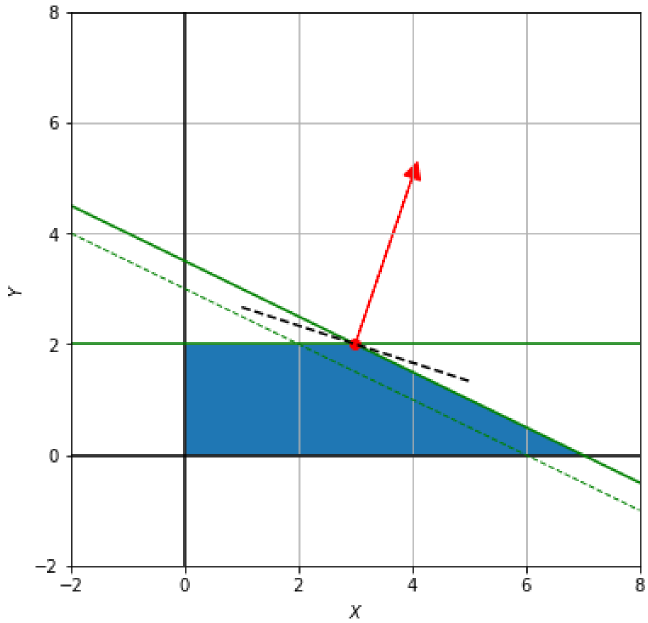
The optimal combination is $(X, Y) = (6, 0)$ and the plot becomes



For part d), the first constraint becomes

$$\text{(Alice)} \qquad X + 2Y \leq 7$$

The updated plot is below:



Hence, the family can make an additional 100 dollars per day if Alice works an additional hour. So the value of this additional hour of work is 100 dollars.

Shadow price of a constraint: The change in the optimal objective value of a LP if the right hand side (RHS) of this constraint increases by 1.

Q1. Graphical Analysis of Linear Program

A small factory can make two products, X and Y. The following table summarizes the required inputs to produce each product and the profit of each.

	Product X	Product Y
Steel	4 kg	1 kg
Plastic	0 kg	2 kg
Labor	1 hour	1 hour

Suppose that each unit of X makes a profit of 100 dollars and each unit of Y a profit of 200 dollars. Moreover, the daily supply of steel is 60kg, of plastic is 48 kg and of labor is 30 hours. What is the optimal production plan and the value of one additional unit of each resource?

Solution to Q1.

Decision Variables:

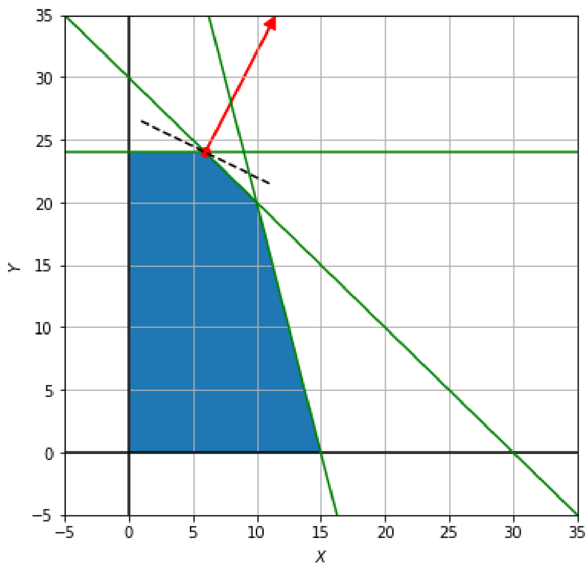
- X: the amount of product X to produce per day.
- Y: the amount of product Y to produce per day.

Objective:

$$\text{Maximize:} \qquad 100X + 200Y$$

Constraints:

$$\begin{aligned} \text{(Steel)} \quad & 4X + Y \leq 60 \\ \text{(Plastic)} \quad & 2Y \leq 48 \\ \text{(Labor)} \quad & X + Y \leq 30 \\ \text{(Non-negativity)} \quad & X, Y \geq 0 \end{aligned}$$



The optimal solution is $(X, Y) = (6, 24)$, with profit $(100)(6) + (200)(24) = 5400$.

For part b), note that an additional unit of steel doesn't help us because it is not a bottleneck.

Given an additional unit of plastic, the optimal solution shifts to $(X, Y) = (5.5, 24.5)$, which has profit $(100)(5.5) + (200)(24.5) = 5450$, which is 50 higher than before.

Given an additional unit of labor, the optimal solution shifts to $(X, Y) = (7, 24)$, which results in a profit gain of $5500 - 5400 = 100$.

Q2. Solve Q1 again using Python and Gurobi

```
[6]: from gurobipy import Model, GRB
     mod=Model()
     X=mod.addVar(lb=0)
     Y=mod.addVar(lb=0)
     mod.setObjective(100*X+200*Y,sense=GRB.MAXIMIZE)
     steel=mod.addConstr(4*X+Y<=60)
     plastic=mod.addConstr(2*Y<=48)
     labor=mod.addConstr(X+Y<=30)
     mod.setParam('OutputFlag',False)
     mod.optimize()
     print('Optimal objective: ',mod.objVal)
     print(f'Optimal plan: X={X.x} Y={Y.x}')
     print(f'Value of additional unit of Steel: {steel.pi}\
      (valid right hand side (RHS): {steel.sarhslow} to {steel.sarhsup})')
     print(f'Value of additional unit of Plastic: {plastic.pi} \
      (valid RHS: {plastic.sarhslow} to {plastic.sarhsup})')
     print(f'Value of additional unit of Labor: {labor.pi} \
      (valid RHS: {labor.sarhslow} to {labor.sarhsup})')
```

```
Optimal objective:  5400.0
Optimal plan: X=6.0 Y=24.0
Value of additional unit of Steel: 0.0 (valid right hand side (RHS): 48.0 to inf)
Value of additional unit of Plastic: 50.0  (valid RHS: 40.0 to 60.0)
Value of additional unit of Labor: 100.0  (valid RHS: 24.0 to 33.0)
```

Q3. Additional Practice

(DMD Ex. 7.2) The Gemstone Tool Company (GTC) produces wrenches and pliers. Each product is made of steel, and requires using a Molding Machine and an Assembly Machine. The daily availability of each resource, as well as the resources required to produce one units of each tool, are shown below.

	Wrench (1 unit)	Plier (1 unit)	Daily Availability
Steel	1.5 lbs	1.0 lbs	27,000 lbs
Molding Machine	1.0 hours	1.0 hours	21,000 hours
Assembly Machine	0.3 hours	0.5 hours	9,000 hours

There is demand for 16,000 wrenches and 15,000 pliers per day. Each wrench earns a profit of .10 dollars and each plier earns a profit of .13 dollars.

a) How much of each product should GTC produce each day and what is the maximum possible profit?

b) How much additional profit can the company obtain if it had one additional unit of each of the three resources?

Solutions to Q3.

The LP is as follows.

Decision Variables:

- W: the number of wrenches to produce.
- P: the number of pliers to produce.

Objective:

$$\text{Maximize:} \quad .1W + .13P$$

Constraints:

$$\begin{aligned}
\text{(Steel)} \quad & 1.5W + P \leq 27000 \\
\text{(Molding)} \quad & W + P \leq 21000 \\
\text{(Assembly)} \quad & .3W + .5P \leq 9000 \\
\text{(Demand W)} \quad & W \leq 16000 \\
\text{(Demand P)} \quad & P \leq 15000 \\
\text{(Non-negativity)} \quad & W, P \geq 0
\end{aligned}$$

[7]:
```python
import gurobipy as grb
mod=grb.Model()
W=mod.addVar()
P=mod.addVar()
mod.setObjective(.1*W+.13*P,sense=grb.GRB.MAXIMIZE)
steel=mod.addConstr(1.5*W+P <= 27000)
molding=mod.addConstr(W+P <=21000)
assembly=mod.addConstr(.3*W+.5*P<=9000)
mod.addConstr(W<=16000)
mod.addConstr(P<=15000)
mod.optimize()
```

```
Gurobi Optimizer version 9.0.1 build v9.0.1rc0 (linux64)
Optimize a model with 5 rows, 2 columns and 8 nonzeros
Model fingerprint: 0xe8f414a1
Coefficient statistics:
  Matrix range     [3e-01, 2e+00]
  Objective range  [1e-01, 1e-01]
  Bounds range     [0e+00, 0e+00]
  RHS range        [9e+03, 3e+04]
Presolve removed 2 rows and 0 columns
Presolve time: 0.01s
Presolved: 3 rows, 2 columns, 6 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    3.5100000e+03   3.375000e+03   0.000000e+00      0s
       3    2.5050000e+03   0.000000e+00   0.000000e+00      0s

Solved in 3 iterations and 0.02 seconds
Optimal objective  2.505000000e+03
```

[8]:
```
print('Optimal profit:',mod.objval)
print('W:',W.x)
print('P:',P.x)
print('\nValue of additional unit of each resource:')
print(f'Steel {steel.pi} \t valid right hand side (RHS): {steel.sarhslow} to {steel.sarhsup}')
print(f'Molding {molding.pi:.3f} \t valid RHS: {molding.sarhslow} to {molding.sarhsup}')
print(f'Assembly {assembly.pi:.3f} \t valid RHS: {assembly.sarhslow} to {assembly.sarhsup}')
```

```
Optimal profit: 2505.0
W: 7500.0
P: 13500.0

Value of additional unit of each resource:
Steel 0.0          valid right hand side (RHS): 24750.0 to inf
Molding 0.055          valid RHS: 20000.0 to 22000.0
Assembly 0.150          valid RHS: 8100.0 to 9300.0
```